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# Lightcone Averaging and Precision Cosmology

Fabien Nugier

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THÈSE DE DOCTORAT  
DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

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(École Doctorale 107)

Présentée par

FABIEN NUGIER

Pour obtenir le grade de

Docteur de l'Université Pierre et Marie Curie

Sujet de thèse

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MOYENNES SUR LE CÔNE DE LUMIÈRE  
ET COSMOLOGIE DE PRÉCISION

LIGHTCONE AVERAGING AND PRECISION COSMOLOGY

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Soutenue le

4 SEPTEMBRE 2013

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# MOYENNES SUR LE CÔNE DE LUMIÈRE ET COSMOLOGIE DE PRÉCISION

LIGHTCONE AVERAGING AND PRECISION COSMOLOGY

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FABIEN NUGIER

This thesis was written in  $\text{\LaTeX}$ .  
A donation for trees plantation was done to compensate for my PhD printings.  
May these trees grow tall and old.

# Contents

	Page
<b>Notations</b>	<b>1</b>
<b>1 Introduction, motivations and outline of this thesis</b>	<b>3</b>
1.1 Cosmology today . . . . .	3
1.2 Homogeneous cosmology . . . . .	4
1.3 Type Ia supernovae . . . . .	8
1.4 The large scale structure . . . . .	15
1.5 Dealing with the inhomogeneous Universe . . . . .	18
1.6 Introduction & Outline of this thesis . . . . .	20
<b>2 Gauge-invariant spatial &amp; lightcone averaging</b>	<b>25</b>
2.1 Covariant spatial averaging . . . . .	25
2.2 Light-cone averaging . . . . .	35
2.3 Buchert-Ehlers commutation rules on the light-cone . . . . .	39
2.4 The FLRW Universe . . . . .	41
<b>3 The GLC metric and its properties</b>	<b>43</b>
3.1 Observational coordinates . . . . .	43
3.2 The geodesic light-cone gauge . . . . .	44
3.3 Redshift in the GLC metric . . . . .	47
3.4 Luminosity distance in the GLC gauge . . . . .	48
3.5 Relations of the GLC gauge with other metrics . . . . .	50
3.6 Applications to the lightcone average . . . . .	54
<b>4 The luminosity-redshift relation at second order</b>	<b>59</b>
4.1 From Poisson to GLC gauge at second order . . . . .	59
4.2 Detailed expression for $d_L(z, \theta^a)$ . . . . .	63
<b>5 Combining light-cone and stochastic averaging</b>	<b>71</b>
5.1 Recalling the first order equations . . . . .	71
5.2 Combining lightcone and ensemble averages . . . . .	74
5.3 Averaging the luminosity flux at second order . . . . .	83
<b>6 Effect of inhomogeneities on cosmological observables</b>	<b>95</b>
6.1 Induced backreaction in CDM and IR & UV convergence . . . . .	95
6.2 Full calculation for the CDM model . . . . .	99
6.3 The $\Lambda$ CDM model: power spectrum in the linear regime . . . . .	103
6.4 The $\Lambda$ CDM model: power spectrum in the non-linear regime . . . . .	108
<b>7 Summary and conclusions</b>	<b>115</b>
<b>Appendices</b>	

<b>Appendix A Complementary topics</b>	<b>121</b>
A.1 ADM formalism (Arnowitt-Deser-Misner) . . . . .	121
A.2 Linear theory of structure formation . . . . .	126
A.3 Gauge invariance and perturbation theory . . . . .	132
A.4 The power spectrum in the linear regime . . . . .	144
A.5 The HaloFit model . . . . .	146
A.6 Distance modulus for a Milne model . . . . .	153
<b>Appendix B Résumé (en français)</b>	<b>155</b>
B.1 Introduction, motivations et plan de la thèse . . . . .	155
B.2 Moyennes sur le cône de lumière . . . . .	161
B.3 Jauge géodésique sur le cône de lumière (GLC) . . . . .	165
B.4 La distance de luminosité au second ordre en perturbations . . . . .	168
B.5 Moyenne stochastique sur le cône de lumière . . . . .	170
B.6 Résultats pour le spectre non-linéaire et un Univers $\Lambda$ CDM . . . . .	178
B.7 Résumé et Conclusions . . . . .	188
<b>PhD publications</b>	<b>197</b>
<b>Bibliography</b>	<b>199</b>
<b>Glossary</b>	<b>206</b>

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*Ceux qui vivent, ce sont ceux qui luttent.*  
Victor Hugo



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# Notations

For the comfort of the reader, we summarize here the different conventions used in this thesis.

We will adopt the convention of Weinberg [1,2] and Misner, Thorne & Wheeler [3] for the signature of the 4-dimensional metric, namely we will take it to be  $(-, +, +, +)$ . As a consequence, the line element defined in the 4-dimensional Minkowski spacetime (sometimes denoted by  $\mathcal{M}_4$ ), will be  $ds^2 = -(cdt)^2 + \delta_{ij}dx^i dx^j$  and we will take the usual convention  $c = 1$ . Latin indices will denote the spatial values 1, 2, 3 and greek indices will correspond to spacetime indices 0, 1, 2, 3 (where 0 will denote the time component). The flat metric will be written as  $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  and we will adopt the usual convention that repeated indices are summed over. For example,  $x_\mu x^\mu$  will denote  $\eta_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$  in  $\mathcal{M}_4$  and  $g_{\mu\nu} x^\mu x^\nu$  in a more general spacetime. The angles over an homogeneous sphere will be denoted by  $\theta^a$  (with  $a = 1, 2$ ) or  $(\theta, \phi)$ .

For 3-dimensional spatial vectors, we will use bold letters instead of putting arrows over them. As an example, we will find  $\mathbf{x}$  instead of  $\vec{x}$  (except for the null vector that we will denote by  $\vec{0}$  and the usual nabla operator  $\vec{\nabla} \equiv \partial_i$ ). The scalar products of two such vectors will be denoted by a dot:  $\mathbf{x} \cdot \mathbf{k} = x^1 k^1 + x^2 k^2 + x^3 k^3$ . Bold letters will also be used to denote tensorial quantities in their general form. The variable dependence ( $x$ ) will designate  $(t, x^i) \equiv (t, \mathbf{x})$ .

The covariant derivative with respect to a coordinate  $x^\mu$  in a 4-dimensional spacetime will be denoted by  $\nabla_\mu$  and sometimes denoted by a semi-colon, e.g.  $\nabla_\mu Q \equiv Q_{;\mu}$ . We will denote by  $\partial_\mu Q \equiv \partial Q / \partial x^\mu \equiv Q_{,\mu}$  the partial  $x^\mu$ -derivative of a quantity  $Q$ . We will also encounter the covariant derivation of a quantity with respect to the 3-dimensional spatial metric in a 4-dimensional spacetime. We will use a vertical stroke for a simple space-derivative:  $\partial_i Q \equiv Q_{|i}$ ; and a double stroke for the covariant derivative:  ${}^{(3)}\nabla_i Q \equiv Q_{||i}$ . The simple time derivation with respect to  $t$ , the cosmic time or the synchronous gauge time, will often be denoted by a dot:  $\partial_t(\dots) \equiv (\dots)'$ . The partial derivative with respect to the conformal time  $\eta$  will be denoted sometimes by a prime:  $\partial_\eta(\dots) \equiv (\dots)'$ .

As it will be defined later, we will use an overline to denote the stochastic average over inhomogeneities:  $\overline{(\dots)}$ . The spacetime averages, in their different forms, will be denoted by angle brackets:  $\langle \dots \rangle$  (where the literature, e.g. [4], usually employs that symbol for the stochastic average).

Let us remark that several quantities are described by the same symbol. This is the case for  $\phi$  which describes the Bardeen potential ( $\Phi$ ,  $\phi^{(2)}$ ), the luminosity flux ( $\Phi$  and  $\Phi^{(0),(1),(2)}$ ), the second angle on the sphere ( $\phi$ ,  $\tilde{\phi}$ ), the inflaton field ( $\phi$ ) and the morphon field ( $\Phi_{\mathcal{D}}$ ). Note also that  $\Theta$  is used for the expansion scalar and for the Heaviside step function. Symbols  $(\theta, \theta)$  are the first angle on the sphere.  $\delta$  is used for the matter density contrast and  $\delta_{\mathcal{D}}$  for the Dirac delta function. No confusion can be made between these different notations and it will be precised only when ambiguous.

Last, we will use sometimes shortcuts of the English language, using ‘‘LHS’’ and ‘‘RHS’’ for *left* and *right hand side* or ‘‘w.r.t.’’ for *with respect to*, or from Latin (like i.e., e.g.). Sometimes also acronyms will be used to lighten the discussions, most of them are summarized in the glossary of this thesis.



# Chapter 1

## Introduction, motivations and outline of this thesis

### 1.1 Cosmology today

Cosmology is by far one of the oldest fields of research and at the same time, maybe one of the youngest. Indeed, human beings have always observed the sky to measure time and seasons and wondered about its composition. Nevertheless, most of the breakthroughs in the field have been done thanks to new ways to observe the sky, and only recently we have been able to gather enough knowledge to build up what we now call the [standard model of cosmology \(SMC\)](#).

We know from the cosmic microwave background (CMB) that the Universe is almost flat today and that it was created about 13.7 *Gyrs* ago. The precise mechanism behind its creation is not known, but we do know that its temperature was very high and its size very small at this time. We also know that a process of rapid expansion, the so-called inflation mechanism, must have occurred to make our Universe so flat and so close to homogeneity on large angular scales. This period was then followed by a phase where photons were tightly coupled to baryonic matter (the *radiation dominated era*) and no matter structure formed. But after some time, expansion cooling down the Universe, neutrons and protons decoupled from light and started to form atoms, and progressively gave birth to stars, heavier elements, galaxies and guys who live in. This is called the *matter dominated era*. We know indeed that matter is nowadays organized in galaxies, regrouped in clusters and superclusters (which size<sup>1</sup> is 2 – 10 Mpc) and these last entities are linked by filaments made of galaxies. All these structures are separated by voids which have a size of about 10 – 80 Mpc and we observe a statistical homogeneity above a scale of about 100 – 150 Mpc (the value of this scale being still debated). One amazing fact about this structure that we observe around us, the [large scale structure](#), is that it has evolved from tiny quantum fluctuations in the early stages of the Universe (when it has a size of the order of the Planck length  $\sim 10^{-33}$ cm), which have been stretched out by inflation.

Unfortunately, or fortunately for scientists, this standard model of cosmology still has some dark clouds making us unsatisfied. The biggest clouds in this picture are the true nature of dark matter and dark energy. These two parts of the energy content of the Universe are dark as we cannot observe them directly but only through indirect ways. The first one, [dark matter \(DM\)](#), was discovered by measurements of the velocity of stars in rotating galaxies. Scientists discovered indeed that stars had a much higher velocity than they should have from the measurement of the mass of their galaxy by light emission. It was thus necessary to add about five times more matter than we could infer from light observations to explain these profiles. Dark matter was also needed actually to explain the rapid formation of structures in simulations compared to the structures we observed, and lately became also needed in the measurements of dark energy. Though we still do not know the nature of dark matter, a lot of scientists believe that we may soon enough discover one or more particles (the so-called weakly interacting massive particles, or WIMPS) to explain it.

---

1. We recall that 1pc  $\sim 3.26$  *ly* (light years) with 1 *ly* = 0.9461  $10^{16}$ m.

On the other hand, **dark energy (DE)** is a complete mystery. It was first discovered experimentally in 1998 by the observation of Type Ia supernovae (SNe Ia) as the fact that the expansion of the Universe seemed to be accelerating. In its simplest version, this exotic type of energy which has a tendency of repulsion under gravity could be simply a term proportional to the metric authorized by Einstein's equations, but we don't understand why its value is so small and comparable to the energy amount of dark+baryonic matter (this is the so-called **coincidence problem**). This type of energy could also be a new particle following a certain potential, a bit like the particle believed to create inflation (the *inflaton*). This scalar field is certainly possible and there has been a very great number of such solutions proposed in the literature (see e.g. the review of [5]). We never directly observed a scalar particle yet, but the discovery at the Large Hadron Collider of a good candidate for the Higgs particle is a strong indication that they may indeed be present in nature. The new game to explain dark energy, when we assume the existence of such a particle, is to lead the expansion of our Universe as it is observed today. We can search for a good potential to explain the evolution of the field (such as in quintessence) or try to couple the field with others, such as matter itself (like with the chameleon field). Direct modifications of the theory of general relativity are another alternative which, in some cases, can be related to these particle models. A lot of involved computations has been done in this direction and especially to learn how we can accommodate these extensions with data.

Theoretical physicists have thus developed a lot of different proposals to explain this wonderful 15-years old discovery of dark energy, but observations have not yet reached enough precision to test these different models and distinguish among them. In science, having a good model to describe a phenomenon is always a good thing. Having two good models is always bad, because it means that we still do not understand something. And when we have tenths of relatively close models to explain this phenomenon, it is natural to start looking for completely new ideas. It is also what happened to dark energy for several reasons. One thing that we should know about dark energy is that it made the expansion of the Universe to accelerate only a few billion years ago ( $\sim 5 \text{ Gyrs}$ ), and this is close to the time when the large scale structure has formed. Some researchers thus came to the idea that the formation of structure, in other words the content of the Universe, could be responsible for this late time evolution, that is to say with this particular dynamics. In fact this isn't a stupid guess at all, Einstein himself told us that the geometry was no different from its content, that energy influence spacetime. So why matter could not "mimic" the role of a dark energy? This idea was even made more exciting from the emphasize by George Ellis, in the 80's, that looking at an inhomogeneous spacetime and its smoothed out equivalent doesn't lead to the same evolution prediction. Another branch of explanations of dark energy has thus been developed, and it has been regrouped in a maybe not so well adapted term of "backreaction", with the idea behind that word being that small inhomogeneities may backreact on spacetime and create a real or an apparent acceleration of the expansion.

## 1.2 Homogeneous cosmology

### 1.2.1 Cosmological principles

Cosmology is a particular branch of physics where the experiment – our whole observable Universe itself – is not reproducible. It thus imposes to make reasonable but philosophical assumptions to interpret our observations and get reliable predictions. Let us present some of these assumptions.

The usual assumption made in cosmology, and more generally in physics, is the **Copernican principle**. This principle states that the place of experiment, e.g. the solar system, is in no privileged position in the Universe. In the elaboration of cosmological models, one can derive another principle from the Copernican one, which is the **cosmological principle**. In [6], this hypothesis on our position in the Universe, and consequently on the data that we gather, is declined in two versions. The *strong cosmological principle* assumes that the Universe is homogeneous and isotropic at all scales. It is the combination between the Copernican principle and the isotropy assumption which leads to the global homogeneity. This principle is *stricto sensu* not satisfied in Nature and we have to assume the more realistic version of it, the *weak cosmological principle*, which says that we live in a Universe having an homogeneity scale and which is almost isotropic on distances above that scale. In any case, we

should remark that the cosmological principle is more strict than the Copernican one as it implies also assumptions on the non-observable parts of our Universe.

We should notice that, though the Copernican principle has proved to be very useful for all physics experiments so far, it may not be true on cosmological scales. Indeed, one cannot contradict the fact that we are, up to a certain degree, *privileged observers*. What we mean by this degree is that human beings couldn't have been born outside a galaxy, and so in parts of the Universe where the density contrast cannot be random. Moreover, the observations that we have at our disposal to understand our past are all so far light signals on our past lightcone (a 2+1 spacetime hypersurface) and few geological observations giving information on our past timelike world line. These facts must be kept in mind when building up realistic cosmological models and they are important for the discussions of inhomogeneous cosmologies.

### 1.2.2 Evolution equations

Let us present the general approach of the standard model of cosmology, based on the cosmological principle. The most general form of the line element of a free falling observer that observes an isotropic and homogeneous universe around him is given by the [Friedmann-Lemaître-Robertson-Walker \(FLRW\)](#) line element, which in cosmic time  $t$  and spherical coordinates  $(\mathbf{x}) = (\chi, \theta, \phi)$  reads :

$$\begin{aligned} ds_{\text{FLRW}}^2 &= -dt^2 + a^2(t)\gamma_{ij}(|\mathbf{x}|)dx^i dx^j \\ &= -dt^2 + a^2(t) \left[ d\chi^2 + \left( \frac{1}{\sqrt{K}} \sin(\sqrt{K}\chi) \right)^2 d\Omega^2 \right] , \\ \text{with } d\Omega^2 &= d\theta^2 + (\sin\theta)^2 d\phi^2 , \end{aligned} \quad (1.1)$$

and where  $K$  is a constant setting the geometry of spacelike hypersurfaces:  $K \in \{-1, 0, 1\}$  in the cases of a { open, flat, closed } Universe<sup>2</sup>. We call  $a(t)$  the scale factor in cosmic time and  $\gamma_{ij}$  is the 3-dimensional metric describing the Universe on a given spatial hypersurface. In this metric,  $a(t)$  has the dimension of a distance. We can also use the so-called conformal time  $\eta$  such that  $d\eta = dt/a(t)$ . Defining now a new radial coordinate  $r$  such that

$$r = \sin(\sqrt{K}\chi) / \sqrt{K} , \quad (1.2)$$

one can show that the FLRW line element is simply written as

$$ds_{\text{FLRW}}^2 = -dt^2 + a(t)^2 \left( \frac{1}{1 - Kr^2} dr^2 + r^2 d\Omega^2 \right) , \quad (1.3)$$

where  $a(t)$  can be considered as a dimensionless quantity if  $r$  is now expressed in units of a distance (in which case  $K$  has the dimension of  $r^{-2}$ ).

Considering the form of the metric presented here, we can compute the [Christoffel symbols](#) which are at the basis of the connection in general relativity and that have for general definition :

$$\Gamma_{\sigma\mu\nu} = \frac{1}{2}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \quad \text{and} \quad \Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\sigma\mu\nu} . \quad (1.4)$$

These symbols are easily computed for this FLRW metric in spherical form and read

$$\Gamma^0_{ij} = Ha^2\gamma_{ij} , \quad \Gamma^i_{j0} = H\delta_j^i , \quad \Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk} , \quad (1.5)$$

2. Indeed, the notation conventionally used in the literature is

$$\gamma_{ij}(|\mathbf{x}|)dx^i dx^j = d\chi^2 + f_K^2(\chi)d\Omega^2 \quad \text{with} \quad f_K(\chi) \equiv \begin{cases} \sin(\sqrt{K}\chi) / \sqrt{K} & \text{for } K = 1 \\ \chi & \text{for } K = 0 \\ \sinh(\sqrt{-K}\chi) / \sqrt{-K} & \text{for } K = -1 \end{cases} ,$$

but  $f_K(\chi)$  is correctly captured by the unique equality  $f_K(\chi) = \sin(\sqrt{K}\chi) / \sqrt{K}$  if we authorize the square root to receive negative arguments, given that  $\sqrt{-K} = i\sqrt{K}$ , and by considering the complex expression of the sine function.

where we have defined the Hubble parameter  $H \equiv \dot{a}/a$  (with the dot denoting the derivative w.r.t. the cosmic time  $t$ ). The  ${}^{(3)}\Gamma^i{}_{jk}$  are the Christoffel symbols associated to the spatial metric  $\gamma_{ij}$ . They define the connection inside the spatial hypersurfaces parametrized by the spherical coordinates  $\{x^i\}$ .

The **Riemann tensor** is generally defined as:

$$R^\delta{}_{\alpha\beta\gamma} = \partial_\alpha \Gamma^\delta{}_{\beta\gamma} - \partial_\beta \Gamma^\delta{}_{\alpha\gamma} + \Gamma^\delta{}_{\alpha\epsilon} \Gamma^\epsilon{}_{\beta\gamma} - \Gamma^\delta{}_{\beta\epsilon} \Gamma^\epsilon{}_{\alpha\gamma} \quad (1.6)$$

and has the following properties<sup>3</sup>:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta} \quad (1.7)$$

We then obtain the **Ricci tensor** from the contraction of two indices in the Riemann tensor:

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} = \partial_\mu \Gamma^\lambda{}_{\nu\lambda} - \partial_\lambda \Gamma^\lambda{}_{\mu\nu} + \Gamma^\eta{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\eta} - \Gamma^\lambda{}_{\lambda\eta} \Gamma^\eta{}_{\mu\nu} \quad (1.8)$$

In the FLRW metric, this leads to the non-zero components

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = \left(2H^2 + \frac{\ddot{a}}{a} + 2\frac{K}{a^2}\right) a^2 \gamma_{ij} \quad (1.9)$$

The final quantity that we can derive is the 4-dimensional scalar curvature, which is

$$R \equiv R^\mu{}_\mu = 6 \left( H^2 + \frac{\ddot{a}}{a} + \frac{K}{a^2} \right) \quad (1.10)$$

In this geometry, we then get the **Einstein tensor**:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \quad (1.11)$$

with the following non-zero components

$$G_{00} = 3 \left( H^2 + \frac{K}{a^2} \right), \quad G_{ij} = - \left( H^2 + 2\frac{\ddot{a}}{a} + \frac{K}{a^2} \right) a^2 \gamma_{ij} \quad (1.12)$$

The assumption of isotropy and homogeneity of the Universe, as seen from an observer in geodesic motion, imposes the **stress energy tensor** to have the following components:

$$T^{00} = \rho(t), \quad T^{0i} = T_{0i} = 0, \quad T^{ij} = p(t) a^{-2}(t) \delta_{ij}, \quad (1.13)$$

where  $\rho(t)$  and  $p(t)$  are respectively the energy density and pressure of the background fluid of matter. Expressing this tensor in terms of the velocity of an observer  $u^\mu$  which can be taken as  $u^\mu = (1, \vec{0})$  in a local comoving frame, we obtain:

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}, \quad (1.14)$$

and this is the general expression of the stress energy tensor of a perfect fluid. Finally, the Einstein equations in the presence of a cosmological constant (see e.g. [7, 8]), given by:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (1.15)$$

3. Other properties of the Riemann tensor are given by the symmetries of its indices and the Bianchi identities:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= R_{[\alpha\beta]\gamma\delta} \text{ and } R_{\alpha\beta\gamma\delta} = R_{\alpha\beta[\gamma\delta]}, \\ 3R_{\alpha[\beta\gamma\delta]} &= R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \quad (1^{st} \text{ Bianchi Id.}), \\ 3R_{\alpha\beta[\mu\nu;\tau]} &= R_{\alpha\beta\mu\nu;\tau} + R_{\alpha\beta\nu\tau;\mu} + R_{\alpha\beta\tau\mu;\nu} = 0 \quad (2^{nd} \text{ Bianchi Id.}), \end{aligned}$$

with the definition of symmetrized and anti-symmetrized tensors:

$$T_{[\mu_1\mu_2\dots\mu_n]} = \frac{1}{n!} \epsilon_{\mu_1\mu_2\dots\mu_n} \sum_{\text{permutations}} T_{\mu_1\mu_2\dots\mu_n}, \quad T_{(\mu_1\mu_2\dots\mu_n)} = \frac{1}{n!} \sum_{\text{permutations}} T_{\mu_1\mu_2\dots\mu_n}.$$

and where  $\epsilon_{\mu_1\mu_2\dots\mu_n} = \pm 1$  depending on the parity (non-parity) of permutations from  $(1, 2, \dots, n)$  to  $(\mu_1, \mu_2, \dots, \mu_n)$ .

take a very simple form given by the  $(0, 0)$  and  $(i, j)$  components, known as the Friedmann equations :

$$H^2(t) = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} , \quad (1.16)$$

$$\frac{\ddot{a}}{a}(t) = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} . \quad (1.17)$$

We should add, to these two equations above, the equation of conservation of the fluid given by  $\nabla_\mu T^{\mu\nu} = 0$ , which reads :

$$\dot{\rho} + 3H(\rho + p) = 0 . \quad (1.18)$$

We remark that only two out of these three equations are independent.

We can also define a density for the cosmological constant and a pressure in order to have the same dependence as matter on the RHS of these equations. We have

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} , \quad p_\Lambda = -\frac{\Lambda}{8\pi G} \quad \text{imposing that} \quad w_\Lambda \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1 . \quad (1.19)$$

It is a well-known result that the equation of state  $w$  for a cosmological constant is  $-1$  and that makes the cosmological constant the most simple explanation for the acceleration of the Universe expansion. Nevertheless, we don't have an explanation for the smallness of this constant, a problem referred to as the *cosmological constant problem* in cosmology<sup>4</sup>. It could be that the cosmological constant is smaller than expected, or even (almost) exactly zero, and that the true explanation for the acceleration of the Universe comes from a particle (in most of the models a scalar field). In that case the equation of state  $w_{DE}$  of dark energy can differ from  $-1$  and even evolve with time.

We can also recast the first Friedmann equation (1.16) by the introduction of the energy density parameters :

$$\Omega_m = \frac{8\pi G \rho}{3H^2} , \quad \Omega_\Lambda = \frac{\Lambda}{3H^2} , \quad \Omega_K = -\frac{K}{H^2 a^2} , \quad (1.20)$$

where here  $\rho$  accounts for both dark and baryonic matter. In these notations, this equation (1.16) is simply written as :

$$\Omega_m + \Omega_\Lambda + \Omega_K \equiv \Omega_{\text{tot}} + \Omega_K = 1 . \quad (1.21)$$

To this equation we could add a radiation contribution  $\Omega_r$  which has the same definition as  $\Omega_m$  (with  $\rho_r$  replacing  $\rho$ ). This contribution is important during the radiation dominated stage of the Universe but is subleading after decoupling. We thus neglect this contribution which is irrelevant for our work. Let us notice also that for a barotropic fluid with a constant *equation of state*  $w = p/\rho$ , the Eq. (1.18) is easily solved and gives the following dependence of the density parameters :

$$\Omega_m(z) = \Omega_{m0} \left( \frac{a}{a_0} \right)^{-3(1+w)} \left( \frac{H_0}{H(z)} \right)^2 . \quad (1.22)$$

For a dust (i.e pressureless) fluid for which  $w = 0$ , one sees that the density decays as  $\sim a^{-3}$ . For a cosmological constant, one has a similar relation giving  $\Omega_\Lambda \sim a^0$ .

In conformal time  $\eta$ , one can similarly show that the Friedmann equations are written as :

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2 - K + \frac{\Lambda}{3}a^2 , \quad (1.23)$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\rho + 3p)a^2 + \frac{\Lambda}{3}a^2 , \quad (1.24)$$

$$0 = \rho' + 3\mathcal{H}(\rho + p) , \quad (1.25)$$

where we have used the conformal Hubble parameter  $\mathcal{H} \equiv a(\eta)'/a(\eta) = aH$ . These equations, which we emphasize have only two independent equations, describe the evolution of the density  $\rho$  and pressure  $p$  of a fluid. In a  $\Lambda$ CDM model, one then has a specific expression for the Hubble parameter  $H$  (or  $\mathcal{H}$ ) and an explicit form for  $\rho$  and  $p$  in terms of the parameters of the model  $\Omega_{m0}$  and  $\Omega_{\Lambda 0}$ .

4. Indeed, the consideration of the energy of the vacuum in quantum field theory predicts a density of energy which is about 120 orders of magnitude greater than the observed density of dark energy (see [5]).

## 1.3 Type Ia supernovae

The idea of dark energy, in its minimal version of a cosmological constant, had been proposed by Albert Einstein in order to make the Universe static and was forgotten after the discovery of Hubble that the Universe was expanding [9]. The acceleration of the expansion of the Universe, or broadly speaking the existence of dark energy (DE), was not considered until 1998 when two different groups of experimentalists studying supernovae (SNe) arrived to the same conclusion that most of the Universe content was of an exotic nature. These groups were the High- $z$  Supernova Search Team [10] and the Supernova Cosmology Project [11] and they recently received the Nobel Prize in physics (in 2011).

### 1.3.1 Main features

First of all, one should think about supernovae as really rare events happening only a few times every thousand years per galaxy. They are phenomena emerging after the explosion of a star which is not completely understood. The best explanation we have is that they consist in the result of matter accretion by a white dwarf (a star made of a plasma of unbound nuclei and electrons) in a binary system (i.e. where a star is feeding up the white dwarf) and up to the point where it reaches a mass of  $1.4 - 1.5 M_{\odot}$ , the [Chandrasekhar mass](#)<sup>5</sup>. This second star can also be a white dwarf. A consequence of this nature of supernovae is seen in their chemical composition, as we will see in Fig. 1.2. What is important in this process is that supernovae are believed to have all roughly the same mass, and that makes them be *standard candles*. Indeed, above the Chandrasekhar mass, the pressure from gravitation is stronger than the electron degeneracy pressure inside the star and that leads to a collapse up to a certain radius fixed by the nuclear degeneracy pressure of the star. We then assist to a violent increase in temperature and the conversion of  $^{12}\text{C}$  and  $^{16}\text{O}$  into  $^{56}\text{Ni}$ , starting a thermonuclear explosion and the ejection of external cores of the star. The amount of energy emitted by this explosion (and though 99% of it is formed of neutrinos and 1% only by light) is so large that a supernova can be more brilliant than its host galaxy, reaching  $\sim 10^{10} L_{\odot}$ . As a consequence, supernovae can be seen from really far distances, making them useful for cosmology.

Another important point is that SNe must be observed at their peak in luminosity to be standardized and thus to be used. We hence need to detect them at an early stage of their explosive life. Indeed, we observe supernovae in different wavelength bands called U, V, B, R and I (other photometric systems can be used) and we follow their intensity, or their magnitude  $m_B$ , in the B-band as they are brighter in this band. We see that their intensity follows a growing time period during 15-20 days and is followed by a more or less slow decrease of  $\sim 2$  months (corresponding to the radioactive decay of  $^{56}\text{Ni}$  into  $^{56}\text{Co}$ , decaying itself into  $^{56}\text{Fe}$ ). This is what we call the *light curve* of a supernova and it has been noticed that the brighter the supernova, the longer the decreasing time. As a consequence, we have classified SNe in different types from their light curves, as can be seen in Fig. 1.1. The main two types are I and II, and we split them in subtypes. These different types are actually confirmed by spectroscopy, showing that supernovae also differ significantly in their composition and that is originating from the variety of physical phenomena giving rise to the explosion. The main feature of SNe Ia is that, like white dwarfs, they do not present any trace of hydrogen in their composition. Their second important feature is that they contain silicon (see Fig. 1.2).

### 1.3.2 Observations and standardization

Supernovae have been studied during many years now and their frequency of detection has significantly grown during the last decade with the advance of fast computing and automatic detection surveys (cf. the *rolling search* method of SNLS [13]). The automatic coverage of the sky (or part of it) has greatly enhanced the number of detected SNe. Spectra are also fundamental in the current use of SNe Ia and the reason for that is because they furnish the composition of the star, necessary to confirm its type. They also determine the redshift of the star (by a direct measurement of the

5. We define the solar mass  $M_{\odot}$  by  $1 M_{\odot} \equiv \frac{4\pi^2 \times (1 \text{ a.u.})^3}{G \times (1 \text{ year})^2} \simeq 1.98 \cdot 10^{30} \text{ kg}$  where *a.u.* denotes the astronomical unit (equal to the Sun-Earth distance). The solar luminosity is about  $L_{\odot} \simeq 3.84 \cdot 10^{26} \text{ W} = 3.84 \cdot 10^{33} \text{ erg s}^{-1}$ .

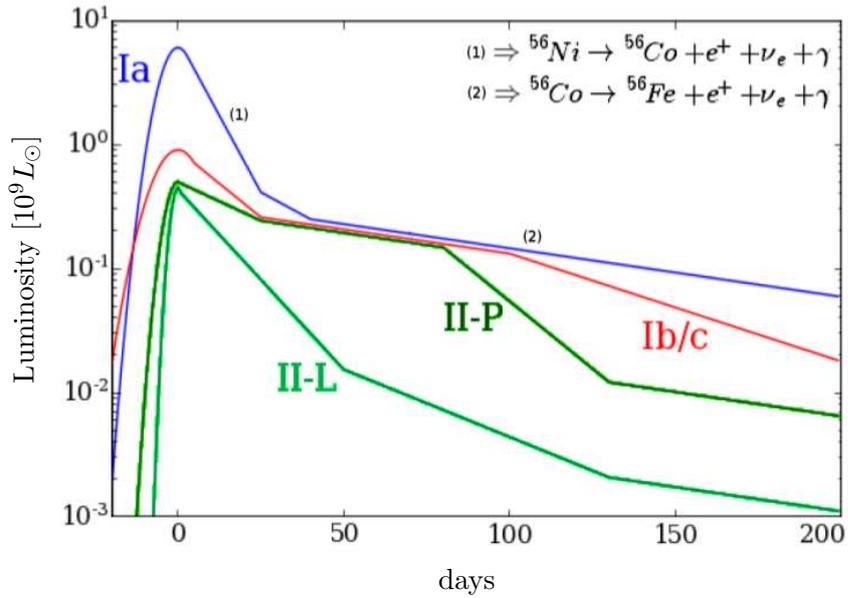


Figure 1.1: Idealized light curves for different types of supernovæ. We notice that SNe Ia are the brightest and the longest lasting case. Time is given in the rest frame of the supernovæ (which for a redshift  $z \sim 1$  corresponds to the one half of the observed time on Earth). Figure taken from [12].

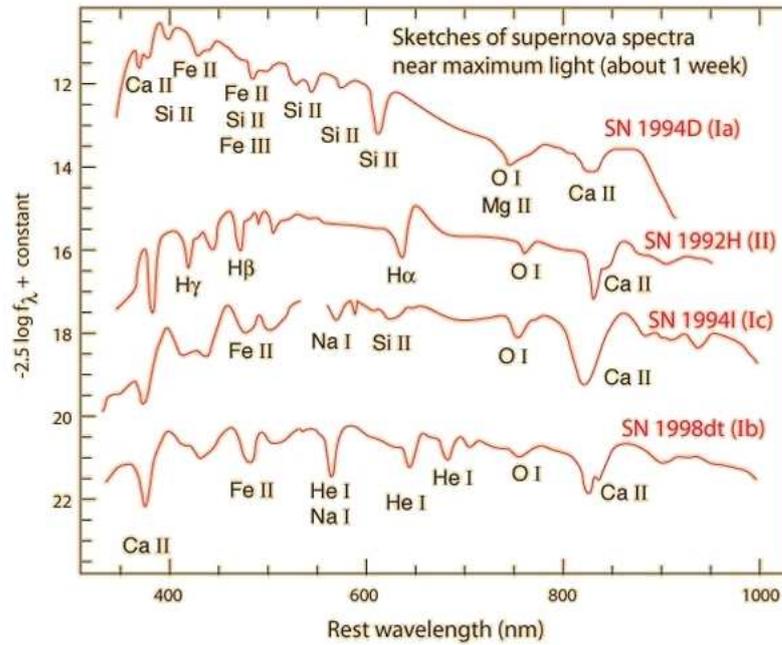


Figure 1.2: Spectra of different types of supernovæ. We can remark the absence of hydrogen and the presence of Si-II for the type Ia. Spectra are shown in the rest frame of the supernovæ, i.e. corrected from redshift. It is actually such kind of Si-II absorption bands which are used [13] to estimate the redshift of the star. Spectra are usually taken near the light curve maximum for a better accuracy. This figure is taken from [14].

wavelengths of absorption bands). Note however that spectroscopy measurements are very costly in terms of observational time, limiting seriously the number of SNe Ia detections. To cure this problem and prepare the next generation of surveys (such as the Large Synoptic Survey, the Dark Energy Survey, or Pan-STARRS), the SNLS and SDSS experiments have tried to include the estimation of the redshift in the fitting process from photometry only [15]. Indeed, a supernova which is highly

redshifted is more luminous in low frequency spectral bands. We can thus estimate – though less accurately but very fast – their redshift by the comparison of their magnitude in these different bands, i.e. from the color itself instead of spectroscopy.

The light curve of a SN is a very important source of information and it is used to calibrate the SNe and standardize them. Several methods have been used for their standardization, one can cite particularly the *Multicolor Light Curve Shape* (MLCS) and *Spectral Adaptive Lightcurve Template* (SALT, see e.g. [12, 16]). From these analyses of the light curves, we can extract two parameters of major importance: the stretch factor  $s$  and the color parameter  $c$ . The first one is linked to the lapse in time of the SNe light curve and the second one is a parameter in the SALT fit of the light curves. We can see in Fig. 1.3 that the different light curves of SNe Ia can be reshaped by applying to them the simple **stretch factor** to significantly reduce their dispersion. The **color parameter** is more precisely the excess of color between the averaged color of the supernova and its maximal color :

$$c \equiv (B - V)_{\max} - \text{Moy}[B - V] . \quad (1.26)$$

We also observe the magnitude of the supernova in different frequency bands and the total luminosity is thus obtained by reconstruction. For this, the magnitude in the B band is used to correct the measurements from the stretch and color parameters. A relation of the following type is used :

$$m_B^{\text{CORR}} = m_B - \alpha (s - 1) + \beta c , \quad (1.27)$$

where  $\alpha$  and  $\beta$  are two parameters evaluated to reduce the dispersion between the light curves of different SNe Ia. When one considers these corrections, we get a reduction from  $\sim 40\%$  to  $\sim 15\%$  on the dispersion of the distance modulus, i.e. directly on the Hubble diagram.

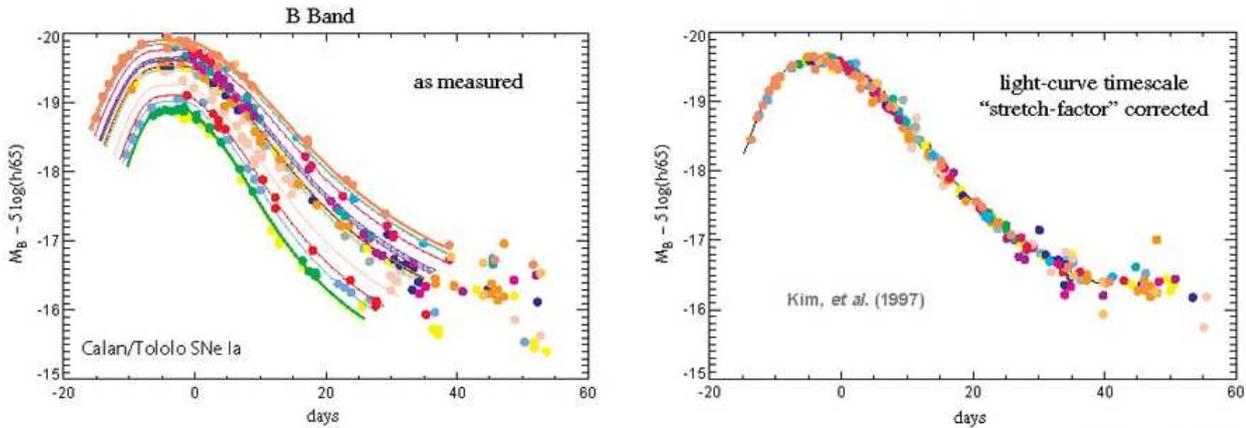


Figure 1.3: SNe Ia light curves of the Calán-Tololo survey. Left : as directly measured in the B-band, replacing them in their supernova rest-frame. Right : same light curves after applying the stretch correction. We note the significant reduction in dispersion induced by this correction. Figures borrowed from [17].

### 1.3.3 Magnitudes

Magnitudes are widely used in astronomy. Let us first consider a non-expanding Universe to explain this notion. One defines the *apparent magnitude*  $m$  of a star at the actual distance  $d_L$ (a.u.) (expressed in arbitrary units a.u.) in terms of its *luminosity flux*  $\Phi$  (using the same distance units) by

$$m = -2.5 \log_{10}(\Phi) + \text{cst} \equiv -2.5 \log_{10} \left( \frac{\Phi(\text{a.u.})}{\Phi_{\text{ref}}(\text{a.u.})} \right) . \quad (1.28)$$

The constant  $\text{cst} \equiv 2.5 \log_{10}(\Phi_{\text{ref}}(\text{a.u.}))$  appearing in this expression makes dimensionless the apparent magnitude. That also means that measuring a luminosity flux is useless if one cannot compare it to a

reference source (being another star or a calibration of the apparatus used for the measurement). We also define the *absolute magnitude* which is the apparent magnitude for the observed star if we were at a distance of 10 pc, in that case we denote it by  $M$ , and we have

$$M = -2.5 \log_{10} \left( \frac{\Phi(10 \text{ pc})}{\Phi_{\text{ref}}(\text{pc})} \right) . \quad (1.29)$$

Using now the relationship between flux and distance, at a distance expressed in a.u., in a homogeneous Universe :

$$\Phi(\text{a.u.}) = \frac{L}{4\pi d_L(\text{a.u.})^2} , \quad (1.30)$$

where  $L$  is the intrinsic luminosity of the source (assumed to be constant), we obtain that

$$m - M = 5 \log_{10} \left( \frac{d_L(\text{a.u.})}{10 \text{ pc}} \right) . \quad (1.31)$$

This combination that we call the *distance modulus*  $\mu \equiv m - M$ , is thus explicitly dependent on the unit used for the distance  $d_L$ . It can be written as :

$$\mu(\text{pc}) = -5 + 5 \log_{10}(d_L(\text{pc})) \quad \text{or} \quad \mu(\text{Mpc}) = 25 + 5 \log_{10}(d_L(\text{Mpc})) . \quad (1.32)$$

Let us now see how this relation is changed in an expanding Universe. In the case of an expanding homogeneous and flat Universe, the luminosity distance can be written in terms of the redshift  $z = a(t_0)/a(t) - 1$  and it can be written as

$$d_L(z) = (1 + z)a(t_0)r(z) , \quad (1.33)$$

where  $r(z)$  is the conformal distance from the source to the observer. Choosing a radial lightcone geodesic reaching the observer, the trajectory of light is given by  $ds^2 = -dt^2 + a(t)^2 dr^2 = 0$  and we simply have the naïve relation<sup>6</sup>  $dr = -a(t)^{-1} dt$  that describes the propagation of the photon. This gives

$$r = - \int_{t_0}^t \frac{dt}{a(t)} = \int_{a(t)}^{a(t_0)} \frac{da}{a^2 H(a)} = a_0^{-1} \int_0^z \frac{dz'}{H(z')} , \quad (1.34)$$

where  $t_0$  is “today” (and  $a_0 \equiv a(t_0)$ ) and we used  $dt/a = da/(a^2 H(a))$  and  $dz = -a_0 da/a^2 = -(a_0 H/a) dt$ . In a general Universe which contains different fluids, all in a pure homogenous state, the function  $H(z)$  is not simple (see e.g. Eq. (1.47)).

Staying at relatively short observable distances, more precisely in a regime where  $z \ll 1$ , we can expand  $H(z)$  in perturbations of  $z$ . The Taylor expansion gives

$$H(z) = H_0 + \left( \frac{dH}{dz} \right)_{t=t_0} z + \mathcal{O}(z^2) , \quad (1.35)$$

where we can use

$$\frac{dH}{dz} = - \frac{1}{(1+z)H(z)} \frac{dH(t)}{dt} = - \frac{1}{(1+z)H(z)} \left( \frac{\ddot{a}}{a} - H^2 \right) , \quad (1.36)$$

which at the observer position (today) is

$$\left( \frac{dH}{dz} \right)_{t=t_0} = -H_0 \left( \frac{a\ddot{a}}{\dot{a}^2} - 1 \right)_{t=t_0} \equiv H_0(1 + q_0) , \quad (1.37)$$

and where we have introduced the so-called *deceleration parameter*  $q_0$  :

$$q_0 \equiv - \left( \frac{a\ddot{a}}{\dot{a}^2} \right)_{t=t_0} = \left( 1 + \frac{\dot{H}}{H^2} \right)_{t=t_0} . \quad (1.38)$$

6.  $t$  being future directed and the system of coordinates centered on the observer, that explains the minus sign here.

We thus obtain that

$$H(z) = H_0[1 + (1 + q_0)z + \mathcal{O}(z^2)] \quad (1.39)$$

which, after being introduced into Eq. (1.34), implies the following perturbative expression for the conformal distance  $r$

$$r(z) = (a_0 H_0)^{-1} \left( z - (1 + q_0) \frac{z^2}{2} + \mathcal{O}(z^3) \right) . \quad (1.40)$$

This gives at the end, from Eq. (1.33), the perturbative expression of the luminosity distance

$$d_L(z) = H_0^{-1} \left[ z + (1 - q_0) \frac{z^2}{2} + \mathcal{O}(z^3) \right] . \quad (1.41)$$

We notice in passing that a stationary (i.e. constantly evolving) Milne Universe implies  $r(z) = (a_0 H_0)^{-1} z$  and thus  $d_L^{\text{Milne}}(z) = z(1 + z/2)H_0^{-1}$  (see Sec. A.6). We then have  $d_L > d_L^{\text{Milne}}$  if  $q_0 > -1$ .

We can now come back on the definition of the distance modulus and write a definition which involves the product  $H_0 d_L$  in order to have a function of redshift only. This is done by writing:

$$\mu(z) = \mathcal{M} + 5 \log_{10}(H_0 d_L(z)) \quad \text{with} \quad \mathcal{M}(\text{Mpc}) \equiv 25 - 5 \log_{10}(H_0) , \quad (1.42)$$

and where  $d_L(z)$  is still in Mpc and  $H_0$  is expressed in  $\text{Mpc}^{-1}$ . More correctly:

$$H_0 \equiv h \, 100 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} = 3.33 \, 10^{-4} \, h \, c \, \text{Mpc}^{-1} , \quad (1.43)$$

where we have used the speed of light  $c = 299792,458 \, \text{km} \, \text{s}^{-1}$  which is taken to be  $c = 1$  when everything is expressed in Mpc (like here). For  $h = 0.7$ , we get  $-5 \log_{10}(H_0) \simeq 18.16$ , leading to

$$\mu(\text{Mpc}) = 43.16 + 5 \log_{10}(H_0(\text{Mpc}^{-1})d_L(\text{Mpc})) . \quad (1.44)$$

We remark that this expression is very well adapted to SNe observations which are sensitive to the product  $H_0 d_L$  rather than  $d_L$  itself.

### 1.3.4 The Hubble diagram

Let us first recall that the luminosity distance  $d_L$  of a source at redshift  $z$  is related to the angular distance  $d_A$  of this source (as seen from the observer) by the [Etherington law](#) (or reciprocity law) [18]:

$$d_L = (1 + z)^2 d_A . \quad (1.45)$$

This relation is valid in any geometry as it relies on the reversibility of light trajectories. In the particular case of an unperturbed, spatially flat FLRW background, and for a source with redshift  $z_s$ , the [angular distance](#)  $d_A$  is simply given by

$$d_A^{\text{FLRW}}(z_s) = a_s r_s = a_s(\eta_o - \eta_s) , \quad (1.46)$$

where  $a_s \equiv a(\eta_s)$ , while  $\eta_o - \eta_s \equiv \Delta\eta$  denotes the conformal time interval between the emission and observation of the light signal.

For an unperturbed metric also we have  $1 + z = a_0/a(t)$ , and  $d\eta = dt/a = -a_0^{-1} dz/H$ , where  $H = d(\ln a)/dt$ . Hence, using the standard (spatially flat) Friedmann equation for  $H$  from Eq. (1.16), assuming that the homogeneous model has perfect fluid sources with present fractions of the critical density  $\Omega_{n0}$  and barotropic parameters  $w_n = p_n/\rho_n = \text{cst}$  as given by Eq. (1.22), i.e. using:

$$H^2(z) = H_0^2 \left[ \sum_n \Omega_{n0} (1 + z)^{3(1+w_n)} \right] , \quad (1.47)$$

we get the general expression<sup>7</sup> in terms of the redshift of a source  $z_s$  :

$$\begin{aligned} d_L^{\text{FLRW}}(z_s) &= (1+z_s)a_0 \int_{\eta_s}^{\eta_0} d\eta = (1+z_s) \int_0^{z_s} \frac{dz}{H(z)} \\ &= \frac{1+z_s}{H_0} \int_0^{z_s} dz \left[ \sum_n \Omega_{n0} (1+z)^{3(1+w_n)} \right]^{-1/2}. \end{aligned} \quad (1.48)$$

Expanding in the limit  $z_s \rightarrow 0$  we also obtain the expression

$$\begin{aligned} d_L^{\text{FLRW}}(z_s) &\simeq \frac{1}{H_0} \left[ z_s + \frac{1}{4} \left( 1 - 3 \sum_n w_n \Omega_{n0} \right) z_s^2 + O(z_s^3) \right] \\ &\equiv \frac{1}{H_0} \left[ z_s + \frac{1}{2} (1 - q_0) z_s^2 + O(z_s^3) \right], \end{aligned} \quad (1.49)$$

which shows the well-known sensitivity of the term quadratic in  $z_s$  to the composition of the cosmic fluid through the deceleration parameter  $q_0$  (see Eq. (1.38)).

If we assume a flat (i.e.  $K = 0$ ) FLRW model with only matter and a cosmological constant, we get the theoretical luminosity distance :

$$d_L^{\Lambda\text{CDM}}(z_s) = \frac{1+z_s}{H_0} \int_0^{z_s} \frac{dz'}{[\Omega_{\Lambda 0} + \Omega_{m0}(1+z')^3]^{1/2}}. \quad (1.50)$$

This relation is not trivially integrable when its two parameters  $\Omega_{\Lambda 0}$  and  $\Omega_{m0}$  are non-zero. One thus has to integrate it numerically until we find its best fit with the data. Measuring magnitudes of SNe Ia with their corresponding redshifts, one can plug the result of Eq. (1.50) into Eq. (1.31) and plot the so-called *Hubble diagram*, as shown in Fig. 1.4 (see [19] for a short history). From the binned data points, we get that  $\Omega_{\Lambda 0}$  is about 73% of the total energy content in the Universe and that  $\Omega_{m0}$  is about 27% (including baryons)<sup>8</sup>. The densities of radiation and curvature are negligible today. This is such a fit which led to the discovery of dark energy. We can see in Fig. 1.4 that the data dispersion is very important. That shows that SNe are very complex objects which are difficult to interpret even after being standardized by a long procedure.

To improve the estimation of cosmological parameters, we usually use the combination of SNe Ia observations with two other probes which give a precise estimation of the parameters in this homogeneous context. These two other probes are the cosmic microwave background (*Cosmic Microwave Background (CMB)*) and the baryonic acoustic oscillations (*Baryonic Acoustic Oscillations (BAO)*). One can indeed see in Fig. 1.5 the crossing of their different predictions and its great increase in the precision of the estimation of  $\Omega_{\Lambda 0}$  and  $\Omega_{m0}$ . We also remark that these data support a Universe very close to be spatially flat.

7. Actually, the most general expression that we can obtain when we consider a non-flat FLRW model with dark matter and a general form of dark energy, i.e. non-necessarily a cosmological constant, is :

$$d_L(z_s) = \frac{c(1+z_s)}{H_0 \sqrt{|\Omega_{K0}|}} S_K \left( \sqrt{|\Omega_{K0}|} \int_0^{z_s} \frac{dz}{\left[ \Omega_{m0}(1+z)^3 + \Omega_{K0}(1+z)^2 + \Omega_{DE,0} \exp \left( 3 \int_0^z \frac{1+w_{DE}(z')}{1+z'} dz' \right) \right]^{1/2}} \right),$$

with the possible addition in the RHS of  $\Omega_r \equiv 8\pi G \rho_r / 3H^2$  the density parameter of radiation (for which  $w_r = 1/3$ ), and where we used  $\Omega_{DE} \equiv 8\pi G \rho_{DE} / 3H^2$ . The exponential contribution is obtained by considering the conservation of the dark matter fluid:  $\rho_{DE} + 3H(\rho_{DE} + p_{DE}) = 0$  with a general (time dependent) equation of state such that  $p_{DE} = w_{DE}(\rho_{DE}) \rho_{DE}$ . We used  $S_K(x) = \sin(x), x, \sinh(x)$  depending on the sign of the curvature  $K > 0, = 0, < 0$ .

8. See however the new reevaluation of these parameters by the Planck satellite first release [20].

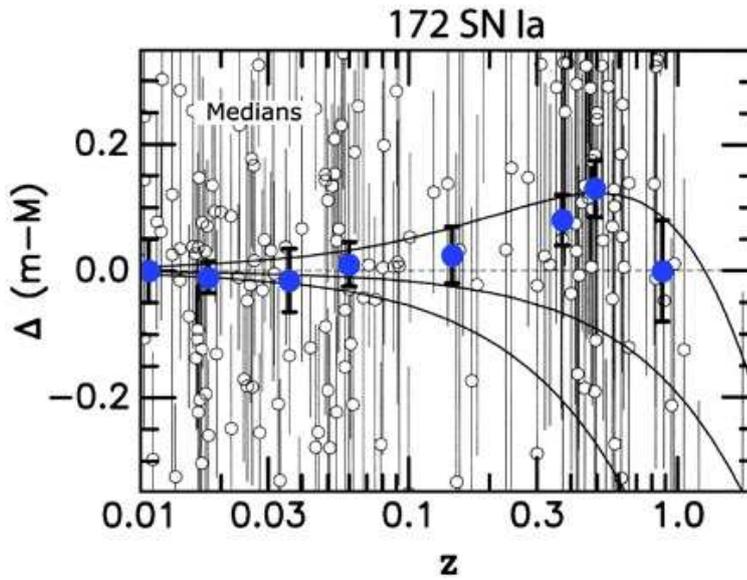


Figure 1.4: Hubble diagram with 172 SNe Ia with median values in 8 redshift bins. Solid lines (from bottom to top) correspond to the evaluation from Eq. (1.50) with  $(\Omega_{m0}, \Omega_{\Lambda0}) \simeq (1, 0)$ ,  $(\Omega_{m0}, \Omega_{\Lambda0}) \simeq (0.3, 0)$  and  $(\Omega_{m0}, \Omega_{\Lambda0}) \simeq (0.3, 0.7)$ . The increase at large redshift in the binned data points shows the existence of a dark energy component. Figure taken from [19].

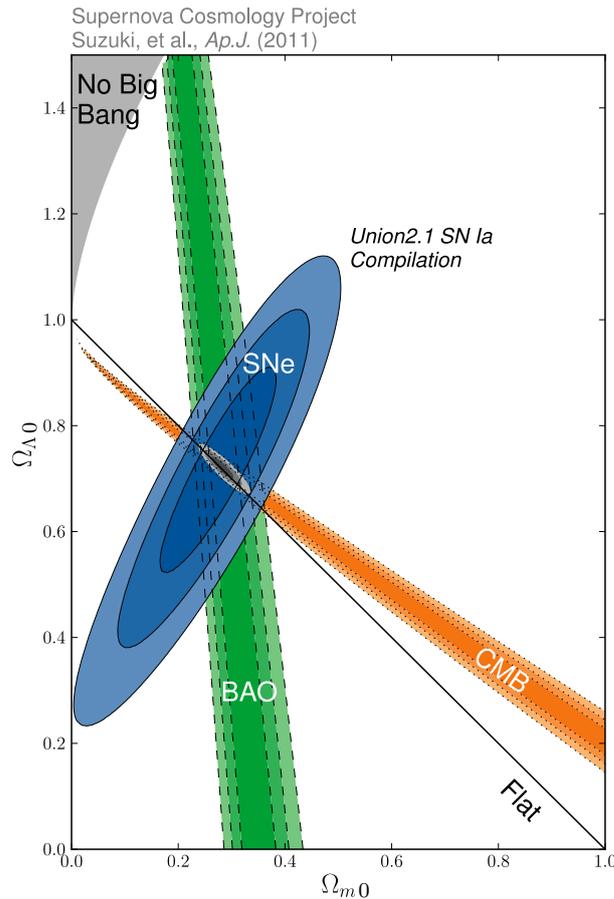


Figure 1.5: Cross plot from the Union2 data set and other probes. One can notice (within the 68%, 95%, and 99.7% confidence regions) the nice combination of the different observables which are SNe Ia, the CMB and BAOs. The predicted values, with the highest confidence, are  $(\Omega_{m0}, \Omega_{\Lambda0}) \simeq (0.27, 0.73)$ . Credit : Supernova Cosmology Project (adapted from [supernova.lbl.gov/Union/](http://supernova.lbl.gov/Union/)), see also [21].

## 1.4 The large scale structure

### 1.4.1 Observations

The SDSS (Sloan Digital Sky Survey) and 2dFGRS (Two degree field galaxy redshift survey) are major surveys. SDSS covered about one quarter of the sky and observed millions of objects, with more than a million of galaxies and their distances. On the other hand the 2dFGRS observed a very narrow part of the sky with 250 000 galaxies and their redshift up to  $\sim 0.3$ . As one can see on Fig. 1.6, matter is organized in a filamentous structure which looks relatively homogeneous at large scales.

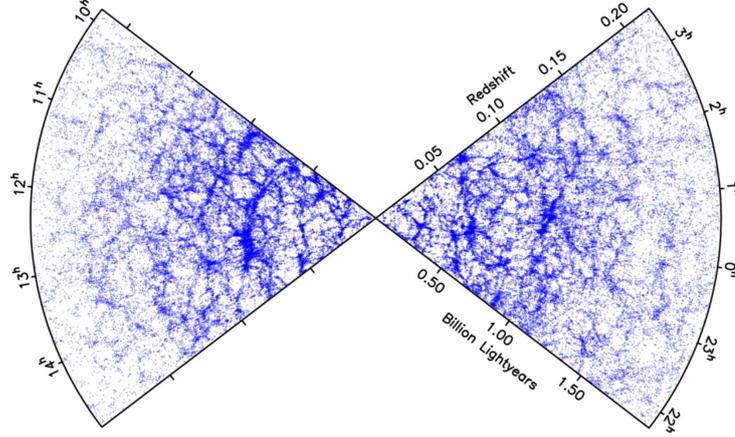


Figure 1.6: The 2dF Galaxy Redshift Survey (2dFGRS). Two narrow bands on the sky are represented. Data has been stacked in the thin ( $\sim 4^\circ$ ) angular direction of the survey to get a two dimensional picture. One can see the homogeneity of structures at large scales. Credit : 2dFGRS (adapted from [www2.aao.gov.au/~TDFgg/](http://www2.aao.gov.au/~TDFgg/)).

Nevertheless, several complications must be addressed to obtain a real picture of the matter distribution in the Universe. First, what we observe is light coming from our past lightcone, so we can only probe the galaxy distribution, not the true matter distribution containing dark baryonic objects as well as cold dark matter itself. The link between the galaxy distribution and the true matter distribution is not trivial and struggles with our lack of understanding of the galaxies formation. The density contrast of galaxies  $\delta_{\text{gal}}$  is related to the density contrast of matter by

$$\delta_{\text{gal}}(t, \mathbf{x}) = b \delta(t, \mathbf{x}) \quad \text{with} \quad \delta(t, \mathbf{x}) = \frac{\rho(t, \mathbf{x}) - \rho(t)}{\rho(t)} \equiv \frac{\delta\rho(t, \mathbf{x})}{\rho(t)}, \quad (1.51)$$

where  $\rho(t)$  is the background homogeneous density field and  $\rho(t, \mathbf{x})$  the true matter field. The “parameter”  $b(t, \mathbf{x})$  is called the **bias** and has been extensively studied in the literature, including its possible spacetime dependence and the complications involved in the non-linear regime.

One other complication is that galaxies are not observed in the real spacetime, namely by two angles on the sky and a spacetime combination as  $\eta - r$  on the past lightcone, but are observed in the redshift space, i.e. that radial distances are measured through the change in light frequency. As a consequence, velocity and position are entangled in the measurements (like in the so-called *fingers of God* effect). Nevertheless, several methods have been confronted to describe matter and evaluate its power spectrum. For example, measurements from weak lensing are free from bias effects as they measure the real distribution of matter.

Considering these effects, one can measure the matter density spectrum  $\delta(t, \mathbf{x})$  and go in Fourier space. By assuming an ergodic principle, one can take the volume average of the 2-point correlation functions and consider that this measurement is equal to the stochastic average  $\overline{\delta_{\mathbf{k}} \delta_{\mathbf{k}}^*}$ . One then obtains the definition of the dimensionless matter power spectrum by the definition :

$$\Delta^2(t, \mathbf{k}) = \frac{k^3}{2\pi^2} |\delta_{\mathbf{k}}(t)|^2. \quad (1.52)$$

From this expression it is possible to plot the experimental power spectrum of Fig. 1.7. We notice the variety of probes composing this incredible measurement over five orders of magnitude in scales.

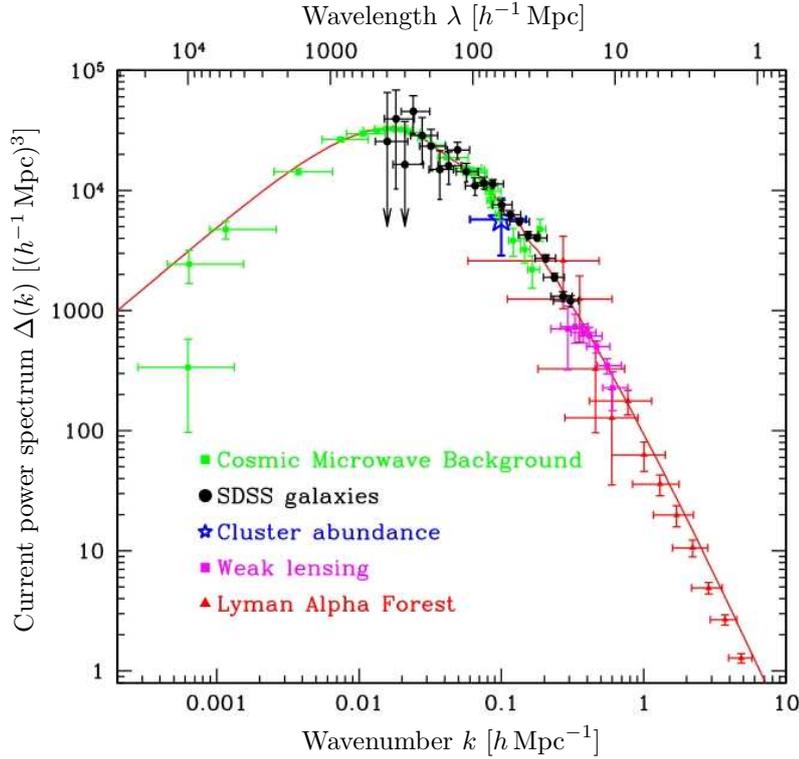


Figure 1.7: The matter power spectrum as measured from different sources. We notice that the linear part of the spectrum at large scales (small  $k$ ) is reconstructed from CMB. Similarly, the linear spectrum shown at small scales (large  $k$ ) is a reconstruction from the non-linear spectrum (as one observes the total, non-linear, spectrum) and is thus model dependent (see Sec. 1.4.3). Figure adapted from [22].

### 1.4.2 The linear power spectrum

What we will be interested by is the computation of the power spectrum for the gravitational potential in its linear and non-linear regimes. The reason for this is that metric perturbations, that we will study later, are expressed in terms of the gravitational potential and not the matter density. As a consequence, one of the most important equation for us will be the [Poisson equation](#) which links the two quantities. In synchronous gauge (see Sec. A.3.3), this equation is written as :

$$\nabla^2 \psi(x) = 4\pi G \rho(t) \delta(x) \quad , \quad (1.53)$$

where  $x = (t, \mathbf{x})$ , and  $\rho(t)$  is the background matter density. Taking the Fourier transform of this expression brings  $k^2 \psi_k \sim H_0^2 \Omega_{m0} \delta_k$ , where  $\delta_k$  is the spectral function associated to the matter density contrast  $\delta(x)$  and  $\psi_k$  the spectral function associated to the gravitational (Bardeen) potential  $\psi(x)$  (at first order in perturbations) that we will define in Chapter 4.

The 2-point correlation function of the gravitational (Bardeen) potential, written as  $\overline{\psi(x)\psi(x)}$  with the overbar denoting a statistical (or stochastic) average<sup>9</sup>, is a dimensionless quantity. Taking its Fourier transform teaches us that  $[\psi_k] = [k^{-3}]$  (where square brackets denote the dimension of the quantity). Thus,  $|\delta_k|^2$  has the same dimension as  $|\psi_k|^2$  when we set  $c = 1$  (in that case  $[k] = [H_0]$ ). More precisely, the relation between  $|\delta_k|^2$  and  $|\psi_k|^2$  is :

$$|\psi_k|^2 = \left(\frac{3}{2}\right)^2 \frac{\Omega_m^2(z) \mathcal{H}^4}{k^4} |\delta_k|^2 \quad , \quad (1.54)$$

9. In textbooks, the stochastic average is usually denoted by  $\langle \dots \rangle$ . In our work this notation denotes sky-average and for that reason we prefer the notation  $\overline{(\dots)}$  to denote stochastic averaging.

where for a CDM flat FLRW Universe we have  $\mathcal{H} = aH = 2/\eta$  (with  $\eta$  the conformal time,  $a(\eta) = C\eta^2$ ) and hence  $\mathcal{H}^4 = \left(\frac{a_0}{a(\eta)}\right)^2 \mathcal{H}_0^4$ .

In a more general context (independent from the Universe model), we have seen in Sec. 1.2.2 that the equation of conservation for a perfect fluid of matter, satisfying  $p = w\rho$  with  $w = \text{cst}$ , gives rise to the solution of Eq. (1.22). Using that  $\mathcal{H} = Ha$ , this relation becomes:

$$\Omega_m(z) = \Omega_{m0}(1+z) \left(\frac{\mathcal{H}_0}{\mathcal{H}}\right)^2, \quad (1.55)$$

and we get, in Fourier space, the Poisson equation at an arbitrary redshift:

$$|\psi_k(z)|^2 = \left(\frac{3}{2}\right)^2 \frac{\Omega_{m0}^2 \mathcal{H}_0^4}{k^4} (1+z)^2 |\delta_k(z)|^2. \quad (1.56)$$

We notice that there is an easy one-to-one relation between  $\eta$  and  $z$  in the case of a CDM flat FLRW Universe (where  $\Omega_{m0} = 1$ ). In the case of a  $\Lambda$ CDM model, the relation  $\eta(z)$  is in some way degenerated as it depends on the values of  $(\Omega_{m0}, \Omega_{\Lambda 0})$  (see Sec. 1.3.4).

Moreover, the dimensionless power spectrum  $\mathcal{P}_\psi(k)$  is related to  $|\psi_k|^2$  by its definition:

$$\mathcal{P}_\psi(k, \eta) \equiv \frac{k^3}{2\pi^2} |\psi_k(\eta)|^2, \quad (1.57)$$

where the time dependence has been written explicitly (but can be expressed in terms of  $z$ ).

The (linear) power spectrum  $\mathcal{P}_\psi(k, z)$  is first obtained from the [inflationary spectrum](#) (i.e. sourced by perturbations at the end of the inflationary era) where its expression is  $\left(\frac{3}{5}\right)^2 \Delta_{\mathcal{R}}^2$ , with:

$$\Delta_{\mathcal{R}}^2 = A \left(\frac{k}{k_0}\right)^{n_s-1} \quad \text{and} \quad A = 2.45 \cdot 10^{-9}, \quad n_s = 0.96, \quad k_0/a_0 = 0.002 \text{ Mpc}^{-1}. \quad (1.58)$$

We remark that the value of the spectral index  $n_s$  can be related to the [slow-roll parameters](#)  $\epsilon$  and  $\eta$  of the single field inflation model (see Sec. A.2.1, Eq. (A.55)) by

$$n_s - 1 = -6\epsilon + 2\eta. \quad (1.59)$$

We have to add a transfer function  $T(k)$  to describe the sub-horizon evolution of the different modes re-entering during the radiation-dominated era (see Sec. A.2.4 and A.4.2 for explicit forms). We also add a growth factor  $g(z)$  which takes into account the evolution of  $\psi_k$  during the dark energy era (assuming here a  $\Lambda$ CDM Universe). Finally, the expression of the linear power spectrum is:

$$\mathcal{P}_\psi(k) = \left(\frac{3}{5}\right)^2 \Delta_{\mathcal{R}}^2 T^2(k) \left(\frac{g(z)}{g_\infty}\right)^2. \quad (1.60)$$

### 1.4.3 Short insight in the non-linear regime

At small scales, matter is organized in structures and the density contrast can take values which are more important than on cosmological scales. As opposed to the last section, we cannot work here solely in terms of the gravitational potential and we need to go through the matter density spectrum to estimate the non-linear regime. The justification for this change is that the non-linear regime has been studied by [N-body simulations](#) in which the density is the direct measurable quantity.

So we define what is called the fractional density variance per unit  $\ln k$ , namely  $\Delta^2(k)$ , by:

$$\Delta^2(k) = \frac{d\sigma^2}{d \ln k} \quad \text{or equivalently:} \quad \sigma^2 = \int \Delta^2(k) d \ln k \quad (1.61)$$

with the definition of the [density variance](#):

$$\sigma^2 \equiv \overline{\delta(x)\delta(x)} = \int \frac{d^3k}{(2\pi)^3} |\delta_{\mathbf{k}}|^2 = \int \frac{k^2 dk}{2\pi^2} |\delta_k|^2. \quad (1.62)$$

Here the quantities depend on time (or redshift) and the overbar denotes again the usual stochastic averaging. These two expressions bring the simple relation between their integrands :

$$\Delta^2(k, z) = \frac{k^3}{2\pi^2} |\delta_k(z)|^2 \quad , \quad (1.63)$$

and confirms that  $\Delta^2(k, z)$  is a dimensionless quantity. Besides this, [23] defines another dimensionless power spectrum,  $P(k)$ , which is called the “initial power spectrum” (in the sense of structure formation simulations) and is taken to be a simple power law :

$$P(k) = \mathcal{A} k^n \quad . \quad (1.64)$$

This spectrum is linked to  $\Delta^2(k)$  through the expression (see Eq. (5) in [23]) :

$$\Delta_L^2(k) = \frac{V}{(2\pi)^3} 4\pi k^3 P(k) \quad , \quad \text{i.e.} \quad \Delta_L^2(k) \sim k^{n+3} \quad , \quad (1.65)$$

now taken at the linear level and where the so-called  $V$  factor is a normalization volume.

Nevertheless, this is not the approach that we will take here. A very first reason for this is that these quantities are only adapted to simulations (e.g. we have no a priori criterion to choose  $V$ ). Furthermore, what we are interested in, here, is the non-linear power spectrum obtained from the linear power spectrum. Both should hence match at large scales. For this reason, the linear power spectrum that we will use is the one from Eq. (1.60), and we will now call it  $\mathcal{P}_\psi^L(k)$ . We should also recall that the Poisson equation is valid both in the linear and non-linear regimes, so that we have :

$$\mathcal{P}_\psi^{L,NL}(k) = \frac{9}{4} \frac{\Omega_{m0}^2 \mathcal{H}_0^4}{k^4} (1+z)^2 \Delta_{L,NL}^2(k) \quad . \quad (1.66)$$

We will thus obtain the linear density spectrum using the Poisson equation at the linear level :

$$\Delta_L^2(k) = \frac{4}{9} \frac{k^4}{\Omega_{m0}^2 \mathcal{H}_0^4 (1+z)^2} \mathcal{P}_\psi^L(k) = \frac{4}{25} \frac{A \Omega_{m0}^{-2} \mathcal{H}_0^{-4}}{(1+z)^2} \frac{k^{n_s+3}}{k_0^{n_s-1}} \left( \frac{g(z)}{g_0} \right)^2 T^2(k) \quad . \quad (1.67)$$

We will then use the HaloFit model, as presented in Sec. A.4 and A.5, to generate from  $\Delta_L^2(k)$  the density power spectrum in the non-linear regime, i.e.  $\Delta_{NL}^2(k)$ . And as a last step, we will use the Eq. (1.66) again, in the non-linear regime, to obtain the desired quantity  $\mathcal{P}_\psi^{NL}(k)$ .

## 1.5 Dealing with the inhomogeneous Universe

### 1.5.1 Real nature of the Universe

As we have seen, the analysis of the standard model of cosmology takes place within a homogeneous context. The low precision of observations not so far in the past has justified this approach and yet brought amazingly consistent results. This is mainly due to the relative smallness of inhomogeneities as supported by the CMB. Nevertheless, we can notice that SNe Ia observations suffer from a very large dispersion which make their discovery of dark energy not straightforward. On the other hand the true nature of the Universe is inhomogeneous as revealed by the large scale structure below a certain scale. It is thus natural to ask to which extent observations, and in particular SNe Ia, are free from a contamination by the presence of matter (and thus gravitational) inhomogeneities. At the end, these probes are local astrophysical objects in virialized domains of our Universe and light signals that we receive from them propagate over very large distances, encountering possibly a large number of sources of dispersion<sup>10</sup>. We also know that SNe Ia measurements can be equally well fitted by a LTB model (see e.g. [25]) in which we are close to the center of a less dense region of matter.

10. Another problem is the one of curvature as the Universe is mostly filled by voids and thus photons propagating in it are sensitive to the Weyl curvature instead of the Riemann curvature (see [24]). We will not address this here.

However, experimental constraints on our position inside this underdensity make us unreasonably close to the center, disproving this possibility. Nevertheless, the question is still there and one can wonder how big is the effect of inhomogeneities and whether or not they could explain (partially maybe) the observation of the acceleration in the expansion. The possibility to explain dark energy with the inhomogeneous Universe is very tempting as it is the most conservative explanation at our disposal, assuming no extra physics, and only reconsidering our interpretation of general relativity. It has also the advantage to solve the coincidence problem, by linking the formation of structures with the appearance of dark energy, and the smallness of the dark energy density.

### 1.5.2 The backreaction problem

In relation to the problem of inhomogeneity of our Universe, different questions related to the non-linear nature of general relativity are still unsolved [26]. The question of how to average small-scales, or coarse-grain structures, to describe their large-scale geometry and dynamics, is called the *averaging problem* of cosmology. The effect of inhomogeneities on the large-scale dynamics of the Universe is difficult to evaluate. Solving this problem is also solving the *backreaction problem* which can be illustrated by the following reasoning. Imagine that we know exactly the metric  $g_{\mu\nu}^{(loc)}$  at a (local) subgalactic scale. By the knowledge of a correct smoothing procedure, we could average this metric to get the metric at a larger scale, like the galactic scale:  $g_{\mu\nu}^{(loc)} \rightarrow g_{\mu\nu}^{(gal)}$ . Let us imagine also that this smoothing procedure is applicable to the stress energy tensor so that we have:  $T_{\mu\nu}^{(loc)} \rightarrow T_{\mu\nu}^{(gal)}$ . Assuming that the Einstein equations at the local scale were:

$$G_{\mu\nu}^{(loc)} = 8\pi G T_{\mu\nu}^{(loc)} \quad , \quad (1.68)$$

the non-linear nature of the *Einstein equations*, i.e. the fact that  $G \sim \Gamma^2$ ,  $\partial\Gamma \sim (g\partial g)^2$ ,  $\partial(g\partial g)$  (in crude notations), makes that these previous equations will become at galactic scales:

$$G_{\mu\nu}^{(gal)} = 8\pi G T_{\mu\nu}^{(gal)} + E_{\mu\nu}^{(gal)} \quad . \quad (1.69)$$

In this expression, the extra contribution  $E_{\mu\nu}^{(gal)} \neq 0$  will emerge from the non-commutation between the averaging procedure of  $g_{\mu\nu}^{(loc)}$  and the non-linear aspect of Einstein's equations. By repeating this procedure up to the large scale structure and cosmic scales, we will ideally reach an homogeneous description where the effect of inhomogeneities have been properly taken into account.

These two problems are also related to the *fitting problem*, i.e. the question of how to relate observations made from a lumpy environment to an idealized model with a (not really existing) "background" geometry. We still do not know, nowadays, how to match an inhomogeneous Universe (what we really observe) with an homogeneous FLRW model (used to interpret the data), as illustrated in Fig. 1.8.

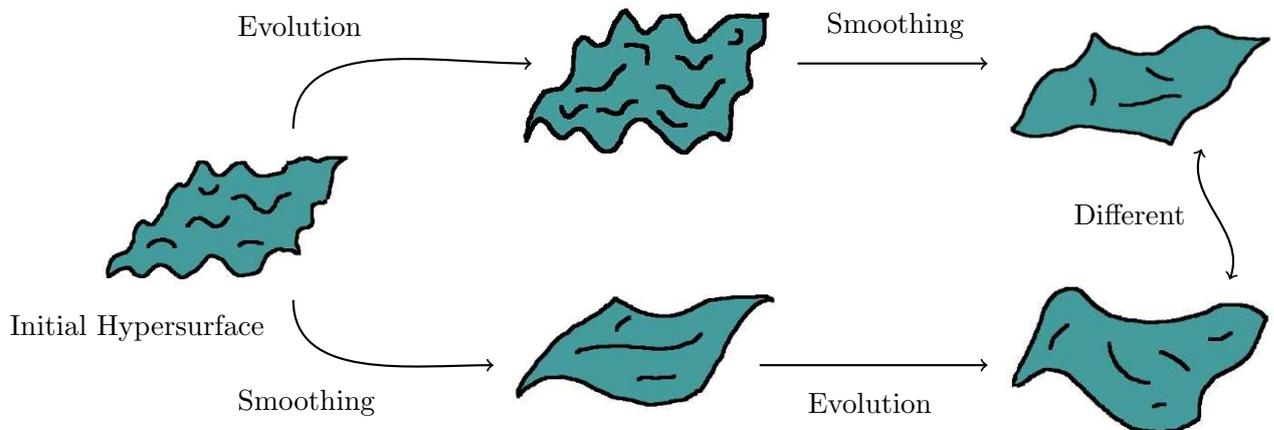


Figure 1.8: Illustration of the backreaction problem : we see the non-commutation between the smoothing, or averaging, and time evolution. This is related to the non-linearity of Einstein's equations.

## 1.6 Introduction & Outline of this thesis

The so-called concordance (or  $\Lambda$ CDM) model is based on a suitable combination of dark matter, dark energy and baryons for an overall critical density and has become the reference paradigm for the late, i.e. post-equality epoch, evolution of our Universe (see e.g. [27]). It accounts equally well for the CMB data, the Large Scale Structure and, even more significantly, for the supernovae data in terms of a cosmic acceleration [10, 11]. However, these three tests of the concordance model are not strictly speaking at the same level of theoretical rigor. While the first two have to do, by definition, with the inhomogeneities present in our Universe, the third is based on an ideal homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. It is thus a very amazing fact that these three probes finally give a very consistent picture and this especially as we still lack of a satisfying theoretical explanation for the real nature of dark energy. Establishing the existence of this unknown component and determining its parameters is thus one of the central issues in modern cosmology.

As we said, the analysis of SNe Ia (taken as standard candels) is usually made in the simplified context of a homogeneous and isotropic (FLRW) cosmology. The issue has then been raised about whether inhomogeneities may affect the conclusion of such a theoretically naive assumption. Inhomogeneous models in which we occupy a privileged position in the Universe, for instance, can mimic dark energy (as first pointed out in [25]), but look both unrealistic and highly fine-tuned. Nevertheless, we cannot deny that the Hubble diagram is affected by a large dispersion on the data, some SNe Ia being so faint that they look to contradict the reasonable condition  $\Omega_{\Lambda 0} \leq 1$ . This kind of effect is certainly related to the non-standard nature of some supernovae, but we can still wonder how much is the contamination coming from the inhomogeneous aspect of our Universe. At the end, cosmology can be summarized as the study of two things: the content of the Universe and the dynamics of this content. It is thus a very important question to ask how much structures affect the measurement of distances. It is also clear that, at least for the sake of precision, a better treatment of cosmic acceleration should take inhomogeneities into account, and this at least in the presence of stochastically isotropic and homogeneous perturbations of the kind predicted by inflation. Only when this is done we can establish in a convincing way whether  $\Lambda$ CDM gives a simultaneous consistent description of the above-mentioned body of cosmological data.

In additions, it is by now well-known (see e.g. [28]) that averaging solutions of the full inhomogeneous Einstein equations leads, in general, to different results from those obtained by solving the averaged (i.e. homogeneous) Einstein equations. In particular, the averaging procedure does not commute with the non-linear differential operators appearing in the Einstein equations. As a result, the dynamics of the averaged geometry is affected by so-called “backreaction” terms, originating from the inhomogeneities present in the geometry and in the various components of the cosmic fluid. Following the discovery of cosmic acceleration on large scales, interest in the possible effects of inhomogeneities for interpreting the data themselves has considerably risen (see [6, 26, 29, 30] for recent reviews). Indeed, though there is by now general agreement that super-horizon perturbations cannot mimic dark-energy effects [31–33], the impact of sub-horizon perturbations is by contrast still unsettled [34–37], owing to the appearance of ultraviolet divergences while computing their “backreaction” on certain classes of large-scale averages [36, 37]. The possibility that these effects may simulate a substantial fraction of dark energy, or that they may at least play some role in the context of near-future precision cosmology, has to be seriously considered. According to some authors [38–44] present inhomogeneities might explain, by themselves, cosmic acceleration without any need for dark-energy contributions, thereby providing an elegant solution to the well-known “coincidence problem”. According to others [31, 33–35, 45, 46] the effect of inhomogeneities is, instead, completely negligible. The truth may lie somewhere in between, in the sense that a quantitative understanding of inhomogeneities effects could be important in order to put precise constraints on dark-energy parameters, such as the critical fraction of dark-energy density,  $\Omega_{\Lambda}$ , and the evolution of its effective equation of state,  $w_{\Lambda}(z)$ .

If inhomogeneities do have an important impact, the actual observations must also correspond in some way to the measurement of averaged quantities. As a consequence of this, and because of the backreaction issue, much work has been done in these last few years on trying to formulate a suitable “averaged” description of inhomogeneous cosmologies. In most of these works, following

Buchert’s seminal papers [47–50], the effective geometry emerging after the smoothing-out of local inhomogeneities has been determined by integrating over three-dimensional spacelike hypersurfaces and computing the ensuing “backreaction” on the averaged geometry. However, as pointed out long ago [51, 52], a phenomenological reconstruction of the spacetime metric and of its dynamic evolution on a cosmological scale – revealed through the experimental study of the luminosity–distance to redshift relation [10, 11] – is necessarily based on past light-cone observations (since most of the relevant signals travel with the speed of light). Hence, the averaging procedure should be possibly referred to a null hypersurface coinciding with a past light-cone rather than to some fixed-time spacelike hypersurface. Nonetheless, such a light-cone averaging procedure, whose importance has been repeatedly stressed in the specialized literature (see e.g. [6, 51–57]), has never been implemented in practice.

In this thesis we introduce a general gauge invariant prescription for averaging scalar quantities on null hypersurfaces, solving thus the long-standing problem of lightcone averaging. We apply it to the past light-cone of a generic observer in the context of an inhomogeneous cosmological metric, and provide the analog of the Buchert-Ehlers commutation rule [58] for the derivatives of light-cone averaged quantities. We also take advantage of the gauge invariance of our formalism by introducing an adapted system of coordinates (defining what we call a “geodesic light-cone (GLC) frame”, which can be seen as a particular specification of the “observational coordinates” introduced in [51, 59]). In these coordinates our averaging prescription greatly simplifies while keeping all the required degrees of freedom for applications to general inhomogeneous metric backgrounds. These GLC coordinates turn out to have other interesting properties for calculations on the past light-cone. In particular, the expression of the redshift  $z$  and the luminosity distance  $d_L$  are very simple and easy to interpret in this system of coordinates.

We then turn our attention to the case of a flat FLRW model filled by scalar perturbations of primordial (inflationary) origin. It is both a realistic model for observations and a simple enough case to authorize an analytical approach. Such perturbations are often conveniently parametrized in the longitudinal (or Newtonian) gauge [60], we thus derive the transformation of coordinates between the GLC gauge and the Newtonian gauge. As a stochastic average is necessary to obtain a prediction which is independent from a given realization of inhomogeneities and because these latter give an effect from the second order in perturbations, we compute the luminosity distance up to that order. The transformation of coordinates must then be pushed to second order. This transformation authorizes us then to compute the expression of the distance-redshift relation at second order in scalar perturbations. This very general result <sup>11</sup> has an interest on its own (i.e. irrespectively of its subsequent application to light-cone/ensemble averaging) and can possibly find many other applications in precision cosmology. Furthermore, the result presented here for  $d_L$  is valid in general, i.e. for any given background model (except if caustics form [66], in which case the area distance is modified).

Applying these two notions that we developed, we study the average of scalar observables around the flat FLRW background and at second order in perturbations. For this we make use of an ensemble average over the inhomogeneities, inserting a realistic power spectrum of stochastic perturbations. The combination of the angular lightcone average and this stochastic average leads to characteristic contributions designated as “induced backreaction” terms, arising from a generic correlation between the inhomogeneities present in the variable we want to average (e.g. the luminosity distance) and those appearing in the covariant integration measure <sup>12</sup>. Applying this developments to scalar quantities such as the luminosity flux or distance of a light source lying on our past light-cone, we compute the effects of a stochastic background of inhomogeneities on the determination of dark-energy parameters in precision cosmology. Unlike the analyses in [36, 37], made on spatial hypersurfaces, we find a result always free from ultraviolet divergences and with no significant infrared contributions either. Nevertheless, the induced backreaction terms are not accounting for all the effects engendered by

11. Following the pioneering work of [61, 62],  $d_L$  has been already computed to first order in the longitudinal gauge (for a CDM model in [63], CDM and  $\Lambda$ CDM in [64]), and to second order in synchronous gauge, but only for a dust-dominated Universe, in [42]. See also [65].

12. This integration measure will also induce “backreaction” terms of the type usually discussed in the literature [47, 48], namely terms that arise from (generalized) commutation rules between differential operators and averaging integrals.

inhomogeneities and a full calculation, including all the second order terms, is made.

Proceeding with the simple model of CDM, we find in particular that the energy flux  $\Phi \sim d_L^{-2}$  is practically unaffected by inhomogeneities, while the most commonly used variables (like  $d_L$  or the distance modulus  $\mu \sim 5 \log_{10} d_L$ ) receive much larger corrections (much bigger than we could have naïvely expected). This shows that there are (at least in principle) intrinsic ambiguities in the measurement of the dark-energy parameters, unless the backreaction of stochastic inhomogeneities is properly taken into account. Actually, the advantages of flux averaging for minimizing biases on dark-energy parameters was first pointed out in [67], where it was shown how the binning of data in appropriate redshift intervals can reduce the bias due to systematic effects such as weak lensing. It is intriguing that the preferred role played by the flux variable also comes out in this work where we perform a completely different averaging procedure, *at fixed redshift*. Our conclusions are not due to a binning of data, but to an application of our covariant spacetime average to different functions of the luminosity distance. It is thus safe to believe that this interesting property of the flux is to be related to the conservation of light along the past lightcone.

Our first conclusions of that analysis, based on the use of a perturbation spectrum valid in the linear regime in a CDM model [68], are as follows. On the one hand, such kind of perturbations cannot simulate a substantial fraction of dark energy: their contribution to the averaged flux-redshift relation is both too small (at large values of the redshift  $z$ ) and has the wrong  $z$ -dependence. On the other hand, stochastic fluctuations add a new and relatively important dispersion with respect to the prediction of the homogeneous and isotropic FLRW cosmology. This dispersion is independent of the experimental apparatus, of the observational procedure, of the intrinsic fluctuations in absolute luminosity, and may prevent a determination of the dark-energy parameter  $\Omega_\Lambda(z)$  down to the percent level – at least if we are using the luminosity-redshift relation alone. Another important conclusion is that (light-cone averages of) different functions of the same observable get biased in different ways, with the energy flux sticking out as the observable which gets minimally affected by inhomogeneities, irrespectively of the redshift binning utilized. We should recall here that other possible sources of uncertainty, bias and scatter in the Hubble diagram have been studied in many previous papers (see e.g. [69–73]).

The power spectrum used in this first study is valid in the linear perturbative regime. We also extend our treatment by considering a  $\Lambda$ CDM model, by adding the effect of baryons, and by considering two parametrizations of the HaloFit model [23, 74], describing the density power spectrum in the non-linear regime. It is thus possible, by this choice, to attain a higher level of accuracy by considering the effect from small scale inhomogeneities. We hence use the power spectrum in  $\Lambda$ CDM to compute the effect of inhomogeneities as in the previous study and restricting ourself to the so-called “enhanced terms” (i.e the dominant ones in the relevant range of  $z$ ), already identified in the CDM case. We find that the effect of inhomogeneities is enhanced at large redshifts because of the dominant (lensing) term involved. Despite this increase in the size of the effect, inhomogeneities are not strong enough to mimic a dark energy effect. The dispersion, however, is significantly increased and corresponds to a possible statistical deviation of  $\sim 10\%$  on  $\Omega_{\Lambda 0}$ . This result underlines the importance of considering properly the effect of inhomogeneities on the dispersion of SNe Ia data. We find, furthermore, that this dispersion is in very good agreement with the Union2 data at small redshifts (where peculiar velocity effects are involved) but too small at large redshifts (where lensing is leading their effect). This constitutes a rigorous proof of the robustness of the standard model of cosmology, but still emphasize the non-negligible aspect of inhomogeneities. Finally, our calculations lead to a precise prediction for the size of the lensing dispersion within the Hubble diagram, a prediction which is in accordance with the recent attempts to detect this signal [75, 76] and that should be confirmed in the next few years.

In this **Chapter 1** we have presented the basic facts about homogeneous cosmology, stressing the link between the astrophysical object that the SNe Ia are and the inhomogeneous nature of the Universe they live in. We have also presented the equations of evolution of the Universe in the homogeneous case with some of their assumptions. We shall now give the outline of this thesis.

In **Chapter 2** we treat the aspects relative to the averaging of scalars. We first recall the historical approach, à la Buchert, and present the interesting properties that emerge from this effective formalism. We then recall the notion of gauge invariance in averaging and give a more general definition of the average on spatial hypersurfaces. We then present the definition of a gauge invariant light-cone average prescription which has all the required properties to average scalars on null hypersurfaces. This allows us to derive relations analogous to the Buchert-Ehlers commutation rule. We also rigorously show that the average has, as expected, no effect on an homogeneous FLRW model. In **Chapter 3** we introduce a system of coordinates, the “Geodesic Light-Cone” (GLC) coordinates, which is very well adapted to calculations on the past light-cone. This system corresponds to a complete gauge fixing of the (recalled) observational coordinates, but it also has characteristic differences with it. In these GLC coordinates, the redshift and the luminosity distance take a very simple form and the null geodesics that photons travel on are parametrized by only one coordinate. We also discuss the link between this system and the synchronous gauge and other of its properties. We finally make use of this GLC coordinates to express our averaging prescriptions in a simple way and consider the example of the redshift drift. The absence of averaging effects in the homogeneous case leads us to compute, in **Chapter 4**, the expression of the luminosity-redshift relation at second order in perturbations around a flat FLRW model. For this, we compute the full second order transformation between the Poisson gauge and the GLC gauge and then make use of the simple expression taken by the distance in this last coordinates. Vector and tensor perturbations are taken into account at second order (as imposed by inflation). The final result, though very involved, encloses terms with a clearly defined physical meaning. The role of **Chapter 5** is to present the combination of the lightcone angular average with a stochastic average over inhomogeneities. We first recall the computation of the luminosity-redshift relation by staying at the first order for a better understanding of the physical terms involved. We then present the stochastic average and the power spectrum describing matter inhomogeneities. The formal combination of this average with the lightcone one gives rise to terms created by the mixing of the averaged scalar  $S$  and the measure of integration. These terms are interesting as they authorize an effect of inhomogeneities from the first order of  $S$  and they are called induced backreaction (IBR) terms. Using this recall at first order, we are able to compute the effect of inhomogeneities on the variance of  $d_L$ , an incomplete result, and rigorously derive its dispersion (depending only from first order). We then extend our analysis to the full second order calculation, reorganizing the numerous contributions, and present the calculation of the luminosity flux which turns out to be the quantity which is the least affected by inhomogeneities. We also show that vector and tensor perturbations exactly cancel within our average. We then link the averages of the flux and the distance and address the averaging of the distance modulus. We also present the computation of their respective dispersions and explain how genuine second order contributions are dealt with. The **Chapter 6** presents the numerical computation of the terms previously derived. To that purpose we introduce an explicit form for the transfer function in the power spectrum. This simple expression allows to compute the IBR terms and gives a first estimate of the dispersion on  $d_L$  in a CDM model. It also gives an understanding of the dominant effects coming from inhomogeneities in our calculation, turning out to be the peculiar velocities at small redshift and the lensing at large ones. We also explain why our integrals over the wavenumber  $k$  of perturbations is finite both in the infrared and ultraviolet regimes. We then proceed to the calculation, still in CDM, of the contributions of the luminosity flux. This derivation gives us an insight on the dominant terms at the quadratic and genuine second orders. It also shows that this latter is always smaller for the redshifts we are interested by. We then move to the study of the realistic  $\Lambda$ CDM model by the introduction of a growth factor and the use of a more complete transfer function. Using the relations presented in the previous chapter, we compute also the impact of inhomogeneities on the luminosity distance and the distance modulus. Introducing an extra description of the power spectrum in its non-linear regime, through the so-called HaloFit model, we are able to study the enhanced effects from very small scales. Comments are given on the relative size of the effects and are compared with observations of the dispersion of SNe Ia. We then recall the main results and achievements of this thesis in **Chapter 7**. Further details on the ADM formalism, structure formation, gauge invariance and the HaloFit model are given in **Appendix A**. A French Résumé of this thesis is given in **Appendix B**.



## Chapter 2

# Gauge-invariant spatial & lightcone averaging

This chapter is devoted to the mathematical framework of averaging scalar quantities as part of the [averaging problem](#). We remark that no easy procedure exists at the moment to average vector and tensor quantities in general relativity. Nevertheless, it is important to stress the existence of a very general formalism, historically developed by R. Zalaletdinov and known as *Macroscopic Gravity*, which addresses the question of averaging and evolving general manifolds [77–80]. This work is intimately related to the backreaction issue and deserves a very high consideration. Nevertheless, because of its highly technical aspect and its relative distance with our approach of averaging, it will not be described in this thesis. In a first section, we recall the historical approach of volume averaging over inhomogeneous spacetimes, as mainly developed by T. Buchert [6,30,47,81] and written in synchronous gauge. We also present the notion of gauge invariant averaging and generalize the volume average to a coordinate-independent formulation. Both sections rely mainly on [82,83]. We then present the formalism of light-cone averaging developed in this thesis and show some of its general properties. We derive in particular the generalized Buchert-Ehlers commutation rules on the light cone. As a final check, and to justify the perturbative calculation of Chapter 4, we show that the average effects disappear in the FLRW homogeneous case.

## 2.1 Covariant spatial averaging

### 2.1.1 Buchert formalism

In this formalism, we assume the weak cosmological principle in order to build up effective Friedmann equations inside a simply-connected compact domain  $\mathcal{D}(t)$ , in a foliated spacetime given by constant- $t$  hypersurfaces. We also assume that the matter content of the Universe is made of a pressureless fluid (dust) and a possible cosmological constant, and we work in synchronous gauge :

$$ds_{\text{SG}}^2 = -dt^2 + g_{ij}dx^i dx^j \quad . \quad (2.1)$$

The volume  $\mathcal{D}$ , fixed in these coordinates, conserve the mass  $M_{\mathcal{D}}$  inside it at all times and the average of a scalar quantity  $S$  is defined through a Riemannian (i.e. weighted by the metric) volume average :

$$\langle S(t, \mathbf{x}) \rangle_{\mathcal{D}}(t) = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} S(t, \mathbf{x}) d\mu_g \quad \text{with} \quad V_{\mathcal{D}} = \int_{\mathcal{D}} d\mu_g \quad \text{and} \quad d\mu_g \equiv J d^3\mathbf{x} \equiv \sqrt{\det g_{ij}} d^3\mathbf{x} \quad . \quad (2.2)$$

As  $g_{ij}$ , the measure of integration  $J$  is a function of  $(t, x^i)$ . On the other hand, the volume  $\mathcal{D}$  is assumed to keep the same  $x^i$  coordinates along its way. In synchronous gauge, the matter geodesics are given by  $x^i = \text{cst}$  and the volume is thus comoving with the matter fluid. In that case, assumed in this section, the domain of integration is independent of time (although the measure  $J$  is not).

Within synchronous gauge and the assumptions employed here, the Einstein equations can be recast in terms of geometrical quantities such as the [expansion scalar](#), the [shear tensor](#) and the [vorticity tensor](#), defined as follows :

$$\Theta \equiv \nabla_\mu n^\mu \quad , \quad \sigma_{\mu\nu} \equiv h^\alpha_\mu h^\beta_\nu \left( \nabla_{(\alpha} n_{\beta)} - \frac{1}{3} h_{\alpha\beta} \Theta \right) \quad , \quad \omega_{\mu\nu} \equiv h^\alpha_\mu h^\beta_\nu \nabla_{[\alpha} n_{\beta]} \quad , \quad (2.3)$$

and where  $n^\mu$  is the normal vector to these spatial hypersurfaces given by  $n_\mu = (-1, \vec{0})$  and  $n^\mu = (1, \vec{0})$ . As we consider a fluid which is at rest in these coordinates, we can also consider  $n^\mu$  to be its velocity. The tensor  $h_{\mu\nu}$  presented here is the projection tensor on spatial hypersurfaces :

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad , \quad h_{\mu\nu} n^\nu = 0 \quad . \quad (2.4)$$

The Einstein equations then turn into a system of two constraint and three dynamical equations given by (see Appendix [A.1.3](#)) :

$$\frac{1}{2} \mathcal{R} + \frac{1}{3} \Theta^2 - \sigma^2 - \omega^2 = 8\pi G\rho + \Lambda \quad , \quad (2.5)$$

$$\sigma^i_{j||i} + \omega^i_{j||i} = \frac{2}{3} \Theta_{|j} \quad , \quad (2.6)$$

$$\dot{\rho} = -\Theta\rho \quad , \quad (2.7)$$

$$(g_{ij})^\cdot = 2g_{ik}\sigma^k_j + 2g_{ik}\omega^k_j + \frac{2}{3}\Theta g_{ij} \quad , \quad (2.8)$$

$$(\sigma^i_j + \omega^i_j)^\cdot = -\Theta(\sigma^i_j + \omega^i_j) - \mathcal{R}^i_j + \frac{2}{3}\delta^i_j \left[ \sigma^2 + \omega^2 - \frac{\Theta^2}{3} + 8\pi G\rho + \Lambda \right] \quad . \quad (2.9)$$

The rates of shear and vorticity appearing in these equations are defined as follows<sup>1</sup> :

$$\sigma^2 \equiv \frac{1}{2} \sigma^i_j \sigma^j_i \quad \text{and} \quad \omega^2 \equiv \frac{1}{2} \omega^i_j \omega^j_i \quad . \quad (2.10)$$

$\mathcal{R} \equiv \mathcal{R}^i_i$  is the 3-dimensional spatial curvature of constant  $t$  hypersurfaces. These new forms of the Einstein equations emerge from a more general set of equations in the ADM formalism which is presented in Appendix [A.1](#). They are obtained by taking the [extrinsic curvature](#) tensor, describing the embedding of spatial hypersurfaces in the 4-dimensional  $\mathcal{M}_4$  spacetime, as the opposite value of the [expansion tensor](#) :

$$K_{ij} = -\Theta_{ij} = -\left( \frac{1}{3} h_{ij} \Theta + \sigma_{ij} + \omega_{ij} \right) \quad . \quad (2.11)$$

It can also be defined in terms of the spatial metric by :

$$K^i_j = -\frac{1}{2} g^{ik} (g_{kj})^\cdot \quad . \quad (2.12)$$

From this framework we also obtain a very important equation, not independent from the system ([2.5-2.9](#)), known as the [Raychaudhuri's equation](#) :

$$\dot{\Theta} + \frac{\Theta^2}{3} + 2\sigma^2 + 2\omega^2 + 4\pi G\rho - \Lambda = 0 \quad . \quad (2.13)$$

This relation describes the evolution of the geometry of a bundle of geodesic world lines.

Inspired by these forms of the Hamiltonian constraint, Eq. ([2.5](#)), and Raychaudhuri's equation ([2.13](#)) in the ADM formalism, we define a volume scale factor and its associated Hubble parameter as

$$a_{\mathcal{D}}(t) \equiv \left( \frac{V_{\mathcal{D}}(t)}{V_{\mathcal{D}}(t_i)} \right)^{1/3} \quad , \quad \text{which implies} \quad \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} = 3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \equiv 3H_{\mathcal{D}} \quad . \quad (2.14)$$

1. Notice the difference in the literature where we sometimes define these rates as  $\sigma^2 \equiv \frac{1}{2} \sigma_{ij} \sigma^{ij}$  and  $\omega^2 \equiv \frac{1}{2} \omega_{ij} \omega^{ij}$ . In this case,  $\omega^{ij}$  being anti-symmetric, we get the opposite sign for  $\omega^2$  in the equations.

The relationship between this effective scale factor and the homogeneous scale factor is a bit hard to interpret. Indeed, one can see that  $a_{\mathcal{D}}(t)$  becomes equivalent to  $a(t)$  if the matter content is taken to be homogeneous, and this whatever the size and form of the domain  $\mathcal{D}$ , but on the other hand very different domains can give the same scale factor in an inhomogeneous Universe. One can nevertheless find equations of evolution for  $a_{\mathcal{D}}$  and this is the purpose of the following considerations.

Using the quantity  $J(t, \mathbf{x}) \equiv \sqrt{\det g_{ij}}$  defined in Eq. (2.2), we obviously have

$$(\ln J)^\cdot = \frac{1}{2} g^{ik} (g_{ki})^\cdot = -K \quad \text{so that} \quad \dot{J} = -KJ = \Theta J \quad . \quad (2.15)$$

From this last equality we obtain that

$$\langle \Theta \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \frac{\dot{J}}{J} J d^3 \mathbf{x} = \frac{1}{V_{\mathcal{D}}} \left( \int_{\mathcal{D}} J d^3 \mathbf{x} \right)^\cdot = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} \quad . \quad (2.16)$$

One can see then that the Eq. (2.7) is simply written as

$$(\rho J)^\cdot = 0 \quad \text{with the solution} \quad \rho(t, \mathbf{x}) = \frac{\rho(t_0, \mathbf{x}) J(t_0, \mathbf{x})}{J(t, \mathbf{x})} \quad . \quad (2.17)$$

This identity implies that the mass contained in the domain  $\mathcal{D}$  is, as requested, constant in time :

$$M_{\mathcal{D}} = \int_{\mathcal{D}} \rho J d^3 \mathbf{x} = \text{cst} \quad , \quad (2.18)$$

and so the averaged density, starting with an initial volume  $V_{\mathcal{D}_0} \equiv V_{\mathcal{D}}(t_0)$ , becomes

$$\langle \rho \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \rho J d^3 \mathbf{x} = \frac{M_{\mathcal{D}}}{V_{\mathcal{D}}} = \frac{M_{\mathcal{D}}}{V_{\mathcal{D}_0} a_{\mathcal{D}}^3} \quad . \quad (2.19)$$

For all the scalar quantities  $S$  encountered, such as  $\rho$ , we can take the time derivative of its average  $\langle S \rangle_{\mathcal{D}}$  and we are lead to the *commutation rule* :

$$\langle S \rangle_{\mathcal{D}}^\cdot - \langle \dot{S} \rangle_{\mathcal{D}} = \langle S \Theta \rangle_{\mathcal{D}} - \langle S \rangle_{\mathcal{D}} \langle \Theta \rangle_{\mathcal{D}} \quad . \quad (2.20)$$

This relation shows that the time derivation of  $S$ , in other words its evolution in time, does not commute with its average  $\langle \dots \rangle_{\mathcal{D}}$ . This is related to the long-standing problem of backreaction, as presented by Ellis in the 80's (see [28]), and it shows that the spatially averaged quantity  $S$  does not follow, on average, the evolution of  $S$  when we have a space-dependent expansion scalar (or equivalently an extrinsic curvature).

### Averaging the Einstein equations

Averaging [Raychaudhuri's equation](#) (2.13) of a pressureless fluid ( $p = 0$ ), and using the commutation rule (2.20) with  $S = \Theta$ , leads to

$$\langle \Theta \rangle_{\mathcal{D}}^\cdot - \frac{2}{3} \langle \Theta^2 \rangle_{\mathcal{D}} + \langle \Theta \rangle_{\mathcal{D}}^2 + 2 \langle \sigma^2 \rangle_{\mathcal{D}} + 2 \langle \omega^2 \rangle_{\mathcal{D}} + 4\pi G \langle \rho \rangle_{\mathcal{D}} - \Lambda = 0 \quad , \quad (2.21)$$

which is equivalent to

$$3 \frac{\ddot{a}_{\mathcal{D}}(t)}{a_{\mathcal{D}}(t)} + 4\pi G \frac{M_{\mathcal{D}}}{V_{\mathcal{D}_0} a_{\mathcal{D}}^3} - \Lambda = \mathcal{Q}_{\mathcal{D}} \quad , \quad (2.22)$$

with the so-called *backreaction parameter* :

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3} \left[ \langle \Theta^2 \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}}^2 \right] - 2 \langle \sigma^2 \rangle_{\mathcal{D}} - 2 \langle \omega^2 \rangle_{\mathcal{D}} \quad . \quad (2.23)$$

Using now the [Hamiltonian constraint](#) of Eq. (2.5) and taking its average, one obtains

$$\frac{1}{2} \langle \mathcal{R} \rangle_{\mathcal{D}} + \frac{1}{3} \langle \Theta^2 \rangle_{\mathcal{D}} - 8\pi G \langle \rho \rangle_{\mathcal{D}} - \Lambda = \langle \sigma^2 \rangle_{\mathcal{D}} + \langle \omega^2 \rangle_{\mathcal{D}} \quad . \quad (2.24)$$

Removing  $\frac{1}{3} \left[ \langle \Theta^2 \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}}^2 \right]$  on both sides, and using that  $\langle \Theta \rangle_{\mathcal{D}} = 3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}$ , one gets

$$3 \left( \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 + \frac{1}{2} \langle \mathcal{R} \rangle_{\mathcal{D}} - 8\pi G \frac{M_{\mathcal{D}}}{V_{\mathcal{D}_0} a_{\mathcal{D}}^3} - \Lambda = -\frac{\mathcal{Q}_{\mathcal{D}}}{2} . \quad (2.25)$$

We can remark that the [vorticity tensor](#) plays a similar role as the [shear tensor](#) in these calculations.

We shall assume now that the considered fluid has no vorticity, so:  $\omega_{\mu\nu} = 0 \quad \forall \mu, \nu$  and  $\omega = 0$ . Gathering Raychaudhuri's equation and the Hamiltonian constraint, and allowing both for a constant curvature  $K_{\mathcal{D}_i}$  of a FLRW model, one gets the general expressions :

$$3 \frac{\ddot{a}_{\mathcal{D}}(t)}{a_{\mathcal{D}}(t)} = -4\pi G \langle \rho \rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}} + \Lambda , \quad (2.26)$$

$$3H_{\mathcal{D}}^2 = 8\pi G \langle \rho \rangle_{\mathcal{D}} - \frac{3K_{\mathcal{D}_i}}{a_{\mathcal{D}}^2} - \frac{1}{2} (\mathcal{W}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}) + \Lambda . \quad (2.27)$$

We defined here the deviation of the curvature from the constant curvature by the quantity :

$$\mathcal{W}_{\mathcal{D}} = \langle \mathcal{R} \rangle_{\mathcal{D}} - 6 \frac{K_{\mathcal{D}_i}}{a_{\mathcal{D}}^2} . \quad (2.28)$$

As in the classical Friedmann derivation, these two equations and the conservation equation

$$\langle \rho \rangle_{\mathcal{D}} \dot{\phantom{\rho}} + 3H_{\mathcal{D}} \langle \rho \rangle_{\mathcal{D}} = 0 \quad (2.29)$$

only form two independent ones. Differentiating Eq. (2.27) with respect to the cosmic time  $t$  and then making use of Eqs. (2.26) and (2.29), one gets the consistency relation – or so-called [integrability condition](#) – involving only geometrical quantities :

$$\dot{\mathcal{Q}}_{\mathcal{D}} + 6H_{\mathcal{D}} \mathcal{Q}_{\mathcal{D}} + \dot{\mathcal{W}}_{\mathcal{D}} + 2H_{\mathcal{D}} \mathcal{W}_{\mathcal{D}} = 0 \iff a_{\mathcal{D}}^{-6} (a_{\mathcal{D}}^6 \mathcal{Q}_{\mathcal{D}}) \dot{\phantom{Q}} + a_{\mathcal{D}}^{-2} (a_{\mathcal{D}}^2 \mathcal{W}_{\mathcal{D}}) \dot{\phantom{W}} = 0 . \quad (2.30)$$

This condition guaranties the Friedmann-like form of the above equations, especially for Eq. (2.29), and gives a dependence between the averaged 3-dimensional curvature  $\langle \mathcal{R} \rangle_{\mathcal{D}}$  (an intrinsic curvature invariant) and the backreaction term  $\mathcal{Q}_{\mathcal{D}}$  (an [extrinsic curvature](#) invariant) which involves expansion, shear and vorticity scalars induced by the inhomogeneous aspect of the compact domain  $\mathcal{D}$ .

One can see from Eq. (2.30) that  $\mathcal{Q}_{\mathcal{D}} = 0$  gives  $\langle \mathcal{R} \rangle_{\mathcal{D}} \propto a_{\mathcal{D}}^{-2}$ . This case contains the particular one of a FLRW model where there is no structure and the spatial curvature goes like  $\mathcal{R} \propto a^{-2}$ . One does not encounter any backreaction also in the case where  $\mathcal{Q}_{\mathcal{D}} \propto a_{\mathcal{D}}^{-6}$ . On the other hand, when none of these cases is satisfied, the relation between extrinsic and intrinsic curvatures is non-trivial. It is possible (i.e. not scientifically rejected) that small perturbations of the averaged spatial curvature and the averaged expansion rate could grow to lead finally to a global instability of the perturbed FLRW model. Our real Universe tends to be dominated by voids and we can suspect that the averaged curvature tends to be negative. It is believed that this situation, lead from the development of structures, could affect the evolution of the averaged expansion rate and, maybe, explain dark energy.

The backreaction is a profound issue which has also been addressed in [N-body simulations](#). The problem of these approaches is that backreaction is also closely related to topological issues. Indeed, when we use a Newtonian description to study structures formation in a spatially flat background, there is no spatial curvature and there are six different orientable space forms out of which we usually take the 3-torus. The backreaction in that case is equivalent to a boundary contribution. Such a study is thus believed to directly kill any possible backreaction mechanism and cannot self-generate an acceleration of the expansion. To be fair, we should also stress that the idea of backreaction could be completely wiped out by mathematics (like a “*No-backreaction*” theorem). Indeed, the consideration of general relativity in 2+1 dimensions and the use of the Gauss-Bonnet theorem proves that the volume integral of the spatial curvature in a closed Riemannian 2-space is a topological invariant directly related to the Euler characteristic of the 2-dimensional manifold (see [6]). It follows that  $\langle \mathcal{R} \rangle_{\mathcal{D}} \propto a_{\mathcal{D}}^{-2}$  and  $\mathcal{Q}_{\mathcal{D}} \propto a_{\mathcal{D}}^{-4}$  (the right exponent to have no effect in a 2+1 geometry, where the coefficient 6 in Eq. (2.30) is replaced by 4). Thus the backreaction is exactly canceled. Nevertheless, the relation between curvature and topology is more involved in 3+1 gravity, and no theorem has been proved. So backreaction is still an open question in 4-dimensional spacetimes.

### Rewriting of the terms, morphon field

One can define the following cosmological-like parameters :

$$\Omega_m^{\mathcal{D}} \equiv \frac{8\pi G}{3} \frac{\langle \rho \rangle_{\mathcal{D}}}{H_{\mathcal{D}}^2} \quad , \quad \Omega_{\Lambda}^{\mathcal{D}} \equiv \frac{\Lambda}{3H_{\mathcal{D}}^2} \quad , \quad \Omega_K^{\mathcal{D}} \equiv -\frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6H_{\mathcal{D}}^2} \quad , \quad \Omega_{\mathcal{Q}}^{\mathcal{D}} \equiv -\frac{\mathcal{Q}_{\mathcal{D}}}{6H_{\mathcal{D}}^2} \quad , \quad (2.31)$$

and see that the Hamiltonian constraint (to compare with Eq. (1.21)) is simply written as

$$\Omega_m^{\mathcal{D}} + \Omega_{\Lambda}^{\mathcal{D}} + \Omega_K^{\mathcal{D}} + \Omega_{\mathcal{Q}}^{\mathcal{D}} = 1 \quad . \quad (2.32)$$

These parameters directly depend on the domain of integration  $\mathcal{D}$  and this last equation has to be interpreted with care. We can also rewrite the averaged Friedmann-like equations derived above in terms of a scalar field which has been called the **mophon**<sup>2</sup>, denoted by  $\Phi_{\mathcal{D}}$ . One then defines the effective density and pressure for this field by :

$$\rho_{\mathcal{D}}^{\Phi} \equiv -\frac{1}{16\pi G} (\mathcal{Q}_{\mathcal{D}} + \mathcal{W}_{\mathcal{D}}) \quad , \quad p_{\mathcal{D}}^{\Phi} \equiv -\frac{1}{16\pi G} \left( \mathcal{Q}_{\mathcal{D}} - \frac{\mathcal{W}_{\mathcal{D}}}{3} \right) \quad . \quad (2.33)$$

The generalized Friedmann equations then read :

$$3 \frac{\ddot{a}_{\mathcal{D}}(t)}{a_{\mathcal{D}}(t)} = -4\pi G (\langle \rho \rangle_{\mathcal{D}} + \rho_{\mathcal{D}}^{\Phi} + 3p_{\mathcal{D}}^{\Phi}) + \Lambda \quad , \quad (2.34)$$

$$3H_{\mathcal{D}}^2 = 8\pi G \langle \rho \rangle_{\mathcal{D}} - \frac{3K_{\mathcal{D}_i}}{a_{\mathcal{D}}^2} - \frac{1}{2} (\mathcal{W}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}) + \Lambda \quad , \quad (2.35)$$

and we have two equations of conservation :

$$\langle \rho \rangle_{\mathcal{D}} \dot{\quad} + 3H_{\mathcal{D}} \langle \rho \rangle_{\mathcal{D}} = 0 \quad , \quad \rho_{\mathcal{D}}^{\Phi} \dot{\quad} + 3H_{\mathcal{D}} (\rho_{\mathcal{D}}^{\Phi} + p_{\mathcal{D}}^{\Phi}) = 0 \quad . \quad (2.36)$$

We can easily check that this last condition is equivalent to the integrability condition expressed in Eq. (2.30). It is an amazing fact to see that the field  $\Phi_{\mathcal{D}}$  defined here has all the properties to effectively describe a cosmological constant.

We can also re-express these quantities by defining directly the field  $\Phi_{\mathcal{D}}$  and its potential  $U_{\mathcal{D}}$  :

$$\rho_{\mathcal{D}}^{\Phi} = \frac{e}{2} \dot{\Phi}_{\mathcal{D}}^2 + U_{\mathcal{D}} \quad , \quad p_{\mathcal{D}}^{\Phi} = \frac{e}{2} \dot{\Phi}_{\mathcal{D}}^2 - U_{\mathcal{D}} \quad , \quad (2.37)$$

where  $e = +1$  if  $\Phi_{\mathcal{D}}$  is a standard scalar field and  $e = -1$  if it is a phantom scalar field. One can also show the correspondence between the terms

$$\mathcal{Q}_{\mathcal{D}} = -8\pi G \left( e \dot{\Phi}_{\mathcal{D}}^2 - U_{\mathcal{D}} \right) \quad , \quad \mathcal{W}_{\mathcal{D}} = -24\pi G U_{\mathcal{D}} \quad , \quad (2.38)$$

and show that the integrability condition can be recast in a Klein-Gordon equation for  $\Phi_{\mathcal{D}}$  :

$$\ddot{\Phi}_{\mathcal{D}} + 3H_{\mathcal{D}} \dot{\Phi}_{\mathcal{D}} + e \frac{\partial U_{\mathcal{D}}}{\partial \Phi_{\mathcal{D}}} (\Phi_{\mathcal{D}}, \langle \rho \rangle_{\mathcal{D}}) = 0 \quad . \quad (2.39)$$

This equation is scale dependent through its dependence in  $\mathcal{D}$ . Last, we can define effective kinetic and potential energies of the morphon field by

$$E_{\mathcal{D}}^{\text{kin}} = e \dot{\Phi}_{\mathcal{D}}^2 V_{\mathcal{D}} \quad , \quad E_{\mathcal{D}}^{\text{pot}} = -U_{\mathcal{D}} V_{\mathcal{D}} \quad , \quad (2.40)$$

and we get the simple relation

$$2E_{\mathcal{D}}^{\text{kin}} + E_{\mathcal{D}}^{\text{pot}} = -\frac{\mathcal{Q}_{\mathcal{D}} V_{\mathcal{D}}}{8\pi G} \quad . \quad (2.41)$$

We deduce that an homogeneous Universe, for which  $\mathcal{Q}_{\mathcal{D}} = 0$ , is equivalent to the *virial equilibrium condition* in this effective formalism.

---

2. This scalar field is indeed sourced by inhomogeneities and thus captures the ‘‘morphological’’ signature of structures.

### Kullback-Leibler relative entropy

Applying the commutation rule, Eq. (2.20), to the density of matter brings the relation :

$$\langle \dot{\rho} \rangle_{\mathcal{D}} - \langle \rho \Theta \rangle_{\mathcal{D}} = \langle \rho \Theta \rangle_{\mathcal{D}} - \langle \rho \rangle_{\mathcal{D}} \langle \Theta \rangle_{\mathcal{D}} . \quad (2.42)$$

To describe the relative difference between the inhomogeneous function  $\rho(t, x^i)$  inside the domain  $\mathcal{D}$  with respect to the homogeneous function  $\langle \rho \rangle_{\mathcal{D}}$ , one can use the concept of relative information given by the **Kullback-Leibler entropy** [30]. This entropy is defined, for the density function, as<sup>3</sup> :

$$\mathcal{S}_{\mathcal{D}} = \int_{\mathcal{D}} \rho \ln \left( \frac{\rho}{\langle \rho \rangle_{\mathcal{D}}} \right) J d^3x . \quad (2.43)$$

One can then show that :

$$-\frac{\dot{\mathcal{S}}_{\mathcal{D}}}{V_{\mathcal{D}}} = - \left\langle \frac{(\rho J)^{\cdot}}{J} \ln \left( \frac{\rho}{\langle \rho \rangle_{\mathcal{D}}} \right) \right\rangle_{\mathcal{D}} - \langle \rho \rangle_{\mathcal{D}} \left\langle \left( \frac{\rho}{\langle \rho \rangle_{\mathcal{D}}} \right)^{\cdot} \right\rangle_{\mathcal{D}} , \quad (2.44)$$

and use that  $(\rho J)^{\cdot} = 0$ , which eliminates the first term. The second one is developed, using Eq. (2.15), (2.17) and (2.29), as :

$$-\frac{\dot{\mathcal{S}}_{\mathcal{D}}}{V_{\mathcal{D}}} = - \langle \rho \rangle_{\mathcal{D}} \left\langle \frac{\dot{\rho}}{\langle \rho \rangle_{\mathcal{D}}} - \rho \frac{\langle \dot{\rho} \rangle_{\mathcal{D}}}{\langle \rho \rangle_{\mathcal{D}}^2} \right\rangle_{\mathcal{D}} = \langle \rho \Theta \rangle_{\mathcal{D}} - \langle \rho \rangle_{\mathcal{D}} \langle \Theta \rangle_{\mathcal{D}} . \quad (2.45)$$

We thus proved that the time variation of the Kullback-Liebler entropy is proportional to the non-commutation terms linking the density of matter and the expansion scalar. One can thus understand that the definition of averaged quantities in cosmology has the first obvious consequence of a loss in information, but secondly has the drawback to introduce a difference in the evolution of averaged quantities with respect to their true non-averaged equivalents.

### 2.1.2 Gauge invariance

The average of a quantity naturally depends on the physical hypervolume used to define it. It should not, on the other hand, depend on the chosen coordinates. This fact, called **gauge invariance**, is a general concept which has some subtleties. Here we will solely present the necessary relations to understand the gauge problem that appears when we try to define averaged quantities. More details about the notion of gauge invariance are presented in Appendix A.3 .

Let us consider a spacetime  $\mathcal{M}_4$  equipped with a system of coordinates  $\{x^\mu\}$  and a Riemannian metric  $g_{\mu\nu}$  to describe its geometry. We also consider a scalar function  $S(x)$  on these coordinates. Under a **General Coordinate Transformation (GCT)**, where  $h$  is a matrix function that we suppose invertible, we have the transformation

$$x \mapsto \hat{x} = h(x) \quad : \quad S(x) \mapsto \hat{S}(\hat{x}) \quad \text{such that} \quad \hat{S}(\hat{x}) = S(x) = S(h^{-1}(\hat{x})) . \quad (2.46)$$

A **Gauge Transformation (GT)**  $f$  – or “local reparametrization of the fields”, supposed invertible – on the other hand consists in the evaluation of the field at the same physical point, thus being given by

$$x \mapsto \tilde{x} = f(x) \quad : \quad S(x) \mapsto \tilde{S}(\tilde{x}) = S(f^{-1}(x)) . \quad (2.47)$$

We can remark that the evaluation of this last equality at  $\tilde{x}$  gives  $\tilde{S}(\tilde{x}) = S(x)$  which is equivalent to Eq. (2.46) if  $f \equiv h$  (and thus  $\hat{S} \equiv \tilde{S}$ ). Under the gauge transformation, the metric is modified in the following way :

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(x) = \left[ \frac{\partial x^\alpha}{\partial f^\mu} \frac{\partial x^\beta}{\partial f^\nu} \right]_{f^{-1}(x)} g_{\alpha\beta}(f^{-1}(x)) \quad (2.48)$$

3. We should notice that this relative entropy can be related to the Weyl entropy, as recently proved in [84].

where the squared brackets contain the product of inverse matrices  $(\partial f/\partial x)^{-1}$  in which we finally replace  $x$  by  $f^{-1}(x)$ . The square root of the determinant  $g \equiv \det g_{\mu\nu}$  is then transformed as

$$\sqrt{-g(x)} \rightarrow \sqrt{-\tilde{g}(x)} = \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))} , \quad (2.49)$$

being thus multiplied by the inverse Jacobian determinant of the gauge transformation  $x \mapsto f(x)$ .

Let us consider now the integration of a scalar quantity  $S(x)$  on a spacetime domain  $\mathcal{D} \subset \mathcal{M}_4$  spanned by the coordinates  $\{x^\mu\}$ . The change of this integration under the gauge transformation  $x \mapsto f(x)$ , when the domain  $\mathcal{D}$  is kept unchanged by this gauge transformation, is given by :

$$I(S, \mathcal{D}) = \int_{\mathcal{D}(x)} d^4x \sqrt{-g(x)} S(x) \mapsto \tilde{I}(\tilde{S}, \mathcal{D}) = \int_{\mathcal{D}(x)} d^4x \sqrt{-\tilde{g}(x)} \tilde{S}(x) . \quad (2.50)$$

We can now proceed to a change of coordinates. Using  $\bar{x} = f^{-1}(x)$ , we write :

$$\tilde{I}(\tilde{S}, \mathcal{D}) = \int_{\mathcal{D}(f(\bar{x}))} d^4\bar{x} \left| \frac{\partial f}{\partial \bar{x}} \right|_{\bar{x}} \sqrt{-\tilde{g}(f(\bar{x}))} \tilde{S}(f(\bar{x})) = \int_{\overline{\mathcal{D}}(\bar{x})} d^4\bar{x} \sqrt{-\tilde{g}(\bar{x})} S(\bar{x}) = I(S, \overline{\mathcal{D}}) , \quad (2.51)$$

where we used the relations (2.49) and (2.47), and introduced at the end the new spacetime domain  $\overline{\mathcal{D}}$  such that  $\overline{\mathcal{D}}(\bar{x}) = \mathcal{D}(x)$ . We conclude that the integral  $I(S, \mathcal{D})$  is not gauge invariant if the domain  $\mathcal{D}$  is unchanged under the gauge transformation. In other words, the domain of integration must be gauge-dependent and transform in a precise way to keep the integral  $I(S, \mathcal{D})$  gauge invariant.

The transformation of this domain under gauge transformations can easily be found. Indeed, let us assume that the domain  $\mathcal{D}$  depends on the coordinates  $\{x^\mu\}$  and let us implement its shape in  $\mathcal{M}_4$  by the use of a [window function](#)  $W_{\mathcal{D}}(x)$  in the integral :

$$I(S, \mathcal{D}) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} W_{\mathcal{D}}(x) S(x) . \quad (2.52)$$

If we assume that this function transforms as

$$W_{\mathcal{D}}(x) \mapsto \widetilde{W}_{\mathcal{D}}(x) = W_{\mathcal{D}}(f^{-1}(x)) , \quad (2.53)$$

under the gauge transformation  $f$ , we then have the new transformation of the integral :

$$I(S, \mathcal{D}) \mapsto \tilde{I}(\tilde{S}, \mathcal{D}) = \int_{\mathcal{M}_4} d^4\bar{x} \sqrt{-\tilde{g}(\bar{x})} W_{\mathcal{D}}(\bar{x}) S(\bar{x}) = I(S, \mathcal{D}) , \quad (2.54)$$

which shows the gauge invariance of the integral as expected. We can understand that if  $W_{\mathcal{D}}$  is only made of scalars, such that it is still invariant under  $x \mapsto f(x)$ , then the gauge invariance is respected (and the GCT invariance is also satisfied). On the other hand, if  $W_{\mathcal{D}}$  is not invariant, we say that the domain of integration breaks gauge invariance.

One such example that we can easily visualize is a cylindrical-type domain delimited by two spacelike hypersurfaces (i.e. defined by timelike normal vectors)  $A(x) = A_1$  and  $A(x) = A_2 > A_1$ . In properly chosen coordinates, this hypervolume can be restricted by the addition of a timelike hypersurface  $B(x) = r_0$  (defined by radial spacelike normal vectors), as we will illustrate in Fig. 2.1.  $W_{\mathcal{D}}$  is then simply given in terms of Heaviside step functions by :

$$W_{\mathcal{D}}(x) = \Theta(A(x) - A_1) \Theta(A_2 - A(x)) \Theta(r_0 - B(x)) , \quad (2.55)$$

and we have here a domain  $\mathcal{D}$  which is gauge invariant if  $A$  and  $B$  are scalars. Otherwise, the physical domain is not anymore the same after a gauge transformation and breaks gauge invariance of quantities defined on this domain (such as  $I(S, \mathcal{D})$ ).

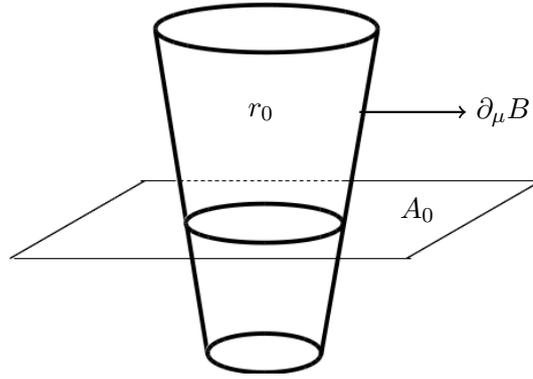


Figure 2.1: A graphic illustration of the introduction of a new parametrized hypersurface  $B(x) = r_0$  to create a boundary inside the constant- $A$  hypersurface. This newly introduced hypersurface has for spacelike gradient  $\partial_\mu B$ .

### 2.1.3 Gauge invariant generalization of Buchert's spatial average

In this section, we wish to present in a more general way the definition of an average on a spatial hypersurface, i.e. the hypersurface orthogonal to a vector  $n^\mu$  which is timelike. Indeed, Buchert's definition of the average presented in Sec. 2.1 has been written in a given gauge (a synchronous gauge) and we intend to give a definition which does not depend on a gauge choice. We want to derive then the Buchert-Ehlers relation associated to this hypersurface, as it was done in [83].

Let us define a spatial hypersurface  $\Sigma(A)$  whose points share the same value of a scalar field  $A(x)$ . This hypersurface can be defined by its (past-directed) normal vector :

$$n_\mu \equiv -\frac{\partial_\mu A}{\sqrt{-\partial_\mu A \partial^\mu A}} \quad (2.56)$$

which is a timelike vector as  $n_\mu n^\mu = -1$ . We choose a finite spatial domain  $\mathcal{D}$  inside this hypersurface by adding up a scalar field  $B(x)$  which can be defined by its spacelike vector  $\propto \partial_\mu B$ , see Fig. 2.1. We can then write the integration of a scalar  $S$ , as seen in Eq. (2.52), by

$$I(S; A_0, r_0) = \int_{\mathcal{D}(x)} d^4x \sqrt{-g(x)} S(x) \equiv \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} S(x) W_{\mathcal{D}}(x) \quad , \quad (2.57)$$

where the window function is

$$W_{\mathcal{D}}(x) = n^\mu \nabla_\mu \Theta(A(x) - A_0) \Theta(r_0 - B(x)) = \sqrt{-\partial_\mu A \partial^\mu A} \delta_D(A(x) - A_0) \Theta(r_0 - B(x)) \quad , \quad (2.58)$$

and we have used that  $n^\mu \nabla_\mu \Theta(A(x) - A_0) = n^\mu \partial_\mu A \delta_D(A(x) - A_0)$ . As previously shown, this integral is gauge invariant as long as  $A$  and  $B$  are gauge invariant scalars. We can thus define the gauge invariant average of a scalar  $S(x)$  by the volume-weighted ratio :

$$\langle S \rangle_{A_0, r_0} = \frac{I(S; A_0, r_0)}{I(1; A_0, r_0)} \quad . \quad (2.59)$$

Let us now compute the derivative of this average with respect to the time parameter  $A_0$  and obtain a generalization of the so-called Buchert-Ehlers commutation rules [58].

Taking the derivative of  $I(S; A_0, r_0)$  with respect to the timelike parameter  $A_0$ , and using the following relation

$$\partial_{A_0} \delta(A(x) - A_0) = -\delta'_D(A(x) - A_0) \quad , \quad (2.60)$$

we get

$$\frac{\partial I(S; A_0, r_0)}{\partial A_0} = - \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \sqrt{-\partial_\mu A \partial^\mu A} \delta'_D(A(x) - A_0) \Theta(r_0 - B(x)) S(x) \quad . \quad (2.61)$$

We can then use an adapted system of coordinates such as ADM coordinates as it is done in [83]. Nevertheless, a more direct derivation is found as follows. Let us consider the identity :

$$n^\mu \nabla_\mu \delta_D(A(x) - A_0) = \sqrt{-\partial_\mu A \partial^\mu A} \delta'_D(A(x) - A_0) . \quad (2.62)$$

From this we get :

$$\begin{aligned} \frac{\partial I(S; A_0, r_0)}{\partial A_0} &= + \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta_D(A(x) - A_0) \nabla_\mu [n^\mu \Theta(r_0 - B(x)) S(x)] \\ &= \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta_D(A(x) - A_0) \Theta(r_0 - B(x)) \frac{n^\mu \nabla_\mu S(x)}{\sqrt{-\partial_\mu A \partial^\mu A}} \sqrt{-\partial_\mu A \partial^\mu A} \\ &\quad + \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta_D(A(x) - A_0) \Theta(r_0 - B(x)) \frac{\Theta(x) S(x)}{\sqrt{-\partial_\mu A \partial^\mu A}} \sqrt{-\partial_\mu A \partial^\mu A} \\ &\quad - \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta_D(A(x) - A_0) n^\mu \nabla_\mu \Theta(r_0 - B(x)) S(x) , \end{aligned} \quad (2.63)$$

where in the first equality we used Eq. (2.62) followed by an integration by parts, with the identity  $\int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \nabla_\mu(\dots) = 0$ . This is true because the contribution (...) is zero at infinity, either from the fact that the domain of integration is rendered finite by the distributions, or simply because  $S$  can be taken to be zero at infinity. The last integral contains the following function :

$$n^\mu \nabla_\mu \Theta(r_0 - B(x)) = -n^\mu \nabla_\mu \delta_D(r_0 - B(x)) = -2n^\mu \partial_\mu B \Theta(r_0 - B(x)) \delta_D(r_0 - B(x)) , \quad (2.64)$$

where the factor two in the last equality can be understood from the definition of the delta-functions as the zero-width limit of a Gaussian distribution centered on zero. We understand also that this last integral is related to an integration on the boundary  $\partial\mathcal{D}$  of the spatial domain  $\mathcal{D}$ .

Finally, using the definition of the integral in Eqs. (2.57) and (2.58), we obtain :

$$\begin{aligned} \frac{\partial I(S; A_0, r_0)}{\partial A_0} &= I\left(\frac{\partial_\mu A \partial^\mu S(x)}{\partial_\mu A \partial^\mu A}; A_0, r_0\right) + I\left(\frac{\Theta(x) S(x)}{\sqrt{-\partial_\mu A \partial^\mu A}}; A_0, r_0\right) \\ &\quad - 2I\left(\frac{\partial_\mu A \partial^\mu B}{\partial_\mu A \partial^\mu A} \delta_D(r_0 - B(x)) S(x); A_0, r_0\right) , \end{aligned} \quad (2.65)$$

where we have done trivial changes (such as  $\partial^\mu A \nabla_\mu S \equiv \nabla^\mu A \nabla_\mu S = \nabla_\mu A \nabla^\mu S = \partial_\mu A \partial^\mu S$  as both  $A$  and  $S$  are scalars). Dividing now by the volume  $I(1; A_0, r_0)$  and taking the limit  $r_0 \rightarrow \infty$ , i.e. integrating over the whole spatial hypersurface fixed by  $A(x) = A_0$ , we have that  $\langle \dots \rangle_{A_0, r_0} \rightarrow \langle \dots \rangle_{A_0}$  and we obtain the generalization of the Buchert-Ehlers [commutation rule](#) :

$$\frac{\partial \langle S \rangle_{A_0}}{\partial A_0} = \left\langle \frac{\partial_\mu A \partial^\mu S(x)}{\partial_\mu A \partial^\mu A} \right\rangle_{A_0} + \left\langle \frac{\Theta(x) S(x)}{\sqrt{-\partial_\mu A \partial^\mu A}} \right\rangle_{A_0} - \left\langle \frac{\Theta(x)}{\sqrt{-\partial_\mu A \partial^\mu A}} \right\rangle_{A_0} \langle S(x) \rangle_{A_0} . \quad (2.66)$$

We understand the direct interpretation of this relation by going in the [Arnowitt-Deser-Misner \(ADM\)](#) coordinates. The spacetime  $\mathcal{M}_4$  in ADM is foliated by hypersurfaces of constant time  $t$  and the line element is given by :

$$ds_{ADM}^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt) , \quad (2.67)$$

and the normal vectors  $n^\mu$  and  $n_\mu$  (see Sec. [A.1.1](#)) are written as

$$n_\mu = N(-1, 0, 0, 0) , \quad n^\mu = \frac{1}{N}(1, -N^i) . \quad (2.68)$$

One can then show that the volume [expansion scalar](#)  $\Theta(x)$  in that case is

$$\Theta(x) = \frac{1}{N} \partial_t \ln \left( \sqrt{\det(g_{ij})} \right) , \quad (2.69)$$

where  $g_{ij}$  is the spatial part of the metric  $g_{\mu\nu}^{ADM}$  encoding the geometry inside the constant- $t$  hypersurface. In those coordinates  $A$  is homogeneous on  $t = \text{cst}$  hypersurfaces. So we can take  $A_0 \equiv t$  and we find that the Buchert-Ehlers commutation rule of Eq. (2.20) reduces to the simple relation :

$$\frac{\partial \langle S \rangle_{A_0}}{\partial A_0} = \left\langle \frac{\partial S}{\partial A_0} \right\rangle_{A_0} + \left\langle \frac{\Theta N}{\partial_t A} S \right\rangle_{A_0} - \left\langle \frac{\Theta N}{\partial_t A} \right\rangle_{A_0} \langle S \rangle_{A_0} , \quad (2.70)$$

where  $S = S(A_0, \mathbf{x})$  and we have used  $\partial_\mu A \partial^\mu A \equiv g_{ADM}^{\mu\nu} \partial_\mu A \partial_\nu A = g_{ADM}^{tt} (\partial_t A)^2 = -N^{-2} (\partial_t A)^2$ . We used here that in ADM coordinates, where  $A$  is homogeneous, we can always choose a vanishing shift vector (and so  $g_{tt}^{ADM} = -N^2$  and  $g_{ti}^{ADM} = 0$ ) corresponding to a partial gauge fixing.

The interpretation of Eq. (2.70) is similar to the commutation rule derived in Eq. (2.20). Indeed, we see that the time derivative of the average of  $S$  is equal to the average of the time derivative of  $S$  (first term on the RHS) followed by the addition of two terms which involve the volume expansion  $\Theta(x)$  and the scalar  $S$  in this schematic form:  $\langle \Theta S \rangle - \langle \Theta \rangle \langle S \rangle$ . If  $A(x)$  is taken to be simply  $t$  and if we fix the gauge to be synchronous, i.e.  $N = 1$ , we get exactly the relation presented in Eq. (2.20).

### 2.1.4 Generalization of Friedmann equations in a gauge invariant form

Let  $u^\mu$  be the fluid velocity field and  $n^\mu$  the field orthonormal to the hypersurface of averaging. This vector  $n^\mu$  can be seen as the peculiar velocity of the observer if this latter one is not comoving with the fluid (as assumed in Sec. 2.1.1) but is in general different from it. We define a tilt angle between these two fields by

$$u_\mu n^\mu = -[1 + \sinh^2(\alpha_T)]^{1/2} = -\cosh(\alpha_T) , \quad (2.71)$$

where the sign is chosen so that we have the normalizations  $u_\mu u^\mu \equiv -1$  and  $n_\mu n^\mu \equiv -1$  (i.e. when  $\alpha_T = 0$ , we want a timelike vector). In this section the fluid of matter is assumed to be a perfect fluid of local density  $\rho$  and pressure  $p$ :

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} . \quad (2.72)$$

From [83] we have the following contractions (ADM energy density and ADM pressure) :

$$\epsilon \equiv T_{\mu\nu} n^\mu n^\nu = \rho + (\rho + p) \sinh^2(\alpha_T) , \quad \pi \equiv \frac{1}{3} T_{\mu\nu} h^{\mu\nu} = p + \frac{1}{3} (\rho + p) \sinh^2(\alpha_T) . \quad (2.73)$$

Here we have used the projection tensor  $h_{\mu\nu}$  orthogonal to the vector  $n^\mu$ , with the properties :

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu , \quad h_{\mu\rho} h_\nu^\rho = h_{\mu\nu} , \quad h_{\mu\nu} n^\nu = 0 . \quad (2.74)$$

This system can be inverted and has a unique solution :

$$\begin{aligned} \rho &= \left[ 1 + \frac{4}{3} \sinh^2(\alpha_T) \right]^{-1} \left[ \left( 1 + \frac{1}{3} \sinh^2(\alpha_T) \right) \epsilon - \sinh^2(\alpha_T) \pi \right] , \\ p &= \left[ 1 + \frac{4}{3} \sinh^2(\alpha_T) \right]^{-1} \left[ -\frac{1}{3} \sinh^2(\alpha_T) \epsilon + (1 + \sinh^2(\alpha_T)) \pi \right] , \end{aligned} \quad (2.75)$$

that we can insert in the expression of  $T_{\mu\nu}$  in Eq. (2.72). We can rewrite it by making use of

$$u_\mu h^{\mu\nu} = u^\nu - \cosh(\alpha_T) n^\nu , \quad (2.76)$$

to get an expression depending on  $\epsilon$  and  $\pi$  :

$$T_{\mu\nu} = \epsilon n_\mu n_\nu + \pi h_{\mu\nu} + \frac{(\epsilon + \pi)}{\left[ 1 + \frac{4}{3} \sinh^2(\alpha_T) \right]} \left[ u_\mu u_\nu - \cosh^2(\alpha_T) n_\mu n_\nu - \frac{1}{3} \sinh^2(\alpha_T) h_{\mu\nu} \right] . \quad (2.77)$$

Here the third term on the RHS gives zero when contracted with  $n^\mu n^\nu$  or  $h^{\mu\nu}$ .

We also have the generalized Friedmann's equations for an averaged perfect fluid (i.e. in the general case where  $n_\mu \neq u_\mu$ , see [83]):

$$\begin{aligned} \left( \frac{1}{\tilde{a}} \frac{\partial \tilde{a}}{\partial A_0} \right)^2 &= \frac{8\pi G}{3} \left\langle \frac{\epsilon}{[-(\partial A)^2]} \right\rangle_{A_0} - \frac{1}{6} \left\langle \frac{\mathcal{R}_S}{[-(\partial A)^2]} \right\rangle_{A_0} \\ &\quad - \frac{1}{9} \left[ \left\langle \frac{\Theta^2}{[-(\partial A)^2]} \right\rangle_{A_0} - \left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0}^2 \right] + \frac{1}{3} \left\langle \frac{\sigma^2}{[-(\partial A)^2]} \right\rangle_{A_0} , \end{aligned} \quad (2.78)$$

and

$$\begin{aligned} -\frac{1}{\tilde{a}} \frac{\partial^2 \tilde{a}}{\partial A_0^2} &= \frac{4\pi G}{3} \left\langle \frac{\epsilon + 3\pi}{[-(\partial A)^2]} \right\rangle_{A_0} - \frac{1}{3} \left\langle \frac{\nabla^\nu (n^\mu \nabla_\mu n_\nu)}{[-(\partial A)^2]} \right\rangle_{A_0} + \frac{1}{6} \left\langle \frac{\partial_\mu A \partial^\mu [(\partial A)^2] \Theta}{[-(\partial A)^2]^{5/2}} \right\rangle_{A_0} \\ &\quad - \frac{2}{9} \left[ \left\langle \frac{\Theta^2}{[-(\partial A)^2]} \right\rangle_{A_0} - \left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0}^2 \right] + \frac{2}{3} \left\langle \frac{\sigma^2}{[-(\partial A)^2]} \right\rangle_{A_0} , \end{aligned} \quad (2.79)$$

where  $\tilde{a}$  is an effective scale factor defined by

$$\frac{1}{\tilde{a}} \frac{\partial \tilde{a}}{\partial A_0} \equiv \frac{1}{3I(1, A_0)} \frac{\partial I(1, A_0)}{\partial A_0} = \frac{1}{3} \left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0} , \quad (2.80)$$

and where in the last equality we took the ADM coordinates and assumed that  $B \subset \Sigma_{A_0}$ .

In fact one can give the generalization of other quantities that have been derived in the literature. For example, we can define an effective density and an effective pressure to consider the backreaction terms as an effective scalar field. We can also generalize the integrability condition to have a gauge invariant definition of it. Nevertheless, because these expressions are really complicated, and their use very limited, we will not present them in this thesis<sup>4</sup>.

## 2.2 Light-cone averaging

### 2.2.1 Choice of a right scalar

Let us first emphasize again the approach given in [82, 83] to gauge invariant averaging on a 3-dimensional spacelike hypersurface  $\Sigma(A)$ , embedded in our 4-dimensional spacetime  $\mathcal{M}_4$ . Assuming the hypersurface (or a spacelike foliation) to be defined by an equation involving a scalar field with timelike gradients  $A(x)$ :

$$A(x) - A_0 = 0 \quad , \quad (2.81)$$

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4. Let us finally notice that the commutation rule for the energy density, necessary to obtain the integrability condition, is derived in [83]. This derivation makes use of the conservation of the stress-energy tensor for a perfect fluid along the geodesics of  $n_\mu$ , i.e.:  $n_\nu \nabla_\mu T^{\mu\nu} = 0$ . Developing  $T^{\mu\nu}$  from Eq. (2.72) in this equation, we get:

$$u^\mu \partial_\mu [(\rho + p)u^\rho n_\rho] + n^\mu \partial_\mu p + (\rho + p)(\nabla_\mu u^\mu)(u^\rho n_\rho) + (\rho + p)[n^\rho u^\mu \nabla_\mu u_\rho - u^\mu \nabla_\mu (u^\rho n_\rho)] = 0 \quad ,$$

where this last term can be written  $-(\rho + p)u^\mu u^\rho \nabla_\mu n_\rho$ . From the definition of  $\Theta^{\mu\nu}$  we have:

$$\begin{aligned} -\Theta^{\mu\nu} u_\mu u_\nu &= -\Theta_{\mu\nu} u^\mu u^\nu = -(\delta_\mu^\alpha + n_\mu n^\alpha)(\delta_\nu^\beta + n_\nu n^\beta)(\nabla_\alpha n_\beta) u^\mu u^\nu \\ &= -u^\mu u^\nu (\nabla_\mu n_\nu) - (u^\rho n_\rho) n^\mu u^\nu \nabla_\mu n_\nu \quad , \end{aligned}$$

where we made use of the equality  $n^\mu n^\nu \nabla_\mu n_\nu = \frac{1}{2} n^\mu \nabla_\mu (n^\nu n_\nu) = 0$  (from  $n^\nu n_\nu = -1$ ). As a consequence, the final equation we obtain is

$$u^\mu \partial_\mu [(\rho + p)u^\rho n_\rho] + n^\mu \partial_\mu p + (\rho + p)[(\nabla_\mu u^\mu)u^\rho n_\rho - \Theta^{\mu\nu} u_\mu u_\nu + (u^\rho n_\rho) n^\mu u^\nu \nabla_\mu n_\nu] = 0 \quad .$$

This last term on the LHS was missing in Eq. (2.25) of [83]. This footnote thus correct this typo.

the gauge (and hypersurface-parametrization) invariant definition of the integral of an arbitrary scalar  $S(x)$  and of its average on such hypersurface is given by<sup>5</sup> :

$$\langle S \rangle_{A_0} = \frac{I(S; A_0)}{I(1; A_0)} \quad \text{with} \quad I(S; A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta_{\text{D}}(A(x) - A_0) \sqrt{-\partial_\mu A \partial^\mu A} S(x) . \quad (2.82)$$

Here the spatial hypersurface has no boundary, it goes to infinity. However, as shown in [82, 83], a possible spatial boundary can be added through the following extension of the previous integral :

$$I(S; A_0; r_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta_{\text{D}}(A(x) - A_0) \Theta(r_0 - B(x)) \sqrt{-\partial_\mu A \partial^\mu A} S(x) , \quad (2.83)$$

and similarly for the corresponding average ( $\Theta$  is the Heaviside step function, and  $B$  is a positive function of the coordinates, with spacelike gradient). As already discussed in Sec. 2.1.3 (see also [82, 86]), and as illustrated in Fig. 2.1, this is still a gauge invariant expression if  $B(x)$  transforms as a scalar, while it gives violations of gauge invariance if  $B$  is not a scalar and keeps the same form in different coordinate systems. Even in that case, however, gauge invariance violations go to zero when we choose  $r_0$  in such a way that the size of the spatial region goes to infinity [82, 86].

The above procedure unfortunately fails if  $A(x) = A_0$  defines a null (light-like) hypersurface, since in that case  $\partial_\mu A \partial^\mu A = 0$ . In order to circumvent this problem let us start with a spacetime integral where the four-dimensional integration region is bounded by two hypersurfaces, one spacelike and the other one null (corresponding e.g. to the past light-cone of some observer). Let us choose, in particular, the region inside the past light-cone of the observer bounded in the past by the hypersurface  $A(x) = A_0$ . Clearly a gauge invariant definition of the integral of a scalar  $S(x)$  over such a hypervolume can be written (in a useful notation generalizing the one used above) as

$$I(S; -; A_0, V_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \Theta(A - A_0) S(x) , \quad (2.84)$$

where  $V(x)$  is a (generalized advanced-time) null scalar satisfying  $\partial_\mu V \partial^\mu V = 0$  and  $V_0$  specifies the past light cone of a given observer. The LHS symbol “-” denotes the absence of delta-like window functions. We should remark the analogy between the scalars  $V$  and  $B$ . Indeed, this gauge invariant definition corresponds to the replacement of the timelike hypersurface  $B(x) = r_0$  by a null-like hypersurface  $V(x) = V_0$ .

### 2.2.2 Definition of the averages

Starting with this hypervolume integral we can construct gauge invariant hypersurface and surface integrals by applying to it appropriate differential operators – or, equivalently, by applying Gauss’s theorem to the volume integral of a covariant divergence. An example of the latter, if we are interested by the variations of the volume averages along the flow lines normal to the reference hypersurface  $\Sigma(A)$ , is obtained by replacing the scalar  $S$  with the divergence of the unit normal to  $\Sigma$ ,

$$n_\mu = -\frac{\partial_\mu A}{\sqrt{-\partial_\nu A \partial^\nu A}} , \quad n_\mu n^\mu = -1 , \quad (2.85)$$

and leads to the identity :

$$\begin{aligned} \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \Theta(A - A_0) \nabla^\mu n_\mu &= - \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \delta_{\text{D}}(A - A_0) \sqrt{-\partial_\mu A \partial^\mu A} \\ &+ \int_{\mathcal{M}_4} d^4x \sqrt{-g} \delta_{\text{D}}(V_0 - V) \Theta(A - A_0) \frac{-\partial_\mu V \partial^\mu A}{\sqrt{-\partial_\nu A \partial^\nu A}} . \end{aligned} \quad (2.86)$$

5. In [83] the prescription introduced in [82] is used to give a covariant and gauge invariant generalization of the effective equations presented in [49, 50]. Such a generalization has been recently used to deal with the backreaction of quantum fluctuations in an inflationary model [85].

Hence, if we start from Eq. (2.84), and we consider the variation of the average integral by shifting the light-cone  $V = V_0$  along the flow lines defined by  $n_\mu$ , we are led to define the hypersurface integral (with positive measure):

$$I(1; V_0; A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \delta_D(V_0 - V) \Theta(A - A_0) \frac{|\partial_\mu V \partial^\mu A|}{\sqrt{-\partial_\nu A \partial^\nu A}} . \quad (2.87)$$

Similarly, if we consider the variation of the average integral by shifting the hypersurface  $A = A_0$  (along the same flow lines defined by  $n_\mu$ ), we are led to another hypersurface integral:

$$I(1; A_0; V_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \Theta(V_0 - V) \delta_D(A - A_0) \sqrt{-\partial_\mu A \partial^\mu A} . \quad (2.88)$$

In the first case, Eq. (2.87), the integration region is on the light-cone itself, and it is spanned by the variation of  $\Sigma(A_0)$  along its normal, at fixed light-cone  $V_0$  (see Fig. 2.2, (a)). In the second case of Eq. (2.88) – which gives exactly the same integral as in Eq. (2.83) with  $V$  replacing  $B$  – the hypersurface  $\Sigma(A_0)$  is kept fixed, and the integration region describes the causally connected section of  $\Sigma$  spanned by the variation of the light-cone hypersurface (see Fig. 2.2, (b)).

Further differentiation also leads to the following invariant surface integral

$$I(1; V_0, A_0; -) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} \delta_D(V_0 - V) \delta_D(A - A_0) |\partial_\mu V \partial^\mu A| , \quad (2.89)$$

with a compact, 2-dimensional integration region defined by the intersection of  $\Sigma(A_0)$  with the light-cone  $V_0$  (Fig. 2.2, (c)). This integral, as well as the integrals of Eqs. (2.87), (2.88), is not only covariant and gauge invariant but also invariant under separate reparametrizations of the scalar fields  $A \rightarrow \tilde{A}(A)$  and  $V \rightarrow \tilde{V}(V)$ . Eq. (2.89), in addition, is a particular case of an invariant integration over an arbitrary codimension-2 hypersurface defined by the conditions  $A^{(n)}(x) = 0$ ,  $n = 1, 2$ . In general, and in  $D$  spacetime dimensions, such an integral can be written as

$$\int_{\mathcal{M}_D} d^Dx \sqrt{-g} \prod_n \delta_D(A^{(n)}(x)) \sqrt{|\det \bar{g}^{pq}|} \quad \text{with} \quad \bar{g}^{pq} \equiv \partial_\mu A^{(p)} \partial_\nu A^{(q)} g^{\mu\nu} , \quad (2.90)$$

(as can be shown by considering the induced metric on the  $(D - 2)$ -hypersurface), and is invariant under the more general reparametrizations  $A^{(1)} \rightarrow \tilde{A}^{(1)}(A^{(1)}, A^{(2)})$  and  $A^{(2)} \rightarrow \tilde{A}^{(2)}(A^{(1)}, A^{(2)})$ . It can be easily checked that Eq. (2.90) reduces to Eq. (2.89) if  $D = 4$  and if  $A^{(1)} = A - A_0$  and  $A^{(2)} = V - V_0$  are scalar functions with timelike and null gradient, respectively.

In order to make contact with Eqs. (2.87), (2.88), it may be useful to remark that the integral of Eq. (2.89) can also be obtained starting from the hypervolume integral of Eq. (2.84) by considering the variation of the volume average along the flow lines normal to  $\Sigma(A)$  for both  $\Theta(A)$  and  $\Theta(V)$ , namely by using the following window function:

$$\begin{aligned} & -n^\mu \nabla_\mu \Theta(A(x) - A_0) n^\mu \nabla_\mu \Theta(V_0 - V(x)) \\ &= \sqrt{-\partial_\mu A \partial^\mu A} \delta_D(A(x) - A_0) \frac{-\partial_\mu V \partial^\mu A}{\sqrt{-\partial_\mu A \partial^\mu A}} \delta_D(V_0 - V(x)) . \end{aligned} \quad (2.91)$$

We note, finally, that averages of a scalar  $S$  over different (hyper)surfaces are trivially defined, with self explanatory notation, by:

$$\langle S \rangle_{V_0, A_0} = \frac{I(S; V_0, A_0; -)}{I(1; V_0, A_0; -)} , \quad (2.92)$$

$$\langle S \rangle_{V_0}^{A_0} = \frac{I(S; V_0; A_0)}{I(1; V_0; A_0)} , \quad (2.93)$$

$$\langle S \rangle_{A_0}^{V_0} = \frac{I(S; A_0; V_0)}{I(1; A_0; V_0)} . \quad (2.94)$$