

Moyennes sur le Cône de Lumière et Cosmologie de Précision

Lightcone Averaging & Precision Cosmology

Soutenance de Thèse

Fabien NUGIER

pour le grade de Docteur de l'UPMC

Jury :

Directeur de thèse: G. VENEZIANO

Rapporteurs: L. AMENDOLA , D. SCHWARZ

Examineurs: R. DURRER , D. LANGLOIS , J.-P. UZAN

Content of the Thesis

This presentation is based on my collaboration with:

G. Veneziano M. Gasperini G. Marozzi I. Ben-Dayan
(CdF, Cern) (U. of Bari) (U. Geneva) (Desy)



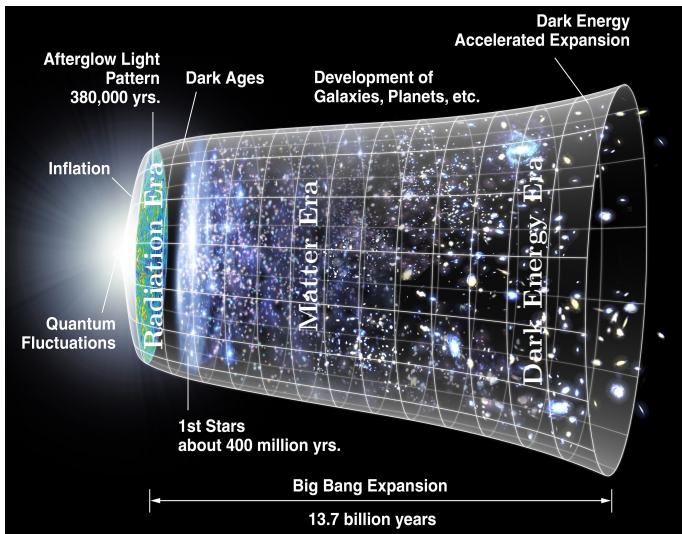
- *Light-cone averaging in cosmology: Formalism and applications*, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, JCAP **1107** (2011) 008, [arXiv:1104.1167 [astro-ph.CO]]. [1]
- *Backreaction on the luminosity-redshift relation from gauge invariant light-cone averaging*, I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, JCAP **1204** (2012) 036, [arXiv:1202.1247 [astro-ph.CO]]. [2]
- *Do stochastic inhomogeneities affect dark-energy precision measurements?*, I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, PRL **110**, 021301 (2013), [arXiv:1207.1286 [astro-ph.CO]]. [3]
- *The second-order luminosity-redshift relation in a generic inhomogeneous cosmology*, I. Ben-Dayan, G. Marozzi, F. Nugier and G. Veneziano, JCAP **1211** (2012) 045, [arXiv:1209.4326 [astro-ph.CO]]. [4]
- *Average and dispersion of the luminosity-redshift relation in the concordance model*, I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, JCAP **06** (2013) 002, [arXiv:1302.0740 [astro-ph.CO]]. [5]

This thesis work has led to seminars in EU and USA.

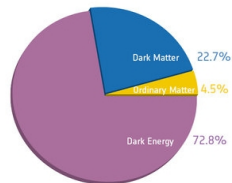
Most of them can be found at: www.phys.ens.fr/~nugier/



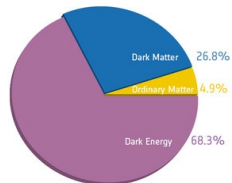
Cosmology Today: the dynamics



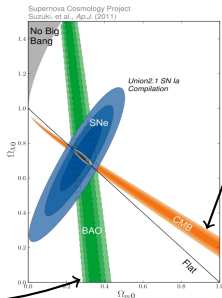
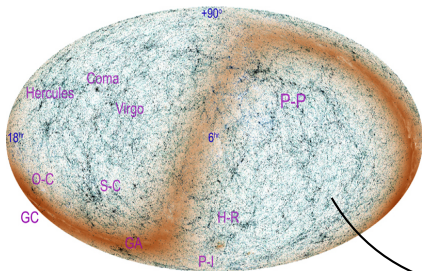
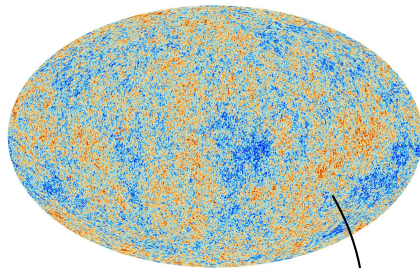
Before Planck :



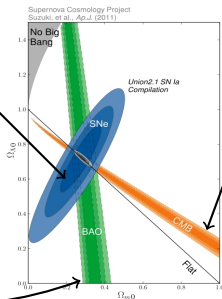
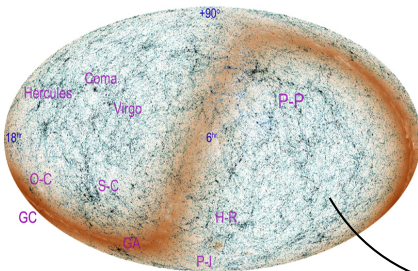
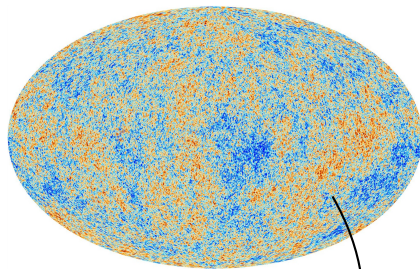
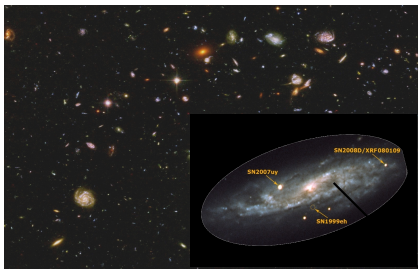
After Planck :



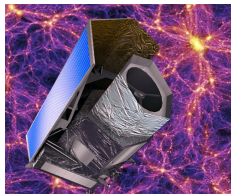
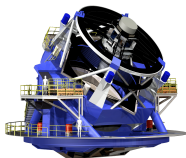
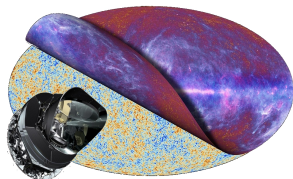
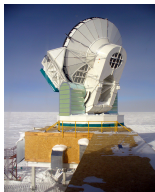
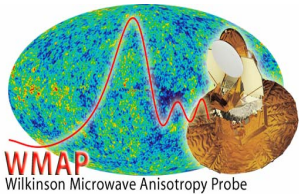
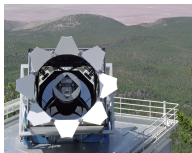
Three main Probes: the content



Three main Probes: the content

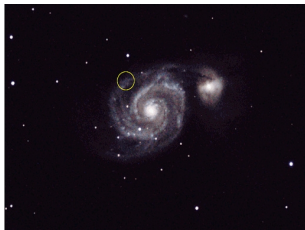
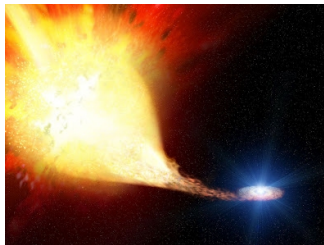


Experiments for cosmology



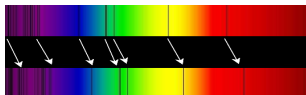
Supernovæ (Nobel 2011)

- Process not well understood
- but same \sim initial conditions

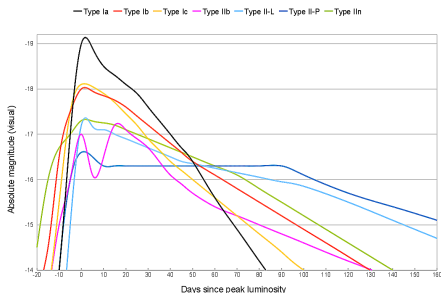


- SNe Ia are very bright !
- and standardizable objects !

Redshift z :

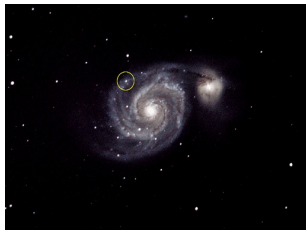


\Rightarrow we can determine the type of Supernova (SN) : Ia, Ib/c, II, ...



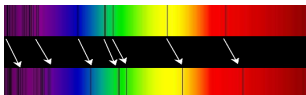
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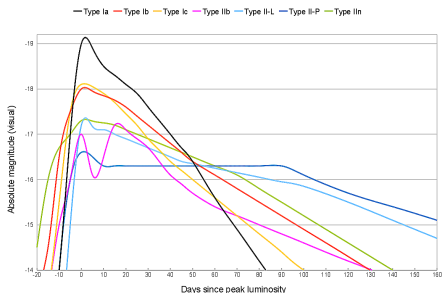


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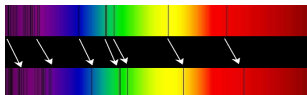


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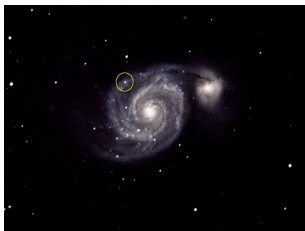
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Redshift z :



\Rightarrow we can determine the type of Supernova (SN) : Ia, Ib/c, II, ...



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Magnitude:

$$m = -2.5 \log_{10}\left(\frac{\Phi}{\Phi_{\text{ref}}}\right)$$

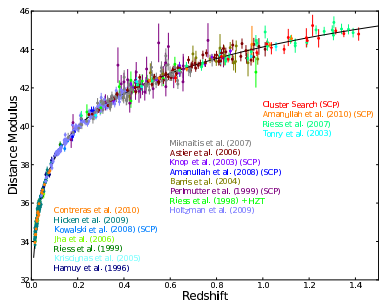
Flux:

$$\Phi = \frac{L}{4\pi d_L^2}$$



Absolute Mag.: $M = -2.5 \log_{10}\left(\frac{\Phi(10\text{pc})}{\Phi_{\text{ref}}(\text{pc})}\right)$

The distance from SNe Ia



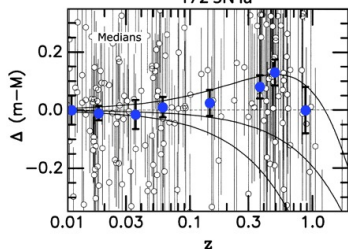
The SM of cosmology has 2 assumptions:

- GR is valid to all observed scales
- Isotropy and Homogeneity

⇒ FLRW model:

$$\begin{aligned}
 ds^2 &= -dt^2 + a^2(t) \left[d\chi^2 + \left[\frac{1}{\sqrt{K}} \sin(\sqrt{K}\chi) \right]^2 d\Omega^2 \right] \\
 &= -dt^2 + a^2(t) \left[\frac{1}{1 - Kr^2} dr^2 + r^2 d\Omega^2 \right], \\
 d\Omega^2 &= d\theta^2 + (\sin\theta)^2 d\phi^2, \quad K = -1, 0, 1.
 \end{aligned}$$

172 SN Ia



Luminosity Distance (for $K = 0$):

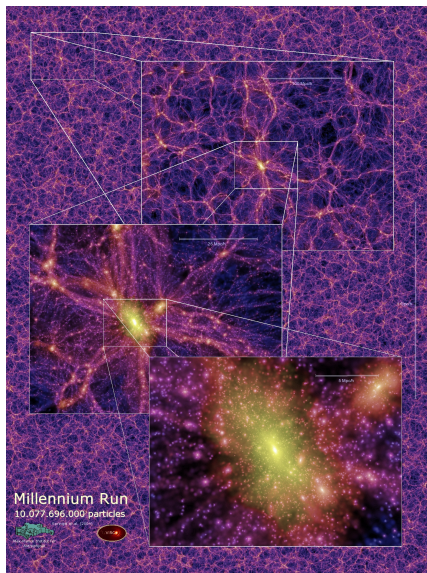
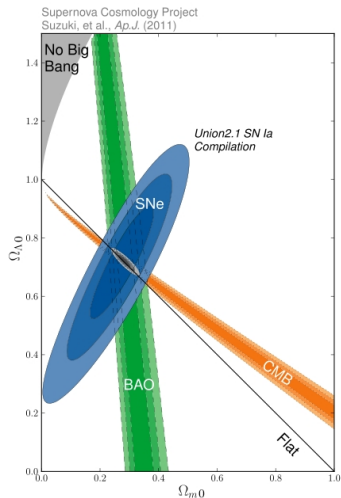
$$d_L^{FLRW}(z) = \frac{1+z}{H_0} \int_0^z \frac{dz'}{[\Omega_{\Lambda 0} + \Omega_{m0}(1+z')^3]^{1/2}}$$

Distance Modulus:

$$\mu = m - M = 5 \log_{10}(d_L) + cst$$

Does large scale structure affect the standard model ?

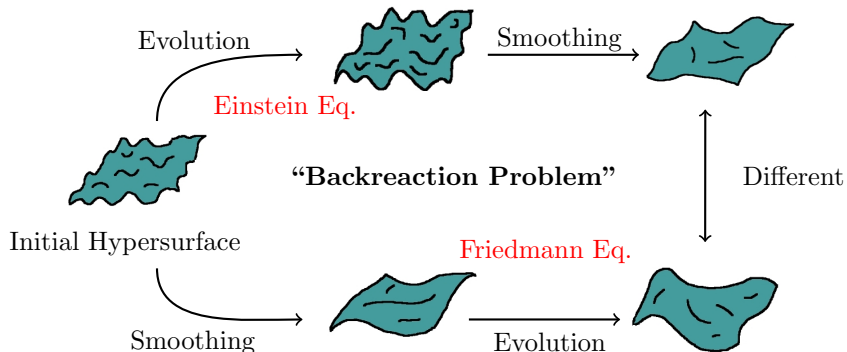
Combining probes $\Rightarrow \Lambda$ CDM:



See e.g.: Clarkson, Ellis, Faltenbacher, Maartens, Umeh, Uzan (2011)

In principle, yes ! Because of General Relativity.

Ellis in the 80's:



⇒ averaging does not commutes with evolution :

$$\langle \dot{S} \rangle_{\mathcal{D}} - \dot{\langle S \rangle}_{\mathcal{D}} = \langle S \Theta \rangle_{\mathcal{D}} - \langle S \rangle_{\mathcal{D}} \langle \Theta \rangle_{\mathcal{D}} \quad .$$

⇒ inhomogeneities do, *in principle*, influence the evolution of the Universe.

Motivation to this Thesis (I) – Physics

What is the theoretical effect of inhomogeneities on the Hubble diagram, and so on dark energy parameters ?

Questions :

- How can we **average** inhomogeneities (on the past light-cone) ?
- What is the impact on **measurements** ? Are there nice **observables** ?
- What is the influence on **dark energy** parameters ?
- What is the **z-dependence** of the effect ? Can we rule it out?

Starting **hypotheses** :

- **H1** : we use a spatially flat **FLRW Universe** with (scalar) perturbations,
- **H2** : perturbations are given by the **power spectrum**, only physical input.

Remark : we do not define an **effective scale factor** (usual with backreaction)

Motivation to this Thesis (II) – Mathematics

Almost all we observe is on our past light cone.

The **average** definition :

- \exists spatial but no lightcone averaging (though A. Coley, S. Räsänen),
- we need a gauge invariant definition [[gr-qc/0901.1303](#) , [0912.3244](#)],
- how simple ? huge simplification with the [GLC metric](#) !

How to deal with the **inhomogeneities** :

- full non-linear models (swiss cheeses, LTB, ...) ?
- perturbation theory around an homogeneous model,
- combining [lightcone averaging](#) and a [stochastic](#) average.

Which **scalar quantity** is interesting ?

- matter density, redshift, luminosity flux, distance, modulus, ...
- cautiously : [distance-redshift](#) relation up to $\mathcal{O}(2)$ in perturbations,

Outline

1 Introduction

- State of the art
- Supernovæ measurements
- Effect of inhomogeneities

2 Formalism

- Light-cone averaging
- Geodesic light-cone coordinates
- Distance-redshift relation

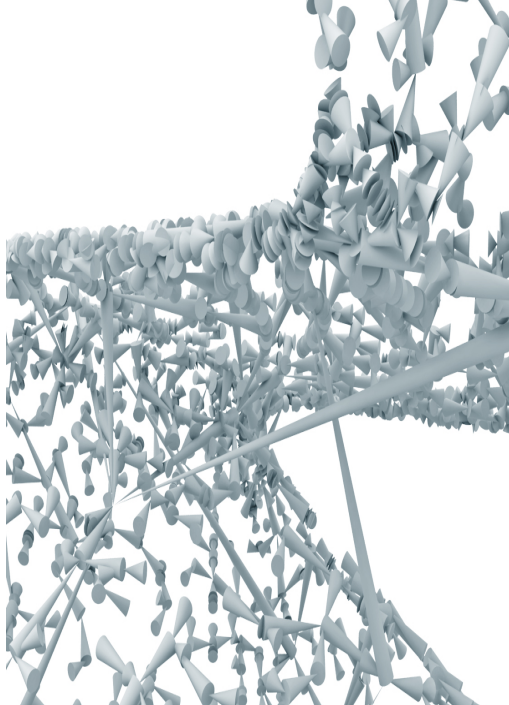
3 Combining averages

- (Non-)linear power spectrum
- Examples

4 Numerical results

- Averaging the flux
- Importance of observables
- Distance and modulus
- Intrinsic dispersion

5 Conclusions

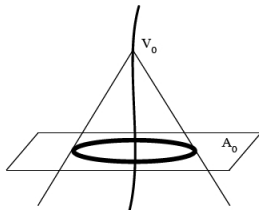
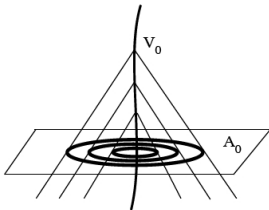
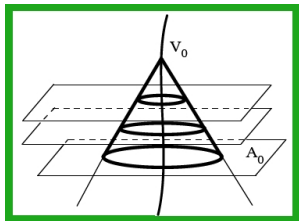


Light-cone Averaging



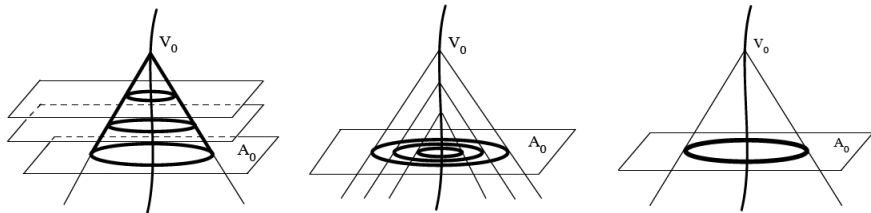
Light-cone Averaging: definition [1]

Averaged scalars S : $\langle S \rangle (V_0, A_0) = \frac{I(S; V_0, A_0)}{I(1; V_0, A_0)}$, $S \sim \rho$, Θ , d_L , d_L^{-2}



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We can define the integral for averaging by :

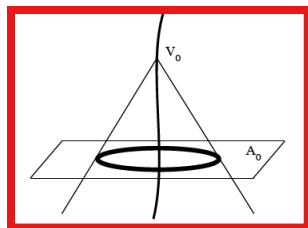
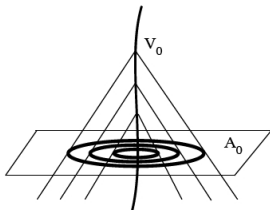
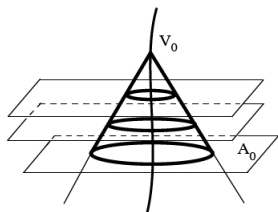
$$I(S; V_0, A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g} D(V_0 - V) D(A - A_0) \mathcal{N}(V, A, \partial_\mu) S(x) .$$

where $\mathcal{N}(V, A, \partial_\mu)$ is a normalization, $D(X) = \delta_D(X)$ or $\Theta(X)$ (Heaviside function).

Average Θ_{δ_D}	$\langle S \rangle_{V_0}^{A_0}$	$\langle S \rangle_{A_0}^{V_0}$	$\langle S \rangle_{V_0, A_0}$
$\mathcal{N}(V, A, \partial_\mu)$	$\frac{ \partial_\mu V \partial^\mu A }{\sqrt{-\partial_\nu A \partial^\nu A}}$	$\sqrt{-\partial_\mu A \partial^\mu A}$	$ \partial_\mu V \partial^\mu A $

Light-cone Averaging: definition [1]

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Light-cone Averaging : properties [1]

These definitions are **gauge invariant**, invariant under $(A, V) \rightarrow (\tilde{A}(A), \tilde{V}(V))$, and general coordinate transformations.

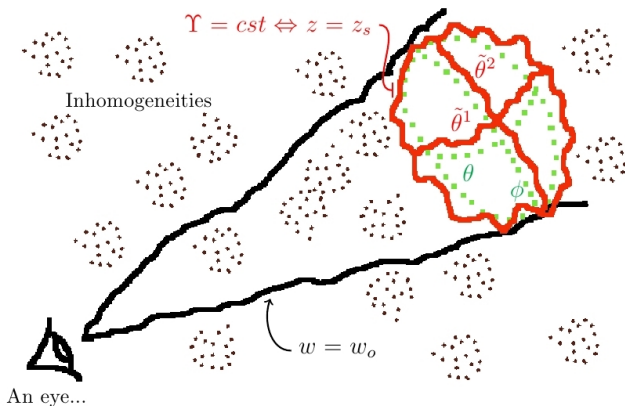
We can derive Buchert-Ehlers commutation rules, e.g. :

$$\frac{\partial}{\partial A_0} \langle S \rangle_{V_0, A_0} = \left\langle \frac{k \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0, A_0} + \left\langle \frac{\nabla \cdot k}{k \cdot \partial A} S \right\rangle_{V_0, A_0} - \left\langle \frac{\nabla \cdot k}{k \cdot \partial A} \right\rangle_{V_0, A_0} \langle S \rangle_{V_0, A_0} ,$$

$$\begin{aligned} \frac{\partial}{\partial V_0} \langle S \rangle_{V_0, A_0} &= \left\langle \frac{\partial A \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0, A_0} - \left\langle k \cdot \partial S \frac{(\partial A)^2}{(k \cdot \partial A)^2} \right\rangle_{V_0, A_0} \\ &+ \left\langle \left[\square A - \nabla_\mu \left(k^\mu \frac{(\partial A)^2}{k \cdot \partial A} \right) \right] \frac{S}{k \cdot \partial A} \right\rangle_{V_0, A_0} \\ &- \left\langle \left[\square A - \nabla_\mu \left(k^\mu \frac{(\partial A)^2}{k \cdot \partial A} \right) \right] \frac{1}{k \cdot \partial A} \right\rangle_{V_0, A_0} \langle S \rangle_{V_0, A_0} , \end{aligned}$$

Here we will average on the deformed **2-sphere embedded** in $V = V_0$ (**light-cone**) $A = A_0$ (**spatial hyp.**). In GLC (see after) : $V \rightarrow w$, $A \rightarrow \tau, z$.

The geodesic light-cone gauge



Geodesic light-cone coordinates [1]

A **light-cone** adapted metric (close to “*observational coordinates*”):

$$ds_{GLC}^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)$$

(6 arbitrary functions : $\Upsilon, U^a, \gamma_{ab}$)

Properties :

- w is a **null coordinate** : $\partial_\mu w \partial^\mu w = 0$,
- $\partial_\mu \tau$ defines a **geodesic flow** : $(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) \equiv 0$ (from $g_{GLC}^{\tau\tau} = -1$) ,
- an observer defined by constant τ spacelike hyp. is in geodesic motion.
- **photons travel at** $(w, \tilde{\theta}^a) = \overrightarrow{cst}$ and their path is orthogonal to $\Sigma(w, z)$

FLRW : $w = \eta + r$, $\tau = t$ (exact if $t =$ synchronous gauge time)

$$\tilde{\theta}^1, \tilde{\theta}^2 = \theta, \phi \quad , \quad \gamma_{ab} d\tilde{\theta}^a d\tilde{\theta}^b = a^2 r^2 d^2 \Omega \quad ,$$
$$\Upsilon = a(t) \quad , \quad U^a = 0 \quad ,$$

Remark : There is no backreaction effect in a (pure) FLRW geometry !

Analogy with the ADM formalism

- In ADM formalism :

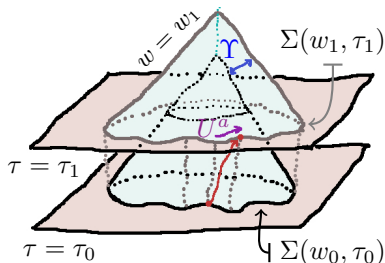
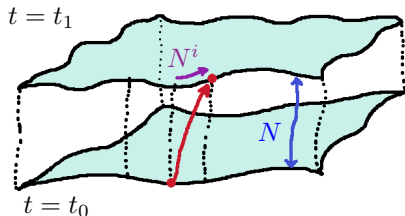
$$ds_{ADM}^2 = -(N^2 - g_{ij}N^iN^j)dt^2 + 2g_{ij}dx^iN^jdt + g_{ij}dx^i dx^j$$

with N the **lapse function** and N^i the **shift vector**.

- Here our metric is :

$$ds_{GLC}^2 = (\Upsilon^2 + \gamma_{ab}U^aU^b)dw^2 - 2\gamma_{ab}d\tilde{\theta}^aU^bdw + \gamma_{ab}d\tilde{\theta}^ad\tilde{\theta}^b - 2\Upsilon dw d\tau$$

$\Rightarrow \Upsilon \sim$ lapse function, $U^a \sim$ shift vector for the “foliation” by lightcones.



It makes life easier ! [1]

$$ds_{GLC}^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)$$

⇒ Redshift perturbation :

$$(1 + z_s) = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{(\partial^\mu w \partial_\mu \tau)_s}{(\partial^\mu w \partial_\mu \tau)_o} = \frac{\Upsilon(w_o, \tau_o, \tilde{\theta}^a)}{\Upsilon(w_o, \tau_s, \tilde{\theta}^a)}$$

where $u_\mu = -\partial_\mu \tau$ is the **peculiar velocity** of the comoving observer/source and $k_\mu = \partial_\mu w$ is the **photon momentum**.

⇒ (exact) **Luminosity distance** :

$$d_L = (1 + z)^2 \gamma^{1/4} \left(\sin \tilde{\theta}^1 \right)^{-1/2} \quad \text{with } \gamma \equiv \det(\gamma_{ab})$$

which, combined with the redshift, gives the distance-redshift relation.

Remark : The **Jacobi map** (Bonvin etal '11, Fleury etal '13) takes a simple and exact form in the GLC gauge (see Fanizza etal '13).

It simplifies the averages :

$$\begin{aligned} \langle S \rangle_{w_o, \tau_s} &= \frac{\int_{\Sigma} d^4x \sqrt{-g} \delta_D(w - w_o) \delta_D(\tau - \tau_s) |\partial_{\mu} \tau \partial^{\mu} w| S(\tau, w, \tilde{\theta}^a)}{\int_{\Sigma} d^4x \sqrt{-g} \delta_D(w - w_o) \delta_D(\tau - \tau_s) |\partial_{\mu} \tau \partial^{\mu} w|} \\ &= \left(\int d^2\tilde{\theta} \sqrt{\gamma(w_o, \tau_s, \tilde{\theta}^a)} S(w_o, \tau_s, \tilde{\theta}^a) \right) / \left(\int d^2\tilde{\theta} \sqrt{\gamma(w_o, \tau_s, \tilde{\theta}^a)} \right) , \end{aligned}$$

$$\boxed{\langle S \rangle_{w_o, z_s} = \frac{\int d^2\tilde{\theta} \sqrt{\gamma(w_o, \tau(z_s, w_o, \tilde{\theta}^a), \tilde{\theta}^b)} S(w_o, \tau(z_s, w_o, \tilde{\theta}^a), \tilde{\theta}^b)}{\int d^2\tilde{\theta} \sqrt{\gamma(w_o, \tau(z_s, w_o, \tilde{\theta}^a), \tilde{\theta}^b)}}$$

and Buchert-Ehlers commutation rules :

$$\begin{aligned} \partial_{\tau_s} \langle S \rangle_{w_o, \tau_s} &= \langle \partial_{\tau} S \rangle_{w_o, \tau_s} + \left\langle S \partial_{\tau} \ln \sqrt{|\gamma|} \right\rangle_{w_o, \tau_s} - \left\langle \partial_{\tau} \ln \sqrt{|\gamma|} \right\rangle_{w_o, \tau_s} \langle S \rangle_{w_o, \tau_s} , \\ \partial_{w_o} \langle S \rangle_{w_o, \tau_s} &= \langle \partial_w S + U^a \partial_a S \rangle_{w_o, \tau_s} + \left\langle \left[\partial_w \ln \sqrt{|\gamma|} + \partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right] S \right\rangle_{w_o, \tau_s} \\ &\quad - \left\langle \partial_w \ln \sqrt{|\gamma|} + \partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right\rangle_{w_o, \tau_s} \langle S \rangle_{w_o, \tau_s} \end{aligned}$$

The distance-redshift relation up to second order



Defining inhomogeneities : Newtonian Gauge [1,2]

How scalar **inhomogeneities** affect the spacetime :

$$ds_{NG}^2 = a^2(\eta) \left(-(1 + 2\Phi)d\eta^2 + (1 - 2\Psi)(dr^2 + \gamma_{ab}^{(0)} d\theta^a d\theta^b) \right)$$

$$\text{with } \gamma_{ab}^{(0)} = r^2 \text{diag} (1, \sin^2 \theta) \quad , \quad \Phi = \phi + \frac{1}{2}\phi^{(2)} \quad , \quad \Psi = \psi + \frac{1}{2}\psi^{(2)} \quad ,$$

and Φ, Ψ are the gauge invariant **Bardeen potentials** $= f(\eta, r, \theta, \phi)$.

At **first order**, $\Phi = \Psi \equiv \psi$ (no anisotropic stress).

FULL transformation **GLC** \leftrightarrow **NG** at second order in PT.

$$\Rightarrow (\tau, w, \tilde{\theta}^1, \tilde{\theta}^2) = f(\eta, r, \theta, \phi) \Rightarrow (\Upsilon, U^a, \gamma^{ab}) = f(\psi, \psi^{(2)}, \phi^{(2)})$$

\Rightarrow we compute d_L up to second order :

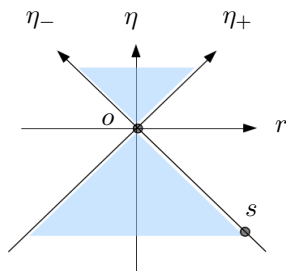
$$\frac{d_L(z_s, \theta^a)}{d_L^{FLRW}(z_s)} = \frac{d_L(z_s, \theta^a)}{(1 + z_s)a_o(\eta_o - \eta_s)} = 1 + \delta_S^{(1)}(z_s, \theta^a) + \delta_S^{(2)}(z_s, \theta^a) \quad ,$$

Ref. : see **Bonvin etal '11** and **Di Dio, Durrer '12**

Luminosity distance at $\mathcal{O}(1)$ [2]

First order corrections are :

$$\delta_S^{(1)}(z_s, \theta^a) = \Xi_s J - \frac{Q_s}{\Delta\eta} - \psi_s^{(1)} - J_2^{(1)}$$



with :

$$\Xi_s = 1 - \frac{1}{\mathcal{H}_s \Delta\eta} \quad , \quad \Delta\eta = \eta_o - \eta_s^{(0)} \quad , \quad J = ([\partial_+ Q]_s - [\partial_+ Q]_o) - ([\partial_r P]_s - [\partial_r P]_o)$$

$$Q(\eta_+, \eta_-, \theta^a) = \int_{\eta_+}^{\eta_-} dx \hat{\psi}(\eta_+, x, \theta^a) \quad , \quad P(\eta, r, \theta^a) = \int_{\eta_{in}}^{\eta} d\eta' \frac{a(\eta')}{a(\eta)} \psi(\eta', r, \theta^a)$$

$$J_2^{(1)} = \frac{1}{2} \nabla_a \tilde{\theta}^{a(1)} = \frac{\cot \theta \tilde{\theta}^{(1)} + \partial_a \tilde{\theta}^{a(1)}}{2} = \int_{\eta_s^{(0)}}^{\eta_o} \frac{d\eta}{\Delta\eta} \frac{\eta - \eta_s^{(0)}}{\eta_o - \eta} \Delta_2 \psi(\eta, \eta_o - \eta, \theta^a)$$

where we use the $\mathcal{O}(0)$ light-cone variables : $\eta_{\pm} = \eta \pm r$, $\partial_{\pm} = \frac{1}{2}(\partial_{\eta} \pm \partial_r)$.

Physical interpretation of terms from $\mathcal{O}(1)$ [2]

- **Doppler :**

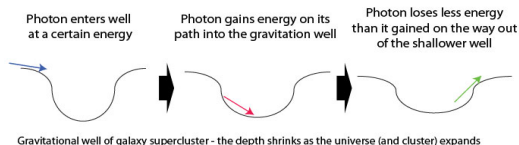
$$\partial_r P = \vec{v} \cdot \hat{n}$$

\hat{n} = unit tangent vector along the null geodesic (from $s \rightarrow o$) ;
the **peculiar velocity** associated to a geodesic config. at $\mathcal{O}(1)$ in NG is

$$\vec{v} = - \int_{\eta_{in}}^{\eta} d\eta' \frac{a(\eta')}{a(\eta)} \vec{\nabla} \psi(\eta', r, \theta^a)$$

- **Sachs-Wolfe (SW) & Integrated Sachs-Wolfe (ISW) :**

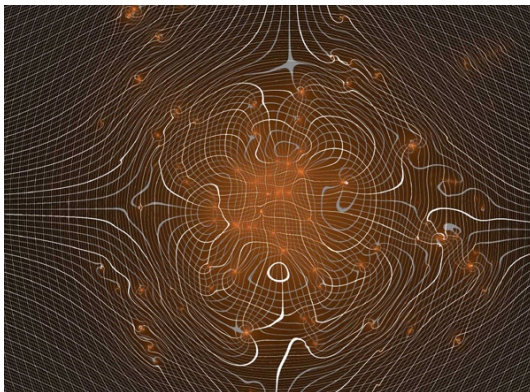
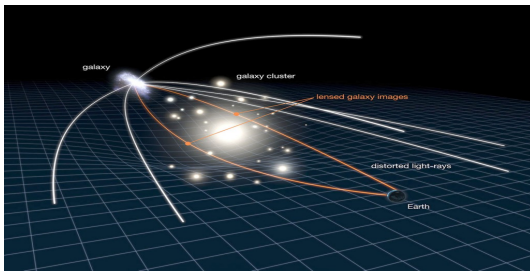
$$\partial_+ Q_s - \partial_+ Q_o \sim \psi_o - \psi_s - 2 \int_{\eta_s}^{\eta_o} d\eta \partial_\eta \psi(\eta, r, \theta^a)$$



- **Lensing :** given by

$$J_2 = \int_{\eta_s^{(0)}}^{\eta_o} \frac{d\eta}{\Delta\eta} \frac{\eta - \eta_s^{(0)}}{\eta_o - \eta} \Delta_2 \psi(\eta, \eta_o - \eta, \theta^a) + \mathcal{O}(2) \quad ,$$

We ♥ ♥ Lensing...



Luminosity distance at $\mathcal{O}(2)$ [4]

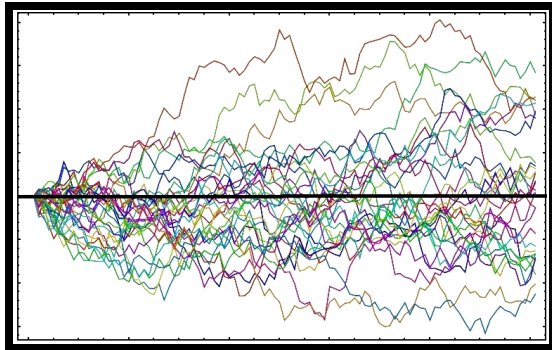
$$\delta_S^{(2)}(z_s, \theta^a) = - \left(\Xi_s J - \frac{Q_s}{\Delta\eta} \right) (\psi_s + J_2) + \dots$$

where ... are $\mathcal{O}(2)$ -corrections of $d_L / [(1 + z_s)^2 a(\eta_s) r_s]$ and $a(\eta_s) r_s / a(\eta_s^{(0)}) \Delta\eta$.

In these corrections, we find :

- Combinations of $\mathcal{O}(1)$ -terms : ψ_s^2 , (ISW)², ISW \times Doppler, $(\psi_s, Q_s) \times$ (Lensing, ISW, Doppler) ...
- (Doppler)², (Lensing)² !!!
- Genuine $\mathcal{O}(2)$ -terms : $\psi_s^{(2)}$, Lensing⁽²⁾ = $\frac{1}{2} \nabla_a \tilde{\theta}^{a(2)}$, $Q_s^{(2)}$...
- New integrated effects : $\frac{1}{4\Delta\eta} \int_{\eta_s^{(0)+}}^{\eta_s^{(0)-}} dx \left[4\hat{\psi} \partial_+ Q + \hat{\gamma}_0^{ab} \partial_a Q \partial_b Q \right] (\eta_s^{(0)+}, x, \theta^a)$
- Angle deformations : $(\gamma_0)_{ab} \partial_+ \tilde{\theta}^{a(1)} \partial_- \tilde{\theta}^{b(1)}$, $\partial_a \tilde{\theta}^{b(1)} \partial_b \tilde{\theta}^{a(1)}$
- Redshift perturbations : from $1 + z_s = \Upsilon_o / \Upsilon_s$, many terms... involving also : $\gamma_0^{ab} (\partial_a P \partial_b P, \partial_a Q \partial_b Q, \partial_a Q \partial_b P)$, $\partial_+ \int_{\eta_+}^{\eta_-} dx \left[4\hat{\psi} \partial_+ Q + \hat{\gamma}_0^{ab} \partial_a Q \partial_b Q \right]$, $\int_{\eta_{in}}^{\eta} d\eta' \frac{a(\eta')}{a(\eta)} \partial_r \left[(\partial_r P)^2 + \gamma_0^{ab} \partial_a P \partial_b P \right]$ (\subset transverse peculiar velocity)
- Other terms: Lens-Lens coupling, Born corrections, ...

Stochastic Average



Describing inhomogeneities

Doing a Fourier expansion :

$$\psi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \psi_k(\eta) E(\vec{k})$$

with E a unit random variable satisfying :

- $E^*(\vec{k}) = E(-\vec{k})$ (statistically homogeneous),
- $\overline{E(\vec{k})} = 0$, $\overline{E(\vec{k}_1)E(\vec{k}_2)} = \delta(\vec{k}_1 + \vec{k}_2)$ (ensemble-average conditions).

Linear power spectrum :

$$\mathcal{P}_\psi(k) \equiv \frac{k^3}{2\pi^2} |\psi_k(\eta)|^2 = g^2(z) T^2(k) \frac{9}{25} A \left(\frac{k}{k_0} \right)^{n_s-1},$$

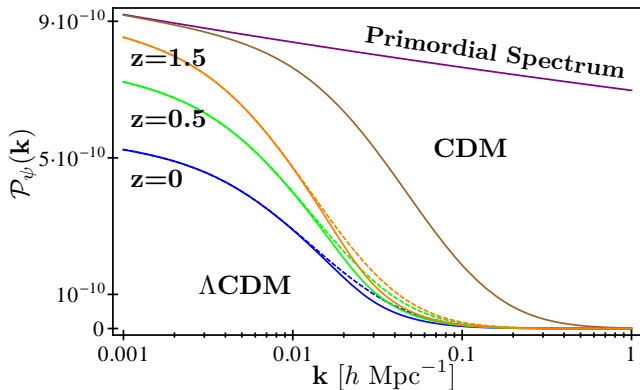
with :

$$A = 2.45 \cdot 10^{-9} \quad , \quad n_s = 0.96 \quad , \quad \frac{k_0}{a_0} = 0.002 \text{ Mpc}^{-1}$$

$$g(z) = \frac{5g_\infty}{2} \Omega_m(z) \left[\Omega_m(z)^{4/7} - \Omega_\Lambda(z) + \left[1 + \frac{\Omega_m(z)}{2} \right] \left[1 + \frac{\Omega_\Lambda(z)}{70} \right] \right]^{-1}$$

and $T(k)$ is a transfer function given by Eisenstein & Hu (1997), including a baryonic component (\Rightarrow Silk damping).

We obtain the **linear power spectrum** for the gravitational potential :



- ΛCDM – dashed : simple $T(k)$ with only dark matter
- ΛCDM – solid : $T(k)$ with dark matter and baryons (Silk damping)

Non-Linear Power Spectrum

From **Poisson equation** in Λ CDM, we can link the gravitational power spectrum to the matter density spectrum :

$$\Delta^2(k) = \frac{k^3}{2\pi^2} |\delta_k|^2 \quad , \quad \mathcal{P}_{\psi}^{\text{L,NL}}(k) = \frac{9}{4} \frac{\Omega_{m0}^2 \mathcal{H}_0^4}{(1+z)^{-2} k^4} \Delta_{\text{L,NL}}^2(k)$$

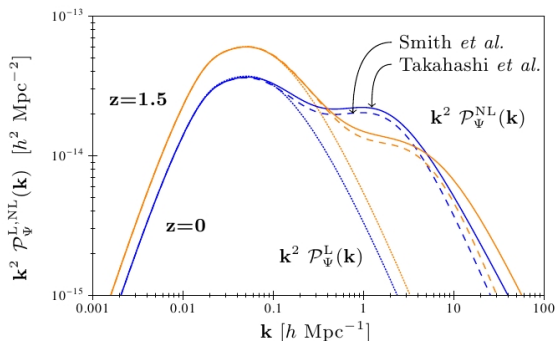
$$\Rightarrow \Delta_{\text{L}}^2(k) = \frac{4}{25} \frac{A \Omega_{m0}^{-2} \mathcal{H}_0^{-4}}{(1+z)^2} \frac{k^{n_s+3}}{k_0^{n_s-1}} \left(\frac{g(z)}{g_0} \right)^2 T^2(k) .$$

Using **Smith et al.** :
 “*HaloFit model*”

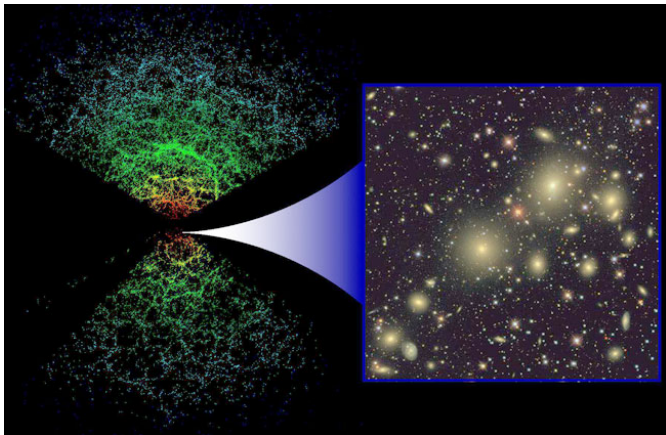
$$\Delta_{\text{NL}}^2(k) = \Delta_{\text{Q}}^2(k) + \Delta_{\text{H}}^2(k)$$

$\Delta_{\text{Q}}^2(k)$: 2-halo term

$\Delta_{\text{H}}^2(k)$: 1-halo terms



Combining averages : Flux v.s. Distance



Our goal in few words [2,3,5]

Goal 1: In **CDM** we can compute exactly the spectral coefficients :

$$\overline{\langle d_L \rangle} = \int_0^\infty \frac{dk}{k} \mathcal{P}_\psi(k) C(k\Delta\eta)$$

Goal 2: We want to do the same type of calculation \forall for all \forall the terms in $\overline{\langle \delta_S^{(1)} \rangle}$ and $\overline{\langle \delta_S^{(2)} \rangle}$ in Λ CDM... approximations are needed !

Complication: We deal with the genuine second order $(\psi^{(2)}, \phi^{(2)})$ by relating it to $\mathcal{O}(1)$ (*Bartolo, Matarrese, Riotto, (0512481)*) :

The genuine $\mathcal{O}(2)$ Bardeen potentials are :

... with the operators :

$$\begin{aligned} \phi^{(2)} &= 2\psi^2 - 6 \mathcal{O}_1^{ij} \partial_i \psi \partial_j \psi + \frac{\eta^2}{14} \mathcal{O}_2^{ij} \partial_i \psi \partial_j \psi & \mathcal{O}_1^{ij} &= \nabla^{-2} \left(\frac{\partial^i \partial^j}{\nabla^2} - \frac{1}{3} \delta^{ij} \right) \\ \psi^{(2)} &= -2\psi^2 + 4 \mathcal{O}_1^{ij} \partial_i \psi \partial_j \psi + \frac{\eta^2}{14} \mathcal{O}_2^{ij} \partial_i \psi \partial_j \psi & \mathcal{O}_2^{ij} &= \frac{10}{3} \frac{\partial^i \partial^j}{\nabla^2} - \delta^{ij} . \end{aligned}$$

Just an example to give intuition...

We compute $\overline{\langle d_L(z, \vec{n}) d_L(z, \vec{n}) \rangle}$ and not $\overline{\langle d_L(z, \vec{n}) d_L(z, \vec{n}') \rangle}$ or $\overline{\langle d_L(z, \vec{n}) d_L(z', \vec{n}) \rangle}$.

Doing the “**angular** + **stochastic**” average of ψ^2 is the simplest term we have :

$$\overline{\langle \psi_s^2 \rangle} = \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^3} \psi_{k_1} \psi_{k_2} \overline{E(\vec{k}_1) E(\vec{k}_2)} \left\langle e^{i(\vec{k}_1 + \vec{k}_2) \cdot \hat{x} \Delta \eta} \right\rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\psi(k)$$

A slightly more complicated example, in CDM :

$$\begin{aligned} \overline{\left\langle \psi_s \frac{Q_s}{\Delta \eta} \right\rangle} &= -\frac{2}{\Delta \eta} \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^3} \psi_{k_1} \psi_{k_2} \overline{E(\vec{k}_1) E(\vec{k}_2)} \left\langle e^{i\vec{k}_1 \cdot \hat{x} \Delta \eta} \int_{\eta_s}^{\eta_0} d\eta e^{i\vec{k}_2 \cdot \hat{x} (\eta_0 - \eta)} \right\rangle \\ &= -\frac{2}{\Delta \eta} \int_0^\infty \frac{dk}{4\pi k} \mathcal{P}_\psi(k) \int_{\eta_s}^{\eta_0} d\eta \left\langle e^{ik(\eta_s - \eta) \cos \theta} \right\rangle \\ &= \int_0^\infty \frac{dk}{k} \mathcal{P}_\psi(k) \left[-\frac{2}{k\Delta \eta} \text{SinInt}(k\Delta \eta) \right] \end{aligned}$$

Identically :

$$\overline{\langle \psi_s \rangle} \overline{\left\langle \frac{Q_s}{\Delta \eta} \right\rangle} = \int_0^\infty \frac{dk}{k} \mathcal{P}_\psi(k) \left[-\text{Sinc}(k\Delta \eta) \frac{2}{k\Delta \eta} \text{SinInt}(k\Delta \eta) \right]$$

Importance of observables :

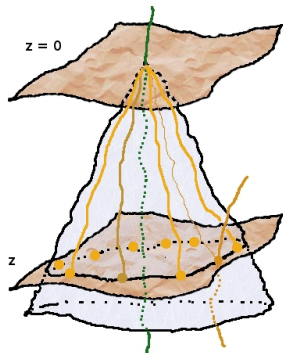
Flux v.s. Distance



Averaging the flux [2,3,4,5]

Luminosity flux:

$$\Phi = \frac{L}{4\pi d_L^2} = \Phi_0 + \Phi_1 + \Phi_2$$



There is a [nice shortcut](#) for the flux (only):

$$\langle d_L^{-2} \rangle(z_s, w_o) = (1 + z_s)^{-4} \left[\int \frac{d^2 \tilde{\theta}^a}{4\pi} \sqrt{\gamma(w_o, \tau_s(z_s, \tilde{\theta}^a), \tilde{\theta}^b)} \right]^{-1} \equiv \frac{I_\phi(w_o, z_s)^{-1}}{(d_L^{FLRW})^2}$$

where

$$\begin{aligned} I_\phi(w_o, z_s) &\equiv \frac{\mathcal{A}(w_o, z_s)}{4\pi (d_A^{FLRW})^2} = (d_A^{FLRW})^{-2} \int \frac{d^2 \tilde{\theta}^a}{4\pi} \sqrt{\gamma(w_o, \tau_s(z_s, \tilde{\theta}^a), \tilde{\theta}^b)} \quad . \\ &= \frac{\mathcal{A}(w_o, z_s)}{4\pi (a(\eta_s^{(0)}) \Delta \eta)^2} = \int \frac{d^2 \tilde{\theta}^a}{4\pi} \sin \tilde{\theta} (1 + \mathcal{I}_1 + \mathcal{I}_{1,1} + \mathcal{I}_2) \quad . \end{aligned}$$

Evaluation of $I_\phi(w_o, z_s)$ [3,4,5]

$$\begin{aligned}
 \mathcal{I}_{1,1} &= 2\Xi_s \left\{ \frac{1}{2} [\psi_s^2 - \psi_o^2] + \frac{1}{2} ([\partial_r P]_s)^2 - \frac{1}{2} ([\partial_r P]_o)^2 - (\psi_s + [\partial_+ Q]_s) [\partial_r P]_s \right. \\
 &+ \frac{1}{4} (\gamma_0^{ab})_s (2\partial_a P_s \partial_b P_s + \partial_a Q_s \partial_b Q_s - 4\partial_a Q_s \partial_b P_s) - \frac{1}{2} \lim_{r \rightarrow 0} \left[\gamma_0^{ab} \partial_a P \partial_b P \right] \\
 &+ Q_s \left(-[\partial_+^2 Q]_s + [\partial_+ \hat{\psi}]_s + [\partial_r^2 P]_s \right) + \frac{J}{\mathcal{H}_s} \left([\partial_\eta \psi]_s + \mathcal{H}_s [\partial_r P]_s + [\partial_r^2 P]_s \right) \\
 &- \frac{1}{2} \int_{\eta_{in}}^{\eta_s^{(0)}} d\eta' \frac{a(\eta')}{a(\eta_s^{(0)})} \partial_r \left[-\psi^2 + (\partial_r P)^2 + \gamma_0^{ab} \partial_a P \partial_b P \right] (\eta', \Delta\eta, \tilde{\theta}^a) \\
 &+ \frac{1}{2} \int_{\eta_{in}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)} \partial_r \left[-\psi^2 + (\partial_r P)^2 + \gamma_0^{ab} \partial_a P \partial_b P \right] (\eta', 0, \tilde{\theta}^a) \\
 &+ \int_{\eta_s^{(0)+}}^{\eta_s^{(0)-}} dx \partial_+ \left[\hat{\psi} \partial_+ Q + \frac{1}{4} \hat{\gamma}_0^{ab} \partial_a Q \partial_b Q \right] (\eta_s^{(0)+}, x, \tilde{\theta}^a) \left. \right\} \\
 &+ \left[\Xi_s^2 - \frac{1}{\mathcal{H}_s \Delta\eta} \left(1 - \frac{\mathcal{H}'_s}{\mathcal{H}_s^2} \right) \right] J^2 - 4\psi_s J + 2\Xi_s \left(\psi_o - \frac{Q_s}{\Delta\eta} + [\partial_r P]_o \right) J + \left(\frac{Q_s}{\Delta\eta} \right)^2 \\
 &+ 2(\psi_s - \psi_o + [\partial_r P]_s - [\partial_r P]_o) \frac{Q_s}{\Delta\eta} + (\gamma_0^{ab})_s \partial_a Q_s \partial_b \left(\frac{Q_s}{2} + \frac{J}{\mathcal{H}_s} \right) - 2 \frac{J}{\mathcal{H}_s} [\partial_\eta \psi]_s \\
 &+ 2 \left(\frac{J}{\mathcal{H}_s} + Q_s \right) [\partial_r \psi]_s - \frac{2}{\Delta\eta} \int_{\eta_s^{(0)+}}^{\eta_s^{(0)-}} dx \left[\hat{\psi} \partial_+ Q + \frac{1}{4} \hat{\gamma}_0^{ab} \partial_a Q \partial_b Q \right] (\eta_s^{(0)+}, x, \tilde{\theta}^a) \\
 &+ \frac{1}{8} \frac{1}{\sin \tilde{\theta}} \frac{\partial}{\partial \tilde{\theta}} \left\{ \cos \tilde{\theta} \left(\int_{\eta_s^{(0)+}}^{\eta_s^{(0)-}} dx [\hat{\gamma}_0^{1b} \partial_b Q] (\eta_s^{(0)+}, x, \tilde{\theta}^a) \right)^2 \right\},
 \end{aligned}$$

$$\mathcal{I}_1 = 2\Xi_s J - \frac{2}{\Delta\eta} Q_s - 2\psi(\eta_s^{(0)}, r_s^{(0)}, \theta^a) ,$$

$$\begin{aligned} \mathcal{I}_2 = & 2\Xi_s \left\{ -\frac{1}{4} (\phi_s^{(2)} - \phi_o^{(2)}) + \frac{1}{4} (\psi_s^{(2)} - \psi_o^{(2)}) - \frac{1}{2} \int_{\eta_{in}}^{\eta_s^{(0)}} d\eta' \frac{a(\eta')}{a(\eta_s^{(0)})} [\partial_r \phi^{(2)}](\eta', r_s^{(0)}, \tilde{\theta}^a) \right. \\ & + \left. \frac{1}{2} \int_{\eta_{in}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)} [\partial_r \phi^{(2)}](\eta', 0, \tilde{\theta}^a) + \frac{1}{4} \int_{\eta_s^{(0)+}}^{\eta_s^{(0)-}} dx \partial_+ [\hat{\phi}^{(2)} + \hat{\psi}^{(2)}](\eta_s^{(0)+}, x, \tilde{\theta}^a) \right\} \\ & - \psi_s^{(2)} - \frac{2}{\Delta\eta} \int_{\eta_s^{(0)+}}^{\eta_s^{(0)-}} dx \left[\frac{\hat{\phi}^{(2)} + \hat{\psi}^{(2)}}{4} \right](\eta_s^{(0)+}, x, \tilde{\theta}^a) . \end{aligned}$$

We have the relation between \mathcal{I} integrands and corrections of d_L :

$$\begin{aligned} \mathcal{I}_1 &= 2\bar{\delta}_S^{(1)} + (\text{t. d.})^{(1)} , \quad \text{with } (\text{t. d.})^{(1)} = 2J_2^{(1)} \\ \mathcal{I}_{1,1} + \mathcal{I}_2 &= 2\bar{\delta}_S^{(2)} + (\bar{\delta}_S^{(1)})^2 + (\text{t. d.})^{(2)} . \end{aligned}$$

$$\bullet \quad \boxed{\langle d_L^{-2} \rangle(z, w_o) = \frac{4\pi(1+z)^{-4}}{\int d^2\tilde{\theta}^a \sqrt{\gamma(w_o, \tau(z, \tilde{\theta}^a), \tilde{\theta}^b)}}} \Rightarrow \begin{cases} \overline{\langle d_L^{-2} \rangle} \equiv (d_L^{FLRW})^{-2} [1 + f_\phi(z)] \\ \text{and } f_\phi(z) \equiv \overline{\langle \mathcal{I}_1 \rangle}^2 - \overline{\langle \mathcal{I}_{1,1} + \mathcal{I}_2 \rangle} \end{cases}$$

Averaging the distance [2,3,4,5]

- Flux: $\overline{\langle d_L^{-2} \rangle} \equiv (d_L^{FLRW})^{-2} [1 + f_\phi(z)]$

where $f_\phi(z) = \int_0^\infty \frac{dk}{k} \mathcal{P}_\psi(k) [f_{1,1}(k, z) + f_2(k, z)] \simeq [\tilde{f}_{1,1}(z) + \tilde{f}_2(z)] \int_0^\infty \frac{dk}{k} \left(\frac{k}{\mathcal{H}_0}\right)^2 \mathcal{P}_\psi(k)$

and $\tilde{f}_{1,1}(z) = \frac{10 - 12\sqrt{1+z} + 5z(2 + \sqrt{1+z})}{27(1+z)(\sqrt{1+z} - 1)^2}$, $\tilde{f}_2(z) = -\frac{1}{189} \left[\frac{2 - 2\sqrt{1+z} + z(9 - 2\sqrt{1+z})}{(1+z)(\sqrt{1+z} - 1)} \right]$.

- Distance: $\overline{\langle d_L \rangle}(z) = d_L^{FLRW} [1 + f_d(z)]$

In general $\overline{\langle F(S) \rangle} \neq F(\overline{\langle S \rangle})$ and simple Taylor expansion gives:

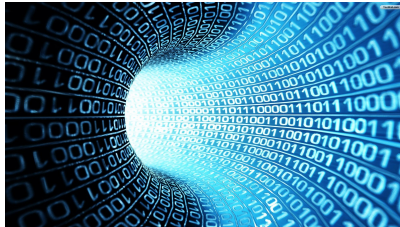
$$\overline{\langle F(S) \rangle} = F(S_0) + F'(S_0) \overline{\langle S - S_0 \rangle} + F''(S_0) \overline{\langle S_1^2/2 \rangle}$$

Applied to $F(\phi) = \Phi^{-1/2} \sim d_L$, it gives:

$$f_d = -(1/2)f_\phi + (3/8) \overline{\langle (\Phi_1/\Phi_0)^2 \rangle}$$

$$\overline{\langle (\Phi_1/\Phi_0)^2 \rangle} = 4 \overline{\langle (\delta_S^{(1)})^2 \rangle} \simeq 4 \left\{ \Xi_s^2 \left[\overline{\langle ([\partial_r P]_s)^2 \rangle} + \overline{\langle ([\partial_r P]_o)^2 \rangle} \right] + \overline{\langle (J_2^{(1)})^2 \rangle} \right\}.$$

Results from Numerical Calculations



Importance of observable: Flux vs Distance (I) [3]

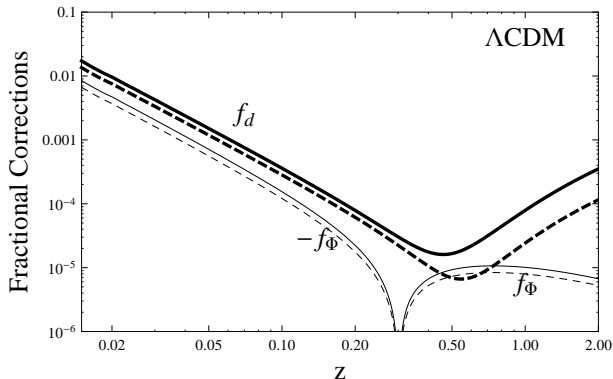
$$f_d = -(1/2)f_\phi + (3/8)\overline{\langle(\Phi_1/\Phi_0)^2\rangle}$$

With linear spectrum :

Plot for :

$k_{UV} = 1 \text{ Mpc}^{-1}$ (solid)

$k_{UV} = 0.1 \text{ Mpc}^{-1}$ (dashed)



Observations :

- $f_d, f_\phi \sim 1 - 10\%$ at small z due to peculiar velocity effect,
- $f_d \gg f_\phi$ at large z due to $\overline{\langle(\Phi_1/\Phi_0)^2\rangle} \sim \overline{\langle(\delta_S^{(1)})^2\rangle}$ involving lensing.

Importance of observable: Flux vs Distance (II) [5]

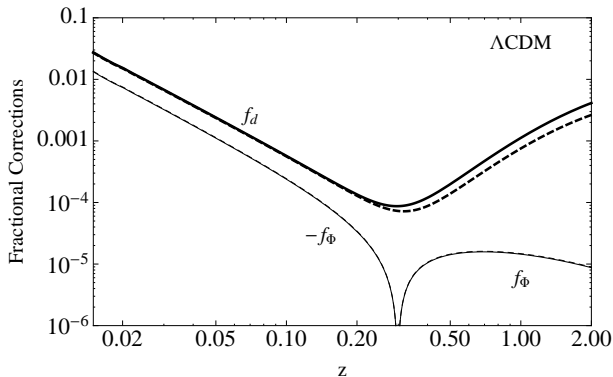
Using the [non-linear power spectrum](#), we can have higher cutoffs.

Plot for :

$k_{UV} = 10 \text{ hMpc}^{-1}$ (solid)

$k_{UV} = 30 \text{ hMpc}^{-1}$ (dashed)

Remark : results are [free](#)
from UV or IR divergences



Observations :

- f_d, f_ϕ at small z are slightly changed,
- $f_d \gg f_\phi$ at large z due to lensing is strongly enhanced : $\sim 1\%$.

Effect on DE measurements (I) [2,3,5]

The same analysis can be applied to the distance modulus:

$$F(\Phi) = -2.5 \log_{10} \Phi + \text{cst} \sim \mu = 5 \log_{10} d_L .$$

We find the **average** effect on μ :

$$\overline{\langle \mu \rangle} = \mu^{FLRW} - 1.25(\log_{10} e) \left[2f_\phi - \overline{\langle (\Phi_1/\Phi_0)^2 \rangle} \right],$$

and the **standard deviation**

$$\sigma_\mu = \sqrt{\overline{\langle \mu^2 \rangle} - \overline{\langle \mu \rangle}^2} = 2.5(\log_{10} e) \sqrt{\overline{\langle (\Phi_1/\Phi_0)^2 \rangle}} .$$

Remark : we can write the more general formula:

$$f_{d^\beta} = -\frac{\beta}{2} f_\phi + \frac{(2+\beta)\beta}{2} \overline{\langle (\delta_S^{(1)})^2 \rangle} ,$$

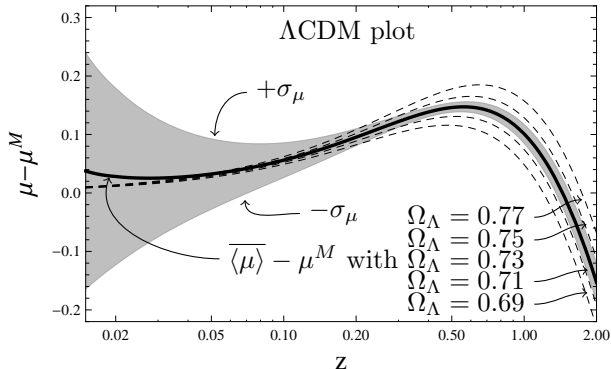
where the flux correction f_ϕ is the minimally affected one at large z .

Effect on DE measurements (II) [2,3,5]

Linear Spectrum :

Plot $k_{UV} = 1 \text{ Mpc}^{-1} \Rightarrow$

$\mu^M = 5 \log_{10} \left[\frac{(2+z)z}{2H_0} \right]$ is
 μ for Milne Universe



Observations :

- Negligible average at large z ($\sim 0.1 - 0.01\%$)
- only a small average shift and big standard deviation at $z \ll 1$ (Doppler)
- BUT standard deviation due to lensing terms is $\sim 1\%$!

Effect on DE measurements (III) [3,5]

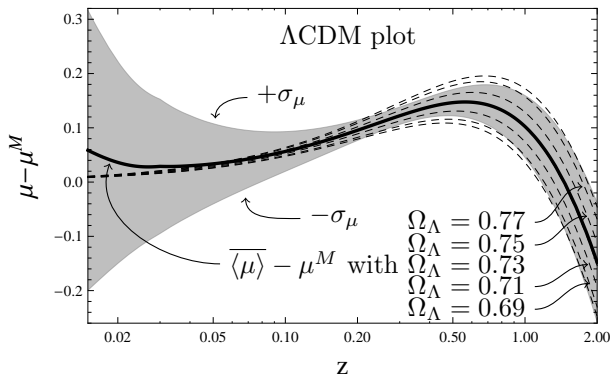
Non-linear Spectrum :

$$k_{UV} = 30 \text{ hMpc}^{-1} \Rightarrow$$

$$\mu^M = 5 \log_{10} \left[\frac{(2+z)z}{2H_0} \right] \text{ is}$$

μ for Milne Universe

Again : no UV or IR divergences

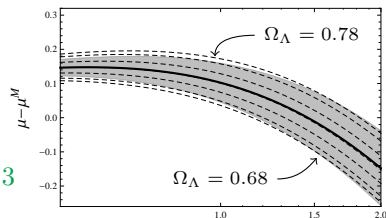


Observations :

- Average at large z still small ($\sim 0.1\%$)
- lensing dispersion bigger : $\sim 10\%$!

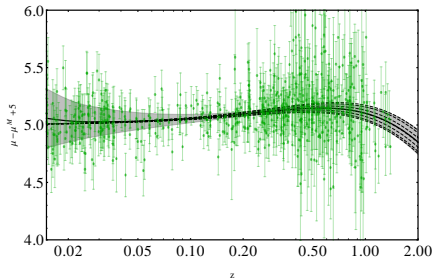
In accordance with other studies :

Li & Schwarz '08, Valkenburg, Kunz, Marra '13



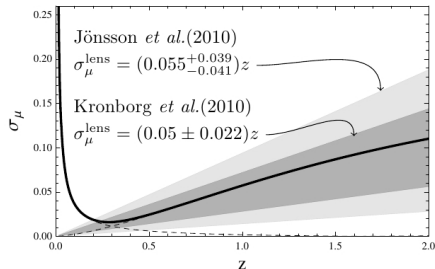
Data and dispersion [5]

With the Union 2 dataset :



- small z : **Velocities** explain quite well the scatter.
- large z : **Lensing** (dominant) term is too weak to explain the large scatter in the data : other effects involved...

$$(\sigma_{\mu}^{\text{obs}})^2 = (\sigma_{\mu}^{\text{fit}})^2 + (\sigma_{\mu}^z)^2 + [(\widehat{\sigma_{\mu}^{\text{int}}})^2 + (\sigma_{\mu}^{\text{lens}})^2]$$



- The total effect is well fitted by **Doppler** ($z \leq 0.2$) + **Lensing** ($z > 0.5$),
- **Doppler** prediction is a bit bigger than in the literature,
- **Lensing** prediction in great agreement with experiments so far!

See recent study: [Marra, Quartin, Amendola '13](#)

Conclusions

Conclusions (I)

This thesis work lead to the development of mathematical tools :

- A gauge invariant **light-cone average** (especially on the 2-sphere) and some applications (like the Buchert-Ehlers commutation rules).
- A system of coordinates, the “**geodesic light-cone**” **gauge**, adapted to light propagation (redshift, distance, Jacobi map, ...)
- The **distance-redshift relation** $d_L(z)$ at $\mathcal{O}(2)$ in the NG, with all its physical terms : SW, ISW, Doppler, Lensing, and new contributions...

We have addressed backreaction from a conservative viewpoint, assuming :

- a flat **FLRW/ Λ CDM** background plus perturbations at $\mathcal{O}(2)$
- one only physical input given by the **power spectrum** $\mathcal{P}_\psi(k)$

Conclusions (II)

Our physical conclusions are the following :

- Effect of inhomogeneities **depends on observables**, e.g. : d_L , d_L^{-2} , μ , ...
The distance is 10-100 times more sensitive than the flux !
- Amount of DE (almost) doesn't change because of an **apparent effect from inhomogeneities** on the light-cone \Rightarrow backreaction negligible.
- Using a **non-linear power spectrum**, the effect of dispersion on the Hubble diagram is **large**, and in good **agreement** with experiments so far.

Inhomogeneities will very probably become an important aspect in the next decades of precision cosmology, especially to understand dark energy.

Thank you!