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Dan Goreac

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LAMA UMR 8050
UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE

Habilitation Thesis

(Mémoire d'Habilitation à Diriger des Recherches)

by

Dan GOREAC

Some topics in deterministic and stochastic control LP methods, PDMP associated to gene networks, controllability

(Quelques sujets en contrôle déterministe et stochastique : méthodes de
type LP, PDMP associés aux réseaux de gènes, contrôlabilité)

Defended on September, 16th 2013

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Mamei mele, Elena

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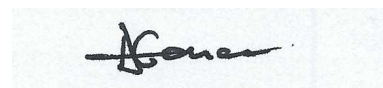
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Présentation des résultats

Résumé

Le but de cette synthèse est de présenter mon activité de recherche couvrant la période de temps écoulée à partir de l'année terminale de ma thèse (c'est à dire, la période octobre 2008 -février 2013). Mes thèmes de recherche correspondent, en majeure partie, à trois directions principales, chacune présentée dans une section dédiée :

- méthodes de programmation linéaire dans l'étude des problèmes de contrôle déterministe ou stochastique ;
- méthodes de contrôle des processus Markoviens déterministes par morceaux et leurs applications dans la théorie des réseaux stochastiques de gènes.
- propriétés de contrôlabilité des systèmes linéaires stochastiques et sujets connexes.

Dans le premier chapitre, nous étudions plusieurs classes de problèmes de contrôle déterministe ou stochastique à coût discontinu. Dans le contexte stochastique, nous considérons le problème de type Mayer et l'arrêt optimal des diffusions contrôlées (correspondant à l'article [G10]), les principes de la programmation dynamique (correspondant à l'article [G6]), ainsi qu'une classe de problèmes de contrôle impliquant des contraintes d'état (correspondant à l'article [G2]). Nous étudions également : des problèmes de contrôle à coût escompté et en horizon infini, ainsi que la moyennisation en temps long (correspondant à [G12]), des systèmes régis par des inégalités variationnelles stochastiques (dans [G3]) et une caractérisation de type Zubov pour les domaines de stabilité asymptotique (toujours dans [G3]). Nous investiguons l'existence d'une fonction valeur limite pour une classe de problèmes de contrôle stochastique sous des hypothèses de non-expansivité, ainsi que des théorèmes Tauberiennes uniformes (correspondant à [G19]). Dans le cadre déterministe, nous considérons la linéarisation et les principes de la programmation dynamique pour des problèmes de type coût supremum (ce qui correspond à [G9]) et pour des systèmes à contraintes d'état (dans [G1]). Nous proposons une méthode de linéarisation pour des problèmes de type min-max (correspondant à [G18]). Le point commun entre ces articles réside dans la méthode employée basée sur des formulations linéaires et des techniques de viscosité. Nous présentons également des résultats de viabilité pour les perturbations singulières des systèmes contrôlés (correspondant à [G13]).

Le deuxième chapitre est axé sur quelques contributions à la théorie des processus de Markov déterministes par morceaux (PDMP, acronyme anglais de "piecewise deterministic Markov process"). Nous investiguons des conditions géométriques pour la viabilité et l'invariance des ensembles fermés par rapport aux dynamiques PDMP contrôlées (correspondant à l'article [G5]). Nous proposons également des formulations linéaires pour certains problèmes de contrôle dans ce contexte (correspondant aux articles [G8] et [G4]). Ces résultats permettent d'en inférer certaines conditions d'atteignabilité (dans l'article [G5]) ainsi que de caractériser les domaines de stabilité asymptotique en généralisant la méthode de Zubov (dans l'article [G4]). Les résultats théoriques sont appliqués à une classe de systèmes associés à des réseaux stochastiques de gènes (des modèles On/Off, le modèle proposé par Cook pour l'haploinsuffisance, ainsi que le modèle de Hasty pour la bistabilité du phage lambda).

Le dernier chapitre présente l'étude de différentes classes de contrôlabilité pour des systèmes linéaires de type diffusion à sauts (correspondant à l'article [G7]) ou des systèmes linéaires de contrôle à dynamique champs-moyen (correspondant à l'article [G20]). Les arguments font intervenir des techniques de viabilité ainsi que des équations différentielles de type Riccati. Une première étape dans l'étude des propriétés de contrôlabilité des systèmes ayant comme espace d'état un espace d'Hilbert est franchie dans l'article [G11]. Nous y proposons une approche de type quasi-tangence dans l'étude de la propriété de (presque)viabilité des systèmes semi-linéaires dans un cadre infini-dimensionnel.

Nous avons essayé de rendre le manuscrit aussi autonome que possible. Pour en assurer la lisibilité, nous avons également essayé de garder l'indépendance des chapitres. Afin de garder une dimension raisonnable du manuscrit, nous avons fait le choix de limitation de la redondance. Pour cette raison, les problèmes de contrôle sous contraintes d'état ont été présentés uniquement dans le contexte stochastique. Aussi, les détails précis de la méthode de Zubov ont été spécifiés uniquement dans le cas des processus Markoviens déterministes par morceaux et les contributions aux diffusions Browniennes ont été seulement mentionnées.

Chapitre 1. Méthodes de programmation linéaire pour des problèmes de contrôle déterministe et stochastique

Les formulations de type programmation linéaire sont très utiles dans l'étude des problèmes de contrôle en horizon fini ou infini. Ces méthodes ont été employées avec succès pour les systèmes à dynamique déterministe (dans [Art02], [GL99], [GN02], [GQ09]) ou à dynamique réfléchie (dans [QS10]). La technique peut être étendue à un cadre stochastique (voir [FV89], [BB96], [BBG97], [KS98], [Sto90], [YZ97], [Gai04], [BG05], [Bor06], [GR06], [BG07], [DS12] et les contributions qui y sont mentionnées). L'un des nombreux avantages de cette méthode réside dans la possibilité d'obtenir des résultats d'approximation pour la fonction valeur (voir [FGL07] et [GQ09]) en approchant les mesures d'occupation par des masses de Dirac.

Dans la section 1.1 intitulée "Methodes LP pour des problèmes de contrôle à dynamique Brownienne", nous étudions plusieurs classes de problèmes de contrôle dont la fonction coût est semi-continue : le problème de type Mayer, l'arrêt optimal pour des diffusions contrôlées ([G10]), des principes de la programmation dynamique ([G6]), des problèmes de contrôle à contraintes d'état ([G2]), des problèmes de contrôle stochastique en horizon infini et à coût escompté ainsi que la moyennisation à long terme ([G12]), des systèmes gouvernés par des inégalités variationnelles stochastiques ([G3]), la caractérisation de type Zubov des domaines de stabilité asymptotique ([G3]), l'existence d'une fonction valeur limite pour une classe de problèmes non-linéaires de contrôle sous une hypothèse de non-expansivité ainsi que des théorèmes de type Tauberien ([G19]).

Dans le cas des problèmes de contrôle à coût discontinu et dynamique gouvernée par une équation déterministe

$$x'(t) = f(t, x(t), u(t)),$$

une hypothèse essentielle est la convexité de la dynamique. Dans ce cas, la fonction valeur est également discontinue (voir, par exemple, [BJ90, Fra93]). Si la fonction coût est continue, alors la fonction valeur associée à la dynamique ci-dessus coïncide avec la fonction valeur associée à l'inclusion différentielle convexifiée

$$x'(t) \in F(t, x(t)), \quad F(t, x) = \overline{\text{co}}(f(t, x, u) : u \in U),$$

à l'aide du théorème de Fillipov (voir, par exemple, [AC84]). Toutefois, si la dynamique n'est pas convexe et le coût est seulement semi-continu inférieurement, alors les deux fonctions valeur ne coïncident plus. Par conséquent, afin de caractériser la fonction valeur par rapport à l'équation de Hamilton-Jacobi associée, on doit modifier la définition en considérant l'inclusion différentielle convexifiée. Dans le cadre stochastique, on peut argumenter de la même façon en prenant la limite faible des solutions (ou encore l'enveloppe convexe fermée de l'ensemble des mesures d'occupation). Nous souhaitons souligner que la procédure utilisée est en quelque sorte inversée : nous considérons la formulation explicite de l'ensemble de contraintes et nous montrons que cet ensemble coïncide avec l'enveloppe convexe fermée des mesures d'occupation. A notre avis, cette méthode est le chemin naturel permettant d'obtenir de bonnes propriétés pour la fonction valeur quand la fonctionnelle de coût est semi-continue (ou, même, seulement mesurable). Ceci constitue la raison principale pour l'utilisation des méthodes LP pour les problèmes de contrôle optimal à coût discontinu et dynamique Brownienne.

Le deuxième avantage de notre méthode est le fait que celle-ci offre une alternative à l'utilisation d'arguments de viabilité. Ceci est particulièrement intéressant lorsqu'on ne suppose aucune condition de convexité sur la dynamique. Ainsi, on ne peut guère garantir la viabilité stricte (mais seulement la viabilité approchée) et on ne peut plus appliquer des techniques d'épigraphe.

Finalement, l'ensemble de contraintes impliqué dans la formulation linéaire bénéficie de propriétés de compacité. Ces dernières jouent un rôle important dans l'étude de problèmes asymptotiques.

Des méthodes similaires s'appliquent au cadre déterministe dans l'étude des problèmes hautement non-linéaires. Dans la section 1.2 intitulée "Les méthodes LP dans les problèmes de contrôle déterministe", nous étudions : des problèmes de contrôle à coût discontinu sous contraintes d'état ([G1]), la linéarisation de problèmes de contrôle à coût classique, de type \mathbb{L}^p ou encore \mathbb{L}^∞ ([G9]). Nous étudions également des principes de la programmation dynamique dans des problèmes à coût discontinu de type classique ou bien de type supremum (dans [G9]) ainsi que pour des dynamiques sous contraintes d'état (dans [G1]). D'autres résultats sont obtenus pour les problèmes min/max (correspondant à l'article [G18]) et pour la viabilité des systèmes contrôlés avec perturbations singulières (correspondant à l'article [G13]).

Chapitre 2. Problèmes de contrôle pour des processus Markoviens déterministes par morceaux et leurs applications aux réseaux stochastiques de gènes

Le deuxième chapitre est dédié aux problèmes de contrôle optimal pour des systèmes ayant une dynamique Markovienne déterministe par morceaux ainsi que les applications de ces problèmes à la théorie des réseaux stochastiques de gènes. Les processus de Markov ont été intensivement utilisés afin de décrire la variabilité des processus cellulaires. A notre connaissance, des outils Markoviens ont été pour la première fois employés en rapport avec la biologie moléculaire dans [Del40]. L'idée naturelle a été d'associer à chaque système de réactions un modèle à saut pure. A cause du grand nombre d'espèces moléculaires présentes dans les réactions, une simulation directe de ce type de système est particulièrement lente. Afin

d'accroître la vitesse, les auteurs de [CDR09] adoptent des modèles hybrides. Ils distinguent les composantes discrètes de celles continues. L'utilisation d'une expansion Kramers-Moyal permet aux auteurs de remplacer le processus initial (à saut pur) par un PDMP (processus Markovien déterministe par morceaux ; l'acronyme provenant de l'anglais). Les modèles hybrides ainsi obtenus peuvent être appliqués à l'étude « in silico » d'une grande variété de systèmes biologiques : le modèle de Cook (voir [CGT98]) pour l'expression de gènes et ses implications sur l'haploinsuffisance, le modèle de Hasty (voir [HPDC00]) pour le phage lambda, etc.

Dans la section 2.3.1, nous investiguons des conditions géométriques pour la viabilité et l'invariance des ensembles fermés par rapport aux dynamiques PDMP contrôlées (correspondant à l'article [G5]). La connaissance des compacts invariants par rapport à la dynamique PDMP permet l'implémentation efficace des algorithmes nécessitant une discrétisation de l'espace d'état. Les méthodes utilisées sont inspirées par la littérature sur le concept de viabilité dans le cadre des diffusions Browniennes et ont comme principale technique les solutions de viscosité des équations Hamilton-Jacobi intégrodifférentielles associées. Elles font apparaître le cône normal à l'ensemble de contraintes.

Dans la section 2.3.2 (correspondant aux articles [G8] et [G4]), nous obtenons des méthodes de type LP analogues au premier chapitre. Ces résultats permettent d'en inférer certaines conditions d'atteignabilité (dans la section 2.3.3, correspondant à l'article [G5]) ainsi que de caractériser les domaines de stabilité asymptotique en généralisant la méthode de Zubov (dans la section 2.3.4, correspondant à l'article [G4]). En particulier, la technique LP permet d'éviter la stabilité des processus de contrôle par concaténation.

Dans la section 2.4, nous illustrons les résultats techniques sur des modèles issus de la biologie des microorganismes : nous présentons les propriétés d'invariance et d'atteignabilité pour le modèle de Cook sur l'haploinsuffisance, ainsi que la zéro-contrôlabilité du modèle de Cook multi-régime (dans le cas instable). Nous illustrons également les techniques de viabilité par rapport au modèle de Hasty sur la bi-stabilité du phage lambda. En particulier, nous montrons que ce dernier système possède un unique point de stabilité et que la deuxième composante stable est nécessairement un ensemble qui ne se réduit pas à un singleton (tel que supposé par les méthodes inspirées par le cadre déterministe).

Chapitre 3. Propriétés de contrôlabilité pour des systèmes linéaires stochastiques et sujets connexes

Le troisième chapitre propose quelques aspects de contrôlabilité des systèmes linéaires ayant une composante infini-dimensionnelle. Pour les diffusions Browniennes à espace d'état fini-dimensionnel, la contrôlabilité exacte a été caractérisée dans [Pen94] (voir également [LP02], [LP10]). La contrôlabilité (terminale) exacte nécessite un opérateur de rang plein agissant sur le contrôle dans le terme de bruit (voir [Pen94]). La méthode utilise la théorie des équations différentielles stochastiques rétrogrades (introduites par [PP90]). La contrôlabilité approchée dans un cadre Brownien fini-dimensionnel a été étudiée dans [BQT06] lorsque le contrôle n'agit pas sur le bruit (voir également [Gor08] pour le cas général). En généralisant la condition de Kalman, les auteurs fournissent un critère de contrôlabilité approchée à l'aide de techniques de dualité. En dimension infinie, des méthodes similaires conduisent à des résultats partiels (voir [Gor09]). De nombreuses méthodes alternatives sont disponibles dans le cadre Hilbertien (voir [FCGAR99], [ST01], [BRT03]).

Dans le cadre fini-dimensionnel, le critère d'observabilité stochastique utilise de façon essentielle les propriétés de viabilité du système adjoint. Dans les articles [BQT06] et [G15, G16], on obtient l'équivalence entre les notions de contrôlabilité approchée et 0-contrôlabilité approchée. Un critère d'invariance est obtenu à l'aide de techniques backward (rétrogrades) et de propriétés de viabilité.

Afin d'obtenir une meilleure compréhension des systèmes infini-dimensionnels, nous avons suivi deux pistes. La première conduit à l'investigation de systèmes de contrôle stochastique ayant un espace d'état fini-dimensionnel mais présentant une composante infini-dimensionnelle. Dans cette direction, nous avons considéré :

la propriété de contrôlabilité approchée pour une classe d'équations stochastiques linéaires gouvernées par un mouvement Brownien et une mesure aléatoire de Poisson indépendante (dans l'article [G7]) ;

la contrôlabilité terminale exacte, la contrôlabilité approchée ainsi que la 0-contrôlabilité approchée pour des systèmes linéaires de type champs-moyen (correspondant à l'article [G20]).

Pour la première classe de systèmes, nous montrons que la notion de contrôlabilité approchée et celle de 0-contrôlabilité approchée coïncident (tout comme c'était déjà le cas pour les systèmes Browniens). Nous obtenons également un critère d'invariance conditionnelle. En général, ce critère explicite fait intervenir un espace de type \mathbb{L}^2 . Toutefois, pour des cas particuliers, nous proposons un schème itératif fini.

Dans le contexte des systèmes de type champs-moyen, nous montrons que les deux notions de contrôlabilité approchée ne coïncident plus. A l'aide d'équations de type Riccati, nous étudions les critères équivalents à la 0-contrôlabilité approchée. Cela nous conduit à l'obtention de conditions d'invariance suffisantes pour garantir cette propriété ainsi que des conditions équivalentes. Nous proposons également quelques hypothèses sous lesquelles la 0-contrôlabilité implique la contrôlabilité approchée.

La deuxième approche consiste à étudier la propriété de viabilité des systèmes semi-linéaires stochastiques contrôlés ayant comme espace d'état un espace de Hilbert. Ce sujet est traité dans l'article [G11] où l'on montre que la viabilité est équivalente à une propriété de quasi-tangence. Les arguments essentiels sont une caractérisation séquentielle de la notion centrale de quasi-tangence et la construction de solutions mild maximales à l'aide du Théorème de Brezis-Browder. Le résultat principal est appliqué au cas linéaire et pour la généralisation d'un résultat de Nagumo pour des ensembles de contraintes à frontière régulière.

Abstract

The aim of this synthesis is to present my research activity covering the period elapsed since the terminal year of my PhD program (i.e. the period October 2008 - February 2013). My subjects of research roughly correspond to three main directions, each one presented in a separate section :

- Linear programming (LP) methods in deterministic and stochastic control problems;
- Control methods in piecewise deterministic Markov processes and their applications to stochastic gene networks;
- Controllability properties for linear stochastic systems and related topics.

In the first section, we study several classes of deterministic/stochastic control problems with semicontinuous cost. In the stochastic framework, we consider the Mayer problem and optimal stopping for controlled diffusions (corresponding to the paper [G10]), dynamic programming principles (corresponding to the paper [G6]) and control problems with state constraints (corresponding to the paper [G2]). We also study optimal control problems with discounted payoffs in an infinite horizon setting and long-time averaging (corresponding to [G12]), systems driven by stochastic variational inequalities ([G3]), and Zubov's characterization of asymptotic stability domains ([G3]). We investigate the existence of the limit value function for a class of nonlinear stochastic control problems under nonexpansive assumptions and uniform Tauberian theorems (corresponding to [G19]). In the deterministic setting, we consider the linearization and the dynamic programming principles for \mathbb{L}^∞ -control problems (corresponding to the paper [G9]) and for systems with state constraints (in [G1]). We propose a linearization method for min-max control problems (corresponding to [G18]). The common point between these articles is the method employed relying on linearized formulation and viscosity tools. We also present some viability results for singularly perturbed control systems (corresponding to [G13]).

The second section is devoted to some contributions to the theory of controlled piecewise deterministic Markov processes (PDMP). We investigate geometric conditions for viability and invariance with respect to controlled PDMPs (corresponding to [G5]). We also provide linear formulations for control problems in this framework (corresponding to [G8] and [G4]). This allows to infer some reachability conditions (cf. [G5]) and to characterize asymptotic stability domains by generalizing Zubov's method (in [G4]). The theoretical results are applied to a class of systems connected to stochastic gene networks (On/Off models, Cook's model for haploinsufficiency and Hasty's model of bistability in bacteriophage λ).

The last section presents the study of different types of controllability in connection to: linear jump-diffusions (corresponding to [G7]) and linear control systems of mean-field type (corresponding to [G20]). The arguments involve some viability criteria. A first step towards the study of controllability properties in a Hilbert setting is done by the paper [G11]. In this article, we propose a quasi-tangency approach to stochastic (near)viability with respect to semilinear systems in an infinite-dimensional framework.

We have tried to keep the manuscript as self-contained as possible. In order to insure better readability, we have also tried to make the sections independent. However, to keep the manuscript's proportions, we have chosen to limit the redundancy. This is the reason why control problems with state constraints are only presented in the stochastic framework. Also, details for Zubov's method are solely given in the PDMP framework and we only mention the contribution for Brownian diffusions.

List of papers

Publications

After PhD

- [G1] Goreac, D., Ivascu, C., *Discontinuous control problems with state constraints : linear formulations and dynamic programming principles*, Journal of Mathematical Analysis and Applications vol. 402 (2) (2013), pp. 635-647, DOI: 10.1016/j.jmaa.2012.12.043.
- [G2] Goreac, D., Ivascu, C., Serea, O.S., *An LP Approach to Dynamic Programming Principles for Stochastic Control Problems with State Constraints*, Nonlinear Analysis Series A: Theory, Methods & Applications 77 (2013), pp. 59-73, DOI: 10.1016/j.na.2012.09.002.
- [G3] Goreac, D., Serea, O.S., *Some applications of linear programming formulations in stochastic control*, Journal of Optimization Theory and Applications Vol. 155, No.2 (2012), pp. 572-593, DOI: 10.1007/s10957-012-0080-z.
- [G4] Goreac, D., Serea, O.-S., *Linearization techniques for controlled piecewise deterministic Markov processes; application to Zubov's method*, Applied Mathematics & Optimization: Volume 66, Issue 2 (2012), pp. 209-238, DOI: 10.1007/s00245-012-9169-x.
- [G5] Goreac, D., *Viability, invariance and reachability for controlled piecewise deterministic Markov processes associated to gene networks*, ESAIM: Control, Optimisation and Calculus of Variations April 2012 18 : pp. 401-426, DOI: 10.1051/cocv/2010103.
- [G6] Goreac, D., Serea, O.-S., *A note on linearization methods and dynamic programming principles for stochastic discontinuous control problems*. Electron. Commun. Probab., 17:no. 12, 1.12, (2012), DOI: 10.1214/ECP.v17-1844.
- [G7] Goreac, D., *A Note on the Controllability of Jump-Diffusions with Linear Coefficients*, IMA Journal of Mathematical Control and Information (2012) 29(3): 427-435 , DOI: 10.1093/imamci/dns001.
- [G8] Goreac, D., *On Linearized Formulations for Control Problems with Piecewise Deterministic Markov Dynamics*, Bulletin of the Transilvania University of Brasov Series III: Mathematics, Informatics, Physics, Vol 5(54) 2012, Special Issue: *Proceedings of the Seventh Congress of Romanian Mathematicians*, 131-144, published by Transilvania University Press, Brasov and Publishing House of the Romanian Academy.
- [G9] Goreac, D., Serea, O.-S., *Linearization Techniques for L^∞ -Control Problems and Dynamic Programming Principles in Classical and L^∞ -Control Problems*, ESAIM: Control, Optimisation and Calculus of Variations, Volume 18, Issue 03, juillet 2012, pp 836 - 855, DOI: 10.1051/cocv/2011183.
- [G10] Goreac, D., Serea, O.-S., *Mayer and optimal stopping stochastic control problems with discontinuous cost*, Journal of Mathematical Analysis and Applications, vol. 380 (1) (2011), pp. 327-342, DOI:10.1016/j.jmaa.2011.02.039.
- [G11] Goreac, D., *Viability of Stochastic Semilinear Control Systems via the Quasi-Tangency Condition*, IMA Journal of Mathematical Control and Information (2011), 28: 391-415; DOI:10.1093/imamci/dnr003.

[G12] Buckdahn, R., Goreac, D., Quincampoix, M., *Stochastic Optimal Control and Linear Programming Approach*, Appl. Math. Optim., vol. 63 (2011), no. 2, pp. 257-276, DOI: 10.1007/s00245-010-9120-y.

[G13] Goreac, D., Serea, O.-S., *Discontinuous control problems for non-convex dynamics and near viability for singularly perturbed control systems*, Nonlinear Anal. 73 (2010), no. 8, pp. 2699-2713, DOI: 10.1016/j.na.2010.06.050.

During PhD

[G14] Goreac, D., *Approximate controllability for linear stochastic differential equations in infinite dimensions*. Appl. Math. Optim. 60 (2009), no. 1, pp. 105-132, DOI: 10.1007/s00245-009-9068-y.

[G15] Goreac, D., *A Kalman-type condition for stochastic approximate controllability*. C. R. Math. Acad. Sci. Paris 346 (2008), no. 3-4, pp. 183-188, DOI:10.1016/j.crma.2007.12.008.

[G16] Goreac, D., *Approximate controllability for linear stochastic differential equations with control acting on the noise*. Applied analysis and differential equations, 153-164, World Sci. Publ., Hackensack, NJ, 2007.

[G17] Goreac, D., *Non-compact-valued stochastic control under state constraints*, Bull. Sci. Math. 131 (2007), no. 8, pp. 716-737, DOI:10.1016/j.bulsci.2006.08.001

Preprints

[G18] D. Goreac, O.S. Serea, *Min-max control problems via occupational measures (2012)*, 22 pp. (accepted to Optimal Control, Applications and Methods).

[G19] R. Buckdahn, D. Goreac, M. Quincampoix, *Existence of Asymptotic Values for Non-expansive Stochastic Control Systems (2013)*, 22 pp. (submitted).

[G20] D. Goreac, *Controllability properties for controlled linear systems of mean-field type (2013)*, 15 pp. (submitted).

[G21] D. Goreac, O.S., Serea, *An Abel-type result for controlled piecewise deterministic Markov processes (2013)*, 12 pp. (submitted).

[G22] D. Goreac, O.S., Serea, *Optimality conditions for stochastic reflecting boundary control problems (2013)* (working paper).

[G23] D. Goreac, M. Kobylanski, *Viability and viability kernels for BSDEs in infinite dimensions (2013)*, (working paper).

Other documents

[G24] D. Goreac, *Insurance, Reinsurance and Dividend Payment* (2008) (unpublished)

[G25] D. Goreac, *Problèmes de contrôle stochastique : contrôle sous contrainte, contrôlabilité et application à la réassurance*, (Thèse de Doctorat, UBO, Brest, 2005-2008).

- [G26] V. Amarandei, N. Buzduga, D. Goreac, L. Iacob, *Problems ... have problems? (in Romanian : Probleme... cu probleme?)*, ISBN: 973-8034-10-8, Ed. AXA, Botosani, (2000), 114 pp. (problems for high-school mathematical competitions).

1 Linear programming (LP) methods for deterministic and stochastic control problems

Linear programming formulations are very useful in the study of finite and infinite horizon control problems. These methods have been successfully applied to classical deterministic systems (cf. [Art02], [GL99], [GN02], [GQ09]) or reflected systems (cf. [QS10]). The technique can be extended to the stochastic setting (e.g. [FV89], [BB96], [BBG97], [KS98], [Sto90], [YZ97], [Gai04], [BG05], [Bor06], [GR06], [BG07], [DS12] and references therein). One of the advantages consists in the possibility to obtain approximation results for the value function (e.g. [FGL07] and [GQ09]) by approximating the occupational measures by Dirac measures.

1.1 LP methods in control problems with Brownian dynamics

1.1.1 Introduction

We study several classes of stochastic control problems with semicontinuous cost : Mayer problem, optimal stopping for controlled diffusions ([G10]), dynamic programming principles ([G6]), control problems with state constraints([G2]), stochastic optimal control problems with constraints and discounted payoffs in an infinite horizon setting; long-time averaging ([G12]), systems driven by stochastic variational inequalities ([G3]), Zubov's characterization of asymptotic stability domains ([G3]), existence of the limit value function for a class of non-linear stochastic control problems under non expansive assumptions and uniform Tauberian theorems ([G19]).

In the case of control problems with discontinuous costs and governed by a deterministic equation

$$x'(t) = f(t, x(t), u(t)),$$

a standard assumption is convexity of the dynamics. In this case, the value function (for Mayer problem with/without constraints) is also discontinuous (see for instance [BJ90, Fra93]). Moreover, if the cost is continuous, the value function associated to the previous dynamics coincide with the value function associated to the convexified differential inclusion

$$x'(t) \in F(t, x(t)), \quad F(t, x) = \overline{\text{co}}(f(t, x, u) : u \in U).$$

due to Fillipov's Theorem (see for instance [AC84]). However, when the dynamics is not convex and the cost is l.s.c., the value functions associated to the previous dynamics do not coincide. Consequently, in order to characterize the value function with respect to the associated Hamilton-Jacobi system, one has to modify the definition by considering the convexified differential inclusion. In the stochastic framework, one can reason in a similar way, by taking the weak limit of solutions (or the closed convex hull of occupational measures). We emphasize that the actual procedure is somewhat reversed: we consider the explicit set and deduce that it is the closed convex hull of occupational measures. It is our opinion that this is the natural method allowing to obtain good properties of the value function whenever the cost is semicontinuous (or just bounded). This is the first reason for our LP approach to control problems with discontinuous costs and Brownian dynamics.

The second advantage of our method is that it offers an alternative to the use of viability arguments. This is particularly interesting if convexity of the dynamics is not assumed and one can only guarantee near viability (hence, not being able to apply epigraph techniques).

Finally, the sets of constraints involved in the linear formulations turn out to enjoy compactness properties. This turns out to be very useful in the study of asymptotes ([G12], [G19]).

The value functions are introduced via a linear optimization problem on appropriate sets of probability measures. These sets of constraints are described deterministically with respect to the coefficient functions. The linearization methods rely on occupational measures and viscosity techniques.

To fix the framework, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual assumptions and W be a standard, d -dimensional Brownian motion with respect to this filtration. We denote by $T > 0$ a finite time horizon and we let U be a compact metric space. We consider the stochastic control system

$$(1) \quad \begin{cases} dX_s^{t,x,u} = b(X_s^{t,x,u}, u_s) ds + \sigma(X_s^{t,x,u}, u_s) dW_s, & \text{for all } s \in [t, T], \\ X_t^{t,x,u} = x \in \mathbb{R}^N, \end{cases}$$

where $t \in [0, T]$. We use the following standard assumption on the coefficient functions $b : \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ and $\sigma : \mathbb{R}^N \times U \rightarrow \mathbb{R}^{N \times d}$:

$$(2) \quad \begin{cases} \text{(i) the functions } b \text{ and } \sigma \text{ are bounded and uniformly continuous on } \mathbb{R}^N \times U, \\ \text{(ii) there exists a real constant } c > 0 \text{ such that} \\ |b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| \leq c|x - y|, \end{cases}$$

for all $(x, y, u) \in \mathbb{R}^{2N} \times U$. We let \mathcal{U} denote the class of all admissible (progressively measurable) control processes on $[0, T] \times \Omega$ and \mathcal{U}_t the class of admissible control processes independent of \mathcal{F}_t . Under the assumption (2), for every $(t, x) \in [0, T] \times \mathbb{R}^N$ and every admissible control process $u \in \mathcal{U}$, there exists a unique solution to (1) starting from (t, x) denoted by $X^{t,x,u}$.

To every $0 < r \leq T$, $(t, x) \in [0, r] \times \mathbb{R}^N$ and every $u \in \mathcal{U}$, we associate the (expectation of the) occupational measures

$$\gamma_1^{t,r,x,u}(A \times B \times C) = \frac{1}{r-t} \mathbb{E} \left[\int_t^r 1_{A \times B \times C}(s, X_s^{t,x,u}, u_s) ds \right], \gamma_2^{t,r,x,u}(D) = \mathbb{E} [1_D(X_r^{t,x,u})],$$

for all Borel subsets $A \times B \times C \times D \subset [t, r] \times \mathbb{R}^N \times U \times \mathbb{R}^N$. Also, we can define $\gamma_1^{r,r,x,u}(A \times B \times C) = \delta_{r,x}(A \times B) \mathbb{P}(u_r \in C)$, $\gamma_2^{r,r,x,u} = \delta_x$, for all Borel subsets $A \times B \times C \subset [t, r] \times \mathbb{R}^N \times U$. We denote by

$$\Gamma(t, r, x) = \{ \gamma^{t,r,x,u} = (\gamma_1^{t,r,x,u}, \gamma_2^{t,r,x,u}) \in \mathcal{P}([t, r] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : u \in \mathcal{U} \}.$$

Here, $\mathcal{P}(\mathcal{X})$ stands for the set of probability measures on the metric space \mathcal{X} . Due to the Assumption (2), there exists a positive constant C_0 such that, for every $T > 0$, every $(t, x) \in [0, r] \times \mathbb{R}^N$ and every $u \in \mathcal{U}$, one has

$$(3) \quad \sup_{s \in [t, r]} \mathbb{E} \left[|X_s^{t,x,u}|^2 \right] \leq e^{C_0(r-t)} (|x|^2 + C_0(r-t)).$$

Therefore,

$$(4) \quad \begin{cases} \int_{\mathbb{R}^N} |y|^2 \gamma_1^{t,r,x,u}([t,r], dy, U) \leq e^{C_0(r-t)} (|x|^2 + C_0(r-t)), \\ \int_{\mathbb{R}^N} |y|^2 \gamma_2^{t,r,x,u}(dy) \leq e^{C_0(r-t)} (|x|^2 + C_0(r-t)). \end{cases}$$

We define

$$(5) \quad \Theta(t, r, x) = \left\{ \begin{array}{l} \gamma \in \mathcal{P}([t, r] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : \forall \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N), \\ \int_{[t,r] \times \mathbb{R}^N \times U \times \mathbb{R}^N} \begin{bmatrix} (r-t) \mathcal{L}^v \phi(s, y) \\ + \phi(t, x) - \phi(T, z) \end{bmatrix} \gamma_1(ds, dy, dv) \gamma_2(dz) = 0, \end{array} \right\}$$

where

$$\mathcal{L}^v \phi(s, y) = \frac{1}{2} Tr [(\sigma \sigma^*)(y, v) D^2 \phi(s, y)] + \langle b(y, v), D \phi(s, y) \rangle + \partial_t \phi(s, y),$$

for all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^N$, $v \in U$ and all $\phi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$.

If $g : \mathbb{R} \times \mathbb{R}^N \times U \rightarrow \mathbb{R}$, $h : \mathbb{R}^N \rightarrow \mathbb{R}$ are bounded, measurable functions, we introduce the usual value function

$$(6) \quad V^r(t, x) = \inf_{u \in \mathcal{U}_t} \mathbb{E} \left[\int_t^r g(s, X_s^{t,x,u}, u_s) ds + h(X_T^{t,x,u}) \right],$$

and the primal linearized value function

$$(7) \quad \Lambda^r(t, x) = \inf_{\gamma \in \Gamma(t,r,x)} \left((r-t) \int_{[t,r] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds, dy, du) + \int_{\mathbb{R}^N} h(y) \gamma_2(dy) \right),$$

for all $(t, x) \in [0, r] \times \mathbb{R}^N$. It is clear that

$$V^r(t, x) \geq \inf_{\gamma \in \Gamma(t,r,x)} \left((r-t) \int_{[t,r] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds, dy, du) + \int_{\mathbb{R}^N} h(y) \gamma_2(dy) \right) \geq \Lambda^r(t, x),$$

for all $(t, x) \in [0, r] \times \mathbb{R}^N$. We also consider the dual value function

$$(8) \quad \Lambda^{*,r}(t, x) = \sup \left\{ \begin{array}{l} \mu \in \mathbb{R} : \exists \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N) \\ \text{s.t. } \forall (s, y, v, z) \in [t, r] \times \mathbb{R}^N \times U \times \mathbb{R}^N, \\ \mu \leq (r-t) (\mathcal{L}^v \phi(s, y) + g(s, y, u)) + h(z) - \phi(r, z) + \phi(t, x), \end{array} \right\}$$

for all $(t, x) \in [0, r] \times \mathbb{R}^N$.

The first result in [G10] states the connection between these value functions.

Theorem 1 ([G10], Theorem 4). *Under the assumptions (2), if g is bounded, uniformly continuous and Lipschitz-continuous in time and space uniformly w.r.t. the control and h is bounded and Lipschitz-continuous, then*

$$V^r = \Lambda^r = \Lambda^{*,r}.$$

This result (under equivalent formulations) is already known in the literature (cf. [FV89], see also [BB96]). The method used in the cited papers relies on a convex analysis approach. In [G10] we provide a short proof based entirely on viscosity arguments. The definition of Θ and $\Lambda^{*,r}$ yield the inequality

$$V^r \geq \Lambda^r \geq \Lambda^{*,r}.$$

To prove that $\Lambda^{*,r}(x) \geq V^r(x)$, we adapt the so-called "shaking the coefficients method" introduced in [Kry00] and the stability results in [BJ02b]. This allows to approximate V^r by a sequence of regular subsolutions of the associated Hamilton-Jacobi-Bellman equation. For further details, the reader is referred to [G10], Theorem 4.

Since this result holds true for arbitrary (regular) functions g and h , a standard separation argument yields:

Corollary 2 *The set of constraints $\Theta(t, r, x)$ is the closed, convex hull of $\Gamma(t, r, x)$:*

$$(9) \quad \Theta(t, r, x) = \overline{\text{co}}\Gamma(t, r, x).$$

The closure is taken with respect to the usual (weak) convergence of probability measures.

In particular, together with (4) and Prohorov's theorem, this corollary implies the compactness of $\Theta(t, r, x)$.

1.1.2 Mayer stochastic control problems with discontinuous cost

In the case when g and h are only semicontinuous, one can still define V^T , Λ^T and $\Lambda^{*,T}$. One can also (formally) consider the associated Hamilton-Jacobi-Bellman equation

$$(10) \quad -\partial_t V(t, x) + H(t, x, DV(t, x), D^2V(t, x)) = 0,$$

for all $(t, x) \in (0, T) \times \mathbb{R}^N$, and $V(T, \cdot) = h(\cdot)$ on \mathbb{R}^N , where the Hamiltonian is given by

$$(11) \quad H(t, x, p, A) = \sup_{u \in U} \left\{ -\frac{1}{2} \text{Tr}(\sigma \sigma^*(x, u) A) - \langle b(x, u), p \rangle - g(t, x, u) \right\},$$

for all $(t, x, p, A) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}^N$. We recall that \mathcal{S}^N stands for the family of symmetric $N \times N$ -type matrix. The following Theorem states the connection between these various functions and the Equation (10). Proofs for these results can be found in Section 4 of [G10]. They are based on inf/sup-convolutions and the compactness of $\Theta(t, T, x)$.

Theorem 3 *If the functions g and h are bounded and lower semicontinuous, then the primal and dual value functions coincide $\Lambda^T = \Lambda^{*,T}$. The common value is the smallest bounded l.s.c. viscosity supersolution of (10).*

If the functions g and h are bounded and upper semicontinuous, then V^T coincides with Λ^T . The common value is the largest bounded u.s.c. viscosity subsolution of (10).

In the l.s.c. case, the assertions are given by Theorem 7 in [G10]. The u.s.c. case is covered by Theorem 8 in [G10].

If the functions g and h are bounded and lower semicontinuous, $V^T \geq \Lambda^T$. Under the usual convexity assumption (cf. [EKNJP87])

$$F(x, u) = \{(b(x, u), \sigma \sigma^*(x, u), g(t, x, u)) : u \in U\} \text{ is convex } \forall x \in \mathbb{R}^N,$$

Λ^T coincides with the weak formulation of V^T (i.e.

$$\Lambda^T(t, x) = \inf_{\pi=(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, u)} \mathbb{E}^\pi \left[\int_t^T g(s, X_s^{t,x,u}, u_s) ds + h(X_T^{t,x,u}) \right].$$

This equality is established by Proposition 12 in [G10]. However, if the dynamics are not convex, the two value functions may not coincide (cf. Example 13 in [G10]). If the functions g and h are bounded and upper semicontinuous, then V^r and $\Lambda^{*,r}$ may not coincide (cf. Example 9 in [G10]).

1.1.3 Adding stopping times

As we have seen in the definition of the stochastic set (7), the main ingredients rely on Itô's formula applied to regular test functions. Furthermore, regularity properties of this set (e.g. compactness) are obtained as a consequence of second order moment estimates (4).

Bearing this in mind, the construction can be extended to the optimal stopping problems for controlled diffusions. Let us consider $(t, x) \in [0, T] \times \mathbb{R}^N$, $u \in \mathcal{U}$ and $\tau \in \mathcal{T}_{t,T}$. Here, $\mathcal{T}_{t,T}$ stands for the family of all stopping times between t and T . Whenever $\mathbb{P}(\tau > t) > 0$, one associates the (expectation of the) occupational measure

$$\nu_{t,x,u,\tau} \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N)$$

defined by

$$(12) \quad \begin{aligned} & \nu_{t,x,u,\tau}(A \times B \times C \times D \times E) \\ &= \frac{1}{\mathbb{E}[\tau] - t} \mathbb{E} \left[\int_t^\tau 1_{A \times B \times C}((s, X_s^{t,x,u}, u_s)) ds 1_{\tau \in D} 1_E(X_\tau^{t,x,u}) \right], \end{aligned}$$

for all Borel set $(A \times B \times C \times D \times E) \subset [t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N$. If $\tau = t$, \mathbf{P} -a.s., we define $\nu_{t,x,u,\tau}$ by setting

$$\nu_{t,x,u,t}(A \times B \times C \times D \times E) = \delta_{t,x}(A \times B) \mathbb{P}(u_t \in C) \delta_{t,x}(D \times E),$$

for all Borel set $(A \times B \times C \times D \times E) \subset [t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N$

For $\nu \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N)$, we denote by $\pi_1(\nu) \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U)$ and $\pi_2(\nu) \in \mathcal{P}([t, T] \times \mathbb{R}^N)$ the "marginal" probability measures given by

$$\begin{aligned} \pi_1(\nu)(ds dy du) &= \nu(ds dy du, [t, T] \times \mathbb{R}^N), \\ \pi_2(\nu)(dr dz) &= \nu([t, T] \times \mathbb{R}^N \times U, dr dz). \end{aligned}$$

We define the (non empty, compact) set of constraints for stopping times (s.t.)

$\Theta^{s.t.}(t, x) :=$

$$\left\{ \begin{array}{l} \nu \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N) : \\ \forall \phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N), \\ \int_{[t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N} \begin{pmatrix} \phi(t, x) + (r_2 - t) \mathcal{L}^u \phi(r_1, y) \\ -\phi(r_2, z) \end{pmatrix} \pi_1(\nu)(dr_1 dy du) \pi_2(\nu)(dr_2 dz) = 0, \\ \int_{[t, T] \times [t, T]} 1_{r_1 > r_2} \nu(dr_1, \mathbb{R}^N \times U, dr_2, \mathbb{R}^N) = 0. \\ \int_{[t, T] \times \mathbb{R}^N \times U \times [t, T] \times \mathbb{R}^N} (|y|^2 + |z|^2) \nu(dr_1 dy du dr_2 dz) \leq C_0 (|x|^2 + 1). \end{array} \right\},$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$.

We study the value function

$$(13) \quad V_h(t, x) = \inf_{u \in \mathcal{U}_t} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} [h(\tau, X_\tau^{t,x,u})],$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$. As for the classical case, let us assume that the cost functional has at most quadratic growth and it is either

$$(14) \quad \begin{cases} \text{(a) lower semicontinuous and bounded from below or} \\ \text{(b) upper semicontinuous and bounded from above.} \end{cases}$$

The linearized value function (for the Mayer problem) is given by

$$\Lambda_h^{s.t.}(t, x) = \inf_{\nu \in \Theta^{s.t.}(t,x)} \int_{[t,T] \times \mathbb{R}^N} h(r, z) \nu([t, T] \times \mathbb{R}^N \times U, dr, dz),$$

for all $(t, x) \in [0, T) \times \mathbb{R}^N$ and $V_h(T, \cdot) = h(T, \cdot)$ on \mathbb{R}^N , while its dual is given by

$$(15) \quad \Lambda_h^{*,s.t.}(t, x) = \sup \left\{ \begin{array}{l} \eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N) \text{ s.t.,} \\ \eta \leq \phi(t, x) + (r_2 - t) \mathcal{L}^u \phi(r_1, y) \mathbf{1}_{r_1 \leq r_2} + (M(r_2, z) - \phi(r_2, z)) \mathbf{1}_{r_1' \leq r_2} \\ \text{for all } r_1, r_1', r_2, r_2' \in [t, T], y, z \in \mathbb{R}^N, \text{ and all } u \in U. \end{array} \right\}$$

Combining the compactness property of $\Theta^{s.t.}(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^N$ with an inf(sup)-convolution method and the results in [Pha98], one easily proves the following result:

Proposition 4 ([G10], Theorem 16) (a) If h is bounded and Lipschitz continuous, $V_h = \Lambda_h^{s.t.} = \Lambda_h^{*,s.t.}$ on $(0, T) \times \mathbb{R}^N$.

(b) In the l.s.c. case (14(a)), the value function $\Lambda_h^{s.t.}$ is the smallest lower semicontinuous viscosity supersolution of the associated HJB variational inequality. Moreover,

$$\Lambda_h^{s.t.}(t, x) = \Lambda_h^{*,s.t.}(t, x),$$

for all $(t, x) \in (0, T) \times \mathbb{R}^N$.

(c) In the u.s.c. case (14(b)), the value function $\Lambda_h^{s.t.}$ is the largest upper semicontinuous viscosity subsolution of the associated HJB variational inequality.

1.1.4 Adding state constraints

We now turn our attention to the case when the solution is constrained to some closed set K . We consider the classical value function for control problems under constraints

$$\inf_{\substack{u \in \mathcal{U}_{t_0} \\ X^{t_0;x_0,u} \in K}} \left[\int_{t_0}^T g(t, X_t^{t_0,x_0,u}) dt + h(X_T^{t_0,x_0,u}) \right],$$

where $g : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $h : \mathbb{R}^N \rightarrow \mathbb{R}$ are bounded and measurable functions. The dependency on the control variable in g has been dropped to simplify the presentation. The results of this subsection correspond to the paper [G2] (see also [G1] for the deterministic setting).

To our best knowledge, the constrained optimal control problem with continuous cost was studied for the first time in [Son86a]. The main assumption is the so-called inward pointing qualification condition. In this framework, the value function of an infinite horizon control problem with space constraints was characterized as a continuous solution to a corresponding Hamilton–Jacobi–Bellman equation. The reader is referred to [CDL90], [Mot95] for weaker inward assumptions.

For discontinuous cost functionals, the deterministic control problem with state constraints was studied in [FP98], [FV00], using viability theory tools. The authors prove their main results using an outward pointing assumption and characterize the value function as the unique lower semicontinuous viscosity solution of the associated HJB equation. Viability methods can also be employed in connection to constrained games (cf. [PQ01]). One can, alternatively, characterize the value function via its epigraph. The method relies on viability theory. For further details, the reader is referred, for instance, to [Aub11], [ABZ12] and references therein.

The literature on the probabilistic counterpart of the subject is also very rich (cf. [BG90], [BPQR98], [MP01], [BJ02a], [PZ08], [DS12] and references therein).

As we have already pointed out in the introduction, most of these methods rely on the convexity of the dynamics, Fillipov’s Theorem and viability arguments. We do not make any convexity assumption. We are going to impose a (non-restrictive) geometric condition of near-viability of the set of constraints. This will guarantee that, starting from K , one is able to find control processes keeping the trajectory close to K (not necessarily in K). Otherwise, it is obvious that the value function will explode.

We recall the definition of the second order normal cone to K .

Definition 5 *Given a point $x \in K$, the first order normal cone to the set K is given by*

$$\mathcal{N}_K^1(x) = \{p \in \mathbb{R}^N : \langle p, y - x \rangle \leq o(|y - x|) \text{ as } K \ni y \rightarrow x\}.$$

The second order normal cone is given by

$$\mathcal{N}_K^2(x) = \left\{ \begin{array}{l} (p, A) \in \mathbb{R}^N \times \mathcal{S}^N : \text{as } K \ni y \rightarrow x, \\ \langle p, y - x \rangle + \frac{1}{2} \langle A(y - x), y - x \rangle \leq o(|y - x|^2) \end{array} \right\}.$$

Here, \mathcal{S}^N stands for the set of symmetric matrix of $N \times N$ type.

We assume that, for every $x_0 \in \partial K$ and every $(p, A) \in \mathcal{N}_K^2(x_0)$, one has

$$(16) \quad \min_{u \in U} \left(\langle p, b(x_0, u) \rangle + \frac{1}{2} Tr(\sigma \sigma^*(x_0, u) A) \right) \leq 0.$$

The following result is standard in the viability theory. It is similar to the main results (in infinite horizon) of [BPQR98] and [BJ02a]. For further results on the subject, the reader is referred to [AF90].

Proposition 6 *Whenever the geometric condition (16) holds true, the set K is near viable, i.e. for every $\varepsilon > 0$, every $r > 0$, $t_0 \in [0, r]$ and every initial datum $x_0 \in K$, one can find an admissible control u^ε such that*

$$\mathbb{E} \left[\int_{t_0}^r d_K(X_s^{t_0, x_0, u^\varepsilon}) \wedge 1 ds + d_K(X_r^{t_0, x_0, u^\varepsilon}) \wedge 1 \right] \leq \varepsilon.$$

One classical way of dealing with the control problem with state constraints and l.s.c. costs would be to consider, for every $n \geq 1$, the penalized value function

$$(17) \quad V_{K,g,h}^n(t_0, x_0) = \inf_{u \in \mathcal{U}_{t_0}} J_{K,g,h}^n(t_0, x_0, u),$$

for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$, where

$$J_{K,g,h}^n(t_0, x_0, u) = \mathbb{E} \left[\int_{t_0}^T \left[g_n(t, X_t^{t_0, x_0, u}) + n(d_K(X_t^{t_0, x_0, u}) \wedge 1) \right] dt + h_n(X_T^{t_0, x_0, u}) + n(d_K(X_T^{t_0, x_0, u}) \wedge 1) \right],$$

for all $u \in \mathcal{U}$. Here, g_n and h_n are the usual inf-convolution functions. Then, one would allow the parameter $n \rightarrow \infty$.

The second approach would be to adapt the definition of Θ by taking into account the state constraints $x \in K$ and define

$$\Theta_K(t_0, r, x_0) := \left\{ (\gamma_1, \gamma_2) \in \Theta(t_0, r, x_0), \text{ Supp}(\gamma_1) \subset [t_0, r] \times K \times U, \text{ Supp}(\gamma_2) \subset K. \right\},$$

for every $r \in [0, T]$, $t_0 \in [0, r]$ and every $x_0 \in K$. This set is nonempty due to the near-viability condition (16) (for further details, the reader is referred to [G2], Remark 10). It is also a convex and compact set. The linearized value functions are

$$(18) \quad \Lambda_{K,g,h}(t_0, x_0) := \inf_{(\gamma_1, \gamma_2) \in \Theta_K(t_0, T, x_0)} \left((T - t_0) \int_{[t_0, T] \times \mathbb{R}^N \times U} g(s, y) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} h(z) \gamma_2(dz) \right),$$

and its dual

$$(19) \quad \Lambda_{K,g,h}^*(t_0, x_0) := \sup \left\{ \begin{array}{l} \mu \in \mathbb{R} : \exists \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N) \text{ s.t. } \forall (s, y, v, z) \in [t_0, T] \times K \times U \times K, \\ \mu \leq (T - t_0) [\mathcal{L}^v \phi(s, y) + g(s, y)] + h(z) - \phi(T, z) + \phi(t_0, x_0). \end{array} \right\},$$

for all $(t_0, x_0) \in [0, T] \times K$.

It turns out that both methods lead to the same result. The connection between these functions is given by the following result (cf. Theorem 11 in [G2]; for the deterministic version, see [G1]).

Theorem 7 ([G2], Theorem 11) 1. *If g and h are bounded, lower semicontinuous functions, then*

$$(20) \quad \Lambda_{K,g,h}(t_0, x_0) = \lim_{n \rightarrow \infty} V_{K,g,h}^n(t_0, x_0) = \Lambda_{K,g,h}^*(t_0, x_0),$$

for all $(t_0, x_0) \in [0, T] \times K$.

2. *Moreover, if g and h are bounded, Lipschitz continuous and the set*

$$F(t, x) := \{(b(x, u), \sigma \sigma^*(x, u)) : u \in U\} \text{ is convex,}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$, then $\Lambda_{K,g,h}(t_0, x_0) = V_{K,g,h}^w(t_0, x_0)$ for all $(t_0, x_0) \in [0, T] \times K$, where

$$V_{K,g,h}^w(t_0, x_0) := \inf_{\substack{\pi = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, u) \\ X_t^{t_0, x_0, u} \in K, \mathbb{P}^\pi - a.s., \forall t \in [t_0, T]}} \mathbb{E}^\pi \left[\int_{t_0}^T g(t, X_t^{t_0, x_0, u}) dt + h(X_T^{t_0, x_0, u}) \right].$$

We would like to point out that $\Lambda_{K,g,h}^*$ describes the value function ($\Lambda_{K,g,h}$) as the pointwise maximum of regular subsolutions of the associated HJB system on K (independently of convexity assumptions on the dynamics).

1.1.5 Abstract and linearized dynamic programming principles (DPP)

For regular cost functionals, the stochastic dynamic programming principle (without constraints) has been extensively studied (e.g. [Bor89], [FS06], [YZ99], [Bor05], [Pha05], [Pha07], etc.). In general, dynamic programming principles are rather difficult to prove if the regularity of the value function is not known a priori. An alternative is to use a weak formulation where the value function is replaced by a test function (cf. [BT11]). This method has been recently employed in [BN11] to provide a dynamic programming principle for stochastic optimal control problems with expectation constraints. We propose an alternative to these methods in [G2] and [G6].

Using the characterization of the set of constraints in the linear formulation, we prove a semigroup property. This property follows naturally from the structure of the sets of constraints. It allows one to derive dynamic programming principles with/without state constraints for general bounded cost functionals. In the bounded case one can only provide an abstract dynamic programming principle. In the semicontinuous setting, further linearized programming principle can be proven. This is just a first step in studying the equations that would characterize the general value function. It suggests that the test functions in the definition of the viscosity solution might be defined on the space of probability measures.

The second application we had in mind when we studied these properties are numerical aspects in the optimal control in the classical or the constrained framework.

We fix $x_0 \in \mathbb{R}^N$. Let us consider $t_1, t_2 \geq 0$ such that $0 < t_1 + t_2 \leq T$, where $T > 0$ is a fixed terminal time. By analogy to $\Theta(t, T, x)$ for $x \in \mathbb{R}^N$, we define sets of constraints starting from measures. If $\gamma \in \mathcal{P}([0, t_1] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$, we define the sets

$$\Theta(t_1, t_1 + t_2, \gamma) := \left\{ \begin{array}{l} \eta = (\eta_1, \eta_2) \in \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : \forall \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N), \\ \int_{\mathbb{R}^N} \phi(t_1, x) \gamma_2(dx) + \int_{[t_1, t_1 + t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \eta_1(ds dy du) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, z) \eta_2(dz) = 0, \\ \int_{\mathbb{R}^N} |y|^2 \eta_1([t_1, t_1 + t_2], dy, U) \leq e^{C_0(t_1 + t_2)} (|x_0|^2 + C_0(t_1 + t_2)), \\ \int_{\mathbb{R}^N} |z|^2 \eta_2(dz) \leq e^{C_0(t_1 + t_2)} (|x_0|^2 + C_0(t_1 + t_2)), \end{array} \right\}$$

and

$$\Theta_K(t_1, t_1 + t_2, \gamma) = \{\eta \in \Theta(t_1, t_1 + t_2, \gamma) : \text{Supp}(\eta_1) \subset [t_1, t_1 + t_2] \times K \times U, \text{Supp}(\eta_2) \subset K\}.$$

The reader is invited to notice that x can be identified with $\gamma^{t,t,x,u}$, for all $u \in \mathcal{U}$. The properties of these sets are given by the following proposition.

Proposition 8 ([G2], Proposition 14) *For every $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$ and every $\gamma = (\gamma_1, \gamma_2) \in \Theta_K(0, t_1, x_0)$, the sets $\Theta(t_1, t_1 + t_2, \gamma)$ and $\Theta_K(t_1, t_1 + t_2, \gamma)$ are nonempty, convex and compact with respect to the usual convergence of probability measures.*

We have a natural way of composing measures.

Definition 9 Whenever $\gamma \in \mathcal{P}([0, t_1] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ and $\eta \in \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$, we define

$$\eta \circ \gamma \in \mathcal{P}([0, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) \text{ by setting:}$$

$$\begin{cases} (\eta \circ \gamma)_1(A, dydu) = \frac{t_1}{t_1+t_2} \gamma_1(A \cap [0, t_1], dydu) + \frac{t_2}{t_1+t_2} \eta_1(A \cap [t_1, t_1 + t_2], dydu), \\ (\eta \circ \gamma)_2 = \eta_2, \end{cases}$$

for all Borel sets $A \subset [0, t_1 + t_2]$.

With the obvious notation

$$\Theta_K(t_1, t_1 + t_2, \cdot) \circ \Theta_K(0, t_1, x_0) = \{\eta \circ \gamma : \gamma \in \Theta_K(0, t_1, x_0), \eta \in \Theta_K(t_1, t_1 + t_2, \gamma)\},$$

we can prove the following result.

Proposition 10 ([G2], Proposition 17) We have the following semigroup property

$$\Theta_K(t_1, t_1 + t_2, \cdot) \circ \Theta_K(0, t_1, x_0) = \Theta_K(0, t_1 + t_2, x_0),$$

for all $x_0 \in K$, $t_1 \geq 0$, $t_2 > 0$ such that $t_1 + t_2 \leq T$.

Both the proof of Proposition 8 and of Proposition 10 are constructive and rather technical. They rely on the near-viability condition (16), the convex hull characterization of Θ (Corollary 2) and compactness arguments.

We extend the definition of the value function to initial measures $\gamma \in \Theta_K(t_0, t, x_0)$ by setting

$$\Lambda_{K,g,h}(t, \gamma) = \inf_{(\eta_1, \eta_2) \in \Theta_K(t, T, \gamma)} \left((T - t) \int_{[t, T] \times \mathbb{R}^N \times U} g(s, y) \eta_1(dsdydu) + \int_{\mathbb{R}^N} h(z) \eta_2(dz) \right),$$

whenever $x_0 \in K$, $0 \leq t_0 < t < T$ and $\gamma \in \Theta_K(t_0, t, x_0)$. The main result consists in the following dynamic programming principles (cf. Theorem 18, Proposition 19 in [G2], see also [G1] for the deterministic, constrained setting and [G6] for the stochastic, unconstrained framework).

Theorem 11 ([G2], Theorem 18, Proposition 19) 1. Let us assume that g and h are bounded measurable functions. Then, for every $x_0 \in K$, $0 \leq t_0 < t < T$,

$$\Lambda_{K,g,h}(t_0, x_0) = \inf_{\gamma = (\gamma_1, \gamma_2) \in \Theta_K(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U} g(s, y) \gamma_1(dsdydu) + \Lambda_{K,g,h}(t, \gamma) \right).$$

2. If the functions g and h are bounded and l.s.c., one also has

$$\Lambda_{K,g,h}(t_0, x_0) = \inf_{\gamma \in \Theta_K(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U} g(s, y) \gamma_1(dsdydu) + \int_{\mathbb{R}^N} \Lambda_{K,g,h}(t, x) \gamma_2(dx) \right),$$

for all $x_0 \in K$ and all $t \in (t_0, T)$.

3. If $K = \mathbb{R}^N$ and g and h are bounded and u.s.c., one also has

$$\Lambda_{\mathbb{R}^N, g, h}(t_0, x_0) \geq \inf_{\gamma \in \Theta(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U} g(s, y) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} \Lambda_{\mathbb{R}^N, g, h}(t, x) \gamma_2(dx) \right),$$

for all $x_0 \in K$ and all $t \in (t_0, T)$.

In the deterministic setting, the inequality in the assertion 3 becomes equality, providing a linear DPP for u.s.c. costs (cf. [G6]).

1.1.6 Asymptotic values and uniform Tauberian results for nonexpansive control systems

In connection to the system (1), we investigate, in [G19], the behavior of the value functions

$$V_t(x) = \inf_{u \in \mathcal{U}} t^{-1} \mathbb{E} \left[\int_0^t g(X_r^{x, u}) dr \right],$$

$$V^C(x) = \inf_{u \in \mathcal{U}} C \mathbb{E} \left[\int_0^\infty e^{-Cr} g(X_r^{x, u}) dr \right], \text{ for all } x \in \mathbb{R}^N,$$

as the time parameter t goes to infinity (respectively as the discount factor $C \rightarrow 0$). The cost function g is assumed to be Lipschitz-continuous in space. In fact, our results are more general and they allow to deal with cost functions depending on the control parameter as well. We have chosen this framework for presentation purposes.

The literature on the subject relies mainly on two approaches :

- passing to the limit in the HJB equation associated to the value function or
- estimations on the trajectories of the control system.

The PDE approach essentially originates from [LPV] and needs coercivity assumptions on the Hamiltonian associated with the control system. This was extended in [Ari98] for the deterministic control context and in [AL98] for the stochastic framework (see also further references in these articles). It is worth pointing out that the coercivity of the Hamiltonian can be ensured by suitable controllability assumptions on the control system (e.g. in [AG00]).

The second approach is based on assumptions on the control system itself. These are mainly controllability or dissipativity assumptions. The reader is referred to [AG00], [Bet05], [QR11] for the deterministic setting. For the stochastic framework, we refer to [BBG97], [BG07], [BI05], [Ric09] and references therein.

In the cited references, independently of the approach, one uses arguments ensuring that the limit value is independent of the initial data. This approach may fail to work for state dependent limit value functions (which is the general case as explained in [QR11]).

In the paper [G19], we use the following (square mean) nonexpansivity condition

$$(21) \quad \begin{cases} \text{For every } T > 0, \varepsilon > 0, x, y \in \mathbb{R}^N, \text{ and every } u \in \mathcal{U}, \text{ there exists } v \in \mathcal{U} \text{ s.t.} \\ \mathbb{E} [|X_r^{x, u} - X_r^{y, v}|^2] \leq |x - y|^2 + \varepsilon, \text{ for all } r \in [0, T], \mathbb{P}\text{-}a.s. \end{cases}$$

It corresponds to the usual nonexpansivity condition (in the case $\varepsilon = 0$). Such a condition was introduced by the authors of [QR11] in the deterministic framework.

We show that the following criterion

$$\sup_{v \in U} \inf_{u \in U} \left(\begin{aligned} & \langle b(x, u) - b(y, v), x - y \rangle \\ & + \frac{1}{2} Tr [(\sigma(x, u) - \sigma(y, v)) (\sigma^*(x, u) - \sigma^*(y, v))] \end{aligned} \right) \leq 0,$$

for every $x, y \in K$ ensures the nonexpansivity condition (21). The proof is constructive and it relies on piecewise constant control processes (cf. [Kry99]) and measurable selection arguments. For further details, the interested reader can consult Lemma 3 in [G19].

This assumption implies the Lipschitz continuity of the value functions V_t , uniformly with respect to the time parameter $t > 0$, when the stochastic system (1) leaves the compact set K invariant.

Proposition 12 ([G19], Proposition 4) *If (21) holds true, then, for every $t > 0$, and every $x, y \in K$,*

$$|V_t(x) - V_t(y)| \leq c_0 |x - y|,$$

where c_0 is the Lipschitz constant of the cost function g .

For every $t > 0$ and $s \geq 0$, we consider the value function $V_{s,t}$ defined by

$$V_{s,t}(x) = \inf_{u \in \mathcal{U}} t^{-1} \mathbb{E} \left[\int_s^{s+t} g(X_r^{x,u}) dr \right], \text{ for all } x \in \mathbb{R}^N.$$

The first main result of [G19] is

Theorem 13 ([G19], Theorem 8) *We assume that the nonexpansion condition (21) holds true. We also assume that there exists a compact set $K \subset \mathbb{R}^N$ invariant with respect to the control system (1). Then, for every $x \in K$, the limit value function exists and*

$$\lim_{t \rightarrow \infty} V_t(x) = V^*(x) := \sup_{t > 0} \inf_{s \geq 0} V_{s,t}(x).$$

One begins by proving that, for every $s_0 \in \mathbb{R}_+^*$, the following inequalities hold true:

$$\sup_{t > 0} \inf_{s \leq s_0} V_{s,t}(x) \geq \limsup_{t \rightarrow \infty} V_t(x) \geq \liminf_{t \rightarrow \infty} V_t(x) \geq \sup_{t > 0} \inf_{s \geq 0} V_{s,t}(x),$$

for all $x \in \mathbb{R}^N$. To complete the proof, one has to show that, for every $\varepsilon > 0$, there exists some s_0 such that

$$\sup_{t > 0} \inf_{s \leq s_0} V_{s,t}(x) \leq \sup_{t > 0} \inf_{s \geq 0} V_{s,t}(x) + \varepsilon.$$

To do this, we need some compactness arguments on reachable sets. In the deterministic setting, the authors of [QR11] employ standard viability results. The stochastic counterpart relies on linearization formulations for control problems. Namely, we use compactness of the sets of constraints w.r.t. the Wasserstein metric W_1 and linearized formulations for dynamic programming principles (see previous subsections).

Since the set K is assumed to be compact and the family of value functions $(V_t)_{t > 1}$ is equicontinuous and uniformly bounded, using the Arzelà-Ascoli Theorem, one gets that V_t converges to V^* uniformly on K as t goes to infinity.

In the second part of the paper [G19], we consider the relation between the existence of the (uniform) limits $\lim_{t \rightarrow \infty} V_t$ and $\lim_{C \rightarrow 0^+} V^C$. In fact we show that

Theorem 14 ([G19], Theorems 10 and 13) *The value function V_t converges uniformly as $t \rightarrow \infty$ if and only if V^C converges uniformly as $C \rightarrow 0$. Moreover, the two limits coincide (when they exist).*

In the deterministic control case with discrete time parameter, this type of result was obtained by [LS92]. In the deterministic control case, similar results have been obtained in [OBV13]. We refer the reader to this article for a nice description of the history of the problem. Our second main result ([G19], Theorems 10 and 13) is a generalization of these results to stochastic control systems.

1.1.7 Miscellaneous

Zubov's characterization of asymptotic stability domains In the first part of [G3], we aim at characterizing the set of initial points that can be steered into a target set with positive probability (domain of stability) using the so-called Zubov method. This method has been introduced in [Zub64] and generalized in several directions: to systems with time varying deterministic perturbation in [CGW01a, Mal05], control deterministic systems in [CGW01a, Son83, GW00, Grü02, CGW08], differential games in [GS11], systems with Brownian perturbation in [CG03, CCGW06, GSB08].

It is classical to assume that the domain of stability contains at least some open neighborhood of the target set (local stability assumption). The basic idea is to stop the trajectory as soon as it reaches the neighborhood and continue with convenient stabilizing measures. To complete this program with classical controls, an assumption of stability of control processes under concatenation is needed [CCGW06]. This assumption is naturally satisfied under convexity conditions. On the other hand, lack of convexity on the dynamics can be compensated by linearization techniques: We associate a finite positive Radon measure to the stopped trajectory and continue it with locally stabilizing measures after this stopping time. This defines a new probability measure that stabilizes the initial trajectory. Both the locally stabilizing measures and the new measure belong to the set of constraints for the linear problem, but, they might fail to be associated to controls. This method enables us to characterize the (extended) stability domain without the additional stability assumption. Similar to the result of [CCGW06], the (extended) domain is shown to coincide with the set of initial data for which the neighborhood is reached in finite time with positive probability. A Zubov-type criterion is obtained by considering the value function associated to the (u.s.c.) indicator of a convenient set. This value function might fail to be continuous and cannot be covered by the results in [CCGW06]. To keep the present manuscript within reasonable limits, we have chosen to give the details of this method only for controlled piecewise deterministic Markov systems (see Section 2). The methods are similar and the results comparable.

LP formulations for reflected control systems The second part of the paper [G3] is dedicated to the study of a linear formulation for control problems where the state equation is a stochastic variational inequality (see, for example, [DI93, AR96, Zal02, Bou08]) :

$$(22) \quad \begin{cases} dX_s^{t,x,u} + \partial\varphi(X_s^{t,x,u}) ds \ni b(s, X_s^{t,x,u}, u_s) ds + \sigma(s, X_s^{t,x,u}, u_s) dW_s, & s \in [t, T], \\ X_t^{t,x,u} = x \in D(\varphi) \subset \mathbb{R}^N, \end{cases}$$

where b and σ are uniformly continuous and have at most linear growth in the space component. The function $\varphi : \mathbb{R}^N \rightarrow \bar{\mathbb{R}}$ is assumed to be convex, lower semicontinuous and

$$\text{Int}(D) \neq \emptyset,$$

where Int stands for the interior and $D := \{x \in \mathbb{R}^N : \varphi(x) < \infty\}$ is assumed to be closed.

To complete our program, we perturb the state equation using the Yosida regularization. The penalized equation has Lipschitz-continuous coefficients. However, previous results no longer apply, since the coefficients are not bounded. The first step consists in generalizing the method in [G10] to coefficients that satisfy a linear-growth condition. The second difficulty

is to adapt the definition of the set of constraints to our case. We introduce generalized occupational measures with a supplementary variable taking into account the approached gradients (or the subdifferential term)

$$\Theta_\varepsilon(t, x) := \left\{ \begin{array}{l} \gamma \in \mathcal{P}([t, T] \times \mathbb{R}^N \times \mathbb{R}^N \times U \times \mathbb{R}^N), \forall \phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N), \\ \int_{[t, T] \times \mathbb{R}^{2N} \times U \times \mathbb{R}^N} \left[(T-t)(\mathcal{L}^v \phi(s, y_1) - \langle y_2, D\phi(s, y_1) \rangle) \right. \\ \quad \left. + \phi(t, x) - \phi(T, z) \right] \gamma(ds, dy_1, dy_2, dv, dz) = 0 \\ \int_{[t, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N} (|y_1|^4 + |z|^4 + |y_2|) \gamma(ds, dy_1, dy_2, dv, dz) \leq C(|x|^4 + 1), \\ \text{Supp}(\gamma) \subset \{(s, y_1, y_2, u, z) : \forall p \in \mathbb{R}^N, \langle p - y_1, y_2 \rangle + \varphi_\varepsilon(y_1) \leq \varphi_\varepsilon(p)\}; \\ \int_{[t, T] \times \mathbb{R}^{2N} \times U \times \mathbb{R}^N} d_D^2(y_1) \wedge 1\gamma(ds, dy_1, dy_2, dv, dz) \leq C(|x|^4 + 1)\varepsilon^{\frac{1}{8}}. \end{array} \right\}$$

We state and prove the equality between the standard value function, its primal linearized version and some dual formulation following similar patterns to Theorem 1 (cf. Theorem 3.1 in [G3]). The proof relies mainly on viscosity techniques and the "shaking the coefficients" method introduced in [Kry00]. The good candidate for the set of constraints in the initial problem is the usual outer limit of sets of constraints for the penalized problems in a convenient space of probability measures with the narrow convergence topology. We prove that minimizing the cost functional with respect to this set of constraints leads to the same value as the classical approach (Proposition 3.3 in [G3]). Further properties of the set of constraints are given (in Proposition 3.2 in [G3]). In particular, the supplementary variable is shown to be in the subdifferential.

The infinite horizon setting In the paper [G12], we study the infinite horizon counterpart of stochastic control problems. We should emphasize that, chronologically, this is the first paper in which we employ linear programming techniques in connection to stochastic control. We have chosen to give less space than it deserves in this manuscript because of notations. In the case of discounted payoff in an infinite horizon setting, the occupational measures have a different definition (although close to the one in the introduction). The first main result ([G12], Theorem 2) is the analogous of Theorem 1 in the infinite horizon setting. As the discount factor decreases to 0, the limit discounted optimal value is interpreted in connection to the long time average problems of stochastic optimal control ([G12], Proposition 5).

1.2 LP methods in deterministic control problems

1.2.1 Introduction

We deal with several control problems for deterministic systems : discontinuous control problems under state constraints ([G1]), linearization of classical, \mathbb{L}^p and \mathbb{L}^∞ control problems and dynamic programming principles in classical, \mathbb{L}^∞ -control and state constrained problems with semicontinuous costs ([G9], [G1]), min/max control problems ([G18]), viability results for singularly perturbed control systems (corresponding to [G13]). We consider the following control system

$$(23) \quad \begin{cases} dx_t^{t_0, x_0, u} = f(t, x_t^{t_0, x_0, u}, u_t) dt, & t_0 \leq t \leq T, \\ x_{t_0}^{t_0, x_0, u} = x_0 \in \mathbb{R}^N, \end{cases}$$

where $T > 0$ is a finite time horizon, $t_0 \in [0, T]$ and the control u takes its values in a compact, metric space U . We recall that a control $(u_t)_{t_0 \leq t \leq T}$ is said to be admissible on $[t_0, T]$ if it is measurable on $[t_0, T]$ and we let \mathcal{U} denote the family of all admissible controls on $[0, T]$. We assume that the coefficient function $f : \mathbb{R} \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ satisfies

$$(24) \quad \begin{cases} f \text{ is bounded and uniformly continuous on } \mathbb{R} \times \mathbb{R}^N \times U, \\ |f(s, x, u) - f(t, y, u)| \leq c(|x - y| + |s - t|), \end{cases}$$

for all $(s, t, x, y, u) \in \mathbb{R}^2 \times \mathbb{R}^{2N} \times U$, for some positive real constant $c > 0$.

The reader who is already familiar with the construction of occupational measures in connection to stochastic systems should skip the following notations. They are similar to those in Subsection 1.1.

We fix $t > 0$ and $x_0 \in \mathbb{R}^N$. To every $r > t$ and $u \in \mathcal{U}$, one can associate a couple of occupational measures $\gamma^{t, r, x_0, u} = (\gamma_1^{t, r, x_0, u}, \gamma_2^{t, r, x_0, u}) \in \mathcal{P}([t, r] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ defined by

$$\begin{cases} \gamma_1^{t, r, x_0, u}(A \times B \times C) = \frac{1}{r-t} \int_t^r \mathbf{1}_{A \times B \times C}(s, x_s^{t, x_0, u}, u_s) ds, \\ \gamma_2^{t, r, x_0, u} = \delta_{x_r^{t, x_0, u}}, \end{cases}$$

for all Borel sets $A \subset [t, r]$, $B \subset \mathbb{R}^N$ and $C \subset U$. Here, δ . stands for the Dirac measure. One can also define $(\gamma_1^{t, t, x_0, u}, \gamma_2^{t, t, x_0, u}) \in \mathcal{P}(\{t\} \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ by setting

$$\gamma_1^{t, t, x_0, u} = \delta_{t, x_0, u_t}, \gamma_2^{t, t, x_0, u} = \delta_{x_0}.$$

For every $r \geq t$, the family of occupational measures

$$(25) \quad \Gamma(t, r, x_0) = \{ (\gamma_1^{t, r, x_0, u}, \gamma_2^{t, r, x_0, u}) \text{ for all } u \in \mathcal{U} \}.$$

can be embedded into a larger set

$$(26) \quad \Theta(t, r, x_0) := \left\{ \begin{array}{l} (\gamma_1, \gamma_2) \in \mathcal{P}([t, r] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N), \forall \phi \in C_2^{1,1}(\mathbb{R}_+ \times \mathbb{R}^N), \\ \int_{[t, r] \times \mathbb{R}^N \times U \times \mathbb{R}^N} (\phi(t, x_0) + (r-t) \mathcal{L}^u \phi(s, y) - \phi(r, z)) \gamma_1(ds dy du) \gamma_2(dz) = 0, \\ \int_{\mathbb{R}^N} |y|^{2+\delta} \gamma_1([t, r], dy, U) \leq c_{T, x_0}, \int_{\mathbb{R}^N} |z|^{2+\delta} \gamma_2(dz) \leq c_{T, x_0}, \end{array} \right\}$$

where

$$\mathcal{L}^u \phi(s, y) = \langle f(s, y, u), D\phi(s, y) \rangle + \partial_t \phi(s, y),$$

for all $\phi \in C^{1,1}(\mathbb{R}_+ \times \mathbb{R}^N)$ and all $s \geq 0, y \in \mathbb{R}^N$. Here, $\delta > 0$ is fixed and c_{T,x_0} is a positive constant depending on x_0 and, eventually, on T, δ . This constant can be chosen of the form $ce^{cT} (1 + |x_0|^{2+\delta})$, where $c > 0$ depends only on the Lipschitz coefficients of b and σ (but not on T , nor x_0). The set $C_2^{1,1}(\mathbb{R}_+ \times \mathbb{R}^N)$ stands for the set of continuously differentiable functions such that the function and its first order derivatives with respect to t and x have at most quadratic growth.

We consider two bounded, measurable functions $g : \mathbb{R}_+ \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}$ and $g' : \mathbb{R}^N \longrightarrow \mathbb{R}$. To any $(t, x) \in [0, r] \times \mathbb{R}^N$, one associates the criterion

$$J_{g,g'}^r(t, x, u) = \int_t^r g(s, x_s^{t,x,u}, u_s) ds + g'(x_T^{t,x,u})$$

and the corresponding value function

$$(27) \quad V_{g,g'}^r(t, x) = \inf_{u \in \mathcal{U}} J_{g,g'}^r(t, x, u).$$

We also consider the linearized problems

$$\Lambda_{g,g'}^r(t, x) = \inf_{(\gamma_1, \gamma_2) \in \Theta(t, r, x)} \left((r-t) \int_{[t,r] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} g'(z) \gamma_2(dz) \right),$$

and its dual

$$(28) \quad \eta_{g,g'}^r(t, x) = \sup \left\{ \begin{array}{l} \eta \in \mathbb{R} : \exists \phi \in C_2^{1,1}(\mathbb{R}_+ \times \mathbb{R}^N) \text{ s.t. } \forall (s, y, v, z) \in [t, r] \times \mathbb{R}^N \times U \times \mathbb{R}^N, \\ \eta \leq (r-t) [\mathcal{L}^v \phi(s, y) + g(s, y, v)] + g'(z) - \phi(T, z) + \phi(t, x). \end{array} \right\},$$

for all $(t, x) \in [0, r] \times \mathbb{R}^N$. The following result links the three quantities.

Proposition 15 ([G9], Proposition 2.4) *If (24) holds true and the functions g and g' are bounded, uniformly continuous and Lipschitz-continuous in time and space uniformly w.r.t. the control parameter, then, for every $(t, x) \in [0, r] \times \mathbb{R}^N$, one has*

$$V_{g,g'}^r(t, x) = \Lambda_{g,g'}^r(t, x) = \eta_{g,g'}^r(t, x).$$

Identification of standard formulation to linearized problems leads to characterize the set of constraints as the closed convex envelope of the family of occupational measures associated to the dynamic system :

Corollary 16 ([G9], Corollary 2.5) *The set $\Theta(t, r, x_0)$ is the closed convex hull of the family of occupational couples $\Gamma(t, r, x_0)$*

$$\Theta(t, r, x_0) = cl (co(\Gamma(t, r, x_0))) = cl_{W_2} (co(\Gamma(t, r, x_0))),$$

for all $r \geq t \geq 0$. The operator cl designate the closure with respect to the topology induced by the weak convergence of probability measures and cl_{W_2} designates the closure w.r.t. the Wasserstein metric W_2 .

1.2.2 Dynamic Programming Principles in classical control problems

The reader who is already familiar with the results on dynamic programming principles presented in the stochastic framework (subsection 1.1.5) may want to skip this part. The results in the deterministic setting are quite similar and they have been adapted to diffusion systems and improved.

In the first part of the paper [G9], we investigate further properties of the set of constraints appearing in classical (deterministic) control problems. We prove a semigroup behavior of the set of constraints. As application we provide a simple proof for the Dynamic Programming Principle in classical control problems with bounded cost. In the semicontinuous and continuous framework, more precise assertions are proved.

We fix $x_0 \in \mathbb{R}^N$. Let us consider $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$, where $T > 0$ is a terminal time. If $\gamma \in \mathcal{P}([0, t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$, we define the set

$$\Theta(t_1, t_1 + t_2, \gamma) = \left\{ \begin{array}{l} \tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in \mathcal{P}([0, t_1] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : \forall \phi \in C_2^{1,1}(\mathbb{R}_+ \times \mathbb{R}^N) \\ \int_{\mathbb{R}^N} \phi(0, x) \gamma_2(dx) + \int_{[0, t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \tilde{\gamma}_1(ds dy du) \\ - \int_{\mathbb{R}^N} \phi(t_2, z) \tilde{\gamma}_2(dz) = 0, \\ \int_{\mathbb{R}^N} |y|^{2+\delta} \tilde{\gamma}_1([0, t_2], dy, U) \leq c_{T, x_0}, \int_{\mathbb{R}^N} |z|^{2+\delta} \tilde{\gamma}_2(dz) \leq c_{T, x_0}. \end{array} \right\}$$

Proposition 17 ([G9], Proposition 2.7) *For every $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$ and every $\gamma = (\gamma_1, \gamma_2) \in \Theta(0, t_1, x_0)$, the set $\Theta(t_1, t_1 + t_2, \gamma)$ is nonempty, convex and compact (with respect to the weak convergence of probability measures and to W_2).*

The following definition provides a natural operator to concatenate measures.

Definition 18 *Whenever $\gamma \in \Theta(0, t_1, x_0)$ and $\tilde{\gamma} \in \Theta(t_1, t_1 + t_2, \gamma)$, we define*

$\tilde{\gamma} \circ \gamma \in \mathcal{P}([0, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ *by setting:*

$$\begin{cases} (\tilde{\gamma} \circ \gamma)_1(A, dy du) = \frac{t_1}{t_1 + t_2} \gamma_1(A \cap [0, t_1], dy du) + \frac{t_2}{t_1 + t_2} \tilde{\gamma}_1((A - t_1) \cap [0, t_2], dy du), \\ (\tilde{\gamma} \circ \gamma)_2 = \tilde{\gamma}_2, \end{cases}$$

for all Borel set $A \subset [0, t_1 + t_2]$.

In a natural way we define

$$\Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) = \{\tilde{\gamma} \circ \gamma : \gamma \in \Theta(0, t_1, x_0), \tilde{\gamma} \in \Theta(t_1, t_1 + t_2, \gamma)\}.$$

Proposition 19 ([G9], Proposition 2.9) *We have the following semigroup property*

$$\Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) = \Theta(0, t_1 + t_2, x_0),$$

for all $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$.

We extend the definition of the value function to initial measures $\gamma \in \Theta_K(t_0, t, x_0)$ by setting

$$\Lambda_{g, g'}^T(t, \gamma) = \inf_{(\tilde{\gamma}_1, \tilde{\gamma}_2) \in \Theta(t, T, \gamma)} \left((T - t) \int_{[t, T] \times \mathbb{R}^N \times U} g(s, y) \tilde{\gamma}_1(ds dy du) + \int_{\mathbb{R}^N} g'(z) \tilde{\gamma}_2(dz) \right),$$

whenever $x_0 \in \mathbb{R}^N$, $0 \leq t_0 < t < T$ and $\gamma \in \Theta(t_0, t, x_0)$. We infer a simple proof for the dynamic programming principle for the value function (27).

Theorem 20 (*Dynamic Programming Principles*) ([G9], Theorem 2.11)

1. Let us suppose that the functions g and g' are bounded. Then, the following equality holds true

$$(29) \quad \Lambda_{g,g'}^T(0, x_0) = \inf_{\gamma \in \Theta(0, t, x_0)} \left(t \int_{[0, t] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds dy du) + \Lambda_{g,g'}^T(t, \gamma) \right),$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (0, T)$. Also

$$\Lambda_{g,g'}^T(0, x_0) \leq \inf_{\gamma \in \Gamma(0, t, x_0)} \left(t \int_{[0, t] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} \Lambda_{g,g'}^T(t, x) \gamma_2(dx) \right),$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (0, T)$.

2. If the functions g and g' are bounded and l.s.c., one also has

$$\Lambda_{g,g'}^T(0, x_0) \leq \inf_{\gamma \in \text{co}\Gamma(0, t, x_0)} \left(t \int_{[0, t] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} \Lambda_{g,g'}^T(t, x) \gamma_2(dx) \right),$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (0, T)$.

3. If g and g' are bounded and upper semicontinuous, one also has

$$\Lambda_{g,g'}^T(0, x_0) = \inf_{\gamma \in \Gamma(0, t, x_0)} \left(t \int_{[0, t] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} \Lambda_{g,g'}^T(t, x) \gamma_2(dx) \right),$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (0, T)$.

4. If g and g' are bounded and continuous,

$$\Lambda_{g,g'}^T(0, x_0) = \inf_{\gamma \in \Theta(0, t, x_0)} \left(t \int_{[0, t] \times \mathbb{R}^N \times U} g(s, y, u) \gamma_1(ds dy du) + \int_{\mathbb{R}^N} \Lambda_{g,g'}^T(t, x) \gamma_2(dx) \right),$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (0, T)$.

The results have been improved for l.s.c costs in [G2], Theorem 18. In this framework, one gets equality by replacing $\text{co}\Gamma(0, t, x_0)$ with $\Theta(0, t, x_0)$.

1.2.3 Linearization techniques and dynamic programming principles for \mathbb{L}^∞ -control problems

In the second part of [G9], we aim at linearizing the value function of an \mathbb{L}^∞ -control problem with either continuous or lower semicontinuous cost. We let $h : \mathbb{R} \times \mathbb{R}^N \times U \rightarrow \mathbb{R}$ be a bounded function. We are interested in characterizing the following value function:

$$V_h^\infty(t_0, x_0) = \inf_{u \in \mathcal{U}} \text{esssup}_{t \in [t_0, T]} h(t, x_t^{t_0, x_0, u}, u_t),$$

for every $t_0 \in [0, T]$ and every $x_0 \in \mathbb{R}^N$.

We assume that the function $h : \mathbb{R} \times \mathbb{R}^N \times U \rightarrow \mathbb{R}$ satisfies

$$(30) \quad \begin{cases} h^0 \geq \sup_{(t,x,u) \in \mathbb{R} \times \mathbb{R}^N \times U} |h(t, x, u)| \geq \inf_{(t,x,u) \in \mathbb{R} \times \mathbb{R}^N \times U} |h(t, x, u)| \geq 2h_0 > 0, \\ |h(t, \cdot, u)| \text{ is lower semicontinuous, for all } (t, u) \in \mathbb{R} \times U, \\ \sup_x |h(s, x, u) - h(t, x, v)| \leq c|t - s| + \omega(|u - v|), \end{cases}$$

for some real constant $c > 0$ and some continuity modulus ω . The \mathbb{L}^∞ -value function Λ_h^∞ is given by

$$(31) \quad \Lambda_h^\infty(t_0, x_0) = \inf_{\gamma \in \Theta(t_0, T, x_0)} \|h\|_{\mathbb{L}^\infty([t_0, T] \times \mathbb{R}^N \times U, \gamma_1)},$$

for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$ and its dual problem

$$(32)$$

$$\begin{aligned} & \eta_h^\infty(t_0, x_0) \\ &= \sup \left\{ \begin{array}{l} \phi(t_0, x_0) - \sup_{z \in \mathbb{R}^N} \phi(T, z) : \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R}_+) \text{ s.t.} \\ \phi < (1 + T)(2T \vee 1)h^0, \quad \inf_{(s,y) \in [0, T] \times \mathbb{R}^N} \phi(s, y) > 0, \\ 0 \leq \mathcal{L}^v \phi(s, y), \quad \forall s, y, v, z \in [t_0, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N \text{ s.t. } |h(s, y, v)| \leq \phi(s, y), \end{array} \right\} \end{aligned}$$

for $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$.

The study of this kind of problems is a very difficult task because, in general, it leads to strongly non-linear equations for which classical solutions may not and generally do not exist. Nevertheless, we succeed to deal with this problem by using an \mathbb{L}^p approximating method. The first step consists in providing appropriate linearized primal and dual formulations for \mathbb{L}^p -problems. The dual formulations are somewhat different from those in the first part. However, we shall skip the exact description. The main feature is that they allow the passage to the limit as $p \rightarrow \infty$. Both Lipschitz-continuous and the lower semicontinuous \mathbb{L}^p -problems are considered. Identifying the limit of primal and dual problems gives an alternative characterization of the value function in \mathbb{L}^∞ -control problems. In the Lipschitz-continuous case, the standard \mathbb{L}^∞ -value function, its primal and dual formulation coincide.

The main result is

Theorem 21 ([G9], Theorem 3.5)

1. (L.s.c. case) We assume that (24) and (30) hold true. Then,

$$\Lambda_h^\infty(t_0, x_0) = \eta_h^\infty(t_0, x_0),$$

for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$.

2. (Continuous case) Moreover, if h is uniformly continuous on $\mathbb{R} \times \mathbb{R}^N \times U$ and Lipschitz-continuous in time and space uniformly w.r.t. the control parameter, then

$$V_h^\infty(t_0, x_0) = \Lambda_h^\infty(t_0, x_0) = \eta_h^\infty(t_0, x_0),$$

for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$.

The dual problem is a linear formulation of the \mathbb{L}^∞ -control problem. The techniques in the proofs rely mainly on the theory of Hamilton-Jacobi equations and occupational measures. One begins by providing linear formulations for \mathbb{L}^p -problems. Second, one employs the compactness properties for the set of constraints and passes to the limit as $p \rightarrow \infty$ in both the primal and the dual formulations.

The assertion 2 is generally not valid for nonconvex dynamics and semicontinuous cost functions. We provide an Example in this direction:

Example 22 We consider the case of \mathbb{R}^2 , the control set is $U = \{-1, 1\}$ and $f : \mathbb{R}^3 \times U \rightarrow \mathbb{R}^2$ is given by

$$f(t, x, y, u) = (u, x^2 \wedge 1),$$

for all $t, x, y \in \mathbb{R}$ and all $u \in U$. We consider the control system

$$\begin{cases} dx_t^{t_0, x_0, y_0, u(\cdot)} = u_t dt, \\ dy_t^{t_0, x_0, y_0, u(\cdot)} = (x_t^{t_0, x_0, y_0, u(\cdot)})^2 \wedge 1 dt, \end{cases}$$

for every $t_0 \in [0, 1]$, all $(x_0, y_0) \in \mathbb{R}^2$ and all U -valued measurable functions $u(\cdot)$. We also consider the lower semicontinuous cost function $h : \mathbb{R} \times \mathbb{R}^2 \times U \rightarrow \mathbb{R}$ defined by

$$h(t, x, y, u) = \begin{cases} 1, & \text{if } (x, y) \neq (0, 0), \\ \frac{1}{2}, & \text{if } (x, y) = (0, 0), \end{cases}$$

for all $t \in \mathbb{R}$ and all $u \in U$. One notices that, $(0, 0) \notin \{x_t^{0,0,0,u(\cdot)} : t \in (0, 1]\}$. Thus,

$$V_h^\infty(0, 0, 0) = 1.$$

On the other hand, for $n \geq 1$, we introduce the control

$$u^n = \sum_{k=0}^{n-1} 1_{\left[\frac{2k}{2n}, \frac{(2k+1)}{2n}\right)} - 1_{\left[\frac{(2k+1)}{2n}, \frac{(2k+2)}{2n}\right)} - 1_{\{1\}}.$$

Then $\text{Supp}(\gamma_1^{0,0,0,u^n}) \subset [0, t_0] \times [0, \frac{1}{n}] \times [0, \frac{1}{n^2}] \times U$. We recall that $\Theta(0, 1, (0, 0))$ is compact and get the existence of some subsequence $(\gamma^{0,0,0,u^n})_n$ converging to some $\gamma \in \Theta(0, 1, (0, 0))$. It follows that $\text{Supp}(\gamma) \subset [0, t_0] \times \{(0, 0)\} \times U$ and

$$\Lambda_h^\infty(0, 0, 0) = \frac{1}{2} < V_h^\infty(0, 0, 0).$$

If $0 < t < T$, $x_0 \in \mathbb{R}^N$ and $\gamma \in \Theta(0, t, x_0)$, we extend the value function Λ_h^∞ to depend on measures as initial data by setting

$$\Lambda_h^\infty(t, \gamma) = \inf_{\tilde{\gamma} \in \Theta(t, T, \gamma)} \|h(\cdot + t, \cdot, \cdot)\|_{\mathbb{L}^\infty([0, T-t] \times \mathbb{R}^N \times U, \tilde{\gamma}_1)}.$$

Finally, we use the semigroup property of the set of constraints to derive a Dynamic Programming Principle in bounded or continuous setting.

Theorem 23 ([G9], Theorem 3.8)

1. Let us assume that h is a bounded, measurable function. For every $0 < t < T$, and every $x_0 \in \mathbb{R}^N$,

$$\Lambda_h^\infty(0, x_0) = \inf_{\gamma \in \Theta(0, t, x_0)} \max \left(\|h\|_{\mathbb{L}^\infty([0, t] \times \mathbb{R}^N \times U, \gamma_1)}, \Lambda_h^\infty(t, \gamma) \right).$$

Moreover,

$$\Lambda_h^\infty(0, x_0) \leq \inf_{\gamma \in \Gamma(0, t, x_0)} \max \left(\|h\|_{\mathbb{L}^\infty([0, t] \times \mathbb{R}^N \times U, \gamma_1)}, \int_{\mathbb{R}^N} \Lambda_h^\infty(t, x) \gamma_2(dx) \right).$$

2. Moreover, if h is uniformly continuous on $\mathbb{R} \times \mathbb{R}^N \times U$ and Lipschitz-continuous in time and space uniformly w.r.t. the control parameter, then,

$$\Lambda_h^\infty(0, x_0) = \inf_{\gamma \in \Gamma(0, t, x_0)} \max \left(\|h\|_{\mathbb{L}^\infty([0, t] \times \mathbb{R}^N \times U, \gamma_1)}, \int_{\mathbb{R}^N} \Lambda_h^\infty(t, x) \gamma_2(dx) \right),$$

for every $0 < t < T$, and every $x_0 \in \mathbb{R}^N$.

We should point out that further linearized DPP can be obtained using the (stronger, but ulterior) results in [G2].

1.2.4 Discontinuous control problems with state constraints : linear formulations and dynamic programming principles

The aim of the paper [G1] is to provide linearized formulations for the general control problem with state constraints and deduce linearized formulations of the dynamic programming principles in the discontinuous framework. The solution is constrained to some closed set K . It appears natural to modify the linearized primal value function by only minimizing with respect to probability measures whose support is (included in) K . Whenever the cost functions are bounded and lower semicontinuous, the linearized value function can also be obtained as the limit of (classical) penalized problems. Under classical convexity assumptions, the penalized value functions also converge to the classical value function. In general, without any convexity assumptions, we provide a dual formulation similar to the unconstrained framework. This dual formulation gives an intuitive idea of the relation between the primal value function and the Hamilton-Jacobi equation associated to the control problem.

Using the characterization of the set of constraints in the linear formulation, we prove a semigroup property. This property follows naturally from the structure of the sets of constraints. We derive dynamic programming principles under state constraints for general bounded cost functionals. In the bounded case, an abstract DPP is given. In the lower semicontinuous setting, we provide a further linearized programming principle. For details, the reader may consult the stochastic counterpart (Subsections 1.1.4 and 1.1.5).

1.2.5 Min-max control problems

Motivated by the results in linear programming methods in control, we consider a min-max control problem, where the two players have separated dynamics. The non-convexity of the

Hamiltonian(s) associated to this problem does not allow one to apply the arguments used in the linearization of (one player) control problems. However, we consider the (convex, weak *-compact) sets of constraints associated to the dynamics of the two players, and define, in a natural way, a joint set of constraints. The value functions allow the two adversaries to play (occupational) measure against measure and are given in linearized form. Using such a formulation seems to be a good approach for obtaining dual characterization for the value functions.

We give an affirmative answer to the existence of the value and of saddle points. We consider both the Lipschitz continuous and the semicontinuous frameworks. We investigate the dual formulations of the linearized value. To our best knowledge, within the min-max control problems/ differential game framework, these are the first results connecting the value to a dual formulation involving regular test functions. The dual is naturally connected to the Hamilton-Jacobi-Isaacs equation. Using the semigroup properties of the sets of constraints, we provide a linear dynamic programming principle.

One of the advantages of transforming a nonlinear control problem into a linear optimization problem consists in the possibility of obtaining approximation results for the value function. Following the methods presented in [FGL07] and [GQ09], one can approximate the occupational measures by Dirac measures and construct an optimal feedback control. Moreover, when considering the ergodic control problem (see, e.g. [BCD97]), the study of the behavior of the value function is simplified whenever this value is expressed by a linear problem.

The contributions of these section come from the paper [G18]. We emphasize that, although mathematically challenging, this is just a toy model allowing a better understanding of game problems with general dynamics.

Throughout the subsection, we will be considering the following bi-controlled system

$$(33) \quad \begin{cases} dx_t^{t_0, x_0, u} = f(t, x_t^{t_0, x_0, u}, u_t) dt, \\ dy_t^{t_0, y_0, v} = g(t, y_t^{t_0, y_0, v}, v_t) dt, \quad t \geq t_0, \\ x_{t_0}^{t_0, x_0, u} = x_0 \in \mathbb{R}^N, \quad y_{t_0}^{t_0, y_0, v} = y_0 \in \mathbb{R}^M. \end{cases}$$

We denote by $T > 0$ a finite time horizon. Here $t_0 \in [0, T]$, the control u (respectively v) takes its values in a compact, metric space U (respectively V) and is associated to the first (respectively the second) player. The functions f and g are assumed to be globally uniformly continuous and Lipschitz continuous in time and space, uniformly w.r.t. the control.

Defining joint occupational measures

To any admissible control u (resp. v) we associate occupational measures and the sets $\Gamma^1(t_0, r, x_0)$, $\Theta^1(t_0, r, x_0)$ (resp Γ^2, Θ^2) similarly to (25) and (26).

The measures in $\Theta^1(t_0, r, x_0)$ are supported by $[t_0, T] \times K_{x_0, c} \times U \times K_{x_0, c}$, where $K_{x_0, c} = \{x \in \mathbb{R}^N : |x| \leq c(1 + |x_0|)\}$, for a fixed generic constant $c > 0$. Similar assertions hold true for the second player. Whenever $\gamma \in \Theta^1(t_0, T, x_0)$, the disintegration theorem yields the existence of some

$$\gamma_{1, \cdot} \in \mathbb{L}^1([t_0, T]; \mathcal{P}(\mathbb{R}^N \times U))$$

such that

$$\gamma_1(Adxdu) = \frac{1}{T - t_0} \int_A \gamma_{1, t}(dxdu) dt,$$

for all Borel set $A \subset [t_0, T]$. Whenever $\eta \in \mathcal{P}([t_0, T] \times K_{y_0, c} \times V) \times \mathcal{P}(K_{y_0, c})$, we define the measure $\gamma \otimes \eta \in \mathcal{P}([t_0, T] \times K_{x_0, c} \times U \times K_{y_0, c} \times V) \times \mathcal{P}(K_{x_0, c} \times K_{y_0, c})$ by setting

$$(34) \quad \begin{cases} (\gamma \otimes \eta)_1(Adxdudydv) = \int_A \gamma_{1,t}(dxdu) \eta_1(dt dydv), \\ (\gamma \otimes \eta)_2(dxdy) = \gamma_2(dx) \eta_2(dy), \end{cases}$$

for all Borel set $A \subset [t_0, T]$. We introduce the sets

$$(35) \quad \begin{aligned} \gamma \otimes \Theta^2(t_0, T, y_0) &= \{\gamma \otimes \eta : \eta \in \Theta^2(t_0, T, y_0)\}, \\ \gamma \otimes \Gamma^2(t_0, T, y_0) &= \{\gamma \otimes \eta : \eta \in \Gamma^2(t_0, T, y_0)\}, \text{ whenever } \gamma \in \Theta^1(t_0, T, x_0). \end{aligned}$$

As a consequence of the Corollary 16, we have

Proposition 24 ([G18], Proposition 3)

1. Let us consider $\gamma \in \Theta^1(t_0, T, x_0)$ and $(\eta^n)_n \subset \Theta^2(t_0, T, y_0)$ a sequence that converges weakly to some $\eta \in \Theta^2(t_0, T, y_0)$. Then $\gamma \otimes \eta^n$ converges weakly to $\gamma \otimes \eta$.
2. Whenever $\gamma \in \Theta^1(t_0, T, x_0)$, one has $\gamma \otimes \Theta^2(t_0, T, y_0) = \overline{co}(\gamma \otimes \Gamma^2(t_0, T, y_0))$.
3. Whenever $\eta \in \Theta^2(t_0, T, y_0)$, one has $\Theta^1(t_0, T, x_0) \otimes \eta = \overline{co}(\Gamma^1(t_0, T, x_0) \otimes \eta)$.

Value function and saddle points

We are going to consider two bounded, measurable cost functions $h : \mathbb{R} \times \mathbb{R}^N \times U \times \mathbb{R}^M \times V \rightarrow \mathbb{R}$ and $l : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}$. Let us now consider the cost functional

$$J(t_0, \gamma, \eta) = \left[\begin{array}{l} (T - t_0) \int_{[t_0, T] \times \mathbb{R}^{N+M}} h(s, x, u, y, v) (\gamma \otimes \eta)_1(ds dx U dy V) \\ + \int_{\mathbb{R}^{N+M}} l(x, y) (\gamma \otimes \eta)_2(dxdy) \end{array} \right],$$

for every $t_0 \in [0, T]$, $\gamma \in \Theta^1(t_0, T, x_0)$ and every $\eta \in \mathcal{P}([t_0, T] \times \mathbb{R}^M \times V) \times \mathcal{P}(\mathbb{R}^M)$. The upper and lower value functions are defined by

$$\begin{aligned} \Lambda^+(t_0, x_0, y_0) &= \inf_{\gamma \in \Theta^1(t_0, T, x_0)} \sup_{\eta \in \Theta^2(t_0, T, y_0)} J(t_0, \gamma, \eta), \text{ and} \\ \Lambda^-(t_0, x_0, y_0) &= \sup_{\eta \in \Theta^2(t_0, T, y_0)} \inf_{\gamma \in \Theta^1(t_0, T, x_0)} J(t_0, \gamma, \eta), \end{aligned}$$

for all $0 \leq t_0 \leq T$, and all $(x_0, y_0) \in \mathbb{R}^{N+M}$.

In the semicontinuous framework, we prove the existence of a common value function and of saddle points.

Theorem 25 ([G18], Theorem 6) *We assume that the functions $h : \mathbb{R} \times \mathbb{R}^N \times U \times \mathbb{R}^M \times V \rightarrow \mathbb{R}$ and $l : \mathbb{R}^{N+M} \rightarrow \mathbb{R}$ are bounded and continuous in time, uniformly w.r.t. x, u, y, v , lower semicontinuous in (x, u) and upper semicontinuous in (y, v) . Then :*

1. The sets

$$\begin{aligned} \Theta^+(t_0, x_0, y_0) &= \{(\gamma^+, \eta^+) \in \Theta^1(t_0, T, x_0) \times \Theta^2(t_0, T, y_0) : \Lambda^+(t_0, x_0, y_0) = J(t_0, \gamma^+, \eta^+)\}, \\ \Theta^-(t_0, x_0, y_0) &= \{(\gamma^-, \eta^-) \in \Theta^1(t_0, T, x_0) \times \Theta^2(t_0, T, y_0) : \Lambda^-(t_0, x_0, y_0) = J(t_0, \gamma^-, \eta^-)\} \end{aligned}$$

are nonempty for every $(t_0, x_0, y_0) \in [0, T] \times \mathbb{R}^{N+M}$.

2. The linearized min-max control problem has a value $\Lambda = \Lambda^+ = \Lambda^-$.

Duality

We introduce

$$h^\varepsilon(s, t, x, y) = \frac{T - t_0}{\min(2\varepsilon, t + \varepsilon - t_0, T + \varepsilon - t)} h(s, x, y), \text{ for } t \in (t_0, T), x \in \mathbb{R}^N.$$

Using duality techniques, we prove general results on the relations between the value functions and the associated HJI systems. In particular, we have the following lower and upper bounds for the value function

Proposition 26 ([G18], Corollary 12) *The following inequalities hold true*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \inf_{\phi \in C_b^{1,1}([t_0, T] \times \mathbb{R}^M)} \sup_{\psi \in C_b^{1,1}([t_0, T] \times \mathbb{R}^N)} \sup_{\substack{(y_1, v, y_2) \in K_{y_0, c} \times V \times K_{y_0, c} \\ t \in [t_0, T]}} \inf_{\substack{(x_1, u, x_2) \in K_{x_0, c} \times U \times K_{x_0, c} \\ s \in [t_0, T]}} \mathcal{H}^\varepsilon(t, s) \\ & \leq \Lambda^+(t_0, x_0, y_0) = \Lambda^-(t_0, x_0, y_0) \\ & \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \sup_{\psi \in C_b^{1,1}([t_0, T] \times \mathbb{R}^N)} \inf_{\phi \in C_b^{1,1}([t_0, T] \times \mathbb{R}^M)} \lim_{\substack{(x_1, u, x_2) \in K_{x_0, c} \times U \times K_{x_0, c} \\ s \in [t_0, T]}} \inf_{\substack{(y_1, v, y_2) \in K_{y_0, c} \times V \times K_{y_0, c} \\ t \in [t_0, T]}} \sup \mathcal{H}^\varepsilon(s, t), \end{aligned}$$

where

$$\mathcal{H}^\varepsilon(s', t') = \left[\begin{array}{c} (T - t_0) \left(\begin{array}{c} \partial_t \phi(t, y_1) + \langle D\phi(t, y_1), g(t, y_1, v) \rangle \\ + 2\varepsilon h^\varepsilon(s', t', x_1, y) \mathbf{1}_{|s' - t'| \leq \varepsilon} \\ + \partial_t \psi(s, x_1) + \langle D\psi(s, x_1), f(s, x_1, u) \rangle \end{array} \right) \\ + 2\varepsilon l(x_2, y_2) - \psi(T, x_2) - \phi(T, y_2) + \phi(t_0, y_0) + \psi(t_0, x_0) \end{array} \right]$$

for all $(t_0, x_0, y_0) \in [0, T] \times \mathbb{R}^{N+M}$.

Dynamic programming principles

As for the classical value functions (with one control), we prove the following dynamic programming principle:

Theorem 27 ([G18], Theorem 15) *We assume that the functions $h : \mathbb{R} \times \mathbb{R}^N \times U \times \mathbb{R}^M \times V \rightarrow \mathbb{R}$ and $l : \mathbb{R}^{N+M} \rightarrow \mathbb{R}$ are bounded and measurable. Then :*

$$\begin{aligned} & \Lambda^+(t_0, x_0, y_0) \\ & \geq \inf_{\gamma \in \Theta^1(t_0, t, x_0)} \sup_{\eta \in \Theta^2(t_0, t, y_0)} \left(\begin{array}{c} (t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U \times \mathbb{R}^M \times V} h(s, x, u, y, v) (\gamma \otimes \eta)_1(ds dx du dy dv) \\ + \Lambda^+(t, \gamma, \eta) \end{array} \right) \\ & \geq \sup_{\eta \in \Theta^2(t_0, t, y_0)} \inf_{\gamma \in \Theta^1(t_0, t, x_0)} \left(\begin{array}{c} (t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U \times \mathbb{R}^M \times V} h(s, x, u, y, v) (\gamma \otimes \eta)_1(ds dx du dy dv) \\ + \Lambda^-(t, \gamma, \eta) \end{array} \right) \\ & \geq \Lambda^-(t_0, x_0, y_0). \end{aligned}$$

Moreover, if h and l are continuous in time, uniformly w.r.t. x, u, y, v , lower semicontinuous in (x, u) and upper semicontinuous in (y, v) , then the previous inequalities become equalities. The functions $\Lambda^{+,-}(t, \gamma, \eta)$ have similar definitions to those in subsection 1.2.2.

1.2.6 Viability results for singularly perturbed control systems

In the paper [G13], we investigate two classes of deterministic control problems for which no convexity assumption is required on the dynamics. First, we provide viscosity characterizations for the value function in both Mayer and supremum cost setting, whenever the cost functional is discontinuous. In the convex case, this problem has been solved using viability theory for instance in [BJ90, BI89, Fra93, PQ01, Ser02]. The main difficulties in our setting are the lack of convexity in the dynamics and the discontinuity of the costs. We overcome this by considering an ε -viability approach and by modifying the definition of the value functions and taking the closure of the reachable sets. This allows one to use Fillipov's Theorem (see for instance [AC84]). We are able to consider several cases when the value function is given with respect to either bounded or semicontinuous cost functions. Illustrating examples are given in order to clarify how the value function should be defined for lower semicontinuous cost.

Second, we obtain ε -viability (near viability) properties for a singularly perturbed control system. This method relies on the averaging method. Using the characterization of ε -viable domains via \mathbb{L}^∞ control problems, we are able to considerably simplify the proof of the main result in [Gai04] and to generalize it to the case with additional constraints.

Singularly perturbed control systems

We consider the following dynamics:

$$(36) \quad \begin{cases} dx_s^{t,x,y,u;\varepsilon} = f(x_s^{t,x,y,u;\varepsilon}, y_s^{t,x,y,u;\varepsilon}, u_s) ds, \\ \varepsilon dy_s^{t,x,y,u;\varepsilon} = g(x_s^{t,x,y,u;\varepsilon}, y_s^{t,x,y,u;\varepsilon}, u_s) ds, \end{cases}$$

for all $s \geq t$, where $(t, x, y) \in [0, \infty) \times \mathbb{R}^M \times \mathbb{R}^N$ and ε is a small real parameter. The evolutions of the two state variables x and y of the system are of different scale. We call x the "slow" variable and y the "fast" variable. When $h : \mathbb{R}^M \rightarrow \mathbb{R}$ is a given bounded function, we set

$$(37) \quad W_{\varepsilon,h}(t, x, y) = \inf_{u \in \mathcal{U}} \sup_{s \in [t, T]} h(x_s^{t,x,y,u;\varepsilon})$$

for all $(t, x, y) \in [0, T] \times \mathbb{R}^M \times \mathbb{R}^N$.

The asymptotic behavior of the value function (37) when $\varepsilon \rightarrow 0$ is a very interesting problem. Whenever the control system (36) has some stability properties, it is possible to prove that the trajectories $\left(x_{(\cdot)}^{t,x,y,u;\varepsilon}, y_{(\cdot)}^{t,x,y,u;\varepsilon}\right)$ of (36) converge towards some solution of some system obtained by formally replacing ε by 0 in (36). This is the so called Tikhonov approach which has been successfully developed in [Tic52], [Vel97] for instance.

When $dy_s^{t,x,y,u} = g(x_s^{t,x,y,u}, y_s^{t,x,y,u}, u_s) ds$ is not stable, another approach consists in investigating relationships between the system (36) and a new differential inclusion

$$(38) \quad dx_s^{t,x} \in F(x_s^{t,x})$$

obtained by an averaging method. It is important to notice that only the behavior of the "slow" variable $x_{(\cdot)}^{t,x,y,u;\varepsilon}$ is concerned by this approach.

The sequence of functions $(W_{\varepsilon,h}(\cdot, \cdot, y))_{\varepsilon > 0}$ converges to the value function $W_{F,h} : [0, T] \times \mathbb{R}^M \rightarrow \mathbb{R}$ defined by

$$(39) \quad W_{F,h}(t, x) = \inf_{x_{(\cdot)}^{t,x} \in S_F(t,x)} \sup_{s \in [t, T]} h(x_s^{t,x}),$$

for all $(t, x) \in [0, T] \times \mathbb{R}^M$ if h is uniformly continuous and bounded. If h is only lower semicontinuous, the equality in the limit has to be replaced by an inequality.

Here, $S_F(t, x)$ stands for the set of solutions of (38) starting from $(t, x) \in [0, T] \times \mathbb{R}^M$ and $S_\varepsilon(t, x, y)$ for the set of all the trajectories of (36) $\left(x_{(\cdot)}^{t,x,y,u;\varepsilon}, y_{(\cdot)}^{t,x,y,u;\varepsilon}\right)$ starting from $(t, x, y) \in [0, T] \times \mathbb{R}^M \times \mathbb{R}^N$. Moreover, we define $S_F(0, x) =: S_F(x)$ and by $S_\varepsilon(0, x, y) =: S_\varepsilon(x, y)$, for all $(x, y) \in \mathbb{R}^{M+N}$.

We recall the notion of near viability.

Definition 28 *A family of solutions $\left(x_{(\cdot)}^{0,x,y,u;\varepsilon}, y_{(\cdot)}^{0,x,y,u;\varepsilon}\right) \in S_\varepsilon(x, y)$, for all $\varepsilon > 0$ is said to be near viable in some closed set $D \subset \mathbb{R}^{M+N}$ if*

$$\max_{t \in [0, T]} d_D \left(\left(x_t^{0,x,y,u;\varepsilon}, y_t^{0,x,y,u;\varepsilon} \right) \right) \leq \omega_T(\varepsilon),$$

for some function $\omega_T(\cdot)$ satisfying $\lim_{\varepsilon \rightarrow 0} \omega_T(\varepsilon) = 0$.

When ε tends to 0, making the change of variables $\tau = \frac{s}{\varepsilon}$, we are led to consider the following associated system:

$$(40) \quad dy_s^{t,y,u} = g(x, y_s^{t,y,u}, u_s) ds,$$

for $s \in [0, +\infty)$, where x is fixed in \mathbb{R}^M . For every fixed $x \in \mathbb{R}^M$, we denote by $y_{(\cdot)}^{t,y,u;x}$ the unique solution of (40) corresponding to the control u and to the initial value y .

We follow an averaging method (cf. for instance [Gai92], [Gra97]): we set, for $(x, y) \in \mathbb{R}^M \times \mathbb{R}^N$, $S > 0$, and $u \in \mathcal{U}$,

$$\begin{aligned} A(x, y, S, u) &= \frac{1}{S} \int_0^S f(x, y_s^{t,y,u;x}, u_s) ds, \\ F(x, y, S) &= \{A(x, y, S, u); u \in \mathcal{U}\}. \end{aligned}$$

We shall make the following assumption on the system:

$$(41) \quad \left\{ \begin{array}{l} \forall R > 0, \text{ there exist nonempty bounded subsets } N_R \text{ and } \Omega_R \text{ of } \mathbb{R}^N \text{ such that:} \\ \mathbf{1}) \forall (x, y) \in B(0, R) \times N_R, \forall \left(x_{(\cdot)}^{0,x,y,u;\varepsilon}, y_{(\cdot)}^{0,x,y,u;\varepsilon}\right) \in S_\varepsilon(0, x, y), \\ y_s^{0,x,y,u;\varepsilon} \in \Omega_R, \text{ for all } s \in [0, T]. \\ \mathbf{2}) \forall (x, y) \in B(0, R) \times N_R, \forall u \in \mathcal{U}, y_s^{0,y,u;x} \in \Omega_R, \text{ for all } s \geq 0. \end{array} \right.$$

Under an assumption of either total controllability or stability (e.g. dissipativity in the y component) of the associated system (40), the set $F(x, y, S)$ converges, in the sense of the Hausdorff metric, towards a compact convex set $F(x)$ of \mathbb{R}^M . If the set-valued map F is Lipschitz, then the set of slow solutions $x_{(\cdot)}^{t,x,y,u;\varepsilon}$ converges towards the set of solutions of the differential inclusion.

We consider two closed subsets $C, D \subset \mathbb{R}^N$. The main result in this setting of [G13] is the following.

Proposition 29 ([G13], Proposition 19) *We assume that (41) is satisfied, that one has either total controllability or stability, and F is locally Lipschitz. Moreover, we assume that D is a viability domain for (38). Then, for every $R_0 > 0$, such that $D \subset B(0, R_0)$ and every*

$R \geq R_0 + T \|f\|_\infty$, there exists a function $\nu_R : (0, +\infty) \rightarrow \mathbb{R}$, such that $\lim_{\varepsilon \rightarrow 0} \nu_R(\varepsilon) = 0$ and, for every $\varepsilon > 0$ and every $(x, y) \in B(0, R_0) \times N_R$:

-for every $\left(x_{(\cdot)}^{0,x,y,u_\varepsilon;\varepsilon}, y_{(\cdot)}^{0,x,y,u_\varepsilon;\varepsilon}\right) \in S_\varepsilon(x, y)$ which are near viable in $D \times \Omega_R$ and contained in $C \times \Omega_R$, there exists $x_{(\cdot)}^{0,x} \in S_F(x)$ which is viable in D and contained in C and:

$$(42) \quad \sup_{s \in [0, T]} |x_s^{0,x,y,u_\varepsilon;\varepsilon} - x_s^{0,x}| \leq \nu_R(\varepsilon);$$

-for any $x_{(\cdot)}^{0,x} \in S_F(x)$ which is viable in D and contained in C , there exist

$$\left(x_{(\cdot)}^{0,x,y,u_\varepsilon;\varepsilon}, y_{(\cdot)}^{0,x,y,u_\varepsilon;\varepsilon}\right) \in S_\varepsilon(x, y)$$

which are near viable in $D \times \Omega_R$, and contained in a neighborhood of $C \times \Omega_R$ such that the previous inequality holds true. The neighborhood of C is of the form $C + \sigma_R(\varepsilon)$ where $\sigma_R(\varepsilon) \rightarrow 0$ when ε tends to 0. Moreover, if $D \cap \partial C = \emptyset$, then the sequence $\left(x_{(\cdot)}^{0,x,y,u_\varepsilon;\varepsilon}, y_{(\cdot)}^{0,x,y,u_\varepsilon;\varepsilon}\right) \in S_\varepsilon(x, y)$ is near viable in $D \times \Omega_R$, and contained in $C \times \Omega_R$.

2 Control of Piecewise Deterministic Markov Processes and applications to stochastic gene networks

Markov processes have been intensively used to describe variability features of various cellular processes. To the best of our knowledge, Markovian tools have first been employed in connection to molecular biology in [Del40]. The natural idea was to associate to each reaction network a pure jump model. Due to the large number of molecular species involved in the reactions, direct simulation of these models turns out to be very slow. To increase proficiency, hybrid models are adopted in [CDR09]. They distinguish the discrete components from the "continuous" ones. Using partial Kramers-Moyal expansion, the authors of [CDR09] replace the initial pure jump process with an appropriate piecewise deterministic Markov process (PDMP). The resulting (hybrid) models can then be applied to "in silico" studies of a variety of biologic systems: Cook's model (cf. [CGT98]) for stochastic gene expression and its implications on haploinsufficiency (as particular case of continuous PDMP with switching), Hasty et al. (cf. [HPDC00]) model for λ -phage, etc. For further biological models, the reader is referred to [CDR09] and references therein.

2.1 Construction of controlled PDMPs

We consider U to be a compact metric space (the control space) and \mathbb{R}^N be the state space, for some $N \geq 1$.

Piecewise deterministic control processes have been introduced by Davis [Dav84] (see also [Dav93]). Such processes are given by their local characteristics: a vector field $f : \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ that determines the motion between two consecutive jumps, a jump rate $\lambda : \mathbb{R}^N \times U \rightarrow \mathbb{R}_+$ and a transition measure $Q : \mathbb{R}^N \times U \rightarrow \mathcal{P}(\mathbb{R}^N)$. We denote by $\mathcal{B}(\mathbb{R}^N)$ the Borel σ -field on \mathbb{R}^N and $\mathcal{P}(\mathbb{R}^N)$ the family of probability measures on \mathbb{R}^N . For every $A \in \mathcal{B}(\mathbb{R}^N)$, the function $(x, u) \mapsto Q(x, u, A)$ is assumed to be measurable and, for every $(x, u) \in \mathbb{R}^N \times U$, $Q(x, u, \{x\}) = 0$.

We summarize the construction of controlled piecewise deterministic Markov processes (PDMP). We let $\mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$ denote the space of U -valued Borel measurable functions defined on $\mathbb{R}^N \times \mathbb{R}_+$. Whenever $u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^N$, we consider the ordinary differential equation

$$\begin{cases} d\Phi_t^{t_0, x_0, u} = f(\Phi_t^{t_0, x_0, u}, u(x_0, t - t_0)) dt, & t \geq t_0, \\ \Phi_{t_0}^{t_0, x_0, u} = x_0. \end{cases}$$

We choose the first jump time T_1 such that the jump rate $\lambda(\Phi_t^{0, x_0, u}, u(x_0, t))$ satisfies

$$\mathbb{P}(T_1 \geq t) = \exp\left(-\int_0^t \lambda(\Phi_s^{0, x_0, u}, u(x_0, s)) ds\right).$$

The controlled piecewise deterministic Markov processes (PDMP) is defined by

$$X_t^{x_0, u} = \Phi_t^{0, x_0, u}, \text{ if } t \in [0, T_1).$$

The post-jump location Y_1 has $Q(\Phi_\tau^{0, x_0, u}, u(x_0, \tau), \cdot)$ as conditional distribution given $T_1 = \tau$. Starting from Y_1 at time T_1 , we select the inter-jump time $T_2 - T_1$ such that

$$\mathbb{P}(T_2 - T_1 \geq t / T_1, Y_1) = \exp\left(-\int_{T_1}^{T_1+t} \lambda(\Phi_s^{T_1, Y_1, u}, u(Y_1, s - T_1)) ds\right).$$

We set

$$X_t^{x_0, u} = \Phi_t^{T_1, Y_1, u}, \text{ if } t \in [T_1, T_2).$$

The post-jump location Y_2 satisfies

$$\mathbb{P}(Y_2 \in A \mid T_2, T_1, Y_1) = Q\left(\Phi_{T_2}^{T_1, Y_1, u}, u(Y_1, T_2 - T_1), A\right),$$

for all Borel set $A \subset \mathbb{R}^N$. And so on.

We assume standard conditions on the local characteristics (uniform continuity, boundedness and Lipschitz behavior in the state component). These assumptions appear under the same form in [Son86b] and are needed to infer the uniform continuity of the value function if the cost function is regular enough. We impose a further natural condition stating that the probability of exiting the ball centered at the initial point is zero as the radius increases to ∞ . This further assumption is needed to provide stability properties of viscosity solutions. In particular, it allows to extend the comparison principle to semicontinuous viscosity solutions. For the exact assumptions, the reader is referred to [G4], [G5] and [G8].

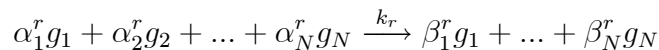
Let us just mention that an extensive literature is available on the theory of continuous or impulsive control for piecewise-deterministic Markov processes. The reader is referred to [Dav86] for the first results on the subject and [Son86b] for the first results in terms of viscosity solutions (see also [DY96]). Several related topics have been studied since (DPP algorithms in [Alm01], impulse control in [Cos89], average continuous control in [CD09], [CD10], control problems with long-run average cost in [CD12], stability in [CD08], numerical methods in [dSD12], [BdSD12], etc.)

2.2 Stochastic gene networks

2.2.1 From biochemical reactions to PDMPs

We recall some rudiments on PDMPs associated to gene networks. For further contributions on gene networks modelling the reader is referred to [CDR09]. For reader's sake, the construction will not involve control arguments in the coefficients. However : the speed of these reactions depends on various external factors (temperature, catalyzers). Whenever a specific regime is wanted, it suffices to modify these external factors. From the theoretical point of view, this is an exogenous control parameter.

We suppose that the biological evolution is given by a family of genes $\mathcal{G} = \{g_i : i = 1, N\}$ interacting through a finite set of reactions \mathcal{R} . Every reaction $r \in \mathcal{R}$ can be represented as



and it specifies that α_i^r molecules of i type (with $1 \leq i \leq N$) called reactants interact in order to form the products (β_i^r molecules of i type, with $1 \leq i \leq N$). The reaction does not occur instantaneously and one needs to specify the reaction speed $k_r > 0$. Also, the presence of all species is not required ($\alpha_i^r, \beta_i^r \in \mathbb{N}$, for all $1 \leq i \leq N$). The species are partitioned in two classes called continuous, respectively discrete component. This partition (for further considerations, see [CDR09]) induces a partition of the reactions. In sum, we distinguish between reactions contributing to the continuous flow ($\mathcal{C} = \{1, 2, \dots, M_1\}$) and jump reactions ($\mathcal{J} = \{M_1 + 1, \dots, \text{card}(\mathcal{R})\}$). To every reaction $r \in \mathcal{R}$, one associates

- 1) a stoichiometric column vector $\theta^r = \beta^r - \alpha^r \in \mathbb{R}^N$,
- 2) a propensity function $\lambda_r : \mathbb{R}^N \longrightarrow \mathbb{R}_+$.

For a \mathcal{C} -type reaction, $\lambda_r(x) = k_r \prod_{i=1}^N x_i^{\alpha_i^r}$, for all $x \in \mathbb{R}^N$.

For a \mathcal{J} -type reaction, one should require further regularity as $x_i \rightarrow 0$. The jump mechanism will specify that the number of molecules of type i diminishes by α_i^r . Therefore, in order to insure positive components, rather than introducing $\lambda_r(x)$ as for continuous reactions, one could consider

$$\lambda_r(x) = k_r \prod_{\substack{i=1 \\ \alpha_i^r > 0}}^N x_i^{\alpha_i^r} \chi\left(\frac{x_i}{\alpha_i^r}\right),$$

for some regular function χ such that $0 \leq \chi \leq 1$, $\chi(y) = 0$, for $0 \leq y \leq 1$ and $\chi(y) = 1$, for $y \geq 1 + \text{err}$ (where err is a positive constant). This construction may also apply to \mathcal{J} -type reactions to take into account specific thresholds levels.

We associate two matrix M_1 whose columns are the vectors α^r , where $r \in \mathcal{C}$, respectively M_2 whose columns are the vectors α^r , where $r \in \mathcal{J}$. The flow is given by

$$f(x) = M_1 \times (\lambda_1(x), \lambda_2(x), \dots, \lambda_{M_1}(x)),$$

the jump intensity

$$\lambda(x) = \sum_{r \in \mathcal{J}} \lambda_r(x)$$

and, whenever $\lambda(x) > 0$, the transition measure Q is given by

$$Q(x, dz) = \sum_{r \in \mathcal{J}} \frac{\lambda_r(x)}{\lambda(x)} \delta_{x+\theta^r}(dz).$$

2.2.2 General On/Off Models

A two-state model is often employed to describe different situations in the molecular biology. Usually, the two states describe either the presence or the absence of some rare molecular specie. Whenever the gene γ is inactive (represented by $\gamma = 0$), the molecule X degrades at a rate r_0 , whileas, whenever γ is active ($\gamma = 1$), the molecule X increases at a rate proportional to some given r_1 .

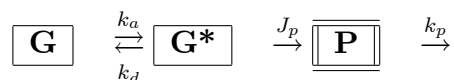
From the mathematical point of view, the system will be given by a process $(X(t), \gamma(t))$ on the state space $E = \mathbb{R} \times \{0, 1\}$. The component $X(t)$ follows a differential dynamic depending on the hidden variable

$$\frac{dX}{dt} = \begin{cases} -r_0(X), & \text{if } \gamma(t) = 0, \\ r_1(X), & \text{if } \gamma(t) = 1, \end{cases}$$

where $r_0(x) \geq 0$ is a bounded, Lipschitz-continuous consumption term and $r_1(x) \geq 0$ is a bounded, Lipschitz continuous production term. To be more precise, the PDMP associated to the model has the characteristic (f, λ, Q) given by $f_\gamma(x) = -r_0(x)(1 - \gamma) + r_1(x)\gamma$, $\lambda_\gamma(x) = \lambda_\gamma$, $Q(\gamma, x; A) = Q((\gamma, x); A) = \delta_{((1-\gamma), x)}(A)$, for all $\gamma \in \{0, 1\}$, $x \in \mathbb{R}$, and all $A \subset \mathbb{R}^2$. The vector field for the γ component can be considered to be 0. One should expect 0-consumption whenever $X = 0$ and $\gamma = 0$ i.e. $r_0(0) = 0$, and no production whenever $X = \alpha_{\max}$ (some maximum level) and $\gamma = 1$, i.e. $r_1(\alpha_{\max}) = 0$.

2.2.3 Cook's model

We now focus on Cook's model introduced in [CGT98] for stochastic gene expression and its implications on haploinsufficiency. This basic model of gene expression, product accumulation and product degradation can be given by the following reaction system:



This model considers a gene to switch randomly between inactive state (G) and active state (G*). The activation (respectively deactivation) rate is denoted by k_a (respectively k_d). When active, each gene expresses a product (P) at a rate J_p . The product is degraded at rate k_p . One can represent this model as a particular case of the On/Off system by considering

$$(43) \quad r_0(x) = k_p x, \quad r_1(x) = J_p - k_p x, \quad \lambda_0 = k_a, \quad \lambda_1 = k_d, \quad \text{and} \quad \alpha_{\max} = \frac{J_p}{k_p}.$$

2.2.4 Cook's model with several regimes

Although many variables may affect the product stability, this model collapses the production and degradation kinetics of both RNA and protein into a single product pool. The authors of [CGT98] distinguish several regimes (stable, slow unstable, fast unstable), controlling the activation rates. To take this into account, one should consider that k_a and/or k_d depend of an external control. The product P is assumed to act as its own expression stimulus such that expression is irreversibly terminated if its level falls below an autoactivation threshold (r). This leads to considering some

$$j_p(x) = \begin{cases} J_p, & \text{if } x \geq 2r; \\ 0, & \text{if } x \leq r. \end{cases}$$

For example, one can consider $j_p(x) = J_p \left[\left(\frac{1}{r}x - 1 \right)^+ \wedge 1 \right] = \begin{cases} J_p, & \text{if } x \geq 2r; \\ J_p \left(\frac{1}{r}x - 1 \right), & \text{if } x \in (r, 2r) . \\ 0, & \text{if } x \leq r. \end{cases}$

2.2.5 Bacteriophage λ

We consider the model introduced in [HPDC00] to describe the regulation of gene expression. The model is derived from the promoter region of bacteriophage λ . The simplification proposed by the authors of [HPDC00] consists in considering a mutant system in which only two operator sites (known as OR2 and OR3) are present. The gene cI expresses repressor (CI), which dimerizes and binds to the DNA as a transcription factor in one of the two available sites. The site OR2 leads to enhanced transcription, while OR3 represses transcription. Using the notations in [HPDC00], we let X stand for the repressor, X_2 for the dimer, D for the DNA promoter site, DX_2 for the binding to the OR2 site, DX_2^* for the binding to the OR3 site and DX_2X_2 for the binding to both sites. We also denote by P the RNA polymerase concentration and by n the number of proteins per mRNA transcript. The dimerization, binding, transcription and degradation reactions are summarized by

$$\begin{cases} 2X & \xrightleftharpoons{K_1} X_2, \\ D + X_2 & \xrightleftharpoons{K_2} DX_2, \\ D + X_2 & \xrightleftharpoons{K_3} DX_2^*, \\ DX_2 + X_2 & \xrightleftharpoons{K_4} DX_2X. \\ DX_2 + P & \xrightarrow{K_i} DX_2 + P + nX \\ X & \xrightarrow{K_d} . \end{cases}$$

To this biological system we associate a piecewise deterministic process on the state space $E = \{\nu \in \{0, 1\}^4 : \sum_{i=1}^4 \nu_i = 1\} \times \mathbb{R}^2$. The characteristic is given by

$$\begin{aligned} f_\nu(x_1, x_2) &= f(x_1, x_2) = (-2k_1x_1^2 - k_dx_1 + 2k_{-1}x_2, k_1x_1^2 - k_{-1}x_2), \\ f(\nu, x) &= (0, 0, 0, 0, f_\nu(x)), \\ \lambda(\nu, x) &= k_2x_2\chi(x_2)\nu_1 + k_3x_2\chi(x_2)\nu_1 + k_4x_2\chi(x_2)\nu_2 \\ &\quad + k_t\nu_2 + k_{-2}\nu_2 + k_{-3}\nu_3 + k_{-4}\nu_4, \\ \lambda(\nu, x)Q((\lambda, x); dz) &= k_2x_2\chi(x_2)\nu_1\delta_{(x_1, x_2-1, \nu_1-1, \nu_2+1, \nu_3, \nu_4)}(dz) \\ &\quad + k_3x_2\chi(x_2)\nu_1\delta_{(x_1, x_2-1, \nu_1-1, \nu_2, \nu_3+1, \nu_4)}(dz) \\ &\quad + k_4x_2\chi(x_2)\nu_2\delta_{(x_1, x_2-1, \nu_1, \nu_2-1, \nu_3, \nu_4+1)}(dz) \\ &\quad + k_t\nu_2\delta_{(x_1+n, x_2, \nu_1, \nu_2, \nu_3, \nu_4)}(dz) + k_{-2}\nu_2\delta_{(x_1, x_2+1, \nu_1+1, \nu_2-1, \nu_3, \nu_4)}(dz) \\ &\quad + k_{-3}\nu_3\delta_{(x_1, x_2+1, \nu_1+1, \nu_2, \nu_3-1, \nu_4)}(dz) + k_{-4}\nu_4\delta_{(x_1, x_2+1, \nu_1, \nu_2+1, \nu_3, \nu_4-1)}(dz), \end{aligned}$$

for every $(\nu, x) \in E$. The function χ is a smooth function such that $0 \leq \chi \leq 1$, $\chi(y) = 0$ for $y < 1$ and $\chi(y) = 1$ for $y \geq 1 + \text{err}$.

2.3 Mathematical contributions

2.3.1 Geometrical criteria for viability and invariance

One may reduce the complexity of PDMPs by restricting the model to some invariant set containing the initial data, whenever this is known. Compact invariant sets are also needed for efficiently implementing algorithms. Another important issue that can be approached using invariance are the stable points. In particular, a fixed point for which one finds arbitrarily small surrounding invariant sets is stable in the sense of Lyapunov.

We begin by characterizing near-viability of controlled PDMPs via some associated control problem. A closed set of constraints K is said to be viable (or near-viable) with respect to some dynamic control system if, starting from K , one is able to find suitable controls keeping the trajectory in K (or, at least in some arbitrarily small neighborhood of the set of constraints).

Viability properties have been extensively studied in both deterministic and stochastic settings (for Brownian diffusions), starting from the pioneer work of Nagumo. The methods used to describe this property for deterministic or diffusion processes rely either on the Bouligand-Severi contingent cone (cf. [Aub91], [ADP90], [GT93]) or on viscosity solutions ([AF90], [BCD97], [BJ02a], [BPQR98], [ZP08]). Using analytical tools from viscosity theory, we provide a geometrical characterization of near-viability and invariance of some set of constraints K with respect to the controlled piecewise deterministic Markov process. As for

the Brownian diffusion case, the criterion involves the normal cone to the set of constraints and is completely deterministic. Similar arguments allow to characterize the invariance of the set of constraints. We emphasize that these geometrical conditions can be rather easily checked for PDMPs associated to gene networks. The proof of the characterization relies on the viscosity solution concept. We begin by recalling the notion of normal cone.

Definition 30 *Let $K \subset \mathbb{R}^N$ be a closed subset and let x be a point of K . The normal cone to K at x , denoted by $N_K(x)$, is defined as*

$$N_K(x) = \{p \in \mathbb{R}^N : \forall \varepsilon > 0, \exists \eta > 0 \text{ such that } \forall y \in K \cap \overline{B}(x, \eta), \langle p, y - x \rangle \leq \varepsilon |y - x|\}.$$

We recall that $\overline{B}(x, \eta) = \{y \in \mathbb{R}^N : |y - x| \leq \eta\}$.

The first contribution is the following characterization of the near-viability property with respect to the controlled piecewise deterministic Markov process.

Theorem 31 ([G5], Theorem 2.6) *Given a nonempty, closed set $K \subset \mathbb{R}^N$, the following properties are equivalent:*

(i) *K is near-viable (i.e. for every initial point $x \in K$ and every $\varepsilon > 0$, there exists an admissible control process $u^\varepsilon \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$ such that*

$$\mathbb{E} \left[\int_0^\infty e^{-t} \left(d_K \left(X_t^{x, u^\varepsilon} \right) \wedge 1 \right) dt \right] \leq \varepsilon.)$$

(ii) *The following assertions hold simultaneously:*

(a) *for every $x \in \partial K$, and every $p \in N_K(x)$,*

$$\inf_{u \in U} \{ \langle f(x, u), p \rangle + \lambda(x, u) Q(x, u, K^c) \} \leq 0.$$

(b) *for every $x \in \overset{\circ}{K}$,*

$$\inf_{u \in U} \{ \lambda(x, u) Q(x, u, K^c) \} \leq 0.$$

In order to establish this theorem, one has to begin by proving comparison results between discontinuous subsolutions/supersolutions for a certain Hamilton-Jacobi integro-differential system (see Proposition 2.5 in [G5]). The second step consists in proving that the geometric conditions in (ii) are equivalent to $1 - 1_K$ being a viscosity supersolution for this system.

Similarly, one proves a criterion for invariance :

Theorem 32 ([G5], Theorem 2.8) *Let $K \subset \mathbb{R}^N$ be a nonempty, closed subset. The following statements are equivalent:*

(i) *The set K is invariant;*

(ii) *For every $x \in \partial K$, every $p \in N_K(x)$, and every $u \in U$,*

$$\langle f(x, u), p \rangle + \lambda(x, u) Q(x, u, K^c) \leq 0.$$

2.3.2 Linearization methods

We formulate linearized versions of continuous control problems for PDMPs. As for Brownian diffusions, the method is based on viscosity techniques and duality for some associated linearized problem. We embed the set of control processes into a set of probability measures via occupational measures. This set of constraints is explicitly given by a deterministic condition involving the coefficient functions. In the case of Lipschitz-continuous cost functionals, we provide primal and dual linear formulations for the value function. The primal value function is given with respect to the previously introduced set of constraints. Using a Hahn-Banach type argument, it is shown that, in general, this set coincides with the closed, convex hull of occupational measures (corollary 34). Under convexity assumptions, this set is the set of occupational measures.

To any $x \in \mathbb{R}^N$ and any $u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$, we associate the discounted occupational measures

$$(44) \quad \gamma_{x,u}(A) = \mathbb{E} \left[\int_0^\infty e^{-t} \mathbf{1}_A(X^{x,u}(t), u(t)) dt \right],$$

for all Borel subsets $A \subset \mathbb{R}^N \times U$. The set of all discounted occupational measures is denoted by $\Gamma(x)$. We let $\mathcal{P}(\mathbb{R}^N \times U)$ denote the set of all probability measures on $\mathbb{R}^N \times U$ and define

$$\Theta(x) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^N \times U) : \forall \phi \in C_b^1(\mathbb{R}^N) : \int_{\mathbb{R}^N \times U} (\mathcal{U}^u \phi(y) + \phi(x) - \phi(y)) \gamma(dy, du) = 0 \right\},$$

where

$$(45) \quad \mathcal{U}^u \phi(y) = \langle \nabla \phi(y), f(y, u) \rangle + \lambda(y, u) \int_{\mathbb{R}^N} (\phi(z) - \phi(y)) Q(y, u, dz),$$

for all $u \in U$, $\phi \in C_b^1(\mathbb{R}^N)$, and all $y \in \mathbb{R}^N$. The set $\Theta(x)$ will be referred to as the set of constraints.

Whenever $g : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded, Lipschitz continuous function, we let

$$v_g(x) = \inf_{u^1 \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{E} \left[\int_0^\infty e^{-t} g(X_t^{x, u^1, 0}) dt \right],$$

for all $x \in \mathbb{R}^N$. We introduce the primal linear formulation of our control problem with value function v_g

$$\inf_{\gamma \in \Theta(x)} \int_{\mathbb{R}^N \times U} g(y) \gamma(dy, du),$$

where the set of constraints $\Theta(x)$ captures all the occupational measures. We also associate the dual formulation

$$(46) \quad \mu^*(x) = \sup \left\{ \mu \in \mathbb{R} : \exists \varphi \in C_b^1(\mathbb{R}^N) \text{ such that } \forall (y, u) \in \mathbb{R}^N \times U, \right. \\ \left. \mu \leq \mathcal{U}^u \varphi(y) + g(y) + (\varphi(x) - \varphi(y)) \right\},$$

for all $x \in \mathbb{R}^N$. We recognize a Lagrange-type dual. Our main result links the classical value function v_g , its primal and dual formulations.

Theorem 33 ([G8], Theorem 3.1, [G4], Theorem 7) Under standard assumptions, for every $x \in \mathbb{R}^N$, the equality

$$(47) \quad v_g(x) = \inf_{\gamma \in \Theta(x)} \int_{\mathbb{R}^N \times U} g(y) \gamma(dy, du) = \mu^*(x)$$

holds true.

The definition of $\Theta(x)$ and $\mu^*(x)$ yield the inequalities

$$v_g(x) \geq \inf_{\gamma \in \Theta(x)} \int_{\mathbb{R}^N \times U} g(y) \gamma(dy, du) \geq \mu^*(x).$$

To prove that $\mu^*(x) \geq v_g(x)$, we adapt the so-called "shaking the coefficients method" introduced in [Kry00] and the stability results in [BJ02b]. This allows to approximate v_g by a sequence of regular subsolutions of the associated Hamilton-Jacobi integro-differential equation. The inequality follows from the definition of μ^* .

As a consequence, we get the following characterization of the set of constraints.

Corollary 34 ([G4], Corollary 8) We let

$$\begin{aligned} \Gamma_1(x) &= \{\gamma(dy, U) : \gamma \in \Gamma(x)\} \text{ and} \\ \Theta_1(x) &= \{\gamma(dy, U) : \gamma \in \Theta(x)\}. \end{aligned}$$

Under standard assumptions,

$$(48) \quad \Theta_1(x) = \overline{\text{co}}(\Gamma_1(x)).$$

The closure is taken w.r.t. the usual (weak) convergence of probability measures.

Sometimes, compactness of the set of constraints is needed. In this case, one should consider a large enough discount factor c_0 . To any $x \in \mathbb{R}^N$ and any $u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$, one associates the c_0 -discounted occupational measures

$$\gamma_{x,u}(A) = \mathbb{E} \left[\int_0^\infty e^{-c_0 t} 1_A(X^{x,u}(t), u(t)) dt \right],$$

for all Borel subsets $A \subset \mathbb{R}^N \times U$. The set of all discounted occupational measures is denoted by $\Gamma_{c_0}(x)$. Compactness is obtained by imposing, for instance,

$$(A5) \quad \sup_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{E} \left[\int_0^\infty e^{-c_0 t} |X_t^{x,u}| dt \right] < \infty, \text{ for every } x \in \mathbb{R}^N.$$

Remark 35 1) This condition is satisfied if, for some c , $\sup_{u \in U} \int_{\mathbb{R}^N} |y - x| Q(x, u, dy) \leq c$, for all $x \in \mathbb{R}^N$. This guarantees that the function $g(y, s, \omega) = |y - x_{s-}(\omega)|$ is (locally) integrable with respect to the point process associated to the PDMP (see section 26 of [Dav93]).

2) This kind of requirement is not surprising: the "uniform behavior" of

$$Q(x, u, \{y : |y - x| \geq n\}) \text{ as } n \rightarrow \infty$$

is already a key ingredient in order to compare discontinuous viscosity solutions (see [G5]).

3) In the case of biochemical reactions, there is only a finite family of jumps (each corresponding to one reaction). Moreover, they are independent of the pre-jump position. Thus, (A5) is satisfied.

We emphasize that (A5) is only a sufficient condition. As an alternative, one could consider positive Radon measures with mass at most 1 ($\mathcal{M}_1(\mathbb{R}^N \times U)$) and endow this space with the vague convergence topology.

If (A5) holds true, Prohorov's theorem (see [Bil99]) insures that $\Gamma_{c_0}(x)$ and $co(\Gamma_{c_0}(x))$ are relatively compact with respect to the topology of weak convergence of probability measures. Thus, $\overline{co}(\Gamma_{c_0}(x))$ is compact.

2.3.3 Reachability of open sets

Stability issues are very important for biological networks. For deterministic models, one can easily decide whether the system is stable, bistable, etc. However, the behavior is much less obvious for a piecewise deterministic approach. One should expect that the trajectories of the controlled PDMP starting from some region around the stable point converge to it. Alternatively, a point for which arbitrarily small surrounding regions are invariant (or at least viable) is a good candidate for stability. Thus, the issue of stability may be addressed via viability techniques. In the case of multiple stable points, given an arbitrary initial state, it would be interesting to know to which of these regions the trajectories of the PDMP are directed. One of the advantages of the linearization methods is that it allows to address the problem of reachability.

Let us consider an arbitrary nonempty, open set $\mathcal{O} \subset \mathbb{R}^N$.

Definition 36 *Given an initial condition $x \in \mathcal{O}^c$ (or even $x \in \mathbb{R}^N$), the set \mathcal{O} is reachable starting from x if there exists some admissible control process u such that the set*

$$\{X_t^{x,u} \in \mathcal{O}, t \in [0, \infty)\}$$

has positive probability.

In connection to this property, we define the value function

$$(49) \quad v_{d_{\mathcal{O}^c} \wedge 1}(x) = \inf_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{E} \left[\int_0^\infty -e^{-t} (d_{\mathcal{O}^c}(X_t^{x,u}) \wedge 1) dt \right],$$

for all $x \in \mathbb{R}^N$.

It is obvious that, whenever $v_{d_{\mathcal{O}^c} \wedge 1}(x) = 0$, the set \mathcal{O} is not reachable starting from the point x . On the other hand, whenever $v_{d_{\mathcal{O}^c} \wedge 1}(x) < 0$, there exist a constant $\delta > 0$, an admissible control process $u_0 \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$ and $T > 0$ such that $\mathbb{E} \left[\int_0^T e^{-t} (d_{\mathcal{O}^c}(X_t^{x,u_0}) \wedge 1) dt \right] > \delta$. It follows that the set $\{X_t^{x,u_0} \in \mathcal{O}, \text{ for some } t \in [0, T]\}$ must have positive probability. Thus, \mathcal{O} is reachable from x if and only if $v_{d_{\mathcal{O}^c} \wedge 1}(x) < 0$.

Theorem 33 yields the following criterion for reachability.

Criterion 37 ([G5], Criterion 3.8) *Let $x \in \mathbb{R}^N$ be an arbitrary initial state. Then the controlled piecewise deterministic Markov process starting from x reaches \mathcal{O} if and only if there exists $n \in \mathbb{N}^*$ such that for every $\varphi \in C_b^1(\mathbb{R}^N)$ there exists $u \in U, y \in \mathbb{R}^N$ such that*

$$(50) \quad \mathcal{U}^u \varphi(y) - d_{\mathcal{O}^c}(y) \wedge 1 + (\varphi(x) - \varphi(y)) < -n^{-1}.$$

One can reduce the dimension by considering a suitable finite family of test functions $(\varphi_m)_{0 \leq m \leq M}$ and introducing a certain precision of the test.

2.3.4 Zubov's approach to asymptotic stability

Given a target set K assumed to be compact, we seek to characterize the initial data x for which an (open loop) control process u can be found steering the process $X_t^{x,u}$ to K as $t \rightarrow \infty$. Whenever $K = \{0\}$, we have the so-called asymptotic null-controllability property. Zubov's method [Zub64] allows to compute Lyapunov functions and domains of attraction for ordinary differential equations with a locally asymptotically stable equilibrium x^* . Under appropriate conditions and for suitable functions $g : \mathbb{R}^N \rightarrow \mathbb{R}$, Zubov's equation has a unique differentiable solution $W : \mathbb{R}^N \rightarrow [0, 1]$ with $W(x^*) = 0$, which characterizes the domain of attraction \mathcal{D} of x^* via $\mathcal{D} = \{x \in \mathbb{R}^N \mid W(x) < 1\}$ and which is a Lyapunov function on \mathcal{D} .

In recent years, Zubov's method has been generalized in various directions: In [CGW01a] and [Mal05] it was extended to systems with time varying deterministic perturbation and in [CG03] to systems with Brownian perturbation. A first application to control systems, in which asymptotic stability is replaced by asymptotic controllability was given in [Son83], in which Zubov's equation was used in an integral form. The original differential version of Zubov's equation was investigated for control systems in [GW00], [Grü02, Section 7.2] and — under more general assumptions — in [CGW08]. A variant of Zubov's method for systems with both controls and Brownian perturbations was studied in [CCGW06]. Furthermore, [CGW01b] introduced a numerical method based on the Zubov equation which was originally developed for the deterministically perturbed setting from [CGW01a] but was subsequently successfully applied to other settings, see for instance [CGW04] for a controlled inverted pendulum or [GSB08] for a stochastic control problem from mathematical economy. An alternative numerical approach which is, however, so far only applicable to Zubov's original setting without perturbation or control has been presented in [Gie07].

When dealing with Zubov's equation, one immediately realizes that classical solutions can hardly be obtained. Since this equation and its generalizations are Hamilton-Jacobi equations, the concept of viscosity solutions turns out to be the right generalized solution concept, cf., e.g., [BCD97]. In [GS11] a further generalization of Zubov's equation to deterministic differential games is provided.

In the Brownian diffusion setting, an assumption of stability of control processes under concatenation is needed (cf. [CCGW06]). This assumption is naturally satisfied under convexity conditions. On the other hand, lack of convexity on the dynamics can be compensated by linearization techniques.

As $t \rightarrow \infty$, we would expect that $X_t^{x,u}$ be close to K with positive probability, for some u . Thus, for arbitrarily small $\alpha > 0$, $\sup_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{P}(d_K(X_t^{x,u}) < \alpha)$ should be larger than some (strictly) positive quantity. We introduce

Definition 38 *The domain of stabilizability for the controlled piecewise deterministic Markov system is the set*

$$\mathcal{D} = \left\{ x \in \mathbb{R}^N : \inf_{\alpha > 0} \liminf_{T \rightarrow \infty} \sup_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \frac{\int_T^\infty e^{-s} \mathbb{P}(d_K(X_s^{x,u}) < \alpha) ds}{\int_T^\infty e^{-s} ds} > 0. \right\}$$

We denote by $K_r = \{x \in \mathbb{R}^N : d_K(x) < r\}$, for $r > 0$. We let $x \in \mathbb{R}^N$ and u be an admissible (piecewise open-loop) control. We introduce the stopping time

$$\tau_{x,u} = \inf \{t \geq 0 : X_t^{x,u} \in K_r\}.$$

In the Brownian diffusion framework, with a further assumption of "stability under concatenation" for the control processes, the authors of [CCGW06] prove that \mathcal{D} coincides with the set of all x such that $\mathbb{P}(\tau_{x,u} < \infty) > 0$ for some admissible control process u . To characterize the domain of stability, it is standard to assume the so-called local stability: every point x of some local region $\overline{K_r}$ can be steered into K (using some piecewise open loop control u^x). Usually, one asks for locally exponential (almost sure) stability.

If (x, u) are such that $\mathbb{P}(\tau_{x,u} < \infty) > 0$, one would like to extend the control process u by setting something like

$$(51) \quad v(s) = 1_{s \leq \tau_{x,u}} u(s) + 1_{s > \tau_{x,u}, \tau_{x,u} < \infty} u^{X_{\tau_{x,u}}^{x,u}}(s - \tau_{x,u})$$

(i.e. u before $X^{x,u}$ reaches K_r and a stabilizing control after this stopping time). However, we are facing two difficulties : the main difficulty is related to the fact that v should be an admissible (piecewise open-loop) control; the second one concerns stability under measurable selection issues. To deal with the first problem, we do not directly consider concatenated controls, but constraint measures.

Concatenating (occupational) measures

If $\mathbb{P}(\tau_{x,u} < \infty) > 0$ for some u , we introduce the finite stopping times

$$\tau_{x,u,n} = \tau_{x,u} \wedge n, \text{ for all } n \geq 1.$$

Then, for some n (great enough), $\mathbb{P}(\tau_{x,u,n} < n) > \theta > 0$. Inspired by the results for optimal stopping of controlled diffusions, we associate two (positive Radon generalized) occupational measures:

$$\begin{aligned} & \bar{\gamma}_{z,u} \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{R}^N \times U), \bar{\gamma}_{z,u}(A) = \mathbb{E} \left[\int_0^\infty e^{-t} 1_A(t, X_t^{z,u}, u_t) dt \right], z \in \mathbb{R}^N; \\ & \pi_{x,u,n} \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^N \times U), \bar{\pi}_{x,u,n} \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{R}^N) \\ & \begin{cases} \pi_{x,u,n}(A) = \mathbb{E} \left[\int_0^{\tau_{x,u,n}} 1_A(t, X_t^{x,u}, u_t) dt \right], \text{ if } \mathbb{P}(\tau_{x,u,n} > 0) > 0, \\ \pi_{x,u,n} = \delta_{0,x,u_0}, \text{ otherwise.} \end{cases} \\ & \bar{\pi}_{x,u,n}(B \times C) = \mathbb{E} \left[1_{\tau_{x,u,n} \in B} 1_C \left(X_{\tau_{x,u,n}}^{x,u} \right) \right], \end{aligned}$$

for all Borel sets $A \subset \mathbb{R}_+ \times \mathbb{R}^N \times U$, $B \subset \mathbb{R}_+$ and $C \subset \mathbb{R}^N$. Here, $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^N \times U)$ stands for the set of positive Radon measures. Obviously,

$$0 < \pi_{x,u,n}(\mathbb{R}_+ \times \mathbb{R}^N \times U) \leq n.$$

If $s' > 0$, we consider the translated measure $\bar{\gamma}_{s',z,u} \in \mathcal{P}([s', \infty) \times \mathbb{R}^N \times U)$ associated to $X^{s',z,u}$ defined by

$$\bar{\gamma}_{s',z,u}(A) = \mathbb{E} \left[\int_0^\infty e^{-t} 1_A(t + s', X_t^{z,u}, u_t) dt \right],$$

for all borelian set $A \subset [s', \infty) \times \mathbb{R}^N \times U$.

Remark 39 For every $(z, u) \in \mathbb{R}^N \times \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$, the marginal measure $\bar{\gamma}_{z,u}(\mathbb{R}_+, dy, du) \in \Theta(z)$.

The occupational measure formulation of (51) would be somewhat like

$$\begin{aligned} \bar{\gamma}(dsdydu) &= e^{-s} \pi_{x,u,n}(dsdydu) \\ &+ \int_{\mathbb{R}_+ \times \mathbb{R}^N} e^{-s'} \left(\begin{array}{c} 1_{\bar{K}_r^c}(z) \bar{\gamma}_{s',z,u}(dsdydu) \\ + 1_{\bar{K}_r}(z) \bar{\gamma}_{s',z,u^z}(dsdydu) \end{array} \right) \bar{\pi}_{x,u,n}(ds', dz). \end{aligned}$$

This measure has no reason to be in the set of occupational measures (i.e., to be associated to a control process). However, one proves that it belongs to

$$\bar{\Theta}(x) = \left\{ \begin{array}{c} \bar{\gamma} \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{R}^N \times U) : \forall \phi \in C_b^1(\mathbb{R}^n) : \\ \int_{\mathbb{R}_+ \times \mathbb{R}^N \times U} (\mathcal{U}^u \phi(y) + \phi(x) - \phi(y)) \bar{\gamma}(dt, dy, du) = 0 \end{array} \right\}.$$

It is now obvious what the second difficulty would be: the application $(s', z) \mapsto \bar{\gamma}_{s',z,u^z}$ should be measurable. This can be overcome by appropriate approximation (cf. [G4], Section 4.1.2).

The main result

We consider the following value function

$$V(x) = \inf_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{E} \left[\int_0^\infty e^{-s} 1_{K_r^c}(X_s^{x,u}) ds \right] = \inf_{\bar{\gamma} \in \bar{\Theta}(x)} \int_{\mathbb{R}_+ \times \mathbb{R}^N \times U} 1_{K_r^c}(y) \bar{\gamma}(dsdydu).$$

(The last equality follows from the u.s.c. of $1_{K_r^c}$ and the Corollary 34.)

To this value function, we associate the Hamilton-Jacobi integro-differential equation

$$(52) \quad V(x) - 1_{K_r^c}(x) + H(x, \nabla V(x), V) = 0,$$

for all $x \in \mathbb{R}^N$, where the Hamiltonian is given by (11). The main result of the the paper [G4] is the following :

Theorem 40 ([G4], Theorem 18) *The value function V is the largest upper semicontinuous viscosity subsolution of (52). Moreover, under classical locally exponential (almost sure) stability assumptions,*

$$\mathcal{D} = \left\{ x \in \mathbb{R}^N : V(x) < 1 \right\} = \left\{ x : \sup_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{P}(\tau_{x,u} < \infty) > 0 \right\}.$$

In fact, our result holds true under less restrictive assumptions. For reader's sake, we have chosen to state it under this classical form. The main ingredients for its proof rely on LP methods, the concatenation of measures and sup-convolutions.

The dual formulation of continuous control problems and the approximating problems in the proof of our previous result lead to the following.

Criterion 41 ([G4], Criterion 20) *An initial datum x belongs to \mathcal{D} if and only if there exists $n \geq 1$ such that, for every $\varphi \in C_b^1(\mathbb{R}^N)$, there exist $(y, u) \in \mathbb{R}^N \times U$ satisfying*

$$\mathcal{U}^u \varphi(y) + h_n(y) + (\varphi(x) - \varphi(y)) < 1,$$

where h_n are given by sup-convolutions of $1_{K_r^c}$ (i.e.

$$h_n(x) := \sup_{y \in \mathbb{R}^N} (1_{K_r^c}(y) - n|y - x|), \text{ for all } x \in \mathbb{R}^N \text{ and all } n \geq 1.)$$

As for the reachability criterion, one reduces the dimension by considering a suitable finite family of test functions $(\varphi_m)_{0 \leq m \leq M}$ and introducing a certain precision of the test.

2.4 Applications

2.4.1 Cook's model for haploinsufficiency

Computing the first order normal cone to $K = [0, \alpha_{\max}] \times \{0, 1\}$, we establish the following result.

Proposition 42 ([G5], Proposition 4.1) *The set $K = [0, \alpha_{\max}] \times \{0, 1\}$ is invariant with respect to the PDMP associated to the On/Off Model.*

Moreover, using similar arguments, one shows that $[a, b] \times \{0, 1\}$ is invariant if and only if

$$r_0(a) = r_1(b) = 0.$$

Therefore, if a point x_0 is stable, then one is able to find a sequence $\varepsilon \searrow 0$ such that $r_0(x_0 - \varepsilon) = r_1(x_0 + \varepsilon) = 0$. In particular, a necessary condition is that $r_0(x_0) = r_1(x_0) = 0$.

Second, using the Criterion 37, we get, for Cook's model, the following result.

Proposition 43 ([G5], Proposition 4.3) *For every real constants a, b such that $0 < a < b < \alpha_{\max}$, we let $\mathcal{O} = (a, b) \times \{0, 1\}$. Then, for every $x \in (0, \alpha_{\max})$, the set \mathcal{O} is reachable with respect to the PDMP associated with Cook's model starting from $(0, x)$.*

The proof is straightforward, but it relies on fastidious computations (see the proof of [G5], Proposition 4.3).

We now illustrate (Figure 1) the viability result in Proposition 42 and the reachability properties. We use the classical description of the PDMP associated with Cook's model. The invariant set is represented in green ($[0, \alpha_{\max}]$) and we simulate a trajectory starting from a randomly chosen initial value for the protein. The time horizon is chosen very small (100) and the trajectory is represented in red. The reachable set is given by randomly generated $a, b \in (0, \alpha_{\max})$ and is represented by the blue border lines. Whenever the sample remains in the target set for two consecutive time steps, the trajectory is represented in blue.

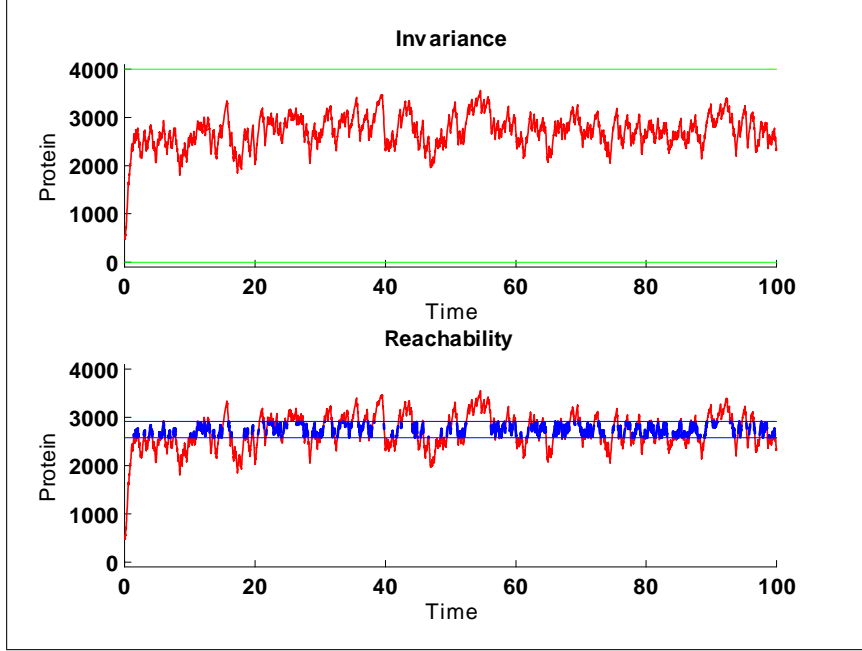


Figure 1. Invariance and reachability properties for Cook's model

2.4.2 Cook's model with several regimes

We apply Zubov's method to characterize the asymptotic null-controllability domain in Cook's model. We consider the target set $K = \{0, 1\} \times \{0\}$. The purpose is to characterize the set of initial states (γ_0, x_0) for which there exists a control such that, starting from (γ_0, x_0) , the system reaches K with positive probability (domain of null-controllability \mathcal{D}).

We notice that, for every $x \leq r$, $f(\gamma, x, u) = -k_p x$. Whenever $(\gamma_0, x_0) \in K_r = \{0, 1\} \times [0, r)$, for every control process u ,

$$(\gamma^{(\gamma_0, x_0), u}(t), X^{(\gamma_0, x_0), u}(t)) \in \{0, 1\} \times [0, e^{-k_p t} r),$$

for all $t \geq 0$. This is the exponential stability property.

As we have already seen, the controllability domain

$$\mathcal{D} = \left\{ (\gamma, x) : \sup_{u \in L^0(\mathbb{R}^{N+1} \times \mathbb{R}_+; U)} \mathbb{P}(\tau_{(\gamma, x), u} < \infty) > 0 \right\}$$

and it is also characterized by the criterion 41. To simplify the argument, let us assume that the activation/ deactivation rates are constant. This is not a restriction; for the controlled case, see remark 44. In this case, $\tau_{(\gamma, x), u}$ does not depend on u and will be denoted $\tau_{(\gamma, x)}$.

We notice that on $\{T_1 > t\}$, $X^{(0, x_0)}(t) = e^{-k_p t} x_0$. We have

$$\mathbb{P}(\tau_{(0, x_0)} < \infty) \geq \mathbb{P}\left(X^{(0, x_0)}(t) < \frac{r}{2}, T_1 > t\right) \geq \mathbb{P}\left(T_1 > \frac{1}{k_p} \ln \frac{2x_0}{r}\right) = \exp\left(-\int_0^{\frac{1}{k_p} \ln \frac{2x_0}{r}} k_a ds\right).$$

Here, $\tau_{(\gamma,x)} = \inf \left\{ t \geq 0 : X_t^{(\gamma,x)} < r \right\}$. Thus,

$$\mathbb{P} \left(\tau_{(0,x_0)} < \infty \right) \geq e^{-\frac{k_d}{k_p} \ln \frac{2x_0}{r}}.$$

On the other hand, if $x_0 \in [r, \alpha_{\max}]$, $X^{(1,x_0)}(t) \leq \alpha_{\max}$ for $t \leq T_1$. On $[T_1, T_2]$, $X^{(1,x_0)}(t) \leq \alpha_{\max} e^{-k_p(t-T_1)}$. We have

$$\mathbb{P} \left(\tau_{(1,x_0)} < \infty \right) \geq \mathbb{P} \left(S_2 > \frac{1}{k_p} \ln \frac{2\alpha_{\max}}{r} \right) \geq e^{-\frac{k_d}{k_p} \ln \frac{2\alpha_{\max}}{r}}.$$

Thus, in general, we have $\mathbb{P} \left(\tau_{(\gamma_0,x_0)} < \infty \right) \geq e^{-\frac{1}{k_p} \ln \frac{2\alpha_{\max}}{r} (k_d+k_a)} > 0$. It follows that $\{0, 1\} \times [0, \alpha_{\max}] \subset \mathcal{D}$.

Remark 44 1. *If the activation and deactivation rates depend on some control term, similar arguments yield*

$$\sup_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{P} \left(\tau_{x,u} < \infty \right) \geq e^{-\frac{1}{k_p} \ln \frac{2\alpha_{\max}}{r} \inf_{u \in U} (k_d(u)+k_a(u))} > 0$$

with similar conclusion.

2. *This means that, in the so-called "unstable" case (positive activation/ deactivation), for all initial protein concentration, the threshold level can be reached with positive probability. Thus, expression is irreversibly terminated with positive probability. These probabilities will obviously depend on the initial protein concentration.*

This is the simplest method to show that all concentrations of interest belong to the asymptotic null-controllability domain. An argument based on the criterion 41 can also be developed. The method would be similar to the reachability result in ([G5], Proposition 4.3). For more complex systems, one is generally not able to provide estimates for hitting time probabilities. In the general cases, the best approach is to numerically compute the value function V solution of (52).

2.4.3 Hasty's model for bacteriophage λ

Again, by computing first order normal cones, we establish the following invariance result.

Proposition 45 ([G5], Proposition 4.5) *For every $\frac{k_d^2}{4k_1k_{-1}} \wedge 1 > \varepsilon > 0$, the set*

$$K_\varepsilon = \{(1, 0, 0, 0)\} \times \left[0, \frac{2k_{-1}}{k_d} \varepsilon \right] \times [0, \varepsilon]$$

is invariant with respect to the PDMP associated to the bacteriophage λ model.

We illustrate the invariance result from the previous proposition. The reader is invited to notice that, in the setting of the previous assertion, the trajectory should be purely deterministic ($\lambda(1, 0, 0, 0, x_1, x_2) = 0$, for every $(x_1, x_2) \in \left[0, \frac{2k_{-1}}{k_d} \varepsilon \right] \times [0, \varepsilon]$). We randomly

simulate the parameter $\varepsilon < \frac{k_d^2}{4k_1k_{-1}} \wedge 1$ and a starting point $(x_1, x_2) \in \left[0, \frac{2k_{-1}}{k_d}\varepsilon\right] \times [0, \varepsilon]$. The simulated trajectory is represented in red and the bounds $\frac{2k_{-1}}{k_d}\varepsilon$ and ε are given in green.

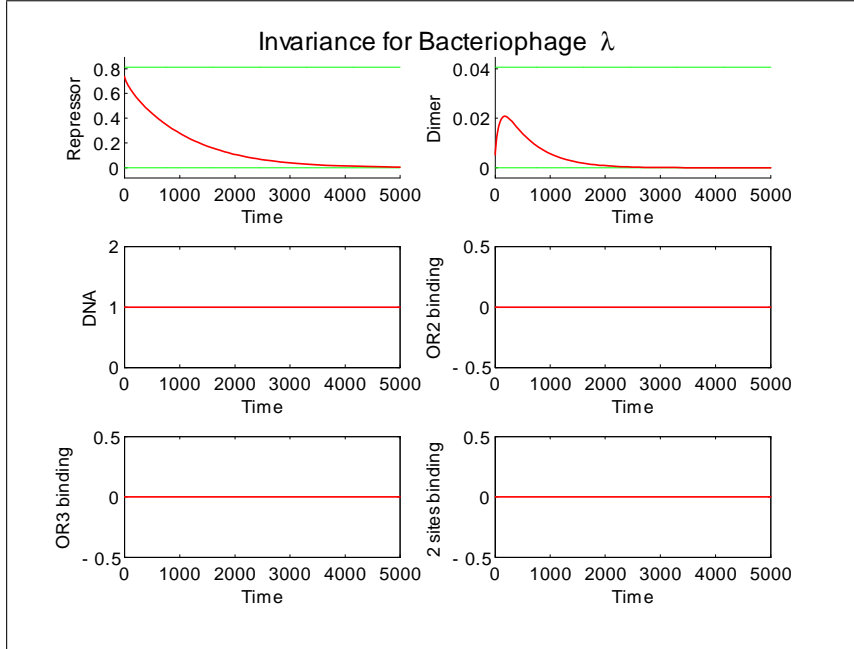


Figure 2. Invariance properties for Bacteriophage λ

We emphasize that similar arguments can be used to infer that no other point has similar stability properties. However, different invariant sets may exist. Thus, bistability of bacteriophage λ should be understood as: a stable state $(0, 0, 1, 0, 0, 0)$ and some stability (invariance) region.

3 Controllability properties of linear stochastic systems and related topics

3.1 Introduction

For Brownian diffusions with a finite-dimensional state space, exact controllability has been characterized in [Pen94] (see also [LP02], [LP10]). Exact (terminal-) controllability requires a full rank operator acting on the control in the noise term (cf. [Pen94]). The method relies on the theory of backward stochastic differential equations (cf. [PP90]). Approximate controllability for the Brownian setting has been studied in [BQT06] (with no control on the noise) and [G15, G16] (general setting). The authors generalize the Kalman condition to some equivalent criterion for approximate controllability using duality techniques. In infinite dimension, a similar approach leads to partial results (cf. [G14]). Various other methods can be employed in Hilbert spaces (cf. [FCGAR99], [ST01], [BRT03]).

In the finite-dimensional framework ([BQT06], [G15, G16]), the stochastic observability criterion uses (in an essential way) viability properties for the dual equation associated to the control system. In these papers, one gets equivalence between approximate controllability and null-approximate controllability. Moreover, an invariance criterion is given and it relies on backward techniques and viability properties.

To understand what happens for systems with infinite-dimensional components, we have followed two possible leads. The first approach consists in investigating stochastic control systems with finite dimensional state space, but an infinite-dimensional component. In this direction, we consider :

- the approximate controllability property for a class of linear stochastic equations driven by independent Brownian motion and Poisson random measure (in the paper [G7]) ;
- the approximate (null-) controllability property for linear systems of mean-field type (in [G20]).

In the first case, as for Brownian diffusions, approximate controllability and approximate null-controllability are proven equivalent. A conditional invariance criterion is given. In general, this (explicit) criterion involves an \mathbb{L}^2 -space, but, for particular cases, an iterative finite scheme is provided.

For the second class of systems, we prove that approximate and approximate null-controllability are not equivalent in all generality. Using Ricatti equations, we study the equivalent criteria for approximate null-controllability. We provide sufficient invariance conditions and study further conditions for equivalence. We also give sufficient assumptions under which approximate null-controllability implies approximate controllability.

The second approach consists in the study of the viability property for stochastic semi-linear control systems on a real, separable Hilbert space. This is covered by the paper [G11].

Although the two approaches are different, the common motivation justifies their presence together in this section.

3.2 Controllability of jump-diffusions with linear coefficients

In the paper [G7], we investigate the approximate controllability property for a class of linear stochastic equations driven by independent Brownian motion and Poisson random measure. The paper generalizes the results in [BQT06], [G15], [G16]. An equivalent conditional invariance criterion is given. In general, this (explicit) criterion involves an \mathbb{L}^2 -space, but, for

particular cases, an iterative finite scheme is provided.

We consider a complete probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. The filtration $(\mathcal{F}_t)_{t \geq 0}$ is assumed to be generated by two mutually independent stochastic processes: a standard 1-dimensional Brownian motion W and a Poisson random measure μ on $\mathbb{R}_+ \times (E = \mathbb{R} \setminus \{0\}, \mathcal{B}(E))$. Here, $\mathcal{B}(E)$ stands for the Borel σ -algebra on E . For some σ -finite Lévy measure on $(E, \mathcal{B}(E))$ denoted by λ such that $\int_E (1 \wedge |e|^2) \lambda(de) < \infty$, the compensator of μ is assumed to be

$$\widehat{\mu}(dtde) = dt\lambda(de).$$

As usual, we let $\widetilde{\mu} = \mu - \widehat{\mu}$ be the compensated random measure associated to μ .

We deal with a linear stochastic differential equation of the form

$$(53) \quad \begin{cases} dX_s^{x,u} = (AX_s^{x,u} + Bu_s) ds + CX_s^{x,u} dW_s + \int_E D(e) X_s^{x,u} \widetilde{\mu}(dsde) \\ X_0^{x,u} = x \in \mathbb{R}^n. \end{cases}$$

Here, $A, C \in \mathbb{R}^{n \times n}$, $D : E \rightarrow \mathbb{R}^{n \times n}$ is $\mathcal{B}(E)$ -measurable and $B \in \mathbb{R}^{n \times d}$. Under the assumption

$$(54) \quad \int_E |D(e)|^2 \lambda(de) < \infty,$$

for every admissible control process $u \in \mathcal{U}$ (i.e. (\mathcal{F}_t) -progressively measurable, \mathbb{R}^d -valued process such that $\mathbb{E} \left[\int_0^T |u_s|^2 ds \right] < \infty$, for all $T > 0$), the system (53) admits a unique adapted, càdlàg solution denoted $X^{x,u}$.

We emphasize that exact controllability cannot be obtained with finite dimensional controls. This is a simple consequence of the incompleteness of the jump-diffusion market (cf. [Mer76]).

We recall the notions of approximate and approximate null-controllability.

Definition 46 *The stochastic linear system (53) is called approximately controllable in time $T > 0$ if, for every $x \in \mathbb{R}^N$, every $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N)$ and every $\varepsilon > 0$, there exists some admissible control process u such that $\mathbb{E} \left[|X_T^{x,u} - \xi|^2 \right] \leq \varepsilon$. The stochastic linear system (53) is called approximately null-controllable in time T if the previous condition holds true for $\xi = 0$.*

Duality

We consider the (dual) control system

$$(55) \quad \begin{cases} dY_t^{y,z,v} = (-A^* Y_t^{y,z,v} - C^* z_t - \int_E D^*(e) v_t(e) \lambda(de)) dt + z_t dW_t + \int_E v_t(e) \widetilde{\mu}(dtde), \\ Y_0^{y,z,v} = y \in \mathbb{R}^n. \end{cases}$$

The process z is progressively measurable, and v is $\mathcal{P} \otimes \mathcal{B}(E)$ -measurable such that, for each $T > 0$,

$$\mathbb{E} \left[\int_0^T |z_t|^2 dt + \int_0^T \int_E |v_t(e)|^2 \lambda(de) dt \right] < \infty.$$

Here, \mathcal{P} stands for the predictable field. Such a couple (z, v) is called admissible control couple. The duality between the systems (53) and (55) leads to

Proposition 47 ([G7], Corollary 2.1)

1. The system (53) is approximately controllable if and only if $B^*Y^{y,z,v} = 0$ implies $Y^{y,z,v} = 0$ (or, equivalently, $z = 0$, $v_t(e) = 0$, $\lambda(de) dt - a.s.$ and $y = 0$).

2. The system (53) is approximately null-controllable if and only if $B^*Y^{y,z,v} = 0$ implies $y = 0$.

Viability

The following notions of viability and viability kernel play an important part in deriving explicit algebraic criteria for approximate controllability.

Definition 48 The viability kernel of a linear subspace $S \subset \mathbb{R}^n$ w.r.t. (55) (denoted by $Viab(S)$) is the family of all initial data $y \in S$ for which one is able to find $T > 0$ and an admissible control couple (z, v) such that $Y_s^{y,z,v} \in S$, ds -a.s. on $[0, T]$. If the viability kernel coincides with S , then S is said to be viable w.r.t. (55).

The notions of viability and viability kernel should read "locally in time", but, for simplicity, we drop this terminology.

The main result of [G7] is the following

Theorem 49 ([G7], Theorem 2.2) The system (53) is approximately controllable if and only if it is approximately null-controllable. Moreover, the necessary and sufficient condition for approximate controllability is that

$$(56) \quad Viab(Ker(B^*)) = \{0\}.$$

The proof of this result relies on Proposition 47. Further ingredients are linear quadratic optimal control problems and the associated Riccati equations.

Algebraic criteria and algorithms

To characterize viable sets with respect to (53), we introduce the linear, bounded operator $\mathcal{D}^* : \mathbb{L}^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ defined by

$$\mathcal{D}^*(\alpha) = \int_E D^*(e) \alpha(e) \lambda(de),$$

for all $\alpha \in \mathbb{L}^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^n)$.

We get the following invariance characterization of approximate controllability :

Criterion 50 ([G7], Proposition 2.3 and Criterion 2.1)

A necessary and sufficient condition for approximate (and null-) controllability of (53) is that $V = \{0\}$ should be the largest linear subspace of $Ker(B^*)$ satisfying

$$A^*V \subset V + C^*V + \mathcal{D}^*(\mathbb{L}^2(E, \mathcal{B}(E), \lambda; V)).$$

The previous inclusion condition is strongly related to the notion of $(A^*; C^*, \mathcal{D}^*)$ -invariance. For further details on the subject, the reader is referred to [SS80], [Cur86] and references therein.

If the jump Lévy measure λ has finite support $E' = \{e_i : i = 1, m\}$ (for some $m \geq 1$), then the corresponding system (53) is approximately controllable if and only if $V = \{0\}$ is the largest linear subspace of $\text{Ker} B^*$ satisfying

$$A^*V \subset V + C^*V + \sum_{i=1}^m D_i^*V.$$

Here, $D_i = D(e_i)$. V is computed via the usual iterative scheme:

$$\left\{ \begin{array}{l} \text{Set } V_0 := \text{Ker} B^* (\subset \mathbb{R}^n); \quad k := 0; \\ \text{repeat} \\ \quad k := k + 1; \quad V_k := (A^*)^{-1} (V_{k-1} + C^*V_{k-1} + \sum_{i=1}^m D_i^*V_{k-1}) \cap V_{k-1}; \\ \quad \text{while } k \leq n \text{ and } V_k \subsetneq V_{k-1} \\ \text{return } V := V_k; \end{array} \right.$$

3.3 Controllability of linear systems of mean-field type

In the paper [G20], we investigate controllability properties for a class of linear stochastic equations of mean-field type in a Brownian setting. We give necessary and sufficient criteria for the exact terminal controllability. We provide examples showing that approximate and approximate null-controllability are not equivalent. Using the Riccati equation approach to linear quadratic control problems, we characterize approximate null-controllability. This property is shown to be equivalent to the fact that a (conditional) viability kernel is trivially reduced to $\{0\}$. We provide invariance criteria and sufficient conditions under which approximate null-controllability implies approximate controllability.

We consider a 1-dimensional Brownian motion W defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by \mathbb{F} the natural filtration generated by W and augmented the \mathbb{P} -null sets. We will be dealing with the following controlled mean-field linear system

$$(57) \quad \left\{ \begin{array}{l} dX_t^{x,u} = (A_1 X_t^{x,u} + A_2 \mathbb{E}[X_t^{x,u}] + B_1 u_t + B_2 \mathbb{E}[u_t]) dt \\ \quad + (C_1 X_t^{x,u} + C_2 \mathbb{E}[X_t^{x,u}] + D_1 u_t + D_2 \mathbb{E}[u_t]) dW_t, \quad t \geq 0, \\ X_0^{x,u} = x \in \mathbb{R}^N. \end{array} \right.$$

where $A_1, A_2, C_1, C_2 \in \mathbb{R}^{N \times N}$, $B_1, B_2, D_1, D_2 \in \mathbb{R}^{N \times d}$, $N, d \in \mathbb{N}^*$. The control process u is called admissible if it is \mathbb{F} -progressively measurable, \mathbb{R}^d -valued and such that $\mathbb{E} \left[\int_0^T |u_s|^2 ds \right] < \infty$, for all $T > 0$. The set of admissible processes is denoted by \mathcal{U} .

Exact terminal controllability

We recall the notion of exact terminal controllability.

Definition 51 *The stochastic linear system (57) is called (exactly terminal) controllable in time $T > 0$ if, for every $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N)$, there exists some admissible control process u and some $x \in \mathbb{R}^N$ such that $X_T^{x,u} = \xi$, \mathbb{P} -a.s.*

In the case of classical stochastic differential systems driven by Brownian motion, the necessary and sufficient condition for exact controllability is that $\text{Rank}(D_1) = N$. This condition is no longer sufficient. In fact, we have

Proposition 52 ([G20], Proposition 2) *A necessary condition in order for (57) to be exactly terminal controllable is*

$$\text{Rank}(D_1 + D_2) = N.$$

If one takes $D_2 = -D_1$, the necessary condition given before is not satisfied (even if D_1 is chosen to be of rank N). To insure exact controllability, one should combine the previous necessary condition with the classical condition $\text{Rank}(D_1) = N$.

Proposition 53 ([G20], Proposition 5) *If $\text{Rank}(D_1) = N$ and $\text{Rank}(D_1 + D_2) = N$, then the system (57) is exactly terminal controllable.*

Approximate controllability vs. approximate null-controllability

Second, we turn our attention to weaker controllability conditions when the control does not act on the noise term. In the remaining of the subsection, we assume that $D_1 = D_2 = 0$. This assumption is made in order to simplify the presentation. Extending these results to general dynamics requires the "conditional invariance" notions from [G15]. The notions of approximate and approximate null-controllability are the same as those in Definition 46.

Example 54 *We consider the state space and the control space to be \mathbb{R}^2 and we take $A_1 = A_2 = C_1 = C_2 = 0$. We also take $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The system (57) writes*

$$dX_T^{x^1, x^2, u^1, u^2} = \begin{pmatrix} x_1 + \int_0^T u_t^1 dt \\ x_2 + \int_0^T \mathbb{E}[u_t^2] dt \end{pmatrix}, \quad x^1, x^2 \in \mathbb{R}.$$

One easily gets

$$\mathbb{E} \left[\left(W_T - \left(x_2 + \int_0^T \mathbb{E}[u_t^2] dt \right) \right)^2 \right] \geq T.$$

Therefore, independently of the initial condition, one is not able to reach arbitrarily small neighborhoods of $\xi := \begin{pmatrix} 0 \\ W_T \end{pmatrix}$. However, it is clear that the system is (exactly) null-controllable.

Dual characterization

In connection to (57), we consider the dual control system

$$(58) \quad \begin{cases} d \begin{pmatrix} y_t^{y,v} \\ Y_t^v \end{pmatrix} = \begin{pmatrix} -(A_1^* + A_2^*) y_t^{y,v} \\ -A_1^* Y_t^v \end{pmatrix} + \begin{pmatrix} -(C_1^* + C_2^*) \mathbb{E}[v_t] \\ -C_1^* (v_t - \mathbb{E}[v_t]) \end{pmatrix} dt + \begin{pmatrix} 0 \\ v_t \end{pmatrix} dW_t, \\ \text{for } 0 \leq t \leq T, \\ \begin{pmatrix} y_0^{y,v} \\ Y_0^v \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \in \mathbb{R}^{2N}. \end{cases}$$

The control processes v are \mathbb{R}^N -valued. The notion of admissibility is similar to the one employed for \mathcal{U} . The duality theorem ([G20], Theorem 7) can be written as follows.

Proposition 55 ([G20], Proposition 8) *The system (57) is approximately null-controllable if and only if, whenever*

$$\begin{pmatrix} y_t^{y,v} \\ Y_t^v \end{pmatrix} \in \text{Ker} \begin{pmatrix} (B_1 + B_2)^* & 0 \\ 0 & B_1^* \end{pmatrix}, \mathbb{P}\text{-a.s.}, \text{ for all } t \in [0, T],$$

one has $y = 0$.

It is approximately controllable if and only if, whenever $\begin{pmatrix} y_t^{y,v} \\ Y_t^v \end{pmatrix} \in \text{Ker} \begin{pmatrix} (B_1 + B_2)^* & 0 \\ 0 & B_1^* \end{pmatrix}$, $\mathbb{P}\text{-a.s.}$, for all $t \in [0, T]$ one has $Y_t^v + y_t^{y,v} = 0$, $\mathbb{P}\text{-a.s.}$, for all $t \in [0, T]$.

The technique of proof relies mainly on duality aspects and Mean-Field Backward Stochastic Differential Equations (introduced in [BDLP09], [BLP09]).

Conditional viability

The following notions of viability and viability kernel allow to obtain explicit algebraic criteria for approximate null-controllability.

Definition 56 *We let $\mathcal{S}, \mathcal{S}' \subset \mathbb{R}^N$ be two linear subspaces. The viability kernel of $\mathcal{S} \subset \mathbb{R}^N$ conditionally to \mathcal{S}' with respect to (58) is the family of all initial data $y \in \mathcal{S}$ such that, for every $T > 0$, one finds an admissible control v satisfying $\bar{Y}_s^{y,v} \left(:= \begin{pmatrix} y_s^{y,v} \\ Y_s^v \end{pmatrix} \right) \in \mathcal{S} \times \mathcal{S}'$, $\mathbb{P}\text{-a.s.}$, for all $s \in [0, T]$. I*

f the viability kernel coincides with \mathcal{S} , then \mathcal{S} is said to be viable conditionally to \mathcal{S}' w.r.t. (58).

The conditional viability kernel is characterized as follows :

Proposition 57 ([G20], Proposition 10) *The viability kernel of $\text{Ker}((B_1 + B_2)^*)$ conditionally to $\text{Ker}(B_1^*)$ with respect to (58) is given by*

$$\begin{aligned} & \text{Viab}(\text{Ker}((B_1 + B_2)^*) / \text{Ker}(B_1^*)) \\ & = \{y \in \text{Ker}((B_1 + B_2)^*) : \text{for all } T > 0, \langle (Q_n(T)y, y) \rangle_n \text{ is bounded} \}. \end{aligned}$$

Here, Q_n is the unique solution of the Ricatti-type system

$$(59) \quad \begin{cases} \dot{P}_n = -P_n A_1^* - A_1 P_n - P_n C_1^* (I + P_n)^{-1} C_1 P_n + n \Pi_{\text{Ker}(B_1^*)^\perp}, \\ \dot{Q}_n = -Q_n (A_1 + A_2)^* - (A_1 + A_2) Q_n - Q_n (C_1 + C_2)^* (I + P_n)^{-1} (C_1 + C_2) Q_n \\ \quad + n \Pi_{\text{Ker}((B_1 + B_2)^*)^\perp} \\ P_n(0) = 0, \quad Q_n(0) = 0, \end{cases}$$

where $\text{Ker}(B^*)^\perp$ denotes the orthogonal of $\text{Ker}(B^*)$ and $\Pi_{\text{Ker}(B^*)^\perp}$ is the usual orthogonal projection. The existence of a unique solution to the previous equation is guaranteed, for example, by Theorem 7.2, Chapter 6 in [YZ99]. The technique of the proof relies mainly on the linear-quadratic control problems for mean-field dynamics (see also [HLY12]).

As a consequence, one proves the following criterion for approximate null-controllability.

Theorem 58 ([G20], Theorem 12) *The controlled mean-field stochastic system (57) is approximately null-controllable if and only if the viability kernel of $\text{Ker}((B_1 + B_2)^*)$ conditionally to $\text{Ker}(B_1^*)$ w.r.t. (58) is trivially reduced to $\{0\}$:*

$$(60) \quad \text{Viab}(\text{Ker}((B_1 + B_2)^*) / \text{Ker}(B_1^*)) = \{0\}.$$

Algebraic criteria and algorithms

We get the following invariance criteria :

Proposition 59 ([G20], Proposition 17)

1. *A sufficient condition for the system (57) to be approximately null-controllable is that the largest subspace \mathcal{S} of $\text{Ker}((B_1 + B_2)^*)$ such that*

$$(61) \quad (A_1 + A_2)^* \mathcal{S} \subset (C_1 + C_2)^* (\text{Ker}(B_1^*)) + \mathcal{S},$$

(i.e. \mathcal{S} is $((A_1 + A_2)^; (C_1 + C_2)^* \Pi_{\text{Ker}((B_1 + B_2)^*)})$ -strictly invariant), is trivially reduced to $\{0\}$.*

2. *Whenever*

$$A_1^* (\text{Ker}(B_1^*)) \subset \text{Ker}(B_1^*) + C_1^* \text{Ker}(B_1^*),$$

(i.e. the space $\text{Ker}(B_1^)$ is $(A_1^*; C_1^* \Pi_{\text{Ker}(B_1^*)})$ -strictly invariant), the system (57) is approximately null-controllable if and only if the largest subspace \mathcal{S} of $\text{Ker}((B_1 + B_2)^*)$ satisfying (61) is trivially reduced to $\{0\}$.*

3. *If*

$$(62) \quad \text{Rank} \begin{pmatrix} C_1 & B_1 & 0 & 0 \\ I_N & 0 & B_1 & 0 \\ A_1 & 0 & 0 & B_1 \end{pmatrix} = 3N,$$

then the system (57) is approximately controllable if and only if it is approximately null-controllable.

We can give a partial algorithm :

```

=====
Set  $\mathcal{S}_0 := \text{Ker}((B_1 + B_2)^*)$ ;
  AppNullCtrl:=UNKNOWN; AppCtrl:=UNKNOWN;
If  $\mathcal{S}_0 = \{0\}$ ,
  Then AppNullCtrl:=TRUE;
Else
  Repeat
     $k := k + 1$ ;  $\mathcal{S}_k := ((A_1 + A_2)^*)^{-1} (\mathcal{S}_{k-1} + (C_1 + C_2)^* \text{Ker}(B_1^*)) \cap \mathcal{S}_{k-1}$ ;
  While  $k \leq N$  and  $\mathcal{S}_k \subsetneq \mathcal{S}_{k-1}$ 
  EndRepeat
  If  $\mathcal{S}_k = \{0\}$ ,
    Then AppNullCtrl:=TRUE;
  Else
    If  $A_1^* \text{Ker}(B_1^*) \subset \text{Ker}(B_1^*) + C_1^* \text{Ker}(B_1^*)$ 
      Then AppNullCtrl:=FALSE;

```



```

    EndIf
  EndIf
EndIf
If AppNullCtrl≠TRUE,
Then AppCtrl:=AppNullCtrl;
Else
  If Rank  $\begin{pmatrix} C_1 & B_1 & 0 & 0 \\ I_N & 0 & B_1 & 0 \\ A_1 & 0 & 0 & B_1 \end{pmatrix} = 3N,$ 
  Then AppCtrl:=TRUE;
  EndIf
EndIf
Return AppCtrl, AppNullCtrl;
=====

```

3.4 Viability of Stochastic Semilinear Control Systems via the Quasi-Tangency Condition

We consider two separable real Hilbert spaces $(H, \langle \cdot, \cdot \rangle_H)$, $(\Xi, \langle \cdot, \cdot \rangle_\Xi)$. We let $\mathcal{L}(\Xi; H)$ be the space of continuous linear operators and $L_2(\Xi; H)$ be the space of Hilbert-Schmidt linear operators endowed with its usual norm. We consider a linear operator $A : D(A) \subset H \rightarrow H$ which generates a C_0 -semigroup of linear operators $(S(t))_{t \geq 0}$. We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The process W will denote a cylindrical Wiener process with values in Ξ . The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with the natural, complete filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by W . We consider $(G, \langle \cdot, \cdot \rangle_G)$ a real separable Hilbert space and a closed, bounded subset $U \subset G$. For a finite time horizon $T > 0$, we let \mathcal{A} denote the space of all predictable processes $u : [0, T] \times \Omega \rightarrow U$. The coefficient functions $f : H \times U \rightarrow H$ and $g : H \times U \rightarrow L_2(\Xi; H)$ will be supposed to satisfy standard assumptions. Finally, we consider a closed set $K \subset H$.

Given a stochastic control system

$$(63) \quad \begin{cases} dX^{t,\xi,u}(s) = (AX^{t,\xi,u}(s) + f(X^{t,\xi,u}(s), u(s))) ds + g(X^{t,\xi,u}(s), u(s)) dW_s, \text{ for all } s \in [t, T], \\ X^{t,\xi,u}(t) = \xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; H), \end{cases}$$

the aim of the paper [G11] is to give necessary and sufficient conditions for which, for every $t \in [0, T]$, and every $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; K)$, one can find an admissible control process $u \in \mathcal{A}$ such that the mild solution of (63) associated to u remains inside the set K , or, at least, in an arbitrarily small neighborhood of K . These properties are called viability, respectively near-viability, and they have been extensively studied both in deterministic and stochastic setting. In the finite-dimensional deterministic framework, the first result on viability goes back to Nagumo [Nag42] and it has been rediscovered several times in the late sixties. For stochastic finite-dimensional systems, the methods used to characterize viability rely either on stochastic contingent cones (e.g. [Aub91]-[AF90], [GT93]) or on viscosity solutions (e.g. [BPQR98], etc.). We also recall [BQT08] for a necessary condition for the viability of semilinear evolution systems using viscosity solutions of a class of fully nonlinear Hamilton-Jacobi-Bellman equations in abstract Hilbert spaces.

We recall the definitions of near-viability and viability.

Definition 60 (a) A nonempty, closed set $K \subset H$ is called (mildly) viable with respect to the control system (63) if, for every $t \in [0, T]$ and every initial condition $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; K)$, there exists an admissible control process u such that

$$X^{t, \xi, u}(s) \in K, \mathbb{P}\text{-a.s.}, \text{ for all } s \in [t, T].$$

(b) A nonempty, closed set $K \subset H$ is called (mildly) ε -viable with respect to the control system (63) if, for every $t \in [0, T]$ and every initial condition $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; K)$,

$$\inf_{u \in \mathcal{A}} \sup_{s \in [t, T]} d(X^{t, \xi, u}(s), \mathbb{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}; K)) = 0.$$

3.4.1 Quasi-tangency characterization

The central notion needed to characterize (near) viability is the concept of stochastic quasi-tangency.

Definition 61 (Quasi-tangency condition) A closed set $K \subset H$ satisfies the quasi-tangency condition with respect to the control system (63) if, for every $t \in [0, T)$ and every $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; K)$, we have

$$(64) \quad \liminf_{h \searrow 0} \inf \left\{ \frac{1}{h} \mathbb{E} [|\zeta - \eta|^2] + \frac{1}{h^2} \mathbb{E} \left[\left| \mathbb{E}^{\mathcal{F}_t} [\zeta - \eta] \right|^2 \right] : \right. \\ \left. (\zeta, \eta) \in \mathcal{S}(t, h) \xi \times \mathbb{L}^2(\Omega, \mathcal{F}_{t+h}, \mathbb{P}; K) \right\} = 0,$$

where

$$\mathcal{S}(t, h) \xi := S(h) \xi + \int_t^{t+h} S(t+h-s) f(\xi, u(s)) ds + \int_t^{t+h} S(t+h-s) g(\xi, u(s)) dW_s, \quad u \in \mathcal{A}.$$

Remark 62 The term involving the conditional expectation in (64) corresponds to the deterministic quasi-tangency condition. The term h is specific to the stochastic part of the equation (1).

The main result of [G11] is the following.

Theorem 63 ([G11], Theorems 1 and 2) Under standard regularity assumptions on f and g , a nonempty, closed set $K \subset H$ is near-viable with respect to the control system (63), if and only if the quasi-tangency condition (64) holds true.

In fact, our result is more general and it covers the case when g takes its values in $\mathcal{L}(\Xi; H)$ (and may not be in $L_2(\Xi; H)$). The main arguments in the proof rely on :

- sequential characterizations of quasi-tangency;
- construction of maximal approximate mild solutions via Brezis-Browder Theorem.

3.4.2 Applications

The linear case

In the case of linear coefficients, we prove that near-viability and viability are equivalent. More precisely, we consider the particular case when

$$(65) \quad \begin{cases} dX^{t,\xi,u}(s) = (AX^{t,\xi,u}(s) + Bu(s)) ds + (CX^{t,\xi,u}(s) + Du(s)) dW_s, \\ \text{for } s \in [t, T], \\ X^{t,\xi,u}(t) = \xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; H). \end{cases}$$

and U is a closed, bounded and convex subset of G . Here, A is a linear unbounded operator on H that generates a C_0 -semigroup of linear operators $(S(t))_{t \geq 0}$, $B \in \mathcal{L}(G; H)$, C is an $\mathcal{L}(\Xi; H)$ -valued linear operator on H and there exist $\gamma \in [0, \frac{1}{2})$ and $c > 0$ such that, for every $t > 0$, $S(t)C \in \mathcal{L}(H; L_2(\Xi; H))$ and $|S(t)C|_{\mathcal{L}(H; L_2(\Xi; H))} \leq c(t \wedge 1)^{-\gamma}$, and D is an $L_2(\Xi; H)$ -valued linear bounded operator on G .

In this context, we prove the following.

Theorem 64 ([G11], Theorem 4) *We suppose that $K \subset H$ is a nonempty, closed and convex set that satisfies the quasi-tangency condition with respect to the control system (65). Then K is viable with respect to the control system (65).*

The proof of this result relies on weak convergence properties and the convexity of the square distance function.

Nagumo's Theorem and deterministic criteria for the viability of smooth sets

We consider the (uncontrolled) stochastic semilinear equation

$$(66) \quad \begin{cases} dX^{t,\xi}(s) = (AX^{t,\xi}(s) + F(X^{t,\xi}(s))) ds + G(X^{t,\xi}(s)) dW_s, \text{ for all } s \in [t, T], \\ X^{t,\xi}(t) = \xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; H), \end{cases}$$

where $F : H \rightarrow H$ and $G : H \rightarrow L_2(\Xi; H)$ are Lipschitz-continuous.

We get the following version of Nagumo's stochastic theorem.

Theorem 65 (Nagumo's stochastic theorem, [G11], Theorem 5)

A nonempty, closed set $K \subset H$ is viable with respect to (66) if and only if, for every $t \in [0, T)$ and every $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; K)$, there exist a sequence $h_n \searrow 0$ and a sequence of random variables $p_n \in \mathbb{L}^2(\Omega, \mathcal{F}_{t+h_n}, \mathbb{P}; H)$ such that

$$\lim_n \mathbb{E} [|p_n|^2] + \frac{1}{h_n} \mathbb{E} \left[\left| \mathbb{E}^{\mathcal{F}_t} [p_n] \right|^2 \right] = 0$$

and $S(h_n)\xi + h_n F(\xi) + G(\xi)(W_{t+h_n} - W_t) + \sqrt{h_n} p_n \in K$, \mathbb{P} -almost surely, for all $n \geq 1$.

As a consequence, we deduce the following explicit conditions for the viability of unit balls.

Corollary 66 *If the unit ball $\overline{B}(0, 1)$ is viable with respect to (66), then, for every $x \in D(A)$ such that $|x| = 1$,*

$$(67) \quad 0 \geq \langle x, Ax \rangle + \langle x, F(x) \rangle + \frac{1}{2} |G(x)|_{L_2(\Xi; H)}^2, \text{ and } G^*(x)x = 0.$$

Conversely, if A is a diagonal operator, this condition is also sufficient for the viability of $\overline{B}(0, 1)$.

In fact, our result is more general and gives deterministic criteria for sets of constraints K that can be written as

$$K = \{x \in H : \varphi(x) \leq 0\},$$

for some C^2 -function $\varphi : H \rightarrow \mathbb{R}$ with bounded, Lipschitz-continuous second order derivative.

Present and future work

LP methods for deterministic and stochastic control problems

As we have already hinted before, the dynamic programming principles obtained in both deterministic and stochastic frameworks are just a first step in our program. On one hand, they suggest that the test functions in the definition of the viscosity solution might be defined on the space of probability measures endowed with the appropriate topology. A stronger lead we are following at the present time relies on the use of linearized DPP to find optimal measures in the study of discontinuous control problems with state constraints. We should mention that, in the deterministic framework, similar formulations have already been employed in [FGL07] and [GQ09] to derive numerical methods.

Finally, these papers on DPP have allowed us a better comprehension of the sets of constraints associated to controlled trajectories. As mentioned before, the work on min-max control problems only deals with a special toy model. Within the framework of differential games, our research aims at replacing the Varaya-Elliot-Kalton type strategies by regular selections with values in these sets of constraints in order to deal with general deterministic/stochastic dynamics.

We also plan to deepen the study of control processes associated to reflected dynamics. One direction (in collaboration with O.S. Serea) seeks to adapt the LP methods to Pontryagin's maximum principle for stochastic problems with reflection. The second lead is the study of control problems for systems with oblique reflection (partially inspired by the recent paper [GRR12]).

Piecewise deterministic Markov processes

As we have already seen, the linear formulations of control problems in the PDMP framework are strongly inspired by the results obtained for Brownian diffusions. This remark should allow to obtain linear formulations for control problems with state constraints. Inspired by [FGL07] and [GQ09], we plan on using the dual formulations to derive numerical methods in order to compute optimal controls for problems with state constraints. This approach may provide a continuous-time version for [DPR12b] and [DPR12a].

A second topic that interests us is the asymptotic behavior of control problems driven by piecewise deterministic Markov dynamics. We hope to extend the results on nonexpansive (possibly nondissipative) systems from the Brownian setting to PDMPs. The main difficulty lies in the presence of jumps depending on the trajectory. To complete the program, adequate coupling should be exhibited (cf. [BCG⁺13], [BLBMZ12] and references therein).

A first step in this sense has been done in the note [G21]. We have proved a partial Abelian-Tauberian result. More precisely, in the framework of continuous control problems for piecewise deterministic Markov processes, the existence of a uniform limit for discounted value functions as the discount factor vanishes implies (without any further assumption) the uniform convergence of the value functions with long run average cost as the time horizon increases to infinity. The two limit values coincide. However, we would also like to prove the converse implication. We are able to exhibit a very simple example in which the dynamics are not dissipative, however discounted values converge uniformly to a non-constant limit function. We would like to prove that this kind of result holds for every nonexpansive system.

With M. Kobylanski and M. Martinez, we are equally interested in PDMP models associated to traffic problems. We are currently investigating junction problems when the

road functionality is modeled by a probabilistic switch component. Our Hamilton-Jacobi approach is inspired by [IMZ13]. A similar motivation has led us to consider processes with piecewise PDE dynamics and their applications to traffic problems regulated by probabilistic traffic lights. This is an ongoing work with E. Rotenstein and is inspired by the results of [Goa09] in the deterministic setting.

Controllability properties of linear stochastic systems and related topics

In the finite dimensional framework, the stochastic criterion for controllability uses, in an essential way, the viability of the dual system. Its interpretation as a forward control system is no longer possible in a Hilbert framework. However, the viability of the dual BSDE is well-posed. In view of the results obtained in [G20], we know that the "good" notion that may be characterized with viability tools is the approximate null-controllability. Using quasi-tangency conditions (as the one introduced in [G11]), one can exhibit viability criteria (with respect to backward differential systems in a Hilbert setting). Partial results in this sense have been obtained with M. Kobylanski. Nevertheless, we still need a reasonable characterization for viability kernels.

References

- [ABZ12] A. Altarovici, O. Bokanowski, and H. Zidani, *A general Hamilton-Jacobi framework for nonlinear state-constrained control problems*, ESAIM: Control, Optimisation, and Calculus of Variations (2012).
- [AC84] J.P. Aubin and A. Cellina, *Differential inclusions. Set-valued maps and viability theory*, Springer-Verlag, Berlin, 1984.
- [ADP90] J.-P. Aubin and G. Da Prato, *Stochastic viability and invariance*, Annali Scuola Normale di Pisa (1990), no. 27, 595–694.
- [AF90] J.P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990.
- [AG00] Z. Artstein and V. Gaitsgory, *The value function of singularly perturbed control systems*, Applied Mathematics and Optimization **41** (2000), no. 3, 425–445 (English).
- [AL98] M. Arisawa and P.-L. Lions, *On ergodic stochastic control*, Commun. Partial Differ. Equations **23** (1998), no. 11-12, 2187–2217 (English).
- [Alm01] A. Almudevar, *A dynamic programming algorithm for the optimal control of piecewise deterministic Markov processes*, SIAM J. Control Optim. **40** (2001), no. 2, 525–539 (English).
- [AR96] I. Asiminoaei and A. Rascanu, *Approximation and simulation of stochastic variational inequalities-splitting up method*, Numer. Funct. Anal. Optim. **18** (1996), no. 3-4, 251–282.
- [Ari98] M. Arisawa, *Ergodic problem for the Hamilton-Jacobi-Bellman equation. II*, Annales de l’Institut Henri Poincaré (C) Non Linear Analysis **15** (1998), no. 1, 1 – 24.
- [Art02] Zvi Artstein, *An occupational measure solution to a singularly perturbed optimal control problem*, Control Cybernet. **31** (2002), no. 3, 623–642, Well-posedness in optimization and related topics (Warsaw, 2001). MR 1978744 (2004c:49060)
- [Aub91] J.P. Aubin, *Viability theory*, Birkhäuser, Boston, 1991.
- [Aub11] J.-P. Aubin, *Viability solutions to structured Hamilton-Jacobi equations under constraints*, SIAM J. Control. Optim. **49** (2011), 1881–1915.
- [BB96] A.G. Bhatt and V.S. Borkar, *Occupation measures for controlled Markov processes: Characterization and optimality*, Ann. of Probability **24** (1996), 1531–1562.
- [BBG97] G.K. Basak, V.S. Borkar, and M.K. Ghosh, *Ergodic control of degenerate diffusions*, Stochastic Analysis and Applications **15** (1997), no. 1, 1–17 (English).
- [BCD97] M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of hamilton-jacobi-bellman equations*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 1997.

- [BCG⁺13] Jean-Baptiste Bardet, Alejandra Christen, Arnaud Guillin, Florent Malrieu, and Pierre-André Zitt, *Total variation estimates for the tcp process*, Electron. J. Probab. **18** (2013), no. 10, 1–21.
- [BDLP09] Rainer Buckdahn, Boualem Djehiche, Juan Li, and Shige Peng, *Mean-field backward stochastic differential equations: a limit approach*, Ann. Probab. **37** (2009), no. 4, 1524–1565. MR 2546754 (2010j:60138)
- [BdSD12] Adrien Brandejsky, Benoîte de Saporta, and François Dufour, *Numerical method for expectations of piecewise deterministic Markov processes*, Commun. Appl. Math. Comput. Sci. **7** (2012), no. 1, 63–104. MR 2979517
- [Bet05] P. Bettiol, *On ergodic problem for Hamilton-Jacobi-Isaacs equations*, ESAIM-Control Optimisation and Calculus of Variations **11** (2005), no. 4, 522–541 (English).
- [BG90] Vivek S. Borkar and Mrinal K. Ghosh, *Controlled diffusions with constraints*, J. Math. Anal. Appl. **152** (1990), no. 1, 88–108. MR 1072929 (91j:93121)
- [BG05] Vivek Borkar and Vladimir Gaitsgory, *On existence of limit occupational measures set of a controlled stochastic differential equation*, SIAM J. Control Optim. **44** (2005), no. 4, 1436–1473 (electronic). MR 2178039 (2006f:93103)
- [BG07] V. Borkar and V. Gaitsgory, *Averaging of singularly perturbed controlled stochastic differential equations*, Appl. Math. Optimization **56** (2007), no. 2, 169–209.
- [BI89] E.N. Barron and H. Ishii, *The Bellman equation for minimizing the maximum cost*, Nonlinear Anal., Theory Methods Appl. **13** (1989), no. 9, 1067–1090.
- [BI05] R. Buckdahn and N. Ichihara, *Limit theorem for controlled backward SDEs and homogenization of Hamilton-Jacobi-Bellman equations*, Applied Mathematics and Optimization **51** (2005), no. 1, 1–33 (English).
- [Bil99] Patrick Billingsley, *Convergence of probability measures*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999, A Wiley-Interscience Publication. MR 1700749 (2000e:60008)
- [BJ90] E.N. Barron and R. Jensen, *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians*, Commun. Partial Differ. Equations **15** (1990), no. 12, 1713–1742.
- [BJ02a] M. Bardi and R. Jensen, *A geometric characterization of viable sets for controlled degenerate diffusions*, Set-Valued Analysis **10** (2002), 129–141.
- [BJ02b] G. Barles and E. R. Jakobsen, *On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations*, ESAIM, Math. Model. Numer. Anal. **36** (2002), no. 1, M2AN, Math. Model. Numer. Anal.

- [BLBMZ12] M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt, *Quantitative ergodicity for some switched dynamical systems*, Electron. Commun. Probab. **17** (2012), no. 56, 1–14.
- [BLP09] Rainer Buckdahn, Juan Li, and Shige Peng, *Mean-field backward stochastic differential equations and related partial differential equations*, Stochastic Process. Appl. **119** (2009), no. 10, 3133–3154. MR 2568268 (2011d:60167)
- [BN11] B. Bouchard and M. Nutz, *Weak dynamic programming for generalized state constraints*, preprint (2011).
- [Bor89] V.S. Borkar, *Optimal control of diffusion processes*, Pitman Research Notes in Math. 203, Longman Scientific and Technical, Harlow, UK, 1989.
- [Bor05] Vivek S. Borkar, *Controlled diffusion processes*, Probab. Surv. **2** (2005), 213–244. MR 2178045 (2006k:93146)
- [Bor06] ———, *Ergodic control of diffusion processes*, International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 1299–1309. MR 2275729 (2007k:93144)
- [Bou08] B. Bouchard, *Optimal reflection of diffusions and barrier options pricing under constraints*, SIAM J. Control Optimization **47** (2008), no. 4, 1785–1813.
- [BPQR98] R. Buckdahn, S. Peng, M. Quincampoix, and C. Rainer, *Existence of stochastic control under state constraints*, C. R. Acad. Sci. Paris, Sér. I Math **327** (1998), 17–22.
- [BQT06] R. Buckdahn, M. Quincampoix, and G. Tessitore, *A characterization of approximately controllable linear stochastic differential equations*, Stochastic partial differential equations and applications—VII, Lect. Notes Pure Appl. Math., vol. 245, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 53–60. MR 2227219 (2007h:60052)
- [BQT08] ———, *Controlled stochastic differential equations under constraints in infinite dimensional spaces*, SIAM JOURNAL ON CONTROL AND OPTIMIZATION **47** (2008), no. 1, 218–250 (English).
- [BRT03] V. Barbu, A. Răşcanu, and G. Tessitore, *Carleman estimates and controllability of linear stochastic heat equations*, Appl. Math. Optim. **47** (2003), no. 2, 97–120. MR 1954986 (2003m:93004)
- [BT11] B. Bouchard and N. Touzi, *Weak dynamic programming principle for viscosity solutions*, SIAM J. Control Optimization **49** (2011), no. 3, 948–962.
- [CCGW06] F. Camilli, A. Cesaroni, L. Grüne, and F. Wirth, *Stabilization of controlled diffusions and Zubov’s method*, Stoch. Dyn. **6** (2006), no. 3, 373–393. MR MR2258491 (2007k:93113)
- [CD08] O. L. V. Costa and F. Dufour, *Stability and ergodicity of piecewise deterministic Markov processes*, SIAM J. Control Optim. **47** (2008), no. 2, 1053–1077. MR 2385873 (2009b:93163)

- [CD09] ———, *The vanishing discount approach for the average continuous control of piecewise deterministic Markov processes*, *Journal of Applied Probability* **46** (2009), no. 4, 1157–1183 (English).
- [CD10] ———, *Average continuous control of piecewise deterministic Markov processes*, *SIAM J. Control Optim.* **48** (2010), no. 7, 4262–4291 (English).
- [CD12] ———, *Continuous control of piecewise deterministic Markov processes with long run average cost*, *Stochastic processes, finance and control*, *Adv. Stat. Probab. Actuar. Sci.*, vol. 1, World Sci. Publ., Hackensack, NJ, 2012, pp. 415–449. MR 2985448
- [CDL90] I. Capuzzo Dolcetta and P.-L. Lions, *Hamilton-Jacobi equations with state constraints*, *Trans. Amer. Math. Soc.* **318** (1990), 643–683.
- [CDR09] A. Crudu, A. Debussche, and O. Radulescu, *Hybrid stochastic simplifications for multiscale gene networks*, *BMC Systems Biology* (2009), 3:89.
- [CG03] F. Camilli and L. Grüne, *Characterizing attraction probabilities via the stochastic Zubov equation*, *Discrete Contin. Dyn. Syst. Ser. B* **3** (2003), no. 3, 457–468. MR 1 974 158
- [CGT98] D. L. Cook, A. N. Gerber, and S. J. Tapscott, *Modelling stochastic gene expression: Implications for haploinsufficiency*, *Proc. Natl. Acad. Sci. USA* **95** (1998), 15641–15646.
- [CGW01a] F. Camilli, L. Grüne, and F. Wirth, *A generalization of Zubov’s method to perturbed systems*, *SIAM J. Control Optim.* **40** (2001), no. 2, 496–515. MR 2002k:93058
- [CGW01b] ———, *A regularization of Zubov’s equation for robust domains of attraction*, *Nonlinear control in the year 2000*, Vol. 1 (Paris), *Lecture Notes in Control and Inform. Sci.*, vol. 258, Springer, London, 2001, pp. 277–289. MR 2001k:93055
- [CGW04] F. Camilli, L. Grüne, and F. Wirth, *Construction of Lyapunov functions on the domain of asymptotic nullcontrollability: numerics*, *Proceedings of NOLCOS 2004*, Stuttgart, Germany, 2004, pp. 883–888.
- [CGW08] F. Camilli, L. Grüne, and F. Wirth, *Control Lyapunov functions and Zubov’s method*, *SIAM J. Control Optim.* **47** (2008), no. 1, 301–326.
- [Cos89] O. Costa, *Average impulse control of piecewise deterministic processes*, *IMA Journal of Mathematical Control and Information* **6** (1989), no. 4, 375–397 (English).
- [Cur86] R. F. Curtain, *Invariance concepts in infinite dimensions*, *SIAM J. Control and Optim.* **24** (1986), no. 5, 1009–1030 (English).
- [Dav84] M. H. A. Davis, *Piecewise-deterministic Markov-processes - A general-class of non-diffusion stochastic-models*, *Journal of the Royal Statistical Society Series B-Methodological* **46** (1984), no. 3, 353–388 (English).

- [Dav86] ———, *Control of Piecewise-deterministic processes via discrete-time dynamic-programming*, Lect. Notes Control Inf. Sci. **78** (1986), 140–150 (English).
- [Dav93] ———, *Markov models and optimization*, Monographs on Statistics and Applied Probability, vol. 49, Chapman & Hall, London, 1993. MR 1283589 (96b:90002)
- [Del40] M. Delbruck, *Statistical fluctuations in autocatalytic reactions*, J. Chem. Phys. **8** (1940), no. 1, 120–124.
- [DI93] P. Dupuis and H. Ishii, *SDEs with oblique reflection on nonsmooth domains*, Annals of Probability **1** (1993), no. 21, 554–580.
- [DPR12a] F. Dufour and T. Prieto-Rumeau, *Conditions for the solvability of the linear programming formulation for constrained discounted markov decision processes*, preprint (2012).
- [DPR12b] ———, *Finite linear programming approximations of constrained discounted markov decision processes*, preprint (2012).
- [DS12] Francois Dufour and Richard H. Stockbridge, *On the existence of strict optimal controls for constrained, controlled Markov processes in continuous time*, Stochastics **84** (2012), no. 1, 55–78. MR 2876473 (2012m:49031)
- [dSD12] Benoîte de Saporta and François Dufour, *Numerical method for impulse control of piecewise deterministic Markov processes*, Automatica J. IFAC **48** (2012), no. 5, 779–793. MR 2912800
- [DY96] M.A.H. Dempster and J.J. Ye, *Generalized Bellman-Hamilton-Jacobi optimality conditions for a control problem with a boundary condition*, Appl. Math. Optim. **33** (1996), no. 3, 211–225 (English).
- [EKNJP87] N. El Karoui, H. Nguyen, and M. Jeanblanc-Picqué, *Compactification methods in the control of degenerate diffusions: existence of an optimal control*, Stochastics **20** (1987), no. 3, 169–219.
- [FCGAR99] E. Fernández-Cara, M. J. Garrido-Atienza, and J. Real, *On the approximate controllability of a stochastic parabolic equation with a multiplicative noise*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 8, 675–680. MR 1680801 (99m:93005)
- [FGL07] L. Finlay, V. Gaitsgory, and I. Lebedev, *Linear programming solutions of periodic optimization problems: approximation of the optimal control*, J. Ind. Manag. Optim. **3** (2007), no. 2, 399–413.
- [FP98] H. Frankowska and S. Plaskacz, *Semicontinuous solutions of Hamilton-Jacobi-Bellman equations with state constraints*, Differential Inclusions and Optimal Control, vol. 2, Lecture Notes in Nonlinear Anal. (1998), 145–161.
- [Fra93] H. Frankowska, *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optimization **31** (1993), no. 1, 257–272.

- [FS06] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, Applications of Mathematics (New York), vol. 25, Springer-Verlag, New York, 2006. MR 94e:93004
- [FV89] W. Fleming and D. Vermes, *Convex duality approach to the optimal control of diffusions*, SIAM J. Control Optimization **36** (1989), no. 2, 1136–1155.
- [FV00] H. Frankowska and R. Vinter, *Existence of neighbouring trajectories: applications to dynamic programming for state constraints optimal control problems*, Journal of Optimization Theory and Applications **104** (2000), no. 1, 20–40.
- [Gai92] V. Gaitsgory, *Suboptimization of singularly perturbed control systems*, SIAM J. Control and Optim. **30** (1992), no. 5, 1228–1249.
- [Gai04] ———, *On a representation of the limit occupational measures set of a control system with applications to singularly perturbed control systems*, SIAM J. Control Optim. **43** (2004), no. 1, 325–340.
- [Gie07] P. Giesl, *Construction of global Lyapunov functions using radial basis functions*, Lecture Notes in Mathematics, vol. 1904, Springer, Berlin, 2007. MR MR2313542
- [GL99] V. Gaitsgory and A. Leizarowitz, *Limit occupational measures set for a control system and averaging of singularity perturbed control systems*, J. Math. Anal. Appl. **233** (1999), no. 2, 461–475.
- [GN02] V. Gaitsgory and M.-T. Nguyen, *Multiscale singularly perturbed control systems: Limit occupational measures sets and averaging*, SIAM J. Control Optimization **41** (2002), no. 3, 954–974.
- [Goa09] P. Goatin, *Analyse et approximation numérique de quelques modèles macroscopiques de trafic routier*, Ph.D. thesis, 2009.
- [Gor08] D. Goreac, *A Kalman-type condition for stochastic approximate controllability*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 3-4, 183–188. MR 2393638 (2009b:93016)
- [Gor09] ———, *Approximate controllability for linear stochastic differential equations in infinite dimensions*, Appl. Math. Optim. **60** (2009), no. 1, 105–132. MR 2511788 (2010g:93012)
- [GQ09] V. Gaitsgory and M. Quincampoix, *Linear programming approach to deterministic infinite horizon optimal control problems with discounting*, SIAM J. Control Optimization **48** (2009), no. 4, 2480–2512.
- [GR06] V. Gaitsgory and S. Rossomakhine, *Linear programming approach to deterministic long run average problems of optimal control*, SIAM J. Control Optim. **44** (2006), no. 6, 2006–2037.
- [Gra97] G. Grammel, *Averaging of singularly perturbed systems*, Averaging of singularly perturbed systems **28** (1997), no. 11, 1851–1865.

- [GRR12] Anouar M. Gassous, Aurel Rascanu, and Eduard Rotenstein, *Stochastic variational inequalities with oblique subgradients*, STOCHASTIC PROCESSES AND THEIR APPLICATIONS **122** (2012), no. 7, 2668–2700 (English).
- [Grü02] L. Grüne, *Asymptotic behavior of dynamical and control systems under perturbation and discretization*, Lecture Notes in Mathematics, vol. 1783, Springer-Verlag, Berlin, 2002. MR 2003d:37001
- [GS11] L. Grüne and O.S. Serea, *Differential games and Zubov’s method*, preprint (2011).
- [GSB08] L. Grüne, W. Semmler, and L. Bernard, *Firm value, diversified capital assets, and credit risk: towards a theory of default correlation*, J. Credit Risk **3** (2007/08), 81–109.
- [GT93] S. Gautier and L. Thibault, *Viability for constrained stochastic differential equations*, Differential Integral Equations **6** (1993), no. 6, 1395–1414.
- [GW00] L. Grüne and F. Wirth, *Computing control Lyapunov functions via a Zubov type algorithm*, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000, pp. 2129–2134.
- [HLY12] J. Huang, X. Li, and J. Yong, *A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite horizon*, arXiv:1208.5308 (2012).
- [HPDC00] J. Hasty, J. Pradines, M. Dolnik, and J.J. Collins, *Noise-based switches and amplifiers for gene expression*, PNAS **97** (2000), no. 5, 2075–2080.
- [IMZ13] Cyril Imbert, Régis Monneau, and Hasnaa Zidani, *A hamilton-jacobi approach to junction problems and application to traffic flows*, ESAIM: Control, Optimisation and Calculus of Variations **19** (2013), 129–166.
- [Kry99] N. Krylov, *Approximating value functions for controlled degenerate diffusion processes by using piece-wise constant policies*, Electron. J. Probab. **4** (1999), no. 2, 1–19.
- [Kry00] N. V. Krylov, *On the rate of convergence of finite-difference approximations for Bellman’s equations with variable coefficients*, Probab. Theory Related Fields **117** (2000), no. 1, 1–16. MR 1759507 (2001j:65134)
- [KS98] Thomas G. Kurtz and Richard H. Stockbridge, *Existence of Markov controls and characterization of optimal Markov controls*, SIAM J. Control Optim. **36** (1998), no. 2, 609–653 (electronic). MR 1616514 (99b:93051)
- [LP02] YZ. Liu and SG. Peng, *Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems*, Automatica **38** (2002), no. 8, 1417–1423 (English).
- [LP10] Feng Liu and Shige Peng, *On controllability for stochastic control systems when the coefficient is time-variant*, J. Syst. Sci. Complex. **23** (2010), no. 2, 270–278. MR 2653589 (2011e:93012)

- [LPV] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi Equations*, unpublished work.
- [LS92] E. Lehrer and S. Sorin, *A uniform tauberian theorem in dynamic programming*, Mathematics of Operations Research **17** (1992), no. 2, 303–307.
- [Mal05] M. Malisoff, *Further results on Lyapunov functions and domains of attraction for perturbed asymptotically stable systems*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2005), no. 2, 193–225. MR MR2170408 (2006i:49046)
- [Mer76] R. C. Merton, *Option pricing when underlying stock returns are discontinuous*, J. Financ. Econ. **3** (1976), 125–144.
- [Mot95] M. Motta, *On nonlinear optimal control problems with state constraints*, SIAM J. Control Optim. **33** (1995), 1411–1424.
- [MP01] Mohammed Mnif and Huyên Pham, *Stochastic optimization under constraints*, Stochastic Process. Appl. **93** (2001), no. 1, 149–180. MR 1819488 (2002e:91069)
- [Nag42] M. Nagumo, *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Phys. Math. Soc. Japan **24** (1942), 551–559.
- [OBV13] M. Oliu-Barton and G. Vigeral, *A uniform Tauberian theorem in optimal control*, Annals of the International Society of Dynamic Games vol 12 : Advances in Dynamic Games (P.Cardaliaguet and R.Cressman, eds.), Birkhäuser Boston, 2013, 14 pages (Anglais).
- [Pen94] S. Peng, *Backward stochastic differential equation and exact controllability of stochastic control systems*, Progr. Natur. Sci. (English Ed.) **4** (1994), 274–284.
- [Pha98] Huyên Pham, *Optimal stopping of controlled jump diffusion processes: a viscosity solution approach*, J. Math. Systems Estim. Control **8** (1998), no. 1, 27 pp. (electronic). MR 1650147 (99h:49043)
- [Pha05] ———, *On some recent aspects of stochastic control and their applications*, Probab. Surv. **2** (2005), 506–549. MR 2203679 (2006j:93004)
- [Pha07] ———, *Optimisation et contrôle stochastique appliqués à la finance*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 61, Springer, Berlin, 2007. MR 2344014 (2008g:93200)
- [PP90] E. Pardoux and S.G. Peng, *Adapted solution of a backward stochastic differential equation*, Syst. Control Lett. **14** (1990), no. 1, 55 – 61.
- [PQ01] S. Plaskacz and M. Quincampoix, *Value-functions for differential games and control systems with discontinuous terminal cost*, SIAM J. Control Optimization **39** (2001), no. 5, 1485–1498.
- [PZ08] Shi Ge Peng and Xue Hong Zhu, *The viability property of controlled jump diffusion processes*, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 8, 1351–1368. MR 2438306 (2009g:60081)

- [QR11] M. Quincampoix and J. Renault, *On the existence of a limit value in some non-expansive optimal control problems*, SIAM Journal on Control and Optimization **49** (2011), no. 5, 2118–2132.
- [QS10] M. Quincampoix and O.S. Serea, *The problem of optimal control with reflection studied through a linear optimization problem stated on occupational measures*, Nonlinear Analysis: Theory, Methods and Applications **72** (2010), 2803–2815.
- [Ric09] A. Richou, *Ergodic BSDEs and related PDEs with Neumann boundary conditions*, Stochastic Processes and Their Applications **119** (2009), no. 9, 2945–2969 (English).
- [Ser02] O.S. Serea, *Discontinuous differential games and control systems with supremum cost*, J. Math. Anal. Appl. **270** (2002), no. 2, 519–542.
- [Son83] E. D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim. **21** (1983), no. 3, 462–471. MR 84e:93015
- [Son86a] H. M. Soner, *Optimal control with state-space constraint. I*, SIAM J. Control Optim. **24** (1986), no. 6, 552–561.
- [Son86b] ———, *Optimal control with state-space constraint. II*, SIAM J. Control Optim. **24** (1986), no. 6, 1110–1122.
- [SS80] E. J. P. G. Schmidt and R. J. Stern, *Invariance theory for infinite dimensional linear control systems*, Applied Mathematics and Optimization **6** (1980), no. 2, 113–122 (English).
- [ST01] M. Sirbu and G. Tessitore, *Null controllability of an infinite dimensional SDE with state- and control-dependent noise*, Syst. Control Lett. **44** (2001), no. 5, 385–394 (English).
- [Sto90] R.H. Stockbridge, *Time-average control of a martingale problem. Existence of a stationary solution*, Ann. of Probability **18** (1990), 190–205.
- [Tic52] A.N. Tichonov, *Systems of differential equations containing small parameter near derivatives*, Mat. Sbornik. **31** (1952), 575–586.
- [Vel97] V. Veliov, *A generalization of the tikhonov theorem for singularly perturbed differential inclusions*, J. Dynam. Control Systems **3** (1997), no. 3, 291–319.
- [YZ97] G.G. Yin and Q. Zhang, *Continuous-time Markov chains and applications. a singular perturbation approach*, Springer-Verlag, New York, 1997.
- [YZ99] J. Yong and X.Y. Zhou, *Stochastic controls. hamiltonian systems and hjb equations*, Springer-Verlag, New York, 1999.
- [Zal02] A. Zalinescu, *Second-order Hamilton-Jacobi-Bellman inequalities*, C.R. Acad. Sci. Paris, Serie I **335** (2002), 591–596.
- [ZP08] X. Zhu and S Peng, *The viability property of controlled jump diffusion processes*, Acta Math. Sinica **24** (2008), no. 8, 1351–1368.

[Zub64] V.I. Zubov, *Methods of A. M. Lyapunov and their application.*, Translation prepared under the auspices of the United States Atomic Energy Commission; edited by Leo F. Boron. P.Noordhoff Ltd, Groningen (1964).