

Continuous quasi-LPV Systems : how to leave the quadratic framework?

Abdelhafidh Jaadari

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Systèmes quasi-LPV continus : comment dépasser le cadre du quadratique ?

JURY

Président du jury
- Dambrine, Michel. Professeur, Université de Valenciennes et du Hainaut Cambrésis.

Rapporteurs
- Ariño Latorre, Carlos. Profesor Contratado Doctor, Universitat Jaume I, Castello de la Plana, Espagne.
- El Hajjaji, Ahmed. Professeur, Université Picardie Jules Verne.
- Maquin, Didier. Professeur, Université de Lorraine.

Examinateurs
- Salcedo, José Vicente. Profesor Titular, Universitat Politecnica de Valencia, Espagne.
- Bernal, Miguel. Profesor Investigador Titular, Instituto Tecnológico de Sonora, Mexique.

Directeur de thèse
- Guerra, Thierry-Marie. Professeur, Directeur du Lamih. Université de Valenciennes et du Hainaut Cambrésis.

Co-directeur de thèse : Sala Piqueras, Antonio. Profesor, Universitat Politecnica de Valencia, Espagne.

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Abstract

This thesis deals with the problem of stability analysis and control design for nonlinear systems in the form of continuous-time Takagi-Sugeno models. The approach to stability analysis is usually based on the direct Lyapunov method. Several approaches in the literature, based on quadratic Lyapunov functions, are proposed to solve this problem; the results obtained using such functions introduce a conservatism that can be very detrimental. To overcome this problem, various approaches based on non-quadratic Lyapunov functions have also been recently presented; however, these approaches are based on very conservative bounds or too restrictive conditions. The idea developed in this work is to use non-quadratic Lyapunov functions and non-PDC controller in order to derive less conservative stability and stabilization conditions. The main proposals are: using local bounds in partial derivatives instead of time derivatives of the memberships, decoupling the controller gain from the Lyapunov function decision variables, using fuzzy Lyapunov functions in polynomial settings and proposing the synthesis of controller ensuring *a priori* known time-derivative bounds are fulfilled in a modelling region instead of checking them a posteriori. These new approaches allow proposing local conditions to stabilize continuous T-S fuzzy systems including those that do not admit a quadratic stabilization. Several simulation examples are chosen to verify the results given in this dissertation.

Key words: Takagi-Sugeno Models, non-quadratic Stability, non-quadratic Stabilization, Lyapunov function, Linear Matrix inqualities, Sum Of Squares.

Resumen

Esta tesis aborda el problema del análisis de estabilidad y diseño de control para sistemas no lineales expresados en forma de modelos Takagi-Sugeno de tiempo continuo. El análisis de estabilidad se basa, por lo general, en el método directo de Lyapunov. Existen varios enfoques en la literatura, sobre la base de las funciones cuadráticas de Lyapunov, que se proponen para resolver este problema; los resultados obtenidos usando tales funciones introducen un conservadurismo que puede ser excesivo. Para superar este problema, diversos enfoques basados en funciones de Lyapunov no cuadráticas se han presentado recientemente; estos enfoques se basan en límites muy conservadores o condiciones demasiado restrictivas. La idea desarrollada en este trabajo es el uso de funciones de Lyapunov no cuadráticas y un controlador no PDC con el fin de obtener condiciones de estabilidad y estabilización menos conservadoras. Las principales propuestas son: el uso de límites locales en derivadas parciales en lugar de derivadas temporales de las funciones de pertenencia, el desacoplo del controlador respecto de las variables de decisión de la función de Lyapunov, el uso de funciones de Lyapunov difusas en modelos polinomiales y finalmente, proponer la síntesis de controladores garantizando ciertos límites de la derivada-temporal, conocidos a priori, en una región de modelado en lugar de comprobarlos a posteriori. Estos nuevos enfoques permiten proponer condiciones locales para estabilizar los sistemas continuos difusos T-S incluyendo aquéllos que no admiten una estabilización cuadrática. Varios ejemplos de simulación han sido seleccionados para verificar los resultados obtenidos en esta tesis.

Palabras clave: modelos Takagi-Sugeno, Estabilidad no cuadrática, Estabilización no cuadrática, función de Lyapunov Desigualdades Matriciales Lineales, Suma de Cuadrados.

Résumé

Cette thèse aborde le problème de l'analyse de la stabilité et de la conception des lois de commande pour les systèmes non linéaires mis sous la forme de modèles flous continus de type Takagi-Sugeno. L'analyse de stabilité est généralement basée sur la méthode directe de Lyapunov. Plusieurs approches existent dans la littérature, basées sur des fonctions de Lyapunov quadratiques sont proposées pour résoudre ce problème, les résultats obtenus à l'aide des telles fonctions introduisent un conservatisme qui peut être très préjudiciable. Pour surmonter ce problème, différentes approches basées sur des fonctions de Lyapunov non quadratiques ont été proposées, néanmoins ces approches sont basées sur des conditions très restrictives. L'idée développée dans ce travail est d'utiliser des fonctions de Lyapunov non quadratiques et des contrôleurs non-PDC afin d'en tirer des conditions de stabilité et de stabilisation moins conservatives. Les propositions principales sont: l'utilisation des bornes locales des dérivées partielles au lieu des dérivés des fonctions d'appartenances, le découplage du gain du régulateur des variables de décision de la fonction Lyapunov, l'utilisation des fonctions de Lyapunov floues polynomiales dans l'environnement des polynômes et la proposition de la synthèse de contrôleur vérifiant certaines limites de dérivés respectées dans une région de la modélisation à la place de les vérifier a posteriori. Ces nouvelles approches permettent de proposer des conditions locales afin de stabiliser les modèles flous continus de type T-S, y compris ceux qui n'admettent pas une stabilisation quadratique et obtenir des domaines de stabilité plus grand. Plusieurs exemples de simulation sont choisis afin de vérifier les résultats présentés dans cette thèse.

Mots clés : modèles flous de types Takagi-Sugeno, Stabilité non-quadratique, stabilisation non-quadratique, Fonction de Lyapunov, Inégalités matricielle linéaires, somme des carrées.

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Notations and abbreviations

Notations

$\mathbb{R}$: Real numbers

$\mathbb{R}^n$: n Dimensional Euclidean space

$\mathbb{R}^{m\times n}$: The set of all real $m\times n$ matrices

I : Identity Matrix

$(*)$: Symmetric block

A^{-1} : Inverse of matrix A

A^T : Transpose of the matrix A

A^{-T} : Transpose of the inverse of A

$\lfloor\ \rfloor$: the floor function

$$Y_z = \sum_{i=1}^{r} h_i\left(z(t)\right) Y_i$$

$$Y_{zz} = \sum_{i=1}^{r}\sum_{i=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right) Y_{ij}$$

$$Y_z^{-1} = \left(\sum_{i=1}^{r} h_i\left(z(t)\right) Y_i\right)^{-1}$$

Abbreviations

LPV : Linear Parameter Varying

Q-LPV: Quasi Linear Parameter Varying

T-S : Takagi-Sugeno

PDC : Parallel Distributed Compensation

MF : Membership Functions

LMI : Linear Matrix Inequality

BMI : Bilinear Matrix inequality

SOS : Sum of squares

NP : Non-positivity

PWLF : Piecewise Lyapunov function

NQFLF: Non-quadratic Lyapunov function

PF : Polynomial fuzzy

PFLF : Polynomial fuzzy Lyapunov Function

SDP : Semi-definite program

1. Chapter 1: Introduction

Synopsys

In this introductory chapter, we provide an overview of the main purpose of this thesis. We briefly give a review of the fuzzy control and we point out the most important existing works dealing with stability analysis and controller design for fuzzy systems in the literature, then we explain the motivations concepts, the objectives and contributions, by giving the chapters outline of this thesis, we end the chapter.

This Ph.D. thesis considers the problems of non-quadratic stability analysis and control design for continuous-time Takagi-Sugeno models. The goal is to develop new approaches to overcome the drawbacks of existing approaches in fuzzy control theory.

1.1. Background and Motivation

Physical systems are generally described by nonlinear models, which make stability analysis a goal difficult to reach; classical approaches tend to approximate them by linear systems. However, the major drawback is that the linearized systems fail to completely represent the real plants that are highly nonlinear. Researchers have proposed several ways to deal with nonlinear systems; a linear parameter varying (LPV) presentation has been proposed by [Shamma, 1988] in order to approximate nonlinear systems, An LPV system is essentially a linear time-varying system which can be written in the form

$$\begin{cases} \dot{x} = A\big(\theta(t)\big)x + B\big(\theta(t)\big)u \\ y = C\big(\theta(t)\big)x + D\big(\theta(t)\big)u \end{cases} \tag{1.1}$$

Where θ is a bounded time varying parameter vector. As such it has a structure which is similar to a linear time-invariant state space system, and control design methods with some similarity to linear state space methods can indeed be used. Although these models do not capture the nonlinear behavior of the original models [Bernal & Guerra, 2010].

Another alternative has been introduced by [Shamma & Cloutier, 1993] to write nonlinear systems in the form of quasi-LPV models, this representation is obtained through an exact transformation of the nonlinear states. A quasi-LPV system is defined as a system where the state realization can be put in the following form:

$$\begin{cases} \dot{x}(t) = A\big(x(t)\big)x(t) + B\big(x(t)\big)u(t) \\ y(t) = C\big(x(t)\big)x(t) + D\big(x(t)\big)u(t) \end{cases} \tag{1.2}$$

This class of models is known also as Takagi-Sugeno models [Takagi & Sugeno, 1985] which consists in a set of linear models blended together with nonlinear functions called membership functions (MFs) which hold the convex-sum property [Tanaka & Wang, 2001]. It allows then to exactly represent a nonlinear model in a compact set of the state variables [Taniguchi & al, 2001].

T-S models may be extended to polynomial fuzzy models which consists in a convex sum of polynomials models. It has been recently proposed in [Sala & Ariño, 2008] , [Tanaka & al,

2009b] and [Lo, 2011] to represent efficiently a nonlinear system, especially when nonlinear terms are polynomials.

In this thesis, nonlinear systems represented in the form of both T-S and polynomial fuzzy models are considered, continuous case will be analysed.

1.2. Review of previous Works

Over the last three decades, the so-called Takagi-Sugeno models [Takagi & Sugeno, 1985] have reached a great attention in the control community. Since, they allow a systematic stability analysis and controller design via linear matrix inequalities (LMIs) [Tanaka & Wang, 2001] which can be efficiently solved by convex programming techniques already implemented in commercially available software [Boyd & al, 1994]. Several results for stability, stabilization, estimation [Tanaka & Wang, 2001], [Lendek & al, 2010], [Lendek & al, 2011], [Feng, 2006] have been obtained.

T-S models are combined with different control laws, among which parallel distributed compensation (PDC) is considered a natural option since it is based on linear state feedbacks blended together using the same MFs of the T-S representation. Once a T-S model and a control law are proposed, the direct Lyapunov method is applied to obtain, when possible, LMI conditions for stability analysis, control and observer design [Tanaka & Wang, 2001], [Sala & al, 2005] (see references therin). The stability of a T-S model is based on the Lyapunov theory, proving the existence of a common matrix $P>0$ such that $\dot{V}<0$, where $V(t)=x(t)^T Px(t)$ is a Lyapunov candidate function [Tanaka & al, 1996], [Wang & al, 1996]. Nonetheless, the quadratic approach presents serious limitations because its solutions are inherently conservative, i.e., there are stable or stabilizable models which do not have a quadratic solution [Sala & al, 2005], this conservativeness comes from different sources [Guerra & al, 2012]: the type of T-S model [Guerra & al, 2007], [Bouarar & al, 2010], the way the membership functions are dropped-off to obtain LMI expressions [Tuan & al, 2001], [Sala & Ariño, 2007, 2007a], the integration of membership-function information [Sala & Guerra, 2008], [Bernal & al, 2009], or the choice of Lyapunov function [Johansson & al, 1999], [Tanaka & al, 2001c], there was room for reducing this conservativeness by changing the choice of the Lyapunov function.

Researchers have proposed several Lyapunov functions to deal with these drawbacks:

In [Blanco & al, 2001], [Tanaka & al, 2003], [Bernal & Husek, 2005] Non-Quadratic Fuzzy Lyapunov functions (NQFLFs) were proposed, thus constituting the first non-quadratic framework for T-S models. Nevertheless, the time-derivative of the membership functions of

the T-S model appears in the derivative of the Lyapunov function which makes the resulting conditions non LMIs, for that, several results propose just to bound them a priori [Bernal & al, 2006], [Mozelli & al, 2009]. This way of doing is not satisfactory because the verification of these bounds can only be done a posteriori with a case by case approach, especially when compared with the discrete-case [Guerra & Vermeiren, 2004], [Ding & al, 2006], [Krusewski & al, 2008], [Guerra & al, 2009b]. Another drawback rises from the fact that authors bound the time-derivatives of the MFs assuming that they do not depend on the input, which turns out to be very restrictive. Moreover, the proposed control law makes use of the time-derivatives of the MFs through a classical PDC scheme, thus ignoring the non-quadratic nature of the involved Lyapunov function.

In [Johansson & al, 1999], [Othake & al, 2003], [Feng & al, 2004], [Feng & al, 2005] researchers proved that the use of piecewise Lyapunov functions (PWLFs) have effectively relaxed the referred pessimism, though they require the MFs to induce a polyhedral partition of the state space. Unfortunately, this condition on the MFs of those TS models obtained by sector nonlinearity approach is not fulfilled; moreover, the piecewise approach leads to bilinear matrix inequalities in the continuous-time context which cannot be optimally solved [Feng & al, 2005].

In [Rhee & Won, 2006], a line-integral Lyapunov function is proposed to circumvent the MFs' time-derivative obstacle, though the line integral is asked to be path-independent thus significantly reducing its applicability [Guelton & al, 2010].

All these approaches consider the problem of global stability which is far to be the general rule for nonlinear systems. Although they present some improvements which are particularly important and allow dealing with problems that was unfeasible before, a change of perspective for non-quadratic stability analysis of T-S models has been proposed in [Guerra & Bernal, 2009]. This approach employing a non-quadratic Fuzzy Lyapunov function (NQFLF) and priori known bounds [Guerra & Bernal, 2009], [Bernal & Guerra, 2010], [Bernal & al, 2010] and [Guerra & al, 2011], reduces global goals to less exigent conditions, thereby showing that an estimation of the region of attraction can be found (local stability); this approach may provide a local solution for nonlinear models that do not admit a global solution [Khalil, 2002].

1.3. Objetcives and Contributions

The subject of this work is to develop new non-quadratic stability and stabilization conditions for continuous T-S fuzzy systems. Based on non-quadratic Lyapunov functions, new non-quadratic stability conditions are derived in order to overcome the drawbacks of the quadratic approaches and the existing ones.

A first motivation for the work of this thesis arises from the fact that most of stability conditions are based on quadratic Lyapunov functions which means that the aim can be reached by finding a common Lyapunov matrix $P > 0$ for all the vertices of the polytopes – or sub-models. This renders stability results conservative and even a large number of systems can be stable without the existence of a quadratic Lyapunov function.

A second motivation is that in most of existing approaches dealing with stability and stabilization, the properties of the membership functions are not taking into account except the convexity property. In other approaches, it is taking in consideration the upper bound for the time derivative of the premise membership function as assumed by [Tanaka & al, 2001a, 2001b, 2001c, 2003].

A third motivation is that it has been shown that reducing global stability goals to something less restrictive will give a nice solution by providing an estimation of the stability domain (local asymptotic conditions), as it is usually the case for nonlinear models for which stability and/or stabilization cannot be reached globally.

The main contributions of this thesis are in both stability analysis and controller design:

The first contribution is concerned with a relaxation in the latter sense which demands a change of perspective from global to local conditions. Non-quadratic Lyapunov functions has been proposed to analyze the stability of continuous-time Takagi-Sugeno models which means that the objective can be reached after finding a number of $P_i > 0$.

The second contribution consists in a sum of squares (SOS) approach based first on polynomial fuzzy modeling providing a more effective representation of the nonlinear systems and second more relaxed stability conditions based on polynomial fuzzy Lyapunov function comparing to the LMI-Based approach. These SOS conditions can be solved numerically using the Matlab toolbox SOSTOOLS [Prajna & al, 2004a].

The third contribution is the extension of the local results obtained for stability analysis to the control design of continuous-time Takagi-Sugeno models, based on non-PDC control law according to the non-quadratic nature of the Lyapunov function, new Local stabilization conditions have been obtained. The well-known problem of handling time-

derivatives of membership functions (MFs) as to obtain conditions in the form of linear matrix inequalities (LMIs) is overcome by reducing global goals to the estimation of a region of attraction.

A last contribution results in a novel approach proposing the design of a robust local H_∞ controller for disturbed continuous-time Takagi-Sugeno based on non-quadratic Lyapunov function, the method is based on a new form of non-PDC controller and by the mean of Finsler's Lemma, LMIs conditions can be obtained, the idea does not require a bound for the input control, it only needs a priori known bound of the states which is given from the domain of definition of the T-S models.

1.4. Chapters outline

This thesis is organized as follows:

Chapter 1 provides an introduction de the study.

Part I presents the state of the art.

Chapter 2 introduces Takagi-Sugeno models followed by the method used to the design of these models. A recall of the basic concepts and definitions of the theory of stability in the Lyapunov sense is given. Quadratic stability and stabilization conditions for continuous-time Takagi-Sugeno models are then presented. Semi definite programming techniques and a number of tools and properties are cited. The chapter ends with a discussion of the drawbacks of existing approaches trying to overcome the problems encountered when using classical approaches for stability and stabilization.

In Part II, we develop the contributions of this thesis and it is organized in four chapters

Chapter 3 is devoted to the first major contribution; it presents new solutions for stability analysis problems for continuous-time Takagi-Sugeno models. This chapter is based on a method first proposed by [Guerra & Bernal, 2009] to obtain local results and better estimation of the region of attraction via non-quadratic Lyapunov functions. An improvement of this approach is then given in order to obtain better relaxed stability conditions followed by illustrative examples to show the advantages of the proposed LMIs conditions.

In chapter 4, we present polynomial fuzzy modeling and stability analysis. The stability conditions based on polynomial Lyapunov functions are represented in terms of SOS and can be numerically (partially symbolically) solved via the recently developed SOSTOOLS. To illustrate the validity and applicability of the proposed approach, a number of analysis and design examples are provided.

Chapter 5 is devoted to the second major contribution; it extends the results obtained in chapter 3 for stability analysis to stabilization. New non-quadratic approaches based on non-PDC controller and non-quadratic Lyapunov functions are proposed in order to obtain more relaxed results comparing with recent existing methods in non-quadratic control design and to prove stabilization of a large number of continuous-time Takagi-Sugeno models which do not admit a quadratic stabilization. Simulation results are then presented to show the effectiveness of the proposed approaches during this chapter.

Chapter 6 studies the design of a robust non-quadratic controller based on non-quadratic Lyapunov function, the goal in this chapter is to take into account during the controller design of the different perturbations and unknown inputs that can affect a nonlinear system, in order to obtain sufficient local conditions allowing to stabilize the proposed models with better attenuation of the external perturbations. In then, a robust H infinity controller is designed for the proposed model showing that the link between the controller gain and the Lyapunov function can be cut in a convenient manner via Finsler's lemma. Simulation examples are given to highlight the method's advantages.

Part III ends the thesis with some concluding remarks and recommendations for future work.

Part I: State of the art

2. Chapter 2: State of the art

Synopsys

This chapter is devoted to the presentation of the context of our work and the definition of basic concepts of our study. We first introduce Takagi-Sugeno models known also as Quasi Linear Parameter Varying (Quasi-LPV) models and the approach used to design such models. Second, we give an overview of the Lyapunov theory and the different Lyapunov functions used to study stability and stabilization of continuous-time Takagi-Sugeno models. Two techniques of Semi-definite programming: Linear matrix inequalities and Sum of squares programming are presented, and some matrix proprieties which will be useful in the following chapters, are recalled. Finally the drawbacks of the use of a quadratic Lyapunov functions are discussed and the existing approaches and results to overcome these problem are studied.

2.1. Takagi-Sugeno (quasi-LPV) models

A nonlinear system can be represented by the so called Takagi-Sugeno (T-S) fuzzy model first proposed by Takagi and Sugeno [Takagi & Sugeno, 1985]. The T-S fuzzy model is based on IF-THEN rules, which represent the local input-output relations of a nonlinear system.
Consider the nonlinear system

$$\begin{cases} \dot{x}(t) = f\left(z(t)\right)x(t) + g\left(z(t)\right)u(t) \\ y(t) = d\left(z(t)\right)x(t) \end{cases} \tag{2.1}$$

With $f\left(z(t)\right)$, $g\left(z(t)\right)$, $d\left(z(t)\right)$ being nonlinear functions, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ is the output vector and $z\left(x(t)\right) \in \mathbb{R}^p$ is the premise vector bounded and smooth in a compact set C of the state space including the origin.

The $i-th$ rules of a so-called T-S Fuzzy model [Takagi & Sugeno, 1985] are given under the following form:
Model Rule i :

$$\text{If } z_1(t) \text{ is } M_{i1} \text{ AND } \cdots \text{ AND } z_p(t) \text{ is } M_{ip}$$

Then

$$\begin{cases} \dot{x}_i(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases}, \; i = 1, 2, \ldots, r \tag{2.2}$$

Where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ and $C_i \in \mathbb{R}^{q \times n}$, $i \in \{1, \ldots, r\}$ are matrices of proper dimensions, $r = 2^p \in \mathbb{N}$ is the number of linear models, M_{i1} is the fuzzy set and r is the number of model rules; $z_1(t), \ldots, z_p(t)$ are known premise variables that may be functions of the state variables, external disturbances, and/or time.
Each equation in (2.2) is called "Sub-model", to each rule corresponds a set of weight $w_i\left(z(t)\right)$ which depends on the degree of membership of the premises variables to the fuzzy set M_{ik}.
Equations (2.2) are evaluated with the following T-S defuzzification formula:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right) \\ y(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) C_i x(t) \end{cases} \tag{2.3}$$

the T-S model (2.3) can be an exact representation of the original nonlinear system (2.1) in C if the following systematic methodology is used to get the T-S model.

2.1.1. Sector non linearity Approach

The idea of using sector nonlinearity in fuzzy Takagi-Sugeno model construction first appeared in [Kawamoto & al, 1992] and expanded in [Tanaka & Wang, 2001], Sector nonlinearity is based on the following idea: Consider a simple nonlinear system $\dot{x} = f\left(x(t)\right)$ where $f(0) = 0$, the aim is to find the global sector such that $\dot{x} = f(x) \in [s_1 \quad s_2]\, x(t)$, where $s_1 x(t)$ and $s_2 x(t)$ are lines as shown in Figure 2.1, this approach guarantees an exact fuzzy Takagi-Seguno model construction. However sometimes it is difficult to find a global sector for general nonlinear systems. In this case, we consider local sector nonlinearity as depicted in Figure 2.2.

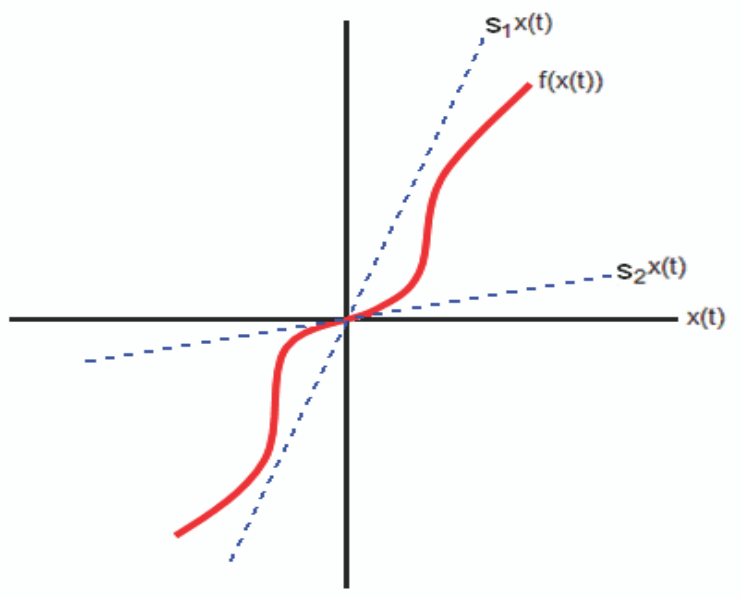


Figure 2.1: Global sector

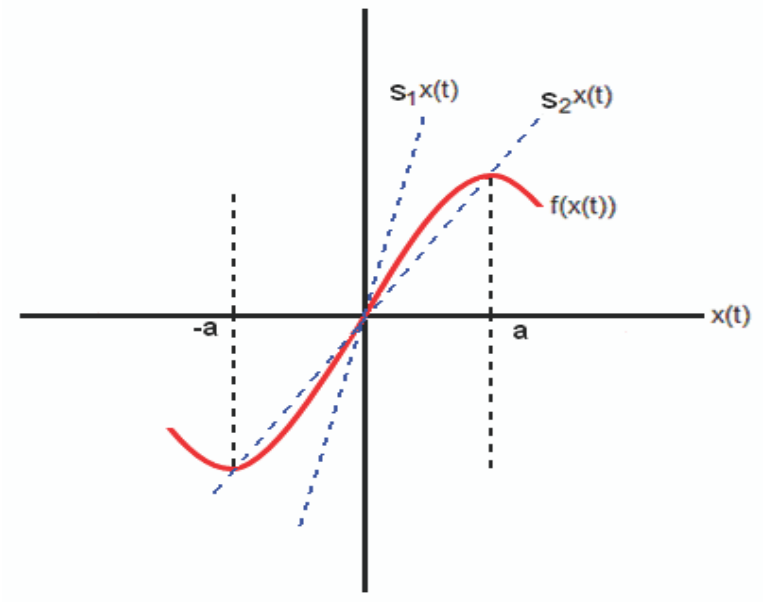


Figure 2.2: Local sector

Remark 2.1:

Consider the nonlinear system $\dot{x} = f\left(x(t)\right), f\left(0\right) = 0$. According to the properties of nonlinear terms encountered in the non-linear mathematical model, we distinguish two types of representative T-S:

- If all the nonlinearities of the system are continuous and bounded on $\mathbb{R}^n$ then the T-S model allows an exact representation of the nonlinear system over the entire state space $\mathbb{R}^n$.
- If all the nonlinearities of the system are only continuous, then the T-S model allows an exact representation of the nonlinear system on a compact subset of the state space $C \subset \mathbb{R}^n$.

Example 2.1:

Consider the autonomous nonlinear model given by:

$$\dot{x}(t) = x(t)\cos\left(x(t)\right) \tag{2.4}$$

Note that $f\left(x(t)\right) = \cos\left(x(t)\right)$ is continuous and bounded in $\left[-1 \quad 1\right]$, then we can write

$$\cos\left(x(t)\right) = \underbrace{\frac{\cos\left(x(t)\right)+1}{2}}_{h_1(x(t))} \times 1 + \underbrace{\frac{1-\cos\left(x(t)\right)}{2}}_{h_2(x(t))} \times (-1)$$

and the Takagi-Sugeno model corresponding to the nonlinear model (2.4) can be written as:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i\left(z(t)\right) A_i x(t) \tag{2.5}$$

where $A_1 = 1$ and $A_2 = -1$.

Tensor-Product structure:

Let $nl_j\left(z(t)\right) \in \left[\underline{nl_j}, \overline{nl_j}\right]$, $j \in \{1, \cdots, p\}$ be the set of bounded nonlinearities in (2.1) belonging to *C*. Employing the sector nonlinearity approach [Tanaka & Wang, 2001], the following weighting functions can be constructed

$$w_0^j\left(z(t)\right) = \frac{\overline{nl_j} - nl_j\left(z(t)\right)}{\overline{nl_j}(\cdot) - \underline{nl_j}}, \quad w_1^j\left(z(t)\right) = 1 - w_0^j\left(z(t)\right), \; j \in \{1, \cdots, p\} \tag{2.6}$$

From the previous weights, the following MFs are defined:

$$h_i = h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}} = \prod_{j=1}^{p} w_{i_j}^j\left(z_j(t)\right) \tag{2.7}$$

with $i \in \left\{1, \cdots, 2^p\right\}$, $i_j \in \{0,1\}$. These MFs satisfy the convex sum property $\sum_{i=1}^{r} h_i\left(z(t)\right) = 1$, $h_i\left(z(t)\right) \geq 0$ in *C*.

Remark 2.2:

T-S models obtained via nonlinear sector approach depend directly on the number of nonlinearities to be cut. Thus, when one has nl nonlinear terms, then the T-S model contains 2^{nl} fuzzy rules.

2.1.2. Polynomial Takagi-Sugeno model

By the mean of the sector nonlinearity approach, a nonlinear system can be modeled by the so called Polynomail fuzzy model which allows an exact representation of the system (2.1) in a compact set of the state space, where the polynomial fuzzy model has a polynomial model consequence as developed thereinafter.

Model Rule i :

If $z_1(t)$ is M_{i1} AND $\cdots$ AND $z_p(t)$ is M_{ip}

Then

$$\begin{cases} \dot{x}(t) = A_i\left(x(t)\right)\hat{x}\left(x(t)\right) + B_i\left(x(t)\right)u(t) \\ y(t) = C_i\left(x(t)\right)\hat{x}\left(x(t)\right) \end{cases}, \; i = 1,2,\ldots,r \tag{2.8}$$

Where $A_i\left(x(t)\right)$, $B_i\left(x(t)\right)$ and $C_i\left(x(t)\right)$, $i \in \{1,\ldots,r\}$ are polynomial matrices in $x(t)$, $\hat{x}\left(x(t)\right)$ is a column vector of monomials in $x(t)$. This family of models will be the subject of chapter 4, in which further development will be given.

2.2. Lyapunov theory

Stability and stabilization analysis are usually based on Lyapunov theory [Vidyasagar, 1993], a large number of results have been obtained for continuous-time Takagi-Sugeno models, in this section, we will give some notions and types of the Lyapunov functions used in the literature.

Theorem 2.1:

An equilibrium point of a time-invariant dynamical system is stable (in the sense of Lyapunov) if there exists a continuously differentiable sector function $V(x)$ such that along the system trajectories the following is satisfied

$$V(x) > 0, \quad V(0) = 0 \tag{2.9}$$

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} \leq 0 \tag{2.10}$$

If the condition (2.10) is a strict inequality then the system is asymptotically stable.

In the following some definitions related to Lyapunov stability will be given:

Theorem 2.2:

Considering the non-linear system

$$\dot{x}(t) = f\left(x(t)\right) \tag{2.11}$$

with an isolated equilibrium point $x^* = 0 \in \Omega \subset \mathbb{R}^n$. If there exist a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ that has continuous partial derivatives and two $\mathbf{K}$ functions[1] α and β such that:

[1] A function $\alpha : [0,a) \to [0,\infty)$ is a k function, if it is strictly decreasing and $\alpha(0) = 0$, It is a $\mathbf{K}_\infty$ function if $a = \infty$ and $\lim_{t\to\infty} \varphi(t) = \infty$

$$\alpha(\|x\|) \le V(x) \le \beta(\|x\|), \quad \forall x^* = 0 \in \Omega \subset \mathbb{R}^n,$$

The origin $x = 0$ of system (2.11) is

- Stable if

$$\frac{\partial V(x)}{\partial t} \le 0, \quad \forall x \in \Omega, \quad x \neq 0;$$

- Asymptotically stable if there exists a K function φ such that

$$\frac{\partial V(x)}{\partial t} \le -\varphi(\|x\|), \quad \forall x \in \Omega, \quad x \neq 0;$$

- Exponentially stable if there exists four positive constant scalars $\bar{\alpha}, \bar{\beta}, \gamma, p$ such that

$$\alpha(\|x\|) = \bar{\alpha}\|x\|^p, \quad \beta(\|x\|) = \bar{\beta}\|x\|^p, \quad \varphi(\|x\|) = \gamma\|x\|;$$

The extension of this theorem for the case of non-autonomous systems is given in [Khalil, 2001].

2.2.1. Lyapunov functions:

Several Lyapunov functions candidate are usually proposed to solve the stability problem

a) Quadratic Lyapunov functions:

A classical Lyapunov function candidate is based on a quadratic form as:

$$V(x(t)) = x(t)^T P x(t), \quad P \in \mathbb{R}^{n \times n}, \quad P = P^T > 0 \tag{2.12}$$

Thus finding a Lyapunov function returns to find a definite positive matrix P.

It is well known that the existence of a quadratic Lyapunov function is only sufficient for asymptotic stability.

b) Piecewise Lyapunov functions:

The piecewise Lyapunov functions are more relaxed than the original designs because the quadratic Lyapunov function can be regarded as a special case of piecewise Lyapunov function, nevertheless, these designs always need certain restrictive boundary conditions or attach some extra constraints or assumptions, which greatly reduce the applicability.

A piecewise Lyapunov function is defined as [Johansson & al, 1999]:

$$V(x)=\begin{cases} x(t)^T P_i x(t), & \mathrm{x}\in \mathrm{S}_i, i\in L_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, & \mathrm{x}\in \mathrm{S}_i, i\in L_1 \end{cases} \tag{2.13}$$

Where $\{S_i\}_{i\in L} \subseteq \mathbb{R}^n$ is a polyhedral partition, L is the set of cell indexes, L_0 denotes the set of indexes of cells that contain the origin and L_1 denotes the set of indexes of cells that do not contain the origin.

This Lyapunov function is parameterized to be continuous across cell boundaries. This condition is fulfilled by means of constraint matrices $\bar{F}_i = [F_i \quad f_i]$ with $f_i = 0$ for $i \in I_0$ satisfying

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in \mathrm{S}_i \cap S_j$$

So we can parameterize Lyapunov functions as

$$P_i = F_i^T T F_i, \quad i \in L_0,$$

$$\bar{P}_i = \bar{F}_i^T T \bar{F}_i, \quad i \in L_1$$

Where free parameters are collected in symmetric matrix T, allowing LMI formulation. This Lyapunov function combines the power of quadratic Lyapunov functions near an equilibrium point with the flexibility of piecewise linear functions in the large.

c) Non-quadratic fuzzy Lyapunov functions

Non-quadratic fuzzy Lyapunov functions are generally given by

$$V(x(t)) = \sum_{i=1}^{r} h_i(x(t))\, x(t)^T P_i x(t), \tag{2.14}$$

Where P_i is a positive definite matrix and $h_i(z(t)) \geq 0$, $\sum_{i=1}^{r} h_i(z(t)) = 1$.

This function allows relaxing the constraints imposed by the quadratic approach. Indeed, finding a Lyapunov matrix for each local model is easier than find a common Lyapunov matrix for all local models. To find the matrices P_i, a convex optimization procedure was proposed by Johansson [Johansson & al, 1999] in the case of nonlinear systems continuously differentiable. Note that this function reduces to the quadratic case, if we simply choose $P_i = P$.

Several studies using this type of functions either in the continuous case [Jadbabaie & al, 1999], [Morère, 2001], [Blanco et al., 2001], [Tanaka et al., 2001c], or in the discrete case [Morère, 2001], [Kruzewski & al, 2008].

d) Polynomial Lyapunov functions

To check the stability and control design for nonlinear systems described by polynomial fuzzy models, polynomial Lyapunov functions as defined in the following, can be used

$$V\left(x(t)\right)=\hat{x}\left(x(t)\right)^{T}P\left(x(t)\right)\hat{x}\left(x(t)\right), \tag{2.15}$$

Where $P\left(x(t)\right)$ is a symmetric polynomial matrix in $x(t)$, $\hat{x}\left(x(t)\right)$ is a column vector whose entries are all monomials of $x(t)$.

This representation is more general than the quadratic one since if $\hat{x}(t)=x(t)$ and $P\left(x(t)\right)$ is a constant matrix, the polynomial Lyapunov function reduces to the quadratic ones, interesting results have been obtained overcoming the problem of conservativeness since polynomial fuzzy models are convex combinations of polynomial models instead of convex combinations of linear ones.

Consider the autonomous nonlinear system of the form

$$\dot{x}=f\left(x\right) \tag{2.16}$$

where $x\in\mathbb{R}^{n}$ and for which we assume without loss of generality that $f\left(0\right)=0$, i.e. the origin is an equilibrium of the system. A Lyapunov function can be found to prove the stability under some conditions which can be formulated as SOS program stated in the following proposition and solved using semi definite programming.

Proposition 2.1: [Papachristodoulou & Prajna, 2002]

Suppose that for the system (2.16) there exists a polynomial function $V\left(x\right)$ such that

$$V\left(0\right)=0, \tag{2.17}$$

$$V\left(x\right)-\phi\left(x\right)>0, \tag{2.18}$$

$$-\frac{\partial V}{\partial x}f\left(x\right)>0, \tag{2.19}$$

with $\phi\left(x\right)>0$ for $x\neq 0$. Then the zero equilibrium of the system is stable.

Proof: [Papachristodoulou & Prajna, 2002] Condition (2.18) enforces $V(x)$ to be positive definite. Since condition (2.19) implies that $\dot{V}(x)$ is negative semi definite, it follows that $V(x)$ is a Lyapunov function that proves stability of the origin

In the above proposition, the function $\phi(x)$ is used to enforce positive definiteness of $V(x)$. If $V(x)$ is a polynomial of degree $2d$, then $\phi(x)$ may be chosen as follows:

$$\phi(x) = \sum_{i}^{n} \sum_{j}^{d} \varepsilon_{ij} x_i^{2j},$$

Where ε's satisfy

$$\sum_{j}^{m} \varepsilon_{ij} > \gamma, \quad \forall \text{i=1,…,n,}$$

with γ a positive number, and $\varepsilon_{ij} \geq 0$ for all i and j. In fact, this choice of $\phi(x)$ will force $V(x)$ to be radially unbounded, and hence the stability property holds globally if the conditions in Proposition 1 are met.

2.2.2. Computationnal tools: Semi definite programming

Several techniques have been used in control theory in order to solve problems related to stability analysis and controller design for nonlinear systems. In the following, we define two powerful tools: Linear matrix inequalities (LMI Toolbox for Matlab) [Boyd & al, 1994] and Sum of squares (SOSTOOLS) [Prajna & al, 2004a].

a) Linear Matrix inequalities (LMI)

A linear matrix inequality or LMI is a matrix inequality of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{2.20}$$

Where $x \in \mathbb{R}^m$ is the variable; and $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $i \in 0,\ldots,m$ are known. The inequality given in (2.20) means that $F(x)$ is positive-definite, i.e., $u^T F(x) u > 0$ for all non-zero $u \in \mathbb{R}^n$. An LMI is a Set of n polynomial inequalities in x. The multiple LMIs $F_1(x) > 0, \ldots, F_n(x) > 0$ can be expressed as a single LMI:

$$\begin{bmatrix} F_1(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & F_n(x) \end{bmatrix} > 0 \tag{2.21}$$

Nowadays, LMI tools are well-known [Boyd & al, 1994] and we just recall thereinafter the properties necessary for the work presented in this thesis.

b) Sum of Squares (SOS)

A multivariable polynomial $p(x_1,\ldots,x_n) \triangleq p(x)$ is a sum of squares if there exist polynomials $f_1(x),\ldots,f_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x) \tag{2.22}$$

It is clear that $f(x)$ being an SOS naturally implies $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Though, a positive polynomial may not be written as SOS, except the some special cases (see [Reznick, 2000])

In the general multivariable case SOSTOOLS can solve two kinds of sum of squares programs: the feasibility and optimization problems [Prajna & al, 2004a].

Proposition 2.2:

A polynomial $p(x)$ of degree $2d$ is an SOS if and only if there exists a semi definite positive matrix Q and a vector of monomials $Z(x)$ containing all monomials in x of degree $\leq d$ such that

$$p(x) = Z(x)^T QZ(x)$$

The proof of this proposition is based on the eigenvalue decomposition and can be found in [Parrilo, 2000]. In general the monomials $Z(x)$ are not algebraically independent. Expanding $Z(x)^T QZ(x)$ and expanding the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine relations in the elements of Q. Since $p(x)$ being SOS is equivalent to $Q \geq 0$, the problem of finding a Q which proves that $p(x)$ is an SOS can be cast as a semi-definite program (SDP) [Parrilo, 2000] [Papachristodoulou & Prajna, 2002].

2.3. Key properties and lemmas

In this thesis, the following widely known properties from literature will be frequently used

Property 1: Schur complement:

Let $P \in \mathbb{R}^{n\times n}$ a positive definite matrix, $X \in \mathbb{R}^{m\times n}$ a full rang matrix and $Q \in \mathbb{R}^{n\times n}$, the following two inequalities are equivalent

$$1.\, Q(s) - X(s)^T P(s)^{-1} X(s) > 0,\ P(s) > 0$$

$$2.\begin{bmatrix} Q(s) & (*) \\ X(s) & P(s) \end{bmatrix} > 0$$

Property 2: Finsler's Lemma [Boyd & al, 1994]

Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $rank(R) < n$; the following expressions are equivalent:

a) $x^T Q x < 0$, $\forall x \in \{x \in \mathbb{R}^n : x \neq \mathbf{0}, Rx = 0\} < 0$

b) $\exists X \in \mathbb{R}^{n\times m} : Q + XR + R^T X^T < 0$.

c) $\exists \mu \in \mathbb{R} : Q - \mu R^T R < 0$.

Property 3: For $y \in \mathbb{R}^n$ and a scalar $\alpha > 0$, the following equivalence holds:

$$y^T y - \alpha < 0 \Leftrightarrow y y^T - \alpha I < 0 \tag{2.23}$$

Property 4: Inequality Lemma: Consider X, Y two matrices of appropriate dimension, for a scalar $\varepsilon > 0$, the following statement holds:

$$X^T Y + Y^T X \leq \varepsilon X^T X + \frac{1}{\varepsilon} Y^T Y \tag{2.24}$$

The same holds with a matrix $Q > 0$:

$$X^T Y + Y^T X \leq X^T Q X + Y^T Q^{-1} Y \tag{2.25}$$

Consider X, Y matrices of appropriate dimension, $\lambda, \beta \in \mathbb{R}$ with λ a varying parameter and the following inequalities problem:

$$|\lambda| < \beta,\ Y + \lambda X \le 0 \tag{2.26}$$

For purpose of proofs, a specific need of finding solutions to (2.26) will be necessary. Among the various possibilities next two properties will be useful.

Property 5: Sufficient conditions for (2.26) to hold are:

$$Y \pm \beta X \le 0. \tag{2.27}$$

Proof: as $|\lambda| < \beta$ thus via sector nonlinearity technique: $\lambda = \frac{\lambda+\beta}{2\beta} \times \beta + \frac{\beta-\lambda}{2\beta} \times (-\beta)$ with obviously: $\frac{\lambda+\beta}{2\beta} \ge 0$ and $\frac{\beta-\lambda}{2\beta} \ge 0$. Therefore, whatever are X and Y:

$$Y + \lambda X = \frac{\lambda+\beta}{2\beta} \times (Y + \beta X) + \frac{\beta-\lambda}{2\beta} \times (Y - \beta X) \tag{2.28}$$

Thus if (2.27) holds the conclusion (2.26) is obvious.

Property 6: A sufficient condition for (2.26) to hold, with $S = S^T > 0$ is:

$$Y + \frac{1}{2}\left(\beta^2 S + X S^{-1} X\right) \le 0 \ \Leftrightarrow \ \begin{bmatrix} -Y - \frac{1}{2}\beta^2 S & X \\ X & 2S \end{bmatrix} \ge 0 \tag{2.29}$$

Proof: using property 4 with any $S = S^T > 0$ of appropriate size gives:

$$Y + \frac{1}{2}(\lambda X + \lambda X) \le Y + \frac{1}{2}\left(\lambda^2 S + X S^{-1} X\right) \tag{2.30}$$

And using $|\lambda| < \beta$ gives directly the sufficient condition (2.29). The second part is just Schur's complement direct application.

Property 7: S-procedure

Consider matrices $T_i = T_i^T > 0$, $i \in \{1, \cdots, p\}$, the following two expressions are equivalent:

$$1. X^T T_0 X > 0, \quad \forall X \ne 0 \text{ such that } X^T T_i X \ge 0, \forall i \in \{1, \ldots, p\}$$

$$2. \exists \sigma_1, \ldots, \sigma_p \ge 0 \text{ such that } T_0 - \sum_{i=1}^{p} \sigma_i T_i > 0$$

a) Relaxation Lemmas

As it will be shown later in (2.40) one source of conservatism relies on the way the multiple sums are considered. To summarize therein various results, consider the following double sum problem with Υ_{ij} $(i,j)\in\{1,\cdots,r\}^2$ symmetric matrices of appropriate dimensions:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right)h_j\left(z(t)\right)\Upsilon_{ij} > 0 \tag{2.31}$$

A "nice" solution to (2.31) without adding slack variables is recalled.

Theorem 2.3: [Tuan & al, 2001]**:**

Sufficient conditions for (2.31) to hold are:

$$\begin{aligned} &\Upsilon_{ii} > 0, \quad \forall i\in\{1,\cdots,r\} \\ &\frac{2}{r-1}\Upsilon_{ii}+\Upsilon_{ij}+\Upsilon_{ji}\geq 0, \quad \forall (i,j)\in\{1,\cdots,r\}^2, i\neq j. \end{aligned} \tag{2.32}$$

Other conditions can be obtained introducing slack variables, for example [Liu & Zhang, 2003].

Theorem 2.4: [Liu & Zhang, 2003]:

Sufficient conditions for (2.31) to hold are:

If there exist matrices $Q_i = Q_i^T, i\in\{1,\cdots,r\}$ and $Q_{ij} = Q_{ji}^T, (i,j)\in\{1,\cdots,r\}^2$ such that:

$$\begin{aligned} &\Upsilon_{ii}\geq Q_i, \quad \forall i\in\{1,\cdots,r\} \\ &\Upsilon_{ij}+\Upsilon_{ji}\geq Q_{ij}+Q_{ji}, \quad \forall (i,j)\in\{1,\cdots,r\}^2, i>j. \end{aligned}$$

$$\begin{bmatrix} Q_1 & Q_{12} & \cdots & Q_{1r} \\ Q_{21} & Q_2 & & \vdots \\ \vdots & & \ddots & Q_{(r-1)r} \\ Q_{r1} & \cdots & Q_{r(r-1)} & Q_r \end{bmatrix} > 0$$

Conditions in [Liu & Zhang, 2003] have been further improved, at the expense of higher computational cost. Consider a multi-dimensional index variable $i\in\{1,\ldots,r\}^n$ where r is the number of rules and n is an arbitrary complexity parameter. Then the result in [Liu & Zhang, 2003] is a particular case of the following theorem.

Theorem 2.5: [Sala & Ariño, 2007]**:**

Sufficient conditions for (2.31) to hold are:

The following inequality (with complexity $n-2$) holds

$$\sum_{\mathbf{k}\in B_{n-2}} h_{\mathbf{k}}\xi^T \begin{bmatrix} Q_{(\mathbf{k},1,1)} & \cdots & Q_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ Q_{(\mathbf{k},r,1)} & \cdots & Q_{(\mathbf{k},r,r)} \end{bmatrix} \xi > 0, \quad \text{for } \xi = \begin{pmatrix} h_1 x^T & h_2 x^T & \dots & h_r x^T \end{pmatrix}^T \neq 0$$

if there exist matrices $X_{\mathbf{j}}$, $\mathbf{j}\in B_n$ so that

$$\sum_{\mathbf{j}\in P(\mathbf{i})} \Upsilon_{j_1 j_2} > \sum_{\mathbf{j}\in P(\mathbf{i})} \frac{1}{2}\left(Q_{\mathbf{j}} + Q_{\mathbf{j}}^Y\right), \qquad \forall \mathbf{i}\in B_n^+$$

where $P(\mathbf{i})$ denotes all the permutations of $\mathbf{i}$, $\mathbf{k} = (\mathbf{i},\mathbf{j})$,

$B_n = \left\{\mathbf{i} = (i_1, i_2, \dots, i_n)\in \mathbb{N}^p \,/\, 1\le i_j \le r, \forall j = 1,\dots,n)\right\}$ and $B_n^+ = \left\{\mathbf{i}\in B_n \,/\, i_k \le i_{k+1},\ k = 1,\dots,n-1)\right\}$

In a suitable recursive framework, it can be proved that the above conditions become necessary and sufficient with $n\to\infty$, and establish some tolerance parameter for finite n [Sala & Ariño, 2007].

In the following part of this chapter, we recall of some existing approaches proposed to overcome the drawbacks of the quadratic approach in stability analysis and controller design using convex optimization techniques (LMIs Toolbox) and sum of squares (SOS) tools.

2.4. Quadratic Lyapunov function approach forT-S models

2.4.1. Quadratic Stability of Takagi-Sugeno models

Consider the Open loop T-S fuzzy system:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t) \tag{2.33}$$

Based on the quadratic Lyapunov function given in (2.12), the following stability theorem can be formulated.

Theorem 2.6: [Tanaka & Sugeno, 1992]

Consider the model (2.33), if there exists a matrix $P = P^T > 0$, such the following holds

$$A_i^T P + P A_i < 0, \quad \forall i=1,\ldots,\mathrm{N}$$

The T-S fuzzy model given in (2.33) is globally asymptotically stable.

The stability conditions in Theorem 2.6 are only sufficient since the membership functions i.e. $h_i\left(z(t)\right)$ are not taken into account.

2.4.2. Quadratic stabilization of Takagi-Sugeno models:

Several control laws have been proposed in the literature, to deal with the stability of the closed loop of Takagi-Sugeno models, the most used control law is the parallel distributed compensation controller.

a) PDC control law (Parallel distributed compensation):

Takagi-Sugeno models can be stabilized using the parallel distributed compensation (PDC) controller [Wang & al, 1996]. Each control rule is designed from the corresponding rule of T-S fuzzy model. Moreover, the linear control technique can be used to design the consequent parts of a T-S fuzzy controller, because the consequent parts are described by linear state equations.

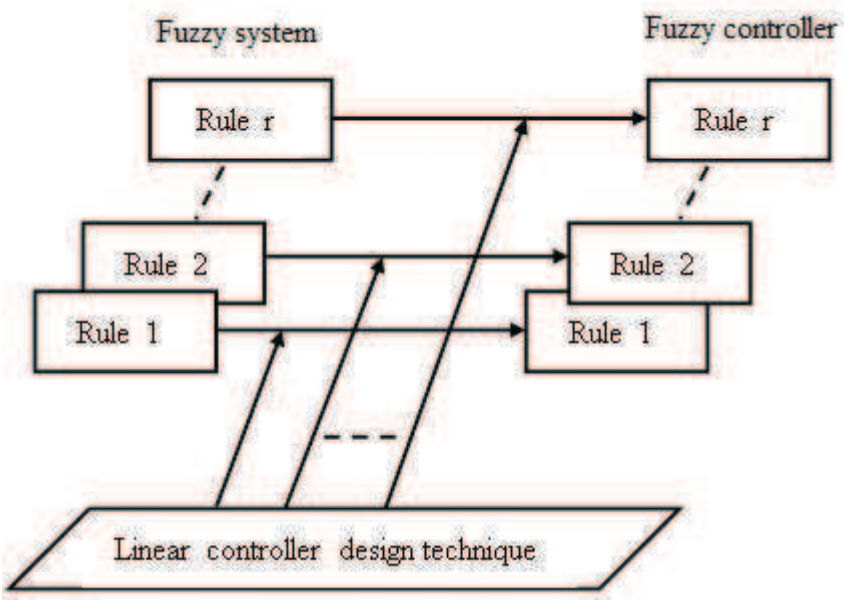


Figure 2.3: Parallel distributed compensation controller design [Wang et al, 1996]

Controller Rule i :

$$\text{If } z_1(t) \text{ is } M_{i1} \text{ AND } \cdots \text{ AND } z_p(t) \text{ is } M_{ip}$$

Then

$$u_i(t) = F_i x(t),\ i = 1, 2, \ldots, r \tag{2.34}$$

The PDC controller shares the same fuzzy sets with the T-S fuzzy model in the premise parts, this mirrored structure is necessary for the LMI-Based analysis and the design procedures.

b) Quadratic Controller design

Consider the following Takagi-Sugeno model given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right) \tag{2.35}$$

The PDC controller is

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) F_i x(t) \tag{2.36}$$

By substituting (2.36) in (2.35), we obtain the Takagi-Sugeno closed loop as follows:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t))\left(A_i + B_i F_j\right) x(t) \tag{2.37}$$

The design of the controller (2.36) returns to calculate the local gains F_i which ensure the stability of the closed loop (2.37), we consider a quadratic Lyapunov function candidate as in (2.12) with $P = P^T > 0$, thus its derivative writes:

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \\ &= x^T(t)\left(\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t))\left(A_i^T P + P A_i + F_j^T B_i^T P + P B_i F_j\right)\right) x(t) < 0 \end{aligned} \tag{2.38}$$

or

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t))\left(A_i^T P + P A_i + F_j^T B_i^T P + P B_i F_j\right) < 0 \tag{2.39}$$

Note that inequality (2.39) is a bilinear matrix inequality (BMI) due to the existence of bilinear terms $F_j^T B_i^T P$ and $P B_i F_j$. In order to obtain linear matrix inequality constraints, left and right product with $X = P^{-1}$ and the classical change of variables $P = X^{-1}$ and $M_j = F_j X$ [Wang & al, 1996] give the following stabilization conditions:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t))\left(X A_i^T + A_i X + M_j^T B_i^T + B_i M_j\right) < 0 \tag{2.40}$$

Thus next result is directly based on $h_i(z(t)) h_j(z(t)) = h_j(z(t)) h_i(z(t))$.

Theorem 2.7: [Tanaka & al, 1998]

The equilibrium of the continuous fuzzy control system described by (2.35) is asymptotically stable in the large if there exist a common positive definite matrix $X = X^T > 0$ and matrices M_i such that $M_i = F_i X$ for which the following holds:

$$\Upsilon_{ii} < 0 \tag{2.41}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad i < j \tag{2.42}$$

With

$$\Upsilon_{ij} = XA_i^T + A_i X + M_j^T B_i^T + B_i M_j$$

for all i and j excepting the pairs (i, j) such that $h_i\left(z(t)\right)h_j\left(z(t)\right) = 0, \forall t$.

Moreover, if a solution holds, the control gains are derived using: $F_i = M_i X^{-1}$.

The goal here is to find a common matrix P and gains F_i simultaneously by solving some conditions which can be formulated as Linear matrix inequalities (LMI) that can be easily solved with convex programming techniques i.e. LMI toolbox of Matlab.

2.5. Non-quadratic fuzzy Lyapunov function (NQ) approaches

2.5.1. Non-quadratic stability analysis

Consider the Takagi-Sugeno model given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t) \tag{2.43}$$

Consider the non-quadratic Lyapunov function candidate:

$$V\left(x(t)\right) = x^T(t)\left(\sum_{i=1}^{r} h_i\left(z(t)\right) P_i\right) x(t) \tag{2.44}$$

With $P_i = P_i^T > 0$. The derivative of the Lyapunov function writes [Tanaka & al., 2003]:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right)h_j\left(z(t)\right)\left(A_j^T P_i + P_i A_j + \sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) P_k\right) < 0 \tag{2.45}$$

Theorem 2.8: [Tanaka & al, 2003]

Assume that $\left|\dot{h}_k\left(z(t)\right)\right| \le \phi_k$, the fuzzy system is stable if there exist $\phi_k \ge 0$, $k = 1,\ldots,r$ such that:

$$P_i > 0, i = 1,\ldots,r$$

$$\sum_{k=1}^{r} \phi_k P_k + \frac{1}{2}\left\{A_j^T P_i + P_i A_j + A_i^T P_j + P_j A_i\right\} < 0,\ i \le j$$

Note that due to the convex sum property $\sum_{\rho=1}^{r} h_\rho\left(z(t)\right) = 1$ it follows directly:

$$\sum_{\rho=1}^{r} \dot{h}_\rho\left(z(t)\right) = 0, \qquad \forall z(t)$$

This property allows extra term addition that relaxes Theorem 2.8.

Theorem 2.9: [Tanaka & al, 2003]

Assume that $\left|\dot{h}_k\left(z(t)\right)\right| \le \phi_k$, the fuzzy system is stable if t$\phi_k \ge 0$here exist , $k = 1,\ldots,r-1$ such that:

$$P_i > 0, i = 1,\ldots,r$$

$$P_k \ge P_r,\ k = 1,\ldots,r-1$$

$$\sum_{k=1}^{r-1} \phi_k \left(P_k - P_r\right) + \frac{1}{2}\left\{A_j^T P_i + P_i A_j + A_i^T P_j + P_j A_i\right\} < 0,\ i \le j$$

The proposed approaches use priori known bounds of the time-derivative of the membership functions which are not always readily available, thing that turns out to be very restrictive.

Another way to overcome the disadvantages of the quadratic approach is to consider the line-integral Lyapunov function [Rhee & Won, 2006]:

$$V(x) = 2\int_{\Gamma(0,x)} f(\psi)\cdot d\psi \tag{2.46}$$

Where $\Gamma(0,x)$ a path from the origin to the current state is x, $\psi \in \mathbb{R}^n$ is a dummy vector for the integral, $f(x) \in \mathbb{R}^n$ is a vector function of the state x, $(\cdot)$ denotes an inner product, and $d\psi \in \mathbb{R}^n$ is an infinitesimal displacement vector. Thus, the application to T-S models follows Non-quadratic approaches such as (2.44) and a possibility is:

$$f(x) = P(x)x$$

Where

$$P(x) = \sum_{i=1}^{r} h_i(x) P_i > 0$$

Nevertheless, it restricts to specific T-S model of the form [Rhee & Won, 2006]: $z(t) = x(t)$ and h_i must only depend on x_i. For example $x_1 \times x_2$ or $x_1^2 \times \sin(x_2)$ could not be used as premise variables. Moreover, the degrees of freedom of the P_i $i \in \{1,\ldots,r\}$ are also very restricted due to the necessary and sufficient conditions for path-independency.

Theorem 2.10: [Rhee & Won, 2006]

The T-S fuzzy system (2.43) is asymptotically stable if there exist $\overline{P}$, D_i and $X \geq 0$ satisfying

$$P_i = \overline{P} + D_i > 0, \quad i = 1,\ldots,r \tag{2.47}$$

$$P_i A_i + A_i^T P_i + (s-1) X < 0, \quad i = 1,\ldots,r \tag{2.48}$$

$$P_i A_j + A_j^T P_i + P_j A_i + A_i^T P_j - 2X \leq 0, \quad i, j = 1,\ldots,r, \text{i<j} \tag{2.49}$$

where $\overline{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n} \\ p_{12} & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & 0 \end{bmatrix}$, $D_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^{\alpha_{in}} \end{bmatrix}$

Although the restrictions depicted, the approach is interesting as it gives LMI conditions in a global sense without any bounds on the membership functions derivative such as needed for (2.45). At last, for stabilization, unfortunately the problem cannot be written in LMI constraints [Rhee & Won, 2006] and solution using two-path algorithms are required.

To overcome the difficulties listed before, A systematic approach has been proposed by [Mozelli & al, 2009a] improving the results those obtained by [Tanaka & al, 2003], the method consist on introducing slack variables into the LMIs in order to separate the system matrices from the Lyapunov matrices, providing more relaxed LMI Conditions

Based on multiple Lyapunov function defined as in (2.44), Stability of the TS model is guaranteed via the following theorem:

Theorem 2.11: [Mozelli & al, 2009a]

The fuzzy system is stable if there exist symmetric matrices P_i, M_3 and matrices M_1, M_2 such that the following set of LMIs:

$$P_i > 0, \quad i = 1,\ldots,r, \tag{2.50}$$

$$\Xi_i < 0, \quad i = 1,\ldots,r, \tag{2.51}$$

$$P_i + M_3 > 0, \quad i = 1,\ldots,r, \tag{2.52}$$

Hold with

$$\Xi_i \triangleq \begin{bmatrix} P_\phi - M_1 A_i - A_i^T M_1^T & (*) \\ P_i - \left(M_2 A_i - M_1^T\right) & M_2 + M_2^T \end{bmatrix} \tag{2.53}$$

$$P_\phi \triangleq \sum_{\rho=1}^{r} \phi_\rho \left(P_\rho + M_3\right) \tag{2.54}$$

$\left|\dot{h}_\rho\left(z(t)\right)\right| \le \phi_\rho$ with $h_\rho\left(z(t)\right) \in C^1$ and $\phi_\rho \ge 0, \rho = 1,\ldots,r$ are given scalars

2.5.2. Non-quadratic stabilization of T-S models

Results obtained for stability are not directly exploitable for stabilization, especially to derive LMI constraints. Consider the Takagi-Sugeno fuzzy system:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right) \tag{2.55}$$

Consider the following PDC controller:

$$u(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) F_i x(t) \tag{2.56}$$

The closed loop writes:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right)\left(A_i + B_i F_j\right) x(t) \tag{2.57}$$

Therefore, bounds such that $\left|\dot{h}_k\left(z(t)\right)\right| \le \phi_k$ become difficult to justify. To illustrate this point consider an example issued from [Tognetti, 2010]. Consider a 2-rules TS model with:

$$A_1 = \begin{bmatrix} 3.6 & -1.6 \\ 6.2 & -4.3 \end{bmatrix},\ A_2 = \begin{bmatrix} -a & -1.6 \\ 6.2 & -4.3 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.45 \\ -3 \end{bmatrix},\ B_2 = \begin{bmatrix} -b \\ -3 \end{bmatrix},$$

The MFs $h_1 = w_0^1 = \frac{1-\sin x_1}{2}$ and $h_2 = w_1^1 = \frac{1+\sin x_1}{2}$ are defined in the compact set $C = \{x : |x_i| \le \pi/2\}, i = 1,2$. Note that the time-derivative of MFs can be developed as:

$$\begin{aligned} \left|\dot{h}_1\right| &= \left|\dot{h}_2\right| = \left|\frac{\cos x_1}{2}\dot{x}_1\right| \\ &= \left|\frac{\cos x_1}{2}\sum_{i=1}^{2} h_i\left(z(t)\right)\left(\left[A_i\right]_{1*} x(t) + \left[B_i\right]_{1*} u(t)\right)\right| \\ &= \left|\frac{\cos x_1}{2}\left(\left(\frac{1-\sin x_1}{2}\right)\left(3.6x_1 - 1.6x_2 - 0.45u\right) + \left(\frac{1+\sin x_1}{2}\right)\left(3.6x_1 - 1.6x_2 - bu\right)\right)\right| \end{aligned} \tag{2.58}$$

which depends on the input $u(t)$ that cannot be known beforehand. Therefore, the conditions in [Tanaka & al, 2003] assuming $\left|\dot{h}_i\right| \le \phi_i = 1$ is somewhat difficult to uphold. This is the major flaw of these approaches and will be highly discussed in the next chapters.
Nevertheless, we give some results found in the literature.

A different result uses an extended control law [Tanaka & al, 2003]:

$$u(t) = -\sum_{i=1}^{r} h_i\left(z(t)\right) F_i x(t) - \sum_{i=1}^{r} \dot{h}_i\left(z(t)\right) T_i x(t) \tag{2.59}$$

Therefore, the closed loop writes

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right)\left(A_i - B_i F_j - \sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) B_i T_k\right) x(t) \tag{2.60}$$

Based on the non-quadratic Lyapunov functions given in (2.44) the following theorem is obtained.

Theorem 2.12: [Tanaka & al, 2003]
The fuzzy system is stable via the new PDC controller if there exist $\varepsilon > 0$, $\gamma > 0$, s_i, positive definite matrices $P_i = P_i^T > 0$, F_i, T_i, $i = 1,\ldots,r$ such that:

$$P_i \ge s_i I,\ s_i \ge 1 \tag{2.61}$$

$$\begin{bmatrix} \left(\begin{array}{c} \frac{1}{3\varepsilon^2(r-1)}(s_i+s_j+s_k)I_{n\times n} \\ +\frac{1}{\gamma^2}(s_i+s_j)I_{n\times n}-\mu_{\rho m}(P_\rho-P_r) \end{array}\right) & \Omega_\upsilon & \Lambda_\upsilon & \Pi_\upsilon \\ \Omega_\upsilon^T & 6(r-1)I_{6n\times 6n} & 0 & 0 \\ \Lambda_\upsilon^T & 0 & 2I_{2n\times 2n} & 0 \\ \Pi_\upsilon^T & 0 & 0 & 2I_{2n\times 2n} \end{bmatrix} > 0 \qquad (2.62)$$

$i \le j \le k$, $\rho = 1,\ldots,r-1$, $m = 1,2$

Where

$$\Omega_\upsilon = \begin{bmatrix} \Omega_{ijk} & \Omega_{ikj} & \Omega_{jik} & \Omega_{jki} & \Omega_{kij} & \Omega_{kji} \end{bmatrix}$$

$$\Lambda_\upsilon = \begin{bmatrix} \Lambda_{ij\rho m} & \Lambda_{ji\rho m} \end{bmatrix},\ \Pi_\upsilon = \begin{bmatrix} \Pi_{ijrm} & \Pi_{jirm} \end{bmatrix},\ \Omega_{ijk} = \varepsilon\left(A_i - B_i F_j\right)^T + \frac{1}{\varepsilon}P_k$$

$$\Lambda_{ij\rho m} = \gamma\mu_{\rho m}\left(B_j T_\rho\right)^T - \frac{1}{\gamma}P_i,\ \Pi_{ijrm} = \gamma\mu_{rm}\left(B_j T_r\right)^T + \frac{1}{\gamma}P_i$$

Apart the fact already mentioned on the bounds $\left|\dot{h}_k\left(z(t)\right)\right| \le \phi_k$, note that (2.59) corresponds to an algebraic loop as soon as $\dot{h}_i\left(z(t)\right)$ is control dependent such as in the example (2.58), therefore its use is highly restricted.

The path-independence property has also been used in [Rhee & Won, 2006]. By the means of line integral Lyapunov function presented in (2.46) for the non-quadratic controller design, it results the following theorem:

Theorem 2.13: [Rhee & Won, 2006]

The T-S fuzzy control system (2.55) with the fuzzy controller (2.56) is asymptotically stable if there exist $\overline{P}$, D_i, F_i and $X \ge 0$ satisfying

$$P_i = \overline{P} + D_i > 0,\quad i = 1,\ldots,r \qquad (2.63)$$

$$G_{iii} + G_{iii}^T + (s-1)X < 0,\quad i = 1,\ldots,r \qquad (2.64)$$

$$G_{iij} + G_{iij}^T + \frac{1}{3}(s-3)X \le 0,\quad i,j = 1,\ldots,r,\ i \ne j \qquad (2.65)$$

$$G_{ijk} + G_{ijk}^T - X \le 0,\quad i,j,k = 1,\ldots,r,\ i<j<k \qquad (2.66)$$

Where

$$G_{iii} = P_i\left(A_i + B_i F_i\right)$$

$$G_{iij} = \frac{1}{3}\left\{P_i\left(A_i + A_j + B_iF_j + B_jF_i\right) + P_j\left(A_i + B_iF_i\right)\right\}$$

$$G_{ijk} = \frac{1}{6}\left\{P_i\left(A_j + A_k + B_jF_k + B_kF_j\right) + P_j\left(A_i + A_k + B_iF_k + B_kF_i\right)P_k\left(A_i + A_j + B_iF_j + B_jF_i\right)\right\}$$

$$\bar{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n} \\ p_{12} & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & 0 \end{bmatrix}, \; D_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^{\alpha_{in}} \end{bmatrix}$$

As already mentioned, conditions (2.63) to (2.66) are not LMI as soon as the control gains F_i $i \in \{1,\ldots,r\}$ are searched. Therefore a 2-step algorithm is proposed in [Rhee & Won, 2006]. Firstly, using a locally-available conventional quadratic Lyapunov function approach, feedback gains guaranteeing the local stability are selected. Then, being fixed (2.63) to (2.66) becomes LMI and can be solved. Several loops can be necessary and there is no guarantee of convergence towards a solution. Nevertheless, it always includes the quadratic case as shown in [Rhee & Won, 2006].

In [Mozelli & al, 2009a], A new PDC based fuzzy control design is proposed with:

$$u(t) = -\sum_{i=1}^{r} h_i\left(z(t)\right)F_i x(t) \tag{2.67}$$

Based on the non-quadratic Lyapunov functions given in (2.44) the following theorem is obtained.

Theorem 2.14: [Mozelli & al, 2009a]

Given a scalar $\mu > 0$, The fuzzy system is stabilizable by the fuzzy controller if there exist symmetric matrices T_i, Y and any matrices R, S_i satisfying the following set of LMIs:

$$T_i > 0, \tag{2.68}$$

$$T_i + Y > 0, \;\; i = 1,\ldots,r, \tag{2.69}$$

$$\Xi_{ii} < 0, \;\; i = 1,\ldots,r, \tag{2.70}$$

$$\overline{\Xi}_{ij} < 0, \;\; i < j = 1,\ldots,r, \tag{2.71}$$

where

$$\Xi_{ij} \triangleq \begin{bmatrix} T_\phi - A_i R^T - R A_i^T + B_i S_j^T + S_j B_i^T & (*) \\ T_i - \mu\left(A_i R^T - B_i S_j^T\right) & \mu\left(R + R^T\right) \end{bmatrix} \tag{2.72}$$

$$\overline{\Xi}_{ij} \triangleq \Xi_{ij} + \Xi_{ji} \tag{2.73}$$

$$T_\phi \triangleq \sum_{\rho=1}^{r} \phi_\rho \left(T_\rho + Y\right) \tag{2.74}$$

The upper bounds for the time-derivative of the membership functions are considerd available.

2.6. Polynomial Lyapunov function approach for T-S models

2.6.1. Non-quadratic stability analysis: SOS approach

Consider the following polynomial fuzzy model

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i\left(x(t)\right) \hat{x}\left(x(t)\right) \tag{2.75}$$

where $A_i\left(x(t)\right)$ is a polynomial matrix in $x(t)$. The term $\hat{x}\left(x(t)\right)$ is a column vector which entries are all monomials in $x(t)$ that is, $\hat{x}\left(x(t)\right) \in \mathbb{R}^N$ is an $N \times 1$ vector of monomials in $x(t)$ and consider a candidate of polynomial Lyapunov function

$$V(x) = \hat{x}^T\left(x(t)\right) P\left(x(t)\right) \hat{x}\left(x(t)\right) \tag{2.76}$$

Where $P\left(x(t)\right) \in \mathbb{R}^{N \times N}$ is a symmetric polynomial matrix.

Theorem 2.15: [Tanaka & al, 2009b]
The zero equilibrium of the system (2.75) is stable if there exists a symmetric polynomial matrix $P(x) \in \mathbb{R}^{N \times N}$ such that (2.77) and (2.78) are satisfied, where $\varepsilon_1(x)$ and $\varepsilon_{2i}(x)$ are nonnegative polynomials such that $\varepsilon_1(x) > 0$ for $x \neq 0$ and $\varepsilon_{2i}(x) \geq 0$ for all x :

$$\hat{x}^T(x)\left(P(x) - \varepsilon_1(x) I\right) \hat{x}(x) \text{ is SOS} \tag{2.77}$$

$$-\hat{x}^T(x) \begin{pmatrix} P(x) T(x) A_i(x) + A_i^T(x) T^T(x) P(x) + \\ \sum_{k=1}^{n} \frac{\partial P}{\partial x_k}(x) A_i^k(x) \hat{x}(x) + \varepsilon_{2i}(x) I \end{pmatrix} \hat{x}(x), \ \forall i, \text{is SOS} \tag{2.78}$$

where $T(x) \in \mathbb{R}^{N \times n}$ is a polynomial matrix whose (i, j) the entry is given by

$$T^{ij}(x) = \frac{\partial \hat{x}_i}{\partial x_j}(x) \tag{2.79}$$

In addition, if (2.78) holds with $\varepsilon_{2i}(x) > 0$ for $x \neq 0$, then the zero equilibrium is asymptotically stable. If $P(x)$ is a constant matrix, then the stability holds globally.

2.6.2. Non-quadratic stabilization analysis: SOS approach

Consider the following Takagi-Sugeno model [Tanaka & al, 2009b]

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) \left\{ A_i(x(t)) \hat{x}(x(t)) + B_i(x(t)) u(t) \right\} \tag{2.80}$$

The overall fuzzy controller is given by:

$$u(t) = -\sum_{i=1}^{r} h_i(z(t)) F_i(x(t)) \hat{x}(x(t)) \tag{2.81}$$

From (2.80) and (2.81), the closed loop system can be represented as

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \left\{ A_i(x(t)) - B_i(x(t)) F_j(x(t)) \right\} \hat{x}(x(t)) \tag{2.82}$$

And consider a candidate of polynomial Lyapunov function

$$V(x) = \hat{x}^T(x(t)) X^{-1}(\tilde{x}) \hat{x}(x(t)) \tag{2.83}$$

Where $X^{-1}(\tilde{x}) \in \mathbb{R}^{N \times N}$ is a symmetric polynomial matrix.

Let $A_i^k(x)$ denote the $k-th$ row of $A_i(x)$, $K = (k_1, k_2, \ldots, k_m)$ denote the row indices of $B_i(x)$ whose corresponding row is equalto zero, and define $\tilde{x} = (x_{k_1}, x_{k_2}, \ldots, x_{k_m})$.

Theorem 2.16: [Tanaka & al, 2009b]
The control system consisting of (2.80) and (2.81) is stable if there exist a symmetric polynomial matrix $X(\tilde{x}) \in \mathbb{R}^{N \times N}$ and a polynomial matrix $M_i(x) \in \mathbb{R}^{m \times N}$ such that (2.84) and (2.85) are satisfied where $\varepsilon_1(x)$ and $\varepsilon_{2ij}(x)$ are nonnegative polynomials such that $\varepsilon_1(x) > 0$ for $x \neq 0$ and $\varepsilon_{2ij}(x) \geq 0$ for all x:

$$v^T \left(X(\tilde{x}) - \varepsilon_1(x) I \right) v \text{ is SOS} \tag{2.84}$$

$$-\upsilon^T \begin{pmatrix} T(x)A_i(x)X(\tilde{x}) - T(x)B_i(x)M_j(x) + X(\tilde{x})A_i^T(x)T^T(x) \\ -M_j^T(x)B_i^T(x)T^T(x) + T(x)A_j(x)X(\tilde{x}) - T(x)B_j(x)M_i(x) \\ +X(\tilde{x})A_j^T(x)T^T(x) - M_i^T(x)B_j^T(x)T^T(x) \\ -\sum_{k\in K}\frac{\partial X}{\partial x_k}(\tilde{x})A_i^k(x)\hat{x}(x) - \sum_{k\in K}\frac{\partial X}{\partial x_k}(\tilde{x})A_j^k(x)\hat{x}(x) + \varepsilon_{2ij}(x)I \end{pmatrix} \upsilon, \quad \text{is SOS}, \forall i \le j \tag{2.85}$$

Where $\upsilon \in \mathbb{R}^N$ is a vector that is independent of x. $T(x) \in \mathbb{R}^{N\times n}$ is a polynomial matrix whose $(i,j)th$ entry is given by

$$T^{ij}(x) = \frac{\partial \hat{x}_i}{\partial x_j}(x) \tag{2.86}$$

In addition, if (2.78) holds with $\varepsilon_{2ij}(x) > 0$ for $x \neq 0$, then the zero equilibrium is asymptotically stable. If $X(\tilde{x})$ is a constant matrix, then the stabilization holds globally.

2.7. Conclusion

In this chapter, we presented a state of art of Takagi-Sugeno models and an introduction to some of the basic concepts used in this thesis. Classical stability and stabilization conditions based on Lyapunov theory have been discussed.

Several relaxation schemes and existing approaches used in the literature to overcome the drawbacks of the quadratic approach have been presented. Clearly the so-called non-quadratic approaches, especially for stabilization, are not satisfactory: no "pure" LMI constraints, *a priori* assumptions that may be impossible to fulfill. Mainly, these approaches present many drawbacks due to the way the time-derivative of the membership functions are dropped and the type of the Lyapunov function used to prove the stability, new non-quadratic approaches for stability analysis and controller design allowing to obtain less conservative results for continuous-time Takagi-Sugeno models will be successively proposed in the following part.

Part II: Contributions

3. Chapter 3: Non-quadratic stability of T-S models: Bounding the MF partial derivatives

Synopsys

This chapter presents the first contribution of this thesis dealing with Local stability analysis for continuous-time Takagi-Sugeno models. New LMI conditions for non-quadratic stability will be derived to overcome the drawbacks of global quadratic solutions. The major contribution is to take into account all the structural information of the membership function when dealing with its time derivative, in particular the partial derivatives of the memberships with respect to the states instead of time. Parts of this chapter are inspired from a publication in which I contributed [Sala & al, 2010].

3.1. Introduction

In the previous chapter, we presented different existing approaches in the literature studying the stability analysis and controller design, thus these approaches generally gives sufficient conditions which lead to obtain conservative results, this problem is due to different reasons, the principal reasons are the type of candidate Lyapunov functions, the way the sums are dropped or also the choice of the relaxation lemmas, these sources and some recent alternatives to overcome them will be discussed in the following sections.

3.2. New local condition for stability analysis of T-S models

This work is based on a new approach first proposed by [Guerra & Bernal, 2009] to deal with the stability analysis of continuous-time T-S models, by the mean of non-quadratic Lyapunov functions, new local conditions have been obtained and proved to be less restrictive than the global ones, moreover this approach has led to an estimation of the stability domain, which is usually the case for nonlinear models [Khalil, 2002].

Consider the following continuous-time T-S model:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t) = A_z x(t) \tag{3.1}$$

where $h_i\left(z(t)\right)$ fulfil asumptions in (2.7). In [Guerra & Bernal, 2009] stability of T-S model (3.1) is investigated using the following non-quadratic Lyapunov function candidate:

$$V(x) = x(t)^T \sum_{i=1}^{r} h_i\left(z(t)\right) P_i x(t) = x(t)^T P_z x(t) \tag{3.2}$$

Where $P_z = P_z^T > 0$.

Its time-derivative along the trajectories of the T-S model (3.1) is:

$$\dot{V}(x) = x^T(t)\left(P_z A_z + A_z^T P_z + \dot{P}_z\right) x(t). \tag{3.3}$$

To ensure the stability of the T-S model (3.1), $\dot{V}(x) < 0$ should be fulfilled, which is equivalent to :

$$P_z A_z + A_z^T P_z + \dot{P}_z < 0. \tag{3.4}$$

Remark 3.1: LMI conditions for stability analysis are usually derived from inequalities as (3.4). However, obtaining LMIs from (3.4) for global stability is no longer possible since the term $\dot{P}_z = \sum_{i=1}^{r} \dot{h}_i P_i$ depends on the time-derivatives of MFs.

Theorem 3.1: (Local stability) The T-S model (3.1) is locally asymptotically stable in a domain D including the origin, if there exist matrices of proper dimension $P_i > 0$, $i \in \{1, \cdots, r\}$ such that the following holds

$$P_z A_z + A_z^T P_z < 0. \tag{3.5}$$

Proof: The non-quadratic Lyapunov function candidate (3.2) satisfies $V(0) = 0$, $V(x) \geq 0$ in $\mathbb{R}$. Its time-derivative (3.4) holds $\dot{V}(0) = 0$. Provided that $P_z A_z + A_z^T P_z < 0$, it is implied that there exists a sufficiently small $\lambda > 0$ such that $P_z A_z + A_z^T P_z + \lambda I < 0$ which can be used to define $D = \left\{ x : x \in B, \left\| \dot{P}_z \right\| < \lambda \right\}$. The origin belongs to domain D since

$$\begin{aligned} \dot{P}_z &= \sum_{i=1}^{r} \dot{h}_i P_i = \sum_{i=1}^{r} \left(\frac{\partial h_i}{\partial z} \right)^T \dot{z} P_i = \sum_{i=1}^{r} \left(\frac{\partial h_i}{\partial z} \right)^T \left(\frac{\partial z^T}{\partial x} \right) \dot{x} P_i \\ &= \sum_{i=1}^{r} \left(\frac{\partial h_i}{\partial z} \right)^T \left(\frac{\partial z^T}{\partial x} \right) A_z x P_i \end{aligned} \tag{3.6}$$

depends on the state vector $x(t)$. Since $V(x) > 0$ and $\dot{V}(x) < 0$ in $D - \{0\}$, the equilibrium point $x = 0$ is locally asymptotically stable, thus concluding the proof. □

In [Guerra & Bernal, 2009], a new approach is proposed to overcome the difficulty mentioned in Remark 3.1 via a local approach, which has allowed to obtain a better region of attraction (local stability). Taking into account all the information contained in the membership functions definition, $\dot{P}_z$ is developed as follows:

$$\begin{aligned} \dot{P}_z &= \sum_{i=1}^{r} \dot{h}_i P_i = \sum_{i=1}^{r} \left(\frac{\partial h_i}{\partial z} \right)^T \dot{z} P_i = \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\partial h_i}{\partial z_k} \dot{z}_k P_i \\ &= \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\partial}{\partial z_k} \left(\prod_{j=1}^{p} w_{i_j}^j (z_j) \right) \dot{z}_k P_i = \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\partial w_{i_k}^k}{\partial z_k} \left(\prod_{\substack{j=1 \\ j \neq k}}^{p} w_{i_j}^j \right) \dot{z}_k P_i. \end{aligned} \tag{3.7}$$

In order to reconstruct the membership functions, each k summand is multiplied by $w_{i_k}^k + \left(1 - w_{i_k}^k\right) = 1$, expression (3.7) gives:

$$\dot{P}_z = \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\partial w_{i_k}^k}{\partial z_k} \left(w_{i_k}^k \prod_{\substack{j=1 \\ j \neq k}}^{p} w_{i_j}^j + \left(1 - w_{i_k}^k\right) \prod_{\substack{j=1 \\ j \neq k}}^{p} w_{i_j}^j \right) \dot{z}_k P_i$$

Knowing that $h_i = \prod_{j=1}^{p} w_{i_j}^j$, $\exists h_\mu : h_\mu = \left(1 - w_{i_k}^k\right) \prod_{\substack{j=1 \\ j \neq k}}^{p} w_{i_j}^j = \prod_{j=1}^{p} w_{\mu_j}^j$ where $\frac{\partial w_{i_k}^k}{\partial z_k} = -\frac{\partial w_{\mu_k}^k}{\partial z_k}$, $i_k = 0$, $\mu_k = 1$, then $\dot{P}_z$ can be rewritten as:

$$\dot{P}_z = \sum_{i=1}^{r} \sum_{\substack{k=1 \\ i_k = 0}}^{p} \frac{\partial w_0^k}{\partial z_k} \left(h_i + h_{\mu(i,k)}\right) \left(P_i - P_{\mu(i,k)}\right) \dot{z}_k = \sum_{\alpha=1}^{r} \sum_{k=1}^{p} h_\alpha \frac{\partial w_0^k}{\partial z_k} \left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right) \dot{z}_k \tag{3.8}$$

where $g_1(\alpha, k) = \left\lfloor (\alpha - 1) / 2^{p+1-k} \right\rfloor \times 2^{p+1-k} + 1 + (\alpha - 1) \bmod 2^{p-k}$ and $g_2(\alpha, k) = g_1(\alpha, k) + 2^{p-k}$, $\lfloor \ \rfloor$ being the floor function.

Considering the premise vector as a linear combination of the states, i.e., $z(t) = Lx(t)$ with $L \in \mathbb{R}^{p \times n}$. This assumption preserves the approximation capabilities of T-S models obtained by sector nonlinearity approach [Tanaka & Wang, 2001] while allowing to write

$$\dot{z}_k = \sum_{\gamma=1}^{n} \left(LA_z\right)_{k\gamma} x_\gamma = \sum_{\beta=1}^{r} \sum_{\gamma=1}^{n} h_\beta \left(LA_\beta\right)_{k\gamma} x_\gamma \tag{3.9}$$

Substituting (3.9) in (3.8) the following is obtained:

$$\dot{P}_z = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{k=1}^{p} \sum_{\gamma=1}^{n} h_\alpha h_\beta \frac{\partial w_0^k}{\partial z_k} x_\gamma \left(LA_\beta\right)_{k\gamma} \left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right) \tag{3.10}$$

Assuming, now, that $\left| \frac{\partial w_0^k}{\partial z_k} x_\gamma \right| \leq \lambda_{k\gamma}$, $\lambda_{k\gamma} > 0$, for $k \in \{1, \cdots, p\}$ and $\gamma \in \{1, \cdots, n\}$, non-quadratic Stability conditions are resumed in the following theorem:

Theorem 3.2: [Guerra & Bernal, 2009]

If there exist symmetric matrices $P_i > 0$, $i \in \{1, \cdots, r\}$, such that LMIs

$$\begin{aligned} &\Upsilon_{\alpha\alpha}^{m} < 0, && \alpha \in \{1,\cdots,r\}, m \in \{1,\cdots,2^{p\times n}\} \\ &\frac{2}{r-1}\Upsilon_{\alpha\alpha}^{m} + \Upsilon_{\alpha\beta}^{m} + \Upsilon_{\beta\alpha}^{m} < 0, && (\alpha,\beta) \in \{1,\cdots,r\}^2, \alpha \neq \beta, \\ & && m \in \{1,\cdots,2^{p\times n}\}, \end{aligned} \tag{3.11}$$

$\Upsilon_{\alpha\beta}^{m} = P_\alpha A_\beta + A_\beta^T P_\alpha + \sum_{k=1}^{p}\sum_{\gamma=1}^{n}(-1)^{d_{k\gamma}^{m}} \lambda_{k\gamma}\left(LA_\beta\right)_{k\gamma}\left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right)$ hold with $d_{k\gamma}^{m}$ defined from the binary representation of $m-1 = d_{pn}^{m} + d_{p(n-1)}^{m}\times 2 + \cdots + d_{11}^{m}\times 2^{p\times(n-1)}$ and $g_1(\alpha,k)$, $g_2(\alpha,k)$ defined as in (3.7), then $x(t)$ tends to zero exponentially for any trajectory satisfying (3.1) in the outermost Lyapunov level contained in $\overline{R} = \bigcap_{k,\gamma}\left\{x : \left|\frac{\partial w_0^k}{\partial z_k} x_\gamma\right| \leq \lambda_{k\gamma}\right\}$.

Remark 3.2: Theorem 3.2 provides non-quadratic Stability conditions which are generally local since they apply for the outermost Lyapunov level in region $\overline{R}$, which is an estimation of the region of attraction of T-S model (3.1). As for the original nonlinear model, $\overline{R} \cap C$ is an estimation of its region of attraction (recall C is the compact on which the T-S model exactly represents the original one) [Tanaka & Wang, 2001].

Remark 3.3: The quadratic case is included in this approach, provided the same relaxation, due to the fact that if a quadratic solution holds then conditions of Theorem 3.2 are satisfied with $P_i = P$. This is direct from expression of $\Upsilon_{\alpha\beta}^{m}$ as $P_i = P$, whatever are the $\lambda_{k\gamma}$ it results in: $\Upsilon_{\alpha\beta}^{m} = PA_\beta + A_\beta^T P$, therefore (3.11) exactly corresponds to the quadratic case conditions.

Remark 3.4: Inequalities in (3.11) are LMIs since $\lambda_{k\gamma} > 0$ are given. Note that if conditions (3.11) of Theorem 3.2 are satisfied, we know that local stability exists, i.e. $\overline{R} \neq \varnothing$ and there exists sufficiently small $\lambda_{k\gamma} > 0$. Therefore, if the initially chosen $\lambda_{k\gamma}$ do not satisfy (3.11), quick algorithms can be used. For example, a simple bisection by successively dividing $\lambda_{k\gamma}$ by a common value $\lambda > 0$ comes at hand [Guerra & Bernal 2009].

Consider the fact that any nonlinear dependence of $z(x(t))$ can be written in terms of $x(t)$ allows us to alternatively write $\dot{P}_z$ as:

$$\begin{aligned}\dot{P}_z &= \sum_{\alpha=1}^{r}\sum_{k=1}^{p} h_\alpha \frac{\partial w_0^k}{\partial z_k}\left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right)\dot{z}_k \\ &= \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{k=1}^{p}\sum_{\gamma=1}^{n}\sum_{v=1}^{n} h_\alpha h_\beta \frac{\partial w_0^k}{\partial x_v} x_\gamma \left(A_\beta\right)_{v\gamma}\left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right)\end{aligned} \tag{3.12}$$

with $g_1(\alpha,k) = \left\lfloor (\alpha-1)/2^{p+1-k} \right\rfloor \times 2^{p+1-k} + 1 + (\alpha-1) \bmod 2^{p-k}$ and $g_2(\alpha,k) = g_1(\alpha,k) + 2^{p-k}$.

In order to clarify the way to obtain the expression of $\dot{P}_z$, consider a T-S model with 4 rules, whose MFs are based on functions $w_0^1(x_1)$, $w_0^2(x_2)$, $w_1^1(x_1) = 1 - w_0^1(x_1)$, and $w_1^2(x_2) = 1 - w_0^2(x_2)$ as $h_1 = w_0^1 w_0^2$, $h_2 = w_0^1 w_1^2$, $h_3 = w_1^1 w_0^2$ and $h_4 = w_1^1 w_1^2$. Therefore, expression (3.12) can be obtained from $\dot{P}_z$ as follows:

$$\begin{aligned}\dot{P}_z &= \sum_{i=1}^{4} \dot{h}_i P_i = \sum_{i=1}^{4}\sum_{k=1}^{2} \frac{\partial h_i}{\partial x_k}\dot{x}_k P_i = \sum_{i=1}^{4}\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(\prod_{j=1}^{1} w_{i_j}^j(x_j)\right)\dot{x}_k P_i \\ &= \sum_{i=1}^{4}\sum_{k=1}^{2} \frac{\partial w_{i_k}^k}{\partial x_k}\left(\prod_{\substack{j=1 \\ j\neq k}}^{1} w_{i_j}^j\right)\dot{x}_k P_i = \sum_{i=1}^{4}\left(\frac{\partial w_{i_1}^1}{\partial x_1} w_{i_2}^2 \dot{x}_1 + \frac{\partial w_{i_2}^2}{\partial x_2} w_{i_1}^1 \dot{x}_2\right) P_i \\ &= \left(\frac{\partial w_0^1}{\partial x_1} w_0^2 \dot{x}_1 + \frac{\partial w_0^2}{\partial x_2} w_0^1 \dot{x}_2\right) P_1 + \left(\frac{\partial w_0^1}{\partial x_1} w_1^2 \dot{x}_1 + \frac{\partial w_1^2}{\partial x_2} w_0^1 \dot{x}_2\right) P_2 \\ &+ \left(\frac{\partial w_1^1}{\partial x_1} w_0^2 \dot{x}_1 + \frac{\partial w_0^2}{\partial x_2} w_1^1 \dot{x}_2\right) P_3 + \left(\frac{\partial w_1^1}{\partial x_1} w_1^2 \dot{x}_1 + \frac{\partial w_1^2}{\partial x_2} w_1^1 \dot{x}_2\right) P_4\end{aligned}$$

Since $\dfrac{\partial w_0^k}{\partial x_k} = -\dfrac{\partial w_1^k}{\partial x_k}$, the previous expression is rewritten as

$$\dot{P}_z = \frac{\partial w_0^1}{\partial x_1} w_0^2 \dot{x}_1 (P_1 - P_3) + \frac{\partial w_0^2}{\partial x_2} w_0^1 \dot{x}_2 (P_1 - P_2) + \frac{\partial w_0^1}{\partial x_1} w_1^2 \dot{x}_1 (P_2 - P_4) + \frac{\partial w_0^2}{\partial x_2} w_1^1 \dot{x}_2 (P_3 - P_4)$$

Multiplying each term $\dfrac{\partial w_0^k}{\partial x_k}$ by $w_0^k + \left(1 - w_0^k\right) = w_0^k + w_1^k = 1$, it gives

$$\begin{aligned}\dot{P}_z &= \frac{\partial w_0^1}{\partial x_1}\left(w_0^1 w_0^2 + w_1^1 w_0^2\right)\dot{x}_1 (P_1 - P_3) + \frac{\partial w_0^2}{\partial x_2}\left(w_0^1 w_0^2 + w_0^1 w_1^2\right)\dot{x}_2 (P_1 - P_2) \\ &+ \frac{\partial w_0^1}{\partial x_1}\left(w_0^1 w_1^2 + w_1^1 w_1^2\right)\dot{x}_1 (P_2 - P_4) + \frac{\partial w_0^2}{\partial x_2}\left(w_1^1 w_0^2 + w_1^1 w_1^2\right)\dot{x}_2 (P_3 - P_4) \\ &= \frac{\partial w_0^1}{\partial x_1}(h_1 + h_3)\dot{x}_1 (P_1 - P_3) + \frac{\partial w_0^2}{\partial x_2}(h_1 + h_2)\dot{x}_2 (P_1 - P_2) \\ &+ \frac{\partial w_0^1}{\partial x_1}(h_2 + h_4)\dot{x}_1 (P_2 - P_4) + \frac{\partial w_0^2}{\partial x_2}(h_3 + h_4)\dot{x}_2 (P_3 - P_4)\end{aligned}$$

from which expression (3.12) is obtained by introducing $\dot{x}_k = \sum_{\beta=1}^{4} h_\beta \sum_{\gamma=1}^{2}\sum_{v=1}^{2} \left(A_\beta\right)_{v\gamma} x_\gamma$ and regrouping terms as shown below:

$$\begin{aligned}\dot{P}_z &= \frac{\partial w_0^1}{\partial x_1}\dot{x}_1\left(\left(h_1+h_3\right)\left(P_1-P_3\right)+\left(h_2+h_4\right)\left(P_2-P_4\right)\right)\\ &\quad+\frac{\partial w_0^2}{\partial x_2}\dot{x}_2\left(\left(h_1+h_2\right)\left(P_1-P_2\right)+\left(h_3+h_4\right)\left(P_3-P_4\right)\right)\\ &= \sum_{\alpha=1}^{4}\sum_{k=1}^{2} h_\alpha \frac{\partial w_0^k}{\partial x_k}\dot{x}_k\left(P_{g_1(\alpha,k)}-P_{g_2(\alpha,k)}\right)\end{aligned}$$

$$\dot{P}_z = \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{k=1}^{2}\sum_{\gamma=1}^{2}\sum_{v=1}^{2} h_\alpha h_\beta \frac{\partial w_0^k}{\partial x_v} x_\gamma \left(A_\beta\right)_{v\gamma}\left(P_{g_1(\alpha,k)}-P_{g_2(\alpha,k)}\right) \tag{3.13}$$

LMIs conditions for stability analysis can now be sorted, coming back to (3.4) and using (3.12), the following conditions can be obtained:

$$P_z A_z + A_z^T P_z + \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{k=1}^{p}\sum_{\gamma=1}^{n}\sum_{v=1}^{n} h_\alpha h_\beta \frac{\partial w_0^k}{\partial x_v} x_\gamma \left(A_\beta\right)_{v\gamma}\left(P_{g_1(\alpha,k)}-P_{g_2(\alpha,k)}\right) < 0 \tag{3.14}$$

Assuming $\left|\frac{\partial w_0^k}{\partial x_v} x_\gamma\right| \le \lambda_{k\gamma v}$ for any $k=1,\ldots,p$, $\gamma, v = 1,\ldots,n$, property 5, (2.27) can be repeatedly applied to (3.14) in order to obtain the following sufficient conditions:

$$\Upsilon_{zz}^m = P_z A_z + A_z^T P_z + \sum_{k=1}^{p}\sum_{\gamma=1}^{n}\sum_{v=1}^{n}(-1)^{d_{k\gamma v}^m}\lambda_{k\gamma v}\left(A_z\right)_{v\gamma}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right) < 0, \quad m=1,\ldots,2^{pn^2} \tag{3.15}$$

with $d_{k\gamma v}^m$ defined from the binary representation of $m-1 = d_{pnn}^m + d_{pn(n-1)}^m \times 2 + \cdots + d_{111}^m \times 2^{pn^2-1}$.

Several sum relaxation scheme can be applied to double-sum expression (3.15). Using relaxation (2.32) the following alternative formulation of Theorem 3.2 can be stated:

Theorem 3.3: [Sala & al, 2010]

If $\exists P_i = P_i^T > 0$, $i \in \{1,\cdots,r\}$, such that

$$\begin{aligned} &\Upsilon_{\alpha\alpha}^m < 0, && \alpha \in \{1,\cdots,r\}, m \in \left\{1,\cdots,2^{p\times n^2}\right\}\\ &\frac{2}{r-1}\Upsilon_{\alpha\alpha}^m + \Upsilon_{\alpha\beta}^m + \Upsilon_{\beta\alpha}^m < 0, && (\alpha,\beta)\in\{1,\cdots,r\}^2, \alpha\neq\beta,\\ & && m \in \left\{1,\cdots,2^{p\times n^2}\right\}, \end{aligned} \tag{3.16}$$

$\Upsilon^m_{\alpha\beta} = P_\alpha A_\beta + A_\beta^T P_\alpha + \sum_{k=1}^{p}\sum_{\gamma=1}^{n}\sum_{v=1}^{n}(-1)^{d^m_{k\gamma v}}\,\lambda_{k\gamma v}\left(A_\beta\right)_{v\gamma}\left(P_{g_1(\alpha,k)} - P_{g_2(\alpha,k)}\right)$ hold with $d^m_{k\gamma v}$ defined from the binary representation of $m-1 = d^m_{pnn} + d^m_{pn(n-1)}\times 2 + \cdots + d^m_{111}\times 2^{pn^2-1}$ and $g_1(\alpha,k)$, $g_2(\alpha,k)$ defined as in (3.12), then $x(t)$ tends to zero exponentially for any trajectory satisfying (3.1) in the outermost Lyapunov level $R_o = \{x : x^T P_z x \le c\}$ contained in the modeling region C_0 and $\bar{R}_0 = \bigcap_{k,\gamma,v}\left\{x : \left|\frac{\partial w_0^k}{\partial x_v}x_\gamma\right| \le \lambda_{k\gamma v}\right\}$.

***Proof*:** It follows immediately from the preceding discussion that established that LMIs (3.16) imply $\dot{V}(x) < 0$. □

Remark 3.5: LMI conditions in Theorem 3.3 are local. They test whether T-S model (3.1) is stable in $R_0 \subseteq \left(C_0 \cap \bar{R}_0\right)$ or not. Nevertheless, a different approach allows the same conditions to be used to estimate a region of attraction in C_0 if $P_z A_z + A_z^T P_z < 0$. To do so, substitute any constant $\lambda_{k\gamma v}$ in (3.16) by $\lambda \times \lambda_{k\gamma v}$ and search via bisection for the maximum $\lambda > 0$ that renders (3.16) feasible in C_0. Once the maximum value λ has been found, it means that T-S model (3.1) and its original nonlinear equivalent model (3.1) are stable in the outermost Lyapunov level $R_o = \{x : x^T P_z x \le c\}$ contained in C_0 and $\bar{R}_0 = \bigcap_{k,\gamma,v}\left\{x : \left|\frac{\partial w_0^k}{\partial x_v}x_\gamma\right| \le \lambda \times \lambda_{k\gamma v}\right\}$.

Remark 3.6: As expected, Theorem 3.3 reduces to the quadratic case if $P_i = P$, i.e., if there is a common quadratic Lyapunov function. In other words, the quadratic case is included in the new approach.

3.3. Improvements on local non-quadratic stability of T-S models: iterative remodeling

This approach illustrates the way to get progressively better estimates of the region of attraction: if a nonlinear model is available, an algorithm alternating closer modeling areas

and bigger estimates is employed; if not, an algorithm alternating closer polytopes (thus modifying the T-S model matrices) and bigger estimates is applied.

Following the discussion in Remark 3.5, further improvements can be done to augment the estimation of the stability domain in case $R_0 \subset C_0$. The intuition behind the method is that a modelling area tighter than C_0 but including estimation R_0 can lead to better estimates since the model matrices will be "closer". Two ways of achieving this goal are proposed.

Algorithm 1:

Step 0: Initialize $k = 1$.

Step 1: Define a polyhedral region $C_k : C_{k-1} \supset C_k \supset R_{k-1}$ and get the maxima and minima of MFs $h_i \in \left[\underline{h}_i, \overline{h}_i\right]$ in C_k. With these bounds, use results in [Sala & Ariño, 2006] to get new matrices $A_j^{(k)}$, $j = 1,\ldots,r_k$ such that in region C_k: $A_j^{(k)} = \sum_{i=1}^{r} v_{ji} A_i$, $h_i = \sum_{j=1}^{r_k} \mu_j\left(z(t)\right) v_{ji}$, $\sum_{j=1}^{r_k} \mu_j\left(z(t)\right) = 1$, $\mu_j\left(z(t)\right) \geq 0$. These matrices are "closer" than those in the original polytope and are likely to provide (once results for stability in Theorem 3.3 are reapplied) larger bounds $\lambda \times \lambda_{k\gamma v}$ leading to greater $\overline{R}_k$ and R_k. Note that only vertices v_{ji} are needed to calculate $A_j^{(k)}$.

Step 2: If more refinement is needed, increase k in 1 and go to Step 1, if not, end the algorithm. Obviously, each iteration adds progressively smaller refinements to the previous estimations of the region of attraction. The original stability domain is augmented by the cumulative refinements.

Algorithm 2: (Only if the nonlinear model is available).

Step 0: Initialize $k = 1$.

Step 1: Define a new modelling compact region $C_k : C_{k-1} \supset C_k \supset R_{k-1}$ and get a new T-S model representation by sector nonlinearity approach, taking into account the new maxima and minima of the model nonlinearities to define the MFs. Reapply the results on stability/stabilization to the new set of matrices $A_i^{(k)}$, $i = 1,\ldots,r$ thus obtained. As in the previous case, these matrices are "closer" than those in the original polytope and are likely to provide larger bounds $\lambda \times \lambda_{k\gamma v}$ and greater regions $\overline{R}_k$ and R_k.

***Step 2**:* If more refinement is needed, increase $k \leftarrow k+1$ and repeat Step 1, if not, otherwise stop the algorithm. As with the first algorithm, this iterative procedure also leads to a limit in the estimation of the region of attraction.

Remark 3.7: Note that the algorithms above circumscribe themselves to estimates in the original modelling region C_0, so their progressively better stability domains are also valid for the original nonlinear models the T-S models come from.

Remark 3.8: Over the quadratic case, Theorem 3.3 increases the number of LMI constraints from $r+1$ to $r+r^2\times 2^{pn^2}$.

3.4. Examples

This section presents two examples: the first one illustrates how algorithm 1 allows "closer" matrices to be obtained for a given T-S model via the procedures detailed in [Sala & Arino, 2006], thus increasing the size of the stability domain; the second example shows how algorithm 2 can be employed to recast a given nonlinear model as a "tighter" T-S one, thereby augmenting the size of the region of attraction.

Example 3.1:

Consider the following T-S model whose matrices are taken from [Tanaka & al, 2003]:

$$\dot{x}(t) = A_z x(t) = \sum_{i=1}^{2} h_i\left(z(t)\right) A_i x(t) \tag{3.17}$$

With

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix},$$

$z_1(t) = x_2(t)$, $h_1 = w_0^1 = -0.5x_2^3 + 0.5$, and $h_2 = w_1^1 = 1 - w_0^1$ defined in the compact region $C_0 = \{x : |x_i| \le 1\}$.

Quadratic stability fails for T-S model (3.17). As indicated in Remark 3.5, conditions (3.16) can be used to estimate the region of attraction because $P_z A_z + A_z^T P_z < 0$ holds for some $P_i > 0$, $i = 1,2$. To do so, initial values $\lambda_{k\gamma v} = \frac{3}{2}$ for $k = 1$, $\gamma = 1,2$, and $v = 2$ can be calculated from the fact that $|x_i| \le 1$ and $\left|\frac{\partial w_0^1}{\partial x_2}\right| = \left|\frac{\partial w_1^1}{\partial x_2}\right| = \left|\frac{3}{2}x_2^2\right| \le \frac{3}{2}$; otherwise $\lambda_{k\gamma v} = 0$.

Moreover, from (3.12) $g_1(1,1) = g_1(2,1) = 1$, $g_2(1,1) = g_2(2,1) = 2$, and $d_{k\gamma v}^m$ are the digits of the binary representation of $m-1 = d_{122}^m + d_{121}^m \times 2 + d_{112}^m \times 2^2 + d_{111}^m \times 2^3$, for example the quadruplet $(1,0,0,0)$ for $m = 2$. With these values, Theorem 3.3 and bisection, stability of T-S model (3.17) is established for the region of attraction R_o whose borders are shown by a closed solid line in Figure 3.1.

At the same figure, the different borders of region $\bar{R}_0$ are shown with dashed lines and the borders of C_0 are shown with dotted lines.

Algorithm 1 can be used to augment the stability domain R_o. To begin with, define the encapsulated regions $C_k = \{x : |x_i| \le 1 - 0.03k\}$, $k = 1,2$, i.e. $C_0 \supset C_k \supset R_0$. In C_1 the maximum and minimum of MFs h_1 and h_2 are given by 0.9563 and 0.0437. Following the procedures in [Sala & Arino, 2006], the new vertices v_{ji} can be written as $v_{11} = 0.9563$, $v_{12} = 0.0437$, and $v_{21} = 0.9563$, $v_{22} = 0.0437$, from which the new matrices $A_j^{(1)}$, $j = 1,2$ are calculated, so Theorem 3.3 can be reapplied.

In Figure 3.2 all estimations R_i, $i = 0,1,2$ are compared (concentric pseudo-ellipsoids) as well as all the borders of C_i, $i = 0,1,2$ (concentric rectangles). Note that as R_i gets larger, C_i gets smaller, which reflects the fact that these algorithms reach a limit. Indeed, the last estimation R_2 corresponds to a quadratic one since matrices $A_j^{(2)}$, $j = 1,2$ are getting "close" enough such that quadratic conditions are feasible (i.e., $\lambda \to \infty$); that explains the fact that R_2 is only bounded by C_2, since $\bar{R}_2 = \mathbb{R}^2$..

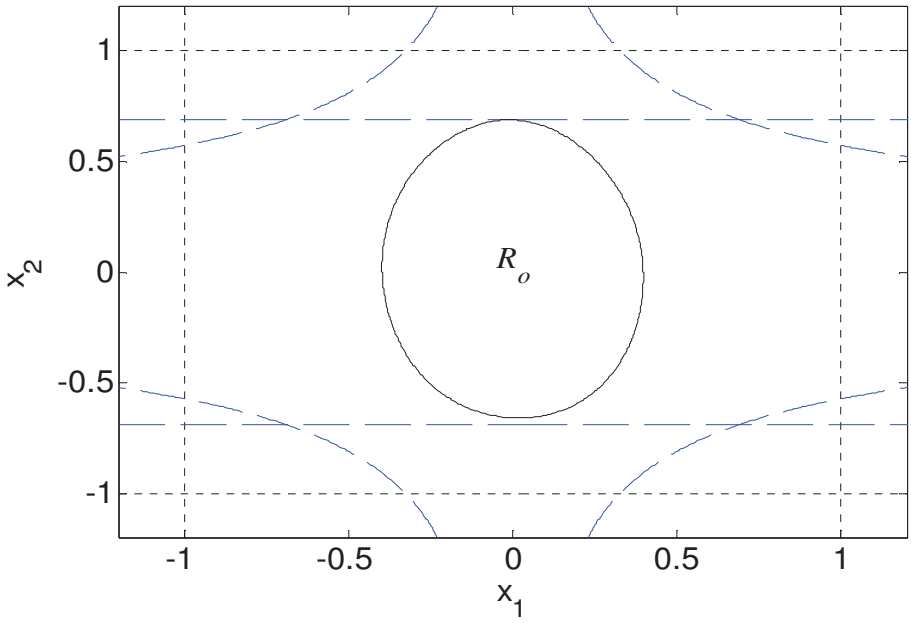


Figure 3.1: First estimation of the region of attraction for T-S model (3.17)**.**

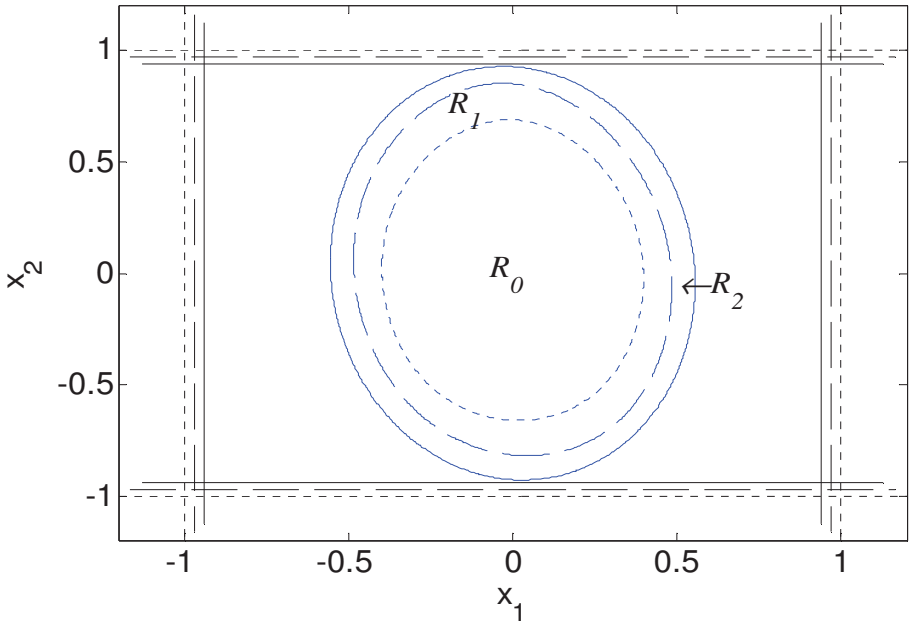


Figure 3.2: Comparing estimations of the stability domains of T-S model (3.17).

Example 3.2:

Consider the following nonlinear model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + 0.275x_1^3 + 0.55x_1x_2^2 - 0.0125x_1^3x_2^2 + 2x_2 + 0.25x_2^3 \\ -0.125x_1^3 - 0.9x_2 + 0.25x_1^2x_2 - 0.1313x_1^2x_2^3 \end{bmatrix} \tag{3.18}$$

The stability properties of (3.18) in $|x_i| \le 2$, $i = 1,2$ are to be investigated via algorithm 2. To this end, we define $C_k = \{x : |x_i| \le 2 - 0.02k\}$, $k = 0,1,2$ as the encapsulated regions in which the following T-S model representations of (3.18) are defined via sector nonlinearity methodology. Recall also that these models are not approximations, but *equivalent* to the nonlinear one in C_k:

$$\dot{x}(t) = A_z x(t) = \sum_{i=1}^{4} h_i\left(z(t)\right) A_i x(t) \tag{3.19}$$

with $m_1 = 0$, $M_1 = (2 - 0.02k)^2$, $m_2 = 0$, $M_2 = (2 - 0.02k)^2$, $z_1(t) = x_1(t)$, $z_2(t) = x_2(t)$, $w_0^1 = \dfrac{M_1 - x_1^2}{M_1 - m_1}$, $w_0^2 = \dfrac{M_2 - x_2^2}{M_2 - m_2}$, $w_1^1 = 1 - w_0^1$, $w_1^2 = 1 - w_0^2$, $h_1 = w_0^1 w_0^2$, $h_2 = w_0^1 w_1^2$, $h_3 = w_1^1 w_0^2$, $h_4 = w_1^1 w_1^2$, and the following model matrices

$$A_1 = \begin{bmatrix} -3 + 0.275m_1 + 0.55m_2 - 0.0125m_1m_2 & 2 + 0.25m_2 \\ -0.125m_1 & -0.9 + 0.25m_1 - 0.1313m_1m_2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -3 + 0.275m_1 + 0.55M_2 - 0.0125m_1M_2 & 2 + 0.25M_2 \\ -0.125m_1 & -0.9 + 0.25m_1 - 0.1313m_1M_2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -3 + 0.275M_1 + 0.55m_2 - 0.0125M_1m_2 & 2 + 0.25m_2 \\ -0.125M_1 & -0.9 + 0.25M_1 - 0.1313M_1m_2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -3 + 0.275M_1 + 0.55M_2 - 0.0125M_1M_2 & 2 + 0.25M_2 \\ -0.125M_1 & -0.9 + 0.25M_1 - 0.1313M_1M_2 \end{bmatrix}$$

Note that region C_0 coincides with the region of interest and its resulting T-S model (equation (3.19) with $k = 0$) can be found in [Guerra & Bernal, 2009]. Quadratic stability fails for this model as well as for those in C_1 and $C_2 \subset C_1 \subset C_0$. Since Theorem 3.1 conditions $P_z A_z + A_z^T P_z < 0$ hold for some $P_i > 0$, $i = 1,2$, Theorem 3.3 can be applied to obtain a first estimation of the stability domain in C_0, as suggested in Remark 3.5. Note also that expression of $\dot{P}_z$ is similar to (3.13).

Since $\frac{\partial w_0^1}{\partial z_1} = -\frac{x_1}{2} = -\frac{\partial w_1^1}{\partial z_1}$, $\frac{\partial w_0^2}{\partial z_2} = -\frac{x_2}{2} = -\frac{\partial w_1^2}{\partial z_2}$ and $|x_i| \le 2$ for any C_k, $\lambda_{k\gamma v} = 2$ is a valid set of values to hold inequality $\left| \left(\frac{\partial w_0^k}{\partial x_v} \right) x_\gamma \right| \le \lambda_{k\gamma v}$, $k, \gamma, v \in \{1, 2\}$.

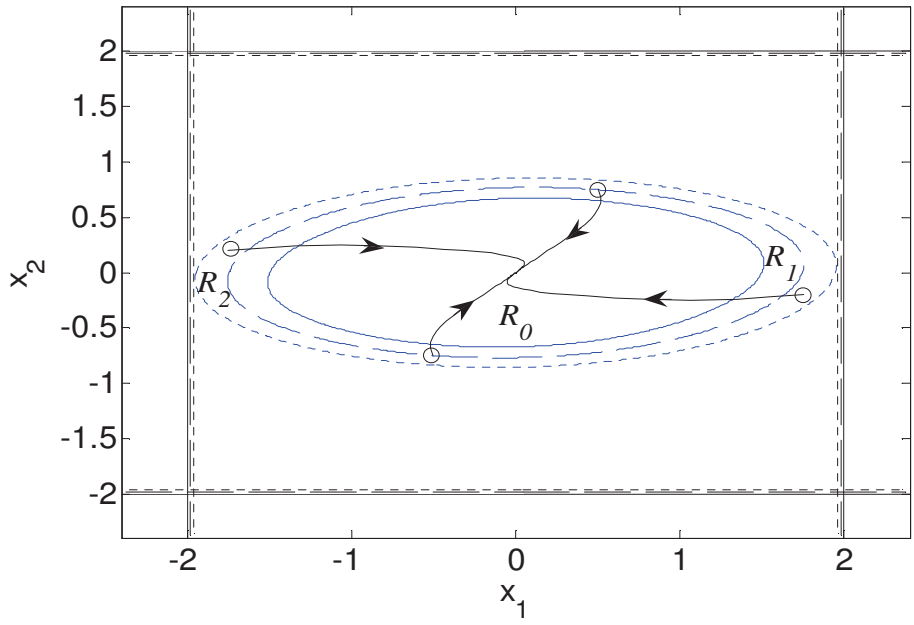


Figure 3.3: Estimates of the region of attraction of T-S model (3.19)

Region C_0 is denoted in Figure 3.3as the interior of a solid-line rectangle, while R_0 is the interior of a solid-line pseudo-ellipsoid.

To ameliorate this estimate, algorithm 2 asks for nonlinear model (3.18) to be recast as a T-S one in a region $C_1 : C_0 \supset C_1 \supset R_0$, which is consistent with the definition of C_k given above. Therefore, applying algorithm 2 for $k = 1, 2$ the regions of attraction R_1 and R_2 shown in Figure 3.3 with dashed and dotted-line pseudo-ellipsoids, respectively, are obtained. Their corresponding modelling regions C_1 and C_2 are depicted in the same figure with dashed- and dotted-line rectangles, respectively. As expected, they increase the quality of the estimations for the region of attraction. In addition, four model trajectories have been showed to illustrate the behaviour of the equilibrium point at the origin (solid lines with arrows pointing to the origin).

3.5. Conclusion

In this chapter, a way to escape the quadratic framework for stability analysis has been presented. This approach is based on reducing the global stability goals to find local conditions that allow estimating the region of attraction via LMIs while taking into account MFs' information and Tensor product structure. The results provide an answer to problems that, otherwise, were previously unsolved for T-S models. Improvements of this approach based on fuzzy Lyapunov functions have been proposed in the second section. These ameliorations are based on taking advantage of the possible gaps between a first estimation of the region of attraction and the modeling area, by recasting the T-S model a) from a redefinition of its MFs leading to new model matrices, or b) from a redefinition of the T-S model out of the nonlinear one, when available. Some illustrative simulation examples have been included that clearly show the advantages of the proposed method.

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4. Chapter 4: Non-quadratic stability of T-S models: polynomial fuzzy lyapunov function

Synopsys

This chapter presents a polynomial fuzzy modeling for nonlinear systems approach based on fuzzy polynomial Lyapunov function, SOS Stability conditions are formulated which may be solved by the mean of the sum of squares (SOS) approach. Parts of this chapter are exposed in paper I contributed [Bernal & al, 2011].

4.1. Introduction

In this chapter, a new contribution dealing with the stability of continuous-time polynomial fuzzy models by means of a polynomial generalization of fuzzy Lyapunov functions. Based on a Taylor-series approach which allows a polynomial fuzzy model to exactly represent a nonlinear model in a compact set of the state space, it is shown that a refinement of the polynomial Lyapunov function so as to make it share the fuzzy structure of the model proves advantageous. Conditions thus obtained are tested via SOS tools which are efficiently solved by semi-definite programming algorithms [Prajna & al, 2004a, 2004b].

Polynomial fuzzy (PF) models have established a new paradigm that overcomes many of the aforementioned problems of conservativeness since they are convex combinations of polynomial models instead of convex combinations of linear ones [Tanaka & al, 2009a], [Tanaka & al, 2009b]. Moreover, conditions derived under this new framework can also be checked with semi-definite programming using Sum-of-Squares (SOS) tools.

This new approach is based on two recent works: the first one [Sala, 2009], [Sala & Arino, 2009] provides a systematic way of obtaining exact polynomial fuzzy representations of nonlinear models via a Taylor-series approach, thus generalizing sector nonlinearity approach; the second one [Guerra & Bernal, 2009], [Bernal & Guerra, 2010] shows how to escape from the quadratic framework by combining local analysis and fuzzy Lyapunov functions for continuous-time T-S models. Since local analysis can be easily included via Lagrange multipliers and the *Positivstellensatz* argumentation in the polynomial framework [Prajna, 2004a], [Sala & Arino, 2009], the use of more general Lyapunov functions such as the polynomial fuzzy ones is investigated in this section as a generalization of the one employed in the previous section.

4.1.1. Polynomial fuzzy modeling and notations

Consider a nonlinear model $\dot{x}(t) = f(x)$ having the origin as an equilibrium point, and assume that it can be expressed in the form:

$$\dot{x}(t) = \pi\left(h^1\left(z_1(x)\right), \ldots, h^\gamma\left(z_\gamma(x)\right), x(t)\right) \tag{4.1}$$

being $\pi(\cdot): \mathbb{R}^{n+\gamma} \to \mathbb{R}^n$ a vector of polynomial functions, $x(t) \in \mathbb{R}^n$ the state vector, $z(x(t)) \in \mathbb{R}^\gamma$ another vector of polynomial functions of the state (denoted as the premise

vector), and a set of functions $h^k(\cdot):\mathbb{R}\to\mathbb{R}$, $k\in\{1,\ldots,\gamma\}$ representing possible non-polynomial nonlinearities in (4.1), such as trigonometric, exponential, etc., functions nonlinearities $h^k(\cdot)$ are assumed bounded and smooth in a region of interest given by a compact set $\Omega\supset 0$. Any compact region of interest Ω can be included into a semi-algebraic set with a piecewise polynomial boundary (for instance, a ball). This fact will be later used for SOS relaxations.

For instance, a model equation $\dot{x}_1=\left(\sin\left(x_1^2-x_2\right)\right)^2 x_2+x_1$ can be expressed in the above form by considering $\pi(h,x_1,x_2)=h^2x_2+x_1$, $h(z)=\sin(z)$, and $z=x_1^2-x_2$. As discussed below, if functions $h^k(z)$ are C^d [2] they admit a representation as a fuzzy combination of polynomials of degree *d*, to be denoted as "polynomial fuzzy" model. The case $d=1$ amounts to the well-known Takagi-Sugeno models.

Once a nonlinear system in the above general form is assumed, fuzzy techniques will be used to analyse its stability. The first step is converting the system to a fuzzy model (a polynomial fuzzy one, in fact). In order to carry out such conversion, consider a particular non-polynomial nonlinearity $h(z)$ as those defined above (subscripts and arguments are omitted for simplicity). Employing the polynomial fuzzy modeling described in [Sala, 2009], [Sala & Arino, 2009] (which is a generalization of sector nonlinearity in [Tanaka & Wang, 2001]), this function can be rewritten as a convex sum of polynomials. Indeed, in order to do so, let us denote the *d*-th degree Taylor approximation of $h(z)$ as $h_d(z)=\sum_{i=0}^{d-1}\frac{h^{(i)}(0)}{i!}z^i$, $d\in\mathbb{N}$, the residual term $T_d(z)=\frac{h(z)-h_d(z)}{z^d}$, with $T_d(0)=\lim_{z\to 0}T_d(z)$, and the bounds $\overline{T_d}=\sup_{z\in\Omega}T_d(z)$, $\underline{T_d}=\inf_{z\in\Omega}T_d(z)$, assuming the arbitrarily chosen degree *d* is low enough such that the required derivatives exist and $T_d(z)$ is continuous.

This notation allows defining the pair of MFs:

$$\overline{w}_0(z)=\frac{T_d(z)-\underline{T_d}}{\overline{T_d}-\underline{T_d}},\ \overline{w}_1(z)=1-\overline{w}_0(z),\ \overline{w}_0(z),\overline{w}_1(z)\geq 0 \tag{4.2}$$

It is straightforward to see that the nonlinearity $h(z)$ can now be written as

[2] C^d : first through d^{th} derivatives are continous

$$h(z)=\overline{w}_0(z)q_0(z)+\overline{w}_1(z)q_1(z)=\sum_{i=0}^{1}\overline{w}_i(z)q_i(z), \tag{4.3}$$

with two vertex polynomials of degree *d* given by:

$$q_0(z)=h_d(z)+\overline{T_d}z^d$$

$$q_1(z)=h_d(z)+\underline{T_d}z^d$$

For details, see [Sala, 2009], [Sala & Arino, 2009]. On the sequel, arguments will be omitted when convenient for brevity, for instance, w_i will stand for $w_i(z)$. Basically, replacing (4.3) into the polynomial π in (4.1) will yield the overall fuzzy polynomial model. However, if the polynomial π is not linear in $h^k(\cdot)$, say it appears with degree d_k, it gives rise to multi-dimensional (nested) tensor-product convex sums. Indeed, in that case, every function $h^k(\cdot)$, $k\in\{1,\cdots,\gamma\}$ can be written as the product of its d_k elementary convex sums of the form (4.3) . Thus, expression (4.1) can be rewritten as the following PF model:

$$\begin{aligned}\dot{x}(t)&=\pi\left(\left(\sum_{i_1=0}^{1}\overline{w}_{i_1}^1q_{i_1}^1\right)^{d_1},\left(\sum_{i_2=0}^{1}\overline{w}_{i_2}^2q_{i_2}^2\right)^{d_2},\cdots,\left(\sum_{i_\gamma=0}^{1}\overline{w}_{i_\gamma}^\gamma q_{i_\gamma}^\gamma\right)^{d_\gamma},x\right)\\&=\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1}w_{i_1}^1w_{i_2}^2\cdots w_{i_p}^p\mathbf{q}_{i_1i_2\cdots i_p}=\sum_{i\in\mathbb{I}_p}w_{\mathbf{i}}\mathbf{q}_{\mathbf{i}}\end{aligned} \tag{4.4}$$

with :

- p being the sum of the degrees in $\pi(\cdot)$ of each of the γ nonlinearities in (4.1), i.e., $p=\sum_{j=1}^{\gamma}d_\gamma$.
- $\mathbb{I}_p=\left\{\mathbf{i}=(i_1,i_2,\ldots,i_p):i_j\in\{0,1\},j\in\{1,\ldots,p\}\right\}$ is the set of all *p*-bit binary numbers, being its elements, $\mathbf{i}$, multidimensional index variables whose *k*-th bit is denoted as i_k.
- $w_{\mathbf{i}}=w_{i_1}^1w_{i_2}^2\cdots w_{i_p}^p=\prod_{j=1}^{p}w_{i_j}^j(z_j)$ is a product of elementary MFs obtained from those $\overline{w}_i$ describing each nonlinearity in (4.3) (h_i for T-S models in section 3.2).
- and $\mathbf{q}_{\mathbf{i}}(x)$ is a polynomial vector of the proper size.

Example 4.1:

To illustrate the modelling process above, consider the model

$$\dot{x} = \sin^2(x) + e^{-x}x$$

It will have a polynomial model for $\sin(x) = \sum_{i_1=0}^{1} \overline{w}_{i_1}^1 q_{i_1}^1$, and another one for $e^{-x} = \sum_{i_2=0}^{1} w_{i_2}^2 q_{i_2}^2$, giving rise to an overall model in the form:

$$\dot{x} = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1} \overline{w}_{i_1}^1 w_{i_2}^2 \overline{w}_{i_3}^1 \mathbf{q}_{i_1 i_2 i_3}, \qquad \text{with } \mathbf{q}_{i_1 i_2 i_3} = q_{i_1}^1 q_{i_3}^1 + q_{i_2}^2 x$$

Defining $w_i^3 = w_i^1 = \overline{w}_i^1$ yields an expression in the form (4.4), i.e., a three-dimensional tensor product combination of vertex polynomials.

Recall that PF model (4.4) is equivalent to the original nonlinear model (4.1) in the compact set Ω of the state space including the origin; moreover, T-S models are a subclass of the PF ones. A PF model is said to be of order *d* if the maximum order found in its Taylor approximations is *d*. This procedure generalizes those in [Arino & Sala, 2007], [Bernal & Guerra, 2010] to the polynomial case. From the modelling procedure, it is clear that many of the 2-rule memberships $w_{\mathbf{i}}$ in (4.4) may be repeated, as in the above example, this fact can be used to remove conservativeness (applying the multi-sum relaxations in chapter 2). This issue will be disregarded in the sequel, for simplicity.

Once a polynomial fuzzy model has been obtained, consider now the following polynomial-fuzzy Lyapunov function candidate:

$$V(x) = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p p_{i_1 i_2 \cdots i_p}(x) = \sum_{\mathbf{i}\in\mathbb{I}_p} w_{\mathbf{i}} p_{\mathbf{i}}(x) \tag{4.5}$$

where $p_{\mathbf{i}}(x) \in \mathbb{R}$ are polynomials to be determined, and the MFs $w_{i_j}^j$ are those in the PF model (4.4). This function is a generalization of the fuzzy Lyapunov function in [Blanco & al, 2001], [Tanaka & al, 2003] where $p_{\mathbf{i}}(x)$ are restricted to be homogeneous quadratic polynomials in the state.

Asking this function to be a valid Lyapunov candidate means to ask $V(x)$ to be positive and radially unbounded; since $w_{\mathbf{i}} \geq 0$, it is enough to guarantee $p_{\mathbf{i}}(x) \geq 0$ to have $V(x) \geq 0$. As naturally follows from the polynomial nature of the PF model and the PFLF, positiveness will

be tested by the sum-of-squares condition, i.e., $p_{\mathbf{i}}(x)$ is SOS $\Rightarrow$ $p_{\mathbf{i}}(x) \geq 0$. Radial unboundedness is achieved by replacing zero in the right-hand side with an arbitrary radially-unbounded polynomial, such as $\varepsilon\left(x_1^2 + x_2^2\right)$, with $\varepsilon > 0$ an arbitrary scalar. In the next section, a solution is proposed to the problem of deriving conditions to make (4.5) a valid PFLF for PF model (4.4) incorporating locality and membership-shape information (bounds on partial derivatives).

4.1.2. Stability conditions: SOS formulation

Note that, as (4.4) has the structure in assumption (2.7), the time-derivative of $w_{\mathbf{i}}$ in (4.4) can be rewritten as shown in the previous section or [Guerra & Bernal, 2009], [Bernal & Guerra, 2010]:

$$\dot{w}_{\mathbf{i}} = \frac{\partial w_{\mathbf{i}}}{\partial z}\dot{z} = \sum_{k=1}^{p}\frac{\partial w_{\mathbf{i}}}{\partial z_k}\dot{z}_k = \sum_{k=1}^{p}\frac{\partial}{\partial z_k}\left(\prod_{j=1}^{p} w_{i_j}^{j}\left(z_j\right)\right)\dot{z}_k = \sum_{k=1}^{p}\frac{\partial w_{i_k}^{k}}{\partial z_k}\left(\prod_{\substack{j=1\\ j\neq k}}^{p} w_{i_j}^{j}\left(z_j\right)\right)\dot{z}_k ,$$

where the fact that each factor in $w_{\mathbf{i}}$ depends on only one premise variable has been used. Multiplying by $w_{i_k}^{k} + \left(1 - w_{i_k}^{k}\right) = 1$ gives

$$\dot{w}_{\mathbf{i}} = \sum_{k=1}^{p}\frac{\partial w_{i_k}^{k}}{\partial z_k}\left(w_{i_k}^{k}\prod_{\substack{j=1\\ j\neq k}}^{p} w_{i_j}^{j} + \left(1 - w_{i_k}^{k}\right)\prod_{\substack{j=1\\ j\neq k}}^{p} w_{i_j}^{j}\right)\dot{z}_k = \sum_{k=1}^{p}\frac{\partial w_{i_k}^{k}}{\partial z_k}\left(w_{\mathbf{i}} + w_{\bar{\mathbf{i}}(k)}\right)\dot{z}_k , \qquad (4.6)$$

where $\bar{\mathbf{i}}(k)$ is defined as the p-bit binary index resulting from changing the k-th bit of $\mathbf{i}$ to its complement (Note that if we define α as the integer representation of the set of binary digits $\mathbf{i}$, we would be in the setting of the previous chapter where functions $g_1(\alpha, k)$, $g_2(\alpha, k)$ were used for an equivalent purpose, details are omitted for brevity).

In order to clarify the new notation, consider

$$w_{\mathbf{i}} = w_{(1,0,1)} = w_1^1\left(z_1\right) w_0^2\left(z_2\right) w_1^3\left(z_3\right).$$

To obtain expression (4.6) the expression

$$\dot{w}_{(1,0,1)} = \sum_{k=1}^{3}\frac{\partial w_{(1,0,1)}}{\partial z_k}\dot{z}_k = \frac{\partial w_1^1}{\partial z_1}\left(w_0^2\left(z_2\right) w_1^3\left(z_3\right)\right)\dot{z}_1 + \frac{\partial w_0^2}{\partial z_2}\left(w_1^1\left(z_1\right) w_1^3\left(z_3\right)\right)\dot{z}_2 + \frac{\partial w_1^3}{\partial z_3}\left(w_1^1\left(z_1\right) w_0^2\left(z_2\right)\right)\dot{z}_3$$

must be written. Omitting arguments, the previous expression can be written as in (4.6) by multiplying each summand by the proper term of the form $w_{i_k}^k + \left(1 - w_{i_k}^k\right) = 1$, i.e.:

$$\begin{aligned}\dot{w}_{(1,0,1)} &= \frac{\partial w_1^1}{\partial z_1}\left(w_0^2 w_1^3\right)\left(w_0^1 + w_1^1\right)\dot{z}_1 + \frac{\partial w_0^2}{\partial z_2}\left(w_1^1 w_1^3\right)\left(w_0^2 + w_1^2\right)\dot{z}_2 + \frac{\partial w_1^3}{\partial z_3}\left(w_1^1 w_0^2\right)\left(w_0^3 + w_1^3\right)\dot{z}_3 \\ &= \frac{\partial w_1^1}{\partial z_1}\left(w_0^1 w_0^2 w_1^3 + w_1^1 w_0^2 w_1^3\right)\dot{z}_1 + \frac{\partial w_0^2}{\partial z_2}\left(w_1^1 w_0^2 w_1^3 + w_1^1 w_1^2 w_1^3\right)\dot{z}_2 + \frac{\partial w_1^3}{\partial z_3}\left(w_1^1 w_0^2 w_0^3 + w_1^1 w_0^2 w_1^3\right)\dot{z}_3 \\ &= \frac{\partial w_1^1}{\partial z_1}\left(w_{(0,0,1)} + w_{(1,0,1)}\right)\dot{z}_1 + \frac{\partial w_0^2}{\partial z_2}\left(w_{(1,0,1)} + w_{(1,1,1)}\right)\dot{z}_2 + \frac{\partial w_1^3}{\partial z_3}\left(w_{(1,0,0)} + w_{(1,0,1)}\right)\dot{z}_3.\end{aligned}$$

This form as in section 3.2, will allow convex expressions to be recovered on the Lyapunov method analysis: taking derivatives of the PFLF in (4.5) along the trajectories of PF model (4.4) and taking (4.6) into account gives

$$\begin{aligned}\dot{V}(x) &= \sum_{\mathbf{i}\in\mathbb{I}_p}\left(w_{\mathbf{i}}\dot{p}_{\mathbf{i}} + \dot{w}_{\mathbf{i}}p_{\mathbf{i}}\right) = \sum_{\mathbf{i}\in\mathbb{I}_p}\left(w_{\mathbf{i}}\dot{p}_{\mathbf{i}} + \sum_{k=1}^{p}\frac{\partial w_{i_k}^k}{\partial z_k}\left(w_{\mathbf{i}} + w_{\bar{\mathbf{i}}(k)}\right)\dot{z}_k p_{\mathbf{i}}\right), \\ &= \sum_{\mathbf{i}\in\mathbb{I}_p} w_{\mathbf{i}}\left(\dot{p}_{\mathbf{i}} + \sum_{k=1}^{p}\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\left(p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}\right)\right)\end{aligned} \tag{4.7}$$

where the straightforward identity $\sum_{\mathbf{i}\in\mathbb{I}_p} w_{\bar{\mathbf{i}}(k)}p_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathbb{I}_p} w_{\mathbf{i}}p_{\bar{\mathbf{i}}(k)}$ has been used to write the rightmost expression.

Example 4.2:

Continuing with our previous example, note that according to (4.7), the polynomials $p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}$ sharing the same MF $w_{\mathbf{i}} = w_{(1,0,1)}$ are $p_{(1,0,1)} - p_{(0,0,1)}$ for $k=1$, $p_{(1,0,1)} - p_{(1,1,1)}$ for $k=2$, and $p_{(1,0,1)} - p_{(1,1,0)}$ for $k=3$. It is important to emphasize that should a stability problem have a quadratic solution, these terms will vanish since $\forall \mathbf{i},\mathbf{j}$, $p_{\mathbf{i}} = p_{\mathbf{j}}$, thus proving the generalization ability behind the proposal in this paper.

Consider now expressions $\dot{z}_k = \left(\frac{\partial z_k}{\partial x}\right)^T \dot{x}$ and $\dot{p}_{\mathbf{i}} = \left(\frac{\partial p_{\mathbf{i}}}{\partial x}\right)^T \dot{x}$ which are fuzzy polynomials (z_k and $p_{\mathbf{i}}$ are polynomials by assumption and $\dot{x}$ is taken from its PF representation in (4.4)). The result of substituting them in (4.7) is:

$$\begin{aligned}\dot{V}(x) &= \sum_{\mathbf{i}\in\mathbb{I}_p} w_{\mathbf{i}}\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x}\right)^T \sum_{\mathbf{l}\in\mathbb{I}_p} w_{\mathbf{l}}\mathbf{q}_{\mathbf{l}} + \sum_{k=1}^{p}\frac{\partial w_0^k}{\partial z_k}\left(\left(\frac{\partial z_k}{\partial x}\right)^T \sum_{\mathbf{l}\in\mathbb{I}_p} w_{\mathbf{l}}\mathbf{q}_{\mathbf{l}}\right)\left(p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}\right)\right) \\ &= \sum_{\mathbf{i}\in\mathbb{I}_p}\sum_{\mathbf{l}\in\mathbb{I}_p} w_{\mathbf{i}}w_{\mathbf{l}}\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x}\right)^T \mathbf{q}_{\mathbf{l}} + \sum_{k=1}^{p}\left(\left(\frac{\partial w_0^k}{\partial z_k}\cdot\frac{\partial z_k}{\partial x}\right)^T \mathbf{q}_{\mathbf{l}}\right)\left(p_{\mathbf{i}} - p_{\bar{\mathbf{i}}(k)}\right)\right)\end{aligned} \tag{4.8}$$

All terms in the above expression are either MFs or polynomials, except possibly for $\frac{\partial w_0^k}{\partial z_k}$.

The basic idea is that, in the same way as the nonlinearities were fuzzified, $\frac{\partial w_0^k}{\partial z_k}$ can be recast again as a convex sum of polynomials, following the polynomial fuzzy modeling technique already described in (4.2) and (4.3) [Sala, 2009] and [Sala & Arino, 2009].

Example 4.3:

Given a scalar nonlinearity $h(x) = \sin x$ in $\Omega = [-1,1]$, it is easy to see that

$$h(x) = 0.8414 w_0 - 0.8414(1 - w_0)$$

with $w_0 = 0.5942 \sin x_1 + 0.5$, from which it follows that

$$\frac{dw_0}{dx} = 0.5942 \cos x$$

These functions are all infinitely differentiable in the chosen region of interest $\Omega = [-1,1]$. The latter one, $\frac{dw_0}{dx}$, can also be written as a convex sum of polynomials in Ω, for instance:

$$\frac{dw_0}{dx} = 0.5942\mu_0 + 0.3211(1 - \mu_0)$$

With $\mu_0 = 2.1755 \cos x - 1.1755$.

Actually, polynomials of degree zero have been chosen in this example, but the methodology applies to any arbitrary chosen degree.

Since $\frac{\partial z_k}{\partial x} \in \mathbb{R}^{n\times 1}$ is assumed to be a polynomial vector, using a PF model of $\frac{\partial w_0^k}{\partial z_k}$, every expression $\frac{\partial w_0^k}{\partial z_k}\cdot\frac{\partial z_k}{\partial x} \in \mathbb{R}^{n\times 1}$ in (4.8) can be written as

$$\frac{\partial w_0^k}{\partial z_k}\cdot\frac{\partial z_k}{\partial x}=\sum_{\mathbf{v}_k\in\mathbb{I}_{s_k}}\mu_{\mathbf{v}_k}^k(x)\mathbf{r}_{\mathbf{v}_k}^k(x),\ k=1,\ldots,p, \tag{4.9}$$

with s_k being the number of possible non-polynomial nonlinearities in $\frac{\partial w_0^k}{\partial z_k}$, and $\mu_{\mathbf{v}_k}^k=\mu_{v_k^1}^k\cdots\mu_{v_k^{s_k}}^k$, $\sum_{v_k^i=0}^1\mu_{v_k^i}^k(\cdot)=1$, $\mu_{v_k^i}^k(\cdot)\geq 0$ being the MFs associated with each modelled nonlinearity, and $\mathbf{r}_{\mathbf{v}_k}^k(x)\in\mathbb{R}^{n\times 1}$ being the resulting polynomial vector.

Substituting (4.9) in (4.8) yields

$$\begin{aligned}\dot{V}(x)&=\sum_{\mathbf{i}\in\mathbb{I}_p}\sum_{\mathbf{l}\in\mathbb{I}_p}w_{\mathbf{i}}w_{\mathbf{l}}\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x}\right)^T\mathbf{q}_{\mathbf{l}}+\sum_{k=1}^{p}\left(\left(\sum_{v_k\in\mathbb{I}_{s_k}}\mu_{v_k}^k\mathbf{r}_{\mathbf{v}_k}^k\right)^T\mathbf{q}_{\mathbf{l}}\right)\left(p_{\mathbf{i}}-p_{\bar{\mathbf{i}}(k)}\right)\right)\\&=\sum_{\mathbf{i}\in\mathbb{I}_p}\sum_{\mathbf{l}\in\mathbb{I}_p}\sum_{\mathbf{v}_1\in\mathbb{I}_{s_1}}\cdots\sum_{\mathbf{v}_p\in\mathbb{I}_{s_p}}w_{\mathbf{i}}w_{\mathbf{l}}\mu_{\mathbf{v}_1}^1\cdots\mu_{\mathbf{v}_p}^p\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x}\right)^T\mathbf{q}_{\mathbf{l}}+\sum_{k=1}^{p}\left(\left(\mathbf{r}_{\mathbf{v}_k}^k\right)^T\mathbf{q}_{\mathbf{l}}\right)\left(p_{\mathbf{i}}-p_{\bar{\mathbf{i}}(k)}\right)\right)\end{aligned}$$

Defining the polynomial vector $\hat{\mathbf{p}}_{\mathbf{i}}=\begin{bmatrix}p_{\mathbf{i}}-p_{\bar{\mathbf{i}}(1)}\\ \vdots\\ p_{\mathbf{i}}-p_{\bar{\mathbf{i}}(p)}\end{bmatrix}\in\mathbb{R}^{1\times p}$, the polynomial matrix $\mathbf{R}_{\mathbf{v}}=\begin{bmatrix}\left(\mathbf{r}_{\mathbf{v}_1}^1\right)^T\\ \vdots\\ \left(\mathbf{r}_{\mathbf{v}_p}^p\right)^T\end{bmatrix}\in\mathbb{R}^{p\times n}$, and the multi-index $\mathbf{v}=\left(\mathbf{v}_1,\cdots,\mathbf{v}_p\right)$, the previous expression can be rewritten as

$$\dot{V}(x)=\sum_{\mathbf{i}\in\mathbb{I}_p}\sum_{\mathbf{l}\in\mathbb{I}_p}\sum_{\mathbf{v}\in\mathbb{I}_\sigma}w_{\mathbf{i}}w_{\mathbf{l}}\mu_{\mathbf{v}}\left(\left(\frac{\partial p_{\mathbf{i}}}{\partial x}\right)^T\mathbf{q}_{\mathbf{l}}+\hat{\mathbf{p}}_{\mathbf{i}}^T\mathbf{R}_{\mathbf{v}}\mathbf{q}_{\mathbf{l}}\right) \tag{4.10}$$

with $\sigma=s_1+\ldots+s_p$.

The main result can now be stated:

Theorem 4.1: [Bernal & al, 2011]

The PF model (4.4) with MF-derivatives as in (4.9) is asymptotically stable if there exist polynomials $p_{\mathbf{i}}(x)\in\mathbb{R}$, and non-negative, radially unbounded polynomials $\varepsilon_1(x)$, $\varepsilon_2(x)>0$ such that

$$p_{\mathbf{i}}(x)-\varepsilon_1(x) \text{ is SOS}$$

and

$$-\left(\left(\partial p_{\mathbf{i}}/\partial x\right)^T \mathbf{q}_{\mathbf{l}}+\hat{\mathbf{p}}_{\mathbf{i}}^T\mathbf{R}_{\mathbf{v}}\mathbf{q}_{\mathbf{l}}\right)-\varepsilon_2(x) \text{ is SOS}$$

for all $\mathbf{i},\mathbf{l}\in\mathbb{I}_p$, $\mathbf{v}\in\mathbb{I}_\sigma$ with $\hat{\mathbf{p}}_{\mathbf{i}}$ and $\mathbf{R}_{\mathbf{v}}$ defined as in (4.9)-(4.10).

Proof: It follows immediately from the fact that $p_{\mathbf{i}}(x)-\varepsilon_1(x)$ being SOS enforces the Lyapunov function candidate (4.5) to be non-negative and radially unbounded, whereas $-\left(\left(\partial p_{\mathbf{i}}/\partial x\right)^T \mathbf{q}_{\mathbf{l}}+\hat{\mathbf{p}}_{\mathbf{i}}^T\mathbf{R}_{\mathbf{v}}\mathbf{q}_{\mathbf{l}}\right)-\varepsilon_2(x)$ being SOS assures the time-derivative of the Lyapunov function to be strictly negative outside the origin, i.e., $\dot{V}(x)<0$, as can be deduced from (4.10). □

Remark 4.1: In order to reduce conservativeness of the above result, any relaxation scheme can be applied to the tensor-product double fuzzy summation in $w_{\mathbf{i}}w_{\mathbf{l}}$ that appears in (4.10), for example, grouping those terms sharing the same factorization of $w_{\mathbf{i}}w_{\mathbf{l}}$ [Tanaka & Wang, 2001], [Sala & Ariño, 2007] and [Ariño & Sala, 2007].

Remark 4.2: As originally explained in [Prajna & al, 2004a], [Parrilo, 2003] and illustrated in [Sala & Ariño, 2009], the *Positivstellensatz* argumentation extends the use of Lagrange multipliers and S-procedure in the LMI framework to the polynomial-SOS case, thus permitting local information to be included as constraints in SOS conditions. Assume that *m* known restrictions arranged as a vector $F(x)\geq 0$, $F(x)\in\mathbb{R}^m$ hold in Ω. Then, conditions in Theorem 4.1 are valid in Ω if $p_i(x)-\sum_j u_j(x)\phi_j(x)$ and $-\left(\left(\partial p_{\mathbf{i}}/\partial x\right)^T q_l+\hat{p}_i^T R_v q_l\right)-\sum_k \tilde{u}_k(x)\phi_k(x)$ are SOS with $u_j(x)$, $\tilde{u}_k(x)$ being SOS polynomials (multipliers) and $\phi_j(x)$ being arbitrary polynomials composed by products of those in F. These sufficient conditions may be easier to fulfil than those without local restrictions.

Remark 4.3: Polynomial-programming techniques, even if convex for a fixed degree of the polynomials, are computationally hard in the fuzzy-control context. The basic drawbacks are: (a) a high-degree Taylor series is needed to approximate the nonlinearities in a large domain;

(b) the number of rules is two to the power of the number of nonlinearities and the degree of them in π;

(c) as polynomials diverge wildly, many times, the obtained results are worse than ordinary T-S ones unless *Positivstellensatz* multipliers are used;

(d) as model and Lyapunov function's degrees increase, so does the needed degree of the *Positivstellensatz* multipliers. Hence, even if the conditions are asymptotically exact under some uniform convergence assumptions, there are severe limitations in applying the approach to realistic problems. In the authors' opinion, polynomial approaches, even if theoretically elegant, they should be used in practice only if ordinary T-S ones fail.

4.2. Examples

Example 4.4:

Consider the following nonlinear model [Tanaka & al, 2003], [Tanaka & al, 2009b]:

$$\dot{x}(t)=\begin{bmatrix} -\frac{7}{2}x_1-4x_2-\frac{3}{2}x_1\sin x_1 \\ \frac{19}{2}x_1-2x_2-\frac{21}{2}x_1\sin x_1 \end{bmatrix}. \tag{4.11}$$

The stability properties of the previous model in $\Omega=\{|x_i|\le 1\}$ will be investigated. To do so, nonlinearity $\sin x_1$ is written as a convex sum of polynomials following the techniques described above with $z_1=x_1$ (for more details, see [Sala, 2009], [Sala & Ariño, 2009]), leading to the following PF model structure:

$$\dot{x}(t)=\sum_{\mathbf{i}\in\mathbb{I}_1} w_{\mathbf{i}}\mathbf{q}_{\mathbf{i}}=w_0^1\begin{bmatrix} -\frac{7}{2}x_1-4x_2-\frac{3}{2}x_1\left(q_0^1(x)\right) \\ \frac{19}{2}x_1-2x_2-\frac{21}{2}x_1\left(q_0^1(x)\right) \end{bmatrix}+w_1^1\begin{bmatrix} -\frac{7}{2}x_1-4x_2-\frac{3}{2}x_1\left(q_1^1(x)\right) \\ \frac{19}{2}x_1-2x_2-\frac{21}{2}x_1\left(q_1^1(x)\right) \end{bmatrix}. \tag{4.12}$$

where $q_0^1(x)$, $q_1^1(x)$ are polynomials of certain degree, and $w_0^1(x)$, $w_1^1(x)$ are the corresponding MFs.

Consider a 0-degree PF model: in this case, $q_0^1(x)=0.8414$ and $q_1^1(x)=-0.8414$ are, plainly, constants while $w_0^1=0.5942\sin x_1+0.5$ and $w_1^1=1-w_0^1$ are the corresponding MFs. For expression $\frac{\partial w_0^1}{\partial x_1}=0.5942\cos x_1$, consider a 0-degree modeling as in (4.9), i.e., bounds

$r_0^1(x)=0.5942$, $r_1^1(x)=0.3211$, and MFs $\mu_0^1=2.1755\cos x_1-1.1755$, $\mu_1^1=1-\mu_0^1$. Theorem 4.1 is now used to analyse stability for a degree-2 PFLF candidate of the form

$$V(x)=w_0^1 p_1(x)+w_1^1 p_2(x).$$

When no Lagrange multipliers are used (global analysis) the SOS problem is unfeasible. In order to make local analysis as pointed out in Remark 4.2, a set of second order polynomial Lagrange multipliers multiplied by the following constraints (valid in Ω with $\bar{x}=1$) are included:

$$\begin{aligned} &x_1^2-\bar{x}^2<0,\quad x_2^2-\bar{x}^2<0,\quad -\left(x_1^2-\bar{x}^2\right)\left(x_2^2-\bar{x}^2\right)<0,\quad -\left(x_1^2-\bar{x}^2\right)\left(x_1-\bar{x}^2\right)<0,\\ &\left(x_1^2-\bar{x}^2\right)\left(x_1+\bar{x}^2\right)<0,\quad -\left(x_2^2-\bar{x}^2\right)\left(x_2-\bar{x}^2\right)<0,\quad \left(x_2^2-\bar{x}^2\right)\left(x_2+\bar{x}^2\right)<0. \end{aligned} \tag{4.13}$$

Via SOSTools, conditions in Theorem 4.1 are then satisfied for:

$$p_1(x)=3.9106x_1^2+1.863x_1x_2+4.1858x_2^2$$

and

$$p_2(x)=10.692x_1^2+1.2375x_1x_2+2.569x_2^2.$$

In Figure 4.1, some level curves of this PFLF are displayed in dashed-lines; the outermost Lyapunov level is in bold-dashed. Some trajectories in solid lines are also included.

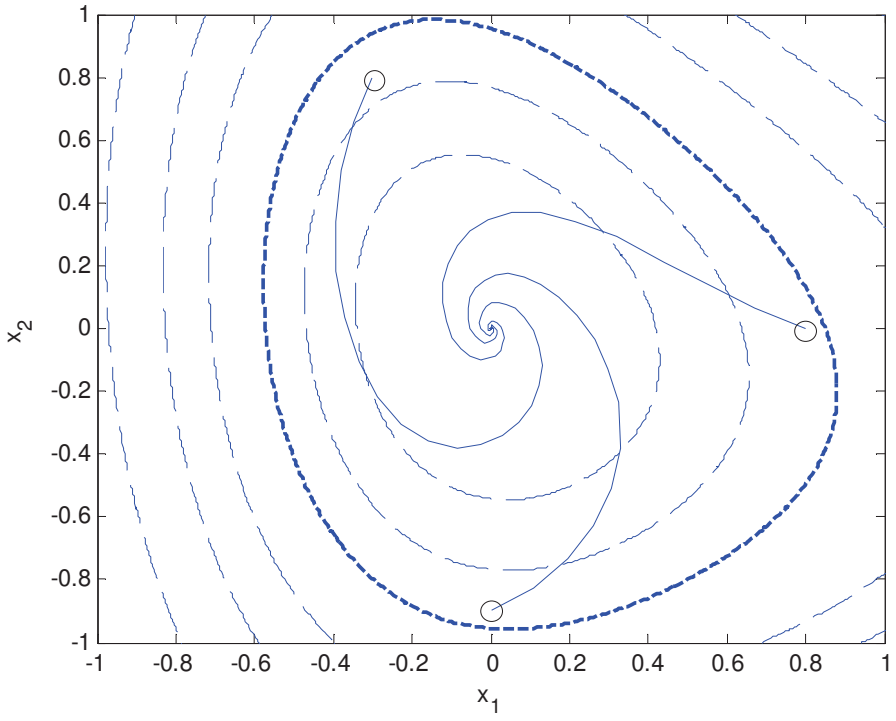


Figure 4.1 : Lyapunov levels for the 0-degree PFLF in Example 4.4

Now consider a 3rd-degree PF model in (4.12) with polynomials

$$q_0^1(x) = x_1 - 0.1585x_1^3,\ q_1^1(x) = x_1 - 0.1667x_1^3,$$

and MFs

$$w_0^1 = 122.9\frac{(\sin x_1 - x_1)}{x_1^3} + 20.48,\ w_1^1 = -\left(19.48 + 122.9\frac{(\sin x_1 - x_1)}{x_1^3}\right)$$

It can be checked that

$$\frac{\partial w_0^1}{\partial x_1} = 122.9\left(\frac{x_1^3(\cos x_1 - 1) - 3x_1^2(\sin x_1 - x_1)}{x_1^6}\right) = 122.9\left(\frac{\cos x_1}{x_1^3} - \frac{3\sin x_1}{x_1^4} + \frac{2}{x_1^3}\right), \quad (4.14)$$

which can be written as follows from the Taylor-series representation of its components

$$\begin{aligned}\frac{\partial w_0^1}{\partial x_1} &= 122.9\left(\frac{1}{x_1^3}\left(1 - \frac{x_1^2}{2!} + \frac{x_1^4}{4!} - \cdots\right) - \frac{3}{x_1^4}\left(x - \frac{x_1^3}{3!} + \frac{x_1^5}{5!} - \cdots\right) + \frac{2}{x_1^3}\right) \\ &= 122.9\left(\frac{1}{x_1^3}\left(\frac{x_1^4}{4!} - \cdots\right) - \frac{3}{x_1^4}\left(\frac{x_1^5}{5!} - \cdots\right)\right) = 122.9\left(\left(\frac{1}{4!} - \frac{3}{5!}\right)x_1 - \left(\frac{1}{6!} - \frac{3}{7!}\right)x_1^3 + \cdots\right)\end{aligned} \quad (4.15)$$

Thus proving that it can be defined in 0 as the limit of (4.14) and it is therefore a smooth function.

Then, since $z_1 = x_1$ the following third-degree Taylor-based PF model in $x_1 \in [-1,1]$ arises:

$$\frac{\partial w_0^1}{\partial z_1} \cdot \frac{\partial z_1}{\partial x} = \sum_{\mathbf{v}_1=0}^{1} \mu_{\mathbf{v}_1}^1(x)\mathbf{r}_{\mathbf{v}_1}^1(x) = \mu_0^1\begin{bmatrix} 2.0483x_1 - 0.09555x_1^3 \\ 0 \end{bmatrix} + \mu_1^1\begin{bmatrix} 2.0483x_1 - 0.0975x_1^3 \\ 0 \end{bmatrix}$$

with $T(x) = \frac{122.9}{x_1^3}\left(\frac{\cos x_1}{x_1^3} - \frac{3\sin x_1}{x_1^4} + \frac{2}{x_1^3} - 0.0167x_1\right)$, $\mu_0^1 = \frac{T(x) + 0.0975}{-0.09555 + 0.0975}$, $\mu_1^1 = 1 - \mu_0^1$.

Recall that according to definitions (4.9)-(4.10), in this example $\mathbf{v} = \mathbf{v}_1 \in \mathbb{I}_1$, so matrix $\mathbf{R}_{\mathbf{v}} = \left(r_{\mathbf{v}_1}^1\right)^T \in \mathbb{R}^{2\times 1}$. The example is now analysed via Theorem 4.1.

Via SOSTools, polynomials $p_1(x) = p_2(x) = 8.4852x_1^2 + 0.23829x_1x_2 + 2.8658x_2^2$ are found satisfying conditions in Theorem 4.1 under the aforementioned constraints. Note that the corresponding Lyapunov function has lost its fuzzy structure since $p_1(x) = p_2(x)$, i.e., $V(x) = w_0^1 p(x) + (1 - w_0^1)p(x) = p(x)$, a solution which is not ruled out by conditions in Theorem 4.1.

Discussion: Independently of their degree, PF models obtained by the aforementioned methodology are all exact representations of nonlinearities associated to a nonlinear model or the MFs' derivatives. Then, a natural question arises: what is the difference between lower or higher degrees in PF modeling? The answer originates from the previous example: as the PF model degree increases the vertex polynomials converge to the Taylor series under mild assumptions; then, MFs yield their modeling influence only to the corresponding polynomials terms of higher degree. Therefore, the fuzzy character of the PF model becomes less significant for higher degree models. As a consequence of this phenomenon, in the previous example an ordinary quadratic polynomial Lyapunov function could not be found when the PF model was highly fuzzy (degree zero approximations): a non-quadratic PFLF has been found instead. On the other hand, when the PF model degree was increased the family of models thus represented seems to have been reduced in such a way that an ordinary quadratic Lyapunov function was found, thus having no need of the fuzzy structure for it.

Example 4.5:

Consider the following nonlinear model:

$$\dot{x}(t)=\begin{bmatrix} -0.2363x_1^2+0.0985x_1\left((0.1x_1)^2+x_1(0.1x_2)^2\right)-0.9x_2 \\ \sinh x_1-2x_1-0.7097x_2+0.3427x_2\left((0.1x_1)^2+(0.1x_2)^2\right) \end{bmatrix}, \tag{4.16}$$

which, from simulations, has a stable focus at the origin and an unstable limit cycle; it is therefore not globally stable.

For different values of $\bar{x}>0$, let $\Omega=\{|x_i|\le\bar{x}\}$ be a square region of interest in which a decreasing Lyapunov function is to be found. Simulation shows that $\bar{x}=4.15$ is the maximum admissible value for the whole Ω to be in the basin of attraction.

First- and third-degree PF models of (4.16) have been obtained depending on whether first- or third-degree polynomials were used for bounding $\sinh x_1$. The MFs' derivatives corresponding to these PF models have been also bounded by first- and third-degree polynomials with an analogous methodology. Then, under second-order Lagrange multipliers with constraints (4.13), Theorem 4.1 has been used to search the maximum $\bar{x}>0$ for which stability can be proved for each combination of the previous cases.

The test is first run for quadratic non-fuzzy polynomial Lyapunov functions of the form

$$V(x)=p(x)$$

where of course the time-derivatives of the MFs play no role (conditions in Theorem 4.1 have $\hat{p}_i = 0$); these results are then compared with those obtained with a second-order fuzzy polynomial function

$$V(x) = \sum_{\mathbf{i} \in \mathbb{I}_p} w_{\mathbf{i}}(x)\, p_{\mathbf{i}}(x)$$

The results are shown in Table 4.1, the degree of the candidate Lyapunov function was fixed to 4.

$\Omega = \{\lvert x_i \rvert \le \bar{x}\}$	$\deg(\mathbf{q}_{\mathbf{i}}) = 1$	$\deg(\mathbf{q}_{\mathbf{i}}) = 3$
Non-fuzzy PLF	$\bar{x} = 2.1094$	$\bar{x} = 2.6406$
PFLF, $\deg(\mathbf{r}^k_{\mathbf{v}_k}) = 1$	$\bar{x} = 2.500$	$\bar{x} = 2.6875$
PFLF, $\deg(\mathbf{r}^k_{\mathbf{v}_k}) = 3$	$\bar{x} = 2.5313$	$\bar{x} = 2.7344$

Table 4.1: Comparing polynomial Lyapunov functions versus polynomial fuzzy Lyapunov functions in Example 4.5: maximum size of a square region of interest where a decreasing LF is feasible.

As expected, better approximations on the PF model and/or the MFs'derivatives lead to better results. On the other hand, given a particular PF model, PFLFs improve over non-fuzzy ones.

4.3. Conclusion

In this chapter, a new methodology for analyzing the stability of continuous-time nonlinear models in the polynomial fuzzy form has been presented. It combines recent advances on Taylor-based fuzzy polynomial models and local stability via fuzzy polynomial Lyapunov functions, exploiting both polynomial bounds on the model's non-polynomial nonlinearities and, also, polynomial bounds on the partial derivatives of the membership functions. The examples in this chapter illustrate that fuzzy-polynomial Lyapunov functions prove useful in performing better than the unstructured polynomial Lyapunov functions, getting larger estimates of the region of attraction.

5. Chapter 5: Non-quadratic stabilization of T-S models: Using partial-derivative information

Synopsys

This chapter represents the third major contribution of this work in which non-quadratic stabilization of continuous-time Takagi Sugeno models will be discussed. Several new local approaches for controller design based on non-quadratic Lyapunov functions and non-PDC controller will be presented. The chapter is mainly based on contributions I participated in and that appear in the papers [Bernal & al, 2010], [Guerra & al, 2011], [Pan & al, 2012] and [Jaadari & al, 2012].

5.1. Introduction

In the previous chapters, we proposed new local approaches to deal with stability analysis and to overcome the problems of conservativeness. It has been found that reducing the global goals to local ones while employing a non-quadratic Lyapunov function actually leads to reasonable local asymptotic conditions that provide an estimation of the stability domain: an egress from the quadratic framework. In this chapter, we extend these approaches to the stabilization of continuous T-S models. The new obtained solutions overcome the problem of dealing with time-derivatives of the membership functions and lead to stabilize a large family of nonlinear models that do not admit global stabilization. Moreover, the new conditions are expressed as linear matrix inequalities (LMIs) which are efficiently solved by convex optimization techniques. For illustration purposes, examples are developed that clearly point out the advantages of the new approaches over already existing ones.

5.2. Non-quadratic stabilization of T-S models: a local point of view

5.2.1. Problem formulation

Consider the following T-S model

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right) = A_z x(t) + B_z u(t) \tag{5.1}$$

where A_i, B_i, $i \in \{1, \cdots, r\}$ are controllable pairs of matrices of proper dimensions, $r = 2^p \in \mathbb{N}$ is the number of linear models and $h_i\left(z(t)\right)$ are the membership functions fulfilling (2.7).

The following non-PDC control law will be used [Guerra & Vermeiren, 2004]:

$$u(t) = \sum_{i=1}^{r} h_i F_i \left(\sum_{j=1}^{r} h_j P_j \right)^{-1} x(t) = F_z P_z^{-1} x(t) \tag{5.2}$$

Substituting (5.2) in (5.1) gives the closed-loop T-S model

$$\dot{x}(t) = \left(A_z + B_z F_z P_z^{-1}\right) x(t), \tag{5.3}$$

whose stability properties will be investigated through the following non-quadratic fuzzy Lyapunov function (NQFLF) candidate:

$$V\left(x(t)\right) = x(t)^T P_z^{-1} x(t) \tag{5.4}$$

where $P_z = P_z^T > 0$ (then $P_z^{-1} > 0$). Its derivative along the trajectories of the closed-loop T-S model (5.3) is:

$$\dot{V}\left(x(t)\right) = x^T(t)\left(P_z^{-1}\left(A_z + B_z F_z P_z^{-1}\right) + \left(A_z + B_z F_z P_z^{-1}\right)^T P_z^{-1} + \dot{P}_z^{-1}\right)x(t) \tag{5.5}$$

Since $P_z \dot{P}_z^{-1} P_z = -\dot{P}_z$, elementary matrix manipulations show that $\dot{V}(x) < 0$ is verified if

$$A_z P_z + B_z F_z + \left(A_z P_z + B_z F_z\right)^T - \dot{P}_z < 0. \tag{5.6}$$

A preliminary result that, in a sense, justifies the future developments is stated first.

Theorem 5.1:(Local stabilizability): If there exist matrices of the proper size $P_i = P_i^T > 0$, F_i, $i \in \{1,\ldots,r\}$ such that $A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T < 0$, then there exists a domain D, $0 \in D$, such that T-S model (5.1) is locally asymptotically stabilizable under control law (5.2).

Proof: The same procedure applied to obtain the local stability conditions in Theorem 3.1, will be used to derive the new local stabilization conditions.

For control purposes, As $A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T < 0$ it always exists a sufficiently small $\lambda > 0$ such that:

$$A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T + \lambda I < 0. \tag{5.7}$$

Then a domain $D = \left\{x : x \in B,\ \left\|\dot{P}_z\right\| < \lambda\right\}$ containing the origin can be defined since:

$$\dot{P}_z = \sum_{i=1}^{r} \dot{h}_i P_i = \sum_{i=1}^{r} \left(\frac{\partial h_i}{\partial z}\right)^T \frac{\partial z}{\partial x}^T \dot{x} P_i = \sum_{i=1}^{r} \left[\left(\frac{\partial h_i}{\partial z}\right)^T \frac{\partial z}{\partial x}^T \left(A_z + B_z F_z P_z^{-1}\right) x(t)\right] P_i \tag{5.8}$$

As $\dot{P}_z$ (5.8) is a continuous function of $x(t)$, equal to zero at the equilibrium point $x = 0$ it is easy to conclude that D contains a small enough open ball by continuity arguments. Since $V(x) > 0$ and $\dot{V}(x) < 0$ in $D - \{0\}$, the equilibrium point $x = 0$ is locally asymptotically stable, thus concluding the proof. □

Remark 5.1 : conditions for Theorem 5.1 can be ensured, for example, with (2.32), i.e. define $\Upsilon_{ij} = A_i P_j + B_i F_j + P_j A_i^T + F_j^T B_i^T$, find $P_i = P_i^T > 0$, F_i, $i \in \{1,\ldots,r\}$ such that:

$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} > 0,\ i, j \in \{1,\cdots,r\} \tag{5.9}$$

In view of these first results, the whole challenge is now to be able to write and/or to bound $\dot{P}_z$ in a satisfactory way, including the LMI formulation problems. As previously mentioned, the term $\dot{P}_z$ depends on the time-derivatives of the MFs and moreover these derivatives can depend on the control to be calculated. Therefore, obtaining LMI conditions implying (5.6) in a "general" case is challenging. Generally [Blanco & al, 2001], [Tanaka & al, 2003], [Bernal & al, 2006], [Mozelli & al. 2009a] use assumptions such as $\left\|\dot{P}_z\right\| < \phi_z$ or $\left|\dot{h}_i(x)\right| \le \phi_i$. Let us illustrate on a very simple example why this *a priori* assumption can be a major problem of these approaches.

Example 5.1 :

Consider the following nonlinear system form [Tanaka & al. 2007]:

$$\dot{x} = ax + \left(x^3 + b\right)u \tag{5.10}$$

Employing the sector nonlinearity approach, the following T-S model can be obtained:

$$\dot{x} = \sum_{i=1}^{2} h_i(x)\left(A_i x + B_i u\right) \tag{5.11}$$

Where with the defined compact set $C = \{x : |x| \le d\}$: $A_1 = A_2 = a$, $B_1 = d^3 + b$, $B_2 = -d^3 + b$, $h_1(x) = w_0^1 = \dfrac{x^3 + d^3}{2d^3}$ and $h_2(x) = w_1^1 = \dfrac{d^3 - x^3}{2d^3}$. Consider now a condition in the form: $\left|\dot{h}_1(x)\right| \le \phi_1$ which results in:

$$\left|\dot{h}_1(x)\right| = \frac{3}{2d^3}x^2\left|\dot{x}\right| = \frac{3}{2d^3}x^2\left|ax + \left(x^3 + b\right)u\right| \le \phi_1 \tag{5.12}$$

Consider now a particular case with $a = 100$, $b = 21$, $d = 2.71$ and as in [Tanaka & al, 2007], $\phi_1 = 10^4$. Employ conditions [Tanaka & al, 2007] to design controller for system (5.11). Figure 5.1 depicts the trajectories of $\dot{h}_1$ with initial state $x_0 = 2.6 \in C$. It can be seen that the lower bound of $\dot{h}_1$ does not satisfy the assumption $\left|\dot{h}_1(x)\right| \le 10^4$. To satisfy the assumption

$\left|\dot{h}_1(x)\right| \le 10^4$, the initial state must be chosen in the region $\{x : -2.71 \le x \le 2.38\}$ whose restrictions go beyond the original compact set $C = \{x : |x| \le 2.71\}$.

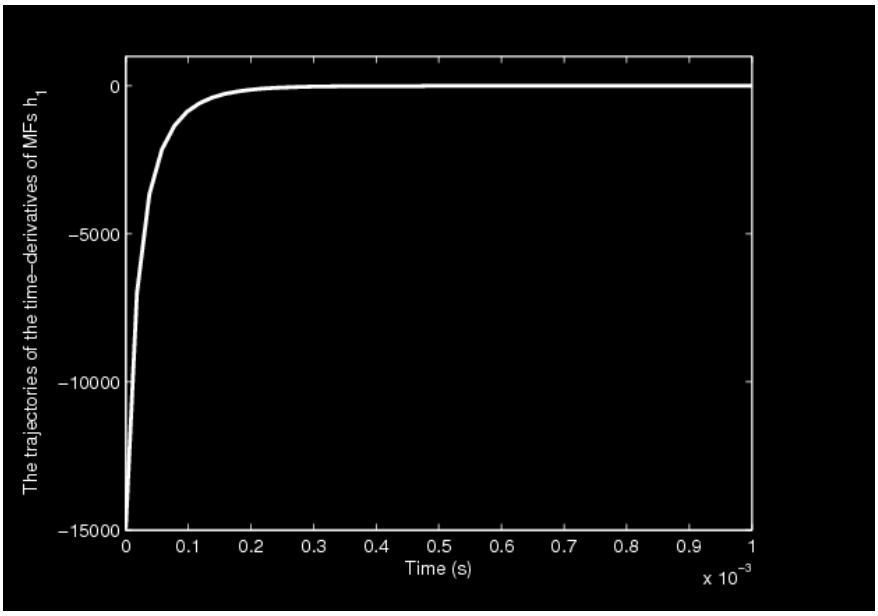


Figure 5.1 : The trajectory of $\dot{h}_1$.

This very simple example shows clearly that the assumption on an *a priori* bound of the time-derivatives of the MFs $\dot{h}_1$ is a major problem of these techniques as they depend on the to-be-designed controller $u(x)$. Their validity must be checked *a posteriori*, which makes their usefulness questionable. Next parts try to overcome this problem whereas keeping LMI constraints problems.

5.2.2. Constraints on the control

Throughout this chapter, $(A)_{v\cdot}$ stands for the v^{th} row of A, $(A)_{\cdot s}$ for the s^{th} column and $(A)_{vs}$ for the element in the v^{th} row and s^{th} column. The interested reader is referred to section 3.2 from chapter 3 or [Guerra & Bernal, 2009] for details concerning the fact that $\dot{P}_z$ can be written as:

$$\dot{P}_z = \sum_{k=1}^{p} \frac{\partial w_0^k}{\partial z_k} \left(P_{g_1(z,k)} - P_{g_2(z,k)} \right) \dot{z}_k = \sum_{j=1}^{r} \sum_{k=1}^{p} h_j \frac{\partial w_0^k}{\partial z_k} \dot{z}_k \left(P_{g_1(j,k)} - P_{g_2(j,k)} \right) \tag{5.13}$$

with $g_1(j,k)=\lfloor (j-1)/2^{p+1-k} \rfloor \times 2^{p+1-k}+1+(j-1) \bmod 2^{p-k}$, and $g_2(j,k)=g_1(j,k)+2^{p-k}$, denoting the floor function as $\lfloor \ \rfloor$. Note that, as $\frac{\partial w_0^k}{\partial z_k}$ are a priori known, the major point will be the writing of $\dot{z}_k$. Consider that:

$$\dot{z}_k=\left(\frac{\partial z_k}{\partial x}\right)^T \dot{x}=\sum_{v=1}^{n}\frac{\partial z_k}{\partial x_v}\dot{x}_v \text{ and } \dot{x}_v=(A_z)_{v\cdot}\,x+(B_z)_{v\cdot}\,u=\sum_{s=1}^{n}(A_z)_{vs}\,x_s+\sum_{e=1}^{m}(B_z)_{ve}\,u_e \quad (5.14)$$

Therefore:

$$\dot{z}_k=\sum_{v=1}^{n}\sum_{s=1}^{n}\frac{\partial z_k}{\partial x_v}(A_z)_{vs}\,x_s+\sum_{v=1}^{n}\sum_{e=1}^{m}\frac{\partial z_k}{\partial x_v}(B_z)_{ve}\,u_e \quad (5.15)$$

Thus, with $\frac{\partial w_0^k}{\partial z_k}\frac{\partial z_k}{\partial x_v}=\frac{\partial w_0^k}{\partial x_v}$, (5.13) writes:

$$\dot{P}_z=\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{s=1}^{n}\frac{\partial w_0^k}{\partial x_v}x_s(A_z)_{vs}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)+\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{e=1}^{m}\frac{\partial w_0^k}{\partial x_v}u_e(B_z)_{ve}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right) \quad (5.16)$$

Now, considering the whole expression (5.6) $\dot{V}(x)<0$ is ensured if:

$$\begin{aligned}A_zP_z+B_zF_z+(A_zP_z+B_zF_z)^T-\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{s=1}^{n}\frac{\partial w_0^k}{\partial x_v}x_s(A_z)_{vs}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)\\ -\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{e=1}^{m}\frac{\partial w_0^k}{\partial x_v}u_e(B_z)_{ve}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)<0\end{aligned} \quad (5.17)$$

Via expressions $\frac{\partial w_0^k}{\partial x_v}x_s$ and $\frac{\partial w_0^k}{\partial x_v}u_e$ locality is now introduced. If these terms have known bounds (derived from the modelling area *C*, for instance), then (5.17) turns out to be an LMI expression to test stability in the region thus induced; if not, some initial bounds can be chosen so bisection can be applied to find the largest region in which (5.17) remains feasible. In both cases, the result is a controller that guarantees the closed-loop T-S model in (5.3) to be stable in a local region around the origin.

Remark 5.2: it is ***very important*** to notice that the expression (5.17) shows that if a quadratic Lyapunov function exists, then $\dot{P}_z=0$ is guaranteed via $P_{g_1(\alpha,k)}=P_{g_2(\alpha,k)}$ for each $i\in\{1,\cdots,r\}$ and $k\in\{1,\cdots,p\}$. This remark will hold also for the LMI constraints thereinafter, lemmas will shown this fact. Therefore, conditions derived ***always*** include the quadratic case.

While bounds are easily found from C for those expressions depending on $x(t)$ like w_k^0, for the control, extra LMI constraints have to be set. A classical way of doing [Tanaka & Wang, 2001] is followed. Assume $\|x(0)\| < x_0$ and $V(x(0)) \leq 1$ therefore:

$$x^T(t) P_z^{-1} x(t) \leq x^T(0) P_z^{-1} x(0) \leq 1 \text{ for } t \geq 0 \tag{5.18}$$

Notice that it means $P_z > x_0^2 I$. Now, to guarantee $\|u(t)\| < \mu$, a sufficient condition is:

$$\frac{1}{\mu^2}\|u\|_2^2 = \frac{1}{\mu^2} x^T(t) P_z^{-1} F_z^T F_z P_z^{-1} x(t) \leq x^T(t) P_z^{-1} x(t) \leq x^T(0) P_z^{-1} x(0) \leq 1 \tag{5.19}$$

Thus middle part of (5.19) can be written:

$$x^T(t)\left(\frac{1}{\mu^2} P_z^{-1} F_z^T F_z P_z^{-1} - P_z^{-1}\right) x(t) \leq 0 \Leftrightarrow \frac{1}{\mu^2} F_z^T F_z - P_z \leq 0 \tag{5.20}$$

and using Schur complement $\|u(t)\| < \mu$ is ensured if:

$$\begin{bmatrix} P_z & F_z^T \\ F_z & \mu^2 I_m \end{bmatrix} > 0 \tag{5.21}$$

The main result can be now stated.

Theorem 5.2: [Bernal & al, 2010]

If there exist matrices of proper size $P_j = P_j^T > x_0^2 I$, F_j, $j \in \{1, \cdots, r\}$, such that the following LMIs

$$\begin{aligned} &\Upsilon_{ii}^{\alpha} < 0, && i \in \{1, \cdots, r\}, \alpha \in \left\{1, \cdots, 2^{pn(m+n)}\right\} \\ &\frac{2}{r-1}\Upsilon_{ii}^{\alpha} + \Upsilon_{ij}^{\alpha} + \Upsilon_{ji}^{\alpha} < 0, && (i, j) \in \{1, \cdots, r\}^2, i \neq j, \\ & && \alpha \in \left\{1, \cdots, 2^{pn(m+n)}\right\}, \end{aligned} \tag{5.22}$$

$$\begin{bmatrix} P_j & F_j^T \\ F_j & \mu^2 I_m \end{bmatrix} > 0, \; j \in \{1, \cdots, r\} \tag{5.23}$$

hold under definitions

$$\begin{aligned}\Upsilon_{ij}^{\alpha} &= A_i P_j + B_i F_j + P_j A_i^T + F_j^T B_i^T \\ &-\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{e=1}^{m}(-1)^{d_{kv(e+n)}^{\alpha}}\eta_{kv}\mu(B_i)_{ve}\left(P_{g_1(j,k)}-P_{g_2(j,k)}\right) \\ &-\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{s=1}^{n}(-1)^{d_{kvs}^{\alpha}}\lambda_{kvs}(A_i)_{vs}\left(P_{g_1(j,k)}-P_{g_2(j,k)}\right),\end{aligned} \tag{5.24}$$

$g_1(j,k)$, $g_2(j,k)$ defined as in (5.13), $d_{kv(e+n)}^{\alpha}$, d_{kvs}^{α} defined from:

$\alpha-1=d_{pn(n+m)}^{\alpha}+d_{pn(n+m-1)}^{\alpha}\times 2+\cdots+d_{111}^{\alpha}\times 2^{pn(n+m)-1}$, then $x(t)$, $|x(0)|<x_0$ tends to zero exponentially for any trajectory satisfying (5.3) in the outermost Lyapunov level $R_0=\{x: x^T P_z^{-1} x\le r_0\}$ contained both in C and $R=\bigcap_{e,k,s,v}\left\{x:\left|\frac{\partial w_0^k}{\partial x_v}u_e\right|\le\mu\eta_{kv},\left|\frac{\partial w_0^k}{\partial x_v}x_s\right|\le\lambda_{kvs}\right\}$.

***Proof*:** from definition of Υ_{ij}^{α} (5.24) and relaxation conditions of (5.22), it holds directly:

$$\begin{aligned}\Upsilon_{zz}^{\alpha} &= A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T \\ &-\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{e=1}^{m}(-1)^{d_{kv(e+n)}^{\alpha}}\eta_{kv}\mu(B_z)_{ve}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right) \\ &-\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{s=1}^{n}(-1)^{d_{kvs}^{\alpha}}\lambda_{kvs}(A_z)_{vs}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)<0\end{aligned} \tag{5.25}$$

Since all the possible sign combinations of the $p\times(n+m)$ terms in the last two summands of (5.25) are taken into account, and given that $\left|\frac{\partial w_0^k}{\partial x_v}u_e\right|\le\mu\eta_{kv}$, $\left|\frac{\partial w_0^k}{\partial x_v}x_s\right|\le\lambda_{kvs}$ in R (LMIs (5.23) guarantee that $\|u(t)\|<\mu$), it follows that:

$$\begin{aligned}&A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T - \sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{e=1}^{m}\frac{\partial w_0^k}{\partial x_v}u_e(B_z)_{ve}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right) \\ &-\sum_{k=1}^{p}\sum_{v=1}^{n}\sum_{s=1}^{n}(A_z)_{vs}\frac{\partial w_0^k}{\partial x_v}x_s\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)<\Upsilon_{zz}^{\alpha}<0\end{aligned} \tag{5.26}$$

thus concluding the proof □

As stated Remark 5.2, it is important to show that these conditions at least always include the quadratic solutions.

Lemma 5.1: Under the same relaxation (5.22) if there exists a solution to quadratic stabilization conditions then Theorem 5.2 conditions are feasible.

Proof: suppose it exists $P = P^T > 0$, F_j, $j \in \{1,\cdots,r\}$, such that:

$$\Upsilon_{zz}^{quad} = A_z P + B_z F_z + P A_z^T + F_z^T B_z^T < 0 \tag{5.27}$$

Consider now $P_j = P$, $j \in \{1,\cdots,r\}$, then as $P_{g_1(j,k)} - P_{g_2(j,k)} = 0$, $j,k \in \{1,\cdots,r\}$ conditions (5.22) exactly match conditions to prove (5.27). Therefore (5.22) is satisfied. If a constraint on the control has to be satisfied, i.e. $\|u(t)\| < \mu$ in the quadratic case therefore it will correspond to [Tanaka & Wang 2001]:

$$\begin{bmatrix} P & F_z^T \\ F_z & \mu^2 I_m \end{bmatrix} > 0 \tag{5.28}$$

Which is equivalent to (5.23) with $P_j = P$. Finally, parameters η_{kv} and λ_{kvs} are obviously free, therefore each set $\left\{ x : \left| \frac{\partial w_0^k}{\partial x_v} u_e \right| \le \mu \eta_{kv}, \left| \frac{\partial w_0^k}{\partial x_v} x_s \right| \le \lambda_{kvs} \right\}$ – with or without constraint on the control – can grow arbitrarily to the large and global stabilization is thus ensured as $R = \bigcup_{e,k,s,v} \left\{ x : \left| \frac{\partial w_0^k}{\partial x_v} u_e \right| \le \mu \eta_{kv}, \left| \frac{\partial w_0^k}{\partial x_v} x_s \right| \le \lambda_{kvs} \right\} \to \mathbb{R}^n$.□

Remark 5.3: Inequalities in Theorem 5.2 are LMIs since $\lambda_{ksv}, \eta_{kv} > 0$ are given. Remember that due to the local stabilizability proof, Theorem 5.1, these values do exist. Of course for some values $\lambda_{ksv}^0, \eta_{kv}^0 > 0$ the conditions could fail. A simple bisection search can be used guaranteeing a solution. For example searching the largest common value $\varepsilon > 0$ such that LMIs (5.22) are feasible with $\lambda_{ksv} = \varepsilon \times \lambda_{ksv}^0$, $\eta_{kv} = \varepsilon \eta_{kv}^0$.

5.3. Design examples

Two examples are developed in this section: the first comes from [Mozelli & al, 2009] as a matter of comparison; a second example illustrates Remark 5.3 and both the effects of the input constraints and the initial conditions.

Example 5.2 :

Consider again (see chapter 2, section 2.7.1) the following family of T-S models with $0 \le a \le 25$, $0 \le b \le 1.8$ [Mozelli & al, 2009]:

$$\dot{x}(t) = A_z x(t) = \sum_{i=1}^{2} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right) \tag{5.29}$$

with

$$A_1 = \begin{bmatrix} 3.6 & -1.6 \\ 6.2 & -4.3 \end{bmatrix},\ A_2 = \begin{bmatrix} -a & -1.6 \\ 6.2 & -4.3 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.45 \\ -3 \end{bmatrix},\ B_2 = \begin{bmatrix} -b \\ -3 \end{bmatrix},$$

$z_1(t) = x_1(t)$, $h_1 = w_0^1 = \dfrac{1-\sin x_1}{2}$ and $h_2 = w_1^1 = \dfrac{1+\sin x_1}{2}$ defined in the compact set $C = \left\{ x : |x_i| \le \dfrac{\pi}{2} \right\}$, $i = 1,2$.

The best results in [Mozelli & al, 2009] for stabilization of T-S models (5.29) are based on direct bounds of the time-derivatives of the MFs $|\dot{h}_i| \le \phi_i$ and an arbitrary parameter μ; in the referred paper these values have been chosen as $\phi_i = 1$, $i = 1,2$ and $\mu = 0.04$ resulting in the feasibility domain shown in Figure 5.2 with cross marks. In order to comment again the bound $|\dot{h}_i| \le \phi_i$ as for Example 5.1, it implies in addition to the assumption $|x_i| \le \dfrac{\pi}{2}$:

$$\left|\dot{h}_1\right| = \left|\dot{h}_2\right| = \left|\frac{\cos x_1}{2}\dot{x}_1\right| = \left|\frac{\cos x_1}{2}\sum_{i=1}^{2} h_i\left(z(t)\right)\left(\left[A_i\right]_{1^*} x(t) + \left[B_i\right]_{1^*} u(t)\right)\right|$$

$$= \left|\frac{\cos x_1}{2}\left(\left(\frac{1-\sin x_1}{2}\right)\left(3.6x_1 - 1.6x_2 - 0.45u\right) + \left(\frac{1+\sin x_1}{2}\right)\left(3.6x_1 - 1.6x_2 - bu\right)\right)\right| \le \phi_i = 1$$

Again the assumption $|\dot{h}_i| \le \phi_i = 1$ is impossible to satisfy *a priori*. Consider now Theorem 5.2 conditions with $|u(t)| \le \mu = 15$, $|x(0)| \le \pi/2$ and $|x_i(t)| \le \pi/2$ (from definition of *C*), the new approach produces the feasibility region shown in Figure 5.2 with circles: it obviously outperforms results in [Mozelli & al, 2009]. Moreover, the trajectories for each point is *a priori* guaranteed to remain in *C*.

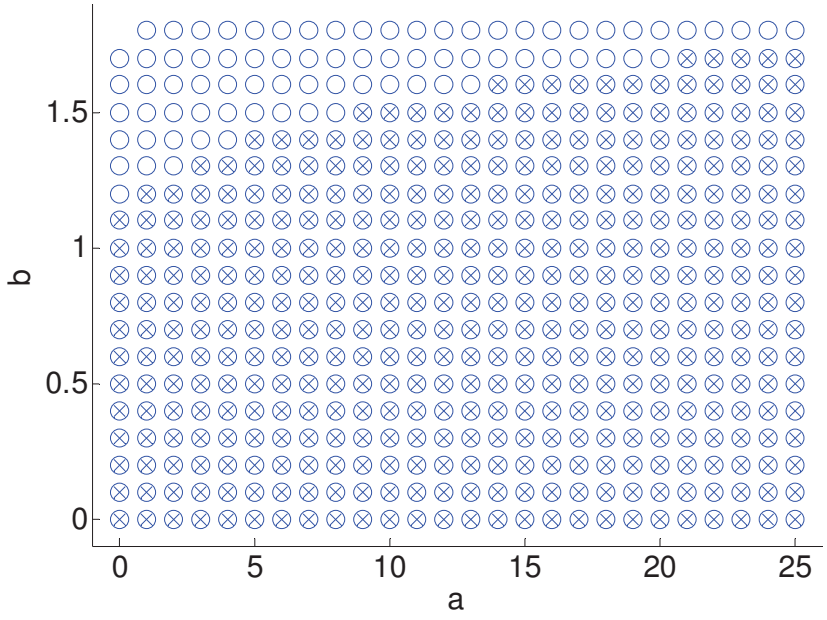


Figure 5.2: Closed-loop stability domains comparison: Theorem 6 [Mozelli & al, 2009] (×) and Theorem 5.2 (o).

Example 5.3 :

Consider the following 4-rule T-S model in the compact set $C=\{x:|x_i|\leq 2\}$ inspired in one of the examples in [Guerra & Bernal, 2009]:

$$\dot{x}(t)=A_z x(t)+B_z u(t)=\sum_{i=1}^{4} h_i\left(z(t)\right)\left(A_i x(t)+B_i u(t)\right) \tag{5.30}$$

with

$$A_1=\begin{bmatrix}-3 & 2\\ 0 & 0.9\end{bmatrix},\ A_2=\begin{bmatrix}-0.8 & 3\\ 0 & -0.9\end{bmatrix},\ A_3=\begin{bmatrix}-1.9 & 2\\ -0.5 & 0.1\end{bmatrix},\ A_4=\begin{bmatrix}0.1 & 3\\ 0 & -2\end{bmatrix},$$

$$B_1=\begin{bmatrix}0\\ 10\end{bmatrix},\ B_2=\begin{bmatrix}0\\ 3\end{bmatrix},\ B_3=\begin{bmatrix}1\\ 1\end{bmatrix},\ B_4=\begin{bmatrix}0\\ 2\end{bmatrix},$$

$w_0^1=\dfrac{4-x_1^2}{4}$, $w_0^2=\dfrac{4-x_2^2}{4}$, $w_1^1=1-w_0^1$, $w_1^2=1-w_0^2$, $h_1=w_0^1w_0^2$, $h_2=w_0^1w_1^2$, $h_3=w_1^1w_0^2$, and $h_4=w_1^1w_1^2$. In that case, the premise variables are the state variables, i.e. $z_1(t)=x_1(t)$, $z_2(t)=x_2(t)$. To exhibit some of the expressions, for this example after some manipulations:

$$\dot{P}_z = \frac{\partial w_0^1}{\partial x_1}\left(h_1+h_3\right)\dot{x}_1\left(P_1-P_3\right)+\frac{\partial w_0^2}{\partial x_2}\left(h_1+h_2\right)\dot{x}_2\left(P_1-P_2\right)$$
$$+\frac{\partial w_0^1}{\partial x_1}\left(h_2+h_4\right)\dot{x}_1\left(P_2-P_4\right)+\frac{\partial w_0^2}{\partial x_2}\left(h_3+h_4\right)\dot{x}_2\left(P_3-P_4\right).$$

And the expression Υ_{zz}^{α} in conditions (5.22) derives directly from:

$$A_zP_z+B_zF_z+P_zA_z^T+F_z^TB_z^T-\sum_{i=1}^{4}\dot{h}_iP_i \le A_zP_z+B_zF_z+P_zA_z^T+F_z^TB_z^T$$
$$-\sum_{k=1}^{2}\sum_{v=1}^{2}\sum_{e=1}^{1}(-1)^{d_{kv(e+n)}^{\alpha}}\eta_{kv}\mu\left(B_z\right)_{ve}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right) \qquad (5.31)$$
$$-\sum_{k=1}^{2}\sum_{v=1}^{2}\sum_{s=1}^{2}(-1)^{d_{kvs}^{\alpha}}\lambda_{kvs}\left(A_z\right)_{vs}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)=\Upsilon_{zz}^{\alpha}$$

Next step is the definition of the initial values λ_{kvs}^0, η_{kv}^0 satisfying, respectively, $\left|\frac{\partial w_0^k}{\partial x_v}x_s\right| \le \lambda_{kvs}$ and $\left|\frac{\partial w_0^k}{\partial x_v}u_e\right| \le \mu\eta_{kv}$. Note that $\frac{\partial w_0^1}{\partial z_1}=-\frac{x_1}{2}=-\frac{\partial w_1^1}{\partial z_1}$ and $\frac{\partial w_0^2}{\partial z_2}=-\frac{x_2}{2}=-\frac{\partial w_1^2}{\partial z_2}$ and with the definition of the compact set $C=\left\{x:\left|x_i\right|\le 2\right\}$, it follows that:

$$\left|\frac{\partial w_0^k}{\partial x_v}x_s\right| \le \lambda_{kvs}=2, \quad \left|\frac{\partial w_0^k}{\partial x_v}\right| \le \eta_{kv}=1, \quad k,v,s\in\{1,2\}, \qquad (5.32)$$

thus $\lambda_{kvs}^0=2$ and $\eta_{kv}^0=1$ can be used as initial values. At last, bound on the control is fixed. First, with $\left|u\left(t\right)\right| \le \mu=0.5$ gives the biggest stabilization region $R_0^1=\left\{x:x^TP_z^{-1}x\le r_0^1\right\}$ presented in Figure 5.3 (dotted line). Second, with $\left|u\left(t\right)\right| \le \mu=0.8$, Figure 5.3 shows the biggest region $R_0^2=\left\{x:x^TP_z^{-1}x\le r_0^2\right\}$ (solid line) with a significant increase. This indicates the importance of this bound. Figure 5.3 exhibits for $\mu=0.8$, four model trajectories as well as gives the borders of the regions of the different constraints in dashed lines:

$$R=\bigcap_{e,k,s,v}\left\{x:\left|\frac{\partial w_0^k}{\partial x_v}u_e\right| \le 0.8125\mu,\ \left|\frac{\partial w_0^k}{\partial x_v}x_s\right| \le 1.625\right\} \qquad (5.33)$$

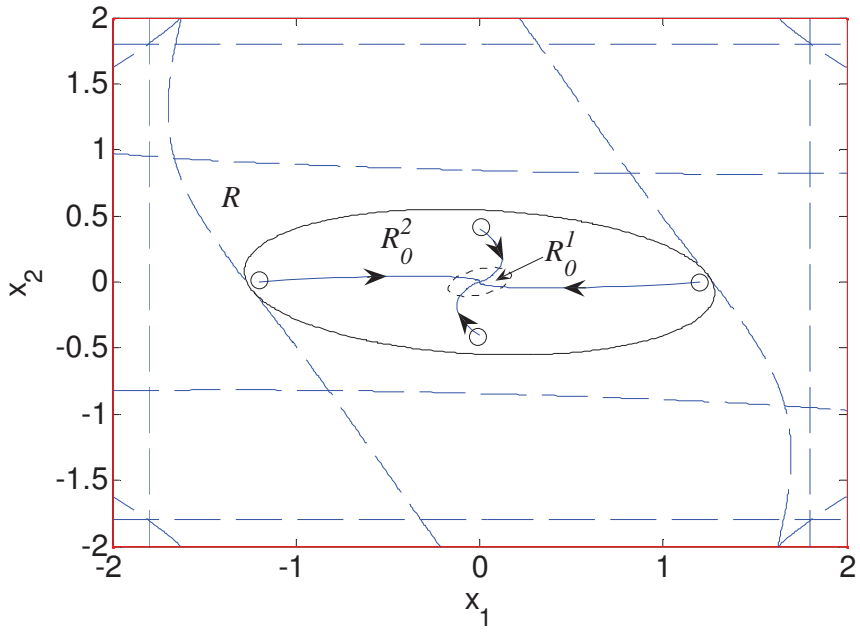


Figure 5.3: Two stabilization regions for T-S model (5.30): with $\mu = 0.5$ (R_0^1 region enclosed by dotted lines) and with $\mu = 0.8$ (R_0^2 enclosed by solid lines).

In order to give some numerical results, the gains and matrices involved in the control law for the case $\mu = 0.8$ are

$$F_1 = \begin{bmatrix} -0.2263 & -2.3089 \end{bmatrix},\ F_2 = \begin{bmatrix} -1.4418 & -2.0579 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 0.8758 & -2.3548 \end{bmatrix},\ F_4 = \begin{bmatrix} -5.1766 & -0.2926 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 47.4617 & -2.8114 \\ -2.8114 & 8.6795 \end{bmatrix},\ P_2 = \begin{bmatrix} 48.6279 & -5.2959 \\ -5.2959 & 8.8808 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 45.2456 & -2.8272 \\ -2.8272 & 8.6803 \end{bmatrix},\ P_4 = \begin{bmatrix} 48.5332 & -5.1353 \\ -5.1353 & 8.6539 \end{bmatrix}.$$

In this case, an example of control law evolution with initial conditions $\begin{bmatrix} 1.2 & 0 \end{bmatrix}$ is presented in Figure 5.4.

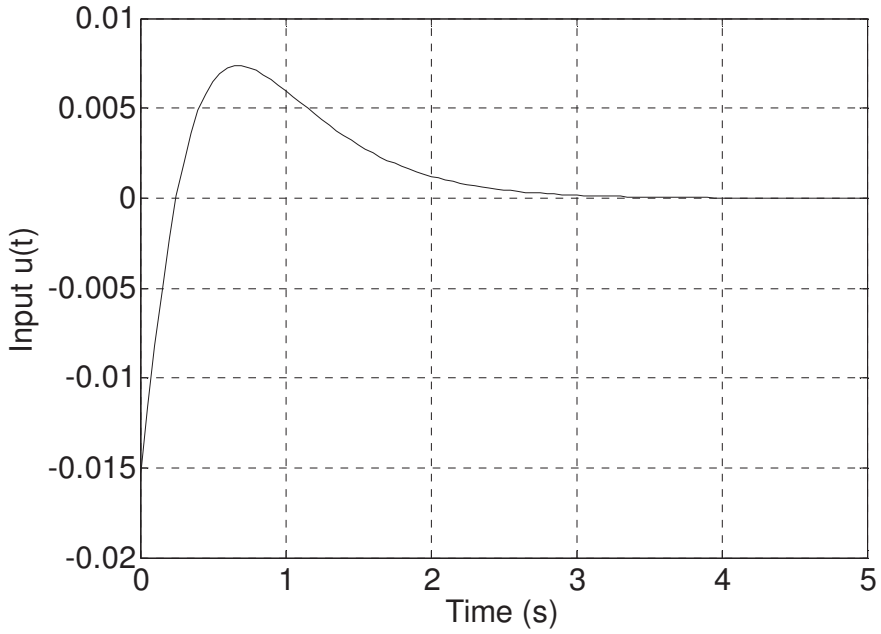


Figure 5.4: Control law evolution.

A local stabilization result has been obtained via Theorem 5.2 nevertheless, bounds on the control law are introduced and the treatments of the problem seem to be more complicated, the work presented in the next section allows solving the stabilization problem in a local framework without this drawback.

5.4. New contributions for non-quadratic stabilization of T-S models

Problem formulation is similar as previous section, i.e. TS model (5.1), control law (5.2) and Lyapunov function (5.4), therefore ending with the same problem (5.6) recalled here:

$$A_z P_z + B_z F_z + \left(A_z P_z + B_z F_z\right)^T - \dot{P}_z < 0 \tag{5.34}$$

With (5.13) $\dot{P}_z = \sum_{k=1}^{p} \frac{\partial w_0^k}{\partial z_k}\left(P_{g_1(z,k)} - P_{g_2(z,k)}\right)\dot{z}_k$. Recall that the major point is to express in a satisfactory way $\dot{z}_k$.

In the previous approach, $\dot{z}_k$ were expressed without expanding the non-PDC control law (5.2) therefore requiring the control bounds: $\|u(t)\| < \mu$. Next, a more interesting way to cope with the stabilization problem is used in order to remove this drawback.

Remark 5.4: In this section the functions $h_i(z)$ and the mapping between the premise vector $z(t)$ and the state vector $x(t)$ are of class C^1 on the defined compact set C; i.e. the partial derivatives are well-defined.

Let us focus on the scalar $\dot{z}_k \in \mathbb{R}$, introducing the control law:

$$\dot{z}_k = \left(\frac{\partial z_k}{\partial x}\right)^T \dot{x} = \left(\frac{\partial z_k}{\partial x}\right)^T \left(A_z + B_z F_z P_z^{-1}\right) x = \left(\frac{\partial z_k}{\partial x}\right)^T \left(A_z P_z + B_z F_z\right) P_z^{-1} x \tag{5.35}$$

Therefore the goal is to find a "nice" bound $\beta_k > 0$ for:

$$\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right| \le \beta_k \tag{5.36}$$

which is equivalent to:

$$\left|\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial x}\right)^T \left(A_z P_z + B_z F_z\right) P_z^{-1} x\right| \le \beta_k \tag{5.37}$$

Remark 5.5: it is also important to note that this is ***not equivalent*** to other approaches found in the literature, [Tanaka et al, 2003] [Bernal & al, 2006], [Mozelli & al, 2009] where the required condition is in the form of: $\left|\dot{h}_i\left(z(t)\right)\right| \le \phi_i$. In this later case, ϕ_i is given *a priori* and can only be checked *a posteriori*, see Example 5.1. In the presented case, the bounds β_k are included in the problem to solve. Said in other words, if (5.37) can be expressed as an LMI problem then its solutions will guarantee *a priori* the future trajectories to remain in the region of attraction.

5.4.1. First trial:

Note that (5.37) holds if:

$$\left(\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial x}\right)^T \left(A_z P_z + B_z F_z\right) P_z^{-1} x\right)^2 \le \beta_k^2 \tag{5.38}$$

Or equivalently:

$$\left(\frac{\partial w_0^k}{\partial z_k}\right)^2 \left(\frac{\partial z_k}{\partial x}\right)^T \left(A_z P_z + B_z F_z\right) P_z^{-1} x x^T P_z^{-1} \left(A_z P_z + B_z F_z\right)^T \frac{\partial z_k}{\partial x} - \beta_k^2 \le 0 \tag{5.39}$$

Remembering that $\|x\|^2 \le \lambda_x^2 \Leftrightarrow xx^T \le \lambda_x^2 I$ (property 3 (2.23)), (5.39) holds if:

$$\lambda_x^2\left(\frac{\partial w_0^k}{\partial z_k}\right)^2\left(\frac{\partial z_k}{\partial x}\right)^T\left(A_zP_z+B_zF_z\right)P_z^{-1}P_z^{-1}\left(A_zP_z+B_zF_z\right)^T\frac{\partial z_k}{\partial x}-\beta_k^2\le 0 \tag{5.40}$$

Applying again property 3 to (5.40) gives:

$$\lambda_x^2\left(\frac{\partial w_0^k}{\partial z_k}\right)^2P_z^{-1}\left(A_zP_z+B_zF_z\right)^T\frac{\partial z_k}{\partial x}\left(\frac{\partial z_k}{\partial x}\right)^T\left(A_zP_z+B_zF_z\right)P_z^{-1}-\beta_k^2I\le 0 \tag{5.41}$$

As the functions w_0^k are known explicitly, as previously mentioned bounding $\frac{\partial w_0^k}{\partial z_k}$ is direct. On the other side, $\frac{\partial z_k}{\partial x}$ represents the linear or nonlinear mapping between $z(t)$ and $x(t)$ and under the assumption made Remark 5.4 is well-defined and therefore known. The bounds can be computed easily and: $\left(\frac{\partial w_0^k}{\partial z_k}\right)^2\left\|\frac{\partial z_k}{\partial x}\right\|^2\le\lambda_k^2\ \Leftrightarrow\left(\frac{\partial w_0^k}{\partial z_k}\right)^2\frac{\partial z_k}{\partial x}\left(\frac{\partial z_k}{\partial x}\right)^T\le\lambda_k^2I$; it follows that:

$$\lambda_x^2\lambda_k^2P_z^{-1}\left(A_zP_z+B_zF_z\right)^T\left(A_zP_z+B_zF_z\right)P_z^{-1}-\beta_k^2I\le 0 \tag{5.42}$$

Then, multiplying left and right with P_z gives:

$$\lambda_x^2\lambda_k^2\left(A_zP_z+B_zF_z\right)^T\left(A_zP_z+B_zF_z\right)-\beta_k^2P_z^2\le 0 \tag{5.43}$$

Considering now that $P_z\ge\delta I$ it follows that $x^TP_zx\ge\delta x^Tx$ and with the change of variable $x=P_z^{1/2}y$, $y^TP_z^2y\ge\delta y^TP_zy$, thus $P_z^2\ge\delta P_z$, thus (5.36) is satisfied if:

$$\lambda_x^2\lambda_k^2\left(A_zP_z+B_zF_z\right)^T\left(A_zP_z+B_zF_z\right)-\beta_k^2\delta P_z\le 0 \tag{5.44}$$

And with the help of the Schur complement, (5.44) is equivalent to:

$$\begin{bmatrix}P_z & (*)\\ \lambda_x\lambda_k\left(A_zP_z+B_zF_z\right) & \beta_k^2\delta I\end{bmatrix}>0 \tag{5.45}$$

Now, we can come back to $\dot{P}_z$ knowing that $\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right|\le\beta_k$. Using property 6 (2.29) and introducing matrices $S_{zz}^k>0$ we can write:

$$\begin{aligned}\dot{P}_z&=\sum_{k=1}^{p}\frac{\partial w_0^k}{\partial z_k}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)\dot{z}_k\\&\le\frac{1}{2}\sum_{k=1}^{p}\left(\beta_k^2S_{zz}^k+\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)\left(S_{zz}^k\right)^{-1}\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)\right)\end{aligned} \tag{5.46}$$

Therefore, let us consider the following quantities:

$$\Gamma_{ij} = \begin{bmatrix} A_iP_j + B_iF_j + P_jA_i^T + F_j^TB_i^T + \frac{1}{2}\sum_{k=1}^{p}\beta_k^2 S_{ij}^k & (*) & \cdots & & (*) \\ P_{g_1(i,1)} - P_{g_2(i,1)} & -2S_{ij}^1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ P_{g_1(i,p)} - P_{g_2(i,p)} & 0 & \cdots & 0 & -2S_{ij}^p \end{bmatrix} \tag{5.47}$$

$$\Upsilon_{ij} = \begin{bmatrix} P_i & (*) \\ \lambda_x\lambda_k\left(A_iP_j + B_iF_j\right) & \beta_k^2\delta I \end{bmatrix} \tag{5.48}$$

Theorem 5.3:[Guerra & al, 2011]

Consider the T-S closed loop model (5.3) and expressions defined in (5.48) and (5.47). If there exist matrices P_i, F_i, $S_{ij}^k > 0$, $i, j \in \{1,\cdots,r\}$, $k \in \{1,\cdots,p\}$ such that the following LMI are satisfied:

$$P_i \geq \delta I,\ i \in \{1,\cdots,r\} \tag{5.49}$$

$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} > 0,\ i, j \in \{1,\cdots,r\} \tag{5.50}$$

$$\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0,\ i, j \in \{1,\cdots,r\} \tag{5.51}$$

Then, the control law (5.2) stabilizes the T-S open loop model (5.1) in a local domain. An estimation of this domain is given by the outermost Lyapunov level contained in the compact set of the state variables C.

Proof: From definition of Υ_{ij} (5.48) (respectively Γ_{ij} (5.47)) and relaxation conditions (5.50) (respectively (5.51)) it follows $\Upsilon_{zz} < 0$ (respectively $\Gamma_{zz} < 0$). $\Upsilon_{zz} < 0$ ensures $\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right| \leq \beta_k$, then, applying the Schur complement on $\Gamma_{zz} < 0$ leads to:

$$A_zP_z + B_zF_z + P_zA_z^T + F_z^TB_z^T + \frac{1}{2}\sum_{k=1}^{p}\frac{\partial w_0^k}{\partial z_k}\left(\beta_k^2 S_{zz}^k + \left(P_{g_1(z,k)} - P_{g_2(z,k)}\right)\left(S_{zz}^k\right)^{-1}\left(P_{g_1(z,k)} - P_{g_2(z,k)}\right)\right) \leq 0$$

Therefore, considering (5.46) it follows:

$$A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T - \dot{P}_z < 0 \tag{5.52}$$

Thus concluding the proof. □

Remark 5.6: notice that (5.49)~(5.51) are LMI constraints only if the scalar β_k^2 is known.

A procedure to determine this value is possible, inspired from [Mehdi & al, 2004] for static output feedback. Recall that if the conditions of Theorem 5.1 are satisfied it ensures the existence of a local stability domain for Theorem 5.3. Therefore, the procedure is based on this remark.

Basic Algorithm

Step 1: consider Theorem 5.1. Find $P_i = P_i^T > 0$, F_i, $i \in \{1,\ldots,r\}$ such that conditions (5.9) hold. The obtained gains give a control law that can be seen as stabilizing the "frozen" time-invariant continuous T-S model. Thus initial bound $\left(\beta_k^2\right)^0$ can be directly obtained from this first step.

Step 2: with $\left(\beta_k^2\right)^0$, find solution to the Theorem 5.3 conditions (5.49)~(5.51).

Many refinements can come at hand, introducing a decay rate to step 1 for example, enforcing step 1 with (5.50). Nevertheless, the weakness of the approach is due to the necessity of the condition $P_z^2 \geq \delta P_z$.

5.4.2. Second trial

Let us begin again from (5.37) recalled thereinafter:

$$\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right| = \left|\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial x}\right)^T \left(A_z P_z + B_z F_z\right) P_z^{-1} x\right| \leq \beta_k \tag{5.53}$$

Note that the vector $\frac{\partial z_k}{\partial x} = \left[\frac{\partial z_k}{\partial x_1} \quad \cdots \quad \frac{\partial z_k}{\partial x_n}\right]^T \in \mathbb{R}^n$ will very often contain empty rows. For example with $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathbb{R}^4$ consider $z_1 = x_1^2$ and $z_2 = 2x_1^2 \cos\left(x_4\right)$ thus directly:

$$\frac{\partial z_1}{\partial x}=\begin{bmatrix}\frac{\partial z_1}{\partial x_1}\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\left[\frac{\partial z_1}{\partial x_1}\right] \text{ and } \frac{\partial z_2}{\partial x}\begin{bmatrix}\frac{\partial z_2}{\partial x_1}\\0\\\frac{\partial z_2}{\partial x_4}\\0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{\partial z_2}{\partial x_1}\\\frac{\partial z_2}{\partial x_4}\end{bmatrix}.$$

Therefore, let us define for each premise variable z_k, a restricted vector of the state $\xi^k=\left\{x_i \backslash \frac{\partial z_k}{\partial x_i}\neq 0\right\}\in\mathbb{R}^{n_k}, n_k\leq n$. Thus, it follows:

$$\frac{\partial z_k}{\partial x}=T_k\frac{\partial z_k}{\partial \xi^k} \text{ with } T_k=\left[\delta_{ij}^k\right]_{i=1,2,\cdots,n;\, j=1,2,\cdots,n_k}, \delta_{ij}^k=\begin{cases}1, & if \quad \frac{\partial z_k}{\partial x_i}=\frac{\partial z_k}{\partial \xi_j^k}\\ 0, & \text{otherwise}\end{cases} \tag{5.54}$$

Note also that the nice property $T_k^T T_k=I\in\mathbb{R}^{n_k\times n_k}$ holds.

With (5.54), expression (5.53) writes:

$$\left|\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)^T T_k^T\left(A_zP_z+B_zF_z\right)P_z^{-1}x\right|\leq\beta_k \tag{5.55}$$

with $T_k^T\left(A_zP_z+B_zF_z\right)P_z^{-1}x\in\mathbb{R}^{n_k\times n}$ whereas $\left(A_zP_z+B_zF_z\right)P_z^{-1}x\in\mathbb{R}^{n\times n}$, thus reducing the size of the matrices. Summarizing; the more $\frac{\partial z_k}{\partial x}$ is "empty", the better the results. Since (5.55) is a scalar expression, it is verified if:

$$\left|\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)^T T_k^T\left(A_zP_z+B_zF_z\right)P_z^{-1}x\right|+\left|x^TP_z^{-1}\left(A_zP_z+B_zF_z\right)^T T_k\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)\right|\leq 2\beta_k \tag{5.56}$$

Applying the completion of square (property 4 (2.25) with slack variable $Q_z=Q_z^T>0$ gives:

$$\begin{aligned}&\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)^T T_k^T\left(A_zP_z+B_zF_z\right)Q_z^{-1}\left(A_zP_z+B_zF_z\right)^T T_k\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)+x^TP_z^{-1}Q_zP_z^{-1}x\\&=\begin{bmatrix}x^TP_z^{-1}Q_z^{\frac{1}{2}} & \frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)^T T_k^T\left(A_zP_z+B_zF_z\right)Q_z^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}Q_z^{\frac{1}{2}}P_z^{-1}x\\ Q_z^{-\frac{1}{2}}\left(A_zP_z+B_zF_z\right)^T T_k\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)\end{bmatrix}\leq 2\beta_k\end{aligned} \tag{5.57}$$

With the help of property 3 (2.23), (5.57) is ensured if

$$\begin{bmatrix} Q_z^{\frac{1}{2}} P_z^{-1} x \\ Q_z^{-\frac{1}{2}} \left(A_z P_z + B_z F_z\right)^T T_k \dfrac{\partial w_0^k}{\partial z_k}\left(\dfrac{\partial z_k}{\partial \xi^k}\right) \end{bmatrix} \begin{bmatrix} x^T P_z^{-1} Q_z^{\frac{1}{2}} & \dfrac{\partial w_0^k}{\partial z_k}\left(\dfrac{\partial z_k}{\partial \xi^k}\right)^T T_k^T \left(A_z P_z + B_z F_z\right) Q_z^{-\frac{1}{2}} \end{bmatrix} \le 2\beta_k I$$

which can be expanded as:

$$\begin{aligned} &\begin{bmatrix} Q_z^{\frac{1}{2}} P_z^{-1} & 0 \\ 0 & Q_z^{-\frac{1}{2}} \left(A_z P_z + B_z F_z\right)^T T_k \end{bmatrix} \begin{bmatrix} x \\ \dfrac{\partial w_0^k}{\partial z_k}\left(\dfrac{\partial z_k}{\partial \xi^k}\right) \end{bmatrix} \\ &\qquad \times \begin{bmatrix} x^T & \dfrac{\partial w_0^k}{\partial z_k}\left(\dfrac{\partial z_k}{\partial \xi^k}\right)^T \end{bmatrix} \begin{bmatrix} P_z^{-1} Q_z^{\frac{1}{2}} & 0 \\ 0 & T_k^T \left(A_z P_z + B_z F_z\right) Q_z^{-\frac{1}{2}} \end{bmatrix} \le 2\beta_k I \end{aligned} \tag{5.58}$$

Locality can now be expressed via known *a priori* bounds. The compact set of the state variables C directly gives $x^T x \le \lambda_x^2$, and $\dfrac{\partial w_0^k}{\partial z_k}\left(\dfrac{\partial z_k}{\partial \xi^k}\right)^T \dfrac{\partial w_0^k}{\partial z_k}\left(\dfrac{\partial z_k}{\partial \xi^k}\right) = \left(\dfrac{\partial w_0^k}{\partial \xi^k}\right)^T \dfrac{\partial w_0^k}{\partial \xi^k} \le \lambda_k^2$. Thus, we can exploit this bound via property 3 (2.23) as:

$$\begin{bmatrix} x^T & \left(\dfrac{\partial z_k}{\partial \xi^k}\right)^T \end{bmatrix} \begin{bmatrix} x \\ \dfrac{\partial z_k}{\partial \xi^k} \end{bmatrix} \le \lambda_x^2 + \lambda_k^2 \Leftrightarrow \begin{bmatrix} x \\ \dfrac{\partial z_k}{\partial \xi^k} \end{bmatrix} \begin{bmatrix} x^T & \left(\dfrac{\partial z_k}{\partial \xi^k}\right)^T \end{bmatrix} \le \left(\lambda_x^2 + \lambda_k^2\right) I \tag{5.59}$$

Therefore, (5.58) holds if

$$\begin{bmatrix} Q_z^{\frac{1}{2}} P_z^{-2} Q_z^{\frac{1}{2}} & 0 \\ 0 & Q_z^{-\frac{1}{2}} \left(A_z P_z + B_z F_z\right)^T T_k T_k^T \left(A_z P_z + B_z F_z\right) Q_z^{-\frac{1}{2}} \end{bmatrix} \le \frac{2\beta_k}{\left(\lambda_x^2 + \lambda_k^2\right)} I \tag{5.60}$$

Multiplying (5.60) on the left and the right side by $\begin{bmatrix} Q_z^{\frac{1}{2}} & 0 \\ 0 & Q_z^{\frac{1}{2}} \end{bmatrix}$ and let, $Q_z = P_z$ and $\varphi_k = \dfrac{2\beta_k}{\left(\lambda_x^2 + \lambda_k^2\right)}$, it results

$$\begin{bmatrix} I & 0 \\ 0 & \left(A_z P_z + B_z F_z\right)^T T_k T_k^T \left(A_z P_z + B_z F_z\right) \end{bmatrix} \le \varphi_k \begin{bmatrix} P_z & 0 \\ 0 & P_z \end{bmatrix} \tag{5.61}$$

Which is equivalent to

$$\begin{cases} I \le \varphi_k P_z \\ \left(A_z P_z + B_z F_z\right)^T T_k T_k^T \left(A_z P_z + B_z F_z\right) \le \varphi_k P_z \end{cases} \tag{5.62}$$

With the use of Schur complement, (5.62) writes

$$\begin{cases} I \le \varphi_k P_z \\ \begin{bmatrix} \varphi_k P_z & (*) \\ T_k^T \left(A_z P_z + B_z F_z\right) & I \end{bmatrix} > 0 \end{cases} \tag{5.63}$$

Thus let us define:

$$\Sigma_{ij}^k = \begin{bmatrix} \varphi_k P_j & (*) \\ T_k^T \left(A_i P_j + B_i F_j\right) & I \end{bmatrix} \tag{5.64}$$

Now, we can come back to $\dot{P}_z$ knowing that $\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right| \le \beta_k$ in the same way as the previous section (5.46) and consider the same quantity (5.47) recalled thereinafter:

$$\Gamma_{ij} = \begin{bmatrix} A_i P_j + B_i F_j + P_j A_i^T + F_j^T B_i^T + \frac{1}{2}\sum_{k=1}^{p} \beta_k^2 S_{ij}^k & (*) & \cdots & & (*) \\ P_{g_1(i,1)} - P_{g_2(i,1)} & -2S_{ij}^1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ P_{g_1(i,p)} - P_{g_2(i,p)} & 0 & \cdots & 0 & -2S_{ij}^p \end{bmatrix} \tag{5.65}$$

LMI constraints can be formulated, as follows:

Theorem 5.4:[Pan & al, 2012]

Given β_k, $k = \{1,\ldots,p\}$ wih $\varphi_k = \frac{2\beta_k}{\left(\lambda_x^2 + \lambda_k^2\right)}$, Γ_{ij} defined in (5.65) and Σ_{ij}^k defined in (5.64), if there exists matrices of proper dimension $P_j = P_j^T > 0$, F_j, $S_{ij}^k > 0$, $i, j = \{1,\ldots,r\}$ satisfying the following conditions:

$$\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0,\ i, j \in \{1,\cdots,r\} \tag{5.66}$$

$$\frac{2}{r-1}\Sigma_{ii}^k + \Sigma_{ij}^k + \Sigma_{ji}^k > 0,\ i, j \in \{1,\cdots,r\} \tag{5.67}$$

$$I < \varphi_k P_j, \ j \in \{1,\cdots,r\} \tag{5.68}$$

Then, the control law (5.2) stabilizes the T-S open loop model (5.1) in a local domain. An estimation of this domain is given by the outermost Lyapunov level contained in the compact set of the state variables C.

Proof: it follows exactly the same line as Theorem 5.3. □

Lemma 5.2: Under the same relaxation, if there exists a solution to quadratic stabilization conditions then Theorem 5.4 conditions are feasible and the result is global.

Proof: suppose it exists $P = P^T > 0$, F_j, $j \in \{1,\cdots,r\}$, such that:

$$\Upsilon_{zz}^{quad} = A_z P + B_z F_z + P A_z^T + F_z^T B_z^T < 0 \tag{5.69}$$

Consider now $P_j = P$, $j \in \{1,\cdots,r\}$, then as $P_{g_1(j,k)} - P_{g_2(j,k)} = 0$, $j,k \in \{1,\cdots,r\}$ and with free slack matrices $S_{ij}^k > 0$, $i, j = \{1,\ldots,r\}$, (5.66) resumes to:

$$A_z P + B_z F_z + P A_z^T + F_z^T B_z^T + \frac{1}{2}\sum_{k=1}^{p} \beta_k^2 S_{zz}^k < 0 \tag{5.70}$$

(5.63) resumes to:

$$\begin{cases} I \le \varphi_k P \\ \begin{bmatrix} \varphi_k P & (*) \\ T_k^T \left(A_z P + B_z F_z \right) & I \end{bmatrix} > 0 \end{cases} \tag{5.71}$$

Note that (5.71) is satisfied whatever are $P = P^T > 0$, F_j, $j \in \{1,\cdots,r\}$, if it exists a "big" enough φ_k, therefore a "big" enough β_k. Thus, $S_{ij}^k > 0$ being free, they can be chosen arbitrarily small and (5.70) is satisfied that concludes the proof. □

5.5. Design examples

Example 5.4 :

Consider a 2-rules T-S fuzzy model of the form, with $a, b \in \mathbb{R}$ free parameters:

$$\dot{x} = \sum_{i=1}^{2} h_i\left(x_1\right)\left(A_i x + B_i u\right) \tag{5.72}$$

$$A_1 = \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ A_2 = \begin{bmatrix} a & -5 \\ 1 & 2 \end{bmatrix},\ B_2 = \begin{bmatrix} b \\ 2 \end{bmatrix}$$

The MFs of the fuzzy model (5.72) are defined as: $h_1\left(x_1\right) = w_0^1 = \dfrac{1-\sin\left(x_1\right)}{2}$ and $h_2\left(x_1\right) = w_1^1 = \dfrac{1+\sin\left(x_1\right)}{2}$ defined in the compact set $C = \left\{x : \left|x_i\right| \le 2\pi\right\},\ i = 1,2$.

Once again computing the derivative of the membership function gives a control dependent results, as shown thereafter:

$$\left|\dot{h}_1\left(x_1\right)\right| = \left|\dot{h}_2\left(x_1\right)\right| = \left|\frac{\cos\left(x_1\right)}{2}\dot{x}_1\right| = \left|\frac{\cos\left(x_1\right)}{2}\left(h_1\left(x_1\right)\left(2x_1 - 10x_2 + u\right) + h_2\left(x_1\right)\left(ax_1 - 5x_2 + bu\right)\right)\right|$$

Therefore, the non-quadratic stabilization conditions proposed by [Tanaka & al, 2003], [Tanaka & al, 2007], [Mozelli & al, 2009] for the above T-S fuzzy model resume in just solving a LMI feasibility problem that does *not guarantee* the future trajectories to remain bounded in *C*. Then, comparing on a grid $\begin{pmatrix} a & b \end{pmatrix}$ the various LMI solutions coming from the approaches where $\left|\dot{h}_1\left(x_1\right)\right| \le \phi_1$ is a priori needed, has no real meaning. Thus the comparison will be done only with quadratic conditions [Tanaka & Wang, 2001], and non-quadratic conditions [Rhee & Won, 2006].

In order to use the results of Theorem 5.4, since $\left|x_i\right| \le 2\pi$ it follows that $\lambda_x^2 = 8\pi^2$, $\lambda_k^2 = 0.5^2$ ($k = 1$ for this example) and as $\xi_1 = x \in \mathbb{R} :\ T_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Consider an example for one pair $\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} -20 & 4 \end{pmatrix}$. With $\beta_1 = 1800$, Figure 5.5 shows the biggest region of attraction based on the solution of LMI constraints problem (5.66)~(5.68). The bounds of region $R = \left\{x : x^T x \le 8\pi^2\right\}$ are shown with dashed lines and the region $R_0 = \left\{x : x^T P_z^{-1} x \le r_0\right\}$ enclosed by a dotted line. Two model trajectories from initial conditions in R_0 are also plotted.

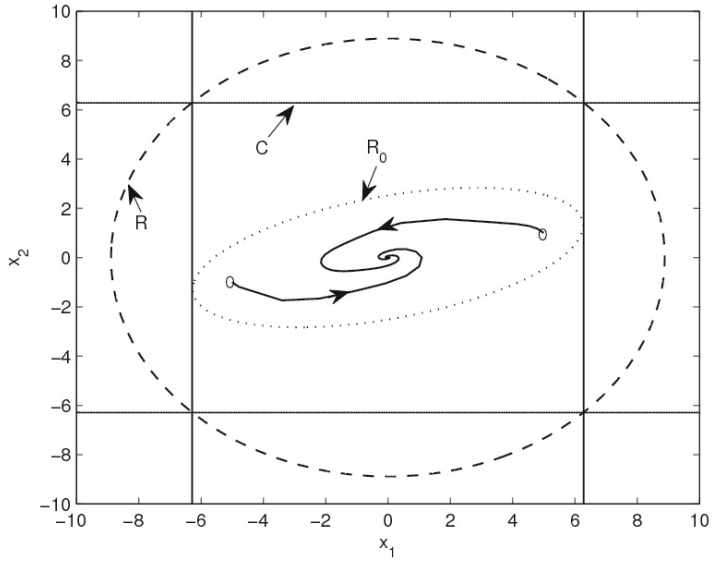


Figure 5.5: Biggest attraction domain for T-S fuzzy model (5.72) with $(a \quad b) = (-20 \quad 4)$

Consider now the gridding of region $a \in [-20 \quad 10]$ and $b \in [3 \quad 25]$. Figure 5.6 exhibits an important increase of the solution compared with quadratic stabilization. Remember also (Lemma 5.2) that whenever a solution is obtained in the quadratic case, it is also global using Theorem 5.4. Figure 5.7 presents a comparison with the conditions of [Rhee & Won, 2006] that use a 2-step algorithm (see chapter 2). These conditions increase the precedent quadratic domain and Theorem 5.4 results are just repeated on the figure. Note that, except the quadratic solutions, at the intersection between [Rhee & Won, 2006] and Theorem 5.4 results we cannot prove that the result tends to a global result. Said in another way, if *global* conditions of 2-step algorithm of [Rhee & Won, 2006] hold, we can only prove that *local* conditions of Theorem 5.4 holds.

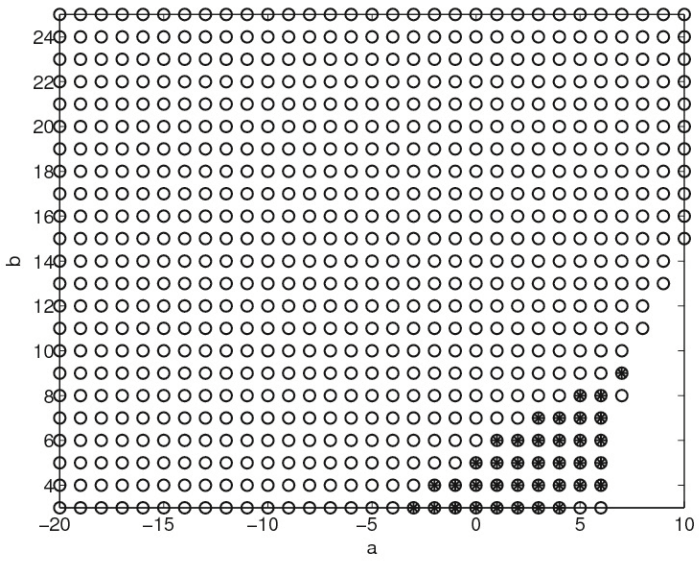


Figure 5.6: Stabilization region given by quadratic conditions [Tanaka & Wang, 2001] (“∗”) and Theorem 5.4 (“o”)

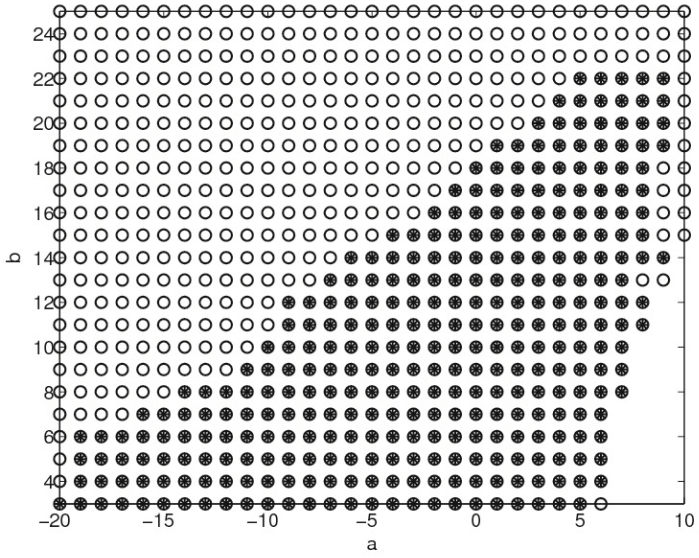


Figure 5.7: Stabilization region given by [Rhee & Won, 2006] (“∗”) and Theorem 5.4 (“o”)

5.6. Finsler’s relaxations for non-quadratic stabilization of T-S models

The previous section presented a new way to deal with non-quadratic stabilization. Knowing that Finsler’s lemma allows relaxing results via adding slack variables, this section tries to take profit from this lemma to derive less conservative results. The central idea followed in the next

pages is to somehow "cut" the link between the Lyapunov function and the control law. It brings some interesting refinements in the quadratic case as well as in the non-quadratic one. Although our works are focused on non-quadratic stabilization, for sake of simplicity, let us begin with the quadratic case.

5.6.1. Quadratic stabilization of T-S models

Consider the following T-S model is derived with $r = 2^p$:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right) \tag{5.73}$$

Consider the following quadratic Lyapunov function candidate with $P = P^T > 0$

$$V\left(x(t)\right) = x(t)^T P^{-1} x(t) \tag{5.74}$$

and the non-PDC control law

$$u(t) = F_z H_z^{-1} x(t) \tag{5.75}$$

Where the matrices P^{-1} from the Lyapunov function are replaced with some slack variables H_z^{-1} to be determined. The closed-loop T-S model writes:

$$\dot{x}(t) = \left(A_z + B_z F_z H_z^{-1}\right) x(t) \tag{5.76}$$

Theorem 5.5: [Jaadari & al, 2012]

The T-S model (5.73) under the control law (5.75) is globally asymptotically stable if $\exists \varepsilon > 0,$ and matrices $P = P^T > 0$, H_i, and F_i, $i = \{1,\ldots,r\}$ of proper dimensions such that (2.32) holds with

$$\Upsilon_{ij} = \begin{bmatrix} A_i H_j + B_i F_j + (*) & (*) \\ H_j - P + \varepsilon\left(A_i H_j + B_i F_j\right) & -2\varepsilon P \end{bmatrix}$$

Proof: Consider the Lyapunov function candidate (5.74); proving that its time-derivative is negative can be written as

$$\dot{V} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^T \begin{bmatrix} 0 & P^{-1} \\ P^{-1} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} < 0 \tag{5.77}$$

which combined with the following expression from (5.76)

$$\begin{bmatrix} A_z + B_z F_z H_z^{-1} & -I \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = 0 \tag{5.78}$$

yields, by Finsler's Lemma, the next inequality:

$$\begin{bmatrix} 0 & P^{-1} \\ P^{-1} & 0 \end{bmatrix} + \begin{bmatrix} U \\ W \end{bmatrix} \begin{bmatrix} A_z + B_z F_z H_z^{-1} & -I \end{bmatrix} + (*) < 0 \tag{5.79}$$

with U and W being matrices of proper dimension. Pre-multiplying by $\begin{bmatrix} H_z^T & 0 \\ 0 & P \end{bmatrix}$ and post-multiplying by $\begin{bmatrix} H_z & 0 \\ 0 & P \end{bmatrix}$ allows the following to be obtained:

$$\begin{bmatrix} 0 & H_z^T \\ H_z & 0 \end{bmatrix} + \begin{bmatrix} H_z^T U \\ PW \end{bmatrix} \begin{bmatrix} A_z H_z + B_z F_z & -P \end{bmatrix} + (*) < 0 \tag{5.80}$$

Let $U = H_z^{-T}$ and $W = \varepsilon P^{-1}$ with $\varepsilon > 0$, so the previous expression renders

$$\begin{bmatrix} A_z H_z + B_z F_z + (*) & (*) \\ H_z - P + \varepsilon \left(A_z H_z + B_z F_z \right) & -2\varepsilon P \end{bmatrix} < 0 \tag{5.81}$$

Applying the Relaxation Lemma to (5.81) ends the proof. □

Remark 5.7: Of course, the problem is not strictly LMI because of the parameter ε. This one is employed in several works concerning linear parameter varying (LPV) systems [Oliveira & Skelton 2001], [Oliveira & al, 2011]. It is normally a prefixed value belonging to a family such as: $\varepsilon \in \mathrm{E} = \left\{ 10^{-6}, 10^{-5}, \ldots, 10^{6} \right\}$. This family logarithmically spaced avoids an exhaustive line search. Why is it interesting to use? In [Oliveira & al, 2011] the authors showed that for 1000s of LPV models and comparing with numerous results – classical quadratic approach, Finsler application, and several variants – this way of doing was outperforming in a large way the existing results. Therefore we will follow the same line. In the next sections parameter ε (or subscript versions of it) will reappear and preserve the same meaning.

Remark 5.8: Another important remark is the necessity of the parameter ε. It is due to a well-known fact about Finsler's lemma applied to continuous state models. Effectively, when choosing the slack variables U and W in (5.79) not only a LMI formulation is important but it must also be kept in mind that the minimum expected is that the obtained results include the ordinary PDC control scheme. To achieve this goal, the term W must be possibly chosen

arbitrarily small. Effectively, consider $H_z = P$ in (5.81), then using Schur complement allows the following expression to be obtained

$$A_z P + B_z F_z + (*) + \frac{1}{2}\varepsilon \left(A_z P + B_z F_z\right)^T P^{-1} \left(A_z P + B_z F_z\right) < 0 \tag{5.82}$$

which for sufficiently small $\varepsilon > 0$ is equivalent to the classical quadratic condition (5.27).

5.6.2. Quadratic stabilization of T-S models : extended constraints

Consider again the quadratic Lyapunov function candidate (5.74) together with the control law (5.75). A way to introduce extra degrees of freedom is to use the control law as another equality constraint via Finsler's lemma.

Theorem 5.6: [Jaadari & al, 2012]

The T-S model (5.73) under the control law (5.75) is globally asymptotically stable if it exists $\varepsilon > 0$ and matrices $P = P^T > 0$, H_i, and F_i, $i = \{1,\ldots,r\}$ of proper dimensions such that (2.32) holds with

$$\Upsilon_{ij} = \begin{bmatrix} A_i H_j + B_i F_j + (*) & (*) & (*) \\ H_j - P + \varepsilon\left(A_i H_j + B_i F_j\right) & -2\varepsilon P & (*) \\ \varepsilon F_j & 0 & -2\varepsilon I \end{bmatrix} < 0 .$$

Proof: Consider the Lyapunov function candidate (5.74); proving that its time-derivative is negative can be written as

$$\dot{V} = \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix}^T \begin{bmatrix} 0 & P^{-1} & 0 \\ P^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix} < 0 \tag{5.83}$$

which combined with the following expressions from (5.1) and (5.75):

$$\begin{bmatrix} A_z & -I & B_z \\ F_z H_z^{-1} & 0 & -I \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix} = 0 \tag{5.84}$$

yields, by Finsler's Lemma, the next inequality:

$$\begin{bmatrix} 0 & P^{-1} & 0 \\ P^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \\ U_3 & V_3 \end{bmatrix} \begin{bmatrix} F_z H_z^{-1} & 0 & -I \\ A_z & -I & B_z \end{bmatrix} + (*) < 0 \tag{5.85}$$

with U_i and $V_i, i = \{1,\ldots,3\}$ being matrices of proper dimension. Pre-multiplying and post-multiplying by $\begin{bmatrix} H_z^T & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I \end{bmatrix}$ and $\begin{bmatrix} H_z & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I \end{bmatrix}$ gives:

$$\begin{bmatrix} 0 & H_z^T & 0 \\ H_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} H_z^T U_1 & H_z^T V_1 \\ P U_2 & P V_2 \\ U_3 & V_3 \end{bmatrix} \begin{bmatrix} F_z & 0 & -I \\ A_z H_z & -P & B_z \end{bmatrix} + (*) < 0 \tag{5.86}$$

Let $U_1 = H_z^{-T} B_z$, $U_2 = \varepsilon P^{-1} B_z$, $U_3 = \varepsilon I$, $V_1 = H_z^{-T}$, $V_2 = \varepsilon P^{-1}$ and $V_3 = 0$ with $\varepsilon > 0$ so the previous expression renders:

$$\begin{bmatrix} A_z H_z + B_z F_z + (*) & (*) & (*) \\ H_z - P + \varepsilon (A_z H_z + B_z F_z) & -2\varepsilon P & (*) \\ \varepsilon F_z & 0 & -2\varepsilon I \end{bmatrix} < 0 \tag{5.87}$$

Applying the Relaxation Lemma to (5.87) ends the proof.

The same discussion as Remark 5.7 and Remark 5.8 holds in this case. This last result will be extended to a non-quadratic Lyapunov function in the next part.

5.6.3. Non-quadratic Stabilization of T-S models

Consider again the following non-quadratic Lyapunov function candidate with $P_i = P_i^T > 0$:

$$V(x(t)) = x(t)^T P_z^{-1} x(t) \tag{5.88}$$

Theorem 5.7: The T-S model (5.73) under the control law (5.75) is locally asymptotically stable according to its initial conditions in the outermost Lyapunov level included in the region $R = \{x \in \mathbb{R}^n, \|x\| \le \lambda_x\}$, if it exists $\varepsilon > 0$ and matrices $P_j = P_j^T > 0$, $Q_j = Q_j^T > 0$, H_j, F_j, $j = \{1,\ldots,r\}$ of proper dimensions such that (2.32) holds with

$$\Upsilon_{ij}=\begin{bmatrix} A_iH_j+B_iF_j+(*) & (*) & (*) & (*) \\ H_j-P_j+\varepsilon\left(A_iH_j+B_iF_j\right) & -2\varepsilon P_j & 0 & 0 \\ \varepsilon F_j & 0 & -2\varepsilon I & 0 \\ \varepsilon H_j & 0 & 0 & -\varepsilon P_j \end{bmatrix}<0$$

$$\Psi_{ij}^{k}=\begin{bmatrix} \varphi_k\left(H_j+H_j^T-Q_j\right) & (*) \\ T_k^T\left(A_iH_j+B_iF_j\right) & I \end{bmatrix}>0,\ I<\varphi_kQ_i$$

And

$$-\sum_{i=1}^{r}\sum_{k=1}^{p}(-1)^{d_i^\gamma}\,\beta_k\left(P_{g_1(i,k)}-P_{g_2(i,k)}\right)\le\varepsilon P_j \tag{5.89}$$

hold with $\varphi_k=\dfrac{2\beta_k}{\lambda_k^2+\lambda_x^2}$ and: $\gamma-1=d_r^\gamma+d_{r-1}^\gamma\times2+\cdots+d_1^\gamma\times2^{r-1}$, $\gamma=\{0,\ldots,2^r-1\}$.

***Proof**:* Consider the Lyapunov function candidate (5.88); in order to guarantee that its time-derivative is negative the following condition must hold:

$$\dot V=\begin{bmatrix}x\\ \dot x\\ u\end{bmatrix}^T\begin{bmatrix}\dot P_z^{-1} & P_z^{-1} & 0\\ P_z^{-1} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot x\\ u\end{bmatrix}<0 \tag{5.90}$$

which combined with the equality constraint (5.84) yields, by a similar procedure of that employed in Theorem 5.6, the next inequality with $\varepsilon>0$:

$$\begin{bmatrix} H_z^T\dot P_z^{-1}H_z+A_zH_z+B_zF_z+(*) & (*) & (*) \\ H_z-P_z+\varepsilon\left(A_zH_z+B_zF_z\right) & -2\varepsilon P_z & (*) \\ \varepsilon F_z & 0 & -2\varepsilon I \end{bmatrix}<0 \tag{5.91}$$

In order to deal with $\dot P_z^{-1}$, consider the following relationship:

$$\dot P_z^{-1}\le\varepsilon P_z^{-1} \tag{5.92}$$

Thus, by Schur complement it can be taken into account to guarantee (5.91) if

$$\begin{bmatrix} A_zH_z+B_zF_z+(*) & (*) & (*) & (*) \\ H_z-P_z+\varepsilon\left(A_zH_z+B_zF_z\right) & -2\varepsilon P_z & 0 & 0 \\ \varepsilon F_z & 0 & -2\varepsilon I & 0 \\ \varepsilon H_z & 0 & 0 & -\varepsilon P_z \end{bmatrix}<0 \tag{5.93}$$

Considering the property $P_z\dot P_z^{-1}P_z=-\dot P_z$, it follows that (5.92) can be rewritten as:

$$P_z \dot{P}_z^{-1} P_z \le P_z \varepsilon P_z^{-1} P_z \Leftrightarrow -\dot{P}_z \le \varepsilon P_z \tag{5.94}$$

Then recalling that:

$$\dot{P}_z = \sum_{j=1}^{r} \sum_{k=1}^{p} h_j \frac{\partial w_0^k}{\partial z_k} \left(P_{g_1(j,k)} - P_{g_2(j,k)} \right) \dot{z}_k \tag{5.95}$$

And writing (i.e. (5.55) replacing P_z with H_z) with vector ξ^k and T_k defined in (5.54):

$$\left| \frac{\partial w_0^k}{\partial z_k} \left(\frac{\partial z_k}{\partial \xi^k} \right)^T T_k^T \left(A_z H_z + B_z F_z \right) H_z^{-1} x \right| \le \beta_k \tag{5.96}$$

The procedure is strictly similar as the one previously explained for Theorem 5.4 as is not repeated therein. After some manipulations considering $Q_z > 0$, (5.96) is ensured if:

$$\begin{bmatrix} Q_z^{-1/2} & 0 \\ 0 & Q_z^{1/2} H_z^{-T} \left(A_z H_z + B_z F_z \right)^T T_k \end{bmatrix} \times \begin{bmatrix} Q_z^{-1/2} & 0 \\ 0 & T_k^T \left(A_z H_z + B_z F_z \right) H_z^{-1} Q_z^{1/2} \end{bmatrix} < \varphi_k I \tag{5.97}$$

with $\varphi_k = \dfrac{2\beta_k}{\lambda_k^2 + \lambda_x^2}$. Thus first row of (5.97) leads directly to $I < \varphi_k Q_z$, whereas second gives after congruence with $Q_z^{-1/2}$:

$$H_z^{-T} \left(A_z H_z + B_z F_z \right)^T T_k T_k^T \left(A_z H_z + B_z F_z \right) H_z^{-1} < \varphi_k Q_z^{-1} \tag{5.98}$$

Or equivalently:

$$\left(A_z H_z + B_z F_z \right)^T T_k T_k^T \left(A_z H_z + B_z F_z \right) < \varphi_k H_z^T Q_z^{-1} H_z \tag{5.99}$$

Recalling that for $Q_z > 0$: $H_z^T Q_z^{-1} H_z \ge H_z + H_z^T - Q_z$, (5.99) is satisfied if:

$$\left(A_z H_z + B_z F_z \right)^T T_k T_k^T \left(A_z H_z + B_z F_z \right) < \varphi_k \left(H_z^T + H_z - Q_z \right) \tag{5.100}$$

And applying a Schur complement on (5.100) gives

$$\Psi_{zz}^k = \begin{bmatrix} \varphi_k \left(H_z^T + H_z - Q_z \right) & (*) \\ T_k^T \left(A_z H_z + B_z F_z \right) & I \end{bmatrix} > 0 \tag{5.101}$$

That is satisfied using Theorem 5.7 conditions.

Recalling the property:

$$Y + \gamma_k X \le 0 \quad \Leftarrow \quad \begin{cases} Y + \lambda_k \times X \le 0 \\ Y - \lambda_k \times X \le 0 \end{cases}, |\gamma_k| \le \lambda_k \tag{5.102}$$

Thus, (5.94) holds if conditions (5.89) hold, which concludes the proof.

Lemma 5.3: Under the same relaxation, if there exists a solution to quadratic stabilization, i.e. Theorem 5.6 conditions, then Theorem 5.7 conditions are feasible and the result is global.

Proof: again consider that it exists $\varepsilon > 0$ and matrices $P = P^T > 0$, H_i, and F_i, $i = \{1,\ldots,r\}$ such that conditions of Theorem 5.6 hold and fix $P_i = P \quad i \in \{1,\ldots,r\}$. Therefore obviously (5.91) corresponds exactly to (5.87). Thus we must prove that the other constraints are always satisfied. First of all (5.89) clearly stands as it remains: $\varepsilon P_j \geq 0$. Now consider:

$$\Psi_{zz}^{k} = \begin{bmatrix} \varphi_k \left(H_z^T + H_z - Q_z \right) & (*) \\ T_k^T \left(A_z H_z + B_z F_z \right) & I \end{bmatrix} > 0 \text{ and } I < \varphi_k Q_z \tag{5.103}$$

Note that (5.103) is satisfied, whatever are H_i, F_i, $i \in \{1,\cdots,r\}$, if it exists a "big" enough φ_k, therefore a "big" enough β_k. Thus it concludes the proof. □

5.7. Design examples

Example 5.5 :

To illustrate the gain obtained with respect to the quadratic conditions, consider the nonlinear model:

$$\dot{x}(t) = \begin{bmatrix} a + b\left(x_1^2 + x_2^2\right) & -1 \\ x_1^2 + 2 & c + d\left(x_1^2 + x_2^2\right) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -x_2^2 \end{bmatrix} u(t) \tag{5.104}$$

with $a = 0.2363$, $b = 0.0985$, $c = 0.7097$, and $d = 0.3427$.

The following T-S model can be constructed from (5.104) in the compact set $C = \{x : |x_1| \leq 1, |x_2| \leq 2\}$:

$$\dot{x}(t) = \sum_{i=1}^{4} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right) \tag{5.105}$$

$$A_1 = \begin{bmatrix} -a + 5b & -1 \\ 3 & -c + 5d \end{bmatrix},\ A_2 = \begin{bmatrix} -a + b & -1 \\ 3 & -c + d \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a + 4b & -1 \\ 2 & -c + 4d \end{bmatrix},\ A_4 = \begin{bmatrix} -a & -1 \\ 2 & -c \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix},\ B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$z_1 = x_1^2,\quad z_2 = x_2^2,\quad w_0^1 = z_1,\quad w_0^2 = \frac{z_2}{4},\quad w_1^1 = 1 - w_0^1,\quad w_1^2 = 1 - w_0^2,\quad h_1 = w_0^1 w_0^2,\quad h_2 = w_0^1 w_1^2,$

$h_3 = w_1^1 w_0^2$, $h_4 = w_1^1 w_1^2$. Using Theorem 5.6 with $\varepsilon = 1$, a non-PDC controller of the form (5.75) can be found via a quadratic Lyapunov function (5.74). The gains and Lyapunov matrix are given by

$$F_1 = \begin{bmatrix} 0.4472 \\ 1.7071 \end{bmatrix}^T,\ F_2 = \begin{bmatrix} 0.2136 \\ 1.5175 \end{bmatrix}^T\ F_3 = \begin{bmatrix} 0.3892 \\ 1.6472 \end{bmatrix}^T,\ F_4 = \begin{bmatrix} 0.1797 \\ 1.5333 \end{bmatrix}^T,$$

$$H_1 = \begin{bmatrix} 0.5147 & 0.3107 \\ 1.2469 & 2.7004 \end{bmatrix},\ H_2 = \begin{bmatrix} 0.3894 & -0.8823 \\ 0.9513 & 2.1158 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0.8551 & 0.5411 \\ 1.39 & 2.8006 \end{bmatrix},\ H_4 = \begin{bmatrix} 0.7453 & -0.7108 \\ 1.1077 & 2.2548 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.8085 & 0.0367 \\ 0.0367 & 4.6949 \end{bmatrix}.$$

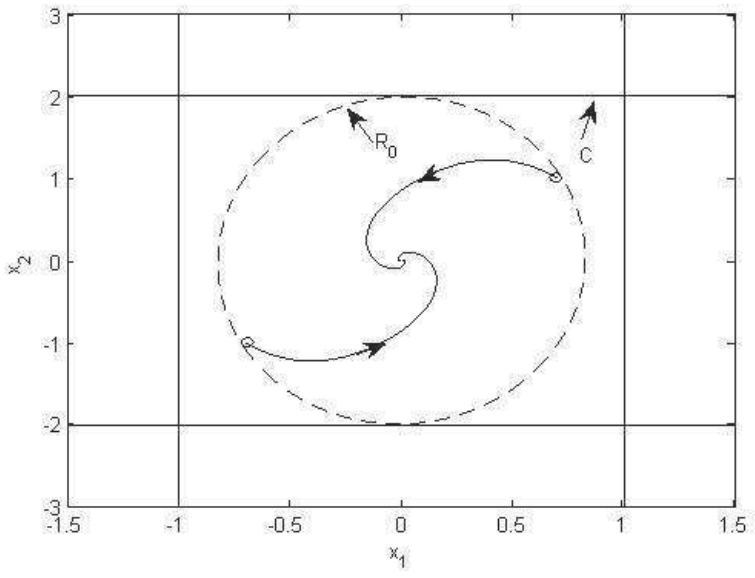


Figure 5.8: Stabilization region with quadratic Lyapunov function and non-PDC controller under Theorem 5.6 scheme.

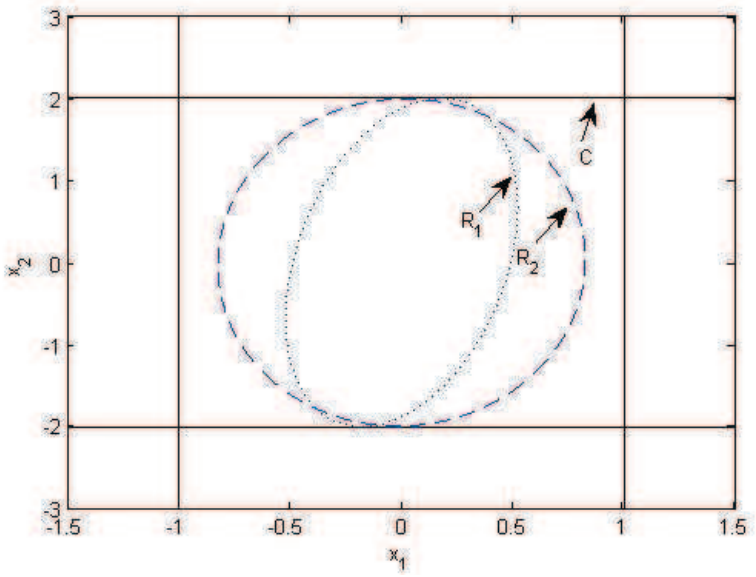


Figure 5.9: Outermost Lyapunov level for Theorem 5.6 R_2 and conditions [Wang & al, 1996] R_1 applied to example (5.104).

Figure 5.8 shows the highest quadratic Lyapunov level R_0 corresponding to this example in the set $C=\{x:|x_1|\le 1,|x_2|\le 2\}$. Two trajectories of the controlled model via a non-PDC control law have been included which show the convergence towards the origin.

In comparison, the classical approaches [Wang & al, 1996] gives the outermost Lyapunov level reduced, showed in Figure 5.9.

Example 5.6 : (continued): To illustrate the new NQ results of Theorem 5.7, we consider again the example of [Mozelli & al, 2009] (5.29). Results show that with $a=0$ and $b=2$, there are no result for a quadratic function neither [Wang & al, 1996], nor Theorem 5.5 & Theorem 5.6. Considering Theorem 5.7 conditions with $\varepsilon=1$, $\lambda_x^2=\dfrac{\pi^2}{2}$, $\lambda_k^2=0.25$ and $\beta=0.2$ provides a feasible solution. The simulation has been performed from initial conditions $x(0)=\begin{bmatrix}1.3 & -0.5\end{bmatrix}^T$. The time evolution of the states is shown in Figure 5.10. The outermost Lyapunov level R_0 in the compact set $C=\left\{x:|x_i|\le\dfrac{\pi}{2}\right\}$, $i=1,2$ is also plotted in Figure 5.11.

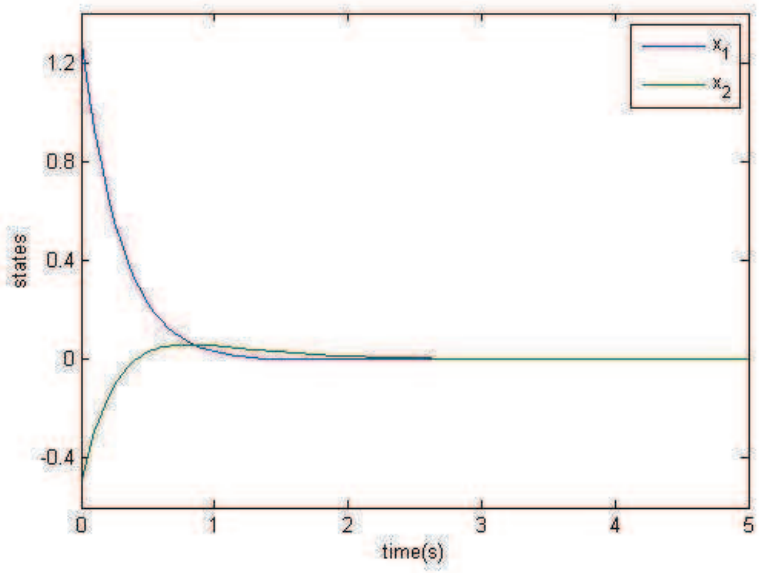


Figure 5.10: Time evolution of the states under Theorem 5.7 scheme.

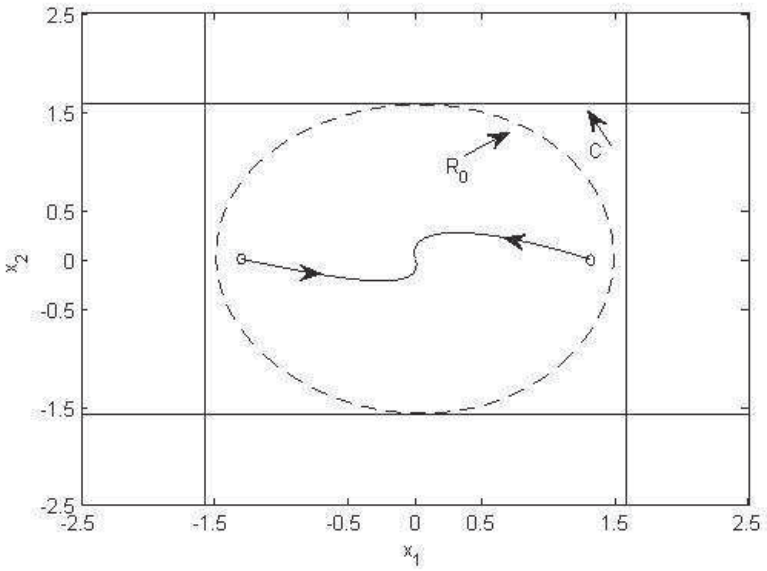


Figure 5.11: The outermost Lyapunov level for Theorem 5.7 applied to example 4.9.

5.8. Conclusion

This chapter summarizes the efforts made when quadratic global conditions fail to find a control law for a continuous TS model. Local based approaches are proposed with the idea to overcome the well-known problem of handling the time-derivatives of membership function as to obtain conditions in the form of linear matrix inequalities. A rewriting of the $\dot{h}_i(z)$ allowed via prescribed bounds to reach this goal in several ways. First attempt kept a bound

on the control that is a reasonable assumption for practical cases. A second way permitted to overcome this drawback via some matrix transformations. At last, the proposition of new control laws, that do not use the matrices involved in the Lyapunov function were done. These new control laws together with the so-called Finsler's lemma brought some new material to solve the problem. The proofs that these results always encapsulate the quadratic results were also done at each step.

Although, examples shown that these methods solve problems unfeasible using the classical results, they are just an initial step. Effectively, the complexity of the LMI involved makes them quickly not tractable for "bigger" TS models than "few" rules and states. Robustness and performances are also to be though and, of course, interconnection with observers to go to output feedback results is also far from being solved.

6. Chapter 6: Non-quadratic H-infinity control of T-S models

Synopsys

This chapter investigates the H_∞ controller design for continuous-time Takagi-Sugeno models. The focus is how to extend the approaches previously developed for stability analysis and controller design to deal with disturbed T-S models. On the basis of a non-quadratic Lyapunov function and with the consideration of the bounds of the partial-derivative of the membership functions, a robust H_∞ control scheme is presented in terms of Linear Matrix inequalities to ensure a local stabilization with external disturbances attenuation, the work presented in this chapter is inspired from [Jaadari & al, 2013]

6.1. Introduction

Recently the problem of H-infinity control design for disturbed nonlinear systems in the form of Takagi-Sugeno models has been widely investigated by many researchers due to the fact that nonlinear systems are frequently affected by unknown inputs and external disturbances. In order to deal with these problems, several works based on quadratic Lyapunov functions were developed [Hong & Langari, 2000], [Lee & al, 2001], [Tanaka & Wang, 2001] and non-quadratic Lyapunov functions [Bernal & al 2011b]. Motivated by the new improvements obtained in recent works in non-quadratic approaches for stability analysis and controller design previously cited. In this chapter, we will consider the design problem of H_∞ controller for continuous-time Takagi-Sugeno models affected by external disturbances. This approach aims to establish relaxed H_∞ control conditions for continuous-time T-S systems based on non-quadratic technique. Both Finsler's lemma and non-quadratic Lyapunov function are employed to further improve the results found in the literature, less conservative stabilization results and better attenuation for the H infinity criterion will be obtained,

6.2. Definitions and Problem statement

Consider a disturbed nonlinear model of the form

$$\begin{cases} \dot{x}(t) = f_1(z(t))x(t) + g_1(z(t))u(t) + d_1(z(t))\phi(t) \\ y(t) = f_2(z(t))x(t) + g_2(z(t))u(t) + d_2(z(t))\phi(t) \end{cases} \tag{6.1}$$

with $f_i(\cdot)$, $g_i(\cdot)$, $d_i(\cdot)$, $i=1,2$ being nonlinear functions, $x(t)\in\mathbb{R}^n$ the state vector, $u(t)\in\mathbb{R}^m$ the input vector, $y(t)\in\mathbb{R}^q$ the output vector, $\phi(t)\in\mathbb{R}^q$ the disturbance vector satisfying $\|\phi(t)\|_2 \le \bar{\phi}$, and $z(x(t))\in\mathbb{R}^p$ the premise vector assumed to be bounded and smooth in a compact set C of the state space including the origin.

Based on the definitions given in previous chapters, an exact representation of (6.1) in a compact set C of the state space, is given by the following continuous-time T-S model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t) + E_i \phi(t)\right) = A_z x(t) + B_z u(t) + E_z \phi(t) \\ y(t) = \sum_{i=1}^{r} h_i(z(t))\left(C_i x(t) + D_i u(t) + G_i \phi(t)\right) = C_z x(t) + D_z u(t) + G_z \phi(t) \end{cases} \tag{6.2}$$

with $r = 2^p \in \mathbb{N}$ representing the number of linear models and $(A_i, B_i, C_i, D_i, E_i, G_i)$, $i = 1, \ldots, r$ a set of matrices of proper dimensions.

The following non-PDC control law, used in section 4.7 of chapter 4 or [Jaadari & al, 2012], is adopted:

$$u(t) = \sum_{i=1}^{r} h_i(z) F_i \left(\sum_{i=1}^{r} h_i(z) H_i \right)^{-1} x(t) = F_z H_z^{-1} x(t) \tag{6.3}$$

with $F_i \in \mathbb{R}^{m \times n}$ the controller gains and $H_i \in \mathbb{R}^{n \times n}$ The closed-loop T-S model is then written as

$$\begin{cases} \dot{x}(t) = \left(A_z + B_z F_z H_z^{-1}\right) x(t) + E_z \phi(t) \\ y(t) = \left(C_z + D_z F_z H_z^{-1}\right) x(t) + G_z \phi(t) \end{cases} \tag{6.4}$$

Expression (6.4) can be written as:

$$\begin{bmatrix} A_z + B_z F_z H_z^{-1} & E_z & -I & 0 \\ C_z + D_z F_z H_z^{-1} & G_z & 0 & -I \end{bmatrix} \begin{bmatrix} x \\ \phi \\ \dot{x} \\ y \end{bmatrix} = 0 \tag{6.5}$$

6.3. H-infinity controller design

In this section, we give a set of conditions to design a robust controller with H_∞ performance. The T-S model (6.2) satisfies the H_∞ attenuation criterion if disturbances $\phi(t)$ are bounded at the output by:

$$\sup_{\|\phi(t)\|_2 \neq 0} \frac{\|y(t)\|_2}{\|\phi(t)\|_2} \leq \gamma \tag{6.6}$$

Where $\|y(t)\|_2$ stands for the L_2 norm of $y(t)$ defined by:

$$\left\| y(t) \right\|_2^2 = \int_0^\infty y^T(t)\, y(t)\, dt$$

In other words, the goal is to minimize the following criterion with $\gamma > 0$:

$$\int_0^\infty \left\| y(t) \right\|^2 dt < \gamma^2 \int_0^\infty \left\| \phi(t) \right\|^2 dt \tag{6.7}$$

Consider the following non-quadratic candidate Lyapunov function:

$$V\left(x(t)\right) = x(t)^T\, P_z^{-1} x(t) = x(t)^T \left(\sum_{i=1}^{r} h_i(z)\, P_i \right)^{-1} x(t) \tag{6.8}$$

with $P_i = P_i^T > 0$, $i \in \{1, \cdots, r\}$.

As shown in [Tanaka & Wang, 2001], condition (6.6) is satisfied if :

$$\dot{V} + y(t)^T\, y(t) - \gamma^2 \phi(t)^T\, \phi(t) \le 0 \tag{6.9}$$

Deriving the Lyapunov function $V\left(x(t)\right)$ and taking into account (6.2), (6.8) and (6.9) can be rewritten as:

$$\begin{aligned} &\dot{V} + y(t)^T\, y(t) - \gamma^2 \phi(t)^T\, \phi(t) \\ &= \dot{x}(t)^T\, P_z^{-1} x(t) + x(t)^T\, P_z^{-1} \dot{x}(t) + x(t)^T\, \dot{P}_z^{-1} x(t) - \gamma^2 \phi(t)^T\, \phi(t) + y(t)^T\, y(t) \le 0, \end{aligned} \tag{6.10}$$

Which taking into account (6.5) can be also put in the following form:

$$\begin{bmatrix} x \\ \phi \\ \dot{x} \\ y \end{bmatrix}^T \begin{bmatrix} \dot{P}_z^{-1} & 0 & P_z^{-1} & 0 \\ 0 & -\gamma^2 I & 0 & 0 \\ P_z^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \phi \\ \dot{x} \\ y \end{bmatrix} \le 0 \tag{6.11}$$

Now, Applying Finsler's Lemma [Guerra & al, 2009a] with (6.11) and constraint (6.5) writes:

$$\begin{bmatrix} \dot{P}_z^{-1} & 0 & P_z^{-1} & 0 \\ 0 & -\gamma^2 I & 0 & 0 \\ P_z^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} + \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \\ U_3 & V_3 \\ U_4 & V_4 \end{bmatrix} \times \begin{bmatrix} A_z + B_z F_z H_z^{-1} & E_z & -I & 0 \\ C_z + D_z F_z H_z^{-1} & G_z & 0 & -I \end{bmatrix} + (*) \le 0 \tag{6.12}$$

Left-multiplying by $diag\left(\begin{bmatrix} H_z^T & I & P_z & I \end{bmatrix}\right)$ and right-multiplying by $diag\left(\begin{bmatrix} H_z & I & P_z & I \end{bmatrix}\right)$ gives:

$$\begin{bmatrix} H_z^T \dot{P}_z^{-1} H_z & 0 & H_z^T & 0 \\ 0 & -\gamma^2 I & 0 & 0 \\ H_z & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} + \begin{bmatrix} H_z^T U_1 & H_z^T V_1 \\ U_2 & V_2 \\ P_z U_3 & P_z V_3 \\ U_4 & V_4 \end{bmatrix} \times \begin{bmatrix} A_z H_z + B_z F_z & E_z & -P_z & 0 \\ C_z H_z + D_z F_z & G_z & 0 & -I \end{bmatrix} + (*) \le 0 \quad (6.13)$$

Let $U_1 = H_z^{-T}$, $U_2 = 0$, $U_3 = \varepsilon_1 P_z^{-1}$, $U_4 = 0$, $V_1 = 0$, $V_2 = 1/2 G_z^T$, $V_3 = 0$, $V_4 = 1/2 I$; thus (6.13) becomes:

$$\begin{bmatrix} \Omega_{zz} + \Omega_{zz}^T + H_z^T \dot{P}_z^{-1} H_z & (*) & (*) & (*) \\ 1/2 G_z^T W_{zz} + E_z^T & -\gamma^2 I + G_z^T G_z & (*) & 0 \\ \varepsilon_1 \Omega_{zz} + H_z - P_z & \varepsilon_1 E_z & -2\varepsilon_1 P_z & 0 \\ 1/2 W_{zz} & 0 & 0 & -I \end{bmatrix} \le 0 \quad (6.14)$$

With $\Omega_{zz} = A_z H_z + B_z F_z$ and $W_{zz} = C_z H_z + D_z F_z$

In order to deal with $\dot{P}_z^{-1}$, consider $Q_z = \varepsilon_2^{-1} P_z$ under the following relationship:

$$\dot{P}_z^{-1} \le Q_z^{-1} \Leftrightarrow \dot{P}_z^{-1} \le \varepsilon_2 P_z^{-1} \quad (6.15)$$

so by Schur complement it can be taken into account to guarantee (6.14) if

$$\begin{bmatrix} \Omega_{zz} + \Omega_{zz}^T & (*) & (*) & (*) & (*) \\ 1/2 G_z^T W_{zz} + E_z^T & -\gamma^2 I + G_z^T G_z & (*) & 0 & 0 \\ \varepsilon_1 \Omega_{zz} + H_z - P_z & \varepsilon_1 E_z & -2\varepsilon_1 P_z & 0 & 0 \\ 1/2 W_{zz} & 0 & 0 & -I & 0 \\ \varepsilon_2 H_z & 0 & 0 & 0 & -\varepsilon_2 P_z \end{bmatrix} \le 0 \quad (6.16)$$

Consider the definition of Q_z above and the property $P_z \dot{P}_z^{-1} P_z = -\dot{P}_z$; it follows that (6.15) can be rewritten as:

$$P_z \dot{P}_z^{-1} P_z \le P_z \varepsilon_2 P_z^{-1} P_z \Leftrightarrow -\dot{P}_z \le \varepsilon_2 P_z \quad (6.17)$$

Remark 6.1: Parameters ε_1 and ε_2 are incorporated due to the fact that results in Theorem 6.1 include ordinary PDC control scheme only if the term U_3 in (6.13) can be fixed arbitrarily small. It follows the same procedure as the one chapter 5.

As developed in section 4.6.3 of chapter 4, LMI conditions can be obtained from (6.17) if it exists a bound $\beta_{k,\, k=1,\ldots,p}$ such that:

$$\left| \frac{\partial w_0^k}{\partial z_k} \dot{z}_k \right| \leq \beta_k \tag{6.18}$$

where

$$\dot{z}_k = \frac{\partial z_k(t)}{\partial x(t)}^T \left(\left(A_z H_z + B_z F_z \right) H_z^{-1} x(t) + E_z \phi(t) \right) \tag{6.19}$$

Since functions w_0^k are explicitly known, thus $\frac{\partial w_0^k}{\partial z_k}$ are known too. On the other side, $\frac{\partial z_k}{\partial x}$ represents the mapping between $z(t)$ and $x(t)$, and it is also known. Therefore we can replace $\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial x}\right)^T$ by the following fuzzy Model [Tanaka & al, 2001c]:

$$\frac{\partial w_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial x}\right)^T = \sum_{l=1}^{n_k} \mu_{kl} v_{kl}^T \,,\ \mu_{kl} \geq 0 \,,\ \sum_{l=1}^{n_k} \mu_{kl} = 1 \tag{6.20}$$

As $\beta_k \in \mathbb{R}$, (6.18) holds if:

$$\left(\sum_{l=1}^{n_k} \mu_{kl} v_{kl}^T \begin{bmatrix} A_z H_z + B_z F_z & E_z \end{bmatrix} \begin{bmatrix} H_z^{-1} x \\ \phi \end{bmatrix} \right)^2 \leq \beta_k^2 \tag{6.21}$$

With $\Phi_{zz} = \begin{bmatrix} A_z H_z + B_z F_z & E_z \end{bmatrix}$, (6.21) writes:

$$\left(\sum_{l=1}^{n_k} \mu_{kl} \begin{bmatrix} x^T H_z^{-T} & \phi^T \end{bmatrix} \Phi_{zz}^T v_{kl} \right) \left(\sum_{l=1}^{n_k} \mu_{kl} v_{kl}^T \Phi_{zz} \begin{bmatrix} H_z^{-1} x \\ \phi \end{bmatrix} \right) \leq \beta_k^2$$

Let $\psi = \begin{bmatrix} H_z^{-1} x \\ \phi \end{bmatrix}$, we obtain:

$$\psi^T \left(\sum_{l=1}^{n_k} \mu_{kl} \Phi_{zz}^T v_{kl} \right) \left(\sum_{l=1}^{n_k} \mu_{kl} v_{kl}^T \Phi_{zz} \right) \psi \leq \beta_k^2 \tag{6.22}$$

Consider that $\begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} P_z^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \leq I$ is equivalent to:

$$\psi^T \begin{bmatrix} H_z^T P_z^{-1} H_z & 0 \\ 0 & I \end{bmatrix} \psi \leq I \tag{6.23}$$

With the S-procedure (property 7) applied on (6.22) and (6.23), we get:

$$\beta_k^{-2}\left(\sum_{l=1}^{n_k}\mu_{kl}\Phi_{zz}^T v_{kl}\right)\left(\sum_{l=1}^{n_k}\mu_{kl}v_{kl}^T\Phi_{zz}\right)-\begin{bmatrix}H_z^T P_z^{-1}H_z & 0\\ 0 & I\end{bmatrix}\le 0 \tag{6.24}$$

And taking into account

$$\left(H_z-P_z\right)^T P_z^{-1}\left(H_z-P_z\right)\ge 0 \Leftrightarrow H_z^T P_z^{-1}H_z+P_z-H_z^T-H_z\ge 0$$

(6.24) holds if:

$$\beta_k^{-2}\left(\sum_{l=1}^{n_k}\mu_{kl}\Phi_{zz}^T v_{kl}\right)\left(\sum_{l=1}^{n_k}\mu_{kl}v_{kl}^T\Phi_{zz}\right)-\begin{bmatrix}H_z+H_z^T-P_z & 0\\ 0 & I\end{bmatrix}\le 0$$

And by the mean of the Schur complement (property 1), we obtain

$$\begin{bmatrix}\beta_k\begin{bmatrix}H_z+H_z^T-P_z & 0\\ 0 & I\end{bmatrix} & (*)\\ \sum_{l=1}^{n_k}\mu_{kl}v_{kl}^T\begin{bmatrix}A_zH_z+B_zF_z & E_z\end{bmatrix} & \beta_k I\end{bmatrix}>0 \tag{6.25}$$

Coming back to (6.17), and based on the development of $\dot{P}_z$ given in the previous chapter, it follows

$$-\sum_{k=1}^{p}\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)\le\varepsilon_2 P_z, \tag{6.26}$$

which can be rewritten, knowing the bound $\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right|\le\beta_k$ and applying property 5:

$$\varepsilon_2 P_z+\sum_{k=1}^{p}(-1)^{d_k^\alpha}\beta_k\left(P_{g_1(z,k)}-P_{g_2(z,k)}\right)>0 \tag{6.27}$$

Where $\alpha\in\{1,\ldots,2^p\}$ and d_k^α defined from the binary representation of $\alpha-1=d_1^\alpha+d_2^\alpha\times 2+\cdots+d_p^\alpha\times 2^p$.

LMI constraints can be then formulated, as follows:

Theorem 6.1:[Jaadari & al, 2013]

Given β_k, $k=\{1,\ldots,p\}$, such that (6.18) holds, the T-S model (6.2) under the control law

(6.3) is locally asymptotically stable with disturbance attenuation γ, if there exists $\varepsilon_1, \varepsilon_2 > 0$ and matrices $P_j = P_j^T > 0, H_j, F_j,\ j = \{1,\ldots,r\}$ of proper dimensions such that

$$\begin{aligned} &\Gamma_{iii} < 0, \Gamma_{ijj} + \Gamma_{jij} + \Gamma_{jji} < 0 \\ &\Gamma_{ijk} + \Gamma_{jik} + \Gamma_{ikj} + \Gamma_{jki} + \Gamma_{kij} + \Gamma_{kji} < 0, \\ &\forall (i,j,k) \in \{1,\cdots,r\}^3 \end{aligned} \tag{6.28}$$

And

$$\frac{2}{r-1}\Sigma_{ii}^{kl} + \Sigma_{ij}^{kl} + \Sigma_{ji}^{kl} > 0,\ \ \forall (i,j) \in \{1,\cdots,r\}^2 \tag{6.29}$$

hold with

$$\Gamma_{ijk} = \begin{bmatrix} \Omega_{ij} + \Omega_{ij}^T & (*) & (*) & (*) & (*) \\ 1/2G_k^T W_{ij} + E_i^T & -\gamma^2 I + G_k^T G_i & (*) & 0 & 0 \\ \varepsilon_1 \Omega_{ij} + H_j - P_j & \varepsilon_1 E_i & -2\varepsilon_1 P_j & 0 & 0 \\ 1/2W_{ij} & 0 & 0 & -I & 0 \\ \varepsilon_2 H_j & 0 & 0 & 0 & -\varepsilon_2 P_j \end{bmatrix} \tag{6.30}$$

$$\Sigma_{ij}^{kl} = \begin{bmatrix} \beta_k \left(H_j + H_j^T - P_j \right) & 0 & (*) \\ 0 & \beta_k I & (*) \\ v_{kl}^T \left(A_i H_j + B_i F_j \right) & v_{kl}^T E_i & \beta_k I \end{bmatrix} \tag{6.31}$$

and

$$\Gamma_j^\alpha = \varepsilon_2 P_j + \sum_{k=1}^{p} (-1)^{d_k^\alpha} \beta_k \left(P_{g_1(j,k)} - P_{g_2(j,k)} \right) > 0 \tag{6.32}$$

with $\Omega_{ij} = A_i H_j + B_i F_j$ and $W_{ij} = C_i H_j + D_i F_j$,

$g_1(j,k) = \left\lfloor (j-1)/2^{p+1-k} \right\rfloor \times 2^{p+1-k} + 1 + (j-1) \bmod 2^{p-k}$ and $g_2(j,k) = g_1(j,k) + 2^{p-k}$, $\lfloor \cdot \rfloor$ standing for the floor function, d_k^α defined from $\alpha - 1 = d_1^\alpha + d_2^\alpha \times 2 + \cdots + d_p^\alpha \times 2^p$, $\alpha = \{1,\ldots,2^p\}$

The T-S model (6.2) satisfies the H_∞ criterion for disturbances rejection provided that $\|\phi(t)\| \le 1$.

***Proof*:**

From (6.28) and respectively the relaxation Lemma (6.29) it follows immediately that inequalities in (6.30) and respectively (6.31) imply the inequalities given by (6.16) and (6.25).

Since all the possible sign combinations in (6.27) are taken into account and inequalities in (6.18) bound the time derivative of the non-quadratic Lyapunov function it follows that (6.27) is a generalization of [Tuan & al, 2001] , thus concluding the proof. □

Remark 6.2: note that it is very often possible to write $G_z = G$, as the external disturbance arrives directly on the output. In this case (6.28) reduces to a double sum and relaxation lemma (6.29) can also be used.

6.4. Design examples

In this section, we demonstrate the effectiveness of the proposed approach using simulation examples. The first example deals with a disturbed continuous-time TS model in which asymptotic stability and disturbance attenuation are guaranteed, the second example shows the advantages of using the new non-PDC controller scheme.

Example 6.1:

Consider the following T-S model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t) + E_i \phi(t)\right) \\ y(t) = \sum_{i=1}^{2} h_i\left(z(t)\right)\left(C_i x(t) + D_i u(t) + G_i \phi(t)\right) \end{cases} \tag{6.33}$$

Where $A_1 = \begin{bmatrix} -6.3 & -3.5 \\ -8.3 & 1.4 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2.7 & 5.4 \\ -6.66 & -3.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 2 & -10 \\ 5 & -1 \end{bmatrix}$,

$C_2 = \begin{bmatrix} -3 & 20 \\ -7 & -2 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $D_2 = \begin{bmatrix} -2 \\ 0.5 \end{bmatrix}$, $E_1 = \begin{bmatrix} -0.3 \\ -0.2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$, $G_1 = G_2 = \begin{bmatrix} -0.35 \\ 0.35 \end{bmatrix}$.

The external disturbance is $\phi(t) = \sin(0.5t)$. The MFs of the fuzzy model are defined as $h_1\left(x_1(t)\right) = \cos x_1^2(t)$, $h_2\left(x_1\right) = 1 - h_1\left(x_1\right)$ in the compact set $C = \left\{ x : \left| x_i(t) \right| \le \frac{\pi}{2} \right\}$.

From the MFs above expression in (6.20) can be written with the following μ 's and v 's:

$$\frac{\partial w_0^1}{\partial z_1}\left(\frac{\partial z_1}{\partial x}\right)^T = \begin{bmatrix} -2\sin x_1(t)\cos x_1(t) & 0 \end{bmatrix}^T = \sum_{l=1}^{2} \mu_{1l} v_{1l}^T$$

where $v_{11} = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$, $v_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\mu_{11} = \dfrac{1+2\sin x_1(t)\cos x_1(t)}{2}$ and $\mu_{12} = 1-\mu_{11}$

Classical conditions based on common Lyapunov function and ordinary PDC controller cannot be solved for this example.

By solving conditions in Theorem 6.1, a local H_∞ controller has been designed. Knowing the bounds corresponding to the external disturbances, it follows $\lambda_w^2 = 1$ and by choosing $\beta_1 = 1.1$, $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 1$.

We have the state feedback gains given by

$$F_1 = \begin{bmatrix} 0.148 & 0.480 \end{bmatrix}, \; F_2 = \begin{bmatrix} -0.047 & 0.066 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0.041 & 0.057 \\ 0.033 & 0.091 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0.025 & 0.012 \\ -0.004 & 0.028 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0.029 & 0.029 \\ 0.029 & 0.073 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0.028 & 0.009 \\ 0.009 & 0.040 \end{bmatrix}$$

The time evolution of the closed loop system states with initial state vector $x(0) = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}^T$ are depicted in Figure 6.1.

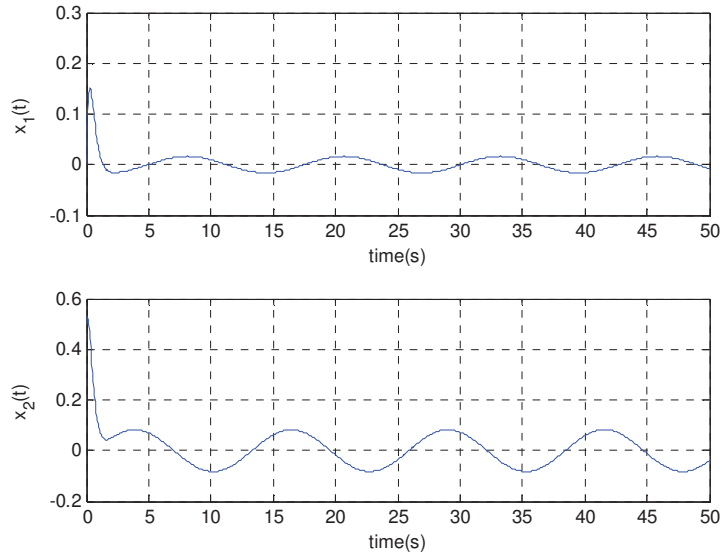


Figure 6.1 : Time evolution of the system states signals

The output signals under the external disturbance $w(t)$ with an attenuation criterion $\gamma = 0.762$ are shown in Figure 6.2.

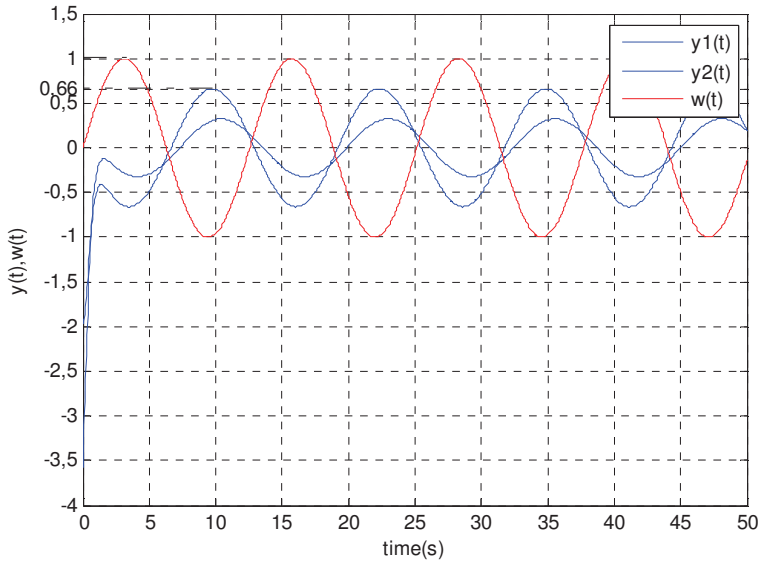


Figure 6.2 : Time evolution of the output and disturbance signals

The simulations results show that the closed loop system is stable with disturbance attenuation γ. Figure 6.3 shows the curve of control input.

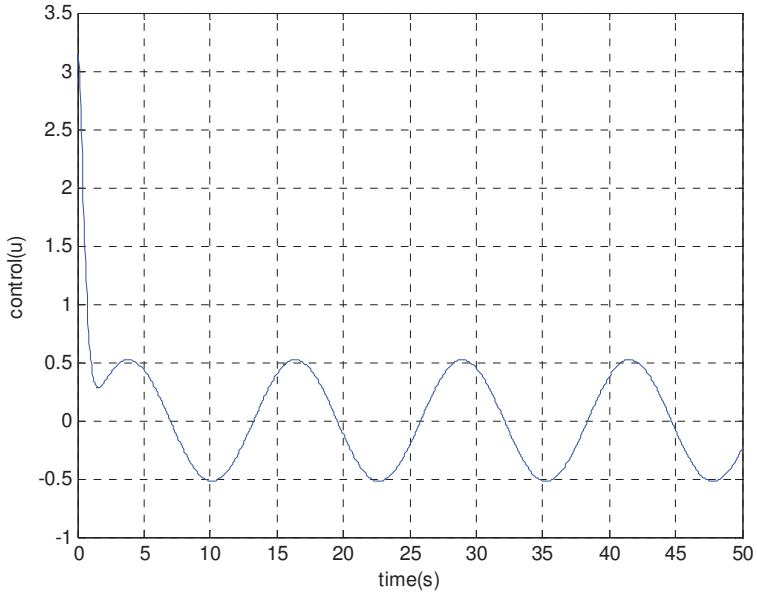


Figure 6.3 : Time evolution of the control input

For the sake of comparison, we propose to compare our approach resumed in Theorem 6.1 with an approach based on the following non-PDC controller

$$u(t)=\sum_{i=1}^{r}h_i(z)F_i\left(\sum_{i=1}^{r}h_i(z)P_i\right)^{-1}x(t)=F_zP_z^{-1}x(t) \tag{6.34}$$

where $P_z=P_z^T>0$.

LMI Conditions ensuring asymptotic stabilization and local H_∞ controller design are stated in the following theorem.

Theorem 6.2:[Jaadari & al, 2013]

The T-S model (6.2) under the control law (6.34) is locally asymptotically stable with an attenuation factor γ, if there exists β_k, $k=\{1,\ldots,p\}$, satisfying (6.18), $\varepsilon_1>0$ and matrices $P_j=P_j^T>0$, F_j, $j=\{1,\ldots,r\}$ of proper dimensions such that,

$$\begin{aligned}&\Upsilon_{iii}^{\alpha}<0,\ \Upsilon_{ijj}^{\alpha}+\Upsilon_{jij}^{\alpha}+\Upsilon_{jji}^{\alpha}<0\\&\Upsilon_{ijk}^{\alpha}+\Upsilon_{jik}^{\alpha}+\Upsilon_{ikj}^{\alpha}+\Upsilon_{jki}^{\alpha}+\Upsilon_{kij}^{\alpha}+\Upsilon_{kji}^{\alpha}<0,\\&\forall(i,j,k)\in\{1,\cdots,r\}^3,\alpha\in\{1,\cdots,2^p\}\end{aligned} \tag{6.35}$$

And

$$\frac{2}{r-1}\Phi_{ii}^{kl}+\Phi_{ij}^{kl}+\Phi_{ji}^{kl}<0,\quad \forall(i,j)\in\{1,\cdots,r\}^2 \tag{6.36}$$

Hold with

$$\Upsilon_{ijk}^{\alpha}=\begin{bmatrix}\Pi_{ij}+\Pi_{ij}^T-\sum_{k=1}^{p}(-1)^{d_k^{\alpha}}\beta_k\left(P_{g_1(j,k)}-P_{g_2(j,k)}\right) & (*) & (*) & (*)\\ 1/2G_k^T\Psi_{ij}+E_i^T & -\gamma^2 I+G_k^TG_i & (*) & 0\\ \varepsilon_1\Pi_{ij} & -2\varepsilon_1P_j & -2\varepsilon_1P_j & 0\\ 1/2\Psi_{ij} & 0 & 0 & -I\end{bmatrix} \tag{6.37}$$

$$\Phi_{ij}^{kl}=\begin{bmatrix}-\beta_kP_j & (*)\\ v_{kl}^T\left(A_iP_j+B_iF_j\right) & -\beta_kI\end{bmatrix} \tag{6.38}$$

Where $\Pi_{ij}=A_iP_j+B_iF_j$ and $\Psi_{ij}=C_iP_j+D_iF_j$,

$g_1(j,k)=\left\lfloor (j-1)/2^{p+1-k}\right\rfloor\times 2^{p+1-k}+1+(j-1)\bmod 2^{p-k}$ and $g_2(j,k)=g_1(j,k)+2^{p-k}$, $\lfloor\cdot\rfloor$ standing for the floor function, d_k^{α} defined from $\alpha-1=d_1^{\alpha}+d_2^{\alpha}\times 2+\cdots+d_p^{\alpha}\times 2^p$, $\alpha=\{1,\ldots,2^p\}$

Proof:

The proof is similar to that of Theorem 6.1, the arbitrary gain H_j used in the controller (6.34) is replaced by a symmetric definite positive matrix $P_j=P_j^T>0$.□

Example 6.2:

Consider the T-S model defined in (6.33)

$$\begin{cases}\dot{x}(t)=\sum_{i=1}^{2}h_i\left(z(t)\right)\left(A_ix(t)+B_iu(t)+E_i\phi(t)\right)\\ y(t)=\sum_{i=1}^{2}h_i\left(z(t)\right)C_ix(t)\end{cases} \tag{6.39}$$

Here, we consider that matrices $D_i=0$ and $G_i=0$

Figure 6.4 shows the behavior of the minimum attenuation factor γ found when the bound β_k varies, a better minimization is obtained using Theorem 6.1 comparing to the attenuation factor obtained via Theorem 6.2.

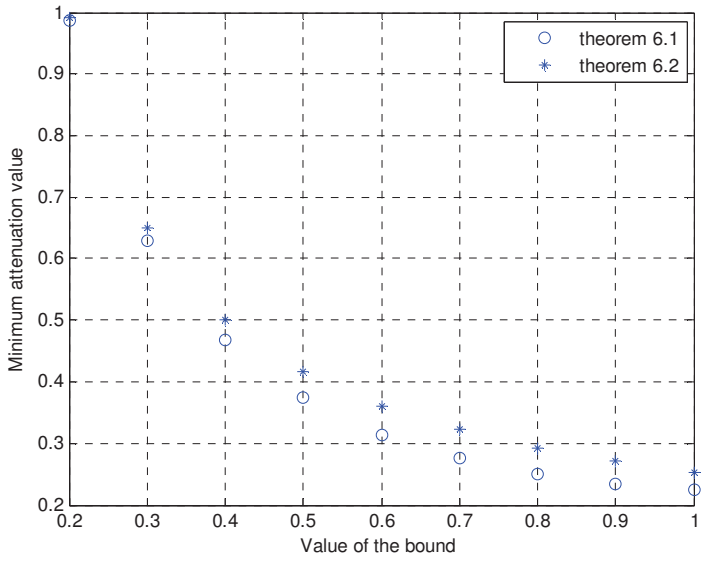


Figure 6.4 : Attenuation factor with different values of the bound $\beta_k, k=1$ (Theorem 6.1 (o)) and Theorem 6.2 (*)

6.5. Conclusion

A new local approach to H-infinity control design for continuous-time T-S models has been presented in this chapter. Based on the notion of non-quadratic stabilization developed in the previous chapter. The problem of the time-derivative of the MFs has been overcome via Finsler's lemma and produced less conservative. New conditions of the asymptotic stabilization problem and the external disturbance attenuation are expressed as LMIs which are easily solved by means of convex optimization techniques. Two examples are included to show the effectiveness of the new approach and its advantages.

Part III: Conclusions

7. Chapter 7: Conclusions

7.1. Thesis summary

This thesis has dealt with stability analysis and controller design of nonlinear systems of the form of continuous–time Takagi-Sugeno models, the main contributions in this work are:

- Non-quadratic stability analysis for continuous-time T-S models
- Non-quadratic stabilization for continuous-time T-S models

Chapter 1 has provided an overview of nonlinear systems modeling, the motivations and a review of previous works have been presented. Next, the contributions of the thesis are described and it ended with an outline of the thesis.

Part I has presented a state of the art for continuous-time Takagi-Sugeno models by introducing the basic concepts used for stability analysis and controller design. In this section, we detailed the principle of stability, stabilization and the Lyapunov theory on which is based this study. Thus, a brief overview of the major works of literature on the stability analysis and synthesis of fuzzy controllers for T-S models based on techniques of convex programming (Linear matrix inequalities and Sum of squares programming) was presented. Then, the drawbacks and the sources of conservatism were discussed. Recent proposed approaches and results to overcome these problems are studied.

Part II contained four chapters developing the contibutions of this thesis
In chapter 3, novel methods were presented for non-quadratic stability analysis of continuous-time T-S models, these methods have taken the full route from an initial idea which consists in a change of perspective for non-quadratic stability analysis of T-S models. This approach reduces global goals to less exigent conditions, thereby showing that an estimation of the region of attraction can be found (local stability); this solution parallelizes nonlinear analysis

and design for models that do not admit a global solution. In this chapter, some improvements on the local stability conditions based on a new way to deal with the membership functions are presented, by the mean of fuzzy Lyapunov functions the results are given of the form of Linear Matrix inequalities.

In chapter 4, a sum of squares (SOS) approach has been presented to deal with the problem of stability analysis of continuous-time nonlinear models. This approach proposed a polynomial fuzzy modeling that is a generalization of the T–S fuzzy model and is more effective in representing nonlinear systems combined with local stability analysis via fuzzy polynomial Lyapunov functions, exploiting both polynomial bounds on the model's non-polynomial nonlinearities and, also, polynomial bounds on the partial derivatives of the membership functions. The simulations have proved less conservative results and better estimation of the region of attraction.

Chapter 5 has represented an extension of the results obtained in previous chapters to the stabilization of T-S models, based on non-quadratic Lyapunov functions. New non-PDC controllers have been designed overcoming the drawbacks of existing results and reducing conservativeness thanks to including the membership-shape information. It is shown that the derived local conditions leads to interesting results comparing to existing quadratic approaches. In the first section, a new way to handle the time derivative of the membership functions is presented by introducing bounds a priori known (derived from the modeling region). Improved results have been shown in a second section of this chapter compared to recent non-quadratic approaches by expressing the bounds used in the previous section in LMI form. In the last section, new non-PDC controller has been designed starting from the possibility to somehow "cut" the link between the Lyapunov function and a non-PDC control law. To that end, quadratic and non-quadratic Lyapunov functions have been considered. This treatment has intended to gradually introduce the use of Finsler's Lemma as to suggest the way the time-derivatives of the MFs can be handled. It has been proved that these non-quadratic approaches reduce to the quadratic cases and include the ordinary PDC control law.

In chapter 6, H-infinity controller design for disturbed continuous-time Takagi-Sugeno models approach has been developed. By the help of non-quadratic Lyapunov function and a new form of non-PDC controller, new local LMI conditions have been derived allowing asymptotic stabilization ans external disturbance attenuation.

In the next section, we discuss the possible future directions of this thesis

7.2. Perspectives

As future research directions, it would be interesting as first step, to improve the results in Chapter 3 and chapter 4 to more general inequalities as develop a new algorithm to optimize the bound of the derivative of the membership functions.

The second step will be to provide an approach dealing with uncertain nonlinear systems in the form of continuous-time Takagi-Sugeno models, the issue is extend the results obtained during this thesis to develop a local robust controller based on non-quadratic Lyapunov functions and including the non-PDC controller structure based on an H infinity criterion

$$\begin{cases} \dot{x}(t) = (A_z + \Delta A_z)\, x(t) + (B_z + \Delta B_z)\, u(t) + E_z \phi(t) \\ y(t) = (C_z + \Delta C_z)\, x(t) + (D_z + \Delta D_z)\, u(t) + G_z \phi(t) \end{cases} \tag{7.1}$$

Where ΔA_z, ΔC_z, ΔB_z and ΔB_z are matrices representing parameter uncertainties of the model (7.1).LMI formulation of the conditions for robust controller design can be derived.

The third issue is to consider the problem of local observer design, the non-quadratic approaches proved in this thesis, can be adapted to the estimation of the unmeasured states of a nonlinear model.

Another issue is to extend the results for stability analysis of nonlinear models modeled in the polynomial fuzzy form developed in chapter 3 to the controller design scheme, a SOS formulation of the SOS Stabilization conditions could be a generalization of the results obtained for Takagi Sugeno models. A more general and relaxed stabilization conditions and better estimation of the region of attraction can be obtained.

Appendices

Positivstellensatz

This section offers more detail on the so-called *Positivstellensatz* argumentation, which extends the Lagrange multipliers and S-procedure form the LMI framework to the SOS context. First appeared in [Parrilo, 2003], this relaxation is derived from real algebraic geometry and states that the solution set of the following problem is a convex set, thus solvable through convex optimization techniques [Prajna, 2004a]:

Find polynomials $p_i(x)$, $i=1,\ldots,\hat{N}$ and sum of squares $p_i(x)$, $i=\hat{N}+1,\ldots,N$ such that

$$a_{0,j}(x)+\sum_{i=1}^{N} p_i(x)\,a_{i,j}(x)=0\text{, for } j=1,\ldots,\hat{J} \tag{7.2}$$

$$a_{0,j}(x)+\sum_{i=1}^{N} p_i(x)\,a_{i,j}(x) \text{ are SOS for } j=\hat{J}+1,\ldots,J\text{,} \tag{7.3}$$

with $a_{i,j}(x)$ are given scalar constant coefficient polynomials.

Restrictions in (7.3) can be understood as inequality constraints which, in the current context, may arise from the modelling region of validity of the PF model, thus taking advantage of its local character. This idea leads to a reformulation of the *Positivstellensatz* argumentation [Sala, 2009b]:

Assume a finite set of known polynomial restrictions $F=\{f_1(x),\ldots,f_m(x)\}$ hold in region Ω. Then, a sufficient condition for a polynomial $\pi(x)$ being positive in Ω is that there exist multiplier SOS polynomials $q_i(x)$, $i=1,\ldots,n$, such that $\pi(x)-\sum_{i=1}^{n} q_i(x)\phi_i(x)$ is SOS, where $\phi_i(x)$ are arbitrary polynomials that are composed of products of those in *F*.

The previous reasoning as well as some practical considerations of polynomial order for SOS tests, leads to a quasi-systematic procedure to include SOS restrictions into the local analysis. Briefly, it can be stated as follows:

1. Define a list of polynomial restrictions holding in the modelling area of the PF model $F=\{f_1(x),\ldots,f_m(x)\}$. Note that this non-unique list is naturally derived and a priori known from the modelling region.
2. Construct polynomials $\phi_i(x)$ as all the product combinations of restrictions in F preserving the same sign up to a certain order.

Polynomial multipliers in Example 1 above have been derived following this procedure up to the double products of the restriction list:

$$F=\left\{x_1^2-\overline{x}^2<0, x_2^2-\overline{x}^2<0, \overline{x}^2-x_1<0, \overline{x}^2-x_2<0, -\left(x_1+\overline{x}^2\right)<0, -\left(x_2+\overline{x}^2\right)<0\right\}.$$

Published contributions

Journal papers

Pan J-T., Guerra T-M., Fei S-M., Jaadari A. (2012). Non-quadratic stabilization of continuous T-S fuzzy models: LMI solution for a local approach. IEEE Transactions on Fuzzy Systems, vol.20, no.3, pp.594–602.

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Titre : Systèmes quasi-LPV continus : comment dépasser le cadre du quadratique ?

Cette thèse aborde le problème de l'analyse de la stabilité et de la conception des lois de commande pour les systèmes non linéaires mis sous la forme de modèles flous continus de type Takagi-Sugeno. L'analyse de stabilité est généralement basée sur la méthode directe de Lyapunov. Plusieurs approches existent dans la littérature, basées sur des fonctions de Lyapunov quadratiques sont proposées pour résoudre ce problème, les résultats obtenus à l'aide des telles fonctions introduisent un conservatisme qui peut être très préjudiciable. Pour surmonter ce problème, différentes approches basées sur des fonctions de Lyapunov non quadratiques ont été proposées, néanmoins ces approches sont basées sur des conditions très restrictives. L'idée développée dans ce travail est d'utiliser des fonctions de Lyapunov non quadratiques et des contrôleurs non-PDC afin d'en tirer des conditions de stabilité et de stabilisation moins conservatives. Les propositions principales sont: l'utilisation des bornes locales des dérivées partielles au lieu des dérivés des fonctions d'appartenances, le découplage du gain du régulateur des variables de décision de la fonction Lyapunov, l'utilisation des fonctions de Lyapunov floues polynomiales dans l'environnement des polynômes et la proposition de la synthèse de contrôleur vérifiant certaines limites de dérivés respectées dans une région de la modélisation à la place de les vérifier a posteriori. Ces nouvelles approches permettent de proposer des conditions locales afin de stabiliser les modèles flous continus de type T-S, y compris ceux qui n'admettent pas une stabilisation quadratique et obtenir des domaines de stabilité plus grand. Plusieurs exemples de simulation sont choisis afin de vérifier les résultats présentés dans cette thèse.

Mots clés : modèles flous de types Takagi-Sugeno, Stabilité non-quadratique, stabilisation non-quadratique, Fonction de Lyapunov, Inégalités matricielle linéaires, somme des carrées.

Title : Continuous quasi-LPV Systems: how to leave the quadratic framework?

This thesis deals with the problem of stability analysis and control design for nonlinear systems in the form of continuous-time Takagi-Sugeno models. The approach to stability analysis is usually based on the direct Lyapunov method. Several approaches in the literature, based on quadratic Lyapunov functions, are proposed to solve this problem; the results obtained using such functions introduce a conservatism that can be very detrimental. To overcome this problem, various approaches based on non-quadratic Lyapunov functions have also been recently presented; however, these approaches are based on very conservative bounds or too restrictive conditions. The idea developed in this work is to use non-quadratic Lyapunov functions and non-PDC controller in order to derive less conservative stability and stabilization conditions. The main proposals are: using local bounds in partial derivatives instead of time derivatives of the memberships, decoupling the controller gain from the Lyapunov function decision variables, using fuzzy Lyapunov functions in polynomial settings and proposing the synthesis of controller ensuring a priori known time-derivative bounds are fulfilled in a modelling region instead of checking them a posteriori. These new approaches allow proposing local conditions to stabilize continuous T-S fuzzy systems including those that do not admit a quadratic stabilization. Several simulation examples are chosen to verify the results given in this dissertation.

Key words: Takagi-Sugeno Models, non-quadratic Stability, non-quadratic Stabilization, Lyapunov function, Linear Matrix inqualities, Sum Of Squares.