



Integrated Production and Inventory Control of Assemble-To-Order Systems with Individual Components Demand

Zhi Li

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Zhi Li. Integrated Production and Inventory Control of Assemble-To-Order Systems with Individual Components Demand. Other. Ecole Centrale de Lille, 2013. English. NNT : 2013ECLI0012 . tel-00866378

HAL Id: tel-00866378

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ECOLE CENTRALE DE LILLE

THESE

présentée en vue d'obtenir le grade de

DOCTEUR

en

Spécialité: Automatique, Génie Informatique, Traitement du Signal et Image

par

Zhi LI

DOCTORAT DELIVRE PAR L'ECOLE CENTRALE DE LILLE

Titre de la thèse:

**Commande optimale (en Production et Stock) de
Systèmes Assemble-To-Order (ATO) avec prise en
compte de demandes en composants individuels**

Integrated Production and Inventory Control of Assemble-To-Order Systems with Individual
Components Demand

Soutenue le 3 septembre 2013 devant le jury d'examen :

Président	Jean-Louis BOIMOND, Professeur, LISA, Université d'Angers
Rapporteur	Alexandre DOLGUI, Professeur, LIMOS, Ecole Nationale Supérieure des Mines de Saint-Etienne
Rapporteur	Bernard GRABOT, Professeur, LGP, Ecole Nationale d'Ingénieurs de Tarbes
Membre	Jean-Louis BOIMOND, Professeur, LISA, Université d'Angers
Directeurs de thèse	Etienne CRAYE, Professeur, LAGIS, Ecole Centrale Lille
Codirecteurs de thèse	Mohsen ELHAFSI, Professeur, University of California, Riverside Hervé CAMUS, Maître de Conférences, LAGIS, Ecole Centrale Lille

Thèse préparée dans le Laboratoire d'Automatique, Génie Informatique et Signal, (UMR CNRS 8219)

Ecole Doctorale SPI 072 (Lille I, Lille III, Artois, ULCO, UVHC, EC Lille)

PRES Université Lille Nord de France

À mes parents,

à ma sœur,

à toute ma famille,

à mes professeurs,

et à mes ch èr(e)s ami(e)s.

Acknowledgements

This PhD research work has been achieved at Laboratoire d'Automatique, Génie Informatique et Signal (LAGIS) in École Centrale de Lille, with the research teams “Optimisation des Système Logistique (OSL)”, and “Système Tolérants aux Fautes”. I have worked in Ecole Centrale de Lille in France for three years, from September 2010 to September 2013. The PhD thesis is fully supported by the China Scholarship Council (CSC), including the Ecoles-Centrales Intergroup.

I start by thanking all the members of the jury for having accepted to examine this work and for their interests and remarks.

Then my deepest gratitude goes to my supervisors Prof. Etienne CRAYE, Prof. Mohsen ELHAFSI, and Mr. Hervé CAMUS for having welcomed me and followed me during my thesis. I would like to thank Prof. Mohsen ELHAFSI, for his professional instructions and suggestions, his personal quality and scientific literacy, and his constant patience and encouragement, that guided me, influenced me and got me through this work. I'd like also to thank Prof. Etienne CRAYE, for his guidance and kindness. His pragmatic and strict attitude toward work set a good example for me in my scientific life. I still remember the words of him “En recherche, il ne faut jamais être ni approximatif, ni superficiel. Avec de la persévérance, de la méthode et de la rigueur, rien n'est impossible.” I'm grateful to my supervisor Mr. Hervé CAMUS during my three years of thesis, who gives me a great help for my research work.

Finally, I would like to thank my families and my friends, who support me to overcome any difficulties in the real life.

Villeneuve d'Ascq, France
August 16th, 2013

Zhi LI

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Chapter 1. General Introduction

The main objective of this work is to study a special case of an assemble-to-order (ATO) manufacturing system that is not only subject to demand for the assembled product but also subject to demand for the individual components. For this purpose, we use a Markov decision process (MDP) framework to formulate the system. In this first chapter, we present a basic knowledge of our work such as the definition of ATO systems, the significant role of optimal control, the principles of the general approach, and the problem setting. Finally, we conclude this chapter with a plan of the thesis.

1.1 Assemble-to-Order Systems

In today's business environment, with the increasing competitiveness of the global market, mass customization has become a major objective for many manufacturing companies. This trend has forced companies to adopt a hybrid operations strategy to better deal with a variety of market environments. Towards this end, an assembly system known as ATO, has emerged and became more popular. An ATO system produces multiple components and assembles them into a variety of final products. Demands occur only for the final products, but the system keeps inventory at the component level (Song and Zipkin, 2003). The products can be assembled from different components while components can be used by different products (see Fig.1.1). An ATO system simplifies the process of manufacturing. It can be regarded as a manufacturing strategy which allows a product to be made or service to be available to meet the needs of a specific customer order.

ATO systems are characterized by short assembly times and high product variety, which have the advantage of decreasing life cycles of products, meeting diverse customer needs, and saving on total cost. It is an efficient strategy that companies have applied to reengineer their production design. The primary application of the ATO strategy is in the computer assembly industry. For instance, companies such as Dell and IBM benefit from using the ATO strategy. The former is famous for controlling inventory levels of components, and the latter is famous for two-stage server computers. Both of them successfully apply the ATO strategy to enhance their competitive position in the global PC market (Agrawal and Cohen, 2001; Cheng et al., 2005).

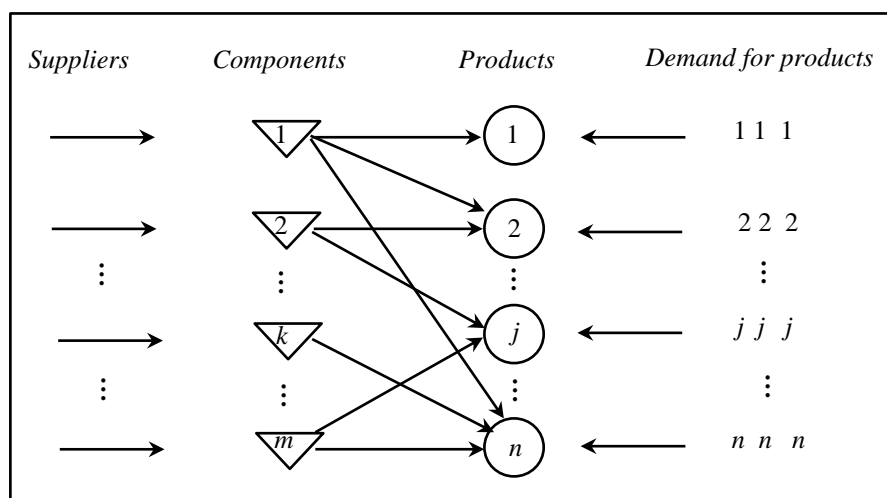


Fig. 1.1. Assemble-to-order system

Generally speaking, the ATO strategy is characterized by flexibility and responsiveness, and it is useful for manufacturing companies to secure market share, improve profits and enjoy a competitive advantage.

In this work, we consider an ATO system that produces n components with a single assembled product. Demand from both the product and the components can be satisfied or rejected/backordered. Components are produced one unit at a time on separate production facilities and held in stock incurring a holding cost. We assume exponentially distributed production times, and demand arrives in the system following independent Poisson processes. In our model, since the final assembly time is considerably short, we neglect it. This assumption is reasonable and applied in most ATO systems (see Song and Zipkin, 2003). Due to the possibility of components stock-out, some orders may not be satisfied immediately. The unsatisfied order may be lost or backordered resulting in a penalty cost. In this study we consider two cases: the pure lost sales and the mixed lost sales and backorders. In the pure lost sales case, an order rejected incurs a lost sale cost. Demand from the assemble product has a higher penalty over the demand from the individual components. Due to limited capacity, it may be desirable to reject a demand from a component even when there is on-hand inventory of components to satisfy future product demand. In the mixed lost sales and backorders case, a component order rejected incurs a lost sale cost and a product order backordered incurs a backorder cost. In this case, the product demand has no priority over the component demand, thus it may be backordered even when there is stock for all the components to satisfy future component demand. For these two cases, a system manager needs to decide which components to produce, and whether to satisfy an incoming component demand or reject it to reserve stock for future product demand, or whether to satisfy an incoming product demand or backorder it to reserve stock for future component demand. The objective is to minimize the expected total operating costs of the system.

1.2 Optimal Control of ATO Systems

ATO systems can be regarded as a multiple resource allocation that induces load distribution, production planning, requirements fulfilling and inventory assignment. The key challenge in the management of ATO systems resides in the difficulty to coordinate components procurement or production as components procurement or production leadtimes are usually stochastic. This is further compounded by the uncertainty of the demand of the assembled product as well as the individual components, if they are sold separately as spares. Another

challenge for ATO systems is to efficiently manage component inventories and make optimal production and allocation decisions. Because of the complexity of such a system, it would tend to be difficult to control and would be uneconomical to operate. In addition, in many business scenarios where manufacturing companies face limited capacity and shortage situations that usually cannot be avoided, it is necessary to adopt a feasible strategy to solve these problems (Akçay, 2002). In this situation, the issue of inventory rationing arises. Because of a limited capacity, it may not be sufficient to produce the total quantity of the order. The manager needs to efficiently manage component inventories and allocation. The problem is how to determine inventory replenishment levels with uncertain demand and how to allocate the components for received demands.

In practice, determining optimal component inventory levels is difficult, especially in a multi-product ATO system. The inventory level of a component at any point in time will depend on the previous allocation decisions. Such decisions depend on the production and consumption of all other components and the demand realizations of all end products. Thus, the problem of determining optimal inventory levels and an allocation policy can be formulated as a dynamic programming with the goal of minimizing the expected long-run system cost. Optimal control is needed to deal with the problem of finding a control policy for a given optimality criterion. For characterizing the structure of optimal policies in the infinite horizon, please refer to the studies by Porteus (1975, 1982), Stidham and Weber (1989).

The main objective of this work is to control an ATO system with demand from both the individual components and the assembled product. In an assembly system, since satisfying a customer order requires multiple available components, the storage of one component delays the fulfillment of the order for product. The optimal control of ATO systems should be correlated across components: the optimal component replenishment policy is applied to the production and the optimal component allocation policy is applied to the inventory. Also, because a customer order requires multiple units of several components, the optimal component allocation policy results in severe computational complexity, especially in the case of multiple demand classes. As mentioned in Ha (1997c), "... as the number of customer classes increases the optimal policy will be difficult to compute because of the curse of dimensionality and will be even more difficult to implement." This implies that as the state space increases in size, the structure of the policy becomes more complex. Because many dimensions must be taken into account when making allocation decision: the inventory level, the number of backorders as well as the production process. For this reason, characterization

of optimal control policies for ATO systems has been regarded as a challenging problem. Various authors have studied this problem including De Véricourt et al. (2000, 2002), Benjaafar and ElHafsi (2006, 2010), Gayon et al. (2009). They showed that the optimal allocation policy is a multi-level rationing policy. In this work, we adopt a similar approach as these authors to analyze the optimal policy for a more general ATO system.

1.3 General Approach

In this work we study an ATO system. In order to determine a control policy, we formulate the problem an MDP. Then we specify the principle and algorithm in the following.

1.3.1 Markov Decision Process

The models we will study in the next few chapters use the MDP framework. Since demand inter-arrival times and production times are uncertain, randomness is one of the key factors that our models must take into account (see Zipkin 2000, section 7.3). Markov Decision Processes, which are also called stochastic dynamic programs or stochastic control problems, provide a mathematical framework for sequential decision making when outcomes are uncertain.

In a dynamical system the state can change over time. At each decision epoch, a decision maker can choose an action that may influence the future state of the system. Markov decision processes are completely determined by a five-tuple $(S, \{A_x | x \in S\}, r_t, p_t, f_t)$, which is defined as follows:

1. S : the set of possible system states.
2. A_x : the set of available actions to the decision maker when the system is in a starting state x .
3. r_t : the cost per unit time. The real-valued function $r_t(x, a)$ for $a \in A_x$ denotes the value at time t of the cost incurred in period t .
4. p_t : the transition probability per unit time. The transition probability function $p_t(\cdot | x, a)$ for $a \in A_x$ denotes the system state at the next decision epoch and is determined by $p_t(\cdot | x, a)$, when action a is chosen in state x , at time t .
5. $f_t(\cdot | x, a)$: the probability density function of time between two changes in states of

the system when action a is chosen in state \mathbf{x} , at time t .

At each instant, the transition probability and the cost function depend on the past only through the current state of the system and an action can be selected in that state. This property is called “Markovian”, which has been widely used in inventory control problems. This is because in this setting a Markovian policy is optimal and properties of the optimal policy are simple to carry out and do not vary with time (Puterman 1994, Chapter 1). In order to choose actions, we must follow some policy. We define a policy to be any decision rule for choosing actions. In other words, a policy π is a sequence of decision rules. Thus, the action chosen by a policy may, for instance, depend on the history of the process up to that state point, or it may be randomized in the sense that it chooses action a with some probability $p_a, a \in A$. When the policy depends only on the current state of the system, it is called Markovian policy. In this case, the control function under a policy π can be defined as A^π in state S and action a . That is, $A^\pi(\mathbf{x})$ is the action selected in \mathbf{x} when a policy π is employed.

Typically, under a policy π and in state \mathbf{x} at time t the decision maker can choose an action a . The cost generated depends on the state of the system at the next decision epoch. At time t for a given probability density function $f_t(\cdot|\mathbf{x},a)$, the system remains in state \mathbf{x} and generates a cost $r_t(\mathbf{x},a)$ per unit of time. When time is divided into periods, a decision epoch is associated with the starting period. Thus, for a time interval τ , at time $t+\tau$, the system changes to a new state \mathbf{x}' , which is determined by the distribution $p_t(\cdot|\mathbf{x},a)$. The total cost generated over this period is equal to $r_t(\mathbf{x}',a) = \int_t^{t+\tau} r_s(\mathbf{x},a)ds$, then a new action is chosen in state \mathbf{x}' and the phenomenon is repeated. Because the action is chosen in the present state incurs a cost that forces the system to move to a new state. Clearly, a new state is determined by the previous action choice. When the distribution f_t is deterministic, the periods τ between two changes of state are constant and equal, corresponding to the representation of periodic time. The decisions in this case are taken at time $t=0,1,2,\dots$, and the specification of the total cost $r_t(\mathbf{x},a)$ over a period is sufficient.

The objective is to determine the optimal policy π^* that minimizes the discounted cost,

$$v^*(\mathbf{x}) = \min_{\pi} E_{\mathbf{x}}^{\pi} \left[\int_0^{\infty} e^{-\alpha t} r(\mathbf{X}(t)) dt \right], \quad (1.1)$$

where, \mathbf{X} is the random process denoting the current state of the system and α is the discount factor ($0 < \alpha < 1$). Since $r(\mathbf{X}(t))$ is the cost generated at time t , it follows that $v^\pi(\mathbf{x})$ represents the expected total discounted cost generated when policy π is applied with the initial state \mathbf{x} .

When decisions are made frequently or the discount factor is not assumed ($\alpha = 0$) the average cost case can be considered,

$$g^\pi(\mathbf{x}) = \min_{\pi} \sup_{T \rightarrow \infty} \frac{E_{\mathbf{x}}^\pi \left[\int_0^T r(\mathbf{X}(t)) dt \right]}{T}, \quad (1.2)$$

where, $g^\pi(\mathbf{x})$ represents the expected average cost per period for any policy π . The objective is to determine the optimal policy π^* that minimizes the average cost. The average cost criterion is simple to implement, because the results of which are independent of the starting state and the discount factor.

In this work, we use these two criteria in our analysis.

1.3.2 The Value Iteration Algorithm

In this sub-section, we consider the computational aspect of MDP. One of the commonly used algorithms in MDP is the value iteration. It is widely used to obtain the optimal policy.

Consider a Markov decision problem (S, A, r, p, f) , which satisfies the assumptions in section 1.3.1. The objective is to find the optimal policy that minimizes the discounted cost in equation (1.1).

Consider a set F of positive real-valued functions defined on S . Under the previous hypotheses, there exists an operator T that reflects the dynamic of the system and equation (1.1) can be written as:

$$v_{n+1}(\mathbf{x}) = Tv_n(\mathbf{x}), \quad (1.3)$$

where, $v_n(\mathbf{x})$ is the n -stage cost function that converges to $v^*(\mathbf{x})$, with $v_0(\mathbf{x}) \in F$. The infinite-horizon optimal cost function satisfies,

$$v^*(\mathbf{x}) = Tv^*(\mathbf{x}). \quad (1.4)$$

The existence of a Markov policy that achieves the minimum of the discounted cost in (1.1) and the convergence of the n -stage policy and cost function to the infinite-horizon optimal policy and cost follow from the fact that only finitely many controls are considered at each state.

An average cost criterion can also be considered. In this case the average cost per stage, g , and the relative cost in state \mathbf{x} , $v(\mathbf{x})$ satisfies,

$$v(\mathbf{x}) + g = Tv(\mathbf{x}). \quad (1.5)$$

Several conditions have to be satisfied for the existence and convergence results for (1.6) by letting $\alpha \rightarrow 0$ in (1.4) (see Cavazos-Cadena, 1992; Weber and Stidham, 1987):

1. there exists a stationary policy π which achieves a finite average cost g^π .
2. the number of states in which the holding cost $h(\mathbf{x}) \leq g^\pi$ is finite.

The value iteration algorithm is the most widely used and best understood algorithm for solving Markov decision problems. It is an easy method to determine the optimal policy. In this work we use the value iteration algorithm, for more details readers can review Puterman (1994, Chapter 8).

1.4 Application of Markov Decision Process in ATO systems

The MDP framework has been used in a wide range of optimization problems. A general application of MDP described in Feinberg and Shwartz (2002, Part III). In this work, we consider an ATO system with limited production capacity, which produces multiple components and assembles them to a signal product. The product is assembled from components only when a customer order is received, and the inventory is kept at the component level. Faced with demands for both, the product and the components, the system manager must determine the optimal policy to minimize the total cost.

In this section, we only present the general characteristics of Markov decision problems. The detailed specification of the models that we study is given in the corresponding chapters.

The state of system

For this work, we assume a discrete-state setting, and use continuous time by converting to an equivalent discrete time. That is, our ATO system produces n types of components with a single assembled product. The product and the components can meet $n+1$ demands. For the pure lost sales case, the state S is a subset of Z^{+^n} (Z^{+} is the set of nonnegative integers). In this case, we define the current state of the system at time t by the vector $\mathbf{X}(t)=(X_1(t), \dots, X_n(t))$, where $X_k(t)$, $k=1, \dots, n$, is a nonnegative integer denoting the on-hand inventory for Component k at time t . For the mixed lost sales and backorders case, the state S is a subset of $Z^{+^{n+1}}$. The current state of the system at time t can be defined by the pair $(\mathbf{X}(t), Y(t))$, where $Y(t)$ is a nonnegative integer denoting the backorder level of the assembled product.

The decisions

For each of the models that we study, the decision maker has to decide which type of components should be produced, and whether to satisfy an incoming component demand or keep stock for future product demand. For instance, in the pure lost sales case: under a policy π for a starting state $\mathbf{x}=(x_1, \dots, x_n)$, the decision maker takes actions $a^\pi(\mathbf{x})=(u_1, \dots, u_n, w_0, \dots, w_n)$, where $u_k=1$ means produce Component k ($k=1, \dots, n$), $u_k=0$ means do not produce Component k , $w_k=1$ means satisfy demand from Component k , and $w_k=0$ means reject demand from Component k , $w_0=1$ means satisfy demand from the assembled product, and $w_0=0$ means reject demand from the assembled product. In the mixed lost sales and backorders case: under a policy π for a starting state $(\mathbf{x}, y)=(x_1, \dots, x_n, y)$, the decision maker takes actions $a'^\pi(\mathbf{x}, y)=(u'_1, \dots, u'_n, w'_0, \dots, w'_n)$, where $u'_k=1$ means produce Component k to stock, $u'_k=2$ means produce Component k to reduce the backorder level of the assembled product, and $u'_k=0$ means do not produce Component k , $w'_k=1$ means satisfy demand from Component k , and $w'_k=0$ means reject demand from Component k , $w'_0=1$ means satisfy demand from the assembled product, and $w'_0=0$ means backorder demand from the assembled product.

The cost structure

The related costs of our system are incurred from two sources: the cost of holding inventory and the cost of backordering. We assume the costs are linear, such as $r_t(\mathbf{x})=r(\mathbf{x})=\sum_{k=1}^n h_k(x_k)$

and $r_t(\mathbf{x}, y) = r(\mathbf{x}, y) = \sum_{k=1}^n h_k(x_k) + b_0(y)$ where both $h_k(\cdot)$ and $b_0(\cdot)$ are increasing convex functions, $h_k(\cdot)$ denotes the holding cost of Component k per unit per unit time, $b_0(\cdot)$ denotes the backorder cost of the assembled product per unit per unit time.

The transition probabilities of the state

Since production times and demand inter-arrival times are stochastic, we focus on these two uncertain sources: production times are exponentially distributed with mean $1/\mu_k$, demands take place continuously over time according to independent Poisson processes with rate λ_k (for Component k) and λ_0 (for the assembled product), respectively.

When the transition times are identically one, it is a Markov decision process, and in general case, it is called a semi-Markov decision process (see Ross 1969, Chapter 7). In the optimal control of exponential queuing systems, we use a semi-Markov decision process. That because a sequential decision process for which the times between transitions are random. In this work, we consider the following two cases:

Pure lost sales case

As mentioned above, in the pure lost sales case the current state of the system at time t can be described by the vector $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$. Under a policy π and a starting state $\mathbf{x} = (x_1, \dots, x_n)$, the decision maker takes the action a . Let $\mathbf{e} = (1, 1, \dots, 1)$, be an n -dimensional vector of ones and \mathbf{e}_k the k th unit vector of dimension n . In the state \mathbf{x} , if the decision maker chooses the action to produce one unit of Component k , the state will transfer to the state $\mathbf{x} + \mathbf{e}_k$ with the transition rate μ_k . If she takes the action to satisfy one unit order from the product, the state will transfer to the state $\mathbf{x} - \mathbf{e}$ with the transition rate λ_0 , or decides to satisfy one unit order from Component k , the state will transfer to the state $\mathbf{x} - \mathbf{e}_k$ with the arrival rate λ_k .

Mixed lost sales and backorders case:

In the backorders case, the current state of the system at time t can be described as the pair $(\mathbf{X}(t), Y(t))$, where $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$. Under a policy π and a starting state $(\mathbf{x}, y) = (x_1, \dots, x_n, y)$, the decision maker takes the action a . When backorders are allowed, the case is more complex than the pure lost sales. Because besides considering the on-hand inventory \mathbf{x} , the backorder level y from the assembled product must be considered. In this case, in the initial state (\mathbf{x}, y) if the decision maker chooses the action with transition rate μ_k ,

which would incur two different results: produce one unit Component k to stock, the state will transfer to the state $(\mathbf{x} + \mathbf{e}_k, y)$; or produce one unit Component k to reduce one unit backorder from the assembled product, the state will transfer to the state $(\mathbf{x} - \sum_{i \neq k} \mathbf{e}_i, y - 1)$. If the decision maker takes the action with transition rate λ_0 , which would lead to satisfy one unit order of the assembled product, the state will transfer to the state $(\mathbf{x} - \mathbf{e}, y)$; or to backorder one unit demand of the assembled product, the state will transfer to the state $(\mathbf{x}, y + 1)$. Similarly, with the rate λ_k , a transition occurs after time t , the next state may be $(\mathbf{x} - \mathbf{e}_k, y)$. Clearly, the distribution of time between two instants of decision depends only on the action specified by the control policy applied by the decision maker. Following Lippman (1975), we uniformize the transition rate by defining the uniform rate $\beta = \sum_{l=0}^n \lambda_l + \sum_{k=1}^n \mu_k$. However, the next state of the system depends on the transition probability. We will discuss them for two cases, pure lost sales and mixed lost sales and backorders:

Pure lost sales case:

In state \mathbf{x} , an action a is selected. If the next state is \mathbf{x}' , the system state at the next decision epoch is determined by the transition probability $p(\mathbf{x}' | \mathbf{x}, a)$, which can be generated as follows:

$$\begin{aligned} p(\mathbf{x}' | \mathbf{x}, a) &= \frac{\mu_k}{\beta} I_{\{u_k=1\}}, & \mathbf{x}' &= \mathbf{x} + \mathbf{e}_k, \\ p(\mathbf{x}' | \mathbf{x}, a) &= \frac{\lambda_0}{\beta} I_{\{\prod_{k=1}^n x_k > 0, \text{ and } w_0=1\}}, & \mathbf{x}' &= \mathbf{x} - \mathbf{e}, \\ p(\mathbf{x}' | \mathbf{x}, a) &= \frac{\lambda_k}{\beta} I_{\{x_k > 0, \text{ and } w_k=1\}}, & \mathbf{x}' &= \mathbf{x} - \mathbf{e}_k, \end{aligned}$$

Mixed lost sales and backorders case:

If an action a is selected in state (\mathbf{x}, y) , the next state is $(\mathbf{x}, y)'$, the system state at the next decision epoch is determined by the transition probability $p((\mathbf{x}, y)' | (\mathbf{x}, y), a)$, which can be generated as follows:

$$p((\mathbf{x}, y)' | (\mathbf{x}, y), a) = \frac{\mu_k}{\beta} I_{\{u'_k=1\}}, \quad (\mathbf{x}, y)' = (\mathbf{x} + \mathbf{e}_k, y),$$

$$p\left((\mathbf{x}, y)' | (\mathbf{x}, y), a\right) = \frac{\mu_k}{\beta} I_{\{\prod_{i \neq k}^n x_i > 0, y > 0, \text{ and } u'_k = 2\}}, \quad (\mathbf{x}, y)' = (\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1),$$

$$p\left((\mathbf{x}, y)' | (\mathbf{x}, y), a\right) = \frac{\lambda_0}{\beta} I_{\{\prod_{k=1}^n x_k > 0, \text{ and } w'_0 = 1\}}, \quad (\mathbf{x}, y)' = (\mathbf{x} - \mathbf{e}, y),$$

$$p\left((\mathbf{x}, y)' | (\mathbf{x}, y), a\right) = \frac{\lambda_k}{\beta} I_{\{x_k > 0, \text{ and } w'_k = 1\}}, \quad (\mathbf{x}, y)' = (\mathbf{x} - \mathbf{e}_k, y),$$

where $I_{\{d\}}$ is the indicator function ($I_{\{d\}} = 1$ if d is true, and $I_{\{d\}} = 0$, otherwise).

The optimal policy

Because the system is memoryless, a Markov policy is optimal. In this study, we formulate the problem as continuous-time MDP. That is, the decisions can be made at any time. Applications in inventory control are modeled by allowing action choice at random times in infinite horizon. The core problem of MDP is to find a policy π in the state S that minimizes the expected discounted (average) cost. For all possible states S , we will find the optimal cost function v^* and use it to determine the optimal policies π^* .

1.5 Problem Setting

In this work, we study an ATO system where we consider demands for both the individual components and assembled product. That is, the product is assembled from multiple components and the components stocked in advance of demand. These products will be used to satisfy the potential orders that arrive later. Components are produced one unit at a time on separate production facilities and held in stock, which incur a holding cost. In addition, both production times and customer inter-arrival times are stochastic. Due to the possibility of shortages, if an order is not satisfied immediately it incurs a lost sale cost or a backorder cost that depends on whether or not the customer is willing to wait for his order. Therefore, the task of the decision maker is to decide whether to satisfy an incoming demand or reject (backorder) it, reserving stock for the future demand from a more valuable type. At the same time, the decision maker also decides which component is needed to produce, if needed, whether to produce it to stock or to produce it to reduce the backorders from a particular demand. The objective is to minimize operating costs while maintaining order fulfillment. In general, this kind of problem can be regarded as a dynamic problem and a decision support tool is needed. In our work, we study the ATO system from an operations' perspective. We

use a Markov decision process framework to determine an optimal policy under both the total expected discounted cost and the average cost per period criteria. We characterize the structure of an optimal policy. We carry out numerical experiments to analyze the structure of the optimal policies. We also offer some managerial insights to control the assembly systems. Furthermore, we show that the optimal production policy is a state-dependent base-stock policy, and the optimal inventory allocation policy is a state-dependent rationing policy. More importantly, we show that the optimal policies are highly sensitive to various system parameters such as the holding and the lost sale/ backorder costs, the demand and production rates.

1.6 Plan of the Thesis

The rest of this thesis is organized as follows:

Chapter 2 provides a brief review of the related literature to this work.

Chapter 3 aims at characterizing the optimal policy of the ATO system with lost sales. We determine the structure of the optimal policy and investigate the impact of different parameters on the optimal policy.

Chapter 4 aims at characterizing the optimal policy of the ATO system with lost sales and backorders. We characterize the optimal production policy and the optimal allocation policy for the components.

In Chapter 5, we develop several heuristic policies for the pure lost sales case and the mixed lost sales and backorders case. For each case, we compare the performance of the heuristics with the optimal policies, and then we find some more efficient heuristics.

Finally, the general conclusion sums up the main results obtained and the perspectives describes some future researches of this work.

1.7 Conclusion

ATO systems are successful strategies that have become increasingly popular in manufacturing. This work studies an ATO system that produces multiple components with a single assembled product. Such a system deals with both product and components demands. In this chapter, we introduced an overview of ATO systems, the basic principles of the general MDP approach, and the value iteration algorithm that is used to compute the optimal

policy. We also presented the application of MDP in ATO systems, based on which we formulate our problem.

Chapter 2. Literature Review

This chapter provides a brief review of the literature related to ATO systems. As mentioned in Chapter 1, the optimal control of ATO systems consist of two decisions: component replenishment and component allocation. These decision problems can be formulated as a single-product or multi- product models, and a single- period or multi-period models. For a comprehensive literature review, we can refer to one classical paper by Song and Zipkin (2003). It covers modeling issues and analytical methods, and a general formulation of ATO systems. From this overview, the literature review on ATO can be classified into the following four categories, which we will discuss in details.

2.1 Make-to-Stock Systems under MDP

Make-to-stock (MTS) systems are manufacturing strategies to manage inventory. In MTS system, products are stocked in advance according to a forecast of customer demand. Because the manager faces a joint production-control and inventory allocation problem, MTS systems can also be called production-inventory systems. A challenging problem in such systems is the dynamic allocation of inventory to different demand classes. This gives rise to an inventory rationing problem which has been widely studied in the literature.

The earlier work on inventory management and production scheduling dates back to Zheng and Zipkin (1990), who studied the optimal control of allocation problem. A simple Markovian behavior is assumed, the problem can be modeled as rationing a fixed production capacity to multiple identical products. More importantly, the authors proposed longest-queue policy and argued that it is always optimal to serve the longest queue under independent base stock policies.

Ha (1997, 2000) pointed out that for a two-dimension state space MTS production system, the optimal production policy is the dynamic “hedging point” policy, and the optimal allocation policy is a “state-dependent rationing” policy. Ha (1997a) is the first to consider rationing in the context of an MTS system. He modeled the system as a single server, single product, M/M/1 make-to-stock queue with multiple demand classes and lost sales. The optimal policy is characterized by a sequence of monotone threshold levels. Under this threshold rationing policy, each class has a rationing threshold below which the demand from that class cannot be satisfied. The system reserves inventory for the orders from the high- priority classes. Ha (1997b) studied hedging point policy with dynamic scheduling problem. By considering single server, two products, make-to-stock queue with backorders, he showed that the optimal rationing policy is of the “switching curve” type. Furthermore, two production switching curves have been obtained: one curve determines when and which product can be produced, and the other curve determines in which region the production can be stopped. In a similar MTS system, Ha (1997c) considered the backorders case but with single product and two priority demand classes. He characterized the optimal production and inventory rationing policies by a single monotone switching curve. He showed that the optimal production policy is of base-stock type and the optimal rationing policy is determined by rationing level, which is decreasing in the number of the low-priority class backorders. In a later article, Ha (2000) extended the results of his work (1997c) to Erlang distributed production times in lost sales

case. Using work storage as a state variable, he indicated that the optimal production policy can be characterized by a critical work storage level. Gayon et al. (2009) analyzed a similar system as Ha (2000) for an $M/E_k/1$ make-to-stock queue with multiple classes and they provided a formulation in the case of backorders and examined the effects of optimal policies under different operating conditions: with and without salvage market value. They showed that the optimal allocation policy with a salvage market is work-storage rationing policy that is characterized by n work-storage rationing thresholds corresponding to n demand classes. Without a salvage market value, they showed that the modified work-storage rationing policy is optimal and is determined by the base-stock level.

De Véricourt et al. (2000, 2002) also considered “hedging point” policies and developed further characterization of the optimal switching curve for the backorders case. The backorders case is more difficult to analyze than the lost-sales case when there are multiple demand classes. One of the major reasons is that backorders of the different demand classes increase dimensionality of the system. Therefore, the analysis is more complex. De Véricourt et al. (2000) showed that in a two-part types production system, it is optimal to produce the expensive item if it has the higher backorder cost. De Véricourt et al. (2002) studied a capacitated supply system with multiple demand classes. By decomposing the problem into n -dimensional control problems and $(n-1)$ -dimensional sub-problem, the optimal policy can be characterized simply by fixed threshold values. In the same vein, De Véricourt et al. (2001) evaluated the benefits of different optimal rationing policies: first come first service (FCFS), strict priority policy (SP) and the multilevel rationing policy (ML), and showed that the ML policy performs better than the other two policies. Gayon et al. (2009) characterized the optimal policy for a production-inventory system with multiple customer classes and imperfect advance demand information (ADI). They showed that in lost sales setting the suppliers benefit more from ADI than customers.

Unlike the pure backorder system or pure lost sales system, Benjaafar et al. (2010a) addressed a more general model, taking into account both features lost sales and backorders. Moreover, this paper initiated a study of the structure of the optimal policy in MTS system with both backorders and lost sales. In their case, the backorder and lost sale costs are similarly ordered. Under this assumption, for each class the optimal production/allocation policy can be characterized as a threshold policy. Benjaafar et al. (2010b) studied a production-inventory system with customer impatience. The unsatisfied customer is either lost or backordered. The impatient customers are willing to wait for delivery until the next replenishment which

depends on the exponentially distributed patience times. That means the customers will wait for an amount of time for fulfilling orders; otherwise, they cancel their orders. In particular, this paper showed that optimal policy base-stock level is non-increasing in the upper bound on the number of backorders, while the optimal policy rationing level is non-decreasing in that. In the same vein, Benjaafar and Elhafsi (2012) studied a two-customer class system: patient and impatient customers. The unsatisfied orders from the patient class can be backordered while the unsatisfied orders from the impatient class can be rejected. The optimal policy can be described by two threshold functions where inventory allocation is not static, which depends on the backorder level of the patient customer class.

There are several studies in the literature that consider production-inventory systems with transshipment/inventory sharing. Benjaafar et al. (2004) discussed the problem of inventory rationing in a system with multiple products and multiple production facilities. Zhao et al. (2005) considered a two-location inventory-sharing system. They used a (S,K) policy, namely base-stock and rationing policy in a decentralized setting. Zhao et al. (2008) also considered a two-location system, while the transshipments can happen in both demand arrivals and production completions. They proved that for each location the optimal production policy is a hedging point policy and the optimal demand filling policy is a state-dependent policy.

There is also a stream of literature that considers the stock rationing problem with batch demand. Huang and Iravani (2008) provided a non-unitary demand system and focused on the problem of rationing quantity. They showed that the order size can affect the benefit of the optimal stock rationing policy. Xu et al. (2010) extended the model of Huang and Iravani (2008) to the multiple-class, batch demand system, where the batch demand can be partially accepted. They showed that the optimal policy is characterized by multiple rationing levels. ElHafsi et al. (2010) studied an integrated production inventory system with multiple non-unitary demand classes. It is assumed that both production times and order inter-arrival times follow the Erlang distributions. They showed that the demand size variability can significantly affect the operating cost of the system.

2.2 ATO Systems under Continuous Review

In contrast to MTS systems, which keep inventory at the end-product level, ATO systems keep inventory at the component level. When the customer order is received, the components can be assembled immediately and delivered to the customer. To our knowledge, most papers

address continuous review models and develop heuristic policies to evaluate or optimize the decisions. In this stream, Song (1998) studied the performance measures for a base-stock system with Poisson demand and constant replenishment leadtimes. She showed that in a multi-item inventory model the order fill rate can be obtained by a series convolution of the batch size distribution and Poisson distribution. Xu (1999; 2001, Chapter 11) studied the effect of arrival correlation on the performance of the ATO system, and discussed how the system responds to different arrival correlations. Gallien and Wein (2001) considered ATO systems with a single-item MTS environment. But in their model, the setting is based on a Poisson demand and an arbitrary distributed processing times. Associated with non-identically stochastic lead times and infinite capacity, the authors developed a simple and effective control policy for an ATO system. That is the structure of the optimal policy is entirely determined by the longest procurement delay and its differences with the other procurement delays. Similar system studied by Song and Yao (2002), who proposed upper and lower bounds for the backorders in a single product case. They showed that it is optimal to keep higher base-stock levels for components with longer mean lead times (and lower unit costs). Lu and Song (2005) formulated an unconstrained cost-minimization model in multiple-product assembly system with order-based backorder costs. They developed an algorithm to approximate the optimal base-stock level. Under the assumption that demands follow a batch Poisson process, Lu et al. (2005) focused on the expected backorder for each product. They solved the optimization problem by minimizing a weighted average of backorders over all products. Later, Lu (2006) extended the model (Lu et al., 2005) to multi-product, multi-component ATO system with general random batch demands. He focused on the average backorder of the system, based on which he developed a new methodology for performance analysis of the system. Zhao (2009) also considered a multi-product and multi-component ATO system with batch ordering. He analyzed and evaluated the impact of the split orders/non-split orders on system performance. Hoen et al. (2011) studied a multiple end-products system with lost sales and deterministic leadtimes. They devised an approximate method for estimating the order fill rates.

Another line of research on ATO systems is base-stock policies with fixed base-stock levels. Song et al. (1999) studied the impact of limited capacity on ATO systems and evaluated performance of base stock policies with stochastic leadtimes. They showed that exact performance measures are a result of multidimensional Markov chains. Glasserman and Wang (1998) considered a system consisting of multiple types of demand, which take place

according to batch Poisson processes. The inventory of each component is controlled by a base-stock policy, and the replenishment leadtimes are i.i.d. (independent and identically distributed) random variables. They focused on a target fill rate and studied the trade-off between inventory levels and delivery leadtimes. Dayanik et al. (2003) considered an ATO system consisting of multiple components and multiple products. They developed lower bounds to estimate the order fill rates for the system. Plambeck and Ward (2007) introduced a separation principle for a class of ATO systems with expediting. They demonstrated that the multidimensional assembly control problem can be separated into a series of single-item inventory control problems. Ko et al. (2011) studied a single product, multiple-component production system under a base-stock policy. They provided explicit approximations of the lead times distributions, from which the base-stock levels can be calculated.

For a more general multi-product ATO system: non-identical production system, where the products differ in characteristics. Lu et al. (2010) focused on the W-, N-, M-system and assumed identical component leadtimes. They used a stochastic program to obtain optimal inventory strategy. Dogru et al. (2010) discussed W-system with identical component lead times and proposed a simple priority allocation policy. Under this environment, Lu et.al. (2012) studied an ATO N-system with non-identical leadtimes. This is the special case of W-system. Under the symmetric structure, the optimal component allocation decision is a no-holdback (NHB) rule and the optimal production decision is a coordinated base-stock (CBS) rule; under the asymmetric structure, the optimal policies depend on the effect of cost asymmetry. Reiman and Wang (2012) considered the model of Dogru et al. (2010), while with non-identical lead times. They developed a multi-stage stochastic program and established a lower bound on the inventory cost of the system, based on which they also discussed the replenishment policy and the allocation policy.

2.3 ATO Systems under Periodic Review

For periodic review models, faced with multi-customer classes and integer-valued correlated random variable in each period, a static threshold production/allocation policy has been considered. In general, as mentioned in Chiang (2003) "...earlier periodic review models, however, have focused on the situation which supply lead times are a multiple of a review period. Such models could be regarded as an approximation of continuous review models, as the review periods can be modeled as small". The earlier literature about this related problem is studied by Cohen et al.(1988) who investigated two demand classes (the emergency

demand and the normal demand) inventory system in a (s,S) policy with lost sales. Assuming the emergency demand has higher priority than the normal demand, the former can be satisfied first. In this setting, a strict priority rule is used for the allocation policy. Furthermore, authors developed and evaluated an efficient and effective solution heuristic for solving the service-constrained optimization problem. Later, Rosling (1989) considered an infinite horizon model with random demands multi-echelon inventory system. Under an assumption of zero setup cost, he showed that a balanced base-stock policy is optimal for multistage assembly systems. Chen (2000) extended this equivalence between assembly and serial systems to the batch-ordering case. He showed that the batch ordering policy is an optimal policy for multi-echelon systems. Cheng et al. (2000) considered a configure-to-order (CTO) system. Such a system takes the ATO concept one step further, and then the customers can select the personalized set of components that assembled to the end product. They used a lower bound on the order fill rate of each product to investigate the optimal inventory-service tradeoff. Karaarslan et al. (2013) considered a single item, two-component system with backorders under two different policies: a pure base-stock policy and a balanced base-stock policy. They showed that the balanced base-stock policy works better under low service levels, low holding cost ratio, and high demand uncertainty. Otherwise, the pure base-stock policy performs well.

Turning to the study of single-period ATO systems, Fu et al. (2006) analyzed the policy of pre-stocking components and a single product ATO system with uncertain demand and limited assembly capacity. They examined the effect of varying component leadtimes on the available assembly capacity. Xiao et al. (2010a) also considered the similar system as Fu et al. (2006), but focused on an ATO system with both uncertain demand and uncertain assembly capacity. By considering assembly-in-advance operations, the authors adopted a profit-maximization model and investigated the optimal production and inventory decisions. Xiao et al. (2010b) extended their study to a two-product production system with two types of uncertain demand. They studied the impact of the uncertain demand patterns on the optimal stocking and allocating decisions.

We found also several review articles on different inventory allocation policies. See, for example: Zhang (1997) considered an assembly system with multiple productions and dependent demands. He proposed a fixed-priority policy with stock commitment for allocating component stocks. Hausman et al. (1998) studied joint demand fulfillment probability in a multi-item inventory system under FCFS policy. Agrawal and Cohen (2001)

studied assembly type production systems with demand uncertainty under a fair-share policy. Several results are used to determine the optimal base-stock policy. Akçay and Xu (2004) considered a multi-component and multi-product ATO system under the independent order-based policy. Duran et al. (2007) focused on limited production capacity inventory system with multi-period time horizon. They considered two customer classes differentiated by their priority level, and showed that a (S, R, B) base-stock policy is optimal, where S is the order-up-to quantity, R is the reserve-up-to amount, and B is the backlog-up-to amount. Feng et al. (2010) worked on a multi-item inventory system under the (r, nQ) policy, where r is the recorder point, nQ is the order size. They showed that the joint inventory positions of the system are stationary, independent and uniformly distributed.

2.4 ATO Systems in Continuous Time

In this section we discuss ATO systems that are managed/operated in continuous time. In this case, the problem is formulated using MDP. In this research branch, an initial view of the optimal control of a system with multiple components on multiple production facilities, and multiple demand classes is given by Benjaafar and ElHafsi (2006), who studied an ATO system consisting of n products assembled from a subset of m distinct components. By assuming that the manager is faced with multiple demand classes, they analyze the optimal production and inventory allocation policies of such system. Unlike the fixed allocation policy, the optimal allocation policies are dependent on the on-hand inventory of the other components. The optimal inventory policy for one component can be described as a state-dependent policy that depends on the on-hand inventory levels of all other components. In this article, although the authors considered a special case of an ATO system, it still can be viewed as a classic one. Because also under continuous review, compared with the other literatures (mentioned in section 2.2) which focus on performance evaluation of heuristic policies, this work determines the optimal production policy.

ElHafsi et al. (2008) also considered an n - product and m -component ATO system where the products have a modular nested design. They showed that the optimal production policy is a base-stock policy and the optimal inventory allocation policy is a multi-level rationing policy. Finally, ElHafsi (2009) studied an ATO system subject to non-unitary multiple demand classes. The author argued that comparing to the effect of order size variability, the optimal average cost rate is more sensitive to the order size. More recently, Benjaafar et al. (2011) discussed a multiple stages, multiple demand classes assembly system with batch production

and batch demand. By considering different items can be produced in different batch sizes, they characterized the optimal policy. In particular, the optimal production quantity for each component and the optimal number of satisfied requirements for each demand class would also be determined.

In this work, we study a continuous time ATO system, and share several common features in the above literature of ATO system under continuous time. That is, the assumption of exponential production time and Poisson process demand. The current work is most closely related to that of Benjaafar and ElHafsi (2006). Similar to their model, we study an ATO system with a single product assembled from n components. However, in our system we consider two types of demand: demand of the assembled product and demand of the individual components, and discuss the demands with the pure lost sales (see Li, 2013a) and the mixed lost sales and backorders (see Li, 2013b). Based on which, we study the optimal control policy.

2.5 Conclusion

We provided a brief overview of the literature involved in this work, which can be classified into four categories: make-to-stock systems under MDP, ATO systems under continuous review, ATO systems under periodic review, and ATO systems in continuous time. Our work is more related to the literature on ATO systems in continuous time. In this line of research, we study the model that deals with two types of demand ATO system: the assembled product demand and the individual components demand. To our knowledge, there is little literature in this area. We hope that the research presented in this work also enriches the current literature on ATO systems with individual components demand.

Chapter 3. ATO System with Individual Components Demand: Lost Sales for Components and Assembled Product

We consider the demand during shortage period as completely lost. In this chapter, we assume that if demand cannot be fulfilled immediately it is lost for both the assembled product as well as the individual components. We propose a dynamic programming model in order to determine the optimal control policy of such a system. The system deals with a single product which is assembled from multiple components. The system faces demand not only from the assembled product but also from the individual components. To determine the optimal operating policy, we formulate the problem using an MDP methodology and using two optimality criteria: discounted cost and average cost per period. Furthermore, we determine the structure of the optimal policy and investigate the impact of different parameters on the optimal policy. We are also interested in the effect of system parameters on the optimal policy. We test the system with a wide range of system parameters and show that the optimal base-stock and rationing levels are sensitive to system parameters

The outline of this chapter is as follows: we start with an introduction in section 3.1. In section 3.2, we formulate the optimal control problem with lost sales and characterize its optimal policy under the discounted cost case. Then we extend our model to the average cost case. In section 3.3, we adopt the average cost criterion in numerical experiments, and use the iteration algorithm to calculate the optimal policy. Based on numerical results, we analyze the structure of the optimal policy. Finally, we conclude this chapter in section 3.4.

3.1 Introduction

This chapter considers an ATO system with a single product assembled from n components. Each component is produced ahead of demand one unit at a time on an independent production facility. Unit held in inventory incurs a unit holding cost. Demand from both the product and the components can be either satisfied or rejected. Demand for the assembled product can be satisfied only if all components are in stock. We assume that demand for the assembled product has a higher priority over demand for the individual components. In this situation, a system manager may need to reject a demand from a component and save the inventory for future assembled product demand. At the same time, a system manager needs to decide when to produce a certain component and when not to produce it. We also assume exponentially distributed production times, and demands arrive to the system according to independent Poisson processes. We assume that the assembly time is instantaneous and there are no setup costs and setup times for production. Our assumption of negligible assembly time is supported by most of the literature on ATO systems.

A system manager must make two types of decisions: one regarding component production and the other regarding inventory allocation. The objective is to determine the optimal control policy that minimizes the expected operating costs of the system. In general, this kind of problem can be regarded as a dynamic problem and a decision support tool is needed. In our study, the problem can be formulated as an MDP resulting in Markovian policies. We show that the optimal production policy is characterized by state-dependent base-stock levels. That is, a component is produced when the on-hand inventory is below the base-stock level, and not be produced otherwise. Moreover, the base-stock level is non-decreasing in the on-hand inventory level of other components. We show that the optimal inventory allocation policy is a rationing policy. An order from a component is satisfied only if its on-hand inventory level is above a certain rationing level. The rationing level for each component is also non-decreasing in the on-hand inventory levels of other components. This is an interesting property since both the base-stock and the rationing levels are non-decreasing in the on-hand inventory level of other components. This result is quite different from the ones in (Benjaafar and ElHafsi 2006, ElHafsi et al. 2008). In our system we discuss two types of demand: demand of the assembled product and demand of the individual components. Benjaafar and ElHafsi 2006 studied a single product ATO system, ElHafsi et al. 2008 studied a nested-multiple-product ATO system, where they consider one type of demand that can only be from

the end product, and focus on different demand classes. However, in our system since demand of the assembled product has a higher priority over demand of individual components, it is always satisfied if all components are in stock. We also assume that demands of the individual components are independent of each other. When the system jointly increases the on-hand inventory of the components, more demands from the assembled product are desired to be satisfied that means several individual components demands must be rejected. Hence, the rationing level of the individual components increases. This implies that it becomes more difficult to satisfy demand of the individual components when the on-hand inventory of components increases. Using a two-component example, we show that the optimal base-stock and rationing levels are sensitive to system parameters, such as holding cost, lost sales cost, production rate and arrival rate. Based on the numerical results, we show the impact of these parameters on the optimal policy and then present some properties in the general case.

3.2 The Optimal Control Problem

3.2.1 Model Formulation

We consider a system with a single product (for which we use the index 0) assembled from n components. Production times are exponentially distributed with mean $1/\mu_k$ for Component $k=1, \dots, n$. Demand for Component k and the assembled product takes place continuously over time according to an independent Poisson process with rate λ_k and λ_0 , respectively. Let c_i ($i=1, \dots, n$) be the lost sale cost for Component i demand. We assume that the lost sale cost for the assembled product is such that $c_0 > c_1 + \dots + c_n$. This condition states that the product demand has priority over the component demand. We define the state of the system, at time t , by the vector $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$, where $X_k(t)$, $k=1, \dots, n$, is a nonnegative integer denoting the on-hand inventory for Component k at time t . The system produces components ahead of demand. Let $h(\mathbf{X}(t)) = \sum_{k=1}^n h_k(X_k(t))$ be the holding cost, where $h_k(\cdot)$ is increasing convex. Under a policy π for each state $\mathbf{x} = (x_1, \dots, x_n)$, the manager needs to decide which components to produce, and how to allocate inventory between components and the assembled product. Because the system is memoryless, these actions taken at a particular decision epoch depend only on the current state of the system. Decisions can be described as two types: production and inventory allocation, which are made simultaneously. In each state, production decision concerns which components to be produced, and inventory allocation

decision concerns which demand should be satisfied immediately or rejected to reserve inventory for future demand of the assembled product.

3.2.2 The Case of Discounted Cost

We formulate the problem as an MDP. The expected discounted cost over an infinite planning horizon obtained under a policy π with a starting state $\mathbf{x} = (x_1, \dots, x_n)$ can be defined as:

$$v^\pi(\mathbf{x}) = E_{\mathbf{x}}^\pi \left[\sum_{k=1}^n \int_0^\infty e^{-\alpha t} h_k(X_k(t)) dt + \sum_{l=0}^n \int_0^\infty e^{-\alpha t} c_l dN_l(t) \right], \quad (3.1)$$

where $\alpha > 0$ is the discount rate, $N_0(t)$ is the number of units of the assembled product demand that have not been satisfied and $N_i(t)$ is the number of units of Component i demand that have not been satisfied up to time t . Our aim is to seek the optimal policy π^* that minimizes the expected discounted cost for all \mathbf{x} . Let v^* denote the optimal cost function (i.e., $v^* \equiv v^{\pi^*}$).

Following Lippman (1975), we define the uniform rate $\beta = \sum_{l=0}^n \lambda_l + \sum_{k=1}^n \mu_k$ and let $\alpha + \beta = 1$.

The dynamic programming equation can be written as:

$$v^*(\mathbf{x}) = h(\mathbf{x}) + \lambda_0 T^0 v^*(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}), \quad (3.2)$$

where the operators T^0 , T^k and T_k , $k=1, \dots, n$ are defined as follows:

$$T^0 v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ \min \{v(\mathbf{x} - \mathbf{e}), v(\mathbf{x}) + c_0\} & \text{otherwise,} \end{cases} \quad (3.3)$$

$$T^k v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_k & \text{if } x_k = 0, \\ \min \{v(\mathbf{x} - \mathbf{e}_k), v(\mathbf{x}) + c_k\} & \text{otherwise,} \end{cases} \quad (3.4)$$

$$T_k v(\mathbf{x}) = \min \{v(\mathbf{x} + \mathbf{e}_k), v(\mathbf{x})\}. \quad (3.5)$$

In our model, \mathbf{e}_k is the k th unit vector of dimension n and $\mathbf{e} = \sum_{k=1}^n \mathbf{e}_k = (1, 1, \dots, 1)$ is an n -dimensional vector of ones. Operator T^0 corresponds to the inventory allocation decision for demand of the assembled product, similarly, operator T^k corresponds to the inventory allocation decision for demand of Component k , operator T_k corresponds the production decisions for Component k .

In Theorem 3.1, we characterize the structure of the optimal policy.

For the further proving we define the following operators

$$\Delta_i v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_i) - v(\mathbf{x}),$$

$$\Delta_{i,i} v(\mathbf{x}) = \Delta_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_i v(\mathbf{x}),$$

$$\Delta_{i,j} v(\mathbf{x}) = \Delta_j v(\mathbf{x} + \mathbf{e}_i) - \Delta_j v(\mathbf{x}), \text{ and } \Delta_{i,j} v(\mathbf{x}) = \Delta_{j,i} v(\mathbf{x}),$$

$$\Delta_{i_1+i_2+\dots+i_k} v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_k}) - v(\mathbf{x}),$$

$$\Delta_{\mathbf{e}} v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}) - v(\mathbf{x}).$$

Definition 3.1. Let \mathcal{V} be the set of functions on Z^{+n} , where Z^+ is the set of non-negative integers, such that if $v \in \mathcal{V}$, we have

$$\text{A1: } \Delta_{i,i} v(\mathbf{x}) \geq 0, \text{ for all } i,$$

$$\text{A2: } \Delta_{i,j} v(\mathbf{x}) \leq 0, \text{ for all } i, j = 1, 2, \dots, n, j \neq i,$$

$$\text{A3: } \Delta_{i+j_1+j_2+\dots+j_p,i} v(\mathbf{x}) \geq 0, \text{ for all } i, j_1, j_2, \dots, j_p \neq i, \text{ and } 1 \leq p \leq n-1,$$

$$\text{A4: } \Delta_{\mathbf{e}} v(\mathbf{x}) \geq -c_0, \text{ for all } \prod_{k=1}^n x_k > 0.$$

Property A1 shows that $\Delta_i v(\mathbf{x})$ is non-decreasing in each of the state variables, and it implies that the function $v(\mathbf{x})$ is convex in x_i . Property A2 shows that $\Delta_i v(\mathbf{x})$ is non-increasing in x_j ($j \neq i$), and it implies that the function $v(\mathbf{x})$ is submodular in the direction (x_i, x_j) . For the definition of submodularity and supermodularity readers can refer to Topkis (1978). Property A3 shows that $\Delta_i v(\mathbf{x})$ is non-decreasing with joint increases in x_i, x_{j_1}, \dots , and x_{j_p} and it implies that the function $v(\mathbf{x})$ is supermodular in the direction $(x_i, x_i + x_{j_1} + \dots + x_{j_p})$. Property A4 implies that it is always optimal to satisfy an order from the assembled product, if all the components are available.

Lemma 3.1. If $v \in \mathcal{V}$, then $Tv \in \mathcal{V}$, where $Tv(\mathbf{x}) = h(\mathbf{x}) + \lambda_0 T^0 v(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v(\mathbf{x})$.

Proof of Lemma 3.1

In order to prove Lemma 3.1, we need to show that if $v \in \mathcal{V}$ then Tv must satisfy A1-A4. In order to do so, we prove that if $v \in \mathcal{V}$ then $T^0v \in \mathcal{V}$, $T_kv \in \mathcal{V}$ and $T^kv \in \mathcal{V}$ for all k . In other words, we show that $T^0v \in \mathcal{V}$, $T_kv \in \mathcal{V}$ and $T^kv \in \mathcal{V}$ satisfy A1-A4.

Operator T_k

$$T_kv(\mathbf{x}) = \min\{v(\mathbf{x} + \mathbf{e}_k), v(\mathbf{x})\} = v(\mathbf{x}) + \min\{\Delta_kv(\mathbf{x}), 0\} \quad (3.6)$$

Property A1

From (3.6) we have the following two cases:

Case 1 if $i=k$,

$$\Delta_{i,i}T_iv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x}) + \min\{\Delta_iv(\mathbf{x} + 2\mathbf{e}_i), 0\} - 2\min\{\Delta_iv(\mathbf{x} + \mathbf{e}_i), 0\} + \min\{\Delta_iv(\mathbf{x}), 0\}$$

by A1 we have

$\Delta_iv(\mathbf{x} + 2\mathbf{e}_i) \geq \Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_iv(\mathbf{x})$, which leads to four possible sub-cases:

1. $\Delta_iv(\mathbf{x} + 2\mathbf{e}_i) \geq \Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_iv(\mathbf{x}) \geq 0 \Rightarrow \Delta_{i,i}T_iv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x}) \geq 0$.
2. $\Delta_iv(\mathbf{x} + 2\mathbf{e}_i) \geq \Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq 0 \geq \Delta_iv(\mathbf{x}) \Rightarrow \Delta_{i,i}T_iv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x}) + \Delta_iv(\mathbf{x}) = \Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq 0$
3. $\Delta_iv(\mathbf{x} + 2\mathbf{e}_i) \geq 0 \geq \Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_iv(\mathbf{x}) \Rightarrow$

$$\Delta_{i,i}T_iv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x}) - 2\Delta_iv(\mathbf{x} + \mathbf{e}_i) + \Delta_iv(\mathbf{x}) = -\Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq 0.$$

4. $0 \geq \Delta_iv(\mathbf{x} + 2\mathbf{e}_i) \geq \Delta_iv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_iv(\mathbf{x}) \Rightarrow$

$$\Delta_{i,i}T_iv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x}) + \Delta_iv(\mathbf{x} + 2\mathbf{e}_i) - 2\Delta_iv(\mathbf{x} + \mathbf{e}_i) + \Delta_iv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x} + \mathbf{e}_i) \geq 0.$$

Case 2 if $i \neq k$,

$$\Delta_{i,i}T_kv(\mathbf{x}) = \Delta_{i,i}v(\mathbf{x}) + \min\{\Delta_kv(\mathbf{x} + 2\mathbf{e}_i), 0\} - 2\min\{\Delta_kv(\mathbf{x} + \mathbf{e}_i), 0\} + \min\{\Delta_kv(\mathbf{x}), 0\},$$

by A2 we have

$\Delta_kv(\mathbf{x} + 2\mathbf{e}_i) \leq \Delta_kv(\mathbf{x} + \mathbf{e}_i) \leq \Delta_kv(\mathbf{x})$, leading to three possible sub-cases:

1. $\Delta_kv(\mathbf{x} + 2\mathbf{e}_i) \leq \Delta_kv(\mathbf{x} + \mathbf{e}_i) \leq \Delta_kv(\mathbf{x}) \leq 0 \Rightarrow$

$$\begin{aligned}
 \Delta_{i,i} T_k v(\mathbf{x}) &= \Delta_{i,i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) - 2\Delta_k v(\mathbf{x} + \mathbf{e}_i) + \Delta_k v(\mathbf{x}) \\
 &= v(\mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_k) - 2v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k) + v(\mathbf{x} + \mathbf{e}_k) \\
 &= \Delta_{i,i} v(\mathbf{x} + \mathbf{e}_k) \geq 0.
 \end{aligned}$$

$$2. \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) \leq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \leq 0 \leq \Delta_k v(\mathbf{x}) \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,i} T_k v(\mathbf{x}) &= \Delta_{i,i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) - 2\Delta_k v(\mathbf{x} + \mathbf{e}_i) \\
 &= \Delta_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_i v(\mathbf{x}) + \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) - 2\Delta_k v(\mathbf{x} + \mathbf{e}_i) \\
 &= \Delta_{i+k} v(\mathbf{x} + \mathbf{e}_i) - \Delta_i v(\mathbf{x}) - 2\Delta_k v(\mathbf{x} + \mathbf{e}_i) \\
 &= \Delta_{i+k,i} v(\mathbf{x}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq 0.
 \end{aligned}$$

$$3. \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) \leq 0 \leq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x}) \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,i} T_k v(\mathbf{x}) &= \Delta_{i,i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) \\
 &= \Delta_{i+k,i} v(\mathbf{x}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) + 2\Delta_k v(\mathbf{x} + \mathbf{e}_i) \\
 &= \Delta_{i+k,i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq 0.
 \end{aligned}$$

$$4. 0 \leq \Delta_k v(\mathbf{x} + 2\mathbf{e}_i) \leq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x}) \Rightarrow \Delta_{i,i} T_k v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x}) \geq 0.$$

Hence, $T_k v$ satisfies A1.

Property A2

From (3.6) we have the following two cases:

Case 1 if $i = k, j \neq k$ ($j = k, i \neq k$),

$$\begin{aligned}
 T_i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x}) + \min\{\Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), 0\} - \min\{\Delta_i v(\mathbf{x} + \mathbf{e}_j), 0\} \\
 &\quad - \min\{\Delta_i v(\mathbf{x} + \mathbf{e}_i), 0\} + \min\{\Delta_i v(\mathbf{x}), 0\}.
 \end{aligned}$$

By A1-A3 we have:

$\Delta_i v(\mathbf{x} + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x}) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i)$ there are five possible sub-cases can be considered:

$$1. \Delta_i v(\mathbf{x} + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x}) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \leq 0 \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,j} T_i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x}) + \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - \Delta_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_i v(\mathbf{x} + \mathbf{e}_j) + \Delta_i v(\mathbf{x}) \\
 &= \Delta_{i,j} v(\mathbf{x}) + \Delta_{i,j} v(\mathbf{x} + \mathbf{e}_i) - \Delta_{i,j} v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x} + \mathbf{e}_i) \leq 0.
 \end{aligned}$$

$$2. \Delta_i v(\mathbf{x} + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x}) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq 0 \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\Delta_{i,j}T_i v(\mathbf{x}) = \Delta_{i,j}v(\mathbf{x}) + \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - \Delta_i v(\mathbf{x} + \mathbf{e}_j) + \Delta_i v(\mathbf{x}) = \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq 0.$$

$$3. \Delta_i v(\mathbf{x} + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x}) \leq 0 \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\Delta_{i,j}T_i v(\mathbf{x}) = \Delta_{i,j}v(\mathbf{x}) - \Delta_i v(\mathbf{x} + \mathbf{e}_j) + \Delta_i v(\mathbf{x}) = 0$$

$$4. \Delta_i v(\mathbf{x} + \mathbf{e}_j) \leq 0 \leq \Delta_i v(\mathbf{x}) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\Delta_{i,j}T_i v(\mathbf{x}) = \Delta_{i,j}v(\mathbf{x}) - \Delta_i v(\mathbf{x} + \mathbf{e}_j) = -\Delta_i v(\mathbf{x}) \leq 0.$$

$$5. 0 \leq \Delta_i v(\mathbf{x} + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x}) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \Rightarrow \Delta_{i,j}T_i v(\mathbf{x}) = \Delta_{i,j}v(\mathbf{x}) \leq 0.$$

Case 2 if $i \neq k, j \neq k$,

$$\begin{aligned} \Delta_{i,j}T_k v(\mathbf{x}) &= \Delta_{i,j}v(\mathbf{x}) + \min\{\Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), 0\} - \min\{\Delta_k v(\mathbf{x} + \mathbf{e}_j), 0\} \\ &\quad - \min\{\Delta_k v(\mathbf{x} + \mathbf{e}_i), 0\} + \min\{\Delta_k v(\mathbf{x}), 0\} \end{aligned}$$

by A2 we have

$$\Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \text{ or}$$

$$\Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j).$$

If $\Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j)$, under this condition, we have the following five cases:

$$1. \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \geq 0 \Rightarrow \Delta_{i,j}T_k v(\mathbf{x}) = \Delta_{i,j}v(\mathbf{x}) \leq 0.$$

$$2. \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \Rightarrow$$

$$\Delta_{i,j}T_k v(\mathbf{x}) = \Delta_{i,j}v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \leq 0.$$

$$3. \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \Rightarrow$$

$$\begin{aligned} \Delta_{i,j}T_k v(\mathbf{x}) &= \Delta_{i,j}v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - \Delta_k v(\mathbf{x} + \mathbf{e}_j) \\ &= \Delta_{i,j}v(\mathbf{x}) + \Delta_{i,k}v(\mathbf{x} + \mathbf{e}_j) \leq 0. \end{aligned}$$

$$4. \Delta_k v(\mathbf{x}) \geq 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,j}T_kv(\mathbf{x}) &= \Delta_{i,j}v(\mathbf{x}) + \Delta_kv(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - \Delta_kv(\mathbf{x} + \mathbf{e}_j) - \Delta_kv(\mathbf{x} + \mathbf{e}_i) \\
 &= v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) - v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j) - v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i) + v(\mathbf{x}) \\
 &= \Delta_jv(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i) - \Delta_jv(\mathbf{x} + \mathbf{e}_k) - \Delta_kv(\mathbf{x}) \\
 &= \Delta_{i,j}v(\mathbf{x} + \mathbf{e}_k) - \Delta_kv(\mathbf{x}) \leq 0.
 \end{aligned}$$

$$0 \geq \Delta_kv(\mathbf{x}) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_j) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,j}T_kv(\mathbf{x}) &= \Delta_{i,j}v(\mathbf{x}) + \Delta_kv(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - \Delta_kv(\mathbf{x} + \mathbf{e}_j) - \Delta_kv(\mathbf{x} + \mathbf{e}_i) + \Delta_kv(\mathbf{x}) \\
 &= \Delta_{i,j}v(\mathbf{x} + \mathbf{e}_k) \leq 0
 \end{aligned}$$

If $\Delta_kv(\mathbf{x}) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_j) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j)$, under this condition, we also consider the other case:

$$5. \quad \Delta_kv(\mathbf{x}) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_j) \geq 0 \geq \Delta_kv(\mathbf{x} + \mathbf{e}_i) \geq \Delta_kv(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,j}T_kv(\mathbf{x}) &= \Delta_{i,j}v(\mathbf{x}) + \Delta_kv(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - \Delta_kv(\mathbf{x} + \mathbf{e}_i) \\
 &= \Delta_{i,j}v(\mathbf{x}) + \Delta_{j,k}v(\mathbf{x} + \mathbf{e}_i) \leq 0.
 \end{aligned}$$

Hence, T_kv satisfies A2.

Property A3

In order to prove the operator $T_kv(x)$ satisfies A3, we need to show that $\Delta_{e,i}T_kv(\mathbf{x}) \geq 0$ which from the following observation (Ref.2. Observation 1):

Observation1. if $v \in \mathcal{V}$ and $\Delta_{e,i}v(\mathbf{x}) \geq 0$ then $\Delta_{i+j_1+\dots+j_p,i}v(\mathbf{x}) \geq 0$ for all $j_1, j_2, \dots, j_p \neq i$, and $1 \leq p \leq n-1$.

Proof: Using property A2, we have

$$\Delta_{i+j_1+\dots+j_p,i}v(\mathbf{x}) = \Delta_iv(\mathbf{x} + \mathbf{e} + \mathbf{e}_i + \mathbf{e}_{j_1} + \dots + \mathbf{e}_{j_p}) - \Delta_iv(\mathbf{x}) \geq \Delta_iv(\mathbf{x} + \mathbf{e}) - \Delta_iv(\mathbf{x}).$$

Then if $\Delta_{e,i}v(\mathbf{x}) = \Delta_iv(\mathbf{x} + \mathbf{e}) - \Delta_iv(\mathbf{x}) \geq 0$, we also have $\Delta_{i+j_1+\dots+j_p,i}v(\mathbf{x}) \geq 0$.

In order to show that $\Delta_{i+j_1+\dots+j_p,i}v(\mathbf{x}) \geq 0$, we use Observation 1 and show that $\Delta_{e,i}v(x) \geq 0$, directly.

From (3.6) we have the following two cases:

Case 1 if $i=k$,

$$\begin{aligned}\Delta_{e,i}T_i v(\mathbf{x}) &= \Delta_{e,i}v(\mathbf{x}) + \min\{\Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}), 0\} - \min\{\Delta_i v(\mathbf{x} + \mathbf{e}), 0\} \\ &\quad - \min\{\Delta_i v(\mathbf{x} + \mathbf{e}_i), 0\} + \min\{\Delta_i v(\mathbf{x}), 0\}.\end{aligned}$$

By A2 and A3 we have

$$\begin{aligned}\Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) &\geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x}) \text{ or} \\ \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) &\geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x}).\end{aligned}$$

If $\Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x})$, there are five possible sub-cases:

$$1. \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x}) \geq 0 \Rightarrow \Delta_{e,i}T_i v(\mathbf{x}) = \Delta_{e,i}v(\mathbf{x}) \geq 0.$$

$$2. \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq 0 \geq \Delta_i v(\mathbf{x}) \Rightarrow$$

$$\Delta_{e,i}T_i v(\mathbf{x}) = \Delta_{e,i}v(\mathbf{x}) + \Delta_i v(\mathbf{x}) = \Delta_i v(\mathbf{x} + \mathbf{e}) \geq 0.$$

$$3. \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x}) \Rightarrow$$

$$\Delta_{e,i}T_i v(\mathbf{x}) = \Delta_{e,i}v(\mathbf{x}) - \Delta_i v(\mathbf{x} + \mathbf{e}) + \Delta_i v(\mathbf{x}) = 0.$$

$$4. \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x}) \Rightarrow$$

$$\Delta_{e,i}T_i v(\mathbf{x}) = \Delta_{e,i}v(\mathbf{x}) - \Delta_i v(\mathbf{x} + \mathbf{e}) - \Delta_i v(\mathbf{x} + \mathbf{e}_i) + \Delta_i v(\mathbf{x}) = -\Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq 0.$$

$$5. 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x}) \Rightarrow$$

$$\begin{aligned}\Delta_{e,i}T_i v(\mathbf{x}) &= \Delta_{e,i}v(\mathbf{x}) + \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) - \Delta_i v(\mathbf{x} + \mathbf{e}) - \Delta_i v(\mathbf{x} + \mathbf{e}_i) + \Delta_i v(\mathbf{x}) \\ &= \Delta_{e,i}v(\mathbf{x}) + \Delta_{e,i}v(\mathbf{x} + \mathbf{e}_i) - \Delta_{e,i}v(\mathbf{x}) = \Delta_{e,i}v(\mathbf{x} + \mathbf{e}_i) \geq 0.\end{aligned}$$

If $\Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x})$, under this condition, we also consider the other case:

$$6. \Delta_i v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}) \geq 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x}) \Rightarrow$$

$$\begin{aligned}\Delta_{e,i}T_i v(\mathbf{x}) &= \Delta_{e,i}v(\mathbf{x}) - \Delta_i v(\mathbf{x} + \mathbf{e}_i) + \Delta_i v(\mathbf{x}) \\ &= \Delta_i v(\mathbf{x} + \mathbf{e}) - \Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq 0.\end{aligned}$$

Case 2: if $i \neq k$,

$$\begin{aligned}\Delta_{e,i}T_k v(\mathbf{x}) &= \Delta_{e,i}v(\mathbf{x}) + \min\{\Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}), 0\} - \min\{\Delta_k v(\mathbf{x} + \mathbf{e}), 0\} \\ &\quad - \min\{\Delta_k v(\mathbf{x} + \mathbf{e}_i), 0\} + \min\{\Delta_k v(\mathbf{x}), 0\}.\end{aligned}$$

By A2 and A3 we have

$$\Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \text{ or}$$

$$\Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i)$$

If $\Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i)$, under this condition, we have the following five cases:

$$1. \Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq 0 \Rightarrow \Delta_{\mathbf{e},i} T_k v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x}) \geq 0.$$

$$2. \Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \Rightarrow \Delta_{\mathbf{e},i} T_k v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) \geq 0.$$

$$3. \Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq 0 \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\Delta_{\mathbf{e},i} T_k v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) + \Delta_k v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x}) - \Delta_{k,i} v(\mathbf{x}) \geq 0.$$

$$4. \Delta_k v(\mathbf{x} + \mathbf{e}) \geq 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\begin{aligned} \Delta_{\mathbf{e},i} T_k v(\mathbf{x}) &= \Delta_{\mathbf{e},i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) + \Delta_k v(\mathbf{x}) \\ &= v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) - v(\mathbf{x} + \mathbf{e}) - v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k) + v(\mathbf{x} + \mathbf{e}_k) \\ &= \Delta_{\mathbf{e}} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i) - \Delta_{\mathbf{e}} v(\mathbf{x} + \mathbf{e}_k) + \Delta_k v(\mathbf{x} + \mathbf{e}) \\ &= \Delta_{\mathbf{e},i} v(\mathbf{x} + \mathbf{e}_k) + \Delta_k v(\mathbf{x} + \mathbf{e}) \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\Delta_{\mathbf{e},i} T_k v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) - \Delta_k v(\mathbf{x} + \mathbf{e}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) + \Delta_k v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x} + \mathbf{e}_k) \geq 0.$$

If $\Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i)$, under this condition, we also consider the other case:

$$6. \Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq 0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) \geq \Delta_k v(\mathbf{x} + \mathbf{e}_i) \Rightarrow$$

$$\begin{aligned} \Delta_{\mathbf{e},i} T_k v(\mathbf{x}) &= \Delta_{\mathbf{e},i} v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}) - \Delta_k v(\mathbf{x} + \mathbf{e}_i) \\ &= \Delta_{\mathbf{e},i} v(\mathbf{x}) + \Delta_{\mathbf{e},k} v(\mathbf{x} + \mathbf{e}_i) \geq 0. \end{aligned}$$

Hence, $T_k v$ satisfies A3.

Property A4

$$\begin{aligned} \Delta_{\mathbf{e}} T_k v(\mathbf{x}) &= \min\{v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}), v(\mathbf{x} + \mathbf{e})\} - \min\{v(\mathbf{x} + \mathbf{e}_k), v(\mathbf{x})\} \\ &= \Delta_{\mathbf{e}} v(\mathbf{x}) + \min\{\Delta_k v(\mathbf{x} + \mathbf{e}), 0\} - \min\{\Delta_k v(\mathbf{x}), 0\} \end{aligned}$$

By A3 we have $\Delta_{i+j_1+\dots+j_p,i} v(\mathbf{x}) = \Delta_{\mathbf{e},i} v(\mathbf{x}) \geq 0$ for $p=n-1$.

So $\Delta_{e,k}v(\mathbf{x}) = \Delta_k v(\mathbf{x} + \mathbf{e}) - \Delta_k v(\mathbf{x}) \geq 0$, then following the three cases:

Case1. $\Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \geq 0 \Rightarrow \Delta_e T_k v(\mathbf{x}) = \Delta_e v(\mathbf{x}) \geq -c_0$.

Case2. $\Delta_k v(\mathbf{x} + \mathbf{e}) \geq 0 \geq \Delta_k v(\mathbf{x}) \Rightarrow \Delta_e T_k v(\mathbf{x}) = \Delta_e v(\mathbf{x}) - \Delta_k v(\mathbf{x}) \geq \Delta_e v(\mathbf{x})$.

Case3. $0 \geq \Delta_k v(\mathbf{x} + \mathbf{e}) \geq \Delta_k v(\mathbf{x}) \Rightarrow \Delta_e T_k v(\mathbf{x}) = \Delta_e v(\mathbf{x}) + \Delta_k v(\mathbf{x} + \mathbf{e}) - \Delta_k v(\mathbf{x})$
 $= \Delta_e v(\mathbf{x}) + \Delta_{e,k} v(\mathbf{x}) \geq \Delta_e v(\mathbf{x}) \geq -c_0$.

Hence, $T_k v$ satisfies A4.

Operator T^0

$$T^0 v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}), v(\mathbf{x}) + c_0\} & \text{otherwise,} \end{cases}$$

$$= \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ v(\mathbf{x} - \mathbf{e}) + \min\{0, \Delta_e v(\mathbf{x}) + c_0\} & \text{otherwise.} \end{cases}$$

By A4, already $T^0 v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ v(\mathbf{x} - \mathbf{e}) & \text{otherwise,} \end{cases}$ hence it satisfies all 4 properties.

Operator T^k

$$T^k v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_k & \text{if } x_k = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}_k), v(\mathbf{x}) + c_k\} & \text{otherwise,} \end{cases}$$

$$= \begin{cases} v(\mathbf{x}) + c_k & \text{if } x_k = 0 \\ v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\} & \text{otherwise,} \end{cases} \quad (3.7)$$

Property A1

From (3.7) we have the following three cases:

Case 1: if $x_k = 0$,

$$\Delta_{i,i} T^k v(x) = \Delta_{i,i} v(x) \geq 0.$$

Case 2: if $x_k > 0, i = k$,

$$T^i v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_i) + \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i\},$$

$$\Delta_{i,i} T^i v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_i) + \min\{0, \Delta_i v(\mathbf{x} + \mathbf{e}_i) + c_i\} - 2 \min\{0, \Delta_i v(\mathbf{x}) + c_i\} + \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i\}.$$

By A1 we have $\Delta_i v(\mathbf{x} + \mathbf{e}_i) \geq \Delta_i v(\mathbf{x}) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i)$, leading to four possible sub-cases:

$$1. \quad \Delta_i v(\mathbf{x} + \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq 0 \Rightarrow \Delta_{i,i} T^i v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_i) \geq 0.$$

$$2. \quad \Delta_i v(\mathbf{x} + \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\Delta_{i,i} T^i v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i = \Delta_i v(\mathbf{x}) + c_i \geq 0.$$

$$3. \quad \Delta_i v(\mathbf{x} + \mathbf{e}_i) + c_i \geq 0 \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\Delta_{i,i} T^i v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_i) - 2\Delta_i v(\mathbf{x}) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) - c_i = -\Delta_i v(\mathbf{x}) - c_i \geq 0.$$

$$4. \quad 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\Delta_{i,i} T^i v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} + \mathbf{e}_i) - 2\Delta_i v(\mathbf{x}) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) = \Delta_i v(\mathbf{x} + \mathbf{e}_i) - \Delta_i v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x}) \geq 0.$$

Case 3: if $x_k > 0, i \neq k$,

$$T^k v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\begin{aligned} \Delta_{i,i} T^k v(\mathbf{x}) &= \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) + c_k\} \\ &\quad - 2 \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k\} + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\}. \end{aligned}$$

By A2 we have $\Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k)$, leading to four possible sub-cases:

$$1. \quad \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq 0 \Rightarrow$$

$$\Delta_{i,i} T^k v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) - 2\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + \Delta_k v(\mathbf{x} - \mathbf{e}_k) = \Delta_{i,i} v(\mathbf{x}) \geq 0.$$

$$2. \quad \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_{i,i} T^k v(\mathbf{x}) &= \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) - 2\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \\ &= \Delta_i v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - \Delta_i v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) - 2\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \\ &= \Delta_{i+k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - \Delta_i v(\mathbf{x} - \mathbf{e}_k) - 2\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \\ &= \Delta_{i+k,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - 2\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \\ &= \Delta_{i+k,i} v(\mathbf{x} - \mathbf{e}_k) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \geq 0. \end{aligned}$$

$$3. \quad \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,i} T^k v(\mathbf{x}) &= \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) + c_k \\
 &= \Delta_{i+k,i} v(\mathbf{x} - \mathbf{e}_k) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k + 2\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + 2c_k \\
 &= \Delta_{i+k,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \geq 0.
 \end{aligned}$$

$$4. \quad 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + 2\mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_{i,i} T^k v(\mathbf{x}) = \Delta_{i,i} v(\mathbf{x} - \mathbf{e}_k) \geq 0.$$

Hence, $T^k v$ satisfies A1.

Property A2

From (3.7) we have the following three cases:

Case 1: if $x_k = 0$,

$$\Delta_{i,j} T^k v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x}) \leq 0.$$

Case 2: if $x_k > 0$, $i = k, j \neq k$ ($j = k, i \neq k$),

$$T^k v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\begin{aligned}
 \Delta_{i,j} T^i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) + \min\{0, \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i\} - \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i\} \\
 &\quad - \min\{0, \Delta_i v(\mathbf{x}) + c_i\} + \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i\}.
 \end{aligned}$$

By A1 we have

$$\Delta_i v(\mathbf{x}) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) \text{ or}$$

$$\Delta_i v(\mathbf{x}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j).$$

If $\Delta_i v(\mathbf{x}) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)$, we have the following five cases:

$$1. \quad \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i \geq 0 \Rightarrow$$

$$\Delta_{i,j} T^i v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) \leq 0.$$

$$2. \quad \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i \Rightarrow$$

$$\begin{aligned}
 \Delta_{i,j} T^i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) - c_i \\
 &= -\Delta_i v(\mathbf{x} - \mathbf{e}_i) - c_i \leq 0.
 \end{aligned}$$

$$3. \quad \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i \Rightarrow$$

$$\begin{aligned}\Delta_{i,j} T^i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} + \mathbf{e}_j) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) \\ &= \Delta_i v(\mathbf{x} + \mathbf{e}_j) - \Delta_i v(\mathbf{x} - \mathbf{e}_i) \leq 0.\end{aligned}$$

$$4. \Delta_i v(\mathbf{x}) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i \Rightarrow$$

$$\begin{aligned}\Delta_{i,j} T^i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} + \mathbf{e}_j) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \\ &= \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \leq 0.\end{aligned}$$

$$5. 0 \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i \Rightarrow$$

$$\begin{aligned}\Delta_{i,j} T^i v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} + \mathbf{e}_j) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) - \Delta_i v(\mathbf{x}) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) \\ &= \Delta_i v(\mathbf{x} + \mathbf{e}_j) - \Delta_i v(\mathbf{x} - \mathbf{e}_i) \leq 0.\end{aligned}$$

If $\Delta_i v(\mathbf{x}) \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)$, we also have consider the other case:

$$6. \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} + \mathbf{e}_j) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + c_i \Rightarrow$$

$$\Delta_{i,j} T^i v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_i) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) = 0.$$

Case 3: if $x_k > 0, i \neq k, j \neq k$,

$$T^k v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\begin{aligned}\Delta_{i,j} T^k v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k\} - \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k\} \\ &\quad - \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k\} + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\}.\end{aligned}$$

By A2 we have

$$\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) \text{ or}$$

$$\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k)$$

If $\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k)$, we have the following five cases:

$$1. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq 0 \Rightarrow$$

$$\begin{aligned}\Delta_{i,j} T^k v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \\ &\quad - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + \Delta_k v(\mathbf{x} - \mathbf{e}_k) = \Delta_{i,j} v(\mathbf{x}) \leq 0.\end{aligned}$$

$$2. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned}\Delta_{i,j} T^k v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \\ &= \Delta_{i,j} v(\mathbf{x}) - \Delta_k v(\mathbf{x} - \mathbf{e}_k) - c_k \leq 0.\end{aligned}$$

$$3. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned}\Delta_{i,j} T^k v(\mathbf{x}) &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \\ &= \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_{k,i} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \leq 0.\end{aligned}$$

$$4. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_{i,j} T^k v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq 0.$$

$$5. 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_{i,j} T^k v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) \leq 0.$$

If $\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k)$, we also consider the other case:

$$6. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_{i,j} T^k v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}_j) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) = \Delta_{i,j} v(\mathbf{x} - \mathbf{e}_k) + \Delta_{k,j} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j) \leq 0.$$

Hence, $T^k v$ satisfies A2.

Property A3

In order to prove $\Delta_{i+j_1+\dots+j_p,i} T^k v(\mathbf{x}) \geq 0$ we use Observation 1 to prove $\Delta_{e,i} T^k v(\mathbf{x}) \geq 0$, from (3.7)

we have the following three cases:

Case 1 if $x_k = 0$,

$$\Delta_{e,i} T^k v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x}) \geq 0.$$

Case 2 if $x_k > 0$, and $i = k$,

$$T^k v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\begin{aligned}\Delta_{e,i} T^i v(\mathbf{x}) &= \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) + \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i\} - \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i\} \\ &\quad - \min\{0, \Delta_i v(\mathbf{x}) + c_i\} + \min\{0, \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i\}.\end{aligned}$$

By A1- A3 we have

$\Delta_i v(\mathbf{x} + \mathbf{e}) \geq \Delta_i v(\mathbf{x}) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i)$, there are four possible sub-cases:

$$1. \Delta_i v(\mathbf{x} + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \geq 0 \Rightarrow$$

$$\Delta_{e,i} T^i v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) \geq 0.$$

$$2. \Delta_i v(\mathbf{x} + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\Delta_{e,i} T^i v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i = \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i \geq 0.$$

$$3. \Delta_i v(\mathbf{x} + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq 0 \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\Delta_{e,i} T^i v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) = 0.$$

$$4. \Delta_i v(\mathbf{x} + \mathbf{e}) + c_i \geq 0 \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\begin{aligned} \Delta_{e,i} T^i v(\mathbf{x}) &= \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) - \Delta_i v(\mathbf{x}) - c_i + \Delta_i v(\mathbf{x} - \mathbf{e}_i) \\ &= -\Delta_i v(\mathbf{x}) - c_i \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_i v(\mathbf{x} + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) + c_i \geq \Delta_i v(\mathbf{x} - \mathbf{e}_i) + c_i \Rightarrow$$

$$\Delta_{e,i} T^i v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) + \Delta_i v(\mathbf{x} + \mathbf{e}) - \Delta_i v(\mathbf{x} - \mathbf{e}_i + \mathbf{e}) - \Delta_i v(\mathbf{x}) + \Delta_i v(\mathbf{x} - \mathbf{e}_i) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_i) \geq 0.$$

Case 3 if $x_k > 0$, and $i \neq k$,

$$\begin{aligned} \Delta_{e,i} T^k v(\mathbf{x}) &= \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k\} - \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k\} \\ &\quad - \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k\} + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\}. \end{aligned}$$

By A2 and A3 we have

$$\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) \text{ or}$$

$$\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e})$$

If $\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e})$, which leads to five possible sub-cases:

$$1. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \leq 0 \Rightarrow$$

$$\begin{aligned} \Delta_{e,i} T^k v(\mathbf{x}) &= \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + \Delta_k v(\mathbf{x} - \mathbf{e}_k) \\ &= v(\mathbf{x} + \mathbf{e}_i) - v(\mathbf{x} + \mathbf{e}_i) - v(\mathbf{x} + \mathbf{e}) + v(\mathbf{x}) \\ &= \Delta_{e,i} v(\mathbf{x}) \geq 0. \end{aligned}$$

$$2. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \Rightarrow$$

$$\begin{aligned}\Delta_{e,i} T^k v(\mathbf{x}) &= \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \\ &= \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \geq 0.\end{aligned}$$

$$3. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \Rightarrow$$

$$T^k v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) + \Delta_{e,k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \geq 0.$$

$$4. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \Rightarrow$$

$$T^k v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) - c_k \geq 0.$$

$$5. 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \Rightarrow$$

$$T^k v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) \geq 0.$$

If $\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e})$, we can also consider the other case:

$$6. \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \leq 0 \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i + \mathbf{e}) + c_k \leq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \Rightarrow$$

$$T^k v(\mathbf{x}) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) - \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i) + \Delta_k v(\mathbf{x} - \mathbf{e}_k) = \Delta_{e,i} v(\mathbf{x} - \mathbf{e}_k) - \Delta_{k,i} v(\mathbf{x} - \mathbf{e}_k) \geq 0.$$

Hence, $T^k v$ satisfies A3.

Property A4

From (3.7) we have the following 2 cases:

Case 1 $x_k = 0$,

$$\Delta_e T^k v(\mathbf{x}) = \Delta_e v(\mathbf{x}) \geq -c_0.$$

Case 2 $x_k \neq 0$,

$$\Delta_e T^k v(\mathbf{x}) = \Delta_e v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k\} - \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\}.$$

Since $\Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) \geq \Delta_k v(\mathbf{x} - \mathbf{e}_k)$, we have

$$(1) \text{ if } \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \geq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0, \text{ then } \Delta_e T^k v(\mathbf{x}) = \Delta_e v(\mathbf{x} - \mathbf{e}_k) \geq -c_0.$$

$$(2) \text{ if } \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \geq 0 \geq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k, \text{ then}$$

$$\Delta_e T^k v(\mathbf{x}) = \Delta_e v(\mathbf{x} - \mathbf{e}_k) - \Delta_k v(\mathbf{x} - \mathbf{e}_k) - c_k \geq \Delta_e v(\mathbf{x}) \geq -c_0.$$

$$(3) \text{ if } 0 \geq \Delta_k v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}) + c_k \geq \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k, \text{ then}$$

$$\Delta_e T^k v(\mathbf{x}) = \Delta_e v(\mathbf{x} - \mathbf{e}_k) + \Delta_{e,k} v(\mathbf{x} - \mathbf{e}_k) \geq \Delta_e v(\mathbf{x} - \mathbf{e}_k) \geq -c_0.$$

Hence, $T^k v$ satisfies A4.

Operator T

Since $h(\mathbf{x})$ is increasing convex, it satisfies A1-A3 and we can see Tv satisfies Properties A1-A3. We will show that Tv also satisfies Property A4 in the following:

$$\begin{aligned} \Delta_e Tv(\mathbf{x}) &= \Delta_e h(\mathbf{x}) + \lambda_0 \Delta_e T^0 v(\mathbf{x}) + \sum_{k=1}^n \lambda_k \Delta_e T^k v(\mathbf{x}) + \sum_{k=1}^n \mu_k \Delta_e T_k v(\mathbf{x}) \\ &\geq \Delta_e h(\mathbf{x}) - \lambda_0 c_0 - \sum_{k=1}^n \lambda_k c_0 - \sum_{k=1}^n \mu_k c_0 \quad (\Delta_e h(\mathbf{x}) > 0) \\ &\geq -c_0 \left(\lambda_0 + \sum_{k=1}^n (\mu_k + \lambda_k) \right) = -c_0 (1 - \alpha) \geq -c_0. \end{aligned}$$

Hence, $Tv \in \mathcal{V}$. This completes the proof of Lemma 3.1. \square

Definition 3.2. For $v \in \mathcal{V}$ we define the base-stock and the rationing levels as following:

$$\begin{aligned} s_k^*(\mathbf{x}_{-k}) &= \min \{x_k \geq 0 \mid v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x}) \geq 0\}, \\ r_k^*(\mathbf{x}_{-k}) &= \min \{x_k \geq 0 \mid v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k) \geq -c_k\}, \end{aligned}$$

where $\mathbf{x}_{-k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ is $n-1$ dimensional vector consisting of the on-hand inventory levels for Component $i \neq k$, and x_k is the on-hand inventory level of Component k . Definition 3.2 indicates that the base-stock $s_k^*(\mathbf{x}_{-k})$ and rationing level $r_k^*(\mathbf{x}_{-k})$ depend on the on-hand inventory of other components.

From Definition 3.1 and 3.2, and Lemma 3.1 we can characterize the structure of the optimal policy in the following Theorem 3.1.

Theorem 3.1. *The optimal production policy for Component k is a base-stock policy with a state-dependent base-stock level $s_k^*(\mathbf{x}_{-k})$ where it is optimal to produce Component k , if $x_k < s_k^*(\mathbf{x}_{-k})$ and not to produce it otherwise. The optimal inventory allocation policy for Component k is a rationing policy with a state-dependent rationing level $r_k^*(\mathbf{x}_{-k})$ where it is optimal to satisfy the demand of Component k if $x_k > r_k^*(\mathbf{x}_{-k})$ and to reject it otherwise. Both levels are non-decreasing in each of the states x_i , $i \neq k$. Furthermore, it is always optimal to*

satisfy demand of the assembled product whenever on-hand inventory for all components is available.

Proof of Theorem 3.1

The optimal production policy consists of a base-stock policy with the base-stock level $s_k^*(\mathbf{x}_{-k})$. From Property A2, the optimal base-stock level $s_k^*(\mathbf{x}_{-k})$ is non-decreasing in x_j , $j \neq k$. By Property A1 if $\Delta_k v(\mathbf{x}) \geq 0$ it implies $x_k \geq s_k^*(\mathbf{x}_{-k})$, in this case we do not produce any more; in the case $\Delta_k v(\mathbf{x}) < 0$ and $x_k < s_k^*(\mathbf{x}_{-k})$, it is optimal to produce. Property A3 implies that $s_i^*(\mathbf{x}_{-i})$ is non-increasing with joint increases in x_i, x_{j_1}, \dots , and x_{j_n} . Also from property A2, the optimal rationing level $r_k^*(\mathbf{x}_{-k})$ is non-decreasing in each of states x_j , $j \neq k$. By Property A4, it is optimal to satisfy demand of the assembled product if there is at least one unit on-hand inventory for each component. It is optimal to satisfy demand of Component k if $x_k \geq r_k^*(\mathbf{x}_{-k})$, for $k=1,2,\dots,n$. Finally, we will prove $v^* \in \mathcal{V}$ directly. Since $v^* = \min_{n \rightarrow \infty} T^{(n)}$ for any v in \mathcal{V} , where $T^{(n)}$ refers to n compositions of T and v^* is the unique solution of $v = Tv$, from which v^* satisfies Properties A1-A4.

This completes the proof of Theorem 3.1. \square

Theorem 3.1 shows that the optimal policy can be determined by the base-stock level and the rationing level. Both levels are state-dependent and non-decreasing in the on-hand inventory level of other components. Similar to Benjaafar and Elhafsi (2006), the optimal production policy indicates that with an increase in the on-hand inventory level of one component, the inventory requirements for other components also increase. It is always optimal to satisfy product demand if all the components are present. Unlike Benjaafar and Elhafsi (2006), the optimal inventory allocation policy indicates that with an increase in the on-hand inventory level of one component, the rationing level for other components also increase. That is because we discuss two types of demand: demand of the assembled product and demand of the individual components. In our system since demand of the assembled product has a higher priority over demand of individual components, it is always satisfied if all components are in stock. When the system jointly increases the on-hand inventory of the components, more demands from the assembled product are desired to be satisfied that means several individual components demands must be rejected. Hence, the rationing level of the individual

components increases. This implies that it becomes more difficult to satisfy demand of the individual components when the on-hand inventory of components increases.

3.2.3 Influence of System Parameters

Theorem 3.3, we will discuss how the optimal policy is affected by system parameters. In order to do so, we redefine the base-stock and rationing levels with different parameters $\varpi \in \{h_k, c_k, c_0, \lambda_k, \lambda_0, \mu_k\}$ for $k=1, 2, \dots, n$, as:

$$\begin{aligned} s_{k,\varpi}^* (\mathbf{x}_{-k}) &= \min \{x_k \geq 0 \mid \Delta_k v_{\varpi} (\mathbf{x}) \geq 0\}, \\ r_{k,\varpi}^* (\mathbf{x}_{-k}) &= \min \{x_k \geq 0 \mid \Delta_k v_{\varpi} (\mathbf{x}) \geq -c_k\}, \quad (\varpi \neq c_k) \\ r_{k,\varpi}^* (\mathbf{x}_{-k}) &= \min \{x_k \geq 0 \mid \Delta_k v_{\varpi} (\mathbf{x}) \geq -\varpi\}. \quad (\varpi = c_k) \end{aligned}$$

Lemma 3.2. The optimal base-stock and rationing levels of Component k depend on various system parameters ϖ_k , for each $\varpi_k' \geq \varpi_k''$ we have

For the optimal base-stock level

$$\text{B1: } \Delta_k v_{h_k'} (\mathbf{x}) \geq \Delta_k v_{h_k''} (\mathbf{x}) \text{ for } h_k' \geq h_k'',$$

$$\Delta_k v_{c_0'} (\mathbf{x}) \leq \Delta_k v_{c_0''} (\mathbf{x}) \text{ for } c_0' \geq c_0'',$$

$$\Delta_k v_{c_k'} (\mathbf{x}) \leq \Delta_k v_{c_k''} (\mathbf{x}) \text{ for } c_k' \geq c_k''.$$

$$\text{B2: } \Delta_k v_{\lambda_0'} (\mathbf{x}) \leq \Delta_k v_{\lambda_0''} (\mathbf{x}) \text{ for } \lambda_0' \geq \lambda_0'',$$

$$\Delta_k v_{\lambda_k'} (\mathbf{x}) \leq \Delta_k v_{\lambda_k''} (\mathbf{x}) \text{ for } \lambda_k' \geq \lambda_k'',$$

$$\Delta_k v_{\mu_k'} (\mathbf{x}) \geq \Delta_k v_{\mu_k''} (\mathbf{x}) \text{ for } \mu_k' \geq \mu_k''.$$

For the optimal rationing level

$$\text{B3: } \Delta_k v_{h_k'} (\mathbf{x}) \geq \Delta_k v_{h_k''} (\mathbf{x}) \text{ for } h_k' \geq h_k'',$$

$$\Delta_k v_{c_0'} (\mathbf{x}) \leq \Delta_k v_{c_0''} (\mathbf{x}) \text{ for } c_0' \geq c_0'',$$

$$\Delta_k v_{c_k'} (\mathbf{x}) \geq \Delta_k v_{c_k''} (\mathbf{x}) \text{ for } c_k' \geq c_k''.$$

$$\text{B4: } \Delta_k v_{\lambda_0'} (\mathbf{x}) \leq \Delta_k v_{\lambda_0''} (\mathbf{x}) \text{ for } \lambda_0' \geq \lambda_0'',$$

$$\Delta_k v_{\lambda_k'} (\mathbf{x}) \leq \Delta_k v_{\lambda_k''} (\mathbf{x}) \text{ for } \lambda_k' \geq \lambda_k'',$$

$$\Delta_k v_{\mu_k'}(\mathbf{x}) \geq \Delta_k v_{\mu_k''}(\mathbf{x}) \text{ for } \mu_k' \geq \mu_k''.$$

Proof of Lemma 3.2

In order to prove Lemma 3.2, we need to show that $Tv(\mathbf{x})$ satisfies Properties B1-B4. In order to do so, we will prove that $T_kv(\mathbf{x})$, $T^0v(\mathbf{x})$ and $T^kv(\mathbf{x})$ satisfy Properties B1-B4, respectively.

Operator T_k

$$T_kv(\mathbf{x}) = \min\{v(\mathbf{x} + \mathbf{e}_k), v(\mathbf{x})\} = v(\mathbf{x}) + \min\{\Delta_kv(\mathbf{x}), 0\},$$

$$\Delta_k T_kv(\mathbf{x}) = \Delta_kv(\mathbf{x}) + \min\{\Delta_kv(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_kv(\mathbf{x}), 0\}.$$

Property B1

In order to prove B1, we consider three cases:

Case 1 if $h_k' \geq h_k''$.

By B1, $\Delta_kv_{h_k'}(\mathbf{x}) \geq \Delta_kv_{h_k''}(\mathbf{x})$ for $h_k' \geq h_k''$,

where $\Delta_kv_{h_k'}(\mathbf{x})$ and $\Delta_kv_{h_k''}(\mathbf{x})$ are the marginal cost difference due to increasing one unit on-hand inventory of Component k under the condition of h_k' and h_k'' , respectively.

$$\Delta_k T_kv_{h_k'}(\mathbf{x}) = \Delta_kv_{h_k'}(\mathbf{x}) + \min\{\Delta_kv_{h_k'}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_kv_{h_k'}(\mathbf{x}), 0\},$$

$$\Delta_k T_kv_{h_k''}(\mathbf{x}) = \Delta_kv_{h_k''}(\mathbf{x}) + \min\{\Delta_kv_{h_k''}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_kv_{h_k''}(\mathbf{x}), 0\},$$

$$\begin{aligned} \Delta_k T_kv_{h_k'}(\mathbf{x}) - \Delta_k T_kv_{h_k''}(\mathbf{x}) &= \Delta_kv_{h_k'}(\mathbf{x}) - \Delta_kv_{h_k''}(\mathbf{x}) + \min\{\Delta_kv_{h_k'}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_kv_{h_k'}(\mathbf{x}), 0\} \\ &\quad - \min\{\Delta_kv_{h_k''}(\mathbf{x} + \mathbf{e}_k), 0\} + \min\{\Delta_kv_{h_k''}(\mathbf{x}), 0\}. \end{aligned}$$

By A1 we have $\Delta_kv_{h_k'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_kv_{h_k'}(\mathbf{x})$ and $\Delta_kv_{h_k''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_kv_{h_k''}(\mathbf{x})$.

By B1 we have $\Delta_kv_{h_k'}(\mathbf{x}) \geq \Delta_kv_{h_k''}(\mathbf{x})$, from the above we have two possibilities:

$$\Delta_kv_{h_k'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_kv_{h_k'}(\mathbf{x}) \geq \Delta_kv_{h_k''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_kv_{h_k''}(\mathbf{x}) \text{ or}$$

$$\Delta_kv_{h_k'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_kv_{h_k''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_kv_{h_k'}(\mathbf{x}) \geq \Delta_kv_{h_k''}(\mathbf{x}).$$

If $\Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x})$ then we have five sub-cases:

$$1. \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq 0 \Rightarrow$$

$$\Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq 0.$$

$$2. \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq 0.$$

$$3. \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \\ &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0. \end{aligned}$$

$$4. \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \\ &= -\Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \\ &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0. \end{aligned}$$

If $\Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x})$, we also have one sub-case:

$$6. \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) = 0.$$

Case 2 if $c_0^{\cdot} \geq c_0^{\cdot}$.

By B1, $\Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x}) \leq \Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x})$ for $c_0^{\cdot} \geq c_0^{\cdot}$,

where $\Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x})$ and $\Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x})$ are the marginal cost difference due to increasing one unit on-hand inventory of Component k under the condition of c_0^{\cdot} and c_0^{\cdot} , respectively.

$$\Delta_k T_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x}) + \min\{\Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_k v_{c_0^{\cdot}}^{\cdot}(\mathbf{x}), 0\},$$

$$\Delta_k T_k v_{c_0}^-(\mathbf{x}) = \Delta_k v_{c_0}^-(\mathbf{x}) + \min\{\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_k v_{c_0}^-(\mathbf{x}), 0\},$$

$$\begin{aligned} \Delta_k T_k v_{c_0}^-(\mathbf{x}) - \Delta_k T_k v_{c_0}^-(\mathbf{x}) &= \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) + \min\{\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_k v_{c_0}^-(\mathbf{x}), 0\} \\ &\quad - \min\{\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k), 0\} + \min\{\Delta_k v_{c_0}^-(\mathbf{x}), 0\}. \end{aligned}$$

By A1 we have $\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x})$ and $\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x})$.

By B2 we have $\Delta_k v_{c_0}^-(\mathbf{x}) \leq \Delta_k v_{c_0}^-(\mathbf{x})$, from the above we have two possibilities:

$$\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \text{ or}$$

$$\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x}).$$

If $\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x})$, then we have five sub-cases:

$$1. \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq 0 \Rightarrow$$

$$\Delta_k T_k v_{c_0}^-(\mathbf{x}) - \Delta_k T_k v_{c_0}^-(\mathbf{x}) = \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) \leq 0.$$

$$2. \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{c_0}^-(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{c_0}^-(\mathbf{x}) - \Delta_k T_k v_{c_0}^-(\mathbf{x}) = \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) = -\Delta_k v_{c_0}^-(\mathbf{x}) \leq 0.$$

$$3. \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq 0 \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{c_0}^-(\mathbf{x}) - \Delta_k T_k v_{c_0}^-(\mathbf{x}) &= \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) + \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_0}^-(\mathbf{x}) \\ &= -\Delta_k v_{c_0}^-(\mathbf{x}) + \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \leq 0. \end{aligned}$$

$$4. \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{c_0}^-(\mathbf{x}) - \Delta_k T_k v_{c_0}^-(\mathbf{x}) &= \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) + \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_0}^-(\mathbf{x}) + \Delta_k v_{c_0}^-(\mathbf{x}) \\ &= \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \leq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{c_0}^-(\mathbf{x}) - \Delta_k T_k v_{c_0}^-(\mathbf{x}) &= \Delta_k v_{c_0}^-(\mathbf{x}) - \Delta_k v_{c_0}^-(\mathbf{x}) + \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_0}^-(\mathbf{x}) + \Delta_k v_{c_0}^-(\mathbf{x}) \\ &= \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \leq 0. \end{aligned}$$

If $\Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^-(\mathbf{x}) \geq \Delta_k v_{c_0}^-(\mathbf{x})$, we also have one sub-case:

$$6. \Delta_k v_{c_0}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_0}^*(\mathbf{x}) \geq 0 \geq \Delta_k v_{c_0}^*(\mathbf{x}) \geq \Delta_k v_{c_0}^*(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{c_0}^*(\mathbf{x}) - \Delta_k T_k v_{c_0}^*(\mathbf{x}) = \Delta_k v_{c_0}^*(\mathbf{x}) - \Delta_k v_{c_0}^*(\mathbf{x}) - \Delta_k v_{c_0}^*(\mathbf{x}) + \Delta_k v_{c_0}^*(\mathbf{x}) = 0.$$

Case 3 if $c_k' \geq c_k''$.

$$\Delta_k T_k v_{c_k}^*(\mathbf{x}) - \Delta_k T_k v_{c_k}^*(\mathbf{x}) \leq 0.$$

The proof is the same as Case 2 ($c_0' \geq c_0''$).

Hence, $T_k v$ satisfies B1.

Property B2

In order to prove B2, we consider three cases:

Case 1 if $\lambda_0' \geq \lambda_0''$.

By B2, $\Delta_k v_{\lambda_0'}^*(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}^*(\mathbf{x})$ for $\lambda_0' \geq \lambda_0''$. Here, $\Delta_k v_{\lambda_0'}^*(\mathbf{x})$ and $\Delta_k v_{\lambda_0''}^*(\mathbf{x})$ are the marginal cost difference due to increasing one unit on-hand inventory of component k under the condition of λ_0' and λ_0'' , respectively.

$$\Delta_k T_k v_{\lambda_0'}^*(\mathbf{x}) = \Delta_k v_{\lambda_0'}^*(\mathbf{x}) + \min \left\{ \Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k), 0 \right\} - \min \left\{ \Delta_k v_{\lambda_0'}^*(\mathbf{x}), 0 \right\},$$

$$\Delta_k T_k v_{\lambda_0''}^*(\mathbf{x}) = \Delta_k v_{\lambda_0''}^*(\mathbf{x}) + \min \left\{ \Delta_k v_{\lambda_0''}^*(\mathbf{x} + \mathbf{e}_k), 0 \right\} - \min \left\{ \Delta_k v_{\lambda_0''}^*(\mathbf{x}), 0 \right\},$$

$$\begin{aligned} \Delta_k T_k v_{\lambda_0'}^*(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}^*(\mathbf{x}) &= \Delta_k v_{\lambda_0'}^*(\mathbf{x}) - \Delta_k v_{\lambda_0''}^*(\mathbf{x}) + \min \left\{ \Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k), 0 \right\} - \min \left\{ \Delta_k v_{\lambda_0'}^*(\mathbf{x}), 0 \right\} \\ &\quad - \min \left\{ \Delta_k v_{\lambda_0''}^*(\mathbf{x} + \mathbf{e}_k), 0 \right\} + \min \left\{ \Delta_k v_{\lambda_0''}^*(\mathbf{x}), 0 \right\}. \end{aligned}$$

By A1 we have $\Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x} + \mathbf{e}_k)$ and $\Delta_k v_{\lambda_0'}^*(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x})$.

By B2 we have $\Delta_k v_{\lambda_0'}^*(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}^*(\mathbf{x})$, from the above we have two possibilities:

$$\Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x}) \text{ or}$$

$$\Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x}).$$

If $\Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x})$, then we have five sub-cases:

$$1. \Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}^*(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}^*(\mathbf{x}) \geq 0 \Rightarrow$$

$$\Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) = \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0''}(\mathbf{x}) \leq 0.$$

$$2. \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) = \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0''}(\mathbf{x}) - \Delta_k v_{\lambda_0'}(\mathbf{x}) = -\Delta_k v_{\lambda_0''}(\mathbf{x}) \leq 0.$$

$$3. \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \geq 0 \geq \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) &= \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0''}(\mathbf{x}) + \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \\ &= -\Delta_k v_{\lambda_0''}(\mathbf{x}) + \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \leq 0. \end{aligned}$$

$$4. \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) &= \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0''}(\mathbf{x}) + \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) + \Delta_k v_{\lambda_0''}(\mathbf{x}) \\ &= \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \leq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) &= \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0''}(\mathbf{x}) + \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{\lambda_0'}(\mathbf{x}) + \Delta_k v_{\lambda_0''}(\mathbf{x}) \\ &= \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \leq 0. \end{aligned}$$

If $\Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x})$, we also have one sub-case:

$$6. \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) = \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0''}(\mathbf{x}) - \Delta_k v_{\lambda_0'}(\mathbf{x}) + \Delta_k v_{\lambda_0''}(\mathbf{x}) = 0.$$

Case 2 if $\lambda_k' \geq \lambda_k''$.

$$\Delta_k T_k v_{\lambda_k'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_k''}(\mathbf{x}) \leq 0.$$

The proof is the same as Case 1 ($\lambda_0' \geq \lambda_0''$).

Case 3 if $\mu_k' \geq \mu_k''$.

By B1, $\Delta_k v_{\mu_k'}(\mathbf{x}) \geq \Delta_k v_{\mu_k''}(\mathbf{x})$ for $\mu_k' \geq \mu_k''$,

where $\Delta_k v_{\mu_k'}(\mathbf{x})$ and $\Delta_k v_{\mu_k''}(\mathbf{x})$ are the marginal cost difference due to increasing one unit on-hand inventory of Component k under the condition of μ_k' and μ_k'' , respectively.

If $\Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x})$, we also have one sub-case:

$$6. \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{\mu_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) = 0.$$

Hence, $T_k v$ satisfies B2.

Property B3

In order to prove B3, we consider three cases:

Case 1 if $h_k^{\cdot} \geq h_k^{\cdot}$.

By B3, $\Delta_k v_{h_k^{\cdot}}(\mathbf{x}) \geq \Delta_k v_{h_k^{\cdot}}(\mathbf{x})$ for $h_k^{\cdot} \geq h_k^{\cdot}$.

We have $\Delta_k T_k v_{h_k^{\cdot}}(\mathbf{x}) - \Delta_k T_k v_{h_k^{\cdot}}(\mathbf{x}) \geq 0$. (see the proof of Property B1 Case 1)

Case 2 if $c_0^{\cdot} \geq c_0^{\cdot}$.

By B3, $\Delta_k v_{c_0^{\cdot}}(\mathbf{x}) \leq \Delta_k v_{c_0^{\cdot}}(\mathbf{x})$ for $c_0^{\cdot} \geq c_0^{\cdot}$.

We have $\Delta_k T_k v_{c_0^{\cdot}}(\mathbf{x}) - \Delta_k T_k v_{c_0^{\cdot}}(\mathbf{x}) \leq 0$. (see the proof of Property B1 Case 2)

Case 3 if $c_k^{\cdot} \geq c_k^{\cdot}$.

By B3, $\Delta_k v_{c_k^{\cdot}}(\mathbf{x}) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x})$ for $c_k^{\cdot} \geq c_k^{\cdot}$.

$$\Delta_k T_k v_{c_k^{\cdot}}(\mathbf{x}) = \Delta_k v_{c_k^{\cdot}}(\mathbf{x}) + \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x}), 0\},$$

$$\Delta_k T_k v_{c_k^{\cdot}}(\mathbf{x}) = \Delta_k v_{c_k^{\cdot}}(\mathbf{x}) + \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x}), 0\},$$

$$\begin{aligned} \Delta_k T_k v_{c_k^{\cdot}}(\mathbf{x}) - \Delta_k T_k v_{c_k^{\cdot}}(\mathbf{x}) &= \Delta_k v_{c_k^{\cdot}}(\mathbf{x}) - \Delta_k v_{c_k^{\cdot}}(\mathbf{x}) + \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k), 0\} - \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x}), 0\} \\ &\quad - \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k), 0\} + \min\{\Delta_k v_{c_k^{\cdot}}(\mathbf{x}), 0\}. \end{aligned}$$

By A1 we have $\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x})$ and $\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x})$.

By B3 we have $\Delta_k v_{c_k^{\cdot}}(\mathbf{x}) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x})$, from the above we have two possibilities:

$$\Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x}) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k^{\cdot}}(\mathbf{x}) \text{ or}$$

$$\Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}).$$

If $\Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x})$, then we have five sub-cases:

$$1. \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq 0 \Rightarrow$$

$$\Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq 0.$$

$$2. \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq 0.$$

$$3. \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \\ &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0. \end{aligned}$$

$$4. \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \\ &= -\Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\begin{aligned} \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \\ &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0. \end{aligned}$$

If $\Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x})$ we also have one sub-case:

$$6. \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} + \mathbf{e}_k) \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \Rightarrow$$

$$\Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T_k v_{c_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) = 0.$$

Hence, $T_k v$ satisfies B3.

Property B4

In order to prove B4, we consider three cases:

Case 1 if $\lambda_0' \geq \lambda_0''$.

By B4, $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$ for $\lambda_0' \geq \lambda_0''$.

We have $\Delta_k T_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_0''}(\mathbf{x}) \leq 0$. (see the proof of Property B2 Case 1)

Case 2 if $\lambda_k' \geq \lambda_k''$.

By B4, $\Delta_k v_{\lambda_k'}(\mathbf{x}) \leq \Delta_k v_{\lambda_k''}(\mathbf{x})$ for $\lambda_k' \geq \lambda_k''$.

We have $\Delta_k T_k v_{\lambda_k'}(\mathbf{x}) - \Delta_k T_k v_{\lambda_k''}(\mathbf{x}) \leq 0$. (see the proof of Property B2 Case 2)

Case 3 if $\mu_k' \geq \mu_k''$.

By B4, $\Delta_k v_{\mu_k'}(\mathbf{x}) \geq \Delta_k v_{\mu_k''}(\mathbf{x})$ for $\mu_k' \geq \mu_k''$.

We have $\Delta_k T_k v_{\mu_k'}(\mathbf{x}) - \Delta_k T_k v_{\mu_k''}(\mathbf{x}) \geq 0$. (see the proof of Property B2 Case 3)

Hence, $T_k v$ satisfies B4.

Operator T^0

$$\begin{aligned} T^0 v(\mathbf{x}) &= \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}), v(\mathbf{x}) + c_0\} & \text{otherwise,} \end{cases} \\ &= \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ v(\mathbf{x} - \mathbf{e}) + \min\{0, \Delta_e v(\mathbf{x}) + c_0\} & \text{otherwise.} \end{cases} \end{aligned}$$

$$\text{By A4, already } T^0 v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ v(\mathbf{x} - \mathbf{e}) & \text{otherwise.} \end{cases}$$

Hence $T^0 v$ satisfies all Properties B1-B4.

Operator T^k

$$\begin{aligned} T^k v(\mathbf{x}) &= \begin{cases} v(\mathbf{x}) + c_k & \text{if } x_k = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}_k), v(\mathbf{x}) + c_k\} & \text{otherwise,} \end{cases} \\ &= \begin{cases} v(\mathbf{x}) + c_k & \text{if } x_k = 0 \\ v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.8}$$

From (3.8) we have the following two possibilities:

Possibility 1: if $x_k = 0$,

$T^k v(\mathbf{x}) = v(\mathbf{x}) + c_k$, hence it satisfies all Properties B1-B4.

Possibility 2: if $x_k > 0$,

$$T^k v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\Delta_k T^k v(\mathbf{x}) = \Delta_k v(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v(\mathbf{x} - \mathbf{e}_k) + c_k\}.$$

Property B1

In order to prove B1, we consider three cases:

Case 1 if $h_k' \geq h_k''$.

By B1, $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x})$ for $h_k' \geq h_k''$.

$$\Delta_k T^k v_{h_k'}(\mathbf{x}) = \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{h_k'}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\Delta_k T^k v_{h_k''}(\mathbf{x}) = \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{h_k''}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\begin{aligned} \Delta_k T^k v_{h_k'}(\mathbf{x}) - \Delta_k T^k v_{h_k''}(\mathbf{x}) &= \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{h_k'}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) + c_k\} \\ &\quad - \min\{0, \Delta_k v_{h_k'}(\mathbf{x}) + c_k\} + \min\{0, \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) + c_k\}. \end{aligned}$$

By A1 we have $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k)$ and $\Delta_k v_{h_k''}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k)$.

By B1 we have $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x})$, from above we have two sub-possibilities:

$$\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{h_k''}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) \text{ or}$$

$$\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k).$$

If $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{h_k''}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k)$, then we have five sub-cases:

$$1. \Delta_k v_{h_k'}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{h_k''}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \Rightarrow$$

$$\Delta_k T^k v_{h_k'}(\mathbf{x}) - \Delta_k T^k v_{h_k''}(\mathbf{x}) = \Delta_k v_{h_k'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k''}(\mathbf{x} - \mathbf{e}_k) \geq 0.$$

$$2. \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k = \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0.$$

$$3. \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq 0. \end{aligned}$$

$$4. \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - c_k - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= -\Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - c_k \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq 0. \end{aligned}$$

If $\Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$ we also have one sub-case:

$$6. \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) = 0.$$

Case 2 if $c_0' \geq c_0''$.

By B2, $\Delta_k v_{c_0'}^{\cdot}(\mathbf{x}) \leq \Delta_k v_{c_0''}^{\cdot}(\mathbf{x})$ for $c_0' \geq c_0''$.

$$\Delta_k T^k v_{c_0'}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_0'}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{c_0'}^{\cdot}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{c_0'}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\Delta_k T^k v_{c_0''}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_0''}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{c_0''}^{\cdot}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{c_0''}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k\}$$

$$\begin{aligned} \Delta_k T^k v_{c_0'}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0''}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_0'}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0''}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{c_0'}^{\cdot}(\mathbf{x}) + c_k\} \\ &\quad - \min\{0, \Delta_k v_{c_0'}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k\} - \min\{0, \Delta_k v_{c_0''}^{\cdot}(\mathbf{x}) + c_k\} + \min\{0, \Delta_k v_{c_0''}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k\}. \end{aligned}$$

By A1 we have $\Delta_k v_{c_0'}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0''}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$ and $\Delta_k v_{c_0''}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0''}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$

By B2 we have $\Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \leq \Delta_k v_{c_0}^{\cdot}(\mathbf{x})$, from above we have two possibilities:

$$\Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \text{ or}$$

$$\Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k).$$

If $\Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$, then we consider five sub-cases:

$$1. \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \Rightarrow$$

$$\Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \leq 0.$$

$$2. \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - c_k = -\Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - c_k \leq 0.$$

$$3. \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= -\Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \leq 0. \end{aligned}$$

$$4. \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \\ &= \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \leq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \leq 0. \end{aligned}$$

If $\Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$ we also have one sub-case:

$$6. \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_0}^{\cdot}(\mathbf{x}) = \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_0}^{\cdot}(\mathbf{x} - \mathbf{e}_k) = 0.$$

Case 3 if $c_k' \geq c_k''$.

$$\Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) \leq 0.$$

The proof is similar as Case 2 ($c_0' \geq c_0''$).

Hence, $T^k v$ satisfies B1.

Property B2

In order to prove B2, we consider three cases:

Case 1 if $\lambda_0' \geq \lambda_0''$.

By B2, $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$ for $\lambda_0' \geq \lambda_0''$.

$$\Delta_k T^k v_{\lambda_0'}(\mathbf{x}) = \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{\lambda_0'}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\Delta_k T^k v_{\lambda_0''}(\mathbf{x}) = \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{\lambda_0''}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + c_k\}$$

$$\begin{aligned} \Delta_k T^k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0''}(\mathbf{x}) &= \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{\lambda_0'}(\mathbf{x}) + c_k\} \\ &\quad - \min\{0, \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) + c_k\} - \min\{0, \Delta_k v_{\lambda_0''}(\mathbf{x}) + c_k\} + \min\{0, \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + c_k\}. \end{aligned}$$

By A1 we have $\Delta_k v_{\lambda_0'}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k)$ and $\Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k)$

By B2 we have $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$, from above we have two possibilities:

$$\Delta_k v_{\lambda_0'}(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) \text{ or}$$

$$\Delta_k v_{\lambda_0''}(\mathbf{x}) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k).$$

If $\Delta_k v_{\lambda_0'}(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k)$, then we have five sub-cases:

$$1. \Delta_k v_{\lambda_0''}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \Rightarrow$$

$$\Delta_k T^k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0''}(\mathbf{x}) = \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) \leq 0.$$

$$2. \Delta_k v_{\lambda_0''}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0''}(\mathbf{x}) = \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) - c_k = -\Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) - c_k \leq 0.$$

$$3. \Delta_k v_{\lambda_0''}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \geq \Delta_k v_{\lambda_0'}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{\lambda_0'}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0''}(\mathbf{x}) &= \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\lambda_0'}(\mathbf{x}) - \Delta_k v_{\lambda_0'}(\mathbf{x} - \mathbf{e}_k) \\ &= -\Delta_k v_{\lambda_0''}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\lambda_0'}(\mathbf{x}) \leq 0. \end{aligned}$$

$$4. \Delta_k v_{\lambda_0^-}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\lambda_0^+}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{\lambda_0^+}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0^-}(\mathbf{x}) &= \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\lambda_0^+}(\mathbf{x}) - \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) + c_k \\ &= \Delta_k v_{\lambda_0^+}(\mathbf{x}) + c_k \leq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{\lambda_0^-}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\lambda_0^+}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{\lambda_0^+}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0^-}(\mathbf{x}) &= \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\lambda_0^+}(\mathbf{x}) - \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0^+}(\mathbf{x}) + \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{\lambda_0^+}(\mathbf{x}) - \Delta_k v_{\lambda_0^-}(\mathbf{x}) \leq 0. \end{aligned}$$

If $\Delta_k v_{\lambda_0^+}(\mathbf{x}) \geq \Delta_k v_{\lambda_0^-}(\mathbf{x}) \geq \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k)$ we also have one sub-case:

$$6. \Delta_k v_{\lambda_0^-}(\mathbf{x}) + c_k \geq \Delta_k v_{\lambda_0^+}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{\lambda_0^+}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0^-}(\mathbf{x}) = \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\lambda_0^+}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\lambda_0^-}(\mathbf{x} - \mathbf{e}_k) = 0.$$

Case 2 if $\lambda_k^+ \geq \lambda_k^-$.

$$\Delta_k T^k v_{\lambda_k^+}(\mathbf{x}) - \Delta_k T^k v_{\lambda_k^-}(\mathbf{x}) \leq 0.$$

The proof is the same as Case 1 ($\lambda_0^+ \geq \lambda_0^-$).

Case 3 if $\mu_k^+ \geq \mu_k^-$.

By B1, $\Delta_k v_{\mu_k^+}(\mathbf{x}) \geq \Delta_k v_{\mu_k^-}(\mathbf{x})$ for $\mu_k^+ \geq \mu_k^-$.

$$\Delta_k T^k v_{\mu_k^+}(\mathbf{x}) = \Delta_k v_{\mu_k^+}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{\mu_k^+}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{\mu_k^+}(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\Delta_k T^k v_{\mu_k^-}(\mathbf{x}) = \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{\mu_k^-}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k) + c_k\},$$

$$\begin{aligned} \Delta_k T^k v_{\mu_k^+}(\mathbf{x}) - \Delta_k T^k v_{\mu_k^-}(\mathbf{x}) &= \Delta_k v_{\mu_k^+}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{\mu_k^+}(\mathbf{x}) + c_k\} - \min\{0, \Delta_k v_{\mu_k^+}(\mathbf{x} - \mathbf{e}_k) + c_k\} \\ &\quad - \min\{0, \Delta_k v_{\mu_k^-}(\mathbf{x}) + c_k\} + \min\{0, \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k) + c_k\}. \end{aligned}$$

By A1 we have $\Delta_k v_{\mu_k^+}(\mathbf{x}) \geq \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k)$ and $\Delta_k v_{\mu_k^-}(\mathbf{x}) \geq \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k)$.

By B1 we have $\Delta_k v_{\mu_k^+}(\mathbf{x}) \geq \Delta_k v_{\mu_k^-}(\mathbf{x})$, from above we have two sub-possibilities:

$$\Delta_k v_{\mu_k^+}(\mathbf{x}) \geq \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\mu_k^-}(\mathbf{x}) \geq \Delta_k v_{\mu_k^-}(\mathbf{x} - \mathbf{e}_k) \text{ or}$$

$$\Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k).$$

If $\Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$, then we have five sub-cases:

$$1. \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \Rightarrow$$

$$\Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq 0.$$

$$2. \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k = \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0.$$

$$3. \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq 0 \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq 0. \end{aligned}$$

$$4. \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - c_k - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= -\Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) - c_k \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{\mu_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{\mu_k}^{\cdot}(\mathbf{x}) \geq 0. \end{aligned}$$

If $\Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$ we also have one sub-case:

$$6. \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x}) + c_k \geq 0 \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \geq \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k \Rightarrow$$

$$\Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{h_k}^{\cdot}(\mathbf{x}) = \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{h_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) = 0.$$

Hence, $T^k v$ satisfies B2.

Property B3

In order to prove B3, we consider three cases:

Case 1 if $h_k^{\cdot} \geq h_k^{\cdot}$.

By B3, $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x})$ for $h_k' \geq h_k''$.

We have $\Delta_k T^k v_{h_k'}(\mathbf{x}) - \Delta_k T^k v_{h_k''}(\mathbf{x}) \geq 0$. (see the proof of Property B1 Case 1)

Case 2: if $c_0' \geq c_0''$.

By B3, $\Delta_k v_{c_0'}(\mathbf{x}) \leq \Delta_k v_{c_0''}(\mathbf{x})$ for $c_0' \geq c_0''$.

We have $\Delta_k T^k v_{c_0'}(\mathbf{x}) - \Delta_k T^k v_{c_0''}(\mathbf{x}) \leq 0$. (see the proof of Property B1 Case 2)

Case 3 if $c_k' \geq c_k''$.

By B3, $\Delta_k v_{c_k'}(\mathbf{x}) \leq \Delta_k v_{c_k''}(\mathbf{x})$ for $c_k' \geq c_k''$,

$$\Delta_k T^k v_{c_k'}(\mathbf{x}) = \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{c_k'}(\mathbf{x}) + c_k'\} - \min\{0, \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) + c_k'\},$$

$$\Delta_k T^k v_{c_k''}(\mathbf{x}) = \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{c_k''}(\mathbf{x}) + c_k''\} - \min\{0, \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + c_k''\},$$

$$\begin{aligned} \Delta_k T^k v_{c_k'}(\mathbf{x}) - \Delta_k T^k v_{c_k''}(\mathbf{x}) &= \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + \min\{0, \Delta_k v_{c_k'}(\mathbf{x}) + c_k'\} - \min\{0, \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) + c_k'\} \\ &\quad - \min\{0, \Delta_k v_{c_k''}(\mathbf{x}) + c_k''\} + \min\{0, \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + c_k''\}. \end{aligned}$$

By A1 we have $\Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k)$ and $\Delta_k v_{c_k''}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k)$.

By B3 we have $\Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x})$, from above we have two possibilities:

$$\Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_k''}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) \text{ or}$$

$$\Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k)$$

If $\Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_k''}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k)$, then we have five sub-cases:

$$1. \Delta_k v_{c_k'}(\mathbf{x}) + c_k' \geq \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) + c_k' \geq \Delta_k v_{c_k''}(\mathbf{x}) + c_k'' \geq \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + c_k'' \geq 0 \Rightarrow$$

$$\Delta_k T^k v_{c_k'}(\mathbf{x}) - \Delta_k T^k v_{c_k''}(\mathbf{x}) = \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) \geq 0.$$

$$2. \Delta_k v_{c_k'}(\mathbf{x}) + c_k' \geq \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) + c_k' \geq \Delta_k v_{c_k''}(\mathbf{x}) + c_k'' \geq 0 \geq \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + c_k'' \Rightarrow$$

$$\Delta_k T^k v_{c_k'}(\mathbf{x}) - \Delta_k T^k v_{c_k''}(\mathbf{x}) = \Delta_k v_{c_k'}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_k''}(\mathbf{x} - \mathbf{e}_k) + c_k''$$

$$= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k'' \geq 0.$$

$$3. \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k' \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k'' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k'' \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq 0. \end{aligned}$$

$$4. \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k' \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k'' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k'' \Rightarrow$$

$$\begin{aligned} \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - c_k' - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= -\Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - c_k' \geq 0. \end{aligned}$$

$$5. 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k'' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k'' \Rightarrow$$

$$\begin{aligned} T^k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \\ &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq 0. \end{aligned}$$

If $\Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k)$ and $c_k' = c_k''$, we also have one sub-case:

$$6. \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x}) + c_k'' \geq 0 \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k' \geq \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k''$$

$$\begin{aligned} \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{c_k}^{\cdot}(\mathbf{x}) &= \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) - c_k' + \Delta_k v_{c_k}^{\cdot}(\mathbf{x} - \mathbf{e}_k) + c_k'' \\ &= c_k'' - c_k' = 0. \end{aligned}$$

Hence, $T^k v$ satisfies B3.

Property B4

In order to prove B4, we consider three cases:

Case 1 if $\lambda_0' \geq \lambda_0''$.

By B4, $\Delta_k v_{\lambda_0'}^{\cdot}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}^{\cdot}(\mathbf{x})$ for $\lambda_0' \geq \lambda_0''$.

We have $\Delta_k T^k v_{\lambda_0'}^{\cdot}(\mathbf{x}) - \Delta_k T^k v_{\lambda_0''}^{\cdot}(\mathbf{x}) \leq 0$. (see the proof of Property B2 Case 1)

Case 2 if $\lambda_k' \geq \lambda_k''$.

By B4, $\Delta_k v_{\lambda_k'}^{\cdot}(\mathbf{x}) \leq \Delta_k v_{\lambda_k''}^{\cdot}(\mathbf{x})$ for $\lambda_k' \geq \lambda_k''$.

We have $\Delta_k T^k v_{\lambda_k'}(\mathbf{x}) - \Delta_k T^k v_{\lambda_k''}(\mathbf{x}) \leq 0$. (see the proof of Property B2 Case 2)

Case 3 if $\mu_k' \geq \mu_k''$.

By B4, $\Delta_k v_{\mu_k'}(\mathbf{x}) \geq \Delta_k v_{\mu_k''}(\mathbf{x})$ for $\mu_k' \geq \mu_k''$.

We have $\Delta_k T^k v_{\mu_k'}(\mathbf{x}) - \Delta_k T^k v_{\mu_k''}(\mathbf{x}) \geq 0$. (see the proof of Property B2 Case 3)

Hence, $T^k v$ satisfies B4.

Operator T

Since $h(\mathbf{x})$ is increasing convex in x_i , it satisfies B1-B4 and we can see that Tv satisfies B1-B4. Cil et al. (2009) presented a general framework for investigating the effect of system parameters on the optimal policy for inventory control problem. The interested reader can refer to it for the overview of this problem.

This completes the proof of Lemma 3.2. \square

From Lemma 3.2 we define Theorem 3.2.

THEOREM 3.2. *The optimal base-stock and rationing levels satisfy the following properties.*

- (1) *The base-stock level $s_k^*(\mathbf{x}_{-k})$, for Component k , is non-increasing in the holding cost h_k and non-decreasing in the lost sales costs c_0 and c_k , for $k=1,2,\dots,n$.*
- (2) *The base-stock level $s_k^*(\mathbf{x}_{-k})$, for Component k , is non-decreasing in the arrival rates λ_0 and λ_k and non-increasing in the production rate μ_k , for $k=1,2,\dots,n$.*
- (3) *The rationing level $r_k^*(\mathbf{x}_{-k})$, for Component k , is non-increasing in the holding cost h_k non-decreasing in the lost sales cost c_0 and non-increasing in c_k for $k=1,2,\dots,n$.*
- (4) *The rationing level $r_k^*(\mathbf{x}_{-k})$, for Component k , is non-decreasing in the arrival rates λ_0 and λ_k and non-increasing in the production rate μ_k , for $k=1,2,\dots,n$.*

Proof of Theorem 3.2

From Lemma 3.2, the base-stock and the rationing levels as:

$$s_{k,\varpi}^*(\mathbf{x}_{-k}) = \min \{x_k \geq 0 \mid \Delta_k v_{\varpi}(\mathbf{x}) \geq 0\},$$

$$r_{k,\varpi}^*(\mathbf{x}_{-k}) = \min \{x_k \geq 0 \mid \Delta_k v_{\varpi}(\mathbf{x}) \geq -c_k\}, \quad (\varpi \neq c_k)$$

$$r_{k,\varpi}^*(\mathbf{x}_{-k}) = \min \{x_k \geq 0 \mid \Delta_k v_{\varpi}(\mathbf{x}) \geq -\alpha\}. \quad (\varpi = c_k)$$

Property 1:

From Property B1, we consider three cases:

$$1. \Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x}) \text{ for } h_k' \geq h_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different holding costs $h_k' \geq h_k''$, from $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x})$ we have $v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x})$ is non-decreasing in h_k .

$$2. \Delta_k v_{c_0'}(\mathbf{x}) \leq \Delta_k v_{c_0''}(\mathbf{x}) \text{ for } c_0' \geq c_0''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different lost sale costs $c_0' \geq c_0''$, from $\Delta_k v_{c_0'}(\mathbf{x}) \leq \Delta_k v_{c_0''}(\mathbf{x})$ we have $v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x})$ is non-increasing in c_0 .

$$3. \Delta_k v_{c_k'}(\mathbf{x}) \leq \Delta_k v_{c_k''}(\mathbf{x}) \text{ for } c_k' \geq c_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different lost sale costs $c_k' \geq c_k''$, from $\Delta_k v_{c_k'}(\mathbf{x}) \leq \Delta_k v_{c_k''}(\mathbf{x})$ we have $v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x})$ is non-increasing in c_k .

Property 2

From Property B2, we consider three cases:

$$1. \Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x}) \text{ for } \lambda_0' \geq \lambda_0''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different arrival rates $\lambda_0' \geq \lambda_0''$, from $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$ we have $v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x})$ is non-increasing in λ_0 .

$$2. \Delta_k v_{\lambda_k'}(\mathbf{x}) \leq \Delta_k v_{\lambda_k''}(\mathbf{x}) \text{ for } \lambda_k' \geq \lambda_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different arrival rates $\lambda_k' \geq \lambda_k''$, from $\Delta_k v_{\lambda_k'}(\mathbf{x}) \leq \Delta_k v_{\lambda_k''}(\mathbf{x})$ we have $v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x})$ is non-increasing in λ_k .

$$3. \Delta_k v_{\mu_k'}(\mathbf{x}) \geq \Delta_k v_{\mu_k''}(\mathbf{x}) \text{ for } \mu_k' \geq \mu_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different production rates $\mu_k' \geq \mu_k''$, from $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$ we have $v(\mathbf{x} + \mathbf{e}_k) - v(\mathbf{x})$ is non-decreasing in μ_k .

Property 3:

From Property B3, we consider three cases:

$$1. \Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x}) \text{ for } h_k' \geq h_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different holding costs $h_k' \geq h_k''$, from $\Delta_k v_{h_k'}(\mathbf{x}) \geq \Delta_k v_{h_k''}(\mathbf{x})$ we have $v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k)$ is non-decreasing in h_k .

$$2. \Delta_k v_{c_0'}(\mathbf{x}) \leq \Delta_k v_{c_0''}(\mathbf{x}) \text{ for } c_0' \geq c_0''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different holding costs $c_0' \geq c_0''$, from $\Delta_k v_{c_0'}(\mathbf{x}) \leq \Delta_k v_{c_0''}(\mathbf{x})$ we have $v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k)$ is non-increasing in c_0 .

$$3. \Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x}) \text{ for } c_k' \geq c_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different holding costs $c_k' \geq c_k''$, from $\Delta_k v_{c_k'}(\mathbf{x}) \geq \Delta_k v_{c_k''}(\mathbf{x})$ we have $v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k)$ is non-decreasing in c_k .

Property 4:

From Property B4, we consider three cases:

$$1. \Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x}) \text{ for } \lambda_0' \geq \lambda_0''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different arrival rates $\lambda_0' \geq \lambda_0''$, from $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$ we have $v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k)$ is non-increasing in λ_0 .

$$2. \Delta_k v_{\lambda_k'}(\mathbf{x}) \leq \Delta_k v_{\lambda_k''}(\mathbf{x}) \text{ for } \lambda_k' \geq \lambda_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different arrival rates $\lambda_k' \geq \lambda_k''$, from $\Delta_k v_{\lambda_k'}(\mathbf{x}) \leq \Delta_k v_{\lambda_k''}(\mathbf{x})$ we have $v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k)$ is non-increasing in λ_k .

$$3. \Delta_k v_{\mu_k'}(\mathbf{x}) \geq \Delta_k v_{\mu_k''}(\mathbf{x}) \text{ for } \mu_k' \geq \mu_k''.$$

Since the function $v(\mathbf{x})$ is convex, for a fixed level x_k and two different production rates $\mu_k' \geq \mu_k''$, from $\Delta_k v_{\lambda_0'}(\mathbf{x}) \leq \Delta_k v_{\lambda_0''}(\mathbf{x})$ we have $v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_k)$ is non-decreasing in μ_k .

This completes the proof of Theorem 3.2. \square

Theorem 3.2 shows that the base-stock and the rationing levels are sensitive to various system parameters. Property 1 indicates that it is optimal to hold less inventory of Component k in stock as holding cost h_k increases, but to hold more inventory of Component k in stock as lost sale cost either c_0 or c_k increases. Property 2 indicates that the more demands of the assembled product or of Component k arrive to the system the more inventory of Component k should be held in stock. However, the higher production rate μ_k leads to the lower inventory requirements for Component k . Property 3 indicates that the increase of holding cost h_k or lost sale cost c_k may also increase the probability that demand of Component k would be satisfied, while the increase of lost sale cost c_0 may reduce this probability. For a similar reason as shown in Property 3, Property 4 indicates that it is optimal to satisfy more demand of Component k as the production rate μ_k increases, but to satisfy less demand of Component k as the arrival rates λ_0 and λ_k increase.

3.2.4 The Case of Average Cost per Period

In this section we extend our analysis to the case of the average cost per period. Our objective is to minimize the expected long-run average cost of the system. Under a control policy π and a starting state \mathbf{x} , the average cost is given by $g^\pi(\mathbf{x})$:

$$g^\pi(\mathbf{x}) = \limsup_{N \rightarrow \infty} \frac{E_{\mathbf{x}}^\pi \left[\sum_{i=0}^{N-1} (h(\mathbf{X}(t_i)) + \sum_{l=0}^n c_l N_l(t_i)) \right]}{N}. \quad (3.9)$$

Our aim is to seek the optimal production policy π^* that minimizes the average cost per period $g^*(\mathbf{x}) = \inf_{\pi} g^\pi(\mathbf{x})$ for all states \mathbf{x} . The optimality equation in this case is as follows:

$$v^*(\mathbf{x}) + g^* = h(\mathbf{x}) + \lambda_0 T^0 v^*(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}), \quad (3.10)$$

where g^* is a finite constant denoting the average cost per period.

In Theorem 3.3, we show that the optimal policy under the average cost criterion retains all the properties observed in Theorem 3.1 and Theorem 3.2 under the discounted cost criterion.

THEOREM 3.3. *The optimal policy under the average cost criterion retains all the properties of the optimal policy under the discounted cost criterion, namely that the optimal policy can be described by two types of state-depend thresholds: a production base-stock level and an inventory rationing level. The base-stock and the rationing levels satisfy all the properties in Theorem 3.1 and Theorem 3.2.*

Proof of Theorem 3.3

We deal with the average cost problem through the limit of discounted cost problem with $\alpha \rightarrow 0$. Since our problem can be formulated as an MDP, there are two conditions must be held for the existence of average optimal policy. As showed by Cavazos-Cadena (1992), Weber and Stidham (1987): 1. there exists a stationary policy π which achieves a finite average cost g^π . 2. the number of states in which the holding cost $h(\mathbf{x}) \leq g^\pi$ is finite.

For our model, consider a policy π that is described by two types of thresholds: a fixed base-stock level s_k and a fixed rationing level r_k . Each Component k can be produced if its on-hand inventory x_k is below s_k , and each demand of Component k can be satisfied if the on-hand inventory x_k is above r_k . The operators T^0 , T^k and T_k in equation (3.10) can be written as:

$$T^0 v^*(\mathbf{x}) = \begin{cases} v^*(\mathbf{x} - \mathbf{e}) & \text{if } \prod_{k=1}^n x_k > 0, \\ v^*(\mathbf{x}) + c_0 & \text{otherwise,} \end{cases}$$

$$T^k v^*(\mathbf{x}) = \begin{cases} v^*(\mathbf{x} - \mathbf{e}_k) & \text{if } x_k > r_k, \\ v^*(\mathbf{x}) + c_k & \text{otherwise,} \end{cases}$$

$$T_k v^*(\mathbf{x}) = \begin{cases} v^*(\mathbf{x} + \mathbf{e}_k) & \text{if } x_k < s_k, \\ v^*(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Since by Property A1 function $v^*(\mathbf{x})$ is convex in each of the state variables x_k , there exists the minimum value that $g^* = \min v^*(\mathbf{x})$. It is not hard to see that policy π is stationary, which achieves a finite average cost and condition 1 holds. For condition 2, since the holding cost $h(\mathbf{x}) = \sum_{k=1}^n h_k(x_k)$ is increasing convex function in each x_k , there exists a positive integer γ , the number of states in which the cost $h(\mathbf{x}) \leq \gamma$ is finite. Thus, condition 2 holds.

Based on the above conditions, Weber and Stidham (1987) proposed that there exists a positive constant g^* and the expected discounted cost $v(\mathbf{x})$, then for all sates \mathbf{x} :

$$v(\mathbf{x}) + g^* \geq h(\mathbf{x}) + \lambda_0 T^0 v(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v(\mathbf{x}).$$

So, the average cost per period g^* can be determined from the above inequation.

This completes the proof of Theorem 3.3. \square

3.3 Numerical Experiments

In this section, the average cost per period criterion is adopted in our numerical experiments. We investigate how the optimal control policies depend on different system parameters. We have used the value iteration algorithm to solve the dynamic programming equation in section 3.2. Readers are referred to the work of Puterman (1994, Chapter 8) for the details of the iteration algorithm, which we adapted directly.

3.3.1 Value Iteration Algorithm for Average Cost Criterion.

We consider a system with two components and one single product. We apply results for the average cost criterion. The details are as following:

Since the state space of our system is infinite, truncating the state space is necessary. The state space is truncated at $\{0, n_1^{\max}\} \times \{0, n_2^{\max}\}$, where $n_k^{\max}, k=1,2$, are positive integers that are gradually increased by five at each time. The iteration algorithm will stop when the minimum of average cost per period is obtained. We use the value iteration algorithm to find a stationary ε -optimal policy, the decision rule $(d_\varepsilon)^\infty$, and an approximation to its value.

Step 1. Input the parameters: $\mu_i, \lambda_i, h_i, c_i, n_i, \varepsilon, inc$, $i=1,2$; $l=0,1,2$, and $inc=5$.

Step 2. Select $g^*=0$ (the optimal average cost rate), specify $\varepsilon > 0$, and $n=0$.

Step 3. For each state \mathbf{x} : $x_1=0,1,\dots,n_1, x_2=0,1,\dots,n_2$, compute $v_{n+1}(\mathbf{x})$ by

$$v_{n+1}(\mathbf{x}) = h(\mathbf{x}) + \lambda_0 T^0 v_n(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v_n(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v_n(\mathbf{x}),$$

$$g = \min v_{n+1}(\mathbf{x}),$$

$$v_{n+1}(\mathbf{x}) = v_{n+1}(\mathbf{x}) - g,$$

$$\Delta = v_{n+1}(\mathbf{x}) - v_n(\mathbf{x}).$$

Step 4.

If $sp = \max|\Delta| - \min|\Delta| < \varepsilon$,

go to Step 5. Otherwise, increment n by 1 and return to Step 3.

Step 5.

If $|g - g^*|/g < \varepsilon$, for each state \mathbf{x} , choose

$$d_\varepsilon(\mathbf{x}) \in \arg \left(h(\mathbf{x}) + \lambda_0 T^0 v_{n+1}(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v_{n+1}(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v_{n+1}(\mathbf{x}) \right), \text{ and stop.}$$

Otherwise, go to Step 6.

Step 6.

$$g^* = g,$$

$$n_1 = n_1 + inc,$$

$$n_2 = n_2 + inc.$$

Go to step 3.

From the above algorithm we obtain the optimal policy, we will illustrate its structure in the following subsection.

3.3.2 The Structure of the Optimal Policy

To simplify the computations, we focus on a two-component ATO system. We test this system with a wide range of parameter values. The structure of the optimal policy is illustrated in Fig. 3.1 and Fig.3.2.

Fig.3.1 and Fig.3.2 show that the optimal production policy is similar to that in Benjaafar and ElHafsi (2006), but the allocation policy is quite different. Fig.3.2 displays the optimal rationing decision in seven regions. In region 1, all demands are satisfied. In region 2, only demand of the assembled product is satisfied. In region 3, demand of Component 2 and demand of the assembled product are satisfied. In region 4, only demand of Component 2 is satisfied. In region 5, demands of Component 1 and demand of the assembled product are satisfied. In region 6, only demand of Component 1 is satisfied. In region 7, neither demand is satisfied. In fact, different regions correspond to different allocation decisions. In region 1, the amount of the on-hand inventory for both Component 1 and Component 2 is large, thus the system has enough capacity to satisfy all the demands. But in region 2, the amount of the on-

hand inventory for both Component 1 and Component 2 is small, thus the optimal policy tends to use the limited capacity only to satisfy demand of the assembled product. In region 3 and region 5, the on-hand inventory of one component is much larger than that of the other. Thus, it is optimal to satisfy demand of the assembled product and the demand of component that has a larger on-hand inventory. In region 4 and region 6, only one component is available while the other is not. When both the end product and component demands arrive, the component demand for the available one can be satisfied directly, but the other demands cannot be. In region 7, both Component 1 and Component 2 are not available, that is the system has no available capacity to satisfy any demand.

Also as seen from Fig.3.2, there are two rationing levels r_1^* and r_2^* . Each level is non-decreasing in the on-hand inventory of the other component. This result is quite different from the result of Benjaafar and ElHafsi (2006). Note that the assembled product demand has higher priority, in order to satisfy demand of the assembled product all the components must be held in stock. If the on-hand inventory of one component increases, it is more desired to reserve other components for demand of the assembled product. As a result, the rationing level for each component increases. In summary, the inventory allocation policy is a state-dependent rationing policy for demand of the individual components. For demand of the assembled product, it can be always satisfied if all the components are available.

3.3.3 The Effect of System Parameters on the Optimal Policy

In this section, we will briefly discuss how the different system parameters impact on the optimal policy. The following figures indicate that the base-stock and rationing levels are sensitive to the values of parameters such as holding cost, lost sales cost, production rate and arrival rate. From Fig. 3.3(a) and Fig. 3.3(b), it is easy to see that both s_1^* and r_1^* are non-increasing in h_k . Since a higher level of h_k may incur a greater average cost, the larger holding cost leads to less production of Component 1. Hence, the base-stock level s_1^* decreases with h_1 . The holding cost also influences the optimal allocation policy that can be seen in Fig. 3.3(b). As the holding cost h_1 increases it is optimal to keep less Component 1 in stock. This leads to the increase of the probability that demand of Component 1 would be satisfied, thus the rationing level r_1^* decreases.

Fig. 3.4(a) and Fig. 3.4(b) show the effect of the lost sale cost c_0 on both the base-stock and rationing levels. Both s_1^* and r_1^* are non-decreasing in c_0 . This result implies that when c_0

increases, it is optimal to hold more inventories for both Component 1 and Component 2, thus the base-stock level of Component 1 increases. Also, the more production of Component 1 increases the probability that demand of the assembled product would be satisfied, which leads to more demand of Component 1 would be rejected. Hence, demand of the individual components is more difficult to satisfy, and the rationing level of Component 1 increases.

Fig. 3.5(a) and Fig. 3.5(b) show the effect of the lost sale cost c_1 on both the base-stock and rationing levels. In Fig. 3.5(a), the base-stock level of Component 1 is non-decreasing in the lost sale cost c_1 . When c_1 increases that would increase the average cost of the system, in order to reduce this effect more Components 1 are desired to produce. Since it is optimal to hold more inventory of Component 1, the base-stock level of Component 1 increases. As seen from Fig. 3.5(b), the rationing level of Component 1 is also non-increasing in the lost sale cost c_1 . Clearly, the increase of the lost sale cost c_1 leads to more demands of Component 1 being satisfied, thus the rationing level of Component 1 decreases.

Fig. 3.6(a) and Fig. 3.6(b) show the effect of the arrival rate λ_0 on both the base-stock and rationing levels. Both s_1^* and r_1^* are non-decreasing in λ_0 . Obviously, when the number of the assembled product demands increases per unit of time, more demands enter in the system and more components are needed. Hence, the base-stock level of Component 1 increases. Also, the result implies that the more production of Component 1 leads to more demands of the assembled product can be satisfied, which reduces the probability that demand of Component 1 would be satisfied. Thus, the rationing level of Component 1 increases.

Fig. 3.7(a) and Fig. 3.7(b) show the effect of the arrival rate λ_1 on both the base-stock and rationing levels. In Fig. 3.7(a), the base-stock level of Component 1 is non-decreasing in arrival rate λ_1 . It is not difficult to understand the fact that when the number of Component 1 demands increases per unit time, it is optimal to hold more inventory of Component 1, thus the base-stock level of Component 1 increases. Similarly, in Fig. 3.7(b), the rationing level of Component 1 increases with the arrival rate λ_1 . It means that the increase of the inventory of Component 1 leads to more demands of the assembled product being satisfied, while more demands of Component 1 can be rejected.

Fig. 3.8(a) and Fig. 3.8(b) show the effect of the production rate μ_1 on both the base-stock and rationing levels. Both s_1^* and r_1^* are non-increasing in μ_1 . Due to the increase of the production rate of Component 1, the system has higher production efficiency per unit of time.

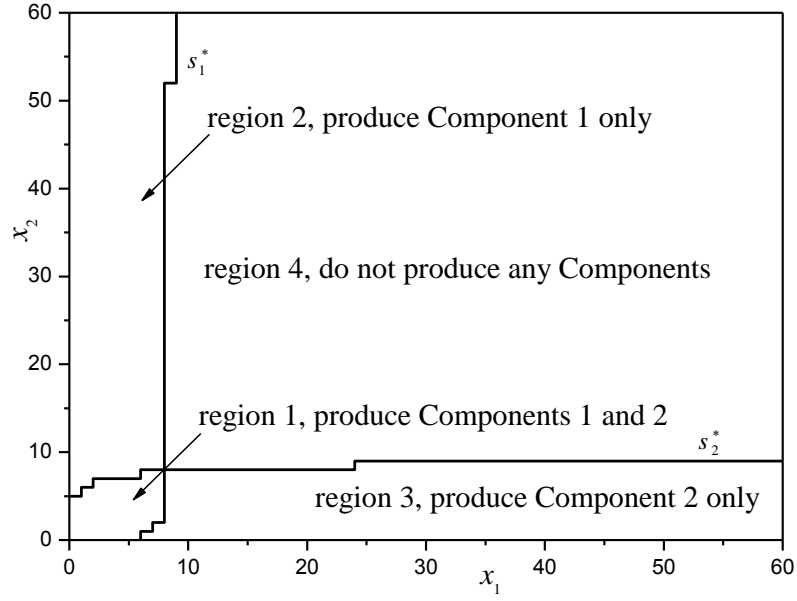


Fig. 3.1. The structure of the optimal production policy with lost sales

$(\mu_1 = \mu_2 = 1.0, \lambda_0 = 0.7, \lambda_1 = 0.25, \lambda_2 = 0.2, c_0 = 750, c_1 = 200, c_2 = 50, h_1 = 10, h_2 = 5)$

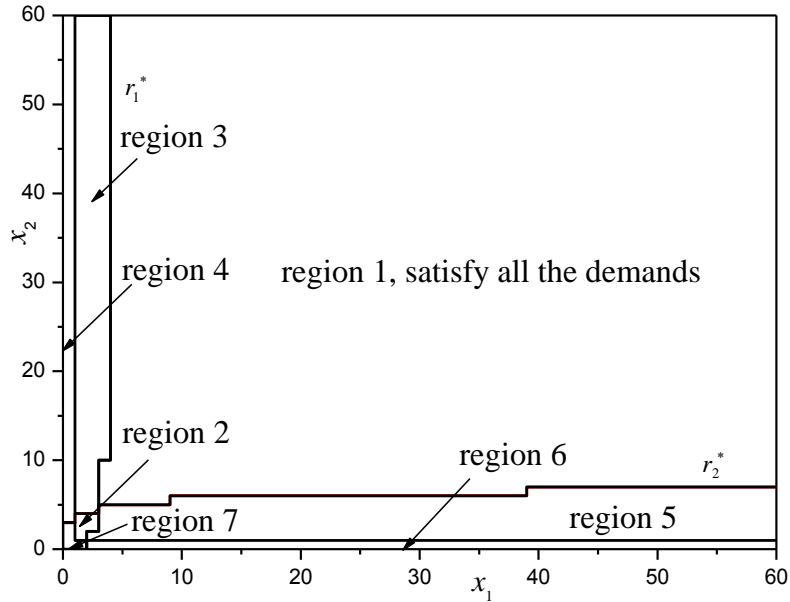
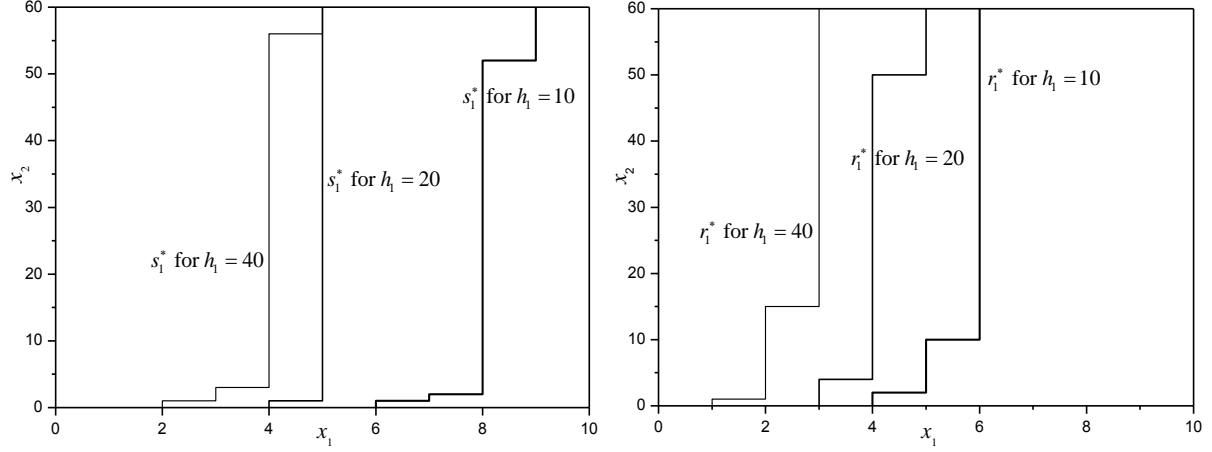


Fig. 3.2. The structure of the optimal allocation policy with lost sales

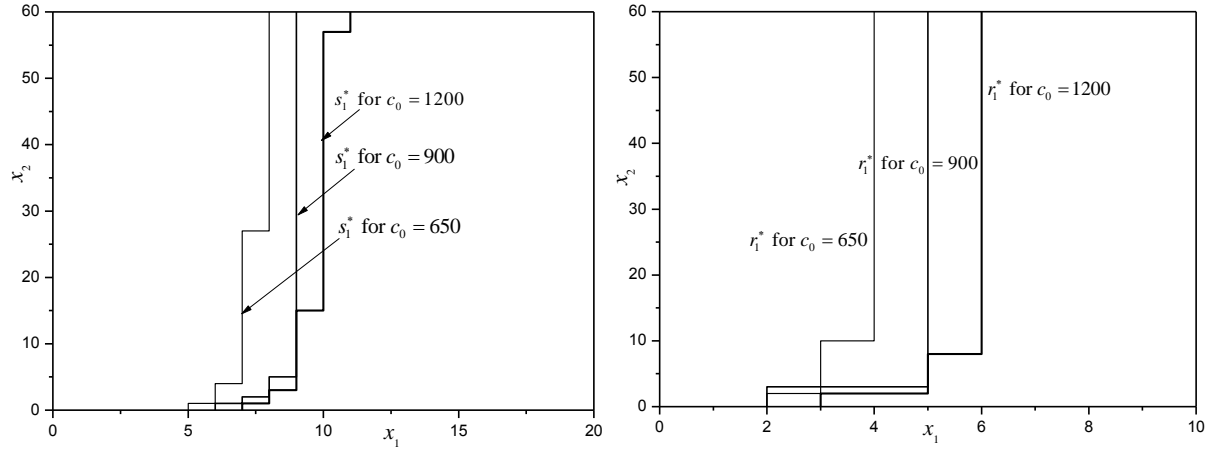
$(\mu_1 = \mu_2 = 1.0, \lambda_0 = 0.7, \lambda_1 = 0.25, \lambda_2 = 0.2, c_0 = 750, c_1 = 200, c_2 = 50, h_1 = 10, h_2 = 5)$



(a) the optimal production policy for fixed values of h_1 (b) the optimal allocation policy for fixed values of h_1

Fig. 3.3 The effect of holding cost h_1 on the optimal policy for Component 1 with lost sales

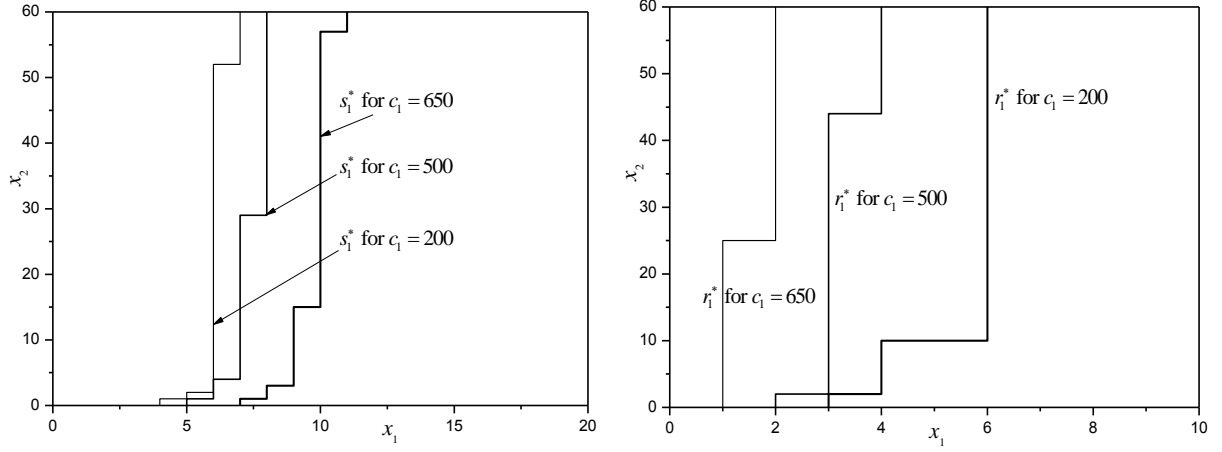
$$(\mu_1 = \mu_2 = 1.0, \lambda_0 = 0.7, \lambda_1 = 0.25, \lambda_2 = 0.2, c_0 = 750, c_1 = 200, c_2 = 50)$$



(a) the optimal production policy for fixed values of c_0 (b) the optimal allocation policy for fixed values of c_0

Fig. 3.4. The effect of lost sale cost c_0 on the optimal policy for Component 1 with lost sales

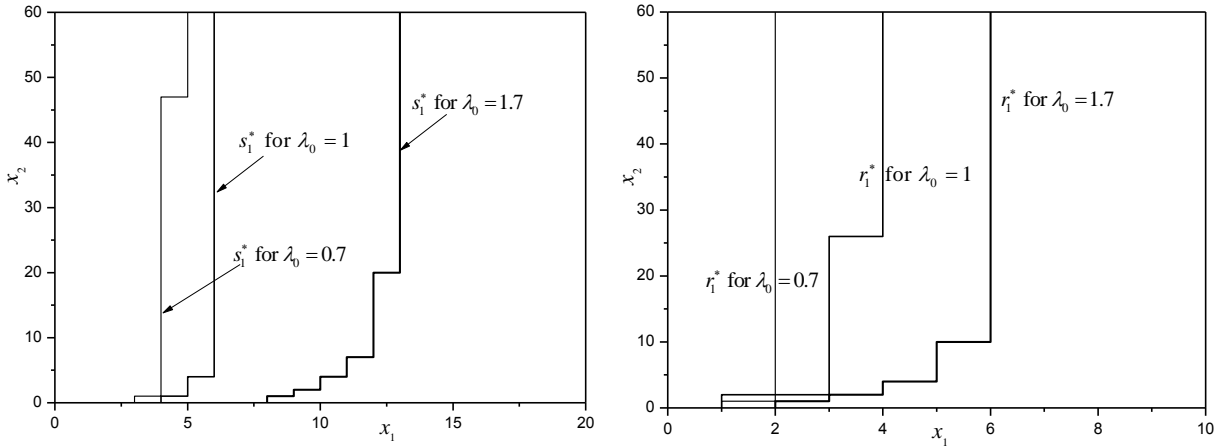
$$(\mu_1 = \mu_2 = 1.0, \lambda_0 = 0.7, \lambda_1 = 0.25, \lambda_2 = 0.2, c_1 = 200, c_2 = 50, h_1 = 10, h_2 = 5)$$



(a) the optimal production policy for fixed values of c_1 (b) the optimal allocation policy for fixed values of c_1

Fig. 3.5. The effect of lost sale cost c_1 on the optimal policy for Component 1 with lost sales

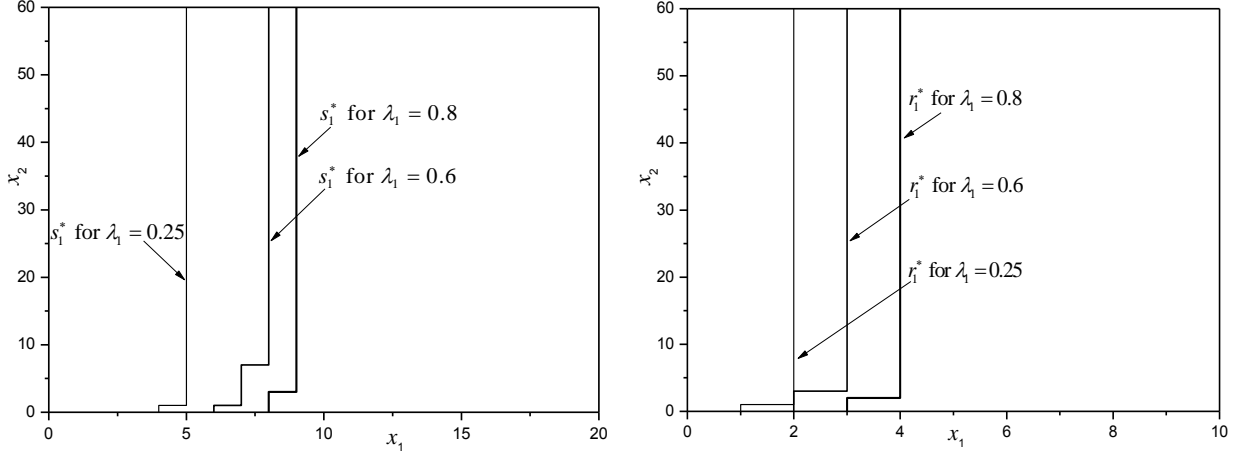
$$(\mu_1 = \mu_2 = 1.0, \lambda_0 = 0.7, \lambda_1 = 0.25, \lambda_2 = 0.2, c_0 = 750, c_2 = 50, h_1 = 10, h_2 = 5)$$



(a) the optimal production policy for fixed values of λ_0 (b) the optimal allocation policy for fixed values of λ_0

Fig. 3.6. The effect of arrival rate λ_0 on the optimal policy for Component 1 with lost sales

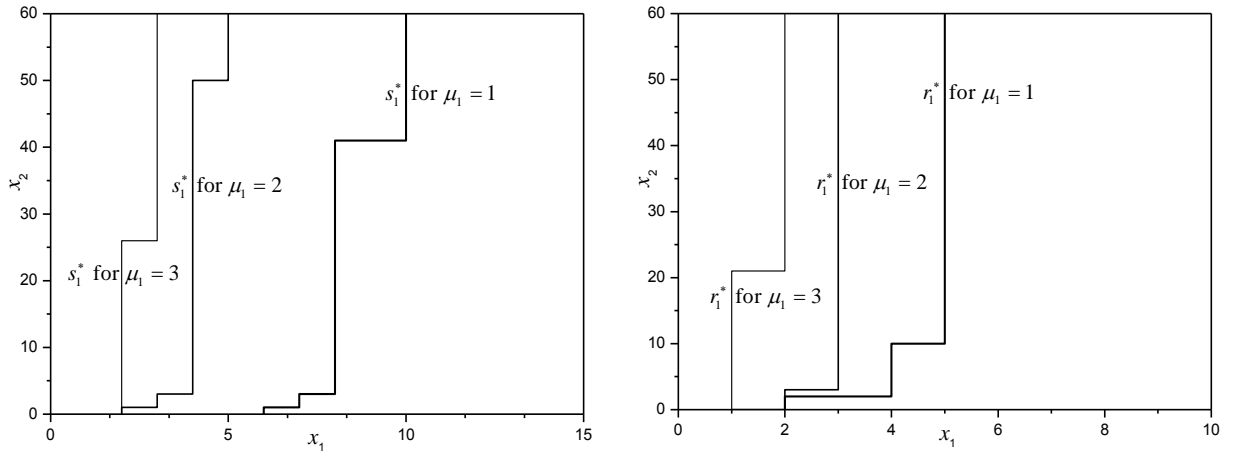
$$(\mu_1 = \mu_2 = 2.0, \lambda_1 = 0.25, \lambda_2 = 0.2, c_0 = 750, c_1 = 200, c_2 = 50, h_1 = 10, h_2 = 5)$$



(a) the optimal production policy for fixed values of λ_1 (b) the optimal allocation policy for fixed values of λ_1

Fig. 3.7. The effect of arrival rate λ_1 on the optimal policy for Component 1 with lost sales

$$(\mu_1 = \mu_2 = 2.0, \lambda_0 = 1, \lambda_2 = 0.2, c_0 = 750, c_1 = 200, c_2 = 50, h_1 = 10, h_2 = 5)$$



(a) the optimal production policy for fixed values of μ_1 (b) the optimal allocation policy for fixed values of μ_1

Fig. 3.8. The effect of production rate μ_1 on the optimal policy for Component 1 with lost sales

$$(\lambda_0 = 0.7, \lambda_1 = 0.25, \lambda_2 = 0.2, c_0 = 750, c_1 = 200, c_2 = 50, h_1 = 10, h_2 = 5)$$

Thus, there is no need to hold a high-level inventory of Component 1 in stock, and the base-stock level of Component 1 decreases. Also, high productivity can meet the coming demand and a low rationing level is optimal for the system.

3.4 Conclusion

In this chapter, we have considered an ATO system with a single product, multi-component and multi-demand, where demand from both the assembled product and the individual components can be satisfied or rejected. Demand of the assembled product is assumed to have a higher priority over demand of the individual components. The problem is modeled as an MDP. We showed that the optimal policy consists of a base-stock policy and a rationing policy where both the base-stock level and the rationing level for a component are non-decreasing in the on-hand inventory of the other components. Finally, we adopted the average cost criterion in the numerical experiments and explored the impact on the optimal policy of different system parameters, such as holding cost, lost sales cost, production rate and arrival rate.

Chapter 4. ATO System with Individual Components Demand: Lost Sales for Components and Backorders for Assembled Product

We consider the ATO system introduced in Chapter 3, however in this chapter we assume that shortages result in both lost sale and backorder penalties. The unsatisfied demands for the assembled product will wait for delivery until the next replenishment but there is a cost associated with dissatisfaction that is independent of the waiting time. The unsatisfied demands for the individual components will be lost to seek some other source to satisfy their demands immediately. Thus, the system involves a mixed lost sales and backorders. Managing such a system is known to be difficult given the correlated component demands. To solve this problem, we formulate it as an MDP and characterize its optimal policy. Finally, we investigate the impact of different penalty cost parameters on the optimal policy.

The outline of this chapter is as follows: a general introduction is given in section 4.1. In section 4.2, we formulate the optimal control model with lost sales and backorders under the discounted cost case. Then we analyze the structure of the optimal policy, and obtain several optimal properties. We also extend our model to the average cost case. In section 4.3, the numerical experiments are provided, based on numerical results we analyze the effect of the cost parameters on the optimal policy. Section 4.4 summarizes our results.

4.1 Introduction

In this chapter, we consider a variation of the system studied in Chapter 3. In particular, demand for the assembled product is satisfied only if all components are available, and a stock-out of at least one component leads to the product demand being backordered. Demand for the individual components is satisfied only if the needed component is available, and the unsatisfied component demand would be lost. Managing such a system is known to be difficult given the correlated component demands. Besides considering the on-hand inventory $\mathbf{X}(t)$ of the system, the backorder level $Y(t)$ of the assembled product must also be considered. Therefore, our state space can be extended to the pair $(\mathbf{X}(t), Y(t))$. In each state, the manager can decide what to produce and how much to produce. Here, the production decision depends on both the on-hand inventory and the backorder level. For each component, several options could be chosen: produce one unit to stock, produce one unit to reduce one unit backorder of the assembled product (if all other components are available), or do not produce at all. In this case, we assume that the product demand has no priority over the components demand. Therefore, when there are backorders from the end product demand, it is not always optimal to produce the components to reduce the backorder level. At the same time, when the demands arrive, the manager has to decide whether to allocate the inventory of the components to the individual components demand, or allocate this inventory to demand of the assembled product.

To solve this problem, we formulate it as MDP and characterize its optimal operating production and inventory allocation policies. In a general ATO system, the base-stock level and rationing level of one component depend on the on-hand inventory of other components. In our system, determination of the above threshold levels of one component should take into account both the backorder level of the assembled product and the on-hand inventory of all components. This chapter shows that for components, the optimal policy can be characterized by a production base-stock level and an inventory rationing level, while for the assembled product, the optimal policy is characterized by a state-dependent rationing level. Furthermore, we show that, the production base-stock level of components (for specific states) can be interpreted as a rationing level of the assembled product with respect to component inventory level.

4.2 The Optimal Control Problem

4.2.1 Model Formulation and Structure of the Optimal Policy

In this section, we describe the sequence of events of the ATO system. We consider a system consisting of a single product (the assembled product) assembled from n components. This chapter is based on the general assumptions in Chapter 3: component production times are exponentially distributed with mean $1/\mu_k$ for $k=1, \dots, n$; demand for Component k and the assembled product arrive according to a Poisson process with rates λ_k and λ_0 , respectively. The current state of the system at time t can be defined by the pair $(\mathbf{X}(t), Y(t))$, where $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$ and $X_k(t)$ is a nonnegative integer denoting the on-hand inventory level of Component k at time t , $Y(t)$ the backorder level for demand of the assembled product is a nonnegative integer. Each demand can be satisfied by only one single product or one component. If a demand of Component k is rejected, a lost sale cost c_k per unit per unit of time is incurred. If a demand of the assembled product is unsatisfied, a backorder cost $b_0(\cdot)$ per unit per unit of time is incurred. Each unit of Component k incurs an inventory carrying cost $h_k(\cdot)$ per unit per unit of time. We assume both $b_0(\cdot)$ and $h_k(\cdot)$ are increasing convex functions.

The expected discounted cost over an infinite planning horizon obtained under a policy π with a starting state $(\mathbf{x}, y) = (x_1, \dots, x_n, y)$ can be defined as

$$v^\pi(\mathbf{x}, y) = E_{(\mathbf{x}, y)}^\pi \left[\sum_{k=1}^n \int_0^\infty e^{-\alpha t} h_k(X_k(t)) dt + \sum_{k=1}^n \int_0^\infty e^{-\alpha t} c_k dN_k(t) + \int_0^\infty e^{-\alpha t} b_0(Y(t)) dt \right], \quad (4.1)$$

where $\alpha > 0$ is the discount rate, $N_k(t)$ is the number of units of Component k demand that have not been satisfied up to time t . Following Lippman (1975), we define the uniform rate $\beta = \sum_{l=0}^n \lambda_l + \sum_{k=1}^n \mu_k$, and let $\alpha + \beta = 1$.

The dynamic programming equation can be written as:

$$v^*(\mathbf{x}, y) = h(\mathbf{x}) + b_0(y) + \lambda_0 T^0 v^*(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}, y), \quad (4.2)$$

where, operators T^0 , T^k and T_k , $k=1, \dots, n$ are defined as follows:

$$T^0 v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y+1) & \text{if } \prod_{k=1}^n x_k = 0 \\ \min \{v(\mathbf{x} - \mathbf{e}, y), v(\mathbf{x}, y+1)\} & \text{otherwise,} \end{cases} \quad (4.3)$$

$$T^k v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y) + c_k & \text{if } x_k = 0, \\ \min \{v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k\} & \text{otherwise,} \end{cases} \quad (4.4)$$

$$T_k v(\mathbf{x}, y) = \begin{cases} \min \{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} & \text{if } y = 0 \\ \min \{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} & \text{if } y > 0, \prod_{i \neq k}^n x_i = 0 \\ \min \left\{ v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) \right\} & \text{if } y > 0, \prod_{i \neq k}^n x_i > 0. \end{cases} \quad (4.5)$$

Here, \mathbf{e}_k is the k^{th} unit vector of dimension n and $\mathbf{e} = \sum_{k=1}^n \mathbf{e}_k$ is an n -dimensional vector. The system chooses the decision that leads to the lowest system cost. Operator T^0 is the decision of how to control demand orders from the assembled product. If $\prod_{k=1}^n x_k > 0$ (i.e., all components are in stock), we consider whether to satisfy the assembled product demand or delay it for later by backordering it. If, on the other hand, $\prod_{k=1}^n x_k = 0$ (i.e., not all the components are available), the assembled product must be backordered. Operator T^k corresponds to the decision of how to control demand orders from Component k . If $x_k > 0$, we have two options: satisfy demand of Component k immediately or reject it. If $x_k = 0$, Component k is not available, thus the demand of Component k is lost. Operator T_k corresponds to the production decision of Component k . If $y = 0$ meaning that there are no backorders of the assembled product, we must choose between: do not produce Component k or produce a unit of Component k to increase its inventory. If $y > 0, \prod_{i \neq k}^n x_i = 0$ in this case there are backorders of the assembled product, and there is at least one component that is not available, the decision, in this case, is either not to produce Component k or produce a unit of Component k to increase its inventory. If $y > 0, \prod_{i \neq k}^n x_i > 0$ meaning there are backorders of the assembled product and all other components are available, the decision is either to produce a unit of Component k to stock or use the produced unit to reduce the backlog of the assembled product (through assembly with other components).

4.2.2 The Structure of the Optimal Policy

We use the dynamic programming formulation (4.1) to analyze the structure of the optimal policy for this system. We show that the optimal function $v^*(\mathbf{x}, y)$ for all states (\mathbf{x}, y) satisfies properties specified in Definition 4.1. In order to do so, we make the following important definition.

Definition 4.1: Let \mathfrak{A} be the set of functions defined on $Z^{+^{n+1}}$, where Z^+ is the set of non-negative integers, such that if $v \in \mathfrak{A}$, we have

$$\text{C1: } v(\mathbf{x} + 2\mathbf{e}_j, y) - v(\mathbf{x} + \mathbf{e}_j, y) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y) \text{ for all } \mathbf{x}, y.$$

$$\text{C2: } v(\mathbf{x}, y + 2) - v(\mathbf{x}, y + 1) \geq v(\mathbf{x}, y + 1) - v(\mathbf{x}, y) \text{ for all } \mathbf{x}, y.$$

$$\text{C3: } v(\mathbf{x} + \mathbf{e}_j, y + 1) - v(\mathbf{x}, y + 1) \leq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y) \text{ for all } \mathbf{x}, y.$$

$$\text{C4: } v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) - v(\mathbf{x} + \mathbf{e}_i, y) \leq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y) \text{ for all } \mathbf{x}, y \text{ and } i \neq j.$$

$$\text{C5: } v(\mathbf{x} + 2\mathbf{e}_j + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) - v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y)$$

for all \mathbf{x}, y and j , where $i_1, i_2, \dots, i_p \neq j$, and $1 \leq p \leq n-1$.

$$\text{C6: } v(\mathbf{x} + 2\mathbf{e}_j, y + 1) - v(\mathbf{x} + \mathbf{e}_j, y + 1) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y) \text{ for all } \mathbf{x}, y.$$

$$\text{C7: } v(\mathbf{x} + \mathbf{e}_j, y + 1) - v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \geq v(\mathbf{x}, y + 1) - v(\mathbf{x} - \mathbf{e}, y) \text{ for all } \mathbf{x}, y \text{ and } \prod_{j=1}^n x_j > 0.$$

$$\text{C8: } v(\mathbf{x}, y + 2) - v(\mathbf{x} - \mathbf{e}, y + 1) \geq v(\mathbf{x}, y + 1) - v(\mathbf{x} - \mathbf{e}, y) \text{ for all } \mathbf{x}, y \text{ and } \prod_{j=1}^n x_j > 0.$$

Property C1 indicates that the optimal cost function v^* is component-wise convex in each of the state variables x_j . Property C2 indicates that the optimal cost function v^* is component-wise convex in the backorder level y . Property C3 indicates that the marginal cost difference due to increasing the on-hand inventory of Component j is non-increasing in the backorder level y . Property C4 indicates that the marginal cost difference due to increasing the on-hand inventory of Component j is non-increasing in x_i . Property C5 indicates that the marginal cost difference due to increasing the on-hand inventory of Component j (given a fixed backorder level y) is non-decreasing with joint increases in x_j, x_{i_1}, \dots , and x_{i_p} . Property C6 indicates that the marginal cost difference due to increasing the on-hand inventory of Component j is non-decreasing with joint increases in x_j and the backorder level y . In particular, Properties C3, C4 and C5 reflect the submodularity and supermodularity of function $v(\mathbf{x}, y)$ (see Koole (2006,

1998) and Hajek (1984)).

Property C7 indicates that the marginal cost difference due to jointly increasing on-hand inventory of all components and backorder of the product, by one unit each is non-decreasing in the on-hand inventory level x_j . Property C8 indicates that the marginal cost difference due to jointly increasing on-hand inventory of all the components and the backorder level of product, by one unit each is non-decreasing in the backorder level y .

Lemma 4.1. If $v \in \mathfrak{A}$, then $Tv \in \mathfrak{A}$, where

$$Tv(\mathbf{x}, y) = h(\mathbf{x}) + b_0(y) + \lambda_0 T^0 v(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^k v(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k v(\mathbf{x}, y).$$

Proof of Lemma 4.1

In order to simplify the proof, we first define the following notation:

$$\Delta_{x_j} v(\mathbf{x}, y) = v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y),$$

$$\Delta_{x_j, x_j} v(\mathbf{x}, y) = \Delta_{x_j} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_j} v(\mathbf{x}, y),$$

$$\Delta_{x_j, x_i} v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x}, y) = \Delta_{x_j} v(\mathbf{x} + \mathbf{e}_i, y) - \Delta_{x_j} v(\mathbf{x}, y),$$

$$\Delta_{x_{i_1} + x_{i_2} + \dots + x_{i_p}} v(\mathbf{x}, y) = v(\mathbf{x} + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) - v(\mathbf{x}, y),$$

$$\Delta_y v(\mathbf{x}, y) = v(\mathbf{x}, y+1) - v(\mathbf{x}, y),$$

$$\Delta_{y, y} v(\mathbf{x}, y) = \Delta_y v(\mathbf{x}, y+1) - \Delta_y v(\mathbf{x}, y),$$

$$\Delta_{x_j, y} v(\mathbf{x}, y) = \Delta_{y, x_j} v(\mathbf{x}, y) = \Delta_{x_j} v(\mathbf{x}, y+1) - \Delta_{x_j} v(\mathbf{x}, y),$$

$$\Delta_{\sum_{l=1}^n x_l + y} v(\mathbf{x}, y) = v(\mathbf{x} + \mathbf{e}, y+1) - v(\mathbf{x}, y).$$

We also show the following two observations

Observation 1. $\Delta_{x_j} v\left(\mathbf{x} + \sum_{i \neq j}^n \mathbf{e}_i, y+1\right) \leq \Delta_{x_j} v(\mathbf{x}, y)$, for $\forall i, j \in n, i \neq j$.

Using C3 and C4, $\Delta_{x_j} v\left(\mathbf{x} + \sum_{i \neq j}^n \mathbf{e}_i, y+1\right) \leq \Delta_{x_j} v(\mathbf{x}, y+1) \leq \Delta_{x_j} v(\mathbf{x}, y)$.

Observation 2. $\Delta_y v\left(\mathbf{x} + \sum_{i \neq k}^n \mathbf{e}_i, y+1\right) \geq \Delta_y v(\mathbf{x}, y)$, for $\forall i, k \in n, i \neq k$.

Using C3 and C8, $\Delta_y v\left(\mathbf{x} + \sum_{i \neq k}^n \mathbf{e}_i, y+1\right) \geq \Delta_y v(\mathbf{x} + \mathbf{e}, y+1) \geq \Delta_y v(\mathbf{x}, y)$.

In order to prove Lemma 4.1, we need to show that if $v \in \mathfrak{A}$ then Tv must satisfy C1-C8. In order to do so, we prove that if $v \in \mathfrak{A}$ then $T^0 v \in \mathfrak{A}$, $T_k v \in \mathfrak{A}$ and $T^k v \in \mathfrak{A}$ for all k . In other words, we show that $T^0 v \in \mathfrak{A}$, $T_k v \in \mathfrak{A}$ and $T^k v \in \mathfrak{A}$ satisfy C1-C8.

First, we note that properties C1, C2, C5 and C6 are implied by properties C3, C4, C7 and C8.

To see this consider the following

Using C4 and C5 we have

$$v(\mathbf{x} + 2\mathbf{e}_j, y) - v(\mathbf{x} + \mathbf{e}_j, y) \geq v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_i, y) - v(\mathbf{x} - \mathbf{e}_i, y) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y).$$

Hence, $v(\mathbf{x} + 2\mathbf{e}_j, y) - v(\mathbf{x} + \mathbf{e}_j, y) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y)$ and C1 holds.

Using C3 and C8 we have

$$v(\mathbf{x}, y+1) - v(\mathbf{x}, y) \leq v(\mathbf{x} - \mathbf{e}_j, y+1) - v(\mathbf{x} - \mathbf{e}_j, y) \leq v(\mathbf{x}, y+2) - v(\mathbf{x}, y+1).$$

Hence, $v(\mathbf{x}, y+2) - v(\mathbf{x}, y+1) \geq v(\mathbf{x}, y+1) - v(\mathbf{x}, y)$ and C2 holds.

Using C3, C4 we have

$$v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}, y) - v(\mathbf{x} + \mathbf{e}, y) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y) \text{ and by C4 we have}$$

$$v(\mathbf{x} + 2\mathbf{e}_j + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) - v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) \geq v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}, y) - v(\mathbf{x} + \mathbf{e}, y).$$

Hence, $v(\mathbf{x} + 2\mathbf{e}_j + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) - v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_p}, y) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y)$, and C5 holds.

Using C4 and C7,

$$v(\mathbf{x} + 2\mathbf{e}_j, y+1) - v(\mathbf{x} + \mathbf{e}_j, y+1) \geq v\left(\mathbf{x} + \mathbf{e}_j - \sum_{k \neq j}^n \mathbf{e}_k, y\right) - v\left(\mathbf{x} - \sum_{k \neq j}^n \mathbf{e}_k, y\right) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y).$$

Hence, $v(\mathbf{x} + 2\mathbf{e}_j, y+1) - v(\mathbf{x} + \mathbf{e}_j, y+1) \geq v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y)$ and C6 holds.

As a consequence, we only need to show that Tv satisfies Properties C3, C4, C7, and C8.

Operator T_k

Property C3: we need to show that

$$\Delta_{x_j,y} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_j, y+1) - T_k v(\mathbf{x} + \mathbf{e}_j, y) - T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x}, y) \leq 0.$$

Since the Operator T_k depends on y and $\prod_{i \neq k}^n x_i$, we distinguish the following cases

I. $y=0$ or $y>0$ and $\prod_{i \neq k}^n x_i = 0$. In this case, Operator T_k can be rewritten as

$$T_k v(\mathbf{x}, y) = \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} = v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}.$$

Hence,

$$\begin{aligned} \Delta_{x_j,y} T_k v(\mathbf{x}, y) &= \Delta_{x_j,y} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y)\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x}, y+1)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}. \end{aligned}$$

Case $j=k$. In this case,

$$\begin{aligned} \Delta_{x_k,y} T_k v(\mathbf{x}, y) &= \Delta_{x_k,y} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x}, y+1)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}. \end{aligned}$$

By B3, we have $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)$, and $\Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y)$. By B1 and B3, we have $\Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)$, by B6 $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y)$. Hence, we have $\Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)$, which leads to five sub-cases.

$$1. \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \leq 0 \Rightarrow$$

$$\begin{aligned} \Delta_{x_k,y} T_k v(\mathbf{x}, y) &= \Delta_{x_k,y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x}, y+1) + \Delta_{x_k} v(\mathbf{x}, y) \\ &= \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) - \Delta_{x_k} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x}, y) \\ &= \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \leq 0. \end{aligned}$$

$$2. \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\begin{aligned} \Delta_{x_k,y} T_k v(\mathbf{x}, y) &= \Delta_{x_k,y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) - \Delta_{x_k} v(\mathbf{x}, y+1) + \Delta_{x_k} v(\mathbf{x}, y) \\ &= \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) - \Delta_{x_k} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x}, y) = \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq 0. \end{aligned}$$

$$3. \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\Delta_{x_k,y} T_k v(\mathbf{x}, y) = \Delta_{x_k,y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x}, y+1) = 0.$$

$$4. \Delta_{x_k} v(\mathbf{x}, y+1) \leq 0 \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\Delta_{x_k, y} T_k v(\mathbf{x}, y) = \Delta_{x_k, y} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x}, y+1) = -\Delta_{x_k} v(\mathbf{x}, y) \leq 0.$$

$$5. 0 \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\Delta_{x_k, y} T_k v(\mathbf{x}, y) = \Delta_{x_k, y} v(\mathbf{x}, y) \leq 0.$$

Case $j \neq k$. In this case,

by B4, we have $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y+1)$, and $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y)$. By B3, $\Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y)$, and $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)$. Hence, we have

$$\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y), \text{ or}$$

$$\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y).$$

Hence, we consider the following sub-cases.

$$1. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq 0 \Rightarrow$$

$$\begin{aligned} \Delta_{x_j, y} T_k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x}, y+1) + \Delta_{x_k} v(\mathbf{x}, y) \\ &= v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y+1) - v(\mathbf{x} + \mathbf{e}_j, y+1) - v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y) + v(\mathbf{x} + \mathbf{e}_k, y) \\ &= \Delta_y v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y) - \Delta_y v(\mathbf{x} + \mathbf{e}_k, y) \leq 0. \end{aligned}$$

$$2. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\begin{aligned} \Delta_{x_j, y} T_k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x}, y+1) + \Delta_{x_k} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x}, y) \\ &= \Delta_y v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y) - \Delta_y v(\mathbf{x} + \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x}, y) \leq 0. \end{aligned}$$

$$3. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\Delta_{x_j, y} T_k v(\mathbf{x}, y) = \Delta_{x_j, y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq 0.$$

$$4. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq 0 \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\Delta_{x_j, y} T_k v(\mathbf{x}, y) = \Delta_{x_j, y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq 0.$$

$$5. 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\Delta_{x_j, y} T_k v(\mathbf{x}, y) = \Delta_{x_j, y} v(\mathbf{x}, y) \leq 0.$$

$$6. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x}, y+1) \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\Delta_{x_j, y} T_k v(\mathbf{x}, y) = \Delta_{x_j, y} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y + 1) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq 0.$$

II. $y > 0$ and $\prod_{i \neq k}^n x_i > 0$. In this case, we have

$$T_k v(\mathbf{x}, y) = \min \left\{ v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1\right) \right\}$$

To prove this case, we use the same method used Ha (1997c) and Benjaafar et al. (2010). As above, we distinguish two cases.

Case $j = k$. In this case, we define the function $W(u, \mathbf{x}, y)$ on $\{0, 1\} \times Z^+$ as

$$W(u, \mathbf{x}, y) = \begin{cases} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u = 0, \\ v(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1) & \text{if } u = 1. \end{cases}$$

$T_k v$ can then be rewritten as follows:

$$\begin{aligned} T_k v(\mathbf{x}, y) &= \min \left\{ v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1\right) \right\} \\ &= \min_{u \in \{0, 1\}} W(u, \mathbf{x}, y) = (1 - u) v(\mathbf{x} + \mathbf{e}_k, y) + u v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1\right). \end{aligned}$$

Also,

$$\Delta_{x_k} W(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_{x_k} v(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1) & \text{if } u = 1. \end{cases}$$

Hence, by B7, we have $\Delta_{x_k} W(0, \mathbf{x}, y) > \Delta_{x_k} W(1, \mathbf{x}, y)$ and therefore, $W(u, \mathbf{x}, y)$ is submodular

in the direction (u, x_k) . Let $T_k v(\mathbf{x} + \mathbf{e}_k, y) = W(u_1, \mathbf{x} + \mathbf{e}_k, y)$ and $T_k v(\mathbf{x}, y + 1) = W(u_2, \mathbf{x}, y + 1)$ and consider the following two cases.

Case $u_1 \leq u_2$.

$$\begin{aligned} T_k v(\mathbf{x} + \mathbf{e}_k, y + 1) + T_k v(\mathbf{x}, y) &\leq W(u_2, \mathbf{x} + \mathbf{e}_k, y + 1) + W(u_1, \mathbf{x}, y) \\ &\leq W(u_2, \mathbf{x} + \mathbf{e}_k, y) + W(u_2, \mathbf{x}, y + 1) - W(u_2, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\ &\leq W(u_1, \mathbf{x} + \mathbf{e}_k, y) + W(u_2, \mathbf{x}, y + 1) - W(u_1, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\ &= T_k v(\mathbf{x} + \mathbf{e}_k, y) + T_k v(\mathbf{x}, y + 1). \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to B3 and the

third inequality is due to the submodularity of W in (u, x_k) .

Case $u_1 > u_2$ ($u_1 = 1, u_2 = 0$.)

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_k, y+1) + T_k v(\mathbf{x}, y) &\leq W(1, \mathbf{x} + \mathbf{e}_k, y+1) + W(0, \mathbf{x}, y) \\
 &= v\left(\mathbf{x} + \mathbf{e}_k - \sum_{i \neq k}^n \mathbf{e}_i, y\right) + v(\mathbf{x} + \mathbf{e}_k, y) \\
 &\leq v\left(\mathbf{x} + \mathbf{e}_k - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) + v(\mathbf{x} + \mathbf{e}_k, y+1) \\
 &= W(1, \mathbf{x} + \mathbf{e}_k, y) + W(0, \mathbf{x}, y+1) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_k, y) + T_k v(\mathbf{x}, y+1).
 \end{aligned}$$

The third inequality is due to the Observation 2.

Case $j \neq k$. In this case, we use the same function $W(u, \mathbf{x}, y)$ as above. Hence,

$$\Delta_y W(u, \mathbf{x}, y) = \begin{cases} \Delta_y v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_y v(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1) & \text{if } u = 1. \end{cases}$$

Also, by B8, we have $\Delta_y W(0, \mathbf{x}, y) \geq \Delta_y W(1, \mathbf{x}, y)$. Hence, $W(u, \mathbf{x}, y)$ is submodular in the direction (u, y) . Let $T_k v(\mathbf{x}, y+1) = W(u_1, \mathbf{x}, y+1)$ and $T_k v(\mathbf{x} + \mathbf{e}_j, y) = W(u_2, \mathbf{x} + \mathbf{e}_j, y)$ and consider the following cases.

Case $u_1 \leq u_2$.

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j, y+1) + T_k v(\mathbf{x}, y) &\leq W(u_2, \mathbf{x} + \mathbf{e}_j, y+1) + W(u_1, \mathbf{x}, y) \\
 &\leq W(u_2, \mathbf{x} + \mathbf{e}_j, y) + W(u_2, \mathbf{x}, y+1) - W(u_2, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\
 &\leq W(u_2, \mathbf{x} + \mathbf{e}_j, y) + W(u_1, \mathbf{x}, y+1) - W(u_1, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x}, y+1).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to B3 and the third inequality is due to the submodularity of W in (u, y) .

Case $u_1 > u_2$. ($u_1 = 1, u_2 = 0$.)

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j, y + 1) + T_k v(\mathbf{x}, y) &\leq W(1, \mathbf{x} + \mathbf{e}_j, y + 1) + W(0, \mathbf{x}, y) \\
 &= v\left(\mathbf{x} + \mathbf{e}_j - \sum_{i \neq k}^n \mathbf{e}_i, y\right) + v(\mathbf{x} + \mathbf{e}_k, y) \\
 &\leq v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) + v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y\right) \\
 &= W(0, \mathbf{x} + \mathbf{e}_j, y) + W(1, \mathbf{x}, y + 1) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x}, y + 1).
 \end{aligned}$$

The third inequality is due to B5.

Hence, $T_k v$ satisfy B3.

Property C4: we need to show that

$$\Delta_{x_j, x_i} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) - T_k v(\mathbf{x} + \mathbf{e}_i, y) - T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x}, y) \leq 0.$$

Since the Operator T_k depends on y and $\prod_{i \neq k}^n x_i$, we distinguish the following cases

I. $y = 0$ or $y > 0$ and $\prod_{i \neq k}^n x_i = 0$. In this case, Operator T_k can be rewritten as

$$T_k v(\mathbf{x}, y) = \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} = v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}.$$

Hence,

$$\begin{aligned}
 \Delta_{x_j, x_i} T_k v(\mathbf{x}, y) &= \Delta_{x_j, x_i} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y)\} \\
 &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}.
 \end{aligned}$$

Case $i = k \neq j$. In this case,

$$\begin{aligned}
 \Delta_{x_k, x_j} T_k v(\mathbf{x}, y) &= \Delta_{x_k, x_j} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)\} \\
 &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}.
 \end{aligned}$$

By C1 and C4 and C5, $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)$, which leads to five sub-cases:

$$1. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \leq 0 \Rightarrow$$

$$\begin{aligned}
 \Delta_{x_k, x_j} T_k v(\mathbf{x}, y) &= \Delta_{x_k, x_j} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + \Delta_{x_k} v(\mathbf{x}, y) \\
 &= \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \leq 0.
 \end{aligned}$$

$$2. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\begin{aligned} \Delta_{x_k, x_j} T_k v(\mathbf{x}, y) &= \Delta_{x_k, x_j} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + \Delta_{x_k} v(\mathbf{x}, y) \\ &= \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq 0. \end{aligned}$$

$$3. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\Delta_{x_k, x_j} T_k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + \Delta_{x_k} v(\mathbf{x}, y) = 0.$$

$$4. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\Delta_{x_k, x_j} T_k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) = -\Delta_{x_k} v(\mathbf{x}, y) \leq 0.$$

$$5. 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) \Rightarrow$$

$$\Delta_{x_k, x_j} T_k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x}, y) \leq 0.$$

Case $j = k \neq i$. In this case,

$$\begin{aligned} \Delta_{x_k, x_j} T_k v(\mathbf{x}, y) &= \Delta_{x_k, x_j} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i, y)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}. \end{aligned}$$

Since the proof is similar as Case $i = k \neq j$, we omit it.

Case $i \neq k \neq j$. In this case,

$$\begin{aligned} \Delta_{x_i, x_j} T_k v(\mathbf{x}, y) &= \Delta_{x_i, x_j} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y)\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}. \end{aligned}$$

By C1, C4 and C5 we have

$$\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq \Delta_{x_k} v(\mathbf{x}, y), \text{ or}$$

$$\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y).$$

If $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq \Delta_{x_k} v(\mathbf{x}, y)$, which leads to five sub-cases:

$$1. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \leq 0 \Rightarrow$$

$$\begin{aligned}\Delta_{x_i, x_j} T_k v(\mathbf{x}, y) &= \Delta_{x_i, x_j} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \\ &= \Delta_{x_i, x_j} v(\mathbf{x}, y) + \Delta_{x_j, x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq 0.\end{aligned}$$

$$4. \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\Delta_{x_i, x_j} T_k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq 0.$$

$$5. 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y) \Rightarrow$$

$$\Delta_{x_i, x_j} T_k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x}, y) \leq 0.$$

II. $y > 0$ and $\prod_{i \neq k}^n x_i > 0$. In this case, we have

$$T_k v(\mathbf{x}, y) = \min \left\{ v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{l \neq k}^n \mathbf{e}_l, y - 1\right) \right\}.$$

Case $i = k \neq j$

We use the same function $W(u, \mathbf{x}, y)$ as in the proof of C3 Case1 (2), and have the first differences of W in x_j direction.

$$\Delta_{x_j} W(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_j} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_{x_j} v(\mathbf{x} - \sum_{l \neq k}^n \mathbf{e}_l, y - 1) & \text{if } u = 1. \end{cases}$$

By C7, we have $\Delta_{x_j} W(0, \mathbf{x}, y) \geq \Delta_{x_j} W(1, \mathbf{x}, y)$ and therefore, $W(u, \mathbf{x}, y)$ is submodular in (u, x_j) .

Let $T_k v(\mathbf{x}, y) = W(u_1, \mathbf{x} + \mathbf{e}_j, y)$ and $T_k v(\mathbf{x} + \mathbf{e}_k, y) = W(u_2, \mathbf{x} + \mathbf{e}_k, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}T_k v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) + T_k v(\mathbf{x}, y) &\leq W(u_2, \mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y) + W(u_1, \mathbf{x}, y) \\ &\leq W(u_2, \mathbf{x} + \mathbf{e}_k, y) + W(u_2, \mathbf{x} + \mathbf{e}_j, y) - W(u_2, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\ &\leq W(u_2, \mathbf{x} + \mathbf{e}_k, y) + W(u_1, \mathbf{x} + \mathbf{e}_j, y) - W(u_1, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\ &= T_k v(\mathbf{x} + \mathbf{e}_k, y) + T_k v(\mathbf{x} + \mathbf{e}_j, y).\end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C4 and the third inequality is due to the submodularity of W in (u, x_j) .

Case $u_1 > u_2$, $u_1 = 1, u_2 = 0$.

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y) + T_k v(\mathbf{x}, y) &\leq W(0, \mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y) + W(1, \mathbf{x}, y) \\
 &= v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, y) + v\left(\mathbf{x} - \sum_{l \neq k}^n \mathbf{e}_l, y - 1\right) \\
 &\leq v\left(\mathbf{x} + \mathbf{e}_j - \sum_{l \neq k}^n \mathbf{e}_l, y - 1\right) + v(\mathbf{x} + \mathbf{e}_k, y) \\
 &= W(1, \mathbf{x} + \mathbf{e}_j, y) + W(0, \mathbf{x} + \mathbf{e}_k, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x} + \mathbf{e}_k, y).
 \end{aligned}$$

The third inequality is due to the Observation 1 $\Delta_{x_j} v(\mathbf{x}, y) \leq \Delta_{x_j} v\left(\mathbf{x} - \sum_{l \neq k}^n \mathbf{e}_l, y - 1\right)$, and C4

$$\Delta_{x_j} v(\mathbf{x} + \mathbf{e}_k, y) \leq \Delta_{x_j} v(\mathbf{x}, y). \text{ Hence we have } \Delta_{x_j} v(\mathbf{x} + \mathbf{e}_k, y) \leq \Delta_{x_j} v\left(\mathbf{x} - \sum_{l \neq k}^n \mathbf{e}_l, y - 1\right)$$

Case $j = k \neq i$. In this case,

$$\begin{aligned}
 \Delta_{x_k, x_i} T_k v(\mathbf{x}, y) &= \Delta_{x_k, x_i} v(\mathbf{x}, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i, y)\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y)\} \\
 &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y)\} + \min\{0, \Delta_{x_k} v(\mathbf{x}, y)\}.
 \end{aligned}$$

Since the proof is similar as Case $i = k \neq j$, we omit it.

Case $i \neq k \neq j$

We use the function $W(u, \mathbf{x}, y)$, where $W(u, \mathbf{x}, y)$ is submodular in (u, x_j) .

Let $T_k v(\mathbf{x} + \mathbf{e}_j, y) = W(u_1, \mathbf{x} + \mathbf{e}_j, y)$ and $T_k v(\mathbf{x} + \mathbf{e}_i, y) = W(u_2, \mathbf{x} + \mathbf{e}_i, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + T_k v(\mathbf{x}, y) &\leq W(u_2, \mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + W(u_1, \mathbf{x}, y) \\
 &\leq W(u_2, \mathbf{x} + \mathbf{e}_j, y) + W(u_2, \mathbf{x} + \mathbf{e}_i, y) - W(u_2, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\
 &\leq W(u_1, \mathbf{x} + \mathbf{e}_j, y) + W(u_2, \mathbf{x} + \mathbf{e}_i, y) - W(u_1, \mathbf{x}, y) + W(u_1, \mathbf{x}, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x} + \mathbf{e}_i, y).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C4 and the third inequality is due to the submodularity of W in (u, x_j) .

Case $u_1 > u_2$, $u_1 = 1, u_2 = 0$.

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + T_k v(\mathbf{x}, y) &\leq W(0, \mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + W(2, \mathbf{x}, y) \\
 &= v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_i, y) + v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) \\
 &\leq v\left(\mathbf{x} + \mathbf{e}_j - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) + v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_i, y) \\
 &= W(2, \mathbf{x} + \mathbf{e}_j, y) + W(0, \mathbf{x} + \mathbf{e}_i, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x} + \mathbf{e}_i, y).
 \end{aligned}$$

The third inequality is due to C4 and the Observation 1

$$\Delta_{x_j} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_k, y) \leq \Delta_{x_j} v(\mathbf{x}, y) \leq \Delta_{x_j} v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right)$$

Hence, $T_k v$ satisfies C4.

Property C7: we need to show that

$$\Delta_{\sum_{i=1}^n x_i + y, x_j} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_j, y+1) - T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

Since the Operator T_k depends on y and $\prod_{i \neq k}^n x_i$, we distinguish the following cases

I. $y=0$ or $y>0$ and $\prod_{i \neq k}^n x_i = 0$. In this case, Operator T_k can be rewritten as

$$T_k v(\mathbf{x}, y) = \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\}.$$

Case $j=k$. In this case,

$$\Delta_{\sum_{i=1}^n x_i + y, x_k} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_k, y+1) - T_k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) - T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

Define O on $\{0,1\} \times Z^+$ by

$$O(u, \mathbf{x}, y) = \begin{cases} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u = 0, \\ v(\mathbf{x}, y) & \text{if } u = 1. \end{cases}$$

$T_k v$ can be rewritten as follows:

$$\begin{aligned}
 T_k v(\mathbf{x}, y) &= \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} \\
 &= \min_{u \in \{0,1\}} O\{u, \mathbf{x}, y\} = (1-u)v(\mathbf{x} + \mathbf{e}_k, y) + uv(\mathbf{x}, y).
 \end{aligned}$$

Also

$$\Delta_{x_k} O(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_{x_k} v(\mathbf{x}, y) & \text{if } u = 1. \end{cases}$$

Hence, by C1, we have $\Delta_{x_k} O(0, \mathbf{x}, y) \geq \Delta_{x_k} O(1, \mathbf{x}, y)$ and therefore, $O(u, \mathbf{x}, y)$ is submodular in (u, x_k) . Let $T_k v(\mathbf{x} + \mathbf{e}_k, y + 1) = O(u_1, \mathbf{x} + \mathbf{e}_k, y + 1)$ and $T_k v(\mathbf{x} - \mathbf{e}, y) = O(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned} T_k v(\mathbf{x}, y + 1) + T_k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) &\leq O(u_1, \mathbf{x}, y + 1) + O(u_2, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\ &\leq O(u_1, \mathbf{x} + \mathbf{e}_k, y + 1) + O(u_1, \mathbf{x} - \mathbf{e}, y) - O(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + O(u_2, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\ &\leq O(u_1, \mathbf{x} + \mathbf{e}_k, y + 1) + O(u_2, \mathbf{x} - \mathbf{e}, y) - O(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + O(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\ &= T_k v(\mathbf{x} + \mathbf{e}_k, y + 1) + T_k v(\mathbf{x} - \mathbf{e}, y). \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C7 and the third inequality is due to the submodularity of O in (u, x_k) .

Case $u_1 > u_2$, $u_1 = 1, u_2 = 0$.

$$\begin{aligned} T_k v(\mathbf{x}, y + 1) + T_k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) &\leq O(0, \mathbf{x}, y + 1) + O(1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\ &= v(\mathbf{x} + \mathbf{e}_k, y + 1) + v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\ &= O(1, \mathbf{x} + \mathbf{e}_k, y + 1) + O(0, \mathbf{x} - \mathbf{e}, y) \\ &= T_k v(\mathbf{x} + \mathbf{e}_k, y + 1) + T_k v(\mathbf{x} - \mathbf{e}, y). \end{aligned}$$

Case $j \neq k$. In this case,

$$\Delta_{\sum_{i=1}^n x_i + y, x_j} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_j, y + 1) - T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - T_k v(\mathbf{x}, y + 1) + T_k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

We use the function $O(u, \mathbf{x}, y)$ and have the first differences of $O(u, \mathbf{x}, y)$ in x_j direction.

$$\Delta_{x_j} O(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_j} v(\mathbf{x} + \mathbf{e}_j, y) & \text{if } u = 0, \\ \Delta_{x_j} v(\mathbf{x}, y) & \text{if } u = 1. \end{cases}$$

Hence, by C4 $\Delta_{x_j} O(1, \mathbf{x}, y) \geq \Delta_{x_j} O(0, \mathbf{x}, y)$ we have and therefor, $O(u, \mathbf{x}, y)$ is supermodular in (u, x_j) . Let $T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) = O(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y)$ and $T_k v(\mathbf{x}, y + 1) = O(u_2, \mathbf{x}, y + 1)$ and consider two

cases:

Case $u_1 \geq u_2$,

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) &\geq O(u_2, \mathbf{x} + \mathbf{e}_j, y+1) + O(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &\geq O(u_2, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - O(u_2, \mathbf{x} - \mathbf{e}, y) + O(u_2, \mathbf{x}, y+1) + O(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &\geq O(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - O(u_1, \mathbf{x} - \mathbf{e}, y) + O(u_2, \mathbf{x}, y+1) + O(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + T_k v(\mathbf{x}, y+1).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C7 and the third inequality is due to the submodularity of O in (u, x_j) .

Case $u_1 < u_2, u_1 = 0, u_2 = 1$.

$$\begin{aligned}
 T_k v(\mathbf{x} + \mathbf{e}_j, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) &\geq O(0, \mathbf{x} + \mathbf{e}_j, y+1) + O(1, \mathbf{x} - \mathbf{e}, y) \\
 &= v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j, y+1) + v(\mathbf{x} - \mathbf{e}, y) \\
 &\geq v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j - \mathbf{e}, y) + v(\mathbf{x}, y+1) \\
 &= O(0, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + O(1, \mathbf{x}, y+1) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + T_k v(\mathbf{x}, y+1).
 \end{aligned}$$

The third inequality is due to C7.

II. $y > 0$ and $\prod_{i \neq k}^n x_i > 0$. In this case, we have

$$T_k v(\mathbf{x}, y) = \min \left\{ v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) \right\}$$

Case $j = k$. In this case,

$$\Delta_{\sum_{l=1}^n x_l + y, x_k} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_k, y+1) - T_k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) - T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

We use the function $W(u, \mathbf{x}, y)$, where $W(u, \mathbf{x}, y)$ is submodular in (u, x_k) .

Let $T_k v(\mathbf{x} + \mathbf{e}_k, y+1) = W(u_1, \mathbf{x} + \mathbf{e}_k, y+1)$ and $T_k v(\mathbf{x} - \mathbf{e}, y) = W(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}
 T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) &\leq W(u_1, \mathbf{x}, y+1) + W(u_2, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\
 &\leq W(u_1, \mathbf{x} + \mathbf{e}_k, y+1) + W(u_1, \mathbf{x} - \mathbf{e}, y) - W(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + W(u_2, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\
 &\leq W(u_1, \mathbf{x} + \mathbf{e}_k, y+1) + W(u_2, \mathbf{x} - \mathbf{e}, y) - W(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + W(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_k, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C7 and the third inequality is due to the submodularity of W in (u, x_k) .

Case $u_1 > u_2, u_1 = 1, u_2 = 0$.

$$\begin{aligned}
 T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) &\leq W(1, \mathbf{x}, y+1) + W(0, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\
 &= v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y\right) + v(\mathbf{x} + 2\mathbf{e}_k - \mathbf{e}, y) \\
 &= v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + v\left(\mathbf{x} + \mathbf{e}_k - \sum_{i \neq k}^n \mathbf{e}_i, y\right) \\
 &= W(0, \mathbf{x} - \mathbf{e}, y) + W(1, \mathbf{x} + \mathbf{e}_k, y+1) \\
 &= T_k v(\mathbf{x} - \mathbf{e}, y) + T_k v(\mathbf{x} + \mathbf{e}_k, y+1).
 \end{aligned}$$

Case $j \neq k$. In this case,

$$\Delta \sum_{l=1}^n x_l + y, x_j T_k v(\mathbf{x}, y) = T_k v(\mathbf{x} + \mathbf{e}_j, y+1) - T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

We use the function $W(u, \mathbf{x}, y)$, where $W(u, \mathbf{x}, y)$ is submodular in (u, x_j) .

Let $T_k v(\mathbf{x} + \mathbf{e}_j, y+1) = W(u_1, \mathbf{x} + \mathbf{e}_j, y+1)$ and $T_k v(\mathbf{x} - \mathbf{e}, y) = W(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}
 T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) &\leq W(u_1, \mathbf{x}, y+1) + W(u_2, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\
 &\leq W(u_1, \mathbf{x} + \mathbf{e}_j, y+1) + W(u_1, \mathbf{x} - \mathbf{e}, y) - W(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + W(u_2, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\
 &\leq W(u_1, \mathbf{x} + \mathbf{e}_j, y+1) + W(u_2, \mathbf{x} - \mathbf{e}, y) - W(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + W(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\
 &= T_k v(\mathbf{x} + \mathbf{e}_j, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C7 and the third inequality is due to the submodularity of W in (u, x_j) .

Case $u_1 > u_2, u_1 = 1, u_2 = 0$.

$$\begin{aligned}
 T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) &\leq W(1, \mathbf{x}, y+1) + W(0, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\
 &= v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y\right) + v(\mathbf{x} + \mathbf{e}_k + \mathbf{e}_j - \mathbf{e}, y) \\
 &= v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + v\left(\mathbf{x} + \mathbf{e}_j - \sum_{i \neq k}^n \mathbf{e}_i, y\right) \\
 &= W(0, \mathbf{x} - \mathbf{e}, y) + W(1, \mathbf{x} + \mathbf{e}_j, y+1) \\
 &= T_k v(\mathbf{x} - \mathbf{e}, y) + T_k v(\mathbf{x} + \mathbf{e}_j, y+1).
 \end{aligned}$$

Hence, $T_k v$ satisfies C7.

Property C8: we need to show that

$$\Delta_{\sum_{i=1}^n x_i + y, y} T_k v(\mathbf{x}, y) = T_k v(\mathbf{x}, y+2) - T_k v(\mathbf{x} - \mathbf{e}, y+1) - T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

Since the Operator T_k depends on y and $\prod_{i \neq k}^n x_i$, we distinguish the following cases

I. $y=0$ or $y>0$ and $\prod_{i \neq k}^n x_i = 0$. In this case, Operator T_k can be rewritten as

$$T_k v(\mathbf{x}, y) = \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\}.$$

We use the same function $O(u, \mathbf{x}, y)$ as in the proof of C7 and have the first differences of $O(u, \mathbf{x}, y)$ in y direction.

$$\Delta_y O(u, \mathbf{x}, y) = \begin{cases} \Delta_y v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } u=0, \\ \Delta_y v(\mathbf{x}, y) & \text{if } u=1. \end{cases}$$

Hence, by C3, we have $\Delta_y O(1, \mathbf{x}, y) \geq \Delta_y O(0, \mathbf{x}, y)$ and therefore, $O(u, \mathbf{x}, y)$ is supermodular in (u, y) .

Let $T_k v(\mathbf{x} - \mathbf{e}, y+1) = O(u_1, \mathbf{x} - \mathbf{e}, y+1)$ and $T_k v(\mathbf{x}, y+1) = O(u_2, \mathbf{x}, y+1)$ and consider two cases:

Case $u_1 \geq u_2$,

$$\begin{aligned}
 T_k v(\mathbf{x}, y+2) + T_k v(\mathbf{x} - \mathbf{e}, y) &\geq O(u_2, \mathbf{x}, y+2) + O(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &\geq O(u_2, \mathbf{x}, y+1) - O(u_2, \mathbf{x} - \mathbf{e}, y) + O(u_2, \mathbf{x} - \mathbf{e}, y+1) + O(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &\geq O(u_2, \mathbf{x}, y+1) - O(u_2, \mathbf{x} - \mathbf{e}, y) + O(u_1, \mathbf{x} - \mathbf{e}, y+1) + O(u_2, \mathbf{x} - \mathbf{e}, y) \\
 &= T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y+1).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C8 and the

third inequality is due to the supermodularity of O in (u, y) .

Case $u_1 < u_2, u_1 = 0, u_2 = 1$.

$$\begin{aligned}
 T_k v(\mathbf{x}, y+2) + T_k v(\mathbf{x} - \mathbf{e}, y) &\geq O(1, \mathbf{x}, y+2) + O(0, \mathbf{x} - \mathbf{e}, y) \\
 &= v(\mathbf{x}, y+2) + v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) \\
 &\geq v(\mathbf{x}, y+1) + v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y+1) \\
 &= O(1, \mathbf{x}, y+1) + O(0, \mathbf{x} - \mathbf{e}, y+1) \\
 &= T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y+1).
 \end{aligned}$$

The third inequality is due to the Observation 2.

II. $y > 0$ and $\prod_{i \neq k}^n x_i > 0$. In this case, we have

$$T_k v(\mathbf{x}, y) = \min \left\{ v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) \right\}$$

We use the same function $W(u, \mathbf{x}, y)$, where $W(u, \mathbf{x}, y)$ is submodular in (u, y) .

Let $T_k v(\mathbf{x}, y+2) = W(u_1, \mathbf{x}, y+2)$ and $T_k v(\mathbf{x} - \mathbf{e}, y) = W(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider the following two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}
 T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y+1) &\leq W(u_1, \mathbf{x}, y+1) + W(u_2, \mathbf{x} - \mathbf{e}, y+1) \\
 &\leq W(u_1, \mathbf{x}, y+1) + W(u_2, \mathbf{x} - \mathbf{e}, y) + W(u_2, \mathbf{x}, y+2) - W(u_2, \mathbf{x}, y+1) \\
 &\leq W(u_1, \mathbf{x}, y+1) + W(u_2, \mathbf{x} - \mathbf{e}, y) + W(u_1, \mathbf{x}, y+2) - W(u_1, \mathbf{x}, y+1) \\
 &= T_k v(\mathbf{x}, y+2) + T_k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The first inequality is due to the definition of $T_k v$; the second inequality is due to C8 and the third inequality is due to the supermodularity of W in (u, y) .

Case $u_1 > u_2, u_1 = 1, u_2 = 0$.

$$\begin{aligned}
 T_k v(\mathbf{x}, y+1) + T_k v(\mathbf{x} - \mathbf{e}, y+1) &\leq W(1, \mathbf{x}, y+1) + W(0, \mathbf{x} - \mathbf{e}, y+1) \\
 &= v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y\right) + v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y+1) \\
 &= v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y+1\right) \\
 &\leq W(0, \mathbf{x} - \mathbf{e}, y) + W(1, \mathbf{x}, y+2) \\
 &= T_k v(\mathbf{x} - \mathbf{e}, y) + T_k v(\mathbf{x}, y+2).
 \end{aligned}$$

Hence, $T_k v$ satisfies C8.

Operator T^k

Since $T^k v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y) + c_k & \text{if } x_k = 0, \\ \min\{v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k\} & \text{otherwise,} \end{cases}$ we have two cases to consider:

For case $x_k = 0$,

$T^k v(\mathbf{x}, y) = v(\mathbf{x}, y) + c_k$, hence it satisfies all the properties C1-C8.

For case $x_k > 0$,

$$T^k v(\mathbf{x}, y) = \min\{v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k\} = v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}. \quad (4.6)$$

Property C3: we need to prove that

$$\Delta_{x_j, y} T^k v(\mathbf{x}, y) = T^k v(\mathbf{x} + \mathbf{e}_j, y + 1) - T^k v(\mathbf{x} + \mathbf{e}_j, y) - T^k v(\mathbf{x}, y + 1) + T^k v(\mathbf{x}, y) \leq 0.$$

From (4.6), we have

$$\begin{aligned} \Delta_{x_j, y} T^k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y + 1) + c_k\} - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y + 1) + c_k\} + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}. \end{aligned}$$

Case $j = k$. In this case,

$$\begin{aligned} \Delta_{x_k, y} T^k v(\mathbf{x}, y) &= \Delta_{x_k, y} v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x}, y + 1) + c_k\} - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y + 1) + c_k\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x}, y) + c_k\} + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}. \end{aligned}$$

By C3 and C6, we have

$\Delta_{x_k} v(\mathbf{x}, y) + c_k \geq \Delta_{x_k} v(\mathbf{x}, y + 1) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y + 1) + c_k$, which leads to the following five sub-cases:

$$1. \quad \Delta_{x_k} v(\mathbf{x}, y) + c_k \geq \Delta_{x_k} v(\mathbf{x}, y + 1) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y + 1) + c_k \geq 0 \Rightarrow$$

$$\Delta_{x_k, y} T^k v(\mathbf{x}, y) = \Delta_{x_k, y} v(\mathbf{x} - \mathbf{e}_k, y) \leq 0.$$

$$2. \quad \Delta_{x_k} v(\mathbf{x}, y) + c_k \geq \Delta_{x_k} v(\mathbf{x}, y + 1) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y + 1) + c_k \Rightarrow$$

$$\begin{aligned}\Delta_{x_k,y} T^k v(\mathbf{x}, y) &= \Delta_{x_k,y} v(\mathbf{x} - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) - c_k \\ &= -\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \leq 0.\end{aligned}$$

$$3. \Delta_{x_k} v(\mathbf{x}, y) + c_k \geq \Delta_{x_k} v(\mathbf{x}, y+1) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \Rightarrow$$

$$\Delta_{x_k,y} T^k v(\mathbf{x}, y) = \Delta_{x_k,y} v(\mathbf{x} - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) = 0.$$

$$4. \Delta_{x_k} v(\mathbf{x}, y) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x}, y+1) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \Rightarrow$$

$$\begin{aligned}\Delta_{x_k,y} T^k v(\mathbf{x}, y) &= \Delta_{x_k,y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x}, y+1) + c_k - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \\ &= \Delta_{x_k} v(\mathbf{x}, y+1) + c_k \leq 0.\end{aligned}$$

$$5. 0 \geq \Delta_{x_k} v(\mathbf{x}, y) + c_k \geq \Delta_{x_k} v(\mathbf{x}, y+1) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \Rightarrow$$

$$\begin{aligned}\Delta_{x_k,y} T^k v(\mathbf{x}, y) &= \Delta_{x_k,y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x}, y+1) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) - \Delta_{x_k} v(\mathbf{x}, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \\ &= \Delta_{x_k} v(\mathbf{x}, y+1) - \Delta_{x_k} v(\mathbf{x}, y) = \Delta_{x_k,y} v(\mathbf{x}, y) \leq 0.\end{aligned}$$

Case $j \neq k$. In this case,

$$\begin{aligned}\Delta_{x_j,y} T^k v(\mathbf{x}, y) &= \Delta_{x_j,y} v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k\} - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k\} \\ &\quad - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k\} + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}.\end{aligned}$$

By C3, we have $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y)$ and $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y)$.

By C4, we have $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1)$ and $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y)$.

Finally, we have

$$\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k, \text{ or}$$

$$\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k.$$

If $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k$,

which leads to the following five sub-cases:

$$\begin{aligned}1. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k &\leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq 0 \Rightarrow \\ &\Delta_{x_j,y} T^k v(\mathbf{x}, y) \\ &= \Delta_{x_j,y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \\ &= v(\mathbf{x} + \mathbf{e}_j, y+1) - v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y+1) + v(\mathbf{x}, y) \\ &= \Delta_y v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_y v(\mathbf{x}, y) = \Delta_{x_j,y} v(\mathbf{x}, y) \leq 0.\end{aligned}$$

$$\begin{aligned}
 2. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k &\leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq 0 \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \Rightarrow \\
 \Delta_{x_j, y} T^k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) \\
 &\quad + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \\
 &= v(\mathbf{x} + \mathbf{e}_j, y+1) - v(\mathbf{x} + \mathbf{e}_j, y) - v(\mathbf{x}, y+1) + v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \\
 &= \Delta_{x_j, y} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \leq 0.
 \end{aligned}$$

$$\begin{aligned}
 3. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k &\leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq 0 \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \Rightarrow \\
 \Delta_{x_j, y} T^k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) \leq 0.
 \end{aligned}$$

$$\begin{aligned}
 4. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k &\leq 0 \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \Rightarrow \\
 \Delta_{x_j, y} T^k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k \leq 0.
 \end{aligned}$$

$$\begin{aligned}
 5. 0 &\leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \Rightarrow \\
 \Delta_{x_j, y} T^k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) \leq 0.
 \end{aligned}$$

If $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k$ we also consider the other sub-case:

$$\begin{aligned}
 6. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) + c_k &\leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq 0 \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y+1) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \Rightarrow \\
 \Delta_{x_j, y} T^k v(\mathbf{x}, y) &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y+1) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) \\
 &= \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_j, y} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) \leq 0.
 \end{aligned}$$

Hence, $T^k v$ satisfies C3.

Property C4: we need to prove that

$$\Delta_{x_j, x_i} T^k v(\mathbf{x}, y) = T^k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) - T^k v(\mathbf{x} + \mathbf{e}_j, y) - T^k v(\mathbf{x} + \mathbf{e}_i, y) + T^k v(\mathbf{x}, y) \leq 0.$$

From (4.6), we have

$$T^k v(\mathbf{x}, y) = \min \{ v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k \} = v(\mathbf{x} - \mathbf{e}_k, y) + \min \{ 0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \}.$$

$$\begin{aligned}
 \Delta_{x_j, x_i} T^k v(\mathbf{x}, y) &= \Delta_{x_j, x_i} v(\mathbf{x} - \mathbf{e}_k, y) + \min \{ 0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_i, y) + c_k \} - \min \{ 0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i, y) + c_k \} \\
 &\quad - \min \{ 0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \} + \min \{ 0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \}.
 \end{aligned}$$

Case $i = k \neq j$. In this case,

$$\Delta_{x_k, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k\} - \min\{0, \Delta_{x_k} v(\mathbf{x}, y) + c_k\} \\ - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k\} + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}.$$

By C4, we have $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y)$ and $\Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x}, y)$.

By C1, we have $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y)$.

By C5, we have $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y)$.

Hence, we have $\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x}, y) + c_k$,

which leads to five sub-cases:

$$1. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x}, y) + c_k \leq 0 \Rightarrow$$

$$\Delta_{x_k, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \\ = \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x}, y) \leq 0.$$

$$2. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq 0 \leq \Delta_{x_k} v(\mathbf{x}, y) + c_k \Rightarrow$$

$$\Delta_{x_k, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \\ = \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq 0.$$

$$3. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq 0 \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x}, y) + c_k \Rightarrow$$

$$\Delta_{x_k, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x} - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) = 0.$$

$$4. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq 0 \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x}, y) + c_k \Rightarrow$$

$$\Delta_{x_k, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x} - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) - c_k = -\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \leq 0.$$

$$5. 0 \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \leq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j, y) + c_k \leq \Delta_{x_k} v(\mathbf{x}, y) + c_k \Rightarrow$$

$$\Delta_{x_k, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_j} v(\mathbf{x} - \mathbf{e}_k, y) \leq 0.$$

Case $j = k \neq i$. In this case,

$$\Delta_{x_k, x_i} T^k v(\mathbf{x}, y) = \Delta_{x_k, x_i} v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i, y) + c_k\} - \min\{0, \Delta_{x_k} v(\mathbf{x}, y) + c_k\} \\ - \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k + \mathbf{e}_i, y) + c_k\} + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}.$$

Since the proof is similar to Case $i = k \neq j$, we omit it.

Case $i \neq k \neq j$. In this case,

$$\Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k\} - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k\} \\ - \min\{0, \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k\} + \min\{0, \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k\}.$$

By C4, we have

$$\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y), \text{ and } \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y).$$

Hence, we have two cases:

$$\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k, \text{ or}$$

$$\Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k.$$

$$\text{If } \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k,$$

which leads to the following five sub-cases:

$$1. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq 0 \Rightarrow \\ \Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) \leq 0.$$

$$2. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \Rightarrow \\ \Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \leq 0.$$

$$3. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \Rightarrow \\ \Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) \leq 0.$$

$$4. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \Rightarrow \\ \Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) \\ + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \\ = \Delta_{x_i} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_i} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \\ = \Delta_{x_i, x_j} v(\mathbf{x}, y) - \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) - c_k \leq 0.$$

$$5. 0 \geq \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \Rightarrow \\ \Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) \\ = \Delta_{x_i} v(\mathbf{x} + \mathbf{e}_j, y) - \Delta_{x_i} v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x}, y) \leq 0.$$

$$\text{If } \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k,$$

we must consider the other sub-case:

$$6. \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_k, y) + c_k \geq 0 \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \geq \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) + c_k \Rightarrow \\ \Delta_{x_i, x_j} T^k v(\mathbf{x}, y) = \Delta_{x_i, x_j} v(\mathbf{x} - \mathbf{e}_k, y) + \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k, y) - \Delta_{x_k} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}_k, y) \leq 0.$$

Hence, $T^k v$ satisfies C4.

Property C7: we need to prove that

$$\Delta_{\sum_{l=1}^n x_l + y, x_j} T^k v(\mathbf{x}, y) = T^k v(\mathbf{x} + \mathbf{e}_j, y + 1) - T^k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - T^k v(\mathbf{x}, y + 1) + T^k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

From (4.6), we have

$$T^k v(\mathbf{x}, y) = \min \{ v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k \}.$$

Case $j = k$. In this case, we define the function $Q(u, \mathbf{x}, y)$ as:

$$Q(u, \mathbf{x}, y) = \begin{cases} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } u = 0, \\ v(\mathbf{x}, y) + c_k & \text{if } u = 1. \end{cases}$$

$T^k v$ can be rewritten as follows:

$$T^k v(\mathbf{x}, y) = \min \{ v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k \} \\ = \min_{u \in \{0,1\}} Q(u, \mathbf{x}, y) = (1 - u) v(\mathbf{x} - \mathbf{e}_k, y) + u (v(\mathbf{x}, y) + c_k).$$

Also,

$$\Delta_{x_k} Q(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_k} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_{x_k} v(\mathbf{x}, y) & \text{if } u = 1. \end{cases}$$

Hence, by C1, we have $\Delta_{x_k} Q(1, \mathbf{x}, y) \geq \Delta_{x_k} Q(0, \mathbf{x}, y)$ and therefore, $Q(u, \mathbf{x}, y)$ is supermodular in the direction (u, x_k) .

Let $T^k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) = Q(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y)$ and $T^k v(\mathbf{x}, y + 1) = Q(u_2, \mathbf{x}, y + 1)$ and consider two cases:

Case $u_1 \geq u_2$,

$$\begin{aligned}
 T^k v(\mathbf{x} + \mathbf{e}_k, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y) &\geq Q(u_2, \mathbf{x} + \mathbf{e}_k, y+1) + Q(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &\geq Q(u_2, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) - Q(u_2, \mathbf{x} - \mathbf{e}, y) + Q(u_2, \mathbf{x}, y+1) + Q(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &\geq Q(u_1, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) - Q(u_1, \mathbf{x} - \mathbf{e}, y) + Q(u_2, \mathbf{x}, y+1) + Q(u_1, \mathbf{x} - \mathbf{e}, y) \\
 &= T^k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + T^k v(\mathbf{x}, y+1).
 \end{aligned}$$

The first inequality is due to the definition of $T^k v$; the second inequality is due to C7 and the third inequality is due to the submodularity of Q in (u, x_k) .

Case $u_1 < u_2$, ($u_1 = 0, u_2 = 1$.)

$$\begin{aligned}
 T^k v(\mathbf{x} + \mathbf{e}_k, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y) &\geq Q(0, \mathbf{x} + \mathbf{e}_k, y+1) + Q(1, \mathbf{x} - \mathbf{e}, y) \\
 &= v(\mathbf{x}, y+1) + v(\mathbf{x} - \mathbf{e}, y) + c_k \\
 &= Q(0, \mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + Q(1, \mathbf{x}, y+1) \\
 &= T^k v(\mathbf{x} + \mathbf{e}_k - \mathbf{e}, y) + T^k v(\mathbf{x}, y+1).
 \end{aligned}$$

Case $j \neq k$. In this case, we use the function $Q(u, \mathbf{x}, y)$, and have the first differences of $Q(u, \mathbf{x}, y)$ in x_j direction.

$$\Delta_{x_j} Q(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_j} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_{x_j} v(\mathbf{x}, y) & \text{if } u = 1. \end{cases}$$

By C3, we have $\Delta_{x_j} Q(0, \mathbf{x}, y) \geq \Delta_{x_j} Q(1, \mathbf{x}, y)$. Hence, $Q(u, \mathbf{x}, y)$ is submodular in (u, x_j) .

Let $T^k v(\mathbf{x} + \mathbf{e}_j, y+1) = Q(u_1, \mathbf{x} + \mathbf{e}_j, y+1)$ and $T^k v(\mathbf{x} - \mathbf{e}, y) = Q(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}
 T^k v(\mathbf{x}, y+1) + T^k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) &\leq Q(u_1, \mathbf{x}, y+1) + Q(u_2, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\
 &\leq Q(u_2, \mathbf{x} + \mathbf{e}_j, y+1) + Q(u_1, \mathbf{x}, y+1) + Q(u_2, \mathbf{x} - \mathbf{e}, y) - Q(u_2, \mathbf{x}, y+1) \\
 &\leq Q(u_1, \mathbf{x} + \mathbf{e}_j, y+1) + Q(u_1, \mathbf{x}, y+1) + Q(u_2, \mathbf{x} - \mathbf{e}, y) - Q(u_1, \mathbf{x}, y+1) \\
 &= T^k v(\mathbf{x} + \mathbf{e}_j, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The first inequality is due to the definition of $T^k v$; the second inequality is due to C7 and the third inequality is due to the submodularity of Q in (u, x_j) .

Case $u_1 > u_2$, ($u_1 = 1, u_2 = 0$.)

$$\begin{aligned}
 T^k v(\mathbf{x}, y+1) + T^k v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) &\leq Q(0, \mathbf{x}, y+1) + Q(1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\
 &= v(\mathbf{x} - \mathbf{e}_j, y+1) + v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + c_k \\
 &\leq v(\mathbf{x} + \mathbf{e}_j, y+1) + c_k + v(\mathbf{x} - \mathbf{e}_j - \mathbf{e}, y) \\
 &= Q(1, \mathbf{x} + \mathbf{e}_j, y+1) + Q(0, \mathbf{x} - \mathbf{e}, y) \\
 &= T^k v(\mathbf{x} + \mathbf{e}_j, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The third inequality is due to C7.

Hence, $T^k v$ satisfies C7.

Property C8: we need to prove that

$$\Delta_{\sum_{i=1}^n x_i + y, y} T^k v(\mathbf{x}, y) = T^k v(\mathbf{x}, y+2) - T^k v(\mathbf{x} - \mathbf{e}, y+1) - T^k v(\mathbf{x}, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

From (4.6), we have

$$T^k v(\mathbf{x}, y) = \min \{ v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k \}.$$

We use the same function $Q(u, \mathbf{x}, y)$ as the proof of C7, and define the first differences of $Q(u, \mathbf{x}, y)$ in y direction.

$$\Delta_y Q(u, \mathbf{x}, y) = \begin{cases} \Delta_y v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } u = 0, \\ \Delta_y v(\mathbf{x}, y) & \text{if } u = 1. \end{cases}$$

By C3, we have $\Delta_y Q(0, \mathbf{x}, y) \geq \Delta_y Q(1, \mathbf{x}, y)$. Hence, $Q(u, \mathbf{x}, y)$ is submodular in (u, y) .

Let $T^k v(\mathbf{x}, y+2) = Q(u_1, \mathbf{x}, y+2)$ and $T^k v(\mathbf{x} - \mathbf{e}, y) = Q(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned}
 T^k v(\mathbf{x}, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y+1) &\leq Q(u_1, \mathbf{x}, y+1) + Q(u_2, \mathbf{x} - \mathbf{e}, y+1) \\
 &\leq Q(u_1, \mathbf{x}, y+1) + Q(u_2, \mathbf{x}, y+2) + Q(u_2, \mathbf{x} - \mathbf{e}, y) - Q(u_2, \mathbf{x}, y+1) \\
 &\leq Q(u_1, \mathbf{x}, y+1) + Q(u_1, \mathbf{x}, y+2) + Q(u_2, \mathbf{x} - \mathbf{e}, y) - Q(u_1, \mathbf{x}, y+1) \\
 &= T^k v(\mathbf{x}, y+2) + T^k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The first inequality is due to the definition of $T^k v$; the second inequality is due to C8 and the third inequality is due to the submodularity of Q in (u, y) .

Case $u_1 > u_2$, ($u_1 = 1, u_2 = 0$.)

$$\begin{aligned}
 T^k v(\mathbf{x}, y+1) + T^k v(\mathbf{x} - \mathbf{e}, y+1) &\leq Q(0, \mathbf{x}, y+1) + Q(1, \mathbf{x} - \mathbf{e}, y+1) \\
 &= v(\mathbf{x} - \mathbf{e}_k, y+1) + v(\mathbf{x} - \mathbf{e}, y+1) + c_k \\
 &\leq v(\mathbf{x}, y+2) + c_k + v(\mathbf{x} - \mathbf{e}_k - \mathbf{e}, y) \\
 &= Q(1, \mathbf{x}, y+2) + Q(0, \mathbf{x} - \mathbf{e}, y) \\
 &= T^k v(\mathbf{x}, y+2) + T^k v(\mathbf{x} - \mathbf{e}, y).
 \end{aligned}$$

The third inequality is due to C7 and C8:

$$\Delta_{\sum_{j=1}^n x_j + y} v(\mathbf{x} - \mathbf{e}_k - \mathbf{e}, y) \leq \Delta_{\sum_{j=1}^n x_j + y} v(\mathbf{x} - \mathbf{e}, y) \leq \Delta_{\sum_{j=1}^n x_j + y} v(\mathbf{x} - \mathbf{e}, y+1).$$

Hence, $T^k v$ satisfies C8.

Operator T^0

$$\text{Since } T^0 v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y+1) & \text{if } \prod_{k=1}^n x_k = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}, y), v(\mathbf{x}, y+1)\} & \text{otherwise,} \end{cases} \quad \text{we have two cases:}$$

For case, $\prod_{k=1}^n x_k = 0$,

$T^0 v(\mathbf{x}, y) = v(\mathbf{x}, y+1)$, thus it satisfies all the properties C1-C8.

For case $\prod_{k=1}^n x_k > 0$,

$$T^0 v(\mathbf{x}, y) = \min\{v(\mathbf{x} - \mathbf{e}, y), v(\mathbf{x}, y+1)\} = v(\mathbf{x} - \mathbf{e}, y) + \min\left\{0, \Delta_{\sum_{k=1}^n x_k + y} v(\mathbf{x} - \mathbf{e}, y)\right\}. \quad (4.7)$$

Property C3: we need to prove that

$$\Delta_{x_j, y} T^0 v(\mathbf{x}, y) = T^0 v(\mathbf{x} + \mathbf{e}_j, y+1) - T^0 v(\mathbf{x} + \mathbf{e}_j, y) - T^0 v(\mathbf{x}, y+1) + T^0 v(\mathbf{x}, y) \leq 0.$$

We define the function $I(u, \mathbf{x}, y)$ as:

$$I(u, \mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y+1) & \text{if } u = 0, \\ v(\mathbf{x} - \mathbf{e}, y) & \text{if } u = 1. \end{cases}$$

$T^0 v$ can be rewritten as follows:

$$\begin{aligned} T^0 v(\mathbf{x}, y) &= \min \{v(\mathbf{x} - \mathbf{e}, y), v(\mathbf{x}, y + 1)\} \\ &= \min_{u \in \{0,1\}} I\{u, \mathbf{x}, y\} = (1 - u)v(\mathbf{x}, y + 1) + uv(\mathbf{x} - \mathbf{e}, y). \end{aligned}$$

Also,

$$\Delta_y I(u, \mathbf{x}, y) = \begin{cases} \Delta_y v(\mathbf{x}, y + 1) & \text{if } u = 0, \\ \Delta_y v(\mathbf{x} - \mathbf{e}, y) & \text{if } u = 1. \end{cases}$$

By C8, we have $\Delta_y I(0, \mathbf{x}, y) \geq \Delta_y I(1, \mathbf{x}, y)$ and therefore, $I(u, \mathbf{x}, y)$ is submodular in the direction (u, y) . Let $T^0 v(\mathbf{x} + \mathbf{e}_j, y) = I(u_1, \mathbf{x} + \mathbf{e}_j, y)$ and $T^0 v(\mathbf{x}, y + 1) = I(u_2, \mathbf{x}, y + 1)$ and consider two cases:

Case $u_1 \geq u_2$,

$$\begin{aligned} T^0 v(\mathbf{x} + \mathbf{e}_j, y + 1) + T^0 v(\mathbf{x}, y) &\leq I(u_1, \mathbf{x} + \mathbf{e}_j, y + 1) + I(u_2, \mathbf{x}, y) \\ &\leq I(u_1, \mathbf{x} + \mathbf{e}_j, y + 1) + I(u_2, \mathbf{x} + \mathbf{e}_j, y) + I(u_2, \mathbf{x}, y + 1) - I(u_2, \mathbf{x} + \mathbf{e}_j, y + 1) \\ &\leq I(u_2, \mathbf{x} + \mathbf{e}_j, y + 1) + I(u_1, \mathbf{x} + \mathbf{e}_j, y) + I(u_2, \mathbf{x}, y + 1) - I(u_2, \mathbf{x} + \mathbf{e}_j, y + 1) \\ &= T^0 v(\mathbf{x} + \mathbf{e}_j, y) + T^0 v(\mathbf{x}, y + 1). \end{aligned}$$

The first inequality is due to the definition of $T^0 v$; the second inequality is due to C3 and the third inequality is due to the submodularity of I in (u, y) .

Case $u_1 < u_2, (u_1 = 0, u_2 = 1.)$

$$\begin{aligned} T^0 v(\mathbf{x} + \mathbf{e}_k, y + 1) + T^0 v(\mathbf{x}, y) &\leq I(1, \mathbf{x} + \mathbf{e}_j, y + 1) + I(0, \mathbf{x}, y) \\ &= v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y + 1) + v(\mathbf{x}, y + 1) \\ &\leq v(\mathbf{x} + \mathbf{e}_j, y + 1) + v(\mathbf{x} - \mathbf{e}, y + 1) \\ &= I(0, \mathbf{x} + \mathbf{e}_j, y) + I(1, \mathbf{x}, y + 1) \\ &= T^0 v(\mathbf{x} + \mathbf{e}_j, y) + T^0 v(\mathbf{x}, y + 1). \end{aligned}$$

The third inequality is due to C5.

Hence, $T^0 v$ satisfies C3.

Property C4: we need to prove that

$$\Delta_{x_j, x_i} T^0 v(\mathbf{x}, y) = T^0 v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) - T^0 v(\mathbf{x} + \mathbf{e}_j, y) - T^0 v(\mathbf{x} + \mathbf{e}_i, y) + T^0 v(\mathbf{x}, y) \leq 0.$$

We use the function $I(u, \mathbf{x}, y)$ as in the proof of property C3, and have the first differences of $I(u, \mathbf{x}, y)$ in x_j direction.

$$\Delta_{x_j} I(u, \mathbf{x}, y) = \begin{cases} \Delta_{x_j} v(\mathbf{x}, y+1) & \text{if } u = 0, \\ \Delta_{x_j} v(\mathbf{x} - \mathbf{e}, y) & \text{if } u = 1. \end{cases}$$

By C7, we have $\Delta_{x_j} I(0, \mathbf{x}, y) \geq \Delta_{x_j} I(1, \mathbf{x}, y)$. Hence, $I(u, \mathbf{x}, y)$ is submodular in (u, x_j) .

Let $T^0 v(\mathbf{x} + \mathbf{e}_j, y) = I(u_1, \mathbf{x} + \mathbf{e}_j, y)$ and $T^0 v(\mathbf{x} + \mathbf{e}_i, y) = I(u_2, \mathbf{x} + \mathbf{e}_i, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned} T_k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + T_k v(\mathbf{x}, y) &\leq I(u_2, \mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + I(u_1, \mathbf{x}, y) \\ &\leq I(u_2, \mathbf{x} + \mathbf{e}_j, y) + I(u_2, \mathbf{x} + \mathbf{e}_i, y) - I(u_2, \mathbf{x}, y) + I(u_1, \mathbf{x}, y) \\ &\leq I(u_1, \mathbf{x} + \mathbf{e}_j, y) + I(u_2, \mathbf{x} + \mathbf{e}_i, y) - I(u_1, \mathbf{x}, y) + I(u_1, \mathbf{x}, y) \\ &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x} + \mathbf{e}_i, y). \end{aligned}$$

The first inequality is due to the definition of $T^0 v$; the second inequality is due to C4 and the third inequality is due to the submodularity of I in (u, x_j) .

Case $u_1 > u_2, (u_1 = 1, u_2 = 0)$

$$\begin{aligned} T_k v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + T_k v(\mathbf{x}, y) &\leq I(1, \mathbf{x} + \mathbf{e}_j + \mathbf{e}_i, y) + I(0, \mathbf{x}, y) \\ &= v(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_i - \mathbf{e}, y) + v(\mathbf{x}, y+1) \\ &\leq v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + v(\mathbf{x} + \mathbf{e}_i, y+1) \\ &= I(1, \mathbf{x} + \mathbf{e}_j, y) + I(0, \mathbf{x} + \mathbf{e}_i, y) \\ &= T_k v(\mathbf{x} + \mathbf{e}_j, y) + T_k v(\mathbf{x} + \mathbf{e}_i, y). \end{aligned}$$

The third inequality is due to C4 and C7 $\Delta_{x_i} v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \leq \Delta_{x_i} v(\mathbf{x} - \mathbf{e}, y) \leq \Delta_{x_i} v(\mathbf{x}, y+1)$.

Hence $T^0 v$ satisfies C4.

Property C7: we need to prove that

$$\Delta_{\sum_{l=1}^n x_l + y, x_j} T^0 v(\mathbf{x}, y) = T^0 v(\mathbf{x} + \mathbf{e}_j, y+1) - T^0 v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) - T^0 v(\mathbf{x}, y+1) + T^0 v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

We use the function $I(u, \mathbf{x}, y)$, where $I(u, \mathbf{x}, y)$ is submodular in (u, x_j) .

Let $T^0v(\mathbf{x} + \mathbf{e}_j, y+1) = I(u_1, \mathbf{x} + \mathbf{e}_j, y+1)$ and $T^0v(\mathbf{x} - \mathbf{e}, y) = I(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned} T^0v(\mathbf{x}, y+1) + T^0v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) &\leq I(u_1, \mathbf{x}, y+1) + I(u_2, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\ &\leq I(u_1, \mathbf{x} + \mathbf{e}_j, y+1) + I(u_1, \mathbf{x} - \mathbf{e}, y) - I(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + I(u_2, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\ &\leq I(u_1, \mathbf{x} + \mathbf{e}_j, y+1) + I(u_2, \mathbf{x} - \mathbf{e}, y) - I(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) + I(u_1, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\ &= T^0v(\mathbf{x} + \mathbf{e}_j, y+1) + T^0v(\mathbf{x} - \mathbf{e}, y). \end{aligned}$$

The first inequality is due to the definition of T^0v ; the second inequality is due to C7 and the third inequality is due to the submodularity of I in (u, x_j) .

Case $u_1 > u_2$, ($u_1 = 1, u_2 = 0$.)

$$\begin{aligned} T^0v(\mathbf{x}, y+1) + T^0v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) &\leq I(1, \mathbf{x}, y+1) + I(0, \mathbf{x} + \mathbf{e}_j - \mathbf{e}, y) \\ &= v(\mathbf{x} - \mathbf{e}, y+1) + v(\mathbf{x} + \mathbf{e}_j - \mathbf{e}, y+1) \\ &= I(0, \mathbf{x} - \mathbf{e}, y) + I(1, \mathbf{x} + \mathbf{e}_j, y+1) \\ &= T^0v(\mathbf{x} - \mathbf{e}, y) + T^0v(\mathbf{x} + \mathbf{e}_j, y+1). \end{aligned}$$

Hence, T^0v satisfies C7.

Property C8: we need to prove that

$$\Delta_{\sum_{l=1}^n x_l + y, y} T^0v(\mathbf{x}, y) = T^0v(\mathbf{x}, y+2) - T^0v(\mathbf{x} - \mathbf{e}, y+1) - T^0v(\mathbf{x}, y+1) + T^0v(\mathbf{x} - \mathbf{e}, y) \geq 0.$$

We use the function $I(u, \mathbf{x}, y)$, where $I(u, \mathbf{x}, y)$ is submodular in (u, y) .

Let $T^0v(\mathbf{x}, y+2) = I(u_1, \mathbf{x}, y+2)$ and $T^0v(\mathbf{x} - \mathbf{e}, y) = I(u_2, \mathbf{x} - \mathbf{e}, y)$ and consider two cases:

Case $u_1 \leq u_2$,

$$\begin{aligned} T^0v(\mathbf{x}, y+1) + T^0v(\mathbf{x} - \mathbf{e}, y+1) &\leq I(u_1, \mathbf{x}, y+1) + I(u_2, \mathbf{x} - \mathbf{e}, y+1) \\ &\leq I(u_1, \mathbf{x}, y+1) + I(u_2, \mathbf{x} - \mathbf{e}, y) + I(u_2, \mathbf{x}, y+2) - I(u_2, \mathbf{x}, y+1) \\ &\leq I(u_1, \mathbf{x}, y+1) + I(u_2, \mathbf{x} - \mathbf{e}, y) + I(u_1, \mathbf{x}, y+2) - I(u_1, \mathbf{x}, y+1) \\ &= T^0v(\mathbf{x}, y+2) + T^0v(\mathbf{x} - \mathbf{e}, y). \end{aligned}$$

The first inequality is due to the definition of T^0v ; the second inequality is due to C8 and the third inequality is due to the submodularity of I in (u, y) .

Case $u_1 > u_2, (u_1 = 1, u_2 = 0.)$

$$\begin{aligned}
 T^0 v(\mathbf{x}, y+1) + T^0 v(\mathbf{x} - \mathbf{e}, y+1) &\leq I(1, \mathbf{x}, y+1) + I(0, \mathbf{x} - \mathbf{e}, y+1) \\
 &= v(\mathbf{x} - \mathbf{e}, y+1) + v(\mathbf{x} - \mathbf{e}, y+2) \\
 &= I(0, \mathbf{x} - \mathbf{e}, y) + I(1, \mathbf{x}, y+1) \\
 &= T^0 v(\mathbf{x} - \mathbf{e}, y) + T^0 v(\mathbf{x}, y+2).
 \end{aligned}$$

Hence, $T^0 v$ satisfies C8.

Operator T

Define function $H(\mathbf{x}, y)$ as: $H(\mathbf{x}, y) = h(\mathbf{x}) + b_0(y)$ which is increasing convex function in each x_i and y . Hence, it satisfies Properties C3-C6 (submodularity/supermodularity). Because C1 and C2 (convexity) are implied by C3-C6, function $H(\mathbf{x}, y)$ satisfies C1 and C2. Since $v(\mathbf{x}, y)$ is closed under multiplication by a scalar and addition, Tv satisfies Properties C1-C6. We now prove Tv satisfies Properties C7-C8.

Proof.

Property C7

Property C7 can be rewritten as:

$$\Delta_{x_j} v(\mathbf{x}, y+1) \geq \Delta_{x_j} v(\mathbf{x} - \mathbf{e}, y).$$

$$\begin{aligned}
 \Delta_{x_j} Tv(\mathbf{x}, y+1) &= \Delta_{x_j} h(\mathbf{x}) + \Delta_{x_j} b_0(y+1) + \lambda_0 \Delta_{x_j} T^0 v(\mathbf{x}, y+1) + \sum_{k=1}^n \lambda_k \Delta_{x_j} T^k v(\mathbf{x}, y+1) + \sum_{k=1}^n \mu_k T_k \Delta_{x_j} v(\mathbf{x}, y+1) \\
 &= \Delta_{x_j} h(\mathbf{x}) + \lambda_0 \Delta_{x_j} T^0 v(\mathbf{x}, y+1) + \sum_{k=1}^n \lambda_k \Delta_{x_j} T^k v(\mathbf{x}, y+1) + \sum_{k=1}^n \mu_k T_k \Delta_{x_j} v(\mathbf{x}, y+1) \quad (\Delta_{x_j} b_0(y+1) = 0)
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{x_j} Tv(\mathbf{x} - \mathbf{e}, y) &= \Delta_{x_j} h(\mathbf{x} - \mathbf{e}) + \Delta_{x_j} b_0(y) + \lambda_0 \Delta_{x_j} T^0 v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \lambda_k \Delta_{x_j} T^k v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \mu_k T_k \Delta_{x_j} v(\mathbf{x} - \mathbf{e}, y) \\
 &= \Delta_{x_j} h(\mathbf{x} - \mathbf{e}) + \lambda_0 \Delta_{x_j} T^0 v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \lambda_k \Delta_{x_j} T^k v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \mu_k T_k \Delta_{x_j} v(\mathbf{x} - \mathbf{e}, y) \quad (\Delta_{x_j} b_0(y) = 0)
 \end{aligned}$$

Using C5, $\Delta_{x_j} h(\mathbf{x}) \geq \Delta_{x_j} h(\mathbf{x} - \mathbf{e})$, then we have $\Delta_{x_j} Tv(\mathbf{x}, y+1) \geq \Delta_{x_j} Tv(\mathbf{x} - \mathbf{e}, y)$ (for each operator using C7).

From above Tv satisfies C7.

Property C8

Property C7 can be rewritten as:

$$\Delta_y v(\mathbf{x}, y+1) \geq \Delta_y v(\mathbf{x} - \mathbf{e}, y).$$

$$\begin{aligned} \Delta_y T v(\mathbf{x}, y+1) &= \Delta_y h(\mathbf{x}) + \Delta_y b_0(y+1) + \lambda_0 \Delta_y T^0 v(\mathbf{x}, y+1) + \sum_{k=1}^n \lambda_k \Delta_y T^k v(\mathbf{x}, y+1) + \sum_{k=1}^n \mu_k T_k \Delta_y v(\mathbf{x}, y+1) \\ &= \Delta_y b_0(y+1) + \lambda_0 \Delta_y T^0 v(\mathbf{x}, y+1) + \sum_{k=1}^n \lambda_k \Delta_y T^k v(\mathbf{x}, y+1) + \sum_{k=1}^n \mu_k T_k \Delta_y v(\mathbf{x}, y+1) \quad (\Delta_y h(\mathbf{x}) = 0) \end{aligned}$$

$$\begin{aligned} \Delta_y T v(\mathbf{x} - \mathbf{e}, y) &= \Delta_y h(\mathbf{x} - \mathbf{e}) + \Delta_y b_0(y) + \lambda_0 \Delta_y T^0 v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \lambda_k \Delta_y T^k v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \mu_k T_k \Delta_y v(\mathbf{x} - \mathbf{e}, y) \\ &= \Delta_y b_0(y) + \lambda_0 \Delta_y T^0 v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \lambda_k \Delta_y T^k v(\mathbf{x} - \mathbf{e}, y) + \sum_{k=1}^n \mu_k T_k \Delta_y v(\mathbf{x} - \mathbf{e}, y) \quad (\Delta_y h(\mathbf{x} - \mathbf{e}) = 0) \end{aligned}$$

Using C2, $\Delta_y b_0(y+1) \geq \Delta_y b_0(y)$, then we have $\Delta_y T v(\mathbf{x}, y+1) \geq \Delta_y T v(\mathbf{x} - \mathbf{e}, y)$ (for each operator using C8).

From above $T v$ satisfies C8.

We use the fact that (1) $v^* = \lim_{n \rightarrow \infty} T^{(n)} v$ for any $v \in \mathfrak{A}$, where $T^{(n)}$ refers to n compositions of operator T (see Puterman, 1994, Theorem 6.10.4), and (2) $T^{(n)} v \in T$, v^* is the unique solution of $v = T v$. Hence, $T v \in \mathfrak{A}$, this completes the proof of Lemma 4.1. \square

To describe the optimal policy implied by the aforementioned properties, we define the base-stock and the rationing levels as follows:

Definition 4.2: Letting $\mathbf{x}_{-k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$, we define the following thresholds

$$s_k^*(\mathbf{x}_{-k}, y) = \begin{cases} \min \{x_k \geq 0 \mid v^*(\mathbf{x} + \mathbf{e}_k, y) - v^*(\mathbf{x}, y) \geq 0\}, & y = 0, \text{ or } y > 0, \prod_{i \neq k}^n x_i = 0, \\ \min \left\{ x_k \geq 0 \mid v^*(\mathbf{x} + \mathbf{e}_k, y) - v^* \left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1 \right) \geq 0 \right\}, & y > 0, \prod_{i \neq k}^n x_i > 0, \end{cases}$$

$$r_k^*(\mathbf{x}_{-k}, y) = \min \{x_k \geq 0 \mid v^*(\mathbf{x}, y) - v^*(\mathbf{x} - \mathbf{e}_k, y) \geq -c_k\},$$

$$R_k^*(\mathbf{x}_{-k}, y) = \min \{x_k \geq 0 \mid v^*(\mathbf{x}, y+1) - v^*(\mathbf{x} - \mathbf{e}, y) \geq 0\}.$$

Here, $s_k^*(\mathbf{x}_{-k}, y)$ denotes the base-stock level for Component k only for $y = 0$, or $y > 0, \prod_{i \neq k}^n x_i = 0$, and below in Theorem 4.1 we will show that $s_k^*(\mathbf{x}_{-k}, y)$ is in fact a rationing level for $y > 0, \prod_{i \neq k}^n x_i > 0$. Besides, $r_k^*(\mathbf{x}_{-k}, y)$ denotes the rationing level for demand of Component k , and $R_k^*(\mathbf{x}_{-k}, y)$ denotes the rationing level for demand of the assembled product with respect to Component k .

Note that when $y > 0$, $\prod_{i \neq k}^n x_i > 0$, $s_k^*(\mathbf{x}_{-k}, y) = \min \{x_k \geq 0 \mid v^*(\mathbf{x} + \mathbf{e}_k, y) - v^*(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1) \geq 0\}$.

In this case, the threshold reflects a routing decision rather than a production decision since we produce Component k to stock or produce Component k and use this unit to assemble a unit of the final product and use the latter to reduce the backlog of the final product. As a matter of fact, in this case $s_k^*(\mathbf{x}_{-k}, y)$ is tightly connected to $R_k^*(\mathbf{x}_{-k}, y)$ as shown in the following theorem.

THEOREM 4.1. *For Component k , $k=1, \dots, n$, the base-stock level $s_k^*(\mathbf{x}_{-k}, y)$ can be interpreted as a rationing level of the assembled product at Component k for $y > 0$, $\prod_{i \neq k}^n x_i > 0$. In other words, $s_k^*(\mathbf{x}_{-k}, y) = R_k^*(\mathbf{x}_{-k}, y-1)$.*

Proof:

For state (\mathbf{x}, y) , with $x_k \leq s_k^*(\mathbf{x}_{-k}, y+1)$, we have $x_k - 1 < s_k^*(\mathbf{x}_{-k}, y+1)$, and by definition of $s_k^*(\mathbf{x}_{-k}, y+1)$, we have $v^*(\mathbf{x} + \mathbf{e}_k - \mathbf{e}_k, y+1) < v^*(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i - \mathbf{e}_k, y) \Rightarrow v^*(\mathbf{x}, y+1) < v^*(\mathbf{x} - \mathbf{e}, y)$, which implies that it is optimal to backorder demand of the assembled product, when $x_k \leq s_k^*(\mathbf{x}_{-k}, y+1)$.

For state (\mathbf{x}, y) , with $x_k > s_k^*(\mathbf{x}_{-k}, y+1)$ then we have $x_k - 1 \geq s_k^*(\mathbf{x}_{-k}, y+1)$, and by definition of $s_k^*(\mathbf{x}_{-k}, y+1)$, we have $v^*(\mathbf{x} + \mathbf{e}_k - \mathbf{e}_k, y+1) \geq v^*(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i - \mathbf{e}_k, y) \Rightarrow v^*(\mathbf{x}, y+1) \geq v^*(\mathbf{x} - \mathbf{e}, y)$, which implies that it is optimal to satisfy demand of the assembled product, when $x_k > s_k^*(\mathbf{x}_{-k}, y+1)$.

Hence, when $y > 0$, $\prod_{k=1}^n x_k > 0$, $s_k^*(\mathbf{x}_{-k}, y)$ can be regarded as a rationing level for the assembled product at Component k , namely $s_k^*(\mathbf{x}_{-k}, y) = R_k^*(\mathbf{x}_{-k}, y-1)$.

This completes the proof of Theorem 4.1 \square

Properties C1- C8 together with Definition 4.2 lead to the structure the optimal policy, specified in the following theorem.

THEOREM 4.2. *For Component k , $k=1, \dots, k$, there exists an optimal stationary policy specified in terms of a state-dependent production base-stock level $s_k^*(\mathbf{x}_{-k}, y)$, for $y=0$, or $y > 0, \prod_{i \neq k}^n x_i = 0$ and a state-dependent inventory rationing level $r_k^*(\mathbf{x}_{-k}, y)$. For the assembled product, there exists an optimal stationary inventory rationing policy specified in terms of a*

state-dependent rationing level $R_k^*(\mathbf{x}_{-k}, y)$. In particular, the structure of the optimal policy can be described as follows:

1. Optimal production policy for Component k

Produce to increase Component k inventory if $x_k < s_k^*(\mathbf{x}_{-k}, y)$ for $y=0$, or $y>0$, and $\prod_{i \neq k}^n x_i = 0$.

Produce to reduce backorders of the assembled product if $x_k \geq R_k^*(\mathbf{x}_{-k}, y-1)$ for $y>0$, and $\prod_{i \neq k}^n x_i > 0$.

Do not produce if $x_k \geq s_k^*(\mathbf{x}_{-k}, y)$ for $y=0$, or $y>0$, and $\prod_{i \neq k}^n x_i = 0$.

2. Optimal inventory allocation policy for Component k

Satisfy demand of Component k if $x_k \geq r_k^*(\mathbf{x}_{-k}, y)$.

Reject demand of Component k if $x_k < r_k^*(\mathbf{x}_{-k}, y)$.

3. Optimal inventory allocation policy for the assembled product

Satisfy demand of the assembled product if $x_k \geq R_k^*(\mathbf{x}_{-k}, y)$ for all $k, k=1, 2, \dots, n$.

Backorder demand of the assembled product if at least one component has inventory level $x_k < R_k^*(\mathbf{x}_{-k}, y)$.

Furthermore, the production and rationing levels have the following properties:

P1: $s_k^*(\mathbf{x}_{-k}, y)$ is non-decreasing in both the inventory level x_i of Component $i, i \neq k$, and the backlog level of the assembled product, y .

P2: $r_k^*(\mathbf{x}_{-k}, y)$ is non-decreasing in both the inventory level x_i of Component $i, i \neq k$, and the backlog level of the assembled product, y .

P3: $R_k^*(\mathbf{x}_{-k}, y)$ is non-increasing in both the inventory level x_i of Component $i, i \neq k$, and the backlog level of the assembled product, y .

Proof of Theorem 4.2

Part 1. We use the definition of the base-stock level $s_k^*(\mathbf{x}_{-k}, y)$ and Properties C1 and C7 to prove the theorem. Three cases can be considered:

(1) $y=0$, by Property C1, we have $v^*(\mathbf{x} + \mathbf{e}_k, y) - v^*(\mathbf{x}, y)$ is non-decreasing in x_k . Hence, $v^*(\mathbf{x} + \mathbf{e}_k, y) \leq v^*(\mathbf{x}, y)$ if $x_k < s_k^*(\mathbf{x}_{-k}, 0)$ which indicates that producing Component k to stock is optimal, and $v^*(\mathbf{x}, y) \leq v^*(\mathbf{x} + \mathbf{e}_k, y)$ if $x_k \geq s_k^*(\mathbf{x}_{-k}, 0)$ which indicates that not producing Component k is optimal.

(2) $y>0, \prod_{i \neq k}^n x_i = 0$, by Property C1, $v^*(\mathbf{x} + \mathbf{e}_k, y) \leq v^*(\mathbf{x}, y)$ if $x_k < s_k^*(\mathbf{x}_{-k}, y)$ which indicates that

producing Component k to stock is optimal, and $v^*(\mathbf{x}, y) \leq v^*(\mathbf{x} + \mathbf{e}_k, y)$ if $x_k \geq s_k^*(\mathbf{x}_{-k}, y)$ which indicates that not producing Component k is optimal.

(3) $y > 0, \prod_{i \neq k}^n x_i > 0$, by Property C7, we have $v^*(\mathbf{x} + \mathbf{e}_k, y) - v^*(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1)$ is non-decreasing in x_k . Hence, $v^*(\mathbf{x} + \mathbf{e}_k, y) \leq v^*(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1)$ if $x_k < R_k^*(\mathbf{x}_{-k}, y - 1)$ which indicates that producing Component k to stock is optimal, and $v^*(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1) \leq v^*(\mathbf{x} + \mathbf{e}_k, y) \leq v^*(\mathbf{x}, y)$ if $x_k \geq R_k^*(\mathbf{x}_{-k}, y - 1)$ which indicates that producing Component k to reduce backorders of the assembled product is optimal.

Part 2. By Property C1, we have $v^*(\mathbf{x}, y) - v^*(\mathbf{x} - \mathbf{e}_k, y)$ is non-decreasing in x_k .

Hence, $v^*(\mathbf{x}, y) + c_k \leq v^*(\mathbf{x} - \mathbf{e}_k, y)$ if $x_k \leq r_k^*(\mathbf{x}_{-k}, y)$ which indicates that rejecting demand of Component k is optimal, and $v^*(\mathbf{x} - \mathbf{e}_k, y) \leq v^*(\mathbf{x}, y) + c_k$ if $x_k > r_k^*(\mathbf{x}_{-k}, y)$ which indicates that satisfying demand of Component k is optimal.

Part 3. By Property C8, we have $v^*(\mathbf{x}, y + 1) - v^*(\mathbf{x} - \mathbf{e}, y)$ is non-decreasing in y . Hence, $v^*(\mathbf{x} - \mathbf{e}, y) \leq v^*(\mathbf{x}, y + 1)$ if $x_k > R_k^*(\mathbf{x}_{-k}, y)$ which indicates that satisfying demand of the assembled product is optimal, and $v^*(\mathbf{x} - \mathbf{e}, y) \geq v^*(\mathbf{x}, y + 1)$ if $x_k \leq R_k^*(\mathbf{x}_{-k}, y)$ which indicates that backordering demand of the assembled product is optimal.

Properties P1. By Property C4 we have $v^*(\mathbf{x} + \mathbf{e}_k, y) - v^*(\mathbf{x}, y)$ is non-increasing in x_i and by property C3, we have $v^*(\mathbf{x} + \mathbf{e}_k, y) - v^*(\mathbf{x}, y)$ is non-increasing in y .

Properties P2. By Property C4 we have $v^*(\mathbf{x}, y) - v^*(\mathbf{x} - \mathbf{e}_k, y)$ is non-increasing in x_i and by property C3 we have $v^*(\mathbf{x}, y) - v^*(\mathbf{x} - \mathbf{e}_k, y)$ is non-increasing in y .

Properties P3. By Property C7 we have $v^*(\mathbf{x}, y + 1) - v^*(\mathbf{x}, y - \mathbf{e})$ is non-decreasing in x_i and by C8 we have $v^*(\mathbf{x}, y + 1) - v^*(\mathbf{x}, y - \mathbf{e})$ is non-decreasing in y .

This completes the proof of Theorem 4.2. \square

Theorem 4.2 reveals the structure of the optimal policy. Part 1 indicates that the optimal production decision for Component k is always determined by $s_k^*(\mathbf{x}_{-k}, y)$ and $R_k^*(\mathbf{x}_{-k}, y - 1)$, which depends on the amount of backorder of the assembled product, if any. That is, if $y = 0$, the base-stock level $s_k^*(\mathbf{x}_{-k}, 0)$ is defined as the inventory level below which it is optimal for the system to produce Component k to stock, and stop producing it when the on-hand inventory reaches $s_k^*(\mathbf{x}_{-k}, 0)$. If $y > 0$, there are two cases to be considered. For $\prod_{i \neq k}^n x_i = 0$, there is at least one other component with no stock, thus it is optimal for the system to produce

Component k to stock when the inventory level is below $s_k^*(\mathbf{x}_{-k}, y)$, and do not produce it when the inventory level reaches $s_k^*(\mathbf{x}_{-k}, y)$. For $\prod_{i \neq k} x_i > 0$, there is stock for all other components, thus the production decision can be controlled by $R_k^*(\mathbf{x}_{-k}, y-1)$. In this case, we produce one unit of Component k and use it to reduce the backorder level of the assembled product for $x_k \geq R_k^*(\mathbf{x}_{-k}, y-1)$. Otherwise, add it to its stock. For part 2, $r_k^*(\mathbf{x}_{-k}, y)$ represents the rationing level of demand of Component k . When the on-hand inventory of Component k is larger than the largest $r_k^*(\mathbf{x}_{-k}, y)$ for $k=1, \dots, n$, the system will use the stock of Component k to satisfy the demand of Component k . Otherwise, the system will reserve this stock to satisfy future demand of the assembled product. For part 3, $R_k^*(\mathbf{x}_{-k}, y)$ represents the rationing level of demand of the assembled product. When the on-hand inventory of Component k is larger than the largest $R_k^*(\mathbf{x}_{-k}, y)$ for $k=1, \dots, n$, the system will use the stock of the different components to satisfy the demand of the assembled product. If on the other hand, the stock of at least one component, say Component i , is less than its corresponding threshold $R_i^*(\mathbf{x}_{-i}, y)$ then stock will be reserved to satisfy future demand of components. Properties P1-P3 reflect the monotonicity of the base-stock and rationing levels. Property P1 indicates that when the on-hand inventory of other components and backorder level of the assembled product increase, production of Component k should increase. Property P2 indicates that the more on-hand inventory of other components and backorder of the assembled product, the less demand of Component k would be satisfied. We express this result as follows: since the production decision depends on both the on-hand inventory and the backorder, on the one hand, the increased on-hand inventory of other components also increase the probability that the system would satisfy demand of the assembled product, on the other hand, the increased backorder required more components to balance, thus it is more desired to reserve Component k to reduce the backorders. Property P2 implies that it is less desirable to satisfy the demand of Component k , when the on-hand inventory of other components and the backorder of the end product increase. In contrast, Property P3 indicates that more demand of the assembled product would be satisfied when the stock of other components and the backorder of the assembled product increase.

To close this section, we extend our model to the case of the average cost per period criterion. Under a control policy π and original state (\mathbf{x}, y) the average cost is given by $g^\pi(\mathbf{x}, y)$:

$$g^\pi(\mathbf{x}, y) = \limsup_{N \rightarrow \infty} \frac{E_{(\mathbf{x}, y)}^\pi \left[\sum_{i=0}^{N-1} \left(h(\mathbf{X}(t_i)) + b_0(Y(t_i)) + \sum_{k=1}^n c_k N_k(t_i) \right) \right]}{N}. \quad (4.6)$$

Our aim is to seek the optimal production policy π^* that minimizes the average cost per period $g^*(\mathbf{x}, y) = \inf_{\pi} g^\pi(\mathbf{x}, y)$ for all states (\mathbf{x}, y) . The optimality equation in this case is as follows:

$$v^*(\mathbf{x}, y) + g^* = h(\mathbf{x}) + b_0(y) + \lambda_0 T^0 v^*(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}, y), \quad (4.7)$$

where g^* is a finite constant denoting the average cost per period.

In the following Theorem 4.3, we show that the optimal policy under the average cost criterion retains all of the properties observed in Theorem 4.1 and Theorem 4.2 under the discounted criterion.

THEOREM 4.3. *The optimal stationary policy under the average cost criterion retains all the properties of the optimal policy under the discounted cost criterion. That is, the production policy is controlled by a state-dependent base-stock level $s_k^*(\mathbf{x}_{-k}, y)$, for $y = 0$, or $y > 0, \prod_{i \neq k}^n x_i = 0$ and the allocation policy is controlled by the state-dependent inventory rationing levels $r_k^*(\mathbf{x}_{-k}, y)$ and $R_k^*(\mathbf{x}_{-k}, y)$. The base-stock and the rationing levels satisfy all the properties in Theorem 4.1 and Theorem 4.2.*

Proof of Theorem 4.3

We deal with the average cost problem through the limit of discounted cost problem with $\alpha \rightarrow 0$. Since our problem can be formulated as an MDP, there are two conditions must be held for the existence of average optimal policy. As showed by Cavazos-Cadena (1992), Weber and Stidham (1987): 1. there exists a stationary policy π' which achieves a finite average cost $g^{\pi'}$. 2. the number of states in which the cost $H(\mathbf{x}, y) \leq g^{\pi'}$ is finite.

For our model, consider a policy π' that is described by: a fixed base-stock level s_k for Component k , a fixed rationing level r_k for demand of Component k , and a fixed rationing level R for demand of the assembled product. Each Component k can be produced if its on-hand inventory x_k is below s_k , each demand of Component k can be satisfied if the on-hand

inventory x_k is above r_k , and demand of the assembled product can be satisfied if the backorder level y is above R . The operators T^0 , T^k and T_k in equation (4.7) can be written as:

$$T^0 v^*(\mathbf{x}, y) = \begin{cases} v^*(\mathbf{x} - \mathbf{e}, y) & \text{if } y \geq R, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v^*(\mathbf{x}, y+1) & \text{otherwise,} \end{cases}$$

$$T^k v^*(\mathbf{x}, y) = \begin{cases} v^*(\mathbf{x} - \mathbf{e}_k, y) & \text{if } x_k \geq r_k, \\ v^*(\mathbf{x}, y) + c_k & \text{otherwise,} \end{cases}$$

$$T_k v^*(\mathbf{x}, y) = \begin{cases} v^*(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y = 0, \\ v^*(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y > 0, \text{ and } \prod_{i \neq k}^n x_i = 0, \\ v^*(\mathbf{x} + \mathbf{e}_k, y) & \text{if } y \leq R, y > 0, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v^*\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) & \text{if } y > R, y > 0, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v^*(\mathbf{x}, y) & \text{otherwise.} \end{cases}$$

Since by Properties C1 and C2 function $v^*(\mathbf{x}, y)$ is convex in each of the state variables x_k and y , there exists the minimum value that $g^* = \min v^*(\mathbf{x}, y)$. It is not hard to see that policy π' is stationary, which achieves a finite average cost and condition 1 holds. Condition 2 is due to the fact that, $H(\mathbf{x}, y) = h(\mathbf{x}) + b_0(y)$, where $h(\mathbf{x}) = \sum_{k=1}^n h_k(x_k)$, is increasing convex in x_k and y . There exists a positive integer γ' , the number of states in which the cost $H(\mathbf{x}, y) = h(\mathbf{x}) + b_0(y) \leq \gamma'$ is finite. Thus, condition 2 holds.

Based on the above conditions, Weber and Stidham (1987) proposed that there exists a positive constant g^* and the expected discounted cost $v(\mathbf{x}, y)$, then for all states (\mathbf{x}, y) :

$$v^*(\mathbf{x}, y) + g^* \geq h(\mathbf{x}) + b_0(y) + \lambda_0 T^0 v^*(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}, y).$$

So, the average cost per period g^* can be determined from the above inequation.

This completes the proof of Theorem 4.3. \square

4.3 Numerical Study

In this section, we adopt the average cost per period criterion in our numerical study. In order

to explore the optimal policy, we begin our analysis by studying an example of a system consisting of two components. We investigate the behavior of the optimal policy for a variety of cases, each with a different combination of the system parameters. We use the value iteration algorithm as described in Chapter 3. We fix the system parameters as follows $\mu_1 = \mu_2 = 1.0$, $\lambda_0 = 0.4$, $\lambda_1 = 0.5$, $\lambda_2 = 0.4$, $h_1 = 2$, $h_2 = 1$, and vary b_0 and c_k ($k=1,2$). The following figures illustrate the structure of the optimal production policy for Component 1 and the optimal inventory allocation policy of the assembled product and Component 1. Similar results are obtained for Component 2 and demand of Component 2, hence omitted.

Fig. 4.1(a)-Fig. 4.1(d) show the optimal production policy for Component 1 when the backorder cost rate, b_0 is much lower than the lost sale costs c_1 and c_2 . Here, we consider the following two cases. If $y=0$, as seen in Fig. 4.1(a), there are no backorders of the assembled product, the system can be viewed as a pure lost sales system. The on-hand inventory of Component 1 is controlled by the base-stock policy with a base-stock level $s_1^*(x_2, 0)$. That is, if the on-hand inventory of Component 1 is less than $s_1^*(x_2, 0)$, then produce Component k to increase its inventory up to $s_1^*(x_2, 0)$. Otherwise, do not produce Component k . If $y>0$, as seen in Fig. 4.1(b)-Fig. 4.1(d), the on-hand inventory of Component 1 is controlled by both the base-stock level $s_1^*(x_2, y)$ and the rationing level $R_1^*(x_2, y-1)$. That is, we consider the decision of producing to reduce backorders of the assembled product if $x_1 \geq R_1^*(x_2, y-1)$ and $x_2 \geq R_2^*(x_1, y-1)$, or producing to increase the on-hand inventory of Component 1, otherwise. Fig. 4.1(a)-Fig. 4.1(d) also display the structure of the optimal production policy with different backorder levels of y .

In Fig. 4.1(b), the different optimal production regions are determined by three thresholds: $s_1^*(x_2, 5)$, $R_1^*(x_2, 4)$ and $R_2^*(x_1, 4)$. In region $x_1 \geq s_1^*(x_2, 5)$ given $y=5$, $x_1=35$, $x_2=1$ for example, the optimal production policy tends not produce Component 1. This means, the backorder level is small and the system has a large stock of Component 1 but small stock of Component 2. In this case, the stock is sufficient for Component 1, thus there is no need to produce it. In region $x_1 \geq R_1^*(x_2, 4)$ and $x_2 \geq R_2^*(x_1, 4)$ given $y=5$, $x_1=20$, $x_2=25$ for example, the system has a large stock of both Components 1 and 2, the optimal production policy tends to produce Component 1 to reduce backorders of the assembled product. In the remaining region, given $y=5$, $x_1=2$, $x_2=25$ for example, the system has a small stock of Component 1 but a large stock of Component 2, the optimal production policy tends to produce Component 1 to increase its

on-hand inventory. In this case, both Component 1 and 2 can be reserved to satisfy future demand of the assembled product. Similar results can also be easily seen from Fig. 4.1(c) and Fig. 4.1(d). Comparing these three figures, we observe that the base-stock level $s_1^*(x_2, y)$ increases with x_2 and y . The optimal production decision region that produces Component 1 to increase its on-hand inventory is smaller when backorder level y increases. This result is quite intuitive: since the production decision depends on both the on-hand inventory and backorder level, the increased backorder required more components to balance. Hence, it is desired to produce Component 1 to reduce the backorders rather than produce it to increase its inventory. Also from Fig. 4.1(b)-Fig. 4.1(d), the rationing level $R_k^*(\mathbf{x}_{-k}, y-1)$, $k=1, 2$, can be viewed as a production parameter, thus the optimal production decision region is bounded by the base-stock level $s_1^*(x_2, y)$ and the rationing level $R_k^*(\mathbf{x}_{-k}, y-1)$.

In Fig. 4.2(a)-Fig. 4.2(d), we still let the backorder cost b_0 be lower than the lost sale costs c_1 and c_2 . But, reduce the difference between them by keeping b_0 unchanged and reducing the value of c_1 to 100 and c_2 to 75. In this setting, comparing with Fig. 4.1(a)-Fig. 4.1(d), the optimal production regions have changed. For example, as seen in Fig. 4.2(a), when there are no backorders, the base-stock level $s_1^*(x_2, 0)$ is lower than that in Fig. 4.1(a). The fact that when both c_1 and c_2 decrease, the optimal production policy tends to produce more Component 1 and 2 to reduce backorders of the assembled product. If $y=0$, there is no need to keep a large number of Component 1 in stock that may increase the holding cost, thus the base-stock level $s_1^*(x_2, 0)$ decreases. If $y>0$, as seen from Fig. 4.2(b)-Fig. 4.2(d), the optimal allocation region that produces Component 1 to reduce backorders of the assembled product is larger than that in Fig. 4.1(b)-Fig. 4.1(d) respectively, while the optimal production region that produces Component 1 to increase its on-hand inventory becomes smaller. We discuss two cases. If $x_2>0$, for the same reason as mentioned in Fig. 4.2(a), the low rationing level $R_1^*(x_2, y-1)$ at Component 1 is optimal. On the other hand, when backorder level y increases, the system tends to produce more Components 1 to deal with large backorders from the assembled product, thus the optimal allocation decision region that produces to reduce backorders of the assembled product grows larger. If $x_2 = 0$, the base-stock level $s_1^*(0, y)$ with different y is lower than that in Fig.4.1 respectively. In that region, demand of the assembled product is always backordered whatever the value of x_1 . Component 1 can be produced to satisfy its demand, thus it may not be optimal to keep a high base-stock level for Component 1.

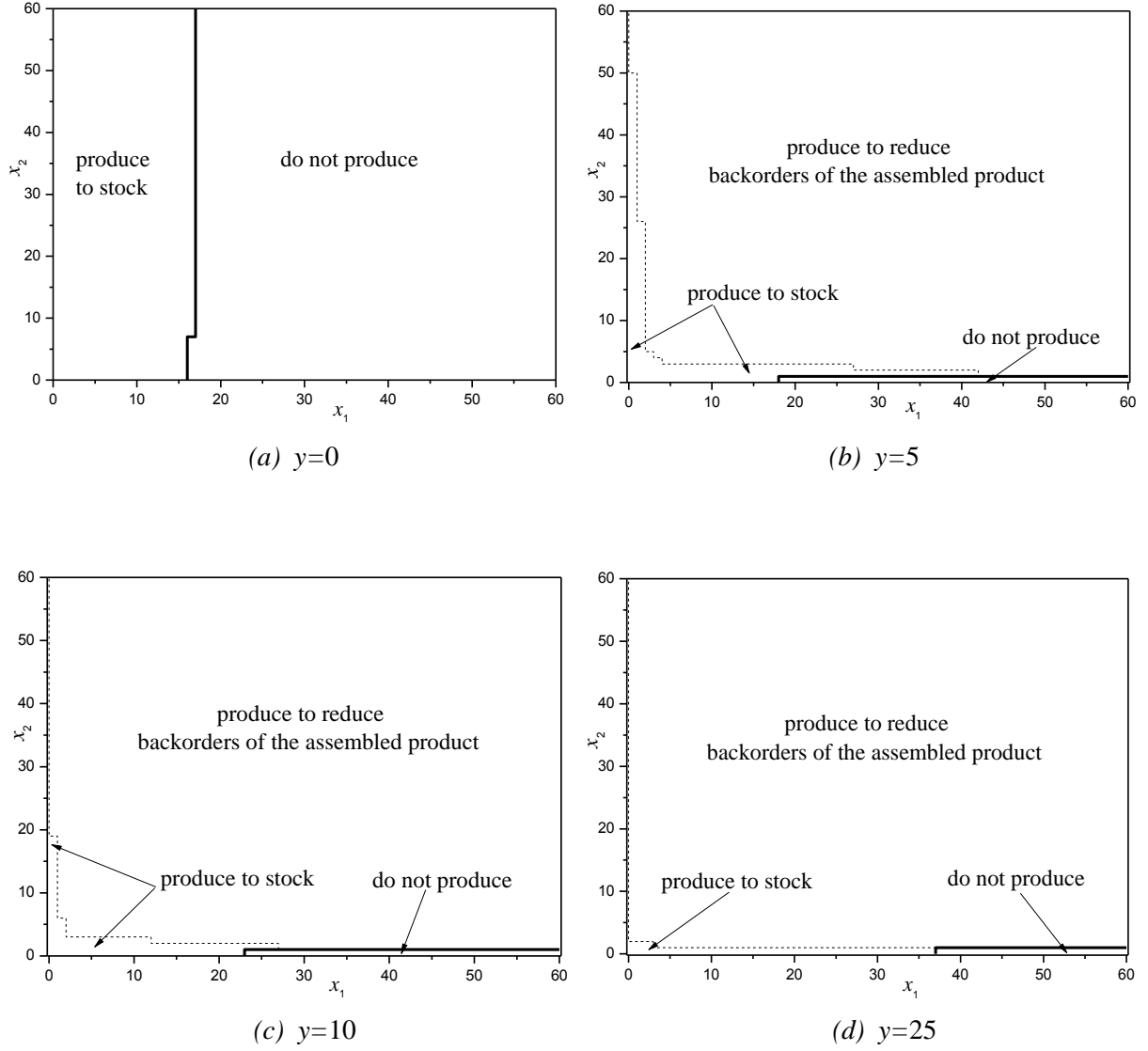


Fig. 4.1. The optimal production policy for Component 1 with lost sales and backorders ($b_0=10, c_1=1000, c_2=800$)

In Fig. 4.3(a)-Fig. 4.3(d), the backorder cost b_0 is larger than the lost sale costs c_1 and c_2 . In this setting, comparing with that in Fig. 4.2, we increase the backorder cost b_0 to 200 and keep lost sale costs c_1 and c_2 unchanged. That means the product demand has priority over the components demand, thus the former one should always be satisfied if possible. For this reason, as displayed in Fig. 4.3(b)-Fig. 4.3(d), it is the more beneficial to produce Component 1 to reduce backorders of the assembled product as long as $y > 0, x_2 > 0$. Even if there are no backorders, see Fig. 4.3(a), the optimal production policy tends to produce more Components 1 to stock for future demand of the assembled product, thus the base-stock level $s_1^*(x_2, 0)$ is higher than that in Fig. 4.2(a). For similar reason, if $y > 0, x_2 = 0$, the base-stock level $s_1^*(0, y)$ with the different value of y is higher than that in Figure 4.2, respectively. Moreover, Fig. 4.2(b)-Fig. 4.2(d) and Fig. 4.3(b)-Fig. 4.3(d) show that the backorder level has a strong impact on the optimal production decision. When there are backorders, the decision that produces as much as possible to reduce the backorder is optimal, thus in these figures the rationing level becomes very low $R_1^*(x_2, y-1) = 0$.

Fig. 4.4(a) and Fig. 4.4(b) show the structure of the optimal allocation policy for demand of the assembled product with larger lost sale costs $c_1=1000$ and $c_2=800$. Fig. 4.4(a) illustrates that there exist a rationing level $R_1^*(x_2, y)$ that divides the state space into two regions: it is optimal to satisfy demand of the assembled product if on-hand inventory for Component 1 is above the rationing level $R_1^*(x_2, y)$, and backorder it otherwise. As we can see the rationing level $R_1^*(x_2, y)$ is non-increasing in x_2 . This is because a high inventory level of Component 2 increases the probability that demand of the assembled product would be satisfied. Hence, the rationing level $R_1^*(x_2, y)$ decreases. Otherwise, the rationing level $R_1^*(x_2, y)$ increases. Fig. 4.4(b) also illustrates that the rationing level $R_1^*(x_2, y)$ is non-increasing in backorder level y . It is known that the increase of backorder level would also increase the total cost of the system. In order to reduce this effect, it is natural to decrease the rationing level $R_1^*(x_2, y)$ when the backorder level increases.

Fig. 4.5(a) and Fig. 4.5(b) show the structure of the optimal allocation policy for demand of the assembled product with lower lost sale costs $c_1=100$ and $c_2=75$. Obviously, the rationing level $R_1^*(x_2, y)$ is lower than that in Fig. 4.4(b), respectively. This is because in Fig.4.5 the

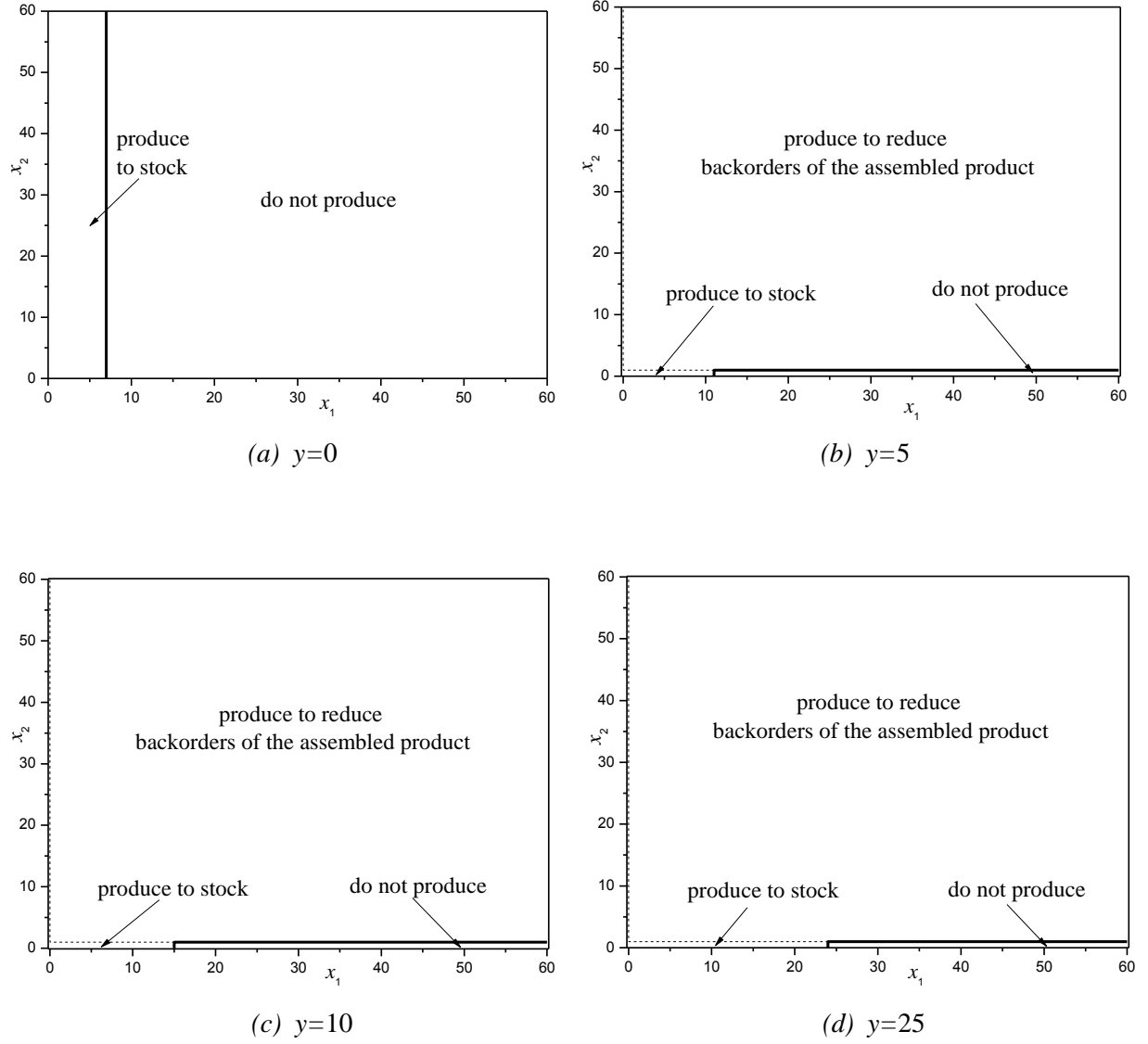


Fig. 4.2. The optimal production policy for Component 1 with lost sales and backorders
($b_0=10, c_1=100, c_2=75$)

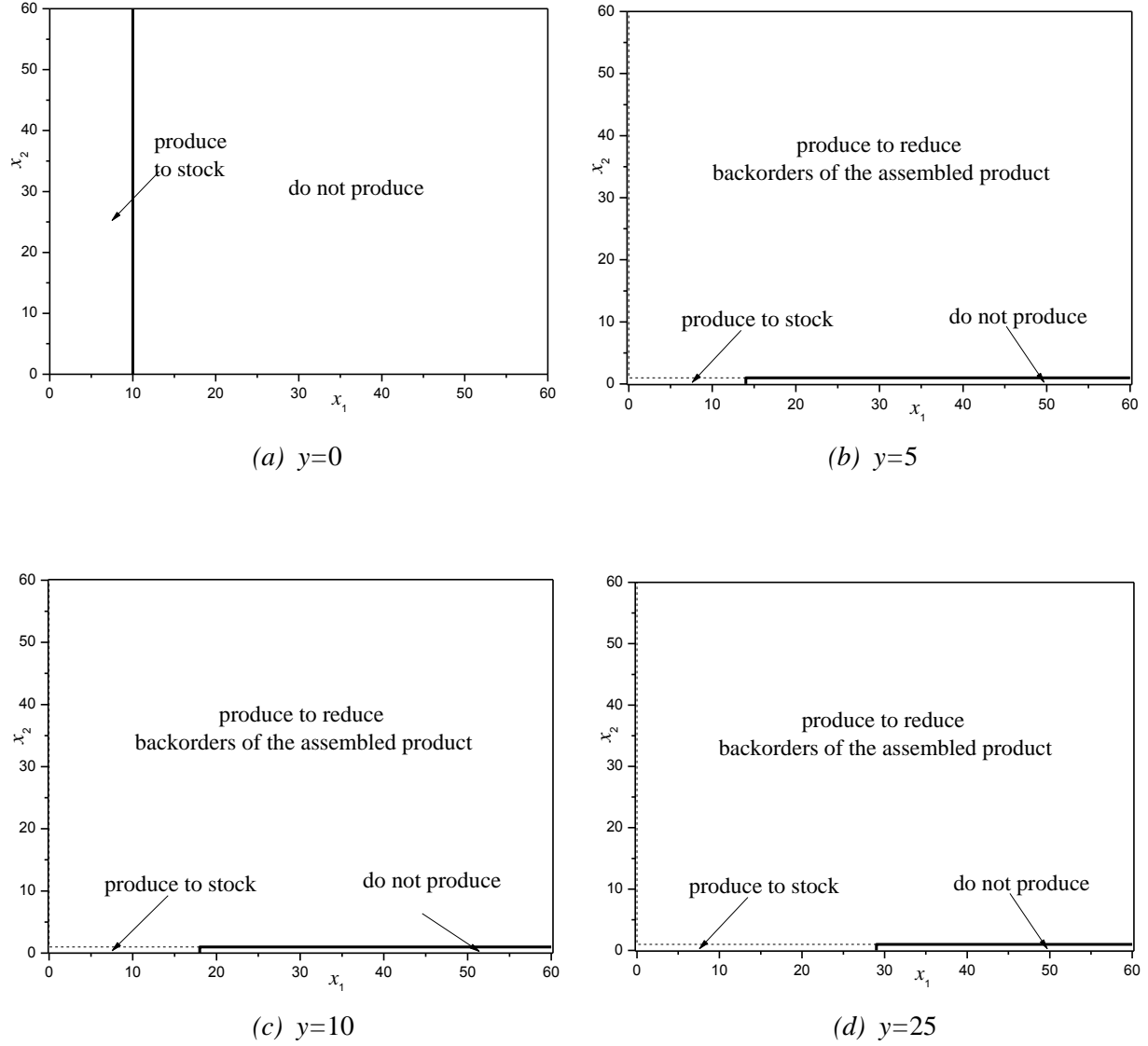


Fig. 4.3. The optimal production policy for Component 1 with lost sales and backorders ($b_0=200$, $c_1=100$, $c_2=75$)

assembled product demand has a relative priority over the individual components demand. Hence, the assembled product demand would be satisfied easily. Also seen from Fig. 4.5(a) and Fig. 4.5(b), the effect of backorder level y is not significant. Especially when $x_2 > 1$, the rationing level becomes to a vertical line with the fixed value $x_1 = 1$. This means if all components are available, it is always optimal to satisfy demand of the assembled product. This result also can be seen from Fig. 4.6, when the backorder cost b_0 is larger than lost sale costs c_1 and c_2 , the optimal allocation region that satisfies demand of the assembled product becomes large enough, and then the rationing level $R_1^*(x_2, y)$ becomes to $x_1 = 1$, for all y .

Fig. 4.7(a) and Fig. 4.7(b) show the structure of the optimal allocation policy for demand of Component 1 under the condition $b_0 = 10$, $c_1 = 1000$, $c_2 = 800$. As displayed in Fig. 4.7(a), the optimal allocation policy divides the state space into two regions: it is optimal to satisfy demand of Component 1 if on-hand inventory x_1 is above the rationing level $r_1^*(x_2, y)$, and reject it otherwise. The rationing level $r_1^*(x_2, y)$ is non-decreasing in both the on-hand inventory x_1 and backorder level y . The explanation is that when backorder level y increases, in order to avoid a large backorder cost penalty, it is desirable to assemble more Component 1 and Component 2 to reduce backorders of the assembled product. Hence, the system tends to keep less Component 1 to stock for its demands that leads to the rationing level $r_1^*(x_2, y)$ increases. Fig. 4.7(b) illustrates that the rationing level $r_1^*(x_2, y)$ is also non-decreasing in the on-hand inventory x_2 . Given a fixed backorder level y , a high inventory level of Component 2 increases the probability that demand of the assembled product would be satisfied. As a result, more Component 1 is needed for demand of the assembled product and demand of Component 1 becomes difficult to satisfy, thus the rationing level $r_1^*(x_2, y)$ increases. In contrast, the rationing level $r_1^*(x_2, y)$ decreases when the system has a low inventory level of Component 2.

Fig. 4.8(a) and Fig. 4.8(b) show the structure of the optimal allocation policy for demand of Component 1 under the condition $b_0 = 10$, $c_1 = 100$, $c_2 = 75$. Obviously, the monotonicity result is similar to that in Fig. 4.7(a) and Fig. 4.7(b), but this dynamic trend is more significant. Relative to the original lost sale costs in Fig. 4.7, both c_1 and c_2 decrease in Fig. 4.8. In other words, the components demand has a lower priority which reduces the probability that demand of Component 1 would be satisfied. Therefore, the rationing level at Component 1 would be satisfied. Therefore, the rationing level at Component 1 with different value of x_2 is

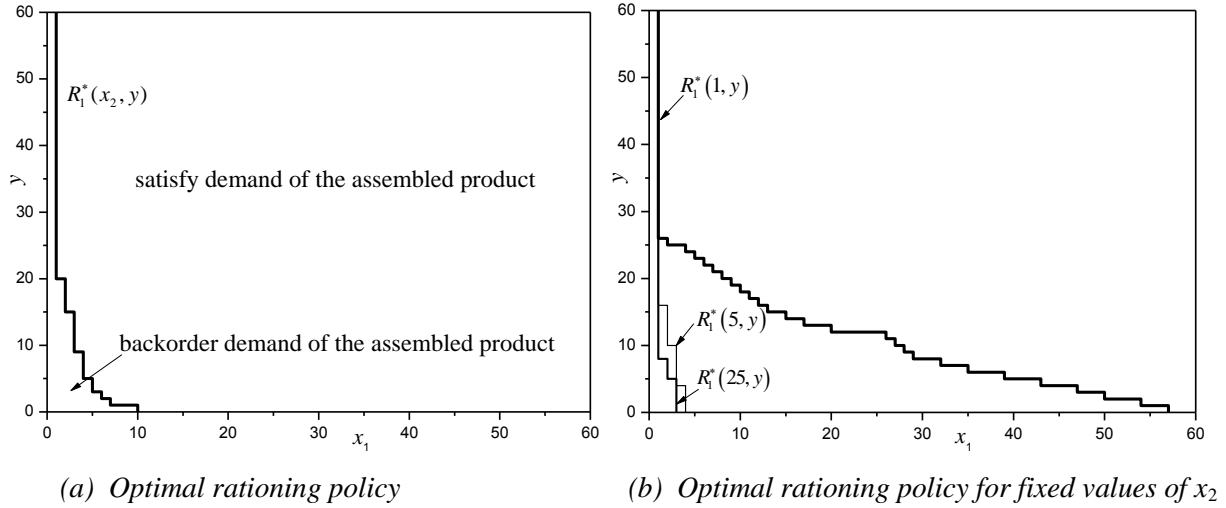


Fig. 4.4. The optimal allocation policy for demand of the assembled product at Component 1 with lost sales and backorders ($b_0=10$, $c_1=1000$, $c_2=800$)

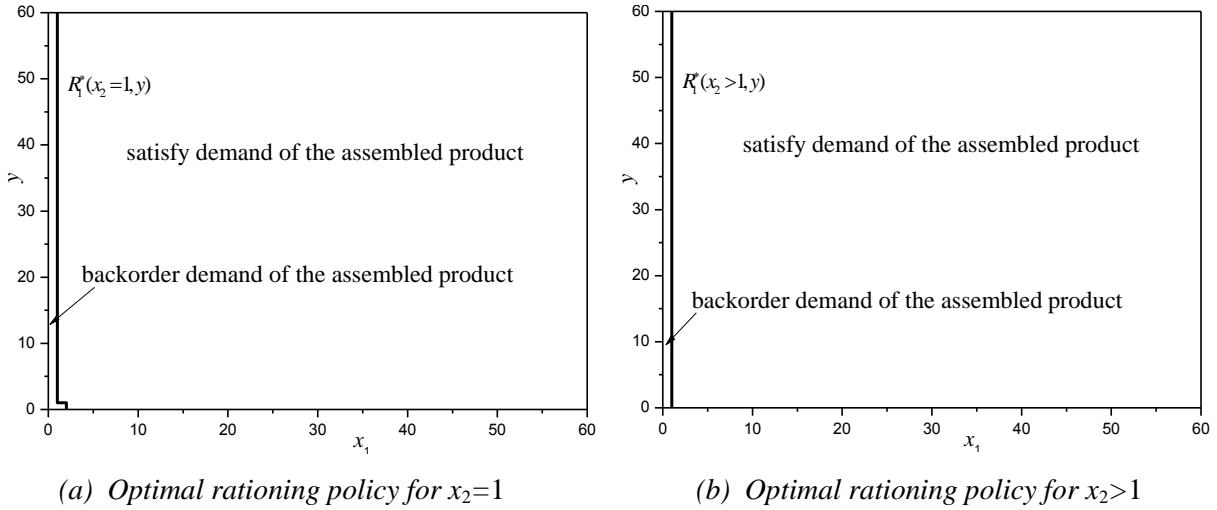


Fig. 4.5. The optimal allocation policy for demand of the assembled product at Component 1 with lost sales and backorders ($b_0=10$, $c_1=100$, $c_2=75$)

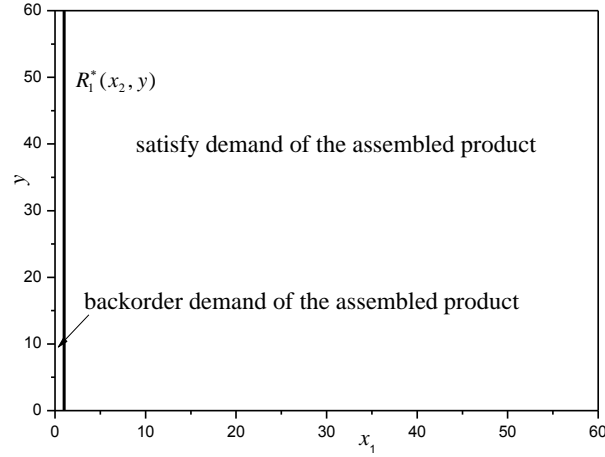


Fig. 4.6. The optimal allocation policy for demand of the assembled product at Component 1 with lost sales and backorders ($b_0=200, c_1=100, c_2=75$)

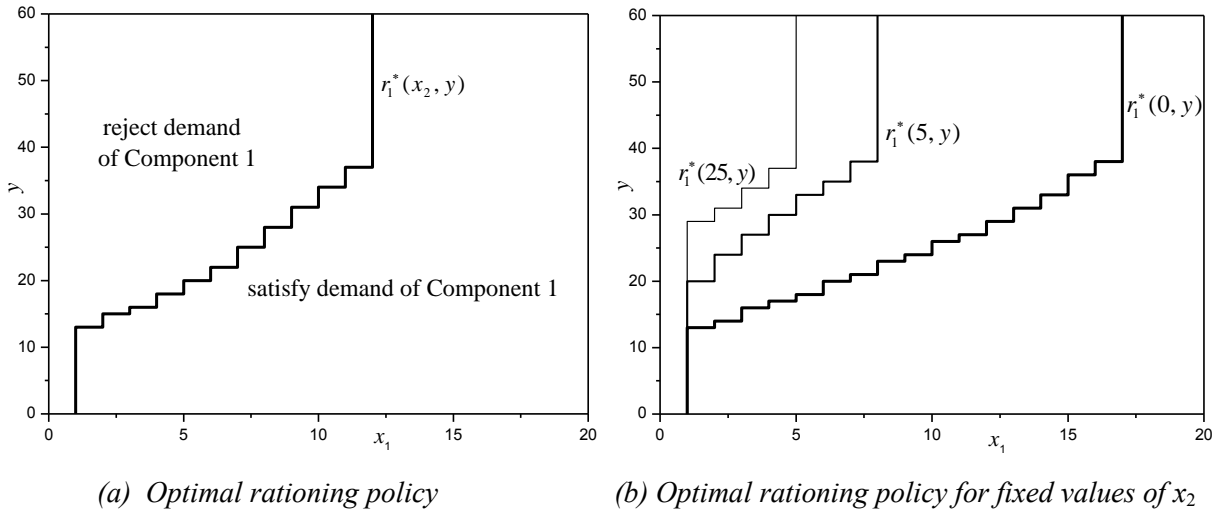


Fig. 4.7. The optimal allocation policy for demand of Component 1 with lost sales and backorders ($b_0=10, c_1=1000, c_2=800$)

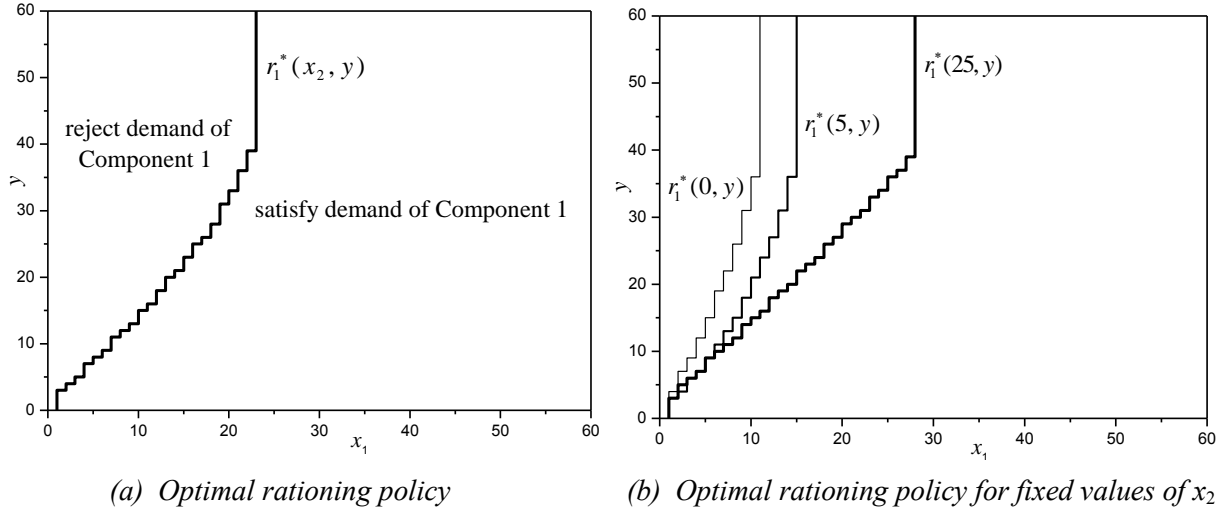


Fig. 4.8. The optimal allocation policy for demand of Component 1 with lost sales and backorders ($b_0=10, c_1=100, c_2=75$)

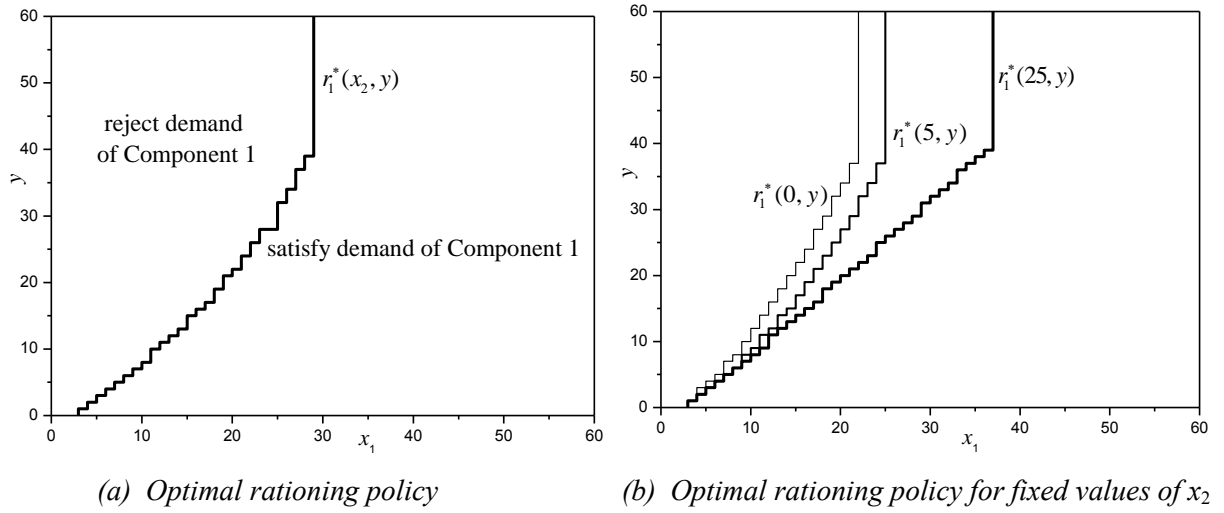


Fig. 4.9. The optimal allocation policy for demand of Component 1 with lost sales and backorders ($b_0=200, c_1=100, c_2=75$)

higher than that in Fig. 4.7, respectively. Such a similar result would be obtained from Fig. 4.9(a) and Fig. 4.9(b).

4.4 Conclusion

We have provided a characterization of the optimal policy for an ATO system with single product with individual components demand. The system defined here considered both lost sales and backorders. Specifically, the unsatisfied component demand is lost and the unsatisfied product demand is backordered if not fulfilled immediately. At any time, for any component, the manager must decide whether to produce it and add it to the stock, to produce and allocate it to fill the backorders of the assembled product if any, or not to produce. Also for any incoming demand, the manager must decide which demand can be filled. Using MDP, we showed the structure of the optimal control policy. In addition, we generated some properties of the base-stock and the rationing levels. Finally, we conducted a comprehensive numerical study with different penalty cost parameters, and analyze the impact of these cost parameters on the optimal policy.

Chapter 5. Heuristic Policies

We present several static heuristic policies based on the dynamic programming model of Chapters 3 and Chapter 4. For our heuristics, we use static thresholds to approximate the optimal state-dependent thresholds. Specifically, these static heuristics provide the simple, yet effective approaches for controlling production and inventory allocation of ATO systems. As we discussed in the General Introduction, due to the complexity of ATO systems, the optimal policy is computationally hard for large number of components. Because of the curse of dimensionality, the optimal policy will be difficult to implement. We therefore need to develop efficient heuristic policies to find a good approximation of the optimal policy. Of course, the heuristics that we proposed here are suboptimal policies, while when these heuristics used effectively they will lead to a performance improvement in ATO system.

The outline of this chapter is as follows: we provide an introduction in section 5.1. In section 5.2, we propose three static heuristic policies with lost sales case, and compare the performance of them. In section 5.3, we propose four static heuristic policies with mixed lost sales and backorders case, and compare the performance of them. Concluding remarks are provided in section 5.4.

5.1 Introduction

In the previous Chapters 3 and 4, we proved that the optimal production policy is a “state-dependent base-stock” policy, and the optimal allocation policy is a “state-dependent rationing” policy. However, such dynamic optimal policies may be difficult to carry out in practice especially when the number of components is large. On the other hand, since the state space considered in our system is infinite, in order to implement the optimal policy we need to truncate the state space. Then we use the value iteration method with the exhaustive searching over the truncated range to compute the cost function for the system. However, the value iteration used in the infinite horizon may lead to an extremely slow convergence. Therefore, we hope to develop simple and effective heuristic policies that should closely approximate to the optimal policy.

In this chapter, we propose several static heuristic policies. Contrary to the optimal policy, these heuristics are characterized by static base-stock and rationing levels. That is, the inventory of each component is controlled by a fixed base-stock policy, and the allocation of each demand can be controlled by a fixed rationing policy or a first-come-first-served (FCFS) policy. Throughout this chapter, let s_k denote the fixed base-stock level for Component k , and $\mathbf{s} = (s_1, s_2, \dots, s_n)$. Let r_k denote the fixed rationing level for demand of Component k , and $\mathbf{r} = (r_1, r_2, \dots, r_n)$. Let R denote the fixed rationing/production level for demand of the assembled product. We define the percentage cost difference between a heuristic and the optimal policy as $PD_{\text{heuristic}} = 100\% \times (g^{\text{heuristic}} - g^*) / g^*$, where $g^{\text{heuristic}}$ is the average cost per period under heuristic, and g^* is the average cost per period under the optimal policy. To simplify the computations, we focus on a two-component ATO system. In this system, we test the static heuristic policies with two cases: the pure lost sales system of Chapter 3, and the mixed lost sales and backorders of Chapter 4. For each case, we conduct a numerical study to compare the performance of the heuristic policies. The performance is measured through the percentage cost difference between the heuristic and the optimal policies.

5.2 The Case of Lost Sales

5.2.1 The Optimal Policy

Under the optimal policy, the dynamic programming equation can be written as:

$$v^*(\mathbf{x}) + g^* = h(\mathbf{x}) + \lambda_0 T^0 v^*(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}),$$

where the operators T^0 , T^k and T_k , $k=1, \dots, n$ are defined as follows:

$$T^0 v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_0 & \text{if } \prod_{k=1}^n x_k = 0 \\ \min \{v(\mathbf{x} - \mathbf{e}), v(\mathbf{x}) + c_0\} & \text{otherwise,} \end{cases}$$

$$T^k v(\mathbf{x}) = \begin{cases} v(\mathbf{x}) + c_k & \text{if } x_k = 0, \\ \min \{v(\mathbf{x} - \mathbf{e}_k), v(\mathbf{x}) + c_k\} & \text{otherwise,} \end{cases}$$

$$T_k v(\mathbf{x}) = \min \{v(\mathbf{x} + \mathbf{e}_k), v(\mathbf{x})\}.$$

As presented in Chapter 3, the optimal production policy for Component k is a base-stock policy with a state-dependent base-stock level $s_k^*(\mathbf{x}_{-k})$. The optimal inventory allocation policy for Component k is a rationing policy with a state-dependent rationing level $r_k^*(\mathbf{x}_{-k})$. Both these levels are non-decreasing in each of the states x_i , $i \neq k$. Furthermore, it is always optimal to satisfy demand of the assembled product whenever on-hand inventory for all components is available.

5.2.2 Three Static Heuristic Policies

Based on the case of pure lost sales in Chapter 3, we develop three static heuristic policies. The key point of these heuristics is to find the fixed base-stock and rationing levels. Once the threshold levels are determined, they can be used in the heuristics to control the production and allocation of ATO system.

The Heuristic H1 policy

We adopt IBR policy developed by Benjaafar and ElHafsi (2006) to our ATO system. Here, we redefine it as Heuristic H1.

Under Heuristic H1, the dynamic programming equation can be written as:

$$v(\mathbf{x}) + g^{\text{H1}} = h(\mathbf{x}) + \lambda_0 T^{\text{H1},0} v(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^{\text{H1},k} v(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k^{\text{H1}} v(\mathbf{x}), \quad (5.1)$$

where the operators $T^{\text{H1},0}$, $T^{\text{H1},k}$ and T_k^{H1} , are defined as follows:

$$T^{\text{H1},0} v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}) & \text{if } \prod_{k=1}^n x_k > 0, \\ v(\mathbf{x}) + c_0 & \text{otherwise,} \end{cases}$$

$$T^{\text{H1},k} v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}_k) & \text{if } x_k > r_k, \\ v(\mathbf{x}) + c_k & \text{otherwise,} \end{cases}$$

$$T_k^{\text{H1}} v(\mathbf{x}) = \begin{cases} v(\mathbf{x} + \mathbf{e}_k) & \text{if } x_k < s_k, \\ v(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Heuristic H1 is a simplified form of the optimal policy which is much easier to implement. In order to find the optimal base-stock and rationing levels, we do an exhaustive search on base-stock levels over the region $\{0, s_1^{\max}\} \times \{0, s_2^{\max}\} \times \dots \times \{0, s_k^{\max}\}$, where s_k^{\max} is the largest base-stock value for Component k under the optimal policy. We also use the same method to determine the rationing levels of each component demand. That is, the rationing levels can be searched exhaustively over the region $\{0, s_1\} \times \{0, s_2\} \times \dots \times \{0, s_k\}$, where s_k is the base-stock value for component k . We examine all feasible combinations of base-stock and rationing values, then determine the minimum average cost per period g^{H1} in equation (5.1), from which the base-stock and the rationing levels can be obtained.

The Heuristic H2 policy

We introduce a heuristic policy with the fixed base-stock and rationing levels, we refer to it as Heuristic H2.

Under Heuristic H2, the dynamic programming equation can be written as:

$$v(\mathbf{x}) + g^{\text{H2}} = h(\mathbf{x}) + \lambda_0 T^{\text{H2},0} v(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^{\text{H2},k} v(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k^{\text{H2}} v(\mathbf{x}), \quad (5.2)$$

where operators $T^{\text{H2},0}$, $T^{\text{H2},k}$ and T_k^{H2} , are defined as follows:

$$T^{H2,0}v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}) & \text{if } \prod_{k=1}^n x_k > 0, \\ v(\mathbf{x}) + c_0 & \text{otherwise,} \end{cases}$$

$$T^{H2,k}v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}_k) & \text{if } x_k > r_k, \\ v(\mathbf{x}) + c_k & \text{otherwise,} \end{cases}$$

$$T_k^{H2}v(\mathbf{x}) = \begin{cases} v(\mathbf{x} + \mathbf{e}_k) & \text{if } x_k < s_k, \\ v(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Here, we use the results of (S,R) policy of Ha (1997a) to determine the base-stock and rationing levels. Ha (1997a) proposed an (S,R) policy in a make-to-stock production system with two demand classes and lost sales. The optimal policy can be described by a base-stock level S and a rationing level R . Since demand class 1 is assumed to have priority over demand class 2, the rationing policy is used only for controlling demand class 2. In the model of Ha (1997 a), there is a single product and two demand classes. In our model, we can decompose the problem with respect to the individual components. That is, each Component k is associated with two demands: demand of the assembled product and demand of itself. Similarly, since priority is given to the product demand, the rationing policy is applied to control Component k demand. Based on the above, our ATO system can be decomposed as a series of queuing systems. For each Component k the system consists an $M/M/1/s_k$ make-to-stock queue (see Buzacott and Shantikumar 1993, section 4.3), with the base-stock level s_k and the rationing level r_k ($r_k < s_k$). This way, a multidimensional optimization problem can be solved as a series of one-dimensional problems.

We define $I(s_k, r_k)$ the inventory level of the system, $p_0(s_k, r_k)$ the probability of demand of the assembled product that is not satisfied, and $p_k(s_k, r_k)$ the probability of demand of Component k that is not satisfied. We use the results of Ha (1997a) as follows:

$$p_0(s_k, r_k) = \frac{(1 - \rho_k)(1 - \rho_{0k})\rho_k^{s_k - r_k}\rho_{0k}^{r_k}}{(1 - \rho_{0k})(1 - \rho_k^{s_k - r_k}) + (1 - \rho_k)\rho_k^{s_k - r_k}(1 - \rho_{0k}^{r_k + 1})},$$

$$p_k(s_k, r_k) = \frac{(1 - \rho_k)(1 - \rho_{0k}^{r_k + 1})\rho_k^{s_k - r_k}}{(1 - \rho_{0k})(1 - \rho_k^{s_k - r_k}) + (1 - \rho_k)\rho_k^{s_k - r_k}(1 - \rho_{0k}^{r_k + 1})},$$

$$I(s_k, r_k) = s_k - \frac{\frac{\rho_k(1-\rho_{0k})}{1-\rho_k} \{1 - \rho_k^{s_k-r_k} - (1-\rho_k)(s_k-r_k)\rho_k^{s_k-r_k-1}\}}{(1-\rho_{0k})(1-\rho_k^{s_k-r_k}) + (1-\rho_k)\rho_k^{s_k-r_k}(1-\rho_{0k}^{r_k+1})} - \frac{\frac{\rho_{0k}(1-\rho_k)}{1-\rho_{0k}} \left(\frac{\rho_k}{\rho_{0k}}\right)^{s_k-r_k} \{ \rho_{0k}^{s_k-r_k} - \rho_{0k}^{s_k+1} + (1-\rho_{0k})[(s_k-r_k)\rho_{0k}^{s_k-r_k-1}] - (s_k+1)\rho_{0k}^{s_k} \}}{(1-\rho_{0k})(1-\rho_k^{s_k-r_k}) + (1-\rho_k)\rho_k^{s_k-r_k}(1-\rho_{0k}^{r_k+1})}.$$

Hence, for each one-dimensional problem the expected cost of the system is obtained by:

$$g(s_k, r_k) = h_k I(s_k, r_k) + c_0 \lambda_0 p_0(s_k, r_k) + c_k \lambda_k p_k(s_k, r_k), \text{ for } r_k \leq s_k.$$

We also define $\rho_k = (\lambda_0 + \lambda_k)/\mu_k$, $\rho_{0k} = \lambda_0/\mu_k$. Indeed, s_k and r_k can be determined from $\min_{(s_k, r_k)} g(s_k, r_k)$, s.t. $r_k \leq s_k$. Once all s_k and r_k are obtained, all of them will be used in equation (5.2) to determine the minimum of average cost per period g^{H2} .

The Heuristic H3 policy

Under Heuristic H3, the production can be controlled by a base-stock policy, while component demands are filled on a first-come-first-served (FCFS) basis. Upon the arrival of component demand, if there is enough on-hand inventory for that demand, then it is satisfied immediately. If there is no available component in stock, the component demand is rejected.

The dynamic programming equation can be written as:

$$v(\mathbf{x}) + g^{H3} = h(\mathbf{x}) + \lambda_0 T^{H3,0} v(\mathbf{x}) + \sum_{k=1}^n \lambda_k T^{H3,k} v(\mathbf{x}) + \sum_{k=1}^n \mu_k T_k^{H3} v(\mathbf{x}), \quad (5.3)$$

where operators $T^{H3,0}$, $T^{H3,k}$ and T_k^{H3} , are defined as follows:

$$T^{H3,0} v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}) & \text{if } \prod_{k=1}^n x_k > 0, \\ v(\mathbf{x}) + c_0 & \text{otherwise,} \end{cases}$$

$$T^{H3,k} v(\mathbf{x}) = \begin{cases} v(\mathbf{x} - \mathbf{e}_k) & \text{if } x_k > 0, \\ v(\mathbf{x}) + c_k & \text{otherwise,} \end{cases}$$

$$T_k^{H3} v(\mathbf{x}) = \begin{cases} v(\mathbf{x} + \mathbf{e}_k) & \text{if } x_k < s_k, \\ v(\mathbf{x}) & \text{otherwise.} \end{cases}$$

We do an exhaustive search on base-stock values over the region $\{0, s_1^{\max}\} \times \{0, s_2^{\max}\} \times \dots \times \{0, s_k^{\max}\}$, and then the base-stock levels $\mathbf{s} = (s_1, s_2, \dots, s_n)$ can be obtained from equation (5.3).

5.2.3 Comparison to the Heuristic Policies

We provide the numerical results on the performance measures with respect to the percentage cost difference between each heuristic and the optimal policies, and compare the performance of the heuristic policies with that of the optimal policy. The system parameters are generated randomly from the 100 examples (see Table 5.1).

Table 5.1 presents results from a set of numerical experiments. In this table, three heuristic policies are tested for their efficiency. Comparing the percentage average cost difference per period between the optimal policy and the heuristic policies, the numerical results in Table 5.1 indicate that Heuristics H1 and H2 are efficient and clearly outperform Heuristic H3. This result is quite intuitive since H1 and H2 have the benefit of rationing, and the purpose of rationing is to reserve inventory for the future arrivals of the more valuable demands. In our numerical result, demand varies from component to component, thus the benefit of rationing is significant. Heuristic H3 does not perform well especially for cases where component demand is relatively high. As we can see in cases 2, 3, 11, 13, 25, 23, 33, 37, 38, 43, 50, 54, 61, 68, 71, 72, 76, 77, 80, 88, 89, and 92. In all these cases, the demand rate for one of the components is higher than the production rate. However, in practice, most systems have sufficient production capacity, thus the demands would be satisfied. In fact, with the advantage of its simplicity and ease of implementation, Heuristic H3 has been widely used in ATO systems. On the other hand, for some special cases Heuristics H1 and H2 are equivalent to Heuristic H3. For instance, we found that in cases 9, 10, 18, 22 the percentage difference of Heuristics H1, H2 and H3 are exactly the same. Note that in these cases, the rationing level for the component demand equals to 0, this means Heuristics H1 and H2 have no benefit of the rationing policy, and these two policies therefor can be regarded as Heuristic H3. Additionally, we found that the difference between Heuristics H1 and H2 is not very large, especially in cases 7, 9, 10, 18, 20, 22, 24, 32, 33, 36, 39, 46, 47, 52, 53, 79, 80, 82, 83, 90, 91, 99, where the percentage difference of these two policies is exactly the same. But in general cases, Heuristic H1 performs better than Heuristic H2. In comparison, Heuristic H1 is more efficient because we do an exhaustive search to determine the base-stock and rationing levels, which needs to take into account the on-hand inventory of other components. In Heuristic H2, the base-stock and rationing levels can be determined from the M/M/1/ s_k make-to-stock queue, and there is therefore no need to consider the on-hand

Table 5.1 Optimal policy versus Heuristic policies with lost sales

(the system parameters are drawn from uniform distributions as $\lambda_0 \sim U(0,9), \lambda_k \sim U(0,9), \mu_k \sim U(1,10),$
 $h_k \sim U(1,10), c_0 \sim U(600,700), c_k \sim U(1,300), k = 1, 2, .)$

	λ_0	λ_1	λ_2	μ_1	μ_2	h_1	h_2	c_0	c_1	c_2	percentage difference from optimal cost %		
											H1	H2	H3
1	0.42	7.65	0.33	8.85	5.68	3.78	3.61	621.90	85.54	252.09	1.34	3.73	5.43
2	1.37	6.40	0.64	9.40	8.18	4.38	1.20	689.68	79.68	193.09	1.46	4.99	21.94
3	2.96	2.73	2.06	6.64	7.29	1.43	2.83	617.42	156.46	93.89	0.41	4.92	15.33
4	2.81	3.04	0.60	6.85	4.97	1.19	8.19	653.43	147.48	212.70	1.26	1.52	6.83
5	0.95	6.48	5.22	7.25	8.40	7.60	2.78	629.88	246.48	147.21	0.52	1.54	2.22
6	3.87	1.79	0.32	9.61	6.89	6.96	4.79	688.79	155.10	76.95	4.28	7.35	9.90
7	0.01	0.13	8.76	7.11	5.91	4.30	4.63	687.75	176.76	105.76	0.23	0.23	0.23
8	0.79	2.08	5.11	7.98	8.53	5.81	5.46	646.11	16.61	176.63	0.81	6.51	9.48
9	0.21	6.52	2.30	9.53	2.83	1.62	7.69	656.43	182.96	271.69	0.17	0.17	0.17
10	1.24	0.86	4.72	2.92	8.52	6.12	8.89	667.81	147.06	171.21	0.16	0.16	0.16
11	2.52	5.34	3.09	8.44	1.52	6.14	9.30	607.75	73.62	274.04	0.17	7.12	17.48
12	4.21	2.18	0.83	8.42	8.01	1.61	9.30	699.05	204.43	276.79	1.09	0.57	3.56
13	2.07	0.28	2.59	9.21	4.78	5.78	9.56	674.77	294.54	244.04	2.35	10.54	14.09
14	0.88	4.49	4.49	5.40	9.45	6.62	1.87	677.02	200.29	210.27	0.03	1.10	1.20
15	3.53	2.26	2.26	7.59	2.00	4.01	4.75	654.66	68.63	158.62	1.56	7.52	10.94
16	5.27	1.01	0.29	9.45	3.18	9.01	3.62	678.91	64.40	223.36	0.18	4.41	5.36
17	0.40	6.11	3.26	3.66	4.98	7.61	9.68	650.56	274.23	204.79	0.34	0.42	4.79
18	0.03	1.69	0.51	3.30	2.20	5.66	5.18	623.48	214.97	249.51	0.22	0.22	0.22
19	1.57	2.47	6.54	6.18	9.17	9.41	9.12	698.83	190.37	268.73	0.30	0.42	0.30
20	1.25	2.17	3.36	8.30	5.97	5.32	7.32	692.95	187.58	175.17	0.91	4.51	4.51
21	1.14	5.90	2.80	9.12	8.30	5.69	1.38	646.24	200.24	179.96	0.76	0.85	0.93
22	0.23	6.30	3.52	8.10	6.85	3.74	9.24	683.23	164.89	97.92	0.71	0.71	0.71
23	0.07	4.51	1.18	1.99	3.78	4.89	5.09	654.64	38.08	50.90	2.94	6.86	12.17
24	1.66	1.24	2.02	8.52	6.28	5.53	4.50	614.32	155.09	216.33	0.75	0.90	0.90
25	0.19	5.42	1.61	3.07	4.59	5.94	7.45	665.59	166.51	117.59	1.23	1.32	6.33
26	1.17	0.95	6.48	6.22	7.25	8.40	7.60	648.90	59.99	246.48	0.51	6.58	7.35
27	8.35	3.87	1.79	1.32	9.61	6.89	6.96	625.40	126.81	155.10	0.26	0.90	4.18
28	0.46	5.15	3.56	5.62	5.00	9.01	2.06	643.12	94.13	163.81	0.49	0.69	6.05
29	0.52	2.28	1.26	6.97	6.37	5.04	5.28	612.22	176.61	86.57	2.28	3.92	7.84
30	6.79	4.46	0.45	4.00	1.12	1.46	4.89	627.13	212.31	139.51	0.01	0.14	0.88
31	4.71	7.75	8.49	7.00	1.54	9.41	9.42	624.89	128.39	296.09	0.06	2.29	4.75
32	2.67	6.47	0.15	9.74	7.57	7.08	6.04	643.52	143.62	79.51	0.14	0.14	8.72
33	2.97	2.74	0.61	8.34	8.81	8.96	4.35	627.14	88.26	124.39	0.06	0.06	10.66
34	4.37	6.29	8.22	4.64	8.01	8.17	6.84	615.27	201.08	215.79	0.15	1.94	9.61
35	2.80	0.96	1.56	6.80	8.52	3.42	4.64	694.79	237.33	221.85	0.69	3.51	3.52
36	4.25	2.34	2.46	9.30	9.22	9.48	7.55	688.78	133.90	174.12	0.35	0.35	6.37
37	0.77	7.52	1.54	1.13	3.16	7.44	9.84	642.04	255.05	73.98	0.38	0.41	10.09
38	7.60	1.07	6.90	8.68	1.94	8.97	8.94	607.74	153.68	96.96	0.13	7.02	11.52
39	1.79	2.47	0.14	6.90	4.84	1.87	3.46	663.30	213.43	164.03	1.77	1.77	2.89
40	0.57	1.94	8.66	4.18	9.31	3.65	6.38	625.54	39.92	249.89	0.18	0.34	8.50
41	8.89	1.92	1.53	1.05	9.69	2.20	8.52	625.90	75.44	251.90	0.03	0.39	2.06
42	1.70	0.52	7.81	7.91	9.92	4.43	6.62	665.95	50.30	174.41	0.67	7.99	9.52
43	0.21	3.93	6.67	9.97	3.58	3.70	7.55	601.30	265.15	99.23	0.29	1.16	10.79
44	4.76	1.13	1.57	1.95	7.20	4.98	2.12	640.60	242.30	203.87	0.43	1.99	6.98
45	8.15	6.12	7.15	1.45	4.41	5.92	7.14	600.12	210.63	29.17	0.07	1.00	5.96
46	0.32	2.29	2.80	9.12	8.30	6.88	4.52	676.55	113.51	127.61	0.16	1.16	0.91
47	0.17	5.70	0.47	4.42	8.25	4.59	8.64	658.56	163.12	70.03	0.28	0.28	6.02
48	8.74	2.30	0.75	4.51	3.71	8.97	5.67	672.18	168.80	196.68	0.13	3.50	4.56
49	0.87	8.25	1.31	7.18	3.00	9.14	2.50	620.89	247.49	72.97	0.18	0.33	9.12
50	6.38	0.11	1.66	8.36	3.14	5.50	4.76	625.64	49.28	181.05	0.22	3.64	12.71

Table 5.1 Optimal policy versus Heuristic policies with lost sales

(the system parameters are drawn from uniform distributions as $\lambda_0 \sim U(0,9), \lambda_k \sim U(0,9), \mu_k \sim U(1,10),$
 $h_k \sim U(1,10), c_0 \sim U(600,700), c_k \sim U(1,300), k = 1, 2, .)$

	λ_0	λ_1	λ_2	μ_1	μ_2	h_1	h_2	c_0	c_1	c_2	percentage difference from optimal cost %		
											H1	H2	H3
51	7.33	7.57	7.68	2.87	1.27	3.71	8.01	676.90	138.16	246.69	0.70	3.51	4.27
52	0.74	1.57	0.29	6.41	7.62	7.48	8.30	683.97	144.75	155.34	2.39	2.39	3.86
53	0.59	0.61	0.30	4.00	7.41	9.28	8.47	667.87	19.13	172.37	0.01	0.01	5.64
54	8.22	1.46	4.76	9.50	5.80	3.08	1.64	642.43	162.20	141.81	1.29	1.78	47.35
55	8.74	7.48	0.45	9.31	3.16	6.05	9.49	646.09	162.68	284.84	0.49	1.11	1.30
56	0.69	1.00	2.48	2.40	5.14	1.64	9.75	636.72	13.03	189.10	1.07	1.82	9.43
57	4.37	4.95	8.50	2.66	1.26	7.02	9.62	661.47	80.17	201.71	1.66	3.90	9.42
58	8.24	6.78	8.10	2.00	7.41	9.83	7.05	619.25	194.02	51.01	0.27	2.48	7.40
59	8.62	5.27	6.20	4.32	6.60	4.31	7.50	647.14	299.73	21.57	0.00	2.29	15.48
60	8.68	2.01	6.73	2.00	6.28	9.89	2.35	603.58	52.17	96.56	0.64	1.35	5.64
61	1.42	6.76	4.05	8.02	2.87	1.34	6.94	617.59	10.75	159.73	1.14	1.22	59.12
62	8.74	2.30	0.75	4.51	3.71	8.97	5.67	672.18	168.80	196.68	1.12	2.63	4.69
63	8.61	4.55	2.06	3.18	5.24	9.22	9.76	647.35	264.68	122.88	0.36	4.86	8.28
64	0.63	2.02	0.03	8.70	9.21	3.38	9.94	658.61	19.07	127.52	0.17	3.79	8.19
65	7.64	4.78	2.18	6.43	1.38	2.72	7.01	604.27	163.62	176.82	0.21	0.82	3.00
66	1.28	8.63	7.43	2.19	2.75	3.36	5.08	660.74	111.31	290.63	0.08	0.39	9.92
67	8.41	7.01	3.64	7.40	1.64	4.85	8.60	663.52	216.59	47.27	0.14	2.34	5.13
68	8.24	1.25	8.97	9.61	2.54	7.12	8.43	673.84	294.51	98.22	0.15	1.59	16.06
69	7.13	1.34	0.70	6.18	3.05	2.23	1.75	624.28	47.77	32.58	0.19	4.35	6.63
70	8.64	2.32	3.98	1.54	4.92	7.49	2.20	691.74	256.80	183.68	0.02	1.75	1.94
71	5.90	7.57	0.96	3.11	3.80	1.96	2.56	626.91	193.78	233.86	0.84	0.84	20.70
72	6.24	1.61	2.90	8.71	2.10	5.89	5.94	630.17	136.97	98.49	0.14	1.37	12.28
73	6.82	1.17	1.19	2.06	1.87	2.09	8.02	653.86	298.12	122.68	0.64	1.38	2.95
74	8.41	2.19	6.97	1.14	2.66	8.01	8.23	628.75	129.05	80.67	0.00	2.52	4.79
75	6.11	8.36	7.36	1.39	9.14	7.44	1.54	609.11	145.12	46.94	0.00	0.33	8.71
76	6.82	3.15	7.82	2.52	9.82	9.13	4.59	657.62	37.06	85.02	0.01	1.41	15.59
77	6.69	1.77	0.76	6.84	4.95	9.02	5.74	668.34	177.26	132.59	1.84	9.49	19.03
78	3.53	1.46	2.11	1.77	6.94	3.27	9.24	612.39	121.97	160.69	2.93	4.37	5.68
79	0.87	1.49	0.39	9.36	8.20	8.42	5.15	627.03	188.74	203.92	0.00	0.00	2.05
80	1.54	4.26	7.20	5.06	4.68	2.78	6.65	664.44	175.31	262.74	0.04	0.04	15.08
81	5.82	8.97	5.63	4.56	4.75	3.81	7.65	605.21	201.76	296.68	0.53	3.54	7.01
82	0.29	7.48	8.20	3.67	3.36	7.70	4.88	667.90	87.84	283.14	0.10	0.10	2.77
83	7.05	0.04	4.38	9.22	1.14	6.39	2.70	686.04	176.24	158.78	1.69	1.69	7.72
84	5.27	2.34	7.36	2.23	5.41	1.62	4.83	617.76	206.65	172.82	1.17	5.23	8.01
85	2.49	5.27	1.64	7.70	6.43	5.50	1.14	663.58	185.51	191.68	0.15	1.33	2.77
86	7.41	2.57	1.22	2.65	2.06	6.49	1.96	670.93	294.82	203.16	0.61	4.83	2.54
87	6.76	3.88	5.80	1.96	9.56	5.78	6.83	613.39	221.17	221.85	0.38	3.30	7.24
88	2.85	6.78	5.22	6.63	3.87	8.73	2.78	611.94	103.82	201.87	0.07	0.21	37.98
89	8.55	3.42	4.95	8.02	4.82	8.25	5.41	660.73	175.64	208.85	0.63	1.62	20.56
90	0.09	3.05	4.97	3.53	3.42	9.86	7.56	601.17	74.76	137.47	0.56	0.56	1.43
91	1.39	2.19	0.59	5.32	7.40	9.36	6.15	624.28	88.58	48.01	0.43	0.43	8.68
92	7.59	1.89	8.81	7.58	3.32	7.44	6.19	653.99	235.54	214.42	0.46	2.13	16.63
93	2.51	8.64	2.37	4.04	8.88	7.11	8.36	632.25	54.26	191.92	0.08	0.60	4.47
94	7.16	7.01	4.62	4.92	1.26	1.26	7.64	635.02	78.96	253.47	0.00	0.41	4.10
95	1.91	3.18	2.73	9.31	9.70	9.22	2.85	688.43	144.63	90.23	0.54	1.51	1.51
96	2.45	2.41	0.92	9.82	8.49	6.81	8.30	640.70	121.87	154.62	0.33	1.43	3.71
97	1.38	1.80	1.64	7.35	4.65	5.71	6.78	653.41	54.54	139.56	0.16	3.27	6.29
98	8.30	4.59	4.94	2.24	3.98	8.55	1.23	609.54	264.97	265.44	0.00	0.29	1.52
99	0.81	3.65	0.11	8.40	3.26	7.85	9.38	613.63	58.21	106.54	0.09	0.09	3.74
100	6.94	8.16	2.97	8.53	2.37	4.90	5.02	614.65	274.20	216.54	0.12	2.08	3.16

inventory of other components. Because of this, the accuracy of approximation under H2 is lower than that under H1. However, Heuristic H2 provides a computational advantage. Instead of requiring a multidimensional search for the base-stock and rationing levels, H2 offers a quite simple and computationally effective way of computing the optimal thresholds.

Also from the Table 5.1, there is very little difference between the optimal policy and Heuristic H1. For example, a maximum of the percentage average cost difference is 4.28%, and a minimum is 0.00%. However, the percentage average cost difference between the optimal policy and Heuristic H3 is larger with a maximum of 59.12% and a minimum of 0.16%. It is clear that Heuristic H1 works best for controlling the ATO system.

Additionally, the optimal average cost is sensitive to the component production utilization $\rho_k = (\lambda_0 + \lambda_k) / \mu_k$. In order to study the combined effect of the related costs (holding cost and lost sale costs) and component production utilization on the average cost. From Fig. 5.1(a) and Fig. 5.1(b) we can see that the optimal average cost increases as ρ_1 increases. When $\rho_1 > 1$, the optimal average cost is increasing in h_1 and c_0 . Clearly, the effect of the related cost on average cost is significant as ρ_1 is large. However, when $\rho_1 \leq 1$, the optimal average cost is decreasing in h_1 and c_0 and the effect of the related cost on average cost is not significant as ρ_1 is small. We can explain this behavior as follows: when $\rho_1 > 1$, the system is busy and has not enough capacity to handle all the demands. In this case, the lost sale cost c_0 has a dominant effect compared to the holding cost h_1 . On one hand, the increase of the lost sale cost c_0 would also incur the average cost increases; on the other hand, as showed in Fig. 5.1(a) the higher value of c_0 increases the value of optimal average cost. When $\rho_1 \leq 1$, the system is not busy and has a relative large capacity to handle the coming demands. In this case, the holding cost h_1 has a dominant effect compared to the lost sale cost c_0 . On one hand, the increase of the holding cost h_1 would also incur the average cost increases; on the other hand, as showed in Fig. 5.1(b) the higher value of h_1 decreases the value of optimal average cost.

To gain more insight, we next examine the effectiveness of the heuristics under different holding cost and lost sale cost structures. We fix some system parameters as $\lambda_0 = 0.75, \lambda_2 = \lambda_2 = 0.2, \mu_1 = 1, \mu_2 = 1$, while vary the value of the other parameters.

Fig.5.2 and Fig.5.3 reflect the sensitivity of the percentage difference from optimal cost in system cost parameters. We compare the average cost of Heuristic H1 to the average cost of

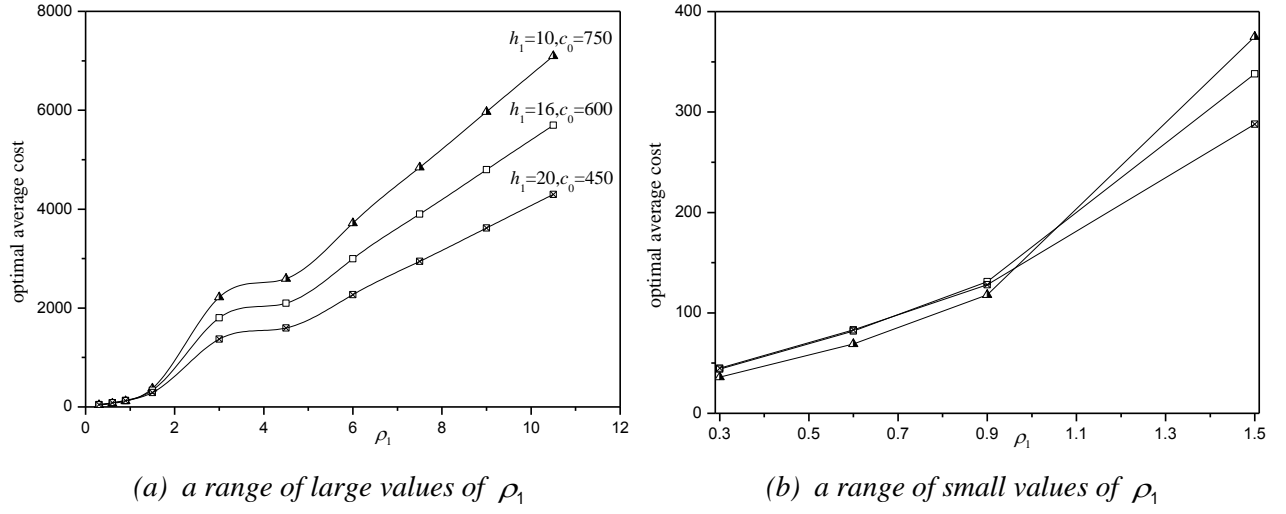


Fig. 5.1. The effect of the relative load on optimal average cost

$$(\lambda_1 = 0.2, \lambda_2 = 0.2, \mu_1 = 1, \mu_2 = 1, h_2 = 5, c_1 = 200, c_2 = 50)$$

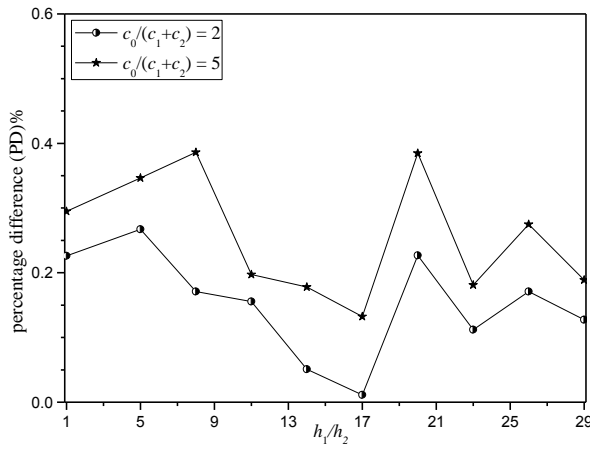


Fig. 5.2. The effect of h_1/h_2 on the system under Heuristic H1

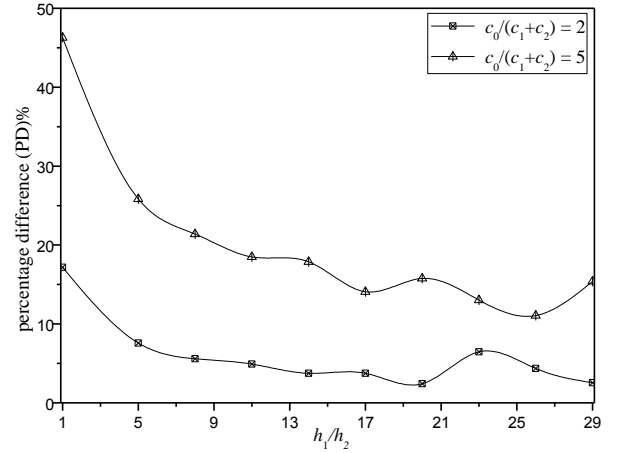
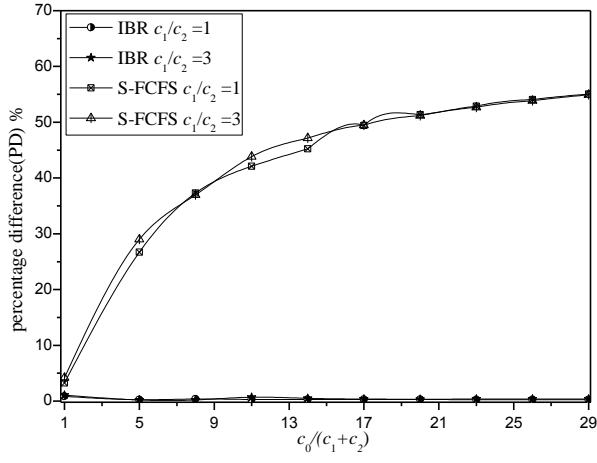
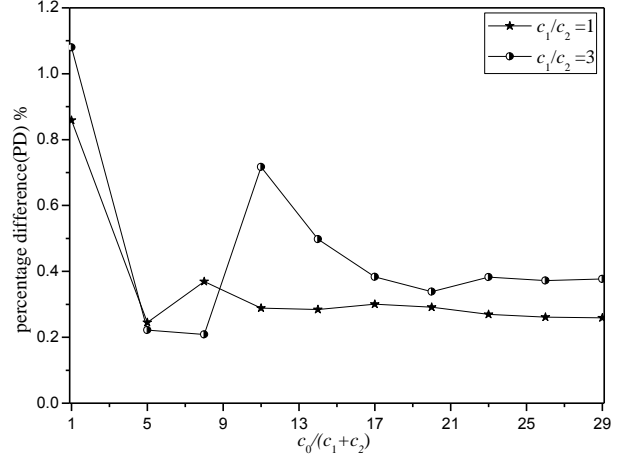


Fig. 5.3. The effect of h_1/h_2 on the system under Heuristic H3

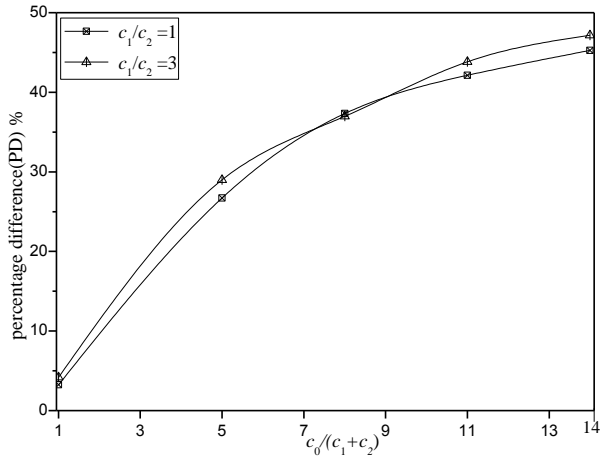
$$(\lambda_0 = 0.75, \lambda_2 = \lambda_2 = 0.2, \mu_1 = 1, \mu_2 = 1, h_1 = 2, c_1 = 200, c_2 = 50)$$



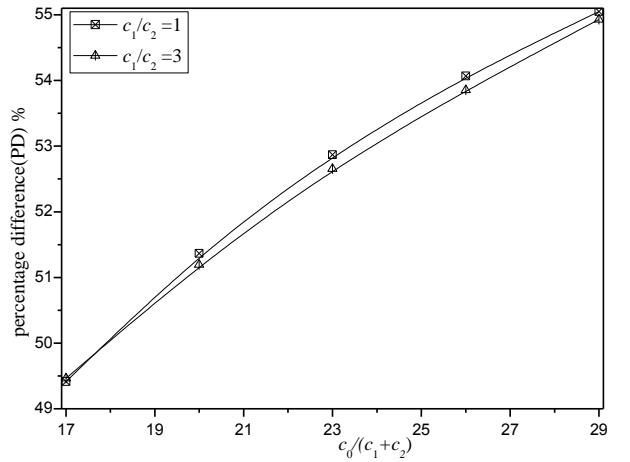
(a) Heuristic H1 vs. Heuristic H3



(b) Heuristic H1



(c) Heuristic H3


 (d) Heuristic H3 for a range of smaller values of $c_0/(c_1+c_2)$
Fig. 5.4 The effect of $c_0 / (c_1+c_2)$ on the system under Heuristics H1 and H3

$$(\lambda_0 = 0.75, \lambda_2 = \lambda_2 = 0.2, \mu_1 = 1, \mu_2 = 1, h_1 = 10, h_1 = 5, c_0 + c_1 + c_2 = 2000)$$

Heuristic H3 with different cost parameters. As shown in Fig.5.2, PD_{H1} is not monotonic in the holding cost ratio h_1/h_2 . For each fixed value of $c_0/(c_1+c_2)$, PD_{H1} fluctuates a lot between 0% to 0.5%. In Fig.5.3, for each fixed value of $c_0/(c_1+c_2)$, PD_{H3} decreases with h_1/h_2 for the range of $1 \leq h_1/h_2 \leq 14$. We also observe that PD_{H3} fluctuates slightly for the range of $15 \leq h_1/h_2 \leq 29$. As can be seen from this figure for a fixed value of h_1/h_2 , we obtain $PD_{c_0/(c_1+c_2)=5} > PD_{c_0/(c_1+c_2)=2}$, that is the percentage difference PD_{H3} is increasing in the relative lost sale cost rate $c_0/(c_1+c_2)$. Comparing these two heuristic policies, the effect of the holding cost rate is obvious under Heuristic H3.

Fig. 5.4(a)-Fig. 5.4(d) show the effect of relative lost sale cost rate $c_0/(c_1+c_2)$ on PD_{H1} and PD_{H3} . We fix the lost sale costs $c_0+c_1+c_2=2000$, but vary c_1/c_2 in different values. Clearly, in Fig.5.4(a) for each fixed value of c_1/c_2 , we can see that the effect of $c_0/(c_1+c_2)$ on PD_{H3} is significant, while this effect on PD_{H1} is slight. That because Heuristic H1 has the advantage of the rationing policy, which can be used to reduce the effect of $c_0/(c_1+c_2)$. However, Heuristic H3 does not have the benefit of rationing, thus the relative lost sale cost rate affects PD_{H3} a lot. Fig. 5.4(b) and Fig. 5.4(c) indicate the trend of PD change under different heuristics. For Heuristic H1, Fig. 5.4(b) shows that $PD_{c_1/c_2=3}$ fluctuates more sharply than $PD_{c_1/c_2=1}$ between 0.0% and 1.2%. For Heuristic H3, Fig. 5.4(c) covers a range of $1 \leq c_0/(c_1+c_2) \leq 14$. We can see that for a fixed value of $c_0/(c_1+c_2)$, the percentage difference between $PD_{c_1/c_2=1}$ and $PD_{c_1/c_2=3}$ is not large. Fig. 5.4(d) covers a range of $17 \leq c_0/(c_1+c_2) \leq 29$. We can see that for a fixed value of $c_0/(c_1+c_2)$, the percentage difference PD_{H3} decreases with the ratio c_1/c_2 . For example, at a value of $c_0/(c_1+c_2)=20$, $PD_{c_1/c_2=3} = 51.19\%$, $PD_{c_1/c_2=1} = 51.36\%$; at a value of $c_0/(c_1+c_2)=23$, $PD_{c_1/c_2=3} = 52.66\%$, $PD_{c_1/c_2=1} = 52.87\%$; at a value of $c_0/(c_1+c_2)=26$, $PD_{c_1/c_2=3} = 53.85\%$, $PD_{c_1/c_2=1} = 54.06\%$. In summary, for each heuristic our numerical results indicate that the difference between $PD_{c_1/c_2=1}$ and $PD_{c_1/c_2=3}$ is not significant.

5.3 The Case of Lost Sales and Backorders

5.3.1 The Optimal Policy

Under the optimal policy, the dynamic programming equation can be written as:

$$v^*(\mathbf{x}, y) + g^* = h(\mathbf{x}) + b_0(y) + \lambda_0 T^0 v^*(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^k v^*(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k v^*(\mathbf{x}, y),$$

where, operators T^0 , T^k and T_k , $k=1, \dots, n$ are defined as follows:

$$T^0 v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y+1) & \text{if } \prod_{k=1}^n x_k = 0 \\ \min\{v(\mathbf{x} - \mathbf{e}, y), v(\mathbf{x}, y+1)\} & \text{otherwise,} \end{cases}$$

$$T^k v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x}, y) + c_k & \text{if } x_k = 0, \\ \min\{v(\mathbf{x} - \mathbf{e}_k, y), v(\mathbf{x}, y) + c_k\} & \text{otherwise,} \end{cases}$$

$$T_k v(\mathbf{x}, y) = \begin{cases} \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} & \text{if } y = 0 \\ \min\{v(\mathbf{x}, y), v(\mathbf{x} + \mathbf{e}_k, y)\} & \text{if } y > 0, \prod_{i \neq k}^n x_i = 0 \\ \min\left\{v(\mathbf{x} + \mathbf{e}_k, y), v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right)\right\} & \text{if } y > 0, \prod_{i \neq k}^n x_i > 0. \end{cases}$$

As presented in Chapter 4, for Component k there exists an optimal stationary policy specified in terms of a state-dependent production base-stock level $s_k^*(\mathbf{x}_{-k}, y)$, for $y=0$, or $y > 0, \prod_{i \neq k}^n x_i = 0$ and a state-dependent inventory rationing level $r_k^*(\mathbf{x}_{-k}, y)$. For the assembled product, there exists an optimal stationary inventory rationing policy specified in terms of a state-dependent rationing level $R_k^*(\mathbf{x}_{-k}, y)$.

5.3.2 Four Static Heuristic Policies

Based on the mixed lost sales and backorders case of Chapter 4, we develop four static heuristic policies. Throughout this sub-section, we use an exhaustive search to construct approximation of the optimal base-stock and rationing levels. Then these threshold levels can be applied for controlling the production and allocation of ATO system.

The Heuristic H4 policy

Heuristic H4 use a similar idea to the IBR policy developed by Benjaafar and ElHafsi (2006). H4 uses fixed thresholds for component and assembled product for production and inventory rationing decisions.

It is not difficult to show that under Heuristic H4, the dynamic programming equation can be written as:

$$v(\mathbf{x}, y) + g^{\text{H4}} = h(\mathbf{x}) + b_0(y) + \lambda_0 T^{\text{H4},0} v(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^{\text{H4},k} v(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k^{\text{H4}} v(\mathbf{x}, y), \quad (5.4)$$

where operators $T^{\text{H4},0}$, $T^{\text{H4},k}$ and T_k^{H4} are defined as follows:

$$\begin{aligned} T^{\text{H4},0} v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} - \mathbf{e}, y) & \text{if } y \geq R, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v(\mathbf{x}, y+1) & \text{otherwise,} \end{cases} \\ T^{\text{H4},k} v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } x_k \geq r_k, \\ v(\mathbf{x}, y) + c_k & \text{otherwise,} \end{cases} \\ T_k^{\text{H4}} v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y = 0, \\ v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y > 0, \text{ and } \prod_{i \neq k}^n x_i = 0, \\ v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } y \leq R, y > 0, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) & \text{if } y > R, y > 0, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v(\mathbf{x}, y) & \text{otherwise.} \end{cases} \end{aligned}$$

We conduct an exhaustive search to find the best base-stock levels s_k and rationing levels r_k and R that minimize the average cost in equation (5.4). That is, we do an exhaustive search on base-stock levels s_k over the region $\{0, s_1^{\max}\} \times \{0, s_2^{\max}\} \times \dots \times \{0, s_k^{\max}\}$, where s_k^{\max} is the largest base-stock level for Component k under the optimal policy. We exhaustively search on rationing levels r_k over the region $\{0, s_1\} \times \{0, s_2\} \times \dots \times \{0, s_k\}$, where s_k the base-stock value for Component k under Heuristic H4. We also do an exhaustive search on rationing level R over the region $\{0, R^{\max}\}$, where R^{\max} the largest rationing value for the assembled product under the optimal policy. Using this technique, the base-stock and the rationing levels and the corresponding optimal average cost per period g^{H4} can be determined.

As we can see, Heuristic H4 has the same structure as the optimal policy except that it uses fixed threshold levels rather than the optimal state-dependent ones. Its advantage is that it is easier to compute and implement compared to the optimal policy.

The Heuristic H5 policy

Under Heuristic H5 the production can be controlled by a fixed base-stock threshold, and demand of the assembled product is always satisfied as long as stock for all components is available. Demand of individual components is rationed through fixed threshold levels similar to Heuristic H4. Hence, Heuristic H5 gives priority to the demand of the assembled product over the demand of the individual components.

The dynamic programming equation can be written as:

$$v(\mathbf{x}, y) + g^{\text{H5}} = h(\mathbf{x}) + b_0(y) + \lambda_0 T^{\text{H5},0} v(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^{\text{H5},k} v(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k^{\text{H5}} v(\mathbf{x}, y), \quad (5.5)$$

where the operators $T^{\text{H5},0}$, $T^{\text{H5},k}$ and T_k^{H5} are defined as follows:

$$T^{\text{H5},0} v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x} - \mathbf{e}, y) & \text{if } \prod_{k=1}^n x_k > 0, \\ v(\mathbf{x}, y + 1) & \text{otherwise,} \end{cases}$$

$$T^{\text{H5},k} v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } x_k \geq r_k, \\ v(\mathbf{x}, y) + c_k & \text{otherwise,} \end{cases}$$

$$T_k^{\text{H5}} v(\mathbf{x}, y) = \begin{cases} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y = 0, \\ v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y > 0, \text{ and } \prod_{i \neq k} x_i = 0, \\ v(\mathbf{x} - \sum_{i \neq k} \mathbf{e}_i, y - 1) & \text{if } y > 0, \prod_{i \neq k} x_i > 0, \\ v(\mathbf{x}, y) & \text{otherwise.} \end{cases}$$

Similar to Heuristic H4, we use dynamic programming to compute H5. That is, we do an exhaustive search on base-stock levels s_k over the region $\{0, s_1^{\max}\} \times \{0, s_2^{\max}\} \times \dots \times \{0, s_k^{\max}\}$, and we can also exhaustively search on rationing levels r_k over the region $\{0, s_1\} \times \{0, s_2\} \times \dots \times \{0, s_k\}$. From equation (5.5), we determine the minimum average cost per period g^{H5} .

The Heuristic H6 policy

Under Heuristic H6 the production can be controlled by a fixed base-stock threshold, and demand of the individual component is always satisfied as long as stock for the needed component is available. Demand of the assembled product is rationed through fixed threshold levels similar to Heuristic H4. Hence, Heuristic H6 gives priority to demand of the individual components over demand of the assembled product.

The dynamic programming equation can be written as:

$$v(\mathbf{x}, y) + g^{\text{H6}} = h(\mathbf{x}) + b_0(y) + \lambda_0 T^{\text{H6},0} v(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^{\text{H6},k} v(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k^{\text{H6}} v(\mathbf{x}, y), \quad (5.6)$$

where the operators $T^{\text{H6},0}$, $T^{\text{H6},k}$ and T_k^{H6} are defined as follows:

$$\begin{aligned} T^{\text{H6},0} v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} - \mathbf{e}, y) & \text{if } y \geq R, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v(\mathbf{x}, y+1) & \text{otherwise,} \end{cases} \\ T^{\text{H6},k} v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } x_k > 0, \\ v(\mathbf{x}, y) + c_k & \text{otherwise,} \end{cases} \\ T_k^{\text{H6}} v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y = 0, \\ v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y > 0, \text{ and } \prod_{i \neq k}^n x_i = 0, \\ v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } y \leq R, y > 0, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v\left(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y-1\right) & \text{if } y > R, y > 0, \text{ and } \prod_{i \neq k}^n x_i > 0, \\ v(\mathbf{x}, y) & \text{otherwise.} \end{cases} \end{aligned}$$

In Heuristic H6, the rationing policy is used only for controlling demand of the assembled product. That is, if the backorder level for the assembled product is below the rationing level R the demand of the assembled product is backordered. Otherwise, demand of the assembled product can be satisfied. The base-stock levels s_k can be exhaustively searched over the region $\{0, s_1^{\max}\} \times \{0, s_2^{\max}\} \times \dots \times \{0, s_k^{\max}\}$, and the rationing level R can be exhaustively searched over the region $\{0, R^{\max}\}$. From equation (5.6), we determine the minimum average cost per period g^{H6} .

The Heuristic H7 policy

Under Heuristic H7, the production can be controlled by a fixed base-stock policy, while a first-come-first-served (FCFS) policy is for both the end product demand and the individual components demand.

The dynamic programming equation can be written as:

$$v(\mathbf{x}, y) + g^{\text{H7}} = h(\mathbf{x}) + b_0(y) + \lambda_0 T^{\text{H7},0} v(\mathbf{x}, y) + \sum_{k=1}^n \lambda_k T^{\text{H7},k} v(\mathbf{x}, y) + \sum_{k=1}^n \mu_k T_k^{\text{H7}} v(\mathbf{x}, y), \quad (5.7)$$

where the operators $T^{H7,0}$, $T^{H7,k}$ and T_k^{H7} are defined as follows:

$$\begin{aligned}
 T^{H7,0}v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} - \mathbf{e}, y) & \text{if } \prod_{i \neq k}^n x_i > 0, \\ v(\mathbf{x}, y + 1) & \text{otherwise,} \end{cases} \\
 T^{H7,k}v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} - \mathbf{e}_k, y) & \text{if } x_k > 0, \\ v(\mathbf{x}, y) + c_k & \text{otherwise,} \end{cases} \\
 T_k^{H7}v(\mathbf{x}, y) &= \begin{cases} v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y = 0, \\ v(\mathbf{x} + \mathbf{e}_k, y) & \text{if } x_k < s_k, y > 0, \text{ and } \prod_{i \neq k}^n x_i = 0, \\ v(\mathbf{x} - \sum_{i \neq k}^n \mathbf{e}_i, y - 1) & \text{if } y > 0, \prod_{i \neq k}^n x_i > 0, \\ v(\mathbf{x}, y) & \text{otherwise.} \end{cases}
 \end{aligned}$$

Heuristic H7 is a simple heuristic, where we only need to consider the base-stock levels for the production decision. For the demand of the individual components and of the assembled product that follow the FCFS rule, if the system has enough stock the coming demand can be satisfied. The base-stock levels s_k can be computed by exhaustively searching over the region $\{0, s_1^{\max}\} \times \{0, s_2^{\max}\} \times \dots \times \{0, s_k^{\max}\}$. From equation (5.7), we determine the minimum average cost per period g^{H7} .

5.3.3 Comparison to the Heuristic Policies

We conduct a numerical study to compare the performance of the heuristic policies. The performance is measured through the percentage cost difference between the heuristic and the optimal policy. The system under study has the following parameters: $\lambda_0, \lambda_k, \mu_k, h_k, b_0$, and c_k for $k=1,2$, which are generated randomly (see Table 5.2). To gain insight, we consider a wide range of parameter values for component production utilization ($0.5 \leq \rho_k \leq 1.2$), and the backorder cost ($b_0 \sim U(5, 15) \times \sum_{k=1}^2 h_k$). Given a combination of the parameters, we compute the value of average cost under the optimal policy and each heuristic policy. Table 5.2 is the comparison of the optimal and heuristic policies and their corresponding average cost. Here, we report the performance of our heuristic policies. As can be seen from Table 5.2, Heuristic H4 performs better than the other three heuristics in each example tested with the percentage difference varies between 0.004% to 1.808%. It is of interest to notice the relative performance of the other three heuristics. Clearly, Heuristics H5, H6 and H7 do not perform

as well as Heuristic H4 as they are further simplification of the Heuristic H4. Except for the cases 10, 47, 56, 67, 68, 72, 86, and 93, Heuristic H5 outperforms Heuristic H6. This is most likely due to the fact that when the arrival rate of end product demand is lower than that of components demand (see cases 67, 68, and 86), or the backorder cost is less than the lost sale cost (see cases 10, 47, 56, 72, and 93), priority is given to the individual components demand over the assembled product demand. In these cases, Heuristic H6 does a better job than Heuristic H5. However, in most cases the effect of backorder is higher than that of lost sale. This implies that the assembled product demand has higher priority than the individual components demand. Thus, Heuristic H5 works better than Heuristic H6. For a particular case 74, the relative load parameter $\rho_k > 1$ and the backorder cost is larger than the lost sale cost. In this case the system is out of stock, which leads to lost sales and backorders most of the time. Heuristic H4 has the benefit of the rationing policy for both demands of the assembled product and the individual components. Since the purpose of rationing is to reserve inventory for future arrivals of the more valuable demands, H4 works better than the other heuristics. Also, it is easy to observe that Heuristic H5 outperforms Heuristic H7 for all cases. In addition, Heuristics H6 and H7 do not perform well especially for cases where component demand is relatively high, as we can see in cases 31, 34, 35, 39, 43, 45, 76, 86, 87, 99, and 100. This reflects the fact that, with high component demand levels, there is a great chance for the system to stock-out. Heuristics H4 and H5 reduce the stock-out probability by the fixed base-stock and rationing levels for components, while Heuristics H6 and H7 are less flexible because there are no rationing levels for components.

By comparing the above four heuristics, it is clear that Heuristic H4 works very well for controlling the ATO system, which is a good approximation of the optimal policy. This is because all the base-stock levels and the rationing levels are considered in this heuristic, and the production and allocation can be well controlled. Although H4 is very efficient, it is computationally intensive because it requires a multi-dimensional search for the optimal thresholds. Hence, developing other heuristics is necessary. As shown in Table 5.2, the difference in performance of H4 and of H5 is very small for the cases 1, 3, 7, 20, 47, 56, 61, 64, 67, 83, 86, 95, 96, 99 and 100, if without considering the special case 74, the performance of H4 and of H5 is exactly same in the rest of the cases. We can also see that there is difference in performance of H6 and of H7 for the cases 1, 7, 47, 56, 61, 86, 95, 96, 99, and 100, and for the other cases the performance of H6 and of H7 is exactly same. This is due to the effect of backorder. In general, if the demand of the assembled product is backordered the

Table 5.2 Optimal policy versus Heuristic policies with lost sales and backorders(the system parameters are drawn from uniform distributions as $\lambda_0 \sim U(0,10), \lambda_k \sim U(0,10), \mu_k \sim U(0,10)$, $0.5 \leq \rho_k \leq 1.2, h_k \sim U(1,10), b_0 \sim U(5,15) \times \sum_{k=1}^2 h_k, c_k \sim U(100,500), \rho_k = (\lambda_0 + \lambda_k) / \mu_k, k = 1, 2.)$

	λ_0	λ_1	λ_2	μ_1	μ_2	h_1	h_2	b_0	c_1	c_2	percentage difference from optimal cost %			
											H4	H5	H6	H7
1	1.11	1.69	4.26	2.69	5.84	7.08	3.30	112.86	340.34	249.73	0.577	1.351	1.804	2.107
2	2.03	2.74	2.38	4.61	4.93	9.93	2.02	94.16	321.38	443.39	0.039	0.039	1.590	1.590
3	3.98	4.02	5.55	8.98	8.13	4.60	1.42	121.49	429.66	178.85	0.268	0.357	1.206	1.206
4	1.67	5.21	6.30	6.72	7.56	3.47	5.95	78.23	138.03	437.72	0.309	0.309	1.845	1.845
5	2.86	3.21	5.63	5.86	7.85	2.49	8.64	144.00	429.40	489.02	0.285	0.285	0.514	0.514
6	1.27	0.84	0.43	2.99	1.99	5.23	9.16	63.83	414.05	241.37	0.627	0.627	2.556	2.556
7	0.46	2.96	5.71	4.51	8.38	4.64	4.73	165.89	353.85	226.61	0.542	1.258	1.356	4.098
8	1.96	0.98	2.22	3.97	5.52	2.52	8.61	85.71	403.39	334.98	0.909	0.909	1.199	1.199
9	2.67	2.33	2.75	4.90	7.93	8.16	6.17	122.62	275.97	114.50	0.190	0.190	0.577	0.577
10	0.07	3.98	2.87	5.85	5.46	4.96	8.49	92.96	123.68	171.41	0.334	0.450	0.334	0.450
11	1.27	1.20	4.07	2.97	7.41	3.89	9.04	168.27	304.65	318.72	0.486	0.486	2.030	2.030
12	1.03	0.96	1.66	2.18	4.64	7.99	1.59	120.66	328.47	452.40	0.537	0.537	2.601	2.601
13	3.01	7.42	0.84	9.21	6.81	7.05	5.90	143.87	148.75	228.29	0.204	0.204	1.336	1.336
14	2.08	5.45	6.84	9.05	8.79	7.25	2.16	139.83	179.62	456.47	0.065	0.065	0.335	0.335
15	0.09	5.32	3.02	8.83	2.93	5.24	3.59	79.70	283.16	227.49	0.669	0.669	2.332	2.332
16	2.30	2.75	2.29	6.85	6.75	4.89	2.94	125.47	285.79	285.08	0.861	0.861	1.352	1.352
17	2.42	2.94	6.46	8.67	7.52	3.01	1.47	148.58	438.44	112.58	0.312	0.312	0.662	0.662
18	1.14	1.58	0.81	4.02	2.24	5.27	6.61	162.89	102.62	281.61	0.860	0.860	2.831	2.831
19	0.70	4.89	2.87	4.87	5.75	6.14	7.11	166.97	304.18	188.88	0.102	0.102	3.353	3.353
20	0.25	4.26	3.66	5.37	6.97	1.05	9.89	89.27	169.85	126.07	0.005	1.206	2.127	2.127
21	0.23	6.41	2.60	7.86	4.28	6.29	1.31	60.60	352.21	172.74	0.440	0.440	2.715	2.715
22	1.07	2.35	3.22	3.39	5.23	1.06	6.50	79.16	178.02	284.10	0.030	0.030	0.665	0.665
23	1.31	6.81	0.73	9.93	1.77	8.58	8.29	106.27	405.39	430.71	0.538	0.538	3.533	3.533
24	0.48	0.83	5.27	1.60	5.21	8.83	1.60	128.24	316.83	121.32	0.356	0.356	2.355	2.355
25	3.71	1.46	0.70	9.03	6.79	2.10	3.12	167.86	430.86	103.79	0.443	0.443	2.570	2.570
26	1.85	6.17	1.04	9.48	2.57	2.30	2.65	69.19	449.21	340.00	0.760	0.760	3.593	3.593
27	2.28	2.04	1.15	5.46	3.13	1.14	2.17	121.83	402.48	483.01	0.658	0.658	3.127	3.127
28	0.06	4.78	1.47	7.75	1.32	3.53	8.94	153.57	215.63	309.19	1.287	1.287	2.659	2.659
29	0.82	6.56	4.87	7.01	4.95	4.91	6.47	118.24	373.69	237.17	1.157	1.157	5.634	5.634
30	2.74	2.26	0.32	6.11	6.00	9.14	7.23	151.83	329.49	220.45	0.178	0.178	1.943	1.943
31	0.78	5.20	3.24	7.15	6.32	3.04	8.30	156.57	341.78	427.60	0.347	0.347	7.879	7.879
32	2.96	4.23	4.83	9.73	6.84	3.81	4.90	162.99	111.36	214.65	0.527	0.527	0.916	0.916
33	0.58	3.34	7.51	3.55	7.32	6.59	4.38	154.75	147.84	253.44	0.411	0.411	4.321	4.321
34	1.18	0.15	1.99	1.82	4.46	2.39	4.73	165.85	235.30	262.86	0.520	0.520	8.562	8.562
35	0.78	9.24	6.15	9.77	5.97	2.08	6.23	71.92	346.96	323.80	0.117	0.117	7.931	7.931
36	1.48	0.88	1.08	2.68	4.17	4.32	3.94	59.88	182.52	360.38	0.327	0.327	3.755	3.755
37	1.86	3.34	3.12	8.33	5.92	1.80	4.27	146.45	421.13	244.95	0.752	0.752	2.768	2.768
38	0.80	2.09	5.38	4.51	8.60	7.03	8.32	161.35	406.12	266.63	0.488	0.488	6.268	6.268
39	5.34	0.25	1.39	6.25	7.29	1.06	7.61	158.48	486.48	326.77	1.414	1.414	23.623	23.623
40	5.25	1.35	0.36	5.83	6.07	6.45	7.16	83.47	134.05	313.26	0.714	0.714	5.765	5.765
41	6.07	2.77	1.18	9.09	7.49	9.93	2.59	154.61	157.41	251.60	0.845	0.845	4.369	4.369
42	8.15	1.51	2.35	9.47	9.25	5.24	8.12	169.05	224.67	285.65	0.427	0.427	3.092	3.092
43	6.25	0.09	3.42	5.42	8.75	2.87	7.81	118.85	338.32	189.28	0.489	0.489	15.937	15.937
44	1.92	1.87	0.48	3.72	3.34	7.31	3.71	96.13	389.94	187.87	0.458	0.458	6.734	6.734
45	0.34	1.39	1.59	2.31	3.77	6.09	1.83	88.94	415.37	196.13	0.154	0.154	7.591	7.591
46	2.03	0.56	0.76	4.94	2.92	3.14	5.93	98.23	185.96	186.00	0.293	0.293	3.947	3.947
47	0.47	0.10	0.49	0.95	1.67	2.74	9.24	41.16	379.96	422.44	0.010	0.723	0.363	5.593
48	0.48	1.73	3.20	2.07	4.26	2.41	6.14	58.13	148.08	158.49	0.354	0.354	6.094	6.094
49	1.21	2.73	1.94	3.41	4.95	2.01	5.98	111.03	424.00	178.18	0.259	0.259	3.589	3.589
50	1.52	1.84	0.85	4.97	2.19	1.79	6.82	79.56	474.18	268.83	0.184	0.184	3.177	3.177

Table 5.2 Optimal policy versus Heuristic policies with lost sales and backorders(the system parameters are drawn from uniform distributions as $\lambda_0 \sim U(0,10)$, $\lambda_k \sim U(0,10)$, $\mu_k \sim U(0,10)$, $0.5 \leq \rho_k \leq 1.2$, $h_k \sim U(1,10)$, $b_0 \sim U(5,15) \times \sum_{k=1}^2 h_k$, $c_k \sim U(100,500)$, $\rho_k = (\lambda_0 + \lambda_k) / \mu_k$, $k = 1, 2$.)

	λ_0	λ_1	λ_2	μ_1	μ_2	h_1	h_2	b_0	c_1	c_2	percentage difference from optimal cost %			
											H4	H5	H6	H7
51	1.05	0.43	1.71	2.43	3.06	5.90	4.63	82.17	117.63	276.33	0.790	0.790	1.574	1.574
52	0.60	1.03	2.45	2.98	3.42	5.59	4.96	69.88	227.82	454.58	0.149	0.149	5.734	5.734
53	1.32	2.60	1.07	3.84	3.22	4.82	4.60	47.74	182.11	199.48	0.618	0.618	1.371	1.371
54	1.70	2.46	2.37	4.95	3.97	5.76	6.74	92.02	310.57	405.30	0.753	0.753	2.429	2.429
55	0.26	5.19	1.66	4.55	3.36	3.51	4.47	86.85	483.11	384.78	0.726	0.726	3.099	3.099
56	0.79	1.81	1.98	4.57	3.18	2.02	5.49	66.18	209.95	332.79	0.018	0.247	0.112	0.686
57	0.69	1.65	1.96	2.63	4.95	6.84	6.69	52.93	427.26	386.80	0.267	0.267	6.115	6.115
58	0.26	3.79	2.59	3.46	3.20	2.16	8.58	69.10	480.45	197.10	0.243	0.243	0.243	0.243
59	1.97	1.04	0.42	4.81	4.01	1.69	5.66	54.73	200.01	164.37	0.454	0.454	5.052	5.052
60	1.59	2.79	1.45	3.80	3.97	6.80	6.55	83.89	277.30	173.66	0.195	0.195	1.627	1.627
61	0.46	1.84	1.63	2.67	1.92	8.07	7.27	49.97	352.29	277.87	0.004	0.857	0.876	1.169
62	0.78	1.88	3.19	4.20	3.99	1.49	7.46	93.58	348.68	465.15	0.162	0.162	0.185	0.185
63	1.01	1.07	1.94	2.57	3.80	3.84	3.57	51.64	297.04	144.26	0.400	0.400	2.873	2.873
64	0.30	0.24	1.31	0.72	2.68	9.42	9.15	40.28	142.12	356.00	0.004	0.358	0.604	0.604
65	1.14	2.47	0.69	4.69	3.54	4.65	7.08	108.09	388.54	305.73	0.568	0.568	4.842	4.842
66	1.42	0.77	6.51	2.44	9.82	8.56	3.66	74.49	318.02	219.48	0.757	0.757	1.798	1.798
67	0.04	4.73	2.18	5.82	2.16	1.47	9.61	210.47	458.20	143.43	0.129	2.200	2.128	2.829
68	1.30	1.38	7.43	4.35	9.46	4.92	2.30	178.88	155.86	188.07	0.128	1.913	1.405	5.380
69	2.16	1.91	4.31	4.96	7.81	5.03	5.83	93.49	365.78	443.31	0.219	0.219	1.373	1.373
70	1.00	7.47	8.12	9.57	8.10	5.47	1.47	155.48	220.94	353.92	0.505	0.505	4.670	4.670
71	2.67	4.58	2.88	8.30	8.02	6.85	5.54	135.60	198.65	492.38	1.414	1.414	1.830	1.830
72	2.36	7.22	2.05	9.95	4.97	7.44	2.72	83.86	100.10	296.27	0.154	0.154	1.096	1.096
73	0.34	7.99	7.54	7.64	7.75	2.13	7.06	206.66	421.96	331.71	0.768	1.182	1.182	1.182
74	1.23	2.41	9.73	6.79	9.50	2.21	3.22	204.06	369.70	195.20	0.262	6.379	6.379	6.379
75	1.20	2.77	3.61	5.97	7.51	9.05	4.40	213.99	474.80	100.03	0.645	0.645	2.233	2.233
76	1.56	6.87	6.54	9.02	8.31	8.58	6.56	166.86	227.97	371.99	0.185	0.185	11.512	11.512
77	0.54	8.36	2.54	9.54	5.38	1.26	1.68	119.78	305.63	255.31	0.126	0.126	0.126	0.126
78	1.50	4.37	9.17	6.43	9.95	2.05	1.90	127.34	393.21	390.50	0.673	0.673	2.209	2.209
79	1.71	6.67	2.80	7.16	8.61	9.97	3.28	103.01	277.92	410.01	0.284	0.284	1.778	1.778
80	1.09	9.29	6.05	9.81	7.01	3.85	9.17	160.18	493.08	439.47	0.049	0.049	1.206	1.207
81	2.35	5.18	2.83	9.79	7.49	7.76	9.99	113.47	240.79	134.63	0.035	0.035	1.158	1.158
82	2.02	1.69	3.53	3.23	6.15	8.60	4.19	103.31	112.05	321.78	0.234	0.234	2.733	2.733
83	0.06	7.62	7.53	8.28	8.29	7.31	8.79	171.79	332.55	476.42	0.124	0.467	0.937	0.937
84	1.88	1.95	1.07	5.53	3.78	3.34	4.39	137.15	423.48	327.29	0.696	0.696	4.845	4.845
85	0.84	3.50	4.97	7.63	6.51	4.91	2.30	131.28	474.00	399.34	0.149	0.149	0.180	0.180
86	1.35	3.23	3.81	4.03	7.15	1.91	9.53	154.88	216.67	342.51	0.274	0.483	0.436	7.136
87	2.36	1.60	3.44	6.15	8.59	2.21	4.51	147.88	381.74	156.01	0.648	0.648	10.786	10.786
88	3.98	2.44	1.38	6.76	7.97	3.23	5.95	169.17	201.51	125.56	0.824	0.824	3.140	3.140
89	3.55	2.65	0.15	5.99	7.04	6.48	8.48	95.45	252.43	358.64	0.321	0.321	3.996	3.996
90	1.39	6.31	2.87	8.44	4.61	8.18	6.95	156.13	385.49	162.64	0.056	0.056	0.315	0.315
91	2.87	6.20	1.22	9.87	5.81	4.06	6.91	157.53	189.58	342.06	0.401	0.401	1.372	1.372
92	0.81	0.84	2.65	2.37	3.62	2.53	4.48	210.84	471.41	148.77	0.554	0.554	4.754	4.754
93	0.01	3.93	4.42	3.95	6.45	8.28	7.14	173.97	468.35	307.94	0.309	0.780	0.309	3.378
94	4.97	2.04	4.66	8.80	8.76	8.36	1.78	101.93	456.99	113.52	0.563	0.563	0.815	0.815
95	0.90	0.47	0.53	2.07	1.99	3.92	9.23	149.65	425.05	474.70	0.560	0.837	4.102	6.342
96	3.24	1.70	3.25	9.14	7.10	4.99	4.85	202.65	446.59	224.88	0.054	0.054	0.088	4.119
97	1.88	5.00	4.23	7.85	5.71	8.10	4.10	90.06	241.20	164.47	0.513	0.513	4.324	4.324
98	2.68	0.70	1.39	4.93	8.06	6.22	3.46	184.15	102.55	282.97	0.268	0.268	1.250	1.250
99	1.19	6.19	8.69	8.58	9.85	1.43	2.31	117.88	352.09	147.67	0.521	1.046	1.315	11.904
100	4.78	4.23	4.78	8.28	1.88	4.35	2.72	191.95	237.26	218.61	0.955	1.548	1.567	27.374

the system tends to satisfy it immediately, there is no need to rationing the demand of the assembled product, thus the rationing level R for the assembled product equals to 0. Consequently, for most cases Heuristic H4 is identical to Heuristic H5, and Heuristic H6 is identical to Heuristic H7. Although H5 does not perform as well as H4, and H7 does not perform as well as H6, the computational time of H5 and of H7 is significantly faster than that of H4 and of H6. Thus, with this advantage Heuristics H5 and H7 are also efficient for controlling the system. Our analysis suggests that the manager should pay more attention to the rationing policy of the individual components. Since the rationing level r_k for Component k depends on the on-hand inventory x_k , and the rationing level R for the assembled product depends on the backorder level y , in comparison, the on-hand inventory level is easier to control.

5.4 Conclusion

In order to well understand the structure of the optimal policy in ATO system studied in this work, we proposed several static heuristic policies for Chapter 3 the pure lost sales and for Chapter 4 the mixed lost sales and backorders. The heuristics are characterized by static base-stock and rationing levels. Note that Heuristics H1 and H4 are similar to the IBR policy in Benjaafar and ElHafsi (2006) for the pure product demand system. Here, we used and redefined the IBR policy to the mixed demand (demand from the assembled product and individual components) system. Based on our numerical experiences, we found that Heuristic H1 is a good approximation of the optimal policy in the pure lost sales case, and Heuristics H4 and H5 are good approximations of the optimal policy in the mixed lost sales and backorders. For the others Heuristics H2, H3, H6 and H7, with the advantage of quite simple formulation and fast computation they are also necessary to be considered for controlling our ATO system.

Conclusions and Future Perspectives

We conclude our work in this chapter. We considered an ATO system with continuous times and infinite-horizon. In this system, both production and demand filling decisions must be made. We have studied the pure lost sales case and the mixed lost sales and backorders case with exponential production times and Poisson demand. Using the dynamic programming formulation for each case, we showed that the structure of the optimal production policy is state-dependent base-stock policy, and the optimal allocation policy is a state-dependent rationing policy. We characterized the structure of the optimal policy. In addition, we developed several simple heuristic policies for the pure lost sales case and the mixed lost sales and backorders case. In each case, we compare the performance of the heuristic policies, and found a good approximation of the optimal policy. Finally, there are several potential avenues for future research.

We studied an ATO system with a single assembled product assembled from n components. Demand from the individual components can be satisfied or rejected, and demand from the assembled product can be satisfied or rejected/backordered. The unsatisfied demand incurs a penalty cost. Components are produced one unit at a time on separate production facilities and held in stock incurring a different holding costs. We assume exponentially distributed production times, and demand arrives in the system as independent Poisson processes. We considered two cases: the pure lost sales and the mixed the lost sales and backorders. For both cases, using an MDP approach, we showed that the optimal production policy is a state-dependent base-stock policy and the optimal allocation policy is a state-dependent rationing policy. We also did some numerical experiments to explore the impact on the optimal policy of different system parameters. Finally, we proposed three heuristic policies in the pure lost sales case and four heuristic policies in the mixed lost sales and backorders case. In each case, we compared the performance of heuristic policies and found a good approximation of the optimal policy. We now outline several important results in our work:

For the pure lost sales case

- The optimal production policy for Component k is a base-stock policy with a state-dependent base-stock level $s_k^*(\mathbf{x}_{-k})$ where it is optimal to produce Component k , if $x_k < s_k^*(\mathbf{x}_{-k})$ and not to produce it otherwise. The optimal inventory allocation policy for Component k is a rationing policy with a state-dependent rationing level $r_k^*(\mathbf{x}_{-k})$ where it is optimal to satisfy the demand of Component k if $x_k > r_k^*(\mathbf{x}_{-k})$ and to reject it otherwise. Both levels are non-decreasing in each of the states x_i , $i \neq k$. Furthermore, it is always optimal to satisfy the demand of the assembled product whenever on-hand inventory for all components is available.
- 1. The base-stock level $s_k^*(\mathbf{x}_{-k})$ for Component k , is non-increasing in the holding cost h_k and non-decreasing in the lost sales costs c_0 and c_k , for $k=1,2,\dots,n$.
- 2. The base-stock level $s_k^*(\mathbf{x}_{-k})$ for Component k , is non-decreasing in the arrival rates λ_0 and λ_k and non-increasing in the production rate μ_k , for $k=1,2,\dots,n$.
- 3. The rationing level $r_k^*(\mathbf{x}_{-k})$ from Component k , is non-increasing in the holding cost h_k , non-decreasing in the lost sales cost c_0 and non-increasing in c_k for $k=1,2,\dots,n$.
- 4. The rationing level $r_k^*(\mathbf{x}_{-k})$ from Component k , is non-decreasing in the arrival rates

λ_0 and λ_k and non-increasing in the production rate μ_k , for $k=1,2,\dots,n$.

- We developed three static heuristic policies: Heuristics H1, H2 and H3 for the pure lost sales case. Comparing the performance of the heuristic policies with that of the optimal policy we found that Heuristics H1 and H2 are efficient and clearly outperform Heuristic H3. Heuristic H3 does not perform well especially for cases where component demand is relatively high. In comparison, Heuristic H1 is more efficient because it needs to take into account the on-hand inventory of other components to determine the optimal base-stock and rationing levels. In Heuristic H2, there is no need to consider the on-hand inventory of other components that may affect the accuracy of approximation. However, the H2 offers a quite simple and computationally effective way of computing the base-stock and rationing levels. Based on the numerical results, it is clear that Heuristic IBR works best for controlling the ATO system.

For the mixed lost sales and backorders case

- For Component k , $k=1,\dots,n$, the base-stock level $s_k^*(\mathbf{x}_{-k}, y)$ can be interpreted as a rationing level of the assembled product at Component k for $y>0$, $\prod_{i \neq k}^n x_i > 0$. In other words, $s_k^*(\mathbf{x}_{-k}, y) = R_k^*(\mathbf{x}_{-k}, y-1)$.
- For Component k , $k=1,\dots,n$, there exists an optimal stationary policy specified in terms of a state-dependent production base-stock level $s_k^*(\mathbf{x}_{-k}, y)$, for $y=0$, or $y>0$, $\prod_{i \neq k}^n x_i = 0$ and a state-dependent inventory rationing level $r_k^*(\mathbf{x}_{-k}, y)$. For the assembled product, there exists an optimal stationary inventory rationing policy specified in terms of a state-dependent rationing level $R_k^*(\mathbf{x}_{-k}, y)$. In particular, the structure of the optimal policy can be described as follows:

1. Optimal production policy for Component k

Produce to increase Component k inventory if $x_k < s_k^*(\mathbf{x}_{-k}, y)$ for $y=0$, or $y>0$ and $\prod_{i \neq k}^n x_i = 0$.

Produce to reduce backorders of the assembled product if $x_k \geq R_k^*(\mathbf{x}_{-k}, y-1)$ for $y>0$, and $\prod_{i \neq k}^n x_i > 0$.

Do not produce if $x_k \geq s_k^*(\mathbf{x}_{-k}, y)$ for $y=0$, or $y>0$, and $\prod_{i \neq k}^n x_i = 0$.

2. Optimal inventory allocation policy for Component k

Satisfy demand of Component k if $x_k \geq r_k^*(\mathbf{x}_{-k}, y)$.

Reject demand of Component k if $x_k < r_k^*(\mathbf{x}_{-k}, y)$.

3. Optimal inventory allocation policy for the assembled product

Satisfy demand of the assembled product if $x_k \geq R_k^*(\mathbf{x}_{-k}, y)$ for all $k, k=1,2,\dots,n$.

Backorder demand of the assembled product if at least one component has inventory level $x_k < R_k^*(\mathbf{x}_{-k}, y)$.

Furthermore, the production and rationing levels have the following properties:

P1: $s_k^*(\mathbf{x}_{-k}, y)$ is non-decreasing in both the inventory level x_i of Component $i, i \neq k$, and the backlog level of the assembled product, y .

P2: $r_k^*(\mathbf{x}_{-k}, y)$ is non-decreasing in both the inventory level x_i of Component $i, i \neq k$, and the backlog level of the assembled product, y .

P3: $R_k^*(\mathbf{x}_{-k}, y)$ is non-increasing in both the inventory level x_i of Component $i, i \neq k$, and the backlog level of the assembled product, y .

- We developed four static heuristic policies: Heuristics H4, H5, H6 and H7 for the mixed lost sales and backorders case. Comparing the performance of the heuristic policies with that of the optimal policy we found that Heuristic H4 performs better than the other three heuristics. Clearly, Heuristics H5, H6 and H7 are the suboptimal policies and they are the further simplification of the Heuristic H7. In comparison, in most cases, Heuristic H5 works better than Heuristic H6. However, when the arrival rate of the assembled product demand is lower than that of components demand or the backorder cost is less than the lost sale cost, Heuristic H6 works better than Heuristic H5. We observed that Heuristic H5 outperforms Heuristic H7 for all cases. In addition, Heuristic H6 and Heuristic H7 do not perform well especially for cases where component demand is relatively high. We were able to obtain the result that Heuristic H4 works very well for controlling the ATO system, which does provide a more precise and a good approximation of the optimal policy.

The results presented in this work encourage us to believe that it will be fruitful to extend the research work in multiple directions. We hope to extend the models and heuristics to the general assembly system. We list several interesting research directions as follows:

Production times have an Erlang distribution

The limitation of our model is that the production times follow an exponential distribution, while this assumption may not be realistic in a real ATO system. In most ATO systems, there exists the multistage manufacturing process that incurs production time variability. In order to deal with this, the production times are assumed with Erlang distribution. Since “information on the production status and production time variability” (Gayon et al., 2009), Erlang distributions have the advantage of flexibility in modeling production processes. Thus one direction for the future research is to study the optimal policy for the general case of Erlang production times. As a continuation of this work, it would be useful to extend our model to Erlang production times with lost sales/backorders.

Batch production

Multiple non-unitary demands

Batch production and multiple non-unitary demands are common in many real ATO systems. In such systems, batch production occurs when many similar components are produced together, and demand not always occurs one unit at a time. We assume that demand inter-arrival times follow a certain distribution, for instance, the Erlang distribution. We consider batch production and multiple non-unitary demands in a single model with multiple products and various demand size.

Develop efficient heuristics for the above models

Due to uncertain product demand, and production leadtimes for different components is different, ATO systems tend to be difficult to manage. The key challenge for ATO systems is to efficiently manage component inventories and make optimal production and allocation decisions. The above models presented are more general cases of ATO systems, which would be more difficult to analyze. We can also use an MDP approach to determine the optimal policy for such systems. However, the structure of the optimal policy would not be simple. Also, the optimal policy would not be easy to implement. Thus, we expect to develop the efficient heuristics for such systems, which should be more easy for system managers to comprehend and implement in practice.

Résumé Etendu en Français

L'objectif principal de ce travail est d'étudier un cas particulier d'un système de fabrication assemble-to-order (ATO) qui est non seulement sujette à la demande pour le produit assemblé mais également sujette à la demande pour les composants individuels. À cette fin, nous utilisons un cadre de processus de décision markovien (MDP) pour formuler le système et nous étudions par la suite la politique optimale.

Introduction générale

Dans l'environnement économique actuel d'aujourd'hui, avec le renforcement de la compétitivité du marché mondial, la personnalisation de masse est devenue un objectif majeur pour de nombreuses entreprises de fabrication. Cette tendance a forcé les entreprises à adopter une stratégie d'exploitation hybride à mieux traiter une variété d'environnements de marché. Dans ce but, un système d'assemblage connu comme ATO, a émergé et est devenu plus populaire. Un système ATO produit plusieurs composants et les assemble en une variété de produits finis. L'information sur les demandes arrive seulement pour les produits finis, mais le système garde l'inventaire au niveau de composants (Song et Zipkin, 2003).

Les systèmes ATO peuvent être considérés comme une affectation de ressources multiples qui induit la planification de production, la satisfaction des contraintes et l'affectation des stocks. Les systèmes ATO représentent une stratégie de logistique populaire utilisée en gestion de fabrication. En raison de la complexité croissante des systèmes de fabrication d'aujourd'hui, le défi pour les systèmes ATO est de gérer efficacement les stocks de composants et de trouver les décisions optimales de production et d'affectation.

Nous étudions un système ATO avec un produit unique qui est assemblé à partir de plusieurs composants. Le système doit répondre à une demande non seulement du produit assemblé mais aussi des composants individuels. Nous considérons le cas avec seulement des *lost sales* puis le cas mixte *lost sales* et *backorders* avec des temps de production suivant des lois de type exponentiel et une demande sous forme de loi de Poisson. Nous formulons le problème

comme un Processus de décision markovien (MDP), et nous considérons deux critères d'optimalité qui sont le coût actualisé et le coût moyen par période. Nous caractérisons la structure de la politique optimale et étudions l'impact des différents paramètres du système sur cette politique. Nous présentons également plusieurs heuristiques pour le cas *lost sales* et le cas mixte *lost sales* et *backorders*. Ces heuristiques fournissent des méthodes simples, mais efficaces pour contrôler la production et l'affectation des stocks du système ATO.

Aperçu de la thèse

Dans chaque chapitre, nous traitons un ATO système avec des demandes en composants individuels. Nous nous concentrons sur le cas *lost sales* et le cas mixte *lost sales* et *backorders*. Pour chaque cas, nous commençons par présenter un aperçu de la revue de la littérature. Ensuite, nous formulons le modèle de problème et étudions la politique optimale du système, puis nous développons les heuristiques pour résoudre le problème. A la fin, nous présentons des exemples numériques pour analyser la performance et démontrer l'efficacité des heuristiques proposés.

Le Chapitre 1 correspond à l'introduction générale qui présente les principes généraux de système ATO et la revue de l'état de l'art. Nous présentons, de même, les notions de base utilisées tout au long de cette thèse, tels que (i) la définition des systèmes ATO, (ii) le rôle important de contrôle optimal, (iii) les principes de l'approche générale, (iv) et les paramètres du problème. Enfin, nous concluons ce chapitre avec un plan de la thèse.

Le Chapitre 2 est une brève revue des travaux existants sur le système ATO. Tous d'abord, nous présentons l'état de l'art des systèmes *make-to-stock* en vertu du MDP et des systèmes ATO en vertu de l'examen continu et de l'examen périodique. Nous nous intéressons plus particulièrement aux systèmes ATO en temps continu qui motive cette partie de la thèse. Dans cette ligne de recherche, nous étudions le modèle qui traite le système ATO avec deux types de demandes: une demande en produit assemblé et une demande en composants individuels. À notre connaissance, il y a actuellement peu de travaux dans ce domaine. Nous espérons que la recherche présentée dans ce travail enrichit la littérature actuelle sur les systèmes ATO avec le type de demande en composants individuels.

Le Chapitre 3 vise à caractériser la politique optimale du système ATO avec le cas *lost sales*. Dans ce chapitre, nous supposons que si une demande en produit assemblé ou en composants individuels ne peut être satisfaite immédiatement, elle sera perdue. Nous proposons un

modèle de programmation dynamique afin de déterminer la politique de contrôle optimal d'un tel système. Nous formulons le problème en utilisant une méthodologie MDP et en utilisant deux critères d'optimalité qui sont le coût actualisé et le coût moyen par période. En outre, nous déterminons la structure de politique optimale et étudions l'impact des différents paramètres sur la politique optimale. Nous testons le système avec un large éventail de paramètres du système et nous montrerons que les niveaux de base-stock et de rationnement sont sensibles aux paramètres du système.

Le Chapitre 4 vise à caractériser la politique optimale du système ATO avec le cas mixte *lost sales* et *backorders*. Dans ce chapitre, nous supposons que (i) si une demande en produit assemblé ne peut être satisfaite immédiatement, elle attendra la future disponibilité (ii) et si une demande en composants individuels ne peut pas être satisfaite immédiatement, elle sera perdue. La gestion efficace d'un tel système est difficile étant donné les exigences de composants corrélés. Pour résoudre ce problème, nous le formulons comme un MDP et nous caractérisons sa politique optimale. Enfin, nous étudions l'impact des différents paramètres de coût de pénalité sur la politique optimale.

Pour le Chapitre 5, nous présentons plusieurs heuristiques qui sont basées sur le modèle de programmation dynamique étudié dans les Chapitres 3 et 4. Pour ces méthodes approchées, nous utilisons l'approche des seuils statiques qui permet de se rapprocher à la politique optimale du système. Plus précisément, ces heuristiques fournissent des méthodes simples, mais efficaces pour contrôler la production et l'affectation des stocks du système ATO. En raison de la complexité de ces systèmes, la politique optimale est mathématiquement difficile pour un grand nombre de composants. Nous devons donc développer d'autres heuristiques plus efficaces pour trouver une bonne approximation de la politique optimale. Evidemment, les heuristiques proposées sont des politiques sous-optimales, mais lorsqu'elles sont utilisées efficacement, elles permettent d'améliorer les performances dans le système ATO.

Nos contributions

Etant donné la revue de la littérature existante, nous pouvons dire qu'il existe peu de travaux concernant les systèmes ATO avec le type de demande en composants individuels. Nous écrivons maintenant plusieurs résultats importants dans notre travail.

Nous avons en premier lieu, étudié un système ATO dans le cas *lost sales* pour des demandes en produit assemblé et composants individuels. Nous avons démontré que la politique

optimale de production pour un Composant k peut être décrite par le niveau de base-stock état dépendant, qui est non-décroissant sur le niveau de l'inventaire d'autres composants. Nous avons aussi démontré que la politique optimale d'affectation pour un Composant k peut être décrite par le niveau de rationnement état dépendant, qui est non-décroissant sur le niveau de l'inventaire d'autres composants. Dans ce cas, nous pouvons produire une unité de Composant k , si le niveau d'inventaire de Composant k est inférieur au niveau de base-stock, sinon nous ne faisons rien. Nous pouvons satisfaire une unité de demande en Composant k , si le niveau d'inventaire de Composant k est inférieur au niveau de rationnement, sinon nous pouvons la refuser.

En deuxième lieu, Nous avons étudié un système ATO dans le cas mixte *lost sales* pour la demande en produit assemblé et *backorders* pour la demande en composants individuels. En plus de considérer l'inventaire du système, le niveau de *backorder* de produit assemblé doit également être envisagé. Nous avons démontré que dans quelques conditions, le niveau de base stock peut être interprété comme un niveau de rationnement du produit assemblé au Composant k . Nous avons aussi démontré que la politique optimale peut être décrite par trois types de seuil état dépendant: (i) pour le Composant k , il existe le niveau de base-stock, et le niveau de rationnement, (ii) pour le produit assemblé il existe le niveau de rationnement. Tous les types de seuil dépendent de l'inventaire du système et du niveau de *backorder* du produit assemblé.

Troisièmement, nous avons développé trois heuristiques pour le cas *lost sales*, et quatre heuristiques pour le cas mixte *lost sales* et *backorders*. Sur la base de nos expériences numériques, nous avons constaté que l'Heuristique H1 est une bonne approximation de la politique optimale dans le cas *lost sales*, et que les Heuristiques H4 et H5 sont de bonnes approximations de la politique optimale dans le cas mixte *lost sales* et *backorders*. Pour les autres Heuristiques H2, H3, H6 et H7, étant donné leur simple formulation et leur calcul rapide, ils sont également nécessaires pour contrôler le système ATO étudié.

Perspectives d'avenir

Les résultats présentés dans ce travail nous incitent à croire qu'il sera utile de prolonger les travaux de recherche dans de multiples directions. Nous espérons étendre les modèles et heuristiques pour le système de l'assemblée générale. Nous listons plusieurs directions de recherche intéressantes comme suit:

Les temps de production ont une distribution Erlang

Toutefois, il existe des limites à nos recherches, tel que les temps de production suivent une distribution exponentielle, alors que cette hypothèse n'est pas réaliste dans un véritable système ATO. Dans la plupart des systèmes ATO, il existe des processus de fabrication par lots de plusieurs étages dans lequel les composants peuvent être produits ce qui engendre la variabilité des temps de production. Afin de régler ce problème, les temps de production sont supposés avec Erlang distribution. Avec les caractéristiques de “*information on the production status and production time variability*” (Gayon et al., 2009), les distributions Erlang ont l'avantage de flexibilité dans la modélisation des processus de production. Ainsi, une direction pour la recherche future est d'étudier la politique optimale pour le cas général d'Erlang temps de production. Dans le prolongement de ce travail, il serait utile d'étendre notre modèle à Erlang temps de production avec le cas *lost sales/backorders*.

Production en lots

Demande non unitaire

La production en lots et la demande non unitaire sont communs dans de nombreux systèmes ATO réels. Dans un tel système, une demande n'arrive pas toujours seule fois mais elle peut avoir lieu continuellement au cours du temps. Nous pouvons supposer que les temps de demande inter-arrivée suivent une certaine distribution, par exemple, la distribution Erlang. Il est nécessaire de considérer un système ATO avec une production en lots ou une demande non unitaire, ou combiner ces deux cas dans un modèle unique. Dans ce cadre, nous pouvons étudier les systèmes avec des produits multiples et différentes tailles de demandes.

Développer des heuristiques efficaces pour les modèles décrits ci-dessus

En raison de l'incertitude des demandes de produits et de la différence des délais de production pour des différents composants, les systèmes ATO ont tendance à être difficiles à gérer. Le principal défi pour les systèmes ATO est de gérer efficacement les stocks de composants et de prendre des décisions de production et d'affectation optimales. Les modèles présentés ci-dessus sont des cas plus généraux des systèmes ATO, qui seront plus difficiles à analyser. Nous pouvons aussi utiliser une approche MDP pour déterminer la politique optimale pour ces systèmes. Cependant, la structure de politique optimale ne serait pas simple. En outre, la politique optimale ne serait pas facile à mettre en œuvre. Ainsi, nous projetons de

développer des heuristiques efficaces pour de tels systèmes, qui devraient être plus faciles à exploiter et à mettre en œuvre dans la pratique pour les gestionnaires du système.

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Commande optimale (en Production et Stock) de Systèmes Assemble-To-Order (ATO) avec prise en compte de demandes en composants individuels

Résumé Les systèmes *assemble-to-order* (ATO) peuvent être considérés comme une affectation de ressources multiples qui induit planification de production, satisfaction des contraintes et affectation des stocks. Les systèmes ATO représentent une stratégie de logistique populaire utilisée en gestion de fabrication. En raison de la complexité croissante des systèmes de fabrication d'aujourd'hui, le défi pour les systèmes ATO est de gérer efficacement les stocks de composants et de trouver les décisions optimales de production et d'affectation.

Nous étudions un système ATO avec un produit unique qui est assemblé à partir de plusieurs composants. Le système doit répondre à une demande non seulement du produit assemblé mais aussi des composants individuels. Nous considérons le cas avec seulement des *lost sales* puis le cas mixte *lost sales* et *backorders* avec des temps de production suivant des lois de type exponentiel et une demande sous forme de loi de Poisson. Nous formulons le problème comme un Processus de décision markovien (MDP), et nous considérons deux critères d'optimalité qui sont le coût actualisé et le coût moyen par période. Nous caractérisons la structure de la politique optimale et étudions l'impact des différents paramètres du système sur cette politique. Nous présentons également plusieurs heuristiques pour le cas *lost sales* et le cas mixte *lost sales* et *backorders*. Ces heuristiques fournissent des méthodes simples, mais efficaces pour contrôler la production et l'affectation des stocks du système ATO.

Mots-clés: systèmes *assemble-to-order*, contrôle optimal, Processus de décision markovien

Integrated Production and Inventory Control of Assemble-To-Order Systems with Individual Components Demand

Abstract: Assemble-to-order (ATO) systems can be regarded as a multiple resource allocation that induces production planning, requirements fulfilling and inventory assignment. ATO is a popular strategy used in manufacturing management. Due to the increasing complexity of today's manufacturing systems, the challenge for ATO systems is to efficiently manage component inventories and make optimal production and allocation decisions.

We study an ATO system with a single product which is assembled from multiple components. The system faces demand not only from the assembled product but also from the individual components. We consider the pure lost sales case and the mixed lost sales and backorders case with exponential production times and Poisson demand. We formulate the problem as a Markov decision process (MDP), and consider it under two optimality criteria: discounted cost and average cost per period. We characterize the structure of the optimal policy and investigate the impact of different system parameters on the optimal policy. We also present several static heuristic policies for the pure lost sales and the mixed lost sales and backorders cases. These static heuristics provide simple, yet effective approaches for controlling production and inventory allocation of ATO system.

Keywords: Assemble-To-Order systems, optimal control, Markov decision processes.