Weak KAM theory and instability for families of Hamiltonians

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Théorie KAM faible et instabilité pour familles d'hamiltoniens

THÈSE

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Résumé

Dans cette thèse nous étudions la dynamique engendrée par une famille de flots hamiltoniens. Un tel système dynamique à plusieurs générateurs est aussi appelé 'polysystème'. Motivés par des questions liées au phénomène de la diffusion d'Arnold, notre objectif est de construire des trajectoires du polysystème qui relient deux régions lointaines de l'espace des phases.

La thèse est divisée en trois parties.

Dans la première partie, nous considérons le polysystème engendré par les flots discrétisés d'une famille d'hamiltoniens Tonelli. En utilisant une approche variationnelle issue de la théorie KAM faible, nous donnons des conditions suffisantes pour l'existence des trajectoires souhaitées.

Dans la deuxième partie, nous traitons le cas d'un polysystème engendré par un couple de flots hamiltoniens à temps continu, dont l'étude rentre dans le cadre de la théorie géométrique du contrôle. Dans ce contexte, nous montrons dans certains cas la transitivité d'un polysystème générique, à l'aide du théorème de transversalité de Thom.

La dernière partie de la thèse est dédiée à obtenir une nouvelle version du théorème de transversalité de Thom s'exprimant en termes d'ensembles rectifiables de codimension positive. Dans cette partie il n'est pas question de polysystèmes, ni d'hamiltoniens. Néanmoins, les résultats obtenus ici sont utilisés dans la deuxième partie de la thèse.

Mots clés: dynamique hamiltonienne et lagrangienne, théorie KAM faible, diffusion d'Arnold, polysystème, semi-groupe de Lax-Oleinik, ensembles d'Aubry et Mañé, propriétés génériques, théorie géométrique du contrôle, ensemble atteignable, théorème de transversalité de Thom, ensemble rectifiable.

Abstract

In this thesis we study the dynamics generated by a family of Hamiltonian flows. Such a dynamical system with several generators is also called 'polysystem'. Motivated by some questions related to the phenomenon of Arnold diffusion, our aim is to construct trajectories of the polysystem which connect two far-apart regions of the phase space.

The thesis is divided into three parts.

In the first part, we consider the polysystem generated by the time-one maps of a family of Tonelli Hamiltonians. By using a variational approach falling within the framework of weak KAM theory, we give sufficient conditions for the existence of the desired trajectories.

In the second part, we address the case of a polysystem generated by two continuous-time Hamiltonian flows. This problem fits into the framework of geometric control theory. In this context, we show in some cases the transitivity of a generic polysystem, by means of Thom's transversality theorem.

The third and last part of the thesis is devoted to the proof of a new version of Thom's transversality theorem, formulated in terms of rectifiable sets of positive codimension. Neither polysystems nor Hamiltonians are explicitly involved in this part. However, the results obtained here are used in the second part of the thesis.

Keywords: Hamiltonian and Lagrangian dynamics, weak KAM theory, Arnold diffusion, polysystem, Lax-Oleinik semigroup, Aubry and Mañé sets, generic properties, geometric control theory, reachable set, Thom's transversality theorem, rectifiable set

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Paris, 01 Mars 2013

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Introduction

Lorsque l'on étudie les propriétés des systèmes hamiltoniens presque-intégrables, et en particulier le phénomène d'instabilité connu sous le nom de *diffusion d'Arnold*, on est amené à considérer des systèmes dynamiques "à plusieurs générateurs". Plus précisément, ces systèmes sont obtenus en itérant dans un ordre quelconque les éléments d'une famille de difféomorphismes hamiltoniens d'une variété symplectique N associée au problème. De tels systèmes, où la dynamique est engendrée par une famille d'applications, sont aussi appelés *polysystèmes*.

Du point de vue de la diffusion d'Arnold, l'intérêt est de chercher des trajectoires du polysystème qui relient deux régions "lointaines" de N. Dans cette optique, nous traitons dans cette thèse la question de l'existence de telles trajectoires dans le contexte d'un polysystème hamiltonien général.

Instabilité pour familles d'hamiltoniens

Dans cette section nous allons d'abord formaliser quelques définitions générales sur les polysystèmes. Cela permettra dans la suite d'avoir un cadre unique pour situer les différents chapitres. On introduira ensuite les objectifs et le contenu de la thèse.

Soient N une variété symplectique et $\mathcal{F}$ une famille arbitraire d'hamiltoniens définis sur N. Pour toute fonction $H\colon N \to \mathbb{R}$, on note $\{\phi_H^t\}_{t\in\mathbb{R}}$ le flot hamiltonien associé. On suppose que tous les flots sont complets. Dans cette thèse nous étudions l'action sur N du semi-groupe engendré par

$$\left\{\phi_H^t : t \geq 0,\ H \in \mathcal{F}\right\}.$$

On appelle le système ainsi obtenu *polysystème hamiltonien à temps continu orienté*. Nous allons aussi considérer le cas où les éléments de $\mathcal{F}$ dépendent du temps de manière périodique (avec période 1), dans ce cas nous étudions l'action sur N du semi-groupe engendré par

$$\left\{\phi_H^n : n \in \mathbb{N},\ H \in \mathcal{F}\right\}.$$

et nous appelons *polysystème hamiltonien à temps discret orienté* le système ainsi obtenu.[1]

La notion de trajectoire d'un système dynamique s'étend naturellement au cas d'un polysystème. Nous utiliserons dans ce dernier cas le terme *polytrajectoire* pour éviter toute ambiguïté. Nous formalisons ci-dessous cette notion.

[1] Les polysystèmes à temps discret orienté sont parfois appelés *systèmes de fonctions iterées*. Les polysystèmes à temps continu sont aussi appelés *switched systems* ou *bang-bang control systems*.

Dans les polysystèmes à temps continu orienté, une polytrajectoire est une courbe $\gamma\colon \mathbb{R} \to N$ telle qu'il existe une suite strictement croissante $(t_j)_{j\in\mathbb{Z}} \subseteq \mathbb{R}$ avec $\lim_{j\to\pm\infty} t_j = \pm\infty$ et une suite $(H_j)_{j\in\mathbb{Z}} \subseteq \mathcal{F}$ avec

$$\gamma(t) = \phi_{H_j}^{t-t_j}(\gamma(t_j)) \qquad \forall\, t \in [t_j, t_{j+1}] \quad \forall\, j \in \mathbb{Z}.$$

Dans les polysystèmes à temps discret orienté, une polytrajectoire est une suite $(z_j)_{j\in\mathbb{Z}} \subset N$ telle qu'il existe une suite $(H_j)_{j\in\mathbb{Z}} \subseteq \mathcal{F}$ avec

$$z_{j+1} = \phi_{H_j}^{1}(z_j) \qquad \forall\, j \in \mathbb{Z}.$$

En remplaçant $\mathcal{F}$ par $\mathcal{F} \cup -\mathcal{F}$ dans les paragraphes précédentes on obtient les définitions analogues pour les polysystèmes à temps *non-orienté* (continu ou bien discret).

Nous abordons dans ces contextes les questions suivantes: existe-t-il des polytrajectoires qui relient deux régions "lointaines" données de la variété N? Sous quelles conditions sur les hamiltoniens de la famille $\mathcal{F}$?

Par exemple, dans le cas particulier du polysystème à temps discret orienté engendré par une famille d'applications *twist* exactes-symplectiques du cylindre $\mathbb{T} \times \mathbb{R}$, les travaux de Moeckel, Jaulent et Le Calvez (cf. [Moe02, Jau, LC07]) montrent que la seule obstruction à l'existence de polytrajectoires reliant deux "hauteurs" différentes du cylindre est représentée par les cercles non contractibles invariants communs à toutes les applications de la famille. Nous reviendrons plus en détail sur leurs résultats dans la suite.

La motivation sous-jacente à ces questions, au-delà de leur intérêt intrinsèque, réside dans le fait que dans certaines situations l'étude de l'instabilité d'un système hamiltonien intégrable perturbé peut être réduit dans une certaine mesure à l'analyse d'un polysystème à temps discret orienté. Comme déjà mentionné dans l'avant-propos, cet aspect est lié au phénomène d'instabilité connu sous le nom de *diffusion d'Arnold*. Ce phénomène constitue à présent l'un des thèmes majeurs de recherche en dynamique hamiltonienne. Nous renvoyons à [Loc99, KL08, DGdlLS08, Ber10a, Che10] pour des articles d'introduction et à [Moe02, MS04, Mar08, BP12] pour des articles où les liens avec les polysystèmes sont explorés.

Structure de la thèse

Cette thèse est divisée en trois parties.

Dans la Partie 1 nous nous plaçons dans le cadre d'un polysystème hamiltonien à temps discret orienté engendré par une famille d'hamiltoniens Tonelli. Pour répondre aux questions précédentes nous utilisons une approche variationnelle issue de la théorie KAM faible.

Dans la Partie 2 nous considérons le cas d'un polysystème hamiltonien à temps continu (orienté et non orienté) engendré par deux hamiltoniens H_1 et H_2. L'approche utilisée ici s'inscrit dans le cadre de la théorie géométrique du contrôle.

Dans la Partie 3, qui a été développée en collaboration avec Patrick Bernard, nous présentons une nouvelle version du théorème classique de transversalité de Thom. Dans cette partie il n'est pas question de polysystèmes, ni d'hamiltoniens. Néanmoins, les résultats obtenus ici sont utilisés dans la Partie 2.

Présentation des résultats

Partie 1: Polysystèmes hamiltoniens à temps discret (approche avec la théorie KAM faible)

Dans cette partie la variété symplectique N introduite précédemment sera toujours l'espace cotangent T^*M d'une variété compacte connexe lisse sans bord M de dimension $d \in \mathbb{N}$. Notons $z = (x, p)$ les éléments de T^*M, avec $x \in M$ et $p \in T^*_x M$.

Soit $\mathcal{F}$ une famille d'hamiltoniens Tonelli sur T^*M. Nous considérons le polysystème à temps discret orienté engendré par les flots au temps un

$$\phi^1_H \colon T^*M \to T^*M, \qquad \text{avec } H \in \mathcal{F}.$$

Notons que les hamiltoniens Tonelli sont les hamiltoniens standard de la théorie KAM faible. Nous renvoyons à [Fat] pour une exposition de cette théorie, et en particulier pour les notions de solution KAM faible et d'ensembles de Mather, Aubry et Mañé, que nous utiliserons dans la suite. Rappelons ici la définition d'hamiltonien Tonelli: une fonction $H \colon T^*M \times \mathbb{T} \to \mathbb{R}$ est un hamiltonien Tonelli si elle est de classe C^2 et si les propriétés suivantes sont vérifiées:

(i) H est convexe dans les fibres et non dégénérée dans le sens suivant: la matrice des dérivées secondes partielles $\partial^2_{pp} H(x, p, t)$ est définie positive en tant que forme quadratique, pour tout $(x, p, t) \in T^*M \times \mathbb{T}$;

(ii) H est superlinéaire dans les fibres, *i.e.*

$$\lim_{|p|_x \to \infty} \frac{H(x, p, t)}{|p|_x} = +\infty \qquad \forall\, (x, t) \in M \times \mathbb{T},$$

où $|\cdot|_x$ est une norme quelconque sur $T^*_x M$ dépendant continûment de $x \in M$. Cette condition est équivalente à la propriété suivante: pour tout $C \in \mathbb{R}$ il existe $D \in \mathbb{R}$ tel que

$$H(x, p, t) \geq C|p|_x + D \qquad \forall\, (x, p, t) \in T^*M \times \mathbb{T}.$$

(iii) le flot de H est complet.

Lorsque $d = 1$ on a nécessairement $M \cong \mathbb{T}$ et donc $T^*M \cong \mathbb{T} \times \mathbb{R}$. Il est connu (cf. [Mos86]) que dans ce cas étudier la dynamique des hamiltoniens Tonelli est essentiellement équivalent à étudier la dynamique des applications twist exactes-symplectiques sur le cylindre $\mathbb{T} \times \mathbb{R}$.

Comme déjà mentionné dans la section précédente, l'objectif de notre travail est de construire des polytrajectoires qui relient deux régions lointaines de T^*M. Pour mieux préciser cette idée, nous regroupons dans l'énoncé suivant certains résultats présents dans la littérature. Notre travail s'interprète comme une généralisation de ces résultats.

Théorème 1. *Supposons $M = \mathbb{T}^d$, et soient $c_0, c_1 \in \mathbb{R}^d$.*

(i) Cas $d = 1$ et $\mathcal{F} = \{H\}$. Si H n'admet pas de cercles invariants non contractibles dont la cohomologie appartient à l'intervalle $[c_0, c_1]$, alors il existe une trajectoire $(x_j, p_j)_{j\in\mathbb{Z}} \subset \mathbb{T}\times\mathbb{R}$ telle que

$$p_0 = c_0 \quad et \quad p_n = c_1 \qquad pour\ un\ certain\ n \in \mathbb{N}.$$

(cf. [Bir32a, KO97, Mat91b, Moe02] pour des versions légèrement différentes de ce résultat dans le cadre des applications twist exactes-symplectiques);

(ii) Cas $d = 1$ et $\operatorname{Card}\mathcal{F} > 1$. S'il n'existe aucun cercle invariant non-contractible commun aux hamiltoniens de $\mathcal{F}$ et dont la cohomologie appartient à $[c_0, c_1]$ alors il existe une polytrajectoire $(x_j, p_j)_{j\in\mathbb{Z}} \subset \mathbb{T} \times \mathbb{R}$ telle que

$$p_0 = c_0 \quad et \quad p_n = c_1 \qquad pour\ un\ certain\ n \in \mathbb{N}.$$

(cf. Moeckel [Moe02], Jaulent [Jau] et Le Calvez [LC07] pour des versions légèrement différentes de ce résultat dans le cadre des applications twist exactes-symplectiques);

(iii) Cas $d \geq 1$ et $\mathcal{F} = \{H\}$. Il existe une fonction

$$\mathbb{R}^d \ni c \mapsto \big(R(c), \varepsilon(c)\big)$$

où $R(c)$ est un sous-espace vectoriel de $\mathbb{R}^d$ défini à partir des solutions KAM faible de H en cohomologie c et $\varepsilon(c)$ est un réel strictement positif, telle que la propriété suivante est satisfaite: s'il existe une suite finie $(c'_k)_{k=0}^K \subset \mathbb{R}^d$ avec $c'_0 = c_0, c'_K = c_1$ et

$$\begin{cases} c'_{k+1} - c'_k \in R(c'_k) \\ |c'_{k+1} - c'_k| < \varepsilon(c'_k) \end{cases} \qquad \forall\ k \in \{0, \dots, K-1\}, \tag{1}$$

alors il existe une polytrajectoire $(x_j, p_j)_{j\in\mathbb{Z}} \subset \mathbb{T}^d \times \mathbb{R}^d$ avec

$$p_0 = c_0 \quad et \quad p_n = c_1 \qquad pour\ un\ certain\ n \in \mathbb{N}.$$

Lorsque $d = 1$ on a $R(c) = \{0\}$ si et seulement si H admet un cercle invariant non contractible de cohomologie c.

Ces résultats ainsi que d'autres plus généraux sont démontrés par Bernard [Ber08]. Des résultats de nature similaire sont démontrés dans [Mat93, CY04, CY09].

Notre contribution concerne le cas $d \geq 1$ et $\operatorname{Card}\mathcal{F} > 1$. Elle est inspirée par l'approche développé par Patrick Bernard dans [Ber08] dont la partie (iii) du Théorème 1 est une conséquence. Comme il est courant en théorie KAM faible, lorsque $M \neq \mathbb{T}^d$ le rôle de $\mathbb{R}^d$ dans le Théorème 1 est joué par le groupe de cohomologie $H^1(M, \mathbb{R})$, c'est-à-dire c_0 et c_1 sont deux classes de cohomologie (rappelons que pour le tore $\mathbb{T}^d$ on a $H^1(\mathbb{T}^d, \mathbb{R}) = \mathbb{R}^d$).

On montre dans la Partie 1 le résultat suivant, qui est une généralisation naturelle au cas Card $\mathcal{F} > 1$ de la Proposition 5.3 dans [Ber08].

Proposition 2. *Il existe une relation d'équivalence $\dashv\vdash_{\mathcal{F}}$ sur $H^1(M,\mathbb{R})$ telle que:*

(i) si $c_0 \dashv\vdash_{\mathcal{F}} c_1$ alors pour tous $H, H' \in \mathcal{F}$ il existe une polytrajectoire $(z_j)_{j\in\mathbb{Z}} \subset T^*M$ qui est α-asymptotique à l'ensemble d'Aubry $\tilde{\mathcal{A}}_H(c_0)$ et ω-asymptotique à $\tilde{\mathcal{A}}_{H'}(c_1)$;

(ii) si $c_0 \dashv\vdash_{\mathcal{F}} c_1$ et si η_0, η_1 sont deux 1-formes de cohomologie c_0 et c_1 respectivement, alors il existe une polytrajectoire $(z_j)_{j\in\mathbb{N}} \subset T^*M$ telle que

$$z_0 \in \mathrm{Graph}\,(\eta_0) \quad \text{et} \quad z_n \in \mathrm{Graph}\,(\eta_1)$$

pour un certain $n \in \mathbb{N}$.

(iii) soient

$$(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \in H^1(M,\mathbb{R}) \times \mathcal{F} \times]0, +\infty[$$

tels que $c_i \dashv\vdash_{\mathcal{F}} c_{i+1}$ pour tout $i \in \mathbb{Z}$. Alors il existe une polytrajectoire $(z_j)_{j\in\mathbb{Z}} \subset T^*M$ qui visite dans l'ordre les ε_i-voisinages des ensembles de Mather $\tilde{\mathcal{M}}_{H_i}(c_i)$.

D'après cette proposition, on est amené à étudier la relation $\dashv\vdash_{\mathcal{F}}$ afin de montrer l'existence de polytrajectoires reliant deux régions de T^*M. Le résultat principal de la Partie 1 fournit des conditions suffisantes assurant l'occurrence de la relation $\dashv\vdash_{\mathcal{F}}$ au voisinage d'une classe $c \in H^1(M,\mathbb{R})$. Ces conditions sont données en termes d'un sous-espace $R_{\mathcal{F}}(c)$ de "directions admissibles". Le résultat est énoncé ci-dessous.

Théorème 3. *Supposons que $\mathcal{F}$ est fini. Il existe une fonction*

$$H^1(M,\mathbb{R}) \ni c \mapsto \big(R_{\mathcal{F}}(c), U(c)\big)$$

où $R_{\mathcal{F}}(c)$ est un sous-espace vectoriel de $H^1(M,\mathbb{R})$ défini à partir du polysystème associé à $\mathcal{F}$ et $U(c)$ est un voisinage de c dans $H^1(M,\mathbb{R})$, telle que:

$$\begin{cases} c', c'' \in U(c) \\ c'' - c' \in R_{\mathcal{F}}(c) \end{cases} \quad \Rightarrow \quad c' \dashv\vdash_{\mathcal{F}} c''.$$

Remarquons que même si ce théorème est un résultat local (en $c \in H^1(M,\mathbb{R})$), il est possible en principe d'en déduire des informations globales grâce à la transitivité de la relation $\dashv\vdash_{\mathcal{F}}$.

Pour que le Théorème 3 soit intéressant il faut avoir plus d'informations sur le sous-espace $R_{\mathcal{F}}(c)$. Sa définition, plutôt abstraite, fait appel aux composantes minimales du polysystème sur $C^0(M)$ engendré par les opérateurs de Lax-Oleinik associés aux hamiltoniens dans $\mathcal{F}$. Nous renvoyons au corps de la thèse (cf. Definition 1.5.2) pour la définition précise. Dans le reste de cette section nous donnons quelques propriétés du sous-espace $R_{\mathcal{F}}(c)$.

Nous commençons par la proposition suivante qui, avec le Théorème 3 et la Proposition 2, permet de retrouver des résultats similaires à ceux de Moeckel, Le Calvez et Jaulent énoncés dans la partie *(ii)* du Théorème 1.

Proposition 4. *Lorsque $M = \mathbb{T}$ on a*

$$R_{\mathcal{F}}(c) = \{0\} \Leftrightarrow \textit{il existe un cercle non-contractible de cohomologie } c \textit{ invariant pour tout flot } \phi_H^1 \textit{ avec } H \in \mathcal{F}.$$

Pour une variété M générale on obtiendra une généralisation partielle de la proposition ci-dessus, à savoir:

Proposition 5. *S'il existe une 1-forme lisse fermée η et une fonction $u \in C^{1,1}(M)$ telles que le graphe de $\eta + du$ est invariant pour tout flot ϕ_H^1, $H \in \mathcal{F}$, alors $R_{\mathcal{F}}([\eta]) = \{0\}$.*

Lorsque $\mathcal{F} = \{H\}$ notre construction coïncide avec celle développée par P. Bernard dans [Ber08]. En particulier on a:

Proposition 6. *Lorsque $\mathcal{F} = \{H\}$ le sous-espace $R_{\mathcal{F}}(c)$ coïncide avec le sous-espace $R(c)$ défini par Bernard dans [Ber08], pour tout $c \in H^1(M, \mathbb{R})$.*

On démontrera aussi:

Proposition 7. *Si les hamiltoniens dans $\mathcal{F}$ commutent on a*

$$R_{\mathcal{F}}(c) = R_{\{H\}}(c) \qquad \forall\ H \in \mathcal{F}.$$

En général, soit $V \subseteq H^1(M, \mathbb{R})$ un sous-espace vectoriel de dimension 1 qui n'est pas inclus dans $R_{\mathcal{F}}(c)$. Qu'est-ce que l'on peut en déduire sur la dynamique des hamiltoniens dans $\mathcal{F}$? Une réponse à cette question est donnée par la proposition suivante. Avant de l'énoncer, nous avons besoin d'introduire deux notations. Pour tout $A \subseteq M$, notons $[A^{\perp}]$ l'ensemble des classes de cohomologie des 1-formes lisses ayant support disjoint de A. De plus, pour toute 1-forme lisse η sur M et toute fonction $u\colon M \to \mathbb{R}$ notons $\mathrm{Graph}\,(\eta + du) \subseteq T^*M$ l'ensemble défini par

$$\mathrm{Graph}\,(\eta + du) = \{(x, \eta_x + d_x u) \in T^*M : x \in M, d_x u \text{ existe}\}.$$

Proposition 8. *Supposons que $V \nsubseteq R_{\mathcal{F}}(c)$. Fixons une 1-forme lisse η_c de cohomologie c.*

(i) Pour toute suite finie $H_1, \dots, H_k$ d'hamiltoniens dans $\mathcal{F}$ il existe un sous-ensemble $S \subseteq T^*M$ invariant par

$$\phi_{H_k}^1 \circ \cdots \circ \phi_{H_1}^1$$

et tel que:

- *S est un graphe Lipschitz sur $\pi(S) \subseteq M$, où $\pi\colon T^*M \to M$ est la projection canonique;*
- *$S \subseteq \mathrm{Graph}\,(\eta_c + du)$ pour une certaine fonction $u\colon M \to \mathbb{R}$ semi-concave;*
- *$V \nsubseteq [\pi(S)^{\perp}]$.*

(ii) Pour tout couple d'hamiltoniens $H_1, H_2 \in \mathcal{F}$ il existe deux fonctions $u_1, u_2\colon M \to \mathbb{R}$ telles que:

- *u_1 est une solution KAM faible pour l'hamiltonien $H_{1,c}(x, p, t) := H_1(x, \eta_c + p, t)$;*
- *u_2 est une solution KAM faible duale pour $H_{2,c}(x, p, t) := H_2(x, \eta_c + p, t)$;*
- *$V \nsubseteq \left[\left(\mathrm{Graph}\,(\eta_c + du_1) \cap \mathrm{Graph}\,(\eta_c + du_2)\right)^{\perp}\right]$.*

Partie 2: Polysystèmes hamiltoniens à temps continu (approche avec la théorie géométrique du contrôle)

La deuxième partie de la thèse porte sur les polysystèmes hamiltoniens à temps continu (orienté aussi bien que non-orienté) engendrés par deux flots hamiltoniens associés à deux hamiltoniens H_1 et H_2 indépendants du temps et définis sur une variété symplectique N quelconque (sans bord).

On dit que le polysystème est *transitif* si pour tout couple de point $(z, z') \in N \times N$ il existe une polytrajectoire qui relie z à z'. Remarquons que dans le cas à temps orienté l'existence d'une polytrajectoire reliant z à z' n'implique pas en général l'existence d'une polytrajectoire reliant z' à z. La question abordée dans la Partie 2 est la suivante: pour un choix "typique" de H_1 et H_2, le polysystème engendré est-il transitif?

Comme souvent en systèmes dynamiques, une partie de la réponse à cette question consiste à donner un sens précis au mot 'typique'. À ce propos, la notion de généricité au sens de Baire est une des plus utilisées. Rappelons brièvement sa définition: on dit qu'un sous-ensemble d'un espace métrique complet est générique (ou résiduel) au sens de Baire s'il contient une intersection dénombrable d'ouverts denses. On dit qu'un sous-ensemble est maigre au sens de Baire si son complémentaire est générique au sens de Baire. Dans notre travail nous allons par contre utiliser une notion différente, celle de sous-ensemble rectifiable de codimension positive dans un espace de Banach ou de Fréchet, développée par Bernard dans [Ber10b] (voir aussi Zajicek [Zaj08]). Il s'agit d'une notion plus forte de "petitesse", dans le sens où si A est rectifiable de codimension positive dans F (où F est un espace de Banach ou de Frechet) alors A est maigre au sens de Baire, tandis que le contraire n'est pas vrai en général. Reprenons de [Ber10b] la définition précise, qui est donnée à partir du cas particulier d'un graphe Lipschitz de codimension positive:

Définition 9 (Graphe Lipschitz)**.** *Soit $A \subset F$ un sous-ensemble de l'espace de Banach F. On dit que A est un graphe Lipschitz de codimension $c \in \mathbb{N}$ s'il existe deux sous-espaces E et G tels que:*

- $\dim G = c$;
- $F = E \oplus G$ *et* E *fermé;*
- *il existe une fonction Lipschitz* $g\colon E \to G$ *telle que*

$$A \subseteq \{x + g(x) : x \in E\}.$$

Définition 10 (Sous-ensemble rectifiable dans un espace de Banach)**.** *Soit $A \subset F$ un sous-ensemble de l'espace de Banach F. On dit que A est rectifiable de codimension c dans F si A peut s'écrire comme une réunion dénombrable $A = \cup_n \varphi_n(A_n)$ où:*

- $\varphi_n\colon U_n \to F$ *est une application de type Fredholm*[2] *d'indice i_n définie sur un ouvert U_n d'un espace de Banach separable F_n;*
- $A_n \subseteq U_n$ *est un graphe Lipschitz de codimension $c + i_n$ dans F_n.*

[2]Une application $\varphi\colon E \to E'$ entre deux espaces de Banach séparables E, E' est de type Fredholm d'indice $i \in \mathbb{Z}$ si elle est de classe C^1 et si, pour tout $x \in E$, on a $i = \dim\ker d_x\varphi - \operatorname{codim}\operatorname{im} d_x\varphi$.

On dit aussi que A est rectifiable de codimension positive dans F s'il est rectifiable de codimension c dans F avec $c \geq 1$.

La définition ci-dessus peut être étendue au cas où F est un espace de Frechet ou une variété de Banach (voir [Ber10b, BM]).

Rappelons que lorsque N est compacte les espaces $C^k(N), k \in \mathbb{N}$, sont des espaces de Banach. Lorsque N n'est pas compacte, nous munissons les espaces $C^k(N)$ de la topologie faible de Whitney, ils sont alors des espaces de Frechet.

Les deux résultats principaux de la Partie 2 sont les suivants. Notons $2d = \dim N$.

Théorème 11 (Cas à temps non-orienté)**.** *Soit $H_1 \in C^{4d+1}(N)$. Supposons que l'ensemble*

$$\{z \in N : d_z H_1 = 0\}$$

est inclus dans une réunion dénombrable $\cup_{l\in\mathbb{N}} N^l$ de sous-variétés $N^l \subseteq N$ de classe C^2 avec

$$\operatorname{codim} N^l \geq \frac{\dim N}{2} + 1 \qquad \forall\, l \in \mathbb{N}.$$

Alors pour tout $k \geq 4d$ l'ensemble

$$\Big\{ H_2 \in C^k(N) : \text{le polysystème à temps continu non orienté engendré par } H_1 \text{ et } H_2 \text{ est transitif} \Big\} \tag{2}$$

a complémentaire rectifiable de codimension positive dans $C^k(N)$. En particulier, il est générique au sens de Baire.

Théorème 12 (Cas à temps orienté)**.** *Supposons N compacte. Soit H_1 comme dans l'énoncé précédent. Alors pour tout $k \geq 4d$ l'ensemble*

$$\Big\{ H_2 \in C^k(N) : \text{le polysystème à temps continu orienté engendré par } H_1 \text{ et } H_2 \text{ est transitif} \Big\} \tag{3}$$

a complémentaire rectifiable de codimension positive dans $C^k(N)$. En particulier, il est générique au sens de Baire.

Des résultats analogues seront obtenus pour des hamiltoniens H_1, H_2 périodiques dans le temps (avec la même période). Remarquons que les conditions demandées à H_1 dans les énoncés ci-dessus sont très faibles: en effet, pour une fonction H_1 générique l'ensemble $\{z \in N : d_z H_1 = 0\}$ est constitué de points isolés, d'après une application immédiate du théorème classique de transversalité de Thom. Remarquons aussi que dans le cas à temps non orienté N n'est pas supposée compacte, et les flots de H_1 et H_2 pourraient donc n'être pas complets.

La strategie de la preuve des résultats ci-dessus est inspirée par celle utilisée par Lobry dans [Lob72], où des questions analogues sont abordées dans le cadre des polysystèmes à temps continu non orienté engendrés par des champs vecteurs généraux (*i.e.* non hamiltoniens).

La preuve se subdivise en trois étapes.

Dans la première étape on utilise le théorème de Rashevsky-Chow en théorie géométrique du contrôle (cf. [AS04, Chapitre 5]) pour obtenir des conditions suffisantes pour que le polysystème à temps continu engendré par H_1 et H_2 soit transitif. Ces conditions sont données en terme des jets de H_1 et H_2. La version du théorème de Rashevsky-Chow pour le cadre à temps orienté est appelée théorème de Krener (cf. [AS04, Chapitre 8]). Pour que ce dernier s'applique de manière fructueuse à la question de la transitivité, il faudra supposer que la dynamique de H_1 et H_2 est récurrente. Cela est toujours vrai lorsque N est compacte (d'après le théorème de récurrence de Poincaré), c'est pourquoi nous rajoutons cette condition dans le Théorème 12.

Dans la deuxième étape on utilise le théorème du redressement d'un flot hamiltonien pour se placer dans un bon système de coordonnées. Quelques calculs explicites montrent alors que les conditions obtenues dans la première étape se reformulent plus précisément de la manière suivante: le polysystème est transitif si le jet de H_2 (H_1 étant fixé) n'intersecte pas certaines sous-variétés (qui dépendent de H_1) dont la codimension dans l'espace des jets est explicitement calculée.

Dans la dernière étape on conclut la preuve en appliquant à ces sous-variétés le théorème de transversalité de Thom dans l'espace des jets. La version classique de ce théorème donne la généricité au sens de Baire des ensembles définis dans (2) et (3). Nous utilisons la version du théorème de Thom qui est démontrée dans la Partie 3 de cette thèse (cf. Théorème 14) et qui implique que les complémentaires des ensembles en question sont rectifiables de codimension positive.

Partie 3: Une version en codimension positive du théorème de transversalité de Thom

Cette partie est issue d'un travail en collaboration avec Patrick Bernard. Rappelons tout d'abord l'énoncé classique (cf. [Hir94, Chapitre 3, Théorème 2.8]) du théorème de transversalité de Thom dans l'espace de jets $J^k(X,Y)$, où k est un nombre naturel et les espaces X, Y sont des variétés lisses de dimension finie. Nous supposons que X est compacte. Nous munissons $C^k(X,Y)$ de la topologie de Whitney, c'est alors une variété de Banach.

Théorème 13 (Théorème de transversalité de Thom, version classique)**.** *Soit $W \subseteq J^k(X,Y)$ une sous-variété lisse. L'ensemble des applications dont le jet d'ordre k n'est pas transverse à W est maigre au sens de Baire dans $C^{k+r}(X,Y)$ pour tout $r \geq 1$.*

Dans la Partie 3 nous obtenons la version suivante du Théorème 13.

Théorème 14 (Théorème de transversalité de Thom, version en codimension positive)**.** *Soit $W \subseteq J^k(X,Y)$ une sous-variété lisse.*

- *si* $\operatorname{codim} W \geq \dim X + 1$ *alors l'ensemble des applications dont le jet d'ordre k n'est pas transverse à W est rectifiable de codimension* $\operatorname{codim} W - \dim X$ *dans $C^{k+r}(X,Y)$ pour tout $r \geq 1$;*
- *si* $\operatorname{codim} W \leq \dim X$ *et si W satisfait au moins une des propriétés suivantes*

(P1) *W est non-dégénérée (voir Définition 15),*

(P2) *W est analytique,*

alors l'ensemble des applications dont le jet d'ordre k n'est pas transverse à W est rectifiable de codimension 1 dans $C^{k+r}(X,Y)$ pour tout $r \geq 2$.

La notion de sous-ensemble rectifiable de codimension positive que l'on utilise ici est celle de la section précédente (voir Définition 10).

Définition 15. *Soit $j = j^k_x f \in W$. La sous-variété W est non-dégénérée en j si*

$$(\pi^k_{k-1})_{|W} \pitchfork_j j^{k-1}f(X),$$

où $\pi^k_{k-1} \colon J^k(X,Y) \to J^{k-1}(X,Y)$ est la projection canonique. Nous disons que W est non-dégénérée si elle est non-dégénérée en tout point $j \in W$. Observons qu'une condition suffisante pour que W soit non-dégénérée est que la projection $(\pi^k_{k-1})_{|W} \colon W \to J^{k-1}(X,Y)$ soit une submersion.

Donnons maintenant un aperçu de la preuve du Théorème 14.

Considérons une sous-variété $W \subseteq J^k(X,Y)$ avec $k \in \mathbb{N}$. Lorsque l'on étudie la transversalité entre W et les k-jets $j^k f$ des applications $f \colon X \to Y$ on peut distinguer, suivant la codimension de W, deux situations différentes: si $\operatorname{codim} W \geq \dim X + 1$ alors le fait que W et $j^k f$ sont transverses veut simplement dire que leur intersection est vide; on est dans le cas dit "d'évitement". On appellera "cas de non évitement" la situation opposée où $\operatorname{codim} W \leq \dim X$.

Focalisons-nous d'abord sur le cas d'évitement. Dans ce cas, le théorème classique de transversalité de Thom peut s'obtenir à partir de la version "facile" suivante du théorème de Sard: si g est une application C^1 de $\mathbb{R}^m$ sur $\mathbb{R}^n$ avec $m < n$, alors l'ensemble $CV(g) \subset \mathbb{R}^n$ des valeurs critiques de g est maigre au sens de Baire. Or, une propriété plus fine est vraie dans cette situation: l'ensemble $CV(g)$ (qui en fait dans ce cas coïncide avec l'image de g), est rectifiable de codimension $n - m \geq 1$ dans $\mathbb{R}^n$, selon la définition usuelle d'ensemble rectifiable dans un espace vectoriel réel de dimension finie. À partir de cette remarque il est possible de déduire la proposition suivante.

Proposition 16. *Soit $W \subseteq J^k(X,Y)$ une sous-variété. Si $\operatorname{codim} W \geq \dim X + 1$, alors l'ensemble des applications dont le jet d'ordre k n'est pas transverse à W est rectifiable de codimension $\operatorname{codim} W - \dim X$ dans $C^{k+r}(X,Y)$ pour tout $r \geq 1$.*

Cela prouve le Théorème 14 dans le cas d'évitement. Qu'en est-il du cas de non évitement? Il s'avère qu'il est possible de le ramener au cas d'évitement de la manière suivante. On remarque d'abord (cf. Gromov [Gro86, page 33] ou Eliashberg et Mishachev [EM02, Chapitre 2]) que, pour une fonction $f \colon X \to Y$ arbitraire, la condition de transversalité entre $j^k f$ et W ne dépend que du jet d'ordre 1 de $j^k f$, c'est-à-dire de $j^{k+1} f$. Il existe donc un sous-ensemble $\tilde{W} \subset J^{k+1}(X,Y)$ tel que, pour toute $f \in C^{k+1}(X,Y)$,

$$j^k f \text{ est transverse à } W \Leftrightarrow j^{k+1} f(X) \cap \tilde{W} = \emptyset. \tag{4}$$

Dans la Partie 3 de la thèse, le résultat technique suivant est alors démontré:

Théorème 17. *Soit $W \subseteq J^k(X,Y)$ une sous-variété lisse satisfaisant au moins une des hypothèses* (P1) *et* (P2). *Alors, le sous-ensemble $\tilde{W} \subset J^{k+1}(X,Y)$ est inclus dans une réunion dénombrable de sous-variétés de codimension $\geq \dim X + 1$.*

Autrement dit, le sous-ensemble $\tilde{W}$ se situe dans le cas d'évitement bien que cela ne soit pas le cas pour W. On peut donc appliquer à $\tilde{W}$ la Proposition 16. D'après l'équivalence (4), cela achève la preuve du Théorème 14.

Remarquons que pour prouver le Théorème 14 seulement la version "facile" du théorème de Sard a été utilisée, mais il a fallu cependant concéder un ordre de différentiabilité supplémentaire (car $\tilde{W}$ appartient à $J^{k+1}(X,Y)$ et non pas à $J^k(X,Y)$).

Dans le reste de cette section nous donnons un aperçu de la preuve du Théorème 17. Rappelons que $\tilde{W} \subseteq J^{k+1}(X,Y)$ est défini par

$$\tilde{W} = \left\{ j^{k+1}_x f \in J^{k+1}(X,Y) : j^k f \text{ n'est pas transverse à } W \text{ en } x \right\},$$

et que l'on cherche à estimer la codimension de $\tilde{W}$, c'est-à-dire le nombre d'équations indépendantes localement nécessaires à définir $\tilde{W}$. Une approche heuristique pour estimer ce nombre est la suivante. D'après la définition de transversalité, le jet $j^{k+1}_x f \in J^k(X,Y)$ appartient à $\tilde{W}$ si et seulement si: (i) sa projection $j^k_x f$ appartient à W et (ii) la matrice suivante (écrite en coordonnées locales) n'est pas de rang maximal:

$$\left[\begin{array}{c|c} & \\ T_{j^k_x f} W & d(j^k f)(T_x X) \\ & \end{array}\right]. \tag{5}$$

Ici, $T_{j^k_x f} W$ est l'espace tangent à W en $j^k_x f$ et $d(j^k f)$ est l'application tangente à l'application $j^k f \colon X \to J^k(X,Y)$.

Étudions le nombre d'équations données par les conditions (i) et (ii). Pour la condition (i), on vérifie aisément qu'elle donne un nombre d'équations indépendantes égal à $\operatorname{codim} W$. Pour la condition (ii), observons que la matrice dans (5) est de taille $\dim J^k(X,Y) \times (\dim X + \dim W)$, et que $\dim X + \dim W \geq \dim J^k(X,Y)$ d'après les hypothèses du théorème. Notons

$$n = \dim X + \dim W, \quad m = \dim J^k(X,Y).$$

On sait que pour qu'une telle matrice ne soit pas de rang maximum il faut imposer au moins $n - m + 1$ équations, correspondants aux déterminants de certaines sous-matrices carrées de taille maximale. Supposons que ces $n - m + 1$ équations soient indépendantes (en tant que fonctions de la variable $j^{k+1}_x f$) lorsque on les impose à la matrice (5), et qu'elles soient également indépendantes vis-à-vis des équations issues de la condition (i). On a dans ce cas que les conditions (i) et (ii) définissant $\tilde{W}$ sont décrites localement par un total de

$$\operatorname{codim} W + n - m + 1 = \dim X + 1$$

équations indépendantes. Cela donne bien le résultat souhaité, *i.e.* $\operatorname{codim} \tilde{W} = \dim X + 1$. Or, il se trouve que lorsque l'hypothèse (P1) est satisfaite on arrive effectivement à traduire cette approche naïve en une preuve rigoureuse, et à démontrer donc le théorème dans cette situation.

Supposons maintenant que l'hypothèse (P1) ne soit pas satisfaite. Dans ce cas, il existe un "ensemble dégénéré" non vide $W_0 \subseteq W$ constitué par les points $j \in W$ tels que W est dégénérée en j. On vérifie que pour un élément $j \in W_0$ la fibre $(\pi_k^{k+1})^{-1}(j)$ est entièrement inclue dans $\tilde{W}$. De plus, on vérifie aisément que W_0 est fermé dans W, donc son complémentaire $W' = W \setminus W_0$ est une sous-variété non-dégénérée, et

$$\tilde{W} = (\pi_k^{k+1})^{-1}(W_0) \ \cup \ \tilde{W'}.$$

La codimension de $\tilde{W'}$ étant déjà connue (grâce au cas non-dégénéré que l'on vient d'examiner), on déduit que pour estimer la codimension de $\tilde{W}$ il suffit d'estimer la codimension de W_0. À ce stade, la propriété (P2) s'avère être utile: elle garantit que l'ensemble W_0 est suffisamment joli (il est en fait analytique), ainsi que sa projection $\pi_{k-1}^k(W_0) \subseteq J^{k-1}(X, Y)$ (qui est subanalytique). Il est possible de conclure maintenant par récurrence sur k.

Chapter 1

Connecting orbits for families of Tonelli Hamiltonians

1.1 Introduction

Much work has been carried out in order to understand the instability properties of Hamiltonian systems, especially for Hamiltonians which are convex in the momenta variables p. The basic case of a periodic Hamiltonian defined on the cotangent space $T^*\mathbb{T} \cong \mathbb{T} \times \mathbb{R}$ of the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ corresponds to exact-symplectic twist maps on the cylinder, see [Mos86]. Quite a lot is known in this case, thanks for instance to the original works of Birkhoff [Bir32a, Bir32b] and to the KAM and Aubry-Mather theories for twist maps. In particular, a general principle is that the non-contractible invariant circles are the unique obstruction to instability phenomena such as the drift in the p-variable.

The situation becomes more complicated when generalizing to higher dimension, namely to Hamiltonians defined on $T^*\mathbb{T}^d$, $d \in \mathbb{N}$, or, more generally, on the cotangent space T^*M of a d-dimensional manifold M. In this setting, among others the variational approach of Mather and Fathi's weak KAM theory has been fruitful, especially in the framework of the so-called Tonelli Hamiltonians. The Mather, Aubry and Mañé sets introduced by Mather and Fathi generalize the invariant circles and the Aubry-Mather sets for twist maps, and provide at the same time both an obstruction and a dynamical skeleton for the instability phenomena. This has allowed a better comprehension of the mechanisms underlying the phenomenon of Arnold diffusion which was firstly exhibited in the seminal paper [Arn64] on a concrete example.

Some studies have also been devoted to the following different generalization: one keeps the dimension $d = 1$, and consider instead a family of several twist maps at once, which can be iterated in any order. Following [Mar08], we shall call such a system a *polysystem*,[1] and *polyorbits* its (discrete-time) trajectories, see Definition 1 for more rigour. Of course, the trajectories of a map in the family are also trajectories for the polysystem, thus the polysystem presents at least the same unstable behaviors as the single maps in the family. Nevertheless, one expects new kinds

[1] The expression *Iterated Functions System* is also used to designate these systems.

of unstable behavior possibly to be created: some obstructions for a map may be circumvented by non-trivial iterations of other maps in the family. Moeckel [Moe02], Le Calvez [LC07] and Jaulent [Jau] have studied this problem, extending some results for single twist maps to the polysystem case. In particular, the general emerging principle is that the unique obstructions to instability phenomena, such as the drift in the p-variable, are the common non-contractible invariant circles.

In this paper, we try to merge both generalizations, i.e. we deal with a family of several Hamiltonians in arbitrary dimension and we investigate the presence of unstable polyorbits (often we will call them "diffusion polyorbits" or "connecting polyorbits"). More precisely, we will consider the polysystem associated to a family $\mathcal{F}$ of one-periodic Tonelli Hamiltonians[2] defined on the cotangent space of a compact d-dimensional manifold M without boundary. Just as in the one-dimensional twist map case, one expects that some new unstable behavior may be created by non-trivial iterations of the time-one maps of the family. On the other hand, unlike the single-Hamiltonian case, there is not a definition of Mather, Aubry and Mañé sets for polysystems, hence one may expect the obstructions to come expressed in terms of some more complicated objects.

Our discussion will be in the framework of weak Kam theory, for which we refer to [Fat]. The ideas will be close to those in Bernard's paper [Ber08], of which the present work may be seen as a generalization to the polysystem case (especially of Section 8 in that paper). We call our method for the construction of unstable polyorbits "Mather mechanism", after the paper [Mat93] which introduced some of the basic ideas of the construction. In [Ber08] a slightly different "Arnold mechanism" is also presented, more reminiscent of the aforementioned paper [Arn64].

The results which we obtain are rather abstract in nature: essentially, they give sufficient conditions in order for the diffusion orbits to occur between two cohomology classes (in the sense of Proposition 2). The conditions are encoded, locally around a cohomology class c, in a subspace $R(c)$ of "allowed cohomological directions for diffusion" (Theorem 3). This subspace is in turn defined (cf. (1.5.2) and Proposition 29) in terms of some sort of generalized Aubry-Mather sets for the polysystem (the sets $\mathcal{I}_\Phi(\mathcal{G})$ defined in Remark 11(i)), which may be in principle quite difficult to decipher. We believe that the generality of our construction may compensate for this abstract character. Moreover, some further study may lead to more transparent conditions, at least in presence of additional hypotheses. For instance, in the twist map case we are able to recover "concrete" and "optimal" results (see Corollary 4), similar to those already proved with different methods by Le Calvez and Moeckel, and extending some other results of Mather in [Mat91b] for a single twist map.

On the negative side, using a result of Cui [Cui10] we show that, if (in arbitrary dimension) the Hamiltonians in the family commute, our mechanism does not give rise to new instability phenomena, which is somehow expected.

As for the interest in studying Hamiltonian polysystems, let us mention that a motivation lies in the fact that the behavior of some complex single-Hamiltonian systems may be to some extent reduced to the analysis of simpler polysystems. We are aware for instance of a work of

[2] We recall that a one-periodic Tonelli Hamiltonian is a C^2 function $H(x,p,t)$ defined on $T^*M\times\mathbb{T}$ which is strictly convex in p (with positive definite Hessian $\partial_p^2 H>0$), superlinear in p, and whose Hamiltonian flow is complete.

Bounemoura and Pennamen [BP12], where the polysystem approach is used in a neighborhood of an invariant normally hyperbolic manifold, and some works of Marco therein cited.

1.1.1 Main results

Before introducing our results, let us review the kind of statements which we want to generalize.

For an exact-symplectic twist map F on the cylinder $\mathbb{T}\times\mathbb{R}$, the archetypal instability result is the following: if, for $A<B$, the annulus $\mathbb{T}\times[A,B]\subset\mathbb{T}\times\mathbb{R}$ does not contain any non-contractible invariant circle, then there exists an orbit $(x_n,p_n)_{n\in\mathbb{Z}}$ such that $p_0<A$ and $p_N>B$ for some $N\in\mathbb{N}$. This dates back to Birkhoff [Bir32a, Bir32b], and has been improved in various ways. Two improvements in the framework of Aubry-Mather theory for twist maps will be relevant to us. The first states that if M_{w_1} and M_{w_2} are two Aubry-Mather sets for F of rotation number w_1 and w_2 respectively, such that there is no non-contractible invariant circle between them, then there exists an orbit $\{z_n=(x_n,p_n)\}_{n\in\mathbb{Z}}\subset\mathbb{T}\times\mathbb{R}$ such that

$$\alpha\text{-lim } z_n\subseteq M_{w_1}\qquad\text{and}\qquad\omega\text{-lim } z_n\subseteq M_{w_2}.$$

The second states that if $(w_i)_{i\in\mathbb{Z}}$ are rotation numbers such that, for any i, there is no non-contractible invariant circle between the Aubry-Mather sets M_{w_i} and $M_{w_{i+1}}$, then for every sequence $(\varepsilon_i)_i$ of positive number there exists an orbit which visits in turn the ε_i-neighborhood of M_{w_i}. Both these results are due to Mather, we refer to [Mat91b] for precise statements.

Of course, for a twist map, non-contractible invariant circles do represent obstructions to the drift in the p-variable, because they disconnect the cylinder, hence the previous statements are optimal. Therefore the principle stemming from these results is that non-contractible invariant circles are the only obstruction to this kind of instability.

For a family of exact-symplectic twist maps on the cylinder, the generalization of the Birkhoff result above obtained by replacing in the statement "non-contractible invariant circle" with "common non-contractible invariant circle" is true. This and other stronger results have been proved by Moeckel, Le Calvez and Jaulent [Moe02, LC07, Jau]. Again, a common non-contractible invariant circle obviously is a real obstruction to the drift in the p-variable, whence the optimality of these results and the principle that, for a polysystem of exact twist maps, the common non-contractible invariant circles are the only obstruction to this kind of instability.

For the case of a single Hamiltonian in higher dimension, usually only sufficient conditions for the existence of unstable orbits can be proved. A great amount of work has been devoted to this topic. Our approach is close to the one of Mather in [Mat93] and of Bernard in [Ber08] (see also [Ber02, CY04, CY09]). Their results are better expressed in terms of cohomology classes rather than rotation vectors: in their papers, the authors define equivalence relations in $H^1(M,\mathbb{R})$ such that equivalence between classes implies existence of diffusing orbits between the corresponding Aubry sets. The obstruction for the equivalence is represented, roughly speaking, by the size of the Mañé sets. Notice however that, unlike the one-dimensional case, the obstructions for the equivalence may not always correspond to real obstructions for the dynamics. Nevertheless, if $d=1$ the obstructions to the equivalence turn out to be exactly the non-contractible invariant circles. Therefore, the results on twist maps mentioned above are recovered, and the equivalence relation is then optimal in this case.

The present paper has the same structure: we define (in terms of pseudographs and of the flows of the Hamiltonians in the family $\mathcal{F}$, see Sections 1.2 and 1.3) an equivalence relation $\dashv\vdash_{\mathcal{F}}$ between cohomology classes, which is a natural adaptation to the polysystem case of the relation $\dashv\vdash$ introduced in [Ber08]. We then prove that the occurrence of such a relation implies the existence of diffusing polyorbits, in the sense of Proposition 2. We find sufficient conditions (in terms of the "homological size" of some sort of generalized Aubry sets) which ensure, locally around a given class c, the occurrence of the relation. If $d = 1$, this conditions turn out to be also necessary, hence the relation is optimal in this case. For $\mathcal{F}$ composed by a single Hamiltonian, our results exactly reduce to the one in Section 8 of [Ber08].

More precisely, let $\mathcal{F}$ be a family of one-periodic Tonelli Hamiltonians on T^*M, where M is a d-dimensional compact manifold without boundary. For $H \in \mathcal{F}$, we denote by

$$\phi_H \colon T^*M \to T^*M$$

the time-one map of the Hamiltonian flow of H. Let us first rigorously define what we mean by polyorbit.

Definition 1 ($\mathcal{F}$-polyorbit)**.** *A bi-infinite sequence $\{z_n\}_{n\in\mathbb{Z}} \subseteq T^*M$ is a* $\mathcal{F}$-polyorbit *(or, simply, a polyorbit) if for every $n \in \mathbb{Z}$ there exists $H \in \mathcal{F}$ such that*

$$\phi_H(z_n) = z_{n+1}.$$

A finite $\mathcal{F}$-polyorbit *(or, simply, a finite polyorbit) is a finite segment $(z_0, z_1, \dots, z_N)$ of a $\mathcal{F}$-polyorbit. We then say that the finite polyorbit joins z_0 to z_N.*

Given two subsets $S, S' \subseteq T^*M$, we say that S is joined to S' by a finite polyorbit if there exist a finite polyorbit joining z to z', for some $z \in S$ and $z' \in S'$.

Let $\tilde{\mathcal{A}}_H(c)$ and $\tilde{\mathcal{M}}_H(c)$ be the Aubry and Mather sets of H of cohomology of c, as defined in [Ber08] or [Fat]. Their definition is also recalled in Subsection 1.4.5.

We have (Section 1.3):

Proposition 2. *There exists an equivalence relation $\dashv\vdash_{\mathcal{F}}$ on $H^1(M, \mathbb{R})$ such that:*

- *if $c \dashv\vdash_{\mathcal{F}} c'$ then for every $H, H' \in \mathcal{F}$ there exists a polyorbit which is α-asymptotic to the Aubry set $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;*

- *if $c \dashv\vdash_{\mathcal{F}} c'$ and if η, η' are one-forms of cohomology c, c' respectively, then there exists a finite polyorbit joining* Graph (η) *to* Graph (η')*;*

- *let $(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \subset H^1(M, \mathbb{R}) \times \mathcal{F} \times]0, +\infty[$ such that $c_i \dashv\vdash_{\mathcal{F}} c_{i+1}$ for every i. Then there exists a polyorbit visiting in turn the ε_i-neighborhoods of the Mather sets $\tilde{\mathcal{M}}_{H_i}(c_i)$. Moreover, if $(c_i, H_i) = (\bar{c}, \bar{H})$ for i small enough (resp. i big enough), then the polyorbit can be taken α-asymptotic to $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$ (resp. ω-asymptotic to $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$).*

The main result is Theorem 31. Let us state it here for finite $\mathcal{F}$, even if it will hold under a weaker assumption.

Theorem 3. *Assume $\mathcal{F}$ is finite. Then for every $c \in H^1(M,\mathbb{R})$ there exist a vector subspace $R(c) \subseteq H^1(M,\mathbb{R})$, a neighborhood W of c and $\varepsilon > 0$ such that*

$$c' \dashv\vdash_{\mathcal{F}} c' + B_\varepsilon R(c) \qquad \forall\, c' \in W.$$

Of course one needs to have information on the subspace $R(c)$ for the result to be interesting. The definition of $R(c)$ is rather abstract and not too easy to handle (cf. the definition given in (1.5.2) and some equivalent expressions given in Proposition 29).

Nevertheless, we are able to prove (Proposition 33) that if there exists a $C^{1,1}$ weak Kam solution of cohomology c which is common to all the Hamiltonians in $\mathcal{F}$, then $R(c) = \{0\}$. In addition, if $d = 1$, the viceversa is true: if $R(c) = \{0\}$ then there exists a $C^{1,1}$ weak Kam solution of cohomology c common to all Hamiltonians in $\mathcal{F}$, i.e. a common non-contractible invariant circle.

This fact, together with Theorem 3 and Proposition 2 yields the following result for families of twist maps (no additional assumptions on $\mathcal{F}$ will be eventually needed):

Corollary 4. *Let us consider the polysystem associated to an arbitrary family $\mathcal{F}$ of one-periodic Tonelli Hamiltonians on $\mathbb{T}\times\mathbb{R}$. Let us make the identification $H^1(\mathbb{T},\mathbb{R}) \cong \mathbb{R}$. If, for some $A < B \in \mathbb{R}$, the family $\mathcal{F}$ does not admit an invariant common circle with cohomology in $[A,B]$, then:*

(i) there exists a polyorbit $(x_n,p_n)_{n\in\mathbb{Z}}$ satisfying $p_0 = A$ and $p_N = B$ for some $N \in \mathbb{N}$;

(ii) for every $H, H' \in \mathcal{F}$ and every $c, c' \in [A,B]$ there exists a polyorbit α-asymptotic to the Aubry set $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;

(iii) for every sequence $(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \subset [A,B]\times\mathcal{F}\times\,]0,+\infty[$ there exists a polyorbit which visits in turn the ε_i-neighborhoods of the Mather sets $\tilde{\mathcal{M}}_{H_i}(c_i)$.

When $d > 1$ some information can still be extracted from the subspace $R(c)$. A sample of what can be obtained is presented in Proposition 35. Roughly speaking, among the obstructions which prevent $R(c)$ from being large, we find:

- for every finite string $H_1,\dots,H_n$ of elements of $\mathcal{F}$, the invariant sets for the map

$$\phi = \phi_{H_n} \circ \cdots \circ \phi_{H_1};$$

- for every pair H_1, H_2 of elements of $\mathcal{F}$, for every c-weak Kam solution u_1 for H_1 and dual c-weak Kam solution u_2 for H_2, the set

$$\mathrm{Graph}\,(du_1) \cap \mathrm{Graph}\,(du_2).$$

However, unlike the twist map case, such obstructions must be intended in a "negative" way: their smallness is a sufficient condition for $R(c)$ to be large, the converse being not necessarily true.

1.1.2 Structure of the paper

The paper is organized as follows. In Section 1.2 we establish some notation and recall some facts about pseudographs and semiconcave functions.

In Section 1.3 we define the forcing relation $\vdash_{\mathcal{F}}$ and the mutual forcing relation $\dashv\vdash_{\mathcal{F}}$, and we show, like in [Ber08], how the occurrence of such relations implies the diffusion for the polysystem (Proposition 7).

In Section 1.4 we present the objects needed later to put in place what we call the Mather mechanism: Lagrangian action, Lax-Oleinik operators, operations on costs (minimum, composition) and families of costs. Eventually we build the semigroup Σ_c^∞ which acts on the space of pseudographs and encodes informations on the underlying polysystem dynamics. The Subsection 1.4.5 gathers some needed results in weak Kam theory, rephrased in the language of pseudographs.

In Section 1.5 the Mather mechanism for the construction of diffusion polyorbits is put in place. The basic step of the mechanism is proved in Subsection 1.5.2. Then we heuristically show the application to the twist map case in Subsection 1.5.3. Finally, in Subsection 1.5.4 we define the subspace $R(c)$ and we prove a general abstract result (Theorem 31) which gives sufficient conditions for the occurrence of the relation $\dashv\vdash_{\mathcal{F}}$ in terms of $R(c)$. We subsequently apply the result to some special cases (such as twist maps and commuting Hamiltonians), and we discuss the properties of $R(c)$ in relation with the dynamics of the polysystem.

1.2 Notation. The space of pseudographs

In this section we recall from [Ber08] some facts about pseudographs. We refer to that article for a more detailed introduction.

Let M be a d-dimensional compact connected manifold without boundary. We denote by Ω the set of smooth closed one-forms on M and by π the projection from the cotangent space T^*M to M. If $\eta \in \Omega$ we denote by $[\eta] \in H^1(M,\mathbb{R})$ its cohomology class and, for $S \subseteq \Omega$, $[S] = \{[\eta] : \eta \in S\}$.

If $u\colon M \to \mathbb{R}$ is a Lipschitz function and $\eta \in \Omega$, then the *pseudograph* $\mathcal{G}_{\eta,u} \subset T^*M$ is defined by

$$\mathcal{G}_{\eta,u} = \{(x, \eta_x + du_x) : x \in M \text{ and } du_x \text{ exists}\}.$$

Let us call E the set of pseudographs:

$$E = \{\mathcal{G}_{\eta,u} : \eta \in \Omega, u \in \mathrm{Lip}(M)\}.$$

Note that

$$\mathcal{G}_{\eta,u} = \mathcal{G}_{\eta+df,u-f}$$

for any smooth function $f\colon M \to \mathbb{R}$. Viceversa, if $\mathcal{G}_{\eta,u} = \mathcal{G}_{\eta',u'}$ then, setting $f = u - u'$, it is not difficult to check that f is smooth, $\eta' = \eta + df$ and $u' = u - f$. In particular, the cohomologies of η and η' are equal. Thus the cohomology of a pseudograph $\mathcal{G}$ is well defined, and we denote it by $c(\mathcal{G})$. If $\mathcal{G} = \mathcal{G}_{\eta,u}$ for some η and u, then

$$c(\mathcal{G}) = [\eta] \in H^1(M,\mathbb{R}).$$

It is not difficult to see that E is a vector space. In fact, it may be regarded as a quotient of $\Omega \times \mathrm{Lip}(M)$ by the subspace $\{(\eta, u) : \eta = -du\} = \{(\eta, u) : \mathcal{G}_{\eta,u} = \mathcal{G}_{0,0}\}$. The operations of sum and scalar multiplication are explicitly given by

$$\mathcal{G}_{\eta,u} + \mathcal{G}_{\nu,v} = \mathcal{G}_{\eta+\nu,u+v} \ , \qquad \lambda\, \mathcal{G}_{\eta,u} = \mathcal{G}_{\lambda\eta,\lambda u} \qquad \text{for } \lambda \in \mathbb{R}$$

(this does not depend on the chosen representatives (η, u) and (ν, v)).

We have the following identification of vector spaces, which will be extensively used throughout the paper:

$$E \ \cong \ H^1(M, \mathbb{R}) \times \big(\mathrm{Lip}(M)/\sim\big)$$

where the relation $\sim$ means up to the addition of constants. Given a linear section $S\colon H^1(M, \mathbb{R}) \to \Omega$ (i.e. $[S(c)] = c$), an isomorphism performing the above identification is given by

$$\begin{aligned} H^1(M, \mathbb{R}) \times \big(\mathrm{Lip}(M)/\sim\big) &\to E \\ (c, u) &\mapsto \mathcal{G}_{S(c),u}. \end{aligned}$$

The space E can be given a norm via the formula

$$\|\mathcal{G}_{S(c),u}\| = \|c\|_{H^1} + |u|,$$

where $|u|$ denotes half the oscillation of u, i.e. $|u| = (\max u - \min u)/2$. Changing the section S or the norm $\|\cdot\|_{H^1}$ gives rise to an equivalent norm. In the rest of the paper, S and $\|\cdot\|_{H^1}$ will be considered as fixed. Everything will be well-defined regardless of this choice. With a little abuse of language, we will often write c in place of $S(c)$, for instance $\mathcal{G}_{c,u}$ in place of $\mathcal{G}_{S(c),u}$.

We will be mostly concerned with a proper subset of E, namely

$$\mathbb{P} = \left\{\mathcal{G}_{c,u} : c \in H^1(M, \mathbb{R}), u\colon M \to \mathbb{R} \text{ semiconcave } \right\}.$$

Here and throughout the paper, the term 'semiconcave' stands for the more accurate expression 'semiconcave with linear modulus'. Some basic properties of semiconcave functions are reviewed in Subsection 1.2.1.

Every $\mathcal{G} \in \mathbb{P}$ is called an *overlapping pseudograph* (the motivation behind this terminology is given in [Ber08, Section 2.9]). The set $\mathbb{P}$ is closed under sum and multiplication by a positive scalar, but not under difference or multiplication by a negative scalar. In fact, the dual set $\breve{\mathbb{P}}$ of *anti-overlapping pseudographs* is defined as

$$\breve{\mathbb{P}} = -\mathbb{P} = \left\{\mathcal{G}_{c,u} : c \in H^1(M, \mathbb{R}), u\colon M \to \mathbb{R} \text{ semiconvex } \right\}.$$

If $c \in H^1(M, \mathbb{R})$ and $C \subseteq H^1(M, \mathbb{R})$, the symbols $\mathbb{P}_c$ and $\mathbb{P}_C$ stand for

$$\mathbb{P}_c = \{\mathcal{G} \in \mathbb{P} : c(\mathcal{G}) = c\}, \qquad \mathbb{P}_C = \bigcup_{c \in C} \mathbb{P}_c.$$

and analogously for $\breve{\mathbb{P}}_c$ and $\breve{\mathbb{P}}_C$.

Given a subset $N \subset M$ and a pseudograph $\mathcal{G}$, we denote by $\mathcal{G}_{|N}$ the restriction of $\mathcal{G}$ above N, that is $\mathcal{G} \cap \pi^{-1}(N)$.

Given $\mathcal{G} = \mathcal{G}_{c,u} \in \mathbb{P}_c$ and $\breve{\mathcal{G}} = \mathcal{G}_{c,v} \in \breve{\mathbb{P}}_c$ (with the same c) the set

$$\mathcal{G} \wedge \breve{\mathcal{G}} \subseteq M$$

is defined as the set of the points of minimum of the difference $u - v$. This is a non-empty compact set because M is compact. Moreover, the semiconcavity of both u and $-v$ implies the following property: for every x in $\mathcal{G} \wedge \breve{\mathcal{G}}$ both du_x and dv_x exist, and they coincide. As a consequence, for any c and any couple $(\mathcal{G}, \breve{\mathcal{G}}) \in \mathbb{P}_c \times \breve{\mathbb{P}}_c$, the following definition yields a non-empty subset of T^*M:

$$\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}} := \mathcal{G}_{|\mathcal{G} \wedge \breve{\mathcal{G}}} = \breve{\mathcal{G}}_{|\mathcal{G} \wedge \breve{\mathcal{G}}} = \mathcal{G} \cap \breve{\mathcal{G}} \cap \pi^{-1}(\mathcal{G} \wedge \breve{\mathcal{G}}) \subseteq \mathcal{G} \cap \breve{\mathcal{G}}$$

and the last inclusion may be strict in general. The set $\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}}$ is compact and is a Lipschitz graph over its projection $\mathcal{G} \wedge \breve{\mathcal{G}}$, by properties of semiconcave functions.

Finally, let us observe that Ω can be naturally regarded as a subset of $\mathbb{P} \cap \breve{\mathbb{P}}$. The inclusion is given by $\eta \mapsto \mathcal{G}_{\eta,0} = \mathrm{Graph}\,(\eta)$.

1.2.1 Semiconcave functions

Let us make a brief digression about semiconcave functions. Recall that for us 'semiconcave' means 'semiconcave with linear modulus'. We refer to [CS04] for a comprehensive exposition in the Euclidean case. On a manifold, the notion of semiconcavity is still meaningful, but the one of semiconcavity constant is chart-dependent. Nevertheless, by taking a finite atlas as shown in [Ber08, Appendix 1] it is still possible to give meaning to the expression "u is C-semiconcave" for a real-valued function defined on a compact manifold and $C \in \mathbb{R}$. Hence we can define the best semiconcavity constant of u as

$$sc(u) = \inf\{C \in \mathbb{R} : u \text{ is } C\text{-semiconcave}\}.$$

It will depend on the particular finite atlas, but this choice will not affect our results.

We now recall some properties which are well-known in the Euclidean case and which hold true in the manifold case as well. We refer to [Ber08, Appendix 1] for a more detailed exposition.

We have

$$sc\,(\inf_\lambda\{u_\lambda\}) \leq \sup_\lambda\{sc(u_\lambda)\}, \tag{1.1}$$

for any family of functions $\{u_\lambda\}_\lambda$, provided that the infimum is finite. Moreover, if u_n converges uniformly to u, then

$$sc(u) \leq \liminf sc(u_n). \tag{1.2}$$

A semiconcave function is differentiable at every point of local minimum (and the differential is 0). We also have: if u and v are semiconcave and if x is a point of local minimum of $u + v$, then both u and v are differentiable at x, and $du_x + dv_x = 0$.

A family of functions $\{u_\lambda\}_\lambda$ is called equi-semiconcave if $sc(u_\lambda) \leq C$ for some constant C independent of λ. We will use a lot the following fact: a family of equi-semiconcave functions is

equi-Lipschitz (this follows for instance by adapting Theorem 2.1.7 and Remark 2.1.8 in [CS04] to the case of a compact manifold).

Finally, the set of semiconcave functions is closed under sum and multiplication by a positive scalar. A function u such that $-u$ is semiconcave is called semiconvex. A function is both semiconcave and semiconvex if and only if it is $C^{1,1}$.

1.3 The forcing relation and diffusion polyorbits

Let $\mathcal{F}$ be an arbitrary family of one-periodic Tonelli Hamiltonians on M. Let us recall that a one-periodic Tonelli Hamiltonian on M is a C^2 function

$$\begin{aligned} H\colon T^*M \times \mathbb{T} &\to \mathbb{R} \\ (x,p,t) &\mapsto H(x,p,t) \end{aligned}$$

which is strictly convex and superlinear in p (for any fixed x and t) and whose Hamiltonian flow is complete. Our goal is to prove existence of diffusion polyorbits for the family $\mathcal{F}$, in the sense discussed in the Introduction. In this section, we first adapt to the polysystem framework the notion of forcing relation, which was introduced in [Ber08] for the case of a single Hamiltonian. Then we show (Proposition 7) how this relation implies the diffusion: roughly speaking, if the cohomology class c forces the class c', then there will exist diffusion polyorbits from the cohomology c to the cohomology c', in a sense which will be made precise in the proposition. The aim of the later sections will then be to give sufficient conditions for the forcing relation to occur between two cohomology classes.

Let us recall that we denote by $\phi_H\colon T^*M \to T^*M$ the time-one map of a Tonelli Hamiltonian H. We define $\phi_{\mathcal{F}}$ of a subset $S \subseteq T^*M$ as follows:

$$\phi_{\mathcal{F}}(S) = \bigcup_{H\in\mathcal{F}} \phi_H(S),$$

and we recursively define $\phi_{\mathcal{F}}^{n+1}(S) = \phi_{\mathcal{F}}(\phi_{\mathcal{F}}^n(S))$. Given two subsets S and S' of T^*M, we write

$$S \vdash_{N,\mathcal{F}} S' \quad \overset{\text{def}}{\iff} \quad S' \subseteq \bigcup_{n=0}^{N} \phi_{\mathcal{F}}^n(S).$$

We write $S \vdash_{\mathcal{F}} S'$, and we say that S forces S', if $S \vdash_{N,\mathcal{F}} S'$ for some $N \in \mathbb{N}$. Note that, for $z, z' \in T^*M$,

$$\{z\} \vdash_{\mathcal{F}} \{z'\} \;\Leftrightarrow\; \text{there exists a finite } \mathcal{F}\text{-polyorbit joining } z \text{ to } z'.$$

(We shall often write $z \vdash_{\mathcal{F}} z'$ to lighten notations.) In fact, we will mainly be interested in the case in which $S = \mathcal{G}$ and $S' = \mathcal{G}'$ are two pseudographs in $\mathbb{P}$. Let us make explicit that

$$\begin{aligned} \mathcal{G} \vdash_{\mathcal{F}} \mathcal{G}' \;\Leftrightarrow\; &\text{for every } z' \in \mathcal{G}' \text{ there exists } z \in \mathcal{G} \text{ and a finite } \mathcal{F}\text{-polyorbit} \\ &\text{joining } z \text{ to } z' \text{ (with an uniform bound on the length of the polyorbit).} \end{aligned}$$

We are now going to extend the definition of $\vdash_{\mathcal{F}}$ to cohomology classes. If $\mathbb{W}$ and $\mathbb{W}'$ are two subsets of $\mathbb{P}$, we write

$$\mathbb{W} \vdash_{N,\mathcal{F}} \mathbb{W}' \quad \overset{\text{def}}{\iff} \quad \forall \mathcal{G} \in \mathbb{W} \quad \exists \mathcal{G}' \in \mathbb{W}' : \mathcal{G} \vdash_{N,\mathcal{F}} \mathcal{G}'.$$

We write $\mathbb{W} \vdash_{\mathcal{F}} \mathbb{W}'$, and we say that $\mathbb{W}$ forces $\mathbb{W}'$, if $\mathbb{W} \vdash_{N,\mathcal{F}} \mathbb{W}'$ for some $N \in \mathbb{N}$. If $\mathbb{W} = \mathbb{P}_c$ or $\mathbb{W} = \mathbb{P}_C$, for some $c \in H^1(M,\mathbb{R})$ or $C \subseteq H^1(M,\mathbb{R})$, we simply write c or C in place of $\mathbb{P}_c$ or $\mathbb{P}_C$. Similarly for $\mathbb{W}' = \mathbb{P}_c$. So, for instance, if c and c' are two cohomology classes, the relation

$$c \vdash_{N,\mathcal{F}} c'$$

means that for every $\mathcal{G} \in \mathbb{P}_c$ there exists $\mathcal{G}' \in \mathbb{P}_{c'}$ such that $\mathcal{G} \vdash_{N,\mathcal{F}} \mathcal{G}'$.

The relation $\vdash_{\mathcal{F}}$ is reflexive and transitive (between subsets of T^*M as well as between subsets of $\mathbb{P}$, and in particular between cohomology classes as well). In the sequel, it will be useful to consider the symmetrized relation $\dashv\vdash_{\mathcal{F}}$ on the cohomology classes defined by

$$c \dashv\vdash_{\mathcal{F}} c' \qquad \overset{\text{def}}{\iff} \qquad c \vdash_{\mathcal{F}} c' \quad \text{and} \quad c' \vdash_{\mathcal{F}} c.$$

If $c \dashv\vdash_{\mathcal{F}} c'$, we say that c and c' *force each other*. The following fact follows directly from the definitions.

Proposition 5. *The relation $\dashv\vdash_{\mathcal{F}}$ is an equivalence relation on $H^1(M,\mathbb{R})$.*

We also have:

Proposition 6. *Let $c \vdash_{\mathcal{F}} c'$. Then for any $\mathcal{G} \in \mathbb{P}_c$ and any $\mathcal{G}' \in \breve{\mathbb{P}}_{c'}$ there exists a finite polyorbit joining $\mathcal{G}$ to $\mathcal{G}'$.*

Proof. Let us fix $\mathcal{G} \in \mathbb{P}_c$ and $\mathcal{G}' \in \breve{\mathbb{P}}_{c'}$. Since $c \vdash_{\mathcal{F}} c'$, there exists $\mathcal{G}'' \in \mathbb{P}_{c'}$ such that $\mathcal{G} \vdash_{\mathcal{F}} \mathcal{G}''$, which means that for every $z'' \in \mathcal{G}''$ there exists a finite polyorbit joining $\mathcal{G}$ to z''. Note that the intersection $\mathcal{G}' \cap \mathcal{G}''$ is not empty, because it contains the non-empty set $\mathcal{G}' \tilde{\wedge} \mathcal{G}''$ (see Section 1.2). Taking z'' in this intersection, we get a finite polyorbit joining $\mathcal{G}$ to $\mathcal{G}'$. □

We can now restate and prove Proposition 2 about the existence of diffusion polyorbits. The proof is essentially the same as in [Ber08, Proposition 5.3].

Proposition 7.

1. *Let $c \vdash_{\mathcal{F}} c'$. Let $H, H' \in \mathcal{F}$ and η, η' be two smooth closed one-forms of cohomology c and c' respectively. Then:*
   - *(i) there exists a polyorbit which is α-asymptotic to $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;*
   - *(ii) there exists a finite polyorbit joining* Graph (η) *to* Graph (η')*;*
   - *(iii) there exists a polyorbit $(z_n)_{n\in\mathbb{Z}}$ which satisfies $z_0 \in$ Graph (η) and is ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;*

(iv) *there exists a polyorbit* $(z_n)_{n\in\mathbb{Z}}$ *which is* α*-asymptotic to* $\tilde{\mathcal{A}}_H(c)$ *and satisfies* $z_0 \in \mathrm{Graph}\,(\eta')$.

2. *Let*

$$(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \;\subseteq\; H^1(M,\mathbb{R}) \times \mathcal{F} \times]0,+\infty[$$

such that $c_i \vdash_{\mathcal{F}} c_{i+1}$. *Then there exists a polyorbit* $(z_n)_{n\in\mathbb{Z}}$ *which visits in turn the* ε_i*-neighborhoods of the Mather sets* $\tilde{\mathcal{M}}_{H_i}(c_i)$. *Moreover, if* $(c_i, H_i) = (\bar{c}, \bar{H})$ *for* $-i$ *large enough (resp.* i *big enough), then the polyorbit can be taken* α*-asymptotic to* $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$ *(resp.* ω*-asymptotic to* $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$*).*

Proof of 1. The proof of any one of the four statements relies on a suitable application of Proposition 6.

Let us start with (i). As it is well-known, there exist a c-weak Kam solution u for H and a dual c'-weak Kam solution u' for H'. It is also known that u is semiconcave and u' is semiconvex. Let us consider the associated pseudographs $\mathcal{G} = \mathcal{G}_{c,u} \in \mathbb{P}_c$ and $\mathcal{G}' = \mathcal{G}_{c',u'} \in \breve{\mathbb{P}}_c$. By Proposition 6, there exists a finite polyorbit $(z_i)_{i=0}^N$ joining $\mathcal{G}$ to $\mathcal{G}'$. Moreover, by a general property of weak Kam solutions (see [Ber08, Proposition 4.3]), every point in $\mathcal{G}$ is α-asymptotic for the flow of H to $\tilde{\mathcal{A}}_H(c)$ and every point in $\mathcal{G}'$ is ω-asymptotic for the flow of H' to $\tilde{\mathcal{A}}_{H'}(c')$. This implies that we can extend the finite polyorbit $(z_i)_{i=0}^N$ to a bi-infinite one satisfying the requirements in (i).

For (ii), let us consider $\mathcal{G} = \mathcal{G}_{\eta,0}$ and $\mathcal{G}' = \mathcal{G}_{\eta',0}$. Since η and η' are smooth, both $\mathcal{G}$ and $\mathcal{G}'$ belong to $\mathbb{P}_c \cap \breve{\mathbb{P}}_c$. Hence the Proposition 6 immediately yields a finite polyorbit joining $\mathcal{G} = \mathrm{Graph}\,(\eta)$ to $\mathcal{G}' = \mathrm{Graph}\,(\eta')$.

The proof of statements (iii) and (iv) is similar. □

Proof of 2. It is a natural adaptation of the proof in [Ber08, Proposition 5.3 (ii)]. □

1.4 Lagrangian action and Lax-Oleinik operators

In this section we introduce the objects needed to put in place, in Section 1.5, the Mather mechanism for the construction of diffusion polyorbits. For this aim, it is more convenient to adopt the Lagrangian point of view on the dynamics rather than the Hamiltonian one.

Let us quickly recall some basic facts about the Lagrangian point of view: to any one-periodic Tonelli Hamiltonian $H\colon T^*M \times \mathbb{T} \to \mathbb{R}$ one can associate a one-periodic Lagrangian $L\colon TM \times \mathbb{T} \to \mathbb{R}$ via the Fenchel-Legendre transform, i.e.

$$L(x,v,t) = \sup_{p\in T_x^*M} \{p(v) - H(x,p,t)\}, \qquad x \in M,\, v \in T_xM,\, t \in \mathbb{T}.$$

The Lagrangian L turns out to be Tonelli as well, in the sense that it is C^2, it is strictly convex in v (with positive definite Hessian $\partial_v^2 L > 0$), superlinear in v, and the associated Euler-Lagrange flow on $TM \times \mathbb{T}$ is complete. This flow is conjugated to the Hamiltonian flow of H on $T^*M \times \mathbb{T}$. Moreover, the Fenchel-Legendre transform applied to L yields H back. We refer to [Fat] for the proofs of all these facts.

Starting from our family $\mathcal{F}$ of one-periodic Tonelli Hamiltonians, we thus dispose of an associated family of one-periodic Tonelli Lagrangians, which we shall still denote by the same symbol $\mathcal{F}$. For instance, depending on context, both expressions $L \in \mathcal{F}$ and $H \in \mathcal{F}$ will be used.

1.4.1 Outline of Section 1.4

The general idea is to translate the dynamics of the family $\mathcal{F}$ into a simpler dynamics on pseudographs, by means of the characterization of the former in terms of minimal action and Lax-Oleinik operators.

Subsection 1.4.2. We recall the definition of time-one action of a Tonelli Lagrangian, along with the properties which are important for the sequel (Proposition 8).

Subsection 1.4.3. We associate to the time-one action (and, more generally, to any cost, i.e. any continuous function on $M \times M$) a Lax-Oleinik operator, in the usual way. This Lax-Oleinik operator can be interpreted also as an operator on pseudographs (formula (1.6)).

The properties of the time-one action, which have been recalled in Subsection 1.4.2, nicely reflect in properties of the corresponding Lax-Oleinik operator (Remark 11). These nice properties are not lost under some operations on costs such as minimums and compositions (Proposition 13).

Subsection 1.4.5. We review how the language of pseudographs allows to concisely rephrase some aspects of the weak Kam theory for one Tonelli Lagrangian. From the viewpoint of the present article, this may be regarded as a special case in which our family $\mathcal{F}$ is a singleton.

Subsection 1.4.6. We generalize the Subsection 1.4.5 to the general case in which $\mathcal{F}$ is not a singleton. The key object is, for every cohomology c, a large semigroup Σ_c^∞ (depending on $\mathcal{F}$) of Lax-Oleinik operators on the space $\mathbb{P}_c$ of pseudographs of cohomology c. This semigroup is essentially the one generated by the time-one actions of the Lagrangians in $\mathcal{F}$ with respect to the operations on costs introduced in Subsection 1.4.3. As we will see, the dynamics on $\mathbb{P}_c$ of the semigroup Σ_c^∞ is related to the dynamics on T^*M of the semigroup generated by the time-one maps ϕ_H, $H \in \mathcal{F}$.

Crucially, the semigroup Σ_c^∞ will contain, after passing to the limit, the operators associated to the Peierls barriers of the Lagrangians in $\mathcal{F}$, along with their successive compositions. This aspect, together with the possibility of "shadowing" these operators with "finite-time" ones, will be at the heart of the Mather mechanism in the next section.

1.4.2 Properties of the Lagrangian action

Given a one-periodic Tonelli Lagrangian L on M and a closed smooth one-form η, the time-one action $A_{L,\eta}\colon M \times M \to \mathbb{R}$ is defined by

$$A_{L,\eta}(y,x) = \min_{\gamma(0)=y,\gamma(1)=x} \int_0^1 L\big(\gamma(t),\dot\gamma(t),t\big) - \eta_{\gamma(t)}(\dot\gamma(t))\,dt \tag{1.3}$$

where the minimum is taken over absolutely continuous curves γ. It is well-known that minimizers exist. The following important properties of A_L are also known.

Proposition 8.

(i) $A_{L,\eta+df}(y,x) = A_{L,\eta}(y,x) + f(y) - f(x)$*; this is immediate from the definition.*

(ii) $\eta \mapsto A_{L,\eta}$ *is continuous if* Ω *is endowed with the topology induced from the space of pseudographs* E *introduced in Section 1.2.*

In view of (i) *above, this is equivalent to the continuity of* $c \mapsto A_{L,S(c)}$*. For a proof of this last fact, see [Ber08, Appendix B.6].*

(iii) $A_{L,\eta}$ *is semiconcave. Even more, if* $C \subset H^1(M,\mathbb{R})$ *is compact, then* $\{A_{L,S(c)}\}_{c\in C}$ *is equi-semiconcave (for a proof see [Ber08, Appendix B.7]).*

(iv) $\partial_x A_{L,\eta}(y,x)$ *exists if and only if* $\partial_y A_{L,\eta}(y,x)$ *exists and in this case we have*

$$\big(x, \eta_x + \partial_x A_{L,\eta}(y,x)\big) = \phi_H\big(y, \eta_y - \partial_y A_{L,\eta}(y,x)\big),$$

where H *is the Hamiltonian associated to* L*.*

The time-n action A_L^n is defined by letting $A_L^1 = A_L$ and by induction

$$A_{L,\eta}^{n+1}(y,x) = \min_{z\in M}\big\{A_{L,\eta}^n(y,z) + A_{L,\eta}^1(z,x)\big\}$$

or, equivalently,

$$A_{L,\eta}^n(y,x) = \min_{\gamma(0)=y,\gamma(n)=x} \int_0^n L\big(\gamma(t),\dot\gamma(t),t\big) - \eta_{\gamma(t)}(\dot\gamma(t))\,dt,$$

the minimum being over absolutely continuous curves.

It is well-known that, given L, there exists an unique function $\alpha\colon H^1(M,\mathbb{R}) \to \mathbb{R}$ such that the function

$$h_{L,\eta}(y,x) = \liminf_{n\to\infty} A_{L,\eta}^n(y,x) + n\alpha([\eta])$$

is real-valued for every η; the family $h_L \equiv \{h_{L,\eta}\}_\eta$ is called the Peierls barrier of L. It clearly satisfies the property (i) of Proposition 8; it also satisfies the property (iii), the proof of this fact will be recalled in Subsection 1.4.5.

1.4.3 Lax-Oleinik operators

For any compact space X, the set of real continuous functions $C(X)$ will be endowed with the standard sup-norm $\|\cdot\|_\infty$. Any continuous function $A \in C(M\times M)$ will be called a *cost*. We will regard the time-one actions $A_{L,\eta}$ of the previous subsection as very special costs.

To any cost A, it is possible to associate the Lax-Oleinik operator $T_A\colon C(M) \to C(M)$ defined by

$$T_A u(x) = \min_{y\in M}\big\{u(y) + A(y,x)\big\}, \qquad u \in C(M)$$

and the dual Lax-Oleinik operator $\breve T_A\colon C(M) \to C(M)$

$$\breve T_A u(y) = \max_{x\in M}\big\{u(x) - A(y,x)\big\}, \qquad u \in C(M).$$

We call $\mathcal{I}_A(u) \subseteq M$ the set of points y such that $T_A u(x) = u(y) + A(y,x)$ for some x. Let us now list without proof some basic properties of these objects. Note that basically every property of T_A has a dual counterpart in a property of $\breve{T}_A$, even though we do not always explicit it. Recall that $|\cdot|$ indicates half the oscillation of a function.

Proposition 9. *Let A be a cost and u be a continuous function on M.*

(i) The minimum and the maximum in the above formulas for $T_A u$ and $\breve{T}_A u$ are actually achieved; $T_A u$ and $\breve{T}_A u$ actually belong to $C(M)$;

(ii) if A' is another cost and u' another continuous function, then

$$\begin{aligned} \|T_{A'}u' - T_A u\|_\infty &\leq \|A' - A\|_\infty + \|u' - u\|_\infty, \\ |T_{A'}u' - T_A u| \ &\leq \ |A' - A| + |u' - u| \end{aligned} \tag{1.4}$$

(iii) $\mathcal{I}_A(u)$ is compact and non-empty;

(iv) the set-valued function $(A,u) \mapsto \mathcal{I}_A(u)$ is upper-semicontinuos;

(v) for every A and u, we have $\breve{T}_A T_A u \leq u$ and

$$\mathcal{I}_A(u) = \{y \in M : \breve{T}_A T_A u(y) = u(y)\} = \arg\min \left\{u - \breve{T}_A T_A u\right\}.$$

(vi) for every A and u, we have

$$T_A \breve{T}_A T_A \, u = T_A \, u \qquad \text{and} \qquad \breve{T}_A T_A \breve{T}_A \, u = \breve{T}_A \, u;$$

(vii) if A is semiconcave, then $T_A u$ is semiconcave for any u, and $sc(u) \leq sc(A)$.

We are going to consider families of costs indexed by closed smooth one-forms. Let us give some definitions.

Definition 10. *Let $A \equiv \{A_\eta\}_{\eta \in \Omega}$ be a family of costs indexed by the closed smooth one-forms. We say that A is:*

(i) geometric *if A_η is Lipschitz for every η and*

$$A_{\eta + df}(y,x) = A_\eta(y,x) + f(y) - f(x) \qquad \forall \ f \in C^\infty(M); \tag{1.5}$$

(ii) continuous *if*

$$\Omega \ni \eta \mapsto \ A_\eta \quad \text{ is continuous}$$

when Ω is endowed with the topology induced from E, see Section 1.2. Note that if a family A is geometric, the continuity of $c \mapsto A_{S(c)}$ is sufficient in order to have the continuity of $\eta \mapsto A_\eta$; here S is the linear section chosen in Section 1.2;

(iii) locally equi-semiconcave *if, for any compact $C \subset H^1(M, \mathbb{R})$, the family $\{A_{S(c)}\}_{c \in C}$ is equi-semiconcave;*

(iv) of $\mathcal{F}$-flow-type *if there exists* $N \in \mathbb{N}$ *such that the following holds:*

$$\text{the partial derivatives } \partial_x A_\eta(y,x) \text{ and } \partial_y A_\eta(y,x) \text{ exist} \\ \Rightarrow \ \big(y, \eta_y - \partial_y A_\eta(y,x)\big) \ \vdash_{N,\mathcal{F}} \ \big(x, \eta_x + \partial_x A_\eta(y,x)\big)$$

for any η *and for any* (y,x). *We say that* A *is of* $N,\mathcal{F}$*-flow-type if we want to specify the* N.

If all the above conditions are satisfied, we say for short that A *is a* $\mathcal{F}$-family.

Observe that the Proposition 8 says that the time-one actions $\{A_{L,\eta}\}_\eta, L \in \mathcal{F}$, are $\mathcal{F}$-families. In the next subsection we are going to introduce some operations on costs which will preserve the property of being an $\mathcal{F}$-family. This will allow to use the Lagrangian time-one actions as "basic bricks" to build many $\mathcal{F}$-families of costs.

The utility of $\mathcal{F}$-families comes from the following remark.

Remark 11.

(i) If $A \equiv \{A_\eta\}_{\eta\in\Omega}$ is a geometric family of costs then

$$T_{A_{\eta+df}}(u-f) = T_{A_\eta}(u) - f.$$

Hence, an induced operator on pseudographs $\Phi_A \colon E \to E$ is well-defined by

$$\Phi_A(\mathcal{G}_{\eta,u}) = \mathcal{G}_{\eta, T_{A_\eta} u} \tag{1.6}$$

as well as its dual counterpart

$$\breve{\Phi}_A(\mathcal{G}_{\eta,u}) = \mathcal{G}_{\eta, \breve{T}_{A_\eta} u}.$$

Note that both operators preserve the cohomology of $\mathcal{G}$, i.e. $c(\Phi_A(\mathcal{G})) = c(\breve{\Phi}_A(\mathcal{G})) = c(\mathcal{G})$.

If A' is another geometric family of costs, and if $\mathcal{G} = \mathcal{G}_{c,u}, \mathcal{G}' = \mathcal{G}_{c',u'} \in E$ are two pseudographs, we have the following inequality:

$$\begin{aligned} \|\Phi_A(\mathcal{G}) - \Phi_{A'}(\mathcal{G}')\|_E &= \|c - c'\|_{H^1} + |T_{A_c} u - T_{A'_{c'}} u'| \\ &\leq \|\mathcal{G} - \mathcal{G}'\|_E + |A_c - A'_{c'}| \end{aligned} \tag{1.7}$$

which follows from (1.4).

In the same spirit, $\mathcal{I}_{A_{\eta+df}}(u-f) = \mathcal{I}_{A_\eta}(u)$, thus the set $\mathcal{I}_{A_\eta}(u)$ is also well-defined on pseudographs, and we will denote it by $\mathcal{I}_A(\mathcal{G})$ or $\mathcal{I}_{\Phi_A}(\mathcal{G})$. Items (v) and (vi) in Proposition 9 translate respectively into

$$\mathcal{I}_A(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}_A \Phi_A(\mathcal{G}). \tag{1.8}$$

and

$$\Phi_A \breve{\Phi}_A \Phi_A = \Phi_A, \qquad \breve{\Phi}_A \Phi_A \breve{\Phi}_A = \breve{\Phi}_A. \tag{1.9}$$

(ii) If $\{A_\eta\}_\eta$ is a continuous geometric family, then Φ_A is continuous thanks to the estimate (1.7). Moreover, $\mathcal{I}_A(\mathcal{G})$ is upper-semicontinuous viewed as a (set-valued) function from E to M. Indeed, the composition

$$(\eta, u) \mapsto (A_\eta, u) \mapsto \mathcal{I}_{A_\eta}(u)$$

is upper-semicontinuous (thanks to Proposition 9(iv)), and this remains true when passing to the quotient space of pseudographs.

(iii) If $\{A_\eta\}_{\eta\in\Omega}$ is a locally equi-semiconcave geometric family, then $\Phi_A(\mathbb{P}) \subseteq \mathbb{P}$ and $\Phi_A(\mathbb{P}_C)$ is relatively compact for all compact $C \subset H^1(M, \mathbb{R})$. This is a consequence of Proposition 9(vii) and the Ascoli-Arzelà Theorem (recall that equi-semiconcave implies equi-Lipschitz). The analogous result holds true for $\breve{\Phi}_A$.

(iv) If $\{A_\eta\}_\eta$ is a $N, \mathcal{F}$-flow-type, locally equi-semiconcave and geometric family of costs, then

$$\mathcal{G}_{|\mathcal{I}_A(\mathcal{G})} \ \vdash_{N,\mathcal{F}} \ \Phi_A(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{P}. \tag{1.10}$$

This important fact is obtained by writing $\mathcal{G} = \mathcal{G}_{\eta,u}$ and then applying Proposition 12 below. The dual statement is also true and is proved analogously. It can be expressed as

$$\mathcal{G}_{|\breve{\mathcal{I}}_A(\mathcal{G})} \ \vdash_{N,-\mathcal{F}} \ \breve{\Phi}_A(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \breve{\mathbb{P}}.$$

Here we have denoted by $-\mathcal{F}$ the family $\{-H : H \in \mathcal{F}\}$; its elements are not Tonelli Hamiltonians but the relation $\vdash_{-\mathcal{F}}$ is still meaningful. We have also denoted by $\breve{\mathcal{I}}_A(\mathcal{G})$ the set of points $x \in M$ such that $\breve{T}_{A_\eta} u(y) = u(x) - A_\eta(y, x)$ for some y (and η and u are such that $\mathcal{G} = \mathcal{G}_{\eta,u}$).

Proposition 12. *Suppose that the family of costs $\{A_\eta\}_\eta$ is of $N, \mathcal{F}$-flow-type, locally equi-semiconcave and geometric. Let $u\colon M \to \mathbb{R}$ be semiconcave and $v = T_{A_\eta} u$. Then, for every x such that dv_x exists and for every y such that $v(x) = u(y) + A_\eta(y, x)$, we have*

$$\text{the derivative } du_y \text{ exists and satisfies} \quad (y, \eta_y + du_y) \ \vdash_{N,\mathcal{F}} \ (x, \eta_x + dv_x).$$

Note that such a point y belongs to $\mathcal{I}_A(\mathcal{G}_{\eta,u})$.

Proof. The proof is essentially the same as in [Ber08, Proposition 2.7] but we report it for completeness. Let x be such that dv_x exists, and let y be such that $v(x) = u(y) + A_\eta(y, x)$. From the definition of T_{A_η}, one gets that the function $y' \mapsto u(y') + A_\eta(y', x)$ has a minimum at y. Being the sum of two semiconcave functions, both of them have to be differentiable at y and

$$du_y + \partial_y A_\eta(y, x) = 0.$$

Similarly, the function $x' \mapsto v(x') - A_\eta(y, x')$ has a maximum at x. Since dv_x exists by assumption and $-A_\eta$ is semiconvex, we get that $\partial_x A_\eta(y, x)$ exists and

$$dv_x - \partial_x A_\eta(y, x) = 0.$$

Thanks to the $N, \mathcal{F}$-flow-type property we can conclude:

$$(y, \eta_y + du_y) = \big(y, \eta_y - \partial_y A_\eta(y, x)\big) \ \vdash_{N,\mathcal{F}} \ \big(x, \eta_x + \partial_x A_\eta(y, x)\big) = (x, \eta_x + dv_x).$$

□

1.4.4 Operations on costs and families of costs

There are three quite natural operations on costs. For A, A' two costs and $\lambda \in \mathbb{R}$, they are defined as follows:

$$\begin{aligned} (A,\lambda) &\mapsto A+\lambda && \textit{(addition of constant)}\\ (A,A') &\mapsto \min\{A,A'\} && \textit{(minimum)}\\ (A,A') &\mapsto A'\circ A(y,x) = \min_{z\in M}\big\{A(y,z)+A'(z,x)\big\} && \textit{(composition).} \end{aligned}$$

It is easily checked that the three of them are continuous in their arguments and that the associated Lax-Oleinik operators are well-behaved in the following sense: for $u \in C(M)$, we have

$$\begin{aligned} T_{A+\lambda}u &= T_A u+\lambda\\ T_{\min\{A,A'\}}u &= \min\{T_A u, T_{A'}u\}\\ T_{A'\circ A}u &= T_{A'}\circ T_A u. \end{aligned} \tag{1.11}$$

We can define the same operations on families of costs: for all $A \equiv \{A_\eta\}_\eta$, $A' \equiv \{A'_\eta\}_\eta$ and all functions $\lambda\colon H^1(M,\mathbb{R}) \to \mathbb{R}$, we define

$$(A+\lambda)_\eta = A_\eta + \lambda([\eta]), \qquad \min\{A,A'\}_\eta = \min\{A_\eta, A'_\eta\}, \qquad (A'\circ A)_\eta = A'_\eta \circ A_\eta.$$

The following proposition shows that these operations preserve the fact of being a $\mathcal{F}$-family.

Proposition 13. *Let A, A' be two $\mathcal{F}$-families of costs, and $\lambda\colon H^1(M,\mathbb{R}) \to \mathbb{R}$ be a continuous function. Then $A+\lambda$, $\min\{A,A'\}$ and $A'\circ A$ are $\mathcal{F}$-families as well. Moreover, the semiconcavity constants are controlled by*

$$\begin{aligned} sc\,(A+\lambda)_\eta &= sc(A_\eta)\\ sc(\min\{A,A'\}_\eta) &\le \max\{sc(A_\eta), sc(A'_\eta)\}\\ sc\,(A'\circ A)_\eta &\le \max\{sc(A_\eta), sc(A'_\eta)\} \end{aligned} \tag{1.12}$$

for each $\eta \in \Omega$.

Proof. We have to verify that the four conditions of Definition 10 hold true for the families $A+\lambda, \min\{A,A'\}$ and $A'\circ A$. Conditions (i) and (ii) are easy to check.

In order to prove (iii) (i.e. the local equi-semiconcavity), it suffices to prove the three relations (1.12). The first is obvious and the second follows from (1.1). For the third, let $\eta \in \Omega$. For any fixed z, each of the functions $(x,y) \mapsto A_\eta(y,z) + A'_\eta(z,x)$ is $\max\{sc(A_\eta), sc(A'_\eta)\}$-semiconcave on $M\times M$. This is a general property for functions on $M\times M$ which have the form $f(y)+g(x)$ with f and g semiconcave. Taking the minimum over z yields $(A'\circ A)_\eta$, without deteriorating the semiconcavity constant due to (1.1). This proves the third relation in (1.12).

As for the condition (iv), i.e. the $\mathcal{F}$-flow-type property, it is obvious for $A+\lambda$. Let us prove it for $\min\{A,A'\}$. Let $\eta \in \Omega$ and x, y be such that $\partial_y \min\{A,A'\}_\eta(y,x)$ and $\partial_x \min\{A,A'\}_\eta(y,x)$ exist. If $A_\eta(y,x) < A'_\eta(y,x)$ then locally $\min\{A,A'\}_\eta = A_\eta$ and the $\mathcal{F}$-flow-type property of

$\min\{A, A'\}$ reduces to the $\mathcal{F}$-flow-type property of A. Similarly if $A'_\eta(y,x) < A_\eta(y,x)$. In the remaining case in which $A_\eta(y,x) = A'_\eta(y,x) = \min\{A, A'\}_\eta(y,x)$, we have by semiconcavity

$$\partial_x \min\{A, A'\}_\eta(y,x) = \partial_x A_\eta(y,x) = \partial_x A'_\eta(y,x)$$
$$\partial_y \min\{A, A'\}_\eta(y,x) = \partial_y A_\eta(y,x) = \partial_y A'_\eta(y,x).$$

thus the $\mathcal{F}$-flow-type property of $\min\{A, A'\}$ reduces to the $\mathcal{F}$-flow-type property of A or A'. From these considerations we conclude that $\min\{A, A'\}$ is a $\mathcal{F}$-flow-type family. From the proof just carried out it is also apparent that, if A is of $N, \mathcal{F}$-flow-type and A' is of $N', \mathcal{F}$-flow-type, then $\min\{A, A'\}$ is of $\max\{N, N'\}, \mathcal{F}$-flow-type.

It remains to prove the $\mathcal{F}$-flow-type property for $A' \circ A$. Let η, y, x be such that $\partial_y (A' \circ A)_\eta(y,x)$ and $\partial_x (A' \circ A)_\eta(y,x)$ exist. Let z be a point of minimum in the expression

$$\big(A' \circ A\big)_\eta(y,x) = \min_{z \in M} \{A_\eta(y,z) + A'_\eta(z,x)\}.$$

Since $A_\eta(y,\cdot) + A'_\eta(\cdot,x)$ is the sum of two semiconcave functions, both $\partial_x A_\eta(y,z)$ and $\partial_y A'_\eta(z,x)$ exist and they satisfy

$$\partial_x A_\eta(y,z) + \partial_y A'_\eta(z,x) = 0.$$

Note also that the function $x' \mapsto (A' \circ A)_\eta(y,x') - A'_\eta(z,x')$ has a maximum at x. Since $\partial_x (A' \circ A)_\eta(y,x)$ exists by assumption and $-A'_\eta$ is semiconvex, we get that $\partial_x A'_\eta(z,x)$ exists and

$$\partial_x (A' \circ A)_\eta(y,x) - \partial_x A'_\eta(z,x) = 0.$$

By similar arguments, one gets that $\partial_y A_\eta(y,z)$ exists and

$$\partial_y (A' \circ A)_\eta(y,x) - \partial_y A_\eta(y,z) = 0.$$

Using the relations just derived and the $\mathcal{F}$-flow-type property of A and A', we finally get

$$\begin{aligned}\big(y, \eta_y - \partial_y (A' \circ A)_\eta(y,x)\big) &= \big(y, \eta_y - \partial_y A_\eta(y,z)\big) \vdash_{\mathcal{F}} \big(z, \eta_z + \partial_x A_\eta(y,z)\big) \\ &= \big(z, \eta_z - \partial_y A'_\eta(z,x)\big) \vdash_{\mathcal{F}} \big(x, \eta_x + \partial_x A'_\eta(z,x)\big) \\ &= \big(x, \eta_x + \partial_x (A' \circ A)_\eta(y,x)\big).\end{aligned}$$

This proves the $\mathcal{F}$-flow-type property for $A' \circ A$. From the proof just carried out it is also clear that, if A is of $N, \mathcal{F}$-flow-type and A' is of $N', \mathcal{F}$-flow-type, then $A' \circ A$ is of $(N+N'), \mathcal{F}$-flow-type. □

Note that (1.11) implies

$$\begin{gathered}\Phi_{A+\lambda} = \Phi_A, \qquad \Phi_{A' \circ A} = \Phi_{A'} \circ \Phi_A \\ \mathcal{I}_{A+\lambda}(\mathcal{G}) = \mathcal{I}_A(\mathcal{G}), \quad \mathcal{I}_{\min\{A,A'\}}(\mathcal{G}) \subseteq \mathcal{I}_A(\mathcal{G}) \cup \mathcal{I}_{A'}(\mathcal{G}), \quad \mathcal{I}_{A' \circ A}(\mathcal{G}) \subseteq \mathcal{I}_A(\mathcal{G}).\end{gathered} \tag{1.13}$$

Instead, we are not able to find an analogous formula for $\Phi_{\min\{A,A'\}}$. Let us notice that even if $\Phi_{A+\lambda} = \Phi_A$, the operation of adding a constant is not completely immaterial: it has a role for operators associated to costs such as $\min\{A + \lambda, A' + \lambda'\}$. If $\lambda' - \lambda$ is sufficiently big, then the corresponding operator will be Φ_A, and if $\lambda - \lambda'$ is sufficiently big, the operator will be $\Phi_{A'}$. Intermediate values of $\lambda' - \lambda$ will correspond to intermediate situations.

1.4.5 Weak KAM theory

In this subsection we consider the special case $\mathcal{F} = \{L\}$ and we rephrase in the language of pseudographs some standard results in weak Kam theory. Some of them have already been used in Proposition 7, and some others will be used in Section 1.5.

From now on we shall often use the symbol c to denote both a cohomology class and (with a little abuse of language) the 1-form $S(c)$. The context should avoid any confusion. Recall that $S(\cdot)$ is a once-for-all fixed linear map which associate to each cohomology class c a smooth closed 1-form having cohomology c. The particular choice of the map $S(\cdot)$ is immaterial for our purposes.

An important role in weak Kam theory is played by the so-called weak Kam solutions. There are several equivalent definitions for them. The one which we are going to use is: given a Tonelli Lagrangian L and a cohomology class c, a c-weak Kam solution for L is a solution $u \in C(M)$ of the equation

$$u = T_{A_{L,c}} u + \alpha_L(c),$$

where $A_{L,c}$ is the c-time-one action and $\alpha_L \colon H^1(M,\mathbb{R}) \to \mathbb{R}$ is Mather's α-function appeared in Subsection 1.4.2. A dual c-weak Kam solution is defined as a solution $u \in C(M)$ of the equation

$$u = \breve{T}_{A_{L,c}} u - \alpha_L(c)$$

We say that u is a weak Kam solution (resp. dual weak Kam solution) if it is a c-weak Kam solution (resp. dual weak Kam solution) for some c. The Weak Kam Theorem (cf. [Fat, Theorem 4.7.1]) states that for any Tonelli Lagrangian L and any cohomology c there exists at least one c-weak Kam solution and one c-dual weak Kam solution. In fact, $\alpha_L(c)$ is the unique constant such that the above equations admit a solution (assuming, as we do, that M is compact).

It is no surprise, in view of the definition of Φ_{A_L} in (1.6), that the language of pseudographs allows to concisely reformulate these concepts. From that definition it is indeed immediate that:

$$u \text{ is a } c\text{-weak Kam solution for } L \;\Leftrightarrow\; \mathcal{G}_{c,u} \text{ is a fixed point of } \Phi_{A_L}. \tag{1.14}$$

In view of this, we shall call weak Kam solutions as well the fixed points of Φ_{A_L}, and c-weak Kam solutions the fixed points in $\mathbb{P}_c$. Analogously for dual weak Kam solutions, with $\breve{\Phi}_A$ in place of Φ_A. Notice that two c-weak Kam solutions u and u' differing by a constant correspond to the same weak Kam solution $\mathcal{G}_{c,u} = \mathcal{G}_{c,u'}$.

Another important object in weak Kam theory is the Peierls barrier h_L, introduced in Subsection 1.4.2. Let us point out that

$$h_{L,c} = \lim_{n\to\infty} \lim_{m\to\infty} \min\{A^n_{L,c} + n\alpha, A^{n+1}_{L,c} + (n+1)\alpha, \dots, A^m_{L,c} + m\alpha\}, \tag{1.15}$$

and that, by Proposition 13, the families of costs appearing in the right-hand side are locally equi-semiconcave in the sense of Definition 10, with a local (in c) common bound for their semiconcavity constants. Hence, they have a local (in c) common bound for their Lipschitz constants. By the Ascoli-Arzelà theorem, this implies that the two limits are uniform (for any fixed c). Since uniform limits preserve semiconcavity constants, we get that the family of costs h_L is locally equi-semiconcave in the sense of Definition 10. In fact, by (1.12) we have

$sc(h_{L,c}) \leq sc(A_{L,c})$ for each c. Remark 11(iii) thus applies, i.e. $\Phi_{h_L}(\mathbb{P}_C)$ is relatively compact in $\mathbb{P}$ for every compact $C \subset H^1(M,\mathbb{R})$.

The next proposition reformulates in our language the well-known identities

$$\begin{aligned}
\min_{z\in M}\big\{h_{L,c}(y,z)+A_{L,c}(z,x)+\alpha_L(c)\big\} &= h_{L,c}(y,x),\\
\min_{z\in M}\big\{A_{L,c}(y,z)+h_{L,c}(z,x)+\alpha_L(c)\big\} &= h_{L,c}(y,x),\\
\min_{z\in M}\big\{h_{L,c}(y,z)+h_{L,c}(z,x)\big\} &= h_{L,c}(y,x) \qquad \forall\, y,x\in M.
\end{aligned}$$

Proposition 14. *Let $h_L \equiv \{h_{L,c}\}_c$ be the family of costs associated to the Peierls barrier of L. The following identities hold true:*

$$\begin{aligned}
\Phi_{A_L}\circ\Phi_{h_L} &= \Phi_{h_L}\\
\Phi_{h_L}\circ\Phi_{A_L} &= \Phi_{h_L}\\
\Phi_{h_L}\circ\Phi_{h_L} &= \Phi_{h_L}
\end{aligned}$$

This proposition has important consequences. Indeed, it implies the following characterizations of weak Kam solutions.

Proposition 15. *Let L be a Tonelli Lagrangian, $c\in H^1(M,\mathbb{R})$ and $u\colon M\to\mathbb{R}$ be a continuous function. The following are equivalent:*

(i) *u is a c-weak Kam solution for L;*

(ii) *$\mathcal{G}_{c,u}$ is a fixed point of Φ_{A_L};*

(iii) *$\mathcal{G}_{c,u}$ is a fixed point of Φ_{h_L};*

(iv) *$\mathcal{G}_{c,u}$ belongs to the image of Φ_{h_L}.*

The dual statement obtained by replacing 'c-weak Kam solution' with 'dual c-weak Kam solution' and Φ with $\check{\Phi}$ is also true.

Proof.
$(i)\Leftrightarrow(ii)$ has been already pointed out in (1.14);
$(iii)\Rightarrow(ii)$: let $\mathcal{G}$ be such that $\Phi_{h_L}(\mathcal{G})=\mathcal{G}$. We then have, by Proposition 14,

$$\Phi_{A_L}(\mathcal{G}) = \Phi_{A_L}\Phi_{h_L}(\mathcal{G}) = \Phi_{h_L}(\mathcal{G}) = \mathcal{G};$$

$(iii)\Rightarrow(iv)$ is obvious;
$(iv)\Rightarrow(iii)$: let $\mathcal{G}\in\Phi_{h_L}(E)$; then there exists $\mathcal{G}'\in E$ such that $\Phi_{h_L}(\mathcal{G}')=\mathcal{G}$. By Proposition 14,

$$\Phi_{h_L}(\mathcal{G}) = \Phi_{h_L}\Phi_{h_L}(\mathcal{G}') = \Phi_{h_L}(\mathcal{G}') = \mathcal{G};$$

$(ii)\Rightarrow(iii)$: for a given $\mathcal{G}\in\mathbb{P}$, the set of costs A such that $\Phi_A(\mathcal{G})=\mathcal{G}$ is closed under addition of constants, finite minima, compositions and uniform limits. From $\Phi_{A_L}(\mathcal{G})=\mathcal{G}$ and expression (1.15) we thus get $\Phi_{h_L}(\mathcal{G})=\mathcal{G}$.

The dual statement is proved analogously. □

Since the image of Φ_{A_L} is contained in $\mathbb{P}$ and the image of $\breve{\Phi}_{A_L}$ is contained in $\breve{\mathbb{P}}$, the previous proposition clearly implies that weak Kam solutions belong to $\mathbb{P}$ and dual weak Kam solutions belong to $\breve{\mathbb{P}}$. In $d = 1$, it is known that the non-contractible invariant circles are exactly the pseudographs which are both weak Kam solutions and dual weak Kam solutions.

The following proposition will be crucial in the proof of Proposition 33. As usual, H denotes the Tonelli Hamiltonian associated to L via the Fenchel-Legendre transform.

Proposition 16.

*(i) A weak Kam solution $\mathcal{G} \subset T^*M$ is subinvariant for ϕ_H^{-1} (*i.e. *$\phi_H^{-1}(\mathcal{G}) \subseteq \mathcal{G}$). A dual weak Kam solution is subinvariant for ϕ_H;*

(ii) if $\mathcal{G}$ is a weak Kam solution belonging to $\breve{\mathbb{P}}$, then automatically $\mathcal{G}$ is a dual weak Kam solution. Analogously, a dual weak Kam solution belonging to $\mathbb{P}$ is a weak Kam solution;

*(iii) if $\mathcal{G}$ is both a weak Kam solution and a dual weak Kam solution, then $\mathcal{G}$ is a Lipschitz graph over M which is invariant for ϕ_H (*i.e. *$\phi_H(\mathcal{G}) = \mathcal{G}$).*

Proof.

(i) Let $\mathcal{G}$ be a weak Kam solution, i.e. $\Phi_{A_L}(\mathcal{G}) = \mathcal{G}$. From Remark 11(iv) and Proposition 8(iv) we know that

$$\mathcal{G}_{|\mathcal{I}_{A_L}(\mathcal{G})} \vdash_{1,\{L\}} \Phi_{A_L}(\mathcal{G}),$$

hence we get

$$\mathcal{G} = \Phi_{A_L}(\mathcal{G}) \subseteq \phi_H\big(\mathcal{G}_{|\mathcal{I}_{A_L}(\mathcal{G})}\big).$$

Applying ϕ_H^{-1} to both sides we get $\phi_H^{-1}(\mathcal{G}) \subseteq \mathcal{G}$, that is the first claim of the statement. The dual claim is obtained analogously, starting from the dual version of Remark 11(iv).

(ii) Let $\mathcal{G}$ be a weak Kam solution belonging to $\breve{\mathbb{P}}$. We have

$$\breve{\Phi}_{A_L}(\mathcal{G}) \subseteq \phi_H^{-1}(\mathcal{G}) \subseteq \mathcal{G},$$

where the first inclusion follows from the dual version of Remark 11(iv), while the second inclusion follows from part (i) of this Proposition.

It is not difficult to prove that if a pseudograph is contained in another one, then the two must coincide. Thus the inclusion above implies $\breve{\Phi}_{A_L}(\mathcal{G}) = \mathcal{G}$, that is $\mathcal{G}$ is a dual weak Kam solution, as desired. The dual statement is analogous.

(iii) Let $\mathcal{G}$ be both a weak Kam and a dual weak Kam solution. It is immediate from part (i) that $\mathcal{G}$ is invariant for ϕ_H. Moreover, $\mathcal{G}$ has to belong to $\mathbb{P} \cap \breve{\mathbb{P}}$, hence it is a Lipschitz graph over M (recall that a function both semiconcave and semiconvex is $C^{1,1}$). □

We have just seen that a weak Kam solution $\mathcal{G}$ is subinvariant for ϕ_H^{-1}. Hence the sequence $\phi_H^{-n}(\mathcal{G})$ is decreasing in n. Moreover, one may prove (see [Ber08], or Proposition 23 in which we are going to prove some analogous statements in more general situations) that its intersection is a compact invariant set, and is given by

$$\bigcap_{n\in\mathbb{N}} \phi_H^{-n}(\mathcal{G}) = \mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})} = \mathcal{G}\tilde{\wedge}\breve{\Phi}_{h_L}(\mathcal{G}).$$

where the second equality is just a direct consequence of (1.8).

We now introduce the c-Aubry set of L, denoted by $\tilde{\mathcal{A}}_L(c)$, which appears in Proposition 7. One of the possible definitions is the following:

$$\tilde{\mathcal{A}}_L(c) = \bigcap \left\{\mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})} : \mathcal{G} \text{ is a } c\text{-weak Kam solution}\right\} \subseteq T^*M.$$

For a weak Kam solution $\mathcal{G}$, the subset of M given by $\mathcal{I}_{h_L}(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}_{h_L}(\mathcal{G})$ is also called the projected Aubry set of $\mathcal{G}$.

If $\mathcal{G} \in \mathbb{P}_c$ and $\mathcal{G}' \in \breve{\mathbb{P}}_c$, it is always true (see Section 1.2) that $\mathcal{G}\tilde{\wedge}\mathcal{G}' \subseteq T^*M$ is a compact set which is a Lipschitz graph over its projection $\mathcal{G} \wedge \mathcal{G}' \subseteq M$, hence the same holds true for each of the sets $\mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})} = \mathcal{G}\tilde{\wedge}\breve{\Phi}_{h_L}(\mathcal{G})$.

It is then clear that $\tilde{\mathcal{A}}_L(c)$ is a compact invariant Lipschitz graph over its projection too, being the intersection of compact invariant Lipschitz graphs. It is less obvious from this description, but true, that $\tilde{\mathcal{A}}_L(c)$ is non-empty.

Let us denote by $\mathbb{V}_L$ and $\breve{\mathbb{V}}_L$ respectively the sets of weak Kam solutions and dual weak solutions for L. The function Φ_{h_L} and $\breve{\Phi}_{h_L}$ are inverse to each other when restricted to these sets. More precisely,

$$\Phi_{h_L} \circ \breve{\Phi}_{h_L|\mathbb{V}_L} = \mathrm{id}, \qquad \breve{\Phi}_{h_L} \circ \Phi_{h_L|\breve{\mathbb{V}}_L} = \mathrm{id}.$$

This is due to the formulas (1.9). A pair of the type $(\mathcal{G}, \breve{\Phi}_{h_L}(\mathcal{G})) \in \mathbb{V} \times \breve{\mathbb{V}}$ is, up to a constant, a conjugate weak Kam pair in the sense of Fathi (see [Fat]). Indeed, we see from Proposition 9(v) that if u and $\breve{u}$ are such that $(\mathcal{G}_{c,u}, \mathcal{G}_{c,\breve{u}}) \in \mathbb{V} \times \breve{\mathbb{V}}$ and $\mathcal{G}_{c,\breve{u}} = \breve{\Phi}_{h_L}(\mathcal{G}_{c,u})$, then $u - \breve{u}$ is constant on the Aubry set of $\mathcal{G}$ (this constant is zero if we choose $\breve{u} = \breve{T}_{h_L} T_{h_L} u$).

The following property (which has been used in the proof of Proposition 7) tells us that weak Kam solutions may be seen as a sort of unstable manifolds of the Aubry set of L, and dual weak Kam solutions as stable manifolds.

Proposition 17. *For every c-weak kam solution $\mathcal{G}$ and every $z \in \mathcal{G}$, the α-limit of z for ϕ_H^1 is contained in $\tilde{\mathcal{A}}(c)$. Analogously, every point in a dual c-weak Kam solution is ω-asymptotic to $\tilde{\mathcal{A}}(c)$.*

Proof. See [Ber08, Proposition 4.3]. □

Let us now give one of the possible definitions of the Mather set $\tilde{\mathcal{M}}_L(c)$: it is the union of the supports of the invariant measures for ϕ_H^1 which are contained in $\tilde{\mathcal{A}}(c)$. It is a compact invariant set. Finally, the following is one of the possible definitions of the Mañé set $\tilde{\mathcal{N}}_L(c)$:

$$\tilde{\mathcal{N}}_L(c) = \bigcup \left\{\mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})} : \mathcal{G} \text{ is a } c\text{-weak Kam solution}\right\}.$$

This can be proved to be a compact invariant set as well. We have

$$\tilde{\mathcal{M}}_L(c) \subseteq \tilde{\mathcal{A}}_L(c) \subseteq \tilde{\mathcal{N}}_L(c) \subseteq T^*M.$$

We refer to [Ber02], [Fat] or [Mat91a] for a detailed analysis.

1.4.6 The semigroup Σ_c^∞

In this part we somehow generalize the previous subsection to the case of more than one Tonelli Hamiltonian. Let us recall that our final aim is to get informations about the forcing relation $\vdash_{\mathcal{F}}$, in order to apply Proposition 7. We have seen that $\mathcal{F}$-families of costs do give us informations about $\vdash_{\mathcal{F}}$, and that the time-one actions of the Tonelli Lagrangians in $\mathcal{F}$ are $\mathcal{F}$-families of costs. Moreover, the Proposition 13 tells us that being a $\mathcal{F}$-family is a property which is preserved by addition of constants, minimums and compositions. This motivates what follows.

Let σ be the unique class of families of costs such that:

(i) σ contains $\{A_L : L \in \mathcal{F}\}$;

(ii) σ is closed under the operations of addition of constants, minimum and composition (defined in Section 1.4.4);

(iii) σ is the smallest among those classes satisfying (i) and (ii).

By property (ii), σ is a semigroup under the operation of composition. Recall that the operator Φ_A associated to a $\mathcal{F}$-family of costs A is the one defined by

$$\Phi_A(\mathcal{G}_{\eta,u}) = \mathcal{G}_{\eta,T_{A_\eta}u}.$$

By Proposition 13 we immediately deduce:

Proposition 18. *Every family $A \in \sigma$ is a $\mathcal{F}$-family according to Definition 10. Hence, all the conclusions of Remark 11 apply to A, and in particular we have*

$$\mathcal{G}_{|\mathcal{I}_A(\mathcal{G})} \vdash_{\mathcal{F}} \Phi_A(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{P}.$$

Let us define

$$\Sigma = \{\Phi_A : A \in \sigma\}.$$

By the formula $\Phi_{A'} \circ \Phi_A = \Phi_{A' \circ A}$ it is clear that Σ is a semigroup with respect to the composition. Let now c belong to $H^1(M, \mathbb{R})$. Recall that we use the symbol A_c to denote the cost $A_{S(c)}$. We then define

$$\sigma_c = \{A_c : A \in \sigma\}$$

and call σ_c^∞ its closure in $C(M \times M)$:

$$\sigma_c^\infty = \mathrm{cl}\,(\sigma_c).$$

Let us stress that the elements of σ_c^∞ are just costs and not families of costs. It is clear that σ_c^∞ is the smallest class containing $\{A_{L,c} : L \in \mathcal{F}\}$ and closed under addition of constants, minimums,

compositions and uniform limits. In particular, it is a semigroup for the composition. Let us point out the important fact that the Peierls barriers $h_{L,c}$ belong to σ_c^∞ as well, since the limits involved in their definition are uniform (see the discussion after relation (1.15)).

In order to have good compactness properties, we will often make the assumption that $\mathcal{F}$ is equi-semiconcave, according to the following definition:

Definition 19. *We say that a family $\mathcal{F}$ of Tonelli Lagrangians is* equi-semiconcave *if, for every fixed $c \in H^1(M,\mathbb{R})$, the time-one actions $\{A_{L,c} : L \in \mathcal{F}\}$ form an equi-semiconcave set of functions on $M \times M$, that is*

$$\sup_{L\in\mathcal{F}} sc(A_{L,c}) < +\infty.$$

Of course, any finite family $\mathcal{F}$ of Tonelli Lagrangians is equi-semiconcave. If $\mathcal{F}$ is equi-semiconcave, then by the estimates (1.12) we have

$$\sup_{A\in\sigma_c^\infty} sc(A) = \sup_{L\in\mathcal{F}} sc(A_{L,c}) < +\infty.$$

hence σ_c^∞ is an equi-semiconcave set of functions. In particular, it is an equi-Lipschitz set of functions. By the Arzeli-Ascolà theorem, σ_c^∞ is then closed under pointwise limits. Being closed under minimums, it is also closed under countable inf and liminf, unless the resulting function is identically $\pm\infty$. Being a separable space, it is actually closed under arbitrary inf and liminf, unless the resulting function is identically $\pm\infty$. The following property is an immediate consequence of the Arzeli-Ascolà theorem:

Proposition 20. *If $\mathcal{F}$ is semiconcave, then σ_c^∞ modulo addition of constants is compact.*

Let us now fix $c \in H^1(M,\mathbb{R})$. Every element $A \in \sigma_c^\infty$ is a cost and not a family of costs, hence in general it is not associated to an operator from $\mathbb{P}$ to $\mathbb{P}$. Nevertheless, we can still define the operator $\Phi_A \colon \mathbb{P}_c \to \mathbb{P}_c$ by

$$\Phi_A(\mathcal{G}_{c,u}) = \mathcal{G}_{c,T_A u}.$$

Since the costs in σ_c^∞ are semiconcave, the image $\Phi_A(\mathbb{P}_c)$ is really contained in $\mathbb{P}_c$. Finally, we define

$$\Sigma_c^\infty = \{\Phi_A : A \in \sigma_c^\infty\},$$

which is a semigroup of operators from $\mathbb{P}_c$ to itself.

The semigroups σ_c^∞ and Σ_c^∞ will play a central role in the sequel. The next proposition states some of their useful properties. Note that item (*iii*) below is a sort of shadowing property. Recall that $|\cdot|$ indicates half the oscillation of a function.

Proposition 21. *Let $\mathcal{F}$ be equi-semiconcave.*

(i) For every $A, A' \in \sigma_c^\infty$ and $\mathcal{G}, \mathcal{G}' \in \mathbb{P}_c$, it holds

$$\|\Phi_A(\mathcal{G}) - \Phi'_{A'}(\mathcal{G}')\|_{\mathbb{P}} \le |A - A'| + \|\mathcal{G} - \mathcal{G}'\|_{\mathbb{P}} \tag{1.16}$$

and in particular every $\Phi \in \Sigma_c^\infty$ is 1-Lipschitz.

(ii) *The function $\mathcal{I}_A(\mathcal{G})$ is upper-semicontinuous in both $A \in \sigma_c^\infty$ and $\mathcal{G} \in \mathbb{P}_c$.*

(iii) *For all $\Phi_A \in \Sigma_c^\infty, \mathcal{G} \in \mathbb{P}_c$ and $\mathbb{U}$ neighborhood of $\Phi_A(\mathcal{G})$ in $\mathbb{P}_c$ there exists $\Phi' \in \Sigma$ such that $\Phi'(\mathcal{G}) \in \mathbb{U}$ (in particular $\mathcal{G} \vdash_{\mathcal{F}} \mathbb{U}$ by Proposition 18).*

Proof. Item (i) is analogous to the estimate (1.7). Item (ii) is an easy consequence of Proposition 9 (iv). For item (iii) note that, by the very definition of Σ_c^∞, there exists a sequence of costs $A_n \in \sigma_c$ converging to A as $n \to +\infty$. Item (iii) then follows from item (i) and the definition of Σ. □

In the next proposition we gather some properties of the minimal subsets of the dynamical system $(\mathbb{P}_c, \Sigma_c^\infty)$ which will be needed in the next section. We recall that a minimal subset is a compact subset of $\mathbb{P}_c$ which is stable by the semigroup Σ_c^∞ and which does not contain any proper subset with the same properties. For compact spaces the existence of minimal subsets is a standard Zorn's Lemma argument (actually, for compact metric spaces the Zorn's Lemma is not needed, see the proof in [HK02, Theorem 2.2.1]). Even if $\mathbb{P}_c$ is not compact, this argument can be easily adapted to our case, as the next proposition shows.

Let us remark that the existence of minimal components is the unique point in our construction for which the equi-semiconcavity of $\mathcal{F}$ seems to be crucial. In Corollary 34 this assumption will be eventually dropped for the case $d = 1$.

In the sequel we say that a subset $\mathbb{S}$ of $\mathbb{P}_c$ is *subinvariant* if $\Phi(\mathbb{S}) \subseteq \mathbb{S}$ for every $\Phi \in \Sigma_c^\infty$.

Proposition 22. *Assume $\mathcal{F}$ is equi-semiconcave. Then:*

(i) *for any $\mathcal{G} \in \mathbb{P}_c$ its orbit $\{\Phi(\mathcal{G}) : \Phi \in \Sigma_c^\infty\}$ is compact;*

(ii) *there exists a minimal set. In fact, the orbit of any $\mathcal{G} \in \mathbb{P}_c$ contains a minimal set;*

(iii) *$\mathcal{G} \in \mathbb{P}_c$ belongs to a minimal component $\mathbb{M}$ if and only if for every $\Phi \in \Sigma_c^\infty$ there exists $\Phi' \in \Sigma_c^\infty$ such that $\Phi'\Phi(\mathcal{G}) = \mathcal{G}$; in this case, $\mathbb{M}$ coincides with the orbit of $\mathcal{G}$. In particular, every minimal component $\mathbb{M}$ is transitive: for every $\mathcal{G}, \mathcal{G}' \in \mathbb{M}$ there exists $\Phi \in \Sigma_c^\infty$ such that $\Phi(\mathcal{G}) = \mathcal{G}'$.*

Proof.

(i) The orbit of a pseudograph $\mathcal{G} \in \mathbb{P}_c$ is the image of the map

$$\sigma_c^\infty \ni A \mapsto \Phi_A(\mathcal{G}).$$

This map is continuous by the estimate 1.16. In addition, $\Phi_{A+\lambda} = \Phi_A$ for every constant λ. By Proposition 20 we know that σ_c^∞ modulo addition of constants is compact, thus the image of the map is compact as well.

(ii) Given $\mathcal{G} \in \mathbb{P}_c$, its orbit is a subinvariant and compact set, by item (i). By a general result in topological dynamics, it contains a minimal set.

(iii) Let $\mathcal{G} \in \mathbb{M}$ with $\mathbb{M}$ minimal, and consider $\Phi \in \Sigma_c^\infty$. The orbit of $\Phi(\mathcal{G})$ contains a minimal component by (ii), and is contained in $\mathbb{M}$ because $\mathbb{M}$ is subinvariant. By minimality of $\mathbb{M}$, the orbit has to coincide with $\mathbb{M}$. Viceversa, suppose that for every $\Phi \in \Sigma_c^\infty$ there exists $\Phi' \in \Sigma_c^\infty$ such that $\Phi'\Phi(\mathcal{G}) = \mathcal{G}$. We know that the orbit of $\mathcal{G}$ contains a minimal set $\mathbb{M}$ by (ii). The assumption says that every subinvariant set contained in the orbit of $\mathcal{G}$ must contain $\mathcal{G}$ as well. We deduce that $\mathbb{M}$ coincides with the orbit of $\mathcal{G}$. □

In order to have a better understanding of the operators in Σ_c^∞ and the minimal components of $\mathbb{P}_c$, let us now further investigate about these objects in some special cases.

- *Case $\mathcal{F} = \{L\}$.* This is the case analyzed in Subsection 1.4.5. In addition to what already said there, one can show that in this case Σ_c^∞ is commutative, and that $\Phi\Phi_{h_{L,c}} = \Phi_{h_{L,c}}\Phi = \Phi_{h_{L,c}}$ for every $\Phi \in \Sigma_c^\infty$. Since the image of $\Phi_{h_{L,c}}$ coincides with its fixed points, it is then easy to verify that $\mathbb{M}$ is a minimal component if and only if $\mathbb{M} = \{\mathcal{G}\}$ for some c-weak Kam solution $\mathcal{G}$ for L.

- *Commuting Hamiltonians.* If the Hamiltonians in $\mathcal{F}$ commute with each other, i.e. their Poisson bracket satisfies

$$\{H, G\} + \partial_t H - \partial_t G = 0 \qquad \forall\, H, G \in \mathcal{F},$$

then it is known (see [Cui10] for the time-periodic case and [CL11, Zav10] for the autonomous case) that the associated Lax-Oleinik semigroups commute and that the Hamiltonians in the family share the same weak Kam solutions and the same Peierls barrier which we denote $\{h_c\}_c$. Thus Σ_c^∞ is commutative and $\Phi\Phi_{h_c} = \Phi_{h_c}\Phi = \Phi_{h_c}$ for every $\Phi \in \Sigma_c^\infty$. It is then easy to verify that the minimal components are exactly the c-weak Kam solutions for one (hence all) Hamiltonian in $\mathcal{F}$.

- *General case.* For every $\Phi_A \in \Sigma_c^\infty$ it is possible to define an analogous of the Peierls barrier. Indeed, arguing as for the case $A = A_{L,c}$, one can show (see [Zav12]) that there exists a unique real number α_A such that the liminf

$$h_A = \liminf_{n \to +\infty} A^n + n\alpha_A \tag{1.17}$$

is real-valued. Exactly as for the Peierls barrier, we have $\Phi_{h_A} \in \Sigma_c^\infty$, and analogous statements to Propositions 14 and 15*(ii)-(iii)-(iv)* hold. In particular the image of Φ_{h_A} coincides with its fixed points and with the fixed points of Φ_A.

The interpretation of an arbitrary operator in Σ_c^∞ in terms of Hamiltonian dynamics is not easy. However, something can be said for particular operators. As a sample, let us pick two Hamiltonians H_1 and H_2 in $\mathcal{F}$, and call A_1, A_2 their time-one actions and h_1, h_2 their Peierls barriers. In the next two propositions we prove some properties of $\Phi_{A_2 \circ A_1}$ and $\Phi_{h_2 \circ h_1}$.

Proposition 23. *Let $H_1, H_2 \in \mathcal{F}$, and call A_1, A_2 their time-one actions and $\phi_1, \phi_2 \colon T^*M \to T^*M$ their time-one maps. Let us also denote $A = A_2 \circ A_1$ and $\phi = \phi_2 \circ \phi_1$. Let us consider the operator $\Phi_A \colon \mathbb{P} \to \mathbb{P}$. The following hold true:*

(i) for every $\mathcal{G} \in \mathbb{P}$ *and every* $n \in \mathbb{N}$ *it holds*

$$\phi^{-n}\big(\Phi_{A^n}(\mathcal{G})\big) \subseteq \mathcal{G}_{|\mathcal{I}_{A^n}(\mathcal{G})};$$

(ii) the fixed points $\mathcal{G}$ *of* Φ_A *are subinvariant for* ϕ^{-1}*; more precisely, they satisfy*

$$\phi^{-n}(\mathcal{G}) \subseteq \mathcal{G}_{|\mathcal{I}_{A^n}(\mathcal{G})};$$

(iii) for every fixed point $\mathcal{G}$ *of* Φ_A*, the set* $\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}$ *is invariant with respect to* ϕ*,* i.e.

$$\phi\left(\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}\right) = \mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})};$$

(iv) for every fixed point $\mathcal{G}$ *of* Φ_A*, every point in* $\mathcal{G}$ *is* α*-asymptotic to* $\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}$ *with respect to* ϕ*.*

Proof.

(i) This is a more precise version of the relation

$$\mathcal{G}_{|\mathcal{I}_{A^n}(\mathcal{G})} \ \vdash_{\mathcal{F}} \ \Phi_{A^n}(\mathcal{G})$$

of Remark 11 (*iv*). It follows by a refinement of the proof of Proposition 12, using property (*iv*) in Proposition 8 and the special form of A (*i.e.* $A = A_2 \circ A_1$);

(ii) it is immediate from item (*i*) since $\Phi_{A^n}(\mathcal{G}) = \mathcal{G}$ for all $n \in \mathbb{N}$;

(iii) let $\mathcal{G}$ be a fixed point of Φ_A. It is easy to check that the set $\cap_n \phi^{-n}(\mathcal{G})$ is invariant. Hence it suffices to show that this intersection is equal to $\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}$. With this aim, let us first notice that

$$\mathcal{I}_{h_A}(\mathcal{G}) = \bigcap_n \mathcal{I}_{A^n}(\mathcal{G}). \tag{1.18}$$

Indeed, from $h_A \circ A^n = h_A$ and relations 1.13, it follows that the left-hand side is included in the right-hand side. Note that the relations 1.13 also imply that the sequence $\mathcal{I}_{A_n}(\mathcal{G})$ is decreasing in n. For the reverse inclusion, write $\mathcal{G} = \mathcal{G}_{c,u}$, consider $\bar{y}$ belonging to the intersection in the right-hand side and let $x_n \in M$ be such that $u(x_n) = u(\bar{y}) + A^n(\bar{y}, x_n) + n\alpha_A$. Then by definition of h_A every accumulation point x of the sequence x_n satisfies

$$u(x) \geq u(\bar{y}) + h_A(\bar{y}, x).$$

Since $\mathcal{G}$ is a fixed point of Φ_A, we also have $u(x) = \min_y\{u(y) + h_A(y, x)\}$. We deduce that the minimum has to be achieved in $\bar{y}$, and thus $\bar{y} \in \mathcal{I}_{h_A}(\mathcal{G})$. This proves (1.18). In order to conclude the proof of item (*iii*), it suffices to prove that

$$\bigcap_n \phi^{-n}(\mathcal{G}) = \mathcal{G}_{|\bigcap_n \mathcal{I}_{A^n}(\mathcal{G})}.$$

The left-hand side is included in the right-hand side by item (*ii*). The reverse inclusion follows from the fact that, if $z \in \mathcal{G}_{|\mathcal{I}_{A^{n+1}}(\mathcal{G})}$, then $\phi^n(z) \in \mathcal{G}$. This follows from property (*iv*) in Proposition 8 and a refinement of the proof of Proposition 12.

(iv) Let $z \in \mathcal{G}$. By equation 1.18, it suffices to prove that any α-limit of z lies in $\mathcal{G}_{|\mathcal{I}_{A^N}(\mathcal{G})}$ for every $N \in \mathbb{N}$. Since $\mathcal{G}_{|\mathcal{I}_{A^N}(\mathcal{G})}$ is a closed set, it suffices to prove that $\phi^{-n}(z) \in \mathcal{G}_{|\mathcal{I}_{A^N}(\mathcal{G})}$ for n large enough. This is indeed true for $n \geq N$ by item (ii). □

Proposition 24. *Let $H_1, H_2 \in \mathcal{F}$, and call h_1, h_2 their Peierls barriers. Let us fix $c \in H^1(M,\mathbb{R})$ and denote $A_c = h_{2,c} \circ h_{1,c}$. Let us consider the operator $\Phi_{A,c} \in \Sigma_c^\infty$ and the subsets $\mathbb{V}_{1,c}, \mathbb{V}_{2,c}, \mathbb{V}_{A,c} \subset \mathbb{P}_c$ constituted respectively by the fixed points of $\Phi_{h_{1,c}}, \Phi_{h_{2,c}}$ and $\Phi_{A,c}$. Then $\mathbb{V}_{A,c}$ is contained in $\mathbb{V}_{2,c}$ and is isometric to a subset of $\mathbb{V}_{1,c}$.*

Proof. Obviously $\mathbb{V}_{A,c}$ is contained in the image of $\Phi_{A,c}$, which is contained in the image of $\Phi_{h_{2,c}}$, that is $\mathbb{V}_{2,c}$. Moreover, since $\Phi_{A,c} = \Phi_{h_{2,c}} \circ \Phi_{h_{1,c}}$ and $\Phi_{A,c}$ is the identity when restricted to $\mathbb{V}_{A,c}$, we get that $\Phi_{h_{2,c}}$ is a left inverse for $\Phi_{h_{1,c}}$ on $\mathbb{V}_{A,c}$. Since both of them are 1-Lipschitz (cf. Proposition 21 (ii)), $\Phi_{h_{1,c}}$ has to be an isometry between $\mathbb{V}_{A,c}$ and $\Phi_{h_{1,c}}(\mathbb{V}_{A,c})$, which is a subset of $\mathbb{V}_{1,c}$. □

Let us point out that, if $d = 1$, the whole of Σ_c^∞ would not be needed for the purposes of this article. Indeed, the heuristic discussion in Section 1.5.3 as well as the proof of Proposition 33 show that the Peierls barrier operators $\Phi_{h_L}, L \in \mathcal{F}$, would suffice to get optimal results. Nevertheless, if $d > 1$, considering the whole of Σ_c^∞ gives stronger (though more abstract) results.

1.5 The Mather mechanism

Throughout the whole section, the family $\mathcal{F}$ is assumed to be equi-semiconcave in the sense of Definition 19, unless otherwise stated. For a subset $S \subseteq M$, we call $S^\perp \subseteq \Omega$ the vector subspace of the smooth closed one-forms whose support is disjoint from S and $[S^\perp]$ its projection on $H^1(M,\mathbb{R})$. It follows from the finite dimensionality of $H^1(M,\mathbb{R})$ that there always exists an open set $U \supseteq S$ such that $[U^\perp] = [S^\perp]$. Such a U will be called an *adapted neighborhood* of S. Let us point out that, if $M = \mathbb{T}$, we have $[S^\perp] = \{0\}$ if and only if $S = \mathbb{T}$, and otherwise $[S^\perp] = H^1(\mathbb{T},\mathbb{R}) \cong \mathbb{R}$. For a vector subspace $V \subseteq H^1(M,\mathbb{R})$, we denote the ε-radius ball centered at the origin by $B_\varepsilon(V)$.

1.5.1 Outline of Section 1.5

We describe a mechanism for the construction of diffusion polyorbits. When $\mathcal{F} = \{L\}$ is a singleton, our construction essentially boils down to the one in [Ber08]. Some of the main ideas come from the seminal paper of Mather [Mat93].

Subsection 1.5.2. We prove the technical results which are at the core of the Mather mechanism. Basically, they show how a pseudograph $\mathcal{G}$ may force nearby cohomologies, with the sets $\mathcal{I}_A(\mathcal{G}), A \in \sigma_c^\infty$ acting as obstructions to this phenomenon.

Subsection 1.5.3. We heuristically show how the results of the previous subsection apply to polysystems of twist maps on the cylinder.

Subsection 1.5.4. We apply the results of Subsection 1.5.2 to prove a general theorem in arbitrary dimension. We discuss some consequences and applications, including the rigorous counterpart of the heuristic picture in Subsection 1.5.3.

Loosely speaking, the mechanism works in the following way: we will be able to associate to every $c \in H^1(M,\mathbb{R})$ a subspace $R(c) \subseteq H^1(M,\mathbb{R})$ of "allowed cohomological directions" for the forcing relation $\dashv\vdash_{\mathcal{F}}$. In view of Proposition 7, this gives allowed cohomological directions for the diffusion: the larger the subspace $R(c)$ is, the more are the directions for which connecting orbits starting at c exist. The obstruction for this subspace to be large will be, roughly, the homological size of the sets $\mathcal{I}_A(\mathcal{G})$, for $\mathcal{G} \in \mathbb{P}_c$ and $A \in \sigma_c^\infty$.

1.5.2 The basic step

Let us introduce some notations: for $c \in H^1(M,\mathbb{R})$, $\mathcal{G} \in \mathbb{P}_c$, $A \in \sigma_c^\infty$, we define

$$R_A(\mathcal{G}) = [\mathcal{I}_A(\mathcal{G})^\perp] = [\mathcal{G} \wedge \breve{\Phi}_A \Phi_A(\mathcal{G})^\perp] \subseteq H^1(M,\mathbb{R}).$$

Here the second equality follows from 1.8. More generally, for $A_1,\dots,A_n \in \sigma_c^\infty$, we define

$$R_{A_1,\dots,A_n}(\mathcal{G}) = \Big[\mathcal{I}_{A_n\circ\cdots\circ A_1}(\mathcal{G})^\perp + \mathcal{I}_{A_n\circ\cdots\circ A_2}(\Phi_{A_1}(\mathcal{G}))^\perp + \cdots + \mathcal{I}_{A_n}\big(\Phi_{A_{n-1}}\circ\cdots\circ\Phi_{A_1}(\mathcal{G})\big)^\perp\Big].$$

We will see in Lemma 27 that the subspace $R_{A_1,\dots,A_n}(\mathcal{G})$ should be intended as a subspace of "allowed cohomological directions for the forcing relation, through the composition $\Phi_{A_n}\circ\cdots\circ\Phi_{A_1}$, starting from $\mathcal{G}$". By taking the union over all finite strings $(A_1,\dots,A_n)$, one should get a space of "allowed cohomological directions for the forcing relation starting from $\mathcal{G}$". Afterward, by intersecting over all $\mathcal{G}$ in $\mathbb{P}_c$, one should get a space of "allowed cohomological directions for the forcing relation starting from c", which is basically what we are looking for in order to apply Proposition 7. This motivates the following definitions:

$$R(\mathcal{G}) = \bigcup_{\substack{A_1,\dots,A_n\in\sigma_c^\infty\\ n\in\mathbb{N}}} R_{A_1,\dots,A_n}(\mathcal{G})$$
$$R(c) = \bigcap_{\mathcal{G}\in\mathbb{P}_c} R(\mathcal{G}) \tag{1.19}$$

At this stage it is not clear whether $R(\mathcal{G})$ or $R(c)$ are vector subspaces. In Proposition 29 several equivalent expressions for $R(c)$ will be given. They will imply that $R(c)$ is indeed a vector subspace, and $R(\mathcal{G})$ is a vector subspace for every $\mathcal{G}$ in a minimal component of $\mathbb{P}_c$. We shall write $R_{\mathcal{F}}(c)$ when we want to emphasize the dependence on the family $\mathcal{F}$.

The following lemma is the basic key step in the accomplishment of the Mather mechanism. Given a family of costs $A \in \sigma$,[3] the lemma shows how a pseudograph $\mathcal{G}$ may force nearby cohomologies, with the set $\mathcal{I}_A(\mathcal{G})$ acting as an obstruction to this phenomenon. Furthermore, the semicontinuity in $\mathcal{G}$ of $\mathcal{I}_A(\mathcal{G})$ allows to extend the conclusion to a whole neighborhood of $\mathcal{G}$.

[3] We recall that the definition of σ is given in Section 1.4.6.

Lemma 25. *Let A be a $\mathcal{F}$-family of costs according to the Definition 10 (in particular, $A \in \sigma$ will work). Let Φ_A be the associated operator on pseudographs. Then, for every $\mathcal{G} \in \mathbb{P}$ and for every neighborhood $\mathbb{U}$ of $\Phi_A(\mathcal{G})$ in $\mathbb{P}$ there exist $N \in \mathbb{N}$, a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and an $\varepsilon > 0$ such that:*

$$\forall\, \mathcal{G}' \in \mathbb{W}\,, \quad \forall\, c \in c(\mathcal{G}') + B_\varepsilon R_A(\mathcal{G}) \qquad \exists\, \mathcal{G}'' \quad \text{such that}$$
$$\mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \vdash_{N,\mathcal{F}} \mathcal{G}'', \qquad c(\mathcal{G}'') = c.$$

Proof. Let us fix $\mathcal{G}$, $\mathbb{U}$ and an adapted neighborhood U of $\mathcal{I}_A(\mathcal{G})$. The set function $\mathcal{G} \mapsto \mathcal{I}_A(\mathcal{G})$ is upper semicontinuous, thus there exists a neighborhood $\mathbb{W}'$ of $\mathcal{G}$ such that $\mathcal{I}_A(\mathcal{G}') \subseteq U$ for all $\mathcal{G}' \in \mathbb{W}'$. Moreover, by continuity of Φ_A, we can suppose that $\Phi_A(\mathbb{W}') \subseteq \mathbb{U}$. The function

$$\mathbb{P} \times U^\perp \ni (\mathcal{G}, \nu) \mapsto \mathcal{G} + \mathcal{G}_{\nu,0}$$

is continuous, hence there exists a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and a neighborhood W of 0 in $U^\perp$ such that $\mathbb{W} + \mathcal{G}_{W,0} \subseteq \mathbb{W}'$. Projections are open maps, thus the projection of W on the cohomology contains a ball $B_\varepsilon[U^\perp]$ centered at 0. With these choices of $\mathbb{W}$ and ε, let $\mathcal{G}' \in \mathbb{W}$ and $c \in c(\mathcal{G}') + B_\varepsilon[U^\perp]$. We can then take as $\mathcal{G}''$ the pseudograph $\Phi_A(\mathcal{G}' + \mathcal{G}_{\nu,0})$ where $\nu \in W$ satisfies $[\nu] = c - c(\mathcal{G}')$. Indeed, by Remark 11(iv) we find N such that

$$\mathcal{G}' \vdash_{0,\mathcal{F}} \mathcal{G}'_{|U} = \big(\mathcal{G}' + \mathcal{G}_{\nu,0}\big)_{|U} \vdash_{N,\mathcal{F}} \Phi_A\big(\mathcal{G}' + \mathcal{G}_{\nu,0}\big) = \mathcal{G}''. \qquad \square$$

The Lemma 25 easily extends to operators in Σ_c^∞.

Proposition 26. *Let $\Phi_A \in \Sigma_c^\infty$. Then, for every $\mathcal{G} \in \mathbb{P}_c$ and for every neighborhood $\mathbb{U}$ of $\Phi_A(\mathcal{G})$ in $\mathbb{P}$ there exist $N \in \mathbb{N}$, a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and an $\varepsilon > 0$ such that:*

$$\forall\, \mathcal{G}' \in \mathbb{W}\,, \ \forall\, c \in c(\mathcal{G}') + B_\varepsilon R_A(\mathcal{G}) \qquad \exists\, \mathcal{G}'' \quad \text{such that}$$
$$\mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \vdash_{N,\mathcal{F}} \mathcal{G}'', \qquad c(\mathcal{G}'') = c.$$

Proof. Let us fix $\mathcal{G}$ and $\mathbb{U}$, and let us consider $\mathcal{I}_A(\mathcal{G})$ and one of its adapted neighborhoods U. By Proposition 21 (iii) there exists $A' \in \sigma$ such that $\Phi_{A'}(\mathcal{G}) \in \mathbb{U}$ and $\mathcal{I}_{A'}(\mathcal{G}) \subseteq U$. This implies

$$R_{A'}(\mathcal{G}) = [\mathcal{I}_{A'}(\mathcal{G})^\perp] \supseteq [U^\perp] = [\mathcal{I}_A(\mathcal{G})^\perp] = R_A(\mathcal{G}).$$

We apply the Lemma 25 and we get the result. $\square$

In the following lemma we prove two similar results which show how Proposition 26 has a good behavior under composition. The second version is in principle stronger but we will see that the first version would eventually lead to the same results, at least for an equi-semiconcave family $\mathcal{F}$. Therefore *a posteriori* the second version is not strictly needed here.

The main point in both results is that, if we compose several operators in Σ_c^∞, the set of allowed directions which we get is greater than just the union of the allowed directions obtained by applying separately Proposition 26 to each operator. In fact, we obtain the vector subspace generated by this union.

Lemma 27. *Let $\Phi_{A_1}, \dots, \Phi_{A_n} \in \Sigma_c^\infty$. Then: for every $\mathcal{G} \in \mathbb{P}$ and for every neighborhood $\mathbb{U}$ of $\Phi_{A_n} \circ \cdots \circ \Phi_{A_1}(\mathcal{G})$ in $\mathbb{P}$ there exist $N \in \mathbb{N}$, a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and an $\varepsilon > 0$ such that:*

$$\begin{aligned} \forall\, \mathcal{G}' \in \mathbb{W}, \quad \forall\ c \in\ & c(\mathcal{G}') + B_\varepsilon R_{A_1,\dots,A_n}(\mathcal{G}) \\ & \exists\, \mathcal{G}'' : \quad \mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \vdash_{N,\mathcal{F}} \mathcal{G}'', \qquad c(\mathcal{G}'') = c. \end{aligned}$$

Proof. Let us suppose $n = 2$ for simplicity. Let us consider $\mathcal{G} \in \mathbb{P}$ and a neighborhood $\mathbb{U}$ of $\Phi_{A_2}\Phi_{A_1}(\mathcal{G})$. Let U_1 and U_2 be adapted neighborhoods in M of $\mathcal{I}_{A_2 \circ A_1}(\mathcal{G})$ and $\mathcal{I}_{A_2}(\Phi_{A_1}(\mathcal{G}))$ respectively. By Proposition 21, there exist $A_1', A_2' \in \sigma$ and a neighborhood $\mathbb{W}'$ of $\mathcal{G}$ in $\mathbb{P}$ such that

$$\Phi_{A_2'}\Phi_{A_1'}(\mathcal{G}') \in \mathbb{U}, \quad \mathcal{I}_{A_2' \circ A_1'}(\mathcal{G}') \subseteq U_1 \quad \text{and} \quad \mathcal{I}_{A_2'}(\Phi_{A_1'}(\mathcal{G}')) \subseteq U_2 \qquad \forall\, \mathcal{G}' \in \mathbb{W}'.$$

Let us now consider $\eta_1 \in U_1^\perp$ and $\eta_2 \in U_2^\perp$. Given $\mathcal{G}' = \mathcal{G}_{\eta,u} \in \mathbb{W}'$, we have

$$\Phi_{A_2'}\big(\Phi_{A_1'}(\mathcal{G}' + \mathcal{G}_{\eta_1,0}) + \mathcal{G}_{\eta_2,0}\big) = \mathcal{G}_{\eta+\eta_1+\eta_2,v}$$

with

$$v := T_{A'_{2,\eta+\eta_1+\eta_2}} w \qquad w := T_{A'_{1,\eta+\eta_1}} u.$$

Let $x \in M$ be a point such that dv_x exists. By Proposition 12, if z is a point which realizes the minimum in the formula for $v(x)$, then dw_z exists and

$$dw_z + \eta_z + \eta_{1,z} + \eta_{2,z} \quad \vdash_{N_2,\mathcal{F}} \quad dv_x + \eta_x + \eta_{1,x} + \eta_{2,x}$$

for some $N_2 \in \mathbb{N}$. In the same way, if y realizes the minimum in the formula for $w(z)$, then

$$du_y + \eta_y + \eta_{1,y} \quad \vdash_{N_1,\mathcal{F}} \quad dw_z + \eta_z + \eta_{1,z}$$

for some $N_1 \in \mathbb{N}$.

A generalization of the upper-semicontinuity result in 11 (*ii*) shows that if $[\eta_1 + \eta_2] \in B_\varepsilon[U_1^\perp + U_2^\perp]$ with ε small enough, then $y \in U_1$ and $z \in U_2$. We thus have $\eta_{1,y} = 0$ and $\eta_{2,z} = 0$ and therefore

$$du_y + \eta_y \quad \vdash_{N_1+N_2,\mathcal{F}} \quad dv_x + \eta_x + \eta_{1,x} + \eta_{2,x}$$

which is to say

$$\mathcal{G}' = \mathcal{G}_{\eta,u} \quad \vdash_{N_1+N_2,\mathcal{F}} \quad \mathcal{G}_{\eta+\eta_1+\eta_2,v}.$$

The proof is now completed with $\mathcal{G}'' = \mathcal{G}_{\eta+\eta_1+\eta_2,v}$, up to choosing $\mathcal{G}'$ in a smaller neighborhood $\mathbb{W} \subseteq \mathbb{W}'$ in such a way that ε can be fixed independently of $\mathcal{G}'$. □

Note that, trivially, Lemma 25 is a particular case of Proposition 26, which in turn is a particular case of Lemma 27, hence just the latter will be used in the sequel.

1.5.3 Heuristic application to twist maps

Even without the main general theorem of the next subsection (Theorem 31), it is possible at this stage, using just the Lemma 27, to derive some results about the presence of diffusion in polysystems of exact-symplectic twist maps on the cylinder. The discussion in this subsection will just be an heuristic one, even if everything could be made rigorous. The corresponding rigorous results will be proven in greater generality in the next subsection (Proposition 33 and Corollary 34).

Let $\mathcal{F} = \{H_1, H_2\}$ be a family of two Tonelli Hamiltonians on $\mathbb{T} \times \mathbb{R}$ and call L_1, L_2 the corresponding Lagrangians. Let us fix $c \in H^1(M, \mathbb{R})$. We now show that just two scenarios are possible:

- there exists a circle of cohomology c which is invariant for both H_1 and H_2 (which obviously provides an obstruction to diffusion); the only cohomology class forced by c is c itself; $R(c) = \{0\}$;
- there is no circle as above; in this case c forces a whole neighborhood of cohomology classes (and thus there exists diffusion in the sense of Proposition 7); $R(c) = H^1(\mathbb{T}, R) \cong \mathbb{R}$.

Indeed, suppose that there exists a circle of cohomology c which is invariant for both H_1 and H_2. It is standard that it can be identified with a pseudograph $\mathcal{G}$ which is invariant for both ϕ_{H_1} and ϕ_{H_2}. In particular, by the very definition of forcing relation in Section 1.3, $\mathcal{G}$ is the only pseudograph forced by $\mathcal{G}$, and thus c is the only cohomology class forced by c. Moreover, we deduce by Lemma 27 and the definition of $R(\mathcal{G})$ that $R(\mathcal{G}) = \{0\}$, thus $R(c) = \{0\}$.

On the other hand, let us suppose that there is no such common invariant circle. Let us consider $\mathcal{G} \in \mathbb{P}_c$, and let us apply $\Phi_{h_2} \circ \Phi_{h_1}$ to it (h_1 and h_2 are the Peierls barrier of H_1, H_2). By the first version of Lemma 27, we get

$$\mathcal{G} \vdash_{\mathcal{F}} c + B_\varepsilon\big(R_{h_1}(\mathcal{G}) + R_{h_2}(\Phi_{h_1}(\mathcal{G}))\big) \qquad \forall\, \mathcal{G} \in \mathbb{P}_c. \tag{1.20}$$

Recall that the image of Φ_{h_1} is contained in $\mathbb{P}$ and consists precisely of the weak Kam solutions for H_1, while the image of $\breve{\Phi}_{h_2}$ is contained in $\breve{\mathbb{P}}$ and consists of the dual weak Kam solutions for H_2. By assumption there are no common invariant circles, hence, in view of Proposition 16,

$$\Phi_{h_1}(\mathcal{G}) \neq \breve{\Phi}_{h_2}\Phi_{h_2}\Phi_{h_1}(\mathcal{G}).$$

This implies (due to $d = 1$) that $R_{h_2}(\Phi_{h_1}(\mathcal{G})) = H^1(\mathbb{T}, \mathbb{R})$ regardless of $\mathcal{G} \in \mathbb{P}_c$. Thus the formula (1.20) implies that every $\mathcal{G} \in \mathbb{P}_c$ forces a whole neighborhood of cohomology classes. In that formula, ε depends in principle on $\mathcal{G}$, but one can show that by compactness it is possible to choose it uniformly in $\mathcal{G}$. Therefore c forces a whole neighborhood of cohomology classes, as claimed. Since $R(\mathcal{G}) \supseteq R_{h_2}(\Phi_{h_1}(\mathcal{G}))$, the discussion also proves that $R(\mathcal{G}) = H^1(\mathbb{T}, \mathbb{R})$ for every $\mathcal{G} \in \mathbb{P}_c$, thus $R(c) = H^1(\mathbb{T}, \mathbb{R})$.

Notice how in this one-dimensional case our construction is optimal, in the following sense: the obstructions to the mechanism (i.e. the "homological size" of the sets $\mathcal{I}_A(\mathcal{G})$) are real obstructions to the diffusion (i.e. the common invariant circles). On the contrary, in $d > 1$ the construction will likely give just sufficient conditions for the diffusion: the obstructions to this mechanism may be circumvented by a different diffusion mechanism (such as, for instance, the Arnold mechanism presented by Bernard in [Ber08]).

1.5.4 A general theorem and some applications

We can summarize the argument used in Subsection 1.5.3 for the case $d = 1$ by saying that we have applied Lemma 27 to $\Phi_{h_2} \circ \Phi_{h_1}$, and the result turned out to be optimal (so that there was no need to consider any other $\Phi \in \Sigma_c^\infty$). Moreover, switching the order and considering $\Phi_{h_1} \circ \Phi_{h_2}$ would have led to the same result. The generalization of this argument to an arbitrary dimension d is not completely straightforward: the choice of the operator could in principle make a difference, and it is less clear if the allowed directions which one obtains are optimal or not.

In order to overcome these difficulties, we will adopt a slightly more abstract approach. This will give stronger conclusions, at the cost of a certain difficulty to interpret the obstructions which we will found.

We start with a "raw" result which follows immediately from Lemma 27.

Proposition 28. *Let c be fixed. For any $\mathcal{G} \in \mathbb{P}_c$ and any finite string $s = (A_1, \dots, A_n)$ of elements of σ_c^∞, there exist $\varepsilon(\mathcal{G}, s) > 0$ such that*

$$c \vdash_{\mathcal{F}} c + \bigcap_{\mathcal{G} \in \mathbb{P}_c} \bigcup_{\substack{s=(A_1,\dots,A_n) \\ n \in \mathbb{N}}} B_{\varepsilon(\mathcal{G},s)}\big(R_{A_1,\dots,A_n}(\mathcal{G})\big). \tag{1.21}$$

Proof. Recall that $c \vdash_{\mathcal{F}} c'$ if and only if $\mathcal{G} \vdash_{\mathcal{F}} c'$ for all $\mathcal{G} \in \mathbb{P}_c$. The result is then a consequence of Lemma 27. □

The general theorem (Theorem 31) will consist in a refined (but at the same time simplified) version of this raw result. Roughly speaking, it will be possible to replace the intersection over $\mathcal{G} \in \mathbb{P}_c$ with an intersection over a smaller set, to replace the union with a sum of vector subspaces and to choose ε uniformly in $(\mathcal{G}, s)$. This will simplify the right-hand side, and will lead in the end to a unique subspace of $H^1(M, \mathbb{R})$ encoding all the information. In fact, $R(c)$ will be such a subspace. Moreover, exploiting some semicontinuity, the result will be proved to hold for c' close enough to c. It will also be possible to replace the forcing relation $\vdash_{\mathcal{F}}$ with the mutual forcing relation $\dashv\vdash_{\mathcal{F}}$, and to have a locally uniform control on the N appearing in its definition.

In order to motivate what follows, let us observe that the map $\mathcal{G} \mapsto R(\mathcal{G})$ is by definition non-increasing along the action of elements of Σ_c^∞. More precisely,

$$R(\Phi(\mathcal{G})) \subseteq R(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{P}_c, \Phi \in \Sigma_c^\infty. \tag{1.22}$$

This can be interpreted by saying that this map is a sort of multi-valued Lyapunov function for the dynamics in $(\mathbb{P}_c, \Sigma_c^\infty)$. Since we are interested in the set $R(c)$, which is the intersection of all the sets $R(\mathcal{G})$, it is natural to look at the minimal components of the dynamics, whose properties have been analysed in Proposition 22.

For a minimal component $\mathbb{M}$ of $(\mathbb{P}_c, \Sigma_c^\infty)$ let us define

$$R(\mathbb{M}) = \bigcap_{\mathcal{G} \in \mathbb{M}} R(\mathcal{G}).$$

Proposition 29 (Equivalent expressions for $R(c)$)**.**

(i) We have:

$$R(c) = \bigcap_{\mathbb{M}\ minimal} R(\mathbb{M}).$$

(ii) We have the following equivalent expressions for $R(\mathbb{M})$:

$$\begin{aligned} R(\mathbb{M}) = R(\mathcal{G}) &= \sum_{\substack{A_1,\dots,A_n\in\sigma_c^\infty \\ n\in\mathbb{N}}} R_{A_1,\dots,A_n}(\mathcal{G}) \qquad \text{for any fixed } \mathcal{G}\in\mathbb{M} \\ &= R_{A_1,\dots,A_n}(\mathcal{G}) \qquad \text{for some } A_1,\dots,A_n \text{ depending on } \mathcal{G}\in\mathbb{M} \\ &= \sum_{\substack{\mathcal{G}\in\mathbb{M} \\ A\in\sigma_c^\infty}} R_A(\mathcal{G}). \end{aligned}$$

In particular, $R(\mathbb{M})$ is a vector subspace for every $\mathbb{M}$, and the same holds for $R(c)$.

Proof. Let us prove item (i). By the definition of $R(c)$ and $R(\mathbb{M})$, it is clear that $R(c)\subseteq R(\mathbb{M})$ for every minimal component $\mathbb{M}$, hence $R(c)\subseteq \cap_{\mathbb{M}} R(\mathbb{M})$. For the reverse inclusion, let us notice that, since $\mathcal{F}$ is equi-semiconcave, by Proposition 22(ii) for every $\mathcal{G}\in\mathbb{P}_c$ there exists $\Phi\in\Sigma_c^\infty$ such that $\Phi(\mathcal{G})$ belongs to a minimal component. By (1.22),

$$R(\mathcal{G}) \supseteq R(\Phi(\mathcal{G})) \supseteq \bigcap_{\mathbb{M}} R(\mathbb{M}).$$

By taking the intersection over all $\mathcal{G}\in\mathbb{P}_c$, one gets the desired inclusion.

Let us now prove item (ii). Thanks to relation (1.22) and the transitivity of minimal components (Proposition 22(iii)), we get that the function $\mathcal{G}\mapsto R(\mathcal{G})$ is constant on every minimal component. This proves that $R(\mathbb{M}) = R(\mathcal{G})$ for any $\mathcal{G}\in\mathbb{M}$. Moreover, for every two strings $(A_1,\dots,A_n)$ and $(A'_1,\dots,A'_{n'})$ the following holds

$$R_{A_1,\dots,A_n}(\mathcal{G}) + R_{A'_1,\dots,A'_{n'}}(\mathcal{G}) \subseteq R_{A_1,\dots,A_n,\bar{A},A'_1,\dots,A'_{n'}}(\mathcal{G}) \subseteq R(\mathcal{G}) \tag{1.23}$$

where $\bar{A}$ is any cost in σ_c^∞ such that $\Phi_{\bar{A}}\circ\Phi_{A_n}\circ\cdots\circ\Phi_{A_1}(\mathcal{G}) = \mathcal{G}$. The existence of such cost $\bar{A}$ is guaranteed once again by the transitivity of the minimal component. This proves that

$$R(\mathcal{G}) \supseteq \sum_{\substack{A_1,\dots,A_n\in\sigma_c^\infty \\ n\in\mathbb{N}}} R_{A_1,\dots,A_n}(\mathcal{G}) \qquad \forall\,\mathcal{G}\in\mathbb{M}$$

and the opposite inclusion is easy from the definitions. Moreover, since the dimension of $H^1(M,\mathbb{R})$ is finite, we can write $R(\mathcal{G})$ as a finite sum

$$R(\mathcal{G}) = R_{A^1_1,\dots,A^1_{n_1}}(\mathcal{G}) + R_{A^2_2,\dots,A^2_{n_2}}(\mathcal{G}) + \cdots + R_{A^N_1,\dots,A^N_{n_N}}(\mathcal{G})$$

and arguing as in (1.23) we get

$$R(\mathcal{G}) = R_{A_1,\dots,A_n}(\mathcal{G}) \qquad \text{for some} \quad A_1,\dots,A_n\in\sigma_c^\infty, n\in\mathbb{N}.$$

The equality $R(\mathbb{M}) = \sum_{\mathcal{G}\in\mathbb{M}}\sum_{A\in\sigma_c^\infty} R_A(\mathcal{G})$ follows by similar arguments. □

Let us mention that, starting from the last expression for $R(\mathbb{M})$ above, one can show that considering just the weaker version of Lemma 27 would eventually lead to the same results.

Remark 30. The function $\mathcal{F} \mapsto R_{\mathcal{F}}(c)$ is increasing. This is natural in view of the interpretation of $R_{\mathcal{F}}(c)$ as a set of allowed directions for diffusion, and follows by an inspection of the definitions (in fact, the map $\mathcal{F} \mapsto \Sigma_c^\infty(\mathcal{F})$ is also increasing). In particular, let us point out that, since $R_{\mathcal{F}}(c)$ is a vector subspace,

$$R_{\mathcal{F}}(c) \supseteq \sum_{H \in \mathcal{F}} R_{\{H\}}(c).$$

The inclusion may be strict though: we will see that this is the case for two twist maps with non-common non-contractible invariant circles of cohomology c.

We can now restate and prove Theorem 3 of the Introduction. It is a generalization of Theorem 0.11 in [Ber08] to the polysystem case.

Theorem 31. *Let $\mathcal{F}$ be a family of one-periodic Tonelli Hamiltonians defined on the cotangent space of a boundaryless compact manifold M. Assume that $\mathcal{F}$ is equi-semiconcave in the sense of Definition 19. Let $c \in H^1(M, \mathbb{R})$. Then there exist a neighborhood W of c in $H^1(M, \mathbb{R})$, $\varepsilon > 0$ and $N \in \mathbb{N}$ such that*

$$c' \dashv\vdash_{N,\mathcal{F}} c' + B_\varepsilon R(c) \qquad \forall\, c' \in W.$$

Proof. We subdivide the proof into four steps.

Step 1. For every $\mathbb{M} \subset \mathbb{P}_c$ minimal and every $\mathcal{G} \in \mathbb{M}$ there exist a neighborhood $\mathbb{W}_\mathcal{G}$ of $\mathcal{G}$ in $\mathbb{P}$, a natural number $N_\mathcal{G}$ and $\varepsilon_\mathcal{G} > 0$ such that

$$\mathcal{G}' \vdash_{N_\mathcal{G},\mathcal{F}} c(\mathcal{G}') + B_{\varepsilon_\mathcal{G}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_\mathcal{G}.$$

Let $\mathbb{M}$ be minimal and $\mathcal{G} \in \mathbb{M}$. Let $A_1, \dots, A_n \in \sigma_c^\infty$ such that $R_{A_1,\dots,A_n}(\mathcal{G}) = R(\mathbb{M})$. This is possible thanks to Proposition 29. Let us then apply Lemma 27 to $\mathcal{G}$ and to the composition $\Phi_{A_n} \circ \cdots \circ \Phi_{A_1}$. Call $\mathbb{W}_\mathcal{G}$, $N_\mathcal{G}$ and $\varepsilon_\mathcal{G}$ the objects yielded by that Proposition. We have

$$\mathcal{G}' \vdash_{N_\mathcal{G},\mathcal{F}} c(\mathcal{G}') + B_{\varepsilon_\mathcal{G}} R(\mathbb{M}) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_\mathcal{G}.$$

In particular, since $R(c) \subseteq R(\mathbb{M})$, we have

$$\mathcal{G}' \vdash_{N_\mathcal{G},\mathcal{F}} c(\mathcal{G}') + B_{\varepsilon_\mathcal{G}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_\mathcal{G}$$

as desired.

Step 2. For every $\mathcal{G} \in \mathbb{P}_c$ there exist a neighborhood $\mathbb{W}_\mathcal{G}$ of $\mathcal{G}$ in $\mathbb{P}$, a natural number $N_\mathcal{G}$ and $\varepsilon_\mathcal{G} > 0$ such that

$$\mathcal{G}' \vdash_{N_\mathcal{G},\mathcal{F}} c(\mathcal{G}') + B_{\varepsilon_\mathcal{G}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_\mathcal{G}.$$

Let $\mathcal{G} \in \mathbb{P}_c$. By Proposition 22, there exists $\Phi \in \Sigma_c^\infty$ such that $\Phi(\mathcal{G})$ is in a minimal component. Moreover, by Proposition 21 (iv) there exists $A \in \sigma$ such that $\Phi_A(\mathcal{G}) \in \mathbb{W}_{\Phi(\mathcal{G})}$. By continuity, $\Phi_A(\mathcal{G}') \in \mathbb{W}_{\Phi(\mathcal{G})}$ if $\mathcal{G}'$ is in a small enough neighborhood $\mathbb{W}_\mathcal{G}$ of $\mathcal{G}$. By Proposition 18 and by Step 1, there exists $N_A \in \mathbb{N}$ such that

$$\mathcal{G}' \vdash_{N_A,\mathcal{F}} \Phi_A(\mathcal{G}') \vdash_{N_{\Phi(\mathcal{G})},\mathcal{F}} c(\mathcal{G}') + B_{\varepsilon_{\Phi(\mathcal{G})}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_\mathcal{G}.$$

Thus we can take $N_\mathcal{G} = N_A + N_{\Phi(\mathcal{G})}$ and $\varepsilon_\mathcal{G} = \varepsilon_{\Phi(\mathcal{G})}$.

Step 3. There exist a neighborhood W' of c in $H^1(M,\mathbb{R})$, a natural number N and $\varepsilon'>0$ such that

$$c' \quad \vdash_{N,\mathcal{F}} \quad c'+B_{\varepsilon'}R(c) \qquad \forall\, c'\in W'.$$

Let us choose A_0 in σ (no matter which one, for instance $A_0=A^1_L$ with $L\in\mathcal{F}$ will work). The closure of $\Phi_{A_0}(\mathbb{P}_c)$ is compact, thus we can extract a finite subfamily $\{\mathcal{G}_j\}_j\subseteq\overline{\Phi_{A_0}(\mathbb{P}_c)}$ such that $\mathbb{W}=\cup_j\mathbb{W}_{\mathcal{G}_j}$ covers $\overline{\Phi_{A_0}(\mathbb{P}_c)}$. Moreover, it is true that $\mathbb{W}$ also covers $\Phi_{A_0}(\mathbb{P}_{W'})$ for a sufficiently small neighborhood W' of c. Indeed, consider an arbitrary neighborhood W'' of c. The function $\mathcal{G}\mapsto c(\mathcal{G})$ is continuous on the compact set $\overline{\Phi_{A_0}(\mathbb{P}_{W''})}\setminus\mathbb{W}$, hence its image is compact too. Since c does not belong to this image, we can take as W' the intersection of W'' with the complement of the image.

In other words, for any $\mathcal{G}'\in\mathbb{P}_{W'}$ there exists $\bar{j}$ such that $\Phi_{A_0}(\mathcal{G}')\in\mathbb{W}_{\mathcal{G}_{\bar{j}}}$. Hence we obtain

$$\mathcal{G}' \quad \vdash_{N_{A_0},\mathcal{F}} \quad \Phi_{A_0}(\mathcal{G}') \quad \vdash_{\max_j N_{\mathcal{G}_j},\mathcal{F}} \quad c(\mathcal{G}')+B_{\min_j\varepsilon_{\mathcal{G}_j}}R(c) \qquad \forall\,\mathcal{G}'\in\mathbb{P}_{W'}.$$

Thus we can take $N=N_{A_0}+\max_j N_{\mathcal{G}_j}$ and $\varepsilon'=\min_j\varepsilon_{\mathcal{G}_j}$, and the Step 3 is proved.

Step 4. There exist a neighborhood W of c in $H^1(M,\mathbb{R})$, a natural number N and $\varepsilon>0$ such that

$$c' \quad \dashv\vdash_{N,\mathcal{F}} \quad c'+B_{\varepsilon}R(c) \qquad \forall\, c'\in W.$$

In order to obtain the mutual forcing relation starting from the one-side forcing relation of Step 3, it suffices to take $W\subseteq W'$ and $\varepsilon\leq\varepsilon'$ small enough in such a way that $W+B_\varepsilon R(c)\subset W'$. This makes possible to apply the one-side forcing in the opposite direction. This concludes the proof of Step 4 (we keep the same N as in the Step 3) and of the Theorem. □

Remark 32. A careful analysis of the proof of the above theorem shows that the multi-valued function $c\mapsto R(c)$ is lower-semicontinuous: for any c there exists a neighborhood Z such that $R(c)\subseteq R(c')$ for every $c'\in Z$. Nevertheless, the statement of the theorem is somehow stronger, because it yields semicontinuity also on N and ε.

In the remainder of this section we draw some relations between the subspace $R(c)$ and the underlying Hamiltonian polysystem dynamics.

Proposition 33. *Assume $\mathcal{F}$ equi-semiconcave. If there exists a $C^{1,1}$ c-weak Kam solution which is common to all $H\in\mathcal{F}$, then $R(c)=\{0\}$. If $d=1$ the viceversa holds: if $R(c)=\{0\}$ then all the Hamiltonians in $\mathcal{F}$ have an invariant circle in common.*

Proof. If there exists such a weak Kam solution as in the statement, we can identify it with a pseudograph $\mathcal{G}\in\mathbb{P}_c\cap\breve{\mathbb{P}}_c$ such that $\Phi_{A_L}(\mathcal{G})=\mathcal{G}$ for every Lagrangian $L\in\mathcal{F}$. Since Σ^∞_c is generated by such operators, we get that every $\Phi\in\Sigma^\infty_c$ satisfies $\Phi(\mathcal{G})=\breve{\Phi}(\mathcal{G})=\mathcal{G}$. The singleton $\{\mathcal{G}\}$ is thus a minimal set for $\mathbb{P}_c$ and, in view of formula (1.8), it satisfies $R(\{\mathcal{G}\})=\{0\}$.

On the other hand, if $d=1$ and $R(c)=\{0\}$, then there exists a minimal set $\mathbb{M}$ such that

$$\sum_{\mathcal{G}\in\mathbb{M},A\in\sigma^\infty_c}[\mathcal{I}_A(\mathcal{G})^\perp]=\{0\},$$

which means, thanks to the fact $d = 1$ and to (1.8), that $\mathcal{G} = \breve{\Phi}_A \Phi_A(\mathcal{G})$ for every $\mathcal{G} \in \mathbb{M}$ and $A \in \sigma_c^\infty$. Let us apply this to the Peierls barrier $h_{L,c}$ of a Lagrangian $L \in \mathcal{F}$. We get

$$\mathcal{G} = \breve{\Phi}_{h_{L,c}} \Phi_{h_{L,c}}(\mathcal{G}) \in \mathrm{Im}(\breve{\Phi}_{h_{L,c}}) \qquad \forall\, L \in \mathcal{F}$$

hence $\mathcal{G}$ is a dual weak Kam solution for every $L \in \mathcal{F}$, which in addition belongs to $\mathbb{P}$. This implies the result, by Proposition 16. □

We now can restate and prove the Corollary 4 about families of exact twist maps. The condition of equi-semiconcavity on $\mathcal{F}$ is dropped.

Corollary 34. *Let $M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\mathcal{F}$ be an arbitrary family of one-periodic Tonelli Hamiltonians on $T^*M \cong \mathbb{T} \times \mathbb{R}$. Let us make the identification $H^1(\mathbb{T}, \mathbb{R}) \cong \mathbb{R}$. If, for some $A < B \in H^1(\mathbb{T}, \mathbb{R})$, the family $\mathcal{F}$ does not admit an invariant common circle with cohomology in $[A, B]$, then:*

(i) there exists an $\mathcal{F}$-polyorbit $(x_n, p_n)_{n\in\mathbb{Z}}$ satisfying $p_0 = A$ and $p_N = B$ for some $N \in \mathbb{N}$;

(ii) for every $H, H' \in \mathcal{F}$ and every $c, c' \in [A, B]$ there exists an $\mathcal{F}$-polyorbit α-asymptotic to the Aubry set $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$

(iii) for every sequence $(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \subset [A, B] \times \mathcal{F} \times \mathbb{R}^+$ there exists an $\mathcal{F}$-polyorbit which visits in turn the ε_i-neighborhoods of the Mather sets $\tilde{\mathcal{M}}_{H_i}(c_i)$.

Proof. If $\mathcal{F}$ is finite, the conclusion is immediate: by Proposition 33, $R(c) = \mathbb{R}$ for every $c \in [A, B]$, hence by Theorem 31 $[A, B]$ is contained in the same equivalence class for $\dashv\vdash_{\mathcal{F}}$. Therefore Proposition 7 applies, and allows to prove the results: for instance, in order to prove item (i) one applies Proposition 7 (ii) with $\eta \equiv A$ and $\eta' \equiv B$.

If $\mathcal{F}$ is arbitrary, we just reduce to the case of $\mathcal{F}$ finite thanks to the following fact: if the family $\mathcal{F}$ does not admit invariant common circles with cohomology in $[A, B]$, then we can extract a finite subfamily $\mathcal{F}' \subset \mathcal{F}$ with the same property. Indeed, suppose on the contrary that every finite subfamily $\mathcal{F}'$ admits an invariant common circle with cohomology in $[A, B]$, and let us arbitrarily pick H_0 in $\mathcal{F}$: then the set $C(\mathcal{F}')$ defined by

$$\begin{aligned} C(\mathcal{F}') &= \left\{\mathcal{G} \in \mathbb{P}_{[A,B]} : \mathcal{G} \text{ is a } C^{1,1} \text{ weak Kam solution for all } H \in \mathcal{F}' \cup \{H_0\}\right\} \\ &= \bigcap_{H\in\mathcal{F}'\cup\{H_0\}} \Big(\{\mathcal{G} : \Phi_{A_H}(\mathcal{G}) = \mathcal{G}\} \cap \{\mathcal{G} : \breve{\Phi}_{A_H}(\mathcal{G}) = \mathcal{G}\}\Big) \ \cap\ \mathbb{P}_{[A,B]} \end{aligned}$$

is non-empty for all finite $\mathcal{F}' \subseteq \mathcal{F}$. Here A_H denotes the time-one action of the Lagrangian associated to H. The second line in the above expression tells us that $C(\mathcal{F}')$ is closed and contained in the set $\Phi_{A_{H_0}}(\mathbb{P}_{[A,B]})$, which is relatively compact by Remark 11(iii). Hence $C(\mathcal{F}')$ is compact and not empty for every finite subfamily $\mathcal{F}'$. We deduce that the sets $C(\mathcal{F}')$ satisfy the finite intersection property, because

$$C(\mathcal{F}'_1) \cap \cdots \cap C(\mathcal{F}'_n) = C(\mathcal{F}'_1 \cup \cdots \cup \mathcal{F}'_n) \neq \emptyset.$$

By compactness, the whole intersection is non-empty too:

$$\bigcap_{\substack{\mathcal{F}'\subseteq\mathcal{F}\\ \mathcal{F}'\text{finite}}} C(\mathcal{F}') \neq \emptyset.$$

Its elements are the invariant circles common to all the Hamiltonians of the family $\mathcal{F}$. This contradicts the assumptions. □

We end this section by discussing the application of Theorem 31 to some special cases.

- *Case* $\mathcal{F} = \{L\}$. This is the case extensively treated in [Ber08]. In that paper, $R(c)$ was defined as

$$R(c) = \bigcap_{\mathcal{G}\ c\text{-weak Kam solution}} [\mathcal{I}_{h_c}(\mathcal{G})^\perp], \tag{1.24}$$

  where h_c is the c-Peierls barrier of L. Let us check that this definition coincides with the one given here. From Section 1.4.6 we know that the minimal components in $\mathbb{P}_c$ are exactly the c-weak Kam solutions for L, and that $h_c \circ \sigma_c^\infty = \sigma_c^\infty \circ h_c = h_c$. Therefore,

$$\mathcal{I}_A(\mathcal{G}) \supseteq \mathcal{I}_{h_c\circ A}(\mathcal{G}) = \mathcal{I}_{h_c}(\mathcal{G}), \qquad \forall\ \mathcal{G}\in\mathbb{P}_c, A\in\sigma_c^\infty$$

  (the first inclusion follows from 1.13). The equality of (1.24) with our definition of $R(c)$ is then easy to verify.

  Note that in this case the obstruction to the diffusion via the Mather mechanism is the homological size of $\mathcal{I}_{h_c}(\mathcal{G})$, for every c-weak Kam solution $\mathcal{G}$. This set is also called the projected Aubry set of $\mathcal{G}$ (see Section 1.4.5), and taking the union over the c-weak Kam solutions $\mathcal{G}$ one gets the projection on M of the Mañé set $\tilde{\mathcal{N}}(c) \subset T^*M$. A relation between $R(c)$ and the homology (in T^*M) of $\tilde{\mathcal{N}}(c)$ is given in [Ber08, Lemma 8.2].

- *Case* $d = 1$. In this case the mutual forcing relation $\dashv\vdash_{\mathcal{F}}$ is well understood thanks to Proposition 33 and Theorem 31: if $B \subseteq H^1(\mathbb{T},\mathbb{R})$ is the closed set of those cohomology classes c for which there exists a common invariant circle of cohomology c, then the equivalence classes for $\dashv\vdash_{\mathcal{F}}$ are the elements of B and the connected components of its complement.

- *Commuting hamiltonians.* By the discussion in Section 1.4.6, we know that for every c there exists a cost $h_c \in \sigma_c^\infty$ such that $h_c \circ \sigma_c^\infty = \sigma_c^\infty \circ h_c = h_c$. We also know that h_c is the common Peierls barrier of all the Hamiltonians in $\mathcal{F}$, and that the minimal components in $\mathbb{P}_c$ are exactly the c-weak Kam solutions for one (hence all) Hamiltonian in $\mathcal{F}$. Arguing as in the case of a single Hamiltonian, one gets

$$R_{\mathcal{F}}(c) = \bigcap_{\mathcal{G}\ c\text{-weak Kam solution}} [\mathcal{I}_{h_c}(\mathcal{G})^\perp],$$

  hence $R_{\mathcal{F}}(c) = R_{\{H\}}(c)$ for every $H \in \mathcal{F}$. Thus the obstructions are the same than those of each single Hamiltonian in $\mathcal{F}$. Therefore, the polysystem does not present any new kind of instability phenomena with respect to each system regarded separately (at least using the Mather mechanism presented here).

- *General case.* The general situation is more complicated. Nevertheless, some information can still be extracted. For instance, let us suppose that V is a one-dimensional subspace of $H^1(M,\mathbb{R})$ not contained in $R(c)$. Then, there must exists a minimal component $\mathbb{M}$ such that V is not contained in $R(\mathbb{M})$. In particular, by Proposition 29 and by invariance of $\mathbb{M}$, we have that

$$V \nsubseteq R_{A_1,\dots,A_n}(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{M},\ \forall\, A_1,\dots,A_n \in \sigma_c^\infty. \tag{1.25}$$

By making different choices of $\mathcal{G}$ and $A_1,\dots,A_n$, one in principle gets a plethora of conditions on the dynamics of the family $\mathcal{F}$. In the next proposition we prove two sample statements, obtained by considering the two operators studied in Propositions 23 and 24.

Let us remark that the condition 1.25 above essentially boils down to a condition on the various sets $\mathcal{I}_A(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}_A \Phi_A(\mathcal{G})$. By property (iv) in Proposition 23 we see that, at least for some choices of $\mathcal{G}$ and A, we can interpret $\mathcal{G}$ as an unstable manifold of some switched flow and $\breve{\Phi}_A\Phi_A(\mathcal{G})$ as a stable manifold of another switched flow. Thus, at least for these choices of $\mathcal{G}$ and A, the size of the obstruction $\mathcal{I}_A(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}_A\Phi_A(\mathcal{G})$ has an interpretation as the size of the intersection between some unstable and stable manifolds.

Proposition 35. *Let $c \in H^1(M,\mathbb{R})$. Suppose that V is a one-dimensional subspace of $H^1(M,\mathbb{R})$ not contained in $R(c)$.*

(i) For every arbitrary finite string $H_1,\dots,H_k$ of Hamiltonians in $\mathcal{F}$, there exists a subset $S \subset T^*M$ such that: S is a Lipschitz graph over its projection on M, it is contained in a pseudograph of cohomology c, it is invariant (both in past and in future) for the switched flow

$$\phi = \phi_{H_k} \circ \cdots \circ \phi_{H_1},$$

and its projection $\pi(S) \subseteq M$ satisfies

$$V \nsubseteq [\pi(S)^\perp].$$

(ii) For every pair of Hamiltonians $H_0, H_1 \in \mathcal{F}$ there exists a c-weak Kam solution $\mathcal{G}_0$ for H_0 and a dual c-weak Kam solution $\mathcal{G}_1$ for H_1 such that

$$V \nsubseteq \left[(\mathcal{G}_0 \wedge \mathcal{G}_1)^\perp\right].$$

Moreover, calling h_0, h_1 the Peierls barriers of H_0 and H_1, we can also suppose that $\Phi_{h_0}\Phi_{h_1}(\mathcal{G}_0) = \mathcal{G}_0$ and $\breve{\Phi}_{h_1}\breve{\Phi}_{h_0}(\mathcal{G}_1) = \mathcal{G}_1$.

Proof. Let $\mathbb{M}$ be a minimal component in $\mathbb{P}_c$ such that (1.25) holds true.

(i) Call $A_1,\dots,A_k$ the time-one actions of $H_1,\dots,H_k$, and consider the composition

$$A = A_k \circ \cdots \circ A_1.$$

Let us apply (1.25) with $n = 1$ to the cost h_{A_c} and to a fixed point $\mathcal{G}$ of $\Phi_{h_{A_c}}$ belonging to $\mathbb{M}$ (let us recall that, by invariance, every minimal component contains such a fixed point). We get

$$V \nsubseteq [\mathcal{I}_{h_{A_c}}(\mathcal{G})^\perp].$$

Set $S = \mathcal{G}_{|\mathcal{I}_{h_{A_c}}(\mathcal{G})}$. By a natural generalization of Proposition 23, S is invariant for ϕ, thus the conclusion of item (i) is achieved.

(ii) Call h_0, h_1 the Peierls barrier at cohomology c of H_0 and H_1 respectively. In (1.25) take $n = 1$, $A_1 = h_0 \circ h_1$ and $\mathcal{G}$ a fixed point of Φ_{A_1} in $\mathbb{M}$. We get

$$V \not\subseteq [\mathcal{I}_{A_1}(\mathcal{G})^\perp] = [\mathcal{G} \wedge \breve{\Phi}_{A_1}(\mathcal{G})^\perp].$$

Note that $\mathcal{G}$ is a c-weak Kam solution for H_0, because it belongs to the image of Φ_{h_0}. Note also that $\breve{\Phi}_{A_1}(\mathcal{G})$ is a dual c-weak Kam solution for H_1, because it belongs to the image of $\breve{\Phi}_{h_1}$ due to $\breve{\Phi}_{A_1} = \breve{\Phi}_{h_0 \circ h_1} = \breve{\Phi}_{h_1} \circ \breve{\Phi}_{h_0}$. Hence the first part of the statement follows by setting $\mathcal{G}_0 = \mathcal{G}$ and $\mathcal{G}_1 = \breve{\Phi}_{A_1}(\mathcal{G})$.

The second part of the statement follows by replacing, in the above argument, the cost $h_0 \circ h_1$ with its Peierls barrier h, i.e.

$$h = \liminf_{m \to \infty} (h_0 \circ h_1)^m + m\, \alpha_{h_0 \circ h_1}$$

where $\alpha_{h_0 \circ h_1}$ is the unique constant such that the $\liminf$ is real valued. The conclusion follows similarly as above, by taking in (1.25) $n = 1$, $\Phi_1 = \Phi_h$ and $\mathcal{G}$ a fixed point of Φ_h in $\mathbb{M}$, and then setting $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_1 = \breve{\Phi}_1(\mathcal{G})$. □

Chapter 2

Generic transitivity for couples of Hamiltonians

All considered manifolds are separable, finite dimensional, smooth, connected and without boundary, unless otherwise stated.

2.1 Introduction

In this paper we consider the switched system associated to a generic couple of Hamiltonians H_1, H_2 on a symplectic manifold N. Our focus is to prove that orbits and reachable sets of such a system are generically the whole of N. Our work is in the same spirit of a paper of Lobry [Lob72] who proved that, for a C^k-generic (in the sense of Baire) couple of vector fields on a manifold M and for k large enough, orbits are the whole of M.

We will come back later to our notion of genericity, which is given in terms of rectifiable subsets of positive codimension in a Banach or Frechet space and which is stronger than the usual one based on Baire Category. Let us now quickly recall the definitions of orbit and reachable set.

Let us denote by $\{\phi^t_{H_1}\}_t$ and $\{\phi^t_{H_2}\}_t$ the Hamiltonian (local) flows of H_1 and H_2 respectively. We assume that H_1 and H_2 are at least C^2. The *orbit* $\mathcal{O}_{H_1,H_2}(z) \subseteq N$ of a point $z \in N$ through the switched system associated to H_1 and H_2 is obtained by applying to z the group[1] generated by the two flows. More explicitly,

$$\mathcal{O}_{H_1,H_2}(z) = \left\{ \phi^{t_n}_{H_{i_n}} \circ \cdots \circ \phi^{t_1}_{H_{i_1}}(z) : i_1, \ldots, i_n \in \{1,2\}, t_1, \ldots, t_n \in \mathbb{R}, n \geq 1 \right\}.$$

The *reachable set* of z, denoted by $\mathcal{O}^+_{H_1,H_2}(z)$, represents instead the "future" of z, and is obtained by applying to z the (pseudo-)semigroup generated by $\{\phi^t_{H_1}\}_{t>0}$ and $\{\phi^t_{H_2}\}_{t>0}$. More explicitly,

$$\mathcal{O}^+_{H_1,H_2}(z) = \left\{ \phi^{t_n}_{H_{i_n}} \circ \cdots \circ \phi^{t_1}_{H_{i_1}}(z) : i_1, \ldots, i_n \in \{1,2\}, t_1, \ldots, t_n > 0, n \geq 1 \right\}.$$

[1] This should be more properly called a pseudogroup, since the flows may not be complete.

By taking $t_1, \dots, t_n < 0$ in the expression above one gets the analogous definition for the negative reachable set $\mathcal{O}^-_{H_1,H_2}(z)$, i.e. the "past" of z.

When interested in orbits, we say that we are in the *time-unoriented case*. We say that we are in the *time-oriented case* when we are interested in reachable sets.

The main result of this paper is that for a C^k-generic couple H_1, H_2 and for k large enough orbits are the whole of N, and the same is true for reachable sets provided that the manifold N is compact. In fact, we prove something more accurate than C^k-genericity of couples: in a first respect, we adopt the notion of rectifiable set of positive codimension in Banach or Frechet spaces, which is a stronger notion[2] than genericity in the sense of Baire: indeed, such a rectifiable set is always Baire-meager whereas the viceversa is not always true (see [Ber10b] for a detailed study; we shall recall the definition and some basic facts in Section 2.2.2, see Definition 41). In a second respect, we make perturbations just in H_2 leaving H_1 fixed (apart from a small subset of highly degenerate H_1, see Assumption 1 below).

Let us set $\dim N = 2d$. Recall that, for each $k \in N$, the space $C^k(N)$ of C^k-real functions on N is a Banach space when N is compact, and a Frechet space otherwise. Our main results are as follows:

Main theorem 1 (Time-unoriented case)**.** *Let $H_1 \in C^{4d+1}(N)$ satisfy the non-degeneracy assumption 1 below. Let $k \geq 4d$. Then, the set*

$$\left\{ H_2 \in C^k(N) : \mathcal{O}_{H_1,H_2}(z) = N \ \forall\, z \in N \right\}$$

has rectifiable complement of codimension ≥ 1 in $C^k(N)$. In particular, it is C^k-generic.

Main theorem 2 (Time-oriented case)**.** *Assume that N is compact. Let H_1 and k be as before. Then, the set*

$$\left\{ H_2 \in C^k(N) : \mathcal{O}^+_{H_1,H_2}(z) = N \ \forall\, z \in N \right\}$$

has rectifiable complement of codimension ≥ 1 in $C^k(N)$, and in particular it is C^k-generic.

The two results above are proved in Section 2.3 (Theorems 46 and 47). They extend quite naturally to time-dependent Hamiltonians as well, this is the content of the last Section 2.4.

The proofs make fundamentally use of three ingredients, namely the Rashevski-Chow Theorem, the Thom transversality Theorem and the Hamiltonian flow-box Theorem. The exact statements serving our purposes will be given in Section 2.2.

The Rashevski-Chow Theorem, as it is well-known, makes a link between the Lie algebra spanned by two vector fields at a point and the orbit or reachable set of that point. The part concerning reachable sets is more precisely called Krener Theorem. In Section 2.2.1 we recall some local and global versions of these theorems. In the global time-oriented version we will make the additional assumption that the flows of the considered vector fields have non-wandering dynamics. Under the assumptions of Krener Theorem, this is a sufficient condition for concluding

[2] It is also stronger than some other notions of translational invariant "smallness" in infinite-dimensional spaces, such as prevalence or Aronszajn-nullity, see [Ber10b].

that reachable sets are equal to the whole manifold. On the other hand, every Hamiltonian flow on a compact symplectic manifold is non-wandering by Poincaré recurrence Theorem; this is the reason for the compactness assumption on N in the statement of the Main Theorem 2.

Concerning the Thom Transversality Theorem, we use a refined version of the classical result which will be proved in Chapter 3 of this thesis (see also [BM]). Such a version yields that the set of maps whose jet is transverse to a submanifold in a jet space not only is generic, but its complement is rectifiable of positive codimension, which, as already mentioned, is a stronger information. We recall the result in Section 2.2.2 along with the notion of rectifiable set of positive codimension in Banach (or Frechet) spaces.

The last ingredient is the Hamiltonian flow-box Theorem, a normal form result which makes computation of iterated Lie (or Poisson) brackets handleable. The exact statement is recalled in Section 2.2.3.

We finish the introduction by stating the non-degeneracy assumption required to H_1 in the main results above. Let us first give a definition.

Definition 36. *Let $S \subseteq M$ be a subset of the manifold M. We say that S has codimension c in M, and we write* $\operatorname{codim}_M S = c$, *if*

$$S \subseteq \bigcup_{l\in\mathbb{N}} S^l$$

for a countable family $\{S^l\}_l$ of C^1-submanifolds of codimension $\geq c$ in M.

Note that, according to the above definition, a subset S of codimension c is also of codimension c' for every $c' \leq c$. In fact, in this paper we are interested in estimate the size of certain subsets, and only lower bounds (rather than sharp values) for codimension will matter.

Assumption 1. *The Hamiltonian H_1 is continuously differentiable and the subset $N_0 \subseteq N$ defined by*

$$N_0 = \{z \in N : d_z H_1 = 0\}$$

is contained in a countable union $\cup_{l\in\mathbb{N}} N^l$ of C^2-submanifolds of N of codimension greater or equal than $\frac{\dim N}{2} + 1$. Given such a family[3] *$\{N^l\}_l$ we define the tangent space TN_0 as*

$$TN_0 = \bigcup_{l\in\mathbb{N}} TN^l.$$

Note that each TN^l is a submanifold of class C^1. According to Definition 36 we have

$$\operatorname{codim}_N N_0 \geq \frac{\dim N}{2} + 1$$
$$\operatorname{codim}_{TN} TN_0 \geq 2 \operatorname{codim}_N N_0 \geq \dim N + 2.$$

The Assumption 1 is generically satisfied, by an immediate application of the classical Thom transversality Theorem.

[3] We will always assume that, given H_1 as above, the family $\{N^l\}_l$ has been chosen once for all. Such a choice will never play any role.

2.2 Notation and preliminaries

2.2.1 The Rashevski-Chow Theorem: time-unoriented and time-oriented versions

We state here several versions of the Rashevski-Chow theorem for two (non-necessarily Hamiltonian) vector fields. For the proofs we refer to the books [Jur97, AS04]. The proofs in these references are sometimes given for smooth vector fields, but they hold unchanged in the C^k case, $k \geq 1$.

Let us first establish some notation. Given two C^k vector fields X_1, X_2 ($k \geq 1$) on a manifold M, the orbit $\mathcal{O}_{X_1,X_2}(z)$ and the reachable set $\mathcal{O}^+_{X_1,X_2}(z)$ of a point $z \in M$ are defined as in the introduction, with the (local) flows of X_1 and X_2 playing the role of the Hamiltonian flows of H_1 and H_2 of the introduction.

We denote by $\mathrm{Lie}^k(X_1, X_2)$ the vector space spanned by the vector fields

$$X_1, \quad X_2, \quad [X_{i_0}, [X_{i_1}, [\dots, [X_{i_{m-1}}, X_{i_m}], \dots]]], \qquad 1 \leq m \leq k,\ i_0, i_1, \dots, i_m \in \{1, 2\}.$$

where $[\cdot, \cdot]$ denotes the usual bracket of vector fields. We also denote by $\mathrm{Lie}^k_1(X_1, X_2)$ the vector space obtained by bracketing just with X_1, i.e. the span of the $k + 2$ vector fields

$$X_1, \quad X_2, \quad \underbrace{[X_1, [X_1, [\dots, [X_1}_{m \text{ times}}, X_2] \dots]]], \qquad 1 \leq m \leq k.$$

We obviously have

$$\mathrm{Lie}^k_1(X_1, X_2) \subseteq \mathrm{Lie}^k(X_1, X_2).$$

Evaluating these vector spaces at a point $z \in M$ yields a vector subspace of T_zM: we will use the notation

$$\mathrm{Lie}^k_1(X_1, X_2)(z), \quad \mathrm{Lie}^k(X_1, X_2)(z)$$

to denote these vector subspaces.

For two Hamiltonians H_1, H_2 defined on a symplectic manifold (N, ω), let us recall the basic identity

$$X_{\{H_1,H_2\}} = [X_{H_1}, X_{H_2}]$$

where $\{\cdot, \cdot\}$ is the usual Poisson bracket of functions and X_{H_1}, X_{H_2} are the Hamiltonian vector fields of H_1, H_2 defined by the usual formula, valid for any function H,

$$\iota_{X_H}\omega = dH.$$

In the next sections we will tacitly use the following immediate consequence of the formulas above and of the non-degeneracy of the symplectic form: for each $z \in N$,

$$\begin{aligned}&\mathrm{Lie}^k_1(X_{H_1}, X_{H_2})(z) = T_zN\\ &\quad \Leftrightarrow \mathrm{Span}_{T^*_zN}\Big\{d_zH_1, d_zH_2, d_z\{H_1, H_2\}, \dots, d_z\underbrace{\{H_1, \{H_1, \{\dots, \{H_1}_{k \text{ times}}, H_2\} \dots\}\}\}\Big\} = T^*_zN.\end{aligned}$$

and of course in this case we also have $\mathrm{Lie}^k(X_{H_1}, X_{H_2})(z) = T_zN$.

The part (ii) of the following statement is also known as Krener's Theorem.

Theorem 37 (Local Rashevski-Chow Theorem)**.** *Let X_1, X_2 be two C^k vector fields $(k \geq 1)$ on the manifold M. Let $z \in M$.*

(i) (time-unoriented case) If $\operatorname{Lie}^{k-1}(X_1, X_2)(z) = T_z M$, then z is contained in the interior of its orbit $\mathcal{O}_{X_1,X_2}(z)$:

$$z \in \operatorname{int} \mathcal{O}_{X_1,X_2}(z).$$

(ii) (time-oriented case) If $\operatorname{Lie}^{k-1}(X_1, X_2)(z) = T_z M$, then z is contained in the closure of the interior of its positive reachable set $\mathcal{O}^+_{X_1,X_2}(z)$, as well as in the closure of the interior of its negative reachable set $\mathcal{O}^-_{X_1,X_2}(z)$:

$$z \in \operatorname{cl}\operatorname{int} \mathcal{O}^+_{X_1,X_2}(z) \cap \operatorname{cl}\operatorname{int} \mathcal{O}^-_{X_1,X_2}(z).$$

We are now going to state the global counterpart to the previous theorem. In the time-oriented case we shall make the additional assumption that X_1 and X_2 are complete vector fields and that all points of M are non-wandering for both X_1 and X_2. Let us recall the precise definition.

Definition 38. *Let X be a complete C^1 vector field on the manifold M with flow $\{\phi^t_X\}_{t\in\mathbb{R}}$. A point $z \in M$ is said to be non-wandering for X if for every neighborhood U of z and every $t > 0$ there exists $t' > t$ such that*

$$\phi^{t'}_X(U) \cap U \neq \emptyset.$$

The set of non-wandering points is closed. Note that a point is non-wandering for X if and only if it is non-wandering for $-X$. Note also that if M is compact then the flow of X is complete; if, in addition, the flow of X preserve a measure of full support then each point is non-wandering for X, by the Poincaré Recurrence Theorem. This is the case for an Hamiltonian vector field on a compact symplectic manifold.

Theorem 39. *Let X_1, X_2 be two C^k vector fields $(k \geq 1)$ on the manifold M.*

(i) (time-unoriented case) If every $z \in M$ satisfies

$$z \in \operatorname{int} \mathcal{O}_{X_1,X_2}(z)$$

then

$$\mathcal{O}_{X_1,X_2}(z) = M \qquad \forall\, z \in M.$$

(ii) (time-oriented case) Assume that X_1 and X_2 are complete and that all points of M are non-wandering for both X_1 and X_2. If

$$z \in \operatorname{cl}\operatorname{int} \mathcal{O}^+_{X_1,X_2}(z) \cap \operatorname{cl}\operatorname{int} \mathcal{O}^-_{X_1,X_2}(z) \qquad \forall\ z \in M \tag{2.1}$$

then

$$\mathcal{O}^+_{X_1,X_2}(z) = M \quad \text{and} \quad \mathcal{O}^-_{X_1,X_2}(z) = M \qquad \forall\, z \in M.$$

Proof. The part (i) is an immediate consequence of the connectedness of M. The part (ii) is a consequence of the Orbit Theorem (see [AS04] or [Jur97]), and of Corollary 8.3 and Proposition 8.14 in [AS04] as we now quickly recall.

By the Orbit Theorem, the orbit of any point $z \in N$ is an immersed submanifold of N. Since the orbit $\mathcal{O}_{X_1,X_2}(z)$ of every point z has non-empty interior (due to assumption (2.1)), we deduce that the orbit of every z is an immersed submanifold of dimension equal to $\dim N$, *i.e.* it is an open subset of M for every z. By the part (i) of Theorem 39 we deduce

$$\mathcal{O}_{X_1,X_2}(z) = M \qquad \forall\, z \in M.$$

From Proposition 8.14 in [AS04] and the non-wandering assumption[4] on X_1 and X_2 we have

$$\mathcal{O}_{X_1,X_2}(z) \subseteq \operatorname{cl} \mathcal{O}^+_{X_1,X_2}(z) \qquad \forall\, z \in M.$$

Putting together the two above relations yields

$$M = \operatorname{cl} \mathcal{O}^+_{X_1,X_2}(z) \qquad \forall\, z \in M.$$

The conclusion now follows by Corollary 8.3 in [AS04]. □

Note that Proposition 8.14 and Corollary 8.3 in [AS04] are therein stated under the assumption that $\operatorname{Lie}(X_1, X_2)(z) = T_z M$ for every $z \in M$, but their proofs hold unchanged under the weaker assumption (2.1). We will need this slightly more general formulation in the sequel.

Remark 40. We will use in the sequel the following basic fact: given two C^1 vector fields X_1, X_2 on M and $z \in M$, we have

$$z \in \operatorname{int} \mathcal{O}_{X_1,X_2}(z) \;\Leftrightarrow\; \operatorname{int} \mathcal{O}_{X_1,X_2}(z) \neq \emptyset.$$

Indeed, let us suppose that the orbit $\mathcal{O}_{X_1,X_2}(z)$ has non-empty interior and let us prove that it is open. By changing the point z if necessary, we can suppose that z belongs to the interior of $\mathcal{O}_{X_1,X_2}(z)$. Let now z' be another point of $\mathcal{O}_{X_1,X_2}(z)$. We want to prove that z' belongs to the interior of $\mathcal{O}_{X_1,X_2}(z)$ as well. By definition of orbit, there exists $\phi = \phi^{t_k}_{X_{i_k}} \circ \cdots \circ \phi^{t_1}_{X_{i_1}}$ for some $k \in \mathbb{N}$, $t_1, \ldots, t_k \in \mathbb{R}$, $i_1, \ldots, i_k \in \{1, 2\}$, such that

$$z \in \operatorname{Dom} \phi \quad \text{and} \quad \phi(z) = z'.$$

Since $\operatorname{Dom} \phi$ is open and $z \in \operatorname{int} \mathcal{O}_{X_1,X_2}(z)$, the set $\operatorname{Dom} \phi \cap \mathcal{O}_{X_1,X_2}(z)$ is a neighborhood of z. Since ϕ is a local diffeomorphism, the set $\phi(\operatorname{Dom} \phi \cap \mathcal{O}_{X_1,X_2}(z))$ is a neighborhood of z' contained in $\mathcal{O}_{X_1,X_2}(z)$. This proves that $z' \in \operatorname{int} \mathcal{O}_{X_1,X_2}(z)$.

[4] In [AS04] the terminology 'Poisson stable' rather than 'non-wandering' is used.

2.2.2 The Thom Transversality Theorem. Rectifiable sets of positive codimension

In this section we recall from [Ber10b] the notion of rectifiable set of positive codimension in Banach and Frechet spaces. It is quite clear from the definitions below that such a set is automatically Baire-meager (i.e. it is contained in a countable union of closed sets with empty interior, or, equivalently, its complement is generic), but the viceversa is not true in general. In this sense, the notion of rectifiable set of positive codimension is a stronger notion of "smallness" than the one of having generic complement.

Let us first give the definition for Banach spaces. We shall present later the extension to Frechet spaces.

Definition 41. *The subset A in the Banach space F is a* Lipschitz graph of codimension $c \in \mathbb{N}$ *if there exists a splitting $F = E \oplus G$ with $\dim G = c$ and E closed, and a Lipschitz map $g : E \longrightarrow G$ such that*

$$A \subseteq \{x + g(x), x \in E\}.$$

A subset $A \subseteq F$ is rectifiable of codimension $c \in \mathbb{N}$ *if it is a countable union $A = \cup_n \varphi_n(A_n)$ where*

- *$\varphi_n : U_n \longrightarrow F$ is a Fredholm map [5] of index i_n defined on an open subset U_n in a separable Banach space F_n.*
- *$A_n \subseteq U_n$ is a Lipschitz graph of codimension $c + i_n$ in F_n.*

Finally, a subset $A \subseteq F$ is rectifiable of positive codimension *if it is rectifiable of codimension $c \in \mathbb{N}$ for some $c \geq 1$.*

In this paper, the Banach spaces under consideration will mostly be the spaces $C^k(M), k \in \mathbb{N}$, of real functions of class C^k on a compact manifold M.

We shall occasionally consider the case of compact manifolds with boundary. More precisely, if M is any such manifold, we will consider the space $C^k(M)$ defined as

$$C^k(M) = \Big\{ f \colon M \to \mathbb{R} : f \text{ is of class } C^k \text{ in } M \setminus \partial M,\ f \text{ is continuous up to the boundary and all partial derivatives of } f \text{ of order } \leq k \text{ extend by continuity to the boundary} \Big\}$$

The space $C^k(M)$ is a separable Banach space when endowed with the norm

$$\|f\|_{C^k(M)} = \max_{0 \leq |\alpha| \leq k} \sup_{x \in M \setminus \partial M} |\partial_\alpha f(x)| \tag{2.2}$$

the maximum being taken over all multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^d$ of length $\leq k$. Here m is the dimension of M. For a function $f \in C^k(M)$, we will regard its k-jet $j^k f$ as a function from $M \setminus \partial M$ to $J^k(M \setminus \partial M, \mathbb{R})$.

[5] A Fredholm map of index i between separable Banach spaces is a C^1 map such that the differential is Fredholm of index i at every point (recall that the index is locally constant).

We will also consider the space $C^k(M)$ when M is a non-compact manifold without boundary. In this case $C^k(M)$ is no more a Banach space, it is however a Frechet space in the usual way, i.e. the Frechet topology is given by the family of seminorms $\left\{\|\cdot\|_{C^k(K_n)}\right\}_{n\in\mathbb{N}}$, where K_n is any increasing sequence of compact sets exhausting M and $\|\cdot\|_{C^k(K_n)}$ is defined as in (2.2). In fact it is not restrictive to assume that each K_n is a smooth manifold with boundary.

The definition of rectifiable subset of positive codimension extends to Frechet spaces as follows (cf. [Ber10b, Section 3], where a different terminology was used: we call 'rectifiable' here what was called 'countably rectifiable' there):

Definition 42. *The subset A of the Frechet space F is rectifiable of codimension c if it is a countable union $A = \cup_n A_n$ where each A_n satisfies: there exists a Banach space B_n and a continuous linear map $P_n\colon F \to B_n$ with dense range such that $P_n(A_n)$ is rectifiable of codimension c in B_n.*

Let us point out from [Ber10b, Prop. 16] the following useful compatibility property among the spaces $C^k(M)$ with respect to different values of k.

Proposition 43. *Let M be a manifold. If $A \subseteq C^k(M)$ is rectifiable of codimension c, and $k' \geqslant k$, then $A \cap C^{k'}(M)$ is rectifiable of codimension c in $C^{k'}(M)$.*

The next theorem can be seen as a finer version of the "avoiding case" of the classical Thom transversality Theorem in jet spaces.

Theorem 44. *Let M be a manifold, and $W \subseteq J^k(M,\mathbb{R})$ a C^1-submanifold such that*

$$\operatorname{codim} W \geq \dim M + 1.$$

Then, for every $r \geq 1$ the set

$$\left\{f \in C^{k+r}(M) : j^k f(M) \cap W \neq \emptyset\right\}$$

is rectifiable of codimension equal to $\operatorname{codim} W - \dim M \geq 1$ in $C^{k+r}(M)$.

Proof. Let $B = \{z \in \mathbb{R}^d : |z| < 1\}$ be the open unit disc in $\mathbb{R}^d$, and let W be a C^1 submanifold of $J^k(B,\mathbb{R})$ with $\operatorname{codim} W \geq d+1$. In Chapter 3 (Proposition 69) it will be proved that for every $r \geq 1$ the set

$$\left\{f \in C^{k+r}(\bar{B}) : j^k f(B) \cap W \neq \emptyset\right\}$$

is rectifiable of codimension equal to $\operatorname{codim} W - d$ in the Banach space $C^{k+r}(\bar{B})$ defined according to (2.2).

The proof of Proposition 69 will actually yield, with essentially no modifications, the following more general result: if M is a compact manifold with or without boundary, and W is a C^1 submanifold of $J^k(M \setminus \partial M, \mathbb{R})$ with $\operatorname{codim} W \geq \dim M + 1$, then for every $r \geq 1$ the set

$$\left\{f \in C^{k+r}(M) : j^k f(M \setminus \partial M) \cap W \neq \emptyset\right\} \tag{2.3}$$

is rectifiable of codimension $\operatorname{codim} W - \dim M$ in the Banach space $C^k(M)$. This proves the present statement in the case of compact M.

In order to end the proof of the theorem it remains to consider the non-compact case. Let M be a non-compact manifold (without boundary), and call $A \subseteq C^{k+r}(M)$ the subset in the statement, namely

$$A = \left\{ f \in C^{k+r}(M) : j^k f(M) \cap W \neq \emptyset \right\}.$$

Let $K_n \subseteq M, n \in \mathbb{N}$, be a sequence of smooth compact sets exhausting M. Each of them is a smooth manifold with boundary. Call $P_n \colon C^k(M) \to C^k(K_n)$ the natural projection. Note that the space $J^k(K_n \setminus \partial K_n, \mathbb{R})$ is naturally included in $J^k(M, \mathbb{R})$, and that $A = \cup_n A_n$ where

$$A_n = P_n^{-1}\Big(\left\{ f \in C^{k+r}(K_n) : j^k f(K_n \setminus \partial K_n) \cap W \neq \emptyset \right\} \Big).$$

For each n the set

$$\left\{ f \in C^{k+r}(K_n) : j^k f(K_n \setminus \partial K_n) \cap W \neq \emptyset \right\}$$

is rectifiable of codimension $\operatorname{codim} W - \dim M$ in $C^{k+r}(K_n)$, because we are in the same situation of the set in (2.3). Moreover, each P_n is a surjective (cf. [Hes41, See64, Bie80] for more general statements) continuous linear operator from the Frechet space $C^k(M)$ to the Banach space $C^k(K_n)$ and, *a fortiori*, it has dense range. Hence, by Definition 42, the set A is rectifiable of codimension $\operatorname{codim} W - \dim M$ in the Frechet space $C^k(M)$, as desired. □

Clearly, the above theorem still holds true if one replaces W by a countable union of C^1 submanifolds of codimension greater or equal than $\dim M + 1$. In fact it is still true when W is a rectifiable set of codimension greater or equal than $\dim M + 1$, which also follows from Proposition 69.

2.2.3 The Hamiltonian Flow-Box Theorem

Theorem 45. *Let (N, ω) be a symplectic $2d$-dimensional manifold, and $H \colon N \to \mathbb{R}$ be a C^k function, $k \geq 2$. Let $z \in N$ be such that $d_z H \neq 0$. Then, there exist a C^{k-1} chart $\psi \colon U \to \mathbb{R}^{2d}$ defined in a neighborhood U of z such that*

$$H \circ \psi^{-1}(x, p) = p_1 \qquad \text{and} \qquad (\psi^{-1})^* \omega_0 = \omega$$

where ω_0 is the standard symplectic form on $\mathbb{R}^{2d}$ and $(x, p) = (x_1, \dots, x_d, p_1, \dots, p_d)$ are the associated Darboux coordinates.

Proof. We refer to [AM78, Theorem 5.2.19], where in fact the function H is assumed to be smooth. However, the diffeomorphism ψ is therein constructed using the flow of H, which is C^{k-1} if H is C^k. □

2.3 The autonomous case

In this section we prove the main results presented in the introduction. For the convenience of the reader, we restate them as Theorem 46 and Theorem 47 below.

Theorem 46 (Time-unoriented, autonomous case)**.** *Let N be a symplectic manifold of dimension $2d$, and let $H_1 \in C^{4d+1}(N)$ satisfy the non-degeneracy assumption (1). Then, for every $k \geq 4d$, the set*

$$\Big\{H_2 \in C^k(N) : \mathcal{O}_{H_1,H_2}(z) = N \ \forall\, z \in N\Big\}$$

has rectifiable complement of codimension ≥ 1 in $C^k(N)$.

Theorem 47 (Time-oriented, autonomous case)**.** *Let N be a compact symplectic manifold of dimension $2d$, and let $H_1 \in C^{4d+1}(N)$ satisfy the non-degeneracy assumption (1). Then, for every $k \geq 4d$, the set*

$$\Big\{H_2 \in C^k(N) : \mathcal{O}^+_{H_1,H_2}(z) = N \ \ \forall\, z \in N\Big\}$$

has rectifiable complement of codimension ≥ 1 in $C^k(N)$.

The proof of both theorems makes use of the following two results.

Lemma 48. *Let N be a symplectic manifold, and let H_1 be of class C^{k+1} ($k \geq 2$) and satisfy the non-degeneracy Assumption (1). Let us set*

$$N_0 = \{z \in N : d_z H_1 = 0\}, \qquad N' = N \setminus N_0$$

and define subsets $W', W_0 \subseteq J^{k-1}(N,\mathbb{R})$ by

$$W' = \Big\{j \in J^{k-1}(N,\mathbb{R}) : \text{if } j = j^{k-1}_z H_2 \text{ then } z \in N' \text{ and } \mathrm{Lie}_1^{k-2}(X_{H_1}, X_{H_2})(z) \subsetneq T_z N\Big\}$$

$$W_0 = \Big\{j \in J^{k-1}(N,\mathbb{R}) : \text{if } j = j^{k-1}_z H_2 \text{ then } z \in N_0 \text{ and } X_{H_2}(z) \in TN_0\Big\}.$$ [6]

We have:

$$\operatorname{codim}_{J^{k-1}(N,\mathbb{R})} W' = k - \dim N + 1$$
$$\operatorname{codim}_{J^{k-1}(N,\mathbb{R})} W_0 \geq \dim N + 2.$$

In particular,

$$\operatorname{codim}(W' \cup W_0) \geq \dim N + 1 \qquad \text{as soon as} \qquad k \geq 2 \dim N.$$

Proposition 49. *Let H_1, N_0, N' be as in Lemma 48 and $H_2 \in C^k(N)$, $k \geq 2$, satisfy*

$$X_{H_2}(z) \notin TN_0 \qquad \forall\, z \in N_0.$$

Then, the set

$$\big\{t \in \mathbb{R} : \phi^t_{H_2}(z) \in N_0\big\}$$

has empty interior for every $z \in N_0$. In particular, each $z \in N_0$ is accumulated by points in $\mathcal{O}^+_{H_1,H_2}(z) \cap N'$ as well as by points in $\mathcal{O}^-_{H_1,H_2}(z) \cap N'$.

[6] Recall that in Assumption 1 we defined TN_0 as the union $\bigcup_{l\in\mathbb{N}} TN^l$ where $\{N^l\}_l$ is a once-for-all fixed countable family of submanifolds of codimension greater or equal than $\operatorname{codim} N_0$ and whose union covers N_0.

Proof of Theorem 46. First of all, let us notice that it suffices to prove the result for $k = 4d$, thanks to Proposition 43.

Let W' and W_0 be as in Lemma 48. By that Lemma and by the positive-codimension version of the Thom transversality theorem (Theorem 44) we get that the set

$$\left\{H_2 \in C^{4d}(N) : j^{4d-1}H_2(N) \cap \left(W' \cup W_0\right) = \emptyset\right\}$$

has rectifiable complement of codimension ≥ 1 in $C^{4d}(N)$. Thus it is enough to prove that any H_2 belonging to the set above satisfies $\mathcal{O}_{H_1,H_2}(z) = N$ for all $z \in N$.

Let H_2 belong to the set above. If the set $N_0 = \{z \in N : d_z H_1 = 0\}$ is empty, then, recalling the definition of W', the conclusion follows from Theorems 37 and 39. If N_0 is not empty, the argument can be adapted as follows.

By connectedness of N, it suffices to prove that the orbit of every point $z \in N$ is open. By Remark 40, an orbit is either open or with empty interior. Hence we are reduced to prove that the orbit of every point $z \in N$ has non-empty interior. This is true if $z \in N' = N \setminus N_0$, by definition of W' and by the Rashevski-Chow Theorem. This is equally true if $z \in N_0$, because in this case we get by Proposition 49 that the orbit of z intersects (hence coincide with) the orbit of some point in N', and we just proved that such an orbit has non-empty interior.

□

Proof of Theorem 47. The proof is analogous to the one of Theorem 46 above. As before, it suffices to prove the result for $k = 4d$. By repeating the first part of that proof, we get that the set

$$\left\{H_2 \in C^{4d}(N) : j^{4d-1}H_2(N) \cap \left(W' \cup W_0\right) = \emptyset\right\} \tag{2.4}$$

has rectifiable complement of codimension ≥ 1 in $C^{4d}(N)$. Hence it suffices to prove that any H_2 belonging to the set above satisfies the properties stated in the theorem.

Let H_2 belong to the set above. If the set $N_0 = \{z \in N : d_z H_1 = 0\}$ is empty, then the conclusion immediately follows from the time-oriented version of the Rashevski-Chow Theorem (see Theorems 37 and 39). Note that any flow is complete on the compact manifold N, and any Hamiltonian flow on N has the property that all points are non-wandering by Poincaré recurrence Theorem, thus part (ii) of Theorem 39 is indeed applicable.

If N_0 is not empty we adapt the argument as follows: by part (ii) of Theorem 39 it suffices to prove that every $z \in N$ satisfies

$$z \in \operatorname{cl}\operatorname{int}\mathcal{O}^+_{X_1,X_2}(z) \cap \operatorname{cl}\operatorname{int}\mathcal{O}^-_{X_1,X_2}(z).$$

This is true if $z \in N' = N \setminus N_0$ by Krener's Theorem, see part (ii) of Theorem 37. This is equally true if $z \in N_0$ as we now show. Indeed, in this case we get by Proposition 49 that z is accumulated by a sequence $(z_n)_{n\in\mathbb{N}}$ of points in $\mathcal{O}^+_{H_1,H_2}(z) \cap N'$. Since each z_n belongs to N', we know by the previous case that z_n is accumulated by a sequence $(z_{n,k})_{k\in\mathbb{N}}$ of points in $\operatorname{int}\mathcal{O}^+_{H_1,H_2}(z_n)$. By definition of reachable set, we have $\operatorname{int}\mathcal{O}^+_{H_1,H_2}(z_n) \subseteq \operatorname{int}\mathcal{O}^+_{H_1,H_2}(z)$ for each n. Hence the subset $\{z_{n,k}\}_{n,k\in\mathbb{N}}$ is contained in $\operatorname{int}\mathcal{O}^+_{H_1,H_2}(z)$ and has z as an accumulation point. This proves that $z \in \operatorname{cl}\operatorname{int}\mathcal{O}^+_{H_1,H_2}(z)$. The proof for $\mathcal{O}^-$ is analogous.

□

Proof of Lemma 48. Let us first prove the inequality about W'. If $k < 2d$ then the condition $\mathrm{Lie}_1^{k-2}(X_{H_1}, X_{H_2}) \subsetneq T_z N$ is trivially satisfied because $\mathrm{Lie}_1^{k-2}(X_{H_1}, X_{H_2})$ is spanned by k vector fields, and the conclusion is true. Let us then suppose $k \geq 2d$.

The set W' is locally defined (above the open set $N' \subseteq N$) by the inequality

$$\operatorname{rank} \begin{pmatrix} d_z H_1 \\ d_z H_2 \\ d_z\{H_1, H_2\} \\ \vdots \\ d_z \underbrace{\{H_1, \{H_1, \{\dots, \{H_1}_{k-2 \text{ times}}, H_2\} \dots\}\}} \end{pmatrix} < 2d. \tag{2.5}$$

By the Hamiltonian Flow-box Theorem (Theorem 45) we can find, near any arbitrary point of N', a local C^k symplectic chart ψ yielding identification with Darboux coordinates $z = (x, p) = (x_1, \dots, x_d, p_1, \dots, p_d)$ such that $H_1(z) = p_1$.[7] A computation then shows that, in these coordinates,

$$\underbrace{\{H_1, \{H_1, \{\dots, \{H_1}_{m \text{ times}}, H_2\} \dots\}\}}(z) = \partial_{x_1^m} H_2(z),$$

for any function H_2 differentiable enough. As a consequence,

$$d_z \underbrace{\{H_1, \{H_1, \{\dots, \{H_1}_{m \text{ times}}, H_2\} \dots\}\}} = \left(\partial_{x_1^{m+1}} H_2 \ , \ \dots \ , \ \partial_{x_d x_1^m} H_2 \ , \ \partial_{p_1 x_1^m} H_2 \ , \ \dots \ , \ \partial_{p_d x_1^m} H_2\right)(z)$$

and the definition (2.5) for W' becomes more explicit:

$$\operatorname{rank} \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \partial_{x_1} H_2 & \dots & \partial_{x_d} H_2 & \partial_{p_1} H_2 & \partial_{p_2} H_2 & \dots & \partial_{p_d} H_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1^{k-1}} H_2 & \dots & \partial_{x_d x_1^{k-2}} H_2 & \partial_{p_1 x_1^{k-2}} H_2 & \partial_{p_2 x_1^{k-2}} H_2 & \dots & \partial_{p_d x_1^{k-2}} H_2 \end{pmatrix}(z) < 2d.$$

This is a $k \times 2d$ – matrix. The first row corresponds to $d_z H_1$. The other rows correspond to the iterated Lie brackets computed above up to $m = k - 2$, and their entries are clearly independent when regarded as jet-coordinates. We deduce that the codimension of the set W' is the same as the codimension of the set of $(k-1) \times (2d-1)$ matrices with non-maximal rank. Since we are assuming $k \geq 2d$, this codimension is well-known to be

$$k - 2d + 1$$

as desired.

[7] Note that ψ also induces a change of coordinates on $J^{k-1}(N, \mathbb{R})$, which is of class C^1 because ψ is of class C^k. Hence the codimension of W' is the same as the codimension of its image under this diffeomorphism. This legitimates the subsequent computations (and accounts for the requirement $H_1 \in C^{k+1}$ rather than just C^k).

Let us now prove the inequality $\operatorname{codim}_{J^{k-1}(N,\mathbb{R})} W_0 \geq 2d+2$. By Assumption 1 we have

$$N_0 \subseteq \bigcup_{l\in\mathbb{N}} N^l$$

where each N^l is a C^2 submanifold of N of codimension greater or equal than $d+1$. Each tangent space TN^l is a C^1 submanifold of TN and

$$\operatorname{codim}_{TN} TN^l \geq 2(d+1).$$

Since the map

$$\begin{aligned} J^{k-1}(N,\mathbb{R}) &\to TN \\ j^{k-1}_z H_2 &\mapsto X_{H_2}(z) \end{aligned}$$

is a submersion for all $k \geq 2$ (due to non-degeneracy of the symplectic form), and W_0 is precisely the preimage of $TN_0 = \cup_{l\in\mathbb{N}} TN^l$ via this map, we get

$$\operatorname{codim}_{J^{k-1}} W_0 = \operatorname{codim}_{TN} TN_0 \geq 2(d+1),$$

as desired. □

Proof of Proposition 49. Recall that N_0 is contained in a countable union of submanifolds $\{N^l\}_l$ and, by definition, $TN_0 = \cup_l TN^l$. We want to prove that for each $z \in N_0$ the set

$$\{t \in \mathbb{R} : \phi^t_{H_2}(z) \in N_0\}$$

has empty interior. By Baire's Theorem, it suffices to prove that, for each fixed l, the closure of the set

$$\{t \in \mathbb{R} : \phi^t_{H_2}(z) \in N^l\}$$

has empty interior. This is easily seen to be true: the assumption

$$X_{H_2}(z) \notin T_z N^l \qquad \forall\, z \in N^l$$

implies that the set above is constituted by isolated points, and the closure of such a set always has empty interior. □

2.4 The non-autonomous case

In this section we extend to the time-dependent case the results obtained in the previous section.

Let N be a symplectic manifold of dimension $2d$, and $H_1, H_2 \colon N \times \mathbb{R} \to \mathbb{R}$ two time-dependent Hamiltonians. They give rise to two time-dependent Hamiltonian vector fields on N denoted

respectively by X_{H_1}, X_{H_2}. Equivalently, they give rise to two time-independent vector fields $\tilde{X}_{H_1}$ and $\tilde{X}_{H_2}$ on $N \times \mathbb{R}$ defined by

$$\tilde{X}_{H_i}(z,t) = (X_{H_i}(z,t), 1) \ \in \ T_{(z,t)}(N \times \mathbb{R}) \cong T_z N \times \mathbb{R}, \qquad (z,t) \in N \times \mathbb{R}, \ i \in \{1,2\}.$$

We shall occasionally denote by M the manifold $N \times \mathbb{R}$. If $(z,t) \in N \times \mathbb{R}$, in this section its orbit $\mathcal{O}_{H_1,H_2}(z,t)$ has to be intended as the orbit of (z,t) through $\tilde{X}_{H_1}$ and $\tilde{X}_{H_2}$. It is therefore a subset of $N \times \mathbb{R}$. We adopt the analogous definition for the reachable set $\mathcal{O}^+_{H_1,H_2}(z,t)$.

We will make the following assumption on H_1:

Assumption 2. *The Hamiltonian H_1 is continuously differentiable and the subset $M_0 \subseteq N \times \mathbb{R}$ defined by*

$$M_0 = \{(z,t) \in N \times \mathbb{R} : d_z H_1(z,t) = 0\},$$

is contained in a countable union $\cup_{l \in \mathbb{N}} M^l$ of C^2-submanifolds of $N \times \mathbb{R}$ of codimension greater or equal than $\frac{\dim N}{2} + 1$. Given such a family $\{M^l\}_l$ we define the tangent space TM_0 as

$$TM_0 = \bigcup_{l \in \mathbb{N}} TM^l.$$

Note that each TM^l is a submanifold of class C^1 in $T(N \times \mathbb{R})$. According to Definition 36 we have

$$\operatorname{codim}_{N \times \mathbb{R}} M_0 \geq \frac{\dim N}{2} + 1$$
$$\operatorname{codim}_{T(N \times \mathbb{R})} TM_0 \geq 2 \operatorname{codim}_{N \times \mathbb{R}} M_0 \geq \dim N + 2.$$

As in the autonomous case, the above assumption is generic by an easy application of the classical Thom transversality Theorem.

Let us now state the time-dependent versions of Theorems 46 and 47.

Theorem 50 (Time-unoriented, non-autonomous case)**.** *Let N be a symplectic manifold of dimension $2d$, and let $H_1 \in C^{4d+2}(N \times \mathbb{R})$ satisfy the non-degeneracy assumption (2). Then, for every $k \geq 4d+1$ the set*

$$\left\{ H_2 \in C^k(N \times \mathbb{R}) : \mathcal{O}_{H_1,H_2}(z,t) = N \times \mathbb{R} \ \ \forall\, (z,t) \in N \right\}$$

has rectifiable complement of codimension ≥ 1 in $C^k(N \times \mathbb{R})$.

Since both vector fields $\tilde{X}_{H_1}$ and $\tilde{X}_{H_2}$ induce the equation $\dot{t} = 1$ on the t-variable, it is obviously impossible for the reachable set of a point to be the whole of $N \times \mathbb{R}$. For this reason we make the assumption that H_1 and H_2 are one-periodic in time, i.e. they are defined on $N \times \mathbb{T}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We then regard the reachable set of a point (z,t) as a subset of $N \times \mathbb{T}$.

Theorem 51 (Time-oriented, periodic case)**.** *Let N be a compact symplectic manifold of dimension $2d$, and let $H_1 \in C^{4d+2}(N \times \mathbb{T})$ satisfy the non-degeneracy Assumption (2). For every $k \geq 4d+1$ the set*

$$\left\{ H_2 \in C^k(N \times \mathbb{T}) : \mathcal{O}^+_{H_1,H_2}(z,t) = N \times \mathbb{T} \ \ \forall\ (z,t) \in N \times \mathbb{T} \right\}$$

has rectifiable complement of codimension ≥ 1 in $C^k(N \times \mathbb{T})$.

Proof of Theorems 50 and 51. The proof is the same as in the autonomous case (Theorems 46 and 47), once the time-dependent counterpart to Lemma 48 has been established. This is the content of Lemma 52 below. Note that for the time-oriented part one has to apply at some point the part (ii) of Theorem 39 which makes the assumption that the two considered flows have non-wandering dynamics. Since $\tilde{X}_{H_1}$ and $\tilde{X}_{H_2}$ preserve the measure $\mu \oplus dt$ on the compact manifold $N \times \mathbb{T}$, this assumption is indeed fulfilled by the Poincaré Recurrence Theorem. □

Lemma 52. *Let N be a symplectic manifold of dimension $2d$, and let $H_1 \colon N \times \mathbb{R}$ be of class C^{k+1} ($k \geq 2$) and satisfy the non-degeneracy assumption (2). Let us define the subsets $M_0, M' \subseteq N \times \mathbb{R}$ by*

$$M_0 = \{(z,t) \in N \times \mathbb{R} : d_z H_1(z,t) = 0\}, \qquad M' = (N \times \mathbb{R}) \setminus M_0$$

and the subsets $W', W_0 \subseteq J^{k-1}(N \times \mathbb{R}, \mathbb{R})$ by

$$W' = \Big\{ j \in J^{k-1}(N \times \mathbb{R}, \mathbb{R}) : \text{if } j = j^{k-1}_{(z,t)} H_2 \text{ then } (z,t) \in M' \text{ and } \mathrm{Lie}_1^{k-2}(\tilde{X}_{H_1}, \tilde{X}_{H_2})(z,t) \subsetneq T_{(z,t)}(N \times \mathbb{R}) \Big\}$$

$$W_0 = \Big\{ j \in J^{k-1}(N \times \mathbb{R}, \mathbb{R}) : \text{if } j = j^{k-1}_{(z,t)} H_2 \text{ then } (z,t) \in M_0 \text{ and } \tilde{X}_{H_2}(z) \in TM_0 \Big\}.$$

We have:

$$\operatorname{codim}_{J^{k-1}(N\times\mathbb{R},\mathbb{R})} W' = k - \dim N$$
$$\operatorname{codim}_{J^{k-1}(N\times\mathbb{R},\mathbb{R})} W_0 \geq \dim(N \times \mathbb{R}) + 1.$$

In particular, $\operatorname{codim}(W' \cup W_0) \geq \dim N + 1$ *as soon as* $k \geq 2 \dim N + 1$.

Remark 53. The lemma above is stated in the non-periodic setting. It holds in the periodic case as well by replacing the jet space $J^{k-1}(N \times \mathbb{R}, \mathbb{R})$ with $J^{k-1}(N \times \mathbb{T}, \mathbb{R})$. The proof remains essentially unchanged.

Proof of Lemma 52. The proof is very similar to the one of Lemma 48. We start with proving the inequality about W', for which we only consider the case $k \geq 2d+1$. In the other cases the result is trivial by a dimensional argument.

Let j_0 be an arbitrary element of W'. We are going to prove that in a neighborhood of j_0 the codimension of W' is bounded below by the desired value $k - \dim N$. The global bound on the codimension of W' will then follow by standard arguments. The subset W' is defined by the inequality

$$\operatorname{rank} \begin{pmatrix} \tilde{X}_{H_1} \\ \tilde{X}_{H_2} \\ [\tilde{X}_{H_1}, \tilde{X}_{H_2}] \\ \vdots \\ \underbrace{[\tilde{X}_{H_1}, [\dots, [\tilde{X}_{H_1}}_{k-2 \text{ times}}, \tilde{X}_{H_2}] \dots]] \end{pmatrix} (z,t) < 2d+1 \tag{2.6}$$

which depends just on the $(k-1)$-jet of H_2. Note that, since H_1 is of class C^{k+1} and the matrix involves up to $k-1$ derivatives of H_1, its entries are of class C^2 in the variable $j = j^{k-1}_{(z,t)}H_2 \in J^{k-1}(N \times \mathbb{R}, \mathbb{R})$.

Let us denote by (z_0, t_0) the source of the jet j_0. Since $j_0 \in W'$, we have $d_z H_1(z_0, t_0) \neq 0$. By the Hamiltonian Flowbox Theorem (Theorem 45) applied at $H_1(\cdot, t_0)$, there exists a local C^k diffeomorphism $\psi \times \mathrm{id}$ yielding identification with coordinates $(z,t) = (x,p,t)$ on $U \times \mathbb{R}$,[8] with $(x,p) = (x_1, \dots, x_d, p_1, \dots, p_d)$ being standard Darboux variables on the open set $U \subseteq \mathbb{R}^{2d}$, and such that the Hamiltonian in the new coordinates satisfies at time t_0

$$H_1(z, t_0) = p_1 \qquad \forall\, z \in U.$$

Using these coordinates it is easy to compute the restriction at $t = t_0$ of the iterated Lie brackets with $\tilde{X}_{H_1}$. Indeed, for any $m \geq 1$ and any function $H_2(z,t)$ differentiable enough, the computation (which we omit) yields

$$\underbrace{[\tilde{X}_{H_1}, [\dots, [\tilde{X}_{H_1}}_{m \text{ times}}, \tilde{X}_{H_2}] \dots]](z, t_0) = \big(X_{K_m}(z, t_0), 0\big) \qquad \forall\, z = (x,p) \in U$$

where X_{K_m} is the Hamiltonian vector field on U associated to the function $K_m \colon U \to \mathbb{R}$ defined by

$$z \mapsto K_m(z) = (\partial_{x_1} - \partial_t)^m H_2(z, t_0) + (\partial_{x_1} - \partial_t)^{m-1} \partial_t H_1(z, t_0).$$

Note that K_m is a sum of two terms, with H_2 appearing just in the first summand and H_1 just in the second.

Let us denote by $j = (z, t, j')$ the elements of $J^{k-1}(U \times \mathbb{R}, \mathbb{R})$, where the variable j' regroups all variables other than the source (z,t), i.e. j' regroups the value of H_2 as well as all its partial derivatives up to order $k-1$. By using the explicit computation above we get that in the new coordinates the inequality (2.6) defining W' becomes

$$\operatorname{rank}\Big(A(j') + B_{H_1}(z, t_0) + C_{H_1}(z, t, j')\Big) < 2d + 1$$

where:

- the matrix $C_{H_1}(z, t, j')$ is a C^1 matrix which is identically zero for $t = t_0$ and its exact expression does not play any role here;

[8] Note that $\psi \times \mathrm{id}$ also induces a local diffeomorphism of class C^1 from (a subset of) $J^{k-1}(N \times \mathbb{R}, \mathbb{R})$ to $J^{k-1}(U \times \mathbb{R}, \mathbb{R})$, thus preserving codimensions of subsets.

- the matrix $A(j')$ is given by[9]

$$A(j') = \begin{pmatrix} 1 & \partial_{p_1} H_2 & \partial_{p_1}\partial_{(x_1-t)} H_2 & \dots & \partial_{p_1}\partial_{(x_1-t)^{k-2}} H_2 \\ 0 & \partial_{p_2} H_2 & \partial_{p_2}\partial_{(x_1-t)} H_2 & \dots & \partial_{p_2}\partial_{(x_1-t)^{k-2}} H_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \partial_{p_d} H_2 & \partial_{p_d}\partial_{(x_1-t)} H_2 & \dots & \partial_{p_d}\partial_{(x_1-t)^{k-2}} H_2 \\ 0 & -\partial_{x_1} H_2 & -\partial_{x_1}\partial_{(x_1-t)} H_2 & \dots & -\partial_{x_1}\partial_{(x_1-t)^{k-2}} H_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\partial_{x_d} H_2 & -\partial_{x_d}\partial_{(x_1-t)} H_2 & \dots & -\partial_{x_d}\partial_{(x_1-t)^{k-2}} H_2 \\ 1 & 1 & 0 & \dots & 0 \end{pmatrix}$$

  (here the first two columns correspond to $\tilde{X}_{H_1}$ and $\tilde{X}_{H_2}$, and the remaining columns account for the H_2-summand in the definition of K_m, for m ranging from 1 to $k-2$);

- the C^1 matrix $B_{H_1}(z, t_0)$, depending just on z but not on t and j', accounts for the H_1-summand in the definition of K_m and its exact expression does not play any role.

From the expressions above it is clear that for $t = t_0$ the variable j' appears just in the matrix A. Let us denote by $J^{k-1}_{(z_0,t_0)}$ the fiber of $J^{k-1}(U \times \mathbb{R}, \mathbb{R})$ over (z_0, t_0). From the expression of $A(j')$ above we deduce that the codimension of $W' \cap J^{k-1}_{(z_0,t_0)}$ in $J^{k-1}_{(z_0,t_0)}$ is the same as the codimension of the set of $2d \times (k-1)$ matrices with non-maximal rank. Since we are assuming $k \geq 2d+1$, this codimension is well-known to be

$$k - 2d.$$

Since B_{H_1} and C_{H_1} are of class C^1, this fiberwise estimate allows to deduce the local bound

$$\operatorname{codim}_{J^{k-1}(U\times\mathbb{R},\mathbb{R})} W' \geq k - 2d \qquad \text{in a neighborhood of } j_0,$$

which is what we wanted to prove.

Let us now prove the inequality about W_0. The set W_0 is the preimage of TM_0 under the map

$$\begin{aligned} J^{k-1}(N \times \mathbb{R}, \mathbb{R}) &\to T(N \times \mathbb{R}) \cong TN \times \mathbb{R} \times \mathbb{R} \\ j^{k-1}_{(z,t)} H_2 &\mapsto \big(X_{H_2}(z), t, 1\big) \end{aligned}$$

Let us write $TM_0 = \cup_{l\in\mathbb{N}} TM^l$ with each M^l being a C^2 submanifold. It is not difficult to check that the map above is transverse to the C^1 submanifold TM^l for each l. Recalling the Assumption 2 on H_1, we then deduce

$$\operatorname{codim}_{J^{k-1}(N\times\mathbb{R},\mathbb{R})} TM_0 = \operatorname{codim}_{T(N\times\mathbb{R})} \bigcup_{l\in\mathbb{N}} TM^l \geq 2d + 2. \square$$

[9] For reasons of page layout, the matrix $A(j')$ written here is rather the transpose of what it should be in according to the matrix in (2.6).

Chapter 3

Some remarks on Thom's transversality Theorem

3.1 Introduction and a conjecture

It is well-known that "most" functions are Morse, which means that their critical points are non-degenerate. Discussing this claim with some details will be an occasion to introduce and motivate the present work. Let us fix some integer $r \geqslant 2$ and a dimension d. Let B^n be the open unit ball in $\mathbb{R}^n$, and $\bar{B}^n$ the closed unit ball. We denote by $C^r(\bar{B}^n, \mathbb{R})$ the space of functions which are C^r on B^n, and whose differentials up to order r extend by continuity to the closed ball $\bar{B}^n$. We endow it with the norm given as the sum of the supremums of the differentials of order less or equal than r. It is then a separable Banach space. Let F be an affine subspace of $C^r(\bar{B}^n, \mathbb{R})$. In most cases F will just be the whole space $C^r(\bar{B}^n, \mathbb{R})$, but it is sometimes useful to consider finite-dimensional spaces F. The map

$$\begin{aligned} e_1 : B^n \times F &\longrightarrow \mathbb{R}^n \\ (x, f) &\longmapsto df(x) \end{aligned}$$

is C^{r-1}, and, when $F = C^r(\bar{B}^n, \mathbb{R})$ it is a submersion, see [AR67], Theorem 10.4. Recall that a C^1 map is a submersion if and only if its differential at each point is onto with a split kernel. Then, it is locally equivalent (by left and right composition by C^1 maps) to a projection with split kernel. We will always assume that F is chosen such that e_1 is a submersion (or at least that it is transverse to $\{0\}$). Let us then denote by Σ_1 the manifold $e_1^{-1}(0)$, and consider the restriction $\pi_{|\Sigma_1}$ to Σ_1 of the projection on the second factor. This map is C^1, and it is Fredholm of index 0.[1] Moreover a map $f_0 \in F$ is Morse (on B^n) if and only if it is a regular value of $\pi_{|\Sigma_1}$, which means that the differential of this map is onto at each point (x, f_0) of Σ_1. These claims are proved in [AR67], Section 19, the argument is recalled in Section 3.2.2 for the convenience of the reader, see Proposition 60. We have proved that the set $N \subset F$ of non-Morse functions can be written

$$N = CV(\pi_{|\Sigma_1}),$$

[1] The needed definitions and results about Fredholm maps will be recalled in Section 3.2.2.

where CV denotes the set of critical values. By the theorem of Sard and Smale (see Section 3.2), this set is Baire-meager, and it has zero measure in F, in a sense that will be made precise in Section 3.2.

Let us now present, for $r \geqslant 3$, a slightly different approach which has the advantage of avoiding the use of the Theorem of Sard and Smale. Denoting by S^n the set of symmetric $d \times d$ matrices, we start with the evaluation map

$$\begin{aligned} e_2 : B^n \times F &\longrightarrow \mathbb{R}^n \times S^n \\ (x, f) &\longmapsto (df(x), d^2 f(x)) \end{aligned}$$

which is C^{r-2} and, when $r \geqslant 3$ and $F = C^r(\bar{B}^n, \mathbb{R})$, is a submersion. Let us denote by $\tilde{A} \subset \mathbb{R}^n \times S^n$ the subset of points $(0, H)$, with H singular. Note that $\tilde{A}$ is an algebraic submanifold of codimension $n+1$ in $\mathbb{R}^n \times S^n$, hence a finite union of smooth submanifolds of codimension at least $n+1$. The set $N \subset F$ of non-Morse functions can be written

$$N = \pi(\Sigma_2), \quad \Sigma_2 = e_2^{-1}(\tilde{A}).$$

It is best here to first consider that F is finite dimensional (but that e_2 is still a submersion). Then, Σ_2 is a finite union of manifolds of dimension less than $\dim F$. This implies that $N = \pi(\Sigma_2)$ is rectifiable of dimension less than $\dim F$, or in other words it is rectifiable of positive codimension in F. This implies that N has zero measure, but is a much more precise information, which was obtained without the use of Sard Theorem. This reasoning can be extended to the case where F is not finite dimensional with the help of an appropriate notion of rectifiable sets recalled in Section 3.2. More precisely, we know that Σ_2 is a finite union of manifolds of codimension at least $n+1$. Since π is obviously Fredholm of index n, we conclude by Proposition 65 that $\pi(\Sigma_2)$ is rectifiable of codimension 1. We obtain:

Theorem 54. *Let $N \subset C^r(\bar{B}^n, \mathbb{R})$ be the set of functions which are not Morse on B^n.*

If $r \geqslant 2$, then N is a countable union of closed sets with empty interior, it has zero measure (in the sense of Haar or Aronszajn).

If $r \geqslant 3$, it is rectifiable of positive codimension.

The concepts of sets of zero measure (Haar null or Aronszajn null sets) in separable Banach spaces used in this statement are recalled in Section 3.2, together with the concept of rectifiable set of positive codimension. Each point in the statement is the result of one of the strategies of proof exposed above, notice that none of these statements contains the other.

As a second illustration, we consider a smooth manifold $A \subset \mathbb{R}^m$ and describe the set $NA \subset F = C^r(\bar{B}^n, \mathbb{R}^m)$ of maps which are not transverse to A on B^n. We consider the evaluation map

$$\begin{aligned} E_0 : B^n \times F &\longrightarrow \mathbb{R}^m \\ (x, f) &\longmapsto f(x). \end{aligned}$$

This map is C^r and, for $r \geqslant 1$, it is a submersion. We then have

$$NA = CV(\pi_{|\Sigma_0}), \quad \Sigma_0 = E_0^{-1}(A),$$

and $\pi_{|\Sigma_0}$ is C^r and Fredholm of index $i = n - c$, where c is the codimension of A, as follows from Proposition 60. If $r \geqslant n - c + 1$, we can apply the theorem of Sard and Smale (see Section 3.2), and obtain that this set is Baire-meager, and has zero measure in F, in a sense that will be made precise in Section 3.2. When $c > n$, we also conclude that $NA = \pi(\Sigma_0)$ is rectifiable of positive codimension.

The second approach, which is useful for $c \leqslant n$, consists in using the evaluation map

$$E_1 : B^n \times F \longrightarrow \mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m)$$
$$(x, f) \longmapsto (f(x), df_x).$$

This map is C^{r-1}, and, for $r \geqslant 2$, it is a submersion. Let us denote by $\tilde{A}$ the set of pairs $(y, l) \in \mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m)$ such that $y \in A$ and $l(\mathbb{R}^n) + T_y A \subsetneq \mathbb{R}^m$. We then have

$$NA = \pi(\tilde{\Sigma}_1), \quad \tilde{\Sigma}_1 = E_1^{-1}(\tilde{A}).$$

We conclude as above that NA is rectifiable of positive codimension in view of the following Lemma:

Lemma 55. *The set $\tilde{A}$ is a countable union of smooth manifolds of codimension more than n.*

Proof. Locally, there exists a submersion $F : \mathbb{R}^m \longrightarrow \mathbb{R}^c$ such that $A = F^{-1}(0)$. Then, the set $\tilde{A}$ is the preimage by the local submersion

$$\mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m) \ni (y, l) \longmapsto (F(y), dF_y \circ l) \in \mathbb{R}^c \times L(\mathbb{R}^n, \mathbb{R}^c)$$

of the set $B := \{0\} \times L_S(\mathbb{R}^n, \mathbb{R}^c)$ where L_S is the set of singular linear maps from $\mathbb{R}^n$ to $\mathbb{R}^c$ (maps of rank less than c). It is well-known that $L_S(\mathbb{R}^n, \mathbb{R}^c)$ is an analytic submanifold of codimension $n - c + 1$ in $L(\mathbb{R}^n, \mathbb{R}^c)$, hence B is an analytic submanifold of codimension $n + 1$. As a consequence, B is a finite union of smooth submanifolds of codimension at least $n+1$, hence so is $\tilde{A}$. □

As a conclusion, we obtain:

Theorem 56. *Let A be a smooth submanifold of $\mathbb{R}^m$ of codimension c.*

- *For $r \geqslant n - c + 1$, The set NA is Baire meager and Aronzajn null (hence Haar null) in $C^r(\bar{B}^n, \mathbb{R}^m)$.*
- *For $r \geqslant 2$, the set NA is rectifiable of positive codimension in $C^r(\bar{B}^n, \mathbb{R}^m)$, it is thus Baire meager and Aronszajn null.*

It is worth observing that the second statement contains the first one, except for the case where $c = n$ and $r = 1$. Our goal in the present paper is to develop an analog of the second strategy presented on the examples above to prove the Thom transversality Theorem in the space of jets. This idea was suggested by Gromov, in [Gro86], page 33, and used in [EM02], Section 2.3., where it is reduced to an appropriate generalization of Lemma 55 above. This Lemma, which is stated there without proof, is our Conjecture 72. The main novelty in the present paper consists in giving a full proof of this conjecture in the analytic case. We also explain that this strategy, as in the examples above, leads to a more precise statement of the Thom transversality Theorem than the usual proof based on the Theorem of Sard:

Theorem 57. *Let A be a smooth submanifold of $J^p(\bar{B}^n, Y)$ of codimension c, where Y is a finite dimensional separable manifold. For $r \geq p+1$, let $NA \subset C^r(\bar{B}^n, Y)$ be the set of maps whose p-jet is not transverse to A.*

- *If $c \geqslant n+1$ and $r \geqslant p+1$, then the set NA is rectifiable of codimension $c-n$ in $C^r(\bar{B}^n, Y)$, it is thus Baire meager and Aronszajn null.*
- *If $c \leqslant n$ and $r \geqslant p+1+n-c$, the set NA is Baire meager and Aronszajn null (hence Haar null) in $C^r(\bar{B}^n, Y)$.*
- *If $c \leqslant n$ and $r \geqslant p+2$ and A is analytic, the set NA is rectifiable of codimension ≥ 1 in $C^r(\bar{B}^n, Y)$, it is thus Baire meager and Aronszajn null.*

Proof. For completeness, we first quickly recall the usual proof of the Thom transversality Theorem, as given in [AR67], which yields the first and the second point of the Theorem. We consider the evaluation map (with $F = C^r(\bar{B}^n, Y)$):

$$\begin{aligned} E_p : B^n \times F &\longrightarrow J^p(\bar{B}^n, Y) \\ (x, f) &\longmapsto j^p_x f. \end{aligned}$$

This map is C^{r-p}, and it is a submersion, see [AR67], Theorem 10.4. We also have

$$NA = CV(\pi_{|\Sigma}), \quad \Sigma = E_p^{-1}(A).$$

It follows from Proposition 60 below that the map $\pi_{|\Sigma}$ is Fredholm of index $i = n - c$. We conclude from the Theorem of Sard and Smale (Theorem 71 below) that NA is Baire meager and Aronszajn null. This proves the second point of the theorem. If, in addition, the codimension of A is strictly larger than n, then so is the codimension of Σ, and we can conclude directly by the "Easy Part" of the theorem of Sard and Smale that NA is rectifiable of positive codimension. This proves the first point of the theorem.

We obtain the proof of the third point of the theorem by considering the evaluation map

$$\begin{aligned} E_{p+1} : B^n \times F &\longrightarrow J^{p+1}(\bar{B}^n, Y) \\ (x, f) &\longmapsto j^{p+1}_x f, \end{aligned}$$

which is a C^{r-p-1} submersion, and the set

$$\tilde{A} \stackrel{\text{def}}{=} \left\{ j^{p+1}_x f \in J^{p+1}(\bar{B}^n, Y) : j^p f \text{ is not transverse to } A \text{ at } x \right\} \subseteq J^{p+1}(\bar{B}^n, Y).$$

By definition, we have

$$NA = \pi(\tilde{\Sigma}), \quad \tilde{\Sigma} = E_{p+1}^{-1}(\tilde{A}).$$

The third point of the theorem, even without the additional restriction on A, would then be a consequence of Conjecture 72 below. We are not going to prove this conjecture in its full generality. Nevertheless, the cases of the conjecture that we will be able to prove (see Theorems 73 and 74) imply the third point of Theorem 57. □

Conjecture 58. *If A is a smooth submanifold of $J^p(\bar{B}^n, Y)$ of codimension $c \leqslant n$, then $\tilde{A}$ is a countable union of smooth submanifolds of codimension strictly larger than n in $J^{p+1}(\bar{B}^n, Y)$.*

In view of Proposition 69, it would even be enough for our applications to prove that $\tilde{A}$ is rectifiable of codimension $n+1$ (in the sense of Section 3.2.3). We come back to this conjecture in Section 3.3, where we obtain some special cases, see Theorem 73 and 74, which are sufficient to imply the third point of Theorem 57. In Section 3.2, we recall several mathematical notions which have been used in this introduction.

3.2 Small sets, rectifiable sets, the theorem of Sard and Smale

3.2.1 Some notions of small sets

Let F be a separable Banach space. We define below three translation invariant σ-ideals of subsets of F. A σ-ideal is a family $\mathcal{F}$ of subsets of F such that

$$A \in \mathcal{F}, A' \subset A \Rightarrow A' \in \mathcal{F},$$
$$\forall n \in \mathbb{N}, A_n \in \mathcal{F} \Rightarrow \cup_{n\in\mathbb{N}} A_n \in \mathcal{F}.$$

A subset $A \subset F$ is called **Baire-meager** if it is contained in a countable union of closed sets with empty interior. The Baire Theorem states that a Baire-meager subset of a Banach space has empty interior.

A subset $A \subset F$ is called **Haar-null** if there exists a Borel probability measure μ on F such that $\mu(A+f) = 0$ for all $f \in F$. The equality $\mu(A+f) = 0$ means that the set $A+f$ is contained in a Borel set $\tilde{A}_f$ such that $\mu(\tilde{A}_f) = 0$. A countable union of Haar-null sets is Haar-null, see [Chr72, BL00] and [HSY92] for the non-separable case.

A subset $A \subset F$ is called **Aronszajn-null** if, for each sequence f_n generating a dense subset of F, there exists a sequence A_n of Borel subsets of F such that $A \subset \cup_n A_n$ and such that, for each $f \in F$ and for each n, the set

$$\{x \in \mathbb{R} : f + xf_n \in A_n\} \subset \mathbb{R}$$

has zero Lebesque measure. A countable union of Aronszajn-null sets is Aronszajn-null, and each Aronszajn-null set is Haar null, see [Aro76, BL00].

The notion of **probe** allows a simple criterion for proving that a Borel set A is Haar or Aronszajn null. A probe for A is a finite dimensional vector space $E \subset F$ such that $(A+f) \cap E$ has Lebesgue measure zero in E for each $f \in F$. It is easy to see that A is Haar null if there exists a probe for A. In the sense of Aronszajn, we have (see [Zaj08], Proposition 4.3):

Lemma 59. *Let $A \subset F$ be a Borel set. If the set of probes for A contains a non-empty open set in the space of finite dimensional subspaces of F, then A is Aronszajn null.*

Proof. Let f_n be a sequence of points of F generating a dense subspace. Under the hypothesis of the lemma, there exists $N \in \mathbb{N}$ and a probe E such that $E \subset \mathrm{Vect}(f_n, n \leqslant N)$. Then, the space

$F_N := \mathrm{Vect}(f_n, n \leqslant N)$ is itself a probe for A. By standard arguments, (see [BL00], Proposition 6.29 or [Zaj08]), we conclude that $A = \cup_{n \leqslant N} A_n$, where each A_n is a Borel set such that the set

$$\{x \in \mathbb{R} : f + xf_n \in A_n\} \subset \mathbb{R}$$

has zero measure for each $f \in F$. Since this holds for each sequence f_n with dense range, we conclude that A is Aronszajn null. □

If X is a separable manifold modeled on a separable Banach space F_X, we also have notions of Baire meager, Haar null and Aronszajn null subsets of X. We say that $A \subset X$ is Baire meager, Haar null or Aronszajn null if, for each C^1 chart $\varphi : B_X \longrightarrow X$, the set $\varphi^{-1}(A)$ is Baire meager, Haar null or Aronszajn null, where B_X is the open unit ball in F_X. Baire meager sets can also be defined directly as subsets of countable unions of closed sets with empty interior in the Baire topological space X.

The situation is slightly more problematic with Haar null or Aronszajn-null sets, because these σ-ideals are not invariant by C^1 diffeomorphisms. As a consequence, being Haar null or Aronszajn null in the Banach space F_X seen as a Banach Manifold is a stronger property than being Haar null or Aronszajn null in F_X seen as a Banach space. This ambiguity in terminology should not cause problems in the sequel. Many other notions of sets of zero measure in nonlinear spaces have been introduced, see for example the survey [HK10].

3.2.2 Fredholm maps

Given Banach spaces F and B, a continuous linear map $L : F \longrightarrow B$ is called Fredholm if its kernel is finite dimensional and if its range has finite codimension. This automatically implies that the range of L is closed, see for this fact Lemma 4.38 in [AA02]). We say that L is a Fredholm linear map of type (k, l) if k is the dimension of the kernel of L and l is the codimension of its range. The index of L is the integer $k - l$. Recall that the set of Fredholm linear maps is open in the space of continuous linear maps (for the norm topology), and that the index is locally constant, although the integers k and l are not. They are lower semi-continuous. When F and B have finite dimension n and m, then the index of all linear maps is $i = n - m$.

The following essentially comes from Section 19 of [AR67].

Proposition 60. *Let F, X be Banach spaces such that X has finite dimension n. Let $l : F \times X \longrightarrow \mathbb{R}^c$ be a surjective continuous linear map, let K be the kernel of l, and let k be the restriction to K of the projection $(f, x) \longmapsto f$.*

Then k is Fredholm of index $n - c$. Moreover, it is onto if and only if the restriction l_0 of l to $\{0\} \times X$ is onto.

Proof. Let us denote by X_0 the space $\{0\} \times X$, by F_0 the space $F \times \{0\}$, and by K_0 the intersection $K \cap X_0$. To prove that the continuous linear map k is Fredholm, we write

$$F \times X = K_1 \oplus K_0 \oplus X_1 \oplus F_1$$

where

- $F_1 \subset F_0$ and $F_1 \oplus (K + X_0) = F \times X$. Such a space exists because $K + X_0$ has finite codimension, and because $F_0 + K + X_0 = F \times X$.
- $K_1 \oplus K_0 = K$.
- $X_1 \oplus K_0 = X_0$.

Denoting by $\pi \colon F \times X \to F$ the projection on the first factor, we see that the restriction of π to $K_1 \oplus F_1$ is an isomorphism onto F. This implies that the map k is conjugated to the linear map:

$$\begin{aligned} K_1 \oplus K_0 &\longrightarrow K_1 \oplus F_1 \\ \kappa_1 + \kappa_0 &\longmapsto \kappa_1 + 0, \end{aligned}$$

which is Fredholm of index $i = \dim K_0 - \dim F_1$. We obtain that

$$i = (\dim K_0 + \dim X_1) - (\dim X_1 + \dim F_1) = n - c.$$

The linear Fredholm map k is onto if and only if its kernel K_0 has dimension $n - c$. On the other hand, the space K_0 is also the kernel of l_0, hence it has dimension $n - c$ if and only if the map l_0 is onto (X_0 has dimension n). We have proved the second part of the statement. □

We now recall a standard Lemma of differential calculus.

Lemma 61. *Let X, Y be separable Banach manifolds modeled respectively on the Banach spaces F_X, F_Y. Let $f : X \longrightarrow Y$ be a C^1 map and x_0 be a point such that df_{x_0} has a closed and split range $I \subset F_Y$ and a split kernel $K \subset F_X$. Let G be a supplement of I in F_Y. Then, for each local diffeomorphism $\phi : (Y, f(x_0)) \longrightarrow (I \times G, 0)$ there exists a local diffeomorphism $\varphi : (I \times K, 0) \longrightarrow (X, x_0)$ such that*

$$\phi \circ f \circ \varphi(x_i, x_k) = (x_i, \psi(x_i, x_k))$$

for some C^1 local map $\psi : I \times K \longrightarrow G$.

This Lemma can be applied in particular to Fredholm maps.

Proof. Let E be a supplement of K in F_X. By considering first an arbitrary local chart $\tilde{\varphi} : (E \times K, 0) \longrightarrow (X, x_0)$, we write

$$\phi \circ f \circ \tilde{\varphi} : (x_e, x_k) \longmapsto (f_i(x_e, x_k), f_g(x_e, x_k)).$$

It follows from the definition of G and I that $\partial_{x_e} f_i$ is an isomorphism, hence the mapping

$$(x_e, x_k) \longmapsto (f_i(x_e, x_k), x_k)$$

is a local diffeomorphism between $(B_X, 0)$ and $(I \times K, 0)$. Denoting by $\hat{\varphi}(x_i, x_k) = (\hat{\varphi}_e(x_i, x_k), x_k)$ its inverse, we see that

$$\phi \circ f \circ \tilde{\varphi} \circ \hat{\varphi}(x_i, x_k) = (x_i, \psi(x_i, x_k))$$

with $\psi(x_i, x_k) = f_g(\hat{\varphi}_e(x_i, x_k), x_k)$. □

We also recall the constant rank (or rather constant corank) theorem.

Lemma 62. *Let $f : X \longrightarrow Y$ be a C^1 map. Assume that there exists an integer c such that, for each $x \in X$, df_x has a closed range $I \subset F_Y$ of codimension c and a split kernel $K \subset F_X$. Let G be a supplement of I in F_Y. Then, near each point $x_0 \in X$ there exists a local diffeomorphism $\phi : (Y, f(x_0)) \longrightarrow (I \times G, 0)$ and a local diffeomorphism $\varphi : (I \times K, 0) \longrightarrow (X, x_0)$ such that*

$$\phi \circ f \circ \varphi(x_i, x_k) = (x_i, 0).$$

Proof. We first apply Lemma 61 and find charts $\tilde{\phi}$ and φ such that $\tilde{\phi} \circ f \circ \varphi(x_i, x_k) = (x_i, \psi(x_i, x_k))$. The differential of this map has corank c (which is the dimension of G) if and only if $\partial_{x_k} \psi = 0$. We conclude that ψ does not depend on x_k. We now set $\hat{\phi}(x_i, x_g) = (x_i, x_g - \psi(x_i))$, and observe that $\hat{\phi} \circ \tilde{\phi} \circ f \circ \varphi(x_i, x_k) = (x_i, 0)$. □

In the sequel we shall make use of nonlinear Fredholm maps, according to the following definition.

Definition 63. *A Fredholm map of index $i \in \mathbb{Z}$ between two separable Banach manifolds X, Y is a function $\varphi \colon X \to Y$ of class C^1 enjoying the following property: for every $x \in X$ the differential $d_x\varphi$ is a linear Fredholm map of index i.*

In the above definition the index of $d_x\varphi$ is assumed to be globally constant as a function of x. Note that this is not very demanding: as we already recalled, the index is locally constant on the set of linear Fredholm maps, which, in addition, is open in the set of continuous linear maps.

3.2.3 Rectifiable sets in Banach manifolds

We use here the definition of rectifiable sets of finite codimension given in [Ber10b], which extrapolates on [Zaj08]. Our terminology, however, differs from that of [Ber10b] : we call rectifiable here what we called countably rectifiable there.

The subset A in the Banach space F is a **Lipschitz graph of codimension d** if there exists a splitting $F = E \oplus G$, with $\dim G = d$ and E closed, and a Lipschitz map $g : E \longrightarrow G$ such that

$$A \subset \{x + g(x), x \in E\}.$$

Let X be a separable manifold modeled on the separable Banach space F_X. A subset $A \subset X$ is **rectifiable of codimension** $d \in \mathbb{N}$ if it is a countable union $A = \cup_n \varphi_n(A_n)$ where

- $\varphi_n : U_n \longrightarrow X$ is a Fredholm map of index i_n defined on an open subset U_n in a separable Banach space F_n.
- $A_n \subset U_n$ is a Lipschitz graph of codimension $d + i_n$ in F_n.

Note that, by definition, if $A \subset X$ is rectifiable of codimension d then it is rectifiable of codimension d' for all $0 \leq d' \leq d$. The following properties are proved in [Zaj08] or [Ber10b].

Proposition 64. *A rectifiable set of positive codimension is Baire meager. More precisely, it is contained in a countable union of closed rectifiable sets of positive codimension. It is also Aronszanjn null, hence Haar null.*

Proposition 65. *Let X and Y be separable Banach manifolds, and let $f : X \longrightarrow Y$ be C^1 Fredholm of index i, and let $A \subset X$ be rectifiable of codimension $d \geqslant i+1$, then the direct image $f(A)$ is rectifiable of codimension $d-i$.*

The following property is almost taken from [Ber10b].

Proposition 66. *Let X and Y be separable Banach manifolds, let $A \subset Y$ be rectifiable of codimension d and let $f : X \longrightarrow Y$ be a C^1 map such that, at each point of $f^{-1}(A)$, the differential df has the following properties with some integer $k \leqslant d-1$:*

- *It has a split kernel.*
- *It has a closed image of codimension at most k.*

Then, $f^{-1}(A)$ is rectifiable of codimension $d-k$. In particular, if f is a submersion, then $f^{-1}(A)$ is rectifiable of codimension d.

Proof. In view of Lemma 61, it is enough to prove the statement for maps of the form $(x_i, x_k) \longmapsto (x_i, \psi(x_i, x_k))$, where $(x_i, x_k) \in I \times K$, and ψ takes value in G, a supplement of I in B_Y (hence $\dim G \leqslant k$). We also consider that $A \subset I \times G$ is rectifiable of codimension d. Then the projection A_I of A on I is rectifiable of codimension $d-k$ in I. In view of the special form of the map we consider, the preimage of A is contained in $A_I \times K$, which is rectifiable of codimension $d-k$ in $I \times K$ because A_I is rectifiable of codimension $d-k$ in I. □

We express the following results in the context of Banach spaces to avoid some technical complications. If F is a separable Banach space, then we define the separable Banach spaces $C^p(\bar{B}^n, F)$ as in the introduction. The following result was proved in [Ber10b]:

Proposition 67. *If $A \subset C^p(\bar{B}^n, F)$ is rectifiable of codimension d, and $p' \geqslant p$, then $A \cap C^{p'}(\bar{B}^n, F)$ is rectifiable of codimension d in $C^{p'}(\bar{B}^n, F)$.*

This results allows to define sets of positive codimension in the Frechet space $C^\infty(\bar{B}^n, F)$, see [AR67]. The following result makes precise the simple fact that "most" n-parameter families avoid sets of codimension d when $d > n$.

Proposition 68. *Let F be a separable Banach space, and $A \subset F$ a rectifiable set of codimension d. For $n < d$, The set $\mathcal{A} \subset C^1(\bar{B}^n, F)$ of maps f such that $f(B^n) \cap A \neq \emptyset$ is rectifiable of codimension $d-n$.*

Proof. This is just a variant of the methods of proof used in the introduction. We consider the evaluation map

$$E_0 : B^n \times C^p(\bar{B}^n, F) \longrightarrow F,$$

which is a C^1 submersion. We conclude from Proposition 66 that $E_0^{-1}(A)$ is rectifiable of codimension d in $B^n \times C^p(\bar{B}^n, F)$. The set $\mathcal{A}$, which is the projection of $E_0^{-1}(A)$ on the second factor, is thus rectifiable of codimension $d-n$. □

The "easy case" of the transversality theorem also has a natural analog in terms of rectifiable sets.

Proposition 69. *Let F be a separable Banach space, and $A \subset J^p(\bar{B}^n, F)$ a rectifiable set of codimension d. For $n < d$, The set $\mathcal{A} \subset C^{p+1}(\bar{B}^n, F)$ of maps f such that $j^p f(B^n) \cap A \neq \emptyset$ is rectifiable of codimension $d - n$.*

Proof. It is the same as above, using the evaluation map $E_p : (x, f) \longmapsto j^p_x f$. □

3.2.4 The Theorem of Sard and Smale

Theorem 70. *Let X be a smooth (separable) manifold of dimension n, Y a smooth manifold of dimension m, and let $f : X \longrightarrow Y$ be a C^r map. If $r \geqslant 1 + (n - m)$, and $m \leqslant n$, then the set $CV(f)$ of critical values of f has zero measure in Y.*

The theorem also holds in the case where $r \geqslant 1$ and $n \leqslant m - 1$, which is sometimes called the easy case of Sard's theorem. In this case, however, the set $CV(f)$ is just the image $f(X)$, which is rectifiable of dimension n in Y. Since $n < m$, this implies the result, but it is a much finer information. The theorem of Sard was extended by Smale to the infinite dimensional case. We give below a more precise statement:

Theorem 71. *Let X and Y be separable smooth manifolds modeled on separable Banach spaces, and let $f : X \longrightarrow Y$ be a C^r Fredholm map of index i.*

- *If $i \geqslant 0$ and $r \geqslant 1 + i$, then the set $CV(f)$ of critical values of f is Aronszajn null (hence Haar null) and Baire meager in Y.*
- *If $i < 0$ and $r \geqslant 1$, then the set $f(X) = CV(f)$ is rectifiable of codimension $-i$ in Y (It is thus Aronszajn-null and Baire meager).*

Proof. The second part of the statement (the "easy case"), is a special case of Proposition 65. Let us focus on the first part. Let P be the set of critical points of f, so that $CV(f) = f(P)$.

We claim that each point x_0 of P has a closed neighborhood $\tilde{P}$ in P such that $f(\tilde{P})$ is Aronszajn null (hence of empty interior) and closed. Since P is a separable metric space, it has the Lindelöf property, and it can be covered by countably many such local sets $\tilde{P}$. As a consequence, the claim implies the statement.

Let F_X and F_Y be the separable Banach spaces on which X and Y are respectively modeled. Let I be the range of df_{x_0}, and let G be a supplement of I in F_Y, note that G has finite dimension l. Let ϕ be any local diffeomorphism between $(Y, f(x_0))$ and $(I \times G, 0)$.

By Lemma 61, there exists a local diffeomorphism $\varphi : (I \times K, 0) \longrightarrow (X, x_0)$ such that

$$\phi \circ f \circ \varphi(x_i, x_k) = x_i + \psi(x_i, x_k),$$

where K is the (finite-dimensional) kernel of df_{x_0} and $\psi : I \times K \longrightarrow G$ is C^r. Let $B \subset I \times K$ be a bounded closed set containing 0 in its interior. Let us then set

$$\tilde{P} = \varphi(B) \cap P.$$

Since φ is a local diffeomorphism, $\tilde{P}$ is a closed neighborhood of x_0. Moreover, we have

$$\phi(f(\tilde{P})) \subset CV(\phi \circ f \circ \varphi) \subset \bigcup_{x_i \in I} \big(x_i + CV(\psi_{x_i})\big)$$

where $\psi_{x_i} : K \longrightarrow G$ is the map $x_k \longmapsto \psi(x_i, x_k)$. In view of the finite dimensional Sard theorem (applied to ψ_{x_i}), we conclude that G is a probe for $\phi(f(\tilde{P}))$. Since it admits a probe, this set is Haar-null. Moreover, since the set of supplements of I is open in the space of l-dimensional subspaces of F_Y, we conclude from Lemma 59 that $\phi(f(\tilde{P}))$ is Aronszajn null. Since this holds for each local chart ϕ, we have proved that $f(\tilde{P})$ is Aronszajn null in Y.

Finally, let us prove that $f(\tilde{P})$ is closed, or equivalently that $\phi(f(\tilde{P}))$ is closed. Let x^n be a sequence in $\tilde{P}$, such that $f(x^n)$ has a limit y^∞, we have to prove that $y^\infty \subset f(\tilde{P})$. Let us denote by (x_i^n, x_k^n) the sequence $\varphi^{-1}(x^n)$. Since K is finite dimensional and $\varphi^{-1}(\tilde{P}) \subset B$ is bounded, we can assume by taking a subsequence that x_k^n has a limit x_k^∞. On the other hand, since $f(x^n) \longrightarrow y^\infty$, we conclude that

$$(x_i^n, \psi(x_i^n, x_k^n)) = \phi \circ f \circ \varphi(x_i^n, x_k^n) = \phi \circ f(x^n) \longrightarrow \phi(y^\infty),$$

hence x_i^n has a limit x_i^∞ (which is the first component of $\phi(y^\infty)$). The sequence (x_i^n, x_k^n) is thus convergent, hence so is $x_n = \varphi(x_i^n, x_k^n)$. Since $\tilde{P}$ is closed, the limit x^∞ belongs to $\tilde{P}$, and $y^\infty = f(x^\infty)$. □

3.3 Some cases of the conjecture

In this section, we consider two finite dimensional smooth manifolds X and Y. For $p \in \mathbb{N}$, we denote by $J^p(X,Y)$ the space of p-jets of functions $X \longrightarrow Y$. See for example [Hir94, GG73] for some details on jet bundles. Note that $J^0(X,Y) = X \times Y$, and it will also be convenient to consider that $J^{-1}(X,Y) = X$. For $p \leqslant p'$ we have a natural projection

$$\pi_p^{p'} : J^{p'}(X,Y) \longrightarrow J^p(X,Y).$$

When $p = -1$, this is just the source map $j_x^{p'} f \longmapsto x$. When $p \in \mathbb{N}$, the bundle

$$\pi_p^{p+1} : J^{p+1}(X,Y) \longrightarrow J^p(X,Y)$$

has a natural affine structure, we denote by $\mathcal{F}_p^{p+1}(a)$ the fiber $(\pi_p^{p+1})^{-1}(a)$, for $a \in J^p(X,Y)$. Given a submanifold $A \subseteq J^p(X,Y)$ of class $C^r, r \geq 1$, we define

$$\tilde{A} \stackrel{\text{def}}{=} \left\{ j_x^{p+1} f \in J^{p+1}(X,Y) : j^p f \text{ is not transverse to } A \text{ at } x \right\} \subseteq J^{p+1}(X,Y). \tag{3.1}$$

If $j_x^p f$ is an element of $J^p(X,Y)$ and $0 \leq k \leq p-1$, we define the vector subspace $E^k(j_x^p f)$ by

$$E^k(j_x^p f) \stackrel{\text{def}}{=} d(j_x^k f)(T_x X) \subseteq T_{j_x^k f}(J^k(X,Y)) \tag{3.2}$$

Here $d(j^k_x f)$ is the tangent map at x of $j^k f\colon X \to J^k(X,Y)$. Note that the subspace $E^k(j^p_x f)$ depends just on $j^{k+1}_x f$ and that its dimension is always equal to $\dim X$. We also extend the definition to $k=-1$ in a trivial way by setting $E^{-1}(j^p_x f) = T_x X$. We have

$$\tilde{A} = \left\{ z \in J^{p+1}(X,Y) : T_{\pi^{p+1}_p{}_z} A + E^p(z) \subsetneq T_{\pi^{p+1}_p{}_z} J^p(X,Y) \right\}.$$

Conjecture 72. *The set $\tilde{A}$ is a countable union of submanifolds of codimension more than $n = \dim X$.*

This conjecture is stated as a Lemma in [EM02], but not proved. The statement is obvious when the codimension c of A is larger than n, we assume from now on that $c \leqslant n$. We will use the notation

$$\tilde{A}_a = \tilde{A} \cap \mathcal{F}^{p+1}_p(a).$$

We say that the point $a \in A$ is degenerate if

$$d\pi^p_{p-1}(T_a A) + E^{p-1}(a) \subsetneq T_{\pi^p_{p-1} a} J^{p-1}(X,Y). \tag{3.3}$$

(for $p=0$ the map π^0_{-1} is the projection from $J^0(X,Y)$ to X). If a is degenerate, then $\tilde{A}_a = \mathcal{F}^{p+1}_p(a)$. The manifold A can be decomposed as the disjoint unions $A = A_0 \cup A_1$, where A_0 is the set of degenerate points of A and A_1 is the set of non-degenerate points (by definition, the other points). The set A_1 is an open submanifold of A, hence it is itself a submanifold of $J^p(X,Y)$, and

$$\tilde{A} = (\pi^{p+1}_p)^{-1}(A_0) \cup \tilde{A}_1. \tag{3.4}$$

Let us first treat the special case where $A = A_1$ (we then say that A is non-degenerate). Note that this condition holds for example if A is transverse to the fibers of the projection π^p_{p-1}. This condition also holds when $p=0$. The following result also implies Lemma 55:

Theorem 73. *Let A be a non-degenerate C^r submanifold in $J^p(X,Y)$. Then $\tilde{A}$ is a countable union of C^{r-1} submanifolds of codimension larger than $n = \dim X$ in $J^{p+1}(X,Y)$.*

This result implies that $\tilde{A}_1$ is a countable union of submanifolds of codimension at least $n+1$. In order to prove the conjecture, we would also need to prove that the manifold A_0 has codimension $n+1$. We are not able to prove this statement in the general case, hence we will restrict to the analytic case. We say that the submanifold $A \subseteq J^p(X,Y)$ is *analytic* if for every $a = j^p_x f \in A$ there exist charts ψ_X and ψ_Y on X and Y, respectively defined on a neighborhood of x in X and a neighborhood of $f(x)$ in Y, such that the induced chart ψ on $J^p(X,Y)$, defined on a neighborhood U_a of a, makes A analytic, i.e.

$$\psi(A \cap U_a) = \bigcap_i F_i^{-1}(0)$$

for a finite family of analytic functions $F_i\colon \psi(U_a) \to \mathbb{R}$. When A is analytic, we manage to study A_0 by recurrence using Theorem 73, and obtain:

Theorem 74. *Let A be an analytic submanifold in $J^p(X,Y)$. Then $\tilde{A}$ is a countable union of analytic submanifolds of codimension larger than $n = \dim X$ in $J^{p+1}(X,Y)$.*

3.3.1 The non-degenerate case

We assume here that the C^r manifold $A \subset J^p(X,Y)$ is non-degenerate, which means that

$$d\pi^p_{p-1}(T_aA) + E^{p-1}(a) = T_{\pi^p_{p-1}a}J^{p-1}(X,Y) \tag{3.5}$$

for each $a \in A$, and prove Theorem 73. To study the set $\tilde{A}_a$, we define, more generally, the set

$$Z_{a,V} = \left\{\hat{a} \in \mathcal{F}^{p+1}_p(a) : V + E^p(\hat{a}) \subsetneq T_aJ^p(X,Y)\right\} \subseteq \mathcal{F}^{p+1}_p(a)$$

associated to a point $a \in J^p(X,Y)$ and a subspace $V \subset T_aJ^p(X,Y)$. Then, we have

$$\tilde{A}_a = Z_{a,T_aA}.$$

We decompose $Z_{a,V}$ as

$$Z_{a,V} = \bigcup_{r=\dim V}^{\dim J^p(X,Y)-1} Z^r_{a,V}$$

where

$$Z^r_{a,V} \stackrel{\text{def}}{=} \left\{\hat{a} \in (\mathcal{F}^{p+1}_p)(a) : \dim\left(V + E^p(\hat{a})\right) = r\right\}.$$

This decomposition obviously yields a decomposition $\tilde{A} = \cup\tilde{A}^r$, where

$$\tilde{A}^r_a := \tilde{A}^r \cap \mathcal{F}^{p+1}_p(a) = Z^r_{a,T_aA}.$$

The following result implies that $\tilde{A}^r$ is a C^{r-1} submanifold of codimension at least $n+1-\operatorname{codim} A$ in $(\pi^{p+1}_p)^{-1}(A)$, hence a submanifold of codimension at least $n+1$ in $J^{p+1}(X,Y)$, which proves Theorem 73.

Proposition 75. *Let $a \in J^p(X,Y)$ and V be a vector subspace of $T_aJ^p(X,Y)$ of dimension m and codimension $c \geq 1$ such that*

$$d\pi^p_{p-1}(V) + E^{p-1}(a) = T_{\pi^p_{p-1}a}J^{p-1}(X,Y). \tag{3.6}$$

We have:

$$\operatorname{codim}_{\mathcal{F}^{p+1}_p(a)} Z_{a,V} \geq n+1-c.$$

More precisely, the set $Z^r_{a,V}$ is locally contained in the preimage $F^{-1}_{a,V}(0)$ of an algebraic submersion

$$F_{a,V} : \mathcal{F}^{p+1}_p(a) \longrightarrow \mathbb{R}^\theta$$

whose coefficients depend smoothly on (a,V), with $\theta = n+1-c+2(\dim J^p(X,Y)-1-r)$.

Proof. Since the result is of local nature, we can suppose without loss of generality that the jet bundles are trivialized. Hence,

$$J^{p+1}(X,Y) = J^p(X,Y) \times \mathcal{F}^{p+1}_p$$

where $\mathcal{F}_p^{p+1}$ is the fiber of the projection $\pi_p^{p+1}\colon J^{p+1}(X,Y) \to J^p(X,Y)$. We have thus the identification $\mathcal{F}_p^{p+1}(a) \cong \mathcal{F}_p^{p+1}$. Hence the sets $Z^r_{a,V}$ can be regarded as subsets of $\mathcal{F}_p^{p+1}$: denoting by z the elements of $\mathcal{F}_p^{p+1}$, we have

$$Z^r_{a,V} = \left\{ z \in \mathcal{F}_p^{p+1} : \dim\Big(V + E^p(a,z)\Big) = r \right\}.$$

We assume from now on that $m \leq r \leq \dim J^p(X,Y) - 1$. Let us pick, for any z in $\mathcal{F}_p^{p+1}$, a function f_z such that

$$j_x^{p+1} f_z = (a,z).$$

Here $x \in X$ is the base-point of a. Let us also choose a basis $v_1, \dots, v_m$ of V. Let us call $M_{a,V}(z)$ (or just $M(z)$) the matrix whose columns are, in the order, the following vectors (belonging to $T_a J^p(X,Y)$)

$$v_1, \dots, v_m, \partial_{x_1} j_x^p f_z, \dots, \partial_{x_n} j_x^p f_z \tag{3.7}$$

expressed in a convenient basis of $T_a J^p(X,Y)$ which we shall explicit shortly. Note that the vectors $\partial_{x_j} j_x^p f_z, 1 \leq j \leq n$ form a basis of $E^p(a,z)$. It is then clear that

$$z \in Z^r_{a,V} \Leftrightarrow \operatorname{rank} M_{a,V}(z) = r,$$

or equivalently

$$z \in Z^r_{a,V} \Leftrightarrow \begin{cases} \det N(z) = 0 & \forall \text{ square submatrix } N \text{ of } M_{a,V} \text{ of size } r+1 \\ \det N(z) \neq 0 & \text{for some square submatrix } N \text{ of size } r. \end{cases}$$

We will now study more precisely these equations with the help of an appropriate system of local coordinates. Locally, we have the identifications

$$\begin{aligned} J^p(X,Y) &= J^{p-1}(X,Y) \times \mathcal{F}_{p-1}^p \\ J^{p+1}(X,Y) &= J^{p-1}(X,Y) \times \mathcal{F}_{p-1}^p \times \mathcal{F}_p^{p+1}, \end{aligned} \tag{3.8}$$

and both $\mathcal{F}_p^{p+1}$ and $\mathcal{F}_{p-1}^p$ can be identified with real vector spaces. More precisely, we fix once for all local coordinates $x_1, \dots, x_n$ and $y^1, \dots, y^q$ on X and Y respectively, this induces the identification $\mathcal{F}_p^{p+1} \cong \mathbb{R}^{\dim \mathcal{F}_p^{p+1}} = \mathbb{R}^{q\binom{n+p}{n-1}}$ via the isomorphism

$$\begin{aligned} \left(y^s_\alpha\right)_{1\leq s\leq q,\, |\alpha|=p+1}\colon \mathcal{F}_p^{p+1} &\to \mathbb{R}^{q\binom{n+p}{n-1}} \\ j_x^p f &\mapsto \left(\partial_\alpha f^s(x)\right)_{1\leq s\leq q,\, |\alpha|=p+1}. \end{aligned}$$

Here $f^s = y^s \circ f$ is the s-th component of f, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index in $\mathbb{N}^n$ of length $|\alpha| = p+1$ and $\partial_\alpha = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$ stands for the associated partial derivative. Note that for the isomorphism to be rigorously defined, one should specify an order on the set of the involved couples (s,α). Since this order will not play any role in the sequel, we do not specify it.

Concerning $\mathcal{F}^p_{p-1}$, we have the analogous identification $\mathcal{F}^p_{p-1} \cong \mathbb{R}^{\dim \mathcal{F}^p_{p-1}} = \mathbb{R}^{q\binom{n+p-1}{n-1}}$ via the isomorphism

$$\begin{aligned} \left(y^s_\alpha\right)_{1\le s\le q,\,|\alpha|=p} \colon \mathcal{F}^p_{p-1} &\to \mathbb{R}^{q\binom{n+p-1}{n-1}} \\ j^p_x f &\mapsto \left(\partial_\alpha f^s(x)\right)_{1\le s\le q,\,|\alpha|=p} \end{aligned}$$

Here the order on the couples (s,α) will play an important role. For reasons which will become clear in the Lemma 79, we adopt the following lexicographic order:[2] if $s, s' \in \{1,\dots,q\}$ and $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha' = (\alpha'_1,\dots,\alpha'_n)$ are multi-indexes of length p, the variable y^s_α strictly precedes the variable $y^{s'}_{\alpha'}$ if and only if

$$s > s' \qquad \text{or} \qquad \left(s = s' \quad \text{and} \quad \exists\, k \ge 1 : \begin{cases} \alpha_h = \alpha'_h & \text{for all } 1 \le h \le k-1 \\ \alpha_k > \alpha'_k \end{cases} \right) \tag{3.9}$$

Summing up the above paragraphs, we will regard $\left(y^s_\alpha\right)_{1\le s\le q,\,|\alpha|=p+1}$ and $\left(y^s_\alpha\right)_{1\le s\le q,\,|\alpha|=p}$ as coordinates respectively on $\mathcal{F}^{p+1}_p$ and $\mathcal{F}^p_{p-1}$, compatible with the affine structure of these spaces. The coordinates on $\mathcal{F}^p_{p-1}$ are ordered according to (3.9). Since we denoted by z the elements of $\mathcal{F}^{p+1}_p$, we have

$$z = \left(y^s_\alpha\right)_{1\le s\le q,\,|\alpha|=p+1} \in \mathcal{F}^{p+1}_p \cong \mathbb{R}^{\dim \mathcal{F}^{p+1}_p}.$$

Let us now write more explicitly the vectors $\partial_{x_j} j^p_x f_z$ appearing in (3.7). According to the decomposition (3.8), an arbitrary p-jet $j^p_x f$ writes

$$j^p_x f = \left(j^{p-1}_x f, \left(\partial_\alpha f^s(x)\right)_{1\le s\le q,\,|\alpha|=p} \right) \quad \in J^{p-1}(X,Y) \times \mathcal{F}^p_{p-1}. \tag{3.10}$$

The vectors $\partial_{x_j} j^p_x f_z$ are elements of the vector space $T_a J^p(X,Y) = T_{\pi^p_{p-1} a} J^{p-1}(X,Y) \times \mathcal{F}^p_{p-1}$. By taking the derivative of (3.10) with respect to x_j we get:

$$\begin{aligned} \partial_{x_j} j^p_x f_z &= \left(\partial_{x_j} j^{p-1}_x f_z, \left(\partial_{\alpha+\delta_j} f^s(x)\right)_{1\le s\le q,\,|\alpha|=p} \right) \\ &= \left(\partial_{x_j} j^{p-1}_x f_z, \left(y^s_{\alpha+\delta_j}\right)_{1\le s\le q,\,|\alpha|=p} \right) \end{aligned}$$

where δ_j is the multi-index $\big(0,\dots,0,\underbrace{1}_{j\text{-th}},0,\dots,0\big) \in \mathbb{N}^n$. Note that the component $\partial_{x_j} j^{p-1}_x f_z$ depends just on partial derivatives of order one of $j^{p-1} f_z$ at x, i.e. it depends just on $j^p_x f_z = a$. This justifies the following notation:

$$e_j(a) \overset{\text{def}}{=} \partial_{x_j} j^{p-1}_x f_z, \qquad 1 \le j \le n.$$

[2] What really matters for our purposes is the lexicographic order with respect to α at fixed s. There are several orders satisfying this condition, but for the sake of definiteness we adopt the one described in (3.9).

Note that

$$\text{Span}\,\{e_1(a),\dots,e_n(a)\} = E^{p-1}(a).$$

We have:

$$\partial_{x_j} j^p_x f_z = \Big(e_j(a), \big(y^s_{\alpha+\delta_j}\big)_{1\le s\le q,\,|\alpha|=p}\Big). \tag{3.11}$$

We can now write the matrix $M_{a,V}(z)$ in the base of $T_a J^p(X,Y) = T_{\pi^p_{p-1}a}J^{p-1}(X,Y) \times \mathcal{F}^p_{p-1}$ obtained as the juxtaposition of an arbitrary base of $T_{\pi^p_{p-1}a}J^{p-1}(X,Y)$ and of the base $\big(y^s_\alpha\big)_{1\le s\le q,\,|\alpha|=p}$ of $\mathcal{F}^p_{p-1}$ in the order which has been specified before:

$$M_{a,V}(z) = \left[\begin{array}{c|c} v_1 \;\vdots\; \dots \;\vdots\; v_m & e_1(a) \;\vdots\; \dots \;\vdots\; e_n(a) \\ \hline & B(z) \end{array}\right] \begin{array}{l} \Big\} \dim J^{p-1}(X,Y) \\ \Big\} \dim \mathcal{F}^p_{p-1} \end{array}$$

Here the vectors $v_1,\dots,v_m$ are a basis of V and the vectors $e_1(a),\dots,e_n(a)$, which have been defined before, form a base of $E^{p-1}(a)$. Finally, the block $B(z)$ depends just on z and is given by

$$B(z) = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ y^s_{\alpha+\delta_1} & y^s_{\alpha+\delta_2} & \cdots & y^s_{\alpha+\delta_{n-1}} & y^s_{\alpha+\delta_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \leftarrow \text{row corresponding to } y^s_\alpha \tag{3.12}$$

where the rows are ordered according to (3.9).

For later use, note that the first m columns of $M(z)$ are linearly independent, as well as the first $\dim J^{p-1}(X,Y)$ rows, as follows from the hypothesis (3.6). Indeed, the first m columns are clearly independent because they represent a basis of V. The fact that the first $\dim J^{p-1}(X,Y)$-rows are independent is equivalent to the assumption $d\pi^p_{p-1}(V) + E^{p-1}(a) = T_{\pi^p_{p-1}a}J^{p-1}(X,Y)$.

An intermede of linear algebra. We prove, for an arbitrary matrix, the existence of a non-singular square submatrix of maximum rank satisfying some special conditions.

Let M be an arbitrary $m \times n$-matrix with real entries. Only in this intermede, m and n are arbitrary integers ≥ 1, with no relation with the values assumed by the same symbols in the rest of the paper.

Let us establish some notations for submatrices of M. The rows of M are $\mathcal{R}(M) = \{1,2,\dots,m\}$ and its columns are $\mathcal{C}(M) = \{1,2,\dots,n\}$. A submatrix N of M is determined by its rows

$$\mathcal{R}(N) \subseteq \{1,2,\dots,m\}$$

and its columns

$$\mathcal{C}(N) \subseteq \{1,2,\dots,n\}.$$

We denote by $|\mathcal{R}(N)|$ and $|\mathcal{C}(N)|$ their cardinality. We also denote by $i_1(N), i_2(N), \dots i_{|\mathcal{R}(N)|}(N)$ the elements of $\mathcal{R}(N)$, and we always assume that the indexes are chosen in such a way that

$$i_1(N) < i_2(N) < \dots < i_{|\mathcal{R}(N)|}(N).$$

Similarly, we denote $\mathcal{C}(N) = \{j_1(N), \dots, j_{|\mathcal{C}(N)|}(N)\}$ with

$$j_1(N) < j_2(N) < \ldots < j_{|\mathcal{C}(N)|}(N).$$

Given two sub-matrices N_1 and N_2, we say that $N_1 \preceq_{\mathcal{R}} N_2$ if the rows of N_1 come "before" the rows of N_2; more rigorously,

$$N_1 \preceq_{\mathcal{R}} N_2 \iff^{\text{def}} |\mathcal{R}(N_1)| \leq |\mathcal{R}(N_2)| \text{ and } i_k(N_1) \leq i_k(N_2) \ \forall\ 1 \leq k \leq |\mathcal{R}(N_1)|.$$

We also give the analogous definition for columns:

$$N_1 \preceq_{\mathcal{C}} N_2 \iff^{\text{def}} |\mathcal{C}(N_1)| \leq |\mathcal{C}(N_2)| \text{ and } i_k(N_1) \leq i_k(N_2) \ \forall\ 1 \leq k \leq |\mathcal{C}(N_1)|.$$

Note that $\preceq_{\mathcal{R}}$ and $\preceq_{\mathcal{C}}$ are preorders (i.e. reflexive and transitive) but not partial orders in general. We write $N_1 \prec_{\mathcal{R}} N_2$ if $N_1 \preceq_{\mathcal{R}} N_2$ and $N_1 \neq N_2$. We define $N_1 \prec_{\mathcal{C}} N_2$ similarly.

It turns out that, when restricted to the set of square submatrices of rank equal to rank M, the relations $\preceq_{\mathcal{R}}$ and $\preceq_{\mathcal{C}}$ admit a unique common minimal element, in a sense made precise by the following lemma.

Lemma 76. *Let M be a $m \times n$-matrix. There exists a submatrix M^* of M such that* $\operatorname{rank} M^* = \operatorname{rank} M$ *and which is minimal in the following sense: any submatrix N with* $\operatorname{rank} N = \operatorname{rank} M$ *satisfies*

$$M^* \preceq_{\mathcal{R}} N \quad \text{and} \quad M^* \preceq_{\mathcal{C}} N. \tag{3.13}$$

The submatrix M^* is uniquely defined by this condition, and is a square matrix.

It is easy to check that if such a submatrix M^* exists, then it is unique and square. Thus we just focus on the existence. We will prove existence in a somehow constructive way, by giving a procedure for finding M^*. In fact, we will give two different procedures and we will show that they yield the same submatrix; as a consequence, this submatrix will satisfy the conditions demanded to M^*.

As a first intermediate step, let us describe two constructions which allow to associate to M two special (non-square in general) submatrices. We call these two sub-matrices $V(M)$ and $H(M)$. Here V stands for *vertical* and H for *horizontal*.

Let us first describe how to construct $V(M)$: it is uniquely defined by the properties

$$\begin{aligned} \mathcal{C}(V(M)) &= \mathcal{C}(M) = \{1, \dots, n\} \\ i \in \mathcal{R}(V(M)) &\Leftrightarrow \text{the } i\text{-th row of } M \text{ is linearly independent from} \\ &\qquad \text{the first } i-1 \text{ rows of } M. \end{aligned}$$

(If $i = 0$, we mean that $i \in \mathcal{R}(V(M))$ if and only if the first row is not identically zero.) This procedure ensures that $V(M)$ is minimal with respect to rows among submatrices of M of maximal rank. More precisely, $V(M)$ satisfies

$$\operatorname{rank} V(M) = \operatorname{rank} M \quad \text{and} \quad V(M) \preceq_{\mathcal{R}} N \quad \forall\ N \text{ submatrix of } M \text{ with } \operatorname{rank} N = \operatorname{rank} M.$$

The construction of $H(M)$ is the same as for $V(M)$, but with the roles of rows and columns inverted. More precisely,

$$\begin{aligned} \mathcal{R}(H(M)) &= \mathcal{R}(M) = \{1, \dots, m\} \\ j \in \mathcal{C}(H(M)) &\Leftrightarrow \text{the } j\text{-th row of } M \text{ is linearly independent from} \\ &\qquad \text{the first } j-1 \text{ rows of } M. \end{aligned}$$

Analogously,

$$\operatorname{rank} H(M) = \operatorname{rank} M \quad \text{and} \quad H(M) \preceq_{\mathcal{C}} N \quad \forall\ N \text{ submatrix of } M \text{ with } \operatorname{rank} N = \operatorname{rank} M.$$

Now that we have introduced the two constructions, we can iterate them. In particular, we can consider $HV(M)$ and $VH(M)$. We regard them as submatrices of M. By construction, they are square non-singular submatrices of size equal to $\operatorname{rank} M$. In fact, it turns out that they coincide, and they are the submatrix M^* which we are looking for. More precisely, the following two claims are true:

(i) $HV(M) = VH(M)$;

(ii) the matrix $M^* := HV(M) = VH(M)$ satisfies the conditions required in the statement.

Proof of (i). We want to prove that $\mathcal{R}(HV(M)) = \mathcal{R}(VH(M))$ and $\mathcal{C}(HV(M)) = \mathcal{C}(VH(M))$. As already pointed out, $HV(M)$ and $VH(M)$ are square submatrices of equal size (equal to $\operatorname{rank} M$). Hence it suffices to prove that $\mathcal{R}(VH(M)) \subseteq \mathcal{R}(HV(M))$ and that $\mathcal{C}(HV(M)) \subseteq \mathcal{C}(VH(M))$. We focus just on the first inclusion, the second being analogous.

By the properties of the constructions H and V described above, we have:

$$\begin{aligned} k \in \mathcal{R}(VH(M)) &\Rightarrow \text{the } k\text{-th row of } H(M) \text{ is linearly independent} \\ &\qquad \text{from the first } k-1 \text{ rows of } H(M) \\ &\Rightarrow \text{the } k\text{-th row of } M \text{ is linearly independent} \\ &\qquad \text{from the first } k-1 \text{ rows of } M \\ &\Rightarrow k \in \mathcal{R}(V(M)) = \mathcal{R}(HV(M)) \end{aligned}$$

as desired.

Proof of (ii). Let N be a sub-matrix of M with $\operatorname{rank} N = \operatorname{rank} M$. By the properties of $V(M)$, $V(M) \preceq_{\mathcal{R}} N$. From $M^* = HV(M)$ we deduce $\mathcal{R}(M^*) = \mathcal{R}(HV(M)) = \mathcal{R}(V(M))$ and thus $M^* \preceq_{\mathcal{R}} N$ as well. The proof of $M^* \preceq_{\mathcal{C}} N$ is similar. □

Let us emphasize the following characterization of M^* which follows directly from the proof above:

$$\begin{aligned} i \in \mathcal{R}(M^*) &\Leftrightarrow \text{the } i\text{-th row of } M \text{ does not belong to the linear span} \\ &\qquad \text{of the first } i-1 \text{ rows of } M. \\ j \in \mathcal{C}(M^*) &\Leftrightarrow \text{the } j\text{-th column of } M \text{ does not belong to the linear span} \\ &\qquad \text{of the first } j-1 \text{ columns of } M. \end{aligned} \tag{3.14}$$

End of the proof of Proposition 75. We fix r between m and $\dim J^p(X,Y) - 1$ and assume that $Z^r_{a,V}$ is not empty (otherwise we have nothing to prove). Let z_0 be an arbitrary element of $Z^r_{a,V}$. Our goal will now be to find θ different square submatrices $[M_{a,V}(z)]_i$ of size $r+1$ in $M_{a,V}(z)$ such that the equations $\det[M_{a,V}(z)]_i = 0$ are independent near z_0. The functions $z \longmapsto \det[M_{a,V}(z)]_i$ are then the components of the map $F_{a,V}(z)$ mentioned in Proposition 75. These functions are clearly polynomials in z, with coefficients depending smoothly on a and V. From now on, we omit to explicitly mention a and V, and we note $M(z)$ instead of $M_{a,V}(z)$.

Let $M^*_{z_0}$ be the square submatrix of size r associated to $M(z_0)$ by the Lemma 76. We take the notational convention that the symbol $M^*_{z_0}$ without further specifications stands for a pattern of rows and columns, i.e. $M^*_{z_0}$ is the datum $(\mathcal{R}(M^*_{z_0}), \mathcal{C}(M^*_{z_0}))$. We can also identify $M^*_{z_0}$ to a matrix-valued function of z, but in this case we explicitly write $M^*_{z_0}(z)$ or $M^*_{z_0}(\cdot)$. This is in order to avoid ambiguities and distinguish, for instance, between $M^*_{z_0}$ and $M^*_{z_0}(z_0)$. We adopt the same convention for all the submatrices encountered below, such as $\hat{M}_{z_0}, M^*_{z_0,\#(i,j)}$, etc., which we shall define shortly.

We have

$$\det M^*_{z_0}(z_0) \neq 0 \quad \text{and} \quad \operatorname{rank} M^*_{z_0}(z_0) = \operatorname{rank} M(z_0) = r.$$

Let us call $\hat{M}_{z_0}$ the submatrix whose rows and columns are exactly the ones *not* appearing in $M^*_{z_0}$, i.e.

$$\mathcal{R}(\hat{M}_{z_0}) = \{1, \dots, \dim J^p(X,Y)\} \setminus \mathcal{R}(M^*_{z_0}), \qquad \mathcal{C}(\hat{M}_{z_0}) = \{1, \dots, m+n\} \setminus \mathcal{C}(M^*_{z_0})$$

As already mentioned above, the first m columns of M are linearly independent as well as the first $\dim J^{p-1}(X,Y)$-rows. By the characterization (3.14) we deduce that $\hat{M}_{z_0}$ is entirely contained in the bottom-right block of M, i.e. it is a submatrix of B:

$$\begin{aligned}\mathcal{R}(\hat{M}_{z_0}) \subseteq \mathcal{R}(B) &= \{\dim J^{p-1}(X,Y) + 1, \dots, \dim J^p(X,Y)\} \\ \mathcal{C}(\hat{M}_{z_0}) \subseteq \mathcal{C}(B) &= \{m+1, \dots, m+n\}.\end{aligned}$$

Let us denote $M^*_{z_0,\#(i,j)}$ the submatrix obtained by adding to $M^*_{z_0}$ the i-th row and the j-th column of M, i.e.

$$\mathcal{R}(M^*_{z_0,\#(i,j)}) = \mathcal{R}(M^*_{z_0}) \cup \{i\}, \qquad \mathcal{C}(M^*_{z_0,\#(i,j)}) = \mathcal{C}(M^*_{z_0}) \cup \{j\}.$$

We are interested to the case when (i,j) belongs to $\mathcal{R}(\hat{M}_{z_0}) \times \mathcal{C}(\hat{M}_{z_0})$. In this case the submatrix $M^*_{z_0,\#(i,j)}$ is a square submatrix of size $r+1$. We are in the following situation:

$$\begin{cases} \det M^*_{z_0}(z_0) \neq 0 \\ \det M^*_{z_0,\#(i,j)}(z_0) = 0 \qquad \forall\ (i,j) \in \mathcal{R}(\hat{M}_{z_0}) \times \mathcal{C}(\hat{M}_{z_0}) \end{cases}$$

Proposition 75 follows from the following two lemma:

Lemma 77. *If $(i_1, j_1), \dots, (i_\theta, j_\theta)$ are pairwise distinct couples in $\mathcal{R}(\hat{M}_{z_0}) \times \mathcal{C}(\hat{M}_{z_0})$ such that*

$$i_1 \leq i_2 \leq \cdots \leq i_\theta \quad \text{and} \quad j_1 \leq j_2 \leq \cdots \leq j_\theta, \tag{3.15}$$

then the differentials evaluated at z_0

$$d_{z_0}\det M^*_{z_0,\#(i_1,j_1)}(\cdot),\ \dots,\ d_{z_0}\det M^*_{z_0,\#(i_\theta,j_\theta)}(\cdot)$$

are linearly independent.

Lemma 78. *For* $\theta = n+1-c+2\big(\dim J^p(X,Y)-1-r\big)$*, there do exist pairwise distinct couples as above.*

Proof of Lemma 78. One may for instance consider the couples of indexes successively encountered along the following "path" in the matrix $\hat{M}_{z_0}$: starting from the upper-left corner of the matrix, and then moving horizontally along the first row until the upper-right corner, and then moving vertically along the last column until the bottom-right corner. It is clear that the couples of indexes successively encountered along this path satisfy the condition (3.15). Their number is the "semi-perimeter" of the matrix, or more rigorously $\big|\mathcal{R}(\hat{M}_{z_0})\big| + \big|\mathcal{C}(\hat{M}_{z_0})\big| - 1$. Recalling the definition of $\hat{M}_{z_0}$, this is the same as

$$\begin{aligned}|\mathcal{R}(M)| - |\mathcal{R}(M^*_{z_0})| + |\mathcal{C}(M)| - |\mathcal{C}(M^*_{z_0})| - 1 &= (\dim J^p(X,Y) - r) + (m+n-r) - 1\\ &= n+1-c+2\big(\dim J^p(X,Y)-1-r\big)\end{aligned}$$

This ends the proof of Lemma 78. □

Proof of Lemma 77. For $(i,j) \in \mathcal{R}(B)\times\mathcal{C}(B)$, we recall that i is the index of a line of B, hence it corresponds to a coordinate $y^{s(i)}_{\alpha(i)}$ of $\mathcal{F}^p_{p-1}$, while j is an integer between $m+1$ and $m+n$. Then, the coefficient of the matrix $B(z)$ at line i and column j is just $y^{s(i)}_{\alpha(i)+\delta_{j-m}}$. It is a component, that we denote by $z_{[i,j]}$, of z. Note however that the same component of z may appear at several different places in the matrix $B(z)$. It can happen that $z_{[i,j]} = z_{[i',j']}$ with $(i,j)\neq(i',j')$, it is the case when $s(i)=s(i')$ and $\alpha(i)+\delta_{j-m} = \alpha(i')+\delta_{j'-m}$. Our order on the coordinates allows us to overcome this difficulty thanks to the following Lemma:

Lemma 79. *Let* (i,j) *and* (h,k) *belong to* $\mathcal{R}(B)\times\mathcal{C}(B)$*, and satisfy*

$$h\geq i,\ \ k\geq j,\ \ (i,j)\neq(h,k).$$

Then, $z_{[h,k]}\neq z_{[i,j]}$ *and, if* (i,j) *and* (h,k) *belong to* $\mathcal{R}(\hat{M}_{z_0})\times\mathcal{C}(\hat{M}_{z_0})$*, then*

$$\begin{aligned}\frac{\partial}{\partial z_{[i,j]}}\det M^*_{z_0,\#(i,j)}(z_0) &= \pm\det M^*_{z_0}(z_0)\neq 0\\ \frac{\partial}{\partial z_{[h,k]}}\det M^*_{z_0,\#(i,j)}(z_0) &= 0.\end{aligned}$$

Lemma 79 implies that $z_{[i_1,j_1]},\dots,z_{[i_\theta,j_\theta]}$ are pairwise distinct components of z, and that the square matrix

$$\begin{bmatrix}\dfrac{\partial}{\partial z_{[i_1,j_1]}}\det M^*_{z_0,\#(i_1,j_1)}(z_0) & \dots & \dfrac{\partial}{\partial z_{[i_\theta,j_\theta]}}\det M^*_{z_0,\#(i_1,j_1)}(z_0)\\ \vdots & \ddots & \vdots\\ \dfrac{\partial}{\partial z_{[i_1,j_1]}}\det M^*_{z_0,\#(i_\theta,j_\theta)}(z_0) & \dots & \dfrac{\partial}{\partial z_{[i_\theta,j_\theta]}}\det M^*_{z_0,\#(i_\theta,j_\theta)}(z_0)\end{bmatrix}$$

has the form

$$\begin{bmatrix} \pm\det M^*_{z_0}(z_0) & 0 & 0 & \dots & 0 \\ * & \pm\det M^*_{z_0}(z_0) & 0 & \dots & 0 \\ * & * & \pm\det M^*_{z_0}(z_0) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & \pm\det M^*_{z_0}(z_0) \end{bmatrix} \tag{3.16}$$

hence is invertible, since $\det M^*_{z_0}(z_0) \neq 0$. This proves Lemma 77, using Lemma 79. □

Proof of Lemma 79. Let us first prove that $z_{[i,j]} \neq z_{[h,k]}$. If $s(i) \neq s(h)$, then the conclusion holds. If $s(i) = s(h)$, then $\alpha(i) \geqslant \alpha(h)$ for the lexicographic order. On the other hand, the inequality $k \geqslant j$ implies that $\delta_{k-m} \leqslant \delta_{j-m}$ for the lexicographic order. These two inequalities do not sum to an equality because they are not both equalities (recall the hypothesis $(i,j) \neq (h,k)$), hence $\alpha(i) + \delta_{j-m} \neq \alpha(h) + \delta_{k-m}$, and then $z_{[i,j]} \neq z_{[h,k]}$.

To prove the equality

$$\frac{\partial}{\partial z_{[h,k]}} \det M^*_{z_0,\#(i,j)}(z_0) = 0, \tag{3.17}$$

let us consider, for every $(h',k') \in \Big(\mathcal{R}(M^*_{z_0,\#(i,j)}) \times \mathcal{C}(M^*_{z_0,\#(i,j)})\Big)$, the submatrix $N_{h',k'}$ defined by

$$\mathcal{R}(N_{h',k'}) = \mathcal{R}(M^*_{z_0,\#(i,j)}) \setminus \{h'\} = \Big(\mathcal{R}(M^*_{z_0}) \cup \{i\}\Big) \setminus \{h'\}$$
$$\mathcal{C}(N_{h',k'}) = \mathcal{C}(M^*_{z_0,\#(i,j)}) \setminus \{k'\} = \Big(\mathcal{C}(M^*_{z_0}) \cup \{j\}\Big) \setminus \{k'\}.$$

We have

$$\frac{\partial}{\partial z_{[h,k]}} \det M^*_{z_0,\#(i,j)}(z_0) = \sum_{(h',k')} \pm\det N_{h',k'}(z_0)$$

where the actual sign is irrelevant and the sum is taken over all couples $(h',k') \in \Big(\mathcal{R}(M^*_{z_0,\#(i,j)}) \times \mathcal{C}(M^*_{z_0,\#(i,j)})\Big)$ such that $z_{[h',k']} = z_{[h,k]}$.

We claim that each square matrix $N_{h',k'}$ is singular, thus proving 3.17. In view of the definition of $M^*_{z_0}$ in Lemma 76, it is enough to observe that we can't have both $M^*_{z_0} \preceq_{\mathcal{R}} N_{h',k'}$ and $M^*_{z_0} \preceq_{\mathcal{C}} N_{h',k'}$. This would imply that we have both $h' \leqslant i$ and $k' \leqslant j$ and then that $h' \leqslant h$ and $k' \leqslant k$. As we have already seen, since $z_{[h',k']} = z_{[h,k]}$, this would imply that $(h',k') = (h,k)$. Since $h' \leq i \leq h$ and $k' \leq j \leq k$, we would finally have $(h,k) = (i,j)$, in contradiction with our hypotheses.

Finally, we have

$$\frac{\partial}{\partial z_{[i,j]}} \det M^*_{z_0,\#(i,j)}(z_0) = \pm\det M^*_{z_0}(z_0) + \sum_{(i',j')} \pm\det N_{i',j'}(z_0)$$

where the sum is taken on all couples $(i',j') \in \Big(\mathcal{R}(M^*_{z_0,\#(i,j)}) \times \mathcal{C}(M^*_{z_0,\#(i,j)})\Big)$ such that $z_{[i',j']} = z_{[i,j]}$. We conclude as above that all the terms in the sum vanish. □

3.3.2 The analytic case

We prove Theorem 74 by recurrence on p. First of all, note that if A is analytic then $\tilde{A}$ is defined by analytic conditions (at least in suitable charts). It is then a stratified set, and each stratum is itself analytic.

When $p = 0$, $A = A_1$ hence the statement follows from Theorem 73.

When $p \geq 1$, the set A_0 may be non-empty. However, since it is defined by analytic conditions (at least in suitable charts), it is a stratified set. Hence it suffices to bound the codimension of the stratum $S \subset A_0$ of maximal dimension. Let us consider the restricted projection

$$\left(\pi^p_{p-1|S}\right) \colon S \to J^{p-1}(X,Y).$$

and the associated rank map

$$S \ni a \mapsto \operatorname{rank} d_a\left(\pi^p_{p-1|S}\right) \quad \in \left\{0, 1, \dots, \min\{\dim S, \dim J^{p-1}(X,Y)\}\right\}.$$

Let us also consider an open subset U of S such that the rank map is constant on U. Such a subset exists, for instance we can take as U the preimage of the maximum value attained by the map (this preimage is open because the rank map is lower-semicontinuous).

It follows from the constant-rank theorem that, up to further restricting U if necessary, $\pi^p_{p-1}(U)$ is a submanifold of $J^{p-1}(X,Y)$. Let us call V this manifold. We claim that

$$U \subseteq \tilde{V}$$

where $\tilde{V} \subseteq J^p(X,Y)$ is defined according to (3.1), i.e.

$$\tilde{V} = \left\{j^p_x f \in J^p(X,Y) : j^{p-1} f \text{ is not transverse to } V \text{ at } x\right\}.$$

Since the conclusion of Theorem 74 is assumed to be true for $p-1$, we have $\operatorname{codim} \tilde{V} \geq n+1$, hence the claim implies $\operatorname{codim} U \geq n+1$. Since U is open in S and S is the stratum of maximal dimension, we get

$$\operatorname{codim} A_0 = \operatorname{codim} S = \operatorname{codim} U \geq n+1,$$

which proves that the conclusion of Theorem 74 is true for p, as desired. Let us now prove the claim $U \subseteq \tilde{V}$. Given any $a = j^p_x f \in U$, we have

$$T_{\pi^p_{p-1}a} V = \left(d\pi^p_{p-1|S}\right)(T_a U) \subseteq d\pi^p_{p-1}(T_a A).$$

Here the first equality follows by the constant-rank theorem, while the inclusion follows from the fact that $U \subseteq S \subseteq A$. Moreover, the very definition of A_0 yields

$$d\pi^p_{p-1}(T_a A) + E^{p-1}(a) \subsetneq T_{\pi^p_{p-1}a} J^{p-1}(X,Y).$$

It follows that

$$T_{\pi^p_{p-1}a} V + E^{p-1}(a) \subseteq d\pi^p_{p-1}(T_a A) + E^{p-1}(a) \subsetneq T_{\pi^p_{p-1}a} J^{p-1}(X,Y)$$

which implies that $a \in \tilde{V}$, as desired. □

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