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Vladimir Shchur

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UNIVERSITÉ PARIS-SUD  
École Doctorale de Mathématiques de la région Paris-Sud  
Laboratoire de Mathématiques de la Faculté des Sciences d'Orsay

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Discipline : Mathématiques  
par Vladimir SHCHUR

Quasi-isometries between hyperbolic metric  
spaces, quantitative aspects

Soutenue le 08 Juillet 2013 devant la commission d'examen:

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## Abstract

In this thesis we discuss possible ways to give quantitative measurement for two spaces not being quasi-isometric. From this quantitative point of view, we reconsider the definition of quasi-isometries and propose a notion of “quasi-isometric distortion growth” between two metric spaces. We revise our article [32] where an optimal upper-bound for Morse Lemma is given, together with the dual variant which we call Anti-Morse Lemma, and their applications.

Next, we focus on lower bounds on quasi-isometric distortion growth for hyperbolic metric spaces. In this class,  $\mathbb{L}^p$ -cohomology spaces provides useful quasi-isometry invariants and Poincaré constants of balls are their quantitative incarnation. We study how Poincaré constants are transported by quasi-isometries. For this, we introduce the notion of a cross-kernel. We calculate Poincaré constants for locally homogeneous metrics of the form  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ , and give a lower bound on quasi-isometric distortion growth among such spaces.

This allows us to give examples of different quasi-isometric distortion growths, including a sublinear one (logarithmic).

**Keywords.** Hyperbolic space, quasi-isometrie, quasi-geodesic, Morse Lemma, Poincaré inequality, Poincaré constant, quasi-isometric distortion growth.

## Résumé

Dans cette thèse, nous considérons les chemins possibles pour donner une mesure quantitative du fait que deux espaces ne sont pas quasi-isométriques. De ce point de vue quantitatif, on reprend la définition de quasi-isométrie et on propose une notion de “croissance de distorsion quasi-isométrique” entre deux espaces métriques. Nous révisons notre article [32] où une borne supérieure optimale pour le lemme de Morse est donnée, avec la variante duale que nous appelons Anti-Morse Lemma, et leurs applications.

Ensuite, nous nous concentrons sur des bornes inférieures sur la croissance de distorsion quasi-isométrique pour des espaces métriques hyperboliques. Dans cette classe, les espaces de  $\mathbb{L}^p$ -cohomologie fournissent des invariants de quasi-isométrie utiles et les constantes de Poincaré des boules sont leur incarnation quantitative. Nous étudions comment les constantes de Poincaré sont transportées par quasi-isométries. Dans ce but, nous introduisons la notion de transnoyau. Nous calculons les constantes de Poincaré pour les métriques localement homogènes de la forme  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ , et donnons une borne inférieure sur la croissance de distorsion quasi-isométrique entre ces espaces.

Cela nous permet de donner des exemples présentant différents type de croissance de distorsion quasi-isométrique, y compris un exemple sous-linéaire (logarithmique).

**Mots-clefs.** Espaces hyperbolique, quasi-isométrie, quasi-géodésique, Lemme de Morse, inégalité de Poincaré, constante de Poincaré, croissance de distorsion quasi-isométrique.

*À mes parents.*

# Contents

<b>Remerciements</b> .....	<b>6</b>
<b>Chapter 1 - Introduction - version française</b> .....	<b>8</b>
<b>1.1</b> - Version quantitative du problème de quasi-isométrie .....	8
1.1.1 Idée générale .....	8
1.1.2 Exemple : croissance polynômiale .....	8
1.1.3 Exemple : arbres .....	9
<b>1.2</b> - Résumé des résultats .....	9
1.2.1 Lemme de Morse .....	9
1.2.2 Le Lemme Anti-Morse .....	10
1.2.3 Bornes inférieures pour des espace localement homogènes à courbure négative .....	11
1.2.4 Bornes supérieures .....	11
<b>Chapter 2 - Introduction - English version</b> .....	<b>13</b>
<b>2.1</b> - The quantitative quasi-isometry problem .....	13
2.1.1 General idea .....	13
2.1.2 Example .....	13
<b>2.2</b> - Summary of results .....	14
2.2.1 Morse Lemma .....	15
2.2.2 Anti-Morse Lemma .....	15
2.2.3 Lower bounds for negatively curved locally homogeneous spaces .....	16
2.2.4 Upper bounds .....	17

<b>2.3</b> - Statement of the quantitative quasi-isometry problem	17
2.3.1 Definition of quasi-isometry	17
2.3.2 Choice of a class of maps	18
2.3.3 Example illustrating the behaviour of $\lambda_1$	19
2.3.4 Example illustrating the behaviour of $\lambda_2$	19
2.3.5 Role of the additive parameter $c$	19
2.3.6 Choice of a numerical measurement of distortion	20
2.3.7 An example of a left-invariant riemannian distance	22
2.3.8 Statement of quantitative problem	23
2.3.9 Example : maps to trees	23
<b>2.4</b> - Quasi-isometric classification - survey	24

## Chapter 3 - Morse Lemma 27

<b>3.1</b> - Basics of hyperbolic geometry	27
3.1.1 Metric definition	27
3.1.2 Case of geodesic metric spaces	28
3.1.3 Divergence	29
3.1.4 Isoperimetry	30
3.1.5 Comparison with trees	31
<b>3.2</b> - The geometry of $\delta$ -hyperbolic spaces	31
<b>3.3</b> - Quasi-geodesics and $\Delta$ -length	37
<b>3.4</b> - Exponential contraction	38
<b>3.5</b> - Quantitative version of the Morse lemma	39
3.5.1 Attempts	40
3.5.2 Proof of the Morse lemma	41
<b>3.6</b> - Optimality of Theorem 2	44
<b>3.7</b> - Anti-Morse lemma	44
<b>3.8</b> - Application of Anti-Morse Lemma	46
3.8.1 Proof of Proposition 1	46
<b>3.9</b> - Geodesically rich spaces	47
<b>3.10</b> - Quasi-isometries fixing the ideal boundary	48



<b>Chapter 4 - Poincaré inequalities and quasi-isometries</b> .....	<b>49</b>
<b>4.1 - Main ideas</b> .....	<b>49</b>
4.1.1 The critical exponent for $L^p$ -cohomology .....	49
4.1.2 Scheme of proof .....	49
<b>4.2 - Regularisation and quasi-isometries</b> .....	<b>50</b>
4.2.1 Kernels .....	50
4.2.2 Cross-kernels .....	53
4.2.3 Transporting cocycles .....	55
4.2.4 Transporting Poincaré constants .....	56
<b>4.3 - Poincaré inequality for exponential metric</b> .....	<b>59</b>
4.3.1 Poincaré inequality for fixed direction .....	59
4.3.2 Poincaré inequality for exponential metric. ....	61
<b>4.4 - Lower bound on Poincaré constant</b> .....	<b>62</b>
4.4.1 Quasi-isometric embeddings and fundamental groups .....	63
4.4.2 Lifting to a double covering space .....	64
4.4.3 Lifting of $\Theta$ .....	65
4.4.4 Proof of the first statement of Theorem 24 - Part 1 .....	67
4.4.5 Proof of the first statement of Theorem 24 - Part 2 .....	68
4.4.6 Proof of the second statement of Theorem 24 .....	71
<b>Chapter 5 - Examples of different distortion growths</b> .....	<b>72</b>
<b>5.1 - Approximation of distances and an example of QI</b> .....	<b>72</b>
<b>5.2 - Examples</b> .....	<b>75</b>
5.2.1 Bi-Hölder maps .....	75
5.2.2 Unipotent locally homogeneous space .....	76
<b>Chapter 6 - Appendix</b> .....	<b>80</b>
<b>6.1 - Poincaré inequality for <math>\mathbb{H}^n</math></b> .....	<b>80</b>
<b>6.2 - Equivalence of three forms of the Poincaré inequality</b> .....	<b>82</b>
<b>Bibliography</b> .....	<b>84</b>

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# Chapter 1

## Introduction - version française

### 1.1 Version quantitative du problème de quasi-isométrie

#### 1.1.1 Idée générale

Le problème de classification des groupes de type fini à quasi-isométrie près [4] à donné lieu à des travaux nombreux dont présenterons un survol en section 4. Lorsqu'on peut montrer que deux groupes ne sont pas quasi-isométriques, on souhaiterait pouvoir le mesurer quantitativement. Il y a sans doute des groupes assez voisins, et d'autres franchement éloignés. L'objet de cette thèse est de mesurer un écart quasi-isométrique à l'échelle  $R$ , pour tout  $R > 0$ , et d'étudier comment cet écart se comporte quand  $R$  tend vers l'infini.

Étant donnés deux espaces métriques  $X$  et  $Y$ , pointés en  $x_0$  et  $y_0$ , notre mesure  $\lambda(R)$  de l'écart à l'échelle est le plus petit  $\lambda$  tel qu'il existe un plongement  $(\lambda, \lambda)$ -quasi-isométrique de la boule de rayon  $R$   $B_X(x_0, R)$  dans  $Y$  envoyant  $x_0$  sur  $y_0$ . Dans la thèse, on essaie quelques autres choix, mais, pour cette introduction, nous nous tiendrons à celui-là.

#### 1.1.2 Exemple : croissance polynômiale

Nous considérons le théorème suivant comme le prototype d'un résultat quantitatif. Y. Shalom et T. Tao ont donné une version quantitative du célèbre théorème de Gromov affirmant que tout groupe à croissance polynômiale est virtuellement nilpotent.

**Théorème 1.** (*Y. Shalom, T. Tao [25]*) *Soit  $G$  un groupe engendré par une partie symétrique finie  $S$ . Supposons qu'il existe un  $R > \exp(\exp(Cd^C))$  tel que*

$$|B_S(R)| \leq R^d.$$

*Alors  $G$  possède un sous-groupe d'indice fini  $H$  qui est nilpotent de classe au plus  $C^d$ . Ici,  $C$  est une constante absolue.*

En voici un corollaire

**Corollaire 1.** *Soit  $(G, S)$  un groupe de type fini. Supposons que  $G$  n'est pas virtuellement nilpotent. Alors, pour tout  $R > 1/\sigma$ ,*

$$|B_S(R)| \geq R^{\sigma(\log \log R)^\sigma}.$$

*Ici,  $\sigma$  est une constante absolue.*

Pour notre problème de quasi-isométrie quantitative, voici ce qu'il en résulte.

*Exemple 1.* Groupes nilpotent versus groupes non-nilpotents.

Soient  $G$  et  $H$  des groupes de type fini, avec  $H$  virtuellement nilpotent et  $G$  non virtuellement nilpotent. Fixons des systèmes générateurs finis  $S \subset G$  et  $S' \subset H$ . On obtient des graphes de Cayley  $G_S$  et  $H_{S'}$ . Alors il existe une constante universelle  $\sigma$  et une constante  $C = C(G, S, H, S')$  dépendant des groupes et des systèmes générateurs telles que, pour tout plongement  $(\lambda, c)$ -quasi-isométrique de  $B_{G_S}(R)$  dans  $H_{S'}$ ,

$$\lambda + c \geq C(\log \log R)^\sigma \log R.$$

Le fait que  $G$  ne soit pas à croissance polynômiale donne seulement la borne  $\lambda(R) \geq \Omega(\log R)$ . Le théorème de Shalom et Tao permet de gagner le facteur  $(\log \log R)^\sigma$ .

### 1.1.3 Exemple : arbres

Un espace dans lequel le complémentaire d'une boule est connexe nécessite une distorsion au moins égale à  $\sqrt{R}$  pour être envoyé dans une droite ou un arbre.

**Proposition 1.** *Soit  $X$  un espace métrique géodésique. On suppose que pour tous points  $x, y$  et tous rayons  $R$  and  $R' \leq R/2$  l'ensemble  $B(x, R) \setminus B(y, R')$  est connexe. Soit  $Y$  un arbre, soit  $f : B(x, R) \rightarrow Y$  un plongement  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isométrique. Alors  $R \leq 12\lambda_2c_1 + 4c_2$ .*

Cette borne inférieure est optimale : Si  $X$  est le plan hyperbolique, une boule de rayon  $R$  s'envoie de façon  $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$ -quasi-isométrique dans un arbre simplicial de valence 3.

## 1.2 Résumé des résultats

### 1.2.1 Lemme de Morse

Le Lemme de Morse affirme que, dans un espace métrique hyperbolique, une  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-géodésique (voir définitions 5, 16)  $\gamma$  est contenue dans le  $H(\lambda_1, \lambda_2, c_1, c_2)$ -voisinage de toute géodésique de mêmes extrémités. Nous donnons une majoration optimale de  $H$ .

**Théorème 2** (Morse lemma). *Soit  $\gamma$  une  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-géodésique dans un espace métrique  $\delta$ -hyperbolique  $E$ . Soit  $\sigma$  une géodésique de mêmes extrémités. Alors  $\gamma$  est contenue dans le  $H$ -voisinage de  $\sigma$ , où*

$$H = A\lambda_1\lambda_2 \left( c_1 + c_2 + \delta + 1 \right),$$

*et  $A$  est une constante absolue.*

La preuve est donnée en Section 3.5.2. Ce résultat est optimal, i.e., il existe dans tout espace métrique hyperbolique un exemple de quasi-géodésique  $\gamma$  possédant un point situé à une distance de toute géodésique  $\sigma$  de mêmes extrémités au moins égale à  $\lambda_1 \lambda_2 \min\{c_1, c_2\}/4$  (voir en Section 3.6).

Le Lemme de Morse joue un rôle important dans l'étude des espaces métriques hyperboliques. Par exemple, on s'en sert pour montrer que l'hyperbolicité est invariante par quasi-isométrie [1] (voir Chapitre 5.2, Théorème 12) : Soient  $E$  et  $F$  des espaces  $\delta_1$ - et  $\delta_2$ -hyperboliques et géodésiques. S'il existe une  $(\lambda, c)$ -quasi-isométrie entre eux, alors

$$\delta_1 \leq 8\lambda(2H + 4\delta_2 + c).$$

Nous espérons que notre borne optimale dans le Lemme de Morse sera utile pour le problème de quasi-isométrie quantitative.

### 1.2.2 Le Lemme Anti-Morse

Voici une seconde illustration. Dans certains espace métriques hyperboliques, les quasi-isométries fixant le bord à l'infini ne déplacent les points que d'une distance bornée. Une application directe du Lemme de Morse donne une borne en  $\lambda^2 c$ , alors que les exemples connus ne réalisent que  $\lambda c$ . Nous parvenons à diminuer cet écart, en utilisant le fait suivant que nous avons baptisé Lemme Anti-Morse.

**Théorème 3** (Anti-Morse lemma). *Soit  $\gamma$  une  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-géodésique dans un espace métrique  $\delta$ -hyperbolique. Soit  $\sigma$  une géodésique de mêmes extrémités. Alors  $\sigma$  est contenue dans le  $H_{am}$ -voisinage de  $\gamma$ , où  $H_{am} = A_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$ . Ici,  $A_3$  est une constante absolue.*

La preuve se trouve en Section 3.7.

Dans la Section 3.10, nous définirons la classe des espaces métriques hyperboliques géodésiquement riches (elle contient tous les groupes hyperboliques) qui possède la propriété suivante.

**Théorème 4.** *Soit  $X$  un espace métrique  $\delta$ -hyperbolique géodésiquement riche. Soit  $f$  une  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isométrie qui fixe le bord à l'infini  $\partial X$ . Alors, pour tout point  $O \in X$ , le déplacement  $d(O, f(O)) \leq \lambda_1(H_{am} + r_1) + 2c_1 + A_4$ , où  $r_1$  et  $A_4$  sont des constantes dépendant seulement de  $X$ .*

Dans la deuxième partie de cette thèse, nous discuterons de la géométrie des espaces hyperboliques et prouverons un lemme important sur la contraction exponentielle des longueurs des courbes par projection sur une géodésique. Muni de cet outil, nous démontrerons le Lemme de Morse et le Lemme Anti-Morse. Nous définirons la classe des espaces métriques hyperboliques géodésiquement riches et nous estimerons le déplacement des points par les quasi-isométries fixant le bord à l'infini. Finalement, nous montrerons que cette classe contient les groupes hyperboliques.

### 1.2.3 Bornes inférieures pour des espace localement homogènes à courbure négative

La troisième partie porte sur le transport des inégalités de Poincaré par les quasi-isométries. A l'aide des résultats obtenus, on obtient une borne inférieure sur la distorsion quasi-isométrique entre des espaces de la forme  $Z_\mu = \mathbb{T}^n \times \mathbb{R}$  munis de métriques de la forme  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ . Notre théorème dit que cette distorsion croît linéairement avec le rayon pour les plongements quasi-isométriques qui sont des équivalences d'homotopie.

**Théorème 5.** (*Version sommaire. Pour un énoncé plus précis, voir Théorème 24.*) *Tout plongement  $(\lambda, c)$ -quasi-isométrique d'une boule de rayon  $R$  de  $Z_\mu$  dans  $Z_{\mu'}$ , qui est une équivalence d'homotopie, satisfait*

$$\lambda + c \geq \left( \frac{\sum \mu_i}{\max \mu_i} - \frac{\sum \mu'_i}{\max \mu'_i} \right) R.$$

La preuve utilise plusieurs résultats intermédiaires qui ont un intérêt indépendant. Nous étudions d'abord le transport des inégalités de Poincaré par les quasi-isométries. Pour cela, nous introduisons les *transnoyaux*, généralisant l'opération de composition d'une fonction, préalablement lissée par convolution, avec une quasi-isométrie.

Ensuite, nous établissons une majoration de la constante de Poincaré de la boule de rayon  $R$  d'un espace  $Z_\mu$ ,

$$C_p(\mu) \leq c \left( p, \sum \mu_i \right) e^{(\max \mu_i)R},$$

où  $c(p, \sum \mu_i)$  est une constante dépendant seulement de  $n$ ,  $p$  et de la somme des  $\mu_i$ .

Un mot sur la preuve du Théorème 5. La particularité de l'espace  $Z_{\mu'}$ , c'est qu'on peut prendre comme fonction test la fonction à valeurs complexes  $e^{2\pi i x_n}$ , dont le module vaut 1 partout. Son gradient est dans  $\mathbb{L}^p$  si et seulement si  $p > \sum \mu'_i / \max \mu'_i$ . En la transportant, on obtient une fonction dont le module est partout proche de 1. Pour en faire une fonction test pour l'inégalité de Poincaré sur une boule de  $Z_\mu$ , il faudrait que sa moyenne soit nulle. C'est là que l'hypothèse que le plongement quasi-isométrique est une équivalence d'homotopie intervient. Elle autorise à relever le plongement à des revêtements doubles  $\tilde{Z}_\mu$  et  $\tilde{Z}_{\mu'}$ , à transporter une fonction impaire  $e^{\pi i x_n}$  sur le revêtement  $\tilde{Z}_{\mu'}$ . La fonction transportée  $v$  étant impaire, elle est d'intégrale nulle. Sa norme  $\mathbb{L}^p$  croît comme  $e^{R \sum \mu_i / p}$ , la norme  $\mathbb{L}^p$  de son gradient est bornée, d'où une minoration de la constante de Poincaré de la boule de rayon  $R$  de  $Z_\mu$ .

C'est en confrontant cette minoration avec la majoration obtenue plus haut (appliquée au revêtement  $\tilde{Z}_\mu$ ) que nous obtenons une minoration de l'écart quasi-isométrique  $\lambda(R)$ . Cette minoration est optimale, voir au paragraphe 1.2.4.

L'hypothèse homotopique est peut-être nécessaire.

### 1.2.4 Bornes supérieures

Dans la quatrième partie, nous donnons une construction de quasi-isométries entre boules de même rayon d'espaces métriques hyperboliques. Pour cela, nous donnons une expression approchée (erreur additive bornée) de la distance dans un espace métrique hyperbolique en

fonction de la distance visuelle sur le bord à l'infini vu d'un point  $P_0$ . Si  $P_1, P_2 \in X$  sont à des distances de  $P_0$  notées  $t_1$  et  $t_2$ , alors

$$d(P_1, P_2) = t_1 + t_2 - 2 \min\{t_1, t_2, t^\infty\},$$

où  $-t^\infty$  est le logarithme de la distance visuelle entre les extrémités des géodésiques issues de  $P_0$  et passant par  $P_1$  et  $P_2$ . Cela permet de borner les constantes de quasi-isométrie des extensions *radiales* des homéomorphismes entre bords à l'infini.

**Théorème 6.** *Soient  $X, Y$  des espaces métriques hyperboliques. Soit  $\theta : \partial X \rightarrow \partial Y$  un homéomorphisme. Pour  $R > 0$ , notons*

$$K(R) = \sup \left\{ \left| \log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)} \right| \mid d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{y_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R} \right\}.$$

*Ici,  $d_{x_0}, d_{y_0}$  sont des métriques visuelles sur les bords à l'infini. Alors l'extension radiale de  $\theta$  est une  $(K(R), K(R))$ -quasi-isométrie de  $B_X(x_0, R)$  sur  $B_Y(y_0, R)$ .*

Pour  $Z_\mu, Z_{\mu'} = \mathbb{T}^n \times \mathbb{R}$ ,  $\theta$  est l'identité sur  $\mathbb{T}^n$ ,  $\Theta$  est l'identité sur  $\mathbb{T}^n \times \mathbb{R}$  (c'est une équivalence d'homotopie), on vérifie que  $K(R) = \max_i |\mu_i/\mu'_i - 1|R$ , ce qui montre que la borne inférieure du paragraphe précédent est optimale. Nous donnons ensuite un exemple de variété homogène à courbure négative non quasi-isométrique à l'espace hyperbolique, mais donnant lieu à une distorsion  $K(R) \lesssim \log R$ .



## Chapter 2

# Introduction - English version

### 2.1 The quantitative quasi-isometry problem

#### 2.1.1 General idea

Gromov's quasi-isometry classification problem for groups [4] has given rise to a large amount of works (for the reader's convenience, we include a survey of the quasi-isometry classification problem in Section 2.4). When two groups are shown to be non-quasi-isometric, it would be desirable to give a quantitative measurement of this (we thank Itai Benjamini for bringing this issue to our attention). The aim of our research is to measure quantitatively how far two spaces are from being quasi-isometric at scale  $R > 0$ , and study on examples what may happen as  $R$  tends to infinity.

Let  $X$  and  $Y$  be two metric spaces not quasi-isometric to each other. Given some positive real number  $R$ , consider quasi-isometries between subsets in  $X$  and  $Y$  respectively of diameter of the order of  $R$ . These subsets are bounded spaces so there exists a  $(\lambda, c)$ -quasi-isometry with minimal  $\lambda = \lambda(R)$ . For simplicity, we shall assume that additive constants  $c$  are much less than  $\lambda(R)$ . We want to study how  $\lambda(R)$  behaves as  $R$  goes to infinity. Later, we shall give precise (and rather cumbersome) definitions, but in this introduction, we content ourselves with a rather vague one.

#### 2.1.2 Example

We consider the following theorem as the prototype of a quantitative result. Y. Shalom and T. Tao gave a quantitative version of Gromov's famous theorem stating that every finitely generated group of polynomial growth is virtually nilpotent.

**Theorem 1.** (*Y. Shalom, T. Tao [25]*) *Let  $G$  be a group generated by a finite (symmetric) set  $S$  and suppose that one has a polynomial growth condition*

$$|B_S(R)| \leq R^d$$

*for some*

$$R > \exp(\exp(Cd^C))$$

for some sufficiently large absolute constant  $C$ . Then  $G$  contains a finite index subgroup  $H$  which is nilpotent of step at most  $C^d$ .

A corollary of this theorem is

**Corollary 1.** *Let  $(G, S)$  be a finitely generated group. Assume that  $G$  is not virtually nilpotent. Then*

$$|B_S(R)| \geq R^{\sigma(\log \log R)^\sigma}$$

for any  $R > 1/\sigma$ , where  $\sigma > 0$  is a sufficiently small absolute constant.

This has the following consequence for our quantitative quasi-isometry problem.

*Example 1.* Nilpotent versus non-nilpotent groups.

Let  $G$  and  $H$  be finitely generated groups, with  $H$  virtually nilpotent and  $G$  not virtually nilpotent group. Pick finite generating systems  $S \subset G$  and  $S' \subset H$  and get metric spaces  $G_S$  and  $H_{S'}$ . If  $\Theta : B_{G_S}(R) \rightarrow H_{S'}$  is a  $(\lambda, c)$ -quasi-isometric embedding, then  $\Theta(B_{G_S}(R)) \subset B_{H_{S'}}(\lambda R + c)$ . Let  $\Lambda$  be a  $\lambda + c$ -lattice in  $B_{G_S}(R)$ . One can pick  $\Lambda$  in such a way that

$$|\Lambda| \geq \frac{|B_{G_S}(R)|}{|B_{G_S}(\lambda + c)|} \geq e^{-C(\lambda+c)} |B_{G_S}(R)|,$$

here we estimate a number of points of  $\Lambda$  in  $B_{G_S}(R)$  as a ratio of volumes. On the other hand, since  $\Theta$  is injective on  $\Lambda$ ,

$$|B_{H_{S'}}(\lambda R + c)| \geq |\Lambda|.$$

Hence,

$$|B_{S'}(\lambda R + c)| \geq e^{-C(\lambda+c)} |B_S(R)|,$$

where  $C = C(G, S)$ .

Now as  $H$  is virtually nilpotent,  $|B_{S'}(R')| \leq K(R')^d$  where  $d$  depends on  $H$  only and  $K$  depends on  $H$  and  $S'$ . So Corollary 1 implies that

$$R^{\sigma(\log \log R)^\sigma} \leq |B_S(\lambda + c)| K(\lambda R + c)^d$$

and for  $R$  big enough we conclude that

$$\lambda + c \geq C(\log \log R)^\sigma \log R,$$

where  $C = C(G, S, H, S')$  is a constant depending on the groups and generating systems, but  $\sigma$  is universal.

The fact that  $G$  does not have polynomial growth gives a mere  $\lambda(R) \geq \Omega(\log R)$ . Shalom and Tao's theorem gives an extra factor of  $(\log \log R)^\sigma$ .

## 2.2 Summary of results

Here we will briefly discuss our results.

### 2.2.1 Morse Lemma

Roughly speaking, the Morse lemma states that in a hyperbolic metric space, a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasigeodesic (see definitions 5, 16)  $\gamma$  belongs to a  $\lambda_1\lambda_2(c_1 + c_2)$ -neighborhood of every geodesic  $\sigma$  with the same endpoints. Our aim is to prove the optimal upper bound for the Morse lemma.

**Theorem 2** (Morse lemma). *Let  $\gamma$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic in a  $\delta$ -hyperbolic space  $E$  and let  $\sigma$  be a geodesic segment connecting its endpoints. Then  $\gamma$  belongs to an  $H$ -neighborhood of  $\sigma$ , where*

$$H = A\lambda_1\lambda_2\left(c_1 + c_2 + \delta + 1\right),$$

and  $A$  is some universal constant.

We will prove this theorem in Section 3.5.2. This result is optimal, i.e., there exists an example of a quasi-geodesic such that the distance of the farthest point of  $\gamma$  from  $\sigma$  is  $\lambda_1\lambda_2 \min\{c_1, c_2\}/4$  (see Section 3.6).

The Morse lemma plays an important role in the geometry of hyperbolic spaces. For example, it is used to prove that hyperbolicity is invariant under quasi-isometries between geodesic spaces [1] (see Chapter 5.2, Theorem 12): let  $E$  and  $F$  be  $\delta_1$ - and  $\delta_2$ -hyperbolic geodesic spaces. If there exists a  $(\lambda, c)$ -quasi-isometry between these two spaces, then

$$\delta_1 \leq 8\lambda(2H + 4\delta_2 + c).$$

We expect our optimal bound in the Morse lemma to be a useful tool in the quantitative quasi-isometric embedding problem for hyperbolic metric spaces.

### 2.2.2 Anti-Morse Lemma

We give a second illustration. In certain hyperbolic metric spaces, self-quasi-isometries fixing the ideal boundary move points a bounded distance. Directly applying the Morse lemma yields a bound of  $H \sim \lambda^2c$ , while the examples that we know achieve merely  $\lambda c$ . For this problem, we can fill the gap partially. Our argument relies on the following theorem, which we call the anti-Morse lemma.

**Theorem 3** (anti-Morse lemma). *Let  $\gamma$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic in a  $\delta$ -hyperbolic metric space and  $\sigma$  be a geodesic connecting the endpoints of  $\gamma$ . Then  $\sigma$  belongs to a  $H_{am}$ -neighborhood of  $\gamma$ , where  $H_{am} = A_3(\delta \ln \lambda_1\lambda_2 + \delta + c_1 + c_2)$ , here  $A_3$  is some universal constant.*

We prove Theorem 3 in Section 3.7.

As an example of an application of Anti-Morse Theorem we show that the center of a ball in a tree cannot be moved very far by a self-quasi-isometry.

**Proposition 1.** *Let  $O$  be the center of a ball of radius  $R$  in a  $d$ -regular metric tree  $T$  ( $d \geq 3$ ). Let  $f$  be  $(\lambda_1, \lambda_2, c_1, c_2)$ -self-quasi-isometry of this ball. Then*

$$d(f(O), O) \leq \min\{R, \lambda_1 H_{am} + c_1 + \lambda_1(c_1 + C_3)\},$$

where  $C_3$  depends on  $d$ .

Because  $\delta = 0$  for a tree, we have  $d(f(O), O) \lesssim \lambda_1(c_1 + c_2)$ . We prove this proposition in Section 3.8.

In Section 3.10, we define the class of geodesically rich hyperbolic spaces (it contains all Gromov hyperbolic groups), for which we can prove the following statement.

**Theorem 4.** *Let  $X$  be a geodesically rich  $\delta$ -hyperbolic metric space and  $f$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -self-quasi-isometry fixing the boundary  $\partial X$ . Then for any point  $O \in X$ , the displacement  $d(O, f(O)) \leq \lambda_1(H_{am} + r_1) + 2c_1 + A_4$ , where  $r_1$  and  $A_4$  are constants depending on the geometry of the space  $X$ .*

In Part 2, we shall first discuss the geometry of hyperbolic spaces and prove a lemma on the exponential contraction of lengths of curves with projections on geodesics. We then discuss the invariance of the  $\Delta$ -length of geodesics under quasi-isometries. Using these results, we prove the quantitative version of the Morse and anti-Morse lemmas. We define the class of geodesically rich spaces; for this class, we estimate the displacement of points by self-quasi-isometries that fix the ideal boundary. Finally, we show that this class includes all Gromov hyperbolic groups.

### 2.2.3 Lower bounds for negatively curved locally homogeneous spaces

The third part is devoted to the study of the transport of Poincaré inequalities by quasi-isometries. Using these results we will give a lower bound for the  $(\lambda, c)$ -quasi-isometric distortion between balls of radius  $R$  in spaces of the form  $Z_\mu = \mathbb{T}^n \times \mathbb{R}$  with exponential metrics  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$  and  $dt^2 + \sum_i e^{2\mu'_i t} dx_i^2$ , where all  $\mu_i, \mu'_i$  are assumed to remain bounded both from below and above. Essentially our theorem states that the quasi-isometric distortion growth function is linear.

**Theorem 5.** *(Rough version. For a precise statement, see Theorem 24). Every  $(\lambda, c)$ -quasi-isometric embedding of an  $R$ -ball in  $Z_\mu$  into  $Z_{\mu'}$  satisfies*

$$\lambda + c \geq \left( \frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_n}{\mu'_n} \right) R.$$

The proof of this theorem involves several results which could have an independent interest and more applications. First, we study the transport of Poincaré inequalities by quasi-isometries. For this purpose we propose to use “cross-kernels”. These objects are naturally obtained as follows. Let  $X$  and  $Y$  be two metric spaces,  $f : X \rightarrow Y$  a quasi-isometry and  $\psi(y_1, y_2)$  a kernel on  $Y$ . The composition relatively to the first argument  $\psi(f(x), y)$  is an example of a cross-kernel. Cross-kernels help us to transport functions from  $Y$  to  $X$  and allow us to control quantitatively their Poincaré constants.

Further, we establish an upper-bound for the Poincaré constant of ball in an exponential metric  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ ,

$$C_p(\mu) \leq c \left( p, \sum \mu_i \right) \left( 1 + (\max_i \mu_i) R \right),$$

where  $c(p, \sum \mu_i)$  is a constant depending only on  $p$  and the sum of  $\mu_i$ .

## 2.2.4 Upper bounds

In Part 4, we shall give a construction of quasi-isometries between balls in hyperbolic metric spaces. We begin with the approximation (up to an additive error depending on hyperbolicity constant) of the distance between two points. Let  $(X, P_0)$  be a hyperbolic metric space with the base points  $P_0$ . Let  $P_1, P_2 \in X$  be two points in this space, the distances to the base point are  $d(P_1, P_0) = t_1$  and  $d(P_2, P_0) = t_2$ . Now consider the geodesics  $P_0P_1$  and  $P_0P_2$ , denote by  $-t^\infty$  the logarithm of visual distance between the ends at infinity of this geodesics. Then up to an additive error

$$d(P_1, P_2) = t_1 + t_2 - 2 \min\{t_1, t_2, t^\infty\}.$$

Using this formula we find quasi-isometry constants for the restriction on balls of a map  $\Theta$  between  $X$  and  $Y$  which is a kind of radial extension of a homeomorphism  $\theta$  between ideal boundaries. The following is a non technical statement of Theorem 25, see Section 5.1 for a complete statement.

**Theorem 6.** *Let  $X, Y$  be hyperbolic metric spaces. Let  $\theta : \partial X \rightarrow \partial Y$  be a homeomorphism. We define the following function. For  $R > 0$ ,*

$$K(R) = \sup \left\{ \left| \log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)} \right| \mid d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{y_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R} \right\}.$$

Here  $d_{x_0}, d_{y_0}$  denote visual metrics on ideal boundaries. Then there exists a  $(K(R), K(R))$ -quasi-isometry between  $B_X(x_0, R)$  and  $B_Y(y_0, R)$ .

For  $Z_\mu, Z_{\mu'} = \mathbb{T}^n \times \mathbb{R}$  with exponential metrics we show that  $K(R) = \max_i |\mu_i/\mu'_i - 1|R$ . Then we give an example of non-quasi-isometric negatively curved homogeneous manifolds with  $K(R) \lesssim \log R$ .

## 2.3 Statement of the quantitative quasi-isometry problem

### 2.3.1 Definition of quasi-isometry

**Definition 1.** Two metric spaces  $X$  and  $Y$  are said to be roughly quasi-isometric if there exists two maps  $f : X \rightarrow Y, g : Y \rightarrow X$  and two constants  $\lambda > 0$  and  $c \geq 0$  such that

- $|f(x) - f(y)| \leq \lambda|x - y| + c$  for every  $x, y \in X$ ,
- $|g(x') - g(y')| \leq \lambda|x' - y'| + c$  for every  $x', y' \in Y$ ,
- $|g(f(x)) - x| \leq c$  for every  $x \in X$ ,
- $|f(g(x')) - x'| \leq c$  for every  $x' \in Y$ .

The word *rough* is often dropped away.

The first two conditions mean that  $f$  and  $g$  are nearly Lipschitz if we are looking from afar. The two latter conditions provide that  $f$  and  $g$  are nearly inverse of each other. It is easy to check that the composition of two quasi-isometries is also a quasi-isometry. So, quasi-isometries provide an equivalence relation on the class of metric spaces.

*Remark 1.* Definition 1 is invariant under taking inverse maps.

**Definition 2.** A map  $f : E \rightarrow F$  between metric spaces is a *rough*  $(\lambda, c)$ -*quasi-isometric embedding* if for any two points  $x, y$  of  $E$

$$\frac{1}{\lambda}(|x - y|_E - c) \leq |f(x) - f(y)|_F \leq \lambda|x - y|_E + c.$$

This definition follows from the definition for two spaces being quasi-isometric but it does not include the existence of a nearly inverse map. We can easily transform Definition 2 to make it equivalent to Definition 1 by adding the condition that  $f$  is nearly surjective. We ask that the image of  $E$  is  $c$ -dense in  $F$ : for every point  $y$  of  $F$  there exists a point  $x$  of  $E$  such that  $d(y, f(x)) < c$ .

We also provide here a definition of  $c$ -connected map.

**Definition 3.** A map  $f : X \rightarrow Y$  between two metric space is called  $c$ -connected if for any point  $x \in X$  and any real number  $\delta > 0$  there exists  $\varepsilon > 0$  such that if a point  $x' \in X$  satisfies  $d(x, x') < \varepsilon$  then  $d(f(x), f(x')) < c + \delta$ ,

**Definition 4.** 1. A metric space  $X$  is called  $c$ -connected if for any two open sets  $U, V \subset X$  such that  $X = U \cup V$ , the intersection of a  $c$ -neighbourhood of  $U$  and  $V$  is not empty:  $(U + c) \cap V \neq \emptyset$ .

2. A metric space  $X$  is called  $c$ -connected by an arc if for any two points  $x, x' \in X$  there exists a  $c$ -connected map  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = x'$ .

First and second definitions are evidently equivalent.

### 2.3.2 Choice of a class of maps

What do we exactly mean by quasi-isometric distortion at scale  $R$  ?

We propose three different settings. Let  $X$  and  $Y$  be metric spaces. Let  $x_0, y_0$  be base point in  $X$  and  $Y$ . Given  $R > 0$ , three families of maps can be considered.

1. Quasi-isometries of  $B_X(x_0, R)$  onto  $B_Y(y_0, R)$ .
2. Quasi-isometries of  $B_X(x_0, R)$  onto  $B_Y(y_0, \rho(R))$ , for some function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ .
3. Quasi-isometric embeddings of  $B_X(x_0, R)$  to  $Y$ .

Neglecting the additive constant  $c$  for a while, these families give rise to distortion functions  $\lambda_1(R)$ ,  $\lambda_2(R)$  and  $\lambda_3(R)$ .

$\lambda_1$  has the advantage of letting  $X$  and  $Y$  play symmetric roles. We shall see next that lower bounds on  $\lambda_1$  can be obtained easily. In fact,  $\lambda_1$  may tend to infinity even if  $X$  and  $Y$  are

quasi-isometric. It is therefore rather surprising that non trivial upper bounds on  $\lambda_1$  can be given (Theorem 6).

$\lambda_2$  seems to be appropriate in certain settings, as examples below will show.

$\lambda_3$  is non-symmetric. It is natural in the sense that it stays bounded if and only if there exists a quasi-isometric embedding of  $X$  to  $Y$ . It looks harder to estimate from below. Nevertheless, this is what is done in Theorem 5.

### 2.3.3 Example illustrating the behaviour of $\lambda_1$

Let  $X$  and  $Y$  be two regular trees  $T_{d_1}$  and  $T_{d_2}$  respectively, suppose that  $d_1 < d_2$ . Consider two balls of radius  $R$  in both of these spaces, denote them by  $B_{d_1}(R)$  and  $B_{d_2}(R)$  respectively. What is the lower bound for the constants of quasi-isometry between them? The volume of  $B_{d_1}(R)$  is roughly  $d_1^R$  and the volume of  $B_{d_2}(R)$  is  $d_2^R$ . A  $(\lambda_R, c_R)$ -quasi-isometry  $f'_R : B_{d_1}(R) \rightarrow B_{d_2}(R)$  should preserve (in quasi-isometric sense) volumes. In our future calculations we will drop some multiplicative constants (which are bounded constants which depend only on a whole space and not on the particular radius  $R$ )

Divide  $B_{d_1}(R)$  in balls of radius  $c_R$ . The image of such a ball has maximal possible radius  $(\lambda_R + 1)c_R$  and the number of such balls is  $Vol(B_{d_1}(R))/Vol(B_{d_1}(c_R)) = d_1^R/d_1^{c_R}$ . By definition of a quasi-isometry  $B_{d_2}(R)$  should be covered by images of these balls, hence  $Vol(B_{d_2}(R)) \leq d_1^R/d_1^{c_R} Vol(B_{d_2}((\lambda_R + 1)c_R))$

$$d_2^R \leq d_1^R/d_1^{c_R} d_2^{(\lambda_R+1)c_R}.$$

From this relation we conclude that  $\lambda_R c_R = \Omega(R)$ . On the other hand, we know from [26] that two regular trees of degrees at least 4 are quasi-isometric.

### 2.3.4 Example illustrating the behaviour of $\lambda_2$

Take a  $d$ -regular tree. Now transform it in a  $d(d-1)$ -regular tree in a following way. Take an origin, drop away all its neighbours and add edges to all their ancestors (all the points of second level). Now we delete all points of third level and connect directly the points of second levels with corresponding points of fourth level. As a result we get a new tree which is evidently  $(2, 1)$ -quasi-isometric to the initial one. Moreover, any ball  $B_d(R)$  is  $(2, 1)$ -quasi-isometric to a ball in a new tree of radius  $R/2$ .

### 2.3.5 Role of the additive parameter $c$

Quasi-isometry constants are pairs  $(\lambda, c)$ . Up to now, we have neglected the additive constant  $c$ . But this cannot be done with impunity, as the following examples show.

*Example 2. Intervals.*

Consider intervals  $I_{\mathbb{R}} = [0, 1]$ ,  $I_{\mathbb{R}}^{\lambda} = [0, \lambda]$  in  $\mathbb{R}$  and  $I_{\mathbb{Z}} = [0, 1]$ ,  $I_{\mathbb{Z}}^{\lambda} = [0, \lambda]$  in  $\mathbb{Z}$ . The  $\lambda$  times stretching of  $I_{\mathbb{R}}$  to  $I_{\mathbb{R}}^{\lambda}$  is a  $(\lambda, 0)$ -quasi-isometry as inner points of  $I_{\mathbb{R}}$  fill the inner points of an image. The natural embeddings of  $I_{\mathbb{Z}}$  in  $I_{\mathbb{R}}$  and  $I_{\mathbb{Z}}^{\lambda}$  in  $I_{\mathbb{R}}^{\lambda}$  are both  $(1, 1)$ -quasi-isometries, though the stretching of  $I_{\mathbb{Z}}$  to  $I_{\mathbb{Z}}^{\lambda}$  is a  $(\lambda, \lambda)$ -quasi-isometry.

$$\begin{array}{ccc}
I_{\mathbb{R}} & \xrightarrow{(\lambda,0)} & I_{\mathbb{R}}^{\lambda} \\
(1,1) \downarrow & & \downarrow (1,1) \\
I_{\mathbb{Z}} & \xrightarrow{(\lambda,\lambda)} & I_{\mathbb{Z}}^{\lambda}
\end{array}$$

*Example 3.* Line versus plane.

Consider  $\mathbb{R}$  and  $\mathbb{R}^2$ . Here we will describe a  $(c_1R, c_2)$ -quasi-isometry between balls in these spaces ( $c_1$  and  $c_2$  are two universal constants). A ball in  $\mathbb{R}$  is just an interval of length  $2R$ . Stretch it  $R$  times and then fill a ball in  $\mathbb{R}^2$  with a serpentine or a zigzag with width 1. It is easy to check that this is indeed a  $(2R, 1)$ -quasi-isometry. Now change  $\mathbb{R}$  by  $\mathbb{Z}$  and  $\mathbb{R}^2$  by  $\mathbb{Z}^2$ . Though there exist evident  $(1, 1)$ -quasi-isometries between balls in  $\mathbb{R}$  and  $\mathbb{Z}$  and balls in  $\mathbb{R}^2$  and  $\mathbb{Z}^2$  provided by natural embeddings, there is no  $(c'_1R, c'_2)$ -quasi-isometry between balls in  $\mathbb{Z}$  and  $\mathbb{Z}^2$ . Moreover, the additive constant should be of order  $R$  with small constant  $c'_2$ . The reason is that by definition, the image should be  $c'_2$ -dense. That is the range should be covered by the balls of radius  $c'_2$  centered in the images of points of the departure space, hence

$$Vol(B_{\mathbb{Z}^2}(R)) \leq |B_{\mathbb{Z}}(R)|Vol(B_{\mathbb{Z}^2}(c'_2)).$$

In  $B_{\mathbb{Z}}(R)$  we have only  $2R$  points and up to some universal multiplicative constants we get

$$R^2 \leq R(c'_2)^2$$

which leads to

$$c'_2 \geq R.$$

$$\begin{array}{ccc}
B_R(\mathbb{R}) & \xrightarrow{(2R,0)} & I_R(\mathbb{R}^2) \\
(1,1) \downarrow & & \downarrow (1,1) \\
I_R(\mathbb{Z}) & \xrightarrow{(2R,2R)} & I_R(\mathbb{Z}^2)
\end{array}$$

We arrive at

*Conclusion 1.* In the quantitative problem both the multiplicative and the additive parameters are important.

### 2.3.6 Choice of a numerical measurement of distortion

Here we want to present a form of definition of quasi-isometries which is more convenient for quantitative problems and to study compositions of quasi-isometries. For this purpose, we shall observe that, under composition, quasi-isometry constants behave like elements of the affine group of the line. We shall introduce a natural distance on the affine group and prove that it is a function of  $\lambda + c^2/\lambda + 1/\lambda$ , where  $\lambda$  and  $c$  are quasi-isometry's constants.

Sometimes it will be useful for us to distinguish constants as follows.



**Definition 5.** We say that a map  $f : X \rightarrow Y$  is a quasi-isometric embedding if there exist constants  $\lambda_1, \lambda_2, c_1, c_2$  such that for any two points  $x_1, x_2 \in X$

$$\frac{1}{\lambda_2}(d_X(x_1, x_2) - c_2) \leq d_Y(f(x_1), f(x_2)) \leq \lambda_1 d_X(x_1, x_2) + c_1.$$

We say that  $X$  and  $Y$  are quasi-isometric if the image  $f(X)$  is  $c_3$ -dense in  $Y$  for some given constant  $c_3$ .

Study compositions of quasi-isometries. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be  $(\lambda_1, \lambda_2, c_1, c_2)$ - and  $(\mu_1, \mu_2, d_1, d_2)$ -quasi-isometries respectively (we use 5 here as definition of quasi-isometries).  $x_1, x_2$  are two points in  $X$ . Hence

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq \lambda_1 d_X(x_1, x_2) + c_1, \\ d_X(f^{-1}(y_1), f^{-1}(y_2)) &\leq \lambda_2 d_Y(y_1, y_2) + c_2. \end{aligned}$$

and

$$\begin{aligned} d_Z(g(y_1), g(y_2)) &\leq \mu_1 d_Y(y_1, y_2) + d_1, \\ d_Y(g^{-1}(z_1), g^{-1}(z_2)) &\leq \mu_2 d_Z(z_1, z_2) + d_2. \end{aligned}$$

hence for  $g \circ f$  we have

$$\begin{aligned} d_Z(g \circ f(x_1), g \circ f(x_2)) &\leq \lambda_1 \mu_1 d_X(x_1, x_2) + \mu_1 c_1 + d_1, \\ d_X((g \circ f)^{-1}(z_1), (g \circ f)^{-1}(z_2)) &\leq \lambda_2 \mu_2 d_Z(z_1, z_2) + \lambda_2 d_2 + c_2. \end{aligned}$$

We see that the distortion of metrics by a quasi-isometry  $f$  can be encoded into two matrices

$$F_1 = \begin{pmatrix} \lambda_1 & c_1 \\ 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} \lambda_2 & c_2 \\ 0 & 1 \end{pmatrix},$$

and in matrix form we can write

$$\begin{pmatrix} d_Y \\ 1 \end{pmatrix} \leq F_1 \begin{pmatrix} d_X \\ 1 \end{pmatrix}.$$

Encode  $g$  with matrices  $G_1$  and  $G_2$ . Hence the composition  $h = g \circ f$  is encoded by matrices

$$G_1 F_1, F_2 G_2.$$

Let  $D$  be a left-invariant distance on  $\mathbb{R} \times \mathbb{R}$ . We set  $D(f) = D((\lambda_1, c_1), (1, 0))$  the distance to an isometry and  $\tilde{D} = \max\{D(f), D(f^{-1})\}$ . It is easily seen that  $\tilde{D}$  satisfies the triangle inequality from the following relation (which uses that  $D$  is left-invariant)

$$\begin{aligned} D(h) &= D((\mu_1 d_1)(\lambda_1, c_1), (1, 0)) \leq D((\mu_1 d_1)(\lambda_1, c_1), (\mu_1, d_1)) + D((\mu_1, d_1), (1, 0)) = \\ &= D((\lambda_1, c_1), (1, 0)) + D((\mu_1, d_1), (1, 0)) = D(f) + D(g). \end{aligned}$$

### 2.3.7 An example of a left-invariant riemannian distance

Call the group of matrices encoding quasi-isometries by  $Aff$

$$Aff^+ = \left\{ \begin{pmatrix} \lambda & c \\ 0 & 1 \end{pmatrix}, \lambda > 0, c \in \mathbb{R} \right\}.$$

Fix the origin  $x_0 \in \mathbb{H}^2$ . For  $A \in Aff^+$  we set

$$D(1, A) = d(x_0, Ax_0).$$

Now,  $\mathbb{H}^2 = SL(2, \mathbb{R})/SO(2)$  and  $x_0 = [1]$ . The following classical formula for the hyperbolic metric can be found in [8]: let  $G \in SL(2, \mathbb{R})$ , let  $G = e^S O$  be the polar decomposition of  $G$ , that is  $S$  is a trace free symmetric

$$S = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

and  $O$  is orthogonal. Then  $d(x_0, Gx_0) = \|S\| = \sqrt{\text{tr}(S^2)} = \sqrt{2(a^2 + b^2)}$ . The action of  $Aff^+$  on  $\mathbb{H}^2$  is the restriction of the action of  $GL^+$  on  $\mathbb{H}^2 = PSL(2, \mathbb{R})/PO(2)$  where  $PSL(2, \mathbb{R}) = GL^+/\mathbb{R}_{*+}$ .

Let  $A \in Aff^+$ , then  $G = A/\sqrt{\det A} \in SL(2, \mathbb{R})$  and  $Gx_0 = Ax_0$ . Therefore

$$D(1, A) = d(x_0, Gx_0) = \|S\|$$

where

$$e^2 \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = AA^T = \frac{1}{\lambda} \begin{pmatrix} \lambda^2 + c^2 & c \\ c & 1 \end{pmatrix}.$$

The matrix

$$\frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

is orthogonal with trace being equal to 0, we conclude that the eigenvalues of its exponential are  $e^{\pm 2\sqrt{a^2 + b^2}}$ . On the other hand eigenvalues of this matrix are roots of the equation

$$x^2 - \frac{\lambda^2 + c^2 + 1}{\lambda}x + 1 = 0.$$

We conclude that the distance  $D = \sqrt{a^2 + b^2}$  is a function of  $D' = \log(\lambda + c^2/\lambda + 1/\lambda)$ .

We will define the measurement of a  $(\lambda_1, c_1, \lambda_2, c_2)$ -quasi-isometric embedding  $f$  as

$$D_0(f) = e^{D'((\lambda_1, c_1), (1, 0))} + e^{D'((\lambda_2, c_2), (1, 0))}.$$

Let us check that  $D_0$  is submultiplicative.

$$\begin{aligned} D_0(g \circ f) &= e^{D'(g \circ f(1, 0))} + e^{D'(f^{-1} \circ g^{-1}(1, 0))} = \\ &e^{D'(g(1, 0))} e^{D'(f(1, 0))} + e^{D'(g^{-1}(1, 0))} e^{D'(f^{-1}(1, 0))} \leq D_0(g) * D_0(f). \end{aligned}$$

### 2.3.8 Statement of quantitative problem

We finally come up with a precise notion of quasi-isometric distortion growth.

**Definition 6.** Let  $(X, x_0)$  be a space with a base point,  $Y$  be another space. Then we call *quasi-isometric distortion growth* the function

$$D_G(X, x_0, Y)(R) = \inf \{d | \exists f : B_{x_0}^X(R) \rightarrow Y \text{ is a quasi-isometric embedding, } d = D_0(f)\},$$

where  $B_{x_0}^X(R)$  is a ball in  $X$  centred at  $x_0$  of radius  $R$ .

### 2.3.9 Example : maps to trees

In the following proposition we can take for example a hyperbolic plane as the space  $X$ .

**Proposition 2.** *Let  $X$  be a geodesic metric space. We suppose that for any points  $x, y$  and any positive real numbers  $R$  and  $R' \leq R/2$  the set  $B_x(R) \setminus B_y(R')$  is connected and non-empty. Let  $Y$  be a tree, let  $f : B_x(R) \rightarrow Y$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding. Then  $R \leq 12\lambda_2c_1 + 4c_2$ .*

*Proof.* We are going to prove that there exist three points  $x_1, x_2$  and  $x$  such that  $x_1, x_2 \in B_x(R)$  and the distance  $d(x_1, x_2)$  is at least  $R$ . Consider a ball of radius  $2R$  centered in  $x_1$ . By hypothesis, the set  $B_{x_1}(2R) \setminus B_{x_1}(R)$  is non-empty, hence there exists a point  $x_2$  such that  $2R > d(x_1, x_2) \geq R$ . The space  $X$  is geodesic, hence now we can take the midpoint of  $x_1x_2$  as  $x$ .

Denote  $y_i = f(x_i)$  for  $i = 1, 2$ .

For any point  $y$  of a geodesic  $(y_1, y_2) \subset Y$  there exists a point  $z \in B_x(R)$  such that  $d(f(z), y) \leq c_1$ . This follows from the fact that the image of  $(x_1, x_2)$  is  $c_1$ -connected by the definition of a quasi-isometric embedding and every  $c_1$ -connected path between  $y_1$  and  $y_2$  includes the geodesic  $(y_1, y_2)$  in its  $c_1$ -neighbourhood.

Now consider a chain of points  $\{\tilde{x}_i\}$  connecting  $x_1, x_2$  and such that  $d(\tilde{x}_i, \tilde{x}_{i+1}) < c_1/\lambda_1$ . Hence, in the image  $d(f(\tilde{x}_i), f(\tilde{x}_{i+1})) < 2c_1$  and so there exists  $i$  such that  $d(f(\tilde{x}_i), y) \leq 2c_1$ . Notice that  $Y \setminus B_y(2c_1)$  has several  $(4c_1 - 2)$ -connected components and the distance between these components is at least  $4c_1$ .

Suppose that a point  $z$  is rather far from both  $x_1$  and  $x_2$ :  $d(z, x_i) > 4\lambda_2c_1 + c_2, i = 1, 2$ . Suppose also that  $R > 2(4\lambda_2c_1 + c_2)$  (if not there is nothing to prove). In the set  $B_x(R) \setminus B_z(4\lambda_2c_1 + c_2)$  we also find a  $c_1/\lambda_1$ -chain. Hence, there exists a point  $z' \notin B_z(4\lambda_2c_1 + c_2)$  of this path such that  $d(f(z'), y) \leq 2c_1$ . Hence,  $d(f(z), f(z')) \leq 4c_1$  and by property of quasi-isometry  $d(z, z') \leq 4\lambda_2c_1 + c_2$ , so  $z' \in B_z(4\lambda_2c_1 + c_2)$ . This leads to a contradiction with our hypothesis. Hence, for any  $y \in (y_1, y_2)$  there exists  $z' \in B_{x_1}(4\lambda_2c_1 + c_2) \cup B_{x_2}(4\lambda_2c_1 + c_2)$  such that  $d(f(z'), y) \leq 2c_1$ .

Consider two points  $y', y''$  on the geodesic  $(y_1, y_2)$  which are close enough to each other (more precisely  $d(y', y'') \leq c_2/\lambda_2$ ) and such that respective points  $z'$  and  $z''$  (which minimise distances to  $y'$  and  $y''$ , that is  $d(y', f(z')) \leq 2c_1$  and  $d(y'', f(z'')) \leq 2c_1$ ) lie in different balls  $z' \in B_{x_1}(4\lambda_2c_1 + c_2)$  and  $z'' \in B_{x_2}(4\lambda_2c_1 + c_2)$ . So, on the one hand  $d(z', z'') \geq R - 8\lambda_2c_1 - 2c_2$

and on the other by triangle inequality  $d(f(z'), f(z'')) \leq c_2/\lambda_2 + 4c_1$ . Hence  $R - 8\lambda_2c_1 - 2c_2 \leq \lambda_2(c_2/\lambda_2 + 4c_1) + c_2 = 4\lambda_2c_1 + 2c_2$ . So we get  $R \leq 12\lambda_2c_1 + 4c_2$ .  $\square$

*Example 4.* Now we will construct an example of a  $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$ -quasi-isometry of a  $R$ -ball in  $\mathbb{H}^2$  to a  $\sqrt{R}$ -ball in a tree which is an illustration for the previous proposition. In this example is that we will consider trees of variable degree which will depend on  $R$ .

Consider a ball  $B_{\mathbb{H}^2}(R, z_0)$  centered at  $z_0$ . We will define a discrete set of point generation by generation in the following way. The 0-generation is the origin  $z_0$ . For each  $k$  we pick a regular polygon of radius  $k\sqrt{R}$  and such that the length  $e_k$  of its edges is bounded by  $\sqrt{R} \leq e_k \leq 2\sqrt{R}$ . The  $k$ th-generation is the set of vertices of this polygon. It is easy to show that every point of  $(k+1)$ th-generation is at distance  $const\sqrt{R}$  from at least one point of  $k$ th-generation.  $k$  provides the order on our tree. We connect each point of  $(k+1)$ th-generation to a closest point of  $k$ th-generation (if the choice is not unique we choose the ancestor arbitrary). Finally we set lengths of all edges of the constructed tree  $T_{\sqrt{R}}$  equal to 1. The radius of  $T_{\sqrt{R}}$  is  $\sim \sqrt{R}$ .

Now we will give the sketch of the proof that the induced map  $f$  is a  $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$ -quasi-isometry. By the construction the discrete set of points in  $B_{\mathbb{H}^2}(R, z_0)$  is  $const\sqrt{R}$ -dense. We also notice that the right-hand quasi-isometric inequality is checked automatically because the radius of  $T_{\sqrt{R}}$  equals to the additive constant. So we need to show that if two points  $x, y$  are far in the source, then their images are also far. The distance from any point to origin  $z_0$  is contracted by factor  $1/\sqrt{R}$  because it is defined only by diameter of generation.

Take the point  $u \in xy$  which is  $\delta$ -close to both  $z_0x$  and  $z_0y$ . Then  $ux$  and  $uy$  are  $\delta$ -close to  $z_0x$  and  $z_0y$  respectively. Let  $u_1$  and  $u_2$  be projections of  $u$  on  $ux$  and  $uy$  respectively. Then their images are near in the tree two what means that  $d(f(x), f(y)) = d(f(x), f(u_1)) + d(f(y), f(u_2))$  up to an additive error. We already have quasi-isometric inequalities for radial distances, so we get immediately the needed result

$$\frac{d(x, y) - const\sqrt{R}}{\sqrt{R}} \leq d(f(x), f(y)),$$

what finishes the proof.

## 2.4 Quasi-isometric classification - survey

One of the first appearances of quasi-isometries was the proof of the famous Mostow rigidity theorem. It is proved by showing that equivariant quasi-isometries are within bounded distance of isometries.

**Theorem 7.** (*G. Mostow [9]*) *Suppose that  $n \geq 3$  and  $\Gamma, \Gamma' \subset Isom(\mathbb{H}^n)$  are lattices and  $\rho: \Gamma \rightarrow \Gamma'$  is an isomorphism. Then  $\rho$  is induced by an isometry, i.e. there exists an isometry  $\alpha \in Isom(\mathbb{H}^n)$  such that  $\alpha \circ \gamma = \rho(\gamma) \circ \alpha$  for all  $\gamma \in \Gamma$ .*

Mostow extended the previous theorem to all rank one symmetric spaces. In the course of the proof, he establishes the following fact.

**Theorem 8.** (*G. Mostow [9]*) Let  $X$  and  $X'$  be two rank 1 symmetric Riemannian spaces of negative curvature. If  $X$  and  $X'$  are quasi-isometric then they are homothetic.

Mostow's theorem was followed by generalizations of P. Pansu [11] (case of rank one) and B. Kleiner and B. Leeb [13] (higher ranks) (see for example the lecture notes of C. Drutu and M. Kapovich [10] for a survey on quasi-isometric rigidity). These generalizations help to proceed in quasi-isometric classification of some important classes of metric spaces.

**Theorem 9.** (*B. Kleiner, B. Leeb [13]*) For  $1 \leq i \leq k, 1 \leq j \leq k'$  let each  $X_i, X'_j$  be either a nonflat irreducible symmetric space of noncompact type (in addition assume that  $X$  has rank 2) or an irreducible thick Euclidean Tits building with cocompact affine Weyl group (in addition assume that  $X$  has Moufang Tits boundary). Let  $X = \mathbb{E}^n \times \prod_{i=1}^k X_i$  and  $X' = \mathbb{E}^{n'} \times \prod_{j=1}^{k'} X'_j$  be metric products. If  $X$  and  $X'$  are quasi-isometric, then  $n = n', k = k'$  and there exists a permutation  $\sigma : \{1, 2, \dots, k\}$  and homotheties  $X_i \rightarrow X'_{\sigma(i)}$ .

The quasi-isometric classification of 3-manifolds is a hard and open problem, only partial results have been achieved yet. For example we do not know if the fundamental groups of all (closed) graph manifolds are quasi-isometric. At least, the following result reduces the problem to the case of non-positively curved manifolds.

**Theorem 10.** (*M. Kapovich, B. Leeb [14]*) Let  $M$  be a Haken manifold of zero Euler characteristic (which is neither Nil nor Sol), equipped with a Riemannian metric. Then there exists a compact non-positively curved 3-manifold  $N$  with totally geodesic flat boundary and a bilipschitz homeomorphism between the universal covers of  $M$  and  $N$  which preserves the canonical decomposition. In particular, the fundamental groups  $\pi_1(M)$  and  $\pi_1(N)$  are quasi-isometric.

Also a special case of Schwartz' theorem (with  $n = 3$ ) gives some results for classification of 3-manifolds.

**Theorem 11.** (*R. Schwartz [16]*) Let  $G \neq \text{Isom}(\mathbb{H}^2)$  be a rank one Lie group. Suppose that  $\Gamma, \Gamma'$  are non-uniform lattices in  $G$  which are quasi-isometric to each other. Then there exists an isometry  $g \in \text{Isom}(\mathbb{H}^n)$  such that the groups  $\Gamma'$  and  $g\Gamma g^{-1}$  are commensurable.

This theorem holds more generally for simple Lie groups of rank 1. For higher ranks we have Wortman's result.

**Theorem 12.** (*K. Wortman [17]*) Let  $K$  be a global field and  $S$  a finite nonempty set of inequivalent valuations containing all of the Archimedean ones. Suppose  $G$  is a connected simple  $K$ -group of adjoint type that is placewise not rank one with respect to  $S$ . Let  $\Lambda$  be a finitely generated group, and assume there is a quasi-isometry  $\phi : \Lambda \rightarrow G(\mathcal{O}_S)$ . If  $G$  is  $K$ -isotropic and  $K$  is a number field, then there exists a finite index subgroup  $\Lambda_S$  of  $\Lambda$  and a homomorphism  $\phi : \Lambda_S \rightarrow G(\mathcal{O}_S)$  with a finite kernel and finite co-image such that

$$\sup_{\lambda \in \Lambda_S} d(\phi(\lambda), \psi(\lambda)) < \infty.$$

Wortman's theorem also covers non  $K$ -isotropic fields and function fields, but the result is not complete in this case.

**Theorem 13.** (*U. Hamenstädt [31]*) *Two negatively curved homogeneous spaces are quasi-isometric if and only if their isometry groups are cocompact subgroups of the same Lie group.*

A lot of results are obtained for solvable groups. For nilpotent groups we have the following theorems of P. Pansu and Y. Shalom.

**Theorem 14.** (*P. Pansu [11]*) *Let  $\Gamma$  and  $\Gamma'$  be two quasi-isometric finitely generated nilpotent groups. The associated graded Lie groups  $gr(\Gamma \otimes \mathbb{R})$  and  $gr(\Gamma' \otimes \mathbb{R})$  are isomorphic.*

**Theorem 15.** (*Y. Shalom [27]*) *Quasi-isometric finitely generated nilpotent groups have the same Betti numbers.*

The theorem of B. Farb and L. Mosher deals with solvable Baumslag-Solitar groups  $BS(1, n)$  ( $n$  is an integer) which are given by the presentations

$$BS(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle .$$

**Theorem 16.** (*B. Farb, L. Mosher [18]*) *Let  $m, n \geq 2$  be two integers, then  $BS(1, n)$  and  $BS(1, m)$  are quasi-isometric if and only if they are commensurable. This holds if and only if there exist integers  $r, i, j$  such that  $n = r^i$  and  $m = r^j$ .*

Further, A. Eskin, D. Fisher and K. Whyte proved the following theorems for solvable groups.

**Theorem 17.** (*A. Eskin, D. Fisher, K. Whyte*) *Let  $\Gamma$  be a finitely generated group quasi-isometric to  $Sol$ . Then  $\Gamma$  is virtually a lattice in  $Sol$ .*

They launched a program for analyzing quasi-isometries of Lie groups of the form  $\mathbb{R}^m \rtimes_M \mathbb{R}^n$  whose completion is still in progress. Here is an instance of the expected results.

**Theorem 18.** *Suppose  $M, M'$  are diagonalisable matrices with no eigenvalues on the unit circle, and  $G = \mathbb{R} \rtimes_M \mathbb{R}^n, G' = \mathbb{R} \rtimes_{M'} \mathbb{R}^n$ . Then  $G$  and  $G'$  are quasi-isometric if and only if  $M'$  has the same absolute Jordan form as  $M^\alpha$  for some  $\alpha \in \mathbb{R}$ .*

Parts and special cases of this theorem are proved in different articles of A. Eskin, D. Fisher, K. Whyte [19, 20, 21], T. Dymarz [22] and I. Peng [23, 24].

An alternate way of proving that two groups are not quasi-isometric is to show that certain algebraic features are quasi-isometry invariants. Results of that kind for solvable groups appear in Y. Shalom's paper [27].

# Chapter 3

## Morse Lemma

Hyperbolic metric spaces have recently appeared in discrete mathematics and computer science (see, e.g., [2]). The notion of  $\delta$ -hyperbolicity turns out to be more appropriate than other previously used notions of approximation by trees (e.g., tree width). This motivates our search for optimal bounds for a cornerstone of hyperbolic group theory like the Morse lemma.

This part is devoted to the quantitative version of the Morse Lemma, its “anti”-variant and their applications.

In the published article [32], a quasi-isometric embedding was defined as

**Definition 7.** A map  $f : E \rightarrow F$  between metric spaces is a *rough*  $(\lambda, c)$ -*quasi-isometric embedding* if for any two points  $x, y$  of  $E$

$$\frac{1}{\lambda}|x - y|_E - c \leq |f(x) - f(y)|_F \leq \lambda|x - y|_E + c.$$

The difference is in the lower bound as the additive constant in it is  $c$  and not  $c/\lambda$ . We revised all proofs and examples using our new definition. All previously obtained results remain true. The main difference in our new proof of Morse Lemma appears in Lemma 10 on exponential contraction. It was previously stated only for  $\Delta$ -connected curves, see Lemma 9 in [32]. Now we do not need to substitute a given quasi-geodesic with a continuous one any more.

### 3.1 Basics of hyperbolic geometry

The contemporary research on hyperbolic groups and hyperbolic spaces was started in 1987 by M.Gromov in his paper [Gr].

#### 3.1.1 Metric definition

In this text we will use following notations for distances between points and sets. Let  $E$  be a metric space with metric  $d$ . We write  $|x - y|$  for the distance  $d(x, y)$  between two points  $x$  and  $y$  of the space  $E$ . For a subset  $A$  of  $E$  and a point  $x$ ,  $d(x, A)$  denotes the distance from  $x$  to  $A$ .

**Definition 8.** Let  $X$  be a metric space and  $x, y, z$  be three points in  $X$ . The Gromov product  $(x|y)_z$  of  $x$  and  $y$  at  $z$  is

$$(x|y)_z = \frac{1}{2} (|x - z| + |y - z| - |x - y|).$$

To explain the geometrical meaning of this definition, we introduce *tripods* which are presented as three points in a metric tree with the branches connecting these points (it is possible that the lengths of some edges is 0).

**Proposition 3.** *Let  $x, y, z$  be three points in some metric space  $X$ . Then there exists a tripod  $T$  and an isometry  $f : x, y, z \rightarrow T$  such that  $f(x), f(y)$  and  $f(z)$  are the endpoints of the tripod  $T$ . Moreover, the lengths of the branches of  $T$  are exactly equals to corresponding Gromov's product.*

The proof is evident, verify it directly by the definition of Gromov's product. Now we are ready to give the definition of  $\delta$ -hyperbolic spaces.

**Definition 9.** A metric space  $X$  is called  $\delta$ -hyperbolic if for any four points  $x, y, z, w$  the inequality

$$(x|z)_w \geq \min(x, y)_w, (y, z)_w - \delta$$

holds.

This definition can be rewritten in another form. There are three ways to divide these four points into pairs. Introduce the corresponding sums of distances

$$\begin{aligned} p &= |x - w| + |z - y| \\ m &= |x - y| + |z - w| \\ g &= |x - z| + |y - w|. \end{aligned}$$

Redenote the points to have that  $p \leq m \leq g$ . Then the definition can be rewritten in the following form

$$g \leq m + 2\delta.$$

That is the greatest sum cannot exceed the mean sum by more than by  $2\delta$ .

### 3.1.2 Case of geodesic metric spaces

**Definition 10.** A geodesic (geodesic segment, geodesic ray)  $\sigma$  in a metric space  $E$  is an isometric embedding of a real line (real interval  $I$ , real half-line  $\mathbb{R}_+$ ) in  $E$ .

We write  $xy$  for a geodesic segment between two points  $x$  and  $y$  (in general, there could exist several geodesic paths between two points; we assume any one of them by this notation). A geodesic metric space is a space such that there exists a geodesic segment  $xy$  between any two points  $x$  and  $y$ . A geodesic triangle  $xyz$  is a union of three geodesic segments  $xy, yz$ , and  $xz$ .

Geodesic  $\delta$ -hyperbolic spaces can be described in terms of "thin triangles".



**Definition 11.** A geodesic triangle  $xyz$  is called  $\delta$ -thin if the distance from any point  $p$  of  $xy$  to the union of  $xz$  and  $yz$  does not exceed  $\delta$ :

$$d(p, xz \cup yz) \leq \delta.$$

**Proposition 4.** (See [1], Proposition 2.3.21)

- A geodesic metric space  $E$  is  $\delta$ -hyperbolic if every geodesic triangle is  $\delta'$ -thin, with  $\delta \leq 2\delta'$ .
- If a geodesic metric space  $E$  is  $\delta$ -hyperbolic, all geodesic triangles are  $\delta'$ -thin for  $\delta' \leq 2\delta$ .

According to M. Bonk and O. Schramm [6], every  $\delta$ -hyperbolic metric space embeds isometrically into a complete  $\delta$ -hyperbolic geodesic metric space. So, many theorems can be reduced to the investigation of geodesic hyperbolic spaces using the definition of hyperbolicity in terms of  $\delta$ -thin triangles. Usually the factor  $\frac{1}{2}$  is dropped in the last definition.

*Example 5.* • One of the most important examples of  $\delta$ -hyperbolic spaces are metric trees, here  $\delta = 0$ .

- Fundamental groups of compact Riemannian manifolds with negative (sectional) curvature are  $\delta$ -hyperbolic.

Take some finitely presented group  $G = \langle X, R \rangle$ . Introduce a word metric on  $G$ . That is the length of any element  $g$  is the minimal length of a word (of generators) which is needed to write  $g$  in  $P$ . It is easy to check that it is indeed a metric. The next theorem shows that in some sense most finite presentations are hyperbolic.

**Theorem 19** (Gromov). *Fix integers  $p$  and  $q$ . Consider all presentations  $P$  with  $p$  generators ( $|X| = p$ ) and  $q$  relators ( $|R| = q$ ). Denote by  $N_{hyp}(n_1, \dots, n_q)$  the number of all hyperbolic presentations with the lengths of relators equal to  $n_1, \dots, n_q$ , by  $N(n_1, \dots, n_q)$  the number of all presentations with the same property. Then*

$$\frac{N_{hyp}(n_1, \dots, n_q)}{N(n_1, \dots, n_q)} \rightarrow 1$$

as  $n_i \rightarrow \infty$  ( $i = 1, \dots, q$ ).

### 3.1.3 Divergence

Now we are going to introduce the notion of *divergence function* which allows us to estimate lengths of paths which leave a ball together with two diverging geodesics. Later this approach will help us show that the length of a curve lying far from a geodesic is very large.

**Definition 12.** Let  $F$  be a metric space. We say that  $e : \mathbb{N} \rightarrow \mathbb{R}$  is a *divergence function* for the space  $F$  if any point  $x \in F$  and any two geodesic segments  $\gamma = (x, y)$  and  $\gamma' = (x, z)$  it holds: for any  $R, r \in \mathbb{N}$  such that  $R + r$  does not exceed the lengths of  $\gamma, \gamma'$  if  $d(\gamma(R), \gamma'(R)) > e(0)$  and  $\sigma$  is a path from  $\gamma(R+r)$  to  $\gamma'(R+r)$  in the closure of the complement of the ball  $B_{R+r}(x)$  (that is in  $\overline{X \setminus B_{R+r}(x)}$ ) then the length of  $\sigma$  is at greater than  $e(r)$ .

While two points move along two geodesic rays, the distance between them grows linearly by the triangle inequality which is true in all metric spaces. Though we will see that if two such geodesics leave some bounded tube then the lengths of paths connecting two points on them and lying in the complement of the ball grow exponentially in any hyperbolic space (for example the length of a circle grows exponentially with the radius). If  $e$  is an exponential function then we say that geodesics diverge exponentially.

**Theorem 20.** *In a hyperbolic space geodesics diverge exponentially.*

An amazing fact is that the opposite statement is also true and even more: a non-linear divergence in a geodesic space implies that the divergence function is exponential and, finally, that the space is hyperbolic. Though here we are not going to prove this result. The reader can find the proof in [7].

### 3.1.4 Isoperimetry

An other important property, characterizing hyperbolic spaces, is that the isoperimetric inequality is linear for them,

$$Area(D) \leq A_{ip}l,$$

where  $l$  is the length of a closed curve filled by an optimal disk  $D$  and  $A_{ip}$  is some constant depending on a particular space. Let us explain how isoperimetric inequalities can be generalized to the case of groups.

Let  $G$  be a finitely presented group,  $P = \langle X | R \rangle$  a finite presentation of  $G$ . Closed curves in the Cayley polyhedron correspond to words  $w \in F(X)$  representing the unity of  $G$  and, hence, they can be expressed in  $F(X)$  in the reduced form

$$w = (u_1^{-1}r_1^{\alpha_1}u_1) \dots (u_n^{-1}r_n^{\alpha_n}u_n)$$

where  $u_i \in F(X)$ ,  $r_i \in R$ ,  $\alpha_i \in \{-1, 1\}$ . Of course, in general there exists infinitely many of such decompositions.

**Definition 13.** The least value of  $n$  is called the *area* of  $w$ .

**Definition 14.** The Dehn (or isoperimetric) function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is

$$f(l) = \max \{Area(w) | w = 1, |w| = l\}.$$

Any finitely presented group has different presentations with different Dehn functions. The following lemma (see [15], Lecture 3, Lemma 5) helps us to establish the relation between them.

**Lemma 1.** *Let  $G$  be a group and let  $P$  and  $Q$  be two finite presentations of  $G$  with Dehn functions  $f$  and  $g$  respectively. Then there exist constants  $a_1, a_2, a_3, a_4 \in \mathbb{N}$  such that for any  $n \in \mathbb{N}$*

$$f(n) \leq a_1g(a_2n + a_3) + a_4.$$

Now we can conclude that if for some presentation, Dehn function is bounded by a linear (polynomial, exponential etc) function, then for any presentation of that group Dehn function is also bounded by a function of the same type. Moreover, the type of isoperimetric inequalities is invariant under quasi-isometries, see Definition 1.

### 3.1.5 Comparison with trees

A metric tree is one of the most important examples of hyperbolic spaces. Most properties of hyperbolic spaces can be illustrated in trees and theorems in this subject should be first verified for them. The following theorem ([1], theorem 2.12) establishes a close relation between general hyperbolic spaces and trees. It says that a finite configurations in hyperbolic spaces can be considered as belonging to a tree. We will write  $|x|$  for the distance from  $x$  to the base point.

**Theorem 21.** *Let  $X$  be a  $\delta$ -hyperbolic metric space with a base point  $w$  and  $k$  be a positive integer.*

- *If  $|X| \leq 2^k + 2$  then there exist a finite metric tree with a base point  $t$  and a map  $\Phi : X \rightarrow T$  such that*

1.  *$\Phi$  preserves distances to the base point,*

$$|\Phi(x) - t| = |x|$$

*for any point  $x$  of  $X$ .*

2.  *$|y - x| - 2k\delta \leq |\Phi(y) - \Phi(x)| \leq |y - x|$  for any two points  $x, y$  of  $X$ .*

- *Let  $X$  be a union of rays  $X_i$  from points  $w_i$  ( $i = 1, \dots, n; n \leq 2^k$ ). Let  $c = \max_i |w_i - w|$ . Then there exists a metric tree  $T$  with a base point  $t$  and a map  $\Phi : X \rightarrow T$  such that for any two points  $x, y \in X$*

$$|y - x| - 2(k + 1)\delta - c \leq |\Phi(y) - \Phi(x)| \leq |y - x|.$$

## 3.2 The geometry of $\delta$ -hyperbolic spaces

In this section we will give some lemmas on geometry of triangles, perpendiculars and projections in  $\delta$ -hyperbolic metric spaces.

**Definition 15.** In a metric space, a *perpendicular* from a point to a curve (in particular, a geodesic) is a shortest path from this point to the curve.

Of course, a perpendicular is not necessarily unique.

Here we state several evident but useful properties of geodesic  $\delta$ -hyperbolic metric spaces.

**Proposition 5.** *Let  $abc$  be a geodesic triangle in a geodesic  $\delta$ -hyperbolic metric space.*

- The function  $d(x, bc)$  of a point  $x \in ab$  is a continuous function.
- There exists a point  $x \in ac$  such that  $d(x, ab) = d(x, bc) \leq \delta$ .
- If  $ab$  is a perpendicular to  $bc$  then the function  $d(x, bc)$  of a point  $x \in ab$  is strictly monotonous. If  $|a - b| > \delta$ , then there exists a point  $x_0 \in ab$  such that  $d(x_0, b) = \delta$  is close to  $ac$ :  $d(x_0, ac) \leq \delta$ .

**Lemma 2.** In a geodesic  $\delta$ -hyperbolic space, let  $b$  be a point and  $\sigma$  be a geodesic. Let  $ba$  be a perpendicular from  $b$  to  $\sigma$ , where  $a \in \sigma$ . Let  $c$  be a point of  $\sigma$  such that  $|b - c| = d(b, \sigma) + 2\Delta$  for some  $\Delta > 0$ . Then  $|a - c| \leq 2\Delta + 4\delta$ .

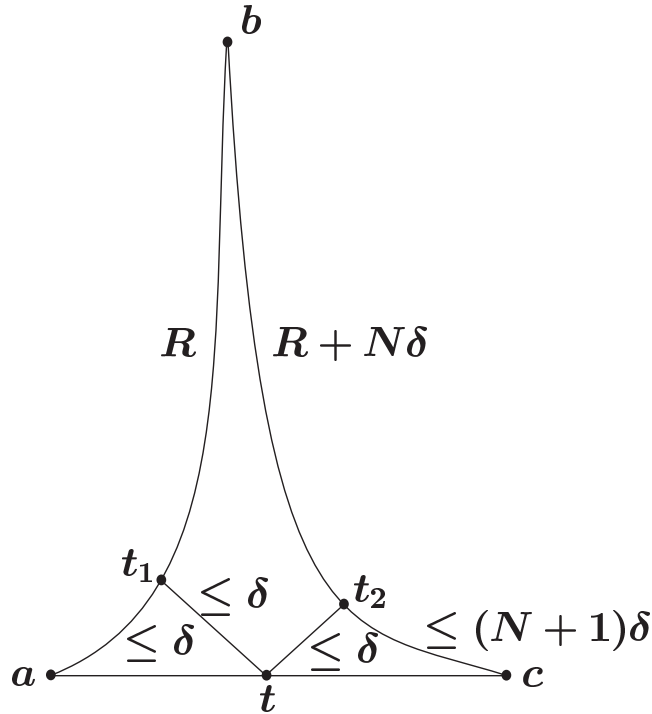


Figure 3.1: Illustration for Lemma 2.

*Proof.* The triangle  $abc$  (see Fig. 3.1) is  $\delta$ -thin by the definition of a  $\delta$ -hyperbolic space. Hence, there exists a point  $t \in \sigma$  such that  $d(t, ba) \leq \delta$  and  $d(t, bc) \leq \delta$ . Let  $t_1$  and  $t_2$  be respective projections of  $t$  on  $ba$  and  $bc$ . We defined  $ba$  as a perpendicular to  $\sigma$ , therefore,  $d(b, \sigma) = |b - a| \leq |b - t_1| + |t_1 - t| \leq |b - t_1| + \delta$  and  $d(b, \sigma) \leq |b - t_2| + |t_2 - t| \leq |b - t_2| + \delta$ . Hence,  $|a - t_1| \leq \delta$  and  $|c - t_2| \leq 2\Delta + \delta$ . By the triangle inequality, we obtain  $|a - c| \leq |a - t_1| + |t_1 - t| + |t - t_2| + |t_2 - c| \leq 2\Delta + 4\delta$ .  $\square$

*Remark 2.* In particular, all orthogonal projections of a point to a geodesic lie in a segment of length  $4\delta$ .

**Lemma 3.** *In a geodesic  $\delta$ -hyperbolic space, let two points  $b$  and  $d$  be such that  $|b - d| = \Delta$ . Let  $\sigma$  be a geodesic and  $a$  and  $c$  be respective orthogonal projections of  $b$  and  $d$  on  $\sigma$ . Let  $|a - b| > \Delta + 3\delta$ , and let  $d(d, \sigma) \geq d(b, \sigma)$ . Let two points  $x_1 \in ab$  and  $x_4 \in cd$  be such that  $\delta < d(x_1, \sigma) = d(x_4, \sigma) < |a - b| - (\Delta + 2\delta)$ . Then  $|x_1 - x_4| \leq 4\delta$  and  $|a - c| \leq 6\delta$ .*

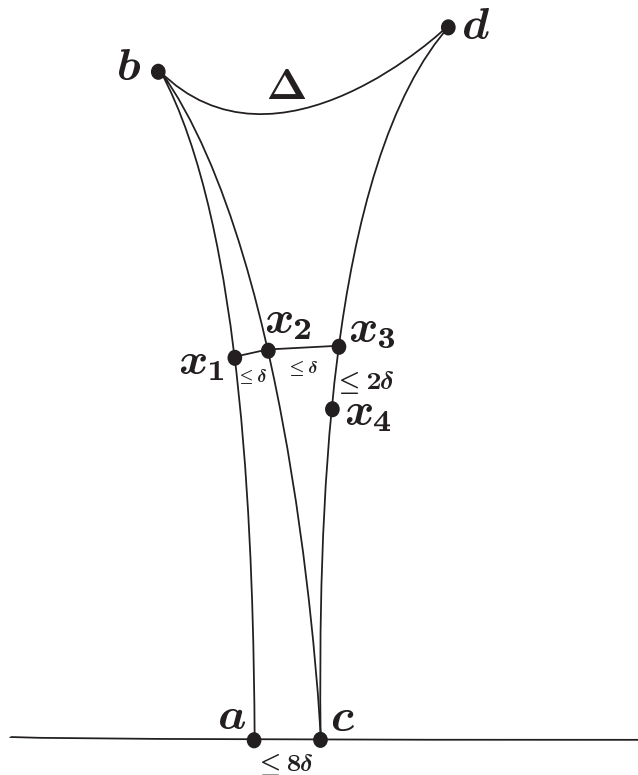


Figure 3.2: Illustration for Lemma 3.

*Proof.* (See Fig. 3.2.)

Because  $d(x_1, ac) = |x_1 - a| > \delta$  and triangle  $abc$  is  $\delta$ -thin,  $d(x_1, bc) \leq \delta$ . Let  $x_2$  denote the point of  $bc$  nearest  $x_1$ . Applying two times triangle inequality we get first  $|b - x_2| \geq |b - x_1| - |x_1 - x_2| > \Delta + \delta$  and then  $d(x_2, bd) \geq |b - x_2| - |b - d| > \delta$ . Because the triangle  $bcd$  is also  $\delta$ -thin, there exists a point  $x_3 \in cd$  such that  $|x_2 - x_3| \leq \delta$ .

It follows from the triangle  $cx_1x_3$  that  $|x_3 - c| \geq |x_1 - c| - 2\delta \geq |x_1 - a| - 2\delta$ . On the other hand, because  $x_3c$  is a perpendicular to  $\sigma$ ,  $|x_3 - c| \leq |x_3 - x_1| + |x_1 - a| \leq |x_1 - a| + 2\delta$ . Now,  $|a - x_1| = |c - x_4|$ , and hence  $|x_4 - x_3| \leq 2\delta$ . Finally, we obtain the statement in the lemma:  $|x_1 - x_4| \leq 4\delta$ .

By the triangle inequality  $|a - c| \leq |a - x_1| + |x_1 - x_4| + |x_4 - c|$ . So taking  $x_1$  and  $x_4$  close to  $\sigma$ , we obtain that  $|a - c| \leq 6\delta$ .  $\square$

**Lemma 4.** *Let  $\sigma$  be a geodesic segment,  $a$  be a point not on  $\sigma$ , and  $c$  be a projection of  $a$  on  $\sigma$ . Let  $b \in \sigma$  be arbitrary, and let  $d$  denote a projection of  $b$  on  $ac$ . Then the  $|c - d| \leq 2\delta$ .*

*Proof.* By hypothesis,  $bd$  minimizes the distance of  $b$  to any point of  $ac$ , and because the triangle  $bcd$  is  $\delta$ -thin, there exists a point  $e \in bd$  such that  $d(e, ac) = |e - d| \leq \delta$  and  $d(e, bc) \leq \delta$ . Because  $ac$  is a perpendicular to  $\sigma$ ,  $|a - c| \leq |a - d| + |d - e| + d(e, bc) \leq |a - d| + 2\delta$ . Hence  $|c - d| \leq 2\delta$ .  $\square$

**Lemma 5.** *As in the preceding lemma, let  $\sigma$  be a geodesic segment,  $a$  be a point not on  $\sigma$ ,  $c$  be a projection of  $a$  on  $\sigma$ , and  $b$  be some point on  $\sigma$ . Let  $d$  denote a point on  $ac$  such that  $|d - c| = \delta$  and  $e$  denote a point on  $bc$  such that  $|e - c| = 3\delta$ . Then*

- $d(d, ab) \leq \delta$ ,  $d(e, ab) \leq \delta$ ,  $d(c, ab) \leq 2\delta$ , and
- the length of  $ab$  differs from the sum of the lengths of the two other sides by at most  $8\delta$ ,

$$|a - c| + |b - c| - 2\delta \leq |a - b| \leq |a - c| + |b - c| + 8\delta.$$

*Proof.* The triangle  $abc$  is  $\delta$ -thin. Therefore, obviously,  $d(d, ab) \leq \delta$  (the distance from a point of  $ac$  to  $ab$  is a continuous function). We take a point  $x \in bc$  such that  $d(x, ca) \leq \delta$ . Using Lemma 4, we obtain  $|b - x| + d(x, ca) \geq |b - c| - 2\delta$ , and hence  $|c - x| \leq d(x, ca) + 2\delta \leq 3\delta$ .

We now let  $d_1$  and  $e_1$  denote respective projections of  $d$  and  $e$  on  $ab$ . Then by the triangle inequality, we have

- $|a - d| - \delta \leq |a - d_1| \leq |a - d| + \delta$ ,
- $|b - e| - \delta \leq |b - e_1| \leq |b - e| + \delta$ , and
- $0 \leq |d_1 - e_1| \leq |d_1 - d| + |d - c| + |c - e| + |e - e_1| \leq 6\delta$ .

Combining all these inequalities, we obtain the second point in the lemma.  $\square$

**Lemma 6.** *Let  $\sigma$  be a geodesic and  $a$  and  $b$  be two points not on  $\sigma$ . Further, let  $a$  and  $b$  have a common projection  $c$  on  $\sigma$ . Let  $d$  be a point of  $\sigma$  and  $c_1$  be a projection of  $d$  on  $ab$ . Then*

$$|d - c| \leq |d - c_1| + 6\delta.$$

*Remark 3.* Lemma 6 deals only with a geodesic segment  $ab$ , not a complete geodesic containing those points. The statement is not true for a complete geodesic passing through  $a$  and  $b$ , as can be seen from Fig. 3.3.

*Proof.* We take a point  $e \in bc$  such that  $|c - e| = \delta$  and consider the triangle  $bcd$  (see Fig. 3.4). Because  $bc$  is a perpendicular to  $dc$ ,  $d(e, bd) \leq \delta$ . Let  $e_1$  denote a projection of  $e$  on  $bd$ . Let  $e_2$  and  $e_3$  be a respective projections of  $e_1$  on the geodesic segments  $dc_1$  and  $bc_1$ . Because the triangle  $dbc_1$  is  $\delta$ -thin, either  $|e_1 - e_2| \leq \delta$  or  $|e_1 - e_3| \leq \delta$ .

**I.** If  $|e_1 - e_2| \leq \delta$ , then  $|d - c| \leq |c - e| + |e - e_1| + |e_1 - e_2| + |e_2 - d| \leq |d - c_1| + 3\delta$ .

**II.** If  $|e_1 - e_2| > \delta$ , then the length of the path  $cee_3$  is at most  $3\delta$ . We apply the same arguments to  $ad$  (we assume that this is possible; otherwise, we could apply the first case to it). We obtain the points  $g$ ,  $g_1$ , and  $g_3$  and the length of the path  $cgg_3$  is also at most  $3\delta$ . If

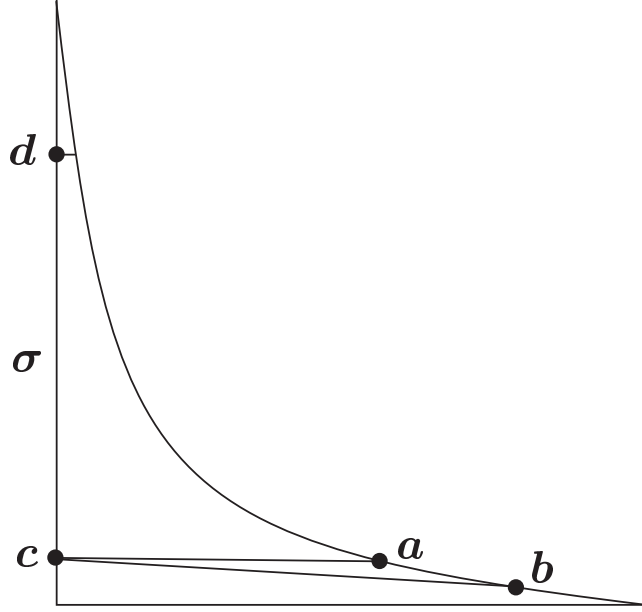


Figure 3.3: Illustration for Remark 3.

neither of these paths intersects  $cc_1$ , then its length does not exceed  $6\delta$  (which follows from consideration of the triangle  $ce_3g_3$ ).  $e_3$  and  $g_3$  lies on the different side of  $c_1$  because by the assumption  $e_1e_2$  and  $g_1g_2$  are rather long.  $\square$

*Remark 4.* This lemma stays true if the distance between projections  $c_a$  and  $c_b$  of  $a$  and  $b$  on  $\sigma$  are close enough. For example if  $|c_a - c_b| \leq \delta$  then  $|d - c_a| \leq |d - c_1| + 7\delta$ .

**Lemma 7.** *Let  $E$  be a geodesic  $\delta$ -hyperbolic metric space and  $abc$  be a triangle in  $E$ . Then the diameter of the set  $S$  of points of the side  $ab$  such that distance to  $bc$  and  $ac$  does not exceed  $d$  is not greater than  $C(d + \delta)$ , where  $C$  is a constant.*

*Proof.* Let  $x$  be a point of  $ab$  such that  $d(x, bc) \leq \delta$  and  $d(x, ac) \leq \delta$  and  $y$  be a point of  $ab$  such that  $d(y, bc) \leq d$  and  $d(y, ac) \leq d$ . Without loss of generality, we assume that  $y \in (a, x)$ . Because the triangle  $abc$  is  $\delta$ -thin, one of these two distances does not exceed  $\delta$ .

We first assume that  $d(y, ac) \leq \delta$ . Let  $x'$  and  $y'$  be points of  $ac$  such that  $d(x, x') \leq \delta$  and  $d(y, y') \leq \delta$ . We let  $t, t', s,$  and  $s'$  denote respective projections of  $x, x', y,$  and  $y'$  on  $bc$ . Because  $x't'$  is a perpendicular to  $bc$ ,  $|x' - t'| \leq |x' - x| + |x - t| \leq 2\delta$ , and hence  $|t - t'| \leq 4\delta$ . If  $y$  and  $y'$  are sufficiently far from  $bc$ , i.e., if  $d(y, s) \geq 4\delta$  and  $d(y', s') \geq 4\delta$ , then  $|s - s'| \leq 6\delta$  by Lemma 3. Otherwise, we can give a rough estimate by the triangle inequality:  $|s - s'| \leq |s - y| + |y - y'| + |y' - s'| \leq 9\delta$ . Hence, in any case,  $|s - s'| \leq 9\delta$ . We consider two cases.

If  $s$  is in the segment  $[b, t']$ , then by applying the triangle inequality several times, we obtain

$$|b - y| \leq |b - s| + |s - y| \leq |b - t'| + |s - y| \leq |b - x| + |x - t| + |t - t'| + |s - y| \leq |b - x| + 5\delta + d.$$

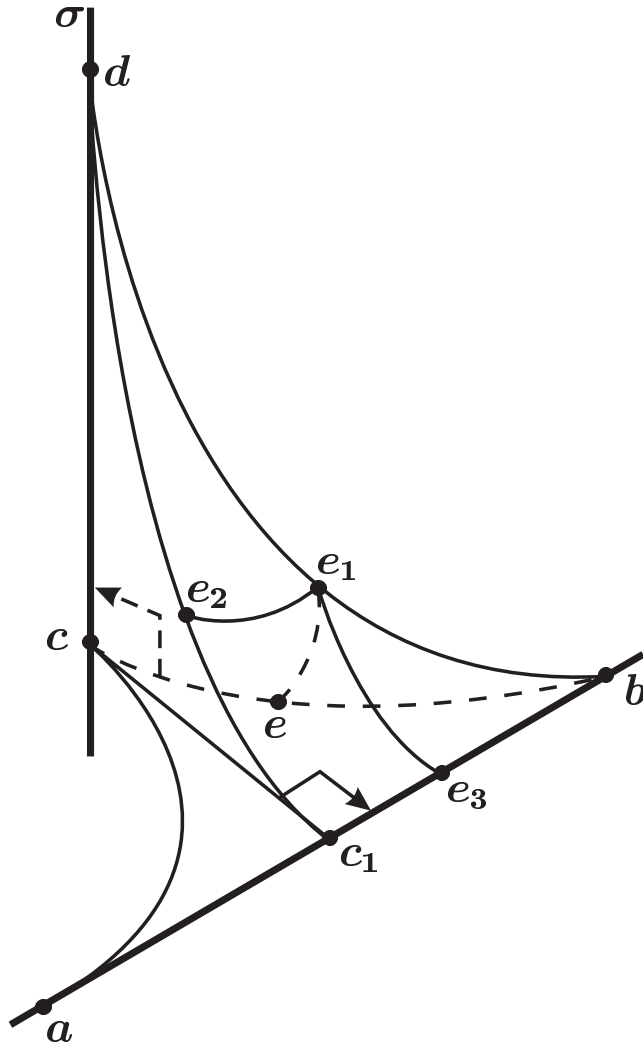


Figure 3.4: Illustration for Lemma 6.

And because  $|b - y| = |b - x| + |x - y|$ , we have  $|x - y| \leq 5\delta + d$ .

We apply the same arguments if  $s \in [t', c]$ . We merely note that we can replace  $y$  with  $y'$  and  $t$  with  $t'$  with respective errors less than  $\delta$  and  $9\delta$ :

$$|c - y'| \leq |c - s'| + |s' - y'| \leq |c - s| + 9\delta + |s - y| + \delta \leq |c - t'| + 10\delta + d,$$

here we use that  $s'$  is a projection of  $y'$ , hence  $|y' - s'| \leq |y' - s| \leq |y - y'| + |y - s|$ .

If  $y' \in cx'$  (that is  $x' \in ay'$ ), we have on the one hand  $|a - y'| \leq |a - y| + |y - y'| \leq |a - y| + \delta$ . On the other hand,  $|a - y'| = |a - x'| + |x' + y'| \geq |a - x| - \delta + |x' + y'| = |a - y| + |y - x| - \delta + |x' + y'| \geq |a - y| + |x' - y'| - 2\delta - \delta + |x' + y'| = |a - y| + 2|x' - y'| - 3\delta$ . Hence, combining these upper and lower bounds for  $|a - y'|$  we arrive at  $|x' - y'| \geq 2\delta$ . Now, suppose  $x' \in cy'$ . Because



$|c - t'| \leq |c - x'| + |x' - t'| \leq |c - x'| + 2\delta$ , we have

$$|c - x'| + |x' - y'| = |c - y'| \leq |c - x'| + 12\delta + d,$$

hence  $|x' - y'| \leq 12\delta + d$ . Finally,  $|x - y| \leq |y - y'| + |y' - x'| + |x - x'| \leq 14\delta + d$ .

The case  $d(y, bc) \leq \delta$  is treated identically with  $d$  and  $\delta$  interchanged.  $\square$

### 3.3 Quasi-geodesics and $\Delta$ -length

**Definition 16.** A  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic in  $F$  is a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding (in the sense of Definition 5) of a real interval  $I = [0, l]$  into  $F$ .

Let  $\gamma : I \rightarrow F$  be a curve. We assume that the interval  $I = [a, a + l]$  of length  $|I| = l$  gives the parametrization of the quasi-geodesic  $\gamma$ . We take a subdivision  $T = (x_0 = a, x_1, \dots, x_{n(T)} = a + l)$  and let  $y_i, i = 0, 1, \dots, n(T)$ , denote  $\gamma(x_i)$ . The *anti-mesh* of  $T$  is  $d(T) = \min_{0 < i \leq n(T)} |y_i - y_{i-1}|$ .

**Definition 17** ( $\Delta$ -length). Let  $\gamma : I \rightarrow F$  be a curve. The value

$$L_\Delta(\gamma) = \sup_{T: d(T) \geq \Delta} \sum_{i=1}^{n(T)} |y_i - y_{i-1}|$$

is called the  $\Delta$ -length of the quasi-geodesic  $\gamma$ .

We note that the values of  $\Delta$ -length and classical length are the same for a geodesic.

**Lemma 8.** Let  $\gamma : I \rightarrow F$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic. For  $\Delta \geq 2c_1$ ,

$$L_\Delta(\gamma) \leq 2\lambda_1 l.$$

*Proof.* By the definition of  $\Delta$ -length,  $\Delta \leq |y_i - y_{i-1}| \leq \lambda_1 |x_i - x_{i-1}| + c_1$ . Hence, because  $\Delta \geq 2c_1$ , we obtain  $|x_i - x_{i-1}| \geq (\Delta - c_1)/\lambda_1 \geq c_1/\lambda_1$ .

Now, by definition of a quasi-geodesic, we have (the supremum is taken over all subdivisions  $T$  of  $I$  with anti-mesh at least  $\Delta$ )

$$\sup_T \sum_i |y_i - y_{i-1}| \leq \sup_T \sum_i (\lambda_1 |x_i - x_{i-1}| + c_1) \leq \sup_T \sum_i 2\lambda_1 |x_i - x_{i-1}| = 2\lambda_1 l,$$

where the last equality follows because the sum of  $|x_i - x_{i-1}|$  for every subdivision of the interval  $I$  is exactly equal to the length of  $I$ .  $\square$

**Lemma 9.** Let  $\gamma : I \rightarrow F$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic. Assume that the distance  $R$  between the endpoints of  $\gamma$  is at least  $c_2/\lambda_2$ , and let  $\Delta \geq 2c_1$ . Then  $L_\Delta(\gamma) \leq 4\lambda_1 \lambda_2 R$ .

*Proof.* By definition of a quasi-isometry,  $(l - c_2)/\lambda_2 \leq R \leq \lambda_1 l + c_1$ . Hence,  $l \leq \lambda_2 R + c_2$ . And by Lemma 8,  $L_\Delta(\gamma) \leq 2(\lambda_2 R + c_2)\lambda_1$ . In particular,  $L_\Delta(\gamma) \leq 4\lambda_1 \lambda_2 R$  for  $R \geq c_2/\lambda_2$ .  $\square$

### 3.4 Exponential contraction

**Lemma 10** (Exponential contraction). *Let  $\Delta > 0$ . In a geodesic  $\delta$ -hyperbolic space  $E$ , let  $\gamma$  be a  $\Delta/2$ -connected curve at a distance not less than  $R \geq \Delta + 3\delta$  from a geodesic  $\sigma$ . Let  $L_\Delta$  be the  $\Delta$ -length of  $\gamma$ . Set*

$$k = \left\lceil \frac{R - \Delta - 42\delta}{19\delta} \right\rceil$$

(assume  $k = \infty$  if  $\delta = 0$ , set  $k = 0$  if the previous expression is negative). Then the diameter of the orthogonal projection of  $\gamma$  on  $\sigma$  is not greater than

$$\max\left(\frac{3\delta}{\Delta}e^{-k \ln 2}L_\Delta, 6\delta\right).$$

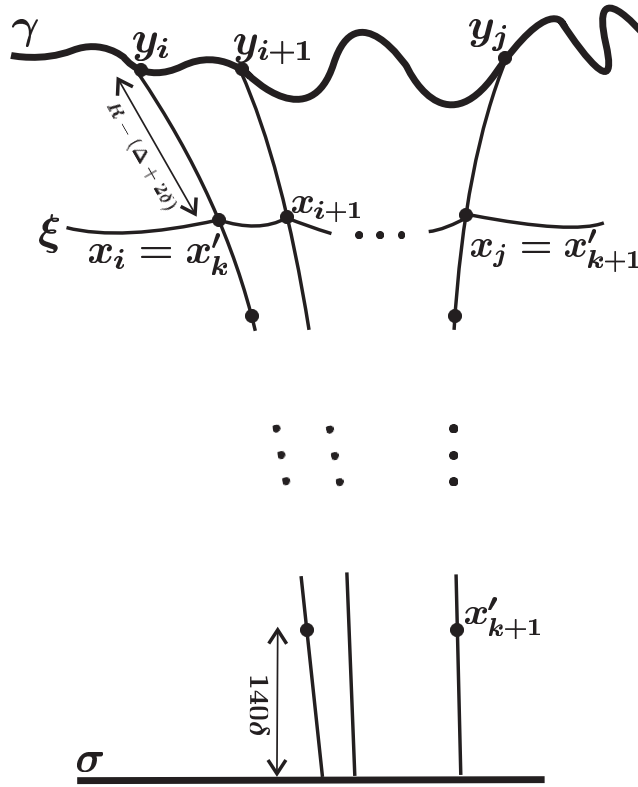


Figure 3.5: Exponential contraction of the length of a curve  $\gamma$  under projection on a geodesic  $\sigma$ .

*Proof.* We will suppose that the projection of the curve lies between projections of its endpoints (otherwise consider a part of a curve between points which correspond to the border of the projection). First assume that  $\delta > 0$ . Let  $y_0, y_1, \dots, y_n$  be points on  $\gamma$  such that  $\Delta \leq |y_i - y_{i-1}| \leq 2\Delta$  for  $i = 1, 2, \dots, n$  and  $y_0$  and  $y_n$  are the endpoints of  $\gamma$ . Let  $y_k$  be the point of this set that is nearest from  $\sigma$ . We take a perpendicular from  $y_k$  to  $\sigma$  and a point  $x_k$  on

it with  $|y_k - x_k| = \Delta + 3\delta$ . Now, on the perpendiculars from all other points  $y_i$ , we take points  $x_i$  such that  $d(x_i, \sigma) = d(x_k, \sigma)$  (see Fig. 3.5). By Lemma 3 applied to points  $y_i, y_{i-1}, x_i, x_{i-1}$  and the geodesic  $\sigma$ ,  $|x_i - x_{i-1}| \leq 4\delta$  for  $i = 1, 2, \dots, n$ . Therefore,

$$\sum_{i=1}^n |x_i - x_{i-1}| \leq n4\delta \leq n\Delta \frac{4\delta}{\Delta} \leq \frac{4\delta}{\Delta} L_\Delta.$$

We set  $\bar{x}_0 = x_0$  and  $\bar{x}_{n^1} = x_n$  and select points  $\bar{x}_i \in \{x_1, x_2, \dots, x_{n-1}\}$  such that  $8\delta \leq |\bar{x}_i - \bar{x}_{i-1}| \leq 16\delta$ . For each  $i = 0, 1, \dots, n^1$ , we choose a perpendicular from  $\bar{x}_i$  to  $\sigma$ , move  $\bar{x}_i$  along it a distance  $19\delta$  towards  $\sigma$ , and obtain  $x_i^1$ . By Lemma 3 applied to points  $\bar{x}_i, \bar{x}_{i-1}, x_i^1, x_{i-1}^1$  and the geodesic  $\sigma$ ,  $|x_i^1 - x_{i-1}^1| \leq 4\delta$  and

$$\sum_{i=1}^{n^1} |x_i^1 - x_{i-1}^1| \leq n^1 4\delta \leq \frac{1}{2} \sum_{i=1}^{n^1} |\bar{x}_i - \bar{x}_{i-1}| \leq \frac{1}{2} \sum_{i=1}^n |x_i - x_{i-1}| \leq \frac{1}{2} \frac{4\delta}{\Delta} L_\Delta.$$

We can continue such a process until one of two events happens.

- at some step  $m$  the diameter of the set of points  $\{x_i^m, i = 0, 1, \dots, n^m\}$  is small;
- at some step  $m$  the distance from the set of points  $\{x_i^m, i = 0, 1, \dots, n^m\}$  to  $\sigma$  is small.

In the first situation we use the second result of Lemma 3 to estimate the diameter of projection by  $6\delta$ .

Consider the second one. After  $k$  steps, we have

$$\sum_{i=1}^{n^k} |x_i^k - x_{i-1}^k| \leq \frac{1}{2^k} \frac{4\delta}{\Delta} L_\Delta = \frac{4\delta}{\Delta} e^{-k \ln 2} L_\Delta.$$

If at some step  $m$   $d(x_i^m, \sigma) \leq 39\delta$ , we do the projection  $Pr_\sigma$  of the subset  $\{\bar{x}_i^m\}$  (we remind that this subset is chosen so that  $8\delta \leq |\bar{x}_i^m - \bar{x}_{i-1}^m| \leq 16\delta$ ) on  $\sigma$  directly. By Lemma 3

$$|Pr_\sigma \bar{x}_i^m - Pr_\sigma \bar{x}_{i-1}^m| \leq 6\delta.$$

So at this step we gain the factor  $3/4$  for the diameter of the projection compared to the diameter of  $\{x_i^m\}$

$$diam\{Pr_\sigma \bar{x}_i^m\} \leq \frac{3\delta}{\Delta} e^{-k \ln 2} L_\Delta.$$

Now we have only to check the case  $\delta = 0$ . Then our space  $E$  is a metric tree and we immediately get that the projection of  $\gamma$  is just the only point. □

### 3.5 Quantitative version of the Morse lemma

We are now ready to state and prove the quantitative version of the Morse lemma. In a  $\delta$ -hyperbolic space  $E$ , any  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic  $\gamma$  belongs to the  $H$ -neighborhood of any geodesic  $\sigma$  connecting its endpoints, where the constant  $H$  depends only on the space  $E$  (in particular, on the constant  $\delta$ ) and the quasi-isometry constants  $\lambda_1, \lambda_2, c_1$  and  $c_2$ .

### 3.5.1 Attempts

To motivate our method, we describe a sequence of arguments yielding sharper and sharper estimates. Here, for simplicity, we will assume that  $\lambda_1 = \lambda_2 = \lambda$  and additive constants are small relatively to  $\lambda$ . We start with the proof in [1], Chapter 5.1, Theorem 6 and Lemma 8, where the upper bound  $H \leq \lambda^8 c^2 \delta$  was obtained (up to universal constants, factors of the order  $\log_2(\lambda c \delta)$ ). The first weak step in this proof is replacing a  $(\lambda, c)$ -quasi-geodesic with a discrete  $(\lambda', c)$ -quasi-geodesic  $\gamma'$  parameterized by an interval  $[1, 2, \dots, l]$  of integers, where  $\lambda' \sim \lambda^2 c$ . For a suitable  $R \sim \lambda'^2$ , we take an arc  $x_u x_v$  of  $\gamma'$  and introduce a partition of that arc  $x_u, x_{u+N}, x_{u+2N}, \dots, x_v$  for some well-chosen  $N \sim \lambda'$ . The approximation of a  $\delta$ -hyperbolic space by a tree (see [1], Chapter 2.2, Theorem 12.ii) is used to obtain an estimate of the form  $|y_{u+iN} - y'_{u+(i+1)N}| \leq c' \sim \ln \lambda'$ . By the triangle inequality,  $|x_u - x_v| \leq |x_u - y_u| + |y_u - y_{u+N}| + \dots + |y_v - x_v| \leq 2(R + \lambda') + (N^{-1}|u - v| + 1)c'$ . On the other hand,  $\lambda'^{-1}|u - v| \leq |x_u - x_v|$ . Combining these two inequalities, we obtain an estimate for  $|u - v|$  and hence for the distance from any point of the arc  $x_u x_v$  to the point  $x_u$ . The second weak step in this argument is in the estimate of the length of projections, which can be improved significantly.

Another proof was given in [7]. It allows to obtain the estimate  $\lambda^2 H_{\text{am}}$ , where  $H_{\text{am}}$  is the constant of the anti-Morse lemma (see Section 3.7) and is given by the equation  $H_{\text{am}} \simeq \ln \lambda + \ln H_{\text{am}}$ .<sup>1</sup> It is very close to an optimal upper bound but still not sharp as the sharp estimate for  $H_{\text{am}} \simeq \ln \lambda$ . The proof uses the fact that in a hyperbolic space the divergence function is exponential.

To prove the anti-Morse lemma, the authors of [7] take a point  $p$  of the geodesic  $\sigma$  that is the distant from the quasi-geodesic  $\gamma$  and construct a path  $\alpha$  between two points of  $\gamma$  such that  $\alpha$  is in the complement of the ball of radius  $d(p, \gamma)$  with the center  $p$ . Finally, they compare two estimates of the length: one estimate follows from the hypothesis that  $\alpha$  is a quasi-geodesic, and the other is given by the exponential geodesic divergence. To prove the Morse lemma, they take a (connected) part  $\gamma_1$  of  $\gamma$  that belongs to the complement of the  $H_{\text{am}}$ -neighborhood of the geodesic  $\sigma$ , and they show that the length of  $\gamma_1$  does not exceed  $2\lambda^2 H_{\text{am}}$  by the definition of a quasi-geodesic. In [7], they also use another definition of a quasi-geodesic, which is less general than our definition because, in particular, it assumes that a quasi-geodesic is a continuous curve. Consequently, some technical work is needed to generalize their results.

To improve these bounds, we use Lemma 10 (exponential contraction) instead of exponential geodesic convergence and Lemma 9, which do not require discretization as in [1] and provide a much more precise estimate for a length of a projection. We can then take  $R = \ln \lambda$  and obtain  $H \leq O(\lambda^2 \ln \lambda)$  by a similar triangle inequality.

Below, we prove the Morse and anti-Morse lemmas independently. We only mention that arguments in [7] can be used to deduce the optimal bound for the Morse lemma from the anti-Morse lemma. We can also obtain an optimal upper bound for  $H$  from Lemma 11.

We now sketch the proof of a stronger result (but still not optimal):  $H \leq O(\lambda^2 \ln^* \lambda)$ , where  $\ln^* \lambda$  is the minimal number  $n$  of logarithms such that  $\underbrace{\ln \dots \ln}_n \lambda \leq 1$ .

---

<sup>1</sup>Be careful while reading [7] because a slightly different definition of quasi-geodesics is used there with  $\lambda_1 = \lambda^2$ ; cf. Lemma 9.

The preceding argument is used as the initial step. It allows assuming that the endpoints  $x$  and  $x'$  of  $\gamma$  satisfy  $|x - x'| \leq O(\ln \lambda)$ . Then comes an iterative step. We prove that if  $xx'$  is an arc on  $\gamma$  and  $|x - x'| = d_1$ , then there exist two points  $y$  and  $y'$  at distance at most  $C_2(c, \delta)\lambda^2$  from a geodesic  $\sigma_1$  connecting  $x$  and  $x'$  such that  $d_2 := |y - y'| \leq C_3(c, \delta) \ln d_1$ . Indeed, we choose a point  $z$  of the arc  $xx'$  that is farthest from  $\sigma_1$  and let  $\sigma'$  denote a perpendicular from  $z$  to  $\sigma_1$ . If all points of the arc  $xx'$  (on either side of  $z$ ) whose projection on  $\sigma'$  is at a distance  $\leq \lambda^2$  from  $\sigma_1$  are at a distance not less than  $\ln d_1$  from  $\sigma'$ , then Lemma 10 implies that the length of the arc is much greater than  $\lambda^2 \ln d_1$ , contradicting the quasi-geodesic assumption. Hence, there are points  $y$  and  $y'$  that are near  $\sigma'$ . We can arrange that their projections on  $\sigma'$  are near each other, which yields  $|y - y'| \leq \ln d_1$ . We apply this relation several times starting with  $d_1 = C_1(c, \delta) \ln \lambda$  until  $d_i \leq 1$  for some  $i = \ln^* \lambda$ .

In summary, we use two key ideas to improve the upper bound of  $H$ : exponential contraction and consideration of a projection of  $\gamma$  on a different geodesic  $\sigma'$ .

### 3.5.2 Proof of the Morse lemma

We use the same ideas to prove the quantitative version of the Morse lemma, but we should do it more accurately.

*Remark 5.* In Section 3.6 we will give examples (properly parametrized and discretized rays in a tree) where  $H = \lambda_1 \lambda_2 \min\{c_1, c_2\}/4$ .

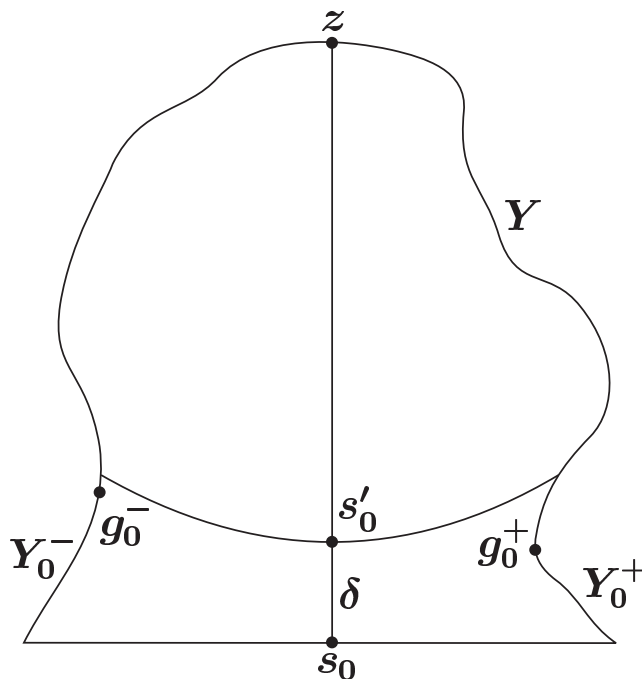


Figure 3.6: Illustration of proof of Theorem 2

*Proof of Theorem 2.* First, we notice that a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic  $\gamma$  is a  $c_1$ -connected curve. We will use Lemma 9 and Lemma 10 to get control on the  $\Delta$ -length of  $\gamma$  with  $\Delta = 2 \max\{c_1, \delta\}$ .

We introduce the following construction for subdividing the quasi-geodesic  $\gamma$ . We let  $z$  denote the point of our quasi-geodesic that is farthest from  $\sigma$ . Let  $\sigma_0 = \sigma$  be the geodesic connecting the endpoints of  $\gamma$ . Let  $\sigma'_0$  be a geodesic minimizing the distance between  $z$  and  $\sigma_0$  (because  $\sigma_0$  is a geodesic segment,  $\sigma'_0$  is not necessarily perpendicular to the complete geodesic carrying  $\sigma_0$ ). Let  $s_0$  denote the point of intersection of  $\sigma_0$  and  $\sigma'_0$ . Let  $s'_0$  be the point of  $\sigma'_0$  such that the length of the segment  $[s_0, s'_0]$  is equal to  $7\delta$ . We consider the set of points of  $\gamma$  whose orthogonal projections (at least one) on  $\sigma'_0$  belong to the segment  $[s_0, s'_0]$ . The point  $z$  separates this set into two subsets  $\gamma_0^+$  and  $\gamma_0^-$  (see Fig. 3.6).

Let  $d_0^\pm$  denote the minimal distance of points of  $\gamma_0^\pm$  to  $\sigma'_0$ . We also introduce the following notation:

- $d_0 = d_0^+ + d_0^- + \delta$ ;
- $\gamma_1$  is the  $c_1$ -connected component of  $\gamma \setminus (\gamma_0^+ \cup \gamma_0^-)$  containing  $z$ ; it is also a quasi-geodesic with the same constants and properties as  $\gamma$ ;
- $\sigma_1$  is a geodesic connecting the endpoints of the sub-quasi-geodesic  $\gamma_1$ ;
- $L_1$  is the  $\Delta$ -length of  $\gamma_1$ .

Applying the same idea to the curve  $\gamma_1$ , the same point  $z$ , and the geodesic  $\sigma_1$ , we obtain the geodesic  $\sigma'_1$ , the parts  $\gamma_1^\pm$  of the quasi-geodesic, and the distances  $d_1^\pm$ . We have  $l(\sigma'_0) \leq l(\sigma'_1) + 8c_1 + 14\delta$ . To show this, we will apply Lemma 6. If the projections onto  $\sigma'_0$  of the endpoints of  $\gamma_1$  coincide with  $s'_0$ , there is nothing to prove. Indeed, in the statement of Lemma 6 assume that  $c = s'_0$ ,  $d = z$ , and  $a$  and  $b$  are the endpoints of  $\gamma_1$ . Otherwise we will show that the same Lemma can be also applied to points at most  $c_1$  far away from the endpoints. Let  $t_1^+$  and  $t_1^-$  be the endpoints of  $\gamma_1$ . There exist points  $r_0^\pm$  of  $\gamma_0^\pm$  at distance at most  $c_1$  from  $t_1^\pm$  because the quasi-geodesic  $\gamma$  is  $c_1$ -connected. Then distances from orthogonal projections of  $t_1^\pm$  on  $\sigma'_0$  to  $s'_0$  are at most  $\max\{c_1, 6\delta\} \leq 6(c_1 + \delta)$ . Find points  $u_1^\pm$  of geodesic segments  $(r_0^- t_1^-)$  and  $(r_0^+ t t_1^+)$  whose projections coincide with  $s'_0$  up to  $\delta$ . Apply to them Lemma 6 (we remind also the remark after this lemma). Now we notice that any point of  $(t_1^+ t t_1^-)$  lies at distance at most  $2(c_1 + \delta)$  from  $(u_1^+ u_1^-)$ . Then  $l(\sigma'_0) \leq l(\sigma'_1) + 6(c_1 + \delta) + 7\delta + 2(c_1 + \delta)$ .

Continuing the process, we obtain a sequence of subsets  $\gamma_i^\pm$  of  $\gamma$  and two families of geodesics  $\sigma_i$  and  $\sigma'_i$ . Finally, for some  $n$ , we obtain  $d_n \leq c_2 + 2\Delta + 4\delta$  (the choice of such a bound will allow us to apply Lemma 10 on exponential contraction for all  $i < n$ ).

The quantity  $L_i$  is the  $\Delta$ -length of the subcurve  $\gamma_i$ , which is also a quasi-geodesic. Hence,  $l(\sigma'_n) \leq L_n \leq 4d_n \lambda_1 \lambda_2$  by construction. Therefore,

$$l(\sigma'_0) \leq \sum_{i=1}^n (8c_1 + 15\delta) + 4(c_2 + 2\Delta + 4\delta) \lambda_1 \lambda_2.$$

Our goal is to estimate  $n$ .

By Lemma 10, we obtain

$$L_\Delta(\gamma_i^+ \cup \gamma_i^-) \geq 7\delta \frac{\Delta}{3\delta} \max(e^{\ln 2(d_{i+1}^+ - \Delta - 42\delta)/19\delta}, e^{\ln 2(d_{i+1}^- - \Delta - 42\delta)/19\delta}) \geq \frac{7\Delta}{3} e^{\ln 2(d_{i+1} - \delta - 2\Delta - 84\delta)/38\delta}.$$

On the other hand,  $L_\Delta(\gamma_i^+ \cup \gamma_i^-) \leq L_i - L_{i+1} + \Delta$ . Hence, setting  $C_0 = (3\Delta/7)e^{-5/2\ln 2}$ , we have

$$C_0 e^{\ln 2(d_{i+1} - \Delta)/38\delta} \leq L_i - L_{i+1} + \Delta. \quad (3.1)$$

Let  $g_i^\pm$  be a point of  $\gamma_i^\pm$  that minimizes the distance to  $\sigma'_i$ . The part of the quasi-geodesic  $\gamma$  between  $g_i^+$  and  $g_i^-$  is also a quasi-geodesic with the same constants and properties. By the triangle inequality,  $|g_i^- - g_i^+| < d_i^+ + d_i^- + 7\delta$ . Therefore, by construction (see the beginning of the proof) and because  $d_i \geq c_2 + 2\Delta + 4\delta \geq c_2$  for  $i < n$ ,

$$L_i \leq 4\lambda_1\lambda_2 d_i \leq 8\lambda_1\lambda_2(d_i - \Delta). \quad (3.2)$$

The function  $de^{-d}$  is decreasing. Therefore, because  $d_i \geq \frac{1}{4\lambda_1\lambda_2}L_i$ , we obtain

$$\frac{\ln 2}{38\delta}(d_i - \Delta)e^{-\ln 2(d_i - \Delta)/38\delta} \leq \frac{\ln 2}{2\delta} \frac{1}{8\lambda_1\lambda_2} L_i e^{-(\ln 2/(304\delta\lambda_1\lambda_2))L_i}.$$

We are now ready to estimate  $n$ :

$$\begin{aligned} n &= \sum_{i=1}^n 1 = \frac{1}{C_0} \sum_{i=1}^n e^{-\ln 2(d_i - \Delta)/38\delta} C_0 e^{\ln 2(d_i - \Delta)/38\delta} \\ &\leq \frac{1}{C_0} \frac{16\lambda_1\lambda_2\delta}{\ln 2} \sum_{i=1}^n e^{-(\ln 2/304\delta\lambda_1\lambda_2)L_i} \frac{\ln 2}{16\lambda_1\lambda_2\delta} (L_{i-1} - L_i + 2\Delta). \end{aligned}$$

Setting  $X_i = (\ln 2/16\lambda_1\lambda_2\delta)L_i$ , we have

$$\sum_{i=1}^n 1 \leq \frac{16\lambda_1\lambda_2\delta}{C_0 \ln 2} \sum_{i=1}^n e^{-X_i} (X_{i-1} - X_i) + 2\Delta/C_0 \sum_{i=1}^n e^{-X_i},$$

and because the function  $e^{-X}$  is decreasing for  $X \geq 0$ , we can use the estimate

$$\sum_{i=1}^n e^{-X_i} (X_{i-1} - X_i) \leq \int_0^\infty e^{-X} dX = -e^{-X}|_0^\infty = 1.$$

Summarizing all facts, we finally obtain the claimed result

$$H = 4\lambda_1\lambda_2 \left( \frac{4\delta}{C_0 \ln 2} + c_2 + 4c_1 + 8\delta \right) + 14/3e^{5/2\ln 2}(8c_1 + 15\delta),$$

we recall that  $C_0 = (3\Delta/7)e^{-5/2\ln 2}$ . Lastly we notice that  $\delta/\Delta \leq 1$  and  $14/3e^{5/2\ln 2}(8c_1 + 15\delta) \leq \lambda_1\lambda_2 14/3e^{5/2\ln 2}(8c_1 + 15\delta)$ .  $\square$

### 3.6 Optimality of Theorem 2

**Proposition 6.** *Let  $T$  be a geodesic ray. Then for any constants  $\lambda_1, \lambda_2 \geq 1$  and  $c_1, c_2 \geq 0$  there exists a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic  $\gamma$  such that Morse constant  $H \geq \lambda_1 \lambda_2 \min\{c_1, c_2\}/4 - \min\{c_1, c_2\}$ .*

*Proof.* We will construct explicitly such a quasi-geodesic  $\gamma : I \rightarrow T$ , where  $I$  is a parametrization interval.

Let  $s_1$  be the base point of  $T$ . Denote by  $c = \min\{c_1, c_2\}$ . Let  $I$  be an interval of length  $l = \lambda_2 c/2$ . Divide  $I$  by intervals  $I_1, I_2, \dots, I_n$  of length  $l_1 = c/\lambda_1$ . Let  $s_2, \dots, s_n$  be consequent points of  $\sigma$  such that  $|s_i - s_{i-1}| = c/2$  for  $i = 2, \dots, n$ . Set

- $\gamma(I_i) = s_{2i-1}$  for any  $i \leq n/2$ ,
- $\gamma(I_i) = s_{2(n-i+1)}$  otherwise.

First, we check that  $\gamma$  is indeed a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry. If two points  $x_1, x_2$  of  $I$  are in the same little interval  $I_i$  then

$$\frac{|x_1 - x_2| - c_2}{\lambda_2} \leq \frac{l_1 - c_2}{\lambda_2} \leq |\gamma(x_1) - \gamma(x_2)| = 0 \leq \lambda_1 |x_1 - x_2| + c_1.$$

If  $x_1$  and  $x_2$  are in different intervals  $I_i$  and  $I_k$  then the distance between their images is at least  $c/2$  and for the left-hand inequality we have

$$\frac{|x_1 - x_2| - c_2}{\lambda_2} \leq \frac{l - c_2}{\lambda_2} \leq \frac{c}{2} \leq |\gamma(x_1) - \gamma(x_2)|.$$

Finally, we prove the right-hand inequality. We have  $(|i - k| - 1)l_1 \leq |x_1 - x_2|$  and also  $|\gamma(x_1) - \gamma(x_2)| \leq c|i - k|$ . Hence,

$$\lambda_1 |x_1 - x_2| + c_1 \geq (|i - k| - 1)c + c_1 \geq |\gamma(x_1) - \gamma(x_2)|.$$

We see easily that  $H \geq 1/2 \cdot c/2 \cdot l/l_1 - c = \lambda_1 \lambda_2 c/4 - c$ . □

### 3.7 Anti-Morse lemma

We have already proved that any quasi-geodesic  $\gamma$  in a hyperbolic space is at distance not more than  $\lambda_1 \lambda_2 (c_1 + c_2 + \delta)$  from a geodesic segment  $\sigma$  connecting its endpoints. This estimate cannot be improved. But the curious thing is that this geodesic belongs to a  $(\delta \ln \lambda_1 \lambda_2 + c_1 + c_2 + \delta)$ -neighborhood of the quasi-geodesic! We can therefore say that any quasi-geodesic is  $\ln \lambda_1 \lambda_2$ -quasiconvex for  $\delta > 0$ . In a particular case of a tree, any quasi-geodesic is  $c_1 + c_2$ -quasiconvex in a tree!

The proof of Theorem 3 (see the introduction) that we give below is based on using

- Lemma 10 (exponential contraction) to prove that at the distance  $\ln \lambda$  from the geodesic the diameter of  $\sigma$  is at most  $\lambda^2 \ln \lambda$  and



- an analogue of Lemma 10 to prove that the length of a circle of radius  $R$  is at least  $e^R$  (up to some constants).

**Lemma 11.** *Let  $X$  be a hyperbolic metric space,  $\gamma$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic, and  $\sigma$  be a geodesic connecting the endpoints of  $\gamma$ . Set  $\Delta = 2 \max\{c_1, \delta\}$ . Let  $(y_u, y_v)$  be an arc of  $\gamma$  such that no point of this arc is at distance less than*

$$R = \frac{19}{\ln 2} \delta \ln (24(\lambda_1 \lambda_2)^2) + \Delta + 42\delta$$

*from  $\sigma$  and  $y_u$  and  $y_v$  are the points of the arc nearest from  $\sigma$ . Then the diameter of the projection of the arc  $(y_u, y_v)$  on  $\sigma$  does not exceed  $\max\{6\delta, C_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)\}$  (with a universal constant  $C_3$ ).*

*Proof.* By the definition of a quasi-geodesic, we have

$$\frac{|u - v| - c_2}{\lambda_2} \leq |y_u - y_v| \leq \lambda_1 |u - v| + c_1,$$

where  $u$  and  $v$  stand for parameters of  $y_u$  and  $y_v$  respectively. On the other hand,

$$|y_u - y_v| \leq |y_u - y'_u| + |y'_u - y'_v| + |y'_v - y_v|,$$

where  $y'_u$  and  $y'_v$  are the projections of  $y_u$  and  $y_v$  on  $\sigma$ . We notice that our choice of  $\Delta$  allows applying Lemma 9. For simplicity of notations we will also suppose that  $d(y_u, y'_u) = d(y_v, y'_v)$ . We apply the lemma on exponential contraction (we assume that the length of the arc is rather large for using the estimate with an exponential factor and not to treat the obvious case where the diameter of the projection is  $6\delta$ ). We let  $l(y_u, y_v)$  denote the  $\Delta$ -length of the arc  $(y_u, y_v)$ :

$$|y'_u - y'_v| \leq l(y_u, y_v) \frac{3\delta}{\Delta} e^{-\ln 2(R - \Delta - 42\delta)/(19\delta)} = \frac{3\delta}{\Delta} \cdot \frac{1}{24(\lambda_1 \lambda_2)^2} l(y_u, y_v).$$

Combining all these inequalities and using Lemma 9 and the fact that  $\delta/\Delta < 1$ , we obtain

$$\begin{aligned} \frac{|u - v| - c_2}{\lambda_2} \leq |y_u - y_v| &\leq 2R + \frac{3\delta}{\Delta} \cdot \frac{1}{24(\lambda_1 \lambda_2)^2} l(y_u, y_v) \\ &\leq 2R + \frac{1}{8(\lambda_1 \lambda_2)^2} 4\lambda_1 \lambda_2 |y_u - y_v| \\ &\leq 2R + \frac{1}{2\lambda_1 \lambda_2} (\lambda_1 |u - v| + c_1). \end{aligned}$$

Hence

$$|u - v| \leq 4R\lambda_2 + 2(c_1 + c_2).$$

We therefore conclude that  $|y_u - y_v| \leq C_1 \lambda_1 \lambda_2 (R + c_1 + c_2) \leq C_2 \lambda_1 \lambda_2 (\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$ , hence  $l(y_u, y_v) \leq C_3 (\lambda_1 \lambda_2)^2 (\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$ , where  $C_1, C_2$  and  $C_3$  are universal constants and, finally, the diameter of the projection of the arc  $(y_u, y_v)$  of  $\gamma$  does not exceed

$$\max\{6\delta, C_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)\}.$$

□

*Proof of Theorem 3.* The proof follows directly from Lemma 11. Because we have already proved in the preceding lemma, for every point  $z' \in \sigma$ , there exists a point  $z \in \gamma$  such that the projection of  $z$  on  $\sigma$  is at distance not more than several times  $c_1 + \delta$  from  $z'$ . For simplicity, we therefore assume that for any point of  $\sigma$ , there exists a point of  $\gamma$  projecting on this point.

Assume  $H_{am} = C_4(\delta \ln \lambda_1 \lambda_2 + \delta + c_1)$  where  $C_4$  is just a universal constant which can be found from Lemma 11. If the distance between  $z$  and  $z'$  is less than  $H_{am}$ , then the statement is already proved. If not, then we take an arc  $(y_u, y_v)$  of  $\gamma$  containing the point  $z$  such that the endpoints  $y_u$  and  $y_v$  are at the distance  $H_{am}$  from  $\sigma$  and these points are the points of this arc that are nearest from  $\sigma$ . Hence, by Lemma 11, the length of the projection (which includes  $z$ ) of the arc  $(y_u, y_v)$  does not exceed  $\max\{6\delta, C_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)\}$ . Therefore, the distance from  $z$  to  $y_u$  (and  $y_v$ ) is not greater than  $C_5(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$  with some universal constant  $C_5$ .  $\square$

### 3.8 Application of Anti-Morse Lemma

**Proposition 7.** *Let  $X, Y$  be two geodesic hyperbolic spaces, let  $f : X \rightarrow Y$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry. Let  $\sigma$  be a geodesic in  $X$ . Then the distance from  $\tilde{\gamma} = f(\sigma) \subset Y$  to any geodesic connecting its ends is at most  $\lambda_1 H_{am}^X + c_1$ , where  $H_{am}^X$  is a anti-Morse constant for the space  $X$ .*

We see that in case of a quasi-isometry instead of a quasi-isometric embedding we have a stronger result than Morse Lemma.

*Proof.* Let  $\tilde{\sigma} \subset Y$  be a geodesic connecting the ends of  $\tilde{\gamma}$ . Define also a quasi-geodesic  $\gamma = f^{-1}(\tilde{\sigma})$  in  $X$ . Because  $\sigma$  and  $\gamma$  share their ends, we can apply the Anti-Morse Lemma to them, so  $\sigma \subset U_{H_{am}}(\gamma)$  lies in  $H_{am}^X$ -neighbourhood of  $\gamma$ . Now applying  $f$  to  $\sigma$  and  $\gamma$  we obtain that  $\tilde{\gamma} \subset U_{\lambda_1 H_{am}^X + c_1}(\tilde{\sigma})$ .  $\square$

#### 3.8.1 Proof of Proposition 1

Here, we prove Proposition 1 (see the introduction). We call any connected component of the ball  $B = B(O, R)$  with deleted center  $O$  a *branch*. We call points that are sent to the branch containing the image of the center  $f(O)$  green points and all other points of  $T$  red points.

*Proof of Proposition 1.* We show that there exist two red points  $r_1$  and  $r_2$  such that  $d(O, r_1 r_2) \leq r = C_3 + 1$  for some constant  $C_3$  which depends on the degree  $d$  (it will be defined later in proof more precisely).

By Definition 1, a  $c$ -neighborhood of every point of the border should contain a point of the image. We must have at least  $(d-1)d^{R-C_3-1}$  red points near the border (we exclude the green part). The number of points in each connected component of the complement of the ball of radius  $r$  is less than  $d^{R-r}$ . Therefore, there is a constant  $C_3$  depending on the tree only such that if  $r \geq C_3$ , then one component contains an insufficient number of points to cover the boundary of  $B$ . Hence, there exist two red points  $r_1$  and  $r_2$  in different components of the complement of  $B(O, r)$ , which means that the geodesic  $r_1 r_2$  passes at a

distance less than  $r$  from the center  $O$ . It follows that the quasi-geodesic  $f(r_1 r_2)$  passes at distance less than  $\lambda_1 r + c_1$  from  $f(O)$  and belongs to a  $(\lambda_1 H_{am} + c_1)$ -neighborhood of the geodesic  $f(r_1) f(r_2)$  by Proposition 7. Because every path from  $f(O)$  to  $f(r_1) f(r_2)$  passes through  $O$ , we conclude that  $d(O, f(O)) < \lambda_1(H_{am} + c_1) + c_1 + \lambda_1 r$ . We need only choose a good value for  $r$ . Simply calculating the number of points in above mentioned components gives the estimate  $1 + d + d^2 + \dots + d^{R-r} \leq (1/\ln d)d^{R-r+1}$ . For  $r = C_3 + 1$ , we have  $(1/\ln d)d^{R-r+1} \leq (d-1)d^{R-C_3-1}$ , which completes the proof.  $\square$

### 3.9 Geodesically rich spaces

**Definition 18.** A metric space  $X$  is said to be geodesically rich if there exist constants  $r_0, r_1, r_2$  such that for every pair of points  $p$  and  $q$  with  $|p - q| \geq r_0$ , there exists a geodesic  $\gamma$  (with ends at infinity) such that  $d(p, \gamma) < r_1$  and  $|d(q, \gamma) - |q - p|| < r_2$ .

*Remark 6.* We introduced the notion of geodesically rich spaces in [32], see Definition 11. Still now we do not need the second condition to estimate a displacement of points under self-quasi-isometries fixing ideal boundary so we change the definition to a weaker form.

*Example 6.* A line and a ray are not geodesically rich.

*Example 7.* Nonelementary hyperbolic groups are geodesically rich. We prove this later.

Any  $\delta$ -hyperbolic metric space  $H$  can be embedded isometrically in a geodesically-rich  $\delta$ -hyperbolic metric space  $G$  (with the same constant of hyperbolicity). Here we give a proof of this fact. Take a 3-regular rooted tree  $(T, O)$ , set  $G = H \times T$ , and set the metric analogously to a real tree. Let  $g_1 = (h_1, t_1), g_2 = (h_2, t_2) \in G$ , then set

$$d(g_1, g_2) = d(t_1, O) + d(t_2, O) + d_H(h_1, h_2).$$

It is easy to show that the space  $G$  is  $\delta$ -hyperbolic and geodesically rich. But such a procedure completely changes the ideal boundary of the space. We, therefore, ask ourselves if it is always possible to embed a  $\delta$ -hyperbolic space in a geodesically rich  $\delta$ -hyperbolic space with the same ideal boundary. The answer is given in the following remark.

*Remark 7.* It is not always possible to embed a  $\delta$ -hyperbolic metric space  $H$  isometrically in a geodesically rich  $\delta$ -hyperbolic metric space  $G$  with an isomorphic boundary. An example can be provided by a  $\delta$ -hyperbolic space with an isolated point at the ideal boundary. As an illustration, consider a real line  $\mathbb{R}$ . Its ideal boundary contains only two points. Now consider a  $\delta$ -hyperbolic space  $H$  with the same ideal boundary  $\partial H = \{\xi_1, \xi_2\}$  and an isometric embedding  $\gamma : \mathbb{R} \rightarrow H$  (hence,  $\gamma$  is a geodesic). We will show that every point  $p \in \gamma$  lies at distance at most  $2\delta$  from any infinite geodesic  $\sigma$  which means that  $H$  is not geodesically rich. Because  $H$  is  $\delta$ -hyperbolic, the triangle  $p\xi_1\xi_2$  with sides coinciding with  $\gamma$  and  $\sigma$  is  $\delta$ -thin. Hence, there exists a point  $q \in \sigma$  such that  $d(q, \gamma(\xi_1, p)) \leq \delta$  and  $d(q, \gamma(\xi_2, p)) \leq \delta$ . And we conclude that  $d(p, \sigma) \leq 2\delta$  because  $\gamma$  is a geodesic.

**Lemma 12.** *Let  $G$  be a nonelementary hyperbolic group. Then for every two points  $p$  and  $q$  in the group  $G$ , there exists a geodesic  $\gamma$  such that  $d(p, \gamma) \leq \delta$  and  $||p - q| - d(q, \gamma)| \leq 7\delta$ .*

We are thankful to Prof. Peter Haïssinsky for pointing out this proof which is clearer and simpler than the one given in [32].

*Proof.* Since  $G$  is not elementary group, there are three distinct points  $\xi_1, \xi_2, \xi_3$  on the ideal boundary. Consider geodesic triangle  $\xi_1\xi_2\xi_3$  and a point  $p$  at distance at most  $\delta$  from all three sides  $\xi_1\xi_2, \xi_2\xi_3, \xi_1\xi_3$ . Take any point  $q$ . First we notice that in the tree the statement is evident. Otherwise apply theorem 21 (second statement) to approximate configuration  $p\xi_1, p\xi_2, p\xi_3$  and  $pq$  with a tree  $T$ , denote corresponding map by  $\Phi$ . The error of this approximation is  $6\delta$ .

In  $T$  for at least two of three points  $\xi_1, \xi_2, \xi_3$   $d(\Phi(q), \Phi(p)\Phi(\xi_i)) = d(\Phi(p), \Phi(q))$ . Without loss of generality we can assume that they are  $\xi_1$  and  $\xi_2$ . Then  $|d(q, \xi_1p \cup p\xi_2) - d(p, q)| \leq 6\delta$ . By the choice of  $p$   $|d(q, \xi_1p \cup p\xi_2) - d(q, \xi_1\xi_2)| \leq \delta$ . Hence,

$$|d(q, \xi_1\xi_2) - d(p, q)| \leq 7\delta.$$

Now we just notice that the action of  $G$  is cocompact, so the statement is true to any point of  $G$ .  $\square$

This Lemma proves that a nonelementary hyperbolic group satisfies the definition of geodesically rich space.

### 3.10 Quasi-isometries fixing the ideal boundary

We now give some estimates of the displacement of points in geodesically rich spaces under quasi-isometries that fix the ideal boundary. We do not yet know whether these results are optimal.

**Theorem** (see Theorem 4 in the introduction). *Let  $X$  be a geodesically rich hyperbolic metric space. Let  $f : X \rightarrow X$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -self-quasi-isometry fixing the boundary  $\partial X$ . Then any point  $O \in X$  can be displaced at most at distance  $d(f(O), O) \leq \lambda_1(H_{am} + r_1) + 2c_1 + r_0 + r_2$ .*

*Proof.* Consider a point  $O$  and its image  $f(O)$ . If  $d(O, f(O)) < r_0$ , there is nothing to prove. Otherwise, let  $\gamma$  be a geodesic such that  $d(O, \gamma) \leq r_1$  and  $|d(f(O), \gamma) - d(O, f(O))| \leq r_2$  and in particular,  $|d(f(O), \gamma) - d(O, f(O))| \geq r_2$ . Such a geodesic exists by definition of geodesically rich.

Because  $f(\gamma)$  is a quasi-geodesic with the same endpoints as  $\gamma$ , the quasi-geodesic lies near  $\gamma$ :  $f(\gamma) \subset U_{\lambda_1 H_{am} + c_1}(\gamma)$  by Proposition 7. Also since  $d(O, \gamma) \leq r_1$ , in the image  $d(f(O), f(\gamma)) \leq \lambda_1 r_1 + c_1$ . Combining all the arguments, we obtain

$$d(O, f(O)) \leq d(f(O), \gamma) + r_2 \leq \lambda_1 H_{am} + c_1 + r_2 + \lambda_1 r_1 + c_1.$$

$\square$

The property of being geodesically rich plays crucial role here. For example a translation of a real line  $\mathbb{R}$  (which is an isometry) fixes its ideal boundary but still moves its point to any pregiven distance.

## Chapter 4

# Poincaré inequalities and quasi-isometries

### 4.1 Main ideas

#### 4.1.1 The critical exponent for $L^p$ -cohomology

$L^p$ -cohomology groups provides invariants for quasi-isometries. The continuous first  $L^p$ -cohomology group of a hyperbolic metric space  $X$  is

$$L^p H_{cont}^1(X) := \{[f] \in L^p H^1(X) \mid f \text{ extends continuously to } X \cup \partial X\},$$

where  $X \cup \partial X$  is Gromov's compactification of  $X$ . Following the works of Pierre Pansu, and Marc Bourdon and Bruce Kleiner [28], we define the following quasi-isometric numerical invariant of  $X$

$$p_{\neq 0}(X) = \inf \{p \geq 1 \mid L^p H_{cont}^1(X) \neq 0\}.$$

If  $p_{\neq 0}$  achieves different values for two spaces  $X$  and  $Y$ , then  $X$  and  $Y$  are not quasi-isometric. We expect that the difference  $|p_{\neq 0}(X) - p_{\neq 0}(Y)|$  also bounds from below the quasi-isometrical distortion growth. We are able to prove this only for a family of examples.

Let  $Z_\mu$  and  $Z_{\mu'}$  be two variants of the space  $\mathbb{T}^n \times (-\infty, \infty)$  with metrics  $dt^2 + \sum e^{2\mu t} dx_i^2$  and  $dt^2 + \sum e^{2\mu' t} dx_i^2$  respectively. The main result of this part is a sharp lower bound for the quasi-isometrical distortion growth between  $Z_\mu$  and  $Z_{\mu'}$ , of the form

$$const (p_{\neq 0}(Z_{\mu'}) - p_{\neq 0}(Z_\mu)) R.$$

#### 4.1.2 Scheme of proof

Constants in Poincaré inequalities are the quantitative incarnation of  $L^p$ -cohomology.

Let  $X$  be a Riemannian manifold. Let  $p$  be a number in  $[1, \infty]$ . Then the  $\mathbb{L}^p$ -norm  $|\cdot|_p$  of functions and vector fields make sense. We will say that  $C = C(X, p)$  is a Poincaré constant for  $X$  and  $\mathbb{L}^p$  if for any smooth function  $f$  in  $\mathbb{L}^p$  there exists such a constant  $c$  (which is in fact the mean value of  $f$  over  $X$ ) such that the Poincaré inequality holds

$$|f - c|_p \leq C |\nabla f|_p.$$

Variants of this definition appear in the litterature. In an appendix 6.2, we shall check that these definitions are equivalent, up to universal constants.

For the family of spaces  $Z_\mu$ , it is known that  $p_{\neq 0}(Z_\mu) = \frac{\sum \mu_i}{\max \mu_i}$  (unpublished result of P.Pansu). In Theorem 24 we show that

- if  $p > p_{\neq 0}(Z_\mu)$ , then the Poincaré constant for a ball of radius  $R$  satisfies

$$C_p(B^{Z_\mu}(R)) \geq \text{const.}(\text{Vol}B(R))^{1/p};$$

- if  $p \leq p_{\neq 0}(Z_\mu)$ , then

$$C_p(B^{Z_\mu}(R)) = o((\text{Vol}B(R))^{1/p}).$$

Next, we show that under transport by a  $(\lambda, c)$ -quasi-isometry,  $C_p$  is multiplied or divided by at most  $e^{(\lambda+c)/a}$  for some positive constant  $a$ . Transport under quasi-isometric embeddings is more delicate, this is why our arguments work only for a family of examples. For these examples, we are able to get a lower bound. Roughly speaking, it states

*Assume that  $p_{\neq 0}(Z_{\mu'}) < p < p_{\neq 0}(Z_\mu)$ . If there exists a  $(\lambda, c)$ -quasi-isometric embedding  $B^{Z_\mu}(R) \rightarrow Z_{\mu'}$ , which induces an isomorphism on fundamental groups, then*

$$C_p(B^{Z_\mu}(R)) \geq \text{const.}e^{-(\lambda+c)/a}C_p(B^{Z_{\mu'}}(R)).$$

This yields

$$\begin{aligned} \lambda + c &\geq a(\log(C_p(B^{Z_{\mu'}}(R))) - \log(C_p(B^{Z_\mu}(R)))) \\ &\sim (p_{\neq 0}(Z_{\mu'}) - p_{\neq 0}(Z_\mu))R. \end{aligned}$$

which is the announced lower bound on quasi-isometric distortion growth.

## 4.2 Regularisation and quasi-isometries

In this section we will study how Poincaré inequalities are transformed under quasi-isometries. For this purpose we will introduce the notion of cross-kernels, which will help us to regularize transported functions.

### 4.2.1 Kernels

First we recall what are classical kernels.

**Definition 19.** Let  $X$  be a geodesic space,  $dx$  a Radon measure on  $X$ . A kernel  $\psi$  is a measurable non-negative function on  $X \times X$  such that

- $\psi$  is bounded,  $\psi \leq S^\psi$ ;
- for every  $x \in X$   $\int_X \psi(x, x')dx' = 1$ ;

- the support of  $\psi$  is concentrated near the diagonal: there exist constants  $\varepsilon^\psi > 0$ ,  $\tau^\psi > 0$  and  $R^\psi < \infty$  such that  $\psi(x, y) > \tau^\psi$  if  $d(x, y) \leq \varepsilon^\psi$ ;  $\psi(x, y) = 0$  if  $d(x, y) > R^\psi$ .

$R^\psi$  is called a *width*,  $\varepsilon^\psi$  a *radius of positivity*,  $S^\psi$  a *supremum* and  $\tau^\psi$  a *margin* of  $\psi$ .

The convolution of two kernels is

$$\psi_1 * \psi_2 = \int_X \psi_1(x, z)\psi_2(z, y) dz,$$

the result is also a kernel. The convolution of a kernel and a function is

$$g * \psi(x) = \int_X g(z)\psi(x, z) dz.$$

**Lemma 13.** *There exists a constant  $c_\tau$  (which depends on the local geometry of the space  $X$ ) such that for any  $\varepsilon > 0$  there exists  $\tau = c_\tau e^{-\varepsilon}$  and a kernel  $\psi$  on  $X \times X$  such that for any two points  $x_1, x_2$  with  $d(x_1, x_2) < \varepsilon$ , we have  $\psi(x_1, x_2) > \tau$ .*

*Proof.* We start from kernel

$$\psi'(x, x') = \text{vol}(B(x, 1))^{-1} \mathbf{1}_{\{d(x, x') \leq 1\}}$$

with radius of positivity  $\varepsilon' = 1$  and margin  $\tau' = v(1)^{-1}$ , where, for  $r > 0$ ,  $v(r)$  denotes the infimum of volumes of balls of radius  $r$  in  $X$ . We know from the proof of Lemma 1.2 in [12] that the  $m$ -th convolution  $\psi'^{*m}$  has radius of positivity  $\varepsilon'_m \geq m(\varepsilon'/2) = m/2$  and margin  $\tau'_m \geq \tau'^m v(\frac{1}{2})^{m-1}$ .  $\square$

**Definition 20.** A *cocycle* on  $Y$  is a measurable map  $a : Y \times Y \rightarrow \mathbb{R}$  such that for every  $y_1, y_2, y_3$  in  $Y$ ,

$$a(y_1, y_2) = a(y_1, y_3) + a(y_2, y_3).$$

The convolution of a cocycle with a kernel is defined by

$$a * \phi(x, x') = \int_{Y \times Y} a(y, y')\phi(x, y)\phi(x', y') dy dy'.$$

**Definition 21.** Let  $\psi$  be a kernel and  $a$  a cocycle on  $X$ . The semi-norm  $N_{p, \psi}$  is defined by

$$N_{p, \psi}(a) = \left( \int_{X \times X} |a(x_1, x_2)|^p \psi(x_1, x_2) dx_1 dx_2 \right)^{1/p}.$$

The following facts are known, see [12].

**Lemma 14.** *Let  $X$  be a geodesic metric space such that the infimum  $\inf\{\text{vol}B(x, r) | x \in X\}$  of volume of balls of radius  $r$  is positive. Semi-norms  $N_{p, \psi}$  are pairwise equivalent. More precisely, let  $\psi_1$  and  $\psi_2$  be two kernels on  $X$ . Then*

$$N_{\psi_2} \leq \hat{C} N_{\psi_1},$$

where

$$\hat{C} = \frac{\sup \psi_1 \sup \psi_2 R^{\psi_2}}{c_\tau \varepsilon^{\psi_1}} (2e)^{R^{\psi_2}/\varepsilon^{\psi_1}}.$$

**Lemma 15.** *Let the space  $X$  be a Riemannian manifold and have the following properties: (1) its injectivity radius is bounded below, (2) its Ricci curvature is bounded from below. Then the volumes of balls are bounded below (Croke inequality [3]) and above (Bishop inequality).*

1) *For any function  $g$  define a cocycle  $u(x, y) = g(x) - g(y)$ . Then for any  $p$  and any kernel  $\psi'$  with bounded derivatives there exists a kernel  $\psi_1$  such that the  $\mathbb{L}^p$ -norm of  $\nabla(g * \psi')$  (we regularise  $g$ ) is bounded from above by a  $\psi_1$ -seminorm of the corresponding cocycle  $u$*

$$\|\nabla(g * \psi')\|_p \leq N_{p, \psi_1}(u)$$

with the kernel  $\psi_1$  defined as follows

$$\psi_1 = \frac{\sup \nabla \psi' \sup \psi'}{\text{vol}(B(z', R^{\psi'}))} 1_{\{d(z, z') \leq R^{\psi'}\}}.$$

2) *Conversely, there also exists a kernel  $\psi_2$  such that*

$$N_{p, \psi_2}(u) \leq C \|\nabla g\|_p,$$

where  $C$  depends only on dimension. Here the kernel  $\psi_2$  can be taken as

$$\psi_2(x, y) = \max\{1, \Theta(x, y)^{-1}\} 1_{\{d(x, y) \leq R\}},$$

where  $\Theta(x, y)$  is the density of the volume element in polar coordinates with origin at  $x$

$$\Theta(x, y)^{-1} dy = dr d\theta$$

and  $R > 0$  can be chosen arbitrarily.

In the third hypothesis we propose to use  $R = 1$ , then  $\psi_2$  is bounded by 1 and the width of its support is also 1. For reader's convenience, we include the proof of the first statement of the last Lemma, following [12].

*Proof.* Denote by  $\alpha$  the cocycle  $u * \psi'$ . Then for any  $y$ ,

$$\nabla(u * \psi')(x) = \frac{\partial \alpha(x, y)}{\partial x} = \int (g(z') - g(z)) d_x \psi'(z, x) \psi'(z', y) dz dz'.$$

Choose  $y = x$ . Then we obtain

$$|\nabla(g * \psi')(x)| \leq \sup \nabla \psi' \sup \psi \int_{B(x, R^\psi) \times B(x, R^\psi)} |g(z') - g(z)| dz dz'.$$

Now applying Hölder inequality we get the needed statement with the kernel

$$\psi_1 = \frac{\sup \nabla \psi' \sup \psi'}{\text{vol}(B(z', R^{\psi'}))} 1_{\{d(z, z') \leq R^{\psi'}\}}.$$

□



This lemma gives us an idea how to generalize Poincaré inequalities for the case of arbitrary metric spaces. Of course, such Poincaré inequality depends on a choice of a kernel  $\psi$ . Let  $f$  be an  $\mathbb{L}_p$ -function on  $X$ ,  $\psi$  a kernel on  $X$ . The Poincaré inequalities for  $f$  associated to  $\psi$  with constants  $c_f$  and  $C_p(f)$  is

$$\|f - c_f\|_p \leq C_p(f) \|N_{p,\psi}(u)\|.$$

The Poincaré constant  $C_p(X, \psi)$  is a constant such that for any  $\mathbb{L}_p$ -function  $f$  Poincaré inequality is checked with  $C_p(f) = C_p(X, \psi)$ . It follows from Lemma 14 that the existence of Poincaré constant does not depend on the choice of a kernel.

### 4.2.2 Cross-kernels

Let  $X, Y$  be two metric spaces, let  $f : X \rightarrow Y$  and  $f' : Y \rightarrow X$  be  $(K, c)$ -quasi-isometries between them such that for any  $x \in X$ ,  $d(x, f' \circ f(x)) \leq c$  and vice versa (that is, they are inverse in the quasi-isometrical sense). Let  $g$  be a measurable function on  $Y$ . We want to find a way to transport  $g$  by our quasi-isometry (using the regularisation) to obtain a similar measurable function on  $X$ . We will take

$$h(x) = \int_Y g(z) \psi(f(x), z) dz$$

as a function on  $X$  corresponding to  $g$ . This integral exists for all  $x$  because  $\psi$  is measurable by the second variable by definition. Still we want  $h$  be also measurable. For that, it will be sufficient if  $f$  is measurable too.

**Proposition 8.** *Let  $f$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding between metric spaces  $X$  and  $Y$ . Then there exists a measurable  $(\lambda_1, \lambda_2, 3c_1, c_2 + 2c_1/\lambda_1)$ -quasi-isometric embedding  $g$  at distance  $2c_1$  from  $f$ .*

*Proof.* Take a measurable partition  $P$  of  $X$  with a mesh  $c_1/\lambda_1$ . For each set  $A \in P$  we choose a base point  $x_A$ . We set  $g$  be constant on  $A$

$$g|_A = f(x_A).$$

Take any two points  $x, x' \in X$ . Assume  $x \in A$  and  $x' \in A'$  where  $A, A' \in P$ . Then

$$\begin{aligned} d(g(x), g(x')) &= d(f(x_A), f(x_{A'})) \leq \lambda_1 d(x_A, x_{A'}) + c_1 \\ &\leq \lambda_1 (d(x, x') + d(x, x_A) + d(x', x_{A'})) + c_1 \leq \lambda_1 d(x, x') + 3c_1. \end{aligned}$$

In the same way we prove the right-hand inequality. □

This proposition gives us an idea that we can always pass to measurable quasi-isometries without significant loss in constants. From now we will consider only measurable quasi-isometries.

We are going to construct a numerical function on  $X \times Y$  which will play the role of a kernel. Indeed, a *cross-kernel* can be considered as the composition (relatively to the first argument) of a quasi-isometry from  $X$  to  $Y$  and a kernel on  $Y \times Y$ . Conversely, a cross-kernel generates a quasi-isometry.

**Definition 22.** A *cross-kernel* is a measurable bounded non-negative function  $\phi : X \times Y \rightarrow \mathbb{R}$  such that

- for all  $x \in X$ ,  $\int_Y \phi(x, y) dy = 1$ ;
- for all  $R > 0$  there exists  $Q_1^\phi > 0$  such that if  $d(y, y') \geq Q_1^\phi$  and  $d(x, x') \leq R$ , then  $\phi(x, y)\phi(x', y') = 0$ ;
- for all  $R > 0$  there exists  $Q_2^\phi > 0$  such that if  $d(y, y') \leq R$  and  $d(x, x') \geq Q_2^\phi$ , then  $\phi(x, y)\phi(x', y') = 0$ ;
- there exists a constant  $S^\phi$  such that for any  $y \in Y$ ,  $\int_X \phi(x, y) dx \leq S^\phi$ ;
- there exist  $\tau^\phi > 0$ ,  $D^\phi > 0$  such that for any  $y \in Y$  the set  $\{x \in X | \phi(x, y) > \tau^\phi\}$  contains a ball of radius  $D^\phi$ .

*Remark 8.* For our purposes, the third axiom could be replaced with a weaker one: there exists  $R > 0$  such that for any  $y \in Y$  there exist  $x_0 \in X$  such that for any  $x \in X$  with  $d(x, x_0) > R$ ,  $\phi(x, y) = 0$ . But we prefer our definition as it is more symmetric and easier to apply.

Before we construct a cross-kernel from a quasi-isometry and a kernel, we will show that a cross-kernel  $\phi$  such that  $Q_1^\phi(R) = c_{11}R + c_{12}$  and  $Q_2^\phi(R) = c_{21}R + c_{22}$  depends linearly on  $R$  (here  $c_{11}, c_{12}, c_{21}, c_{22}$  are some constants) defines a quasi-isometry. Simply let  $f : X \rightarrow Y$  be defined as follows,  $x$  is mapped to a point of the set  $\{y | \phi(x, y) > 0\}$  which is not empty by the first hypothesis of cross-kernels. Let us check that  $f$  is quasi-surjective. First we notice that for any  $y \in Y$  the set  $\{x | \phi(x, y) > 0\}$  is non-empty because the last hypothesis. Now suppose that for two different points  $y_1, y_2$  there exists a point  $x$  such that both  $\phi(x, y_1) > 0$  and  $\phi(x, y_2) > 0$ . We need only check that  $d(y_1, y_2)$  is bounded. We have  $\phi(x, y_1)\phi(x, y_2) > 0$ . So, it follows from the second hypothesis that  $d(y_1, y_2) < Q_1^\phi(0)$  (the setting corresponds to the case  $x = x'$  that is  $R = 0$ ).

We notice that if we remove the last hypothesis in the definition, we get a quasi-isometric embedding instead of a quasi-isometry. Check that quasi-isometric inequalities are satisfied. Let  $x_1, x_2$  be two points and  $y_1 = f(x_1), y_2 = f(x_2)$  their images. We know that  $\phi(x_1, y_1) \neq 0$  and  $\phi(x_2, y_2) \neq 0$ . Hence,  $d(y_1, y_2) \leq Q_1^\phi(d(x_1, x_2)) = c_1 d(x_1, x_2) + c_2$ . In the same way we obtain the lower bound for  $d(y_1, y_2)$ .

**Lemma 16.** *If  $\psi$  is a kernel on  $Y \times Y$  and  $f$  is a measurable  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry or a quasi-isometric embedding from  $X$  to  $Y$ . In case of a quasi-isometry we also assume that the radius of positivity of  $\psi$  is at least  $\zeta\lambda_2 + c_2$  with  $\zeta > 0$ , then  $\phi(x, y) = \psi(f(x), y)$  is a cross-kernel on  $X \times Y$  and  $Q_1^\phi(R) \leq 2R^\psi + \lambda_1 R + c_1$ ,  $Q_2^\phi(R) \leq \lambda_2(2R^\psi + R + c_2)$  and  $S^\phi \leq \sup \text{Vol}B(2\lambda_2 R^\psi + c_2) \sup_{Y \times Y} \psi$ . In case of a quasi-isometry  $D^\phi \geq \zeta$  and  $\tau^\phi = \tau^\psi \geq c_\tau e^{-\varepsilon^\psi}$ .*

*Proof.* 1) Evidently, for any  $x \in X$   $\int_Y \phi(x, y) dy = 1$  by the definition of kernels.

2) Check the second axiom. Take two points  $x_1, x_2$  such that  $d(x_1, x_2) \leq R_1$  and two points  $y_1, y_2$  such that  $d(y_1, y_2) \geq 2R^\psi + \lambda_1 R_1 + c_1$ . If  $d(f(x_1), y_1) \geq R^\psi$ , there is nothing to prove as  $\psi(f(x_1), y_1) = 0$ . Otherwise  $d(f(x_2), y_2) \geq d(y_1, y_2) - d(f(x_2), y_1) \geq d(y_1, y_2) -$

$(d(f(x_2), f(x_1)) + d(f(x_1), y_1)) \geq d(y_1, y_2) - (\lambda_1 d(x_1, x_2) + c_1 + R^\psi) \geq R^\psi$ . Hence,  $\psi(f(x_2), y_2) = 0$ .

3) Check the third axiom. Take two points  $y_1, y_2$  such that  $d(y_1, y_2) \leq Q_1$  and two points  $x_1, x_2$  such that  $d(x_1, x_2) \geq \lambda_2(2R^\psi + Q_1 + c_2)$ . If  $d(f(x_1), y_1) \geq R^\psi$ , there is nothing to prove as  $\psi(f(x_1), y_1) = 0$ . Otherwise  $d(f(x_2), y_2) \geq d(f(x_1), f(x_2)) - d(f(x_1), y_2) \geq d(f(x_1), f(x_2)) - (d(f(x_1), y_1) + d(y_1, y_2)) \geq d(x_1, x_2)/\lambda_2 - c_2 - (R^\psi + Q_1) \geq R^\psi$ . Hence,  $\psi(f(x_2), y_2) = 0$ .

4) Check the fourth axiom. For any  $y \in Y$ , if  $d(f(x), y) > R^\psi$  then  $\psi(f(x), y) = 0$ . Hence, the diameter of the set of points  $X_y \in X$  such that for any  $x \in X_y$   $d(f(x), y) \leq R^\psi$ , is less than  $\lambda_2 2R^\psi + c_2$ . Hence,  $\int_X \phi(x, y) dx \leq \sup \text{Vol} B(2\lambda_2 R^\psi + c_2) \sup_{Y \times Y} \psi$ , here  $\sup \text{Vol} B(2\lambda_2 R^\psi + c_2)$  stands for the supremum of volumes of all balls of radius  $2\lambda_2 R^\psi + c_2$ .

5) If  $d(f(x), y) < \zeta \lambda_2 + c_2$  then  $\phi(x, y) > \tau^\psi$ . Hence, the diameter of the set of points of  $X$  with this property is at least  $2\zeta$ .  $\square$

### 4.2.3 Transporting cocycles

**Definition 23.** Let  $a$  be a cocycle on  $Y$  and  $\phi$  a cross-kernel on  $X \times Y$ . The convolution of  $a$  with  $\phi$  is the cocycle defined on  $X$  by

$$a * \phi(x, x') = \int_{Y \times Y} a(y, y') \phi(x, y) \phi(x', y') dy dy'.$$

**Lemma 17.** Let  $\phi : X \times Y \rightarrow \mathbb{R}$  be a cross-kernel, let  $a$  be a cocycle on  $Y$  and let  $\psi$  be a kernel on  $X$ . Then

$$N_\psi(a * \phi) \leq C N_{\tilde{\psi}}(a),$$

where  $\tilde{\psi}$  is a kernel on  $Y$  and

$$C \leq \left( \frac{\sup \psi}{\tau} \right)^{1/p} (S^\phi)^{2/p},$$

where  $\tau = c_\tau^Y e^{-Q_1^\phi(R^\psi)}$  (for the definition of constant  $c_\tau^Y$  see lemma 13, it depends on the local geometry of the space  $Y$  only).

In particular, if  $\phi$  is associated with a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry or a quasi-isometric embedding,

$$C \leq \frac{1}{c_\tau^Y} (\sup \psi)^{3/p} e^{((2+\lambda_1)R^\psi + c_1)/p} (2\lambda_1 R^\psi + c_1)^{2/p}.$$

*Proof.*

$$\begin{aligned} (N_\psi(a * \phi))^p &= \int_{X \times X} |a * \phi(x, x')|^p \psi(x, x') dx dx' = \\ &= \int_{X \times X} \left| \int_{Y \times Y} a(y, y') \phi(x, y) \phi(x', y') dy dy' \right|^p \psi(x, x') dx dx' \end{aligned}$$

By Hölder inequality

$$\leq \int_{X \times X} \int_{Y \times Y} |a(y, y')|^p |\phi(x, y)\phi(x', y')\psi(x, x')| dx dx'$$

$$\text{Let } \psi'(y, y') = \int_{X \times X} \phi(x, y)\phi(x', y')\psi(x, x') dx dx'$$

$$= \int_{Y \times Y} |a(y, y')|^p \psi'(y, y') dy dy'.$$

Now we need to show that  $\psi'$  is dominated by some kernel  $\psi''$ .

First we will prove that  $\psi'(y, y') = 0$  if  $d(y, y') > R^{\psi'}$  for some  $R^{\psi'}$ .

If  $d(x, x') > R^{\psi}$  then by the definition of kernels  $\psi(x, x') = 0$ , hence

$$\begin{aligned} \psi'(y, y') &= \int_{X \times X} \phi(x, y)\phi(x', y')\psi(x, x') dx dx' = \\ &= \int_{(x, x') \in X \times X, d(x, x') \leq R^{\psi}} \phi(x, y)\phi(x', y')\psi(x, x') dx dx'. \end{aligned}$$

If  $d(x, x') < R^{\psi}$  then by definition of cross-kernels there exists a number  $Q_1^{\phi}(R^{\psi})$  such that if  $d(y, y') > Q_1^{\phi}$  we have  $\phi(x, y)\phi(x', y') = 0$ . We estimate  $\psi'(y, y')$  from above in an evident way

$$\psi'(y, y') \leq \sup \psi \int_{X \times X} \phi(x, y)\phi(x', y') dx dx' \leq \sup \psi (S^{\phi})^2.$$

By Lemma 13 we conclude that there exists a kernel  $\tilde{\psi}$  such that  $\tilde{\psi}(y, y') \geq \tau = c_{\tau}^Y e^{-Q_1^{\phi}}$  whenever the distance between  $y, y'$  does not exceed  $Q_1^{\phi}$ . Hence,

$$\psi'(y, y') \leq \frac{\sup \psi}{\tau} (S^{\phi})^2 \tilde{\psi}(y, y').$$

□

#### 4.2.4 Transporting Poincaré constants

**Theorem 22.** *Let  $X, Y$  be two geodesic metric spaces such that infimums  $v_X(r) = \inf\{\text{vol}B(x, r) | x \in X\}$ ,  $v_Y(r) = \inf\{\text{vol}B(y, r) | y \in Y\}$  of volume of balls of radius  $r$  are positive. Let also  $\phi$  be a cross-kernel on  $X \times Y$ . Suppose that there exists a Poincaré constant  $C_X = C_p(X, \psi)$  associated to a kernel  $\psi$  on  $X$ . Then the Poincaré constant  $C_p(Y, \psi_Y)$  for  $Y$  also exists and depends on parameters characterizing  $\phi, \psi, \psi_Y$  and  $c_{\tau}(Y)$  (which depends on local geometry of  $Y$ , see Lemma 13 for definition).*

*Remark 9.* Lemma 17 allows us to extend this proof to riemannian manifolds as it provides the method to pass from a gradient to a cocycle and vice-versa.

Now we prove the theorem.

*Proof.* For any  $\mathbb{L}_p$ -function  $g$  on  $Y$ , denote the cocycle associated with  $g$  by  $dg(z, y) = g(z) - g(y)$ . The idea of our proof is the following. By Minkowski inequality we will show that

$$\|g\|_p \leq N_\psi(dg) + \|h\|_p$$

where  $h$  is a measurable function on  $X$  defined as before

$$h(x) = \int_Y g(z)\phi(x, z)dz.$$

Then we will apply consequently Poincaré inequality to the function  $h$  and Lemma 17 on transporting cocycles. Briefly,

$$\begin{aligned} \|g\|_p \leq N_{p,\psi}(dg) + \|h\|_p &\leq N_{p,\psi'}(dg) + C_{\text{poincare}}(X)N_{p,\psi}(dh) \\ &\leq N_{p,\psi'}(dg) + C_{\text{cocycle}}C_{\text{poincare}}(X)N_{p,\psi'}(dg) \end{aligned}$$

for some well chosen kernel  $\psi'$  on  $Y$ .

**First step.** Notice that for any  $z \int_X \phi(x, z)dx \geq \tau^\phi \inf \text{Vol}B(D^\phi)$ , where the supremum is taken over all balls in  $X$  of radius  $D^\phi$ . Denote  $V_\phi = (\tau^\phi \inf \text{Vol}B(D^\phi))^{-1}$  (we remind that the infimum is non-zero by the assumption). We have

$$\int_Y |g(z)|^p dz \leq V_\phi \int_{X \times Y} |g(z)|^p \phi(x, z) dx dz.$$

**Second step.** Now by Minkowski inequality applied to  $(\int |g(z)|^p \phi(x, z) dx dz)^{1/p}$  we get

$$\begin{aligned} \left( \int |g(z)|^p \phi(x, z) dz dx \right)^{1/p} &\leq \left( \int |g(z) - h(x)|^p \phi(x, z) dz dx \right)^{1/p} + \left( \int |h(x)|^p \phi(x, z) dz dx \right)^{1/p} = \\ &= \left( \int |g(z) - h(x)|^p \phi(x, z) dz dx \right)^{1/p} + \left( \int |h(x)|^p dx \right)^{1/p} \end{aligned}$$

**Third step.** For any points  $z \in Y$  and  $x \in X$  we have

$$g(z) - h(x) = g(z) - \int_Y g(y)\phi(x, y)dy =$$

by definition of cross-kernel  $\int_Y \phi(x, y)dy = 1$  so we go on

$$= g(z) \int_Y \phi(x, y)dy - \int_Y g(y)\phi(x, y)dy = \int_Y (g(z) - g(y))\phi(x, y)dy.$$

Now we apply Hölder inequality

$$|g(z) - h(x)|^p \leq \int_Y |g(z) - g(y)|^p \phi(x, y)dy.$$

So,

$$\left( \int |g(z) - h(x)|^p \phi(x, z) dz dx \right)^{1/p} \leq \left( \int |g(z) - g(y)|^p \phi(x, y) \phi(x, z) dz dx dy \right)^{1/p}.$$

**Fourth step.** Evidently,  $\int_X \phi(x, y) \phi(x, z) dx$  is uniformly bounded, and it vanishes outside of a strip of width  $Q_1^\phi(0)$  (take  $R = 0$  for the second property of cross-kernel). Hence, there exists a kernel  $\psi'$  on  $Y \times Y$  and the constant  $C_1 = C_1(\phi, Y)$  such that

$$\begin{aligned} & \left( \int |g(z) - g(y)|^p \phi(x, y) \phi(x, z) dz dx dy \right)^{1/p} \\ & \leq C_1 \left( \int |g(z) - g(y)|^p \psi'(z, y) dz dy \right)^{1/p} = C_1 N_{p, \psi'}(dg). \end{aligned}$$

We will find  $\psi'$  and  $C_1$  using the convolution of some model (or typical) kernel on  $Y$  with the margin  $\tilde{\tau}_Y$  and the radius of positivity  $\tilde{\varepsilon}_Y$ . Fix some  $r < \tilde{\varepsilon}_Y$ . Assume  $m$  to be the least integer such that  $m(\tilde{\varepsilon}_Y - r) \geq Q_1^\phi(0)$ . Hence, if we take  $\psi' = (\psi_Y)^{*m}$ , we set

$$C_1 = \frac{(\sup \phi)^2}{\tau_Y^{m/p} v(r)^{(m-1)/p}},$$

where  $v(r)$  is the infimum of volumes of balls of radius  $r$ . We get also that

$$\sup \psi' \leq (\sup \psi_Y)^m.$$

**Fifth step.** We apply Poincaré inequality to  $h$

$$\|h\|_p \leq C_p(X, \psi) N_{p, \psi}(dh).$$

Now Lemma 17 allows us to transport cocycle  $dh$  to  $Y$  with a multiplicative constant  $C_{tr}$

$$N_{p, \psi}(dh) \leq C_{tr} N_{p, \tilde{\psi}'}(dg).$$

We need to pass from the kernel  $\tilde{\psi}'$  to the kernel  $\psi'$ . For this purpose we use Lemma 14

$$N_{p, \tilde{\psi}'}(dg) \leq \hat{C} N_{p, \psi'}(dg),$$

where  $\hat{C}$  can be calculated from the mentioned lemma.

Note that if we return to definition of  $h$ , we obtain

$$dh(x, x') = h(x) - h(x') = \int_Y g(y) (\phi(x, y) - \phi(x', y)) dy = \int_Y g(y) d_X \phi(y)(x, x') dy.$$

**Final step.** Combining all these results we conclude that

$$\begin{aligned} \|g\|_p & \leq V_\phi(\|h\|_p + N_{p, \psi'}(dg)) \leq V_\phi(C_p(X, \psi) C_{tr} \hat{C} N_{p, \psi'}(dg) + C_1 N_{p, \psi'}(dg)) = \\ & = V_\phi \left( C_p(X, \psi') C_{tr} \hat{C} + C_1 \right) N_{p, \psi'}(dg). \end{aligned}$$

□

Now summarizing all the results of this section we know that quasi-isometries preserve Poincaré inequalities.

### 4.3 Poincaré inequality for exponential metric

We will give an upper bound for the Poincaré constant in a ball of radius  $R$  in a space with the metric  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ .

**Theorem 23.** *Let  $\tilde{X} = \mathbb{R}_+ \times \mathbb{R}^n$  with the metric  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ . Let  $X = \tilde{X}/\Gamma$  where  $\Gamma$  is a lattice of translations in the factor  $\mathbb{R}^n$ . Then the Poincaré constant for a ball  $B(R)$  in  $X$  is*

$$C_p(\mu) \leq \frac{p}{\mu} + (A(\mu))^{1/p} C_p(\mathbb{T}^n) e^{\mu_n R},$$

where  $\mu = \sum \mu_i$ ,  $A(\mu)$  is a constant depending only on  $\mu$ ,  $C_p(\mathbb{T}^n)$  is a Poincaré constant for a torus  $\mathbb{T}^n$ .

First, we fix the direction  $\theta = (x_1, \dots, x_n)$ .

#### 4.3.1 Poincaré inequality for fixed direction

**Lemma 18.** *Let  $\tilde{X} = \mathbb{R}_+ \times \mathbb{R}^n$  with the metric  $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ . Let  $X = \tilde{X}/\Gamma$  where  $\Gamma$  is a lattice of translations in the factor  $\mathbb{R}^n$ . Let  $R \in \mathbb{R}^+ \cup \{\infty\}$ . Then for any fixed direction  $\theta = (x_1, \dots, x_n)$*

$$\left( \int_a^R |f(t) - c_\theta|^p e^{\mu t} dt \right)^{1/p} \leq \frac{p}{\mu} \left( \int_a^R |f'(t)|^p e^{\mu t} dt \right)^{1/p},$$

where  $c_\theta = f(R, \theta)$  or  $c_\theta = \lim_{R \rightarrow \infty} f(R, \theta)$ .

*Proof.* Let  $f$  be a function such that its partial derivative  $\partial f / \partial t$  is in  $\mathbb{L}^p(e^{\mu t} dt, [0, +\infty))$  where  $p > 1$ . By Hölder inequality we get

$$\int_0^{+\infty} \left| \frac{\partial f}{\partial t} \right| dt \leq \left( \int_0^{+\infty} \left| \frac{\partial f}{\partial t} \right|^p e^{\mu t} dt \right)^{1/p} \left( \int_0^{+\infty} e^{-(\mu t/p)(p/(p-1))} dt \right)^{1-1/p} < +\infty.$$

Hence, for every fixed direction  $\theta$  there exists a limit  $\lim_{t \rightarrow \infty} f(t, \theta)$ .

First, if  $R = \infty$ , prove that  $|f(t) - c_\theta|^p e^{\mu t} \rightarrow 0$  as  $t \rightarrow \infty$ . Apply the Newton-Leibniz theorem and then Hölder inequality to  $|f(t) - c_\theta|$ . We have

$$\begin{aligned} |f(t) - c_\theta| &= \left| \int_t^\infty \frac{\partial f}{\partial s} ds \right| \leq \int_t^\infty \left| \frac{\partial f}{\partial s} \right| ds \leq \\ &\leq \left( \int_t^\infty \left| \frac{\partial f}{\partial s} \right|^p e^{\mu u} du \right)^{1/p} \left( \int_t^\infty e^{-\mu s/(p-1)} ds \right)^{1-1/p}. \end{aligned} \quad (4.1)$$

Calculate the last integral

$$\int_t^\infty e^{-\mu s/(p-1)} ds = -\frac{p-1}{\mu} e^{-\frac{\mu s}{p-1}} \Big|_t^\infty = \frac{p-1}{\mu} e^{-\frac{\mu t}{p-1}}.$$

Denote the constant  $D_0 = \left(\frac{p-1}{\mu}\right)^{p-1}$

$$|f(t) - c_\theta|^p \leq D_0 e^{-\mu t} \int_t^{+\infty} \left| \frac{\partial f}{\partial s} \right|^p e^{\mu s} ds.$$

Hence

$$|f(t) - c_\theta|^p e^{\mu t} \leq D_0 \int_t^{+\infty} \left| \frac{\partial f}{\partial s} \right|^p e^{\mu s} ds \rightarrow 0$$

as  $t \rightarrow +\infty$ .

Now we integrate by parts

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt = \left[ |f(t) - c_\theta|^p \frac{e^{\mu t}}{\mu} \right]_a^R - \int_a^R f'(t) p |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt. \quad (4.2)$$

As  $c_\theta = f(R)$

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt = -|f(a) - c_\theta|^p \frac{e^{\mu a}}{\mu} - p \int_a^R f'(t) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt.$$

We notice that the integral at the left is positive. At the right part, the first term is negative (for this reason we will drop it soon). Hence, the second term should be positive. By Hölder inequality

$$\int_a^R (-f'(t)) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt \leq \left( \int_a^R |f'(t)|^p \frac{e^{\mu t}}{\mu} dt \right)^{1/p} \left( \int_a^R |f(t) - c_\theta|^p \frac{e^{\mu t}}{\mu} dt \right)^{(p-1)/p}. \quad (4.3)$$

Introduce following notations

$$X = \int_a^R |f(t) - c_\theta|^p e^{\mu t} dt,$$

$$Y = \int_a^R |f'(t)|^p e^{\mu t} dt.$$

Using this notations we return to 4.2. First we drop the term  $-|f(a) - c_\theta|^p e^{\mu a} / \mu$  and then we apply 4.3

$$X \leq \frac{p}{\mu} Y^{1/p} X^{(p-1)/p}.$$

So, we get immediately that

$$X^{1/p} \leq \frac{p}{\mu} Y^{1/p}$$

which proves Poincaré inequality for fixed direction.  $\square$



### 4.3.2 Poincaré inequality for exponential metric.

Here we will finish the proof of Theorem 23. Introduce the following notations  $\tilde{f}_r(t, \theta) = f(r, \theta)$  (the function is considered as a function of two variables),  $f_r(\theta) = f(r, \theta)$  (the function is considered as a function of one variable).

We have already proved that for any  $\theta \in \mathbb{T}^n$

$$\int_0^R |f(t, \theta) - f(R, \theta)|^p e^{\mu t} dt \leq \left(\frac{p}{\mu}\right)^p \int_0^R \left|\frac{\partial f}{\partial t}\right|^p e^{\mu t} dt.$$

We integrate over  $\theta$  and we introduce the volume element for  $\tilde{X}$   $dvol = dr d\theta e^{\sum \mu_i r}$

$$\int_{B(R)} |f - f_R|^p dvol \leq \left(\frac{p}{\mu}\right)^p \int_{B(R)} |\nabla f|^p dvol.$$

Denote the Euclidean gradient by  $\nabla_e$ . By the form of the metric we write that  $e^{2\mu_i t} |dx_i^2| = 1$ . Hence,  $\|\nabla_e f_r\| \leq e^{\mu_n t} |\nabla f|$ . Now we notice that

$$\int_{R-1}^R \|\nabla_e f_r\|_{\mathbb{L}^p(\mathbb{T}^n)}^p e^{\mu t} dt \geq e^{\sum \mu_i (R-1)} \int_{R-1}^R \|\nabla_e f_r\|_{\mathbb{L}^p(\mathbb{T}^n)}^p dt.$$

So we write

$$e^{\sum \mu_i (R-1)} \int_{R-1}^R \|\nabla_e f_r\|_{\mathbb{L}^p(\mathbb{T}^n)}^p dt \leq e^{p\mu_n R} \int_{B(R) \setminus B(R-1)} |\nabla f|^p dvol. \quad (4.4)$$

Fix  $r \in [R-1, R]$ . Write Poincaré inequality on torus for the function  $f_r(\theta)$ . There exists a number  $c_r$  such that

$$\int_{\mathbb{T}^n} |f_r(\theta) - c_r|^p d\theta \leq (C_p(\mathbb{T}^n))^p \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta,$$

where  $C_p(\mathbb{T}^n)$  is a Poincaré constant for  $\mathbb{T}^n$ . Next we consider the function  $f_r(\theta)$  as a function on the ball  $B(R)$  independent on  $t$ . We integrate this inequality over  $t$

$$\begin{aligned} \int_{B(R)} |f_r(\theta) - c_r|^p dvol &\leq (C_p(\mathbb{T}^n))^p \int_0^R \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta e^{\sum \mu_i t} dt \\ &\leq \frac{e^{\sum \mu_i R}}{\sum \mu_i} (C_p(\mathbb{T}^n))^p \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta. \end{aligned}$$

We integrate over  $r$  from  $R-1$  to  $R$  and we remind the inequality 4.4. It gives

$$\int_{R-1}^R \left( \int_{B(R)} |f_r(\theta) - c_r|^p dvol \right) dr \leq A(\mu) (C_p(\mathbb{T}^n))^p e^{p\mu_n R} \int_{B(R) \setminus B(R-1)} |\nabla f|^p dvol,$$

where  $A(\mu)$  is a constant which depends only on  $\mu_i, i = 1, \dots, n$ . Now apply Hölder's inequality

$$\begin{aligned} \int_{R-1}^R \|f_r - c_r\|_{\mathbb{L}^p(B(R))} dr &\leq \left( \int_{R-1}^R \int_{B(R)} |f_r - c_r|^p d\text{vol} dr \right)^{1/p} \\ &\leq \left( A(\mu) (C_p(\mathbb{T}^n))^p e^{p\mu_n R} \int_{B(R) \setminus B(R-1)} |\nabla f|^p d\text{vol} \right)^{1/p} \\ &\leq (A(\mu))^{1/p} C_p(\mathbb{T}^n) e^{\mu_n R} \|\nabla f\|_{\mathbb{L}^p(B(R))} \end{aligned}$$

Set  $c = \int_{R-1}^R c_r dr$ . In the following line of inequalities we will first apply a triangle inequality and then we will use the fact that the norm of the integral is less or equal to the integral of the norm (briefly  $\|\int f dr\| = \int \|f\| dr$ )

$$\begin{aligned} \|f - c\|_{\mathbb{L}^p(B(R))} &= \left\| \int_{R-1}^R (f - c_r) dr \right\|_{\mathbb{L}^p(B(R))} \\ &\leq \left\| \int_{R-1}^R (f - f_r) dr \right\|_{\mathbb{L}^p(B(R))} + \left\| \int_{R-1}^R (f_r - c_r) dr \right\|_{\mathbb{L}^p(B(R))} \\ &\leq \int_{R-1}^R (\|f - f_r\|_{\mathbb{L}^p(B(R))} + \|f_r - c_r\|_{\mathbb{L}^p(B(R))}) dr \\ &\leq \frac{p}{\mu} \|\nabla f\|_{\mathbb{L}^p(B(R))} + (A(\mu))^{1/p} C_p(\mathbb{T}^n) e^{\mu_n R} \|\nabla f\|_{\mathbb{L}^p(B(R))}. \end{aligned}$$

#### 4.4 Lower bound on Poincaré constant

Let  $Z_\mu$  denote  $\mathbb{T}^n \times \mathbb{R}$  equipped with metrics  $dt^2 + \sum e^{2\mu_i t} dx_i^2$ , where we suppose  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ . In this section we will give a lower bound for the quasi-isometric distortion growth between two spaces  $Z = Z_\mu$  and  $Z' = Z_{\mu'}$ , using our results on transported Poincaré inequalities. Let  $O, O' = (0, \dots, 0)$  be base points of  $Z$  and  $Z'$  respectively. First we notice that the "width" of  $\mathbb{T}^n \times (-\infty, 0]$  is finite so it is at finite distance from a ray  $(-\infty, 0]$ , so from now on, we shall focus our attention on the part of  $B_Z(O, R)$  where  $t \geq 0$ .

Our method does not apply to a general quasi-isometric embedding. We will need quasi-isometric embeddings be homotopy equivalences. Therefore we need a variant of the definition of quasi-isometric distortion growth.

**Definition 24.** Let  $X, Y$  be metric spaces,  $x_0, y_0$  their base points respectively. The homotopy quasi-isometric distortion growth is the function

$$D_{hG}(X, x_0, Y, y_0)(R) = \inf \{d | \exists f : B_X(x_0, R) \rightarrow Y \text{ a quasi-isometric embedding such that } f(x_0) = y_0 \text{ and } f \text{ is a homotopy equivalence, } d = D_0(f)\},$$

where  $D_0(f)$  is the quasi-isometric distortion growth, see Definition 6.

**Theorem 24.** *Let  $Z, Z'$  be two locally homogeneous hyperbolic metric spaces with metrics  $dt^2 + \sum e^{2\mu_i t} dx_i^2$  and  $dt^2 + \sum e^{2\mu'_i t} dx_i^2$  respectively,  $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$  and  $0 < \mu'_1 \leq \mu'_2 \leq \dots \leq \mu'_n$ . Assume also that  $\sum \mu_i / \mu_n > \sum \mu'_i / \mu'_n$ . Suppose that there exist constants  $a$  and  $b$  such that for any  $i$   $b \leq \mu_i, \mu'_i \leq a$ . Then there exist constants  $G_0(a, b)$ ,  $G_1(a, b)$  and  $G_2(a, b)$  such that the following holds.*

- *Let  $\Theta : B_Z(R) \rightarrow Z'$  be a continuous  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding, inducing an isomorphism on fundamental groups. Suppose that  $\Theta$  sends base point to base point,  $\Theta(O) = O'$  and that  $R \geq 8(\lambda_1 + c_1) + (\lambda_2 + c_2) + 1$ . If  $p > \sum \mu'_i / \mu'_n$ , up to replacing  $Z$  with a connected 2-sheeted covering, Poincaré constant  $C_p(\mu)$  for a ball of radius  $R$  in the space  $Z$  is bounded from below by*

$$C_p(\mu) \geq (G_0(a, b))^{1/p} (\lambda_1 + c_1)^{-3/p-2/p^2} e^{-(9/p+3/p^2)(\lambda_1+c_1)} e^{(\sum \mu_i/p)R} \left( p - \sum \mu'_i / \mu'_n \right)^{1/p}.$$

- *The homotopy distortion growth (see Definition 24) for quasi-isometrical embedding of  $B_Z(R)$  into  $Z'$  is bounded from below by*

$$D_G(R) \geq \min \left\{ G_1 \left( \frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} \right) R - G_2, \frac{1}{8} R \right\}.$$

We will prove this theorem in several steps. First we introduce non-trivial double-covering spaces  $\tilde{Z}$  and  $\tilde{Z}'$  of  $Z$  and  $Z'$ . We prove that  $\Theta$  lifts to a  $(\lambda_1, 2c_1)$ -"quasi-lipschitz" map. Then we take the test-function  $e^{\pi i x_n}$  on  $\tilde{Z}'$  which depends only on one coordinate  $x_n$ . It varies very slowly outside of some ball, so the absolute value of the transported and regularised function  $v$  on  $\tilde{Z}$  stays near to 1. Lemmas 15 and 17 help us to control how the lower bound of Poincaré constant changes under transport. This helps us get a lower bound for Poincaré constant of  $\tilde{Z}$  in function of  $\{\mu_i\}, \{\mu'_i\}$  and the constants of quasi-isometric embedding. We also have an upper bound for the Poincaré constant of  $\tilde{Z}$  by Theorem 23. The combination of these results provides a lower bound for the homotopy distortion growth for  $Z$  and  $Z'$ .

#### 4.4.1 Quasi-isometric embeddings and fundamental groups

If  $\dim(Z) \geq 3$ , one may believe that the assumption that  $\Theta$  be isomorphic on fundamental groups is not that restrictive. Indeed, in Proposition 9, we shall show that this is automatic, but unfortunately the argument introduces an ineffective constant  $R_0$ , which makes it useless. For instance, if it turns out that  $R_0 = \lambda_1^2$ , Proposition 9 does not help to remove the homotopy assumption in Theorem 24. Nevertheless, it is included for completeness sake.

**Proposition 9.** *Let  $Z, Z'$  be two spaces of the described form with equal dimensions  $n+1 \geq 3$ . Then for any  $\lambda_1 \geq 1, \lambda_2 \geq 1, c_1 \geq 0, c_2 \geq 0$  there exists  $R_0 = R_0(\lambda_1, \lambda_2, c_1, c_2)$  such that if  $R > R_0$  and a continuous map  $f : B_{Z_\mu}(O, R_0) \rightarrow Z_{\mu'}$  is a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding, then  $f$  induces an isomorphism on the fundamental groups  $\pi_1(Z_\mu) \rightarrow \pi_1(Z_{\mu'})$ .*

*Proof.* We provide a proof by contradiction. Assume that for arbitrarily large values of  $R$ , there exists a map  $f_R : B_Z(R) \rightarrow Z'$  which is a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding which is not isomorphic on fundamental groups. Pick a  $2c_1/\lambda_1$ -dense and  $c_1/\lambda_1$ -discrete subset  $\Lambda$  of  $Z$ . Notice that if  $f_R$  is a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry, then  $f_R$  is bi-Lipschitz on  $B_Z(R) \cap \Lambda$ . Conversely, if a map defined on  $B(R) \cap \Lambda$  is bi-Lipschitz, then it can be continuously extended on  $B(R)$  as a quasi-isometric embedding. Indeed, away from a ball,  $Z'$  is contractible up to scale  $c_1$ .

Set  $\rho = d(O', f_R(O))$ . First, consider the case when  $\rho \rightarrow \infty$ . Set  $\sigma = (\rho/4 - c_1)/\lambda_1$ . Then  $f_R(B(O, \sigma))$  is contained in a ball  $B(f_R(O), \rho/4)$  which lies in the complement of  $B(O', \rho/2)$

$$f_R(B(O, \sigma)) \subset B(f_R(O), \rho/4) \subset B(O', \rho/2)^c.$$

The diameter of the image of any loop in  $B(O, \sigma)$  is at most  $\lambda_1\sigma + c_1$ . Because  $\lambda_1\sigma + c_1 < \rho/4$ , these loops are homotopic to 0 (diameters of loops are too short relatively to  $B(O', \rho/2)^c$ ). Hence, the restriction of  $f_R$  on  $B(O, \sigma)$  is homotopic to 0. Hence  $f_R$  lifts to  $\tilde{f}_R : B_Z(\sigma) \rightarrow \tilde{Z}' = X_{\mu'}$  which is homogeneous. Now up to composing  $\tilde{f}_R$  with an isometry we can suppose that it preserves the center  $\tilde{f}_R(O) = O'$ . By Ascoli's theorem, we can find a sequence  $\tilde{f}_{R_j}|_{\Lambda}$  which uniformly converges to  $\tilde{f}|_{\Lambda} : Z \cap \Lambda \rightarrow \tilde{Z}'$  which is also bi-Lipschitz. We continuously extend  $\tilde{f}|_{\Lambda}$  to  $\tilde{f} : Z \rightarrow \tilde{Z}'$ ,  $\tilde{f}$  is a quasi-isometric embedding. Its extension to ideal boundaries is continuous and injective. By the theorem of invariance of domain,  $\partial\tilde{f} : T^n \simeq \partial X_{\mu} = S^n$  is open, and thus a homeomorphism. This provides a contradiction if  $n \geq 2$ .

If  $\rho = d(O', f_R(O))$  stays bounded, we can directly use Ascoli's theorem, and get a limiting continuous quasi-isometric embedding  $f$ . Again,  $f$  extends to the ideal boundary,  $\partial f : \partial Z \rightarrow \partial Z'$ , the map  $\partial f$  is continuous and injective. Because  $\partial Z$  and  $\partial Z'$  have the same dimension,  $\partial f$  is an open map by the theorem of invariance of domain and  $\partial f$  is a homeomorphism. Hence,  $\partial f$  induces an isomorphism on fundamental groups. If  $R_j$  is sufficiently large, then  $f_{R_j}$  is at bounded distance from  $f$  and hence  $f_{R_j}$  also induces an isomorphism  $\pi_1(B_Z(R)) \rightarrow \pi_1(Z')$ . This contradiction completes the proof.  $\square$

*Remark 10.* The proof does not provide an effective value of  $R_0$ .

#### 4.4.2 Lifting to a double covering space

Introduce a double covering of  $Z'$ . Let  $\tilde{Z}' = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1} \times \mathbb{R}/2\mathbb{Z} \times [0; +\infty)$  with the metric defined by the same formula as for  $Z'$ :  $dt^2 + \sum e^{2\mu_i t} dx_i^2$ . Consider the map  $\tilde{Z}' \rightarrow Z'$  defined by

$$(x_1, x_2, \dots, x_n, t) \mapsto (x_1, x_2, \dots, x_n \pmod{1}, t).$$

So we identify  $(x_1, x_2, \dots, x_n, t)$  and  $(x_1, x_2, \dots, x_n + 1, t)$  in  $\tilde{Z}'$ . Consider a complex function  $u(x_1, x_2, \dots, x_n, t) = e^{\pi i x_n}$  on  $\tilde{Z}'$ .

Composition of  $u$  with deck transformation  $\iota' : \tilde{Z}' \rightarrow \tilde{Z}'$

$$\iota' : (x_1, x_2, \dots, x_n, t) \mapsto (x_1, x_2, \dots, x_n + 1, t)$$

gives  $u \circ \iota' = -u$ .

We have  $\Theta : Z \rightarrow Z'$  which is a continuous map inducing an isomorphism in fundamental groups, and we have  $\tilde{Z}'$  which is a covering space of  $Z'$ . We need to show that there exists a non-trivial covering space  $\tilde{Z} \rightarrow Z$  such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\Theta}} & \tilde{Z}' \\ \pi_Z \downarrow & & \downarrow \pi_{Z'} \\ Z & \xrightarrow{\Theta} & Z' \end{array}$$

Define

$$\tilde{Z} = \{(z, \tilde{z}') \mid z \in Z, \tilde{z}' \in \pi_{Z'}^{-1}(\Theta(z))\},$$

that is  $\tilde{Z} \subset Z \times \tilde{Z}'$ . Let  $[\gamma']$  be a loop in  $Z'$  which does not lift to a loop in  $\tilde{Z}'$ . By hypothesis, there exists a loop  $\gamma$  in  $Z$  such that  $\Theta(\gamma)$  is homotopic to  $\gamma'$ . Then  $\gamma$  does not lift to a loop in  $\tilde{Z}$ . There exists an isometry  $\iota$  of order 2 on  $\tilde{Z}$  such that  $\tilde{\Theta} \circ \iota = \iota' \circ \tilde{\Theta}$ .

#### 4.4.3 Lifting of $\Theta$

Here we will prove that in the constructed double coverings  $\Theta$  lifts to a map satisfying the right-hand inequality in the definition of quasi-isometry with constants  $\lambda_1$  and  $2c_1$ . We need two preliminary lemmas concerning distances in two-fold coverings.

**Lemma 19.** *Let  $Z = Z_\mu$  be a locally homogeneous space. There is an effective constant  $c_0(\mu)$  with the following effect. Let  $z$  be a point in  $Z$  in the region where  $t \geq c_0$ . Let  $c = t(z)$ . Every loop based at  $z$  of length less than  $c$  is null-homotopic.*

*Proof.* Let  $\pi_s : Z \rightarrow T^n \times \{s\} \subset Z$  denotes projection onto the first factor. This is a homotopy equivalence. Note that  $\pi_s$  is length decreasing on  $\{(t, x) \in Z; t \geq s\}$ . Moreover, on  $T^n \times \{t\}$ ,  $\pi_s$  decreases length by  $e^{\mu_1(s-t)}$  at least. Let  $\gamma$  be a non null-homotopic geodesic loop at  $z$ . Assume that its length is  $\leq 2c$ . Then  $\gamma \subset \{(t, x) \in Z; t \geq \frac{c}{2}\}$ , therefore

$$\text{length}(\pi_{\frac{c}{2}}(\gamma)) \leq c,$$

thus

$$\text{length}(\pi_0(\gamma)) \leq c e^{-\mu_1 \frac{c}{2}}.$$

Since  $\pi_0(\gamma)$  is not null-homotopic, its length is at least 1, this shows that

$$c \geq e^{\mu_1 \frac{c}{2}}.$$

This can happen only for  $c \leq c_0(\mu_1)$ . □

**Lemma 20.** *Let  $z_1, z_2$  be two points in  $Z$  such that  $d(O', \Theta(z_1)) > c_1$  or  $d(O', \Theta(z_2)) > c_1$  and  $d(z_1, z_2) \leq c_1/\lambda_1$ . Then  $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))$ .*

*Proof.* Let  $\tilde{z}_1 \in \tilde{Z}$  be such that  $d(\tilde{O}, \tilde{z}_1) > c_1$ . Set

$$\begin{aligned} W &= \{\tilde{z}_2 \in \tilde{Z} \mid d(\tilde{z}_1, \tilde{z}_2) \leq c_1\}, \\ U &= \{\tilde{z}_2 \in W \mid d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))\} \subset W, \\ V &= \{\tilde{z}_2 \in W \mid d(\tilde{\Theta}(\tilde{z}_1), \iota' \circ \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))\} \subset W. \end{aligned}$$

By construction,  $W = U \cup V$ . Let us show that the intersection of  $U$  and  $V$  is empty

$$U \cap V = \{\tilde{z}_2 \in W \mid d(\tilde{\Theta}(\tilde{z}_1), \iota' \circ \tilde{\Theta}(\tilde{z}_2)) = d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2))\}.$$

If  $\tilde{z}_2 \in U \cap V$ , then the geodesic segments connecting  $\tilde{\Theta}(\tilde{z}_1)$  with  $\tilde{\Theta}(\tilde{z}_2)$  and  $\tilde{\Theta}(\tilde{z}_1)$  with  $\iota' \circ \tilde{\Theta}(\tilde{z}_2)$  induce a loop  $\gamma$  in  $Z'$  of length  $2d(\Theta(z_1), \Theta(z_2)) \leq 2(\lambda_1(c_1/\lambda_1) + c_1) = 4c_1$  which is not homotopic to 0. According to Lemma 19, this is incompatible with the assumption that  $d(\tilde{O}', \Theta(z_1)) > c_1$ . Hence,  $U \cap V$  is empty. Since  $U$  is non-empty (it contains at least  $\tilde{z}_1$ ) and closed in  $W$ ,  $V$  is closed in  $W$  and  $W$  is connected, we conclude that  $U = W$ , which finishes the proof.  $\square$

**Lemma 21.** *A  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding  $\Theta : Z \rightarrow Z'$  lifts to a "quasi-lipschitz" map  $\tilde{\Theta} : \tilde{Z} \rightarrow \tilde{Z}'$  that is for any two points  $\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}$*

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq \lambda_1 d(\tilde{z}_1, \tilde{z}_2) + 2c_1.$$

*Proof.* Let  $\tilde{\gamma} \subset \tilde{Z}$  be a geodesic between  $\tilde{z}_1$  and  $\tilde{z}_2$ . Let  $t_1$  be the first point such that  $d(\tilde{\Theta}\tilde{\gamma}(t), \tilde{O}') \leq c_1$  and  $t_2$  be the last point with such a property (if such points  $t_1, t_2$  do not exist, then we can apply the following arguments directly to  $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2))$  instead of cutting the curve in three parts and considering  $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_2))$ ). Then

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq d(\tilde{\Theta}\tilde{\gamma}(t_1), \tilde{\Theta}\tilde{\gamma}(t_2)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_2)).$$

By definition of  $t_1$  and  $t_2$   $d(\tilde{\Theta}\tilde{\gamma}(t_1), \tilde{\Theta}\tilde{\gamma}(t_2)) \leq 2c_1$ . Now divide parts of  $\gamma$  between  $\tilde{\Theta}(\tilde{z}_1)$  and  $\tilde{\Theta}\tilde{\gamma}(t_1)$  and between  $\tilde{\Theta}(\tilde{z}_1)$  and  $\tilde{\Theta}\tilde{\gamma}(t_2)$  by segments of length  $c_1/\lambda_1$ . We apply the previous lemma to them, so

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_2)) \leq N \left( \lambda_1 \frac{c_1}{\lambda_1} + c_1 \right),$$

where  $N \leq d(\tilde{z}_1, \tilde{z}_2)/(c_1/\lambda_1)$  is the number of segments in the subdivision. So,

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq 2c_1 + 2\lambda_1 d(\tilde{z}_1, \tilde{z}_2).$$

$\square$

#### 4.4.4 Proof of the first statement of Theorem 24 - Part 1

Let  $\psi'$  be a kernel on  $\tilde{Z}$  invariant by isometry, that is for any isometry  $\iota$

$$\psi'(\iota(\tilde{z}_1), \iota(\tilde{z}_2)) = \psi'(\tilde{z}_1, \tilde{z}_2).$$

As an example of such a kernel we can consider a kernel depending only on the distance between points. Let also  $\phi$  be the cross-kernel constructed with the quasi-isometry  $\tilde{\Theta}$  and a kernel  $\zeta$  on  $\tilde{Z}'$  which is also invariant by isometries. Define a complex function  $v$  on  $\tilde{Z}$  as follows

$$v = (u * \phi) * \psi'.$$

Then  $v \circ \iota = -v$ . Indeed,

$$v \circ \iota = (u * \phi) * \psi' \circ \iota = (u * \phi \circ \iota) * \psi'.$$

On the other hand,

$$\begin{aligned} u * \phi \circ \iota &= \int u(\tilde{z}') \phi(\iota \tilde{z}, \tilde{z}') d\tilde{z}' = \int u(\tilde{z}') \zeta(\iota' \tilde{\Theta}(\tilde{z}), (\iota')^2 \tilde{z}') d\tilde{z}' = \\ &= \int u(\tilde{z}') \zeta(\tilde{\Theta}(\tilde{z}), \iota' \tilde{z}') d\tilde{z}' = \int u(\iota' \tilde{z}') \zeta(\tilde{\Theta}(\tilde{z}), \tilde{z}') d\tilde{z}' = -u * \phi \end{aligned}$$

hence,  $v$  is skewsymmetric with respect to  $\iota$ . We get immediately that  $\int v = 0$ . Now we apply successively Lemma 15 and Lemma 17.

*Step 1.* By Lemma 15 there exists a kernel  $\psi_1$  on  $\tilde{Z}$  which is controlled by  $a$  and  $b$  and such that

$$\left( \int |\nabla(u * \phi * \psi')|^p \right)^{1/p} \leq N_{\psi_1}(u * \phi),$$

where for  $\psi_1$  we have the width of support is  $R^{\psi_1} = R^{\psi'}$  and

$$\sup \psi_1 \leq \frac{\sup \nabla \psi' \sup \psi'}{\inf_z \text{vol} B(\tilde{z}, R^{\psi})}.$$

*Step 2.* By Lemma 17 there exists a kernel  $\zeta_1$  on  $\tilde{Z}'$  such that

$$N_{\psi_1}(u * \phi) \leq \tilde{C} N_{\zeta_1}(u),$$

where the width of support of  $\zeta_1$  is  $2R^\zeta + \lambda_1 R^{\psi'} + c_1$ , the supremum of  $\zeta_1$  is

$$\sup \zeta_1 = \frac{\sup \psi_1}{c_\tau^Y} e^{2R^\zeta + \lambda_1 R^{\psi'} + c_1} (2\lambda_1 R^\zeta + c_1)^2$$

and

$$\tilde{C} = \frac{1}{c_\tau^Y} (\sup \psi_1)^{3/p} e^{((2+\lambda_1)R^{\psi'} + c_1)/p} \left( (2 + \lambda_1)R^{\psi'} + c_1 \right)^{2/p}.$$

*Step 3.* Applying Lemma 15 we get that there exists a kernel  $\zeta_2$  on  $\tilde{Z}'$  such that

$$N_{\zeta_2}(u) \leq C(n) \|\nabla u\|_p,$$

we remind that the constant  $C(n)$  depends only on the dimension of  $\tilde{Z}'$  if the Ricci curvature is bounded from below, that is  $\sup \mu_i$  is bounded.

*Step 4.* Here we merely need to pass from  $N_{\zeta_1}$  to  $N_{\zeta_2}$ . We apply Lemma 15 once more

$$N_{\zeta_1} \leq \hat{C} N_{\zeta_2},$$

where

$$\hat{C} = \frac{\sup \zeta_1 \sup \zeta_2 R^{\zeta_2}}{c_\tau^Y \varepsilon^{\zeta_2}} (2e)^{(2R^\zeta + \lambda_1 R^{\psi'} + c_1)/\varepsilon^{\zeta_2}}.$$

Choose  $\psi'$  and  $\zeta$  such that  $R^{\psi'} = 1$  and  $R^\zeta = 1$ . Then  $\sup \psi'$  and  $\sup \zeta$  are controlled by  $a$  and  $b$ . We note also that  $\varepsilon^{\zeta_2} = 1$ . So combining all inequalities we get

$$\int_{B(R)} |\nabla v|^p \leq C_1(a, b) (\lambda_1 + c_1)^{3+2/p} e^{(9+3/p)(\lambda_1+c_1)} \int_{\mathbb{T}^n \times [0, +\infty]} |\nabla u|^p,$$

where  $C_1(a, b)$  is a constant depending only on  $a, b$  and dimension  $n$ . Denote  $Q = \lambda_1 + c_1$ . The distortion growth  $D_G \geq 1/2Q$  so we will establish a lower bound for  $Q$  now. Assume

$$C(Q) = (\lambda_1 + c_1)^{3+2/p} e^{(9+3/p)(\lambda_1+c_1)}.$$

#### 4.4.5 Proof of the first statement of Theorem 24 - Part 2

We will give a lower bound for the  $\mathbb{L}^p$ -norm of the function  $v = (u * \phi) * \psi'$ . Our aim is to prove that the absolute value of  $v$  is nearly constant. For simplicity of notations we suppose first that the volume growth of  $Z_\mu$  and  $Z_{\mu'}$  is the same, that is  $\sum \mu_i = \sum \mu'_i$ . We will write  $|\mu|$  and  $|\mu'|$  for these sums respectively. We are going to show that there exists a subset  $A$  of a ball  $B(z_0, R)$  such that on the one hand the volume of  $A$  is rather big, that is  $Vol(A) \geq Vol(B(z_0, R))/2$  and on the other hand its image lies rather far from the base point  $\Theta(A) \cap B(z'_0, R - (\lambda_1 + c_1 + \lambda_2 + c_2)) = \emptyset$ .

Denote  $r = \lambda_2 + c_2$ . We will construct a finite subset  $J$  in  $B(z_0, R) \subset Z_\mu$  and its partition  $\{J_k\}_{k=1, \dots, n}$  of cardinality  $e^{|\mu|r}$  in subsets of cardinality  $|J_k| = e^{|\mu|(R-r)}$  with the following property

- **(P)** For any  $k \in \{1, \dots, n\}$  if  $z_1$  and  $z_2$  are points of  $J_k$  then the open balls of radius  $r$  centered at these points are disjoint.

So, let  $z_1, z_2 \in J_k$  be two different points. It follows from **(P)** that

$$2r \leq d(z_1, z_2) \leq \lambda_2 d(\Theta(z_1), \Theta(z_2)) + c_2,$$



hence  $d(\Theta(z_1), \Theta(z_2)) \geq 2$  so the balls  $B(\Theta(z_1), 1)$  and  $B(\Theta(z_2), 1)$  are disjoint. Fix some  $d > 0$  and denote by  $J'_k \subset J_k$  the set of points whose images are not farther than  $R - d$  from  $z'_0$  that is if  $z \in J'_k$  then  $d(z'_0, \Theta(z)) \leq R - d$ . We obtain

$$|J'_k| \text{Vol}(B(\Theta(z), 1)) \leq \text{Vol}(B(z'_0, R - d + 1))$$

and we conclude that  $|J'_k| \leq e^{|\mu|(R-d)}$ . Denote the union of  $J'_k$  by  $J'$  then  $|J'| \leq e^{|\mu|(R-d+r)}$ . Hence, whenever  $d \geq r + 1$

$$\frac{|J'|}{|J|} \leq e^{|\mu|(r-d)} \leq \frac{1}{2}.$$

So, we set  $d = r + 1$ . Now let  $A$  be the union of all 1-balls centered at points of  $J \setminus J'$   $A = \cup_{z \in J \setminus J'} B(z, 1)$ . The volume  $\text{Vol} A \geq 1/2 \text{Vol}(B(z_0, R))$ . By definition of  $A$  for any point  $z \in A$  there exists a point  $z' \in J \setminus J'$  at most 1-far away from  $z$   $d(z, z') \leq 1$ . Applying triangle inequality we get  $d(z'_0, \Theta(z)) \geq d(z'_0, \Theta(z')) - (\lambda_1 + c_1) \geq R - (\lambda_1 + c_1 + \lambda_2 + c_2)$ .

Here we describe the set  $J \subset \{R\} \times \mathbb{R}^n / \mathbb{Z}^n$  (we fix the first coordinate  $t = R$ ). This is the set of points  $z = (R, x_1, \dots, x_n)$  such that for any  $i = 1, \dots, n$   $x_i$  is a whole multiple of  $e^{-\mu_i R}$  modulo 1.  $J_0$  is the subset of points such that for any  $i$   $x_i$  is a whole multiple of  $e^{\mu_i(r-R)}$ . Let  $K$  be the set of vectors  $k = (0, k_1, \dots, k_n)$  such that for any  $i$  the number  $e^{\mu_i R} k_i$  is a whole number between 0 and  $e^{\mu_i(r-R)} - 1$ . For  $k \in K$  we define  $J_k = J_0 + k$ . Then for any two different points  $z_1, z_2$  of  $J_k$

$$d(z_1, z_2) = \max \log(|x_i^1 - x_i^2|^{1/\mu_i}) \geq r.$$

We constructed the needed set. Now we notice that the lifting  $\tilde{A} \subset \tilde{Z}$  of  $A$  has the same properties relatively to  $\tilde{\Theta}$ : the image  $\tilde{\Theta}(\tilde{A})$  lies at distance  $R - (\lambda_1 + c_1 + \lambda_2 + c_2)$  from the base point and the volume of  $\tilde{A}$  is at least a half of the volume of the ball  $B(\tilde{z}_0, R)$ .

Now let us compute  $|v(\tilde{z})|$  for  $\tilde{z} \in \tilde{A}$  (in fact here we will give an upper bound for  $|v|$  which is true for all  $\tilde{z} \in B(z_0, R)$  and a lower bound for  $\tilde{z} \in \tilde{A}$ ). We remind that by the construction  $\tilde{z}$  is sent far from the base point  $d(\tilde{z}_0, \tilde{\Theta}(\tilde{z})) \geq R - (\lambda_1 + c_1 + \lambda_2 + c_2)$ .

$$\begin{aligned} |(u * \phi) * \psi'(\tilde{z})| &= \left| \int_X \int_Y u(\tilde{z}') \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\ &\geq \left| \int_X \int_Y (u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z})) + u(\tilde{\Theta}(\tilde{z}))) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\ &\geq \left| \int_X \int_Y (u(\tilde{\Theta}(\tilde{z}))) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\ &\quad - \left| \int_X \int_Y (u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z}))) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\ &\geq 1 - \int_X \int_Y |u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z}))| \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1. \end{aligned} \quad (4.5)$$

For the last inequality we use following properties:  $|u| = 1$  and an integral of a kernel or a cross-kernel over the second argument equals to 1

$$\begin{aligned} & \left| \int_X \int_Y u(\tilde{\Theta}(\tilde{z})) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\ &= \left| \int_X u(\tilde{\Theta}(\tilde{z})) \psi'(\tilde{z}, \tilde{z}_1) \left( \int_Y \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') d\tilde{z}' \right) d\tilde{z}_1 \right| \\ &= \left| \int_X u(\tilde{\Theta}(\tilde{z})) \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}_1 \right| = |u(\tilde{\Theta}(\tilde{z}))| = 1. \end{aligned}$$

We need to estimate the double integral at formula 4.5.  $\psi'(\tilde{z}, \tilde{z}_1)$  is non-zero if  $d(\tilde{z}, \tilde{z}_1) \leq R^{\psi'} = 1$  and  $\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')$  is non-zero if  $d(\tilde{z}', \tilde{\Theta}(\tilde{z}_1)) \leq R^\zeta = 1$ . So the diameter of the set  $\hat{S}$  of points  $\tilde{z}'$  such that the integrand is non-zero, is at most  $2\lambda_1 + c_1 + 2 \leq 4(\lambda_1 + c_1)$  because  $\lambda_1 \geq 1$ . Hence  $\hat{S}$  is contained in a ball  $B_{\hat{S}}$  of radius  $4(\lambda_1 + c_1)$ . Assume  $\tilde{z}' = \tilde{\Theta}(\tilde{z}) \in \hat{S}$ . Then by the mean value theorem, for any point  $\tilde{z}' \in \hat{S}$ ,

$$\begin{aligned} |u(\tilde{z}') - u(\tilde{z}')| &\leq |\tilde{z}' - \tilde{z}'| \sup_{\tilde{z}' \in B_{\hat{S}}} |\nabla u(\tilde{z}')| \leq 8(\lambda_1 + c_1) \sup_{\tilde{z}' \in B_{\hat{S}}} \left| \frac{\partial u}{\partial \tilde{x}_n} \right| e^{-\mu'_n t} \leq 8\pi(\lambda_1 + c_1) e^{-\mu'_n t} \\ &\leq 8\pi(\lambda_1 + c_1) \sup_{\tilde{z}' \in B_{\hat{S}}} e^{-\mu'_n d(O', \tilde{z}')} \leq 8\pi(\lambda_1 + c_1) e^{-\mu'_n (R - (\lambda_1 + c_1 + \lambda_2 + c_2) - 2(\lambda_1 + c_1))} \leq \frac{1}{2} \end{aligned}$$

for  $R \geq 8(\lambda_1 + c_1) + (\lambda_2 + c_2) = R_0$ . Hence we have proved that

$$\begin{aligned} \frac{1}{2} &\leq |(u * \phi) * \psi'(\tilde{z})| \quad \text{if } \tilde{z} \in \tilde{A} \\ |(u * \phi) * \psi'(\tilde{z})| &\leq 1 \quad \text{if } \tilde{z} \in B(\tilde{z}_0, R). \end{aligned}$$

And we conclude from this relation that for  $R \geq R_0 + 1$

$$\int_{B(R)} |v|^p \geq \frac{1}{2^p} \text{vol}(B(R)) - \text{vol}(B(R_0)) \geq e^{(\sum \mu_i)R} / 2^{p+1}.$$

Let us compute the integral  $\int |\nabla u|^p$ .

$$\int |\nabla u|^p = \int \left| \frac{\partial u}{\partial x_n} \right|^p e^{-\mu'_n p t} e^{(\sum \mu'_i) t} dt dx_n = \pi \int_0^{+\infty} e^{(\sum \mu'_i - p \mu'_n) t} dt = \frac{\mu'_n \pi}{-\sum \mu'_i / \mu'_n + p}.$$

Hence the Poincaré constant  $C_p(\mu)$  for  $Z$  satisfies

$$\begin{aligned} (C_p(\mu))^p &\geq \frac{\|v\|^p}{\|\nabla v\|^p} \geq \frac{\|v\|^p}{C_1(a, b) C(Q) \|\nabla u\|^p} \\ &\geq (\mu'_n \pi 2^{p+1} C_1(a, b) C(Q))^{-1} e^{(\sum \mu_i)R} (p - \sum \mu'_i / \mu'_n). \end{aligned}$$

This proves the first claim in Theorem 24.

#### 4.4.6 Proof of the second statement of Theorem 24

Let  $\Theta : B_Z(R) \rightarrow Z'$  be a  $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding. By hypothesis,  $\Theta$  is isomorphic on fundamental groups. Lemma 19 implies that  $\Theta$  moves the origin a bounded distance away. Indeed, a non null-homotopic loop of length 1 based at  $O$  is mapped to a non null-homotopic loop of length  $\leq Q = \lambda_1 + c_1$  based at  $\Theta(O)$ . This implies that  $t(\Theta(O)) \leq 4Q$  and  $d(O', \Theta(O)) \leq 4Q + 1$ .

The space  $\tilde{Z}$  is of the form  $\tilde{T} \times \mathbb{R}$  where  $\tilde{T} \rightarrow T$  is a connected 2-sheeted covering space of torus, that is  $\tilde{T}$  is also a torus. Hence we can apply Theorem 23. We have  $C_p(\mu) \leq C_2(a, b)e^{\mu_n R}$ . If  $R \leq 8(\lambda_1 + c_1) + (\lambda_2 + c_2)$  there is nothing to prove. Otherwise we arrive to

$$(\mu'_n \pi 2^{p+1} C_1(a, b) C(Q))^{-1/p} e^{(\sum \mu_i / p) R} \left( p - \sum \mu'_i / \mu'_n \right)^{1/p} \leq C_2(a, b) e^{\mu_n R}.$$

Hence with  $C_3(a, b) = (\mu'_n \pi 2^{p+1} C_1(a, b))^{1/p} C_2(a, b)$ ,

$$C_3(a, b) C(Q) \geq e^{(\sum \mu_i / p - \mu_n) R} \left( p - \frac{\sum \mu'_i}{\mu'_n} \right)^{1/p}.$$

We have calculated that  $C(Q) = Q^{3+2/p} e^{(9+3/p)Q}$ . Combine these results and take the logarithm (note that in the following calculations every constant depending on  $\mu$  and  $\mu'$  can be estimated using  $a$  and  $b$ .)

$$\left( 3 + \frac{2}{p} \right) \log Q + \left( 9 + \frac{3}{p} \right) Q \geq G'(a, b) + \left( \frac{\sum \mu_i}{p} - \mu_n \right) R + \frac{1}{p} \log \left( p - \frac{\sum \mu'_i}{\mu'_n} \right)$$

with some constant  $G'$  depending only on  $a, b$ .  $p \geq 1$  hence the left-hand part can be estimated as  $5 \log Q + 12Q < 24Q$ . Setting  $p = \sum \mu'_i / \mu'_n + 1/R$ , we get

$$24Q \geq G'(a, b) + \frac{\mu_n \left( \frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} - \frac{1}{R} \right) R}{\frac{\sum \mu'_i}{\mu'_n} + \frac{1}{R}} + \frac{1}{p} \log \frac{1}{R}.$$

For  $R \geq G''(a, b)$  with some well-chosen constant  $G''$

$$24Q \geq G'(a, b) + \frac{\mu_n \mu'_n}{4 \sum \mu'_i} \left( \frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} \right) R - \frac{\mu'_n}{2 \sum \mu'_i} \log R$$

and finally we can rewrite our inequality under the desired form

$$Q \geq G_1(a, b) \left( \frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} \right) R - G_2(a, b)$$

with  $G_1(a, b)$  and  $G_2(a, b)$  being constants depending only on  $a$  and  $b$ .

This finishes the proof of Theorem 24.

## Chapter 5

# Examples of different distortion growths

### 5.1 Approximation of distances and an example of QI

Let  $X, Y$  be two geodesic hyperbolic metric spaces with base points  $x_0 \in X, y_0 \in Y$ . Let  $\theta : \partial X \rightarrow \partial Y$  be a homeomorphism between ideal boundaries.

*Hypothesis 1.* Assume that there exists a constant  $D$  such that for any  $x \in X$  there exists a geodesic ray  $\gamma$  from the base point  $\gamma(0) = x_0$  and passing near  $x$ :  $d(x, \gamma) < D$ .

We are going to construct approximatively (up to  $D$ ) a map  $\Theta : X \rightarrow Y$  extending the boundary homeomorphism  $\theta$ . Take some point  $x$  and a geodesic ray  $\gamma$  from  $x_0$  passing near  $x$ :  $d(\gamma, x) < D$ . Then  $\gamma(\infty)$  is a point on ideal boundary  $\partial X$ . The corresponding point  $\theta(\gamma(\infty)) \in \partial Y$  defines a geodesic ray  $\gamma'$  such that  $\gamma'(0) = y_0$  and  $\gamma'(\infty) = \theta(\gamma(\infty))$ . Set  $\Theta(x) = \gamma'(d(x_0, x))$ . So, by construction,  $\Theta$  preserves the distance to the base point. Still, it depends on the choices of  $\gamma$  and  $\gamma'$ .

**Definition 25.** Define the following quantity

$$K(R) = \sup \left\{ \left| \log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)} \right| \mid d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{y_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R} \right\}.$$

We are going to prove that the restriction of  $\Theta$  on the ball  $B(R) \subset X$  of radius  $R$  is a  $\left(1 + 2\frac{K(R)}{D+\delta}, D + \delta + 2K(R)\right)$ -quasi-isometry. We begin with a Lemma which gives an approximation (up to an additive constant) of the distance between two points in a hyperbolic metric space. In its proof, all equalities hold with a bounded additive error depending linearly on  $\delta$ .

**Lemma 22.** *Let  $P_1, P_2$  be two points in a hyperbolic metric space  $Z$ . Let  $P_0$  be a base point (possibly at infinity). Let distances (horo-distances if  $P_0$  is at infinity) from  $P_1$  and  $P_2$  to  $P_0$  be  $d(P_1, P_0) = t_1$  and  $d(P_2, P_0) = t_2$ . Assume that there exist points  $P_1^\infty$  and  $P_2^\infty$  such that  $P_1$  (resp.  $P_2$ ) belongs to the geodesic ray defined by  $P_0$  and  $P_1^\infty$  (resp.  $P_2^\infty$ ). Denote by*

$$t_\infty = -\log \text{visdist}_{P_0}(P_1^\infty, P_2^\infty)^1$$

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<sup>1</sup>We define  $\text{visdist}(P_1^\infty, P_2^\infty)$  of two points  $P_1^\infty, P_2^\infty$  at the ideal boundary as the exponential of minus Gromov's product of these points  $e^{-(P_1^\infty | P_2^\infty)}$ . Indeed, it is not a distance as it does not satisfy triangle inequality. But we will never have more than two points at infinity at the same time in our setting, so we will not use this property.

the logarithm of visual distance seen from  $P_0$ . Then up to adding a multiple of  $\delta$ ,

$$d(P_1, P_2) = t_1 + t_2 - 2 \min\{t_1, t_2, t_\infty\}.$$

*Proof.* Let  $P'_0$  be a projection of  $P_0$  on the geodesic  $P_1^\infty P_2^\infty$ . By Lemma 5,  $P'_0$  lies at distance at most  $2\delta$  from both  $P_0 P_1^\infty$  and  $P_0 P_2^\infty$ . Hence, up to an additive constant bounded by  $4\delta$  the distance between  $P_0$  and  $P'_0$  is equal to Gromov's product of  $P_1^\infty$  and  $P_2^\infty$ . It follows that  $t_\infty = d(P_0, P'_0) = -\log \text{visdist}(P_1, P_2)$ .

The triangle  $P_0 P_1^\infty P_2^\infty$  is  $\delta$ -thin. Notice that if  $P_1$  (or  $P_2$ ) lies near the side  $P_1^\infty P_2^\infty$  then  $t_1 \geq t_\infty$ . Otherwise,  $t_1 \leq t_\infty$  (both inequalities are understood up to an additive error  $\delta$ ). This follows from the definition of the point  $P'_0$  as a projection and Lemma 5.

Hence, if  $t_1, t_2 \geq t_\infty$ ,  $d(P_1, P_2) = d(P_1, P_0) + d(P_2, P_0) - 2d(P_0, P'_0) = t_1 + t_2 - 2t_\infty$ .

If  $t_1 \leq t_\infty \leq t_2$ ,  $d(P_1, P_2) = d(P_1, P'_0) + d(P'_0, P_2) = t_2 - t_1$ .

Finally, if  $t_1, t_2 \leq t_\infty$ , we get  $d(P_1, P_2) = |t_1 - t_2| = t_1 + t_2 - 2 \min\{t_1, t_2\}$  as  $P_1$  lies near  $P_0 P_2^\infty$ . □

**Lemma 23.** *Let  $Z$  and  $Z'$  be two hyperbolic metric spaces. Let  $\Theta$  be the radial extension of a boundary homeomorphism  $\theta$ , as described at the beginning of this section. Then for any two points  $P_1, P_2 \in B(P_0, R) \subset Z$  such that  $d(P_1, P_2) > c$ , we have*

$$\frac{d_{Z'}(\Theta(P_1), \Theta(P_2))}{d_Z(P_1, P_2)} \leq 1 + 2 \frac{K(R)}{c}.$$

If  $d(P_1, P_2) < c$ ,

$$d_{Z'}(\Theta(P_1), \Theta(P_2)) < 2K(R) + c.$$

*Proof.* We will use the same notations as in Lemma 22. Visual distance  $d_Z^\infty$  between  $P_1^\infty$  and  $P_2^\infty$  and the (horo-)distance  $t_\infty$  from  $P_0$  to  $P_1^\infty P_2^\infty$  are connected by the relation  $e^{-t_\infty} = d_\infty(P_1^\infty, P_2^\infty)$ . In the same way we define  $t'_\infty$  as the (horo-)distance for corresponding images.

By Lemma 22 we know that  $d(P_1, P_2) = t_1 + t_2 - 2 \min\{t_1, t_2, t_\infty\}$ .

Assume first  $d(P_1, P_2) > c$ . We will write  $d_Z = d(P_1, P_2)$  for the distance between  $P_1$  and  $P_2$  and  $d_{Z'} = d(\Theta(P_1), \Theta(P_2))$  for the distance between their images.

We have to consider four cases depending on the relative sizes of  $t_1, t_2, t_\infty$  and  $t'_\infty$  as they determine values of minima defining  $d_Z$  and  $d_{Z'}$ . Without loss of generality, we may assume that  $t_1 \leq t_2$ .

*1st case* If both  $t_1 < t_\infty$  and  $t_1 < t'_\infty$ , then

$$\frac{d_{Z'}}{d_Z} = \frac{t_2 - t_1}{t_2 - t_1} = 1,$$

and this case is trivial.

*2nd case* If  $t_\infty < t_1$  and  $t'_\infty < t_1$ . We have to give an upper bound for

$$\frac{d_{Z'}}{d_Z} = \frac{t_1 + t_2 - 2t'_\infty}{t_1 + t_2 - 2t_\infty}.$$

Consider

$$t'_\infty - t_\infty = \log \frac{d_\infty(\theta(P_1^\infty), \theta(P_2^\infty))}{d_\infty(P_1^\infty, P_2^\infty)}.$$

Because  $d_Z > c$ , we have  $t_1 + t_2 - 2t_\infty > c$  hence  $e^{(t_1+t_2)/2}e^{-t_\infty} > e^{c/2}$ . And as  $t_1, t_2 \leq R$  we obtain for visual distance  $d_Z^\infty \geq e^{c/2}e^{-R} \geq e^{-R}$ . We conclude that

$$|t'_\infty - t_\infty| \leq K(R).$$

Finally,

$$\frac{d_{Z'}}{d_Z} = \frac{d_{Z'} - d_Z + d_Z}{d_Z} = 1 + \frac{t'_\infty - t_\infty}{t_1 + t_2 - t_\infty} \leq 1 + \frac{1}{c}|t'_\infty - t_\infty|.$$

*3d case* Now let  $t_\infty < t_1 < t'_\infty$ . Then

$$d_{Z'} - d_Z = t_2 - t_1 - (t_1 + t_2 - 2t_\infty) = 2(t_\infty - t_1) \leq 0,$$

which leads to

$$\frac{d_{Z'}}{d_Z} \leq 1.$$

*4th case* Finally if  $t'_\infty < t_1 < t_0^\infty$  then

$$d_{Z'} - d_Z = (t_1 + t_2 - 2t'_\infty) - (t_2 - t_1) = 2(t_1 - t'_\infty) \leq 2(t_0^\infty - t'_\infty).$$

We know that  $t_1 \leq R$  and at the same time we have  $t'_\infty < t_1$ , hence  $t'_\infty < R$  and visual distance between  $P_1^{\infty'}$  and  $P_2^{\infty'}$  is at least  $e^{-R}$ . Now as in the 2nd case we obtain that  $t_0^\infty - t'_\infty \leq K(R)$  and hence

$$\frac{d_{Z'}}{d_Z} \leq 1 + 2\frac{K(R)}{c}.$$

Now assume that  $d_Z(P_1, P_2) \leq c$  (we still suppose  $t_1 \leq t_2$ ), hence the distance  $t_\infty > t_2$  and we are either in first or fourth situation. In the first case,  $t_1 < t_\infty$  and  $t_1 < t'_\infty$  so  $d_{Z'} = d_Z \leq c$ . In the fourth case, we have still  $d_{Z'} - d_Z \leq 2K(R)$  and hence  $d'_Z \leq c + 2K(R)$ .  $\square$

Applying the Lemma both to  $\Theta$  and  $\Theta^{-1}$ , we get the following Theorem.

**Theorem 25.** *Let  $X, Y$  be two geodesic hyperbolic metric spaces with base points  $x_0 \in X$ ,  $y_0 \in Y$ . Assume that there exists a constant  $D$  such that for any  $x \in X$  there exists a geodesic ray  $\gamma$  from the base point  $\gamma(0) = x_0$  and passing near  $x$ :  $d(x, \gamma) < D$  (Hypothesis 1). Let the restriction of  $\Theta : \partial X \rightarrow \partial Y$  be a homeomorphism between ideal boundaries. Then the restriction of  $\Theta$  on a ball  $B(x_0, R) \subset X$  of radius  $R$  is a  $(\lambda, C_q)$ -quasi-isometry to  $B(y_0, R) \subset Y$ , where  $\lambda = 1 + 2\frac{K(R)}{c}$  and  $C_q = 2K(R) + c$ . The constant  $c$  can be chosen as  $c = D + \delta$  where  $\delta$  is the hyperbolicity constant.*

## 5.2 Examples

### 5.2.1 Bi-Hölder maps

Let  $\theta$  be a bi-Hölder map:

$$\begin{aligned} d(\theta(\xi_1), \theta(\xi_2)) &\leq cd(\xi_1, \xi_2)^\alpha, \alpha < 1, \\ d(\theta(\xi_1), \theta(\xi_2)) &\geq \frac{1}{c}d(\xi_1, \xi_2)^\beta, \beta > 1. \end{aligned}$$

Assume first that for two points  $\xi_1, \xi_2$  of the ideal boundary, the visual distance  $d(\xi_1, \xi_2) > e^{-R}$ . Then we have

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \leq \log cd(\xi_1, \xi_2)^{\alpha-1} = -(1-\alpha) \log d(\xi_1, \xi_2) \lesssim (1-\alpha)R.$$

Now, if the visual distance between images of  $\xi_1$  and  $\xi_2$  satisfy  $d(\theta(\xi_1), \theta(\xi_2)) > e^{-R}$ , we get

$$d(\xi_1, \xi_2) \geq \frac{1}{c^{1/\alpha}} e^{-R/\alpha}$$

and hence

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \gtrsim \frac{1-\alpha}{\alpha} R.$$

We obtain the lower bound for  $\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)}$  just in the same way as the upper-bound. If  $d(\xi_1, \xi_2) > e^{-R}$

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \geq \log \frac{1}{c} d(\xi_1, \xi_2)^{\beta-1} = -(1-\beta) \log d(\xi_1, \xi_2) \lesssim (1-\beta)R.$$

If  $d(\theta(\xi_1), \theta(\xi_2)) > e^{-R}$

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \geq \log \frac{1}{c} d(\theta(\xi_1), \theta(\xi_2))^{(\beta-1)/\beta} = -\frac{1-\beta}{\beta} \log d(\theta(\xi_1), \theta(\xi_2)) \gtrsim \frac{1-\beta}{\beta} R.$$

This gives

$$K(R) \lesssim \max\{1-\alpha, 1-\beta\}R.$$

In particular, consider two variants of the space  $T^n \times [0, +\infty)$   $Z$  and  $Z'$  with metrics  $dt^2 + \sum e^{2\mu_i t} dx_i^2$  and  $dt^2 + \sum e^{2\mu'_i t} dx_i^2$  respectively. The visual distance between points  $P_1$  and  $P_2$  is given by

$$d_\infty(P_1, P_2) \sim \max |x_i^1 - x_i^2|^{1/\mu_i}.$$

Pick the identity map  $\theta : \partial Z \rightarrow \partial Z'$ . Then

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \sim \frac{\max_i |x_i^1 - x_i^2|^{1/\mu'_i}}{\max_i |x_i^1 - x_i^2|^{1/\mu_i}} \leq \max_i |x_i^1 - x_i^2|^{1/\mu'_i - 1/\mu_i}.$$

Suppose that  $d(P_1, P_2) > e^{-R}$ . Then

$$\begin{aligned} \left| \log \frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \right| &\leq \left| \log \max_i |x_i^1 - x_i^2|^{1/\mu'_i - 1/\mu_i} \right| = \\ &= \max_i \left( \mu_i \left| \frac{1}{\mu'_i} - \frac{1}{\mu_i} \right| \left| \log |x_i^1 - x_i^2|^{1/\mu_i} \right| \right) \leq \max_i \left| \frac{\mu_i}{\mu'_i} - 1 \right| R. \end{aligned}$$

So, we conclude that  $K(R) = |\max_i(\mu_i/\mu'_i) - 1| R$ .

*Remark 11.* More generally, such bi-Hölder maps exist between boundaries of arbitrary simply connected Riemannian manifolds with bounded negative sectional curvature. The Hölder exponent is controlled by sectional curvature bounds.

### 5.2.2 Unipotent locally homogeneous space

Now assume the space  $Z$  is a quotient  $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}$  of the space  $\mathbb{R}^2 \times \mathbb{R}$  with the metric  $dt^2 + e^{2t}(dx^2 + dy^2)$ . Consider the space  $Z' = \mathbb{R}^2/\mathbb{Z}^2 \rtimes_\alpha \mathbb{R}$ , quotient of the space  $\mathbb{R}^2 \rtimes_\alpha \mathbb{R}$ , where  $\alpha$  is the  $2 \times 2$  matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The locally homogeneous metric is of the form  $dt^2 + g_t$  where  $g_t = (e^{t\alpha})^* g_0$

$$e^{t\alpha} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t x + te^t y \\ e^t y \end{pmatrix}$$

and so  $g_t = d(e^t x + te^t y)^2 + d(e^t y)^2 = e^{2t}(dx^2 + 2tdxdy + (t^2 + 1)dy^2)$ .

Let  $\theta : \partial Z \rightarrow \partial Z'$  be the identity. Consider two points  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$  in  $Z$ . We will write  $x = x_1 - x_2$  and  $y = y_1 - y_2$ . For the visual distance between  $P_1, P_2$  we have

$$d_\infty(P_1, P_2) = \max\{|x|, |y|\}.$$

For their images  $\theta(P_1)$  and  $\theta(P_2)$  (see section 5 of [29] and [30])

$$d_\infty(\theta(P_1), \theta(P_2)) = \max\{|y|, |x - y \log |y|\}.$$

First we will give an upper-bound for  $\log(d_\infty(\theta(P_1), \theta(P_2))/d_\infty(P_1, P_2))$ . We have four different cases.

*1st case.* If  $|x| < |y|$  and  $|x - y \log |y|| < |y|$ ,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = 1.$$

*2nd case.* If  $|x - y \log |y|| < |y| < |x|$ ,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} < 1.$$



3d case. If  $|x| < |y| < |x - y \log |y||$ .

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|y|} \leq \frac{|x|}{|y|} + |\log |y||.$$

If  $d_\infty(P_1, P_2) > e^{-R}$  we have  $e^{-R} < |y| \leq 1$  (the upper bound follows from the fact that  $y$  is a coordinate of a point of a torus) and hence  $|\log |y|| \leq R$  and we finish as follows,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq \frac{|x|}{|y|} + |\log |y|| \leq 1 + R.$$

If  $d_\infty(\theta(P_1), \theta(P_2)) > e^{-R}$  we will consider two situations.

- If  $|x| > |y \log |y||$  then  $|x - y \log |y|| < 2|x|$  and as  $|x| < |y|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq 2.$$

- If  $|x| < |y \log |y||$  then  $e^{-R} < |x - y \log |y|| < 2|y \log |y||$  and hence  $|\log |y|| < R$ , so

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq 1 + R.$$

4th case. Let now  $|y| < |x|$  and  $|y| < |x - y \log |y||$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|x|} \leq 1 + \frac{|y \log |y||}{|x|}.$$

We will check two possibilities.

- If  $|y| \leq |x|^2$  then

$$\frac{|y \log |y||}{|x|} = \frac{|y|^{1/2}}{|x|} \left| |y|^{1/2} \log |y| \right| \leq 1.$$

- Now suppose that  $|y| \geq |x|^2$ . If  $d_\infty(P_1, P_2) > e^{-R}$ , we see easily that  $|y| \geq e^{-2R}$  and hence

$$\frac{|y \log |y||}{|x|} \leq \frac{|x \log |y||}{|x|} \leq |\log |y|| \leq 2R.$$

If  $d_\infty(\theta(P_1), \theta(P_2)) > e^{-R}$  we use the fact that  $|a + b| \geq 2 \max\{|a|, |b|\}$ . Hence, either  $|x| > e^{-R}/2$  or  $|y \log |y|| > e^{-R}/2$  and so  $|y| \gtrsim e^{-R}$  and we finish the estimation as earlier.

So in the fourth case we have also

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq 2R.$$

Here, we have proved that  $\log(d_\infty(\theta(P_1), \theta(P_2))/d_\infty(P_1, P_2)) \leq \log R$ . Now we proceed to give also a lower bound for this expression.

*1st case.* If  $|x| < |y|$  and  $|x - y \log |y|| < |y|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = 1.$$

*2nd case.* If  $|x - y \log |y|| < |y| < |x|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|y|}{|x|}.$$

Without loss of generality, assume  $x > 0$ . By the construction of  $Z$ ,  $|y| < 1$  hence  $\log |y| < 0$ . If  $0 < x \leq y \log |y|$ , we have  $y < 0$ . Now transform  $x \leq y \log |y|$  as  $1 \leq -\log |y|(-y)/x$ , hence

$$-\frac{y}{x} \geq -\frac{1}{\log |y|}.$$

Now either  $d_\infty(\theta(P_1), \theta(P_2)) = |y| > e^{-R}$  or  $e^{-R} \leq d_\infty(P_1, P_2) = |x| \leq y \log |y|$  which also means that  $|y| \gtrsim e^{-R}$ . So,

$$\frac{|y|}{|x|} \geq \frac{1}{R}.$$

If on the contrary  $y \log |y| \leq x$  we have

$$x - y \log |y| < |y| < x. \tag{5.1}$$

First we notice that  $y \log |y| > x - |y| > 0$ . As  $|y| < 1$  for any point of our space,  $\log |y| < 0$  and we conclude that  $y < 0$ . Now from (5.1) we obtain that  $x < -y(1 - \log |y|)$ . As  $1 - \log |y| > 0$  we obtain

$$-\frac{y}{x} > \frac{1}{1 - \log |y|}.$$

If  $d_\infty(\theta(P_1), \theta(P_2)) = |y| > e^{-R}$ , we trivially get that

$$\frac{|y|}{|x|} > \frac{1}{R}.$$

If  $e^{-R} \leq d_\infty(P_1, P_2) = |x|$  we write  $e^{-R} < x < -y(1 - \log |y|)$  and hence  $y \gtrsim e^{-R}$ , so we obtain the same result. So, in both cases we come to the same result

$$\left| \log \frac{|y|}{|x|} \right| < R.$$

*3d case.* Assume  $|x| < |y| < |x - y \log |y||$ , this case is trivial as

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|y|} \geq 1.$$

*4th case.* Let now  $|y| < |x|$  and  $|y| < |x - y \log |y||$ . We also suppose that  $x > 0$  to save notation.

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|x|} = \left| 1 - \frac{y \log |y|}{x} \right|. \quad (5.2)$$

If (5.2) is greater than  $1/2$  then we have nothing to prove. So suppose that (5.2) is less than  $1/2$

$$-\frac{x}{2} \leq x - y \log |y| \leq \frac{x}{2}$$

and so

$$\frac{x}{2} \leq y \log |y| \leq \frac{3x}{2}.$$

The last inequality shows that if either  $d_\infty(\theta(P_1), \theta(P_2)) \geq e^{-R}$  or  $d_\infty(P_1, P_2) \geq e^{-R}$ ,  $|y| \gtrsim e^{-R}$  and so we have

$$\frac{|y \log |y||}{x} \geq \frac{|y \log |y||}{y} = |\log |y|| \geq \frac{1}{R},$$

which completes our discussion of this example. We have proved that

$$K(R) \lesssim \log R.$$

# Chapter 6

## Appendix

### 6.1 Poincaré inequality for $\mathbb{H}^n$

Let  $\mathbb{H}^n$  be  $n$ -dimensional hyperbolic space. The metric is written as  $dr^2 + \sinh^2(r)d\theta^2$  in polar coordinates, this is very close to the exponentially growing metrics studied in section 4.3. From the results of section 4.3, little effort is needed to get the Poincaré inequality for balls in  $\mathbb{H}^n$ ,

$$\left( \int_{B_{\mathbb{H}^n}(R)} |f(x) - c|^p d\mu \right)^{1/p} \leq C_p^{hyp}(R) \left( \int_{B_{\mathbb{H}^n}(R)} |\nabla f|^p d\mu \right)^{1/p}.$$

**Theorem 26.** *Let  $\mathbb{H}^n$  be  $n$ -dimensional hyperbolic space. Then for a ball  $B(R)$  of  $\mathbb{H}^n$  the Poincaré constant does not exceed*

$$C_p^{hyp}(R) \leq C(p, n)(1 + e^R),$$

where  $C(p, n)$  depends only on  $p$  and dimension  $n$ .

*Proof.* We will provide the proof by comparing the hyperbolic metric with an exponential metric  $dr^2 + e^{2r}d\theta^2$ . To pass from the exponential to  $\sinh$ , we will divide the ball  $B(R)$  in two parts: a little ball near the center and its complement. Finally we will compare the initial inequality with the Euclidean Poincaré inequality on this small ball and with our "exponential" inequality (Theorem 23) on the complement.

Let the volume element be  $d\mu = \sinh^{n-1} r dr d\theta$ . We will also write  $dvol_{hyp}$  for  $d\mu$ ,  $dvol_{eucl}$  for euclidean volume element and  $dvol_{exp}$  for exponential volume element  $d\mu_{exp} = e^{(n-1)r} dr d\theta$ . The idea of the proof is following. First we notice that outside of a ball  $B(1)$  exponential and hyperbolic metrics are equivalent. On the other hand inside of a ball  $B(2)$  hyperbolic metric is equivalent with euclidean metric. This motivates us to use the partition of unity to prove the initial Poincaré inequality for hyperbolic metric.

Let  $b = \int_{B(2)} f dvol_{eucl}$ ,  $\chi$  be the continuous function

- $\chi(x) = 1$  if  $x \in B(1)$
- $\chi(x) = 0$  if  $x \in H \setminus B(2)$

- $\chi(x) = 2 - r$  if  $x \in B(2) \setminus B(1)$

We notice that

- $\sinh^{n-1} r \leq e^{(n-1)r}$  for  $r \geq 0$ ,
- $e^{(n-1)r} \leq c_e \sinh^{n-1} r$  for  $r \geq 1$  where the constant  $c_e$  is equal to  $e^{n-1} / \sinh^{n-1} 1$ ,
- in  $B(2)$  ( $r \leq 2$ ) the hyperbolic and euclidean metrics are equivalent

$$1 \leq \frac{\sinh^{n-1} r dr d\theta}{r^{n-1} dr d\theta} \leq \left( \frac{\sinh 2}{2} \right)^{n-1}.$$

Now present  $f - b$  as follow  $f - b = \chi(f - b) + (1 - \chi)(f - b)$ . First we consider the function  $b + (1 - \chi)(f - b)$ . We notice that  $\nabla(b + (1 - \chi)(f - b))$  equals to 0 on  $B(1)$ , hence

$$\int_{B(R)} |\nabla(1 - \chi)(f - b)|^p \sinh^{(n-1)} r dr d\theta = \int_{B(R) \setminus B(1)} |\nabla(1 - \chi)(f - b)|^p \sinh^{(n-1)} r dr d\theta.$$

And we can write

$$\begin{aligned} \int_{B(R)} |b + (1 - \chi)(f - b) - c_1|^p \sinh^{(n-1)} r dr d\theta &\leq \int_{B(R)} |b + (1 - \chi)(f - b) - c_1|^p e^{(n-1)r} dr d\theta \\ &\leq (C_{p,n}^{exp})^p \int_{B(R)} |\nabla(1 - \chi)(f - b)|^p e^{(n-1)r} dr d\theta \\ &\leq c_e (C_{p,n}^{exp}(R))^p \int_{B(R)} |\nabla(1 - \chi)(f - b)|^p \sinh^{(n-1)} r dr d\theta, \end{aligned}$$

where the second inequality is a Poincaré inequality for exponential metric.

Now we will apply to the righthand part of the inequality the following formulas

$$\nabla(1 - \chi)(f - b) = (1 - \chi)\nabla f + (f - b)\nabla(1 - \chi).$$

and

$$\|f_1 + f_2\|_{\mathbb{L}^p}^p \leq 2^p (\|f_1\|_{\mathbb{L}^p}^p + \|f_2\|_{\mathbb{L}^p}^p).$$

We get that

$$\begin{aligned} &\int_{B(R)} |b + (1 - \chi)(f - b) - c_1|^p \sinh^{(n-1)} r dr d\theta \leq \\ &\leq c_e 2^p (C_{p,n}^{exp})^p \left( \int_{B(R)} |\nabla f|^p \sinh^{(n-1)} r dr d\theta + \int_{B(2)} |f - b|^p \sinh^{(n-1)} r dr d\theta \right) \end{aligned}$$

Now we write euclidean Poincare inequality in  $B(2)$  with euclidean constant  $C_{p,n}^{eucl}$  (it depends only on dimension)

$$\begin{aligned} \int_{B(2)} |f - b|^p dvol_{hyp} &\leq \left(\frac{\sinh 2}{2}\right)^{n-1} \int_{B(2)} |f - b|^p dvol_{eucl} \\ &\leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|_{eucl}^p dvol_{eucl} \\ &\leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|_{hyp}^p dvol_{hyp}. \end{aligned}$$

Consider the function  $\chi(f - b)$ . It equals to 0 on the complement of  $B(2)$  so we can easily treat this case involving euclidean Poincaré inequality as two metrics are equivalent there.

$$\begin{aligned} \int_{B(2)} |\chi(f - b) - c_2|^p dvol_{hyp} &\leq \left(\frac{\sinh 2}{2}\right)^{n-1} \int_{B(2)} |\chi(f - b) - c_2|^p dvol_{eucl} \leq \\ &\leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|^p dvol_{eucl} \leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|^p dvol_{hyp}. \end{aligned}$$

Now we need to combine all these results. First, we have

$$\int_{B(R)} |f - c_1 - c_2|^p d\mu \leq p \int_{B(R)} \left( |b + (1 - \chi)(f - b) - c_1|^p + |\chi(f - b) - c_2|^p \right) d\mu,$$

remind that  $d\mu = dvol_{hyp}$ . Further, we note that for big enough  $R$

$$\left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \leq c_e 2^p (C_{p,n}^{exp}(R))^p$$

hence

$$\begin{aligned} \int_{B(R)} |f - c_1 - c_2|^p d\mu &\leq c_e p 2^p (C_{p,n}^{exp}(R))^p 2c_e (C_{p,n}^{eucl})^p \left( \int_{B(R)} |\nabla f|^p d\mu + \int_{B(2)} |\nabla f|^p d\mu \right) \\ &\leq 4c_e^2 p (2C_{p,n}^{exp}(R)C_{p,n}^{eucl})^p \int_{B(R)} |\nabla f|^p d\mu. \end{aligned}$$

□

## 6.2 Equivalence of three forms of the Poincaré inequality

In the literature, we can meet three different definitions of Poincaré inequalities. We will show that they are equivalent.

**Definition 26.** • There exists a constant  $C_p^1$  such that for any function  $f$  with  $\nabla f \in \mathbb{L}^p$  and its mean value  $\tilde{c}_f = \oint f$

$$\|f - \tilde{c}_f\|_{\mathbb{L}^p} \leq C_p^1 \|\nabla f\|_{\mathbb{L}^p};$$

- there exists a constant  $C_p^2$  such that for any function  $f$  with  $\nabla f \in \mathbb{L}^p$  there exists a constant  $c_f$

$$\|f - \tilde{c}_f\|_{\mathbb{L}^p} \leq C_p^2 \|\nabla f\|_{\mathbb{L}^p};$$

- there exists a constant  $C_p^3$  such that for any function  $f$  with  $\nabla f \in \mathbb{L}^p$

$$\left( \oint \oint_{X \times X} |f(x) - f(y)|^p dx dy \right)^{1/p} \leq C_p^3 \left( \oint_X |\nabla f(x)|^p dx \right)^{1/p}.$$

**Proposition 10.** All three definitions are equivalent in the sense that  $C_p^1, C_p^2$  and  $C_p^3$  differs only by universal multiplicative constants.

*Proof.* 1  $\Rightarrow$  2 Evident, just assume  $c_f = \tilde{c}_f$ .

2  $\Rightarrow$  3 Assume  $g = f - c_f$ . Hence  $\nabla g \in \mathbb{L}^p$  and we have  $\|g\|_p \leq C_p^2 \|\nabla g\|_p$ . So,

$$\oint |f(x) - f(y)|^p dx dy \leq 2 \left( \oint |f(x)|^p dx + \oint |f(y)|^p dy \right) \leq 4 \oint C_p^2 |\nabla f|^p.$$

We just proved the third definition with  $C_p^3 \leq 4C_p^2$ .

3  $\Rightarrow$  1 Now consider  $\|f - \tilde{c}_f\|_{\mathbb{L}^p}$

$$\begin{aligned} \|f - \tilde{c}_f\|_{\mathbb{L}^p} &= \left( \int_X \left| f(x) - \frac{\int_X f(y) dy}{\int_X dy} \right|^p dx \right)^{1/p} = \left( \oint_X \left| \int_X (f(x) - f(y)) dy \right|^p dx \right)^{1/p} \\ &\leq \left( \text{Vol}(X) \oint |f(x) - f(y)|^p dx dy \right)^{1/p} \leq \left( \text{Vol}(X) C_p^3 \oint |\nabla f|^p dx \right)^{1/p} = C_p^3 \|\nabla f\|_{\mathbb{L}^p}. \end{aligned}$$

Hence,  $C_p^1 \leq C_p^3$ .

□

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