Bilinear decompositions for the product space H1 X BMO and related problems

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Décomposition bilinéaire du produit H^1-BMO et problèmes liés

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Bilinear decompositions for the product space $H^1 \times BMO$ and related problems

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Résumé

Dans cette thèse, nous étudions le produit (dans le sens des distributions) de fonctions f dans H^1 et g dans BMO, désigné par $f \times g$, et les problèmes connexes. En particulier, nous prouvons qu'il existe deux opérateurs bornés bilinéaires $S : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ et $T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to H^{\log}(\mathbb{R}^n)$ de telle sorte que la décomposition suivante bilinéaire

$$f \times g = S(f, g) + T(f, g) \in L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n) \tag{1}$$

est vrai pour chaque $(f, g) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$. Ici, $H^{\log}(\mathbb{R}^n)$ est un nouveau type d'espace de Hardy-Orlicz défini comme l'espace des distributions f dont la fonction "grand maximale" satisfait

$$\int\limits_{\mathbb{R}^n} \frac{|\mathfrak{M}f(x)|}{\log(e + |x|) + \log(e + |\mathfrak{M}f(x)|)} dx < \infty.$$

L'espace $H^{\log}(\mathbb{R}^n)$ apparaît comme un exemple d'une nouvelle classe de Hardy $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ que nous introduisons et étudions ici. Ils sont appelés espaces de Hardy de type Musielak-Orlicz. Ces espaces généralisent les espaces Hardy-Orlicz de Janson et les espaces de Hardy à poids de García-Cuerva, Strömberg, et Torchinsky. Notant que l'espace $H^{\log}(\mathbb{R}^n)$ est un cas particulier de $\varphi(x, t) \equiv \frac{t}{\log(e+|x|)+\log(e+t)}$, nous prouvons que l'ensemble des multiplicateurs ponctuels de $BMO(\mathbb{R}^n)$ est en fait l'espace dual de $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$. En conséquence, nous montrons que, dans la décomposition bilinéaire (1), l'espace $H^{\log}(\mathbb{R}^n)$ ne peut pas être remplacé par un espace plus petit dans un certain sens.

Soit b une BMO-fonction. Il est bien connu que le commutateur linéaire $[b, T]$ d'un opérateur Calderón-Zygmund T n'est pas, en général, porné de $H^1(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$. Cependant, Pérez a montré que si $H^1(\mathbb{R}^n)$ est remplacé par un sous-espace approprié atomique $\mathcal{H}^1_b(\mathbb{R}^n)$ alors le commutateur est continu de $\mathcal{H}^1_b(\mathbb{R}^n)$ à valeurs $L^1(\mathbb{R}^n)$. Dans cette thèse, nous trouvons le plus grand sous-espace $H^1_b(\mathbb{R}^n)$ de telle sorte que tous les commutateurs des opérateurs Calderón-Zygmund sont continus de $H^1_b(\mathbb{R}^n)$ à valeurs $L^1(\mathbb{R}^n)$. Certaines caractérisations équivalentes de $H^1_b(\mathbb{R}^n)$ sont également données. Nous

étudions également les commutateurs $[b,T]$, où T est dans une classe $\mathcal{K}$ des opérateurs sous-linéaire contenant presque tous les opérateurs importants de l'analyse harmonique. Plus précisément, nous prouvons qu'il existe un opérateur borné sous-bilinéaire $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ de telle sorte que pour tous $(f,b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, nous avons

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|, \tag{2}$$

où $\mathfrak{S}$ est un opérateur borné bilinéaire de $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$ qui ne dépend pas de T. La décomposition sous- bilinéaire (4) nous permet de donner un apercu général de toutes les estimations L^1 faibles ou fortes connues. Ils expliquent pourquoi les commutateurs avec des opérateurs fondamentaux sont de type faible (H^1, L^1), et quand un commutateurs $[b,T]$ est de type fort (H^1, L^1). En particulier, la décomposition sous-bilinéaire (4) permet de voir que, si pour tous les commutateurs avec les opérateurs de Calderón-Zygmund $[b,T]$ est borné de $H^1(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$ alors b est une fonction constante.

Soit $L = -\Delta + V$ un opérateur Schrödinger sur $\mathbb{R}^n$, $n \geq 3$, où V est un potentiel positif, $V \neq 0$, qui appartient à la classe inverse Hölder $RH_{n/2}$. Etant donné $b \in BMO(\mathbb{R}^n)$, nous prouvons que tous les commutateurs des opérateurs Schrödinger-Calderón-Zygmund $[b,T]$ envoie $H^1_L(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$ si et seulement si $b \in BMO^{\log}_L(\mathbb{R}^n)$, cela signifie que

$$\|b\|_{BMO^{\log}_L} = \sup_{B(x,r)} \left(\log\left(e + \frac{\rho(x)}{r}\right) \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}| dy \right) < \infty,$$

où $\rho(x) = \sup\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1\}$. En outre, le commutateur de la transformée de Riesz $[b, \nabla L^{-1/2}]$ est continue sur $H^1_L(\mathbb{R}^n)$ chaque fois que $b \in BMO^{\log}_L(\mathbb{R}^n)$.

Enfin, nous prouvons une version analogue de la décomposition bilinéaire (3) et une version analogue du théorème classique de Jones et Journé sur la convergence faible* dans le cadre de l'opérateur de Schrödinger L qui a été mentionné ci-dessus.

Abstract

In this thesis, we investigate the product (in the distribution sense) of functions f in H^1 and g in BMO, denoted by $f \times g$, and related problems. In particular, we prove that there are two bounded bilinear operators $S : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to H^{\log}(\mathbb{R}^n)$ such that the following bilinear decomposition

$$f \times g = S(f, g) + T(f, g) \in L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n) \tag{3}$$

holds for every $(f, g) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$. Here $H^{\log}(\mathbb{R}^n)$ is a new kind of Hardy-Orlicz space defined as the space of distributions f whose grand maximal function $\mathfrak{M}f$ satisfies

$$\int_{\mathbb{R}^n} \frac{|\mathfrak{M}f(x)|}{\log(e + |x|) + \log(e + |\mathfrak{M}f(x)|)} dx < \infty.$$

The space $H^{\log}(\mathbb{R}^n)$ appears as an example of a new class of Hardy spaces $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ that we introduce and study here. They are called as Hardy spaces of Musielak-Orlicz type, these spaces generalize the Hardy-Orlicz spaces of Janson and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky. Noting that the space $H^{\log}(\mathbb{R}^n)$ is a special case of $\varphi(x, t) \equiv \frac{t}{\log(e+|x|)+\log(e+t)}$, we prove that the set of all pointwise multipliers of $BMO(\mathbb{R}^n)$ is in fact the dual space of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$. As a consequence, we show that, in the bilinear decomposition (3), the space $H^{\log}(\mathbb{R}^n)$ could not be replaced by a smaller space in some sense.

Let b be a BMO-function. It is well-known that the linear commutator $[b, T]$ of a Calderón-Zygmund operator T does not, in general, map continuously $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. However, Pérez showed that if $H^1(\mathbb{R}^n)$ is replaced by a suitable atomic subspace $\mathcal{H}^1_b(\mathbb{R}^n)$ then the commutator is continuous from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. In this thesis, we find the largest subspace $H^1_b(\mathbb{R}^n)$ such that all commutators of Calderón-Zygmund operators are continuous from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Some equivalent characterizations of $H^1_b(\mathbb{R}^n)$ are also given. We also study the commutators $[b, T]$ for T in a class $\mathcal{K}$ of sublinear operators containing almost all important operators in harmonic analysis.

More precisely, we prove that there exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|, \tag{4}$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which does not depend on T. The subbilinear decomposition (4) allows us to give a general overview of all known weak and strong L^1-estimates, which explain why commutators with the fundamental operators are of weak type (H^1, L^1), and when a commutator $[b,T]$ is of strong type (H^1, L^1). In particular, the subbilinear decomposition (4) yields that all commutators of Calderón-Zygmund operators $[b,T]$ map continuously $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if and only if b is a constant function.

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{n/2}$. Given $b \in BMO(\mathbb{R}^n)$, we prove that all commutators of Schrödinger-Calderón-Zygmund operators $[b,T]$ map continuously $H^1_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if and only if $b \in BMO^{\log}_L(\mathbb{R}^n)$, that is,

$$\|b\|_{BMO^{\log}_L} = \sup_{B(x,r)} \left(\log\left(e + \frac{\rho(x)}{r}\right) \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}| dy \right) < \infty,$$

where $\rho(x) = \sup\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1\}$. Furthermore, the commutator of the Riesz transform $[b, \nabla L^{-1/2}]$ is bounded on $H^1_L(\mathbb{R}^n)$ whenever $b \in BMO^{\log}_L(\mathbb{R}^n)$.

Finally, we prove an analogous version of the bilinear decomposition (3) and an analogous version of the classical theorem of Jones and Journé on weak*-convergence in the setting of the Schrödinger operator L as mentioned above.

Contents

Chapter 1

Overview and motivation of the results

1.1 Paper I: Paraproducts and Products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets

For p and p' two conjugate exponents, with $1 < p < \infty$, when we consider two functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n) = (L^p(\mathbb{R}^n))^*$, their product fg is integrable, which means in particular that their pointwise product gives rise to a distribution. When $p = 1$ and $p = \infty$, the right substitute to Lebesgue spaces is, for many problems, the Hardy space $H^1(\mathbb{R}^n)$ and the space $BMO(\mathbb{R}^n)$, respectively.

Let us recall that $H^1(\mathbb{R}^n)$ is the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$, such that the grand maximal function $\mathfrak{M}f$ belongs to $L^1(\mathbb{R}^n)$ equipped with the norm $\|f\|_{H^1} := \|\mathfrak{M}f\|_{L^1}$, where

$$\mathfrak{M}f(x) := \sup_{\phi\in\mathcal{A}} \sup_{|y-x|<t} t^{-n}|f * \phi(y/t)| \tag{1.1}$$

with

$$\mathcal{A} = \Big\{\phi \in \mathcal{S}(\mathbb{R}^n) : |\phi(x)| + |\nabla\phi(x)| \leq (1 + |x|^2)^{-(n+1)}\Big\}.$$

For a ball $B \subset \mathbb{R}^n$, we denote by $|B|$ the Lebesgue measure of B. The average of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ on B is denoted by

$$f_B = \frac{1}{|B|}\int_B f(x)dx.$$

A locally integrable function f is said to have bounded mean oscillation, say $f \in BMO(\mathbb{R}^n)$, if

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|} \int\limits_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

A famous result of Fefferman (1971) states that $BMO(\mathbb{R}^n)$ is in fact the dual of $H^1(\mathbb{R}^n)$. So it is natural to ask for the right definition of the product of $h \in H^1(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$, denoted by $h \times b$, so that this product gives rise to a distribution. The investigation of such distributions are also motivated by recent developments in the geometric function theory and nonlinear elasticity [4, 7, 69, 71, 72, 112]. A typical reason is to study the H^1-theory of Jacobians, the operator $\mathscr{L}(f) = f \log |f|$, and their relations to the Rochberg-Weiss commutator, $T^{\log} f = T(f \log |f|) - Tf \log |Tf|$, where T is a singular integral operator, see the seminal works of Iwaniec and Stein [68, 70, 72, 73, 128]. In this context, the pointwise product is not integrable in general. In order to get a distribution, one has to define the product in a different way. This question and related problems have been studied by Bonami, Iwaniec, Jones and Zinsmeister in [15].

Before giving the definition of the products $h \times b$, we need to recall the characterization of the pointwise multipliers for $BMO(\mathbb{R}^n)$ due to Nakai and Yabuta [116].

Theorem A (Nakai and Yabuta, 1985)**.** *A function g is a pointwise multiplier for $BMO(\mathbb{R}^n)$ if and only if g belong to $L^\infty(\mathbb{R}^n) \cap BMO^{\log}(\mathbb{R}^n)$, where $BMO^{\log}(\mathbb{R}^n)$ is the space of locally integrable functions f such that*

$$\|f\|_{BMO^{\log}} := \sup_{B(a,r)} \frac{|\log r| + \log(e + |a|)}{|B(a,r)|} \int\limits_{B(a,r)} |f(x) - f_{B(a,r)}| dx < \infty.$$

Now, for a function $h \in H^1(\mathbb{R}^n)$ and a function $b \in BMO(\mathbb{R}^n)$, noting that the space Schwartz $\mathcal{S}(\mathbb{R}^n)$ is contained in $L^\infty(\mathbb{R}^n) \cap BMO^{\log}(\mathbb{R}^n)$, Theorem A allows to define the product $h \times b$ as the distribution

$$\langle h \times b, \phi \rangle := \langle \phi b, h \rangle,$$

where the second bracket stands for the duality bracket between $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. In [15], the authors proved that such a distribution can be written as a sum of an integrable function and a function in a weighted Hardy-Orlicz space $H^{\Xi}_{\sigma}(\mathbb{R}^n)$, related to the Orlicz function

$$\Xi(t) = \frac{t}{\log(e + t)} \tag{1.2}$$

and the weight $\sigma(x) = \frac{1}{\log(e+|x|)}$, the space of distributions f such that

$$\int_{\mathbb{R}^n} \frac{\mathfrak{M}f(x)}{\log(e+\mathfrak{M}f(x))} \frac{dx}{\log(e+|x|)} < \infty$$

with the Luxemburg norm

$$\|f\|_{H^{\Xi}_{\sigma}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e+\frac{\mathfrak{M}f(x)}{\lambda}\right)} \frac{dx}{\log(e+|x|)} \leq 1\right\}.$$

More precisely, in [15] the authors established that for each $f \in H^1(\mathbb{R}^n)$, there are two bounded linear operators $L_f : BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $H_f : BMO(\mathbb{R}^n) \to H^{\Xi}_{\sigma}(\mathbb{R}^n)$ such that for every $g \in BMO(\mathbb{R}^n)$,

$$f \times g = L_f(g) + H_f(g).$$

A question (see [15], Conjecture 1.7) by Bonami, Iwaniec, Jones and Zinsmeister is to find two operators L_f and H_f depending linearly on f. Motivated by this question, in Paper I, Bonami, Grellier and the author proved the following result:

Theorem 1. *There exist two continuous bilinear operators on the product space $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, respectively $S : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to H^{\log}(\mathbb{R}^n)$ such that*

$$f \times g = S(f, g) + T(f, g).$$

Here $H^{\log}(\mathbb{R}^n)$ is a new kind of Hardy-Orlicz space consisting of all distributions f such that

$$\int_{\mathbb{R}^n} \frac{\mathfrak{M}f(x)}{\log(e+\mathfrak{M}f(x)) + \log(e+|x|)} dx < \infty$$

with the Luxemburg norm

$$\|f\|_{H^{\log}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e+\frac{\mathfrak{M}f(x)}{\lambda}\right) + \log(e+|x|)} dx \leq 1\right\}.$$

The operators S and T, in Theorem 1, are defined in terms of a wavelet decomposition. The operator T is defined in terms of paraproducts. There is no uniqueness, of course. In fact, the same decomposition of the product fg has already been considered by Dobyinsky and Meyer (see [41, 42, 43], and also [26, 28]). The action of replacing the product by the operator T was called by them a *renormalization* of the product. Namely, T preserves the cancellation properties of the factor, while S does not. Dobyinsky and Meyer considered L^2-data for both factors, and showed that $T(f,g)$ is in the Hardy space $H^1(\mathbb{R}^n)$. What is surprising in our context is that both terms inherit some properties of the factors. Even if the product fg is not integrable, the function $S(f,g)$ is, while $T(f,g)$ inherits cancellation properties of functions in Hardy spaces without being integrable. So, in some way each term has more properties than expected at first glance.

Theorem 1 not only gives an answer for Conjecture 1.7 of [15] but also improves it by showing that the space $H^{\Xi}_{\sigma}(\mathbb{R}^n)$ can be replaced by the smaller space $H^{\log}(\mathbb{R}^n)$.

Another implicit conjecture of [15] concerns bilinear operators with cancellations, such as the ones involved in the div-curl lemma for instance. In this case it is expected that there is no L^1-term.

The second main result of Paper I concerns endpoint estimates for the div-curl lemma.

Let us first recall that the theory of compensated compactness initiated and developed by Tartar [135] and Murat [114] has been largely studied and extended to various setting. The famous paper of Coifman, Lions, Meyer and Semmes [33] gives an overview of this theory in the context of Hardy spaces in the Euclidean space $\mathbb{R}^n$. They prove in particular, that, for $\frac{n}{n+1} < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{n}$, when F is a vector field belonging to the Hardy space $H^p(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{curl} F = 0$ and G is a vector field belonging to $H^q(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{div} G = 0$, then the scalar product $F \cdot G$ can be given a meaning as a distribution of $H^r(\mathbb{R}^n)$ with

$$\|F \cdot G\|_{H^r(\mathbb{R}^n)} \leq C \|F\|_{H^p(\mathbb{R}^n, \mathbb{R}^n)} \|G\|_{H^q(\mathbb{R}^n, \mathbb{R}^n)},$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

We shall consider here the endpoint $q = \infty$. In 2003, Auscher, Russ and Tchamitchian noted firstly in [5] that, for $p = 1$, one has, under the same assumptions of being respectively curl free and divergence free,

$$\|F \cdot G\|_{H^1(\mathbb{R}^n)} \leq C \|F\|_{H^1(\mathbb{R}^n, \mathbb{R}^n)} \|G\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^n)}.$$

Recently, another interesting endpoint estimate has been obtained by Bonami, Feuto and Grellier in [11]. They showed that when $G \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is replaced by $G \in bmo(\mathbb{R}^n, \mathbb{R}^n)$, then the scalar product $F \cdot G$ is in the weighted Hardy-Orlicz space $H^{\Xi}_{\sigma}(\mathbb{R}^n)$ mentioned before (in fact there is an additional assumption on the bmo-factor). Here $bmo(\mathbb{R}^n)$, the dual of the local Hardy $h^1(\mathbb{R}^n)$ studied by Goldberg [56], is the space of locally integrable functions f such that

$$\|f\|_{bmo} := \sup_{|B| \leq 1} \frac{1}{|B|} \int\limits_B |f(x) - f_B| dx + \sup_{|B| \geq 1} \frac{1}{|B|} \int\limits_B |f(x)| dx < \infty,$$

where the supremums are taken over all balls $B \subset \mathbb{R}^n$.

More precisely, they proved in [11] that

$$\|F \cdot G\|_{H^{\Xi}_{\sigma}(\mathbb{R}^n)} \leq C\|F\|_{H^1(\mathbb{R}^n,\mathbb{R}^n)}\|G\|_{bmo(\mathbb{R}^n,\mathbb{R}^n)}. \tag{1.3}$$

By using the same technique as Dobyinsky and Meyer [41, 42, 43] to deal with the terms coming from the operator S in Theorem 1, we improve the estimate (1.3), and give a new proof without any additional assumption. More precisely, the second main result of Paper I can be stated as follows:

Theorem 2. *Let F and G be two vector fields, one of them in $H^1(\mathbb{R}^n, \mathbb{R}^n)$ and the other one in $BMO(\mathbb{R}^n, \mathbb{R}^n)$, such that $\operatorname{curl} F = 0$ and $\operatorname{div} G = 0$.Then their scalar product $F \cdot G$ (in the distribution sense) is in $H^{\log}(\mathbb{R}^n)$, moreover,*

$$\|F \cdot G\|_{H^{\log}(\mathbb{R}^n)} \leq C\|F\|_{H^1(\mathbb{R}^n,\mathbb{R}^n)}\|G\|_{BMO^+(\mathbb{R}^n,\mathbb{R}^n)}.$$

Here, for a function f in $BMO(\mathbb{R}^n)$,

$$\|f\|_{BMO^+} := \|f\|_{BMO} + |f_{\mathbb{Q}}|$$

with $\mathbb{Q} := [0, 1)^n$ the unit cube in $\mathbb{R}^n$. This is a norm, while the *BMO* norm is only a norm on equivalent classes modulo constants. Furthermore, it is easy to see that for any $f \in bmo(\mathbb{R}^n)$, we have

$$\|f\|_{BMO} \leq \|f\|_{BMO^+} \leq C\|f\|_{bmo}.$$

1.2 Paper II: New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators

Motivated by the results of Paper I, an interesting question arises:

Question 1. *Could one replace $H^{\log}(\mathbb{R}^n)$ by a smaller space?*

To answer this question, we introduce a new class of Hardy spaces.

Since Lebesgue theory of integration has taken a center stage in concrete problems of analysis, the need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$-families naturally arose. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ when $p \in (0, 1]$ are good substitutes of $L^p(\mathbb{R}^n)$ when studying the boundedness of operators: for example, the Riesz operators are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. The theory of Hardy spaces H^p on the Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss

[129]. Later, Fefferman and Stein [48] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$, which now plays an important role in various fields of analysis and partial differential equations; see, for example, [32, 33, 113]. A key feature of the classical Hardy spaces is their atomic decomposition characterizations, which were obtained by Coifman [27] when $n = 1$ and Latter [86] when $n > 1$. Later, the theory of Hardy spaces and their dual spaces associated with Muckenhoupt weights have been extensively studied by García-Cuerva [52], Strömberg and Torchinsky [131] (see also [111, 22, 53]); there the weighted Hardy spaces was defined by using the nontangential maximal functions and the atomic decompositions were derived. On the other hand, as another generalization of $L^p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum-Orlicz in [10] and Orlicz in [117], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [4, 70, 104]. Moreover, the Hardy-Orlicz spaces are also good substitutes of the Orlicz spaces in dealing with many problems of analysis, say, the boundedness of operators.

Recall that $\Psi : [0,\infty) \to [0,\infty)$ is an Orlicz function if it is nondecreasing and $\Psi(0) = 0$; $\Psi(t) > 0, t > 0$; $\lim_{t\to\infty} \Psi(t) = \infty$. We also say that Ψ is of positive lower type if there exists $p > 0$ and a positive constant $C = C(p)$ such that

$$\Psi(st) \leq C s^p \Psi(t), \tag{1.4}$$

for all $t \geq 0$ and $s \in (0,1)$.

Let Ψ be a Orlicz function which is (quasi-)concave and of positive lower type. In [75], Janson has considered the Hardy-Orlicz space $H^\Psi(\mathbb{R}^n)$ the space of all distributions f such that the grand maximal function of f defined by

$$f^*(x) = \sup_{\phi\in\mathcal{A}_N} \sup_{|x-y|<t} |f * \phi_t(y)|, \ x \in \mathbb{R}^n, \tag{1.5}$$

belongs to the Orlicz space $L^\Psi(\mathbb{R}^n)$. Here $\phi_t(\cdot) := t^{-n}\phi(t^{-1}\cdot)$ and

$$\mathcal{A}_N = \Big\{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial_x^\alpha \phi(x)| \leq 1 \text{ for } \alpha \in \mathbb{N}^n, |\alpha| \leq N\Big\}$$

with $N = N(n, \Psi)$ taken large enough. Remark that these Hardy-Orlicz type spaces appear naturally when studying the theory of nonlinear PDEs [57, 71, 73] since many cancellation phenomena for Jacobians cannot be observed in the usual Hardy spaces $H^p(\mathbb{R}^n)$. For instance, let $f = (f^1, ..., f^n)$ in the Sobolev class $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and the Jacobians $J(x,f)dx = df^1 \wedge \cdots \wedge df^n$, then (see [73], Theorem 10.2)

$$\mathcal{T}(J(x,f)) \in L^1(\mathbb{R}^n) + H^\Xi(\mathbb{R}^n)$$

where the Orlicz function Ξ is defined as in (1.2) and $\mathcal{T}(f) = f \log|f|$, since $J(x,f) \in H^1(\mathbb{R}^n)$ (cf. [33]) and $\mathcal{T}$ is well defined on $H^1(\mathbb{R}^n)$. We refer the reader to [72, 121] for this interesting nonlinear operator $\mathcal{T}$.

Now, let us return to *Question 1*. By duality with Theorem 1, functions f that are bounded and in the dual of $H^{\log}(\mathbb{R}^n)$ are multipliers of $BMO(\mathbb{R}^n)$, i.e. f belongs to $L^\infty(\mathbb{R}^n) \cap BMO^{\log}(\mathbb{R}^n)$ from the theorem of Nakai and Yabuta. Consequently, *we can conclude that* $H^{\log}(\mathbb{R}^n)$, *in some sense, could not be replaced by a smaller space, once established that the dual space of* $H^{\log}(\mathbb{R}^n)$ *is* $BMO^{\log}(\mathbb{R}^n)$.

Motivated by this, in Paper II, we introduce a new class of Hardy spaces $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$, so-called Hardy spaces of Musielak-Orlicz type, which generalize the Hardy-Orlicz spaces of Janson and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky. Here, $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a function such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt A_∞ weight, namely, either

$$\sup_B \frac{1}{|B|}\int_B \varphi(x,t)dx\Big(\frac{1}{|B|}\int_B (\varphi(x,t))^{-1/(q-1)}dx\Big)^{q-1} < \infty$$

for some $q \in (1,\infty)$ or

$$\sup_B \frac{1}{|B|}\int_B \varphi(x,t)dx\Big(\operatorname*{ess-inf}_{x\in B}\varphi(x,t)\Big)^{-1} < \infty.$$

More precisely, we define $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$, or $H^\varphi(\mathbb{R}^n)$ for simplicity, as the space of distributions f such that $x \mapsto \varphi(x, |f^*(x)|)$ is integrable, where f^* is the grand maximal function of f defined as in (1.5) with $N = N(\varphi)$ large enough. We equip $H^\varphi(\mathbb{R}^n)$ with the norm $\|f\|_{H^\varphi} := \|f^*\|_{L^\varphi}$. Here, for a measurable function f,

$$\|f\|_{L^\varphi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda)dx \le 1\right\}. \tag{1.6}$$

Note that $H^{\log}(\mathbb{R}^n)$ is just a special case of these new Hardy spaces related to the Musielak-Orlicz function $\varphi(x,t) \equiv \frac{t}{\log(e+|x|)+\log(e+t)}$. We then establish their atomic characterizations, which are new even for the classical weighted Hardy-Orlicz spaces related to Musielak-Orlicz functions $\varphi(x,t) = w(x)\Psi(t)$. In order to state the atomic decomposition theorem, we need some new notations.

For B a ball in $\mathbb{R}^n$ and $q \in [1,\infty]$, define $L^q_\varphi(B)$ as the space of all measurable functions f supported in B such that

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup_{t>0}\left(\frac{\int_{\mathbb{R}^n}|f(x)|^q\varphi(x,t)dx}{\int_B \varphi(x,t)dx}\right)^{1/q} < \infty & , \quad 1 \le q < \infty, \\ \|f\|_{L^\infty} < \infty & , \quad q = \infty. \end{cases}$$

Definition 1.2.1. *Let $1 < q < \infty$ and $s \in \mathbb{Z}^+$. A measurable function a is a (φ, q, s)-atom if it satisfies the following three conditions*

i) $a \in L^q_\varphi(B)$ for some ball B,

ii) $\|a\|_{L^q_\varphi(B)} \leq \|\chi_B\|^{-1}_{L^\varphi}$,

iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $|\alpha| \leq s$.

We now define the *atomic Hardy space of Musielak-Orlicz type* $H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ as those distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j b_j$ (in the sense of $\mathcal{S}'(\mathbb{R}^n)$), where b_j's are multiples of (φ, q, s)-atoms supported in the balls B_j's, with the property $\sum_j \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty$; and define the norm of f by

$$\|f\|_{H^{\varphi,q,s}_{\rm at}} = \inf \left\{ \Lambda_q(\{b_j\}) : f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n) \right\},$$

where $\Lambda_q(\{b_j\}) = \inf \left\{ \lambda > 0 : \sum_j \varphi\Big(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\Big) \leq 1 \right\}$.

When q and s are large enough, we prove that

Theorem 3. *$H^\varphi(\mathbb{R}^n) \equiv H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ with equivalent norms.*

We are also interested in duality results. To state them, we introduce BMO type spaces $BMO^\varphi(\mathbb{R}^n)$. Precisely, a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ is said to belong to $BMO^\varphi(\mathbb{R}^n)$ if

$$\|f\|_{BMO^\varphi} := \sup_B \frac{1}{\|\chi_B\|_{L^\varphi}} \int_B |f(x) - f_B| dx < \infty, \tag{1.7}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Theorem 4. *The dual of the space $H^\varphi(\mathbb{R}^n)$ is $BMO^\varphi(\mathbb{R}^n)$.*

When $\varphi(x,t) \equiv \frac{t}{\log(e+|x|)+\log(e+t)}$, we prove that $BMO^\varphi(\mathbb{R}^n)$ is just $BMO^{\log}(\mathbb{R}^n)$. As a consequence of Theorem 4, this gives:

Theorem 5. *The dual of the space $H^{\log}(\mathbb{R}^n)$ is $BMO^{\log}(\mathbb{R}^n)$.*

As mentioned above, Theorem 5 allows to give an answer for *Question 1*: $H^{\log}(\mathbb{R}^n)$, in some sense, could not be replaced by a smaller space.

The last main theorem of this part concerns the boundedness of operators on Hardy spaces. Usually, in order to establish the boundedness of operators on Hardy spaces, one usually appeals to the atomic decomposition characterization, see [27, 86, 133], which means that a function or a distribution in Hardy spaces can be represented as a linear combination of functions of an elementary form, namely, atoms. Then, the boundedness of operators on Hardy spaces can be deduced from their behavior on atoms or molecules in principle. However, caution needs to be taken due to an example constructed in Theorem 2 of [19]. There exists a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1,\infty,0)$-atoms into bounded scalars, but however does not extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that the uniform boundedness of a linear operator T on atoms does not automatically guarantee the boundedness of T from $H^1(\mathbb{R}^n)$ to a Banach space $\mathcal{B}$. Nevertheless, by using the grand maximal function characterization of $H^p(\mathbb{R}^n)$, Meda, Sjögren, and Vallarino [105, 106] proved that if a sublinear operator T maps all (p,q,s)-atoms when $q<\infty$ and continuous (p,∞,s)-atoms into uniformly bounded elements of $L^p(\mathbb{R}^n)$ (see also [144, 20] for quasi-Banach spaces), then T uniquely extends to a bounded sublinear operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In this paper, we study boundedness of sublinear operators in the context of these new Hardy spaces of Musielak-Orlicz type which generalize the main results in [105, 106]. More precisely, under additional assumption on $\varphi(\cdot,\cdot)$, we prove that finite atomic norms on dense subspaces of $H^\varphi(\mathbb{R}^n)$ are equivalent with the standard infinite atomic decomposition norms. As an application, we prove that if T is a sublinear operator and maps all atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^\varphi(\mathbb{R}^n)$ to $\mathcal{B}$.

Recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}}=0$ if and only if $f=0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant κ not less than 1 such that for all $f,g\in\mathcal{B}$, we have $\|f+g\|_{\mathcal{B}}\le\kappa(\|f\|_{\mathcal{B}}+\|g\|_{\mathcal{B}})$.

Definition 1.2.2. *Let $\gamma\in(0,1]$. A quasi-Banach space $\mathcal{B}_\gamma$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is said to be a γ-quasi-Banach space if there exists a positive constant κ not less than* 1 *such that for all $f_j\in\mathcal{B}_\gamma, j=1,2,...,m$, we have*

$$\Big\|\sum_{j=1}^m f_j\Big\|_{\mathcal{B}_\gamma}^\gamma\le\kappa\sum_{j=1}^m\|f_j\|_{\mathcal{B}_\gamma}^\gamma.$$

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces $\ell^p, L^p_w(\mathbb{R}^n)$ and $H^p_w(\mathbb{R}^n)$ with $p\in(0,1]$ are typical p-quasi-Banach spaces. Also, when φ is of uniformly lower type $p\in(0,1]$, the space $H^\varphi(\mathbb{R}^n)$ is a p-quasi-Banach space.

For any given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma\in(0,1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is called $\mathcal{B}_\gamma$-sublinear if there exists a positive constant κ not less than 1 such that for all $f_j\in\mathcal{Y},\lambda_j\in\mathbb{C}, j=1,...,m$, we have

$$\Big\|T\Big(\sum_{j=1}^m\lambda_jf_j\Big)\Big\|_{\mathcal{B}_\gamma}^\gamma\le\kappa\sum_{j=1}^m|\lambda_j|^\gamma\|T(f_j)\|_{\mathcal{B}_\gamma}^\gamma.$$

We remark that if T is linear, then T is $\mathcal{B}_\gamma$-sublinear. We should point out that if the constant κ, in Definition 3.2.5, equal 1, then we obtain the notion of γ-quasi-Banach spaces introduced in [144] (see also [20]).

Under some assumptions on q, s, φ, we get the last main theorem of Paper II as follows:

Theorem 6. *Let $\mathcal{B}_\gamma$ be a γ-quasi-Banach space for some $\gamma \in (0,1]$. Suppose that φ is of uniformly upper type γ, and one of the following holds:*

i) $T : H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n) \to \mathcal{B}_\gamma$, $q < \infty$, is a $\mathcal{B}_\gamma$-sublinear operator such that

$$A = \sup\{\|Ta\|_{\mathcal{B}_\gamma} : a \textit{ is a } (\varphi, q, s)\mathrm{-atom}\} < \infty;$$

ii) T is a $\mathcal{B}_\gamma$-sublinear operator defined on continuous (φ, ∞, s)-atoms such that

$$A = \sup\{\|Ta\|_{\mathcal{B}_\gamma} : a \textit{ is a continuous } (\varphi, \infty, s)\mathrm{-atom}\} < \infty.$$

Then there exists a unique bounded $\mathcal{B}_\gamma$-sublinear operator $\widetilde{T}$ from $H^\varphi(\mathbb{R}^n)$ to $\mathcal{B}_\gamma$ which extends T.

Here a Orlicz function φ is said to be of uniformly upper type γ if there exists a constant $C > 0$ such that

$$\varphi(x, st) \leq C s^\gamma \varphi(x, t)$$

for all $x \in \mathbb{R}^n$, $t > 0$ and $s \in [1, \infty)$.

Very recently, many authors have studied and generalized the theory of new Hardy spaces of Musielak-Orlicz type to many setting, see for example [66, 93, 140, 141, 142]. To be more precise, using the theory of tent spaces, introduced by Coifman, Meyer and Stein [30], together with the classical Calderón reproducing formula, the authors [66, 93] have established the Lusin area function and the molecular characterizations for $H^\varphi(\mathbb{R}^n)$. Under some additional mild restrictions on φ, they also obtained some real-variable characterizations of $H^\varphi(\mathbb{R}^n)$ in terms of the vertical and the non-tangential maximal functions and in terms of the Littlewood-Paley functions. The Carleson-type measure characterization for $BMO^\varphi(\mathbb{R}^n)$ is also considered, and many applications of these new Hardy spaces are also given. Besides, in [140, 141, 142] the authors have also considered these new spaces in the setting of nonnegative selfadjoint operators in L^2 satisfying the Davies-Gaffney estimates and in the setting of local Hardy spaces related to the class of local weights introduced by V. S. Rychkov [123]. Some applications are also given in [140, 141, 142].

1.3 Paper III: Bilinear decompositions and commutators of singular integral operators

By using the results of Paper I and Paper II, we investigate the endpoint theory for commutators of singular integral operators. Let us first recall that giving a function b locally integrable on $\mathbb{R}^n$, and a Calderón-Zygmund operator T, the linear commutator $[b, T]$ is defined for smooth, compactly supported functions f by

$$[b, T](f) = bT(f) - T(bf).$$

A classical result of Coifman, Rochberg and Weiss (see [31]), states that the commutator $[b, T]$ is continuous on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, when $b \in BMO(\mathbb{R}^n)$. Unlike the theory of Calderón-Zygmund operators, the proof of this result does not rely on a weak type $(1, 1)$ estimate for $[b, T]$. In fact, it was shown in [119] and [62] that, in general, the linear commutator is neither of weak type $(1, 1)$ nor of strong type (H^1, L^1), when b is in $BMO(\mathbb{R}^n)$. Instead of this, the weak type estimate (H^1, L^1) for $[b, T]$ is well-known, see for example [96, 101, 139]. More precisely, one has:

Theorem B. *Let $b \in BMO(\mathbb{R}^n)$ and T be a Calderón-Zygmund operator. Then, the commutator $[b, T]$ is bounded from $H^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$.*

It should be pointed out that intuitively one would like to write

$$[b, T](f) = \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) T(a_j) - T\Big(\sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j\Big),$$

where $f = \sum_{j=1}^{\infty} \lambda_j a_j$ a atomic decomposition of f. This is equivalent to ask for a commutation property

$$\sum_{j=1}^{\infty} \lambda_j b_{B_j} T(a_j) = T\Big(\sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j\Big). \tag{1.8}$$

To prove Theorem B, most authors, for instance in [96, 101, 139, 146, 90, 137, 95], implicitely use (4.3). As pointed out in the subsection 1.2, one must be careful at this point. Indeed, Equality (4.3) is not clear since the two series $\sum_{j=1}^{\infty} \lambda_j b_{B_j} T(a_j)$ and $\sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j$ are not yet well-defined, in general. Furthermore, acceptting equalities like Equality (4.3) would follow in particular that boundedness of T on atoms implies boundedness of T which is not true, in general. We refer the reader to [19], Section 3, for a counterexample.

Although the commutator $[b, T]$ does not map continuously, in general, $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, following Pérez [119] one can find a subspace $\mathcal{H}^1_b(\mathbb{R}^n)$ of $H^1(\mathbb{R}^n)$ such that $[b, T]$ maps continuously $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Recall that a function a is a b-atom if

i) supp $a \subset Q$ for some cube Q,

ii) $\|a\|_{L^\infty} \leq |Q|^{-1}$,

iii) $\int_{\mathbb{R}^n} a(x)dx = \int_{\mathbb{R}^n} a(x)b(x)dx = 0$.

The space $\mathcal{H}^1_b(\mathbb{R}^n)$ consists of the subspace of $L^1(\mathbb{R}^n)$ of functions f which can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$ where a_j are b-atoms, and λ_j are complex numbers with $\sum_{j=1}^\infty |\lambda_j| < \infty$.

Then, one has:

Theorem C. *Let $b \in BMO(\mathbb{R}^n)$ and T be a Calderón-Zygmund operator. Then, the commutator $[b,T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

To prove Theorem C, in [119], the author showed that the commutator $[b,T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ by establishing that

$$\sup\{\|[b,T](a)\|_{L^1} : a \,\text{is a}\; b\text{−atom}\} < \infty. \tag{1.9}$$

As we already emphasized this leaves a gap in the proof which we fill here. Note that this difficulty has been mentioned *as a question* in the paper of Hu, Meng and Yang (see [67], page 1132). Actually, in [19], a linear operator U defined on the space of all finite linear combination of $(1,\infty)$-atoms satisfies

$$\sup\{\|U(a)\|_{L^1} : a \,\text{is a}\, (1,\infty)\text{−atom}\} < \infty,$$

but does not admit an extension to a bounded operator from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. From this result, we see that Inequality (4.4) does not suffice to conclude that $[b,T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. In the setting of $H^1(\mathbb{R}^n)$, it is well-known (see [105] or [144] for details) that a linear operator U can be extended to a bounded operator from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if for some $1 < q < \infty$, we have

$$\sup\{\|U(a)\|_{L^1} : a \,\text{is a}\, (1,q)\text{−atom}\} < \infty.$$

It follows from the fact that the finite atomic norm on $H^{1,q}_{\text{fin}}(\mathbb{R}^n)$ is equivalent to the standard infinite atomic decomposition norm on $H^{1,q}_{\text{ato}}(\mathbb{R}^n)$ through the grand maximal function characterization of $H^1(\mathbb{R}^n)$. However, one can not use this method in the context of $\mathcal{H}^1_b(\mathbb{R}^n)$. In order to give a correct proof for Theorem 1.3, we use a different approach.

Now, a natural question arises:

Question 2. *Can one find the largest subspace of $H^1(\mathbb{R}^n)$ such that all commutators of Calderón-Zygmund operators are bounded from this space into $L^1(\mathbb{R}^n)$?*

In order to answer this question, in Paper III, we consider the class $\mathcal{K}$ of all sublinear operators T, bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, satisfying the condition

$$\|(b - b_Q)Ta\|_{L^1} \leq C\|b\|_{BMO}$$

for all BMO-function b, H^1-atom a related to the cube Q. This class $\mathcal{K}$ contains almost all important operators in harmonic analysis: Calderón-Zygmund type operators, strongly singular integral operators, multiplier operators, pseudo-differential operators, maximal type operators, the area integral operator of Lusin, Littlewood-Paley type operators, Marcinkiewicz operators, maximal Bochner-Riesz operators, etc...

We then study the commutators $[b, T]$ for T in the class $\mathcal{K}$. In particular, we prove the following:

Theorem 7 (Subbilinear decomposition)**.** *Let $T \in \mathcal{K}$. There exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have*

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|. \tag{1.10}$$

Here $\mathfrak{S}$ is the bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which does not depend on T. It is defined by

$$\mathfrak{S}(f, b) := -\sum_I \sum_{\sigma \in E} \langle f, \psi_I^\sigma \rangle \langle b, \psi_I^\sigma \rangle (\psi_I^\sigma)^2.$$

Furthermore, when T is linear and belongs to $\mathcal{K}$, we obtain the bilinear decomposition for the linear commutator $[b, T]$ of f, $[b, T](f) = bT(f) - T(bf)$, instead of the subbilinear decomposition as stated in Theorem 4.3.1.

Theorem 8 (Bilinear decomposition)**.** *Let T be a linear operator in $\mathcal{K}$. Then, there exists a bounded bilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have*

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)). \tag{1.11}$$

The bilinear decomposition (1.10) and the subbilinear decomposition (1.11) not only completes the proofs for Theorem B and Theorem C but also allow us to give a general overview of all known weak and strong L^1-estimates. They explain why almost all commutators of the fundamental operators (Calderón-Zygmund operators, strongly singular integral operators, multiplier operators, pseudo-differential operators, maximal type operators, the area integral operator of Lusin, Littlewood-Paley type operators, Marcinkiewicz operators, maximal Bochner-Riesz operators, etc...) are of weak type (H^1, L^1), and when a commutator $[b, T]$ is of strong type (H^1, L^1).

As a consequence, we find the largest subspace $H^1_b(\mathbb{R}^n)$ such that all commutators of Calderón-Zygmund operators are continuous from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. More precisely, for $b \in BMO(\mathbb{R}^n)$, a non-constant function, we define $H^1_b(\mathbb{R}^n)$ as the space consisting of all $f \in H^1(\mathbb{R}^n)$ such that the (sublinear) commutator $[b, \mathfrak{M}]$ of f belongs to $L^1(\mathbb{R}^n)$, where $[b, \mathfrak{M}](f)(x) := \mathfrak{M}(b(x)f(\cdot) - b(\cdot)f(\cdot))(x)$. Recall that $\mathfrak{M}$ is the grand maximal function given in (1.1). The norm on $H^1_b(\mathbb{R}^n)$ is then defined by $\|f\|_{H^1_b} := \|f\|_{H^1}\|b\|_{BMO} + \|[b, \mathfrak{M}](f)\|_{L^1}$. Here we just consider b a non-constant BMO-function since the commutator $[b, T] = 0$ if b is a constant function. Then, we prove that $[b, T]$ is bounded from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for every Calderón-Zygmund singular integral operator T (in fact it holds for all $T \in \mathcal{K}$, see below). Furthermore, $H^1_b(\mathbb{R}^n)$ *is the largest space having this property*, in particular it contains $\mathcal{H}^1_b(\mathbb{R}^n)$ of Pérez, which answers Question 2.

In Paper III, we also consider Hardy estimates for commutators by giving two sufficient conditions for the boundedness from $H^1_b(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$ and from $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$ of commutators $[b, T]$. More precisely, the last two main theorems of this part are as follow:

Theorem 9. *Let b be a non-constant $BMO^{\log}$-function and T be a Calderón-Zygmund operator with $T1 = T^*1 = 0$. Then, the linear commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$.*

Theorem 10. *Let b be a non-constant BMO-function and T be a Calderón-Zygmund operator with $T^*1 = T^*b = 0$. Then, the linear commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.*

Observe that the condition $T^*b = 0$ is "necessary" in the sense that if the linear commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} b(x)Ta(x)dx = 0$ holds for all (q, b)-atoms a, $1 < q \leq \infty$.

Let us give some examples of operators T satisfying the assumption $T^*1 = T^*b = 0$. To have many examples, let us consider Euclidean spaces $\mathbb{R}^n, n \geq 2$. Now, consider all Calderón-Zygmund operators T such that $T^*1 = 0$. As the closure of $T(H^1(\mathbb{R}^n))$ is a proper subset of $H^1(\mathbb{R}^n)$, by the Hahn-Banach theorem (note that $BMO(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$), one may take b a non-constant BMO-function such that $\int_{\mathbb{R}^n} bTadx = 0$ for all H^1-atoms a, i.e. $T^*b = 0$, and thus b and T satisfy the sufficient condition in Theorem 10.

1.4 Paper IV: Endpoint estimates for commutators of singular integrals related to Schrödinger operators

A natural question related to Paper III is whether there exist non-constant functions $b \in BMO(\mathbb{R}^n)$ such that the space $H^1_b(\mathbb{R}^n)$ coincides with $H^1(\mathbb{R}^n)$. Particularly, whether there exist non-constant functions $b \in BMO(\mathbb{R}^n)$ such that the commutators $[b, \mathcal{R}_j]$ are bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where the $\mathcal{R}_j, j = 1, ..., n$, are the classical Riesz transforms. The answer, in this setting, is negative. Actually, by the decomposition (1.10), it is not hard to see that the commutators $[b, \mathcal{R}_j]$ are bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if and only if b is a constant function. It should be pointed out that when $n = 1$, the above result was mentioned in the paper of Harboure, Segovia and Torrea [62], see also Remark 4.1 in the paper of Janson, Peetre and Semmes [76]. There, they proved that the commutator of the Hilbert transform $[b, H]$ is bounded from $H^1(\mathbb{R})$ into $L^1(\mathbb{R})$ if and only if b is a constant function.

In contrast with the Euclidean space $\mathbb{R}^n$, the situation is different in the setting of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Janson, Peetre and Semmes showed in [76] that the commutator of the Hilbert transform $[b, H]$ is bounded on the Hardy space $H^1(\mathbb{T})$ whenever $b \in BMO^{\log}(\mathbb{T})$, that is,

$$\|b\|_{BMO^{\log}(\mathbb{T})} = \frac{1}{2\pi}\Big|\int\limits_{\mathbb{T}} b(z)|dz|\Big| + \sup_I \frac{\log\frac{4}{|I|}}{|I|}\int\limits_I \Big|b(\eta) - \frac{1}{|I|}\int\limits_I b(z)|dz|\Big||d\eta| < \infty,$$

where the supremum is taken over all arcs I of $\mathbb{T}$ and $|I|$ is the length of I.

In this paper, we consider this problem in the setting of Hardy spaces and the Riesz transforms related to Schrödinger operators.

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{n/2}$. Recall that a nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q, $1 < q < \infty$, if there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int\limits_B (V(x))^q dx\Big)^{1/q} \leq \frac{C}{|B|}\int\limits_B V(x)dx \tag{1.12}$$

holds for every balls B in $\mathbb{R}^n$. According to [46], we define $H^1_L(\mathbb{R}^n)$ as the space of all functions $f \in L^1(\mathbb{R}^n)$ such that $\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) = \sup_{t>0} |e^{-tL} f(x)|$ for all $x \in \mathbb{R}^n$.

We look for non-constant functions $b \in BMO(\mathbb{R}^n)$ such that the commutators $[b, R_j]$ are bounded from $H^1_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where the $R_j = \partial_{x_j} L^{-1/2}, j = 1, ..., n$, are the Riesz transforms associated with the Schrödinger operator L. Also, we discuss the conditions on functions $b \in BMO(\mathbb{R}^n)$ which ensure the commutators $[b, R_j]$ to be bounded from $H^1_L(\mathbb{R}^n)$ into itself.

According to [17], for $\theta \geq 0$, we define $BMO_{L,\theta}(\mathbb{R}^n)$ and $BMO^{\log}_{L,\theta}(\mathbb{R}^n)$, respectively, as the spaces of locally integrable functions f satisfying

$$\|f\|_{BMO_{L,\theta}} := \sup_{B(x,r)} \left(\frac{1}{\left(1 + \frac{r}{\rho(x)}\right)^{\theta}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \right) < \infty$$

and

$$\|f\|_{BMO^{\log}_{L,\theta}} := \sup_{B(x,r)} \left(\frac{\log\left(e + \frac{\rho(x)}{r}\right)}{\left(1 + \frac{r}{\rho(x)}\right)^{\theta}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \right) < \infty,$$

respectively, where $\rho(x) := \sup\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1\}$. When $\theta = 0$, we write $BMO^{\log}_L(\mathbb{R}^n)$ instead of $BMO^{\log}_{L,0}(\mathbb{R}^n)$.

Note that the space $BMO_{L,\infty}(\mathbb{R}^n)$ is in general larger than the space $BMO(\mathbb{R}^n)$. Indeed, when $V(x) \equiv |x|^2$, it is easy to check that the functions $b_j(x) = |x_j|^2$, $j = 1, ..., n$, belong to $BMO_{L,\infty}(\mathbb{R}^n)$ but not to $BMO(\mathbb{R}^n)$.

In this paper, we prove the following.

Theorem 11. *Let $b \in BMO_{L,\infty}(\mathbb{R}^n) = \cup_{\theta \geq 0} BMO_{L,\theta}(\mathbb{R}^n)$. Then, the commutators $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^n)$ if and only if $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^n) = \cup_{\theta \geq 0} BMO^{\log}_{L,\theta}(\mathbb{R}^n)$. Furthermore, if $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^n)$ for some $\theta \geq 0$, we have*

$$\|b\|_{BMO^{\log}_{L,\theta}} \approx \|b\|_{BMO_{L,\theta}} + \sum_{j=1}^{n} \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

Remark that the above constants depend on θ.

Next, let us recall the notation of Schrödinger-Calderón-Zygmund operators.

Let $\delta \in (0, 1]$. According to [103], a continuous function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ is said to be a (δ, L)-Calderón-Zygmund singular integral kernel if for each $N > 0$,

$$|K(x, y)| \leq \frac{C(N)}{|x - y|^n} \Big(1 + \frac{|x - y|}{\rho(x)}\Big)^{-N} \tag{1.13}$$

for all $x \neq y$, and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \tag{1.14}$$

for all $2|x - x'| \leq |x - y|$.

A linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is said to be a (δ, L)-Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$ and if there exists a (δ, L)-Calderón-Zygmund singular integral kernel K such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and all $x \notin \operatorname{supp} f$, we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy.$$

We say that T is a L-Calderón-Zygmund operator (or Schrödinger-Calderón-Zygmund operator related to L) if it is a (δ, L)-Calderón-Zygmund operator for some $\delta \in (0,1]$. We say also that T satisfies the condition $T^*1 = 0$ if there are $q \in (1,\infty]$ and $\varepsilon > 0$ so that $\int_{\mathbb{R}^n} Ta(x)dx = 0$ holds for every generalized (H^1_L, q, ε)-atoms a. Here, a function a is called a generalized (H^1_L, q, ε)-atom related to the ball $B(x_0, r)$ if

(a) $\operatorname{supp} a \subset B(x_0, r)$,

(b) $\|a\|_{L^q} \leq |B(x_0,r)|^{1/q-1}$,

(c) $|\int_{\mathbb{R}^n} a(x)dx| \leq \Big(\frac{r}{\rho(x_0)}\Big)^\varepsilon$.

Remark 1.4.1. *i) By Lemma 1.4 of [125], Inequality (5.14) is equivalent to*

$$|K(x,y)| \leq \frac{C(N)}{|x-y|^n}\Big(1 + \frac{|x-y|}{\rho(y)}\Big)^{-N}$$

for all $x \neq y$.

ii) By Theorem 0.8 of [125] and Theorem 1.1 of [126], the Riesz transforms R_j are L-Calderón-Zygmund operators satisfying $R_j^*1 = 0$ whenever $V \in RH_n$.

iii) If T is a L-Calderón-Zygmund operator then it is also a classical Calderón-Zygmund operator, and thus T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and bounded from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Our second main theorem concerns the H^1_L-estimates for commutators of Schrödinger-Calderón-Zygmund operators.

Theorem 12. *i) Let $b \in BMO_L^{\log}(\mathbb{R}^n)$ and T be a L-Calderón-Zygmund operator satisfying $T^*1 = 0$. Then, the linear commutator $[b,T]$ is bounded on $H_L^1(\mathbb{R}^n)$.*

ii) When $V \in RH_n$, the converse holds. Namely, if $b \in BMO(\mathbb{R}^n)$ and $[b,T]$ is bounded on $H_L^1(\mathbb{R}^n)$ for every L-Calderón-Zygmund operator T satisfying $T^*1 = 0$, then $b \in BMO_L^{\log}(\mathbb{R}^n)$. Furthermore,

$$\|b\|_{BMO_L^{\log}} \approx \|b\|_{BMO} + \sum_{j=1}^{n} \|[b,R_j]\|_{H_L^1 \to H_L^1}.$$

Recently, the authors in [23, 92, 134, 138] considered endpoint estimates for commutators of singular integral operators $[b,T]$ with functions $b \in BMO(\mathbb{R}^n)$. In particular, when $b \in BMO(\mathbb{R}^n)$, they proved that the commutators of the Riesz transforms $[b,R_j]$ are bounded from $H_L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$ and from $\mathcal{H}_b^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where $\mathcal{H}_b^1(\mathbb{R}^n)$ is the atomic Hardy space introduced by Pérez (see Paper III before).

This paper explains why, when b is in $BMO(\mathbb{R}^n)$, commutators of singular integral operators related to L (containing the Riesz transforms R_j), say $[b,T]$, are of weak type (H_L^1, L^1), and when a commutator $[b,T]$ is of strong type (H_L^1, L^1). To be more precise, we investigate commutators of singular integral operators T related to the Schrödinger operator L, where T is in the class $\mathcal{K}_L$ of all sublinear operators T, bounded from $H_L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and that there are $q \in (1,\infty]$ and $\varepsilon > 0$ such that

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO}$$

for every $b \in BMO(\mathbb{R}^n)$, any generalized (H_L^1, q, ε)-atom a related to the ball B, where $C > 0$ is a constant independent of b, a. The class $\mathcal{K}_L$ contains the fundamental operators (see Paper III for the classical case $L = -\Delta$) related to the Schrödinger operator L: the Riesz transforms R_j, Schrödinger-Calderón-Zygmund operators, L-maximal operators, L-square operators, etc... Remark that the Riesz transforms R_j are just, in general, Schrödinger-Calderón-Zygmund operators when $V \in RH_n$. In this work, we consider all potentials V which belong to the reverse Hölder class $RH_{n/2}$.

Although Schrödinger-Calderón-Zygmund operators map $H_L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ (see Proposition 4.1 of the paper), it was observed in [92] that, when $b \in BMO(\mathbb{R}^n)$, the commutators $[b,R_j]$ do not map, in general, $H_L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Thus, when $b \in BMO(\mathbb{R}^n)$, it is natural to ask for subspaces of $H_L^1(\mathbb{R}^n)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms map continuously these spaces into $L^1(\mathbb{R}^n)$. Here, we are interested in the following two questions.

Question 3. *For $b \in BMO(\mathbb{R}^n)$. Find the largest subspace $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ of $H^1_L(\mathbb{R}^n)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

Question 4. *Characterize the functions b in $BMO(\mathbb{R}^n)$ so that $\mathcal{H}^1_{L,b}(\mathbb{R}^n) \equiv H^1_L(\mathbb{R}^n)$.*

Let X be a Banach space. We say that an operator $T : X \to L^1(\mathbb{R}^n)$ is a sublinear operator if for all $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$, we have

$$|T(\alpha f + \beta g)(x)| \leq |\alpha||Tf(x)| + |\beta||Tg(x)|.$$

Obviously, a linear operator $T : X \to L^1(\mathbb{R}^n)$ is a sublinear operator. We also say that a operator $\mathfrak{T} : H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is a subbilinear operator if for every $(f, g) \in H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, the operators $\mathfrak{T}(f, \cdot) : BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $\mathfrak{T}(\cdot, g) : H^1_L(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ are sublinear operators.

To answer *Question 3* and *Question 4*, we study commutators of sublinear operators in $\mathcal{K}_L$. More precisely, when $T \in \mathcal{K}_L$ is a sublinear operator, we prove the following theorem.

Theorem 13. *Let $T \in \mathcal{K}_L$. Then, there exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ so that for all $(f, b) \in H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$,*

$$|T(\mathfrak{S}_L(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}_L(f, b))|, \tag{1.15}$$

where $\mathfrak{S}_L$ is a bounded bilinear operator from $H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which does not depend on T (see Proposition 5.2 of the paper)

When T is linear and belongs to $\mathcal{K}_L$, we obtain the bilinear decomposition for the linear commutator $[b, T]$ of f, $[b, T](f) = bT(f) - T(bf)$, instead of the subbilinear decomposition as stated in Theorem 13.

Theorem 14. *Let T be a linear operator in $\mathcal{K}_L$. Then, there exists a bounded bilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f, b) \in H^1_L(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have*

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}_L(f, b)),$$

where $\mathfrak{S}_L$ is as in Theorem 13.

The above theorem gives a general overview and explains why almost commutators of the fundamental operators are of weak type (H^1_L, L^1), and when a commutator $[b, T]$ is of strong type (H^1_L, L^1).

Let b be a non-constant BMO-function (otherwise $[b,T]=0$). We define the space $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ as the set of all f in $H^1_L(\mathbb{R}^n)$ such that $[b,\mathcal{M}_L](f)(x)=\mathcal{M}_L(b(x)f(\cdot)-b(\cdot)f(\cdot))(x)$ belongs to $L^1(\mathbb{R}^n)$, and the norm on $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ is defined by $\|f\|_{\mathcal{H}^1_{L,b}}=\|f\|_{H^1_L}\|b\|_{BMO}+\|[b,\mathcal{M}_L](f)\|_{L^1}$. Then, using the subbilinear decomposition (5.2), we prove the following.

Theorem 15. *Let b be a non-constant BMO-function. Then, the following statements hold:*

i) For every $T\in\mathcal{K}_L$, the commutator $[b,T]$ is bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

ii) Assume that $\mathcal{X}$ is a subspace of $H^1_L(\mathbb{R}^n)$ such that all commutators of the Riesz transforms are bounded from $\mathcal{X}$ into $L^1(\mathbb{R}^n)$. Then, $\mathcal{X}\subset\mathcal{H}^1_{L,b}(\mathbb{R}^n)$.

iii) $\mathcal{H}^1_{L,b}(\mathbb{R}^n)\equiv H^1_L(\mathbb{R}^n)$ if and only if $b\in BMO^{\log}_L(\mathbb{R}^n)$.

The above theorem allows that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, moreover, $\mathcal{H}^1_{L,b}(\mathbb{R}^n)$ *is the largest space having this property*. Also, it allows to characterize functions b in $BMO(\mathbb{R}^n)$ so that $\mathcal{H}^1_{L,b}(\mathbb{R}^n)\equiv H^1_L(\mathbb{R}^n)$. This answers *Question 3* and *Question 4*.

As another interesting application of the subbilinear decomposition (5.2), we find some subspaces of $H^1_L(\mathbb{R}^n)$ which *do not depend* on $b\in BMO(\mathbb{R}^n)$ and $T\in\mathcal{K}_L$, such that $[b,T]$ maps continuously these spaces into $L^1(\mathbb{R}^n)$. For instance, when $L=-\Delta+1$, Theorem 7.4 of the paper state that for every $b\in BMO(\mathbb{R}^n)$ and $T\in\mathcal{K}_L$, the commutator $[b,T]$ is bounded from $H^{1,1}_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Here $H^{1,1}_L(\mathbb{R}^n)$ is the (inhomogeneous) Hardy-Sobolev space considered by Hofmann, Mayboroda and McIntosh in [65], defined as the set of functions f in $H^1_L(\mathbb{R}^n)$ such that $\partial_{x_1}f,...,\partial_{x_n}f\in H^1_L(\mathbb{R}^n)$ with the norm

$$\|f\|_{H^{1,1}_L}=\|f\|_{H^1_L}+\sum_{j=1}^n\|\partial_{x_j}f\|_{H^1_L}.$$

Finally, we give an open question.

Open question. *Find the set of all functions b such that the commutators $[b,R_j]$, $j=1,...,n$, are bounded on $H^1_L(\mathbb{R}^n)$.*

1.5 Paper V: Bilinear decompositions for the product space $H^1_L(\mathbb{R}^n)\times BMO_L(\mathbb{R}^n)$

Let $L=-\Delta+V$ be a Schrödinger operator as in Paper IV. Namely, V is a nonnegative potential, $V\neq 0$, and belongs to the reverse Hölder class $RH_{n/2}$. In [45], Dziubański et al. showed that the dual of $H^1_L(\mathbb{R}^n)$ can be identified with the space $BMO_L(\mathbb{R}^n)$ which consists of all functions $f \in BMO(\mathbb{R}^n)$ with

$$\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x)\leq r} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy < \infty.$$

As for the classical spaces $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, the pointwise products fg of functions $f \in H^1_L(\mathbb{R}^n)$ and functions $g \in BMO_L(\mathbb{R}^n)$ maybe not integrable. However, similarly to the classical setting, Li and Peng showed in [91] that such products can be defined in the sense of distributions which action on the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is

$$\langle f \times g, \varphi\rangle := \langle \varphi g, f\rangle, \tag{1.16}$$

where the second bracket stands for the duality bracket between $H^1_L(\mathbb{R}^n)$ and its dual $BMO_L(\mathbb{R}^n)$. Moreover, they proved that $f \times g$ can be written as the sum of two distributions, one in $L^1(\mathbb{R}^n)$, the other in $H^{\Xi}_{L,\sigma}(\mathbb{R}^n)$ the weighted Hardy-Orlicz space associated with L related to the Orlicz function $\Xi(t) \equiv \frac{t}{\log(e+t)}$ and the weight $\sigma(x) \equiv \frac{1}{\log(e+|x|)}$. Namely, $H^{\Xi}_{L,\sigma}(\mathbb{R}^n)$ is the completion of

$$\{f \in L^2(\mathbb{R}^n) : \mathcal{M}_L f \in L^{\Xi}_{\sigma}(\mathbb{R}^n)\}$$

in the norm

$$\|f\|_{H^{\Xi}_{L,\sigma}} := \left\{ \lambda > 0 : \int\limits_{\mathbb{R}^n} \frac{\frac{\mathcal{M}_L f(x)}{\lambda}}{\log\left(e + \frac{\mathcal{M}_L f(x)}{\lambda}\right)} \frac{1}{\log(e+|x|)} dx \leq 1 \right\}.$$

More precisely, in [91], the authors proved the following.

Theorem D. *For each $f \in H^1_L(\mathbb{R}^n)$, there are two bounded linear operators $L_f : BMO_L(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $H_f : BMO_L(\mathbb{R}^n) \to H^{\Xi}_{L,\sigma}(\mathbb{R}^n)$ such that for every $g \in BMO_L(\mathbb{R}^n)$, we have*

$$f \times g = L_f(g) + H_f(g) \tag{1.17}$$

and the uniform bound

$$\|L_f(g)\|_{L^1} + \|H_f(g)\|_{H^{\Xi}_{L,\sigma}} \leq C\|f\|_{H^1_L}\|g\|_{BMO^+_L}, \tag{1.18}$$

where $\|g\|_{BMO^+_L} = \|g\|_{BMO_L} + |g_{\mathbb{B}}|$, $g_{\mathbb{B}}$ denotes the mean value of g over the unit ball $\mathbb{B}$.

In Paper V, we prove the following theorem.

Theorem 16. *There are two bounded bilinear operators* $S_L : H^1_L(\mathbb{R}^n) \times BMO_L(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ *and* $T_L : H^1_L(\mathbb{R}^n) \times BMO_L(\mathbb{R}^n) \to H^{\log}(\mathbb{R}^n)$ *such that for every* $(f,g) \in H^1_L(\mathbb{R}^n) \times BMO_L(\mathbb{R}^n)$*, we have*

$$f \times g = S_L(f,g) + T_L(f,g) \tag{1.19}$$

and the uniform bound

$$\|S_L(f,g)\|_{L^1} + \|T_L(f,g)\|_{H^{\log}} \leq C\|f\|_{H^1_L}\|g\|_{BMO_L}. \tag{1.20}$$

Note that $H^{\log}(\mathbb{R}^n) \subset H^{\Xi}_{\sigma}(\mathbb{R}^n) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^n)$ with continuous embeddings. Compared with the main result of [91] (Theorem D), Theorem 16 makes an essential improvement in two directions as in the classical case (Theorem 1). The first one consists in proving that the space $H^{\Xi}_{L,\sigma}(\mathbb{R}^n)$ can be replaced by a smaller space $H^{\log}(\mathbb{R}^n)$. Secondly, we give the bilinear decomposition (6.7) for the product space $H^1_L(\mathbb{R}^n) \times BMO_L(\mathbb{R}^n)$ instead of the linear decomposition (6.5) depending on $f \in H^1_L(\mathbb{R}^n)$. Moreover, we just need the BMO_L-norm (see (6.8)) instead of the BMO^+_L-norm as in (6.6).

In applications to nonlinear PDEs, the distribution $f \times g \in \mathcal{S}'(\mathbb{R}^n)$ is used to justify weak continuity properties of the pointwise product fg. It is therefore important to recover fg from the action of the distribution $f \times g$ on the test functions. An idea that naturally comes to mind is to look at the mollified distributions

$$(f \times g)_\epsilon = (f \times g) * \phi_\epsilon, \tag{1.21}$$

and let $\epsilon \to 0$. Here $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$.

In the classical setting of $f \in H^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, Bonami et al. proved in [15] that the limit (6.10) exists and equals fg almost everywhere. An analogous result is also true for the Schrödinger setting. Namely, we prove the following.

Theorem 17. *Let* $f \in H^1_L(\mathbb{R}^n)$ *and* $g \in BMO_L(\mathbb{R}^n)$*. Then, for almost every* $x \in \mathbb{R}^n$*,*

$$\lim_{\epsilon \to 0} (f \times g)_\epsilon(x) = f(x)g(x).$$

1.6 Paper VI: On weak*-convergence in $H^1_L(\mathbb{R}^n)$

A famous and classical result of Fefferman [47] states that the John-Nirenberg space $BMO(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$. It is also well-known that $H^1(\mathbb{R}^n)$ is one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $VMO(\mathbb{R}^n)$ denote the closure of the space $C^\infty_c(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, where $C^\infty_c(\mathbb{R}^n)$ is the set of C^∞-functions with compact support, Coifman and Weiss showed in

[32] that $H^1(\mathbb{R}^n)$ is the dual space of $VMO(\mathbb{R}^n)$, which gives $H^1(\mathbb{R}^n)$ a richer structure than $L^1(\mathbb{R}^n)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^n)$, but bounded on $H^1(\mathbb{R}^n)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^n)$, which is useful in the application of Hardy spaces to compensated compactness (see [33]). More precisely, in [78], Jones and Journé proved the following.

Theorem E. *Suppose that $\{f_j\}_{j\geq 1}$ is a bounded sequence in $H^1(\mathbb{R}^n)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^n$. Then, $f \in H^1(\mathbb{R}^n)$ and $\{f_j\}_{j\geq 1}$ weak*-converges to f, that is, for every $\varphi \in VMO(\mathbb{R}^n)$, we have*

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f_j(x)\varphi(x)dx = \int_{\mathbb{R}^n} f(x)\varphi(x)dx.$$

Let $L = -\Delta + V$ be the Schrödinger operators as in Paper IV. Recently, Deng et al. [37] introduced and developed new function spaces of VMO-type $VMO_A(\mathbb{R}^n)$ associated with some operators A which have a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. When $A \equiv L$, their space $VMO_L(\mathbb{R}^n)$ is just the set of all functions f in $BMO_L(\mathbb{R}^n)$ such that $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^n, t\leq r}\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_2(f) = \lim_{R\to\infty}\left(\sup_{x\in\mathbb{R}^n, t\geq R}\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_3(f) = \lim_{R\to\infty}\left(\sup_{B(x,t)\cap B(0,R)=\emptyset}\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2dy\Big)^{1/2}\right).$$

The authors in [37] further showed that $H^1_L(\mathbb{R}^n)$ is in fact the dual of $VMO_L(\mathbb{R}^n)$, which allows us to study the weak*-convergence in $H^1_L(\mathbb{R}^n)$. This is useful in the study of the Hardy estimates for commutators of singular integral operators related to L, see for example Theorem 7.1 and Theorem 7.3 of [82].

In this paper, we prove the following.

Theorem 18. *The space $C^\infty_c(\mathbb{R}^n)$ is dense in the space $VMO_L(\mathbb{R}^n)$.*

Furthermore, the weak*-convergence is true in $H^1_L(\mathbb{R}^n)$.

Theorem 19. *Suppose that* $\{f_j\}_{j\geq 1}$ *is a bounded sequence in* $H^1_L(\mathbb{R}^n)$*, and that* $f_j(x) \to f(x)$ *for almost every* $x \in \mathbb{R}^n$*. Then,* $f \in H^1_L(\mathbb{R}^n)$ *and* $\{f_j\}_{j\geq 1}$ *weak*-converges to* f*, that is, for every* $\varphi \in VMO_L(\mathbb{R}^n)$*, we have*

$$\lim_{j\to\infty} \int\limits_{\mathbb{R}^n} f_j(x)\varphi(x)dx = \int\limits_{\mathbb{R}^n} f(x)\varphi(x)dx.$$

Chapter 2

Paraproducts and Products of functions in $BMO(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)$ through wavelets

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Résumé

Dans cet article, nous prouvons que le produit (dans le sens des distributions) de deux fonctions, qui sont respectivement dans $BMO(\mathbb{R}^n)$ et $\mathcal{H}^1(\mathbb{R}^n)$, peut être écrit comme la somme de deux opérateurs bilinéaires continus, l'un de $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$, l'autre de $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ à valeurs dans un nouveau type d'espace de Hardy-Orlicz désigné par $\mathcal{H}^{\log}(\mathbb{R}^n)$. Plus précisément, l'espace $\mathcal{H}^{\log}(\mathbb{R}^n)$ est l'ensemble des distributions f dont la fonction "grand maximale" $\mathcal{M}f$ satisfait

$$\int_{\mathbb{R}^n} \frac{|\mathcal{M}f(x)|}{\log(e+|x|)+\log(e+|\mathcal{M}f(x)|)}dx < \infty.$$

Les deux opérateurs bilinéaires peuvent être définis en termes de paraproduits. En conséquence, nous obtenons un lemme div-curl impliquant l'espace $\mathcal{H}^{\log}(\mathbb{R}^n)$.

2.1 Introduction

Products of functions in $\mathcal{H}^1$ and BMO have been considered by Bonami, Iwaniec, Jones and Zinsmeister in [15]. Such products make sense as distributions, and can be written as the sum of an integrable function and a function in a weighted Hardy-Orlicz space. To be more precise, for $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, we define the product (in the distribution sense) fg as the distribution whose action on the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\langle fg, \varphi \rangle := \langle \varphi g, f \rangle \,, \tag{2.1}$$

where the second bracket stands for the duality bracket between $\mathcal{H}^1(\mathbb{R}^n)$ and its dual $BMO(\mathbb{R}^n)$. It is then proven in [15] that

$$fg \in L^1(\mathbb{R}^n) + \mathcal{H}^{\Phi}_{\omega}(\mathbb{R}^n). \tag{2.2}$$

Here $\mathcal{H}^{\Phi}_{\omega}(\mathbb{R}^n)$ is the weighted Hardy-Orlicz space related to the Orlicz function

$$\Phi(t) := \frac{t}{\log(e+t)} \tag{2.3}$$

and with weight $\omega(x) := (\log(e+|x|))^{-1}$.

Our aim is to improve this result in many directions. The first one consists in proving that the space $\mathcal{H}^{\Phi}_{\omega}(\mathbb{R}^n)$ can be replaced by a smaller space. More precisely, we define the Musielak-Orlicz space $L^{\log}(\mathbb{R}^n)$ as the space of measurable functions f such that

$$\int\limits_{\mathbb{R}^n} \frac{|f(x)|}{\log(e+|x|) + \log(e+|f(x)|)} dx < \infty.$$

The space $\mathcal{H}^{\log}(\mathbb{R}^n)$ is then defined, as usual, as the space of tempered distributions for which the grand maximal function is in $L^{\log}(\mathbb{R}^n)$. This is a particular case of a Hardy space of Musielak-Orlicz type, with a variable (in x) Orlicz function that is also called a Musielak-Orlicz function (see [81]). This kind of space had not yet been considered. A systematic study of Hardy spaces of Musielak-Orlicz type has been done separately by the last author [81]. It generalizes the work of Janson [75] on Hardy-Orlicz spaces. In particular, it is proven there that the dual of the space $\mathcal{H}^{\log}(\mathbb{R}^n)$ is the generalized BMO space that has been introduced by Nakai and Yabuta (see [116]) to characterize multipliers of $BMO(\mathbb{R}^n)$. Remark that by duality with our result, functions f that are bounded and in the dual of $\mathcal{H}^{\log}(\mathbb{R}^n)$ are multipliers of $BMO(\mathbb{R}^n)$. By the theorem of Nakai and Yabuta there are no other multipliers, which, in some sense, indicates that $\mathcal{H}^{\log}(\mathbb{R}^n)$ could not be replaced by a smaller space.

Secondly we answer a question of [15] by proving that there exists continuous bilinear operators that allow to split the product into an $L^1(\mathbb{R}^n)$ part and a part in this Hardy-Orlicz space $\mathcal{H}^{\log}(\mathbb{R}^n)$. More precisely we have the following.

Theorem 20. *There exists two continuous bilinear operators on the product space* $\mathcal{H}^1(\mathbb{R}^n)\times BMO(\mathbb{R}^n)$*, respectively* $S : \mathcal{H}^1(\mathbb{R}^n)\times BMO(\mathbb{R}^n) \mapsto L^1(\mathbb{R}^n)$ *and* $T : \mathcal{H}^1(\mathbb{R}^n)\times BMO(\mathbb{R}^n) \mapsto \mathcal{H}^{\log}(\mathbb{R}^n)$ *such that*

$$fg = S(f,g) + T(f,g). \tag{2.4}$$

The operators S and T are defined in terms of a wavelet decomposition. The operator T is defined in terms of paraproducts. There is no uniqueness, of course. In fact, the same decomposition of the product fg has already been considered by Dobyinsky and Meyer (see [43, 41, 42], and also [28, 26]). The action of replacing the product by the operator T was called by them a *renormalization* of the product. Namely, T preserves the cancellation properties of the factor, while S does not. Dobyinsky and Meyer considered L^2-data for both factors, and showed that $T(f,g)$ is in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. What is surprising in our context is that both terms inherit some properties of the factors. Even if the product fg is not integrable, the function $S(f,g)$ is, while $T(f,g)$ inherits cancellation properties of functions in Hardy spaces without being integrable. So, in some way each term has more properties than expected at first glance.

Another implicit conjecture of [15] concerns bilinear operators with cancellations, such as the ones involved in the div-curl lemma for instance. In this case it is expected that there is no L^1-term. To illustrate this phenomenon, it has been proven in [11] that, whenever F and G are two vector fields respectively in $\mathcal{H}^1(\mathbb{R}^n,\mathbb{R}^n)$ and $BMO(\mathbb{R}^n,\mathbb{R}^n)$ such that F is curl-free and G is div-free, then their scalar product $F\cdot G$ is in $\mathcal{H}^{\Phi}_w(\mathbb{R}^n,\mathbb{R}^n)$ (in fact there is additional assumption on the BMO-factor). By using the same technique as Dobyinsky to deal with the terms coming from S, we give a new proof, without any additional assumption. Namely, we have the following.

Theorem 21. *Let F and G be two vector fields, one of them in* $\mathcal{H}^1(\mathbb{R}^n,\mathbb{R}^n)$ *and the other one in* $BMO(\mathbb{R}^n,\mathbb{R}^n)$*, such that* $\operatorname{curl} F = 0$ *and* $\operatorname{div} G = 0$*. Then their scalar product* $F\cdot G$ *(in the distribution sense) is in* $\mathcal{H}^{\log}(\mathbb{R}^n)$*.*

In Section 2 we introduce the spaces $L^{\log}(\mathbb{R}^n)$ and $\mathcal{H}^{\log}(\mathbb{R}^n)$ and give the generalized Hölder inequality that plays a central role when dealing with products of functions respectively in $L^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. In Sections 3 and 4 we give prerequisites on wavelets and recall the L^2-estimates of Dobyinsky. We prove Theorem 20 in Section 5 and Theorem 21 in Section 6.

2.2 The space $\mathcal{H}^{\log}(\mathbb{R}^n)$ and a generalized Hölder inequality

We first define the (variable) Orlicz function

$$\theta(x,t) := \frac{t}{\log(e+|x|)+\log(e+t)}$$

for $x \in \mathbb{R}^n$ and $t > 0$. For fixed x it is an increasing function while $t \mapsto \theta(x,t)/t$ decreases. We have $p < 1$ in the following inequalities satisfied by θ.

$$\theta(x,st) \quad \leq C_p s^p \theta(x,t) \qquad 0 < s < 1 \tag{2.5}$$

$$\theta(x,st) \quad \leq s\theta(x,t) \qquad s > 1. \tag{2.6}$$

These two properties are among the ones that are usually required for (constant) Orlicz functions in Hardy Theory, see for instance [75, 12, 81]. They guarantee, in particular, that $L^{\log}(\mathbb{R}^n)$, defined as the set of functions f such that

$$\int_{\mathbb{R}^n} \theta(x,|f(x)|)dx < \infty$$

is a vector space. For $f \in L^{\log}(\mathbb{R}^n)$, we define

$$\|f\|_{L^{\log}} := \inf\{\lambda > 0 \; ; \int_{\mathbb{R}^n} \theta(x,|f(x)|/\lambda)dx \leq 1\}.$$

It is not a norm, since it is not sub-additive. In place of sub-additivity, there exists a constant C such that, for $f, g \in L^{\log}(\mathbb{R}^n)$,

$$\|f+g\|_{L^{\log}} \leq C(\|f\|_{L^{\log}} + \|g\|_{L^{\log}}).$$

On the other hand, it is homogeneous.

The space $L^{\log}(\mathbb{R}^n)$ is a complete metric space, with the distance given by

$$\mathrm{dist}\,(f,g) := \inf\{\lambda > 0 \; ; \int_{\mathbb{R}^n} \theta(x,|f(x)-g(x)|/\lambda)dx \leq \lambda\}$$

(see [120], from which proofs can be adapted, and [81]). Because of (2.5), a sequence f_k tends to 0 in $L^{\log}(\mathbb{R}^n)$ for this distance if and only if $\|f_k\|_{L^{\log}}$ tends to 0.

Before stating our first proposition on products, we need some notations related to the space $BMO(\mathbb{R}^n)$. For Q a cube of $\mathbb{R}^n$ and f a locally integrable function, we note f_Q the mean of f on Q. We recall that a function f is in $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| dx < \infty.$$

We note $\mathbb{Q} := [0,1)^n$ and, for f a function in $BMO(\mathbb{R}^n)$,

$$\|f\|_{BMO^+} := |f_{\mathbb{Q}}| + \|f\|_{BMO}.$$

This is a norm, while the BMO norm is only a norm on equivalent classes modulo constants.

The aim of this section is to prove the following proposition, which replaces Hölder Inequality in our context.

Proposition 1. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$. Then the product fg is in $L^{\log}(\mathbb{R}^n)$. Moreover, there exists some constant C such that*

$$\|fg\|_{L^{\log}} \leq C\|f\|_{L^1}\|g\|_{BMO^+}.$$

Proof. It is easy to adapt the proof given in [15], which leads to a weaker statement. We prefer to give a complete proof here, which has the advantage to be easier to follow than the one given in [15]. We first restrict to functions f of norm 1 and functions g such that $g_{\mathbb{Q}} = 0$ and $\|g\|_{BMO} \leq \alpha$ for some uniform constant α. Let us prove in this case the existence of a uniform constant δ such that

$$\int_{\mathbb{R}^n} \theta(x, |f(x)g(x)|) dx \leq \delta. \tag{2.7}$$

The constant α is chosen so that, by John-Nirenberg inequality, one has the inequality

$$\int_{\mathbb{R}^n} \frac{e^{|g|}}{(e+|x|)^{n+1}} dx \leq \kappa,$$

with κ a uniform constant that depends only of the dimension n (see [128]). Our main tool is the following lemma.

Lemma 1. *Let $M \geq 1$. The following inequality holds for $s, t > 0$,*

$$\frac{st}{M + \log(e + st)} \leq e^{t-M} + s. \tag{2.8}$$

Proof. By monotonicity it is sufficient to consider the case when $s = e^{t-M}$. More precisely, it is sufficient to prove that

$$\frac{t}{M + \log(e + te^{t-M})} \leq 1.$$

This is direct when $t \leq M$. Now, for $t \geq M$, the denominator is bounded below by $M + t - M$, that is, by t. $\square$

Let us go back to the proof of the proposition. We choose $M := (n+1)\log(e + |x|)$. Then

$$\frac{|f(x)g(x)|}{(n+1)(\log(e+|x|) + \log(e + |f(x)g(x)|))} \leq \frac{e^{|g(x)|}}{(e+|x|)^{n+1}} + |f(x)|.$$

After integration we get (2.7) with $\delta = (n+1)(\kappa+1)$. Let us then assume that $|g_{\mathbb{Q}}| \leq \alpha$ while the other assumptions on f and g are the same. We then write $fg = fg_{\mathbb{Q}} + f(g - g_{\mathbb{Q}})$ and find again the estimate (2.7) with $\delta = (n+1)(\kappa+1)+\alpha$. Using (2.5), this means that, for $\|f\|_{L^1} = 1$ and $\|g\|_{BMO^+} = \alpha$ and for $p < 1$, we have the inequality $\|fg\|_{L^{\log}} \leq (\delta C_p)^{1/p}$. The general case follows by homogeneity, with $C = \delta\alpha^{-1}$.

$\square$

Remark that we only used the fact that g is in the exponential class for the weight $(e + |x|)^{-(n+1)}$.

Finally let us define the space $\mathcal{H}^{\log}(\mathbb{R}^n)$. We first define the grand maximal function of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ as follows. Let $\mathcal{F}$ be the set of functions Φ in $\mathcal{S}(\mathbb{R}^n)$ such that $|\Phi(x)| + |\nabla\Phi(x)| \leq (1 + |x|)^{-(n+1)}$. For $t > 0$, let $\Phi_t(x) := t^{-n}\Phi(\frac{x}{t})$. Then

$$\mathcal{M}f(x) := \sup_{\Phi \in \mathcal{F}} \sup_{t>0} |f * \Phi_t(x)|. \tag{2.9}$$

By analogy with Hardy-Orlicz spaces, we define the space $\mathcal{H}^{\log}(\mathbb{R}^n)$ as the space of tempered distributions such that $\mathcal{M}f$ in $L^{\log}(\mathbb{R}^n)$. We need the fact that $\mathcal{H}^{\log}(\mathbb{R}^n)$ is a complete metric space. Convergence in $\mathcal{H}^{\log}(\mathbb{R}^n)$ implies convergence in distribution. The space $\mathcal{H}^1(\mathbb{R}^n)$, that is, the space of functions $f \in L^1(\mathbb{R}^n)$ such that $\mathcal{M}f$ in $L^1(\mathbb{R}^n)$, is strictly contained in $\mathcal{H}^{\log}(\mathbb{R}^n)$.

2.3 Prerequisites on Wavelets

Let us consider a wavelet basis of $\mathbb{R}$ with compact support. More explicitly, we are first given a $\mathcal{C}^1(\mathbb{R})$ wavelet in Dimension one, called ψ, such that $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ form an $L^2(\mathbb{R})$ basis. We assume that this wavelet basis comes for a multiresolution analysis (MRA) on $\mathbb{R}$, as defined below (see [107]).

Definition 2.3.1. *A multiresolution analysis (MRA) on $\mathbb{R}$ is defined as an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ with the following four properties*

i) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ *and* $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

ii) for every $f \in L^2(\mathbb{R})$ and every $j \in \mathbb{Z}$, $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,

iii) for every $f \in L^2(\mathbb{R})$ and every $k \in \mathbb{Z}$, $f(x) \in V_0$ if and only if $f(x-k) \in V_0$,

iv) there exists a function $\phi \in L^2(\mathbb{R})$, called the scaling function, such that the family $\{\phi_k(x) = \phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

It is classical that, when given an (MRA) on $\mathbb{R}$, one can find a wavelet ψ such that $\{2^{j/2}\psi(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j, the orthogonal complement of V_j in V_{j+1}. Moreover, by Daubechies theorem (see [36]), it is possible to find a suitable (MRA) so that ϕ and ψ are $\mathcal{C}^1(\mathbb{R})$ and compactly supported, ψ has mean 0 and $\int x\psi(x)dx = 0$, which is known as the moment condition. We could content ourselves, in the following theorems, to have ϕ and ψ decreasing sufficiently rapidly at ∞, but proofs are simpler with compactly supported wavelets. More precisely we assume that ϕ and ψ are supported in the interval $1/2 + m(-1/2, +1/2)$, which is obtained from $(0,1)$ by a dilation by m centered at $1/2$.

Going back to $\mathbb{R}^n$, we recall that a wavelet basis of $\mathbb{R}^n$ is found as follows. We call E the set $E = \{0,1\}^n \setminus \{(0,\cdots,0)\}$ and, for $\lambda \in E$, state $\psi^\lambda(x) = \phi^{\lambda_1}(x_1)\cdots\phi^{\lambda_n}(x_n)$, with $\phi^{\lambda_j}(x_j) = \phi(x_j)$ for $\lambda_j = 0$ while $\phi^{\lambda_j}(x_j) = \psi(x_j)$ for $\lambda_j = 1$. Then the set $\{2^{nj/2}\psi^\lambda(2^j x - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,\lambda\in E}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. As it is classical, for I a dyadic cube of $\mathbb{R}^n$, which may be written as the set of x such that $2^j x - k \in (0,1)^n$, we note

$$\psi_I^\lambda(x) = 2^{nj/2}\psi^\lambda(2^j x - k).$$

We also note $\phi_I = 2^{nj/2}\phi_{(0,1)^n}(2^j x - k)$, with $\phi_{(0,1)^n}$ the scaling function in n variables, given by $\phi_{(0,1)^n}(x) = \phi(x_1)\cdots\phi(x_n)$. In the sequel, the letter I always refers to dyadic cubes. Moreover, we note kI the cube of same center dilated by the coefficient k. Because of the assumption on the supports of ϕ and ψ, the functions ψ_I^λ and ϕ_I are supported in the cube mI.

The wavelet basis $\{\psi_I^\lambda\}$, obtained by letting I vary among dyadic cubes and λ in E, comes from an (MRA) in $\mathbb{R}^n$, which we still note $\{V_j\}_{j\in\mathbb{Z}}$, obtained by taking tensor products of the one dimensional ones. The functions ϕ_I, taken for a fixed length $|I| = 2^{-jn}$, form a basis of V_j. As in the one dimensional case we note W_j the orthogonal complement of V_j in V_{j+1}. As it is classical, we note P_j the orthogonal projection onto V_j and Q_j the orthogonal projection onto W_j. In particular,

$$\begin{aligned} f &= \sum_{i\in\mathbb{Z}} Q_i f \\ &= P_j f + \sum_{i\geq j} Q_i f. \end{aligned}$$

2.4 The L^2-estimates for the product of two functions

We summarize here the main results of Dobyinsky [42].

Let us consider two L^2-functions f and g, which we express through their wavelet expansions, for instance

$$f = \sum_{\lambda\in E}\sum_{I}\langle f, \psi_I^\lambda\rangle \psi_I^\lambda.$$

Then, when f and g have a finite wavelet expansion, we have

$$\begin{aligned} fg &= \sum_{j\in\mathbb{Z}}(P_j f)(Q_j g) + \sum_{j\in\mathbb{Z}}(Q_j f)(P_j g) + \sum_{j\in\mathbb{Z}}(Q_j f)(Q_j g) \\ &:= \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g). \end{aligned} \tag{2.10}$$

The two operators Π_1 and Π_2 are called paraproducts. A posteriori each term of Formula (2.10) can be given a meaning for all functions $f, g \in L^2(\mathbb{R}^n)$. Indeed the two operators Π_1 and Π_2, which coincide, up to permutation of f and g, extend as bilinear operators from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $\mathcal{H}^1(\mathbb{R}^n)$, see [42], while the operator Π_3 extends to an operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

The two L^2 estimates are given in the following two lemmas. We sketch their proof for the convenience of the reader as this will be the basis of our proofs in the context of $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Details may be found in [42].

Lemma 2. *The bilinear operator Π_3 is a bounded operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

Proof. The series $\sum_{j\in\mathbb{Z}} Q_j f Q_j g$ is normally convergent in $L^1(\mathbb{R}^n)$, with

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \|Q_j f Q_j g\|_{L^1} &\leq \sum_{j\in\mathbb{Z}} \|Q_j f\|_{L^2}\|Q_j g\|_{L^2} \\ &\leq \Big(\sum_{j\in\mathbb{Z}}\|Q_j f\|_{L^2}^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}\|Q_j g\|_{L^2}^2\Big)^{1/2} \\ &\leq C\|f\|_{L^2}\|g\|_{L^2}. \end{aligned}$$

This concludes for Π_3. □

Lemma 3. *The bilinear operator Π_1, a priori well defined for f and g having a finite wavelet expansion, extends to $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into a bounded operator to $\mathcal{H}^1(\mathbb{R}^n)$.*

Proof. Let us recall that one can write

$$P_j f = \sum_{|I|=2^{-jn}} \langle f, \phi_I \rangle \phi_I .$$

This means that $P_j f Q_j g$ can be written as a linear combination of $\psi_I^\lambda \phi_{I'}$, with $|I| = |I'| = 2^{-jn}$. As before, for fixed I, this function is non zero only for a finite number of I'. More precisely, such I's can be written as $k2^{-j} + I$, with $k \in K$, where K is the set of points with integer coordinates contained in $(-m, +m]^n$. So $\Pi_1(f, g)$ can be written as a sum in $\lambda \in E$ and $k \in K$ of

$$F_{\lambda,k} := \sum_{j\in\mathbb{Z}} \sum_{|I|=2^{-jn}} \langle f, \phi_{k2^{-j}+I} \rangle \langle g, \psi_I^\lambda \rangle \phi_{k2^{-j}+I} \psi_I^\lambda .$$

At this point, we use the fact that the functions $|I|^{1/2} \phi_{k2^{-j}+I} \psi_I^\lambda$ are of mean zero because of the orthogonality of V_j and W_j. Moreover they are of class $\mathcal{C}^1(\mathbb{R}^n)$ and are obtained from the one for which $I = (0, 1)^n$ through the same process of dilation and translation as the wavelets. So they form what is called a system of molecules. It is well-known (see Meyer's book [107]) that such a linear combination of molecules has its $\mathcal{H}^1$-norm bounded by C times the $\mathcal{H}^1$-norm of the linear combination of wavelets with the same coefficients. Namely, we are linked to prove that

$$\| \sum_j \sum_{|I|=2^{-jn}} \sum_{\lambda\in E} \langle f, \phi_{k2^{-j}+I} \rangle \langle g, \psi_I^\lambda \rangle 2^{nj/2} \psi_I^\lambda \|_{\mathcal{H}^1} \leq C \|f\|_{L^2} \|g\|_{L^2} .$$

We use the characterization of $\mathcal{H}^1(\mathbb{R}^n)$ through wavelets to bound this norm by the L^1-norm of its square function, given by

$$\left(\sum_j \sum_{|I|=2^{-jn}} \sum_{\lambda\in E} |\langle f, \phi_{k2^{-j}+I} \rangle \langle g, \psi_I^\lambda \rangle|^2 2^{nj} |I|^{-1} \chi_I \right)^{1/2} .$$

This function is bounded at x by

$$\sup_{I \ni x} |\langle f, |I|^{-1/2} \phi_I \rangle| \times \left(\sum_j \sum_{|I|=2^{-jn}} \sum_{\lambda\in E} |\langle g, \psi_I^\lambda \rangle|^2 |I|^{-1} \chi_I(x) \right)^{1/2} .$$

The first factor is bounded, up to a constant, by the Hardy Littlewood maximal function of f, which we note Mf. We conclude by using Schwarz inequality, then the maximal theorem to bound the L^2-norm of Mf by the L^2-norm of f, then the fact that the L^2-norm of the second factor is the L^2-norm of g. □

We will need the expression of $\Pi_1(f,g)$ and $\Pi_2(f,g)$ when f has a finite wavelet expansion while g in only assumed to be in $L^2(\mathbb{R}^n)$. The following lemma is immediate for g with a finite wavelet expansion, then by passing to the limit otherwise.

Lemma 4. *Assume that f has a finite wavelet expansion and $Q_j f = 0$ for $j \notin [j_0, j_1)$. For $g \in L^2(\mathbb{R}^n)$, one has*

$$\Pi_1(f,g) = \sum_{j=j_0}^{j_1-1} P_j f Q_j g + f \sum_{j \geq j_1} Q_j g \tag{2.11}$$

$$\Pi_2(f,g) = f P_{j_0} g + \sum_{j=j_0}^{j_1-1} Q_j f \left(\sum_{j_0 \leq i \leq j-1} Q_i g \right). \tag{2.12}$$

2.5 Products of functions in $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$

Let us first recall the wavelet characterization of $BMO(\mathbb{R}^n)$: if g is in $BMO(\mathbb{R}^n)$, then for all (not necessarily dyadic) cubes R , we have that

$$\Big(|R|^{-1} \sum_{\lambda \in E} \sum_{I \subset R} |\langle g, \psi_I^\lambda \rangle|^2 \Big)^{1/2} \leq C \|g\|_{BMO},$$

and the supremum over all cubes R of the left hand side is equivalent to the BMO-norm of g.

Remark that the wavelet coefficients of a function g in $BMO(\mathbb{R}^n)$ are well defined since g is locally square integrable. The $\langle g, \phi_I \rangle$'s are well defined as well. So $Q_j g$ makes sense, as well as $P_j g$. Indeed, they are sums of the corresponding series in ψ_I^λ or ϕ_I with $|I| = 2^{-jn}$, and at each point only a finite number of terms are non zero.

Moreover, we claim that (2.11) and (2.12) are well defined for f with a finite wavelet expansion and g in $BMO(\mathbb{R}^n)$. This is direct for $\Pi_2(f,g)$. For $\Pi_1(f,g)$, it is sufficient to see that the series $\sum_{j \geq j_1} Q_j g$ converges in $L^2(R)$, where R is a large cube containing the support of f. This comes from the wavelet characterization of $BMO(\mathbb{R}^n)$. Indeed, on R one has

$$\sum_{j_1 \leq j \leq k} Q_j g = \sum_{\lambda \in E} \sum_{I \subset mR, 2^{-nk} \leq |I| \leq 2^{-nj_1}} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda.$$

This is the partial sum of an orthogonal series, that converges in $L^2(\mathbb{R}^n)$.

As a final remark, we find the same expressions for $\Pi_1(f,g)$, $\Pi_2(f,g)$, $\Pi_3(f,g)$ and fg when g is replaced by ηg, where η is a smooth compactly supported function such that η is equal to 1 on a large cube R. Just take R sufficiently large to contain the supports of f, $Q_j f$, and all functions ϕ_I and ψ_I^λ that lead to a non zero contribution in the expressions of the four functions under consideration. Since ηg is in $L^2(\mathbb{R}^n)$, we have the identity (2.10). This leads to the identity

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g). \tag{2.13}$$

So Theorem 20 will be a consequence of the boundedness of the operators $\Pi_1(f,g)$, $\Pi_2(f,g)$ and $\Pi_3(f,g)$.

Before considering this boundedness, we describe the atomic decomposition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, which will play a fundamental role in the proofs.

We recall that a function a is called a (classical) atom of $\mathcal{H}^1(\mathbb{R}^n)$ related to the (not necessarily dyadic) cube R if a is in $L^2(\mathbb{R}^n)$, is supported in R, has mean zero and is such that $\|a\|_{L^2} \leq |R|^{-1/2}$.

For simplicity we will consider atoms that are adapted to the wavelet basis under consideration. More precisely, we call the function a a ψ-atom related to the dyadic cube Q if it is an L^2-function that may be written as

$$a = \sum_{I \subset R} \sum_{\lambda \in E} a_{I,\lambda} \psi_I^\lambda \tag{2.14}$$

such that, moreover, $\|a\|_{L^2} \leq |R|^{-1/2}$. Remark that a is compactly supported in mR and has mean 0, so that it is a classical atom related to mR, up to the multiplicative constant $m^{n/2}$. It is standard that an atom is in $\mathcal{H}^1(\mathbb{R}^n)$ with norm bounded by a uniform constant. The atomic decomposition gives the converse.

Theorem 2.5.1 (Atomic decomposition)**.** *There exists some constant C such that all functions $f \in \mathcal{H}^1(\mathbb{R}^n)$ can be written as the limit in the distribution sense and in $\mathcal{H}^1$ of an infinite sum*

$$f = \sum_\ell \mu_\ell a_\ell \tag{2.15}$$

with a_ℓ ψ-atoms related to some dyadic cubes R_ℓ and μ_ℓ constants such that

$$\sum_\ell |\mu_\ell| \leq C\|f\|_{\mathcal{H}^1}.$$

Moreover, for f with a finite wavelet series, we can choose an atomic decomposition with a finite number of atoms a_ℓ, which have also a finite wavelet expansion extracted from the one of f.

This theorem is a small variation of a standard statement. The second part may be obtained easily by taking the atomic decomposition given in [63], Section 6.5. Remark that the interest of dealing with finite atomic decompositions has been underlined recently, for instance in [105, 106].

We want now to give sense to the decomposition (2.10) for $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$. We will do it when f has a finite wavelet expansion.

Let us first consider that two operators Π_1 and Π_3.

Theorem 2.5.2. Π_3 *extends into a bounded bilinear operator from* $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ *into* $L^1(\mathbb{R}^n)$.

Proof. We consider f with a finite wavelet expansion and $g \in BMO(\mathbb{R}^n)$, so that $\Pi_3(f,g)$ is well defined as a finite sum in j. Let us give an estimate of its L^1-norm. We use the atomic decomposition of f given in (2.15), that is, $f = \sum_{\ell=1}^{L} \mu_\ell a_\ell$ where each a_ℓ is a ψ-atom related to the dyadic cube R_ℓ and $\sum_{\ell=1}^{L} |\mu_\ell| \leq C\|f\|_{\mathcal{H}^1}$. Recall that each atom has also a finite wavelet expansion extracted from the one of f. From this, it is sufficient to prove that, for a ψ-atom a, which is supported in R and has L^2-norm bounded by $|R|^{-1/2}$, we have the estimate

$$\|\Pi_3(a,g)\|_{L^1} \leq C\|g\|_{BMO}. \tag{2.16}$$

We claim that $\Pi_3(a,g) = \Pi_3(a,b)$, where $b := \sum_{\lambda \in E} \sum_{I \in 2mR} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda$. Indeed, in the wavelet expansion of g we only have to consider at each scale j the terms ψ_I^λ for which $\psi_I^\lambda \psi_{I'}^{\lambda'}$ is not identically 0 for all I' contained in R such that $|I| = |I'| = 2^{-jn}$. In other words we want $mI \cap mI' \neq \emptyset$, which is only possible for I in $2mR$. Now let us recall the wavelet characterization of $BMO(\mathbb{R}^n)$: for all cubes Q, we have that

$$\Big(|Q|^{-1} \sum_{\lambda \in E} \sum_{I \subset Q} |\langle g, \psi_I^\lambda \rangle|^2\Big)^{1/2} \leq C\|g\|_{BMO},$$

and the supremum on all cubes Q of the left hand side is equivalent to the BMO-norm of g. It follows that the L^2-norm of b is bounded by $Cm^{n/2}|R|^{1/2}\|g\|_{BMO}$. This allows to conclude for the proof of (2.16), using Lemma 2. □

Next we look at Π_1.

Theorem 2.5.3. Π_1 *extends into a bounded bilinear operator from* $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ *into* $\mathcal{H}^1(\mathbb{R}^n)$.

Proof. Again, we consider $\Pi_1(f,g)$ for f with a finite wavelet expansion, so that it is well defined by (2.11). As in the previous theorem we can consider separately each atom. So, as before, let a be such a ψ-atom. One can estimate $\Pi_1(a,g)$ as in the previous theorem. We again claim that $\Pi_1(f,g) = \Pi_1(f,b)$, where $b := \sum_{\lambda\in E}\sum_{I\in 2mR}\langle g,\psi_I^\lambda\rangle\psi_I^\lambda$. We then use Lemma 3 to conclude that

$$\|\Pi_1(a,g)\|_{\mathcal{H}^1} \leq C\|g\|_{BMO}, \tag{2.17}$$

which we wanted to prove. □

We now consider the last term.

Theorem 2.5.4. Π_2 *extends into a bounded bilinear operator from* $\mathcal{H}^1(\mathbb{R}^n)\times BMO^+(\mathbb{R}^n)$ *into* $\mathcal{H}^{\log}(\mathbb{R}^n)$.

Proof. The main point is the following lemma.

Lemma 5. *let a be a ψ-atom with a finite wavelet expansion related to the cube R and $g \in BMO(\mathbb{R}^n)$. Then we can write*

$$\Pi_2(a,g) = h^{(1)} + \kappa g_R h^{(2)} \tag{2.18}$$

where $\|h^{(1)}\|_{\mathcal{H}^1} \leq C\|g\|_{BMO}$ and $h^{(2)}$ is an atom related to mR. Here g_R is the mean of g on R and κ a uniform constant, independent of a and g.

Let us conclude from the lemma, which we take for granted for the moment. Let $f = \sum_{\ell=1}^L \mu_\ell a_\ell$ be the atomic decomposition of the function f, which has a finite wavelet expansion. Let us prove the existence of some uniform constant C such that

$$\left\|\mathcal{M}\left(\sum_{\ell=1}^L \mu_\ell \Pi_2(a_\ell,g)\right)\right\|_{L^{\log}} \leq C\|g\|_{BMO^+}\left(\sum_{\ell=1}^L|\mu_\ell|\right). \tag{2.19}$$

With obvious notations, we conclude directly for terms $h_\ell^{(1)}$, using the fact that $L^1(\mathbb{R}^n)$ is contained in $L^{\log}(\mathbb{R}^n)$. So it is sufficient to prove that

$$\left\|\mathcal{M}\left(\sum_{\ell=1}^L \mu_\ell g_{R_\ell} h_\ell^{(2)}\right)\right\|_{L^{\log}} \leq C\|g\|_{BMO^+}\left(\sum_{\ell=1}^L|\mu_\ell|\right).$$

At this point we proceed as in [15]. We use the inequality

$$\mathcal{M}\left(\sum_{\ell=1}^L \mu_\ell g_{R_\ell} h_\ell^{(2)}\right) \leq \sum_{\ell=1}^L |\mu_\ell||g_{R_\ell}|\mathcal{M}\left(h_\ell^{(2)}\right).$$

Then we write $g_{R_\ell} = g + (g_{R_\ell} - g)$. For the first term, that is,

$$|g| \left(\sum_{\ell=1}^{L} |\mu_\ell| \mathcal{M} \left(h_\ell^{(2)} \right) \right),$$

we use the generalized Hölder inequality given in Proposition 1. Indeed, g is in $BMO(\mathbb{R}^n)$ and the function $\mathcal{M}(a)$, for a an atom, is uniformly in $L^1(\mathbb{R}^n)$, so that $\sum_{\ell=1}^{L} |\mu_\ell| \mathcal{M} \left(h_\ell^{(2)} \right)$ has norm in $L^1(\mathbb{R}^n)$ bounded by $C \sum_{\ell=1}^{L} |\mu_\ell|$. To conclude for (2.19), it is sufficient to prove that

$$\left\| \sum_{\ell=1}^{L} |\mu_\ell| |g - g_{R_\ell}| \mathcal{M} \left(h_\ell^{(2)} \right) \right\|_{L^1} \leq C \sum_{\ell=1}^{L} |\mu_\ell|.$$

This is a consequence of the following uniform inequality, valid for $g \in BMO(\mathbb{R}^n)$ and a an atom adapted to the cube R:

$$\int_{\mathbb{R}^n} |g - g_R| \mathcal{M}(a)\, dx \leq C \|g\|_{BMO}.$$

To prove this inequality, by using invariance through dilation and translation, we may assume that R is the cube $\mathbb{Q}$. We conclude by using the following classical lemma.

Lemma 6. *Let a be a classical atom related to the cube $\mathbb{Q}$ and g be in $BMO(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |g - g_{\mathbb{Q}}| \mathcal{M}(a)\, dx \leq C \|g\|_{BMO}.$$

Proof. We cut the integral into two parts. By Schwarz Inequality and the boundedness of the operator $\mathcal{M}$ on $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{|x| \leq 2} |g - g_{\mathbb{Q}}| \mathcal{M}(a)\, dx &\leq C \left(\int_{2\mathbb{Q}} |g - g_{\mathbb{Q}}|^2 dx \right)^{1/2} \|a\|_{L^2} \\ &\leq C \|g\|_{BMO}, \end{aligned}$$

here one used $|g_{2\mathbb{Q}} - g_{\mathbb{Q}}| \leq C \|g\|_{BMO}$. Next, for $|x| > 2$ we have the inequality

$$\mathcal{M}(a)(x) \leq \frac{C}{(1 + |x|)^{n+1}},$$

and the classical inequality (see Stein's book [128])

$$\int_{\mathbb{R}^n} \frac{|g - g_{\mathbb{Q}}|}{(1 + |x|)^{n+1}} dx \leq C \|g\|_{BMO}.$$

We have proven (2.19). □

It remains to prove Lemma 5, which we do now.

Proof of Lemma 5. Let a be a ψ-atom which is related to the dyadic cube R. Let j_0 be such that $|R| = 2^{-nj_0}$. We assume that a has a finite wavelet expansion, so that $\Pi_2(a, g)$ is given by (2.11) for some $j_1 > j_0$. As before, we can write $\Pi_2(a, g) = aP_{j_0}g+\Pi_2(a, b)$, where b is defined by $b := \sum_{\lambda\in E}\sum_{I\in 2mR}\langle g, \psi_I^\lambda\rangle\psi_I^\lambda$. It follows again from the characterization of *BMO*-function through wavelets that the L^2-norm of b is bounded by $C\|g\|_{BMO}|R|^{1/2}$. We use the L^2-estimate given by Lemma 3 to bound uniformly the $\mathcal{H}^1$-norm of $\Pi_2(a, b)$. This term goes into $h^{(1)}$.

It remains to consider $aP_{j_0}g$. By definition of $P_{j_0}g$, it can be written as $a\sum_I\langle g, \phi_I\rangle\phi_I$, where the sum in I is extended to all dyadic cubes such that $|I| = 2^{-nj_0}$ and $mI\cap mR \neq \emptyset$. There are at most $(2m)^n$ such terms in this sum, and it is sufficient to prove that each of them can be written as $h_1 + \kappa|g_R|h_2$, with h_2 a classical atom related to mQ and h_1 such that $\|h_1\|_{\mathcal{H}^1} \leq C\|g\|_{BMO}$. Let us first remark that for each of these $(2m)^n$ terms, the function $h := |I|^{1/2}\phi_I a$ is (up to some uniform constant) a classical atom related to mR: indeed, it has mean value 0 because of the orthogonality of ϕ_I and $\psi_{I'}$ when $|I'| \leq |I|$ and the norm estimate follows at once. In order to conclude, it is sufficient to prove that $h_1 = (g_R - |I|^{-1/2}\langle g, \phi_I\rangle)h$ has the required property. We conclude easily by showing that $g_R - |I|^{-1/2}\langle g, \phi_I\rangle$ is bounded by $C\|g\|_{BMO}$. But this difference may be written as $\langle\gamma, g\rangle$, where $\gamma := |R|^{-1}\chi_R - |I|^{-1/2}\phi_I$. The function γ has zero mean, is supported in $2mR$ and has L^2-norm bounded by $2|R|^{-1/2}$. Thus, up to multiplication by some uniform constant, it is a classical atom related to the cube $2mR$. It has an $\mathcal{H}^1$-norm that is uniformly bounded and its scalar product with g is bounded by the *BMO*-norm of g, up to a constant, as a consequence of the $\mathcal{H}^1 - BMO$ duality.

This concludes for the proof.

□

We have finished the proof of Theorem 2.5.4, and also of the one of Theorem 20. Just take $S = \Pi_3$.

□

2.6 Div-Curl Lemma

The aim of this section is to prove Theorem 21. The methods that we develop are inspired by the papers of Dobyinsky in the case of $L^2(\mathbb{R}^n)$. They are generalized in a forthcoming paper of the last author [82].

Let us first make some remarks. By using the decomposition of each product F_jG_j into $S(F_j, G_j) + T(F_j, G_j)$, we already know that all terms $T(F_j, G_j)$ are in $\mathcal{H}^{\log}(\mathbb{R}^n)$. So we claim that it is sufficient to prove that $\sum_{j=1}^n S(F_j, G_j)$ is also in $\mathcal{H}^{\log}(\mathbb{R}^n)$. We first assume that F is in $\mathcal{H}^1(\mathbb{R}^n, \mathbb{R}^n)$ and G in $BMO(\mathbb{R}^n, \mathbb{R}^n)$. Since F is curl-free, we can assume that F_j is a gradient, or, equivalently, $F_j = R_jf$, where R_j is the j-th Riesz transform and $f = -\sum_{j=1}^n R_j(F_j) \in \mathcal{H}^1(\mathbb{R}^n)$ since $\mathcal{H}^1(\mathbb{R}^n)$ is invariant under Riesz transforms. Next, since G is div-free, we have the identity $\sum_{j=1}^n R_jG_j = 0$. So it is sufficient to prove that $S(R_jf, G_j) + S(f, R_jG_j)$ is in $\mathcal{H}^{\log}(\mathbb{R}^n)$ for each j. So Theorem 21 is a corollary of the following proposition.

Proposition 2. *Let A be an odd Calderón-Zygmund operator. Then, the bilinear operator $S(Af, g) + S(f, Ag)$ maps continuously $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $\mathcal{H}^1(\mathbb{R}^n)$.*

Proof. We make a first reduction, which is done by Dobyinsky in [42]. When considering $S(f, g)$ on $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we can write it as $S(f, g) = h + S_0(f, g)$ with $h \in \mathcal{H}^1(\mathbb{R}^n)$, where

$$S_0(f, g) = \sum_{\lambda \in E} \sum_I \langle f, \psi_I^\lambda \rangle \langle g, \psi_I^\lambda \rangle |\psi_I^\lambda|^2. \tag{2.20}$$

Indeed, $S(f, g) - S_0(f, g)$ may be written in terms of products $\psi_I^\lambda \psi_{I'}^{\lambda'}$, with $|I| = |I'|$, $(I, \lambda) \neq (I', \lambda')$. These functions are of mean 0 because of the orthogonality of the wavelet basis, have L^2 norm bounded, up to a constant, by $|I|^{-1/2}$, and are supported in mI. So they are C times atoms of $\mathcal{H}^1(\mathbb{R}^n)$. Recall that they are non zero only if $I' = k|I|^{1/n} + I$, with $k \in K$, where K is the set of points with integer coordinates contained in $(-m, +m]^n$. So, to prove that $S(f, g) - S_0(f, g)$ is in $\mathcal{H}^1(\mathbb{R}^n)$ it is sufficient to use the fact that, for fixed λ, λ' and k,

$$\sum_I |\langle f, \psi_I^\lambda \rangle| \, |\langle g, \psi_{k|I|^{1/n}+I}^{\lambda'} \rangle| \leq C \|f\|_{\mathcal{H}^1} \|g\|_{BMO}.$$

This is a consequence of the wavelet characterization of f in $\mathcal{H}^1(\mathbb{R}^n)$ and g in $BMO(\mathbb{R}^n)$ and the following lemma, which may be found in [49].

Lemma 7. *There exists a uniform constant C, such that, for $(a_I)_{I \in \mathcal{D}}$ and $(b_I)_{I \in \mathcal{D}}$ two sequences that are indexed by the set $\mathcal{D}$ of dyadic cubes , one has the inequality*

$$\sum_{I \in \mathcal{D}} |a_I||b_I| \leq C \left\| \left(\sum_{I \in \mathcal{D}} |a_I|^2 |I|^{-1} \chi_I \right)^{1/2} \right\|_{L^1} \times \sup_{R \in \mathcal{D}} \left(|R|^{-1} \sum_{I \subset R} |b_I|^2 \right)^{1/2}.$$

Let us come back to the proof of the proposition. From this first step, we conclude that it is sufficient to prove that $B(f,g) := S_0(Af,g) + S_0(f,Ag)$ is in $\mathcal{H}^1(\mathbb{R}^n)$. Using bilinearity as well as the fact that $A^* = -A$, we have

$$B(f,g) := \sum_{\lambda\in E}\sum_{\lambda'\in E}\sum_{I,I'}\langle f,\psi_I^\lambda\rangle\langle g,\psi_{I'}^{\lambda'}\rangle\langle A\psi_I^\lambda,\psi_{I'}^{\lambda'}\rangle(|\psi_{I'}^{\lambda'}|^2 - |\psi_I^\lambda|^2).$$

From this point, the proof is standard. An explicit computation gives that $|\psi_{I'}^{\lambda'}|^2 - |\psi_I^\lambda|^2$ is in $\mathcal{H}^1(\mathbb{R}^n)$, with

$$\||\psi_{I'}^{\lambda'}|^2 - |\psi_I^\lambda|^2\|_{\mathcal{H}^1} \le C\left(\log(2^{-j}+2^{-j'})^{-1} + \log(|x_I - x_{I'}| + 2^{-j} + 2^{-j'})\right).$$

Here $|I| = 2^{-jn}$ and $|I'| = 2^{-j'n}$, while x_I and $x_{I'}$ denote the centers of the two cubes. Next we use the well-known estimate of the matrix of a Calderón-Zygmund operator (see [18, Proposition 1]): there exists some $\delta \in (0,1]$, such that

$$|\langle A\psi_I^\lambda,\psi_{I'}^{\lambda'}\rangle| \le Cp_\delta(I,I')$$

with

$$p_\delta(I,I') = 2^{-|j-j'|(\delta+n/2)}\left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|x_I - x_{I'}|}\right)^{n+\delta}.$$

So, by using the inequality

$$\log\left(\frac{2^{-j}+2^{-j'}+|x_I - x_{I'}|}{2^{-j}+2^{-j'}}\right) \le \frac{2}{\delta}\left(\frac{2^{-j}+2^{-j'}+|x_I - x_{I'}|}{2^{-j}+2^{-j'}}\right)^{\delta/2},$$

we obtain

$$\|B(f,g)\|_{\mathcal{H}^1} \le C\sum_{\lambda,\lambda'\in E}\sum_{I,I'}|\langle f,\psi_I^\lambda\rangle|\,|\langle g,\psi_{I'}^{\lambda'}\rangle|p_{\delta'}(I,I')$$

where $\delta' = \delta/2 > 0$. We conclude by using the fact that the almost diagonal matrix $p_{\delta'}(I,I')$ defines a bounded operator on the space of all sequences $(a_I)_{I\in\mathcal{D}}$ such that $\left(\sum_I |a_I|^2|I|^{-1}\chi_I\right)^{1/2} \in L^1(\mathbb{R}^n)$, which may be found in [49].

This is the end of the proof of Theorem 21 for $F \in \mathcal{H}^1(\mathbb{R}^n,\mathbb{R}^n)$ and $G \in BMO(\mathbb{R}^n,\mathbb{R}^n)$ with $\operatorname{curl} F = 0$ and $\operatorname{div} G = 0$. Assume now that $\operatorname{div} F = 0$ and $\operatorname{curl} G = 0$. Similarly as above, we have $\sum_{j=1}^n R_jF_j = 0$ and $G_j = R_jg$ where $g = -\sum_{j=1}^n R_jG_j \in BMO(\mathbb{R}^n)$ since $BMO(\mathbb{R}^n)$ is invariant under Riesz transforms. Hence,

$$F\cdot G = \sum_{j=1}^n(T(F_j,G_j)+S(F_j,G_j)) = \sum_{j=1}^n T(F_j,G_j) + \sum_{j=1}^n(S(F_j,R_jg)+S(R_jF_j,g)).$$

We conclude as before from the proposition. $\square$

Chapter 3

New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators

Ce chapitre est un prépublication (soumise).

Résumé

Nous introduisons une nouvelle classe d'espaces de Hardy $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$,

appelés espaces de Hardy de type Musielak-Orlicz, qui généralisent les espaces de Hardy-Orlicz de Janson et les espaces Hardy à poids de García-Cuerva, Strömberg, et Torchinsky. Ici, $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ est une fonction telle que $\varphi(x, \cdot)$ est une fonction Orlicz et $\varphi(\cdot, t)$ est un poids Muckenhoupt A_∞. Une fonction f appartient à $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ si et seulement si sa fonction maximale f^* est de telle sorte que $x \mapsto \varphi(x, |f^*(x)|)$ est intégrable. Un tel espace se pose tout naturellement, par exemple dans la description du produit des fonctions dans $H^1(\mathbb{R}^n)$ et $BMO(\mathbb{R}^n)$. Nous caractérisons ces espaces grâce à la fonction de "grand maximale" et nous établissons leur décomposition atomique. Nous caractérisons aussi leurs espaces duaux. La classe de multiplicateurs ponctuels pour $BMO(\mathbb{R}^n)$ caractérisée par Nakai et Yabuta peut être vu comme le dual de $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$ où $H^{\log}(\mathbb{R}^n)$ est l'espace Hardy de type Musielak-Orlicz liée à la fonction Musielak-Orlicz $\theta(x,t) = \dfrac{t}{\log(e + |x|) + \log(e + t)}$.

En outre, sous certaines hypothèses supplémentaires sur $\varphi(\cdot, \cdot)$, nous montrons que si T est un opérateur sous-linéaire qui envoie tous les atomes dans les éléments uniformément bornés d'un quasi-espace de Banach $\mathcal{B}$, alors T se prolonge de manière unique à un opérateur borné sous-linéaire de $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ à valeurs dans $\mathcal{B}$. Ces résultats sont nouveaux,

même pour les espaces de Hardy-Orlicz classiques.

3.1 Introduction

Since Lebesgue theory of integration has taken a center stage in concrete problems of analysis, the need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$-families naturally arose. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ when $p \in (0,1]$ are good substitutes of $L^p(\mathbb{R}^n)$ when studying the boundedness of operators: for example, the Riesz operators are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0,1]$. The theory of Hardy spaces H^p on the Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss [129]. Later, Fefferman and Stein [48] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$, which now plays an important role in various fields of analysis and partial differential equations; see, for example, [32, 33, 113]. A key feature of the classical Hardy spaces is their atomic decomposition characterizations, which were obtained by Coifman [27] when $n = 1$ and Latter [86] when $n > 1$. Later, the theory of Hardy spaces and their dual spaces associated with Muckenhoupt weights have been extensively studied by García-Cuerva [52], Strömberg and Torchinsky [131] (see also [111, 22, 53]); there the weighted Hardy spaces was defined by using the nontangential maximal functions and the atomic decompositions were derived. On the other hand, as another generalization of $L^p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum-Orlicz in [10] and Orlicz in [117], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [4, 70, 104]. Moreover, the Hardy-Orlicz spaces are also good substitutes of the Orlicz spaces in dealing with many problems of analysis, say, the boundedness of operators.

Let Φ be a Orlicz function which is of positive lower type and (quasi-)concave. In [75], Janson has considered the Hardy-Orlicz space $H^\Phi(\mathbb{R}^n)$ the space of all tempered distributions f such that the nontangential grand maximal function of f is defined by

$$f^*(x) = \sup_{\phi \in \mathcal{A}_N} \sup_{|x-y|<t} |f * \phi_t(y)|,$$

for all $x \in \mathbb{R}^n$, here and in what follows $\phi_t(x) := t^{-n}\phi(t^{-1}x)$, with

$$\mathcal{A}_N = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\partial_x^\alpha \phi(x)| \leq 1 \text{ for } \alpha \in \mathbb{N}^n, |\alpha| \leq N \right\}$$

with $N = N(n, \Phi)$ taken large enough, belongs to the Orlicz space $L^\Phi(\mathbb{R}^n)$. Remark that these Hardy-Orlicz type spaces appear naturally when studying the theory of nonlinear PDEs (cf. [57, 71, 73]) since many cancellation phenomena for Jacobians cannot be observed in the usual Hardy spaces $H^p(\mathbb{R}^n)$. For instance, let $f = (f^1, ..., f^n)$ in the Sobolev class $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and the Jacobians $J(x,f)dx = df^1 \wedge \cdots \wedge df^n$, then (see Theorem 10.2 of [73])

$$\mathcal{T}(J(x,f)) \in L^1(\mathbb{R}^n) + H^{\Phi}(\mathbb{R}^n)$$

where $\Phi(t) = t/\log(e+t)$ and $\mathcal{T}(f) = f\log|f|$, since $J(x,f) \in H^1(\mathbb{R}^n)$ (cf. [33]) and $\mathcal{T}$ is well defined on $H^1(\mathbb{R}^n)$. We refer readers to [121, 72] for this interesting nonlinear operator $\mathcal{T}$.

In this paper we want to allow generalized Hardy-Orlicz spaces related to generalized Orlicz functions that may vary in the spatial variables. More precisely the Orlicz function $\Phi(t)$ is replaced by a function $\varphi(x,t)$, called Musielak-Orlicz function (cf. [115, 38]). We then define Hardy spaces of Musielak-Orlicz type. Apart from interesting theoretical considerations, the motivation to study function spaces of Musielak-Orlicz type comes from applications to elasticity, fluid dynamics, image processing, nonlinear PDEs and the calculus of variation (cf. [38, 39]).

A particular case of Hardy spaces of Musielak-Orlicz type appears naturally when considering the products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [14]); and the endpoint estimates for the div-curl lemma (see [11, 14]). More precisely, in [14] the authors proved that product of a $BMO(\mathbb{R}^n)$ function and a $H^1(\mathbb{R}^n)$ function may be written as a sum of an integrable term and of a term in $H^{\log}(\mathbb{R}^n)$, a Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$. Moreover, the corresponding bilinear operators are bounded. This result gives in particular a positive answer to the Conjecture 1.7 in [15]. By duality, one finds pointwise multipliers for $BMO(\mathbb{R}^n)$. Recall that a function g on $\mathbb{R}^n$ is called a pointwise multiplier for $BMO(\mathbb{R}^n)$, if the pointwise multiplication fg belongs to $BMO(\mathbb{R}^n)$ for all f in $BMO(\mathbb{R}^n)$. In [116], Nakai and Yabuta characterize the pointwise multipliers for $BMO(\mathbb{R}^n)$: they prove that g is a pointwise multiplier for $BMO(\mathbb{R}^n)$ if and only if g belong to $L^\infty(\mathbb{R}^n) \cap BMO^{\log}(\mathbb{R}^n)$, where

$$\begin{aligned} &BMO^{\log}(\mathbb{R}^n) = \\ = &\left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{BMO^{\log}} := \sup_{B(a,r)} \frac{|\log r| + \log(e+|a|)}{|B(a,r)|} \int\limits_{B(a,r)} |f(x) - f_{B(a,r)}|dx < \infty \right\}. \end{aligned}$$

By using the theory of these new Hardy spaces and dual spaces, we establish that the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is just the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$. Remark that the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ have also recently been used by Lerner

[88] for solving a conjecture of Diening (see [38]) on the boundedness of the Hardy-Littlewood maximal operator on the generalized Lebesgue spaces $L^{p(x)}(\mathbb{R}^n)$ (a special case of Musielak-Orlicz spaces, for the details see [38, 88]).

Motivated by all of the above mentioned facts, in this paper, we introduce a new class of Hardy spaces $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$, called Hardy spaces of Musielak-Orlicz type, which generalize the Hardy-Orlicz spaces of Janson and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky. Here, $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is a function such that $\varphi(x, \cdot)$ is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt weight A_∞. In the special case $\varphi(x, t) = w(x)\Phi(t)$ with w in the Muckenhoupt class and Φ an Orlicz function, our Hardy spaces are weighted Hardy-Orlicz spaces but they are different from the ones considered by Harboure, Salinas, and Viviani [60, 61].

As an example of our results, let us give the atomic decomposition with bounded atoms. Let φ be a *growth function* (see Section 2). A bounded function a is a φ-atom if it satisfies the following three conditions

i) supp $a \subset B$ for some ball B,

ii) $\|a\|_{L^\infty} \le \|\chi_B\|^{-1}_{L^\varphi}$,

iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $|\alpha| \le [n(\frac{q(\varphi)}{i(\varphi)} - 1)]$,

where $q(\varphi)$ and $i(\varphi)$ are the indices of φ (see Section 2). We next define the *atomic Hardy space of Musielak-Orlicz type* $H^{\varphi(\cdot,\cdot)}_{\mathrm{at}}(\mathbb{R}^n)$ as those distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j b_j$ (in the sense of $\mathcal{S}'(\mathbb{R}^n)$), where b_j*'s* are multiples of φ-atoms supported in the balls B_j*'s*, with the property $\sum_j \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty$; and define the norm of f by

$$\|f\|_{H^{\varphi(\cdot,\cdot)}_{\mathrm{at}}} = \inf\left\{\Lambda_\infty(\{b_j\}) : f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n)\right\},$$

where $\Lambda_\infty(\{b_j\}) = \inf\left\{\lambda > 0 : \sum_j \varphi\Big(B_j, \frac{\|b_j\|_{L^\infty}}{\lambda}\Big) \le 1\right\}$ with $\varphi(B, t) := \int_B \varphi(x, t)dx$ for all $t \ge 0$ and B is measurable. Then we obtain:

Theorem 3.1.1. $H^{\varphi(\cdot,\cdot)}_{\mathrm{at}}(\mathbb{R}^n) = H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ *with equivalent norms.*

The fact that $\Lambda_\infty(\{b_j\})$, which is the right expression for the (quasi-)norm in the atomic Hardy space of Musielak-Orlicz type, plays a central role in this paper. It should be emphasized that, even if the steps of the proof of such a theorem are standard, the adaptation to this context is not standard.

On the other hand, to establish the boundedness of operators on Hardy spaces, one usually appeals to the atomic decomposition characterization, see [27, 86, 133], which means that a function or distribution in Hardy spaces can be represented as a linear combination of functions of an elementary form, namely, atoms. Then, the boundedness of operators on Hardy spaces can be deduced from their behavior on atoms or molecules in principle. However, caution needs to be taken due to an example constructed in Theorem 2 of [19]. There exists a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1, \infty, 0)$-atoms into bounded scalars, but however does not extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that the uniform boundedness of a linear operator T on atoms does not automatically guarantee the boundedness of T from $H^1(\mathbb{R}^n)$ to a Banach space $\mathcal{B}$. Nevertheless, by using the grand maximal function characterization of $H^p(\mathbb{R}^n)$, Meda, Sjögren, and Vallarino [105, 106] proved that if a sublinear operator T maps all (p, q, s)-atoms when $q < \infty$ and continuous (p, ∞, s)-atoms into uniformly bounded elements of $L^p(\mathbb{R}^n)$ (see also [144, 20] for quasi-Banach spaces), then T uniquely extends to a bounded sublinear operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In this paper, we study boundedness of sublinear operators in the context of new Hardy spaces of Musielak-Orlicz type which generalize the main results in [105, 106]. More precisely, under additional assumption on $\varphi(\cdot, \cdot)$, we prove that finite atomic norms on dense subspaces of $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ are equivalent with the standard infinite atomic decomposition norms. As an application, we prove that if T is a sublinear operator and maps all atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^{\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ to $\mathcal{B}$.

In a forecoming paper, using the theory of these new Hardy spaces and ideas from [14], we study and establish some new interesting estimates of endpoint type for the commutators of singular integrals and fractional integrals on Hardy-type spaces.

Our paper is organized as follows. In Section 2 we give the notation and definitions that we shall use in the sequel. For simplicity we write φ for $\varphi(\cdot, \cdot)$. One then introduces Hardy spaces of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ via grand maximal functions, atomic Hardy spaces $H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$, finite atomic Hardy spaces $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ for any admissible triplet (φ, q, s), BMO-Musielak-Orlicz-type spaces $BMO^\varphi(\mathbb{R}^n)$, and generalized quasi-Banach spaces $\mathcal{B}_\gamma$ for $\gamma \in (0, 1]$. In Section 3 we state the main results: the atomic decompositions (Theorem 3.3.1), the duality (Theorem 3.3.2), the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ (Theorem 3.3.3), the finite atomic decomposition (Theorem 3.3.4), and the criterion for boundedness of sublinear operators in $H^\varphi(\mathbb{R}^n)$ (Theorem 3.3.5). In Section 4 we present and prove the basic properties of the growth functions φ since they provide the tools for further work with this type of functions. In Section 5 we generalize the Calderón-Zygmund decomposition associated to the grand maximal function on $\mathbb{R}^n$ in the setting of the spaces of Musielak-Orlicz type. Applying this, we further prove that for any admissible triplet (φ, q, s), $H^\varphi(\mathbb{R}^n) = H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ with equivalent norms (Theorem 3.3.1). In Section 6 we prove the dual theorem. By Theorem 2 in [19], one has to be careful with the argument *"the operator T is uniformly bounded in $H^p_w(\mathbb{R}^n)$ ($H^\varphi(\mathbb{R}^n)$ here $\varphi(x,t) = w(x).t^p$ in our context) on w-(p,∞)-atoms, and hence it extends to a bounded operator on $H^p_w(\mathbb{R}^n)$"* which has been used in [52] and [22]. In Section 7 we introduce *log-atoms* and consider the particular case of $H^{\log}(\mathbb{R}^n)$. Finally, in Section 8 we prove that $\|\cdot\|_{H^{\varphi,q,s}_{\rm fin}}$ and $\|\cdot\|_{H^\varphi}$ are equivalent quasi-norms on $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ when $q<\infty$ and on $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ when $q=\infty$, here and in what follows $C(\mathbb{R}^n)$ denotes the set of all continuous functions. Then, we consider generalized quasi-Banach spaces which generalize the notion of quasi-Banach spaces in [144] (see also [20]), and obtain criterious for boundedness of sublinear operators on $H^\varphi(\mathbb{R}^n)$.

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. The symbol $f \approx g$ means that f is equivalent to g (i.e. $C^{-1}f \le g \le Cf$), and $[\cdot]$ denotes the integer function. By X^* we denote the dual of the (quasi-)Banach space X. In $\mathbb{R}^n$, we denote by $B = B(x,r)$ an open ball with center x and radius $r>0$. For any measurable set E, we denote by χ_E its characteristic function, by $|E|$ its Lebesgue measure, and by E^c the set $\mathbb{R}^n \setminus E$.

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3.2 Notation and definitions

3.2.1 Musielak-Orlicz-type functions

First let us recall notations for Orlicz functions.

A function $\phi : [0,\infty) \to [0,\infty)$ is called *Orlicz* if it is nondecreasing and $\phi(0) = 0$; $\phi(t) > 0, t > 0$; $\lim_{t\to\infty} \phi(t) = \infty$. An Orlicz function ϕ is said to be of *lower type* (resp., *upper type*) p, $p \in (-\infty,\infty)$, if there exists a positive constant C so that

$$\phi(st) \le Cs^p \phi(t),$$

for all $t \ge 0$ and $s \in (0,1)$ (resp., $s \in [1,\infty)$). One say that ϕ is of *positive lower type* (resp., *finite upper type*) if it is of lower type (resp., upper type) p for some $p>0$ (resp., p finite).

Obviously, if ϕ is both of lower type p_1 and of upper type p_2, then $p_1 \le p_2$. Moreover, if ϕ is of lower type (resp., upper type) p then it is also of lower type (resp., upper) $\widetilde{p}$ for

$-\infty < \widetilde{p} < p$ (resp., $p < \widetilde{p} < \infty$). We thus write

$$i(\phi) := \sup\{p \in (-\infty, \infty) : \phi \text{ is of lower type } p\}$$

$$I(\phi) := \inf\{p \in (-\infty, \infty) : \phi \text{ is of upper type } p\}$$

to denote the critical lower type and the critical upper type of the function ϕ.

Let us generalize these notions to functions $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$.

Given a function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ so that for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is Orlicz. We say that φ is of *uniformly lower type* (resp., *upper type*) p if there exists a positive constant C so that

$$\varphi(x, st) \leq C s^p \varphi(x, t), \tag{3.1}$$

for all $x \in \mathbb{R}^n$ and $t \geq 0, s \in (0, 1)$ (resp., $s \in [1, \infty)$). We say that φ is of *positive uniformly lower type* (resp., *finite uniform upper type*) if it is of uniformly lower type (resp., uniform upper type) p for some $p > 0$ (resp., p finite), and denote

$$i(\varphi) := \sup\{p \in (-\infty, \infty) : \varphi \text{ is of uniformly lower type } p\}$$

$$I(\varphi) := \inf\{p \in (-\infty, \infty) : \varphi \text{ is of uniformly upper type } p\}.$$

We next need to recall notations for Muckenhoupt weights.

Let $1 \leq q < \infty$. A nonnegative locally integrable function w belongs to the *Muckenhoupt class* A_q, say $w \in A_q$, if there exists a positive constant C so that

$$\frac{1}{|B|} \int_B w(x) dx \Big(\frac{1}{|B|} \int_B (w(x))^{-1/(q-1)} dx \Big)^{q-1} \leq C, \quad \text{if } 1 < q < \infty,$$

and

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname*{ess-inf}_{x \in B} w(x), \quad \text{if } q = 1,$$

for all balls B in $\mathbb{R}^n$. We say that $w \in A_\infty$ if $w \in A_q$ for some $q \in [1, \infty)$.

It is well known that $w \in A_q$, $1 \leq q < \infty$, implies $w \in A_r$ for all $r > q$. Also, if $w \in A_q$, $1 < q < \infty$, then $w \in A_r$ for some $r \in [1, q)$. One thus write $q_w := \inf\{q \geq 1 : w \in A_q\}$ to denote the critical index of w.

Now, let us generalize these notions to functions $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$.

Let $\varphi : \mathbb{R}^n \times [0, \infty) \to \mathbb{C}$ be so that $x \mapsto \varphi(x, t)$ is measurable for all $t \in [0, \infty)$. We say that $\varphi(\cdot, t)$ is *uniformly locally integrable* if for all compact set K in $\mathbb{R}^n$, the following holds

$$\int_K \sup_{t>0} \frac{|\varphi(x, t)|}{\int_K |\varphi(y, t)| dy} dx < \infty$$

whenever the integral exists. A simple example for such *uniformly locally integrable* functions is $\varphi(x,t) = w(x)\Phi(t)$ with w a locally integrable function on $\mathbb{R}^n$ and Φ an arbitrary function on $[0,\infty)$. Our interesting examples are *uniformly locally integrable* functions $\varphi(x,t) = \frac{t^p}{(\log(e+|x|)+\log(e+t^p))^p}$, $0 < p \leq 1$, since they arise naturally in the study of pointwise product of functions in $H^p(\mathbb{R}^n)$ with functions in $BMO(\mathbb{R}^n)$ (cf. [14]).

Given $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a *uniformly locally integrable* function. We say that the function $\varphi(\cdot,t)$ satisfies the *uniformly Muckenhoupt* condition $\mathbb{A}_q$, say $\varphi \in \mathbb{A}_q$, for some $1 \leq q < \infty$ if there exists a positive constant C so that

$$\frac{1}{|B|}\int_B \varphi(x,t)dx.\Big(\frac{1}{|B|}\int_B \varphi(x,t)^{-1/(q-1)}dx\Big)^{q-1} \leq C, \quad \text{if } 1 < q < \infty,$$

and

$$\frac{1}{|B|}\int_B \varphi(x,t)dx \leq C \operatorname*{ess-inf}_{x\in B} \varphi(x,t), \quad \text{if } q = 1,$$

for all $t > 0$ and balls B in $\mathbb{R}^n$. We also say that $\varphi \in \mathbb{A}_\infty$ if $\varphi \in \mathbb{A}_q$ for some $q \in [1,\infty)$, and denote

$$q(\varphi) := \inf\{q \geq 1 : \varphi \in \mathbb{A}_q\}.$$

Now, we are able to introduce the *growth functions* which are the basis for our new Hardy spaces.

Definition 3.2.1. *We say that $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a* **growth function** *if the following conditions are satisfied.*

1. *The function φ is a* **Musielak-Orlicz function** *that is*

   (a) *the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$,*

   (b) *the function $\varphi(\cdot,t)$ is a Lebesgue measurable function for all $t \in [0,\infty)$.*

2. *The function φ belongs to $\mathbb{A}_\infty$.*

3. *The function φ is of positive uniformly lower type and of uniformly upper type 1.*

   For φ a growth function, we denote $m(\varphi) := \Big[n\Big(\frac{q(\varphi)}{i(\varphi)} - 1\Big)\Big]$.

Clearly, $\varphi(x,t) = w(x)\Phi(t)$ is a *growth function* if $w \in A_\infty$ and Φ is of positive lower type and of upper type 1. Of course, there exists *growth functions* which are not of that form for instance $\varphi(x,t) = \frac{t^\alpha}{[\log(e+|x|)]^\beta + [\log(e+t)]^\gamma}$ for $\alpha \in (0,1]; \beta, \gamma \in (0,\infty)$. More precisely, $\varphi \in \mathbb{A}_1$ and φ is of uniformly upper type α with $i(\varphi) = \alpha$. In this paper, we are especially interested in the *growth functions* $\varphi(x,t)=\frac{t^p}{(\log(e+|x|)+\log(e+t^p))^p}$, $0<p\leq 1$, since the Hardy spaces of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ arise naturally in the study of pointwise product of functions in $H^p(\mathbb{R}^n)$ with functions in $BMO(\mathbb{R}^n)$ (see also [12] in the setting of holomorphic functions in convex domains of finite type or strictly pseudoconvex domains in $\mathbb{C}^n$).

3.2.2 Hardy spaces of Musielak-Orlicz type

Throughout the whole paper, we always assume that φ is a *growth function*.

Let us now introduce the *Musielak-Orlicz-type spaces*.

The Musielak-Orlicz-type space $L^\varphi(\mathbb{R}^n)$ is the set of all measurable functions f such that $\int_{\mathbb{R}^n}\varphi(x,|f(x)|/\lambda)dx<\infty$ for some $\lambda>0$, with Luxembourg (quasi-)norm

$$\|f\|_{L^\varphi}:=\inf\Big\{\lambda>0:\int\limits_{\mathbb{R}^n}\varphi(x,|f(x)|/\lambda)dx\leq 1\Big\}.$$

As usual, $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of test functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions (or distributions for brevity). For $m\in\mathbb{N}$, we define

$$\mathcal{S}_m(\mathbb{R}^n)=\Big\{\phi\in\mathcal{S}(\mathbb{R}^n):\|\phi\|_m=\sup_{x\in\mathbb{R}^n,|\alpha|\leq m+1}(1+|x|)^{(m+2)(n+1)}|\partial_x^\alpha\phi(x)|\leq 1\Big\}.$$

For each distribution f, we define the nontangential grand maximal function f^*_m of f by

$$f^*_m(x)=\sup_{\phi\in\mathcal{S}_m(\mathbb{R}^n)}\sup_{|y-x|<t}|f*\phi_t(y)|,\ x\in\mathbb{R}^n.$$

When $m=m(\varphi)$ we write f^* instead of $f^*_{m(\varphi)}$.

Definition 3.2.2. *The Hardy space of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ is the space of all distributions f such that $f^*\in L^\varphi(\mathbb{R}^n)$ with the (quasi-)norm*

$$\|f\|_{H^\varphi}:=\|f^*\|_{L^\varphi}.$$

Observe that, when $\varphi(x,t)=w(x)\Phi(t)$ with w a Muckenhoupt weight and Φ an Orlicz function, our Hardy spaces are weighted Hardy-Orlicz spaces which include the classical Hardy-Orlicz spaces of Janson [75] ($w\equiv 1$ in this context) and the classical weighted Hardy spaces of García-Cuerva [52], Strömberg and Torchinsky [131] ($\Phi(t)\equiv t^p$ in this context), see also [111, 22, 53] . Recently, the weighted anisotropic Hardy spaces (see [20]) and the Hardy-Orlicz spaces associated with operators (see [77]) have also been studied.

Next, to introduce the *atomic Hardy spaces of Musielak-Orlicz type* below, we need the following new spaces.

Definition 3.2.3. *For each ball B in $\mathbb{R}^n$, we denote $L^q_\varphi(B), 1 \leq q \leq \infty$, the set of all measurable functions f on $\mathbb{R}^n$ supported in B such that*

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup\limits_{t>0} \left(\frac{\int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)dx}{\varphi(B,t)} \right)^{1/q} < \infty & , \quad 1 \leq q < \infty, \\ \|f\|_{L^\infty} < \infty & , \quad q = \infty, \end{cases} \tag{3.2}$$

here and in the future $\varphi(B,t) := \int_B \varphi(x,t)dx$.

Then, it is straightforward to verify that $(L^q_\varphi(B), \|\cdot\|_{L^q_\varphi(B)})$ is a Banach space.

Now, we are able to introduce the *atomic Hardy spaces of Musielak-Orlicz type*.

Definition 3.2.4. *A triplet (φ, q, s) is called admissible, if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{N}$ satisfies $s \geq m(\varphi)$. A measurable function a is a (φ, q, s)-atom if it satisfies the following three conditions*

i) $a \in L^q_\varphi(B)$ for some ball B,
ii) $\|a\|_{L^q_\varphi(B)} \leq \|\chi_B\|^{-1}_{L^\varphi}$,
iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $|\alpha| \leq s$.

In this setting we define the *atomic Hardy space of Musielak-Orlicz type* $H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ as those distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be represented as a sum of multiples of (φ, q, s)-atoms, that is,

$$f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where b_j's are multiples of (φ, q, s)-atoms supported in the balls B_j's, with the property

$$\sum_j \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty.$$

We introduce a (quasi-)norm in $H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$. Given a sequence of multiples of (φ, q, s)-atoms, $\{b_j\}_j$, we denote

$$\Lambda_q(\{b_j\}) = \inf \left\{ \lambda > 0 : \sum_j \varphi\Big(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\Big) \leq 1 \right\} \tag{3.3}$$

and define

$$\|f\|_{H^{\varphi,q,s}_{\rm at}} = \inf \left\{ \Lambda_q(\{b_j\}) : f = \sum_j b_j \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n) \right\}. \tag{3.4}$$

Let (φ, q, s) be an admissible triplet. We denote $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ the vector space of all finite linear combinations of (φ, q, s)-atoms, that is,

$$f = \sum_{j=1}^k b_j,$$

where b_j's are multiples of (φ, q, s)-atoms supported in balls B_j's. Then, the norm of f in $H^{\varphi,q,s}_{\text{fin}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{\varphi,q,s}_{\text{fin}}} = \inf\Big\{\Lambda_q(\{b_j\}_{j=1}^k) : f = \sum_{j=1}^k b_j\Big\}. \tag{3.5}$$

Obviously, for any admissible triplet (φ, q, s), the set $H^{\varphi,q,s}_{\text{fin}}(\mathbb{R}^n)$ is dense in $H^{\varphi,q,s}_{\text{at}}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{\varphi,q,s}_{\text{at}}}$.

We should point out that the theory of atomic Hardy-Orlicz spaces have been first introduced by Viviani [136] in the setting of spaces of homogeneous type. Later, Serra [124] generalized it to the context of the Euclidean space $\mathbb{R}^n$ and obtained the molecular characterization. In the particular case, when $\varphi(x,t) \equiv \Phi(t)$ the space $H^{\varphi,q,s}_{\text{at}}(\mathbb{R}^n)$ is the space considered in [124]. We also remark that when $\varphi(x,t) \equiv w(x).t^p, 0 < p \leq 1$, w a Muckenhoupt weight, the space $H^{\varphi,q,s}_{\text{at}}(\mathbb{R}^n)$ is just the classical weighted atomic Hardy space $H^{p,q,s}_w(\mathbb{R}^n)$ which has been considered by García-Cuerva [52], Strömberg and Torchinsky [131].

3.2.3 BMO-Musielak-Orlicz-type spaces

We also need BMO type spaces, which will be in duality of the Hardy spaces of Musielak-Orlicz type defined above. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to belong to $BMO^\varphi(\mathbb{R}^n)$ if

$$\|f\|_{BMO^\varphi} := \sup_B \frac{1}{\|\chi_B\|_{L^\varphi}} \int_B |f(x) - f_B| dx < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f(x)dx$ and the supremum is taken over all balls B in $\mathbb{R}^n$.

Our typical example is $BMO^\varphi(\mathbb{R}^n)$, called $BMO^{\log}(\mathbb{R}^n)$, related to $\varphi(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$. Clearly, when $\varphi(x,t) \equiv t$, then $BMO^\varphi(\mathbb{R}^n)$ is just the well-known $BMO(\mathbb{R}^n)$ of John and Nirenberg. We remark that when $\varphi(x,t) = w(x).t$ with $w \in A_{(n+1)/n}$, then $BMO^\varphi(\mathbb{R}^n)$ is just $BMO_w(\mathbb{R}^n)$ was first introduced by Muckenhoupt and Wheeden [110, 111]. There, they proved that $BMO_w(\mathbb{R}^n)$ is the dual of $H^1_w(\mathbb{R}^n)$ (see also [22]).

3.2.4 Quasi-Banach valued sublinear operators

Recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant κ no less than 1 such that for all $f, g \in \mathcal{B}$, we have $\|f+g\|_{\mathcal{B}} \leq \kappa(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Definition 3.2.5. *Let $\gamma \in (0,1]$. A quasi-Banach space $\mathcal{B}_\gamma$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is said to be a γ-quasi-Banach space if there exists a positive constant κ no less than 1 such that for all $f_j \in \mathcal{B}_\gamma, j = 1,2,...,m$, we have*

$$\Big\|\sum_{j=1}^m f_j\Big\|_{\mathcal{B}_\gamma}^\gamma \leq \kappa \sum_{j=1}^m \|f_j\|_{\mathcal{B}_\gamma}^\gamma.$$

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces $\ell^p, L^p_w(\mathbb{R}^n)$ and $H^p_w(\mathbb{R}^n)$ with $p \in (0,1]$ are typical p-quasi-Banach spaces. Also, when φ is of uniformly lower type $p \in (0,1]$, the space $H^\varphi(\mathbb{R}^n)$ is a p-quasi-Banach space.

For any given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma \in (0,1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is called $\mathcal{B}_\gamma$-sublinear if there exists a positive constant κ no less than 1 such that for all $f_j \in \mathcal{Y}, \lambda_j \in \mathbb{C}, j = 1,...,m$, we have

$$\Big\|T\Big(\sum_{j=1}^m \lambda_j f_j\Big)\Big\|_{\mathcal{B}_\gamma}^\gamma \leq \kappa \sum_{j=1}^m |\lambda_j|^\gamma \|T(f_j)\|_{\mathcal{B}_\gamma}^\gamma.$$

We remark that if T is linear, then T is $\mathcal{B}_\gamma$-sublinear. We should point out that if the constant κ, in Definition 3.2.5, equal 1, then we obtain the notion of γ-quasi-Banach spaces introduced in [144] (see also [20]).

3.3 Statement of the results

Our main theorems are the following.

Theorem 3.3.1. *Let (φ, q, s) be admissible. Then $H^\varphi(\mathbb{R}^n) = H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ with equivalent norms.*

Denote by $L^\infty_0(\mathbb{R}^n)$ the set of all bounded functions with compact support and zero average. As a consequence of Theorem 3.3.1, we have the following.

Lemma 3.3.1. *Let φ be a growth function satisfying $nq(\varphi) < (n+1)i(\varphi)$. Then, $L^\infty_0(\mathbb{R}^n)$ is dense in $H^\varphi(\mathbb{R}^n)$.*

We now can present our dual theorem as follows

Theorem 3.3.2. *Let φ be a growth function satisfying $nq(\varphi) < (n+1)i(\varphi)$. Then, the dual space of $H^\varphi(\mathbb{R}^n)$ is $BMO^\varphi(\mathbb{R}^n)$ in the following sense*

i) Suppose $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$. Then the linear functional $L_\mathfrak{b} : f \to L_\mathfrak{b}(f) := \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx$, initially defined for $L^\infty_0(\mathbb{R}^n)$, has a bounded extension to $H^\varphi(\mathbb{R}^n)$.

ii) Conversely, every continuous linear functional on $H^\varphi(\mathbb{R}^n)$ arises as the above with a unique element $\mathfrak{b}$ of $BMO^\varphi(\mathbb{R}^n)$. Moreover $\|\mathfrak{b}\|_{BMO^\varphi} \approx \|L_\mathfrak{b}\|_{(H^\varphi)^*}$.

Next result concerns the class of pointwise multipliers for $BMO(\mathbb{R}^n)$.

Theorem 3.3.3. *The class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is the dual of $L^1(\mathbb{R}^n)+H^{\log}(\mathbb{R}^n)$ where $H^{\log}(\mathbb{R}^n)$ is a Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t)=\frac{t}{\log(e+|x|)+\log(e+t)}$.*

In order to obtain the finite atomic decomposition, we need the notion of *uniformly locally dominated convergence condition*. A growth function φ is said to be satisfy *uniformly locally dominated convergence condition* if the following holds:

Given K compact set in $\mathbb{R}^n$. Let $\{f_m\}_{m\geq 1}$ be a sequence of measurable functions s.t $f_m(x)$ tends to $f(x)$ a.e $x\in\mathbb{R}^n$. If there exists a nonnegative measurable function g s.t $|f_m(x)|\leq g(x)$ and $\sup_{t>0}\int_K g(x)\frac{\varphi(x,t)}{\int_K\varphi(y,t)dy}dx<\infty$, then $\sup_{t>0}\int_K|f_m(x)-f(x)|\frac{\varphi(x,t)}{\int_K\varphi(y,t)dy}dx$ tends 0.

We remark that the growth functions $\varphi(x,t)=w(x)\Phi(t)$ and $\varphi(x,t)=\frac{t^p}{(\log(e+|x|)+\log(e+t^p))^p}$, $0<p\leq 1$, satisfy the *uniformly locally dominated convergence condition*.

Theorem 3.3.4. *Let φ be a growth function satisfying uniformly locally dominated convergence condition, and (φ,q,s) be an admissible triplet.*

i) If $q\in(q(\varphi),\infty)$ then $\|\cdot\|_{H^{\varphi,q,s}_{\rm fin}}$ and $\|\cdot\|_{H^\varphi}$ are equivalent quasi-norms on $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$.

ii) $\|\cdot\|_{H^{\varphi,\infty,s}_{\rm fin}}$ and $\|\cdot\|_{H^\varphi}$ are equivalent quasi-norms on $H^{\varphi,\infty,s}_{\rm fin}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$.

As an application, we obtain criterions for boundedness of quasi-Banach valued sublinear operators in $H^\varphi(\mathbb{R}^n)$.

Theorem 3.3.5. *Let φ be a growth function satisfying uniformly locally dominated convergence condition, (φ,q,s) be an admissible triplet, φ be of uniformly upper type $\gamma\in(0,1]$, and $\mathcal{B}_\gamma$ be a quasi-Banach space. Suppose one of the following holds:*

i) $q\in(q(\varphi),\infty)$, and $T:H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)\to\mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that

$$A=\sup\{\|Ta\|_{\mathcal{B}_\gamma}: a \text{ is a } (\varphi,q,s)-\text{atom}\}<\infty;$$

ii) T is a $\mathcal{B}_\gamma$-sublinear operator defined on continuous (φ,∞,s)-atoms such that

$$A=\sup\{\|Ta\|_{\mathcal{B}_\gamma}: a \text{ is a continuous } (\varphi,\infty,s)-\text{atom}\}<\infty.$$

Then there exists a unique bounded $\mathcal{B}_\gamma$-sublinear operator $\widetilde{T}$ from $H^\varphi(\mathbb{R}^n)$ to $\mathcal{B}_\gamma$ which extends T.

3.4 Some basic lemmas on growth functions

We start by the following lemma.

Lemma 3.4.1. *i) Let φ be a growth function. Then φ is uniformly σ-quasi-subadditive on $\mathbb{R}^n \times [0, \infty)$, i.e. there exists a constant $C > 0$ such that*

$$\varphi(x, \sum_{j=1}^{\infty} t_j) \leq C \sum_{j=1}^{\infty} \varphi(x, t_j),$$

for all $(x, t_j) \in \mathbb{R}^n \times [0, \infty)$, $j = 1, 2, ...$

ii) Let φ be a growth function and $\widetilde{\varphi}(x, t) := \int_0^t \frac{\varphi(x,s)}{s} ds$ for $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Then $\widetilde{\varphi}$ is a growth function equivalent to φ, moreover, $\widetilde{\varphi}(x, \cdot)$ is continuous and strictly increasing.

iii) A Musielak-Orlicz function φ is a growth function if and only if φ is of positive uniformly lower type and uniformly quasi-concave, i.e. there exists a constant $C > 0$ such that

$$\lambda\varphi(x, t) + (1 - \lambda)\varphi(x, s) \leq C\varphi(x, \lambda t + (1 - \lambda)s),$$

for all $x \in \mathbb{R}^n, t, s \in [0, \infty)$ and $\lambda \in [0, 1]$.

Proof. i) We just need to consider the case when $\sum_{j=1}^{\infty} t_j > 0$. Then it follows from the fact that

$$\frac{t_k}{\sum_{j=1}^{\infty} t_j} \varphi(x, \sum_{j=1}^{\infty} t_j) \leq C\varphi(x, t_k)$$

by φ is of uniformly upper type 1.

ii) Since φ is a growth function, it is easy to see that $\widetilde{\varphi}(x, \cdot)$ is continuous and strictly increasing. Moreover, there exists $p > 0$ such that φ is of uniformly lower type p. Hence,

$$\widetilde{\varphi}(x, t) = \int_0^t \frac{\varphi(x, s)}{s} ds \leq C \frac{\varphi(x, t)}{t^p} \int_0^t \frac{1}{s^{1-p}} ds \leq C\varphi(x, t). \tag{3.6}$$

On the other hand, since φ is of uniformly upper type 1, we get

$$\widetilde{\varphi}(x, t) = \int_0^t \frac{\varphi(x, s)}{s} ds \geq C^{-1} \int_0^t \frac{\varphi(x, t)}{t} ds \geq C^{-1}\varphi(x, t). \tag{3.7}$$

Combining (3.6) and (3.7), we obtain $\widetilde{\varphi} \approx \varphi$, and thus $\widetilde{\varphi}$ is a growth function.

iii) Suppose φ is a growth function. By (ii), φ is equivalent to $\widetilde{\widetilde{\varphi}}$. On the other hand, $\frac{\partial \widetilde{\widetilde{\varphi}}}{\partial t}(x, t) = \frac{\widetilde{\varphi}(x,t)}{t}$ is uniformly quasi-decreasing in t. Hence, $\widetilde{\widetilde{\varphi}}$ is uniformly quasi-concave, and thus is φ.

The converse is easy by taking $s = 0$. We omit the details. □

Remark 3.4.1. *Let us observe that the results stated in Section 3 are invariant under change of equivalent growth functions. By Lemm 3.4.1, in the future, we always consider a growth function φ of positive uniformly lower type, of uniformly upper type 1 (or, equivalently, uniformly quasi-concave), and so that $\varphi(x,\cdot)$ is continuous and strictly increasing for all $x \in \mathbb{R}^n$.*

Lemma 3.4.2. *Let φ be a growth function. Then*

i) $\int\limits_{\mathbb{R}^n} \varphi\Big(x, \frac{|f(x)|}{\|f\|_{L^\varphi}}\Big)dx = 1$ *for all* $f \in L^\varphi(\mathbb{R}^n) \setminus \{0\}$.

ii) $\lim_{k\to\infty} \|f_k\|_{L^\varphi} = 0$ *if and only if* $\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi(x, |f_k(x)|)dx = 0$.

Proof. Statement (i) follows from the fact that the function

$$\vartheta(t) := \int\limits_{\mathbb{R}^n} \varphi(x, t|f(x)|)dx,$$

$t \in [0,\infty)$, is continuous by the dominated convergence theorem since $\varphi(x,\cdot)$ is continuous.

Statement (ii) follows from the fact that

$$\|f\|_{L^\varphi} \le C \max\Big\{\int\limits_{\mathbb{R}^n} \varphi(x,|f(x)|)dx, \Big(\int\limits_{\mathbb{R}^n} \varphi(x,|f(x)|)dx\Big)^{1/p}\Big\},$$

and

$$\int\limits_{\mathbb{R}^n} \varphi(x,|f(x)|)dx \le C \max\Big\{\|f\|_{L^\varphi}, (\|f\|_{L^\varphi})^p\Big\}$$

for some $p \in (0, i(\varphi))$. □

Lemma 3.4.3. *Given c is a positive constant. Then, there exists a constant $C > 0$ such that*

i) The inequality $\int_{\mathbb{R}^n} \varphi\Big(x, \frac{|f(x)|}{\lambda}\Big)dx \le c$, *for* $\lambda > 0$, *implies*

$$\|f\|_{L^\varphi} \le C\lambda.$$

ii) The inequality $\sum_j \varphi\Big(B_j, \frac{t_j}{\lambda}\Big) \le c$, *for* $\lambda > 0$, *implies*

$$\inf\Big\{\alpha > 0 : \sum_j \varphi\Big(B_j, \frac{t_j}{\alpha}\Big) \le 1\Big\} \le C\lambda.$$

Proof. The proofs are simple since we may take $C = (1 + c.C_p)^{1/p}$, for some $p \in (0, i(\varphi))$, where C_p is such that (3.1) holds. □

Lemma 3.4.4. *Let (φ, q, s) be an admissible triplet. Then there exists a positive constant C such that*

$$\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi} \leq C\Lambda_q(\{b_j\}),$$

for all $f = \sum_{j=1}^{\infty} b_j \in H^{\varphi,q,s}_{at}(\mathbb{R}^n)$ where b_j's are multiples of (φ, q, s)-atoms supported in balls B_j's.

Proof. Since φ is of uniformly upper type 1, there exists a positive constant $c > 0$ such that

$$\varphi\Big(x, \frac{\|b_i\|_{L^q_\varphi(B_i)}}{\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi}}\Big) \geq c \frac{\|b_i\|_{L^q_\varphi(B_i)} \|\chi_{B_i}\|_{L^\varphi}}{\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi}} \varphi\Big(x, \frac{1}{\|\chi_{B_i}\|_{L^\varphi}}\Big)$$

for all $x \in \mathbb{R}^n, i \geq 1$. Hence, for all $i \geq 1$,

$$\varphi\Big(B_i, \frac{\|b_i\|_{L^q_\varphi(B_i)}}{\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi}}\Big) \geq c \frac{\|b_i\|_{L^q_\varphi(B_i)} \|\chi_{B_i}\|_{L^\varphi}}{\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi}}$$

since $\int_{B_i} \varphi\Big(x, \frac{1}{\|\chi_{B_i}\|_{L^\varphi}}\Big)dx = 1$ by Lemma 3.4.2. It follows that

$$\sum_{i=1}^{\infty} \varphi\Big(B_i, \frac{\|b_i\|_{L^q_\varphi(B_i)}}{\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi}}\Big) \geq c.$$

We deduce from the above that

$$\sum_{j=1}^{\infty} \|b_j\|_{L^q_\varphi(B_j)} \|\chi_{B_j}\|_{L^\varphi} \leq C\Lambda_q(\{b_j\}),$$

with $C = (C_p/c)^{1/p}$ for some $p \in (0, i(\varphi))$, where C_p is such that (3.1) holds. $\square$

Lemma 3.4.5. *Let $\varphi \in \mathbb{A}_q, 1 < q < \infty$. Then, there exists a positive constant C such that*

i) For all ball $B(x_0, r), \lambda > 0$, and $t \in [0, \infty)$, we have

$$\varphi(B(x_0, \lambda r), t) \leq C\lambda^{nq}\varphi(B(x_0, r), t).$$

ii) For all ball $B(x_0, r)$ and $t \in [0, \infty)$, we have

$$\int_{B^c} \frac{\varphi(x, t)}{|x - x_0|^{nq}} dx \leq C \frac{\varphi(B, t)}{r^{nq}}.$$

iii) For all ball B, f measurable and $t \in (0, \infty)$, we have

$$\Big(\frac{1}{|B|} \int_B |f(x)|\Big)^q \leq C \frac{1}{\varphi(B, t)} \int_B |f(x)|^q \varphi(x, t) dx.$$

iii) For all f measurable and $t \in [0, \infty)$, we have

$$\int_{\mathbb{R}^n} \mathcal{M}f(x)^q \varphi(x,t)dx \leq C \int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)dx,$$

where $\mathcal{M}$ is the classical Hardy-Littlewood maximal operator defined by

$$\mathcal{M}f(x) = \sup_{x \in B-\text{ball}} \frac{1}{|B|} \int_B |f(y)|dy \ , \quad x \in \mathbb{R}^n.$$

In the setting $\varphi(x,t) = w(x)\Phi(t)$, $w \in A_\infty$ and Φ a Orlicz function, the above lemma is well-known as a classical result in the theory of Muckenhoupt weight (see [54]). Since φ satisfies uniformly Muckenhoupt condition, the proof of Lemma 3.4.5 is a slight modification of the classical result. We omit the details.

3.5 Atomic decompositions

The purpose of this section is prove the atomic decomposition theorem (Theorem 3.3.1). The construction is by now standard, but the estimates require the preliminary lemmas. For the reader convenience, we give all steps of the proof, even if only the generalization to our framework is new.

We first introduce a class of Hardy spaces that the Hardy space of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ containing as a particular case.

Definition 3.5.1. *For $m \in \mathbb{N}$, we denote by $H^\varphi_m(\mathbb{R}^n)$ the space of all distributions f such that $f^*_m \in L^\varphi(\mathbb{R}^n)$ with the (quasi-)norm*

$$\|f\|_{H^\varphi_m} := \|f^*_m\|_{L^\varphi}.$$

Clearly, $H^\varphi(\mathbb{R}^n)$ is a special case associated with $m = m(\varphi)$.

3.5.1 Some basic properties concerning $H^\varphi_m(\mathbb{R}^n)$ and $H^{\varphi,q,s}_{at}(\mathbb{R}^n)$

We start by the following proposition.

Proposition 3.5.1. *For $m \in \mathbb{N}$, we have $H^\varphi_m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. Let $f \in H^\varphi_m(\mathbb{R}^n)$. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in B(0,1)$, we write

$$\langle f, \phi \rangle = f * \widetilde{\phi}(0) = f * \psi(x),$$

where $\psi(y) = \widetilde{\phi}(y - x) = \phi(x - y)$ for all $y \in \mathbb{R}^n$.

It is easy to verify that $\sup\limits_{x \in B(0,1), y \in \mathbb{R}^n} \frac{1+|y|}{1+|y-x|} \leq 2$. Consequently,

$$\begin{aligned} |\langle f, \phi \rangle| = |f * \psi(x)| &\leq 2^{(m+2)(n+1)} \|\phi\|_{\mathcal{S}_m} \inf_{x \in B(0,1)} f_m^*(x) \\ &\leq 2^{(m+2)(n+1)} \|\phi\|_{\mathcal{S}_m} \|\chi_{B(0,1)}\|_{L^\varphi}^{-1} \|f\|_{H_m^\varphi}. \end{aligned}$$

This implies that $f \in \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous. □

The following proposition gives the completeness of $H_m^\varphi(\mathbb{R}^n)$.

Proposition 3.5.2. *The space $H_m^\varphi(\mathbb{R}^n)$ is complete.*

Proof. In order to prove the completeness of $H_m^\varphi(\mathbb{R}^n)$, it suffices to prove that for every sequence $\{f_j\}_{j \geq 1}$ with $\|f_j\|_{H_m^\varphi} \leq 2^{-j}$ for any $j \geq 1$, the series $\sum_j f_j$ converges in $H_m^\varphi(\mathbb{R}^n)$. Let us now take $p > 0$ such that φ is of uniformly lower type p. Then, for any $j \geq 1$,

$$\int\limits_{\mathbb{R}^n} \varphi(x, (f_j)_m^*(x)) dx \leq C(2^{-j})^p \int\limits_{\mathbb{R}^n} \varphi\Big(x, \frac{(f_j)_m^*(x)}{2^{-j}}\Big) dx \leq C 2^{-jp}. \tag{3.8}$$

Since $\{\sum_{i=1}^j f_i\}_{j \geq 1}$ is a Cauchy sequence in $H_m^\varphi(\mathbb{R}^n)$, by Proposition 3.5.1 and the completeness of $\mathcal{S}'(\mathbb{R}^n)$, $\{\sum_{i=1}^j f_i\}_{j \geq 1}$ is also a Cauchy sequence in $\mathcal{S}'(\mathbb{R}^n)$ and thus converges to some $f \in \mathcal{S}'(\mathbb{R}^n)$. This implies that, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, the series $\sum_j f_j * \phi$ converges to $f * \phi$ pointwisely. Therefore $f_m^*(x) \leq \sum_j (f_j)_m^*(x)$ and $(f - \sum_{j=1}^k f_j)_m^*(x) \leq \sum_{j \geq k+1} (f_j)_m^*(x)$ for all $x \in \mathbb{R}^n, k \geq 1$. Combining this and (3.8), we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \varphi(x, (f - \sum_{j=1}^k f_j)_m^*(x)) dx &\leq C \sum_{j \geq k+1} \int\limits_{\mathbb{R}^n} \varphi(x, (f_j)_m^*(x)) dx \\ &\leq C \sum_{j \geq k+1} 2^{-jp} \to 0, \end{aligned}$$

as $k \to \infty$, here we used Lemma 3.4.1. Thus, the series $\sum_j f_j$ converges to f in $H_m^\varphi(\mathbb{R}^n)$ by Lemma 3.4.2. This completes the proof. □

Corollary 3.5.1. *The Hardy space of Musielak-Orlicz type $H^\varphi(\mathbb{R}^n)$ is complete.*

The following lemma and its corollary show that (φ, q, s)-atoms are in $H^\varphi(\mathbb{R}^n)$. Furthermore, it is the necessary estimate for proving that $H_{\text{at}}^{\varphi,q,s}(\mathbb{R}^n) \subset H^\varphi(\mathbb{R}^n)$ and the inclusion is continuous, see Theorem 3.5.1 below.

Lemma 3.5.1. *Let (φ, q, s) be an admissible triplet and $m \geq s$. Then, there exists a constant $C = C(\varphi, q, s, m)$ such that*

$$\int_{\mathbb{R}^n} \varphi(x, f_m^*(x))dx \leq C\varphi(B, \|f\|_{L^q_\varphi(B)}),$$

for all f multiples of (φ, q, s)-atom associated with ball $B = B(x_0, r)$.

Proof. The case $q = \infty$ is easy and will be omitted. We just consider $q \in (q(\varphi), \infty)$. Now let us set $\widetilde{B} = B(x_0, 9r)$, and write

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(x, f_m^*(x))dx &= \int_{\widetilde{B}} \varphi(x, f_m^*(x))dx + \int_{(\widetilde{B})^c} \varphi(x, f_m^*(x))dx \\ &= I + II. \end{aligned}$$

Since φ is of uniformly upper type 1, by Hölder inequality, we get

$$\begin{aligned} I &= \int_{\widetilde{B}} \varphi(x, f_m^*(x))dx \leq C \int_{\widetilde{B}} \Big(\frac{f_m^*(x)}{\|f\|_{L^q_\varphi(B)}} + 1\Big)\varphi(x, \|f\|_{L^q_\varphi(B)})dx \\ &\leq C\varphi(\widetilde{B}, \|f\|_{L^q_\varphi(B)}) \\ &+ C\frac{1}{\|f\|_{L^q_\varphi(B)}}\Big(\int_{\widetilde{B}} |f_m^*(x)|^q \varphi(x, \|f\|_{L^q_\varphi(B)})dx\Big)^{1/q} \varphi(\widetilde{B}, \|f\|_{L^q_\varphi(B)})^{(q-1)/q} \\ &\leq C\varphi(B, \|f\|_{L^q_\varphi(B)}) + C\frac{1}{\|f\|_{L^q_\varphi(B)}}\|f\|_{L^q_\varphi(\widetilde{B})}\varphi(\widetilde{B}, \|f\|_{L^q_\varphi(B)}) \\ &\leq C\varphi(B, \|f\|_{L^q_\varphi(B)}). \end{aligned}$$

We used the fact $f_m^*(x) \leq C(m)\mathcal{M}f(x)$ and Lemma 3.4.5.

To estimate II, we note that since $m \geq s$, there exists a constant $C = C(m)$ such that

$$\Big|\phi\Big(\frac{x-y}{t}\Big) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \phi(\frac{x-x_0}{t})}{\alpha!}\Big(\frac{x_0 - y}{t}\Big)^\alpha\Big| \leq Ct^n \frac{|y - x_0|^{s+1}}{|x - x_0|^{n+s+1}}$$

for all $\phi \in \mathcal{S}_m(\mathbb{R}^n), t > 0, x \in (\widetilde{B})^c, y \in B$. Therefore

$$\begin{aligned} |f * \phi_t(x)| &= \frac{1}{t^n}\Big|\int_B f(y)\Big[\phi\Big(\frac{x-y}{t}\Big) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \phi(\frac{x-x_0}{t})}{\alpha!}\Big(\frac{x_0 - y}{t}\Big)^\alpha\Big]dy\Big| \\ &\leq C\int_B |f(y)|\frac{|y - x_0|^{s+1}}{|x - x_0|^{n+s+1}}dy \\ &\leq C\frac{r^{s+1}}{|x - x_0|^{n+s+1}}\Big(\int_B |f(y)|^q \varphi(y, \lambda)dy\Big)^{1/q}\Big(\int_B [\varphi(y, \lambda)]^{-1/(q-1)}dy\Big)^{(q-1)/q} \\ &\leq C\|f\|_{L^q_\varphi(B)}\Big(\frac{r}{|x - x_0|}\Big)^{n+s+1}. \end{aligned}$$

For any $\lambda > 0$, we used that $\int_B \varphi(y,\lambda)dy(\int_B[\varphi(y,\lambda)]^{-1/(q-1)}dy)^{q-1} \leq C|B|^q$ since $\varphi \in \mathbb{A}_q$. As a consequence, we get

$$f_m^*(x) \leq C(m) \sup_{\phi\in\mathcal{S}_m(\mathbb{R}^n)} \sup_{t>0} |f * \phi_t(x)| \leq C\|f\|_{L^q_\varphi(B)}\Big(\frac{r}{|x-x_0|}\Big)^{n+s+1}.$$

By $s \geq m(\varphi)$, there exists $p \in (0, i(\varphi))$ such that $(n+s+1)p > nq(\varphi)$. Hence, by Lemma 3.4.5,

$$\begin{aligned} II = \int\limits_{(\widetilde{B})^c} \varphi(x, f_m^*(x))dx &\leq C \int\limits_{(\widetilde{B})^c} \Big(\frac{r}{|x-x_0|}\Big)^{(n+s+1)p} \varphi(x, \|f\|_{L^q_\varphi(B)})dx \\ &\leq Cr^{(n+s+1)p}\frac{\varphi(\widetilde{B}, \|f\|_{L^q_\varphi(B)})}{(9r)^{(n+s+1)p}} \\ &\leq C\varphi(B, \|f\|_{L^q_\varphi(B)}). \end{aligned}$$

This completes the proof. □

Corollary 3.5.2. *There exists a constant $C = C(\varphi, q, s) > 0$ such that*

$$\|a\|_{H^\varphi} \leq C,$$

for all (φ, q, s)-atom a.

Theorem 3.5.1. *Let (φ, q, s) be an admissible triplet and $m \geq s$. Then*

$$H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n) \subset H^\varphi_m(\mathbb{R}^n),$$

moreover, the inclusion is continuous.

Proof. For any $0 \neq f \in H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$. Let $f = \sum_j b_j$ be an atomic decomposition of f, with supp $b_j \subset B_j$, $j = 1, 2, ...$ For all $\phi \in \mathcal{S}(\mathbb{R}^n)$, the series $\sum_j b_j * \phi$ converges to $f * \phi$ pointwise since $f = \sum_j b_j$ in $\mathcal{S}'$. Hence $f_m^*(x) \leq \sum_j (b_j)_m^*(x)$. By applying Lemma 3.5.1, we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\Lambda_q(\{b_j\})}\Big)dx &\leq C\sum_j \int\limits_{\mathbb{R}^n} \varphi\Big(x, \frac{(b_j)_m^*(x)}{\Lambda_q(\{b_j\})}\Big)dx \\ &\leq C\sum_j \varphi\Big(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\Lambda_q(\{b_j\})}\Big) \\ &\leq C. \end{aligned}$$

This implies that $\|f\|_{H^\varphi_m} \leq C\Lambda_q(\{b_j\})$ (see Lemma 3.4.3) for any atomic decomposition $f = \sum_j b_j$, and thus, $\|f\|_{H^\varphi_m} \leq C\|f\|_{H^{\varphi,q,s}_{\rm at}}$. □

3.5.2 Calderón-Zygmund decompositions

Throughout this subsection, we fix m and s so that $m, s \geq m(\varphi)$. For a given $\lambda > 0$, we set $\Omega = \{x \in \mathbb{R}^n : f_m^*(x) > \lambda\}$. Observe that Ω is open. Hence by Whitney's Lemma , there exists $x_1, x_2, ...$ in Ω and $r_1, r_2, ... > 0$ such that

(i) $\Omega = \cup_j B(x_j, r_j)$,

(ii) the balls $B(x_j, r_j/4)$, $j = 1, 2, ...$, are disjoint,

(iii) $B(x_j, 18r_j) \cap \Omega^c = \emptyset$, but $B(x_j, 54r_j) \cap \Omega^c \neq \emptyset$, for any $j = 1, 2, ...$,

(iv) there exists $L \in \mathbb{N}$ (depending only on n) such that no point of Ω lies in more than L of the balls $B(x_j, 18r_j)$, $j = 1, 2, ...$

We fix once for all, a function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that supp $\theta \subset B(0, 2)$, $0 \leq \theta \leq 1$, $\theta = 1$ on $B(0, 1)$, and set $\theta_j(x) = \theta((x - x_j)/r_j)$, for j=1,2,... Obviously, supp $\theta_j \subset B(x_j, 2r_j)$, $j = 1, 2, ...$, and $1 \leq \sum_j \theta_j \leq L$ for all $x \in \Omega$. Hence if we set $\zeta_j(x) = \theta_j(x)/\sum_{i=1}^\infty \theta_i(x)$ if $x \in \Omega$ and $\zeta_j(x) = 0$ if $x \in \Omega^c$, $j = 1, 2, ...$, then supp $\zeta_j \subset B(x_j, 2r_j)$, $0 \leq \zeta_j \leq 1$, $\sum_j \zeta_j = \chi_\Omega$, and $L^{-1} \leq \zeta_j \leq 1$ on $B(x_j, r_j)$. The family $\{\zeta_j\}_j$ forms a smooth partition of unity of Ω. let $s \in \mathbb{N}$ be some fixed natural number and $\mathcal{P}_s(\mathbb{R}^n)$ (or simply $\mathcal{P}_s$) denote the linear space of polynomials in n variables of degree less than s. For each j, we consider the inner product $\langle P, Q\rangle_j = \frac{1}{\int_{\mathbb{R}^n} \zeta_j(x)dx} \int_{\mathbb{R}^n} P(x)Q(x)\zeta_j(x)dx$ for $P, Q \in \mathcal{P}_s$. Then $(\mathcal{P}_s, \langle\cdot,\cdot\rangle_j)$ is a finite dimensional Hilbert space. Let $f \in \mathcal{S}'$. Since f induces a linear functional on $\mathcal{P}_s$ via $Q \to \frac{1}{\int_{\mathbb{R}^n} \zeta_j(x)dx} \int_{\mathbb{R}^n} f(x)Q(x)\zeta_j(x)dx$, by the Riesz theorem, there exists a unique polynomial $P_j \in \mathcal{P}_s$ such that for all $Q \in \mathcal{P}_s$, $\langle P_j, Q\rangle_j = \frac{1}{\int_{\mathbb{R}^n} \zeta_j(x)dx} \int_{\mathbb{R}^n} f(x)Q(x)\zeta_j(x)dx$. For each j, $j = 1, 2, ...$, we define $b_j = (f - P_j)\zeta_j$, and note $B_j = B(x_j, r_j)$, $\widetilde{B}_j = B(x_j, 9r_j)$. Then, it is easy to see that $\int_{\mathbb{R}^n} b_j(x)Q(x)dx = 0$ for all $Q \in \mathcal{P}_s$. It turns out, in the case of interest, that the series $\sum_j b_j$ converges in $\mathcal{S}'$. In this case, we set $g = f - \sum_j b_j$, and we call the representation $f = g + \sum_j b_j$ a Calderón-Zygmund decomposition of f of degree s and height λ associated to f_m^*.

For any $j = 1, 2, ...$, we denote $B_j = B(x_j, r_j)$ and $\widetilde{B}_j = B(x_j, 9r_j)$. Then we have the following lemma.

Lemma A (see [17, Chapter 3). *There are four constant c_1, c_2, c_3, c_4, independent of f, j, and λ, such that*

i)

$$\sup_{|\alpha| \leq N, x \in \mathbb{R}^n} r_j^{|\alpha|} |\partial^\alpha \zeta_j(x)| \leq c_1.$$

ii)

$$\sup_{x \in \mathbb{R}^n} |P_j(x)\zeta_j(x)| \leq c_2\lambda.$$

iii)

$$(b_j)^*_m(x) \le c_3 f^*_m(x), \quad \textit{for all } x \in \widetilde{B}_j.$$

iv)

$$(b_j)^*_m(x) \le c_4 \lambda (r_j/|x-x_j|)^{n+m_s}, \qquad \textit{for all } x \notin \widetilde{B}_j,$$

where $m_s = \min\{s+1, m+1\}$.

Lemma 3.5.2. *For all* $f \in H^\varphi_m(\mathbb{R}^n)$*, there exists a geometric constant* C*, independent of* $f, j,$ *and* λ*, such that,*

$$\int_{\mathbb{R}^n} \varphi\Big(x, (b_j)^*_m(x)\Big) dx \le C \int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx.$$

Moreover, the series $\sum_j b_j$ *converges in* $H^\varphi_m(\mathbb{R}^n)$*, and*

$$\int_{\mathbb{R}^n} \varphi\Big(x, (\sum_j b_j)^*_m(x)\Big) dx \le C \int_{\Omega} \varphi(x, f^*_m(x)) dx.$$

Proof. As $m, s \ge m(\varphi)$, $m_s = \min\{s+1, m+1\} > n(q(\varphi)/i(\varphi) - 1)$. Hence, there exist $q > q(\varphi)$ and $0 < p < i(\varphi)$ such that $m_s > n(q/p-1)$, deduce that $(n+m_s)p > nq$. Therefore, $\varphi \in \mathbb{A}_{(n+m_s)p/n}$ and φ is of uniformly lower type p. Thus, there exists a positive constant C, independent of $f, j,$ and λ, such that

$$\begin{aligned}
\int_{(\widetilde{B}_j)^c} \varphi(x, \lambda(r_j/|x-x_j|)^{n+m_s}) dx &\le C \int_{(\widetilde{B}_j)^c} \Big(\frac{r_j}{|x-x_j|}\Big)^{(n+m_s)p} \varphi(x,\lambda) dx \\
&\le C (r_j)^{(n+m_s)p} \frac{\varphi(\widetilde{B}_j, \lambda)}{(9r_j)^{(n+m_s)p}} \\
&\le C \int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx,
\end{aligned}$$

since $r_j/|x-x_j| < 1$ and $f^*_m > \lambda$ on $\widetilde{B}_j$. Combining this and Lemma A, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi\Big(x, (b_j)^*_m(x)\Big) dx &\le C \Big[\int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx + \int_{(\widetilde{B}_j)^c} \varphi(x, \lambda(r_j/|x-x_j|)^{n+m_s}) dx \Big] \\
&\le C \int_{\widetilde{B}_j} \varphi(x, f^*_m(x)) dx.
\end{aligned}$$

As a consequence of the above estimate, since $\sum_j \chi_{\widetilde{B}_j} \leq L$ and $\Omega = \cup_j \widetilde{B}_j$, we obtain

$$\begin{aligned}\sum_j \int\limits_{\mathbb{R}^n} \varphi\Big(x, (b_j)^*_m(x)\Big)dx &\leq C\sum_j \int\limits_{\widetilde{B}_j} \varphi(x, f^*_m(x))dx \\ &\leq C\int\limits_{\Omega} \varphi(x, f^*_m(x))dx.\end{aligned}$$

This implies that the series $\sum_j b_j$ converges in $H^{\varphi}_m(\mathbb{R}^n)$ by completeness of $H^{\varphi}_m(\mathbb{R}^n)$. Moreover,

$$\int\limits_{\mathbb{R}^n} \varphi\Big(x, (\sum_j b_j)^*_m(x)\Big)dx \leq C\int\limits_{\Omega} \varphi(x, f^*_m(x))dx.$$

$\square$

Let $q \in [1, \infty]$. We denote by $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$ the usually weighted Lebesgue space with the Muckenhoupt weight $\varphi(x, 1)$. Then, we have the following.

Lemma B (see [20], Lemma 4.8)**.** *Let $q \in (q(\varphi), \infty]$. Assume that $f \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$, then the series $\sum_j b_j$ converges in $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$ and there exists a constant C, independent of f, j, and λ such that $\| \sum_j |b_j|\|_{L^q_{\varphi(\cdot,1)}} \leq C\|f\|_{L^q_{\varphi(\cdot,1)}}$.*

Remark 3.5.1. *By Lemma B, the series $\sum_j |b_j|$, and thus the series $\sum_j b_j$, converges almost everywhere on $\mathbb{R}^n$.*

Lemma C (see [51], Lemma 3.19)**.** *Suppose that the series $\sum_j b_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Then, there exists a positive constant C, independent of f, j, and λ, such that for all $x \in \mathbb{R}^n$,*

$$g^*_m(x) \leq C\lambda \sum_j \Big(\frac{r_j}{|x - x_j| + r_j}\Big)^{n+m_s} + f^*_m(x)\chi_{\Omega^c}(x).$$

Lemma 3.5.3. *For any $q \in (q(\varphi), \infty)$ and $f \in H^{\varphi}_m(\mathbb{R}^n)$. Then $g^*_m \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$, and there exists a positive constant C, independent of f, j, and λ, such that*

$$\int\limits_{\mathbb{R}^n} [g^*_m(x)]^q \varphi(x, 1)dx \leq C\lambda^q \max\{1/\lambda, 1/\lambda^p\} \int\limits_{\mathbb{R}^n} \varphi(x, f^*_m(x))dx.$$

Proof. For any $j = 1, 2, ...$ and $x \in \mathbb{R}^n$, we have

$$\Big(\frac{r_j}{|x - x_j| + r_j}\Big)^n = \frac{1}{|B(x_j, |x - x_j| + r_j)|} \int\limits_{B(x_j,|x-x_j|+r_j)} \chi_{B_j}(y)dy \leq \mathcal{M}(\chi_{B_j})(x)$$

since $B_j \subset B(x_j, |x - x_j| + r_j)$.

Therefore, by $L^{rq}_{\varphi(\cdot,1)}$-boundedness of vector-valued maximal functions (see [3], Theorem 3.1), where $r := (n + m_s)/n > 1$, we obtain that

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \Big[\sum_j \Big(\frac{r_j}{|x-x_j|+r_j}\Big)^{n+m_s}\Big]^q \varphi(x,1)dx &\leq \int\limits_{\mathbb{R}^n} \Big[\Big(\sum_j (\mathcal{M}(\chi_{B_j})(x))^r\Big)^{1/r}\Big]^{rq} \varphi(x,1)dx \\
&\leq C_{s,q} \int\limits_{\mathbb{R}^n} \Big[\Big(\sum_j (\chi_{B_j}(x))^r\Big)^{1/r}\Big]^{rq} \varphi(x,1)dx \\
&\leq C_{s,q} L \int\limits_{\Omega} \varphi(x,1)dx \\
&\leq C \max\{1/\lambda, 1/\lambda^p\} \int\limits_{\mathbb{R}^n} \varphi(x, f^*_m(x))dx
\end{aligned}$$

for some $p \in (0, i(\varphi))$ since $\varphi \in \mathbb{A}_q \subset \mathbb{A}_{rq}$ and $f^*_m > \lambda$ on Ω. Combine this, Lemma C and Hölder inequality, we obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} [g^*_m(x)]^q \varphi(x,1)dx &\leq C\lambda^q \max\{1/\lambda, 1/\lambda^p\} \int\limits_{\mathbb{R}^n} \varphi(x, f^*_m(x))dx + C \int\limits_{\Omega^c} [f^*_m(x)]^q \varphi(x,1)dx \\
&\leq C\lambda^q \max\{1/\lambda, 1/\lambda^p\} \int\limits_{\mathbb{R}^n} \varphi(x, f^*_m(x))dx,
\end{aligned}$$

since $f^*_m \leq \lambda$ on Ω^c, here one used $\varphi(x,\lambda)/\lambda^q \leq C\varphi(x, f^*_m(x))/[f^*_m(x)]^q$ on Ω^c. □

Proposition 3.5.3. *For any $q \in (q(\varphi), \infty)$ and $m \geq m(\varphi)$. The subspace $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$ is dense in $H^\varphi_m(\mathbb{R}^n)$.*

Proof. Let f be an arbitrary element in $H^\varphi_m(\mathbb{R}^n)$. For each $\lambda > 0$, let $f = g^\lambda + \sum_j b^\lambda_j$ be the Calderon-Zygmund decomposition of f of degree $m(\varphi)$, and height λ associated with f^*_m. Then by Lemma 3.5.2 and Lemma 3.5.3, $g^\lambda \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$, moreover,

$$\int\limits_{\mathbb{R}^n} \varphi(x, (g^\lambda - f)^*_m(x))dx \leq C \int\limits_{f^*_m(x)>\lambda} \varphi(x, f^*_m(x))dx \to 0,$$

as $\lambda \to \infty$. Consequently, $\|g^\lambda - f\|_{H^\varphi_m} \to 0$ as $\lambda \to \infty$ by Lemma 3.4.2. Thus $L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi_m(\mathbb{R}^n)$ is dense in $H^\varphi_m(\mathbb{R}^n)$. □

3.5.3 The atomic decompositions $H^\varphi_m(\mathbb{R}^n)$

Recall that $m, s \geq m(\varphi)$, and f is a distribution such that $f^*_m \in L^\varphi(\mathbb{R}^n)$. For each $k \in \mathbb{Z}$, let $f = g^k + \sum_j b^k_j$ be the Calderón-Zygmund decomposition of f of degree s and height 2^k

associated with f_m^*. We shall label all the ingredients in this construction as in subsection 3.5.2, but with superscript $k's$: for example,

$$\Omega^k = \{x \in \mathbb{R}^n : f_m^*(x) > 2^k\}, \qquad b_j^k = (f - P_j^k)\zeta_j^k, \qquad B_j^k = B(x_j^k, r_j^k).$$

Moreover, for each $k \in \mathbb{Z}$, and i, j, let $P_{i,j}^{k+1}$ be the orthogonal projection of $(f - P_j^{k+1})\zeta_i^k$ onto $\mathcal{P}_s$ with respect to the norm associated to ζ_j^{k+1}, namely, the unique element of $\mathcal{P}_s$ such that for all $Q \in \mathcal{P}_s$,

$$\int_{\mathbb{R}^n} (f(x) - P_j^{k+1}(x))\zeta_i^k(x)Q(x)\zeta_j^{k+1}(x)dx = \int_{\mathbb{R}^n} P_{i,j}^{k+1}(x)Q(x)\zeta_j^{k+1}(x)dx.$$

For convenience, we set $\hat{B}_j^k = B(x_j^k, 2r_j^k)$. Then we have the following lemma.

Lemma D (see [51], Chapter 3). *i) If $\hat{B}_j^{k+1} \cap \hat{B}_i^k \neq \emptyset$, then $r_j^{k+1} < 4r_i^k$ and $\hat{B}_j^{k+1} \subset B(x_i^k, 18r_i^k)$.*
ii) For each j there are at most L (depending only on n as in last section) values of i such that $\hat{B}_j^{k+1} \cap \hat{B}_i^k \neq \emptyset$.
iii) There is a constant $C > 0$, independent of f, i, j, and k, such that

$$\sup_{x \in \mathbb{R}^n} |P_{i,j}^{k+1}(x)\zeta_j^{k+1}(x)| \leq C2^{k+1}.$$

iv) For every $k \in \mathbb{Z}$, $\sum_i(\sum_j P_{i,j}^{k+1}\zeta_j^{k+1}) = 0$, where the series converges pointwise and in $\mathcal{S}'(\mathbb{R}^n)$.

We now give the necessary estimates for proving that $H_m^\varphi(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$, $m \geq s \geq m(\varphi)$, and the inclusion is continuous.

Lemma 3.5.4. *Let $f \in H_m^\varphi(\mathbb{R}^n)$, and for each $k \in \mathbb{Z}$, set*

$$\Omega^k = \{x \in \mathbb{R}^n : f_m^*(x) > 2^k\}.$$

Then for any $\lambda > 0$, there exists a constant C, independent of f and λ, such that

$$\sum_{k=-\infty}^{\infty} \varphi\Big(\Omega^k, \frac{2^k}{\lambda}\Big) \leq C \int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx.$$

Proof. Let $p \in (0, i(\varphi))$ and C_p is such that (3.1) holds. We now set $N_0 = [(\log_2 C_p)/p] + 1$ so that $2^{N_0 p} > C_p$. For each $\ell \in \mathbb{N}, 0 \leq \ell \leq N_0 - 1$, we consider the sequence $U_m^\ell =$

$\sum_{k=-m}^{m} \varphi\Big(\Omega^{N_0 k+\ell}, \frac{2^{N_0 k+\ell}}{\lambda}\Big)$. Obviously, $\{U_m^\ell\}_{m\in\mathbb{N}}$ is an increasing sequence. Moreover, for any $m \in \mathbb{N}$,

$$\begin{aligned}
U_m^\ell &= \sum_{k=-m}^{m} \varphi\Big(\Omega^{N_0(k+1)+\ell}, \frac{2^{N_0k+\ell}}{\lambda}\Big) + \sum_{k=-m}^{m} \Big\{\varphi\Big(\Omega^{N_0k+\ell}, \frac{2^{N_0k+\ell}}{\lambda}\Big) - \varphi\Big(\Omega^{N_0(k+1)+\ell}, \frac{2^{N_0k+\ell}}{\lambda}\Big)\Big\} \\
&\leq C_p \frac{1}{2^{N_0 p}}\Big\{U_m^\ell + \varphi\Big(\Omega^{N_0(m+1)+\ell}, \frac{2^{N_0(m+1)+\ell}}{\lambda}\Big) + \varphi\Big(\Omega^{N_0(-m)+\ell}, \frac{2^{N_0(-m)+\ell}}{\lambda}\Big)\Big\} + \\
&\qquad\qquad + \sum_{k=-m}^{m} \int\limits_{\Omega^{N_0k+\ell}\setminus\Omega^{N_0(k+1)+\ell}} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx \\
&\leq \frac{C_p}{2^{N_0 p}} U_m^\ell + \Big(2\frac{C_p}{2^{N_0p}} + 1\Big) \int\limits_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx.
\end{aligned}$$

This implies that $U_m^\ell \leq \frac{3}{1-C_p/(2^{N_0p})} \int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx$. Consequently,

$$\sum_{k=-\infty}^{\infty} \varphi\Big(\Omega^k, \frac{2^k}{\lambda}\Big) = \sum_{\ell=0}^{N_0-1} \lim_{m\to\infty} U_m^\ell \leq C \int\limits_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda}\Big)dx,$$

where $C = \frac{3N_0}{1-C_p/(2^{N_0p})}$ independent of f and λ. □

Theorem 3.5.2. *Let $m \geq s \geq m(\varphi)$. Then, $H_m^\varphi(\mathbb{R}^n) \subset H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. Suppose first that $f \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H_m^\varphi(\mathbb{R}^n)$ for some $q \in (q(\varphi), \infty)$. Let $f = g^k + \sum_j b_j^k$ be the Calderón-Zygmund decompositions of f of degree s with height 2^k, for $k \in \mathbb{Z}$ associated with f_m^*. By Proposition 3.5.3, $g^k \to f$ in $H_m^\varphi(\mathbb{R}^n)$ as $k \to \infty$, while by Lemma 4.10 of [20], $g^k \to 0$ uniformly as $k \to -\infty$ since $f \in L^q_{\varphi(\cdot,1)}(\mathbb{R}^n)$. Therefore, $f = \sum_{-\infty}^{\infty}(g^{k+1} - g^k)$ in $\mathcal{S}'(\mathbb{R}^n)$. Using Lemma 3.27 of [51] together with the equation $\sum_i \zeta_i^k b_j^{k+1} = \chi_{\Omega^k} b_j^{k+1} = b_j^{k+1}$ by $\mathrm{supp} b_j^{k+1} \subset \Omega^{k+1} \subset \Omega^k$, we get

$$\begin{aligned}
g^{k+1} - g^k &= (f - \sum_j b_j^{k+1}) - (f - \sum_i b_i^k) \\
&= \sum_i b_i^k - \sum_j b_j^{k+1} + \sum_i \sum_j P_{i,j}^{k+1} \zeta_j^{k+1} \\
&= \sum_i \Big[b_i^k - \sum_j \Big(\zeta_i^k b_j^{k+1} - P_{i,j}^{k+1} \zeta_j^{k+1}\Big)\Big] \\
&= \sum_i h_i^k
\end{aligned}$$

where all the series converge in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Furthermore,

$$h_i^k = (f - P_i^k)\zeta_i^k - \sum_j \Big((f - P_j^{k+1})\zeta_i^k - P_{i,j}^{k+1} \Big) \zeta_j^{k+1}. \tag{3.9}$$

From this formula it is obvious that $\int_{\mathbb{R}^n} h_i^k(x) P(x) dx = 0$ for all $P \in \mathcal{P}_s$. Moreover, $h_i^k = \zeta_i^k f \chi_{(\Omega^{k+1})^c} - P_i^k \zeta_i^k + \zeta_i^k \sum_j P_j^{k+1} \zeta_j^{k+1} + \sum_j P_{i,j}^{k+1} \zeta_j^{k+1}$, by $\sum_j \zeta_j^{k+1} = \chi_{\Omega^{k+1}}$. But $|f(x)| \le C(m) f_m^*(x) \le C 2^{k+1}$ for almost every $x \in (\Omega^{k+1})^c$, so by Lemma 3.8 and (3.26) of [51], and $\sum_j \zeta_j^{k+1} \le L$,

$$\|h_i^k\|_{L^\infty} \le C 2^{k+1} + C 2^k + C L 2^{k+1} + C L 2^{k+1} \le C 2^k, \tag{3.10}$$

Lastly, since $P_{i,j}^{k+1} = 0$ unless $\hat{B}_i^k \cap \hat{B}_j^{k+1} \ne \emptyset$, it follows from (3.9) and Lemma 3.24 of [51], that h_i^k is supported in $B(x_i^k, 18 r_i^k)$. Thus h_i^k is a multiple of (φ, ∞, s)-atom. Moreover, by (3.10) and Lemma 3.5.4, for any $\lambda > 0$,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sum_i \varphi\Big(B(x_i^k, 18 r_i^k), \frac{\|h_i^k\|_{L^\infty}}{\lambda} \Big) &\le \sum_{k \in \mathbb{Z}} L \varphi(\Omega^k, C 2^k / \lambda) \\ &\le C \int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\lambda} \Big) dx < \infty. \end{aligned}$$

Thus the series $\sum_{k \in \mathbb{Z}} \sum_i h_i^k$ converges in $H_{\mathrm{at}}^{\varphi,\infty,s}(\mathbb{R}^n)$ and defines an atomic decomposition of f. Moreover,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sum_i \varphi\Big(B(x_i^k, 18 r_i^k), \frac{\|h_i^k\|_{L^\infty}}{\|f\|_{H_m^\varphi}} \Big) &\le C \int_{\mathbb{R}^n} \varphi\Big(x, \frac{f_m^*(x)}{\|f\|_{H_m^\varphi}} \Big) dx \\ &\le C. \end{aligned}$$

Consequently, $\|f\|_{H_{\mathrm{at}}^{\varphi,\infty,s}} \le \Lambda_\infty(\{h_i^k\}) \le C \|f\|_{H_m^\varphi}$ by Lemma 3.4.3.

Now, let f be an arbitrary element of $H_m^\varphi(\mathbb{R}^n)$. There exists a sequence $\{f_\ell\}_{\ell \ge 1} \subset L^q_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H_m^\varphi(\mathbb{R}^n)$ such that $f = \sum_{\ell=1}^\infty f_\ell$ in $H_m^\varphi(\mathbb{R}^n)$ (thus in $\mathcal{S}'(\mathbb{R}^n)$) and $\|f_\ell\|_{H_m^\varphi} \le 2^{2-\ell} \|f\|_{H_m^\varphi}$ for any $\ell \ge 1$. For any $\ell \ge 1$, let $f_\ell = \sum_j b_{j,\ell}$ be the atomic decomposition of f_ℓ, with supp $b_{j,\ell} \subset B_{j,\ell}$ constructed above. Then $f = \sum_{\ell=1}^\infty \sum_j b_{j,\ell}$ is an atomic decomposition of f, and

$$\begin{aligned} \sum_{\ell=1}^\infty \sum_j \varphi\Big(B_{j,\ell}, \frac{\|b_{j,\ell}\|_{L^\infty}}{\|f\|_{H_m^\varphi}} \Big) &\le \sum_{\ell=1}^\infty \sum_i \varphi\Big(B_{j,\ell}, \frac{\|b_{j,\ell}\|_{L^\infty}}{2^{\ell-2} \|f_\ell\|_{H_m^\varphi}} \Big) \\ &\le \sum_{\ell=1}^\infty C_p \frac{1}{(2^{\ell-2})^p} =: C, \end{aligned}$$

where C_p is such that (3.1) holds. Thus $f \in H_{\rm at}^{\varphi,\infty,s}(\mathbb{R}^n)$, moreover,

$$\|f\|_{H_{\rm at}^{\varphi,\infty,s}} \le \Lambda_\infty(\{b_{j,\ell}\}) \le C\|f\|_{H_m^\varphi}$$

by Lemma 3.4.3. This completes the proof. $\square$

Proof of Theorem 3.3.1. By Theorem 3.5.1 and Theorem 3.5.2, we obtain

$$H_{\rm at}^{\varphi,\infty,s}(\mathbb{R}^n) \subset H_{\rm at}^{\varphi,q,s}(\mathbb{R}^n) \subset H_{\rm at}^{\varphi,q,m(\varphi)}(\mathbb{R}^n) \subset H^\varphi(\mathbb{R}^n) \subset H_s^\varphi(\mathbb{R}^n) \subset H_{\rm at}^{\varphi,\infty,s}(\mathbb{R}^n)$$

and the inclusions are continuous. Thus $H^\varphi(\mathbb{R}^n) = H_{\rm at}^{\varphi,q,s}(\mathbb{R}^n)$ with equivalent norms. $\square$

3.6 Dual spaces

In this section, we give the proof of Theorem 3.3.2. In order to do this, we need the below lemma, which can be seen as a consequence of the fact that $\varphi(\cdot,t)$ is *uniformly locally integrable*. We omit the details here.

Lemma 3.6.1. *Given a ball B, and $\{B_j\}_j$ be a sequence of measurable subsets of B such that $\lim\limits_{j\to\infty}|B_j| = 0$. Then the following holds*

$$\lim_{j\to\infty}\sup_{t>0}\frac{\varphi(B_j,t)}{\varphi(B,t)} = 0.$$

We next note that if $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ is real-valued and

$$\mathfrak{b}_N(x) = \begin{cases} N & \text{if} \quad \mathfrak{b}(x) > N, \\ \mathfrak{b}(x) & \text{if} \quad |\mathfrak{b}(x)| \le N, \\ -N & \text{if} \quad \mathfrak{b}(x) < -N, \end{cases}$$

then by using the fact

$$\|f\|_{BMO^\varphi} \le \sup_{B-\text{ball}} \frac{1}{\|\chi_B\|_{L^\varphi}}\frac{1}{|B|}\int_B\int_B |f(x)-f(y)|dxdy \le 2\|f\|_{BMO^\varphi},$$

we obtain that $\|\mathfrak{b}_N\|_{BMO^\varphi} \le 2\|\mathfrak{b}\|_{BMO^\varphi}$ for all $N > 0$.

Proof of Theorem 3.3.2. i) It is sufficient to prove it for $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ real-valued since $\mathfrak{b} \in BMO^\varphi(\mathbb{R}^n)$ iff $\mathfrak{b} = \mathfrak{b}_1 + i\mathfrak{b}_2$ with $\mathfrak{b}_j \in BMO^\varphi(\mathbb{R}^n)$ real-valued, $j = 1, 2$, moreover

$$\|\mathfrak{b}\|_{BMO^\varphi} \approx \|\mathfrak{b}_1\|_{BMO^\varphi} + \|\mathfrak{b}_2\|_{BMO^\varphi}.$$

Suppose first that $\mathfrak{b} \in BMO^{\varphi}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. Then, the functional

$$L_{\mathfrak{b}}(f) = \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx$$

is well defined for any $f \in L_0^{\infty}(\mathbb{R}^n)$ since $\mathfrak{b} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Furthermore, since $f \in L_0^{\infty}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, we remark that the atomic decomposition $f = \sum_{k\in\mathbb{Z}}\sum_i h_i^k$ in the proof of Theorem 3.5.2 is also the classical atomic decomposition of f in $H^1(\mathbb{R}^n)$, so that the series converge in $H^1(\mathbb{R}^n)$ and thus in $L^1(\mathbb{R}^n)$. Combining this with the fact $\mathfrak{b} \in L^{\infty}(\mathbb{R}^n)$, we obtain

$$L_{\mathfrak{b}}(f) = \int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx = \sum_{k\in\mathbb{Z}}\sum_i \int_{\mathbb{R}^n} h_i^k(x)\mathfrak{b}(x)dx.$$

Therefore, by Lemma 3.4.4 and the proof of Theorem 3.5.2,

$$\begin{aligned}
|L_{\mathfrak{b}}(f)| = \Big|\int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx\Big| &\leq \sum_{k\in\mathbb{Z}}\sum_i \Big|\int_{\mathbb{R}^n} h_i^k(x)\mathfrak{b}(x)dx\Big| \\
&= \sum_{k\in\mathbb{Z}}\sum_i \Big|\int_{B(x_i^k,18r_i^k)} h_i^k(x)(\mathfrak{b}(x) - \mathfrak{b}_{B(x_i^k,18r_i^k)}(x))dx\Big| \\
&\leq \|\mathfrak{b}\|_{BMO^{\varphi}} \sum_{k\in\mathbb{Z}}\sum_i \|h_i^k\|_{L^{\infty}}\|\chi_{B(x_i^k,18r_i^k)}\|_{L^{\varphi}} \\
&\leq C\|\mathfrak{b}\|_{BMO^{\varphi}}\Lambda_{\infty}(\{h_i^k\}) \\
&\leq C\|\mathfrak{b}\|_{BMO^{\varphi}}\|f\|_{H^{\varphi}}.
\end{aligned}$$

Now, let $\mathfrak{b}$ be an arbitrary element in $BMO^{\varphi}(\mathbb{R}^n)$. For any $f \in L_0^{\infty}(\mathbb{R}^n)$, it is clear that $|f\mathfrak{b}_{\ell}| \leq |f\mathfrak{b}| \in L^1(\mathbb{R}^n)$ for every $\ell \geq 1$, and $f(x)\mathfrak{b}_{\ell}(x) \to f(x)\mathfrak{b}(x)$, as $\ell \to \infty$, for almost every $x \in \mathbb{R}^n$. Therefore, by the dominated convergence theorem of Lebesgue, we obtain

$$|L_{\mathfrak{b}}(f)| = \Big|\int_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx\Big| = \lim_{\ell\to\infty}\Big|\int_{\mathbb{R}^n} f(x)\mathfrak{b}_{\ell}(x)dx\Big| \leq C\|\mathfrak{b}\|_{BMO^{\varphi}}\|f\|_{H^{\varphi}},$$

since $\|\mathfrak{b}_{\ell}\|_{BMO^{\varphi}} \leq 2\|\mathfrak{b}\|_{BMO^{\varphi}}$ for all $\ell \geq 1$.

Because of the density of $L_0^{\infty}(\mathbb{R}^n)$ in $H^{\varphi}(\mathbb{R}^n)$, the functional $L_{\mathfrak{b}}$ can be extended to a bounded functional on $H^{\varphi}(\mathbb{R}^n)$, moreover, $\|L_{\mathfrak{b}}\|_{(H^{\varphi})^*} \leq C\|\mathfrak{b}\|_{BMO^{\varphi}}$.

ii) Conversely, suppose L is a continuous linear functional on $H^{\varphi}(\mathbb{R}^n) \equiv H^{\varphi,q,0}_{\mathrm{at}}(\mathbb{R}^n)$ for some $q \in (q(\varphi), \infty)$. For any ball B, denote by $L^q_{\varphi,0}(B)$ the subspace of $L^q_{\varphi}(B)$ defined by

$$L^q_{\varphi,0}(B) := \Big\{f \in L^q_{\varphi}(B) : \int_{\mathbb{R}^n} f(x)dx = 0\Big\}.$$

Obviously, if $B_1 \subset B_2$ then

$$L^q_\varphi(B_1) \subset L^q_\varphi(B_2) \quad \text{and} \quad L^q_{\varphi,0}(B_1) \subset L^q_{\varphi,0}(B_2). \tag{3.11}$$

Moreover, when $f \in L^q_{\varphi,0}(B) \setminus \{0\}$, $a(x) = \|\chi_B\|^{-1}_{L^\varphi}\|f\|^{-1}_{L^q_\varphi(B)}f(x)$ is a $(\varphi, q, 0)$-atom, thus $f \in H^{\varphi,q,0}_{\rm at}(\mathbb{R}^n)$ and

$$\|f\|_{H^{\varphi,q,0}_{\rm at}} \leq \|\chi_B\|_{L^\varphi}\|f\|_{L^q_\varphi(B)}.$$

Since $L \in (H^{\varphi,q,0}_{\rm at}(\mathbb{R}^n))^*$, by the above,

$$|L(f)| \leq \|L\|_{(H^{\varphi,q,0}_{\rm at})^*}\|f\|_{H^{\varphi,q,0}_{\rm at}} \leq \|L\|_{(H^{\varphi,q,0}_{\rm at})^*}\|\chi_B\|_{L^\varphi}\|f\|_{L^q_\varphi(B)},$$

for all $f \in L^q_{\varphi,0}(B)$. Therefore, L provides a bounded linear functional on $L^q_{\varphi,0}(B)$ which can be extended by the Hahn-Banach theorem to the whole space $L^q_\varphi(B)$ without increasing its norm. On the other hand, by Lemma 3.6.1 and Lebesgue-Nikodym theorem, there exists $h \in L^1(B)$ such that

$$L(f) = \int\limits_{\mathbb{R}^n} f(x)h(x)dx,$$

for all $f \in L^q_{\varphi,0}(B)$, and thus $f \in L^\infty_{\varphi,0}(B)$ since $L^\infty_{\varphi,0}(B) \subset L^q_{\varphi,0}(B)$.

We now take a sequence of balls $\{B_j\}_{j\geq 1}$ such that $B_1 \subset B_2 \subset \cdots \subset B_j \subset \cdots$ and $\cup_j B_j = \mathbb{R}^n$. Then, there exists a sequence $\{h_j\}_{j\geq 1}$ such that

$$h_j \in L^1(B_j) \quad \text{and} \quad L(f) = \int\limits_{\mathbb{R}^n} f(x)h_j(x)dx,$$

for all $f \in L^\infty_{\varphi,0}(B_j), j = 1, 2, ...$ Hence, for all $f \in L^\infty_{\varphi,0}(B_1) \subset L^\infty_{\varphi,0}(B_2)$ (by (3.11)),

$$\int\limits_{\mathbb{R}^n} f(x)(h_1(x) - h_2(x))dx = \int\limits_{\mathbb{R}^n} f(x)h_1(x)dx - \int\limits_{\mathbb{R}^n} f(x)h_2(x)dx = L(f) - L(f) = 0.$$

As $f_{B_1} = 0$ if $f \in L^\infty_{\varphi,0}(B_1)$, we have

$$\int\limits_{\mathbb{R}^n} f(x)\Big((h_1(x) - h_2(x)) - (h_1 - h_2)_{B_1}\Big)dx = 0$$

for all $f \in L^\infty_{\varphi,0}(B_1)$, and thus for $f \in L^\infty_\varphi(B_1)$. Hence,

$$h_1(x) - h_2(x) = (h_1 - h_2)_{B_1}\ ,\ a.e\ x \in B_1.$$

By the similar arguments, we also obtain

$$h_j(x) - h_{j+1}(x) = (h_j - h_{j+1})_{B_j} \tag{3.12}$$

a.e $x \in B_j, j = 2, 3, ...$ Consequently, if we define the sequence $\{\widetilde{h}_j\}_{j\geq 1}$ by

$$\begin{cases} \widetilde{h}_1 = h_1 \\ \widetilde{h}_{j+1} = h_{j+1} + (\widetilde{h}_j - h_{j+1})_{B_j} \quad , \quad j = 1, 2, ... \end{cases}$$

then it follows from (3.12) that

$$\widetilde{h}_j \in L^1(B_j) \quad \text{and} \quad \widetilde{h}_{j+1}(x) = \widetilde{h}_j(x)$$

a.e $x \in B_j, j = 1, 2, ...$ Thus, we can be define the function $\mathfrak{b}$ on $\mathbb{R}^n$ by

$$\mathfrak{b}(x) = \widetilde{h}_j(x)$$

if $x \in B_j$ for some $j \geq 1$ since $B_1 \subset B_2 \subset \cdots \subset B_j \subset \cdots$ and $\cup_j B_j = \mathbb{R}^n$.

Let us now show that $\mathfrak{b} \in BMO^{\varphi}(\mathbb{R}^n)$ and

$$L(f) = \int\limits_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx,$$

for all $f \in L_0^{\infty}(\mathbb{R}^n)$.

Indeed, for any $f \in L_0^{\infty}(\mathbb{R}^n)$, there exists $j \geq 1$ such that $f \in L^{\infty}_{\varphi,0}(B_j)$. Hence,

$$L(f) = \int\limits_{\mathbb{R}^n} f(x)\widetilde{h}_j(x)dx = \int\limits_{\mathbb{B}_j} f(x)\widetilde{h}_j(x)dx = \int\limits_{\mathbb{R}^n} f(x)\mathfrak{b}(x)dx.$$

On the other hand, for all ball B, one consider $f = \text{sign}(\mathfrak{b} - \mathfrak{b}_B)$ where $\text{sign}\xi = \overline{\xi}/|\xi|$ if $\xi \neq 0$ and $\text{sign}0 = 0$. Then,

$$a = \frac{1}{2}\|\chi_B\|^{-1}_{L^{\varphi}}(f - f_B)\chi_B$$

is a $(\varphi, \infty, 0)$-atom. Consequently,

$$\begin{aligned} |L(a)| &= \frac{1}{2}\|\chi_B\|^{-1}_{L^{\varphi}}\Big|\int\limits_{\mathbb{R}^n} \mathfrak{b}(x)(f(x) - f_B)\chi_B(x)dx\Big| \\ &= \frac{1}{2}\frac{1}{\|\chi_B\|_{L^{\varphi}}}\Big|\int\limits_{B} (\mathfrak{b}(x) - \mathfrak{b}_B)f(x)dx\Big| \\ &= \frac{1}{2}\frac{1}{\|\chi_B\|_{L^{\varphi}}}\int\limits_{B} |\mathfrak{b}(x) - \mathfrak{b}_B|dx \\ &\leq \|L\|_{(H^{\varphi})^*}\|a\|_{H^{\varphi}} \leq C\|L\|_{(H^{\varphi})^*} \end{aligned}$$

since $L \in (H^{\varphi}(\mathbb{R}^n))^*$ and Corollary 5.2. As B is arbitrary, the above implies $\mathfrak{b} \in BMO^{\varphi}(\mathbb{R}^n)$ and

$$\|\mathfrak{b}\|_{BMO^{\varphi}} \leq C\|L\|_{(H^{\varphi})^*}.$$

The uniqueness (in the sense $\mathfrak{b} = \widetilde{\mathfrak{b}}$ if $\mathfrak{b} - \widetilde{\mathfrak{b}} = \text{const}$) of the function $\mathfrak{b}$ is clear. And thus the proof is finished. $\square$

3.7 The class of pointwise multipliers for $BMO(\mathbb{R}^n)$

In this subsection, we give as an interesting application that the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is just the dual of $L^1(\mathbb{R}^n)+H^{\log}(\mathbb{R}^n)$ where $H^{\log}(\mathbb{R}^n)$ is a Hardy space of Musielak-Orlicz type related to the Musielak-Orlicz function $\theta(x,t) = \frac{t}{\log(e+|x|)+\log(e+t)}$.

We first introduce *log-atoms*. A measurable function a is said to be *log-atom* if it satisfies the following three conditions

- a supported in B for some ball B in $\mathbb{R}^n$,
- $\|a\|_{L^\infty} \leq \dfrac{\log(e+\frac{1}{|B|}) + \sup_{x\in B}\log(e+|x|)}{|B|}$,
- $\int_{\mathbb{R}^n} a(x)dx = 0$.

To prove Theorem 3.3.3, we need the following two propositions.

Proposition 3.7.1. *There exists a positive constant C such that if f is a θ-atom (resp., log-atom) then $C^{-1}f$ is a log-atom (resp., θ-atom).*

Proposition 3.7.2. *On $BMO^{\log}(\mathbb{R}^n)$, we have*

$$\|f\|_{BMO^{\log}} \approx \sup_{B-\text{ball}} \frac{\log(e+\frac{1}{|B|}) + \sup_{x\in B}\log(e+|x|)}{|B|}\int_B |f(x) - f_B|dx < \infty.$$

We first note that θ is a *growth function* that satisfies $nq(\theta) < (n+1)i(\theta)$ in Theorem 3.3.2. More precisely, $\theta \in \mathbb{A}_1$ and $\theta(x,\cdot)$ is concave with $i(\theta) = 1$.

Proof of Proposition 3.7.1. Let f be a *log-atom*. By the above remark, to prove that there exists a constant $C > 0$ (independent of f and which may change from line to line) such that $C^{-1}f$ is a θ-atom, it is sufficient to show that there exists a constant $C > 0$ such that

$$\int_B \theta\Big(x, \frac{\log(e+\frac{1}{|B|}) + \sup_{x\in B}\log(e+|x|)}{|B|}\Big)dx \leq C$$

or, equivalently,

$$\frac{\frac{\log(e+\frac{1}{|B|})+\sup_{x\in B}\log(e+|x|)}{|B|}}{\log(e+\frac{\log(e+\frac{1}{|B|})+\sup_{x\in B}\log(e+|x|)}{|B|})+\sup_{x\in B}\log(e+|x|)}|B|\le C,$$

since $\theta\in\mathbb{A}_1$. However, the last inequality is obvious.

Conversely, suppose that f is a θ-atom. Similarly, we need to show that there exists a constant $C>0$ such that

$$\int_B \theta\Big(x, C\frac{\log(e+\frac{1}{|B|})+\sup_{x\in B}\log(e+|x|)}{|B|}\Big)dx\ge 1$$

or, equivalently,

$$\frac{C\frac{\log(e+\frac{1}{|B|})+\sup_{x\in B}\log(e+|x|)}{|B|}}{\log(e+C\frac{\log(e+\frac{1}{|B|})+\sup_{x\in B}\log(e+|x|)}{|B|})+\sup_{x\in B}\log(e+|x|)}|B|\ge 1.$$

However it is true. For instance we may take $C=3$. □

Proof of Proposition 3.7.2. It is sufficient to show that there exists a constant $C>0$ such that

$$C^{-1}(|\log r|+\log(e+|x|))\le\log\Big(e+\frac{1}{|B(x,r)|}\Big)+\sup_{y\in B(x,r)}\log(e+|y|)\le C(|\log r|+\log(e+|x|)).$$

The first inequality is easy and shall be omited. For the second, one first consider the 1 dimensional case. Then by symmetry, we just need to prove that

$$\log(e+1/(b-a))+\sup_{x\in[a,b]}\log(e+|x|)\le C(|\log(b-a)/2|+\log(e+|a+b|/2))$$

for all $b>0, a\in[-b,b)\subset\mathbb{R}$. However, this follows from the basic two inequalities:

$$\log(e+1/(b-a))\le 2(|\log(b-a)/2|+\log(e+|a+b|/2))$$

and

$$\log(e+b)\le 5\log(e+b)/2\le 5(|\log(b-a)/2|+\log(e+|a+b|/2)).$$

For the general case $\mathbb{R}^n$, by the 1-dimensional result, we obtain

$$\begin{aligned}\log\Big(e+\frac{1}{|B(x,r)|}\Big) &\le \frac{2^n}{c_n}\sum_{i=1}^n\log\Big(e+\frac{1}{|[x_i-r,x_i+r]|}\Big)\\ &\le C\sum_{i=1}^n(|\log r|+\log(e+|x_i|))\\ &\le C(|\log r|+\log(e+|x|))\end{aligned}$$

where $c_n = |B(0,1)|$, and

$$\begin{aligned}
\sup_{y\in B(x,r)} \log(e+|y|) &\leq \sum_{i=1}^{n} \sup_{y_i\in[x_i-r,x_i+r]} \log(e+|y_i|) \\
&\leq C\sum_{i=1}^{n}(|\log r| + \log(e+|x_i|)) \\
&\leq C(|\log r| + \log(e+|x|))
\end{aligned}$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$. This finishes the proof. $\square$

Proof of Theorem 3.3.3. By Theorem 3.3.1, Theorem 3.3.2, Proposition 3.7.1, and Proposition 3.7.2, we obtain $(H^{\log}(\mathbb{R}^n))^* \equiv BMO^{\log}(\mathbb{R}^n)$. We deduce that, the class of pointwise multipliers for $BMO(\mathbb{R}^n)$ is the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$. $\square$

3.8 Finite atomic decompositions and their applications

We first prove the finite atomic decomposition theorem.

Proof of Theorem 3.3.4. Obviously, $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n) \subset H^{\varphi}(\mathbb{R}^n)$ and for all $f \in H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$,

$$\|f\|_{H^\varphi} \leq C\|f\|_{H^{\varphi,q,s}_{\rm fin}}.$$

Thus, we have to show that for every $q \in (q(\varphi), \infty)$ there exists a constant $C > 0$ such that

$$\|f\|_{H^{\varphi,q,s}_{\rm fin}} \leq C\|f\|_{H^\varphi}$$

for all $f \in H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ and that a similar estimate holds for $q = \infty$ and all $f \in H^{\varphi,\infty,s}_{\rm fin}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Assume that $q \in (q(\varphi), \infty]$, and by homogeneity, $f \in H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ with $\|f\|_{H^\varphi} = 1$. Notice that f has compact support. Suppose that supp $f \subset B = B(x_0, r)$ for some ball B. Recall that, for each $k \in \mathbb{Z}$,

$$\Omega_k = \{x \in \mathbb{R}^n : f^*(x) > 2^k\}.$$

Clearly, $f \in L^{\overline{q}}_{\varphi(\cdot,1)}(\mathbb{R}^n) \cap H^\varphi(\mathbb{R}^n)$ where $\overline{q} = q$ if $q < \infty$, $\overline{q} = q(\varphi) + 1$ if $q = \infty$. Hence, by Theorem 3.5.2, there exists a atomic decomposition $f = \sum_{k\in\mathbb{Z}}\sum_i h^k_i \in H^{\varphi,\infty,s}_{\rm at}(\mathbb{R}^n) \subset H^{\varphi,q,s}_{\rm at}(\mathbb{R}^n)$ where the series converges in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Moreover,

$$\Lambda_q(\{h^k_i\}) \leq \Lambda_\infty(\{h^k_i\}) \leq C\|f\|_{H^\varphi} = C. \tag{3.13}$$

On the other hand, it follows from the second step in the proof of Theorem 6.2 of [20] that there exists a constant $\widetilde{C} > 0$, depending only on $m(\varphi)$, such that $f^*(x) \le \widetilde{C} \inf_{y\in B} f^*(y)$ for all $x \in B(x_0, 2r)^c$. Hence, we have

$$f^*(x) \le \widetilde{C} \inf_{y\in B} f^*(y) \le \widetilde{C}\|\chi_B\|_{L^\varphi}^{-1}\|f^*\|_{L^\varphi} \le \widetilde{C}\|\chi_B\|_{L^\varphi}^{-1}$$

for all $x \in B(x_0, 2r)^c$. We now denote by k' the largest integer k such that $2^k < \widetilde{C}\|\chi_B\|_{L^\varphi}^{-1}$. Then,

$$\Omega_k \subset B(x_0, 2r) \quad \text{for all } k > k'. \tag{3.14}$$

Next we define the functions g and ℓ by

$$g = \sum_{k\le k'}\sum_i h_i^k \quad \text{and} \quad \ell = \sum_{k>k'}\sum_i h_i^k,$$

where the series converge in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Clearly, $f = g + \ell$ and supp $\ell \subset \cup_{k>k'}\Omega_k \subset B(x_0, 2r)$ by (3.14). Therefore, $g = f = 0$ in $B(x_0, 2r)^c$, and thus supp $g \subset B(x_0, 2r)$.

Let $1 < \widetilde{q} < \frac{q}{q(\varphi)}$, then $\varphi \in \mathbb{A}_{q/\widetilde{q}}$. Consequently,

$$\Big(\frac{1}{|B|}\int_B |f(x)|^{\widetilde{q}}dx\Big)^{1/\widetilde{q}} \le C\left(\frac{1}{\varphi(B,1)}\int_B |f(x)|^q\varphi(x,1)dx\right)^{1/q} < \infty$$

by Lemma 3.4.5 if $q < \infty$ and it is trivial if $q = \infty$. Observe that supp $f \subset B$ and that f has vanishing moments up to order s. By the above, we obtain that f is a multiple of a classical $(1, \widetilde{q}, 0)$-atom and thus $f^* \in L^1(\mathbb{R}^n)$. Hence, it follows from (3.14) that

$$\int_{\mathbb{R}^n}\sum_{k>k'}\sum_i |h_i^k(x)x^\alpha|dx \le C(|x_0| + 2r)^s\sum_{k>k'}2^k|\Omega_k| \le C(|x_0| + 2r)^s\|f^*\|_{L^1} < \infty,$$

for all $|\alpha| \le s$. This together with the vanishing moments of h_i^k implies that ℓ has vanishing moments up to order s and thus so does g by $g = f - \ell$.

In order to estimate the size of g in $B(x_0, 2r)$, we recall that

$$\|h_i^k\|_{L^\infty} \le C2^k \ , \text{ supp } h_i^k \subset B(x_i^k, 18r_i^k) \text{ and } \sum_i \chi_{B(x_i^k,18r_i^k)} \le C. \tag{3.15}$$

Combining the above and the fact $\|\chi_B\|_{L^\varphi} \approx \|\chi_{B(x_0,2r)}\|_{L^\varphi}$, we obtain

$$\|g\|_{L^\infty} \le C\sum_{k\le k'}2^k \le C2^{k'} \le C\widetilde{C}\|\chi_B\|_{L^\varphi}^{-1} \le C\|\chi_{B(x_0,2r)}\|_{L^\varphi}^{-1}.$$

This proves that (see Definition 3.2.4)

$$C^{-1}g \text{ is a } (\varphi,\infty,s)\text{−atom.} \tag{3.16}$$

Now, we assume that $q \in (q(\varphi),\infty)$ and conclude the proof of (i). We first verify $\sum_{k>k'}\sum_i h_i^k \in L^q_\varphi(B(x_0,2r))$. For any $x \in \mathbb{R}^n$, since $\mathbb{R}^n = \cup_{k\in\mathbb{Z}}(\Omega_k \setminus \Omega_{k+1})$, there exists $j \in \mathbb{Z}$ such that $x \in \Omega_j \setminus \Omega_{j+1}$. Since supp $h_i^k \subset \Omega_k \subset \Omega_{j+1}$ for $k \geq j+1$, it follows from (3.15) that

$$\sum_{k>k'}\sum_i |h_i^k(x)| \leq C\sum_{k\leq j} 2^k \leq C2^j \leq Cf^*(x).$$

Since $f \in L^q_\varphi(B) \subset L^q_\varphi(B(x_0,2r))$, we have $f^* \in L^q_\varphi(B(x_0,2r))$. As φ satisfies uniformly locally dominated convergence condition, we further obtain $\sum_{k>k'}\sum_i h_i^k$ converges to ℓ in $L^q_\varphi(B(x_0,2r))$.

Now, for any positive integer K, set $F_K = \{(i,k) : k > k', |i| + |k| \leq K\}$ and $\ell_K = \sum_{(i,k)\in F_K} h_i^k$. Observe that since $\sum_{k>k'}\sum_i h_i^k$ converges to ℓ in $L^q_\varphi(B(x_0,2r))$, for any $\varepsilon > 0$, if K is large enough, we have $\varepsilon^{-1}(\ell - \ell_K)$ is a (φ,q,s)-atom. Thus, $f = g + \ell_K + (\ell - \ell_K)$ is a finite linear atom combination of f. Then, it follows from (3.13) and (3.16) that

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{fin}}} \leq C(C + \Lambda_q(\{h_i^k\}_{(i,k)\in F_K}) + \varepsilon) \leq C,$$

which ends the proof of (i).

To prove (ii), assume that f is a continuous function in $H^{\varphi,\infty,s}_{\mathrm{fin}}(\mathbb{R}^n)$, and thus f is uniformly continuous. Then, h_i^k is continuous by examining its definition. Since f is bounded, there exists a positive integer $k'' > k'$ such that $\Omega_k = \emptyset$ for all $k > k''$. Consequently, $\ell = \sum_{k'<k\leq k''}\sum_i h_i^k$.

Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Write $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$ with

$$\ell_1^\varepsilon \equiv \sum_{(i,k)\in F_1} h_i^k \quad \text{and} \quad \ell_2^\varepsilon \equiv \sum_{(i,k)\in F_2} h_i^k$$

where $F_1 = \{(i,k) : Cr_i^k \geq \delta, k' < k \leq k''\}$ and $F_2 = \{(i,k) : Cr_i^k < \delta, k' < k \leq k''\}$ with $C > 36$ the geometric constant (see [106]). Notice that the remaining part ℓ_1^ε will then be a finite sum. Since the atoms are continuous, ℓ_1^ε will be a continuous function. Furthermore, $\|\ell_2^\varepsilon\|_{L^\infty} \leq C(k''-k')\varepsilon$ (see also [106]). This means that one can write ℓ as the sum of one continuous term and of one which is uniformly arbitrarily small. Hence, ℓ is continuous, and so is $g = f - \ell$.

To find a finite atomic decomposition of f, we use again the splitting $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$. By (3.13), the part ℓ_1^ε is a finite sum of multiples of (φ,∞,s)-atoms, and

$$\|\ell_1^\varepsilon\|_{H^{\varphi,\infty,s}_{\mathrm{fin}}} \leq \Lambda_\infty(\{h_i^k\}) \leq C\|f\|_{H^\varphi} = C. \tag{3.17}$$

By ℓ, ℓ_1^ε are continuous and have vanishing moments up to order s, and thus so does $\ell_2^\varepsilon = \ell - \ell_1^\varepsilon$. Moreover, supp $\ell_2^\varepsilon \subset B(x_0, 2r)$ and $\|\ell_2^\varepsilon\|_{L^\infty} \le C(k'' - k')\varepsilon$. So we can choose ε small enough such that ℓ_2^ε into an arbitrarily small multiple of a continuous (φ, ∞, s)-atom. Therefore, $f = g + \ell_1^\varepsilon + \ell_2^\varepsilon$ is a finite linear continuous atom combination of f. Then, it follows from (3.16) and (3.17) that

$$\|f\|_{H^{\varphi,\infty,s}_{\rm fin}} \le C(\|g\|_{H^{\varphi,\infty,s}_{\rm fin}} + \|\ell_1^\varepsilon\|_{H^{\varphi,\infty,s}_{\rm fin}} + \|\ell_2^\varepsilon\|_{H^{\varphi,\infty,s}_{\rm fin}}) \le C.$$

This finishes the proof of (ii) and hence, the proof of Theorem 3.4.

$\square$

Next we give the proof for Theorem 3.3.5.

Proof of Theorem 3.3.5. Suppose that the assumption (i) holds. For any $f \in H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$, by Theorem 3.3.4, there exists a finite atomic decomposition $f = \sum_{j=1}^k \lambda_j a_j$, where a_j's are multiples of (φ, q, s)-atoms with supported in balls B_j's, such that

$$\Lambda_q(\{\lambda_j a_j\}_{j=1}^k) = \inf\left\{\lambda > 0 : \sum_{j=1}^k \varphi\Big(B_j, \frac{|\lambda_j| \|\chi_{B_j}\|_{L^\varphi}^{-1}}{\lambda}\Big) \le 1\right\} \le C\|f\|_{H^\varphi}.$$

Recall that, since φ is of uniformly upper type γ, there exists a constant $C_\gamma > 0$ such that

$$\varphi(x, st) \le C_\gamma s^\gamma \varphi(x, t) \text{ for all } x \in \mathbb{R}^n, s \ge 1, t \in [0, \infty). \tag{3.18}$$

If there exist $j_0 \in \{1, ..., k\}$ such that $C_\gamma |\lambda_{j_0}|^\gamma \ge \sum_{j=1}^k |\lambda_j|^\gamma$, then

$$\sum_{j=1}^k \varphi\left(B_j, \frac{|\lambda_j| \|\chi_{B_j}\|_{L^\varphi}^{-1}}{C_\gamma^{-1/\gamma}(\sum_{j=1}^k |\lambda_j|^\gamma)^{1/\gamma}}\right) \ge \varphi(B_{j_0}, \|\chi_{B_{j_0}}\|_{L^\varphi}^{-1}) = 1.$$

Otherwise, it follows from (3.18) that

$$\sum_{j=1}^k \varphi\left(B_j, \frac{|\lambda_j| \|\chi_{B_j}\|_{L^\varphi}^{-1}}{C_\gamma^{-1/\gamma}(\sum_{j=1}^k |\lambda_j|^\gamma)^{1/\gamma}}\right) \ge \sum_{j=1}^k \frac{|\lambda_j|^\gamma}{\sum_{j=1}^k |\lambda_j|^\gamma} \varphi(B_j, \|\chi_{B_j}\|_{L^\varphi}^{-1}) = 1.$$

The above means that

$$\Big(\sum_{j=1}^k |\lambda_j|^\gamma\Big)^{1/\gamma} \le C_\gamma^{1/\gamma} \Lambda_q(\{\lambda_j a_j\}_{j=1}^k) \le C\|f\|_{H^\varphi}.$$

Therefore, by assumption (i), we obtain that

$$\|Tf\|_{\mathcal{B}_\gamma} = \left\|T\Big(\sum_{j=1}^k \lambda_j a_j\Big)\right\|_{\mathcal{B}_\gamma} \le C\Big(\sum_{j=1}^k |\lambda_j|^\gamma\Big)^{1/\gamma} \le C\|f\|_{H^\varphi}.$$

Since $H^{\varphi,q,s}_{\text{fin}}(\mathbb{R}^n)$ is dense in $H^{\varphi}(\mathbb{R}^n)$, a density argument gives the desired result.

The case (ii) is similar by using the fact that $H^{\varphi,\infty,s}_{\text{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^{\varphi,\infty,s}_{\text{fin}}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{H^{\varphi}}$, see the below lemma. $\square$

We end the paper by the following lemma.

Lemma 3.8.1. *Let φ be a growth function satisfying uniformly locally dominated convergence condition, and (φ, ∞, s) be an admissible triplet. Then, $H^{\varphi,\infty,s}_{\text{fin}}(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$ is dense in $H^{\varphi,\infty,s}_{\text{fin}}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{H^{\varphi}}$.*

Proof. We take $q \in (q(\varphi), \infty)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\operatorname{supp} \phi \subset B(0,1)$, $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Then, the proof of the lemma is simple since it follows from the fact that for every (φ, ∞, s)-atom a supported in ball $B(x_0, r)$,

$$\lim_{t\to 0} \|a - a * \phi_t\|_{L^q_{\varphi}(B(x_0,2r))} = 0$$

as φ satisfies uniformly locally dominated convergence condition. $\square$

Chapter 4

Bilinear decompositions and commutators of singular integral operators

Ce chapitre est un article qui a été accepté à "Transactions of the American Mathematical Society".

Résumé

Soit b une BMO-fonction. Il est bien connu que le commutateur linéaire $[b, T]$ d'un opérateur de Calderón-Zygmund T ne constitue pas, en général, un opérateur borné de $H^1(\mathbb{R}^n)$ dans $L^1(\mathbb{R}^n)$. Cependant, Pérez a montré que si $H^1(\mathbb{R}^n)$ est remplacé par un sous-espace approprié atomique $\mathcal{H}^1_b(\mathbb{R}^n)$, alors le commutateur est continu de $\mathcal{H}^1_b(\mathbb{R}^n)$ à valeurs $L^1(\mathbb{R}^n)$. Dans cet article, nous trouvons le plus grand sous-espace $H^1_b(\mathbb{R}^n)$ telle que tous les commutateurs des opérateurs Calderón-Zygmund sont continus de $H^1_b(\mathbb{R}^n)$ à valeurs $L^1(\mathbb{R}^n)$. Certaines caractérisations équivalentes de $H^1_b(\mathbb{R}^n)$ sont également données. Nous étudions également les commutateurs $[b, T]$ pour T dans une classe $\mathcal{K}$ des opérateurs sous-linéaire contenant presque tous les opérateurs importants dans l'analyse harmonique. Lorsque T est linéaire, nous montrons qu'il existe des opérateurs bilinéaires $\mathfrak{R} = \mathfrak{R}_T$ borné de $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ à valeurs $L^1(\mathbb{R}^n)$ tels que pour tout $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, nous avons

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)), \tag{4.1}$$

où $\mathfrak{S}$ est un opérateur borné bilinéaire de $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ à valeurs $L^1(\mathbb{R}^n)$, indépendant de T. Dans le cas particulier où T est un opérateur de Calderón-Zygmund satisfaisant $T1 = T^*1 = 0$ et b dans $BMO^{\log}(\mathbb{R}^n)$– l'espace généralisé de type BMO qui a été introduit

par Nakai et Yabuta pour caractériser les multiplicateurs de $BMO(\mathbb{R}^n)$–nous démontrons que le commutateur $[b, T]$ est continu de $H^1_b(\mathbb{R}^n)$ à valeurs $h^1(\mathbb{R}^n)$. En outre, si b est dans $BMO(\mathbb{R}^n)$ et $T^*1 = T^*b = 0$, alors le commutateur $[b, T]$ envoie $H^1_b(\mathbb{R}^n)$ dans $H^1(\mathbb{R}^n)$. Lorsque T est sous-linéaire, nous montrons qu'il existe un opérateur borné sous-bilinéaire $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ tel que pour tout $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, nous avons

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|. \tag{4.2}$$

La décomposition bilinéaire (4.1) et la décomposition sous-bilinéaire (4.2) nous permettent de donner un apercu général de toutes les estimations L^1 faibles ou fortes connues.

4.1 Introduction

Given a function b locally integrable on $\mathbb{R}^n$, and a Calderón-Zygmund operator T, we consider the linear commutator $[b,T]$ defined for smooth, compactly supported functions f by

$$[b,T](f) = bT(f) - T(bf).$$

A classical result of R. Coifman, R. Rochberg and G. Weiss (see [31]), states that the commutator $[b,T]$ is continuous on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, when $b \in BMO(\mathbb{R}^n)$. Unlike the theory of Calderón-Zygmund operators, the proof of this result does not rely on a weak type $(1,1)$ estimate for $[b,T]$. In fact, it was shown in [119] that, in general, the linear commutator fails to be of weak type $(1,1)$, when b is in $BMO(\mathbb{R}^n)$. Instead, an endpoint theory was provided for this operator. It is well-known that any singular integral operator maps $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. However, it was observed in [62] that the commutator $[b,H]$ with b in $BMO(\mathbb{R})$, where H is Hilbert transform on $\mathbb{R}$, does not map, in general, $H^1(\mathbb{R})$ into $L^1(\mathbb{R})$. Instead of this, the weak type estimate (H^1, L^1) for $[b,T]$ is well-known, see for example [96, 101, 139]. Remark that intuitively one would like to write

$$[b,T](f) = \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) T(a_j) - T\Big(\sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j\Big),$$

where $f = \sum_{j=1}^{\infty} \lambda_j a_j$ a atomic decomposition of f and b_{B_j} the average of b on B_j. This is equivalent to ask for a commutation property

$$\sum_{j=1}^{\infty} \lambda_j b_{B_j} T(a_j) = T\Big(\sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j\Big). \tag{4.3}$$

Even if most authors, for instance in [96, 101, 139, 146, 90, 137, 95], implicitely use (4.3), one must be careful at this point. Indeed, the equality (4.3) is not clear since the two series $\sum_{j=1}^{\infty} \lambda_j b_{B_j} T(a_j)$ and $\sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j$ are not yet well-defined, in general. We refer the reader to [19], Section 3, to be convinced that one must be careful with Equality (4.3).

Although the commutator $[b,T]$ does not map continuously, in general, $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, following Pérez [119] one can find a subspace $\mathcal{H}^1_b(\mathbb{R}^n)$ of $H^1(\mathbb{R}^n)$ such that $[b,T]$ maps continuously $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Recall that (see [119]) a function a is a b-atom if

i) supp $a \subset Q$ for some cube Q,

ii) $\|a\|_{L^\infty} \leq |Q|^{-1}$,

iii) $\int_{\mathbb{R}^n} a(x)dx = \int_{\mathbb{R}^n} a(x)b(x)dx = 0$.

The space $\mathcal{H}^1_b(\mathbb{R}^n)$ consists of the subspace of $L^1(\mathbb{R}^n)$ of functions f which can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where a_j are b-atoms, and λ_j are complex numbers with $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.

In [119] the author showed that the commutator $[b,T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ by establishing that

$$\sup\{\|[b,T](a)\|_{L^1} : a\,\text{is a}\; b\text{−atom}\} < \infty. \tag{4.4}$$

This leaves a gap in the proof which we fill here (see below). Indeed, as it is pointed out in [19], there exists a linear operator U defined on the space of all finite linear combination of $(1,\infty)$-atoms satisfying

$$\sup\{\|U(a)\|_{L^1} : a\,\text{is a}\,(1,\infty)\text{−atom}\} < \infty,$$

but which does not admit an extension to a bounded operator from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. From this result, we see that Inequality (4.4) does not suffice to conclude that $[b,T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. In the setting of $H^1(\mathbb{R}^n)$, it is well-known (see [105] or [144] for details) that a linear operator U can be extended to a bounded operator from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if for some $1 < q < \infty$, we have

$$\sup\{\|U(a)\|_{L^1} : a\,\text{is a}\,(1,q)\text{−atom}\} < \infty.$$

It follows from the fact that the finite atomic norm on $H^{1,q}_{\text{fin}}(\mathbb{R}^n)$ is equivalent to the standard infinite atomic decomposition norm on $H^{1,q}_{\text{ato}}(\mathbb{R}^n)$ through the grand maximal function characterization of $H^1(\mathbb{R}^n)$. However, one can not use this method in the context of $\mathcal{H}^1_b(\mathbb{R}^n)$.

Also, a natural question arises: can one find *the largest subspace of* $H^1(\mathbb{R}^n)$ (of course, this space contains $\mathcal{H}^1_b(\mathbb{R}^n)$, see also Theorem 4.5.2) such that all commutators $[b,T]$ of Calderón-Zygmund operators are bounded from this space into $L^1(\mathbb{R}^n)$? For $b \in BMO(\mathbb{R}^n)$, a non-constant function, we consider the space $H^1_b(\mathbb{R}^n)$ consisting of all $f \in H^1(\mathbb{R}^n)$ such that the (sublinear) commutator $[b,\mathfrak{M}]$ of f belongs to $L^1(\mathbb{R}^n)$ where $\mathfrak{M}$ is the nontangential grand maximal operator (see Section 2). The norm on $H^1_b(\mathbb{R}^n)$ is defined by $\|f\|_{H^1_b} := \|f\|_{H^1}\|b\|_{BMO} + \|[b,\mathfrak{M}](f)\|_{L^1}$. Here we just consider b is a non-constant BMO-function since the commutator $[b,T] = 0$ if b is a constant function. Then, we prove that $[b,T]$ is bounded from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ for every Calderón-Zygmund singular integral operator T (in fact it holds for all $T \in \mathcal{K}$, see below). Furthermore, $H^1_b(\mathbb{R}^n)$ *is the largest space having this property* (see Remark 4.5.1). This answers the question above. Besides, we also consider the class $\mathcal{K}$ of all sublinear operators T, bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, satisfying the condition

$$\|(b - b_Q)Ta\|_{L^1} \leq C\|b\|_{BMO}$$

for all BMO-function b, H^1-atom a related to the cube Q. Here b_Q denotes the average of b on Q, and $C > 0$ is a constant independent of b, a. This class $\mathcal{K}$ contains almost all important operators in harmonic analysis: Calderón-Zygmund type operators, strongly singular integral operators, multiplier operators, pseudo-differential operators, maximal type operators, the area integral operator of Lusin, Littlewood-Paley type operators, Marcinkiewicz operators, maximal Bochner-Riesz operators, etc... (See Section 4). When T is linear and belongs to $\mathcal{K}$, we prove that there exists a bounded bilinear operators $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have the following bilinear decomposition

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)), \tag{4.5}$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which does not depend on T (see Section 3). This bilinear decomposition is strongly related to our previous result in [14] on paraproduct and product on $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$.

We then prove that $[b, T]$ is bounded from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ (see Theorem 4.3.3) via Bilinear decomposition (4.5) (see Theorem 4.3.2) and some characterizations of $H^1_b(\mathbb{R}^n)$ (see Theorem 4.5.1). Furthermore, by using the weak convergence theorem in $H^1(\mathbb{R}^n)$ of Jones and Journé, we prove that $\mathcal{H}^1_b(\mathbb{R}^n) \subset H^1_b(\mathbb{R}^n)$ (see Theorem 4.5.2). These allow us to fill the gap mentioned above in [119].

On the other hand, as an immediate corollary of Bilinear decomposition (4.5), we also obtain the weak type estimate (H^1, L^1) for the commutator $[b, T]$, where T is a Calderón-Zygmund type operator, a strongly singular integral operator, a multiplier operator or a pseudo-differential operator. We also point out that weak type estimates and Hardy type estimates for the (linear) commutators of multiplier operators and of strongly singular Calderón-Zygmund operators have been studied recently (see [146, 90, 137] for the multiplier operators and [95] for strongly singular Calderón-Zygmund operators).

Next, two natural questions for Hardy-type estimates of the commutator $[b, T]$ arised: when does $[b, T]$ map $H^1_b(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$ and when does $[b, T]$ map $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$?

This paper gives two sufficient conditions for the above two questions. Recall that $BMO^{\log}(\mathbb{R}^n)$ –the generalized BMO type space that has been introduced by Nakai and Yabuta [116] to characterize multipliers of $BMO(\mathbb{R}^n)$– is the space of all locally integrable functions f such that

$$\|f\|_{BMO^{\log}} := \sup_{B(a,r)} \frac{|\log r| + \log(e + |a|)}{|B(a, r)|} \int\limits_{B(a,r)} |f(x) - f_{B(a,r)}| dx < \infty.$$

We obtain that if T is a Calderón-Zygmund operator satisfying $T1 = T^*1 = 0$ and b is in $BMO^{\log}(\mathbb{R}^n)$, then the linear commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into

$h^1(\mathbb{R}^n)$. This gives a sufficient condition to the first problem. For the second one, we prove that if T is a Calderón-Zygmund operator satisfying $T^*1 = T^*b = 0$ and b is in $BMO(\mathbb{R}^n)$, then the linear commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

A difficult point to prove the first result is that we have to deal directly with $f \in H^1_b(\mathbb{R}^n)$. It would be easier to do it for atomic type Hardy spaces as in the case of $\mathcal{H}^1_b(\mathbb{R}^n)$. However, we do not know *whether there exists an atomic characterization for the space* $H^1_b(\mathbb{R}^n)$. *This is still an open question.*

Let X be a Banach space. We say that an operator $T : X \to L^1(\mathbb{R}^n)$ is a sublinear operator if for all $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$, we have

$$|T(\alpha f + \beta g)(x)| \leq |\alpha||Tf(x)| + |\beta||Tg(x)|.$$

Obviously, a linear operator $T : X \to L^1(\mathbb{R}^n)$ is a sublinear operator. We also say that a operator $\mathfrak{T} : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is a subbilinear operator if for all $(f, g) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ the operators $\mathfrak{T}(f, \cdot) : BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $\mathfrak{T}(\cdot, g) : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ are sublinear operators.

In this paper, we also obtain the subbilinear decomposition for sublinear commutator. More precisely, when $T \in \mathcal{K}$ is a sublinear operator, we prove that there exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ so that for all $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|. \tag{4.6}$$

Then, the strong type estimate (H^1_b, L^1) and the weak type estimate (H^1, L^1) of the commutators of Littlewood-Paley type operators, of Marcinkiewicz operators, and of maximal Bochner-Riesz operators, can be viewed as an immediate corollary of (4.6). When $H^1_b(\mathbb{R}^n)$ is replaced by $\mathcal{H}^1_b(\mathbb{R}^n)$, these type of estimates have also been considered recently (see for example [97, 25, 102, 99, 100, 98]).

Let us emphasize the three main purposes of this paper. First, we want to give the bilinear (resp., subbilinear) decompositions for the linear (resp., sublinear) commutators. Second, we find the largest subspace of $H^1(\mathbb{R}^n)$ such that all commutators of Calderón-Zygmund operators map continuously this space into $L^1(\mathbb{R}^n)$. Finally, we obtain the (H^1_b, h^1) and (H^1_b, H^1) type estimates for commutators of Calderón-Zygmund operators.

Our paper is organized as follows. In Section 2 we present some notations and preliminaries about the Calderón-Zygmund operators, the function spaces we use, and a short introduction to wavelets, a useful tool in our work. In Section 3 we state our two decomposition theorems (Theorem 4.3.1 and Theorem 4.3.2), the (H^1_b, L^1) type estimates for commutators (Theorem 4.3.3), and some remarks. The bilinear type estimates for commutators of Calderón-Zygmund operators (Theorem 4.3.4) and the boundedness of commutators of Calderón-Zygmund operators on Hardy spaces are also given in this section. In Section 4 we give some examples of operators in the class $\mathcal{K}$ and recall our result from [14] which decomposes a product of f in $H^1(\mathbb{R}^n)$ and g in $BMO(\mathbb{R}^n)$ as a sum of images by four bilinear operators defined through wavelets. These operators are fundamental for the two decomposition theorems. In Section 5 we study the space $H^1_b(\mathbb{R}^n)$. Section 6 and 7 are devoted to the proofs of the two decomposition theorems, the (H^1_b, L^1) type estimates of commutators $[b, T]$ with $T \in \mathcal{K}$, and the boundedness results of commutators of Calderón-Zygmund operators. Finally, in Section 8 we present without proofs some results for commutators of fractional integrals.

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In $\mathbb{R}^n$, we denote by $Q = Q[x,r] := \{y = (y_1, ..., y_n) \in \mathbb{R}^n : \sup_{1\leq i\leq n} |y_i - x_i| \leq r\}$ a cube with center $x = (x_1, ..., x_n)$ and radius $r > 0$. For any measurable set E, we denote by χ_E its characteristic function, by $|E|$ its Lebesgue measure, and by E^c the set $\mathbb{R}^n \setminus E$. For a cube Q and f a locally integrable function, we denote by f_Q the average of f on Q.

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4.2 Some preliminaries and notations

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of test functions on $\mathbb{R}^n$, $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, and $C_0^\infty(\mathbb{R}^n)$ the space of C^∞-functions with compact support.

4.2.1 Calderón-Zygmund operators

Let $\delta \in (0, 1]$. A continuous function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ is said to be a δ-Calderón-Zygmund singular integral kernel if there exists a constant $C > 0$ such that

$$|K(x, y)| \leq \frac{C}{|x - y|^n}$$

for all $x \neq y$, and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}}$$

for all $2|x-x'| \leq |x-y|$.

A linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is said to be a δ-Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$ and if there exists a δ-Calderón-Zygmund singular integral kernel K such that for all $f \in C_0^\infty(\mathbb{R}^n)$ and all $x \notin \operatorname{supp} f$, we have

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x,y)f(y)dy.$$

We say that T is a Calderón-Zygmund operator if it is a δ-Calderón-Zygmund operator for some $\delta \in (0,1]$.

We say that the Calderón-Zygmund operator T satisfies the condition $T^*1 = 0$ (resp., $T1 = 0$) if $\int_{\mathbb{R}^n} Ta(x)dx = 0$ (resp., $\int_{\mathbb{R}^n} T^*a(x)dx = 0$) holds for all classical H^1-atoms a. Let b be a locally integrable function on $\mathbb{R}^n$. We say that the Calderón-Zygmund operator T satisfies the condition $T^*b = 0$ if $\int_{\mathbb{R}^n} b(x)Ta(x)dx = 0$ holds for all classical H^1-atoms a.

4.2.2 Function spaces

We first consider the subspace $\mathcal{A}$ of $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\mathcal{A} = \Big\{\phi \in \mathcal{S}(\mathbb{R}^n) : |\phi(x)| + |\nabla\phi(x)| \leq (1+|x|^2)^{-(n+1)}\Big\},$$

where $\nabla = (\partial/\partial x_1, ..., \partial/\partial x_n)$ denotes the gradient. We then define

$$\mathfrak{M}f(x) := \sup_{\phi\in\mathcal{A}} \sup_{|y-x|<t} |f*\phi_t(y)| \quad \text{and} \quad \mathfrak{m}f(x) := \sup_{\phi\in\mathcal{A}} \sup_{|y-x|<t<1} |f*\phi_t(y)|,$$

where $\phi_t(\cdot) = t^{-n}\phi(t^{-1}\cdot)$. The space $H^1(\mathbb{R}^n)$ is the space of all tempered distributions f such that $\mathfrak{M}f \in L^1(\mathbb{R}^n)$ equipped with the norm $\|f\|_{H^1} = \|\mathfrak{M}f\|_{L^1}$. The space $h^1(\mathbb{R}^n)$ denotes the space of all tempered distributions f such that $\mathfrak{m}f \in L^1(\mathbb{R}^n)$ equipped with the norm $\|f\|_{h^1} = \|\mathfrak{m}f\|_{L^1}$. The space $H^{\log}(\mathbb{R}^n)$ (see [81, 14]) denotes the space of all tempered distributions f such that $\mathfrak{M}f \in L^{\log}(\mathbb{R}^n)$ equipped with the norm $\|f\|_{H^{\log}} = \|\mathfrak{M}f\|_{L^{\log}}$. Here $L^{\log}(\mathbb{R}^n)$ is the space of all measurable functions f such that $\int_{\mathbb{R}^n} \frac{|f(x)|}{\log(e+|x|)+\log(e+|f(x)|)}dx < \infty$ with the (quasi-)norm

$$\|f\|_{L^{\log}} := \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^n} \frac{\frac{|f(x)|}{\lambda}}{\log(e+|x|)+\log(e+\frac{|f(x)|}{\lambda})}dx \leq 1\right\}.$$

Clearly, for any $f \in H^1(\mathbb{R}^n)$, we have

$$\|f\|_{h^1} \leq \|f\|_{H^1} \quad \text{and} \quad \|f\|_{H^{\log}} \leq \|f\|_{H^1}.$$

We remark that the local real Hardy space $h^1(\mathbb{R}^n)$, first introduced by Goldberg [56], is larger than $H^1(\mathbb{R}^n)$ and allows more flexibility, since global cancellation conditions are not necessary. For example, the Schwartz space is contained in $h^1(\mathbb{R}^n)$ but not in $H^1(\mathbb{R}^n)$, and multiplication by cutoff functions preserves $h^1(\mathbb{R}^n)$ but not $H^1(\mathbb{R}^n)$. Thus it makes $h^1(\mathbb{R}^n)$ more suitable for working in domains and on manifolds.

It is well-known (see [48] or [128]) that the dual of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$ the space of all locally integrable functions f with

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|} \int\limits_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all balls B. We note $\mathbb{Q} := [0,1)^n$ and, for f a function in $BMO(\mathbb{R}^n)$,

$$\|f\|_{BMO^+} := \|f\|_{BMO} + |f_{\mathbb{Q}}|.$$

We should also point out that the space $H^{\log}(\mathbb{R}^n)$ arises naturally in the study of pointwise product of functions in $H^1(\mathbb{R}^n)$ with functions in $BMO(\mathbb{R}^n)$, and in the endpoint estimates for the div-curl lemma (see for example [11, 14, 81]).

In [56] it was shown that the dual of $h^1(\mathbb{R}^n)$ can be identified with the space $bmo(\mathbb{R}^n)$, consisting of locally integrable functions f with

$$\|f\|_{bmo} := \sup_{|B|\leq 1} \frac{1}{|B|} \int\limits_B |f(x) - f_B| dx + \sup_{|B|\geq 1} \frac{1}{|B|} \int\limits_B |f(x)| dx < \infty,$$

where the supremums are taken over all balls B.

Clearly, for any $f \in bmo(\mathbb{R}^n)$, we have

$$\|f\|_{BMO} \leq \|f\|_{BMO^+} \leq C \|f\|_{bmo}.$$

We next recall that the space $VMO(\mathbb{R}^n)$ (resp., $vmo(\mathbb{R}^n)$) is the closure of $C_0^\infty(\mathbb{R}^n)$ in $(BMO(\mathbb{R}^n), \|\cdot\|_{BMO})$ (resp., $(bmo(\mathbb{R}^n), \|\cdot\|_{bmo})$). It is well-known that (see [32] and [35]) the dual of $VMO(\mathbb{R}^n)$ (resp., $vmo(\mathbb{R}^n)$) is the Hardy space $H^1(\mathbb{R}^n)$ (resp., $h^1(\mathbb{R}^n)$). We point out that the space $VMO(\mathbb{R}^n)$ (resp., $vmo(\mathbb{R}^n)$) considered by Coifman and Weiss (resp., Dafni [35]) is different from the one considered by Sarason. Thanks to Bourdaud [18], it coincides with the space $VMO(\mathbb{R}^n)$ (resp., $vmo(\mathbb{R}^n)$) considered above.

In the study of the pointwise multipliers for $BMO(\mathbb{R}^n)$, Nakai and Yabuta [116] introduced the space $BMO^{\log}(\mathbb{R}^n)$, consisting of locally integrable functions f with

$$\|f\|_{BMO^{\log}} := \sup_{B(a,r)} \frac{|\log r| + \log(e + |a|)}{|B(a,r)|} \int\limits_{B(a,r)} |f(x) - f_{B(a,r)}| dx < \infty.$$

There, the authors proved that a function g is a pointwise multiplier for $BMO(\mathbb{R}^n)$ if and only if g belongs to $L^\infty(\mathbb{R}^n) \cap BMO^{\log}(\mathbb{R}^n)$. Furthermore, it is also shown in [81] that the space $BMO^{\log}(\mathbb{R}^n)$ is the dual of the space $H^{\log}(\mathbb{R}^n)$.

Definition 4.2.1. *Let b be a locally integrable function and $1 < q \leq \infty$. A function a is called a (q,b)-atom related to the cube Q if*

i) supp $a \subset Q$,

ii) $\|a\|_{L^q} \leq |Q|^{1/q-1}$,

iii) $\int_{\mathbb{R}^n} a(x)dx = \int_{\mathbb{R}^n} a(x)b(x)dx = 0$.

The space $\mathcal{H}^{1,q}_b(\mathbb{R}^n)$ consists of the subspace of $L^1(\mathbb{R}^n)$ of functions f which can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$, where a_j's are (q,b)-atoms, $\lambda_j \in \mathbb{C}$, and $\sum_{j=1}^\infty |\lambda_j| < \infty$. As usual, we define on $\mathcal{H}^{1,q}_b(\mathbb{R}^n)$ the norm

$$\|f\|_{\mathcal{H}^{1,q}_b} := \inf\Big\{\sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j\Big\}.$$

Observe that when $q = \infty$, then the space $\mathcal{H}^{1,\infty}_b(\mathbb{R}^n)$ is just the space $\mathcal{H}^1_b(\mathbb{R}^n)$ considered in [119]. Furthermore, $\mathcal{H}^{1,\infty}_b(\mathbb{R}^n) \subset \mathcal{H}^{1,q}_b(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ and the inclusions are continuous.

We next introduce the space $H^1_b(\mathbb{R}^n)$ as follows.

Definition 4.2.2. *Let b be a non-constant BMO-function. The space $H^1_b(\mathbb{R}^n)$ consists of all f in $H^1(\mathbb{R}^n)$ such that $[b,\mathfrak{M}](f)(x) = \mathfrak{M}(b(x)f(\cdot) - b(\cdot)f(\cdot))(x)$ belongs to $L^1(\mathbb{R}^n)$. We equipped $H^1_b(\mathbb{R}^n)$ with the norm $\|f\|_{H^1_b} := \|f\|_{H^1}\|b\|_{BMO} + \|[b,\mathfrak{M}](f)\|_{L^1}$.*

4.2.3 Prerequisites on Wavelets

Let us consider a wavelet basis of $\mathbb{R}$ with compact support. More explicitly, we are first given a $\mathcal{C}^1(\mathbb{R})$-wavelet in Dimension one, called ψ, such that $\{2^{j/2}\psi(2^jx-k)\}_{j,k\in\mathbb{Z}}$ form an $L^2(\mathbb{R})$ basis. We assume that this wavelet basis comes for a multiresolution analysis (MRA) on $\mathbb{R}$, as defined below (see [107]).

Definition 4.2.3. *A multiresolution analysis (MRA) on $\mathbb{R}$ is defined as an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ with the following four properties*

i) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

ii) for every $f \in L^2(\mathbb{R})$ and every $j \in \mathbb{Z}$, $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,

iii) for every $f \in L^2(\mathbb{R})$ and every $k \in \mathbb{Z}$, $f(x) \in V_0$ if and only if $f(x-k) \in V_0$,

iv) there exists a function $\phi \in L^2(\mathbb{R})$, called the scaling function, such that the family $\{\phi_k(x) = \phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

It is classical that, given an (MRA) on $\mathbb{R}$, one can find a wavelet ψ such that $\{2^{j/2}\psi(2^jx-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j, the orthogonal complement of V_j in V_{j+1}. Moreover, by Daubechies Theorem (see [36]), it is possible to find a suitable (MRA) so that ϕ and ψ are $\mathcal{C}^1(\mathbb{R})$ and compactly supported, ψ has mean 0 and $\int x\psi(x)dx=0$, which is known as the moment condition. We could content ourselves, in the following theorems, to have ϕ and ψ decreasing sufficiently rapidly at ∞, but proofs are simpler with compactly supported wavelets. More precisely we can choose $m>1$ such that ϕ and ψ are supported in the interval $1/2+m(-1/2,+1/2)$, which is obtained from $(0,1)$ by a dilation by m centered at $1/2$.

Going back to $\mathbb{R}^n$, we recall that a wavelet basis of $\mathbb{R}^n$ is constructed as follows. We call E the set $E=\{0,1\}^n\setminus\{(0,\cdots,0)\}$ and, for $\sigma\in E$, put $\psi^\sigma(x)=\phi^{\sigma_1}(x_1)\cdots\phi^{\sigma_n}(x_n)$, with $\phi^{\sigma_j}(x_j)=\phi(x_j)$ for $\sigma_j=0$ while $\phi^{\sigma_j}(x_j)=\psi(x_j)$ for $\sigma_j=1$. Then the set $\{2^{nj/2}\psi^\sigma(2^jx-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,\sigma\in E}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. As it is classical, for I a dyadic cube of $\mathbb{R}^n$, which may be written as the set of x such that $2^jx-k\in(0,1)^n$, we note

$$\psi_I^\sigma(x)=2^{nj/2}\psi^\sigma(2^jx-k).$$

We also note $\phi_I=2^{nj/2}\phi_{(0,1)^n}(2^jx-k)$, with $\phi_{(0,1)^n}$ the scaling function in n variables, given by $\phi_{(0,1)^n}(x)=\phi(x_1)\cdots\phi(x_n)$. In the sequel, the letter I always refers to dyadic cubes. Moreover, we note kI the cube of same center dilated by the coefficient k. Because of the assumption on the supports of ϕ and ψ, the functions ψ_I^σ and ϕ_I are supported in the cube mI.

The wavelet basis $\{\psi_I^\sigma\}$, obtained by letting I vary among dyadic cubes and σ in E, comes from an (MRA) in $\mathbb{R}^n$, which we still note $\{V_j\}_{j\in\mathbb{Z}}$, obtained by taking tensor products of the one-dimensional ones.

The following theorem gives the wavelet characterization of $H^1(\mathbb{R}^n)$ (cf. [107, 63]).

Theorem 4.2.1. *There exists a constant $C>0$ such that $f\in H^1(\mathbb{R}^n)$ if and only if $\mathcal{W}_\psi f:=\Big(\sum_I\sum_{\sigma\in E}|\langle f,\psi_I^\sigma\rangle|^2|I|^{-1}\chi_I\Big)^{1/2}\in L^1(\mathbb{R}^n)$, moreover,*

$$C^{-1}\|f\|_{H^1}\le\|\mathcal{W}_\psi f\|_{L^1}\le C\|f\|_{H^1}.$$

A function $a\in L^2(\mathbb{R}^n)$ is called a ψ-atom related to the (not necessarily dyadic) cube R if it may be written as

$$a=\sum_{I\subset R}\sum_{\sigma\in E}a_{I,\sigma}\psi_I^\sigma$$

with $\|a\|_{L^2}\le|R|^{-1/2}$. Remark that a is compactly supported in mR and has mean 0, so that it is a classical atom related to mR, up to the multiplicative constant $m^{n/2}$. It is standard that an atom is in $H^1(\mathbb{R}^n)$ with norm bounded by a uniform constant. The atomic decomposition gives the converse.

Theorem 4.2.2 (Atomic decomposition)**.** *There exists a constant $C > 0$ such that all functions $f \in H^1(\mathbb{R}^n)$ can be written as the limit in the distribution sense and in $H^1(\mathbb{R}^n)$ of an infinite sum*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

with a_j ψ-atoms related to some dyadic cubes R_j and λ_j constants such that

$$C^{-1}\|f\|_{H^1} \leq \sum_{j=1}^{\infty} |\lambda_j| \leq C\|f\|_{H^1}.$$

This theorem is a small variation of a standard statement which can be found in [63], Section 6.5. Remark that the interest of dealing with finite atomic decompositions has been underlined recently, for instance in [105, 81].

Now, we denote by $H^1_{\text{fin}}(\mathbb{R}^n)$ the vector space of all finite linear combinations of ψ-atoms, that is,

$$f = \sum_{j=1}^{k} \lambda_j a_j,$$

where a_j's are ψ-atoms. Then, the norm of f in $H^1_{\text{fin}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^1_{\text{fin}}} = \inf\Big\{\sum_{j=1}^{k} |\lambda_j| : f = \sum_{j=1}^{k} \lambda_j a_j\Big\}.$$

By the atomic decomposition theorem, the set $H^1_{\text{fin}}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ for the norm $\|\cdot\|_{H^1}$.

The following two wavelet characterizations of $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and $BMO(\mathbb{R}^n)$ are well-known (see [107]).

Theorem 4.2.3. *Let $1 < p < \infty$. Then the norms*

$$\|f\|_{L^p},\ \|(\sum_I \sum_{\sigma\in E} |\langle f, \psi_I^\sigma\rangle|^2 |I|^{-1}\chi_I)^{1/2}\|_{L^p} \text{ and } \|(\sum_I \sum_{\sigma\in E} |\langle f, \psi_I^\sigma\rangle|^2 (\psi_I^\sigma)^2)^{1/2}\|_{L^p}$$

are equivalent on $L^p(\mathbb{R}^n)$.

Theorem 4.2.4. *A function $g \in BMO(\mathbb{R}^n)$ if and only if*

$$\frac{1}{|R|}\sum_{I\subset R}\sum_{\sigma\in E} |\langle g, \psi_I^\sigma\rangle|^2 < \infty$$

for all (not necessarily dyadic) cubes R. Moreover, there exists a constant $C > 0$ such that for all $g \in BMO(\mathbb{R}^n)$,

$$C^{-1}\|g\|_{BMO} \leq \sup_R \Big(\frac{1}{|R|}\sum_{I\subset R}\sum_{\sigma\in E}|\langle g, \psi_I^\sigma\rangle|^2\Big)^{1/2} \leq C\|g\|_{BMO},$$

where the supremum is taken over all cubes R.

By Theorem 4.2.3, Theorem 4.2.4 and John-Nirenberg inequality, we obtain the following lemma. The proof is easy and will be omitted.

Lemma 4.2.1. *Let f be a ψ-atom related to the cube R and $b \in BMO(\mathbb{R}^n)$. Then, $\sum_{I\subset R}\sum_{\sigma\in E}\langle f, \psi_I^\sigma\rangle\langle b, \psi_I^\sigma\rangle(\psi_I^\sigma)^2 \in L^q(\mathbb{R}^n)$ for any $q \in (1, 2)$.*

4.3 Bilinear, subbilinear decompositions and commutators

Recall that $\mathcal{K}$ is the set of all sublinear operators T bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ satisfying

$$\|(b - b_Q)Ta\|_{L^1} \leq C\|b\|_{BMO},$$

for all $b \in BMO(\mathbb{R}^n)$, any H^1-atom a supported in the cube Q, where $C > 0$ a constant independent of b, a. This class $\mathcal{K}$ contains almost all important operators in harmonic analysis: Calderón-Zygmund type operators, strongly singular integral operators, multiplier operators, pseudo-differential operators, maximal type operators, the area integral operator of Lusin, Littlewood-Paley type operators, Marcinkiewicz operators, maximal Bochner-Riesz operators, etc... (See Section 4).

Here and in what follows the bilinear operator $\mathfrak{S}$ is defined by

$$\mathfrak{S}(f, g) = -\sum_I\sum_{\sigma\in E}\langle f, \psi_I^\sigma\rangle\langle g, \psi_I^\sigma\rangle(\psi_I^\sigma)^2.$$

In [14], the authors show that $\mathfrak{S}$ is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

4.3.1 Two decomposition theorems and (H^1_b, L^1) type estimates

Let b be a locally integrable function and $T \in \mathcal{K}$. As usual, the (sublinear) commutator $[b, T]$ of the operator T is defined by $[b, T](f)(x) := T\Big((b(x) - b(\cdot))f(\cdot)\Big)(x)$.

Theorem 4.3.1 (Subbilinear decomposition)**.** *Let $T \in \mathcal{K}$. There exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f, b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have*

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|.$$

Corollary 4.3.1. *Let $T \in \mathcal{K}$ be such that T is of weak type $(1,1)$. Then, the bilinear operator $\mathfrak{P}(f,g) = [g,T](f)$ maps continuously $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$. In particular, the commutator $[b,T]$ is of weak type (H^1, L^1) if $b \in BMO(\mathbb{R}^n)$.*

We remark that the class of operators $T \in \mathcal{K}$ of weak type $(1,1)$ contains Calderón-Zygmund operators, strongly singular integral operators, multiplier operators, pseudo-differential operators whose symbols in the Hörmander class $S^m_{\varrho,\delta}$ with $0 < \varrho \leq 1, 0 \leq \delta < 1, m \leq -n((1-\varrho)/2 + \max\{0, (\delta-\varrho)/2\})$, maximal type operators, the area integral operator of Lusin, Littlewood-Paley type operators, Marcinkiewicz operators, maximal Bochner-Riesz operators T^δ_* with $\delta > (n-1)/2$, etc...

When T is linear and belongs to $\mathcal{K}$, we obtain the bilinear decomposition for the linear commutator $[b,T]$ of f, $[b,T](f) = bT(f) - T(bf)$, instead of the subbilinear decomposition as stated in Theorem 4.3.1.

Theorem 4.3.2 (Bilinear decomposition)**.** *Let T be a linear operator in $\mathcal{K}$. Then, there exists a bounded bilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that for all $(f,b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we have*

$$[b,T](f) = \mathfrak{R}(f,b) + T(\mathfrak{S}(f,b)).$$

The following result gives (H^1_b, L^1)-type estimates for commutators $[b,T]$ when T belongs to the class $\mathcal{K}$.

Theorem 4.3.3. *Let b be a non-constant BMO-function and $T \in \mathcal{K}$. Then, the commutator $[b,T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

Remark that in the particular case of T a 1-Calderón-Zygmund operator and $H^1_b(\mathbb{R}^n)$ replaced by $\mathcal{H}^1_b(\mathbb{R}^n)$, Pérez [119] proved

$$\sup\{\|[b,T](a)\|_{L^1} : a \text{ is a } (\infty, b)\text{−atom}\} < \infty. \tag{4.7}$$

Then he concludes that the (linear) commutator $[b,T]$ maps continuously $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Notice that $\mathcal{H}^1_b(\mathbb{R}^n) \subset \mathcal{H}^{1,q}_b(\mathbb{R}^n) \subset H^1_b(\mathbb{R}^n)$, $1 < q \leq \infty$, and the inclusions are continuous (see Section 5). However, as mentioned in the introduction, Inequality

(4.7) does not suffice to conclude that the (linear) commutator $[b,T]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. We should also point out that the (H^1, L^1) weak type estimates and the $(\mathcal{H}^1_b, L^1)$ type estimates for the (linear) commutators of multiplier operators (see [146, 90, 137]), strongly singular Calderón-Zygmund operators (see [95]) and for the (sublinear) commutators of Littlewood-Paley type operators (see [97]), Marcinkiewicz operators (see [102]), maximal Bochner-Riesz operators (see [99, 100, 98]) have been studied recently. However, the authors just prove Inequality (4.4) (that is Inequality (4.7)) and use Equality (4.3) which leaves a gap as pointed out in the introduction.

4.3.2 Boundedness of linear commutators on Hardy spaces

Analogously to Hardy estimates for bilinear operators of Coifman and Grafakos [29] (see also [42]), we obtain the following strongly bilinear estimates which improve Corollary 5.3.1.

Theorem 4.3.4. *Let T be a linear operator in $\mathcal{K}$. Assume that A_i, B_i, $i = 1, ..., K$, are Calderón-Zygmund operators satisfying $A_i 1 = A_i^* 1 = B_i 1 = B_i^* 1 = 0$, and for every f and g in $L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \Big(\sum_{i=1}^K A_i f . B_i g\Big) dx = 0.$$

Then, the bilinear operator $\mathfrak{T}$, defined by

$$\mathfrak{T}(f,g) = \sum_{i=1}^K [B_i g, T](A_i f),$$

maps continuously $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

We now give a sufficient condition for the linear commutator $[b,T]$ to map continuously $H^1_b(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$.

Theorem 4.3.5. *Let b be a non-constant $BMO^{\log}$-function and T be a Calderón-Zygmund operator with $T1 = T^*1 = 0$. Then, the linear commutator $[b,T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$.*

The last theorem gives a sufficient condition for the linear commutator $[b,T]$ to map continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

Theorem 4.3.6. *Let b be a non-constant BMO-function and T be a Calderón-Zygmund operator with $T^*1 = T^*b = 0$. Then, the linear commutator $[b,T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.*

Observe that the condition $T^*b = 0$ is "necessary" in the sense that if the linear commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} b(x)Ta(x)dx = 0$ holds for all (q, b)-atoms a, $1 < q \leq \infty$.

Also, let us give some examples to illustrate the sufficient conditions in Theorem 4.3.6. To have many examples, let us consider Euclidean spaces $\mathbb{R}^n, n \geq 2$. Now, consider all Calderón-Zygmund operators T such that $T^*1 = 0$. As the closure of $T(H^1(\mathbb{R}^n))$ is a proper subset of $H^1(\mathbb{R}^n)$, by the Hahn-Banach theorem (note that $BMO(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$), one may take b a non-constant BMO-function such that $\int_{\mathbb{R}^n} bTadx = 0$ for all H^1-atoms a, i.e. $T^*b = 0$, and thus b and T satisfy the sufficient condition in Theorem 4.3.6.

4.4 The class $\mathcal{K}$ and four bilinear operators on $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$

4.4.1 The class $\mathcal{K}$

The purpose of this subsection is to give some examples of operators in the class $\mathcal{K}$. More precisely, the class $\mathcal{K}$ contains almost all important operators in Harmonic analysis: Calderón-Zygmund type operators, strongly singular integral operators, multiplier operators, pseudo-differential operators with symbols in the Hörmander class $S^m_{\varrho,\delta}$ with $0 < \varrho \leq 1, 0 \leq \delta < 1, m \leq -n((1-\varrho)/2 + \max\{0, (\delta - \varrho)/2\})$ (see [2, 1]), maximal type operators, the area integral operator of Lusin, Littlewood-Paley type operators, Marcinkiewicz operators, maximal Bochner-Riesz operators T^δ_* with $\delta > (n-1)/2$ (cf. [87]), etc... It is well-known that these operators T are bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. So, in order to establish that these ones are in the class $\mathcal{K}$, we just need to show that

$$\|(b - b_Q)Ta\|_{L^1} \leq C\|b\|_{BMO} \tag{4.8}$$

for all BMO-function b, H^1-atom a related to a cube $Q = Q[x_0, r]$ with constant $C > 0$ independent of b, a.

Observe that the nontangential grand maximal operator $\mathfrak{M}$ belongs to $\mathcal{K}$ since it satisfies Inequality (4.8) (cf. [128]). We refer also to [62] for the (sublinear) commutators $[b, M_{\varphi,\alpha}]$ of the maximal operators $M_{\varphi,\alpha}$ –note that $M_{\varphi,0}$ lies in $\mathcal{K}$–.

Here we just give the proofs for Calderón-Zygmund operators (linear operators) and the area integral operator of Lusin (sublinear operator). For the other operators, we leave the proofs to the interested reader.

First recall that $P(x) = \frac{1}{(1+|x|^2)^{(n+1)/2}}$ is the Poisson kernel and $u_f(x,t) := f * P_t(x)$ is the Poisson integral of f. Then the area integral operator S of Lusin is defined by

$$S(f)(x) = \left(\int_{\Gamma(x)} |\nabla u_f(y,t)|^2 t^{1-n} dy dt \right)^{1/2},$$

where $\Gamma(x)$ is the cone $\{(y,t) \in \mathbb{R}_+^{n+1} : |y-x| < t\}$ with vertex at x, while $\nabla u_f = (\partial u_f/\partial x_1, ..., \partial u_f/\partial x_1, \partial u_f/\partial t)$ is the gradient of u_f on $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty)$.

Proposition 4.4.1. *Let $\delta \in (0,1]$ and T be a δ-Calderón-Zygmund operator. Then T satisfies Inequality (4.8), and thus T belongs to $\mathcal{K}$.*

Proof. We cut the integral of $|(b-b_Q)Ta|$ into two parts. By Schwarz inequality and the boundedness of T on $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{2Q} |b(x)-b_Q||Ta(x)|dx &\leq C \left(\int_{2Q} |b(x)-b_Q|^2 dx \right)^{1/2} \|a\|_{L^2} \\ &\leq C\|b\|_{BMO} \end{aligned}$$

here one used the fact $|b_{2Q} - b_Q| \leq C\|b\|_{BMO}$. Next, for $x \notin 2Q$,

$$\begin{aligned} |Ta(x)| &= \left| \int_Q (K(x,y) - K(x,x_0))a(y)dy \right| \\ &\leq C \int_Q \frac{|y-x_0|^\delta}{|x-x_0|^{n+\delta}} |a(y)|dy \\ &\leq C \frac{r^\delta}{|x-x_0|^{n+\delta}}. \end{aligned}$$

Therefore,

$$\int_{(2Q)^c} |b(x)-b_Q||Ta(x)|dx \leq C \int_{Q^c} |b(x)-b_Q| \frac{r^\delta}{|x-x_0|^{n+\delta}} dx \leq C\|b\|_{BMO},$$

since the last inequality is classical (cf. [128]). This finishes the proof.

□

Corollary 4.4.1. *Let $\mathcal{R}_j, j = 1, ..., n$, be the classical Riesz transforms. Then, $\mathcal{R}_j$ belongs to $\mathcal{K}$ for all $j = 1, ..., n$.*

Proposition 4.4.2. *The area integral operator S satisfies Inequality (4.8), and thus S belongs to $\mathcal{K}$.*

Proof. We also cut the integral of $|(b-b_Q)S(a)|$ into two parts. By Schwarz inequality and the boundedness of S on $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\int\limits_{2Q} |b(x)-b_Q||S(a)(x)|dx &\leq C\left(\int\limits_{2Q} |b(x)-b_Q|^2 dx\right)^{1/2} \|a\|_{L^2} \\ &\leq C\|b\|_{BMO}.\end{aligned}$$

Next, for $x \notin 2Q$, by using the equality

$$u_a(y,t) = \int\limits_{\mathbb{R}^n} \frac{1}{t^n}\Big(P\Big(\frac{y-z}{t}\Big) - P\Big(\frac{y-x_0}{t}\Big)\Big)a(z)dz,$$

since $\int_{\mathbb{R}^n} a(z)dz = 0$, it is easy to establish that

$$S(a)(x) = \left(\int\limits_{\Gamma(x)} |\nabla u_a(y,t)|^2 t^{1-n} dydt\right)^{1/2} \leq C\frac{r}{|x-x_0|^{n+1}}.$$

Therefore,

$$\int\limits_{(2Q)^c} |b(x)-b_Q||S(a)(x)|dx \leq C\int\limits_{Q^c} |b(x)-b_Q|\frac{r}{|x-x_0|^{n+1}}dx \leq C\|b\|_{BMO},$$

which ends the proof. $\square$

We should point out that the Littlewood-Paley type operators can be viewed as vector-valued Calderón-Zygmund operators (see [122]). See also [62] in the context of vector-valued commutators.

4.4.2 Four bilinear operators on $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$

We now consider four bilinear operators on $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ which are fundamental for our bilinear decomposition theorem.

We first state some lemmas whose proofs can be found in [14].

Lemma 4.4.1. *The bilinear operator Π_3 defined on $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ by*

$$\Pi_3(f,g) = \sum_I \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \langle g, \psi_I^\sigma\rangle (\psi_I^\sigma)^2$$

is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

Observe that $\mathfrak{S}(f,g) = -\Pi_3(f,g)$ for all $(f,g) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$.

Lemma 4.4.2. *The bilinear operator Π_4, defined on $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ by*

$$\Pi_4(f,g) = \sum_{I,I'} \sum_{\sigma,\sigma' \in E} \langle f, \psi_I^\sigma \rangle \langle g, \psi_{I'}^{\sigma'} \rangle \psi_I^\sigma \psi_{I'}^{\sigma'},$$

the sums being taken over all dyadic cubes I, I' and $\sigma, \sigma' \in E$ such that $(I,\sigma) \neq (I', \sigma')$, is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

Lemma 4.4.3. *The bilinear operator Π_1 defined by*

$$\Pi_1(a,g) = \sum_{|I|=|I'|} \sum_{\sigma \in E} \langle a, \phi_I \rangle \langle g, \psi_{I'}^\sigma \rangle \phi_I \psi_{I'}^\sigma,$$

where a is a ψ-atom and $g \in BMO(\mathbb{R}^n)$, can be extended into a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

Lemma 4.4.4. *The bilinear operator Π_2 defined by*

$$\Pi_2(a,g) = \sum_{|I|=|I'|} \sum_{\sigma \in E} \langle a, \psi_I^\sigma \rangle \langle g, \phi_{I'} \rangle \psi_I^\sigma \phi_{I'},$$

where a is a ψ-atom related to the cube R and $g \in BMO(\mathbb{R}^n)$, can be extended into a bounded bilinear operator from $H^1(\mathbb{R}^n) \times BMO^+(\mathbb{R}^n)$ into $H^{\log}(\mathbb{R}^n)$. Furthermore, we can write

$$\Pi_2(a,g) = h^{(1)} + \kappa g_R h^{(2)} \tag{4.9}$$

where $\|h^{(1)}\|_{H^1} \leq C\|g\|_{BMO}$, $h^{(2)}$ is an atom related to mR, and κ a uniform constant, independent of a and g.

The following remarks are useful in our proofs in Section 6 and Section 7.

Remark 4.4.1. *1. If $g \in BMO(\mathbb{R}^n)$ and $f \in H^1(\mathbb{R}^n)$ such that $fg \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} fg\,dx = -\int_{\mathbb{R}^n} \mathfrak{S}(f,g)\,dx = \sum_I \sum_{\sigma \in E} \langle f, \psi_I^\sigma \rangle \langle g, \psi_I^\sigma \rangle.$$

2. For any $(f,g) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ and c a constant, we have

$$\Pi_i(f,g) = \Pi_i(f, g+c)\,,\ i = 1,3,4.$$

3. As a consequence of Lemma 4.4.4, if $g_R = 0$ then Equality (4.9) gives that $\Pi_2(a,g) \in H^1(\mathbb{R}^n)$. Moreover, $\|\Pi_2(a,g)\|_{H^1} \leq C\|g\|_{BMO}$.

In [14], the authors have shown the following decomposition theorem for the product space $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$.

Theorem 4.4.1 (Decomposition theorem)**.** *Let $f \in H^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$. Then, we have the following decomposition*

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g) + \Pi_4(f,g),$$

that is

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_4(f,g) - \mathfrak{S}(f,g).$$

4.5 The space $H^1_b(\mathbb{R}^n)$

Let b be a non-constant BMO-function. In this section, we study the space $H^1_b(\mathbb{R}^n)$. In particular, we give some characterizations of the space $H^1_b(\mathbb{R}^n)$ (see Theorem 4.5.1), and the comparison with the space $\mathcal{H}^1_b(\mathbb{R}^n)$ of Pérez (see Theorem 4.5.2).

First, let us consider the class $\widetilde{\mathcal{K}}$ of all $T \in \mathcal{K}$ such that T characterizes the space $H^1(\mathbb{R}^n)$, that means $f \in H^1(\mathbb{R}^n)$ if and only if $Tf \in L^1(\mathbb{R}^n)$. Clearly, the class $\widetilde{\mathcal{K}}$ contain the maximal operator $\mathfrak{M}$, the area integral operator S of Lusin, the Littlewood-Paley g-operator (see [48]), the Littlewood-Paley g^*_λ-operator with $\lambda > 3n$ (see [67]), etc...

Here and in what follows, the symbol $f \approx g$ means that $C^{-1} f \leq g \leq Cf$ for some constant $C > 0$. We obtain the following characterization of $H^1_b(\mathbb{R}^n)$.

Theorem 4.5.1. *Let b be a non-constant BMO-function and $T \in \widetilde{\mathcal{K}}$. For $f \in H^1(\mathbb{R}^n)$, the following conditions are equivalent:*

i) $f \in H^1_b(\mathbb{R}^n)$.

ii) $\mathfrak{S}(f,b) \in H^1(\mathbb{R}^n)$.

iii) $[b, \mathcal{R}_j](f) \in L^1(\mathbb{R}^n)$ for all $j = 1, ..., n$.

iv) $[b, T](f) \in L^1(\mathbb{R}^n)$.

Furthermore, if one of these conditions is satisfied, then

$$\begin{aligned}
\|f\|_{H^1_b} &= \|f\|_{H^1}\|b\|_{BMO} + \|[b,\mathfrak{M}](f)\|_{L^1}\\
&\approx \|f\|_{H^1}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1}\\
&\approx \|f\|_{H^1}\|b\|_{BMO} + \sum_{j=1}^n \|[b,\mathcal{R}_j](f)\|_{L^1}\\
&\approx \|f\|_{H^1}\|b\|_{BMO} + \|[b,T](f)\|_{L^1},
\end{aligned}$$

where the constants are independent of f and b.

Remark 4.5.1. *Theorem 4.3.3 and Theorem 4.5.1 give that $[b, T]$ is bounded from $H^1_b(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for every T a Calderón-Zygmund singular integral operator. Furthermore, $H^1_b(\mathbb{R}^n)$ is the largest space having this property.*

Proof of Theorem 4.5.1. (*i*) $\Leftrightarrow$ (*ii*) By Theorem 4.3.1, there exists a bounded subbilinear operator $\mathfrak{R} : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that

$$\mathfrak{M}(\mathfrak{S}(f, b)) - \mathfrak{R}(f, b) \leq |[b, \mathfrak{M}](f)| \leq \mathfrak{R}(f, b) + \mathfrak{M}(\mathfrak{S}(f, b)).$$

Consequently, $\mathfrak{S}(f, b) \in H^1(\mathbb{R}^n)$ if and only if $[b, \mathfrak{M}](f) \in L^1(\mathbb{R}^n)$. Moreover,

$$\|f\|_{H^1_b} \approx \|f\|_{H^1}\|b\|_{BMO} + \|\mathfrak{S}(f, b)\|_{H^1}.$$

(*ii*) $\Leftrightarrow$ (*iii*). By Theorem 4.3.2, there exist n bounded bilinear operators $\mathfrak{R}_j : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $j = 1, ..., n$, such that

$$[b, \mathcal{R}_j](f) = \mathfrak{R}_j(f, b) + \mathcal{R}_j(\mathfrak{S}(f, b)).$$

Consequently, $\mathfrak{S}(f, b) \in H^1(\mathbb{R}^n)$ if and only if $[b, \mathcal{R}_j](f) \in L^1(\mathbb{R}^n)$ for all $j = 1, ..., n$. Moreover,

$$\|f\|_{H^1}\|b\|_{BMO} + \|\mathfrak{S}(f, b)\|_{H^1} \approx \|f\|_{H^1}\|b\|_{BMO} + \sum_{j=1}^{n} \|[b, \mathcal{R}_j](f)\|_{L^1}.$$

(*ii*) $\Leftrightarrow$ (*iv*). By Theorem 4.3.1, there exists a bounded subbilinear operator $\mathfrak{R} : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|.$$

Consequently, $\mathfrak{S}(f, b) \in H^1(\mathbb{R}^n)$ if and only if $[b, T](f) \in L^1(\mathbb{R}^n)$ since $T \in \widetilde{\mathcal{K}}$. Moreover,

$$\|f\|_{H^1}\|b\|_{BMO} + \|\mathfrak{S}(f, b)\|_{H^1} \approx \|f\|_{H^1}\|b\|_{BMO} + \|[b, T](f)\|_{L^1}.$$

□

Remark that the constants in the last equivalence depend on T.

The following lemma is an immediate corollary of the weak convergence theorem in $H^1(\mathbb{R}^n)$ of Jones and Journé. See also [35] in the setting of $h^1(\mathbb{R}^n)$.

Lemma 4.5.1. *Let $\{f_k\}_{k\geq 1}$ be a bounded sequence in $H^1(\mathbb{R}^n)$ (resp., in $h^1(\mathbb{R}^n)$) such that f_k tends to f in $L^1(\mathbb{R}^n)$. Then f in $H^1(\mathbb{R}^n)$ (resp., in $h^1(\mathbb{R}^n)$), and*

$$\|f\|_{H^1} \leq \varliminf_{k\to\infty} \|f_k\|_{H^1} \quad (resp., \|f\|_{h^1} \leq \varliminf_{k\to\infty} \|f_k\|_{h^1}).$$

Theorem 4.5.2. *Let b be a non-constant BMO-function and $1 < q \leq \infty$. Then, $\mathcal{H}_b^{1,q}(\mathbb{R}^n) \subset H_b^1(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. Let a be a (q,b)-atom related to the cube Q. We first prove that $(b - b_Q)a$ is $C\|b\|_{BMO}$ times a classical $(\widetilde{q}+1)/2$-atom. One has supp $(b-b_Q)a \subset$ supp $a \subset Q$ and $\int_{\mathbb{R}^n}(b(x) - b_Q)a(x)dx = \int_{\mathbb{R}^n} b(x)a(x)dx - b_Q \int_{\mathbb{R}^n} a(x)dx = 0$. Moreover, by Hölder inequality and John-Nirenberg inequality, we get

$$\|(b-b_Q)a\|_{L^{(\widetilde{q}+1)/2}} \leq \|(b-b_Q)\chi_Q\|_{L^{\widetilde{q}(\widetilde{q}+1)/(\widetilde{q}-1)}} \|a\|_{L^{\widetilde{q}}} \leq C\|b\|_{BMO}|Q|^{(-\widetilde{q}+1)/(\widetilde{q}+1)},$$

where $\widetilde{q} = q$ if $1 < q < \infty$, $\widetilde{q} = 2$ if $q = \infty$, and $C > 0$ is independent of b, a. Hence, $(b - b_Q)a$ is $C\|b\|_{BMO}$ times a classical $(\widetilde{q}+1)/2$-atom, and $\|(b-b_Q)a\|_{H^1} \leq C\|b\|_{BMO}$.

We now prove that $\mathfrak{S}(a,b)$ belongs to H^1.

By Theorem 4.3.2, there exist n bounded bilinear operators $\mathfrak{R}_j : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $j = 1, ..., n$, such that

$$[b, \mathcal{R}_j](a) = \mathfrak{R}_j(a,b) + \mathcal{R}_j(\mathfrak{S}(a,b)),$$

since $\mathcal{R}_j$ is linear and belongs to $\mathcal{K}$ (see Corollary 4.4.1). Consequently, for all $j = 1, ..., n$, as $\mathcal{R}_j \in \mathcal{K}$,

$$\begin{aligned} \|\mathcal{R}_j(\mathfrak{S}(a,b))\|_{L^1} &= \|(b-b_Q)\mathcal{R}_j(a) - \mathcal{R}_j((b-b_Q)a) - \mathfrak{R}_j(a,b)\|_{L^1} \\ &\leq \|(b-b_Q)\mathcal{R}_j(a)\|_{L^1} + \|\mathcal{R}_j\|_{H^1 \to L^1}\|((b-b_Q)a)\|_{H^1} + \|\mathfrak{R}_j(a,b)\|_{L^1} \\ &\leq C\|b\|_{BMO}. \end{aligned}$$

This proves that $\mathfrak{S}(a,b) \in H^1(\mathbb{R}^n)$ since $\|\mathfrak{S}(a,b)\|_{L^1} \leq C\|b\|_{BMO}$, and moreover that

$$\|\mathfrak{S}(a,b)\|_{H^1} \leq C\|b\|_{BMO}. \tag{4.10}$$

Now, for any $f \in \mathcal{H}_b^{1,q}(\mathbb{R}^n)$, there exists an expansion $f = \sum_{j=1}^\infty \lambda_j a_j$ where the a_j's are (q,b)-atoms and $\sum_{j=1}^\infty |\lambda_j| \leq 2\|f\|_{\mathcal{H}_b^{1,q}}$. Then the sequence $\{\sum_{j=1}^k \lambda_j a_j\}_{k\geq 1}$ converges to f in $\mathcal{H}_b^{1,q}(\mathbb{R}^n)$ and thus in $H^1(\mathbb{R}^n)$. Hence, Lemma 4.4.1 implies that the sequence $\left\{\mathfrak{S}\Big(\sum_{j=1}^k \lambda_j a_j, b\Big)\right\}_{k\geq 1}$ converges to $\mathfrak{S}(f,b)$ in $L^1(\mathbb{R}^n)$. In addition, by (4.10),

$$\Big\|\mathfrak{S}\Big(\sum_{j=1}^k \lambda_j a_j, b\Big)\Big\|_{H^1} \leq \sum_{j=1}^k |\lambda_j| \|\mathfrak{S}(a_j,b)\|_{H^1} \leq C\|f\|_{\mathcal{H}_b^{1,q}}\|b\|_{BMO}.$$

We then use Lemma 4.5.1 to conclude that $\mathfrak{S}(f,b) \in H^1(\mathbb{R}^n)$, and thus $f \in H_b^1(\mathbb{R}^n)$ (see

Theorem 4.5.1). Moreover,

$$\begin{aligned}\|f\|_{H^1_b} &\leq C(\|f\|_{H^1}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1})\\ &\leq C\Big(\|f\|_{\mathcal{H}^{1,q}_b}\|b\|_{BMO} + \varliminf_{k\to\infty}\Big\|\mathfrak{S}\Big(\sum_{j=1}^{k}\lambda_j a_j, b\Big)\Big\|_{H^1}\Big)\\ &\leq C\|f\|_{\mathcal{H}^{1,q}_b}\|b\|_{BMO},\end{aligned}$$

which ends the proof. □

From Theorem 4.3.3 and Theorem 4.5.1, we get the following corollary.

Corollary 4.5.1. *Let b be a BMO-function, $T \in \mathcal{K}$ and $1 < q \leq \infty$. Then the linear commutator $[b,T]$ maps continuously $\mathcal{H}^{1,q}_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

4.6 Proof of Theorem 4.3.1, Theorem 4.3.2, Theorem 4.3.3

In order to prove the decomposition theorems (Theorem 4.3.2 and Theorem 4.3.1), we need the following two lemmas.

Lemma 4.6.1. *Let $T \in \mathcal{K}$ and a be a classical H^1-atom related to the cube mQ. Then, there exists a positive constant $C = C(m)$ such that*

$$\|(g - g_Q)Ta\|_{L^1} \leq C\|g\|_{BMO}, \textit{ for all } g \in BMO(\mathbb{R}^n).$$

Proof. Since $T \in \mathcal{K}$ and since $|g_Q - g_{mQ}| \leq C(m)\|g\|_{BMO}$, we have

$$\|(g - g_Q)Ta\|_{L^1} \leq C(m)\|g\|_{BMO}\|Ta\|_{L^1} + \|(g - g_{mQ})Ta\|_{L^1} \leq C\|g\|_{BMO}.$$

□

Lemma 4.6.2. *The norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^1_{\text{fin}}}$ are equivalent on $H^1_{\text{fin}}(\mathbb{R}^n)$.*

We point out that in the proof below we use the results and notations of Theorem 5.12 of [63]. Even though the proofs in [63] are in the one-dimensional case, they can be easily carried out in higher dimension as well.

The proof of Lemma 4.6.2. Obviously, $H^1_{\text{fin}}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ and for all $f \in H^1_{\text{fin}}(\mathbb{R}^n)$, we have $\|f\|_{H^1} \leq C\|f\|_{H^1_{\text{fin}}}$. We now have to show that there exists a constant $C > 0$ such that for all $f \in H^1_{\text{fin}}(\mathbb{R}^n)$,

$$\|f\|_{H^1_{\text{fin}}} \leq C\|f\|_{H^1}.$$

By homogeneity, we can assume that $\|f\|_{H^1} = 1$. We write $f = \sum_{j=1}^{N_0} \lambda_j a_j$, where the a_j's are ψ-atoms related to the cubes R_j's. Since $f \in L^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, there exists a ψ-atomic decomposition (see [63], Theorem 5.12)

$$f = \sum_I \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma \;=\; \sum_{k\in\mathbb{Z}} \sum_{i\in\Lambda_k} \left(\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma \right)$$

where $\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma = \lambda(k,i) a_{k,i}$ with $a_{k,i}$ ψ-atoms related to the cubes $m\widetilde{I}_k^i$ and

$$\sum_{k\in\mathbb{Z}} \sum_{i\in\Lambda_k} |\lambda(k,i)| \leq C\|f\|_{H^1} = C. \tag{4.11}$$

We note that supp $a_{k,i} \subset \bigcup_{j=1}^{N_0} mR_j$ for all $k\in\mathbb{Z}, i\in\Lambda_k$. Recall that

$$\begin{aligned} \mathcal{W}_\psi f &= \Big(\sum_I \sum_{\sigma\in E} |\langle f, \psi_I^\sigma\rangle|^2 |I|^{-1} \chi_I \Big)^{1/2} \\ &= \Big(\sum_{j=1}^{N_0} \sum_{I\subset R_j} \sum_{\sigma\in E} |\langle f, \psi_I^\sigma\rangle|^2 |I|^{-1} \chi_I \Big)^{1/2} \end{aligned}$$

and $\Omega_k = \{x\in\mathbb{R}^n : \mathcal{W}_\psi f(x) > 2^k\}$ for any $k\in\mathbb{Z}$. Clearly, supp $\mathcal{W}_\psi f \subset \bigcup_{j=1}^{N_0} mR_j$. So, there exists a cube Q such that $\Omega_k \subset \operatorname{supp} \mathcal{W}_\psi f \subset \bigcup_{j=1}^{N_0} mR_j \subset Q$ for all $k\in\mathbb{Z}$. We now denote by k' the largest integer k such that $2^k \leq |Q|^{-1}$. Then, we define the functions g and ℓ by

$$g = \sum_{k\leq k'} \sum_{i\in\Lambda_k} \left(\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma \right) \text{ and } \ell = \sum_{k> k'} \sum_{i\in\Lambda_k} \left(\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma \right).$$

Obviously, $f = g + \ell$, moreover, supp $g \subset Q$ and supp $\ell \subset Q$. On the other hand, it follows from Theorem 5.12 of [63] that $\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} |\langle f, \psi_I^\sigma\rangle|^2 \leq C 2^{2k} |\widetilde{I}_k^i \cap \Omega_k|$. Hence, as the dyadic cubes $\widetilde{I}_k^i$ are disjoint (see also [63]), we get

$$\begin{aligned} \|g\|_{L^2}^2 &\leq C \sum_{k\leq k'} \sum_{i\in\Lambda_k} \sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} |\langle f, \psi_I^\sigma\rangle|^2 \\ &\leq C \sum_{k\leq k'} \sum_{i\in\Lambda_k} 2^{2k} |\widetilde{I}_k^i \cap \Omega_k| \leq C \sum_{k\leq k'} 2^{2k} |\Omega_k| \\ &\leq C 2^{2k'} |Q| \leq C|Q|^{-1}. \end{aligned}$$

This proves that $C^{-1/2} g$ is a ψ-atom related to the cube Q.

Now, for any positive integer K, set $F_K = \{(k,i) : k > k', |k| + |i| \leq K\}$ and $\ell_K = \sum_{(k,i)\in F_K} \Big(\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma\Big)$. Observe that since $f \in L^2(\mathbb{R}^n)$, the series $\sum_{k>k'} \sum_{i\in\Lambda_k} \Big(\sum_{I\subset \widetilde{I}_k^i, I\in\mathcal{B}_k} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma\Big)$ converges in $L^2(\mathbb{R}^n)$. So, for any $\varepsilon > 0$, if K is large enough, $\varepsilon^{-1}(\ell - \ell_K)$ is a ψ-atom related to the cube Q. Therefore, $f = g + \ell_K + (\ell - \ell_K)$ is a finite linear combination of atoms for f, and thus

$$\begin{aligned} \|f\|_{H^1_{\rm fin}} &\leq C(\|g\|_{H^1_{\rm fin}} + \|\ell_K\|_{H^1_{\rm fin}} + \|\ell - \ell_K\|_{H^1_{\rm fin}}) \\ &\leq C\Big(C + \sum_{k\in\mathbb{Z}} \sum_{i\in\Lambda_k} |\lambda(k,i)| + \varepsilon\Big) \leq C \end{aligned}$$

by (4.11). It ends the proof.

□

Proof of Theorem 4.3.1. We define the subbilinear operator $\mathfrak{R}$ by

$$\mathfrak{R}(f,b)(x) := \Big|T\Big(b(x)f(\cdot) - \Pi_2(f,b)(\cdot)\Big)(x)\Big| + |T(\Pi_1(f,b))(x)| + |T(\Pi_4(f,b))(x)|$$

for all $(f,b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$. Then, by Theorem 4.4.1, we obtain that

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|.$$

By Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.4.3, it is sufficient to show that the subbilinear operator

$$\mathfrak{U}(f,b)(x) := \Big|T\Big(b(x)f(\cdot) - \Pi_2(f,b)(\cdot)\Big)(x)\Big|$$

is bounded from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

We first consider $b \in BMO(\mathbb{R}^n)$ and f a ψ-atom related to the cube Q. Then, by Remark 4.4.1, we have

$$\mathfrak{U}(f,b)(x) = \mathfrak{U}(f, b - b_Q)(x) \leq |(b(x) - b_Q)Tf(x)| + |T(\Pi_2(f, b - b_Q))(x)|.$$

Consequently, by Remark 4.4.1, Lemma 4.6.1 and the fact f is C times a classical atom related to the cube mQ, we obtain that

$$\|\mathfrak{U}(f,b)\|_{L^1} \leq \|(b - b_Q)Tf\|_{L^1} + \|T\|_{H^1\to L^1} \|\Pi_2(f, b - b_Q)\|_{H^1} \leq C\|b\|_{BMO}, \qquad (4.12)$$

where $C > 0$ independent of f, b.

Now, let $b \in BMO(\mathbb{R}^n)$ and $f \in H^1_{\rm fin}(\mathbb{R}^n)$. By Lemma 4.6.2, there exists a finite decomposition $f = \sum_{j=1}^k \lambda_j a_j$ such that $\sum_{j=1}^k |\lambda_j| \leq C\|f\|_{H^1}$. Consequently, by (4.12), we obtain that

$$\|\mathfrak{U}(f,b)\|_{L^1} \leq \sum_{j=1}^k |\lambda_j| \|\mathfrak{U}(a_j, b)\|_{L^1} \leq C\|f\|_{H^1}\|b\|_{BMO},$$

which ends the proof as $H^1_{\text{fin}}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ for the norm $\|\cdot\|_{H^1}$.

□

Proof of Theorem 4.3.2. We define the bilinear operator $\mathfrak{R}$ by

$$\mathfrak{R}(f,b) = \Big(bTf - T(\Pi_2(f,b))\Big) - T(\Pi_1(f,b) + \Pi_4(f,b)),$$

for all $(f,b) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$. Then, it follows from Theorem 4.4.1 and the proof of Theorem 4.3.1 that

$$[b,T](f) = \mathfrak{R}(f,b) + T(\mathfrak{S}(f,b)),$$

where the bilinear operator $\mathfrak{R}$ is bounded from $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. This completes the proof.

□

Proof of Theorem 4.3.3. Theorem 4.3.3 is an immediate corollary of Theorem 4.3.1 and Theorem 4.5.1. □

4.7 Proof of Theorem 4.3.4, Theorem 4.3.5 and Theorem 4.3.6

First we recall the following well-known result.

Theorem A (see [29] or [42]). *Let T be a Calderón-Zygmund operator satisfying $T1 = T^*1 = 0$, $1 < q < \infty$ and $1/p + 1/q = 1$. Then, $fTg - gT^*f \in H^1(\mathbb{R}^n)$ for all $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$.*

Now, in order to prove the bilinear type estimates and the Hardy type theorems for the commutators of Calderón-Zygmund operators, we need the following three technical lemmas.

Lemma 4.7.1. *Let $\delta \in (0,1]$, and A, B be two δ-Calderón-Zygmund operators such that $A1 = A^*1 = B1 = B^*1 = 0$. Then, there exists a constant $C = C(n,\delta)$ such that*

$$\sum_{I,I',I''}\sum_{\sigma,\sigma',\sigma''\in E} |\langle f,\psi_I^\sigma\rangle\langle g,\psi_{I'}^{\sigma'}\rangle\langle A\psi_I^\sigma,\psi_{I''}^{\sigma''}\rangle\langle B\psi_{I'}^{\sigma'},\psi_{I''}^{\sigma''}\rangle| \leq C\|f\|_{H^1}\|g\|_{BMO}$$

for all $f \in H^1(\mathbb{R}^n), g \in BMO(\mathbb{R}^n)$.

Lemma 4.7.2. *Let $\delta \in (0,1]$, and A_i, B_i, $i = 1, ..., K$, be δ-Calderón-Zygmund operators satisfying $A_i 1 = A_i^* 1 = B_i 1 = B_i^* 1 = 0$, and for every f and g in $L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{K} A_i f . B_i g \Big) dx = 0.$$

Then, the bilinear operator $\mathfrak{P}$, defined by $\mathfrak{P}(f,g) = \sum_{i=1}^K \mathfrak{S}(A_i f, B_i g)$, maps continuously $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

Corollary 4.7.1. *Let T be a Calderón-Zygmund operator satisfying $T1 = T^*1 = 0$. Then the bilinear operator $\mathfrak{P}$, defined by $\mathfrak{P}(f,g) = \mathfrak{S}(Tf, g) - \mathfrak{S}(f, T^* g)$, maps continuously $H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.*

Lemma 4.7.3. *Let b be a non-constant BMO-function and T be a Calderón-Zygmund operator with $T1 = T^*1 = 0$. Assume that $f \in H^1_b(\mathbb{R}^n)$ has the wavelet decomposition $f = \sum_{j=1}^\infty \sum_{I \subset R_j} \sum_{\sigma \in E} \langle f, \psi_I^\sigma \rangle \psi_I^\sigma$ where the R_j's are dyadic cubes and $\sum_{I \subset R_j} \sum_{\sigma \in E} \langle f, \psi_I^\sigma \rangle \psi_I^\sigma$ are multiples of ψ-atoms related to the cubes R_j. Set $f_k = \sum_{j=1}^k \sum_{I \subset R_j} \sum_{\sigma \in E} \langle f, \psi_I^\sigma \rangle \psi_I^\sigma$, $k = 1, 2, ...$ Then, the sequence $\{[b,T](f_k)\}_{k \geq 1}$ tends to $[b,T](f)$ in the sense of distributions $\mathcal{S}'(\mathbb{R}^n)$.*

Proof of Lemma 4.7.1. We first remark (see [108], Proposition 1) that there exists a constant $C > 0$ such that for all dyadic cubes I, I' and $\sigma, \sigma' \in E$, we have

$$\max\{|\langle A\psi_I^\sigma, \psi_{I'}^{\sigma'} \rangle|, |\langle B\psi_I^\sigma, \psi_{I'}^{\sigma'} \rangle|\} \leq C 2^{-|j-j'|(\delta + n/2)} \Big(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |x_I - x_{I'}|} \Big)^{n+\delta}. \quad (4.13)$$

Consequently,

$$\max\{|\langle A\psi_I^\sigma, \psi_{I'}^{\sigma'} \rangle|, |\langle B\psi_I^\sigma, \psi_{I'}^{\sigma'} \rangle|\} \leq C p_\delta(I, I') \quad (4.14)$$

with

$$p_\delta(I, I') = \frac{2^{-|j-j'|(\delta/2 + n/2)}}{1 + |j - j'|^2} \Big(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |x_I - x_{I'}|} \Big)^{n+\delta/2}.$$

Here $|I| = 2^{-jn}$ and $|I'| = 2^{-j'n}$, while x_I and $x_{I'}$ denote the centers of the two cubes. On the other hand, it follows from Lemma 1.3 in [42] that there exists a constant $C = C(n, \delta) > 0$ such that

$$\sum_{I''} p_\delta(I, I'') p_\delta(I', I'') \leq C p_\delta(I, I'). \quad (4.15)$$

Combining (4.14) and (4.15), we obtain

$$\sum_{I, I', I''} \sum_{\sigma, \sigma', \sigma'' \in E} |\langle f, \psi_I^\sigma \rangle \langle g, \psi_{I'}^{\sigma'} \rangle \langle A\psi_I^\sigma, \psi_{I''}^{\sigma''} \rangle \langle B\psi_{I'}^{\sigma'}, \psi_{I''}^{\sigma''} \rangle| \leq C \sum_{I, I'} \sum_{\sigma, \sigma' \in E} p_\delta(I, I') |\langle f, \psi_I^\sigma \rangle| |\langle g, \psi_{I'}^{\sigma'} \rangle|.$$

It is easy to establish that the matrix $\{p_\delta(I,I')\}_{I,I'}$ is almost diagonal (by taking $\varepsilon = \delta/4$ in the definition (3.1) of Frazier and Jawerth [49]) and thus is bounded on $\dot{f}_1^{0,2}$ the space of all sequences $(a_I)_I$ such that $\left(\sum_I |a_I|^2|I|^{-1}\chi_I\right)^{1/2}$ is in $L^1(\mathbb{R}^n)$. We then use the wavelet characterization of $H^1(\mathbb{R}^n)$ (see Theorem 4.2.1) and the fact that (cf. [49])

$$\sum_{I'}\sum_{\sigma'\in E} |\langle h, \psi_{I'}^{\sigma'}\rangle||\langle g, \psi_{I'}^{\sigma'}\rangle| \le C\|h\|_{H^1}\|g\|_{BMO},$$

for all $h \in H^1(\mathbb{R}^n)$, to conclude that

$$\sum_{I,I',I''}\sum_{\sigma,\sigma',\sigma''\in E} |\langle f, \psi_I^\sigma\rangle\langle g, \psi_{I'}^{\sigma'}\rangle\langle A\psi_I^\sigma, \psi_{I''}^{\sigma''}\rangle\langle B\psi_{I'}^{\sigma'}, \psi_{I''}^{\sigma''}\rangle| \le C\|f\|_{H^1}\|g\|_{BMO}.$$

□

Proof of Lemma 4.7.2. By Lemma 4.7.1, we have

$$\begin{aligned}\mathfrak{P}(f,g) &= \sum_{i=1}^K \mathfrak{S}(A_i f, B_i g)\\ &= \sum_{i=1}^K \sum_{I,I',I''}\sum_{\sigma,\sigma',\sigma''\in E} \langle f, \psi_I^\sigma\rangle\langle g, \psi_{I'}^{\sigma'}\rangle\langle A_i\psi_I^\sigma, \psi_{I''}^{\sigma''}\rangle\langle B_i\psi_{I'}^{\sigma'}, \psi_{I''}^{\sigma''}\rangle(\psi_{I''}^{\sigma''})^2\end{aligned}$$

where all the series converge in $L^1(\mathbb{R}^n)$. For any dyadic cubes I, I', $\sigma, \sigma' \in E$, we have

$$\begin{aligned}&\sum_{i=1}^K \sum_{I''}\sum_{\sigma''\in E} \langle f, \psi_I^\sigma\rangle\langle g, \psi_{I'}^{\sigma'}\rangle\langle A_i\psi_I^\sigma, \psi_{I''}^{\sigma''}\rangle\langle B_i\psi_{I'}^{\sigma'}, \psi_{I''}^{\sigma''}\rangle(\psi_{I''}^{\sigma''})^2\\ &\quad= \sum_{i=1}^K \sum_{I''}\sum_{\sigma''\in E} \langle f, \psi_I^\sigma\rangle\langle g, \psi_{I'}^{\sigma'}\rangle\langle A_i\psi_I^\sigma, \psi_{I''}^{\sigma''}\rangle\langle B_i\psi_{I'}^{\sigma'}, \psi_{I''}^{\sigma''}\rangle\Big((\psi_{I''}^{\sigma''})^2 - (\psi_I^\sigma)^2\Big)\end{aligned}$$

since (see Remark 4.4.1)

$$\sum_{i=1}^K \sum_{I''}\sum_{\sigma''\in E} \langle A_i\psi_I^\sigma, \psi_{I''}^{\sigma''}\rangle\langle B_i\psi_{I'}^{\sigma'}, \psi_{I''}^{\sigma''}\rangle = \int_{\mathbb{R}^n} \Big(\sum_{i=1}^K A_i\psi_I^\sigma . B_i\psi_{I'}^{\sigma'}\Big)dx = 0.$$

An explicit computation gives that $|\psi_{I''}^{\sigma''}|^2 - |\psi_I^\sigma|^2$ is in $H^1(\mathbb{R}^n)$, with

$$\||\psi_{I''}^{\sigma''}|^2 - |\psi_I^\sigma|^2\|_{H^1} \le C\left(\log(2^{-j} + 2^{-j''})^{-1} + \log(|x_I - x_{I''}| + 2^{-j} + 2^{-j''})\right).$$

Here $|I| = 2^{-jn}$ and $|I''| = 2^{-j''n}$, while x_I and $x_{I''}$ denote the centers of the two cubes. Consequently, by (4.13) and (4.15), we get

$$\begin{aligned}
&\Big\|\sum_{i=1}^{K}\sum_{I''}\sum_{\sigma''\in E}\langle f,\psi_I^\sigma\rangle\langle g,\psi_{I'}^{\sigma'}\rangle\langle A_i\psi_I^\sigma,\psi_{I''}^{\sigma''}\rangle\langle B_i\psi_{I'}^{\sigma'},\psi_{I''}^{\sigma''}\rangle(\psi_{I''}^{\sigma''})^2\Big\|_{H^1}\\
\leq\ & \sum_{i=1}^{K}\sum_{I''}\sum_{\sigma''\in E}|\langle f,\psi_I^\sigma\rangle\langle g,\psi_{I'}^{\sigma'}\rangle\langle A_i\psi_I^\sigma,\psi_{I''}^{\sigma''}\rangle\langle B_i\psi_{I'}^{\sigma'},\psi_{I''}^{\sigma''}\rangle|\Big\|(\psi_{I''}^{\sigma''})^2-(\psi_I^\sigma)^2\Big\|_{H^1}\\
\leq\ & C\sum_{i=1}^{K}\sum_{I''}\sum_{\sigma''\in E}|\langle f,\psi_I^\sigma\rangle\langle g,\psi_{I'}^{\sigma'}\rangle|p_\delta(I,I'')p_\delta(I',I'')\\
\leq\ & Cp_\delta(I,I')|\langle f,\psi_I^\sigma\rangle||\langle g,\psi_{I'}^{\sigma'}\rangle|,
\end{aligned}$$

here we used the fact that

$$(1+|j-j''|^2)\log\Big(\frac{|x_I-x_{I''}|+2^{-j}+2^{-j''}}{2^{-j}+2^{-j''}}\Big)\leq C(\delta)2^{|j-j''|\delta/2}\Big(\frac{|x_I-x_{I''}|+2^{-j}+2^{-j''}}{2^{-j}+2^{-j''}}\Big)^{\delta/2}.$$

Thus, the same argument as in the proof of Lemma 4.7.1 allows to conclude that

$$\begin{aligned}
\|\mathfrak{P}(f,g)\|_{H^1} &\leq C\sum_{I,I'}\sum_{\sigma,\sigma'\in E}p_\delta(I,I')|\langle f,\psi_I^\sigma\rangle||\langle g,\psi_{I'}^{\sigma'}\rangle|\\
&\leq C\|f\|_{H^1}\|g\|_{BMO},
\end{aligned}$$

which ends the proof.

$\square$

Before giving the proof of Lemma 4.7.3, let us recall the following lemma. It can be found in [50].

Lemma A. (see [50], Lemma 2.3) Let T be a Calderón-Zygmund operator satisfying $T1 = 0$. Then T maps $\mathcal{S}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$, depending only on T, such that for any $\phi\in\mathcal{S}(\mathbb{R}^n)$ with supp $\phi\subset B(x_0,r)$, we have

$$\|T\phi\|_{L^\infty}\leq C(\|\phi\|_{L^\infty}+r\||\nabla\phi|\|_{L^\infty}).$$

Proof of Lemma 4.7.3. By Theorem 4.3.2, it is sufficient to prove that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}T(\mathfrak{S}(f_k,b))hdx=\int_{\mathbb{R}^n}T(\mathfrak{S}(f,b))hdx,$$

for all $h\in\mathcal{S}(\mathbb{R}^n)$. Because of the hypothesis, we observe that $\mathfrak{S}(f,b)\in H^1(\mathbb{R}^n)$ and $\mathfrak{S}(f_k,b)\in L^q(\mathbb{R}^n)$, $k=1,2,\ldots$, for some $q\in(1,2)$ (see Lemma 4.2.1).

Let $\mathfrak{S}(f,b) = \sum_{j=1}^\infty \lambda_j a_j$ be a classical L^q-atomic decomposition of $\mathfrak{S}(f,b)$. Then, $T(\sum_{j=1}^k \lambda_j a_j)$ tends to $T(\mathfrak{S}(f,b))$ in $L^1(\mathbb{R}^n)$ (in fact, it also holds in $H^1(\mathbb{R}^n)$ since $T^*1 = 0$). Hence, as $h \in \mathcal{S}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \cap L^{q'}(\mathbb{R}^n)$ where $1/q + 1/q' = 1$, $\mathfrak{S}(f_k, b), a_j \in L^q(\mathbb{R}^n)$ and $T^*h \in L^\infty(\mathbb{R}^n)$ since $T^*1 = 0$ (see Lemma A), by Theorem A we get

$$\begin{aligned}
\int_{\mathbb{R}^n} T(\mathfrak{S}(f,b))h dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n} T\Big(\sum_{j=1}^k \lambda_j a_j\Big) h dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \Big(\sum_{j=1}^k \lambda_j a_j\Big) T^* h dx \\
&= \int_{\mathbb{R}^n} \mathfrak{S}(f,b) T^* h dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \mathfrak{S}(f_k,b) T^* h dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} T(\mathfrak{S}(f_k,b)) h dx,
\end{aligned}$$

since $\mathfrak{S}(f_k, b)$ tends to $\mathfrak{S}(f,b)$ in $L^1(\mathbb{R}^n)$ as f_k tends to f in $H^1(\mathbb{R}^n)$ (see Theorem 4.3.2). This finishes the proof.

□

Proof of Theorem 4.3.4. Let $(f,g) \in H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$. By Theorem 4.3.2 and Lemma 4.7.2, we obtain $\mathfrak{T}(f,g) = \sum_{i=1}^K [B_i g, T](A_i f) \in L^1(\mathbb{R}^n)$, moreover,

$$\begin{aligned}
\|\mathfrak{T}(f,g)\|_{L^1} &\leq \sum_{i=1}^K \|\mathfrak{R}(A_i f, B_i g)\|_{L^1} + \Big\| T\Big(\sum_{i=1}^K \mathfrak{S}(A_i f, B_i g)\Big)\Big\|_{L^1} \\
&\leq C \sum_{i=1}^K \|A_i f\|_{H^1} \|B_i g\|_{BMO} + \|T\|_{H^1 \to L^1} \Big\| \sum_{i=1}^K \mathfrak{S}(A_i f, B_i g)\Big\|_{H^1} \\
&\leq C \|f\|_{H^1} \|g\|_{BMO}.
\end{aligned}$$

This completes the proof.

□

Proof of Theorem 4.3.5. Let $f \in H^1_b(\mathbb{R}^n)$, we prove $[b,T](f) \in h^1(\mathbb{R}^n)$ using the fact that $BMO^{\log}(\mathbb{R}^n)$ is the dual of $H^{\log}(\mathbb{R}^n)$ (see [81]). Indeed, by Theorem 4.2.2, there exists a decomposition $f = \sum_{j=1}^\infty \sum_{I\subset R_j} \sum_{\sigma \in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma$ where $\sum_{I\subset R_j} \sum_{\sigma\in E} \langle f, \psi_I^\sigma \rangle \psi_I^\sigma$ are multiples of ψ-atoms related to the dyadic cubes R_j. Set $f_k = \sum_{j=1}^k \sum_{I\subset R_j} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle \psi_I^\sigma$, $k = 1, 2, ...$ Then, the sequence $[b,T](f_k)$ tends to $[b,T](f)$ in the sense of distributions $\mathcal{S}'(\mathbb{R}^n)$ (see Lemma 4.7.3), and thus

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} [b,T](f_k) h dx = \int_{\mathbb{R}^n} [b,T](f) h dx, \tag{4.16}$$

for all $h \in C_0^\infty(\mathbb{R}^n)$. Notice that $[b,T](f_k) \in L^2(\mathbb{R}^n)$ and $[b,T](f) \in L^1(\mathbb{R}^n)$.

Let $h \in C_0^\infty(\mathbb{R}^n)$. By Lemma 4.4.2, Lemma 4.4.3, Lemma 4.4.4, Remark 4.4.1 and Corollary 4.7.1, we have $hT(f_k) - f_k\Big(T^*h - (T^*h)_{\mathbb{Q}}\Big) \in H^{\log}(\mathbb{R}^n)$. More precisely,

$$\begin{aligned}
&\Big\|hT(f_k) - f_k\Big(T^*h - (T^*h)_{\mathbb{Q}}\Big)\Big\|_{H^{\log}} \\
\leq\ & C\Bigg\{\Big\|\mathfrak{S}(T(f_k),h) - \mathfrak{S}\Big(f_k, T^*h - (T^*h)_{\mathbb{Q}}\Big)\Big\|_{H^1} + \\
&\quad + \sum_{j=1,4}\Big(\|\Pi_j(T(f_k),h)\|_{H^1} + \Big\|\Pi_j\Big(f_k, T^*h - (T^*h)_{\mathbb{Q}}\Big)\Big\|_{H^1}\Big) + \\
&\qquad + \|\Pi_2(T(f_k),h)\|_{H^{\log}} + \Big\|\Pi_2\Big(f_k, T^*h - (T^*h)_{\mathbb{Q}}\Big)\Big\|_{H^{\log}}\Bigg\} \\
\leq\ & C\Bigg\{\|f_k\|_{H^1}\|h\|_{BMO} + \|T(f_k)\|_{H^1}\|h\|_{BMO} + \|f_k\|_{H^1}\Big\|T^*h - (T^*h)_{\mathbb{Q}}\Big\|_{BMO} + \\
&\qquad + \|T(f_k)\|_{H^1}\|h\|_{BMO^+} + \|f_k\|_{H^1}\|T^*h - (T^*h)_{\mathbb{Q}}\|_{BMO^+}\Bigg\} \\
\leq\ & C(\|f_k\|_{H^1}\|h\|_{bmo} + \|f_k\|_{H^1}\|T^*h\|_{BMO}) \leq C\|f\|_{H^1}\|h\|_{bmo},
\end{aligned}$$

here one used $\mathfrak{S}\Big(f, T^*h - (T^*h)_{\mathbb{Q}}\Big) = \mathfrak{S}(f, T^*h)$, $\|T^*h - (T^*h)_{\mathbb{Q}}\|_{BMO^+} = \|T^*h\|_{BMO}$ and $\|f_k\|_{H^1} \leq C\|f\|_{H^1}$. As the L^2- functions f_k have compact support, $b \in BMO^{\log}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, we deduce that $bhT(f_k), hT(bf_k), bf_kT^*h \in L^1(\mathbb{R}^n)$. Moreover, $\int_{\mathbb{R}^n} hT(bf_k)dx = \int_{\mathbb{R}^n} bf_kT^*hdx$ since $hT(bf_k) - bf_kT^*h \in H^1(\mathbb{R}^n)$ (see Theorem A). Therefore, as $BMO^{\log}(\mathbb{R}^n)$ is the dual of $H^{\log}(\mathbb{R}^n)$ (see [81]), we get

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n}[b,T](f_k)hdx\Big| &= \Big|\int_{\mathbb{R}^n} b(hT(f_k) - f_kT^*h)dx\Big| \\
&\leq \Big|\int_{\mathbb{R}^n} b\Big(hT(f_k) - f_k\Big(T^*h - (T^*h)_{\mathbb{Q}}\Big)\Big)dx\Big| + |(T^*h)_{\mathbb{Q}}|\Big|\int_{\mathbb{R}^n} bf_kdx\Big| \\
&\leq C\|b\|_{BMO^{\log}}\Big\|hT(f_k) - f_k\Big(T^*h - (T^*h)_{\mathbb{Q}}\Big)\Big\|_{H^{\log}} + |(T^*h)_{\mathbb{Q}}|\Big|\int_{\mathbb{R}^n} bf_kdx\Big| \\
&\leq C\|b\|_{BMO^{\log}}\|f\|_{H^1}\|h\|_{bmo} + |(T^*h)_{\mathbb{Q}}|\Big|\sum_{j=1}^{k}\sum_{I\subset R_j}\sum_{\sigma\in E}\langle f,\psi_I^\sigma\rangle\langle b,\psi_I^\sigma\rangle\Big|.
\end{aligned}$$

The above inequality and (4.16) imply that for all $h \in C_0^\infty(\mathbb{R}^n)$, we obtain

$$\Big|\int_{\mathbb{R}^n}[b,T](f)hdx\Big| \leq C\|b\|_{BMO^{\log}}\|f\|_{H^1}\|h\|_{bmo}$$

since $\mathfrak{S}(f,b) \in H^1(\mathbb{R}^n)$ (see Theorem 4.5.1) and thus (see Remark 4.4.1)

$$\lim_{k\to\infty} \sum_{j=1}^{k} \sum_{I\subset R_j} \sum_{\sigma\in E} \langle f, \psi_I^\sigma\rangle\langle b, \psi_I^\sigma\rangle = \int_{\mathbb{R}^n} \mathfrak{S}(f,b)dx = 0.$$

This proves that $[b,T](f) \in h^1(\mathbb{R}^n)$ since $h^1(\mathbb{R}^n)$ is the dual of $vmo(\mathbb{R}^n)$ (see Section 2). Furthermore,

$$\|[b,T](f)\|_{h^1} \leq C\|b\|_{BMO^{\log}}\|f\|_{H^1} \leq C\|b\|_{BMO^{\log}}\|b\|_{BMO}^{-1}\|f\|_{H^1_b},$$

which ends the proof of Theorem 4.3.5.

□

Proof of Theorem 4.3.6. By Theorem 4.3.2 and Theorem 4.5.1 together with Lemma 4.4.2 and Lemma 4.4.3, it is sufficient to prove that the linear operator

$$f \mapsto \mathfrak{U}(f,b) := bTf - T(\Pi_2(f,b))$$

is bounded from $H^1(\mathbb{R}^n)$ into itself. Similarly to the proof of Theorem 4.3.1, we first consider f a ψ-atom related to the cube $Q = Q[x_0, r]$ and note that

$$\mathfrak{U}(f,b) = \mathfrak{U}(f, b-b_Q) = (b-b_Q)Tf - T(\Pi_2(f, b-b_Q)). \tag{4.17}$$

Let $\varepsilon \in (0,1)$, recall that (see [133]) g is an ε-molecule for $H^1(\mathbb{R}^n)$ centered at y_0 if

$$\int_{\mathbb{R}^n} g(x)dx = 0 \quad \text{and} \quad \|g\|_{L^q}^{1/2}\|g|\cdot - y_0|^{2n\varepsilon}\|_{L^q}^{1/2} =: \mathfrak{N}(g) < \infty,$$

where $q = 1/(1-\varepsilon)$. It is well known that if g is an ε-molecule for $H^1(\mathbb{R}^n)$ centered at y_0, then $g \in H^1(\mathbb{R}^n)$ and $\|g\|_{H^1} \leq C\mathfrak{N}(g)$ where $C > 0$ depends only on n, ε.

We now prove that $(b-b_Q)Tf$ is an ε-molecule for $H^1(\mathbb{R}^n)$ centered at x_0 when T is a δ-Calderón-Zygmund operator for some $\delta \in (0,1]$ and $\varepsilon = \delta/(4n) < 1/2$. Note first that f is C times a classical L^2-atom related to the cube mQ. It is clear that $\int_{\mathbb{R}^n}(b-b_Q)Tf dx = 0$ since $T^*1 = T^*b = 0$. As $q = 1/(1-\varepsilon) < 2$, the fact $|b_Q - b_{2mQ}| \leq C\|b\|_{BMO}$ together with Hölder inequality and John-Nirenberg inequality, give

$$\|(b-b_Q)Tf.\chi_{2mQ}\|_{L^q} \leq C|Q|^{1/q-1}\|b\|_{BMO}. \tag{4.18}$$

It is well-known that $|Tf(x)| \leq C\frac{r^\delta}{|x-x_0|^{n+\delta}}$, for all $x \in (2mQ)^c$, since T is a δ-Calderón-Zygmund operator. Hence

$$\begin{aligned}
\|(b-b_Q)Tf.\chi_{(2mQ)^c}\|_{L^q} &\leq C\left(\int_{(2mQ)^c} |b-b_Q|^q\Big(\frac{r^\delta}{|x-x_0|^{n+\delta}}\Big)^q dx\right)^{1/q} \\
&\leq C|Q|^{1/q-1}\|b\|_{BMO}.
\end{aligned}$$

The last inequality, which can be found in [128], is classical. Combining this and (6.24), we obtain

$$\|(b-b_Q)Tf\|_{L^q} \leq C|Q|^{1/q-1}\|b\|_{BMO}. \tag{4.19}$$

Similarly, we also have

$$\|(b-b_Q)Tf.|\cdot - x_0|^{2n\varepsilon}.\chi_{2mQ}\|_{L^q} \leq C|Q|^{2\varepsilon+1/q-1}\|b\|_{BMO}$$

and as $2n\varepsilon = \delta/2$,

$$\begin{aligned} \|(b-b_Q)Tf.|\cdot - x_0|^{2n\varepsilon}.\chi_{(2mQ)^c}\|_{L^q} &\leq C\left(\int\limits_{(2mQ)^c} |b-b_Q|^q\Big(\frac{r^\delta}{|x-x_0|^{n+\delta/2}}\Big)^q dx\right)^{1/q} \\ &\leq C|Q|^{2\varepsilon+1/q-1}\|b\|_{BMO}. \end{aligned}$$

Consequently,

$$\|(b-b_Q)Tf.|\cdot - x_0|^{2n\varepsilon}\|_{L^q} \leq C|Q|^{2\varepsilon+1/q-1}\|b\|_{BMO}.$$

Combining this and (4.19), we get $(b-b_Q)Tf$ is an ε-molecule for $H^1(\mathbb{R}^n)$ centered at x_0, moreover,

$$\mathfrak{N}((b-b_Q)Tf) \leq C|Q|^{\varepsilon+1/q-1}\|b\|_{BMO} \leq C\|b\|_{BMO},$$

since $q = 1/(1-\varepsilon)$. Thus, by (6.22) and Remark 4.4.1,

$$\|\mathfrak{U}(f,b)\|_{H^1} \leq C\mathfrak{N}((b-b_Q)Tf) + \|T(\Pi_2(f,b-b_Q))\|_{H^1} \leq C\|b\|_{BMO}. \tag{4.20}$$

Now, let us consider $f \in H^1_{\rm fin}(\mathbb{R}^n)$. By Lemma 4.6.2, there exists a finite decomposition $f = \sum_{j=1}^k \lambda_j a_j$ such that $\sum_{j=1}^k |\lambda_j| \leq C\|f\|_{H^1}$. Consequently, by (4.20), we obtain that

$$\|\mathfrak{U}(f,b)\|_{H^1} \leq \sum_{j=1}^k |\lambda_j|\|\mathfrak{U}(a_j,b)\|_{H^1} \leq C\|f\|_{H^1}\|b\|_{BMO},$$

which ends the proof as $H^1_{\rm fin}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ for the norm $\|\cdot\|_{H^1}$.

$\square$

4.8 Commutators of Fractional integrals

Given $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Let b be a locally integrable function. We consider the linear commutator $[b, I_\alpha]$ defined by

$$[b, I_\alpha](f) = bI_\alpha f - I_\alpha(bf).$$

We end this article by presenting some results related to commutators of fractional integrals as follows.

Theorem 4.8.1. *Let $0 < \alpha < n$. There exist a bounded bilinear operator $\mathfrak{R} : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^{n/(n-\alpha)}(\mathbb{R}^n)$ and a bounded bilinear operator $\mathfrak{S} : H^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that*

$$[b, I_\alpha](f) = \mathfrak{R}(f, b) + I_\alpha(\mathfrak{S}(f, b)).$$

Corollary 4.8.1. *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Then, the linear commutator $[b, I_\alpha]$ maps continuously $H^1(\mathbb{R}^n)$ into weak-$L^{n/(n-\alpha)}(\mathbb{R}^n)$.*

Theorem 4.8.2. *Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, and $1 < q \leq \infty$. Then, the linear commutator $[b, I_\alpha]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $L^{n/(n-\alpha)}(\mathbb{R}^n)$.*

The results above can be proved similarly to Theorem 4.3.2 and Theorem 4.3.3. We leave the proofs to the interested readers. When $H^1_b(\mathbb{R}^n)$ is replaced by $\mathcal{H}^1_b(\mathbb{R}^n)$, Theorem 4.8.2 was considered by the authors in [40]. There, they proved that

$$\sup\{\|[b, I_\alpha](a)\|_{L^{n/(n-\alpha)}} : a \,\text{is a}\, (\infty, b)\text{−atom}\} < \infty.$$

However, as pointed out before, this argument does not suffice to conclude that $[b, I_\alpha]$ is bounded from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^{n/(n-\alpha)}(\mathbb{R}^n)$.

Chapter 5

Endpoint estimates for commutators of singular integrals related to Schrödinger operators

Ce chapitre est une prépublication (soumise).

Résumé

Soit $L = -\Delta + V$ un opérateur Schrödinger sur $\mathbb{R}^d$, $d \geq 3$, où V est un potentiel positif, $V \neq 0$, et appartient à la classe Hölder inverse $RH_{d/2}$. Dans cet article, nous étudions les commutateurs $[b, T]$ pour T dans une classe $\mathcal{K}_L$ des opérateurs sous-linéaire contenant les opérateurs fondamentaux en analyse harmonique liée à L. Plus précisément, lorsque $T \in \mathcal{K}_L$, nous prouvons qu'il existe un opérateur borné sous-bilinéaire $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ tell que

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|, \tag{5.1}$$

où $\mathfrak{S}$ est un opérateur borné bilinéaire de $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ à valeurs $L^1(\mathbb{R}^d)$, indépendant de T. La décomposition sous-bilinéaire (5.1) nous permet d'expliquer pourquoi les commutateurs avec les opérateurs fondamentaux sont de type faible (H^1_L, L^1), et quand un commutateur $[b, T]$ est de type fort (H^1_L, L^1).

En outre, nous étudions les estimations H^1_L des commutateurs de la transformée de Riesz associée à l'opérateur de Schrödinger L.

5.1 Introduction

Given a function b locally integrable on $\mathbb{R}^d$, and a (classical) Calderón-Zygmund operator T, we consider the linear commutator $[b,T]$ defined for smooth, compactly supported functions f by

$$[b,T](f) = bT(f) - T(bf).$$

A classical result of Coifman, Rochberg and Weiss (see [31]), states that the commutator $[b,T]$ is continuous on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, when $b \in BMO(\mathbb{R}^d)$. Unlike the theory of (classical) Calderón-Zygmund operators, the proof of this result does not rely on a weak type $(1,1)$ estimate for $[b,T]$. Instead, an endpoint theory was provided for this operator. A general overview about these facts can be found for instance in [82].

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. We recall that a nonnegative locally integrable function V belongs to the reverse Hölder class RH_q, $1 < q < \infty$, if there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int_B (V(x))^q dx\Big)^{1/q} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every balls B in $\mathbb{R}^d$. In [46], Dziubański and Zienkiewicz introduced the Hardy space $H^1_L(\mathbb{R}^d)$ as the set of functions $f \in L^1(\mathbb{R}^d)$ such that $\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) := \sup_{t>0} |e^{-tL} f(x)|$. There, they characterized $H^1_L(\mathbb{R}^d)$ in terms of atomic decomposition and in terms of the Riesz transforms associated with L, $R_j = \partial_{x_j} L^{-1/2}$, $j = 1, ..., d$. In the recent years, there is an increasing interest on the study of commutators of singular integral operators related to Schrödinger operators, see for example [17, 22, 58, 92, 134].

In the present paper, we consider commutators of singular integral operators T related to the Schrödinger operator L. Here T is in the class $\mathcal{K}_L$ of all sublinear operators T, bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ and satisfying for any $b \in BMO(\mathbb{R}^d)$ and a a *generalized atom* related to the ball B (see Definition 5.2.1), we have

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO},$$

where b_B denotes the average of b on B and $C > 0$ is a constant independent of b, a. The class $\mathcal{K}_L$ contains the fundamental operators (we refer the reader to [82] for the classical case $L = -\Delta$) related to the Schrödinger operator L: the Riesz transforms R_j, L-Calderón-Zygmund operators (so-called Schrödinger-Calderón-Zygmund operators), L-maximal operators, L-square operators, etc... (see Section 5.4). It should be pointed out that, by the work of Shen [125] and Definition 5.2.2 (see Remark 5.2.3), one only can conclude that the Riesz transforms R_j are Schrödinger-Calderón-Zygmund operators if $V \in RH_d$. In this work, we consider all potentials V which belong to the reverse Hölder class $RH_{d/2}$.

Although Schrödinger-Calderón-Zygmund operators map $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ (see Proposition 5.4.1), it was observed in [92] that, when $b \in BMO(\mathbb{R}^d)$, the commutators $[b, R_j]$ do not map, in general, $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Thus, when $b \in BMO(\mathbb{R}^d)$, it is natural (see the paper of Pérez [119] for the classical case) to ask for subspaces of $H^1_L(\mathbb{R}^d)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms map continuously these spaces into $L^1(\mathbb{R}^d)$. Here, we are interested in the following two questions.

Question 5. *For $b \in BMO(\mathbb{R}^d)$. Find the largest subspace $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ of $H^1_L(\mathbb{R}^d)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Question 6. *Characterize the functions b in $BMO(\mathbb{R}^d)$ so that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$.*

Let X be a Banach space. We say that an operator $T : X \to L^1(\mathbb{R}^d)$ is a sublinear operator if for all $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$, we have

$$|T(\alpha f + \beta g)(x)| \leq |\alpha||Tf(x)| + |\beta||Tg(x)|.$$

Obviously, a linear operator $T : X \to L^1(\mathbb{R}^d)$ is a sublinear operator. We also say that an operator $\mathfrak{T} : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a subbilinear operator if for every $(f, g) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, the operators $\mathfrak{T}(f, \cdot) : BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $\mathfrak{T}(\cdot, g) : H^1_L(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ are sublinear operators.

To answer *Question 5* and *Question 6*, we study commutators of sublinear operators in $\mathcal{K}_L$. More precisely, when $T \in \mathcal{K}_L$ is a sublinear operator, we prove (see Theorem 5.3.1) that there exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ so that for all $(f, b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$,

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|, \tag{5.2}$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ which does not depend on T (see Proposition 5.5.2). When $T \in \mathcal{K}_L$ is a linear operator, we prove (see Theorem 5.3.2) that there exists a bounded bilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f, b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$,

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)). \tag{5.3}$$

The decompositions (5.2) and (5.3) give a general overview and *explains why* almost commutators of the fundamental operators are of weak type (H^1_L, L^1), and when a commutator $[b,T]$ is of strong type (H^1_L, L^1).

Let b be a function in $BMO(\mathbb{R}^d)$. We assume that b non-constant, otherwise $[b,T]=0$. We define the space $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ as the set of all f in $H^1_L(\mathbb{R}^d)$ such that $[b,\mathcal{M}_L](f)(x) = \mathcal{M}_L(b(x)f(\cdot) - b(\cdot)f(\cdot))(x)$ belongs to $L^1(\mathbb{R}^d)$, and the norm on $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ is defined by $\|f\|_{\mathcal{H}^1_{L,b}} = \|f\|_{H^1_L}\|b\|_{BMO} + \|[b,\mathcal{M}_L](f)\|_{L^1}$. Then, using the subbilinear decomposition (5.2), we prove that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Furthermore, $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ *is the largest space having this property*, and $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$ if and only if $b \in BMO^{\log}_L(\mathbb{R}^d)$ (see Theorem 5.7.2), that is,

$$\|b\|_{BMO^{\log}_L} = \sup_{B(x,r)} \left(\log\left(e + \frac{\rho(x)}{r}\right) \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}|dy \right) < \infty,$$

where $\rho(x) = \sup\{r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)dy \leq 1\}$. This space $BMO^{\log}_L(\mathbb{R}^d)$ arises naturally in the characterization of pointwise multipliers for $BMO_L(\mathbb{R}^d)$, the dual space of $H^1_L(\mathbb{R}^d)$, see [9, 103].

The above answers *Question 5* and *Question 6*. As another interesting application of the subbilinear decomposition (5.2), we find subspaces of $H^1_L(\mathbb{R}^d)$ which *do not depend* on $b \in BMO(\mathbb{R}^d)$ and $T \in \mathcal{K}_L$, such that $[b,T]$ maps continuously these spaces into $L^1(\mathbb{R}^d)$ (see Section 5.7). For instance, when $L = -\Delta + 1$, Theorem 5.7.4 state that for every $b \in BMO(\mathbb{R}^d)$ and $T \in \mathcal{K}_L$, the commutator $[b,T]$ is bounded from $H^{1,1}_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Here $H^{1,1}_L(\mathbb{R}^d)$ is the (inhomogeneous) Hardy-Sobolev space considered by Hofmann, Mayboroda and McIntosh in [65], defined as the set of functions f in $H^1_L(\mathbb{R}^d)$ such that $\partial_{x_1} f, ..., \partial_{x_d} f \in H^1_L(\mathbb{R}^d)$ with the norm

$$\|f\|_{H^{1,1}_L} = \|f\|_{H^1_L} + \sum_{j=1}^d \|\partial_{x_j} f\|_{H^1_L}.$$

Recently, similarly to the classical result of Coifman-Rochberg-Weiss, Gou et al. proved in [58] that the commutators $[b,R_j]$ are bounded on $L^p(\mathbb{R}^d)$ whenever $b \in BMO(\mathbb{R}^d)$ and $1 < p < \frac{dq}{d-q}$ where $V \in RH_q$ for some $d/2 < q < d$. Later, in [17], Bongioanni et al. generalized this result by showing that the space $BMO(\mathbb{R}^d)$ can be replaced by a larger space $BMO_{L,\infty}(\mathbb{R}^d) = \cup_{\theta \geq 0} BMO_{L,\theta}(\mathbb{R}^d)$, where $BMO_{L,\theta}(\mathbb{R}^d)$ is the space of locally integrable functions f satisfying

$$\|f\|_{BMO_{L,\theta}} = \sup_{B(x,r)} \left(\frac{1}{\left(1+\frac{r}{\rho(x)}\right)^{\theta}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| dy \right) < \infty.$$

Let R_j^* be the adjoint operators of R_j and $BMO_L(\mathbb{R}^d)$ be the dual space of $H^1_L(\mathbb{R}^d)$. In [16], Bongioanni et al. established that the operators R_j^* are bounded on $BMO_L(\mathbb{R}^d)$, and thus from $L^\infty(\mathbb{R}^d)$ into $BMO_L(\mathbb{R}^d)$. Therefore, it is natural to ask for a class of functions b so that the commutators $[b, R_j^*]$ are bounded from $L^\infty(\mathbb{R}^d)$ into $BMO_L(\mathbb{R}^d)$. In [17], the authors found such a class of functions. More precisely, they proved in [17] that the commutators $[b, R_j^*]$ are bounded from $L^\infty(\mathbb{R}^d)$ into $BMO_L(\mathbb{R}^d)$ whenever $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d) = \cup_{\theta\geq 0} BMO^{\log}_{L,\theta}(\mathbb{R}^d)$, where $BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ is the space of locally integrable functions f satisfying

$$\|f\|_{BMO^{\log}_{L,\theta}} = \sup_{B(x,r)} \left(\frac{\log\left(e+\frac{\rho(x)}{r}\right)}{\left(1+\frac{r}{\rho(x)}\right)^{\theta}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| dy \right) < \infty.$$

A natural question arises: can one replace the space $L^\infty(\mathbb{R}^d)$ by $BMO_L(\mathbb{R}^d)$?

Question 7. *Are the commutators $[b, R_j^*]$, $j = 1, ..., d$, bounded on $BMO_L(\mathbb{R}^d)$ whenever $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d)$?*

Motivated by this question, we study the H^1_L-estimates for commutators of the Riesz transforms. More precisely, given $b \in BMO_{L,\infty}(\mathbb{R}^d)$, we prove that the commutators $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^d)$ if and only if b belongs to $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ (see Theorem 5.3.4). Furthermore, if $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ for some $\theta \geq 0$, then there exists a constant $C > 1$, independent of b, such that

$$C^{-1}\|b\|_{BMO^{\log}_{L,\theta}} \leq \|b\|_{BMO_{L,\theta}} + \sum_{j=1}^{d} \|[b, R_j]\|_{H^1_L \to H^1_L} \leq C\|b\|_{BMO^{\log}_{L,\theta}}.$$

As a consequence, we get the positive answer for *Question 7.*

Now, an open question is the following:

Open question. *Find the set of all functions b such that the commutators $[b, R_j]$, $j = 1, ..., d$, are bounded on $H^1_L(\mathbb{R}^d)$.*

Let us emphasize the three main purposes of this paper. First, we prove the two decomposition theorems: the subbilinear decomposition (5.2) and the bilinear decomposition (5.3). Second, we characterize functions b in $BMO_{L,\infty}(\mathbb{R}^d)$ so that the commutators of the Riesz transforms are bounded on $H^1_L(\mathbb{R}^d)$, which answers *Question 7*. Finally, we find *the largest subspace* $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ *of* $H^1_L(\mathbb{R}^d)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Besides, we find also the characterization of functions $b \in BMO(\mathbb{R}^d)$ so that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$, which answer *Question 5* and *Question 6*. Especially, we show that there exist subspaces of $H^1_L(\mathbb{R}^d)$ which *do not depend* on $b \in BMO(\mathbb{R}^d)$ and $T \in \mathcal{K}_L$, such that $[b, T]$ maps continuously these spaces into $L^1(\mathbb{R}^d)$.

This paper is organized as follows. In Section 2, we present some notations and preliminaries about Hardy spaces, new atoms, BMO type spaces and Schrödinger-Calderón-Zygmund operators. In Section 3, we state the main results: two decomposition theorems (Theorem 5.3.1 and Theorem 5.3.2), Hardy estimates for commutators of Schrödinger-Calderón-Zygmund operators and the commutators of the Riesz transforms (Theorem 5.3.3 and Theorem 5.3.4). In Section 4, we give some examples of fundamental operators related to L which are in the class $\mathcal{K}_L$. Section 5 is devoted to the proofs of the main theorems. Section 6 is devoted to the proofs of the key lemmas. Finally, in Section 7, we give some subspaces of $H^1_L(\mathbb{R}^d)$ which *do not necessarily* depend on $b \in BMO(\mathbb{R}^d)$ and $T \in \mathcal{K}_L$ (see Theorem 5.7.3 and Theorem 5.7.4), such that the commutator $[b, T]$ maps continuously these spaces into $L^1(\mathbb{R}^d)$. Especially, we find in this section the *largest subspace* $\mathcal{H}^1_{L,b}$ of $H^1_L(\mathbb{R}^d)$ so that all commutators of Schrödinger-Calderón-Zygmund operators and the commutators of the Riesz transforms map continuously this space into $L^1(\mathbb{R}^d)$ (see Theorem 5.7.2).

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. The symbol $f \approx g$ means that f is equivalent to g (i.e. $C^{-1} f \leq g \leq Cf$). In $\mathbb{R}^d$, we denote by $B = B(x, r)$ an open ball with center x and radius $r > 0$, and $tB(x, r) := B(x, tr)$ whenever $t > 0$. For any measurable set E, we denote by χ_E its characteristic function, by $|E|$ its Lebesgue measure, and by E^c the set $\mathbb{R}^d \setminus E$.

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5.2 Some preliminaries and notations

In this paper, we consider the Schrödinger differential operator

$$L = -\Delta + V$$

on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$. As in the works of Dziubański et al [45, 46], we always assume that V belongs to the reverse Hölder class $RH_{d/2}$. Recall that a nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q, $1 < q < \infty$, if there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int\limits_B (V(x))^q dx\Big)^{1/q} \leq \frac{C}{|B|}\int\limits_B V(x)dx$$

holds for every balls B in $\mathbb{R}^d$. By Hölder inequality, $RH_{q_1} \subset RH_{q_2}$ if $q_1 \geq q_2 > 1$. For $q > 1$, it is well-known that $V \in RH_q$ implies $V \in RH_{q+\varepsilon}$ for some $\varepsilon > 0$ (see [55]). Moreover, $V(y)dy$ is a doubling measure, namely for any ball $B(x,r)$ we have

$$\int\limits_{B(x,2r)} V(y)dy \leq C_0 \int\limits_{B(x,r)} V(y)dy. \tag{5.4}$$

Let $\{T_t\}_{t>0}$ be the semigroup generated by L and $T_t(x,y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL}f(x) = \int\limits_{\mathbb{R}^d} T_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

We say that a function $f \in L^2(\mathbb{R}^d)$ belongs to the space $\mathbb{H}^1_L(\mathbb{R}^d)$ if

$$\|f\|_{\mathbb{H}^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty,$$

where $\mathcal{M}_L f(x) := \sup_{t>0} |T_t f(x)|$ for all $x \in \mathbb{R}^d$. The space $H^1_L(\mathbb{R}^d)$ is then defined as the completion of $\mathbb{H}^1_L(\mathbb{R}^d)$ with respect to this norm.

In [45] it was shown that the dual of $H^1_L(\mathbb{R}^d)$ can be identified with the space $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO(\mathbb{R}^d)$ with

$$\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x)\leq r} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy < \infty,$$

where ρ is the auxiliary function defined as in [125], that is,

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{d-2}}\int\limits_{B(x,r)} V(y)dy \leq 1\Big\}, \tag{5.5}$$

$x \in \mathbb{R}^d$. Clearly, $0 < \rho(x) < \infty$ for all $x \in \mathbb{R}^d$, and thus $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n$, where the sets $\mathcal{B}_n$ are defined by

$$\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2}\}. \tag{5.6}$$

The following proposition plays an important role in our study.

Proposition 5.2.1 (see [125], Lemma 1.4)**.** *There exist two constants $\kappa > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$\kappa^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq \kappa\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}.$$

Throughout the whole paper, we denote by $\mathcal{C}_L$ the L-constant

$$\mathcal{C}_L = 8.9^{k_0}\kappa \tag{5.7}$$

where k_0 and κ are defined as in Proposition 7.2.1.

Given $1 < q \leq \infty$. Following Dziubański and Zienkiewicz [46], a function a is called a (H^1_L, q)-atom related to the ball $B(x_0, r)$ if $r \leq \mathcal{C}_L\rho(x_0)$ and

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,

iii) if $r \leq \frac{1}{\mathcal{C}_L}\rho(x_0)$ then $\int_{\mathbb{R}^d} a(x)dx = 0$.

A function a is called a classical (H^1, q)-atom related to the ball $B = B(x_0, r)$ if it satisfies (i), (ii) and $\int_{\mathbb{R}^d} a(x)dx = 0$.

The following characterization of $H^1_L(\mathbb{R}^d)$ is due to Dziubański and Zienkiewicz [46].

Theorem 5.2.1 (see [46], Theorem 1.5)**.** *Let $1 < q \leq \infty$. A function f is in $H^1_L(\mathbb{R}^d)$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H^1_L, q)-atoms and $\sum_j |\lambda_j| < \infty$. Moreover,*

$$\|f\|_{H^1_L} \approx \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\}.$$

Note that a classical (H^1, q)-atom is not a (H^1_L, q)-atom in general. In fact, there exists a constant $C > 0$ such that if f is a classical (H^1, q)-atom, then it can be written as $f = \sum_{j=1}^n \lambda_j a_j$, for some $n \in \mathbb{Z}^+$, where a_j are (H^1_L, q)-atoms and $\sum_{j=1}^n |\lambda_j| \leq C$, see for example [145]. In this work, we need a variant of the definition of atoms for $H^1_L(\mathbb{R}^d)$ which include classical (H^1, q)-atoms and (H^1_L, q)-atoms. This kind of atoms have been used in the work of Chang, Dafni and Stein [24, 34].

Definition 5.2.1. *Given $1 < q \leq \infty$ and $\varepsilon > 0$. A function a is called a generalized (H^1_L, q, ε)-atom related to the ball $B(x_0, r)$ if*

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,

iii) $|\int_{\mathbb{R}^d} a(x)dx| \leq \Big(\frac{r}{\rho(x_0)}\Big)^{\varepsilon}$.

The space $\mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d)$ is defined to be set of all functions f in $L^1(\mathbb{R}^d)$ which can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$ where the a_j are generalized (H^1_L, q, ε)-atoms and the λ_j are complex numbers such that $\sum_{j=1}^\infty |\lambda_j| < \infty$. As usual, the norm on $\mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d)$ is defined by

$$\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,at}} = \inf\Big\{\sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j\Big\}.$$

The space $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is defined to be set of all $f = \sum_{j=1}^k \lambda_j a_j$, where the a_j are generalized (H^1_L, q, ε)-atoms. Then, the norm of f in $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is defined by

$$\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}} = \inf\Big\{\sum_{j=1}^k |\lambda_j| : f = \sum_{j=1}^k \lambda_j a_j\Big\}.$$

Remark 5.2.1. *Let $1 < q \leq \infty$ and $\varepsilon > 0$. Then, a classical (H^1, q)-atom is a generalized (H^1_L, q, ε)-atom related to the same ball, and a (H^1_L, q)-atom is $\mathcal{C}_L{}^{\varepsilon}$ times a generalized (H^1_L, q, ε)-atom related to the same ball.*

Throughout the whole paper, we always use generalized (H^1_L, q, ε)-atoms except in the proof of Theorem 5.3.4. More precisely, in order to prove Theorem 5.3.4, we need to use (H^1_L, q)-atoms from Dziubański and Zienkiewicz (see above).

The following gives a characterization of $H^1_L(\mathbb{R}^n)$ in terms of generalized atoms.

Theorem 5.2.2. *Let $1 < q \leq \infty$ and $\varepsilon > 0$. Then, $\mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d) = H^1_L(\mathbb{R}^d)$ and the norms are equivalent.*

In order to prove Theorem 5.2.2, we need the following lemma.

Lemma 5.2.1 (see [91], Lemma 2)**.** *Let $V \in RH_{d/2}$. Then, there exists $\sigma_0 > 0$ depends only on L, such that for every $|y - z| < |x - y|/2$ and $t > 0$, we have*

$$|T_t(x, y) - T_t(x, z)| \leq C\Big(\frac{|y - z|}{\sqrt{t}}\Big)^{\sigma_0} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{t}} \leq C\frac{|y - z|^{\sigma_0}}{|x - y|^{d+\sigma_0}}.$$

Proof of Theorem 5.2.2. As $\mathcal{M}_L$ is a sublinear operator, by Remark 5.2.1 and Theorem 7.2, it is sufficient to show that

$$\|\mathcal{M}_L(a)\|_{L^1} \leq C \tag{5.8}$$

for all generalized (H^1_L, q, ε)-atom a related to the ball $B = B(x_0, r)$.

Indeed, from the L^q-boundedness of the classical Hardy-Littlewood maximal operator $\mathcal{M}$, the estimate $\mathcal{M}_L(a) \leq C\mathcal{M}(a)$ and Hölder inequality,

$$\|\mathcal{M}_L(a)\|_{L^1(2B)} \leq C\|\mathcal{M}(a)\|_{L^1(2B)} \leq C|2B|^{1/q'}\|\mathcal{M}(a)\|_{L^q} \leq C, \tag{5.9}$$

where $1/q' + 1/q = 1$. Let $x \notin 2B$ and $t > 0$, Lemma 6.3.2 and (3.5) of [46] give

$$\begin{aligned}
|T_t(a)(x)| &= \Big|\int\limits_{\mathbb{R}^d} T_t(x,y)a(y)dy\Big| \\
&\leq \Big|\int\limits_B (T_t(x,y) - T_t(x,x_0))a(y)dy\Big| + |T_t(x,x_0)|\Big|\int\limits_B a(y)dy\Big| \\
&\leq C\frac{r^{\sigma_0}}{|x-x_0|^{d+\sigma_0}} + C\frac{r^{\varepsilon}}{|x-x_0|^{d+\varepsilon}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{M}_L(a)\|_{L^1((2B)^c)} &= \|\sup_{t>0}|T_t(a)|\|_{L^1((2B)^c)} \\
&\leq C\int\limits_{(2B)^c}\frac{r^{\sigma_0}}{|x-x_0|^{d+\sigma_0}}dx + C\int\limits_{(2B)^c}\frac{r^{\varepsilon}}{|x-x_0|^{d+\varepsilon}}dx \\
&\leq C.
\end{aligned} \tag{5.10}$$

Then, (5.8) follows from (5.9) and (5.10). □

By Theorem 5.2.2, the following can be seen as a direct consequence of Proposition 3.2 of [145] and remark 5.2.1.

Proposition 5.2.2. *Let $1 < q < \infty$, $\varepsilon > 0$ and $\mathcal{X}$ be a Banach space. Suppose that $T : \mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d) \to \mathcal{X}$ is a sublinear operator with*

$$\sup\{\|Ta\|_{\mathcal{X}} : a \textit{ is a generalized } (H^1_L, q, \varepsilon) - \textit{atom}\} < \infty.$$

Then, T can be extended to a bounded sublinear operator $\widetilde{T}$ from $H^1_L(\mathbb{R}^d)$ into $\mathcal{X}$, moreover,

$$\|\widetilde{T}\|_{H^1_L \to \mathcal{X}} \leq C\sup\{\|Ta\|_{\mathcal{X}} : a \textit{ is a generalized } (H^1_L, q, \varepsilon) - \textit{atom}\}.$$

Now, we turn to explain the new BMO type spaces introduced by Bongioanni, Harboure and Salinas in [17]. Here and in what follows $f_B := \frac{1}{|B|}\int_B f(x)dx$ and

$$MO(f,B) := \frac{1}{|B|}\int\limits_B |f(y) - f_B|dy. \tag{5.11}$$

For $\theta \geq 0$, following [17], we denote by $BMO_{L,\theta}(\mathbb{R}^d)$ the set of all locally integrable functions f such that

$$\|f\|_{BMO_{L,\theta}} = \sup_{B(x,r)} \left(\frac{1}{\left(1+\frac{r}{\rho(x)}\right)^{\theta}} MO(f, B(x,r)) \right) < \infty,$$

and $BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ the set of all locally integrable functions f such that

$$\|f\|_{BMO^{\log}_{L,\theta}} = \sup_{B(x,r)} \left(\frac{\log\left(e+\frac{\rho(x)}{r}\right)}{\left(1+\frac{r}{\rho(x)}\right)^{\theta}} MO(g, B(x,r)) \right) < \infty.$$

When $\theta = 0$, we write $BMO^{\log}_{L}(\mathbb{R}^d)$ instead of $BMO^{\log}_{L,0}(\mathbb{R}^d)$. We next define

$$BMO_{L,\infty}(\mathbb{R}^d) = \bigcup_{\theta\geq 0} BMO_{L,\theta}(\mathbb{R}^d)$$

and

$$BMO^{\log}_{L,\infty}(\mathbb{R}^d) = \bigcup_{\theta\geq 0} BMO^{\log}_{L,\theta}(\mathbb{R}^d).$$

Observe that $BMO_{L,0}(\mathbb{R}^d)$ is just the classical $BMO(\mathbb{R}^d)$ space. Moreover, for any $0 \leq \theta \leq \theta' \leq \infty$, we have

$$BMO_{L,\theta}(\mathbb{R}^d) \subset BMO_{L,\theta'}(\mathbb{R}^d), \quad BMO^{\log}_{L,\theta}(\mathbb{R}^d) \subset BMO^{\log}_{L,\theta'}(\mathbb{R}^d) \tag{5.12}$$

and

$$BMO^{\log}_{L,\theta}(\mathbb{R}^d) = BMO_{L,\theta}(\mathbb{R}^d) \cap BMO^{\log}_{L,\infty}(\mathbb{R}^d). \tag{5.13}$$

Remark 5.2.2. *The inclusions in (5.12) are strict in general. In particular:*

i) The space $BMO_{L,\infty}(\mathbb{R}^d)$ is in general larger than the space $BMO(\mathbb{R}^d)$. Indeed, when $V(x) \equiv |x|^2$, it is easy to check that the functions $b_j(x) = |x_j|^2$, $j = 1, ..., d$, belong to $BMO_{L,\infty}(\mathbb{R}^d)$ but not to $BMO(\mathbb{R}^d)$.

ii) The space $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ is in general larger than the space $BMO^{\log}_{L}(\mathbb{R}^d)$. Indeed, when $V(x) \equiv 1$, it is easy to check that the functions $b_j(x) = |x_j|$, $j = 1, ..., d$, belong to $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ but not to $BMO^{\log}_{L}(\mathbb{R}^d)$.

Next, let us recall the notation of Schrödinger-Calderón-Zygmund operators.

Let $\delta \in (0,1]$. According to [103], a continuous function $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\} \to \mathbb{C}$ is said to be a (δ, L)-Calderón-Zygmund singular integral kernel if for each $N > 0$,

$$|K(x,y)| \leq \frac{C(N)}{|x-y|^d}\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-N} \tag{5.14}$$

for all $x \neq y$, and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{d+\delta}} \tag{5.15}$$

for all $2|x-x'| \leq |x-y|$.

As usual, we denote by $C_c^\infty(\mathbb{R}^d)$ the space of all C^∞-functions with compact support, by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$.

Definition 5.2.2. *A linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is said to be a (δ, L)-Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(\mathbb{R}^d)$ and if there exists a (δ, L)-Calderón-Zygmund singular integral kernel K such that for all $f \in C_c^\infty(\mathbb{R}^d)$ and all $x \notin$ supp f, we have*

$$Tf(x) = \int\limits_{\mathbb{R}^d} K(x,y)f(y)dy.$$

An operator T is said to be a L-Calderón-Zygmund operator (or Schrödinger-Calderón-Zygmund operator) if it is a (δ, L)-Calderón-Zygmund operator for some $\delta \in (0,1]$.

We say that T satisfies the condition $T^*1 = 0$ (see for example [8]) if there are $q \in (1,\infty]$ and $\varepsilon > 0$ so that $\int_{\mathbb{R}^d} Ta(x)dx = 0$ holds for every generalized (H^1_L, q, ε)-atoms a.

Remark 5.2.3. *i) Using Proposition 7.2.1, Inequality (5.14) is equivalent to*

$$|K(x,y)| \leq \frac{C(N)}{|x-y|^d}\Big(1 + \frac{|x-y|}{\rho(y)}\Big)^{-N}$$

for all $x \neq y$.

ii) By Theorem 0.8 of [125] and Theorem 1.1 of [126], the Riesz transforms R_j are L-Calderón-Zygmund operators satisfying $R_j^*1 = 0$ whenever $V \in RH_d$.

iii) If T is a L-Calderón-Zygmund operator then it is also a classical Calderón-Zygmund operator, and thus T is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and bounded from $L^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$.

5.3 Statement of the results

Recall that $\mathcal{K}_L$ is the set of all sublinear operators T bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ and that there are $q \in (1,\infty]$ and $\varepsilon > 0$ such that

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO}$$

for all $b \in BMO(\mathbb{R}^d)$, any generalized (H^1_L, q, ε)-atom a related to the ball B, where $C > 0$ is a constant independent of b, a.

5.3.1 Two decomposition theorems

Let b be a locally integrable function and $T \in \mathcal{K}_L$. As usual, the (sublinear) commutator $[b,T]$ of the operator T is defined by $[b,T](f)(x) := T\Big((b(x) - b(\cdot))f(\cdot)\Big)(x)$.

Theorem 5.3.1 (Subbilinear decomposition)**.** *Let $T \in \mathcal{K}_L$. There exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have*

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|,$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ which does not depend on T.

Using Theorem 5.3.1, we obtain immediately the following result.

Proposition 5.3.1. *Let $T \in \mathcal{K}_L$ so that T is of weak type $(1,1)$. Then, the subbilinear operator $\mathfrak{T}(f,g) = [g,T](f)$ maps continuously $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$.*

Recall that $R_j = \partial_{x_j} L^{-1/2}$, $j = 1, ..., d$, are the Riesz transforms associated with L. As the Riesz transforms R_j are of weak type $(1,1)$ (see [89]), the following can be seen as a consequence of Proposition 5.3.1 (see also [92]).

Corollary 5.3.1 (see [92], Theorem 4.1)**.** *Let $b \in BMO(\mathbb{R}^d)$. Then, the commutators $[b, R_j]$ are bounded from $H^1_L(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$.*

When T is linear and belongs to $\mathcal{K}_L$, we obtain the bilinear decomposition for the linear commutator $[b,T]$ of f, $[b,T](f) = bT(f) - T(bf)$, instead of the subbilinear decomposition as stated in Theorem 5.3.1.

Theorem 5.3.2 (Bilinear decomposition)**.** *Let T be a linear operator in $\mathcal{K}_L$. Then, there exists a bounded bilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f, b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have*

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)),$$

where $\mathfrak{S}$ is as in Theorem 5.3.1.

5.3.2 Hardy estimates for linear commutators

Our first main result of this subsection is the following theorem.

Theorem 5.3.3. *i) Let $b \in BMO^{\log}_L(\mathbb{R}^d)$ and T be a L-Calderón-Zygmund operator satisfying $T^*1 = 0$. Then, the linear commutator $[b, T]$ is bounded on $H^1_L(\mathbb{R}^d)$.*

ii) When $V \in RH_d$, the converse holds. Namely, if $b \in BMO(\mathbb{R}^d)$ and $[b, T]$ is bounded on $H^1_L(\mathbb{R}^d)$ for every L-Calderón-Zygmund operator T satisfying $T^*1 = 0$, then $b \in BMO^{\log}_L(\mathbb{R}^d)$. Furthermore,

$$\|b\|_{BMO^{\log}_L} \approx \|b\|_{BMO} + \sum_{j=1}^{d} \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

Next result concerns the H^1_L-estimates for commutators of the Riesz transforms.

Theorem 5.3.4. *Let $b \in BMO_{L,\infty}(\mathbb{R}^d)$. Then, the commutators $[b, R_j]$, $j = 1, ..., d$, are bounded on $H^1_L(\mathbb{R}^d)$ if and only if $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d)$. Furthermore, if $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ for some $\theta \geq 0$, we have*

$$\|b\|_{BMO^{\log}_{L,\theta}} \approx \|b\|_{BMO_{L,\theta}} + \sum_{j=1}^{d} \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

Remark that the above constants depend on θ.

Note that $BMO^{\log}_L(\mathbb{R}^d)$ is in general proper subset of $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ (see Remark 5.2.2). When $V \in RH_d$, although the Riesz transforms R_j are L-Calderón-Zygmund operators satisfying $R^*_j 1 = 0$, Theorem 5.3.4 cannot be deduced from Theorem 5.3.3.

As a consequence of Theorem 5.3.4, we obtain the following interesting result.

Corollary 5.3.2. *Let $b \in BMO(\mathbb{R}^d)$. Then, b belongs to $LMO(\mathbb{R}^d)$ if and only if the vector-valued commutator $[b, \nabla(-\Delta + 1)^{-1/2}]$ maps continuously $h^1(\mathbb{R}^d)$ into $h^1(\mathbb{R}^d, \mathbb{R}^d) = (h^1(\mathbb{R}^d), ..., h^1(\mathbb{R}^d))$. Furthermore,*

$$\|b\|_{LMO} \approx \|b\|_{BMO} + \|[b, \nabla(-\Delta + 1)^{-1/2}]\|_{h^1(\mathbb{R}^d) \to h^1(\mathbb{R}^d, \mathbb{R}^d)}.$$

Here $h^1(\mathbb{R}^d)$ is the local Hardy space of D. Goldberg (see [56]), and $LMO(\mathbb{R}^d)$ is the space of all locally integrable functions f such that

$$\|f\|_{LMO} := \sup_{B(x,r)} \left(\log\left(e + \frac{1}{r}\right) MO(f, B(x,r))\right) < \infty.$$

It should be pointed out that LMO type spaces appear naturally when studying the boundedness of Hankel operators on the Hardy spaces $H^1(\mathbb{T}^d)$ and $H^1(\mathbb{B}^d)$ (where $\mathbb{B}^d$ is the unit ball in $\mathbb{C}^d$ and $\mathbb{T}^d = \partial\mathbb{B}^d$), characterizations of pointwise multipliers for BMO type spaces, endpoint estimates for commutators of singular integrals operators and their applications to PDEs, see for example [13, 21, 75, 76, 82, 118, 127, 132].

5.4 Some fundamental operators and the class $\mathcal{K}_L$

The purpose of this section is to give some examples of fundamental operators related to L which are in the class $\mathcal{K}_L$.

5.4.1 Schrödinger-Calderón-Zygmund operators

Proposition 5.4.1. *Let T be any L-Calderón-Zygmund operator. Then, T belongs to the class $\mathcal{K}_L$.*

Proposition 5.4.2. *The Riesz transforms R_j are in the class $\mathcal{K}_L$.*

The proof of Proposition 5.4.2 follows directly from Lemma 5.5.7 and the fact that the Riesz transforms R_j are bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

To prove Proposition 5.4.1, we need the following two lemmas.

Lemma 5.4.1. *Let $1 \leq q < \infty$. Then, there exists a constant $C > 0$ such that for every ball B, $f \in BMO(\mathbb{R}^d)$ and $k \in \mathbb{Z}^+$,*

$$\Big(\frac{1}{|2^k B|}\int\limits_{2^k B} |f(y) - f_B|^q dy\Big)^{1/q} \leq Ck\|f\|_{BMO}.$$

The proof of Lemma 5.4.1 follows directly from the classical John-Nirenberg inequality. See also Lemma 5.6.6 below.

Lemma 5.4.2. *Let $1 < q \leq \infty$ and $\varepsilon > 0$. Assume that T is a (δ, L)-Calderón-Zygmund operator and a is a generalized (H^1_L, q, ε)-atom related to the ball $B = B(x_0, r)$. Then,*

$$\|Ta\|_{L^q(2^{k+1}B \setminus 2^k B)} \leq C2^{-k\delta_0}|2^k B|^{1/q-1}$$

for all $k = 1, 2, ...,$ where $\delta_0 = \min\{\varepsilon, \delta\}$.

Proof. Let $x \in 2^{k+1}B \setminus 2^k B$, so that $|x - x_0| \geq 2r$. Since T is a (δ, L)-Calderón-Zygmund operator, we get

$$\begin{aligned}
|Ta(x)| &\leq \Big|\int_B (K(x,y) - K(x,x_0))a(y)dy\Big| + |K(x,x_0)|\Big|\int_{\mathbb{R}^d} a(y)dy\Big| \\
&\leq C\int_B \frac{|y-x_0|^\delta}{|x-x_0|^{d+\delta}}|a(y)|dy + C\frac{1}{|x-x_0|^d}\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{-\varepsilon}\Big(\frac{r}{\rho(x_0)}\Big)^{\varepsilon} \\
&\leq C\frac{r^\delta}{|x-x_0|^{d+\delta}} + C\frac{r^\varepsilon}{|x-x_0|^{d+\varepsilon}} \leq C\frac{r^{\delta_0}}{|x-x_0|^{d+\delta_0}}.
\end{aligned}$$

Consequently,

$$\|Ta\|_{L^q(2^{k+1}B\setminus 2^k B)} \leq C\frac{r^{\delta_0}}{(2^k r)^{d+\delta_0}}|2^{k+1}B|^{1/q} \leq C2^{-k\delta_0}|2^k B|^{1/q-1}.$$

□

Proof of Proposition 5.4.1. Assume that T is a (δ, L)-Calderón-Zygmund for some $\delta \in (0,1]$. Let us first verify that T is bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. By Proposition 5.2.2, it is sufficient to show that

$$\|Ta\|_{L^1} \leq C$$

for all generalized $(H^1_L, 2, \delta)$-atom a related to the ball B. Indeed, from the L^2-boundedness of T and Lemma 5.4.2, we obtain that

$$\begin{aligned}
\|Ta\|_{L^1} &= \|Ta\|_{L^1(2B)} + \sum_{k=1}^{\infty}\|Ta\|_{L^1(2^{k+1}B\setminus 2^k B)} \\
&\leq C|2B|^{1/2}\|T\|_{L^2\to L^2}\|a\|_{L^2} + C\sum_{k=1}^{\infty}|2^{k+1}B|^{1/2}2^{-k\delta}|2^k B|^{-1/2} \\
&\leq C.
\end{aligned}$$

Let us next establish that

$$\|(f - f_B)Ta\|_{L^1} \leq C\|f\|_{BMO}$$

for all $f \in BMO(\mathbb{R}^d)$, any generalized $(H^1_L, 2, \delta)$-atom a related to the ball $B = B(x_0, r)$.

Indeed, by Hölder inequality, Lemma 5.4.1 and Lemma 5.4.2, we get

$$\begin{aligned}
& \|(f-f_B)Ta\|_{L^1} \\
= \;& \|(f-f_B)Ta\|_{L^1(2B)} + \sum_{k\geq 1} \|(f-f_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)} \\
\leq \;& \|(f-f_B)\chi_{2B}\|_{L^2}\|T\|_{L^2\to L^2}\|a\|_{L^2} + \sum_{k\geq 1} \|f-f_B\|_{L^2(2^{k+1}B)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^kB)} \\
\leq \;& C\|f\|_{BMO} + \sum_{k\geq 1} C(k+1)\|f\|_{BMO}|2^{k+1}B|^{1/2}2^{-k\delta}|2^kB|^{-1/2} \\
\leq \;& C\|f\|_{BMO},
\end{aligned}$$

which ends the proof.

$\square$

5.4.2 Some L-maximal operators

Recall that $\{T_t\}_{t>0}$ is heat semigroup generated by L and $T_t(x,y)$ are their kernels. Namely,

$$T_tf(x) = e^{-tL}f(x) = \int_{\mathbb{R}^d} T_t(x,y)f(y)dy, \quad f\in L^2(\mathbb{R}^d), \quad t>0.$$

Then the "heat" maximal operator is defined by

$$\mathcal{M}_Lf(x) = \sup_{t>0}|T_tf(x)|,$$

and the "Poisson" maximal operator is defined by

$$\mathcal{M}_L^Pf(x) = \sup_{t>0}|P_tf(x)|,$$

where

$$P_tf(x) = e^{-t\sqrt{L}}f(x) = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}}T_uf(x)du.$$

Proposition 5.4.3. *The "heat" maximal operator $\mathcal{M}_L$ is in the class $\mathcal{K}_L$.*

Proposition 5.4.4. *The "Poisson" maximal operator $\mathcal{M}_L^P$ is in the class $\mathcal{K}_L$.*

Here we just give the proof of Proposition 5.4.3. For the one of Proposition 5.4.4, we leave the details to the interested reader.

Proof of Proposition 5.4.3. Obviously, $\mathcal{M}_L$ is bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

Now, let us prove that

$$\|(f - f_B)\mathcal{M}_L(a)\|_{L^1} \leq C\|f\|_{BMO}$$

for all $f \in BMO(\mathbb{R}^d)$, any generalized $(H^1_L, 2, \sigma_0)$-atom a related to the ball $B = B(x_0, r)$, where the constant $\sigma_0 > 0$ is as in Lemma 6.3.2. Indeed, by the proof of Theorem 5.2.2, for every $x \notin 2B$,

$$\mathcal{M}_L(a)(x) \leq C\frac{r^{\sigma_0}}{|x - x_0|^{d+\sigma_0}}.$$

Therefore, using Lemma 5.4.1, the L^2-boundedness of the classical Hardy-Littlewood maximal operator $\mathcal{M}$ and the estimate $\mathcal{M}_L(a) \leq C\mathcal{M}(a)$, we obtain that

$$\begin{aligned}
& \|(f - f_B)\mathcal{M}_L(a)\|_{L^1} \\
= \ & \|(f - f_B)\mathcal{M}_L(a)\|_{L^1(2B)} + \|(f - f_B)\mathcal{M}_L(a)\|_{L^1((2B)^c)} \\
\leq \ & C\|f - f_B\|_{L^2(2B)}\|\mathcal{M}(a)\|_{L^2} + C \int\limits_{|x-x_0|\geq 2r} |f(x) - f_{B(x_0,r)}|\frac{r^{\sigma_0}}{|x - x_0|^{d+\sigma_0}}dx \\
\leq \ & C\|f\|_{BMO},
\end{aligned}$$

where we have used the following classical inequality

$$\int\limits_{|x-x_0|\geq 2r} |f(x) - f_{B(x_0,r)}|\frac{r^{\sigma_0}}{|x - x_0|^{d+\sigma_0}}dx \leq C\|f\|_{BMO},$$

which proof can be found in [48]. This completes the proof of Proposition 5.4.3.

□

5.4.3 Some L-square functions

Recall (see [45]) that the L-square funcfions $\mathfrak{g}$ and $\mathcal{G}$ are defined by

$$\mathfrak{g}(f)(x) = \left(\int\limits_0^\infty |t\partial_t T_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}$$

and

$$\mathcal{G}(f)(x) = \left(\int\limits_0^\infty \int\limits_{|x-y|<t} |t\partial_t T_t(f)(y)|^2 \frac{dydt}{t^{d+1}}\right)^{1/2}.$$

Proposition 5.4.5. *The L-square function $\mathfrak{g}$ is in the class $\mathcal{K}_L$.*

Proposition 5.4.6. *The L-square function $\mathcal{G}$ is in the class $\mathcal{K}_L$.*

Here we just give the proof for Proposition 5.4.5. For the one of Proposition 5.4.6, we leave the details to the interested reader.

In order to prove Proposition 5.4.5, we need the following lemma.

Lemma 5.4.3. *There exists a constant $C > 0$ such that*

$$|t\partial_t T_t(x, y + h) - t\partial_t T_t(x, y)| \leq C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-y|^2}{t}}, \tag{5.16}$$

for all $|h| < \frac{|x-y|}{2}$, $0 < t$. Here and in the proof of Proposition 5.4.5, the constants $\delta, c \in (0, 1)$ are as in Proposition 4 of [45].

Proof. One only needs to consider the case $\sqrt{t} < |h| < \frac{|x-y|}{2}$. Otherwise, (5.16) follows directly from (b) in Proposition 4 of [45].

For $\sqrt{t} < |h| < \frac{|x-y|}{2}$. By (a) in Proposition 4 of [45], we get

$$\begin{aligned} |t\partial_t T_t(x, y + h) - t\partial_t T_t(x, y)| &\leq Ct^{-d/2} e^{-c\frac{|x-y-h|^2}{t}} + Ct^{-d/2} e^{-c\frac{|x-y|^2}{t}} \\ &\leq C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-y|^2}{t}}. \end{aligned}$$

$\square$

Proof of Proposition 5.4.5. The $(H^1_L - L^1)$ type boundedness of $\mathfrak{g}$ is well-known, see for example [45, 64]. Let us now show that

$$\|(f - f_B)\mathfrak{g}(a)\|_{L^1} \leq C\|f\|_{BMO}$$

for all $f \in BMO(\mathbb{R}^d)$, any generalized $(H^1_L, 2, \delta)$-atom a related to the ball $B = B(x_0, r)$. Indeed, it follows from Lemma 5.4.3 and (a) in Proposition 4 of [45] that for every $t > 0$, $x \notin 2B$,

$$\begin{aligned} &|t\partial_t T_t(a)(x)| \\ = &\Big| \int_B (t\partial_t T_t(x, y) - t\partial_t T_t(x, x_0))a(y)dy + t\partial_t T_t(x, x_0) \int_B a(y)dy \Big| \\ \leq &C\Big(\frac{r}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-x_0|^2}{t}} \|a\|_{L^1} + Ct^{-d/2} e^{-c\frac{|x-x_0|^2}{t}} \Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-\delta} \Big(\frac{r}{\rho(x_0)}\Big)^{\delta} \\ \leq &C\Big(\frac{r}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-x_0|^2}{t}}. \end{aligned}$$

Therefore, as $0<\delta<1$, using the estimate $e^{-\frac{c}{2}\frac{|x-x_0|^2}{t}} \leq C(c,d)(\frac{t}{|x-x_0|^2})^{d+2}$,

$$\begin{aligned}
\mathfrak{g}(a)(x) &\leq C\left\{\int_0^\infty \Big(\frac{r^2}{t}\Big)^\delta t^{-d} e^{-\frac{c}{2}\frac{|x-x_0|^2}{t}}\frac{dt}{t}\right\}^{1/2}\\
&\leq C\left\{\int_0^{|x-x_0|^2} \Big(\frac{r^2}{t}\Big)^\delta t^{-d}\Big(\frac{t}{|x-x_0|^2}\Big)^{d+2}\frac{dt}{t} + \int_{|x-x_0|^2}^\infty \Big(\frac{r^2}{t}\Big)^\delta t^{-d}\frac{dt}{t}\right\}^{1/2}\\
&\leq C\frac{r^\delta}{|x-x_0|^{d+\delta}}.
\end{aligned}$$

Therefore, the L^2-boundedness of $\mathfrak{g}$ and Lemma 5.4.1 yield

$$\begin{aligned}
&\|(f-f_B)\mathfrak{g}(a)\|_{L^1}\\
=\ &\|(f-f_B)\mathfrak{g}(a)\|_{L^1(2B)} + \|(f-f_B)\mathfrak{g}(a)\|_{L^1((2B)^c)}\\
\leq\ &\|f-f_B\|_{L^2(2B)}\|\mathfrak{g}(a)\|_{L^2} + C\int_{|x-x_0|\geq 2r} |f(x)-f_{B(x_0,r)}|\frac{r^\delta}{|x-x_0|^{d+\delta}}dx\\
\leq\ &C\|f\|_{BMO},
\end{aligned}$$

which ends the proof.

□

5.5 Proof of the main results

In this section, we fix a non-negative function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with supp $\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d}\varphi(x)dx = 1$. Then, we define the linear operator $\mathfrak{H}$ by

$$\mathfrak{H}(f) = \sum_{n,k}\Big(\psi_{n,k}f - \varphi_{2^{-n/2}} * (\psi_{n,k}f)\Big),$$

where $\psi_{n,k}$, $n\in\mathbb{Z}$, $k=1,2,...$ is as in Lemma 2.5 of [46] (see also Lemma 7.3.2).

Remark 5.5.1. *When $V(x)\equiv 1$, we can define $\mathfrak{H}(f) = f - \varphi * f$.*

Let us now consider the set $\mathcal{E} = \{0,1\}^d \setminus \{(0,\cdots,0)\}$ and $\{\psi^\sigma\}_{\sigma\in\mathcal{E}}$ the wavelet with compact support as in Section 3 of [14] (see also Section 2 of [82]). Suppose that ψ^σ is supported in the cube $(\frac{1}{2}-\frac{c}{2},\frac{1}{2}-\frac{c}{2})^d$ for all $\sigma\in\mathcal{E}$. As it is classical, for $\sigma\in\mathcal{E}$ and I a dyadic cube of $\mathbb{R}^d$ which may be written as the set of x such that $2^jx-k\in(0,1)^d$, we note

$$\psi_I^\sigma(x) = 2^{dj/2}\psi^\sigma(2^jx-k).$$

In the sequel, the letter I always refers to dyadic cubes. Moreover, we note kI the cube of same center dilated by the coefficient k.

Remark 5.5.2. *For every $\sigma \in \mathcal{E}$ and I a dyadic cube. Because of the assumption on the support of ψ^σ, the function ψ_I^σ is supported in the cube cI.*

In [14] (see also [82]), Bonami et al. established the following.

Proposition 5.5.1. *The bounded bilinear operator Π, defined by*

$$\Pi(f,g) = \sum_I \sum_{\sigma \in \mathcal{E}} \langle f, \psi_I^\sigma \rangle \langle g, \psi_I^\sigma \rangle (\psi_I^\sigma)^2,$$

is bounded from $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

5.5.1 Proof of Theorem 5.3.1, Theorem 5.3.2

In order to prove Theorem 5.3.1 and Theorem 5.3.2, we need the following key two lemmas which proofs will given in Section 5.6.

Lemma 5.5.1. *The linear operator $\mathfrak{H}$ is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.*

Lemma 5.5.2. *Let $T \in \mathcal{K}_L$. Then, the subbilinear operator*

$$\mathcal{U}(f,b) := [b,T](f - \mathfrak{H}(f))$$

is bounded from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

By Proposition 5.5.1 and Lemma 5.5.1, we obtain:

Proposition 5.5.2. *The bilinear operator $\mathfrak{S}(f,g) := -\Pi(\mathfrak{H}(f), g)$ is bounded from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

We recall (see [82]) that the class $\mathcal{K}$ is the set of all sublinear operators T bounded from $H^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ so that for some $q \in (1, \infty]$,

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO},$$

for all $b \in BMO(\mathbb{R}^d)$, any classical (H^1, q)-atom a related to the ball B, where $C > 0$ a constant independent of b, a.

Remark 5.5.3. *By Remark 5.2.1 and as $H^1(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$, we obtain that $\mathcal{K}_L \subset \mathcal{K}$, which allows to apply the two classical decomposition theorems (Theorem 3.1 and Theorem 3.2 of [82]). This is a key point in our proofs.*

Proof of Theorem 5.3.1. As $T \in \mathcal{K}_L \subset \mathcal{K}$, it follows from Theorem 3.1 of [82] that there exists a bounded subbilinear operator $\mathcal{V} : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(g, b) \in H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have

$$|T(-\Pi(g,b))| - \mathcal{V}(g,b) \leq |[b,T](g)| \leq \mathcal{V}(g,b) + |T(-\Pi(g,b))|. \tag{5.17}$$

Let us now define the bilinear operator $\mathfrak{R}$ by

$$\mathfrak{R}(f,b) := |\mathcal{U}(f,b)| + \mathcal{V}(\mathfrak{H}(f),b)$$

for all $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, where $\mathcal{U}$ is the subbilinear operator as in Lemma 5.5.2. Then, using the subbilinear decomposition (5.17) with $g = \mathfrak{H}(f)$,

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq |T(\mathfrak{S}(f,b))| + \mathfrak{R}(f,b),$$

where the bounded bilinear operator $\mathfrak{S} : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is given in Proposition 5.5.2.

Furthermore, by Lemma 5.5.2 and Lemma 5.5.1, we get

$$\begin{aligned} \|\mathfrak{R}(f,b)\|_{L^1} &\leq \|\mathcal{U}(f,b)\|_{L^1} + \|\mathcal{V}(\mathfrak{H}(f),b)\|_{L^1} \\ &\leq C\|f\|_{H^1_L}\|b\|_{BMO} + C\|\mathfrak{H}(f)\|_{H^1}\|b\|_{BMO} \\ &\leq C\|f\|_{H^1_L}\|b\|_{BMO}, \end{aligned}$$

where we used the boundedness of $\mathcal{V}$ on $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. This completes the proof.

□

Proof of Theorem 5.3.2. The proof follows the same lines except that now, one deals with equalities instead of inequalities. Namely, as T is a linear operator in $\mathcal{K}_L \subset \mathcal{K}$, Theorem 3.2 of [82] yields that there exists a bounded bilinear operator $\mathcal{W} : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for every $(g,b) \in H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$,

$$[b,T](g) = \mathcal{W}(g,b) + T(-\Pi(g,b))$$

Therefore, for every $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$,

$$[b,T](f) = \mathfrak{R}(f,b) + T(\mathfrak{S}(f,b)),$$

where $\mathfrak{R}(f,b) := \mathcal{U}(f,b) + \mathcal{W}(\mathfrak{H}(f),b)$ is a bounded bilinear operator from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. This completes the proof.

□

5.5.2 Proof of Theorem 5.3.3 and Theorem 5.3.4

First, recall that $VMO_L(\mathbb{R}^d)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$. Then, the following result due to Ky [83].

Theorem 5.5.1. *The space $H^1_L(\mathbb{R}^d)$ is the dual of the space $VMO_L(\mathbb{R}^d)$.*

In order to prove Theorem 5.3.3, we need the following key lemmas, which proofs will be given in Section 5.6.

Lemma 5.5.3. *Let $1 \leq q < \infty$ and $\theta \geq 0$. Then, for every $f \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$, $B = B(x,r)$ and $k \in \mathbb{Z}^+$, we have*

$$\Big(\frac{1}{|2^k B|}\int\limits_{2^k B} |f(y) - f_B|^q dy\Big)^{1/q} \leq Ck \frac{\Big(1 + \frac{2^k r}{\rho(x)}\Big)^{(k_0+1)\theta}}{\log\Big(e + (\frac{\rho(x)}{2^k r})^{k_0+1}\Big)} \|f\|_{BMO^{\log}_{L,\theta}},$$

where the constant k_0 is as in Proposition 7.2.1.

Lemma 5.5.4. *Let $1 < q < \infty$, $\varepsilon > 0$ and T be a L-Calderón-Zygmund operator. Then, the following two statements hold:*

i) If $T^*1 = 0$, then T is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.

ii) For every $f, g \in BMO(\mathbb{R}^d)$, generalized (H^1_L, q, ε)-atom a related to the ball B,

$$\|(f - f_B)(g - g_B)Ta\|_{L^1} \leq C\|f\|_{BMO}\|g\|_{BMO}.$$

Proof of Theorem 5.3.3. (i). Assume that T is a (δ, L)-Calderón-Zygmund operator. We claim that, as, by Lemma 5.5.4, it is sufficient to prove that

$$\|(b - b_B)a\|_{H^1_L} \leq C\|b\|_{BMO^{\log}_L} \tag{5.18}$$

and

$$\|(b - b_B)Ta\|_{H^1_L} \leq C\|b\|_{BMO^{\log}_L} \tag{5.19}$$

hold for every generalized $(H^1_L, 2, \delta)$-atom a related to the ball $B = B(x_0, r)$ with the constants are independent of b, a. Indeed, if (5.18) and (5.19) are true, then

$$\begin{aligned}
\|[b,T](a)\|_{H^1_L} &\leq \|(b - b_B)Ta\|_{H^1_L} + C\|T((b - b_B)a)\|_{H^1} \\
&\leq C\|b\|_{BMO^{\log}_L} + C\|T\|_{H^1_L \to H^1}\|(b - b_B)a\|_{H^1_L} \\
&\leq C\|b\|_{BMO^{\log}_L}.
\end{aligned}$$

Therefore, Proposition 5.2.2 yields that $[b,T]$ is bounded on $H^1_L(\mathbb{R}^d)$, moreover,

$$\|[b,T]\|_{H^1_L \to H^1_L} \leq C,$$

where the constant C is independent of b.

The proof of (5.18) is similar to the one of (5.19) but uses an easier argument, we leave the details to the interested reader. Let us now establish (5.19). By Theorem 5.5.1, it is sufficient to show that

$$\|\phi(b-b_B)Ta\|_{L^1} \leq C\|b\|_{BMO_L^{\log}}\|\phi\|_{BMO_L} \tag{5.20}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. Besides, from Lemma 5.5.4,

$$\|(\phi-\phi_B)(b-b_B)Ta\|_{L^1} \leq C\|b\|_{BMO}\|\phi\|_{BMO} \leq C\|b\|_{BMO_L^{\log}}\|\phi\|_{BMO_L}.$$

This together with Lemma 2 of [45] allow us to reduce (5.20) to showing that

$$\log\left(e+\frac{\rho(x_0)}{r}\right)\|(b-b_B)Ta\|_{L^1} \leq C\|b\|_{BMO_L^{\log}}. \tag{5.21}$$

Setting $\varepsilon = \delta/2$, it is easy to check that there exists a constant $C = C(\varepsilon) > 0$ such that

$$\log(e+kt) \leq Ck^\varepsilon \log(e+t)$$

for all $k \geq 2, t > 0$. Consequently, for all $k \geq 1$,

$$\log\left(e+\frac{\rho(x_0)}{r}\right) \leq C2^{k\varepsilon}\log\left(e+\left(\frac{\rho(x_0)}{2^{k+1}r}\right)^{k_0+1}\right). \tag{5.22}$$

Then, by Lemma 5.4.2 and Lemma 5.5.3, we get

$$\begin{aligned}
&\log\left(e+\frac{\rho(x_0)}{r}\right)\|(b-b_B)Ta\|_{L^1}\\
= \;& \log\left(e+\frac{\rho(x_0)}{r}\right)\|(b-b_B)Ta\|_{L^1(2B)} + \\
&+\sum_{k\geq 1}\log\left(e+\frac{\rho(x_0)}{r}\right)\|(b-b_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\leq \;& C\log\left(e+\left(\frac{\rho(x_0)}{2r}\right)^{k_0+1}\right)\|b-b_B\|_{L^2(2B)}\|Ta\|_{L^2} + \\
&+C\sum_{k\geq 1}2^{k\varepsilon}\log\left(e+\left(\frac{\rho(x_0)}{2^{k+1}r}\right)^{k_0+1}\right)\|b-b_B\|_{L^2(2^{k+1}B)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^kB)}\\
\leq \;& C|2B|^{1/2}\|b\|_{BMO_L^{\log}}\|a\|_{L^2} + C\sum_{k\geq 1}2^{k\varepsilon}(k+1)|2^{k+1}B|^{1/2}\|b\|_{BMO_L^{\log}}2^{-k\delta}|2^kB|^{-1/2}\\
\leq \;& C\|b\|_{BMO_L^{\log}},
\end{aligned}$$

where we used $\delta = 2\varepsilon$. This ends the proof of (i).

(ii). By Remark 5.2.3, (ii) can be seen as a consequence of Theorem 5.3.4 that we are going to prove now.

□

Next, let us recall the following lemma due to Tang and Bi [134].

Lemma 5.5.5 (see [134], Lemma 3.1). *Let $V \in RH_{d/2}$. Then, there exists $c_0 \in (0,1)$ such that for any positive number N and $0 < h < |x-y|/16$, we have*

$$|K_j(x,y)| \leq \frac{C(N)}{\left(1+\frac{|x-y|}{\rho(y)}\right)^N} \frac{1}{|x-y|^{d-1}} \left(\int_{B(x,|x-y|)} \frac{V(z)}{|x-z|^{d-1}} dz + \frac{1}{|x-y|} \right)$$

and

$$|K_j(x,y+h) - K_j(x,y)| \leq \frac{C(N)}{\left(1+\frac{|x-y|}{\rho(y)}\right)^N} \frac{h^{c_0}}{|x-y|^{c_0+d-1}} \left(\int_{B(x,|x-y|)} \frac{V(z)}{|x-z|^{d-1}} dz + \frac{1}{|x-y|} \right),$$

where $K_j(x,y)$, $j = 1, ..., d$, are the kernels of the Riesz transforms R_j.

In order to prove Theorem 5.3.4, we need also the following two technical lemmas, which proofs will be given in Section 5.6.

Lemma 5.5.6. *Let $1 < q \leq d/2$ and c_0 be as in Lemma 5.5.5. Then, $R_j(a)$ is C times a classical (H^1, q, c_0)-molecule (e.g. [126]) for all generalized (H^1_L, q, c_0)-atom a related to the ball $B = B(x_0, r)$. Furthermore, for any $N > 0$ and $k \geq 4$, we have*

$$\|R_j(a)\|_{L^q(2^{k+1}B \setminus 2^k B)} \leq \frac{C(N)}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} 2^{-kc_0} |2^k B|^{1/q-1}, \tag{5.23}$$

where $C(N) > 0$ depends only on N.

Lemma 5.5.7. *Let $1 < q \leq d/2$ and $\theta \geq 0$. Then, for every $f \in BMO(\mathbb{R}^d)$, $g \in BMO_{L,\theta}(\mathbb{R}^d)$ and (H^1_L, q)-atom a related to the ball $B = B(x_0, r)$, we have*

$$\|(g - g_B) R_j(a)\|_{L^1} \leq C \|g\|_{BMO_{L,\theta}}$$

and

$$\|(f - f_B)(g - g_B) R_j(a)\|_{L^1} \leq C \|f\|_{BMO} \|g\|_{BMO_{L,\theta}}.$$

Proof of Theorem 5.3.4. Suppose that $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d)$, i.e. $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ for some $\theta \geq 0$. By Proposition 3.2 of [145], in order to prove that $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^d)$, it is sufficient to show that $\|[b, R_j](a)\|_{H^1_L} \leq C \|b\|_{BMO^{\log}_{L,\theta}}$ for all $(H^1_L, d/2)$-atom a. Similarly to the proof of Theorem 5.3.3, it remains to show

$$\|(b - b_B) a\|_{H^1_L} \leq C \|b\|_{BMO^{\log}_{L,\theta}} \tag{5.24}$$

and

$$\|(b-b_B)R_j(a)\|_{H^1_L} \le C\|b\|_{BMO^{\log}_{L,\theta}} \tag{5.25}$$

hold for every $(H^1_L, d/2)$-atom a related to the ball $B = B(x_0, r)$, where the constants C in (5.24) and (5.25) are independent of b, a.

As before, we leave the proof of (5.24) to the interested reader.

Let us now establish (5.25). Similarly to the proof of Theorem 5.3.3, Lemma 5.5.7 allows to reduce (5.25) to showing that

$$\log\left(e + \frac{\rho(x_0)}{r}\right)\|(b-b_B)R_j(a)\|_{L^1} \le C\|b\|_{BMO^{\log}_{L,\theta}}. \tag{5.26}$$

Setting $\varepsilon = c_0/2$, there is a constant $C = C(\varepsilon) > 0$ such that for all $k \ge 1$,

$$\log\left(e + \frac{\rho(x_0)}{r}\right) \le C2^{k\varepsilon}\log\left(e + \left(\frac{\rho(x_0)}{2^{k+1}r}\right)^{k_0+1}\right). \tag{5.27}$$

Note that $r \le \mathcal{C}_L\rho(x_0)$ since a is a $(H^1_L, d/2)$-atom related to the ball $B(x_0, r)$. In (5.23) of Lemma 5.5.6, we choose $N = (k_0+1)\theta$. Then, Hölder inequality, (5.27) and Lemma 5.5.3 allow to conclude that

$$\begin{aligned}
&\log\left(e + \frac{\rho(x_0)}{r}\right)\|(b-b_B)R_j(a)\|_{L^1}\\
= \;& \log\left(e + \frac{\rho(x_0)}{r}\right)\|(b-b_B)R_j(a)\|_{L^1(2^4B)} +\\
&+\sum_{k\ge 4}\log\left(e + \frac{\rho(x_0)}{r}\right)\|(b-b_B)R_j(a)\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\le\;& C\log\left(e + \left(\frac{\rho(x_0)}{2^4r}\right)^{k_0+1}\right)\|b-b_B\|_{L^{\frac{d}{d-2}}(2^4B)}\|R_j(a)\|_{L^{d/2}} +\\
&+C\sum_{k\ge 4}2^{k\varepsilon}\log\left(e + \left(\frac{\rho(x_0)}{2^{k+1}r}\right)^{k_0+1}\right)\|b-b_B\|_{L^{\frac{d}{d-2}}(2^{k+1}B)}\|R_j(a)\|_{L^{d/2}(2^{k+1}B\setminus 2^kB)}\\
\le\;& C\|b\|_{BMO^{\log}_{L,\theta}} + C\|b\|_{BMO^{\log}_{L,\theta}}\sum_{k\ge 4}k2^{-k\varepsilon}\\
\le\;& C\|b\|_{BMO^{\log}_{L,\theta}}
\end{aligned}$$

where we used $c_0 = 2\varepsilon$. This proves (5.26), and thus $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^d)$.

Conversely, assume that $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^d)$. Then, although b belongs to $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ from a duality argument and Theorem 2 of [17], we would also like to give a direct proof for completeness.

As $b \in BMO_{L,\infty}(\mathbb{R}^d)$ by assumption, there exist $\theta \ge 0$ such that $b \in BMO_{L,\theta}(\mathbb{R}^d)$.

For every $(H^1_L, d/2)$-atom a related to some ball $B = B(x_0, r)$. By Remark 5.2.1 and Lemma 5.5.7,

$$\begin{aligned}\|R_j((b-b_B)a)\|_{L^1} &\leq \|(b-b_B)R_j(a)\|_{L^1} + C\|[b,R_j](a)\|_{H^1_L}\\ &\leq C\|b\|_{BMO_{L,\theta}} + C\|[b,R_j]\|_{H^1_L\to H^1_L}\end{aligned}$$

hold for all $j = 1, ..., d$. In addition, noting that $r \leq \mathcal{C}_L\rho(x_0)$ since a is a $(H^1_L, d/2)$-atom related to some ball $B = B(x_0, r)$, Hölder inequality and Lemma 1 of [17] (see also Lemma 5.6.6 below) give

$$\|(b-b_B)a\|_{L^1} \leq \|b-b_B\|_{L^{\frac{d}{d-2}}(B)}\|a\|_{L^{d/2}(B)} \leq C\|b\|_{BMO_{L,\theta}}.$$

By the characterization of $H^1_L(\mathbb{R}^d)$ in terms of the Riesz transforms (see [46]), the above proves that $(b-b_B)a \in H^1_L(\mathbb{R}^d)$, moreover,

$$\|(b-b_B)a\|_{H^1_L} \leq C\left(\|b\|_{BMO_{L,\theta}} + \sum_{j=1}^d \|[b,R_j]\|_{H^1_L\to H^1_L}\right) \tag{5.28}$$

where the constant $C > 0$ is independent of b, a.

Now, we prove that $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$. More precisely, the following

$$\frac{\log\left(e + \frac{\rho(x_0)}{r}\right)}{\left(1 + \frac{r}{\rho(x_0)}\right)^\theta} MO(b, B(x_0,r)) \leq C\left(\|b\|_{BMO_{L,\theta}} + \sum_{j=1}^d \|[b,R_j]\|_{H^1_L\to H^1_L}\right) \tag{5.29}$$

holds for any ball $B(x_0, r)$ in $\mathbb{R}^d$. In fact, we only need to establish (5.29) for $0 < r < \rho(x_0)/2$ since $b \in BMO_{L,\theta}(\mathbb{R}^d)$.

Indeed, in (5.28) we choose $B = B(x_0, r)$ and $a = (2|B|)^{-1}(f - f_B)\chi_B$, where $f =$ sign $(b - b_B)$. Then, it is easy to see that a is a $(H^1_L, d/2)$-atom related to the ball B. We next consider

$$g_{x_0,r}(x) = \chi_{[0,r]}(|x-x_0|)\log\left(\frac{\rho(x_0)}{r}\right) + \chi_{(r,\rho(x_0)]}(|x-x_0|)\log\left(\frac{\rho(x_0)}{|x-x_0|}\right).$$

Then, thanks to Lemma 2.5 of [103], one has $\|g_{x_0,r}\|_{BMO_L} \leq C$. Moreover, it is clear that $g_{x_0,r}(b - b_B)a \in L^1(\mathbb{R}^d)$. Consequently, (5.28) together with the fact that $BMO_L(\mathbb{R}^d)$ is the dual of $H^1_L(\mathbb{R}^d)$ allows us to conclude that

$$\begin{aligned}
\frac{\log\left(e+\frac{\rho(x_0)}{r}\right)}{\left(1+\frac{r}{\rho(x_0)}\right)^{\theta}} MO(b,B(x_0,r)) &\le 3\log\left(\frac{\rho(x_0)}{r}\right) MO(b,B(x_0,r)) \\
&= 6\Big|\int\limits_{\mathbb{R}^d} g_{x_0,r}(x)(b(x)-b_B)a(x)dx\Big| \\
&\le 6\|g_{x_0,r}\|_{BMO_L}\|(b-b_B)a\|_{H^1_L} \\
&\le C\left(\|b\|_{BMO_{L,\theta}}+\sum_{j=1}^{d}\|[b,R_j]\|_{H^1_L\to H^1_L}\right),
\end{aligned}$$

where we used $r<\rho(x_0)/2$ and

$$\int\limits_{\mathbb{R}^d}(b(x)-b_B)a(x)dx = \frac{1}{2|B(x_0,r)|}\int\limits_{B(x_0,r)}|b(x)-b_{B(x_0,r)}|dx.$$

This ends the proof.

□

5.6 Proof of the key lemmas

First, let us recall some notations and results due to Dziubański and Zienkiewicz in [46]. These notations and results play an important role in our proofs.

Let $P(x)=(4\pi)^{-d/2}e^{-|x|^2/4}$ be the Gauss function. For $n\in\mathbb{Z}$, the space $h^1_n(\mathbb{R}^d)$ denotes the space of all integrable functions f such that

$$\mathcal{M}_n f(x)=\sup_{0<t<2^{-n}}|P_{\sqrt{t}}*f(x)| = \sup_{0<t<2^{-n}}\Big|\int\limits_{\mathbb{R}^d}p_t(x,y)f(y)dy\Big|\in L^1(\mathbb{R}^d),$$

where the kernel p_t is given by $p_t(x,y)=(4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}}$. We equipped this space with the norm $\|f\|_{h^1_n}:=\|\mathcal{M}_n f\|_{L^1}$.

For convenience of the reader, we list here some lemmas used in our proofs.

Lemma 5.6.1 (see [46], Lemma 2.3)**.** *There exists a constant $C>0$ and a collection of balls $B_{n,k}=B(x_{n,k},2^{-n/2})$, $n\in\mathbb{Z}, k=1,2,...,$ such that $x_{n,k}\in\mathcal{B}_n$, $\mathcal{B}_n\subset\bigcup_k B_{n,k}$, and*

$$card\,\{(n',k') : B(x_{n,k},R2^{-n/2})\cap B(x_{n',k'},R2^{-n/2})\neq\emptyset\}\le R^C$$

for all n,k and $R\ge 2$.

Lemma 5.6.2 (see [46], Lemma 2.5)**.** *There are nonnegative C^∞-functions $\psi_{n,k}$, $n \in \mathbb{Z}, k = 1, 2, \ldots$, supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that*

$$\sum_{n,k} \psi_{n,k} = 1 \quad \textit{and} \quad \|\nabla \psi_{n,k}\|_{L^\infty} \leq C 2^{n/2}.$$

Lemma 5.6.3 (see (4.7) in [46])**.** *For every $f \in H^1_L(\mathbb{R}^d)$, we have*

$$\sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \leq C \|f\|_{H^1_L}.$$

To prove Lemma 5.5.1, we need the following.

Lemma 5.6.4. *There exists a constant $C = C(\varphi, d) > 0$ such that*

$$\|f - \varphi_{2^{-n/2}} * f\|_{H^1} \leq C \|f\|_{h^1_n}, \quad \textit{for all}\, n \in \mathbb{Z}, f \in h^1_n(\mathbb{R}^d). \tag{5.30}$$

The proof of Lemma 5.6.4 can be found in [56]. In fact, in [56], Goldberg proved it just for $n = 0$, however, by dilations, it is easy to see that (5.30) holds for every $n \in \mathbb{Z}, f \in h^1_n(\mathbb{R}^d)$ with an uniform constant $C > 0$ depends only on φ and d.

Proof of Lemma 5.5.1. It follows from Lemma 5.6.4 and Lemma 7.3.4 that

$$\begin{aligned} \|\mathfrak{H}(f)\|_{H^1} &= \Big\| \sum_{n,k} (\psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f)) \Big\|_{H^1} \\ &\leq \sum_{n,k} \Big\| \psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f) \Big\|_{H^1} \\ &\leq C \sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \leq C \|f\|_{H^1_L} \end{aligned}$$

for every $f \in H^1_L(\mathbb{R}^d)$. This completes the proof. $\square$

For $1 < q \leq \infty$ and $n \in \mathbb{Z}$. Recall (see [46]) that a function a is said to be a (h^1_n, q)-atom related to the ball $B(x_0, r)$ if $r \leq 2^{1-n/2}$ and

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,

iii) if $r \leq 2^{-1-n/2}$ then $\int_{\mathbb{R}^d} a(x) dx = 0$.

In order to prove Lemma 5.5.2, we need the following lemma.

Lemma 5.6.5. *Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Suppose that $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x, 2^{1-n/2})$. Then, there are (H^1_L, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C \|f\|_{h^1_n}$$

with a positive constant C independent of n and f.

Proof. By Theorem 4.5 of [46], there are (h^1_n, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h^1_n}.$$

Now, let us establish that the a_j's are (H^1_L, q)-atoms related to the balls $B(x_j, r_j)$.

Indeed, as $x_j \in B(x, 2^{2-n/2})$ and $x \in \mathcal{B}_n$, Proposition 7.2.1 implies that $r_j \leq 2^{2-n/2} \leq \mathcal{C}_L \rho(x_j)$, where $\mathcal{C}_L$ is as in (6.13). Moreover, if $r_j < \frac{1}{\mathcal{C}_L}\rho(x_j)$, then Proposition 7.2.1 implies that $r_j \leq 2^{-1-n/2}$, and thus $\int_{\mathbb{R}^d} a_j(x)dx = 0$ since a_j are (h^1_n, q)-atoms related to the balls $B(x_j, r_j)$. These prove that the a_j's are (H^1_L, q)-atoms related to the balls $B(x_j, r_j)$. $\square$

Proof of Lemma 5.5.2. As $T \in \mathcal{K}_L$, there exist $q \in (1, \infty]$ and $\varepsilon > 0$ such that

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO} \tag{5.31}$$

for all $b \in BMO(\mathbb{R}^d)$ and generalized (H^1_L, q, ε)-atom a related to the ball B.

From $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is dense in $H^1_L(\mathbb{R}^d)$, we need only prove that

$$\|\mathcal{U}(f, b)\|_{L^1} = \|[b, T](f - \mathfrak{H}(f))\|_{L^1} \leq C\|f\|_{H^1_L}\|b\|_{BMO}$$

holds for every $(f, b) \in \mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$.

For any $(n, k) \in \mathbb{Z} \times \mathbb{Z}^+$. As $x_{n,k} \in \mathcal{B}_n$ and $\psi_{n,k} f \in h^1_n(\mathbb{R}^d)$, it follows from Lemma 5.6.5 and Remark 5.2.1 that there are generalized (H^1_L, q, ε)-atoms $a^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j)$ such that $B(x^{n,k}_j, r^{n,k}_j) \subset B(x_{n,k}, 2^{2-n/2})$ and

$$\psi_{n,k} f = \sum_j \lambda^{n,k}_j a^{n,k}_j, \quad \sum_j |\lambda^{n,k}_j| \leq C\|\psi_{n,k} f\|_{h^1_n} \tag{5.32}$$

with a positive constant C independent of n, k and f.

Clearly, supp $\varphi_{2^{-n/2}} * a^{n,k}_j \subset B(x_{n,k}, 5.2^{-n/2})$ since supp $\varphi \subset B(0, 1)$ and supp $a^{n,k}_j \subset B(x_{n,k}, 2^{2-n/2})$; the following estimate holds

$$\|\varphi_{2^{-n/2}} * a^{n,k}_j\|_{L^q} \leq \|\varphi_{2^{-n/2}}\|_{L^q}\|a^{n,k}_j\|_{L^1} \leq (2^{-n/2})^{d(1/q-1)}\|\varphi\|_{L^q} \leq C|B(x_{n,k}, 5.2^{-n/2})|^{1/q-1}.$$

Moreover, as $x_{n,k} \in \mathcal{B}_n$,

$$\Big|\int_{\mathbb{R}^d} \varphi_{2^{-n/2}} * a^{n,k}_j dx\Big| \leq \|\varphi_{2^{-n/2}}\|_{L^1}\|a^{n,k}_j\|_{L^1} \leq C\Big(\frac{5.2^{-n/2}}{\rho(x_{n,k})}\Big)^{\varepsilon}.$$

These prove that $\varphi_{2^{-n/2}} * a^{n,k}_j$ is C times a generalized (H^1_L, q, ε)-atom related to $B(x_{n,k}, 5.2^{-n/2})$. Consequently, (5.31) yields

$$\|(b - b_{B(x_{n,k}, 5.2^{-n/2})})T(\varphi_{2^{-n/2}} * a^{n,k}_j)\|_{L^1} \leq C\|b\|_{BMO}. \tag{5.33}$$

By an analogous argument, it is easy to check that $(\varphi_{2^{-n/2}} * a_j^{n,k})(b - b_{B(x_{n,k},5.2^{-n/2})})$ is $C\|b\|_{BMO}$ times a generalized $(H^1_L, \frac{q+1}{2}, \varepsilon)$-atom related to $B(x_{n,k}, 5.2^{-n/2})$. Hence, it follows from (5.32) and (5.33) that

$$\begin{aligned}
\|[b,T](\varphi_{2^{-n/2}} * (\psi_{n,k}f))\|_{L^1} &\leq \|(b - b_{B(x_{n,k},5.2^{-n/2})})T(\varphi_{2^{-n/2}} * (\psi_{n,k}f))\|_{L^1} \\
&\quad + \Big\| T\Big((b - b_{B(x_{n,k},5.2^{-n/2})})(\varphi_{2^{-n/2}} * (\psi_{n,k}f))\Big)\Big\|_{L^1} \\
&\leq C\|\psi_{n,k}f\|_{h^1_n}\|b\|_{BMO}, \qquad (5.34)
\end{aligned}$$

where we used the fact that T is bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ since $T \in \mathcal{K}_L$.

On the other hand, by $f \in \mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$, there exists a ball $B(0,R)$ such that supp $f \subset B(0,R)$. As $\overline{B(0,R)}$ is a compact set, Lemma 7.3.1 allows to conclude that there is a finite set $\Gamma_R \subset \mathbb{Z} \times \mathbb{Z}^+$ such that for every $(n,k) \notin \Gamma_R$,

$$B(x_{n,k}, 2^{1-n/2}) \cap \overline{B(0,R)} = \emptyset.$$

It follows that there are $N, K \in \mathbb{Z}^+$ such that

$$f = \sum_{n,k} \psi_{n,k} f = \sum_{n=-N}^{N} \sum_{k=1}^{K} \psi_{n,k} f.$$

Therefore, (5.34) and Lemma 7.3.4 yield

$$\begin{aligned}
\|\mathcal{U}(f,b)\|_{L^1} &\leq \Big\| \sum_{n=-N}^{N} \sum_{k=1}^{K} \Big|[b,T](\varphi_{2^{-n/2}} * (\psi_{n,k}f))\Big| \Big\|_{L^1} \\
&\leq C\|b\|_{BMO} \sum_{n,k} \|\psi_{n,k}f\|_{h^1_n} \leq C\|f\|_{H^1_L}\|b\|_{BMO},
\end{aligned}$$

which ends the proof.

□

Proof of Lemma 5.5.3. First, we claim that for every ball $B_0 = B(x_0, r_0)$,

$$\Big(\frac{1}{|B_0|}\int_{B_0} |f(y) - f_{B_0}|^q dy\Big)^{1/q} \leq C \frac{\Big(1 + \frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e + (\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)} \|f\|_{BMO^{\log}_{L,\theta}}. \qquad (5.35)$$

Assume that (5.35) holds for a moment. Then,

$$\begin{aligned}
&\Big(\frac{1}{|2^kB|}\int\limits_{2^kB}|f(y)-f_B|^q dy\Big)^{1/q}\\
\le\;& \Big(\frac{1}{|2^kB|}\int\limits_{2^kB}|f(y)-f_{2^kB}|^q dy\Big)^{1/q}+\sum_{j=0}^{k-1}|f_{2^{j+1}B}-f_{2^jB}|\\
\le\;& \frac{\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x)}{2^kr})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}+\sum_{j=0}^{k-1}2^d\frac{\Big(1+\frac{2^{j+1}r}{\rho(x)}\Big)^{\theta}}{\log\Big(e+\frac{\rho(x)}{2^{j+1}r}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}\\
\le\;& Ck\frac{\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x)}{2^kr})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}.
\end{aligned}$$

Now, it remains to prove (5.35).

Let us define the function h on $\mathbb{R}^d$ as follows

$$h(x)=\begin{cases}1, & x\in B_0,\\ \frac{2r_0-|x-x_0|}{r_0}, & x\in 2B_0\setminus B_0,\\ 0, & x\notin 2B_0,\end{cases}$$

and remark that

$$|h(x)-h(y)|\le\frac{|x-y|}{r_0}. \tag{5.36}$$

Setting $\widetilde{f}:=f-f_{2B_0}$. By the classical John-Nirenberg inequality, there exists a constant $C=C(d,q)>0$ such that

$$\begin{aligned}
\Big(\frac{1}{|B_0|}\int\limits_{B_0}|f(y)-f_{B_0}|^q dy\Big)^{1/q}&=\Big(\frac{1}{|B_0|}\int\limits_{B_0}|h(y)\widetilde{f}(y)-(h\widetilde{f})_{B_0}|^q dy\Big)^{1/q}\\
&\le C\|h\widetilde{f}\|_{BMO}.
\end{aligned}$$

Therefore, the proof of the lemma is reduced to showing that

$$\|h\widetilde{f}\|_{BMO}\le C\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}},$$

namely, for every ball $B=B(x,r)$,

$$\frac{1}{|B|}\int\limits_{B}|h(y)\widetilde{f}(y)-(h\widetilde{f})_B|dy\le C\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}. \tag{5.37}$$

Now, let us focus on Inequality (5.37). Noting that supp $h \subset 2B_0$, Inequality (5.37) is obvious if $B \cap 2B_0 = \emptyset$. Hence, we only consider the case $B \cap 2B_0 \neq \emptyset$. Then, we have the following two cases:

The case $r > r_0$: the fact $B \cap 2B_0 \neq \emptyset$ implies that $2B_0 \subset 5B$, and thus

$$\begin{aligned}
\frac{1}{|B|}\int\limits_B |h(y)\widetilde{f}(y) - (h\widetilde{f})_B|dy &\leq 2\frac{1}{|B|}\int\limits_B |h(y)\widetilde{f}(y)|dy \\
&\leq 2.5^d \frac{1}{|2B_0|}\int\limits_{2B_0} |f(y) - f_{2B_0}|dy \\
&\leq C\frac{\left(1+\frac{2r_0}{\rho(x_0)}\right)^\theta}{\log\left(e+\frac{\rho(x_0)}{2r_0}\right)}\|f\|_{BMO^{\log}_{L,\theta}} \\
&\leq C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}.
\end{aligned}$$

The case $r \leq r_0$: Inequality (5.36) yields

$$\begin{aligned}
\frac{1}{|B|}\int\limits_B |h(y)\widetilde{f}(y) - (h\widetilde{f})_B|dy &\leq 2\frac{1}{|B|}\int\limits_B |h(y)\widetilde{f}(y) - h_B\widetilde{f}_B|dy \\
&\leq 2\frac{1}{|B|}\int\limits_B |h(y)(\widetilde{f}(y) - \widetilde{f}_B)|dy + \\
&\quad + 2|\widetilde{f}_B|\frac{1}{|B|}\int\limits_B \frac{1}{|B|}\Big|\int\limits_B (h(x)-h(y))dy\Big|dx \\
&\leq 2\frac{1}{|B|}\int\limits_B |f(y) - f_B|dy + 4\frac{r}{r_0}|f_B - f_{2B_0}|. \qquad (5.38)
\end{aligned}$$

By $r \leq r_0$, $B = B(x,r) \cap B(x_0,r_0) \neq \emptyset$, Proposition 7.2.1 gives

$$\frac{r}{\rho(x)} \leq \frac{r_0}{\rho(x)} \leq \kappa\frac{r_0}{\rho(x_0)}\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{k_0} \leq C\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{k_0+1}.$$

Consequently,

$$\begin{aligned}
\frac{1}{|B|}\int\limits_B |f(y) - f_B|dy &\leq \frac{\left(1+\frac{r}{\rho(x)}\right)^\theta}{\log(e+\frac{\rho(x)}{r})}\|f\|_{BMO^{\log}_{L,\theta}} \\
&\leq C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}, \qquad (5.39)
\end{aligned}$$

and

$$\frac{1}{|B(x,2^3r_0)|}\int\limits_{B(x,2^3r_0)}|f(y)-f_{B(x,2^3r_0)}|dy \leq \frac{\left(1+\frac{2^3r_0}{\rho(x)}\right)^{\theta}}{\log(e+\frac{\rho(x)}{2^3r_0})}\|f\|_{BMO^{\log}_{L,\theta}}$$

$$\leq C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}. \qquad (5.40)$$

Noting that for every $k\in\mathbb{N}$ with $2^{k+1}r\leq 2^3r_0$,

$$\begin{aligned}|f_{2^{k+1}B}-f_{2^kB}| &\leq 2^d\frac{1}{|2^{k+1}B|}\int\limits_{2^{k+1}B}|f(y)-f_{2^{k+1}B}|dy\\ &\leq C\frac{\left(1+\frac{2^3r_0}{\rho(x)}\right)^{\theta}}{\log(e+\frac{\rho(x)}{2^3r_0})}\|f\|_{BMO^{\log}_{L,\theta}}\\ &\leq C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}},\end{aligned}$$

allows us to conclude that

$$|f_{B(x,r)}-f_{B(x,2^3r_0)}|\leq C\log\left(e+\frac{r_0}{r}\right)\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}. \qquad (5.41)$$

Then, the inclusion $2B_0\subset B(x,2^3r_0)$ together with the inequalities (5.38), (5.39), (5.40) and (5.41) yield

$$\begin{aligned}\frac{1}{|B|}\int\limits_B|h(y)\widetilde{f}(y)-(h\widetilde{f})_B|dy &\leq 2\frac{1}{|B|}\int\limits_B|f(y)-f_B|dy+\\ &\quad +4\frac{r}{r_0}\Big(|f_{B(x,r)}-f_{B(x,2^3r_0)}|+4^dMO(f,B(x,2^3r_0))\Big)\\ &\leq C\Big(1+\frac{r}{r_0}\log(e+\frac{r_0}{r})\Big)\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}\\ &\leq C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}},\end{aligned}$$

we have used $\frac{r}{r_0}\log(e+\frac{r_0}{r})\leq\sup_{t\leq 1}t\log(e+1/t)<\infty$. This ends the proof.

□

By an analogous argument, we can also obtain the following, which was proved by Bongioanni et al (see Lemma 1 of [17]) through another method.

Lemma 5.6.6. *Let* $1 \leq q < \infty$ *and* $\theta \geq 0$. *Then, for every* $f \in BMO_{L,\theta}(\mathbb{R}^d)$, $B = B(x,r)$ *and* $k \in \mathbb{Z}^+$, *we have*

$$\Big(\frac{1}{|2^k B|}\int\limits_{2^k B} |f(y) - f_B|^q dy\Big)^{1/q} \leq Ck\Big(1 + \frac{2^k r}{\rho(x)}\Big)^{(k_0+1)\theta} \|f\|_{BMO_{L,\theta}}.$$

Proof of Lemma 5.5.4. i) Assume that T is a (δ, L)-calderón-Zygmund operator for some $\delta \in (0,1]$. For every generalized $(H^1_L, 2, \delta)$-atom a related to the ball B, as $T^*1 = 0$, Lemma 5.4.2 implies that Ta is C times a classical $(H^1, 2, \delta)$-molecule (see for example [126]) related to B, and thus $\|Ta\|_{H^1} \leq C$. Therefore, Proposition 5.2.2 yields T maps continuously $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.

ii) By Lemma 5.4.1, Lemma 5.4.2 and Hölder inequality, we get

$$\begin{aligned}
& \|(f - f_B)(g - g_B)Ta\|_{L^1} \\
= \; & \|(f - f_B)(g - g_B)Ta\|_{L^1(2B)} + \sum_{k\geq 1} \|(f - f_B)(g - g_B)Ta\|_{L^1(2^{k+1}B\setminus 2^k B)} \\
\leq \; & \|f - f_B\|_{L^{2q'}(2B)} \|g - g_B\|_{L^{2q'}(2B)} \|T(a)\|_{L^q} + \\
& + \sum_{k\geq 1} \|f - f_B\|_{L^{2q'}(2^{k+1}B)} \|g - g_B\|_{L^{2q'}(2^{k+1}B)} \|T(a)\|_{L^q(2^{k+1}B\setminus 2^k B)} \\
\leq \; & C\|f\|_{BMO}\|g\|_{BMO} + \sum_{k\geq 1} C(k+1)^2 \|f\|_{BMO}\|g\|_{BMO} |2^{k+1}B|^{1/q'} 2^{-k\delta} |2^k B|^{1/q-1} \\
\leq \; & C\|f\|_{BMO}\|g\|_{BMO},
\end{aligned}$$

where $1/q + 1/q' = 1$.

$\square$

Proof of Lemma 5.5.6. It is well-known that the Riesz transforms R_j are bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$, in particular, one has $\int_{\mathbb{R}^d} R_j(a)(x)dx = 0$. Moreover, by the L^q-boundedness of R_j (see [125], Theorem 0.5) one has $\|R_j(a)\|_{L^q} \leq C|B|^{1/q-1}$. Therefore, it is sufficient to verify (5.23). Thanks to Lemma 5.5.5, as a is a generalized (H^1_L, q, c_0)-atom related to the ball B, for every $x \in 2^{k+1}B \setminus 2^kB$,

$$
\begin{aligned}
&|R_j(a)(x)| \le \Big|\int_B (K_j(x,y) - K_j(x,x_0))a(y)dy\Big| + |K_j(x,x_0)|\Big|\int_B a(y)dy\Big| \\
&\le \int_B \frac{C(N)}{\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{N+4N_0}} \frac{|y-x_0|^{c_0}}{|x-x_0|^{d+c_0-1}} \Big\{ \int_{B(x,|x-x_0|)} \frac{V(z)}{|x-z|^{d-1}}dz + \frac{1}{|x-x_0|}\Big\}|a(y)|dy \\
&\quad + \frac{C(N)}{\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{N+4N_0+c_0}} \frac{1}{|x-x_0|^{d-1}} \Big(\int_{B(x,|x-x_0|)} \frac{V(z)}{|x-z|^{d-1}}dz + \frac{1}{|x-x_0|}\Big)\Big(\frac{r}{\rho(x_0)}\Big)^{c_0} \\
&\le \frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^{N}} \left(\frac{1}{\Big(1+\frac{2^{k+2}r}{\rho(x_0)}\Big)^{N_0}} \frac{r^{c_0}}{(2^kr)^{d+c_0-1}} \int_{B(x,|x-x_0|)} \frac{V(z)}{|x-z|^{d-1}}dz + \frac{2^{-kc_0}}{|2^kB|}\right). \qquad (5.42)
\end{aligned}
$$

Here and in what follows, the constants $C(N)$ depend only on N, but may change from line to line. Note that for every $x \in 2^{k+1}B \setminus 2^kB$, one has $B(x,|x-x_0|) \subset B(x, 2^{k+1}r) \subset B(x_0, 2^{k+2}r)$. The fact $V \in RH_{d/2}$, $d/2 \ge q > 1$, and Hölder inequality yield

$$
\begin{aligned}
&\left\| \int_{B(x,|x-x_0|)} \frac{V(z)}{|x-z|^{d-1}}dz \right\|_{L^q(2^{k+1}B\setminus 2^kB, dx)} \\
&\le C(2^{k+1}r)^{1-\frac{2}{d}} \left\{ \int_{2^{k+1}B\setminus 2^kB} \Big(\int_{B(x,2^{k+1}r)} \frac{|V(z)|^{d/2}}{|x-z|^{d-1}}dz\Big)^{\frac{2q}{d}} dx \right\}^{1/q} \\
&\le C(2^kr)^{1-\frac{2}{d}} |2^{k+1}B|^{\frac{1}{q}-\frac{2}{d}} \left\{ \int_{B(z,2^{k+1}r)} dx \int_{B(x_0,2^{k+2}r)} \frac{|V(z)|^{d/2}}{|x-z|^{d-1}}dz \right\}^{2/d} \\
&\le C2^kr|2^kB|^{1/q-1} \int_{B(x_0,2^{k+2}r)} V(z)dz. \qquad (5.43)
\end{aligned}
$$

Combining (5.42), (5.43) and Lemma 1 of [58], we obtain that

$$
\begin{aligned}
&\|R_j(a)\|_{L^q(2^{k+1}B\setminus 2^kB)} \\
&\le \frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^{N}} \left(\frac{r^{c_0}2^kr|2^kB|^{1/q-1}}{(2^kr)^{d+c_0-1}} \frac{1}{\Big(1+\frac{2^{k+2}r}{\rho(x_0)}\Big)^{N_0}} \int_{B(x_0,2^{k+2}r)} V(z)dz + \frac{2^{-kc_0}}{|2^kB|}|2^{k+1}B|^{1/q}\right) \\
&\le \frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^{N}} 2^{-kc_0}|2^kB|^{1/q-1},
\end{aligned}
$$

where $N_0 = \log_2 C_0 + 1$ with C_0 the constant in (5.4). This completes the proof.

$\square$

Proof of Lemma 5.5.7. Note that $r \leq \mathcal{C}_L \rho(x_0)$ since a is a (H^1_L, q)-atom related to the ball $B = B(x_0, r)$; and a is ${\mathcal{C}_L}^{c_0}$ times a generalized (H^1_L, q, c_0)-atom related to the ball $B = B(x_0, r)$ (see Remark 5.2.1). In (5.23), we choose $N = (k_0+1)\theta$. Then, Hölder inequality and Lemma 5.6.6 give

$$\begin{aligned}
& \|(g - g_B) R_j(a)\|_{L^1} \\
= \;& \|(g - g_B) R_j(a)\|_{L^1(2^4 B)} + \sum_{k=4}^{\infty} \|(g - g_B) R_j(a)\|_{L^1(2^{k+1}B \setminus 2^k B)} \\
\leq \;& \|g - g_B\|_{L^{q'}(2^4 B)} \|R_j\|_{L^q \to L^q} \|a\|_{L^q} + \sum_{k=4}^{\infty} \|g - g_B\|_{L^{q'}(2^{k+1}B\setminus 2^k B)} \|R_j(a)\|_{L^q(2^{k+1}B \setminus 2^k B)} \\
\leq \;& C\|g\|_{BMO_{L,\theta}} + \\
& + C \sum_{k=4}^{\infty} (k+1) |2^{k+1}B|^{1/q'} \Big(1 + \frac{2^{k+1} r}{\rho(x)}\Big)^{(k_0+1)\theta} \|g\|_{BMO_{L,\theta}} \frac{1}{\Big(1 + \frac{2^k r}{\rho(x)}\Big)^{(k_0+1)\theta}} 2^{-kc_0} |2^k B|^{1/q - 1} \\
\leq \;& C\|g\|_{BMO_{L,\theta}},
\end{aligned}$$

where $1/q + 1/q' = 1$. Similarly, we also obtain that

$$\begin{aligned}
& \|(f - f_B)(g - g_B) R_j(a)\|_{L^1} \\
= \;& \|(f - f_B)(g - g_B) R_j(a)\|_{L^1(2^4 B)} + \sum_{k=4}^{\infty} \|(f - f_B)(g - g_B) R_j(a)\|_{L^1(2^{k+1}B \setminus 2^k B)} \\
\leq \;& \|f - f_B\|_{L^{2q'}(2^4 B)} \|g - g_B\|_{L^{2q'}(2^4 B)} \|R_j(a)\|_{L^q} + \\
& + \sum_{k=4}^{\infty} \|f - f_B\|_{L^{2q'}(2^{k+1}B)} \|g - g_B\|_{L^{2q'}(2^{k+1}B)} \|R_j(a)\|_{L^q(2^{k+1}B \setminus 2^k B)} \\
\leq \;& C\|f\|_{BMO} \|g\|_{BMO_{L,\theta}},
\end{aligned}$$

which ends the proof.

$\square$

5.7 Some applications

The purpose of this section is to give some applications of the decomposition theorems (Theorem 5.3.1 and Theorem 5.3.2). To be more precise, we give some subspaces of $H^1_L(\mathbb{R}^d)$, *which do not necessarily depend on* b *and* T, such that all commutators $[b, T]$, for $b \in BMO(\mathbb{R}^d)$ and $T \in \mathcal{K}_L$, map continuously these spaces into $L^1(\mathbb{R}^d)$.

Especially, using Theorem 5.3.1 and Theorem 5.3.2, we find the largest subspace $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ of $H^1_L(\mathbb{R}^d)$ so that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Also, it allows to find all functions b in $BMO(\mathbb{R}^d)$ so that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$.

5.7.1 Atomic Hardy spaces related to $b \in BMO(\mathbb{R}^d)$

Definition 5.7.1. *Let $1 < q \leq \infty$, $\varepsilon > 0$ and $b \in BMO(\mathbb{R}^d)$. A function a is called a $(H^1_{L,b}, q, \varepsilon)$-atom related to the ball $B = B(x_0, r)$ if a is a generalized (H^1_L, q, ε)-atom related to the same ball B and*

$$\Big|\int_{\mathbb{R}^d} a(x)(b(x) - b_B)dx\Big| \leq \Big(\frac{r}{\rho(x_0)}\Big)^{\varepsilon}. \tag{5.44}$$

As usual, the space $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ is defined as $\mathbb{H}^{1,q,\varepsilon}_{L,\rm at}(\mathbb{R}^d)$ with generalized (H^1_L, q, ε)-atoms replaced by $(H^1_{L,b}, q, \varepsilon)$-atoms.

Obviously, $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d) \subset \mathbb{H}^{1,q,\varepsilon}_{L,\rm at}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$ and the inclusion is continuous.

Theorem 5.7.1. *Let $1 < q \leq \infty$, $\varepsilon > 0$, $b \in BMO(\mathbb{R}^d)$ and $T \in \mathcal{K}_L$. Then, the commutator $[b, T]$ is bounded from $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Remark 5.7.1. *The space $H^1_b(\mathbb{R}^d)$ which has been considered by Tang and Bi [134] is a strict subspace of $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ in general. As an example, let us take $1 < q \leq \infty$, $\varepsilon > 0$, $L = -\Delta + 1$, and b be a non-constant bounded function, then it is easy to check that the function $f = \chi_{B(0,1)}$ belongs to $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ but not to $H^1_b(\mathbb{R}^d)$. Thus, Theorem 5.7.1 can be seen as an improvement of the main result of [134].*

We should also point out that the authors in [134] proved their main result (see [134], Theorem 3.1) by establishing that

$$\|[b, R_j](a)\|_{L^1} \leq C\|b\|_{BMO}$$

for all H^1_b-atom a. However, as pointed in [19] and [82], such arguments are not sufficient to conclude that $[b, R_j]$ is bounded from $H^1_b(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ in general.

Proof of Theorem 5.7.1. Let a be a $(H^1_{L,b}, q, \varepsilon)$-atom related to the ball $B = B(x_0, r)$. We first prove that $(b - b_B)a$ is $C\|b\|_{BMO}$ times a generalized $(H^1_L, (\widetilde{q} + 1)/2, \varepsilon)$-atom, where $\widetilde{q} \in (1, \infty)$ will be defined later and the positive constant C is independent of b, a. Indeed, one has supp $(b - b_B)a \subset$ supp $a \subset B$. In addition, from Hölder inequality and John-Nirenberg (classical) inequality,

$$\|(b - b_B)a\|_{L^{(\widetilde{q}+1)/2}} \leq \|(b - b_B)\chi_B\|_{L^{\widetilde{q}(\widetilde{q}+1)/(\widetilde{q}-1)}}\|a\|_{L^{\widetilde{q}}} \leq C\|b\|_{BMO}|B|^{(-\widetilde{q}+1)/(\widetilde{q}+1)},$$

where $\widetilde{q} = q$ if $1 < q < \infty$ and $\widetilde{q} = 2$ if $q = \infty$. These together with (5.44) yield that $(b-b_B)a$ is $C\|b\|_{BMO}$ times a generalized $(H^1_L, (\widetilde{q}+1)/2, \varepsilon)$-atom, and thus $\|(b-b_B)a\|_{H^1_L} \leq C\|b\|_{BMO}$.

We now prove that $\mathfrak{S}(a,b)$ belongs to $H^1_L(\mathbb{R}^d)$.

By Theorem 5.3.2, there exist d bounded bilinear operators $\mathfrak{R}_j : H^1_L(\mathbb{R}^d)\times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$, $j = 1, ..., d$, such that

$$[b, R_j](a) = \mathfrak{R}_j(a, b) + R_j(\mathfrak{S}(a, b)),$$

since R_j is linear and belongs to $\mathcal{K}_L$ (see Proposition 5.4.2). Consequently, for every $j = 1, ..., d$, as $R_j \in \mathcal{K}_L$,

$$\begin{aligned}
\|R_j(\mathfrak{S}(a,b))\|_{L^1} &= \|(b-b_B)R_j(a) - R_j((b-b_B)a) - \mathfrak{R}_j(a,b)\|_{L^1}\\
&\leq \|(b-b_B)R_j(a)\|_{L^1} + \|R_j\|_{H^1_L\to L^1}\|(b-b_B)a\|_{H^1_L} + \|\mathfrak{R}_j(a,b)\|_{L^1}\\
&\leq C\|b\|_{BMO}.
\end{aligned}$$

This together with Proposition 5.5.2 prove that $\mathfrak{S}(a,b) \in H^1_L(\mathbb{R}^d)$, and moreover that

$$\|\mathfrak{S}(a,b)\|_{H^1_L} \leq C\|b\|_{BMO}. \tag{5.45}$$

Now, for any $f \in H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$, there exists an expansion $f = \sum_{k=1}^\infty \lambda_k a_k$ where the a_k are $(H^1_{L,b}, q, \varepsilon)$-atoms and $\sum_{k=1}^\infty |\lambda_k| \leq 2\|f\|_{H^{1,q,\varepsilon}_{L,b}}$. Then, the sequence $\{\sum_{k=1}^n \lambda_k a_k\}_{n\geq 1}$ converges to f in $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ and thus in $H^1_L(\mathbb{R}^d)$. Hence, Proposition 5.5.2 implies that the sequence $\Big\{\mathfrak{S}\Big(\sum_{k=1}^n \lambda_k a_k, b\Big)\Big\}_{n\geq 1}$ converges to $\mathfrak{S}(f,b)$ in $L^1(\mathbb{R}^d)$. In addition, by (5.45),

$$\Big\|\mathfrak{S}\Big(\sum_{k=1}^n \lambda_k a_k, b\Big)\Big\|_{H^1_L} \leq \sum_{k=1}^n |\lambda_k| \|\mathfrak{S}(a_k,b)\|_{H^1_L} \leq C\|f\|_{H^{1,q,\varepsilon}_{L,b}}\|b\|_{BMO}.$$

We then use Theorem 5.3.1 and the weak-star convergence in $H^1_L(\mathbb{R}^d)$ (see [83]) to conclude that

$$\begin{aligned}
\|[b,T](f)\|_{L^1} &\leq \|\mathfrak{R}_T(f,b)\|_{L^1} + \|T\|_{H^1_L\to L^1}\|\mathfrak{S}(f,b)\|_{H^1_L}\\
&\leq C\|f\|_{H^1_L}\|b\|_{BMO} + C\|f\|_{H^{1,q,\varepsilon}_{L,b}}\|b\|_{BMO}\\
&\leq C\|f\|_{H^{1,q,\varepsilon}_{L,b}}\|b\|_{BMO},
\end{aligned}$$

which ends the proof.

$\square$

5.7.2 The spaces $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ related to $b \in BMO(\mathbb{R}^d)$

In this section, we find the largest subspace $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ of $H^1_L(\mathbb{R}^d)$ so that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Also, we find all functions b in $BMO(\mathbb{R}^d)$ so that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$.

Definition 5.7.2. *Let b be a non-constant BMO-function. The space $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ consists of all f in $H^1_L(\mathbb{R}^d)$ such that $[b, \mathcal{M}_L](f)(x) = \mathcal{M}_L(b(x)f(\cdot) - b(\cdot)f(\cdot))(x)$ belongs to $L^1(\mathbb{R}^d)$. We equipped $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ with the norm*

$$\|f\|_{\mathcal{H}^1_{L,b}} = \|f\|_{H^1_L}\|b\|_{BMO} + \|[b, \mathcal{M}_L](f)\|_{L^1}.$$

Here, we just consider non-constant functions b in $BMO(\mathbb{R}^d)$ since $[b, T] = 0$ if b is a constant function.

Theorem 5.7.2. *Let b be a non-constant BMO-function. Then, the following statements hold:*

i) For every $T \in \mathcal{K}_L$, the commutator $[b, T]$ is bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

ii) Assume that $\mathcal{X}$ is a subspace of $H^1_L(\mathbb{R}^d)$ such that all commutators of the Riesz transforms are bounded from $\mathcal{X}$ into $L^1(\mathbb{R}^d)$. Then, $\mathcal{X} \subset \mathcal{H}^1_{L,b}(\mathbb{R}^d)$.

iii) $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$ if and only if $b \in BMO^{\log}_L(\mathbb{R}^d)$.

To prove Theorem 5.7.2, we need the following lemma.

Lemma 5.7.1. *Let b be a non-constant BMO-function and $f \in H^1_L(\mathbb{R}^d)$. Then, the following conditions are equivalent:*

i) $f \in \mathcal{H}^1_{L,b}(\mathbb{R}^d)$.

ii) $\mathfrak{S}(f, b) \in H^1_L(\mathbb{R}^d)$.

iii) $[b, R_j](f) \in L^1(\mathbb{R}^d)$ for all $j = 1, ..., d$.

Furthermore, if one of these conditions is satisfied, then

$$\begin{aligned} \|f\|_{\mathcal{H}^1_{L,b}} &= \|f\|_{H^1_L}\|b\|_{BMO} + \|[b, \mathcal{M}_L](f)\|_{L^1} \\ &\approx \|f\|_{H^1_L}\|b\|_{BMO} + \|\mathfrak{S}(f, b)\|_{H^1_L} \\ &\approx \|f\|_{H^1_L}\|b\|_{BMO} + \sum_{j=1}^{d} \|[b, R_j](f)\|_{L^1}, \end{aligned}$$

where the constants are independent of b and f.

Proof. $(i) \Leftrightarrow (ii)$. As $\mathcal{M}_L \in \mathcal{K}_L$ (see Proposition 5.4.3), by Theorem 5.3.1, there is a bounded subbilinear operator $\mathfrak{R} : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that

$$\mathcal{M}_L(\mathfrak{S}(f,b)) - \mathfrak{R}(f,b) \leq |[b, \mathcal{M}_L](f)| \leq \mathcal{M}_L(\mathfrak{S}(f,b)) + \mathfrak{R}(f,b).$$

Consequently, $[b, \mathcal{M}_L](f) \in L^1(\mathbb{R}^d)$ iff $\mathfrak{S}(f,b) \in H^1_L(\mathbb{R}^d)$, moreover,

$$\|f\|_{\mathcal{H}^1_{L,b}} \approx \|f\|_{H^1_L}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1_L}.$$

$(ii) \Leftrightarrow (iii)$. As the Riesz transforms R_j are in $\mathcal{K}_L$ (see Proposition 5.4.2), by Theorem 5.3.2, there are d bounded subbilinear operator $\mathfrak{R}_j : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$, $j = 1, ..., d$, such that

$$[b, R_j](f) = \mathfrak{R}_j(f,b) + R_j(\mathfrak{S}(f,b)).$$

Therefore, $\mathfrak{S}(f,b) \in H^1_L(\mathbb{R}^d)$ iff $[b, R_j](f) \in L^1(\mathbb{R}^d)$ for all $j = 1, ..., d$, moreover,

$$\|f\|_{H^1_L}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1_L} \approx \|f\|_{H^1_L}\|b\|_{BMO} + \sum_{j=1}^d \|[b, R_j](f)\|_{L^1}.$$

$\square$

Proof of Theorem 5.7.2. By Theorem 5.3.1, there is a bounded subbilinear operator $\mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}_T(f,b) \leq |[b, T](f)| \leq |T(\mathfrak{S}(f,b))| + \mathfrak{R}_T(f,b).$$

Applying Lemma 5.7.1 gives for every $f \in \mathcal{H}^1_{L,b}(\mathbb{R}^d)$,

$$\begin{aligned} \|[b,T](f)\|_{L^1} &\leq \|T\|_{H^1_L \to L^1}\|\mathfrak{S}(f,b)\|_{H^1_L} + \|\mathfrak{R}_T(f,b)\|_{L^1} \\ &\leq C\|f\|_{\mathcal{H}^1_{L,b}} + C\|f\|_{H^1_L}\|b\|_{BMO} \leq C\|f\|_{\mathcal{H}^1_{L,b}}. \end{aligned}$$

Therefore, $[b, T]$ is bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. This ends the proof of (i).

The proof of (ii) follows directly from Lemma 5.7.1.

The proof of (iii) follows directly from Theorem 5.3.4 and Lemma 5.7.1. $\square$

5.7.3 Atomic Hardy spaces $H^{\log}_{L,\alpha}(\mathbb{R}^d)$

Definition 5.7.3. *Let $\alpha \in \mathbb{R}$. We say that the function a is a $H^{\log}_{L,\alpha}$-atom related to the ball $B = B(x_0, r)$ if*

i) supp $a \subset B$,

ii) $\|a\|_{L^2} \leq \Big(\log(e + \frac{\rho(x_0)}{r})\Big)^\alpha |B|^{-1/2}$,

iii) $\int_{\mathbb{R}^d} a(x)dx = 0$.

As usual, the space $H^{\log}_{L,\alpha}(\mathbb{R}^d)$ is defined as $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{at}}$ with generalized (H^1_L, q, ε)-atoms replaced by $H^{\log}_{L,\alpha}$-atoms.

Clearly, $H^{\log}_{L,0}(\mathbb{R}^d)$ is just $H^1(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$. Moreover, $H^{\log}_{L,\alpha}(\mathbb{R}^d) \subset H^{\log}_{L,\alpha'}(\mathbb{R}^d)$ for all $\alpha \leq \alpha'$. It should be pointed out that when $L = -\Delta + 1$ and $\alpha \geq 0$, then $H^{\log}_{L,\alpha}(\mathbb{R}^d)$ is just the space of all distributions f such that

$$\int\limits_{\mathbb{R}^d} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\Big(\log(e + \frac{\mathfrak{M}f(x)}{\lambda})\Big)^{\alpha}} dx < \infty$$

for some $\lambda > 0$, moreover (see [81] for the details),

$$\|f\|_{H^{\log}_{L,\alpha}} \approx \inf \left\{ \lambda > 0 : \int\limits_{\mathbb{R}^d} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\Big(\log(e + \frac{\mathfrak{M}f(x)}{\lambda})\Big)^{\alpha}} dx \leq 1 \right\}.$$

Theorem 5.7.3. *For every $T \in \mathcal{K}_L$ and $b \in BMO(\mathbb{R}^d)$, the commutator $[b, T]$ is bounded from $H^{\log}_{L,-1}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Proof. Let a be a $H^{\log}_{L,-1}$-atom related to the ball $B = B(x_0, r)$. Let us first prove that $(b - b_B)a \in H^1_L(\mathbb{R}^d)$. As $H^1_L(\mathbb{R}^d)$ is the dual of $VMO_L(\mathbb{R}^d)$ (see Theorem 5.5.1), it is sufficient to show that for every $g \in C^\infty_c(\mathbb{R}^d)$,

$$\|(b - b_B)ag\|_{L^1} \leq C\|b\|_{BMO}\|g\|_{BMO_L}.$$

Indeed, using the estimate $|g_B| \leq C \log\Big(e + \frac{\rho(x_0)}{r}\Big)\|g\|_{BMO_L}$ (see Lemma 2 of [45]), Hölder inequality and classical John-Nirenberg inequality give

$$\begin{aligned}
\|(b - b_B)ag\|_{L^1} &\leq \|(g - g_B)(b - b_B)a\|_{L^1} + |g_B|\|(b - b_B)a\|_{L^1}\\
&\leq \|(g - g_B)\chi_B\|_{L^4}\|(b - b_B)\chi_B\|_{L^4}\|a\|_{L^2} +\\
&\quad + C\log\Big(e + \frac{\rho(x_0)}{r}\Big)\|g\|_{BMO_L}\|(b - b_B)\chi_B\|_{L^2}\|a\|_{L^2}\\
&\leq C\|b\|_{BMO}\|g\|_{BMO_L},
\end{aligned}$$

which proves that $(b - b_B)a \in H^1_L(\mathbb{R}^d)$, moreover, $\|(b - b_B)a\|_{H^1_L} \leq C\|b\|_{BMO}$.

Similarly to the proof of Theorem 5.7.1, we also obtain that

$$\|\mathfrak{S}(f, b)\|_{H^1_L} \leq C\|f\|_{H^{\log}_{L,-1}}\|b\|_{BMO}$$

for all $f \in H^{\log}_{L,-1}(\mathbb{R}^d)$. Therefore, Theorem 5.3.1 allows to conclude that

$$\|[b, T](f)\|_{L^1} \leq C\|f\|_{H^{\log}_{L,-1}}\|b\|_{BMO},$$

which ends the proof. $\square$

As a consequence of the proof of Theorem 5.7.3, we obtain the following result.

Proposition 5.7.1. *Let $T \in \mathcal{K}_L$. Then, $\mathfrak{T}(f,b) := [b,T](f)$ is a bounded subbilinear operator from $H^{\log}_{L,-1}(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

5.7.4 The Hardy-Sobolev space $H^{1,1}_L(\mathbb{R}^d)$

Following Hofmann et al. [65], we say that f belongs to the (inhomogeneous) Hardy-Sobolev $H^{1,1}_L(\mathbb{R}^d)$ if $f, \partial_{x_1} f, ..., \partial_{x_d} f \in H^1_L(\mathbb{R}^d)$. Then, the norm on $H^{1,1}_L(\mathbb{R}^d)$ is defined by

$$\|f\|_{H^{1,1}_L} = \|f\|_{H^1_L} + \sum_{j=1}^d \|\partial_{x_j} f\|_{H^1_L}.$$

It should be pointed out that the authors in [65] proved that the space $H^{1,1}_{-\Delta}(\mathbb{R}^d)$ is just the classical (inhomogeneous) Hardy-Sobolev $H^{1,1}(\mathbb{R}^d)$ (see for example [6]), and can be identified with the (inhomogeneous) Triebel-Lizorkin space $F^{1,2}_1(\mathbb{R}^d)$ (see [79]). More precisely, f belongs to $H^{1,1}(\mathbb{R}^d)$ if and only if

$$\mathcal{W}_\psi(f) = \left\{ \sum_I \sum_{\sigma \in \mathcal{E}} |\langle f, \psi^\sigma_I \rangle|^2 (1 + |I|^{-1/d})^2 |I|^{-1} \chi_I \right\}^{1/2} \in L^1(\mathbb{R}^d),$$

moreover,

$$\|f\|_{H^{1,1}} \approx \|\mathcal{W}_\psi(f)\|_{L^1}. \tag{5.46}$$

Here $\{\psi^\sigma\}_{\sigma \in \mathcal{E}}$ is the wavelet as in Section 4.

Theorem 5.7.4. *Let $L = -\Delta + 1$. Then, for every $T \in \mathcal{K}_L$ and $b \in BMO(\mathbb{R}^d)$, the commutator $[b,T]$ is bounded from $H^{1,1}_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Remark 5.7.2. *When $L = -\Delta + 1$, we can define $\mathfrak{H}(f) = f - \varphi * f$ instead of $\mathfrak{H}(f) = \sum_{n,k} (\psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f))$ as in Section 5.5. In other words, the bilinear operator $\mathfrak{S}$ in Theorem 5.3.1 and Theorem 5.3.2 can be defined as $\mathfrak{S}(f,g) = -\Pi(f - \varphi * f, g)$. As $\mathfrak{H}(f) = f - \varphi * f$, it is easy to see that*

$$\partial_{x_j}(\mathfrak{H}(f)) = \mathfrak{H}(\partial_{x_j} f).$$

Here and in what follows, for any dyadic cube $Q = Q(y,r) := \{x \in \mathbb{R}^d : -r \leq x_j - y_j < r \text{ for all } j = 1, ..., d\}$, we denote by B_Q the ball

$$B_Q := \Big\{ x \in \mathbb{R}^d : |x - y| < 2\sqrt{d} r \Big\}.$$

To prove Theorem 5.7.4, we need the following lemma.

Lemma 5.7.2. *Let $L = -\Delta + 1$. Then, the bilinear operator Π maps continuously $H^{1,1}(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $H^1_L(\mathbb{R}^d)$.*

Proof. Note that $\rho(x) = 1$ for all $x \in \mathbb{R}^d$ since $V(x) \equiv 1$. We first claim that there exists a constant $C > 0$ such that

$$\|(1+|I|^{-1/d})^{-1}(\psi_I^\sigma)^2\|_{H^1_L} \leq C \tag{5.47}$$

for all dyadic $I = Q[x_0, r)$ and $\sigma \in \mathcal{E}$. Indeed, it follows from Remark 5.5.2 that supp $(1+|I|^{-1/d})^{-1}(\psi_I^\sigma)^2 \subset cI \subset cB_I$, and it is clear that $\|(1+|I|^{-1/d})^{-1}(\psi_I^\sigma)^2\|_{L^\infty} \leq |I|^{-1}\|\psi\|_{L^\infty} \leq C|cB_I|^{-1}$. In addition,

$$\Big|\int_{\mathbb{R}^d} (1+|I|^{-1/d})^{-1}(\psi_I^\sigma(x))^2 dx\Big| = (1+|I|^{-1/d})^{-1} \leq C\frac{r}{\rho(x_0)}.$$

Hence, $(1+|I|^{-1/d})^{-1}(\psi_I^\sigma)^2$ is C times a generalized $(H^1_L, \infty, 1)$-atom related to the ball cB_I, and thus (5.47) holds.

Now, for every $(f, g) \in H^{1,1}(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, (5.47) implies that

$$\begin{aligned}
\|\Pi(f,g)\|_{H^1_L} &= \|\sum_I \sum_{\sigma\in\mathcal{E}} \langle f, \psi_I^\sigma\rangle\langle g, \psi_I^\sigma\rangle(\psi_I^\sigma)^2\|_{H^1_L}\\
&\leq C\sum_I \sum_{\sigma\in\mathcal{E}} \Big(|\langle f, \psi_I^\sigma\rangle|(1+|I|^{-1/d})\Big)|\langle g, \psi_I^\sigma\rangle|\\
&\leq C\|\mathcal{W}_\psi(f)\|_{L^1}\|g\|_{\dot F^{0,2}_\infty}\\
&\leq C\|f\|_{H^{1,1}}\|g\|_{BMO},
\end{aligned}$$

where we have used the fact that $BMO(\mathbb{R}^d) \equiv \dot F^{0,2}_\infty(\mathbb{R}^d)$ is the dual of $H^1(\mathbb{R}^d) \equiv \dot F^{0,2}_1(\mathbb{R}^d)$, we refer the reader to [49] for more details.

□

Proof of Theorem 5.7.4. Let $(f, b) \in H^{1,1}_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$. Thanks to Lemma 5.7.2, Remark 5.7.2 and Lemma 5.5.1, we get

$$\begin{aligned}
\|\mathfrak{S}(f,b)\|_{H^1_L} &\leq C\|\mathfrak{H}(f)\|_{H^{1,1}}\|b\|_{BMO}\\
&\leq C\|f\|_{H^{1,1}_L}\|b\|_{BMO}.
\end{aligned}$$

Then we use Theorem 5.3.1 to conclude that

$$\begin{aligned}
\|[b,T](f)\|_{L^1} &\leq \|\mathfrak{R}_T(f,b)\|_{L^1} + \|T\|_{H^1_L\to L^1}\|\mathfrak{S}(f,b)\|_{H^1_L}\\
&\leq C\|f\|_{H^{1,1}_L}\|b\|_{BMO},
\end{aligned}$$

which ends the proof.

□

As a consequence of the proof of Theorem 5.7.4, we obtain the following result.

Proposition 5.7.2. *Let $L = -\Delta + 1$ and $T \in \mathcal{K}_L$. Then, $\mathfrak{T}(f, b) := [b, T](f)$ is a bounded subbilinear operator from $H^{1,1}_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Chapter 6

Bilinear decompositions for the product space $H^1_L \times BMO_L$

Ce chapitre est une prépublication (soumise).

Résumé

Dans cet article, nous améliorons un résultat récent de Li et Peng sur les produits de fonctions dans $H^1_L(\mathbb{R}^d)$ et $BMO_L(\mathbb{R}^d)$, où $L = -\Delta + V$ est un opérateur Schrödinger avec V satisfaisant une inégalité Hölder inverse appropriée. Plus précisément, nous prouvons que ces produits peuvent être écrit comme la somme de deux opérateurs bilinéaires continus, l'un de $H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$ à valeurs $L^1(\mathbb{R}^d)$, l'autre de $H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$ à valeurs $H^{\log}(\mathbb{R}^d)$, où l'espace $H^{\log}(\mathbb{R}^d)$ est l'ensemble des distributions f dont la fonction "grand-maximale" $\mathfrak{M}f$ satisfait

$$\int_{\mathbb{R}^d} \frac{|\mathfrak{M}f(x)|}{\log(e + |\mathfrak{M}f(x)|) + \log(e + |x|)} dx < \infty.$$

6.1 Introduction

Products of functions in H^1 and BMO have been firstly considered by Bonami, Iwaniec, Jones and Zinsmeister in [15]. Such products make sense as distributions, and can be written as the sum of an integrable function and a function in a weighted Hardy-Orlicz space. To be more precise, for $f \in H^1(\mathbb{R}^d)$ and $g \in BMO(\mathbb{R}^d)$, we define the product (in the distribution sense) $f \times g$ as the distribution whose action on the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$\langle f \times g, \varphi \rangle := \langle \varphi g, f \rangle \,, \tag{6.1}$$

where the second bracket stands for the duality bracket between $H^1(\mathbb{R}^d)$ and its dual $BMO(\mathbb{R}^d)$. It is then proven in [15] that

$$f \times g \in L^1(\mathbb{R}^d) + H^{\Xi}_{\sigma}(\mathbb{R}^d). \tag{6.2}$$

Here $H^{\Xi}_{\sigma}(\mathbb{R}^d)$ is the weighted Hardy-Orlicz space related to the Orlicz function

$$\Xi(t) := \frac{t}{\log(e+t)} \tag{6.3}$$

and with weight $\sigma(x) := \frac{1}{\log(e+|x|)}$.

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In [45] and [46], Dziubański et al. introduced two kinds of function spaces associated with L. One is the Hardy space $H^1_L(\mathbb{R}^d)$, the other is the space $BMO_L(\mathbb{R}^d)$. They established in [45] that the dual space of $H^1_L(\mathbb{R}^d)$ is just $BMO_L(\mathbb{R}^d)$. Unfortunately, as for the classical spaces $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$, the pointwise products fg of functions $f \in H^1_L(\mathbb{R}^d)$ and functions $g \in BMO_L(\mathbb{R}^d)$ maybe not integrable. However, similarly to the classical setting, Li and Peng showed in [91] that such products can be defined in the sense of distributions which action on the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is

$$\langle f \times g, \varphi \rangle := \langle \varphi g, f \rangle, \tag{6.4}$$

where the second bracket stands for the duality bracket between $H^1_L(\mathbb{R}^d)$ and its dual $BMO_L(\mathbb{R}^d)$. Moreover, they proved that $f \times g$ can be written as the sum of two distributions, one in $L^1(\mathbb{R}^d)$, the other in $H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ the weighted Hardy-Orlicz space associated with L related to the Orlicz function $\Xi(t) \equiv \frac{t}{\log(e+t)}$ and the weight $\sigma(x) \equiv \frac{1}{\log(e+|x|)}$, see Definition 6.2.3.

More precisely, in [91], the authors proved the following.

Theorem 6.1.1. *For each $f \in H^1_L(\mathbb{R}^d)$, there are two bounded linear operators $L_f : BMO_L(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $H_f : BMO_L(\mathbb{R}^d) \to H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ such that for every $g \in BMO_L(\mathbb{R}^d)$, we have*

$$f \times g = L_f(g) + H_f(g) \tag{6.5}$$

and the uniform bound

$$\|L_f(g)\|_{L^1} + \|H_f(g)\|_{H^{\Xi}_{L,\sigma}} \leq C\|f\|_{H^1_L}\|g\|_{BMO^+_L}, \tag{6.6}$$

where $\|g\|_{BMO^+_L} = \|g\|_{BMO_L} + |g_{\mathbb{B}}|$, $g_{\mathbb{B}}$ denotes the mean value of g over the unit ball $\mathbb{B}$.

Our main theorem is as follows.

Theorem 6.1.2. *There are two bounded bilinear operators $S_L : H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $T_L : H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \to H^{\log}(\mathbb{R}^d)$ such that for every $(f,g) \in H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$, we have*

$$f \times g = S_L(f,g) + T_L(f,g) \tag{6.7}$$

and the uniform bound

$$\|S_L(f,g)\|_{L^1} + \|T_L(f,g)\|_{H^{\log}} \leq C\|f\|_{H^1_L}\|g\|_{BMO_L}. \tag{6.8}$$

Here $H^{\log}(\mathbb{R}^d)$ is a new kind of Hardy-Orlicz space consisting of all distributions f such that $\int_{\mathbb{R}^d} \frac{\mathfrak{M}f(x)}{\log(e+\mathfrak{M}f(x))+\log(e+|x|)}dx < \infty$ with the norm

$$\|f\|_{H^{\log}} = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^d} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e + \frac{\mathfrak{M}f(x)}{\lambda}\right) + \log(e+|x|)}dx \leq 1\right\}.$$

Recall that the grand maximal operator $\mathfrak{M}$ is defined by

$$\mathfrak{M}f(x) = \sup_{\phi \in \mathcal{A}} \sup_{|y-x|<t} |f * \phi_t(y)|, \tag{6.9}$$

where $\mathcal{A} = \{\phi \in \mathcal{S}(\mathbb{R}^d) : |\phi(x)| + |\nabla\phi(x)| \leq (1+|x|^2)^{-(d+1)}\}$ and $\phi_t(\cdot) := t^{-d}\phi(t^{-1}\cdot)$.

Note that $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ with continuous embedding, see Section 6.3. Compared with the main result of [91] (Theorem 6.1.1), our main result makes an essential improvement in two directions. The first one consists in proving that the space $H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ can be replaced by a smaller space $H^{\log}(\mathbb{R}^d)$. Secondly, we give the bilinear decomposition (6.7) for the product space $H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$ instead of the linear decomposition (6.5) depending on $f \in H^1_L(\mathbb{R}^d)$. Moreover, we just need the BMO_L-norm (see (6.8)) instead of the BMO^+_L-norm as in (6.6).

In applications to nonlinear PDEs, the distribution $f \times g \in \mathcal{S}'(\mathbb{R}^d)$ is used to justify weak continuity properties of the pointwise product fg. It is therefore important to recover fg from the action of the distribution $f \times g$ on the test functions. An idea that naturally comes to mind is to look at the mollified distributions

$$(f \times g)_\epsilon = (f \times g) * \phi_\epsilon, \tag{6.10}$$

and let $\epsilon \to 0$. Here $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \phi(x)dx = 1$.

In the classical setting of $f \in H^1(\mathbb{R}^d)$ and $g \in BMO(\mathbb{R}^d)$, Bonami et al. proved in [15] that the limit (6.10) exists and equals fg almost everywhere. An analogous result is also true for the Schrödinger setting. Namely, the following is true.

Theorem 6.1.3. *Let $f \in H^1_L(\mathbb{R}^d)$ and $g \in BMO_L(\mathbb{R}^d)$. Then, for almost every $x \in \mathbb{R}^d$,*

$$\lim_{\epsilon \to 0} (f \times g)_\epsilon(x) = f(x)g(x).$$

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line.

The paper is organized as follows. In Section 2, we present some notations and preliminaries about Hardy type spaces associated with L. Section 3 is devoted to prove that $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ with continuous embedding. Finally, the proofs of Theorem 6.1.2 and Theorem 6.1.3 are given in Section 4.

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6.2 Some preliminaries and notations

In this paper, we consider the Schrödinger differential operator

$$L = -\Delta + V$$

on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$. As in the works of Dziubański et al [45, 46], we always assume that V belongs to the reverse Hölder class $RH_{d/2}$. Recall that a nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q, $1 < q < \infty$, if there exists $C > 0$ such that

$$\Big(\frac{1}{|B|} \int\limits_B (V(x))^q dx\Big)^{1/q} \leq \frac{C}{|B|} \int\limits_B V(x)dx$$

holds for every balls B in $\mathbb{R}^d$.

Let $\{T_t\}_{t>0}$ be the semigroup generated by L and $T_t(x,y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^d} T_t(x,y) f(y) dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

We say that a function $f \in L^2(\mathbb{R}^d)$ belongs to the space $\mathbb{H}^1_L(\mathbb{R}^d)$ if

$$\|f\|_{\mathbb{H}^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty,$$

where $\mathcal{M}_L f(x) := \sup_{t>0} |T_t f(x)|$ for all $x \in \mathbb{R}^d$. The space $H^1_L(\mathbb{R}^d)$ is then defined as the completion of $\mathbb{H}^1_L(\mathbb{R}^d)$ with respect to this norm.

In [45] it was shown that the dual of $H^1_L(\mathbb{R}^d)$ can be identified with the space $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO(\mathbb{R}^d)$ with

$$\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x) \leq r} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy < \infty,$$

where ρ is the auxiliary function defined as in [125], that is,

$$\rho(x) = \sup \Big\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y) dy \leq 1 \Big\}, \tag{6.11}$$

$x \in \mathbb{R}^d$. Clearly, $0 < \rho(x) < \infty$ for all $x \in \mathbb{R}^d$, and thus $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}} \mathcal{B}_n$, where the sets $\mathcal{B}_n$ are defined by

$$\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2}\}. \tag{6.12}$$

The following proposition is due to Shen [125].

Proposition 6.2.1 (see [125], Lemma 1.4)**.** *There exist $C_0 > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$C_0^{-1} \rho(x) \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C_0 \rho(x) \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}.$$

Here and in what follows, we denote by $\mathcal{C}_L$ the L-constant

$$\mathcal{C}_L = 8.9^{k_0} C_0 \tag{6.13}$$

where k_0 and C_0 are defined as in Proposition 7.2.1.

Definition 6.2.1. *Given $1 < q \leq \infty$. A function a is called a (H^1_L, q)-atom related to the ball $B(x_0, r)$ if $r \leq \mathcal{C}_L \rho(x_0)$ and*

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q - 1}$,

iii) if $r \leq \frac{1}{\mathcal{C}_L} \rho(x_0)$ then $\int_{\mathbb{R}^d} a(x) dx = 0$.

The following atomic characterization of $H_L^1(\mathbb{R}^d)$ is due to Dziubański and Zienkiewicz [46].

Theorem A. *Let $1 < q \leq \infty$. A function f is in $H_L^1(\mathbb{R}^d)$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H_L^1, q)-atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists $C > 1$ such that for every $f \in H_L^1(\mathbb{R}^d)$, we have*

$$C^{-1}\|f\|_{H_L^1} \leq \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\} \leq C\|f\|_{H_L^1}.$$

Let $1 \leq q < \infty$. A nonnegative locally integrable function w belongs to the *Muckenhoupt class* A_q, say $w \in A_q$, if there exists a positive constant C so that

$$\frac{1}{|B|}\int_B w(x)dx\Big(\frac{1}{|B|}\int_B (w(x))^{-1/(q-1)}dx\Big)^{q-1} \leq C, \quad \text{if } 1 < q < \infty, \tag{6.14}$$

and

$$\frac{1}{|B|}\int_B w(x)dx \leq C \operatorname*{ess-inf}_{x\in B} w(x), \quad \text{if } q = 1, \tag{6.15}$$

for all balls B in $\mathbb{R}^d$. We say that $w \in A_\infty$ if $w \in A_q$ for some $q \in [1, \infty)$.

Remark 6.2.1. *The weight $\sigma(x) \equiv \frac{1}{\log(e+|x|)}$ belongs to the class A_1.*

It is well known that $w \in A_p$, $1 \leq p < \infty$, implies $w \in A_q$ for all $q > p$. For a measurable set E, we note $w(E) = \int_E w(x)dx$ its weighted measure.

Definition 6.2.2. *Let $0 < p \leq 1$. A function Φ is called a growth function of order p if it satisfies the following properties:*

i) The function Φ is a Orlicz function, that is, Φ is a nondecreasing function with $\Phi(t) > 0, t > 0$, $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

ii) The function Φ is of lower type p, that is, there exists a constant $C > 0$ such that for every $s \in (0, 1]$ and $t > 0$,

$$\Phi(st) \leq Cs^p\Phi(t).$$

iii) The function Φ is of upper type 1, that is, there exists a constant $C > 0$ such that for every $s \in [1, \infty)$ and $t > 0$,

$$\Phi(st) \leq Cs\Phi(t).$$

We will also say that Φ is a *growth function* whenever it is a *growth function* of some order $p < 1$.

Remark 6.2.2. *i) Let Φ be a growth function. Then, there exists a constant $C > 0$ such that*

$$\Phi\Big(\sum_{j=1}^{\infty} t_j\Big) \leq C \sum_{j=1}^{\infty} \Phi(t_j)$$

for every sequence $\{t_j\}_{j\geq 1}$ of nonnegative real numbers. See Lemma 4.1 of [81].

ii) The function $\Xi(t) \equiv \frac{t}{\log(e+t)}$ is a growth function of order p for any $p \in (0,1)$.

Now, let us define weighted Hardy-Orlicz spaces associated with L.

Definition 6.2.3. *Given $w \in A_\infty$ and Φ a growth function. We say that a function $f \in L^2(\mathbb{R}^d)$ belongs to $\mathbb{H}^{\Phi}_{L,w}(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} \Phi(\mathcal{M}_L f(x)) w(x) dx < \infty$. The space $H^{\Phi}_{L,w}(\mathbb{R}^d)$ is defined as the completion of $\mathbb{H}^{\Phi}_{L,w}(\mathbb{R}^d)$ with respect to the norm*

$$\|f\|_{H^{\Phi}_{L,w}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^d} \Phi\Big(\frac{\mathcal{M}_L f(x)}{\lambda}\Big) w(x) dx \leq 1\right\}.$$

Remark that when $w(x) \equiv 1$ and $\Phi(t) \equiv t$, the space $H^{\Phi}_{L,w}(\mathbb{R}^d)$ is just $H^1_L(\mathbb{R}^d)$. We refer the reader to the recent work of D. Yang and S. Yang [142] for a complete study of the theory of weighted Hardy-Orlicz spaces associated with operators.

6.3 The inclusion $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$

The purpose of this section is to establish the following embedding.

Proposition 6.3.1. *$H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ and the inclusion is continuous.*

Recall (see [81]) that the weighted Hardy-Orlicz space $H^{\Xi}_{\sigma}(\mathbb{R}^d)$ is defined as the space of all distributions f such that $\int_{\mathbb{R}^d} \frac{\mathfrak{M}f(x)}{\log(e+\mathfrak{M}f(x))} \frac{1}{\log(e+|x|)} dx < \infty$ with the norm

$$\|f\|_{H^{\Xi}_{\sigma}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^d} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e + \frac{\mathfrak{M}f(x)}{\lambda}\right)} \frac{1}{\log(e+|x|)} dx \leq 1\right\}.$$

Clearly, $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{\sigma}(\mathbb{R}^d)$ and the inclusion is continuous. Consequently, *the proof of Proposition 6.3.1* can be reduced to showing that for every $f \in H^{\Xi}_{\sigma}(\mathbb{R}^d)$,

$$\|f\|_{H^{\Xi}_{L,\sigma}} \leq C \|f\|_{H^{\Xi}_{\sigma}}. \tag{6.16}$$

Let $1 < q \leq \infty$. Recall (see [81]) that a function a is called a (H^{Ξ}_{σ}, q)-atom related to the ball B if

i) supp $a \subset B$,
ii) $\|a\|_{L^q_\sigma} \le \sigma(B)^{1/q}\Xi^{-1}(\sigma(B)^{-1})$, where Ξ^{-1} is the inverse function of Ξ,
iii) $\int_{\mathbb{R}^d} a(x)dx = 0$.

In order to prove Proposition 6.3.1, we need the following lemma.

Lemma 6.3.1. *Let $1 < q < \infty$. Then,*

$$\int\limits_{\mathbb{R}^d} \Xi(\mathcal{M}_L f(x))\sigma(x)dx \le C\sigma(B)\Xi(\sigma(B)^{-1/q}\|f\|_{L^q_\sigma}) \tag{6.17}$$

for every f multiples of (H^{Ξ}_σ, q)-atom related to the ball $B = B(x_0, r)$,

To prove Lemma 6.3.1, let us recall the following.

Lemma 6.3.2 (see [91], Lemma 2)**.** *Let $V \in RH_{d/2}$. Then, there exists $\delta > 0$ depends only on L, such that for every $|y - z| < |x - y|/2$ and $t > 0$, we have*

$$|T_t(x, y) - T_t(x, z)| \le C\Big(\frac{|y - z|}{\sqrt{t}}\Big)^{\delta} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{t}} \le C\frac{|y - z|^{\delta}}{|x - y|^{d+\delta}}.$$

Proof of Lemma 6.3.1. First, note that $\sigma \in A_1$ and Ξ is a *growth function* of order p for any $p \in (0, 1)$, see Remark 6.2.1 and Remark 6.2.2. Denote by $\mathcal{M}$ the classical Hardy-Littlewood maximal operator. Then, the estimate $\mathcal{M}_L f \le C\mathcal{M}f$, the L^q_σ-boundedness of $\mathcal{M}$ and Hölder inequality give

$$\begin{aligned}
&\int\limits_{B(x_0,2r)} \Xi(\mathcal{M}_L f(x))\sigma(x)dx \\
\le\ & C \int\limits_{B(x_0,2r)} \Xi(\mathcal{M}f(x) + \sigma(B)^{-1/q}\|f\|_{L^q_\sigma})\sigma(x)dx \\
\le\ & C \int\limits_{B(x_0,2r)} \Big(\frac{\mathcal{M}f(x) + \sigma(B)^{-1/q}\|f\|_{L^q_\sigma}}{\sigma(B)^{-1/q}\|f\|_{L^q_\sigma}}\Big)\Xi(\sigma(B)^{-1/q}\|f\|_{L^q_\sigma})\sigma(x)dx \\
\le\ & C\sigma(B)\Xi(\sigma(B)^{-1/q}\|f\|_{L^q_\sigma}),
\end{aligned} \tag{6.18}$$

where we used the facts that $t \mapsto \frac{\Xi(t)}{t}$ is nonincreasing and $\sigma(B(x_0, 2r)) \le C\sigma(B)$.

Let $x \notin B(x_0, 2r)$ and $t > 0$. By Lemma 6.3.2 and (6.14),

$$\begin{aligned}
|T_t f(x)| = \Big|\int\limits_{\mathbb{R}^d} T_t(x, y)f(y)dy\Big| &= \Big|\int\limits_{B} (T_t(x, y) - T_t(x, x_0))f(y)dy\Big| \\
&\le C\int\limits_{B} \frac{|y - x_0|^{\delta}}{|x - x_0|^{d+\delta}}|f(y)|dy \\
&\le C\sigma(B)^{-1/q}\|f\|_{L^q_\sigma}\frac{r^{d+\delta}}{|x - x_0|^{d+\delta}}
\end{aligned}$$

Therefore, as Ξ is of lower type $\frac{2d+\delta}{2(d+\delta)} < 1$,

$$
\begin{aligned}
&\int\limits_{(B(x_0,2r))^c} \Xi(\mathcal{M}_L f(x))\sigma(x)dx \\
&\leq \; C\Xi(\sigma(B)^{-1/q}\|f\|_{L^q_\sigma}) \int\limits_{(B(x_0,2r))^c} \Big(\frac{r^{d+\delta}}{|x-x_0|^{d+\delta}}\Big)^{\frac{2d+\delta}{2(d+\delta)}}\sigma(x)dx \\
&\leq \; C\sigma(B)\Xi(\sigma(B)^{-1/q}\|f\|_{L^q_\sigma}), \qquad (6.19)
\end{aligned}
$$

where we used (see [54], page 412)

$$
\int\limits_{(B(x_0,2r))^c} \frac{r^{d+\delta/2}}{|x-x_0|^{d+\delta/2}}\sigma(x)dx \leq C\sigma(B(x_0,2r)) \leq C\sigma(B).
$$

Then, (6.17) follows from (6.18) and (6.19). This completes the proof.

□

Proof of Proposition 6.3.1. As mentioned above, it is sufficient to show that

$$
\|f\|_{H^\Xi_{L,\sigma}} \leq C\|f\|_{H^\Xi_\sigma}
$$

for every $f \in H^\Xi_\sigma(\mathbb{R}^d)$. By Theorem 3.1 of [81], there are multiples of $(H^\Xi_\sigma, 2)$-atoms b_j, $j = 1, 2, ...,$ related to balls B_j such that $f = \sum_{j=1}^\infty b_j$ and

$$
\Lambda_2(\{b_j\}) \leq C\|f\|_{H^\Xi_\sigma}, \qquad (6.20)
$$

where

$$
\Lambda_2(\{b_j\}) := \inf\left\{\lambda > 0 : \sum_{j=1}^\infty \sigma(B_j)\Xi\Big(\frac{\sigma(B_j)^{-1/2}\|b_j\|_{L^2_\sigma}}{\lambda}\Big) \leq 1\right\}.
$$

On the other hand, the estimate $\mathcal{M}_L f \leq \sum_{j=1}^\infty \mathcal{M}_L(b_j)$, Remark 6.2.2 and Lemma 6.3.1 give

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} \Xi\Big(\frac{\mathcal{M}_L f(x)}{\Lambda_2(\{b_j\})}\Big)\sigma(x)dx \;&\leq\; C\sum_{j=1}^\infty \int\limits_{\mathbb{R}^d} \Xi\Big(\frac{\mathcal{M}_L(b_j)(x)}{\Lambda_2(\{b_j\})}\Big)\sigma(x)dx \\
&\leq\; C\sum_{j=1}^\infty \sigma(B_j)\Xi\Big(\frac{\sigma(B_j)^{-1/2}\|b_j\|_{L^q_\sigma}}{\Lambda_2(\{b_j\})}\Big) \\
&\leq\; C,
\end{aligned}
$$

which implies that $\|f\|_{H^\Xi_{L,\sigma}} \leq C\Lambda_2(\{b_j\})$. Therefore, (6.20) yields

$$
\|f\|_{H^\Xi_{L,\sigma}} \leq C\|f\|_{H^\Xi_\sigma},
$$

which completes the proof of Proposition 6.3.1.

□

6.4 Proof of Theorem 6.1.2 and Theorem 6.1.3

Let $P(x) = (4\pi)^{-d/2}e^{-|x|^2/4}$ be the Gauss function. For $n \in \mathbb{Z}$, following [46], the space $h^1_n(\mathbb{R}^d)$ denotes the space of all integrable functions f such that

$$\mathcal{M}_n f(x) = \sup_{0<t<2^{-n}} |P_{\sqrt{t}} * f(x)| = \sup_{0<t<2^{-n}} \Big| \int_{\mathbb{R}^d} p_t(x,y) f(y) dy \Big| \in L^1(\mathbb{R}^d),$$

where the kernel p_t is given by $p_t(x,y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}$. We equipped this space with the norm $\|f\|_{h^1_n} := \|\mathcal{M}_n f\|_{L^1}$.

For convenience of the reader, we list here some lemmas used in our proofs.

Lemma 6.4.1 (see [46], Lemma 2.3). *There exists a constant $C > 0$ and a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$, $n \in \mathbb{Z}, k = 1, 2, ...,$ such that $x_{n,k} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B_{n,k}$, and*

$$card\,\{(n', k') : B(x_{n,k}, R2^{-n/2}) \cap B(x_{n',k'}, R2^{-n/2}) \neq \emptyset\} \leq R^C$$

for all n, k and $R \geq 2$.

Lemma 6.4.2 (see [46], Lemma 2.5). *There are nonnegative C^∞-functions $\psi_{n,k}$, $n \in \mathbb{Z}, k = 1, 2, ...,$ supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that*

$$\sum_{n,k} \psi_{n,k} = 1 \quad \textit{and} \quad \|\nabla \psi_{n,k}\|_{L^\infty} \leq C 2^{n/2}.$$

Lemma 6.4.3 (see (4.7) in [46]). *For every $f \in H^1_L(\mathbb{R}^d)$, we have*

$$\sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \leq C \|f\|_{H^1_L}.$$

In this section, we fix a non-negative function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with supp $\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. Then, we define the linear operator $\mathfrak{H}$ by

$$\mathfrak{H}(f) = \sum_{n,k} \Big(\psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f) \Big).$$

In order to prove Theorem 6.1.2, we need two key lemmas.

Lemma 6.4.4. *The operator $\mathfrak{H}$ maps continuously $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.*

The proof of Lemma 6.4.4 can be found in [82] (see Lemma 5.1 of [82]).

Lemma 6.4.5. *There exists a constant $C = C(\varphi, d) > 0$ such that for all $(n, k) \in \mathbb{Z}\times\mathbb{Z}^+$, $g \in BMO_L(\mathbb{R}^d)$ and $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x_{n,k}, 2^{1-n/2})$, we have*

$$\Big\|(\varphi_{2^{-n/2}} * f)g\Big\|_{H^1_L} \leq C\|f\|_{h^1_n}\|g\|_{BMO_L}.$$

To prove Lemma 6.4.5, we need the following.

Lemma 6.4.6 (see [82], Lemma 6.5)**.** *Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Suppose that $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x, 2^{1-n/2})$. Then, there are (H^1_L, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h^1_n}$$

with a positive constant C independent of n and f.

Here and in what follows, for any B a ball in $\mathbb{R}^d$ and f a locally integrable function, we denote by f_B the average of f on B.

Proof of Lemma 6.4.5. As $x_{n,k} \in \mathcal{B}_n$, it follows from Lemma 7.3.3 that there are $(H^1_L, 2)$-atoms $a^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j) \subset B(x_{n,k}, 2^{2-n/2})$ such that

$$f = \sum_j \lambda^{n,k}_j a^{n,k}_j \quad \text{and} \quad \sum_j |\lambda^{n,k}_j| \leq C\|f\|_{h^1_n}, \tag{6.21}$$

where the positive constant C is independent of f, n, k.

Now, let us establish that $\varphi_{2^{-n/2}} * a^{n,k}_j$ is C times a $(H^1_L, 2)$-atom related to the ball $B(x_{n,k}, 5.2^{-n/2})$. Indeed, it is clear that $\frac{1}{\mathcal{C}_L}\rho(x_{n,k}) < 5.2^{-n/2} < \mathcal{C}_L\rho(x_{n,k})$ since $x_{n,k} \in \mathcal{B}_n$; and supp $\varphi_{2^{-n/2}} * a^{n,k}_j \subset B(x_{n,k}, 5.2^{-n/2})$ since supp $\varphi \subset B(0,1)$ and supp $a^{n,k}_j \subset B(x_{n,k}, 2^{2-n/2})$. In addition,

$$\|\varphi_{2^{-n/2}} * a^{n,k}_j\|_{L^2} \leq \|\varphi_{2^{-n/2}}\|_{L^2}\|a^{n,k}_j\|_{L^1} \leq (2^{-n/2})^{-d/2}\|\varphi\|_{L^2} \leq C|B(x_{n,k}, 5.2^{-n/2})|^{-1/2}.$$

These prove that $\varphi_{2^{-n/2}} * a^{n,k}_j$ is C times a $(H^1_L, 2)$-atom related to $B(x_{n,k}, 5.2^{-n/2})$.

By an analogous argument, it is easy to check that $(\varphi_{2^{-n/2}} * a^{n,k}_j)(g - g_{B(x_{n,k}, 5.2^{-n/2})})$ is $C\|g\|_{BMO}$ times a $(H^1_L, 3/2)$-atom related to $B(x_{n,k}, 5.2^{-n/2})$.

Therefore, (6.21) yields

$$\begin{aligned}\Big\|(\varphi_{2^{-n/2}} * f)g\Big\|_{H^1_L} &\leq C\sum_j |\lambda_j|\|(\varphi_{2^{-n/2}} * a_j)(g - g_{B(x_{n,k}, 5.2^{-n/2})})\|_{H^1_L} \\ &\quad + C\sum_j |\lambda_j|\|\varphi_{2^{-n/2}} * a_j\|_{H^1_L}|g_{B(x_{n,k}, 5.2^{-n/2})}| \\ &\leq C\|f\|_{h^1_n}\|g\|_{BMO_L},\end{aligned}$$

where we used $|g_{B(x_{n,k},5.2^{-n/2})}| \leq \|g\|_{BMO_L}$ since $\rho(x_{n,k}) \leq 5.2^{-n/2}$.

$\square$

Our main results are strongly related to the recent result of Bonami, Grellier and Ky [14]. In [14], the authors proved the following.

Theorem 6.4.1. *There exists two continuous bilinear operators on the product space $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, respectively $S : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \mapsto L^1(\mathbb{R}^d)$ and $T : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \mapsto H^{\log}(\mathbb{R}^d)$ such that*

$$f \times g = S(f,g) + T(f,g).$$

Before giving the proof of the main theorems, we should point out that the bilinear operator T in Theorem 6.4.1 satisfies

$$\|T(f,g)\|_{H^{\log}} \leq C\|f\|_{H^1}(\|g\|_{BMO} + |g_{\mathbb{Q}}|) \tag{6.22}$$

where $\mathbb{Q} := [0,1)^d$ is the unit cube. To prove this, the authors in [14] used the generalized Hölder inequality (see also [15])

$$\|fg\|_{L^{\log}} \leq C\|f\|_{L^1}\|g\|_{\mathrm{Exp}}$$

and the fact that $\|g - g_{\mathbb{Q}}\|_{\mathrm{Exp}} \leq C\|g\|_{BMO}$. Here, $L^{\log}(\mathbb{R}^d)$ denotes the space of all measurable functions f such that $\int_{\mathbb{R}^d} \frac{|f(x)|}{\log(e+|f(x)|)+\log(e+|x|)}dx < \infty$ with the norm

$$\|f\|_{L^{\log}} = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^d} \frac{\frac{|f(x)|}{\lambda}}{\log(e + \frac{|f(x)|}{\lambda}) + \log(e+|x|)}dx \leq 1\right\}$$

and $\mathrm{Exp}(\mathbb{R}^d)$ denotes the space of all measurable functions f such that $\int_{\mathbb{R}^d}(e^{|f(x)|} - 1)\frac{1}{(1+|x|)^{2d}}dx < \infty$ with the norm

$$\|f\|_{\mathrm{Exp}} = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^d} \left(e^{|f(x)|/\lambda} - 1\right)\frac{1}{(1+|x|)^{2d}}dx \leq 1\right\}.$$

In fact, Inequality (6.22) also holds when we replace the unit cube $\mathbb{Q}$ by $B(0,r)$ for every $r > 0$ since $\|g - g_{B(0,r)}\|_{\mathrm{Exp}} \leq C\|g\|_{BMO}$. More precisely, there exists a constant $C > 0$ such that

$$\|fg\|_{L^{\log}} \leq C\|f\|_{L^1}(\|g\|_{BMO} + |g_{B(0,\rho(0))}|) \leq C\|f\|_{L^1}\|g\|_{BMO_L} \tag{6.23}$$

for all $f \in L^1(\mathbb{R}^d)$ and $g \in BMO_L(\mathbb{R}^d)$. As a consequence, we obtain

$$\|T(f,g)\|_{H^{\log}} \leq C\|f\|_{H^1}\|g\|_{BMO_L} \tag{6.24}$$

for all $f \in H^1(\mathbb{R}^d)$ and $g \in BMO_L(\mathbb{R}^d)$.

Now, we are ready to give the proof of the main theorems.

Proof of Theorem 6.1.2. We define two bilinear operators S_L and T_L by

$$S_L(f,g) = S(\mathfrak{H}(f), g) + \sum_{n,k} \Big(\varphi_{2^{-n/2}} * (\psi_{n,k} f)\Big) g$$

and

$$T_L(f,g) = T(\mathfrak{H}(f), g)$$

for all $(f,g) \in H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$. Then, it follows from Theorem 6.4.1, Lemma 7.3.4, Lemma 6.4.4 and Lemma 6.4.5 that

$$\begin{aligned} \|S_L(f,g)\|_{L^1} &\leq \|S(\mathfrak{H}(f), g)\|_{L^1} + C\sum_{n,k} \Big\|\Big(\varphi_{2^{-n/2}} * (\psi_{n,k} f)\Big) g\Big\|_{H^1_L} \\ &\leq C\|g\|_{BMO}\|\mathfrak{H}(f)\|_{H^1} + C\|g\|_{BMO_L}\sum_{n,k}\|\psi_{n,k} f\|_{h^1_n} \\ &\leq C\|f\|_{H^1_L}\|g\|_{BMO_L}, \end{aligned}$$

and as (6.24),

$$\begin{aligned} \|T_L(f,g)\|_{H^{\log}} = \|T(\mathfrak{H}(f), g)\|_{H^{\log}} &\leq C\|\mathfrak{H}(f)\|_{H^1}\|g\|_{BMO_L} \\ &\leq C\|f\|_{H^1_L}\|g\|_{BMO_L}. \end{aligned}$$

Furthermore, in the sense of distributions, we have

$$\begin{aligned} & S_L(f,g) + T_L(f,g) \\ = & \left(\sum_{n,k}\Big(\psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f)\Big)\right) \times g + \sum_{n,k}\Big(\varphi_{2^{-n/2}} * (\psi_{n,k} f)\Big) g \\ = & \left(\sum_{n,k}\psi_{n,k} f\right) \times g = f \times g, \end{aligned}$$

which ends the proof of Theorem 6.1.2.

□

Proof of Theorem 6.1.3. By the proof of Theorem 6.1.2, the function $\sum_{n,k}(\varphi_{2^{-n/2}} * (\psi_{n,k} f))g$ belongs to $H^1_L(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. This implies that $\Big(\sum_{n,k}(\varphi_{2^{-n/2}} * (\psi_{n,k} f))g\Big) * \phi_\epsilon$ tends to $\sum_{n,k}(\varphi_{2^{-n/2}} * (\psi_{n,k} f))g$ almost everywhere, as $\epsilon \to 0$. Therefore, applying Theorem 1.8 of [15], we get

$$
\begin{aligned}
\lim_{\epsilon \to 0}(f \times g)_\epsilon(x) &= \lim_{\epsilon \to 0}(\mathfrak{H}(f) \times g)_\epsilon(x) + \lim_{\epsilon \to 0}\Big(\sum_{n,k}(\varphi_{2^{-n/2}} * (\psi_{n,k} f))g\Big) * \phi_\epsilon(x) \\
&= \mathfrak{H}(f)(x)g(x) + \left(\sum_{n,k}(\varphi_{2^{-n/2}} * (\psi_{n,k} f))(x)\right) g(x) \\
&= f(x)g(x)
\end{aligned}
$$

for almost every $x \in \mathbb{R}^d$, which completes the proof of Theorem 6.1.3.

$\square$

Chapter 7

On weak*-convergence in $H^1_L(\mathbb{R}^d)$

Ce chapitre est une prépublication (soumise).

Résumé

Soit $L = -\Delta + V$ un opérateur Schrödinger sur $\mathbb{R}^d$, $d \geq 3$, où V est une fonction positive, $V \neq 0$, qui appartient à la classe Hölder inverse $RH_{d/2}$. Dans cet article, nous prouvons une version du théorème classique de Jones et Journé sur la convergence faible* dans l'espace de Hardy $H^1_L(\mathbb{R}^d)$.

7.1 Introduction

A famous and classical result of Fefferman [47] states that the John-Nirenberg space $BMO(\mathbb{R}^d)$ is the dual of the Hardy space $H^1(\mathbb{R}^d)$. It is also well-known that $H^1(\mathbb{R}^d)$ is one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $VMO(\mathbb{R}^d)$ denote the closure of the space $C_c^\infty(\mathbb{R}^d)$ in $BMO(\mathbb{R}^d)$, where C_c^∞ is the set of C^∞-functions with compact support, Coifman and Weiss showed in [32] that $H^1(\mathbb{R}^d)$ is the dual space of $VMO(\mathbb{R}^d)$, which gives $H^1(\mathbb{R}^d)$ a richer structure than $L^1(\mathbb{R}^d)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^d)$, but bounded on $H^1(\mathbb{R}^d)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^d)$, which is useful in the application of Hardy spaces to compensated compactness (see [33]). More precisely, in [78], Jones and Journé proved the following.

Theorem 7.1.1. *Suppose that $\{f_j\}_{j\geq 1}$ is a bounded sequence in $H^1(\mathbb{R}^d)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H^1(\mathbb{R}^d)$ and $\{f_j\}_{j\geq 1}$ weak*-converges to f, that is, for every $\varphi \in VMO(\mathbb{R}^d)$, we have*

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} f_j(x)\varphi(x)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

The aim of this paper is to prove an analogous version of the above theorem in the setting of function spaces associated with Schrödinger operators.

Let $L = -\Delta + V$ be a Schrödinger differential operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In the recent years, there is an increasing interest on the study of the problems of harmonic analysis associated with these operators, see for example [37, 45, 46, 82, 98, 125, 145]. In [46], Dziubański and Zienkiewicz considered the Hardy space $H^1_L(\mathbb{R}^d)$ as the set of functions $f \in L^1(\mathbb{R}^d)$ such that $\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) := \sup_{t>0} |e^{-tL}f(x)|$. There, they characterized $H^1_L(\mathbb{R}^d)$ in terms of atomic decomposition and in terms of the Riesz transforms associated with L. Later, in [45], Dziubański et al. introduced a BMO-type space $BMO_L(\mathbb{R}^d)$ associated with L, and established the duality between $H^1_L(\mathbb{R}^d)$ and $BMO_L(\mathbb{R}^d)$. Recently, Deng et al. [37] introduced and developed new function spaces of VMO-type $VMO_A(\mathbb{R}^d)$ associated with some operators A which have a bounded holomorphic functional calculus on $L^2(\mathbb{R}^d)$. When $A \equiv L$, their space $VMO_L(\mathbb{R}^d)$ is just the set of all functions f in $BMO_L(\mathbb{R}^d)$ such that $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\leq r} \Big(\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy\Big)^{1/2}\right),$$

$$\gamma_2(f) = \lim_{R\to\infty} \left(\sup_{x\in\mathbb{R}^d, t\geq R} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \Big)^{1/2} \right),$$

$$\gamma_3(f) = \lim_{R\to\infty} \left(\sup_{B(x,t)\cap B(0,R)=\emptyset} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \Big)^{1/2} \right).$$

The authors in [37] further showed that $H^1_L(\mathbb{R}^d)$ is in fact the dual of $VMO_L(\mathbb{R}^d)$, which allows us to study the weak*-convergence in $H^1_L(\mathbb{R}^d)$. This is useful in the study of the Hardy estimates for commutators of singular integral operators related to L, see for example Theorem 7.1 and Theorem 7.3 of [82].

Our main result is the following theorem.

Theorem 7.1.2. *Suppose that $\{f_j\}_{j\geq 1}$ is a bounded sequence in $H^1_L(\mathbb{R}^d)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H^1_L(\mathbb{R}^d)$ and $\{f_j\}_{j\geq 1}$ weak*-converges to f, that is, for every $\varphi \in VMO_L(\mathbb{R}^d)$, we have*

$$\lim_{j\to\infty} \int\limits_{\mathbb{R}^d} f_j(x)\varphi(x)dx = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In $\mathbb{R}^d$, we denote by $B = B(x,r)$ an open ball with center x and radius $r > 0$. For any measurable set E, we denote by $|E|$ its Lebesgue measure.

The paper is organized as follows. In Section 2, we present some notations and preliminary results. Section 3 is devoted to the proof of Theorem 7.1.2. In the last section, we prove that $C^\infty_c(\mathbb{R}^d)$ is dense in the space $VMO_L(\mathbb{R}^d)$.

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7.2 Some preliminaries and notations

In this paper, we consider the Schrödinger differential operator

$$L = -\Delta + V$$

on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$. As in the works of Dziubański et al [45, 46], we always assume that V belongs to the reverse Hölder class $RH_{d/2}$. Recall that a nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q, $1 < q < \infty$, if there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^d$,

$$\Big(\frac{1}{|B|}\int\limits_B (V(x))^q dx\Big)^{1/q} \leq \frac{C}{|B|}\int\limits_B V(x)dx.$$

Let $\{T_t\}_{t>0}$ be the semigroup generated by L and $T_t(x,y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL}f(x) = \int\limits_{\mathbb{R}^d} T_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \leq T_t(x,y) \leq \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}. \tag{7.1}$$

According [46], the space $H^1_L(\mathbb{R}^d)$ is defined as the completion of

$$\{f \in L^2(\mathbb{R}^d) : \mathcal{M}_L f \in L^1(\mathbb{R}^d)\}$$

in the norm

$$\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1},$$

where $\mathcal{M}_L f(x) := \sup_{t>0} |T_t f(x)|$ for all $x \in \mathbb{R}^d$.

In [45] it was shown that the dual space of $H^1_L(\mathbb{R}^d)$ can be identified with the space $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO(\mathbb{R}^d)$ with

$$\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x)\leq r} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy < \infty, \tag{7.2}$$

where ρ is the auxiliary function defined as in [125], that is,

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{d-2}}\int\limits_{B(x,r)} V(y)dy \leq 1\Big\}, \tag{7.3}$$

$x \in \mathbb{R}^d$. Clearly, $0 < \rho(x) < \infty$ for all $x \in \mathbb{R}^d$, and thus $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n$, where the sets $\mathcal{B}_n$ are defined by

$$\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2}\}. \tag{7.4}$$

The following fundamental property of the function ρ is due to Shen [125].

Proposition 7.2.1 (see [125], Lemma 1.4)**.** *There exist $C_0 > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$C_0^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C_0\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}.$$

Let $VMO_L(\mathbb{R}^d)$ be the subspace of $BMO_L(\mathbb{R}^d)$ consisting of those functions f satisfying $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy\Big)^{1/2}\right),$$

$$\gamma_2(f) = \lim_{R\to\infty}\left(\sup_{x\in\mathbb{R}^d, t\ge R}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy\Big)^{1/2}\right),$$

$$\gamma_3(f) = \lim_{R\to\infty}\left(\sup_{B(x,t)\cap B(0,R)=\emptyset}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy\Big)^{1/2}\right).$$

In [37] it was shown that $H^1_L(\mathbb{R}^d)$ is the dual space of $VMO_L(\mathbb{R}^d)$.

In the sequel, we denote by $\mathcal{C}_L$ the L-constant

$$\mathcal{C}_L = 8.9^{k_0} C_0$$

where k_0 and C_0 are defined as in Proposition 7.2.1.

Following Dziubański and Zienkiewicz [46], we define atoms as follows.

Definition 7.2.1. *Given $1 < q \le \infty$. A function a is called a (H^1_L, q)-atom related to the ball $B(x_0, r)$ if $r \le \mathcal{C}_L \rho(x_0)$ and*

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \le |B(x_0,r)|^{1/q-1}$,

iii) if $r \le \frac{1}{\mathcal{C}_L}\rho(x_0)$ then $\int_{\mathbb{R}^d} a(x)dx = 0$.

We have then the following atomic characterization of $H^1_L(\mathbb{R}^d)$.

Theorem A (see [46], Theorem 1.5)**.** *Let $1 < q \le \infty$. A function f is in $H^1_L(\mathbb{R}^d)$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H^1_L, q)-atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists a constant $C > 0$ such that*

$$\|f\|_{H^1_L} \le \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\} \le C\|f\|_{H^1_L}.$$

Let $P(x) = (4\pi)^{-d/2} e^{-|x|^2/4}$ be the Gauss function. According to [46], the space $h^1_n(\mathbb{R}^d)$, $n \in \mathbb{Z}$, denotes the space of all integrable functions f such that

$$\mathcal{M}_n f(x) = \sup_{0<t<2^{-n/2}} |P_t * f(x)| \in L^1(\mathbb{R}^d),$$

where $P_t(\cdot) := t^{-d}P(t^{-1}\cdot)$. The norm on $h^1_n(\mathbb{R}^d)$ is then defined by

$$\|f\|_{h^1_n} := \|\mathcal{M}_n f\|_{L^1}.$$

It was shown in [56] that the dual space of $h^1_n(\mathbb{R}^d)$ can be identified with $bmo_n(\mathbb{R}^d)$ the space of all locally integrable functions f such that

$$\|f\|_{bmo_n} = \|f\|_{BMO} + \sup_{x\in\mathbb{R}^d, 2^{-n/2}\leq r} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy < \infty.$$

Here and in what follows, for a ball B and a locally integrable function f, we denote by f_B the average of f on B. Following Dafni [35], we define $vmo_n(\mathbb{R}^d)$ as the subspace of $bmo_n(\mathbb{R}^d)$ consisting of those f such that

$$\lim_{\sigma\to 0}\left(\sup_{x\in\mathbb{R}^d, r<\sigma} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|dy\right) = 0$$

and

$$\lim_{R\to\infty}\left(\sup_{B(x,r)\cap B(0,R)=\emptyset, r\geq 2^{-n/2}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy\right) = 0.$$

Recall that $C^\infty_c(\mathbb{R}^d)$ is the space of all C^∞-functions with compact support. Then, the following was established by Dafni [35].

Theorem B (see [35], Theorem 6 and Theorem 9)**.** *Let $n \in \mathbb{Z}$. Then,*

i) The space $vmo_n(\mathbb{R}^d)$ is the closure of $C^\infty_c(\mathbb{R}^d)$ in $bmo_n(\mathbb{R}^d)$.

ii) The dual of $vmo_n(\mathbb{R}^d)$ is the space $h^1_n(\mathbb{R}^d)$.

Furthermore, the weak*-convergence is true in $h^1_n(\mathbb{R}^d)$.

Theorem C (see [35], Theorem 11)**.** *Let $n \in \mathbb{Z}$. Suppose that $\{f_j\}_{j\geq 1}$ is a bounded sequence in $h^1_n(\mathbb{R}^d)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in h^1_n(\mathbb{R}^d)$ and $\{f_j\}_{j\geq 1}$ weak*-converges to f, that is, for every $\varphi \in vmo_n(\mathbb{R}^d)$, we have*

$$\lim_{j\to\infty} \int\limits_{\mathbb{R}^d} f_j(x)\varphi(x)dx = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

7.3 Proof of Theorem 7.1.2

We begin by recalling the following two lemmas due to [46]. These two lemmas together with Proposition 7.2.1 play an important role in our study.

Lemma 7.3.1 (see [46], Lemma 2.3)**.** *There exists a constant $C > 0$ and a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$, $n \in \mathbb{Z}, k = 1, 2, ...$, such that $x_{n,k} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B_{n,k}$, and*

$$card\{(n', k') : B(x_{n,k}, R2^{-n/2}) \cap B(x_{n',k'}, R2^{-n/2}) \neq \emptyset\} \leq R^C$$

for all n, k and $R \geq 2$.

Lemma 7.3.2 (see [46], Lemma 2.5)**.** *There are nonnegative C^∞-functions $\psi_{n,k}$, $n \in \mathbb{Z}, k = 1, 2, ...$, supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that*

$$\sum_{n,k} \psi_{n,k} = 1 \quad \text{and} \quad \|\nabla \psi_{n,k}\|_{L^\infty} \leq C2^{n/2}.$$

The following corollary is useful, which proof follows directly from Lemma 7.3.1. We leave the details to the reader (see also Corollary 1 of [45]).

Corollary 7.3.1. *i) Let $\mathbb{K}$ be a compact set. Then, there exists a finite set $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ such that $\mathbb{K} \cap B(x_{n,k}, 2^{1-n/2}) = \emptyset$ whenever $(n, k) \notin \Gamma$.*

ii) There exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$,

$$card\ \{(n, k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{1-n/2}) \cap B(x, 2\rho(x)) \neq \emptyset\} \leq C.$$

iii) There exists a constant $C > 0$ such that for every ball $B(x, r)$ with $\rho(x) \leq r$, we have

$$|B(x, r)| \leq \sum_{B(x_{n,k}, 2^{-n/2}) \cap B(x,r) \neq \emptyset} |B(x_{n,k}, 2^{-n/2})| \leq C|B(x, r)|.$$

The key point in the proof of Theorem 7.1.2 is the theorem.

Theorem 7.3.1. *The space $C_c^\infty(\mathbb{R}^d)$ is dense in the space $VMO_L(\mathbb{R}^d)$.*

The proof of Theorem 7.3.1 will be given in the last section.

To prove Theorem 7.1.2, we need also the following two lemmas.

Lemma 7.3.3 (see [82], Lemma 6.5)**.** *Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Suppose that $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x, 2^{1-n/2})$. Then, there are (H^1_L, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_{j=1}^\infty \lambda_j a_j, \quad \sum_{j=1}^\infty |\lambda_j| \leq C\|f\|_{h^1_n}$$

with a positive constant C independent of n and f.

Lemma 7.3.4 (see (4.7) in [46])**.** *For every $f \in H^1_L(\mathbb{R}^d)$, we have*

$$\sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \le C\|f\|_{H^1_L}.$$

Now, we are ready to give the proof of the main theorem.

Proof of Theorem 7.1.2. By assumption, there exists $\mathscr{M} > 0$ such that

$$\|f_j\|_{H^1_L} \le \mathscr{M}\ , \quad \text{for all } j \ge 1.$$

Let $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$. Then, for almost every $x \in \mathbb{R}^d$, $\psi_{n,k}(x) f_j(x) \to \psi_{n,k}(x) f(x)$ since $f_j(x) \to f(x)$. By Theorem C, this yields that $\psi_{n,k} f$ belongs to $h^1_n(\mathbb{R}^d)$ and $\{\psi_{n,k} f_j\}_{j \ge 1}$ weak*-converges to $\psi_{n,k} f$ in $h^1_n(\mathbb{R}^d)$, that is,

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} \psi_{n,k}(x) f_j(x)\phi(x)dx = \int_{\mathbb{R}^d} \psi_{n,k}(x) f(x)\phi(x)dx, \tag{7.5}$$

for all $\phi \in C^\infty_c(\mathbb{R}^d)$. Furthermore,

$$\|\psi_{n,k} f\|_{h^1_n} \le \varliminf_{j\to\infty} \|\psi_{n,k} f_j\|_{h^1_n}. \tag{7.6}$$

As $x_{n,k} \in \mathcal{B}_n$ and $\operatorname{supp} \psi_{n,k} f \subset B(x_{n,k}, 2^{1-n/2})$, by Lemma 7.3.3, there are $(H^1_L, 2)$-atoms $a^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j) \subset B(x_{n,k}, 2^{2-n/2})$ such that

$$\psi_{n,k} f = \sum_j \lambda^{n,k}_j a^{n,k}_j, \quad \sum_j |\lambda^{n,k}_j| \le C\|\psi_{n,k} f\|_{h^1_n}.$$

Let $N, K \in \mathbb{Z}^+$ be arbitrary. Then, the above together with (7.6) and Lemma 7.3.4 follow that there exists $m_{N,K} \in \mathbb{Z}^+$ such that

$$\begin{aligned}
\sum_{n=-N}^{N} \sum_{k=1}^{K} \sum_j |\lambda^{n,k}_j| &\le \sum_{n=-N}^{N} \sum_{k=1}^{K} C\Big(\frac{\mathscr{M}}{(1+n^2)(1+k^2)} + \|\psi_{n,k} f_{m_{N,K}}\|_{h^1_n}\Big) \\
&\le C\sum_{n,k} \frac{\mathscr{M}}{(1+n^2)(1+k^2)} + C\|f_{m_{N,K}}\|_{H^1_L} \\
&\le C\mathscr{M},
\end{aligned}$$

where the constants C are independent of N, K. By Theorem A, this allows to conclude that

$$f = \sum_{n,k} \psi_{n,k} f \in H^1_L(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{H^1_L} \le \sum_{n,k} \sum_j |\lambda^{n,k}_j| \le C\mathscr{M}.$$

Finally, we need to show that for every $\phi \in VMO_L(\mathbb{R}^d)$,

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} f_j(x)\phi(x)dx = \int_{\mathbb{R}^d} f(x)\phi(x)dx. \tag{7.7}$$

By Theorem 7.3.1, we only need to prove (7.7) for $\phi \in C_c^\infty(\mathbb{R}^d)$. In fact, by (i) of Corollary 7.3.1, there exists a finite set $\Gamma_\phi \subset \mathbb{Z}\times\mathbb{Z}^+$ such that

$$f\phi = \sum_{(n,k)\in\Gamma_\phi} \psi_{n,k} f\phi \quad \text{and} \quad f_j\phi = \sum_{(n,k)\in\Gamma_\phi} \psi_{n,k} f_j\phi$$

since supp $\psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. This together with (7.5) give

$$\begin{aligned}
\lim_{j\to\infty} \int_{\mathbb{R}^d} f_j(x)\phi(x)dx &= \lim_{j\to\infty} \int_{\mathbb{R}^d} \sum_{(n,k)\in\Gamma_\phi} \psi_{n,k}(x) f_j(x)\phi(x)dx \\
&= \sum_{(n,k)\in\Gamma_\phi} \lim_{j\to\infty} \int_{\mathbb{R}^d} \psi_{n,k}(x) f_j(x)\phi(x)dx \\
&= \sum_{(n,k)\in\Gamma_\phi} \int_{\mathbb{R}^d} \psi_{n,k}(x) f(x)\phi(x)dx \\
&= \int_{\mathbb{R}^d} f(x)\phi(x)dx,
\end{aligned}$$

which ends the proof of Theorem 7.1.2.

□

7.4 Proof of Theorem 7.3.1

The main point in the proof of Theorem 7.3.1 is the theorem.

Theorem 7.4.1. *Let $CMO_L(\mathbb{R}^d)$ be the closure of $C_c^\infty(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$. Then, $H_L^1(\mathbb{R}^d)$ is the dual space of $CMO_L(\mathbb{R}^d)$.*

To prove Theorem 7.4.1, we need the following three lemmas.

Lemma 7.4.1. *There exists a constant $C > 0$ such that*

$$2^{-n/2} \le Cr$$

whenever $B(x_{n,k}, 2^{1-n/2}) \cap B(x,r) \neq \emptyset$ and $\rho(x) \le r$.

The proof of Lemma 7.4.1 follows directly from Proposition 7.2.1. We omit the details.

Lemma 7.4.2. *Let $\psi_{n,k}$, $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$, be as in Lemma 7.3.2. Then, there exists a constant C independent of $n, k, \psi_{n,k}$, such that*

$$\|\psi_{n,k} f\|_{bmo_n} \leq C\|f\|_{bmo_n} \tag{7.8}$$

for all $f \in bmo_n(\mathbb{R}^d)$, and

$$\|\psi_{n,k}\phi\|_{BMO_L} \leq C\|\phi\|_{bmo_n} \tag{7.9}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Lemma 7.4.3. *For every $f \in BMO_L(\mathbb{R}^d)$, we have*

$$\|f\|_{BMO_L} \approx \sup_{r\leq\rho(x)} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|dy + \sup_{\rho(x)\leq r\leq 2\rho(x)} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy.$$

Proof of Lemma 7.4.2. Noting that $\psi_{n,k}$ is a multiplier of $bmo_n(\mathbb{R}^d)$ and $\|\psi_{n,k}\|_{L^\infty} \leq 1$, Theorem 2 of [116] allows us to reduce (7.8) to showing that

$$\frac{\log\left(e + \frac{2^{-n/2}}{r}\right)}{|B(x,r)|} \int\limits_{B(x,r)} \left|\psi_{n,k}(y) - \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} \psi_{n,k}(z)dz\right|dy \leq C \tag{7.10}$$

holds for every ball $B(x,r)$ which satisfies $r \leq 2^{-n/2}$. In fact, from $\|\nabla\psi_{n,k}\|_{L^\infty} \leq C2^{n/2}$ and the estimate $\frac{r}{2^{-n/2}} \log\left(e + \frac{2^{-n/2}}{r}\right) \leq \sup_{0<t\leq 1} t\log(e+1/t) < \infty$,

$$\begin{aligned}
&\frac{\log\left(e + \frac{2^{-n/2}}{r}\right)}{|B(x,r)|} \int\limits_{B(x,r)} \left|\psi_{n,k}(y) - \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} \psi_{n,k}(z)dz\right|dy \\
\leq\ & \frac{\log\left(e + \frac{2^{-n/2}}{r}\right)}{|B(x,r)|^2} \int\limits_{B(x,r)} \int\limits_{B(x,r)} |\psi_{n,k}(y) - \psi_{n,k}(z)|dzdy \\
\leq\ & \log\left(e + \frac{2^{-n/2}}{r}\right)\|\nabla\psi_{n,k}\|_{L^\infty} 2r \\
\leq\ & C\frac{r}{2^{-n/2}} \log\left(e + \frac{2^{-n/2}}{r}\right) \leq C,
\end{aligned}$$

which proves (7.10), and thus (7.8) holds.

As (7.8) holds, we get

$$\|\psi_{n,k}\phi\|_{BMO} \leq \|\psi_{n,k}\phi\|_{bmo_n} \leq C\|\phi\|_{bmo_n}.$$

Therefore, to prove (7.9), we only need to show that

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\psi_{n,k}(y)\phi(y)|dy\leq C\|\phi\|_{bmo_n} \tag{7.11}$$

holds for every $x\in\mathbb{R}^d$ and $r\geq\rho(x)$. Since supp $\psi_{n,k}\subset B(x_{n,k},2^{1-n/2})$, (7.11) is obvious if $B(x,r)\cap B(x_{n,k},2^{1-n/2})=\emptyset$. Otherwise, as $\rho(x)\leq r$, Lemma 7.4.1 gives $2^{-n/2}\leq Cr$. As a consequence, we get

$$\begin{aligned}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\psi_{n,k}(y)\phi(y)|dy &\leq C\sup_{2^{-n/2}\leq r}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\phi(y)|dy\\ &\leq C\|\phi\|_{bmo_n},\end{aligned}$$

which proves (7.11), and hence (7.9) holds.

□

Proof of Lemma 7.4.3. Clearly, it is sufficient to prove that

$$\sup_{\rho(x)\leq r}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy\leq C\sup_{\rho(x)\leq r\leq 2\rho(x)}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy. \tag{7.12}$$

In fact, for every ball $B(x,r)$ which satisfies $\rho(x)\leq r$, setting

$$G=\{(n,k)\in\mathbb{Z}\times\mathbb{Z}^+ : B(x_{n,k},2^{-n/2})\cap B(x,r)\neq\emptyset\},$$

one has

$$B(x,r)\subset\cup_{(n,k)\in G}B(x_{n,k},2^{-n/2})\quad\text{and}\quad\sum_{(n,k)\in G}|B(x_{n,k},2^{-n/2})|\leq C|B(x,r)|$$

since $\mathbb{R}^d=\cup_{n\in\mathbb{Z}}\mathcal{B}_n\subset\cup_{n,k}B(x_{n,k},2^{-n/2})$ and (iii) of Corollary 7.3.1. Therefore,

$$\begin{aligned}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy &\leq \frac{1}{|B(x,r)|}\int\limits_{\cup_{(n,k)\in G}B(x_{n,k},2^{-n/2})}|f(y)|dy\\ &\leq \frac{1}{|B(x,r)|}\sum_{(n,k)\in G}|B(x_{n,k},2^{-n/2})|\sup_{\rho(z)\leq s\leq 2\rho(z)}\frac{1}{|B(z,s)|}\int\limits_{B(z,s)}|f(y)|dy\\ &\leq C\sup_{\rho(z)\leq s\leq 2\rho(z)}\frac{1}{|B(z,s)|}\int\limits_{B(z,s)}|f(y)|dy,\end{aligned}$$

which implies that (7.12) holds.

□

Proof of Theorem 7.4.1. Since $CMO_L(\mathbb{R}^d)$ is a subspace of $BMO_L(\mathbb{R}^d)$, which is the dual of $H^1_L(\mathbb{R}^d)$, every function f in $H^1_L(\mathbb{R}^d)$ determines a bounded linear functional on $CMO_L(\mathbb{R}^d)$ of norm bounded by $\|f\|_{H^1_L}$.

Conversely, given a bounded linear functional $\mathcal{T}$ on $CMO_L(\mathbb{R}^d)$. Then, for every $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$, from (7.9) and density of $C^\infty_c(\mathbb{R}^d)$ in $vmo_n(\mathbb{R}^d)$, the linear functional $\mathcal{T}_{n,k}(g) \mapsto \mathcal{T}(\psi_{n,k} g)$ is continuous on $vmo_n(\mathbb{R}^d)$. Consequently, by Theorem B, there exists $f_{n,k} \in h^1_n(\mathbb{R}^d)$ such that for every $\phi \in C^\infty_c(\mathbb{R}^d)$,

$$\mathcal{T}(\psi_{n,k}\phi) = \mathcal{T}_{n,k}(\phi) = \int\limits_{\mathbb{R}^d} f_{n,k}(y)\phi(y)dy, \tag{7.13}$$

moreover,

$$\|f_{n,k}\|_{h^1_n} \leq C\|\mathcal{T}_{n,k}\|, \tag{7.14}$$

where C is a positive constant independent of $n, k, \psi_{n,k}$ and $\mathcal{T}$.

Noting that $\operatorname{supp} \psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$, (7.13) implies that $\operatorname{supp} f_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. Consequently, as $x_{n,k} \in \mathcal{B}_n$, Lemma 7.3.3 yields that there are $(H^1_L, 2)$-atoms $a^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j)$ such that

$$f_{n,k} = \sum_{j=1}^{\infty} \lambda^{n,k}_j a^{n,k}_j, \quad \sum_{j=1}^{\infty} |\lambda^{n,k}_j| \leq C\|f_{n,k}\|_{h^1_n} \tag{7.15}$$

with a positive constant C independent of $\psi_{n,k}$ and $f_{n,k}$.

Since $\operatorname{supp} f_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$, by Lemma 7.3.1, the function

$$x \mapsto f(x) = \sum_{n,k} f_{n,k}(x)$$

is well defined, and belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, for every $\phi \in C^\infty_c(\mathbb{R}^d)$, by (i) of Corollary 7.3.1, there exists a finite set $\Gamma_\phi \subset \mathbb{Z} \times \mathbb{Z}^+$ such that

$$\mathcal{T}(\phi) = \sum_{(n,k)\in\Gamma_\phi} \mathcal{T}(\psi_{n,k}\phi) = \sum_{(n,k)\in\Gamma_\phi} \int\limits_{\mathbb{R}^d} f_{n,k}(y)\phi(y)dy = \int\limits_{\mathbb{R}^d} f(y)\phi(y)dy.$$

Next, we need to show that $f \in H^1_L(\mathbb{R}^d)$.

We first claim that there exists $C > 0$ such that

$$\sum_{n,k} \|f_{n,k}\|_{h^1_n} \leq C\|\mathcal{T}\|. \tag{7.16}$$

Assume that (7.16) holds for a moment. Then, from (7.15), there are $(H^1_L, 2)$-atoms $a^{n,k}_j$ and complex numbers $\lambda^{n,k}_j$ such that

$$f = \sum_{n,k}\sum_j \lambda^{n,k}_j a^{n,k}_j \quad \text{and} \quad \sum_{n,k}\sum_j |\lambda^{n,k}_j| \leq C\sum_{n,k}\|f_{n,k}\|_{h^1_n} \leq C\|\mathcal{T}\|.$$

By Theorem A, this proves that $f \in H^1_L(\mathbb{R}^d)$, moreover, $\|f\|_{H^1_L} \leq C\|\mathcal{T}\|$.

Now, we return to prove (7.16).

Without loss of generality, we can assume that $\mathcal{T}$ is a real-valued functional. By (7.14), for each $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$, there exists $\phi_{n,k} \in C^\infty_c(\mathbb{R}^d)$ such that

$$\|\phi_{n,k}\|_{vmo_n} \leq 1 \quad \text{and} \quad \|f_{n,k}\|_{h^1_n} \leq C\mathcal{T}(\psi_{n,k}\phi_{n,k}). \tag{7.17}$$

For any $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ a finite set, let $\phi = \sum_{(n,k)\in\Gamma} \psi_{n,k}\phi_{n,k} \in C^\infty_c(\mathbb{R}^d)$. We prove that $\|\phi\|_{BMO_L} \leq C$. Indeed, let $B(x,r)$ be an arbitrary ball satisfying $r \leq 2\rho(x)$. Then, by (ii) of Corollary 7.3.1, we get

$$card\,\{(n,k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{1-n/2}) \cap B(x,r) \neq \emptyset\} \leq C.$$

This together with (7.8) and (7.17) give

$$\begin{aligned}
\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\phi(y) - \phi_{B(x,r)}|dy &\leq C \sup_{(n,k)\in\Gamma} \|\psi_{n,k}\phi_{n,k}\|_{BMO} \\
&\leq C \sup_{(n,k)\in\Gamma} \|\psi_{n,k}\phi_{n,k}\|_{bmo_n} \\
&\leq C \sup_{(n,k)\in\Gamma} \|\phi_{n,k}\|_{bmo_n} \leq C
\end{aligned}$$

if $r \leq \rho(x)$, and as (7.11),

$$\begin{aligned}
\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\phi(y)|dy &\leq C \sup_{(n,k)\in\Gamma} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\psi_{n,k}(y)\phi_{n,k}(y)|dy \\
&\leq C \sup_{(n,k)\in\Gamma} \|\phi_{n,k}\|_{bmo_n} \leq C
\end{aligned}$$

if $\rho(x) \leq r \leq 2\rho(x)$. Therefore, Lemma 7.4.3 yields

$$\begin{aligned}
\|\phi\|_{BMO_L} &\leq C\Big\{ \sup_{r\leq\rho(x)} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\phi(y) - \phi_{B(x,r)}|dy + \\
&\qquad + \sup_{\rho(x)\leq r\leq 2\rho(x)} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\phi(y)|dy\Big\} \\
&\leq C
\end{aligned}$$

since $B(x,r)$ is an arbitrary ball satisfying $r \leq 2\rho(x)$. This implies that

$$\begin{aligned}
\sum_{(n,k)\in\Gamma} \|f_{n,k}\|_{h^1_n} &\leq C \sum_{(n,k)\in\Gamma} \mathcal{T}(\psi_{n,k}\phi_{n,k}) = C\mathcal{T}(\phi) \\
&\leq C\|\mathcal{T}\|\|\phi\|_{BMO_L} \leq C\|\mathcal{T}\|.
\end{aligned}$$

Consequently, (7.16) holds since $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^{+}$ is an arbitrary finite set and the constants C are dependent of Γ. This ends the proof of Theorem 7.4.1.

$\square$

To prove Theorem 7.3.1, we need to recall the following lemma.

Lemma 7.4.4 (see [46], Lemma 3.0)**.** *There is a constant $\varepsilon > 0$ such that for every C' there exists $C > 0$ such that for every $t > 0$ and $|x - y| \le C'\rho(x)$,*

$$\Big|\frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} - T_t(x,y)\Big| \le C \frac{1}{|x-y|^d}\Big(\frac{|x-y|}{\rho(x)}\Big)^{\varepsilon}.$$

Proof of Theorem 7.3.1. As $H^1_L(\mathbb{R}^d)$ is the dual space of $VMO_L(\mathbb{R}^d)$ (see Theorem 4.1 of [37]), by Theorem 7.4.1 and Hahn-Banach theorem, it suffices to show that $C_c^\infty(\mathbb{R}^d) \subset VMO_L(\mathbb{R}^d)$. In fact, for every $f \in C_c^\infty(\mathbb{R}^d)$ with supp $f \subset B(0, R_0)$, one only needs to establish the following three steps:

Step 1. By (7.1), one has $\|e^{-tL} f\|_{L^2} \le \|f\|_{L^2}$ for all $t > 0$. Therefore,

$$\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy \le \frac{1}{|B(x,t)|} 4\|f\|^2_{L^2}$$

for all $x \in \mathbb{R}^d$ and $t > 0$. This implies that

$$\gamma_2(f) = \lim_{R\to\infty} \left(\sup_{x\in\mathbb{R}^d, t\ge R} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy\Big)^{1/2} \right) = 0.$$

Step 2. For every $R > 2R_0$ and $B(x,t) \cap B(0,R) = \emptyset$, by (7.1) again,

$$\begin{aligned} &\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy \\ \le\ & \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} \Big(\frac{1}{(4\pi t)^{d/2}} \int\limits_{B(0,R_0)} e^{-\frac{(R-R_0)^2}{4t}} |f(z)| dz\Big)^2 dy \\ \le\ & (4\pi)^{-d} \|f\|^2_{L^1} \frac{1}{t^d} e^{-\frac{R^2}{8t}} \le (4\pi)^{-d} \|f\|^2_{L^1} \Big(\frac{8d}{R^2}\Big)^d e^{-d}. \end{aligned}$$

Therefore,

$$\gamma_3(f) = \lim_{R\to\infty} \left(\sup_{B(x,t)\cap B(0,R)=\emptyset} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy\Big)^{1/2} \right) = 0.$$

Step 3. Finally, we need to show that

$$\gamma_1(f) = \lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right) = 0. \qquad (7.18)$$

For every $x\in\mathbb{R}^d$ and $t>0$, we have

$$\left\{\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big|f(y)-\frac{1}{(4\pi t)^{d/2}}\int\limits_{\mathbb{R}^d}e^{-\frac{|y-z|^2}{4t}}f(z)dz\Big|^2dy\right\}^{1/2}$$
$$\le \sup_{|y-z|<t^{1/4}}|f(y)-f(z)| + 2\|f\|_{L^\infty}\frac{1}{(4\pi t)^{d/2}}\int\limits_{|z|\ge t^{1/4}}e^{-\frac{|z|^2}{4t}}dz.$$

By the uniformly continuity of f, the above implies that

$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big|f(y)-\frac{1}{(4\pi t)^{d/2}}\int\limits_{\mathbb{R}^d}e^{-\frac{|y-z|^2}{4t}}f(z)dz\Big|^2dy\Big)^{1/2}\right) = 0.$$

Therefore, we can reduce (7.18) to showing that

$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big[\int\limits_{\mathbb{R}^d}\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|y-z|^2}{4t}}-T_t(y,z)\Big||f(z)|dz\Big]^2dy\Big)^{1/2}\right) = 0. \qquad (7.19)$$

From supp $f\subset B(0,R_0)$ and $\mathbb{R}^d\equiv\cup_{n,k}B(x_{n,k},2^{-n/2})$, there exists a finite set $\Gamma_f\subset\mathbb{Z}\times\mathbb{Z}^+$ such that supp $f\subset\cup_{(n,k)\in\Gamma_f}B(x_{n,k},2^{-n/2})$. As a consequence, (7.19) holds if we can prove that for each $(n,k)\in\Gamma_f$,

$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big[\int\limits_{B(x_{n,k},2^{-n/2})}\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|y-z|^2}{4t}}-T_t(y,z)\Big||f(z)|dz\Big]^2dy\Big)^{1/2}\right) = 0. \qquad (7.20)$$

We now prove (7.20). Let $x\in\mathbb{R}^d$ and $0<t<2^{-2n}$. As $x_{n,k}\in\mathcal{B}_n$, by Proposition 7.2.1, there is a constant $C>1$ such that $C^{-1}2^{-n/2}\le\rho(z)\le C2^{-n/2}$ for all $z\in B(x_{n,k},2^{-n/2})$. This together with (7.1) and Lemma 7.4.4, give

$$\left\{\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big[\int\limits_{B(x_{n,k},2^{-n/2})}\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|y-z|^2}{4t}}-T_t(y,z)\Big||f(z)|dz\Big]^2dy\right\}^{1/2}$$
$$\le 2\|f\|_{L^\infty}\frac{1}{(4\pi t)^{d/2}}\int\limits_{|z|\ge t^{1/4}}e^{-\frac{|z|^2}{4t}}dz + C2^{n\varepsilon/2}\|f\|_{L^\infty}\int\limits_{|z|<t^{1/4}}\frac{1}{|z|^{d-\varepsilon}}dz,$$

which implies that (7.20) holds. The proof of Theorem 7.3.1 is thus completed.

$\square$

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Dang Ky LUONG

Décomposition bilinéaire du produit H^1-BMO et problèmes liés

Résumé :

Dans cette thèse, nous étudions le produit (au sens des distributions) des fonctions de l'espace de Hardy H^1 par des fonctions à variations moyennes bornées BMO ainsi que des problèmes connexes. En particulier, nous démontrons qu'il existe deux opérateurs bilinéaires S et T tels que $f \times g = S(f,g) + T(f,g)$, f dans H^1, g dans BMO où S est continu de $H^1 \times BMO$ à valeurs dans L^1 et T est continu de $H^1 \times BMO$ et à valeurs dans un nouvel espace de type Hardy-Orlicz noté $H^{\log}$. Ce nouvel espace $H^{\log}$ appartient à une classe plus large d'espaces de Hardy de type Musielak-Orlicz que nous introduisons et étudions. En utilisant une méthode analogue à celle de la décomposition du produit H^1-BMO, nous établissons une décomposition bilinéaire des commutateurs $[b, T]$, T dans une large classe d'opérateurs sous-linéaires– classe contenant tous les opérateurs classiques de l'analyse harmonique. Nous généralisons ensuite nos résultats aux espaces de Hardy associés à un opérateur de Schrödinger.

Mots clés: espaces de Hardy, ondelettes, BMO, commutateur, opérateur de Schrödinger.

Bilinear decompositions for the product space $H^1 \times BMO$ and related problems

Abstract:

In this thesis, we investigate the product (in the distribution sense) of functions f in H^1 and g in BMO, denoted by $f \times g$, and related problems. In particular, we prove that there are two bounded bilinear operators $S : H^1 \times BMO \to L^1$ and $T : H^1 \times BMO \to H^{\log}$ such that $f \times g = S(f,g) + T(f,g) \in L^1 + H^{\log}$ holds for every $(f,g) \in H^1 \times BMO$. Here $H^{\log}$ is a new kind of Hardy-Orlicz space. This new space $H^{\log}$ appears as an example of a new class of Hardy spaces of Musielak-Orlicz type which we introduce and study. As an application, we give (sub)bilinear decompositions for commutators of singular integral operators which include almost all fundmental operators in harmonic analysis. Some Hardy estimates for commutators are also studied here. Finally, we investigate some related problems in the setting of Schrödinger harmonic analysis.

Keywords: Hardy spaces, wavelet, BMO, commutator, Schrödinger operator.

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