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Cyril Labbé

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Cyril Labbé. Flots stochastiques et représentation lockdown. Probability [math.PR]. Université Pierre et Marie Curie - Paris VI, 2013. English. NNT: . tel-00874551

**HAL Id: tel-00874551**

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# THÈSE DE DOCTORAT DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

Spécialité : Mathématiques

École Doctorale de Sciences Mathématiques de Paris Centre

présentée par

**Cyril LABBÉ**

pour obtenir le grade de

Docteur de l'Université Pierre et Marie Curie

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## FLOTS STOCHASTIQUES ET REPRÉSENTATION LOOKDOWN

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"Après de mon arbre, je vivais heureux,  
J'aurais jamais dû m'éloigner de mon arbre.  
Après de mon arbre, je vivais heureux,  
J'aurais jamais dû le quitter des yeux."

*Georges Brassens.*



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## Remerciements

Mes premiers remerciements sont adressés aux membres du jury, et d'abord à Amaury et Julien. Ils ont su me mettre à l'aise dès le début de la thèse et se sont toujours montrés optimistes vis-à-vis de mon travail. Je tiens également à leur dire que leur confiance et leurs encouragements tout au long de ces trois années ont été des facteurs importants pour moi. Je voudrais remercier Donald Dawson et Jean-François Le Gall d'avoir rapporté ma thèse. C'est une très grande joie de voir Jean Bertoin et Jean-François Le Gall parmi les membres du jury : leurs travaux sur les flots stochastiques ont été une référence ainsi qu'une source d'inspiration dans mon travail. Je suis également très heureux qu'Alison Etheridge ait accepté de participer au jury, et je tiens à la remercier pour les cinq mois passés à Oxford - séjour si enrichissant qui m'a permis de partir sur de nouveaux sujets. Enfin je ne remercierai jamais assez Thomas Duquesne pour tout ce qu'il m'a si patiemment appris en probabilités lors de notre collaboration.

Il y a six ou sept ans, j'étais loin d'imaginer que je continuerais mes études en mathématiques. Je suis profondément reconnaissant envers Erick Herbin pour m'avoir fait découvrir les probabilités durant mes années à Centrale et m'avoir fortement incité à poursuivre sur un master à Orsay. Je souhaiterais également remercier Grégory Miermont, qui en plus d'avoir été un enseignant exceptionnel à Orsay, a su m'orienter vers un sujet de mémoire riche et passionnant qui fut une excellente préparation pour ma thèse.

Au premier abord, le laboratoire peut sembler intimidant. Mais je dois dire que chaque fois que j'ai fini par oser frapper à une porte pour poser une question - évidemment triviale - j'ai toujours été très gentiment accueilli. Je pense en particulier à Jean Bertoin, Thomas Duquesne et Lorenzo Zambotti. Je voudrais également remercier Florence Deschamps, Maria Pochot et Josette Saman pour leur efficacité et leur gentillesse, Jacques Portès pour ses conseils avisés ainsi que Philippe Macé pour la chaleureuse ambiance dans la bibliothèque. En plus de cela, le LPMA compte un groupe de thésards qui décape. La liste est longue et je vais certainement en oublier. D'abord, ceux avec qui j'ai partagé mon bureau où la vie a été très agréable : Andreea, Karim, Reda, Bastien, Nelo, Guillaume et Pablo. Ceux du bureau juste au-dessus : Pascal, Xinxin et Julien. Mais également tout au bout du couloir : Mathieu, Eric, Raoul, Nikos, Antoine, Cécile, Alexandre(s) B. et G., Florian, Vincent, Pierre-Antoine, Yvain, Max. Je souhaiterais en particulier remercier Clément pour les nombreuses discussions que nous avons eues au cours de ces trois années mais également pour son humour ravageur et son *inévitabile* penchant pour le perrier. Un mot également pour Xan afin de le remercier pour tous les bons moments partagés ici, à Saint-Flour ou sur les éreintants deux-kilomètres qui encerclent le jardin du Luxembourg. Je voudrais saluer David Kelly - joyeux "ranga" - et le remercier pour son chaleureux accueil à Warwick. Merci également à Leif Doering, Mathieu Merle et Damien Simon pour de nombreuses discussions mathématiques.

A mes 3+1 colocs, Lio-FF-Chhoeu-Ben, avec qui j'ai partagé deux années au 54' : lever au son du youpi-matin, petit-déj devant le palais des papes, gratin de chou-fleur au menu et tisane de grand-mère pour abreuver un énième débat avant de s'endormir dans la cabane au fond du salon. Je n'oublie rien et je leur donne rendez-vous dans le Warwickshire pour de nouvelles aventures.

Je pense évidemment à mes parents et à mes soeurs, je tiens à les remercier pour leur soutien et leur bienveillance. Enfin, je ne serais probablement pas arrivé au bout de cette thèse sans Laure : son humeur joyeuse qui a si souvent remonté mon moral en berne, son optimisme inébranlable, son affection et ... ses récits de l'hôpital qui me rappellent qu'après tout, je n'ai entre les mains *que* des epsilon et des mouvements Browniens.

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## Résumé

Cette thèse est consacrée à l'étude de quelques propriétés mathématiques de deux modèles de population : le processus Fleming-Viot généralisé d'une part et le processus de branchement d'autre part. Dans les deux cas, la population est composée d'une infinité d'individus, chacun étant caractérisé par un type génétique. Au cours du temps les fréquences asymptotiques de ces types évoluent de façon aléatoire au travers d'événements de reproduction où un individu tiré aléatoirement donne naissance à une descendance portant le même type génétique.

Mathématiquement ces deux modèles sont décrits par des processus aléatoires à valeurs mesures. Afin de donner un sens à la généalogie de la population sous-jacente, plusieurs approches ont été proposées au cours des quinze dernières années. La contribution principale de cette thèse consiste en l'unification de deux constructions : la représentation lookdown définie par Peter Donnelly et Thomas Kurtz en 1999 et les flots stochastiques de ponts (ou de subordinateurs) introduits au début des années 2000 par Jean Bertoin et Jean-François Le Gall. Cette unification nécessite l'introduction d'objets nouveaux (les Eves, les flots stochastiques de partitions) et repose sur une étude fine des comportements asymptotiques des deux modèles mentionnés précédemment.

En particulier, nous définissons la propriété d'Eve comme suit : si la fréquence asymptotique d'un type génétique tend vers 1 lorsque  $t$  devient grand alors la population descend asymptotiquement d'un seul individu au temps initial, appelé l'Eve de la population. Dans le cas des processus de branchement nous obtenons une condition nécessaire et suffisante sur le paramètre du modèle (aussi appelé mécanisme de branchement) qui assure que cette propriété d'Eve est vérifiée. Nous obtenons également une classification complète de tous les autres comportements possibles. Dans le cas des processus Fleming-Viot généralisés, nous obtenons une classification partielle des comportements possibles en fonction du paramètre du modèle. Enfin, lorsque la propriété d'Eve est vérifiée, nous construisons de façon trajectorielle la représentation lookdown à partir d'un flot stochastique de ponts (ou de subordinateurs).

Nous présentons également une étude complète du processus de branchement explosif conditionné à la non-explosion et faisons apparaître une famille infinie de mesures quasi-stationnaires pour ce processus. Finalement nous nous intéressons au processus des longueurs du coalescent de Kingman dynamique et présentons une construction alternative à celle de Pfaffelhuber, Wakolbinger et Weisshaupt.



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## Abstract

This thesis focuses on mathematical properties of two population models, namely the generalised Fleming-Viot process and the branching process. In both cases, the population is composed of infinitely many individuals characterised by a genetic type. As time passes, the asymptotic frequencies of the types within the population evolve stochastically through reproduction events where a uniformly chosen individual gives birth to a progeny with the same genetic type.

Mathematically these two models are defined by measure-valued processes. In order to give a meaning to the genealogy of the underlying population, several approaches have been proposed these last fifteen years. One of the main contributions of this thesis is to unify two constructions: the lookdown representation introduced by Peter Donnelly and Thomas Kurtz in 1999 and the stochastic flow of bridges (or subordinators) introduced by Jean Bertoin and Jean-François Le Gall in 2000. This unification relies on the definition of new objects (the Eves, the stochastic flow of partitions) and necessitates a fine study of the asymptotic behaviours of the two aforementioned population models.

In particular we define the Eve property as follows: if there is a genetic type whose asymptotic frequency tends to 1 as  $t$  becomes large then the population asymptotically descends from a single ancestor called the Eve of the population. In the case of the branching process, we obtain a necessary and sufficient condition on the branching mechanism ensuring the Eve property. We also provide a complete classification of all the possible asymptotic behaviours according to the branching mechanism. In the case of the generalised Fleming-Viot process, we obtain a partial classification of the possible asymptotic behaviours. Finally when the Eve property is fulfilled we present a pathwise construction of the lookdown representation from a stochastic flow of bridges (or subordinators).

We also present a complete study of the explosive branching process conditioned to the non-explosion and provide an infinite collection of quasi-stationary distributions for this conditioned process. Finally we study the process of lengths of the evolving Kingman coalescent and propose an alternative construction to that of Pfaffelhuber, Wakolbinger and Weisshaupt.

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In this thesis, we study two classes of measure-valued processes:

- The generalised Fleming-Viot processes.
- The measure-valued branching processes.

These objects describe the evolution of infinite populations and can be obtained as scaling limits of elementary finite population models. In population genetics, one is naturally interested in tracking the genealogy of the population backward in time. Although it could seem a simple matter, giving a mathematical meaning to the genealogy of these measure-valued processes is not straightforward and has given rise to several approaches. In the present work, we consider two constructions: the so-called lookdown representation, introduced in 1999 by Peter Donnelly and Thomas Kurtz [24], and the stochastic flow of subordinators (or bridges), defined in 2000-2003 by Jean Bertoin and Jean-François Le Gall [12, 13].

We present a new framework that allows to unify these constructions. This framework relies on two key objects: the sequence of so-called Eves, which is a relevant ordering of the atomic support of the measure-valued processes, and the stochastic flow of partitions, which appears as the genealogical structure shared by both representations. These objects are interesting in their own right and motivate a fine study of the asymptotic behaviour of the two classes of measure-valued processes. In the introduction, this new framework is explained in detail and the main results are presented, sometimes in an informal way. However we shall always provide precise references to the formal statements.

After this introduction, the thesis is composed of five chapters that we now briefly present:

**Chapter I** focuses on the asymptotic properties of the generalised Fleming-Viot process and presents the coupling between the two representations. It corresponds to the article [52].

**Chapter II** presents several properties of the measure-valued branching process together with the coupling of the two representations in this case. This chapter forms the article [53] accepted for publication in the *Annales de l'Institut Henri Poincaré*.

**Chapter III** provides a classification of the asymptotic behaviours of the measure-valued branching process normalised by its total mass. It has been taken from the article [25] written in collaboration with Thomas Duquesne.

Chapter IV presents a complete study of the quasi-stationary distributions associated with an explosive continuous-state branching process. It has been taken from the article [54] published in Electronic Communications in Probability.

Chapter V proposes an alternative construction of the tree length process of the evolving Kingman coalescent. The original construction is due to Pfaffelhuber, Wakolbinger and Weisshaupt [65].

Finally, an Appendix gathers additional results and some technical proofs.

## 1 Two classes of measure-valued processes

Two classes of measure-valued processes have received a particular interest for many years: the generalised Fleming-Viot processes, which can be seen as the infinite population limit of the Moran model (see for instance Etheridge [29] and Fleming and Viot [32]); and the measure-valued branching processes, that generalise one-dimensional branching processes. We will see that these two classes present a lot of similarities, though the tools used to deal with them are different (duality for generalised Fleming-Viot processes, Laplace transform for measure-valued branching processes).

We mention that more elaborate versions of these processes, where the type of each individual evolves independently according to a Markov process, have been studied for many years (see Dawson [21], Etheridge [30] or Le Gall [60]). In this work, we do not consider these elaborate versions.

Let us now introduce the basic framework for the definition of these measure-valued processes. We are going to deal with populations of individuals that possess a type, assumed to be for simplicity a point in  $[0, 1]$ . Although the population is uncountably infinite, we characterize its *size* by a positive real number. Henceforth the population is described by a finite measure on  $[0, 1]$ : the mass given to any subinterval  $[a, b] \subset [0, 1]$  corresponds to the size of the subpopulation whose types lie in  $[a, b]$ . The total mass of this measure is then the size of the population. We denote by  $\mathcal{M}_f$  the set of finite measures on  $[0, 1]$ , and by  $\mathcal{M}_1$  the subset of probability measures on  $[0, 1]$ .

### 1.1 Generalised Fleming-Viot processes

The generalised Fleming-Viot processes take values in  $\mathcal{M}_1$  and are in duality with a famous class of coalescent processes introduced by Pitman [66] and Sagitov [67], called  $\Lambda$  coalescents. Therefore, these measure-valued processes are also called  $\Lambda$  Fleming-Viot.

Let  $\Lambda$  be a finite measure on  $[0, 1]$ . We start with a brief definition of the  $\Lambda$  coalescent. We denote by  $\mathcal{P}_n$  the set of partitions of  $[n] := \{1, 2, \dots, n\}$  for all  $n \in \mathbb{N} \cup \{\infty\}$ . The  $\Lambda$  coalescent is a  $\mathcal{P}_\infty$ -valued Markov process  $(\Pi_t, t \geq 0)$  with the following dynamics. For all  $n \geq 1$  and all  $t \geq 0$ , let  $\Pi_t^{[n]}$  be the restriction to  $\mathcal{P}_n$  of  $\Pi_t$ : if  $\Pi_t^{[n]}$  has  $m \in \{2, \dots, n\}$  blocks, then any  $k$  of them merge into a single block at rate

$$\lambda_{m,k} := \int_{[0,1]} u^k (1-u)^{m-k} u^{-2} \Lambda(du)$$

for all  $k \in \{2, \dots, m\}$ . One should think of a  $\Lambda$  coalescent as the genealogy of an infinite population: blocks involved in a coalescence event can be interpreted as lineages finding their common ancestor backward in time.

The  $\Lambda$  Fleming-Viot process is the corresponding forward-in-time population model.

**DEFINITION 0.1.** (Informal) Let  $\mathcal{N}$  be a Poisson point process on  $\mathbb{R}_+ \times [0, 1]$  with intensity  $dt \otimes u^{-2} \Lambda(du)$ . The  $\Lambda$  Fleming-Viot process  $(\rho_t, t \geq 0)$  can be constructed as follows:

- Initially, the population is composed of a continuum of types:  $\rho_0(dx) = dx$ .

- At each jump  $(t, u) \in \mathcal{N}$ , choose a parent  $y$  according to the distribution  $\rho_{t-}(\cdot)$  just before the jump, kill a fraction  $u$  of individuals uniformly chosen among the population and add a fraction  $u$  of individuals with the same type as the parent:

$$\rho_t(dx) = (1 - u)\rho_{t-}(dx) + u \delta_y(dx)$$

This definition is informal and does not cover the case where  $\Lambda$  has an atom at 0. A rigorous definition is given by the following martingale problem.

**DEFINITION 0.2.** (Bertoin-Le Gall [13]) *The  $\Lambda$  Fleming-Viot process  $\rho = (\rho_t, t \geq 0)$  is an  $\mathcal{M}_1$ -valued Markov process completely characterised by the following martingale problem. For every integer  $n \geq 1$ , let  $f$  be a continuous function on  $[0, 1]^n$  and  $\mu$  be an element of  $\mathcal{M}_1$ . Define*

$$\begin{aligned} G_f(\mu) &:= \int_{[0,1]^n} \mu(dx_1) \dots \mu(dx_n) f(x_1, \dots, x_n) \\ LG_f(\mu) &:= \sum_{\substack{K \subset [n] \\ k = \#K \geq 2}} \lambda_{n,k} \int_{[0,1]} \mu(dx_1) \dots \mu(dx_n) (f(R_K(x_1, \dots, x_{n-k+1})) - f(x_1, \dots, x_n)) \end{aligned}$$

where  $R_K(x_1, \dots, x_{n-k+1}) = (y_1, \dots, y_n)$  with  $y_i = x_{\min K}$  for all  $i \in K$  and  $y_i, i \notin K$  are the values  $x_1, \dots, x_{\min K-1}, x_{\min K+1}, \dots, x_{n-k+1}$  in the same order.

Then

$$G_f(\rho_t) - \int_0^t LG_f(\rho_s) ds$$

is a martingale.

To prove that this martingale problem characterises the law of a Markov process, Bertoin and Le Gall showed a duality relation between the one-dimensional marginals of the  $\Lambda$  Fleming-Viot and the  $\Lambda$  coalescent, we refer to [13] for further details.

**REMARK 0.3.** *We will always assume that  $\Lambda(\{1\}) = 0$  to avoid trivial behaviours. Indeed, an atom at 1 implies reproduction events involving all the individuals at once.*

From now on,  $\rho_0$  is the uniform measure on  $[0, 1]$  so that initially all the individuals have distinct types. This implies, together with the definition, that the distribution of  $(\rho_t, t \geq 0)$  is invariant under bijections from  $[0, 1]$  onto  $[0, 1]$  that preserve the Lebesgue measure. Therefore, the associated distribution function  $F_t(x) := \rho_t([0, x])$  verifies for all  $t \geq 0$ :

- $F_t(0) = 0, F_t(1) = 1$ .
- $x \mapsto F_t(x)$  is non-decreasing.
- $x \mapsto F_t(x)$  has exchangeable increments.

This makes  $F_t$  a bridge in the sense of Kallenberg [47], and therefore ensures that  $\rho_t$  has the following form

$$\rho_t(dx) = \sum_{i \geq 1} \rho_t(\{x_i\}) \delta_{x_i}(dx) + \left(1 - \sum_{i \geq 1} \rho_t(\{x_i\})\right) dx \quad (1)$$

where  $(x_i)_{i \geq 1}$  is a sequence of i.i.d. uniform  $[0, 1]$  r.v. Notice that this sequence can be finite. In Section 3 we will see how Bertoin and Le Gall exploited this fact to define the stochastic flow of bridges. We now provide some important examples of  $\Lambda$  Fleming-Viot processes, that will be of use in the sequel.



**Kingman coalescent / standard Fleming-Viot.**  $\Lambda(du) = \delta_0(du)$  corresponds to the celebrated standard Fleming-Viot process [32] whose genealogy is given by the Kingman coalescent [49], that is, the coalescent where only two blocks merge at once.

**Beta coalescent / Beta Fleming-Viot.** Take for  $\Lambda$  the distribution of a  $\text{Beta}(2 - \alpha, \alpha)$  r.v. with  $\alpha \in (0, 2)$

$$\Lambda(du) = \frac{u^{1-\alpha}(1-u)^{\alpha-1}}{\Gamma(2-\alpha)\Gamma(\alpha)} du$$

These  $\text{Beta}(2 - \alpha, \alpha)$  Fleming-Viot are in duality with the  $\text{Beta}(2 - \alpha, \alpha)$  coalescents that have been extensively studied, see [6, 7, 18] for instance.

**Bolthausen-Sznitman coalescent.** When  $\Lambda(du) = du$ , the  $\Lambda$  coalescent is called Bolthausen-Sznitman coalescent. Notice that it corresponds to the  $\text{Beta}(2 - \alpha, \alpha)$  coalescent with  $\alpha = 1$ . This object arose in the mathematical physics literature, see in particular [4, 12, 19, 37].

Some contributions of this thesis.

Here are two results on the standard Fleming-Viot / Kingman coalescent. The first concerns the number of atoms / blocks of these processes, the second focuses on the length of the evolving Kingman coalescent.

Denote by  $\#\mu$  (resp.  $\#\pi$ ) the number of atoms of a given measure  $\mu$  (resp. the number of blocks of a given partition of integers  $\pi$ ). It is well-known that  $\#\rho_t$  and  $\#\Pi_t$  are equal in law at any given time  $t \geq 0$ , whenever  $\rho$  and  $\Pi$  are a  $\Lambda$  Fleming-Viot process and a  $\Lambda$  coalescent.

**THEOREM (cf TH A.1)** *Suppose that  $\rho$  is the standard Fleming-Viot process and  $(\Pi_t, t \geq 0)$  is a Kingman coalescent. The processes  $(\#\rho_t, t \geq 0)$  and  $(\#\Pi_t, t \geq 0)$  have the same distribution.*

We refer to Appendix A for a discussion on a similar identity for other measures  $\Lambda$ .

One can naturally associate a real tree to a Kingman coalescent. We denote by  $\epsilon$ -erased Kingman coalescent the tree obtained after having erased a length  $\epsilon > 0$  from each leaf. Note also that the evolving Kingman coalescent is a consistent collection of Kingman coalescents obtained from the lookdown construction (see the next section).

**THEOREM (cf TH V.2 AND V.3)** *The process of length of the evolving  $\epsilon$ -erased Kingman coalescent can be suitably compensated so that it converges in the Skorohod's topology as  $\epsilon \downarrow 0$  toward a non-trivial limit, which is the same as the one obtained by Pfaffelhuber, Wakolbinger and Weisshaupt in [65].*

## 1.2 Measure-valued branching processes

We start with the definition of the continuous-state branching process (CSBP for short) introduced by Jirina [46] and Lamperti [57]. A CSBP is a  $[0, \infty]$ -valued Markov process that verifies the branching property: for any two given initial values  $x, x' \in [0, \infty]$ , the process starting from  $x + x'$  has the same distribution as the sum of two independent copies starting from  $x$  and  $x'$  respectively. Such a process is uniquely characterized by a so-called branching mechanism  $\Psi$ , which is a convex function of the form

$$\forall u \geq 0, \quad \Psi(u) = \gamma u + \frac{\sigma^2}{2} u^2 + \int_{(0, \infty)} (e^{-hu} - 1 + hu \mathbf{1}_{\{h < 1\}}) \nu(dh)$$

where  $\gamma \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\nu$  is a measure on  $(0, \infty)$  such that  $\int_{(0, \infty)} (1 \wedge h^2) \nu(dh) < \infty$ . Observe that  $\Psi$  is the Laplace exponent of a spectrally positive Lévy process.

**DEFINITION 0.4.** A  $\Psi$ -CSBP is a Markov process  $(Z_t, t \geq 0)$  whose semigroup is characterized by

$$\forall t \geq 0, \forall \lambda > 0, \quad \mathbb{E}[e^{-\lambda Z_t} | Z_0] = e^{-Z_0 u(t, \lambda)}$$

where

$$u(t, \lambda) = \lambda - \int_0^t \Psi(u(s, \lambda)) ds, \quad \forall t \geq 0, \lambda > 0$$

This semigroup is Feller so that the  $\Psi$ -CSBP can be constructed as a r.v. on  $\mathbb{D}([0, \infty), [0, \infty])$ . It is easy to deduce from the definition that a CSBP admits two absorbing states, namely 0 and  $+\infty$ . Hence we introduce the *lifetime* of  $Z$  as the stopping time  $T := T_0 \wedge T_\infty$  where

$$\begin{aligned} (\text{Extinction}) \quad T_0 &:= \inf\{t \geq 0 : Z_t = 0\} \\ (\text{Explosion}) \quad T_\infty &:= \inf\{t \geq 0 : Z_t = +\infty\} \end{aligned}$$

We refer to [60, 62] or Chapter II Subsection 2.2 for further details on CSBPs. Let us now introduce the measure-valued branching process (MVBP for short). In the setting of measure-valued processes, we say that the branching property is verified if for any given  $m, m' \in \mathcal{M}_f$ , the process starting from  $m + m'$  has the same distribution as the sum of two independent copies starting from  $m$  and  $m'$  respectively. In other terms, two disjoint subpopulations have independent and identically distributed (w.r.t. their initial conditions) evolutions. Processes verifying the branching property can see their total mass explode in finite time, therefore we need to define a point  $\Delta$  that stands for all the infinite measures. The space  $\overline{\mathcal{M}}_f := \mathcal{M}_f \cup \{\Delta\}$  is endowed with the so-called Watanabe topology, see [77] or Chapter II Subsection 2.3 for further details.

**DEFINITION 0.5.** A Markov process  $(m_t, t \geq 0)$  taking values in  $\overline{\mathcal{M}}_f$  and such that

$$\forall f \in \mathcal{B}^{++}([0, 1]), \quad \mathbb{E}[\exp(-\langle m_t, f \rangle) | m_0] = \exp\left(-\langle m_0, u(t, f(\cdot)) \rangle\right)$$

is called a *measure-valued branching process associated with  $\Psi$*  ( $\Psi$ -MVBP for short).

Recall that  $\mathcal{B}^{++}([0, 1])$  denotes the set of bounded Borel functions on  $[0, 1]$  with a strictly positive infimum. Note that  $\langle \Delta, f \rangle = +\infty$ . From now on, we assume that  $m_0$  is the Lebesgue measure on  $[0, 1]$ . From the definition, we derive that the size  $(m_t([0, x]), t \geq 0)$  of the subpopulation  $[0, x]$ , for any  $x \in [0, 1]$ , is a  $\Psi$ -CSBP starting from  $x$ . In particular, we denote by  $Z_t := m_t([0, 1]), t \geq 0$  the total-mass of  $m$  and we define the lifetime of  $m$ , say  $T$ , as the lifetime of  $Z$ .

Intuitively a  $\Psi$ -MVBP is a coupling of an infinity of  $\Psi$ -CSBPs, each of them starting from an initial condition in  $[0, 1]$ . An important fact is that  $x \mapsto m_t([0, x])$  is a (possibly killed) subordinator with Laplace exponent  $u(t, \cdot)$ , for every  $t \geq 0$ . Using the Lévy-Khintchine formula, there exist  $a_t, d_t \geq 0$  and a measure  $w_t$  verifying  $\int_{(0, \infty)} (1 \wedge h) w_t(dh) < \infty$  such that

$$\forall \lambda > 0, \quad u(t, \lambda) = a_t + d_t \lambda + \int_{(0, \infty)} (1 - e^{-h\lambda}) w_t(dh)$$

When  $a_t$  is strictly positive, the  $\Psi$ -CSBP has a positive probability to explode in finite time: we say that the branching mechanism is non-conservative (see Grey [40] or Subsection 2.2 for a necessary and sufficient condition on  $\Psi$  for being non-conservative). From the Lévy-Khintchine decomposition, we deduce that  $m_t$  (conditional on the event that it has not exploded yet) has the following form

$$m_t(dx) = \sum_{i \geq 1} m_t(\{x_i\}) \delta_{x_i}(dx) + d_t dx \tag{2}$$

where  $(x_i)_{i \geq 1}$  is a sequence of i.i.d. uniform $[0, 1]$  r.v. Observe the similarity with (1).

One should notice that our previous definition characterizes a unique Markov process: the semigroup is completely characterized by the Laplace functional, see for instance Section 3.5 in [21]. This characterization from Laplace transforms is fundamentally different from that of  $\Lambda$  Fleming-Viot via a martingale problem and involving duality. However, as it will be useful in the lookdown construction, we recall in Appendix D a martingale problem that characterizes this Markov process. We end this subsection with a selection of important examples.

**Feller diffusion.** When  $\Psi(u) = \frac{\sigma^2}{2}u^2$ , the  $\Psi$ -CSBP is called the Feller diffusion and we have

$$u(t, \lambda) = \frac{2\lambda}{2 + \sigma^2\lambda t}, \quad \forall t \geq 0, \lambda > 0.$$

**$\alpha$ -stable branching mechanism,  $\alpha \in (1, 2)$ .** Take  $\Psi(u) = k u^\alpha$  with  $\alpha \in (1, 2)$  and  $k > 0$ , then  $\Psi$  is the Laplace exponent of an  $\alpha$ -stable Lévy process. In that case

$$\nu(dh) = \frac{k \alpha (\alpha - 1)}{\Gamma(2 - \alpha) h^{1+\alpha}} dh, \quad \gamma = \int_{[1, \infty)} h \nu(dh), \quad \sigma = 0.$$

The associated  $\Psi$ -CSBP is self-similar with index  $\alpha - 1$  and we have

$$u(t, \lambda) = (\lambda^{1-\alpha} + (\alpha - 1)kt)^{-1/(\alpha-1)}, \quad \forall t \geq 0, \lambda > 0.$$

**Neveu branching mechanism.** Take  $\Psi(u) = u \log u$ . In that case

$$\nu(dh) = h^{-2}dh, \quad \gamma = 1 - \int_{(0, \infty)} (\mathbf{1}_{\{h < 1\}} - e^{-h})h^{-1}dh, \quad \sigma = 0.$$

This  $\Psi$ -CSBP is due to Neveu and we have

$$u(t, \lambda) = \lambda e^{-t}, \quad \forall t \geq 0, \lambda > 0.$$

**$\alpha$ -stable branching mechanisms,  $\alpha \in (0, 1)$ .** Take  $\Psi(u) = -k u^\alpha$  with  $\alpha \in (0, 1)$  and  $k > 0$ , then  $\Psi$  is the Laplace exponent of an  $\alpha$ -stable Lévy process. In that case

$$\nu(dh) = \frac{k \alpha (1 - \alpha)}{\Gamma(2 - \alpha) h^{1+\alpha}} dh, \quad \gamma = - \int_{(0, 1)} h \nu(dh), \quad \sigma = 0.$$

We have

$$u(t, \lambda) = (\lambda^{1-\alpha} + (1 - \alpha)kt)^{1/(1-\alpha)}, \quad \forall t \geq 0, \lambda > 0.$$

Some contributions of this thesis.

The following lemma gives the formula of the drift term that appears in the Lévy-Khintchine decomposition of  $u(t, \cdot)$ : Silverstein proved in [72] that this drift equals 0 whenever  $\Psi$  is of infinite variation form (i.e.  $\sigma > 0$  or  $\int_{(0, 1)} h \nu(dh) = +\infty$ ), but it seems that the expression of this drift in the finite variation case does not appear in the literature.

**LEMMA.** (cf **LEMMA III.2 OR PROP IV.3**) *When  $\Psi$  is the Laplace exponent of a Lévy process with infinite variation paths,  $d_t = 0$ . Otherwise, for all  $t \geq 0$  we have*

$$d_t = e^{-Dt}, \quad \text{where } D := \gamma + \int_{(0, 1)} h \nu(dh).$$

The next theorem is a complete description of the quasi-stationary distributions associated with CSBPs that explode in finite time. We call quasi-stationary distribution (associated with the  $\Psi$ -CSBP  $Z$ ) a probability measure  $\mu$  on  $(0, \infty)$  such that

$$\mathbb{P}_\mu(Z_t \in \cdot \mid T > t) = \mu(\cdot)$$

where  $\mathbb{P}_\mu$  is the distribution on  $\mathbb{D}([0, \infty), [0, \infty])$  of the  $\Psi$ -CSBP with initial distribution  $\mu$ .

**THEOREM** (cf **TH IV.1, IV.2 AND IV.3**) *Consider a branching mechanism  $\Psi$  such that  $T < \infty$  and  $Z_T = \infty$  almost surely and set*

$$\Phi(\lambda) := \int_\lambda^0 \frac{du}{\Psi(u)}, \quad \forall \lambda \geq 0$$

*For any  $\beta > 0$  there exists a unique quasi-stationary distribution  $\mu_\beta$  associated to the rate of decay  $\beta$ . This distribution is infinitely divisible and is characterized by*

$$\int_{(0, \infty)} \mu_\beta(dr) e^{-r\lambda} = e^{-\beta\Phi(\lambda)}, \quad \forall \lambda \geq 0$$

*Additionally, the following dichotomy holds true:*

- (i)  $\Psi(+\infty) \in (-\infty, 0)$ . *The limiting conditional distribution is given by*

$$\lim_{t \rightarrow \infty} \mathbb{P}_x(Z_t \in \cdot \mid T > t) = \mu_{x\nu(0, \infty)}(\cdot), \quad \forall x \in (0, \infty)$$

- (ii)  $\Psi(+\infty) = -\infty$ . *The limiting conditional distribution is trivial:*

$$\lim_{t \rightarrow \infty} \mathbb{P}_x(Z_t \leq a \mid T > t) = 0, \quad \forall a, x \in (0, \infty)$$

*However, under some regularity assumptions on  $\Psi$ , there exists a function  $f$  such that  $Z_t/f(t)$  conditioned on non-explosion converges to a non-trivial limit.*

*Finally, in any case the  $Q$ -process associated with the  $\Psi$ -CSBP is itself a CSBP with branching mechanism*

$$u \mapsto Du.$$

*This is the drift part of  $\Psi$  as identified in the previous lemma.*

We refer to Chapter IV for more details.

### 1.3 Classification of the behaviours

The specific forms (1) and (2) taken by  $\rho_t$  and  $m_t$  are very close. Actually many connections have been established between these two classes of measure-valued processes. We recall them briefly, and then we present a classification that we develop in Chapters I and II.

*Connection for stable processes.* A well-known and beautiful result due to Birkner and al. [18] shows a pathwise connection between  $\alpha$ -stable MVBPs and Beta( $2 - \alpha, \alpha$ ) Fleming-Viot processes. Consider a  $\Psi$ -MVBP  $m$  and let  $\tau : [0, T) \rightarrow [0, \infty)$  be a random time change (adapted to the natural filtration of  $m$ ). Then the process

$$\rho_t := \frac{m_{\tau^{-1}(t)}}{m_{\tau^{-1}(t)}[0, 1]}$$

is a  $\Lambda$  Fleming-Viot process if and only if we are in one the following cases

(a) Feller diffusion / standard Fleming-Viot

$$\Psi(u) = \frac{\sigma^2}{2} u^2, \quad \tau(t) = \int_0^t \frac{\sigma^2}{Z_s} ds, \quad \Lambda(du) = \delta_0(du)$$

(b)  $\alpha$ -stable MVBP / Beta( $2 - \alpha, \alpha$ ) Fleming-Viot

$$\Psi(u) = k u^\alpha, \quad \tau(t) = \frac{1}{k \alpha (\alpha - 1) \Gamma(\alpha)} \int_0^t \frac{ds}{Z_s^{\alpha-1}}, \quad \Lambda \text{ is a Beta}(2 - \alpha, \alpha), \text{ with } \alpha \in (1, 2)$$

or

$$\Psi(u) = -k u^\alpha, \quad \tau(t) = \frac{-1}{k \alpha (\alpha - 1) \Gamma(\alpha)} \int_0^t Z_s^{1-\alpha} ds, \quad \Lambda \text{ is a Beta}(2 - \alpha, \alpha), \text{ with } \alpha \in (0, 1)$$

(c) Neveu branching mechanism / Bolthausen-Sznitman coalescent

$$\Psi(u) = u \log u, \quad \tau(t) = t, \quad \Lambda(du) = du$$

*Connection for small populations.* Consider a critical branching mechanism, that is, a branching mechanism  $\Psi$  of the form

$$\Psi(u) = \frac{\sigma^2}{2} u^2 + \int_{(0, \infty)} (e^{-hu} - 1 - hu) \nu(dh)$$

with  $\sigma \geq 0$  and  $\int_{(0, \infty)} (h \wedge h^2) \nu(dh) < \infty$ . For every  $a > 0$ , let  $\nu^{(a)} := \nu \mathbf{1}_{(0, a)}$  be the restriction of  $\nu$  to  $(0, a)$  and  $\tilde{\nu}^{(a)}$  its pushforward through the contraction  $r \mapsto r/a$  so that  $u^2 \tilde{\nu}^{(a)}(du)$  is a finite measure on  $(0, 1)$ . Therefore, one can consider the generalised Fleming-Viot process  $\tilde{\rho}^{(a)}$  associated with the measure  $u^2 \tilde{\nu}^{(a)}(du)$ . Bertoin and Le Gall proved in [15] the following convergence

$$(a \tilde{\rho}_{at}^{(a)}([0, a^{-1}], t \geq 0) \xrightarrow{a \rightarrow \infty} (Z_t, t \geq 0)$$

where  $Z$  is a  $\Psi$ -CSBP starting from 1 and the convergence holds in  $\mathbb{D}([0, \infty), [0, \infty))$ .

Let us make the following simple comment. The process  $t \mapsto \tilde{\rho}_t^{(a)}([0, a^{-1}])$  makes both positive and negative jumps. The latter are due to the constraint on the population size in the Fleming-Viot model and form the major difference with the behaviour of a CSBP. The result of Bertoin and Le Gall shows that a small subpopulation tends to feel less this constraint.

*Connection for small times.* Berestycki, Berestycki and Limic [5] constructed a coupling between a  $\Lambda$  Fleming-Viot (when  $\Lambda(\{0\}) = 0$ ) and a  $\Psi$ -MVBP where

$$\Psi(u) := \int_{(0, 1)} (e^{-hu} - 1 - hu) \nu(dh) \quad \text{and} \quad \nu(dh) := h^{-2} \Lambda(dh), \quad \forall h \in (0, 1).$$

They remarked that these two processes can be defined from a same Poisson point process in the look-down representation (up to a random time change for the MVBP). For small times, this random time change is close to the identity, and therefore, the two measure-valued processes are close. In particular, this coupling gives a probabilistic explanation to the condition of Bertoin and Le Gall [15] for the coming down from infinity of the  $\Lambda$  coalescent (see below).

**A classification into four regimes.** These three connections have been a source of motivation for a systematic comparison of the two classes of measure-valued processes. In order to present the results in a unified way, we adopt the notation  $\gamma = 0$ ,  $\sigma^2 = \Lambda(\{0\})$  and  $\nu(dh) = h^{-2}\Lambda(dh)$  along with  $T = \infty$  in the  $\Lambda$  Fleming-Viot case and so

$$\Psi(u) := \frac{\Lambda(\{0\})}{2} u^2 + \int_{(0,1)} (e^{-hu} - 1 - hu)h^{-2}\Lambda(dh), \quad \forall u \geq 0.$$

For the MVBP, these quantities keep the same meaning. Recall the general form (1), (2) taken by  $\rho_t, m_t$ : it is composed of an atomic part and a Lebesgue part, that we now call dust component. Our classification relies on the following two criteria:

- a) The number of atoms for all  $t \in (0, T)$ : finite or infinite.
- b) The dust component for all  $t \in (0, T)$ : positive or null.

According to  $(\gamma, \sigma, \nu)$ , we identify four regimes for the two classes of measure-valued processes:

- REGIME 1:  $\nu((0, 1)) < \infty$  and  $\sigma = 0$ .
- REGIME 2:  $\nu((0, 1)) = \infty$ ,  $\int_{(0,1)} h\nu(dh) < \infty$  and  $\sigma = 0$ .
- REGIME 3:  $\int_{(0,1)} h\nu(dh) = \infty$  and  $\int^\infty \frac{du}{\Psi(u)} = \infty$ .
- REGIME 4:  $\int^\infty \frac{du}{\Psi(u)} < \infty$ .

Each regime corresponds to a given behaviour in terms of our two criteria above.

		REGIME 1	REGIME 2	REGIME 3	REGIME 4
		<i>Finitely many events</i> $\nu((0, 1)) < \infty$ and $\sigma = 0$	<i>Infinitely many events</i> $\nu((0, 1)) = \infty$ or $\sigma > 0$		
		<i>Finite variation</i> $\int_{(0,1)} h\nu(dh) < \infty$ and $\sigma = 0$	<i>Infinite variation</i> $\int_{(0,1)} h\nu(dh) = \infty$ or $\sigma > 0$		
		<i>No CDI</i> $\int^\infty \frac{du}{\Psi(u)} = \infty$			<i>CDI</i> $\int^\infty \frac{du}{\Psi(u)} < \infty$
$\Lambda$ F-V	# of atoms	$< \infty$	$\infty$	$\infty$	$< \infty$
	Dust	✓	✓		
$\Psi$ -MVBP	# of atoms	$< \infty$	$\infty$	$\infty$	$< \infty$
	Dust	✓	✓		

Let us make some comments on this table. We start with the  $\Lambda$  F-V. The condition for *Finitely many events* / *Infinitely many events* is due to Freeman [35]: it ensures that the number of non-singleton blocks in a  $\Lambda$  coalescent is finite / infinite in REGIME 1 / REGIME 2. The duality with  $\Lambda$  F-V ensures that the same holds when considering the number of atoms of the  $\Lambda$  F-V instead of the number of blocks of the  $\Lambda$  coalescent. Notice that Freeman's proof relies on the representation through flows of bridges: we propose another proof of this result based on the lookdown representation in Theorem B.1. The condition for the split *Finite variation* / *Infinite variation* is due to Pitman [66]: it ensures that the  $\Lambda$  coalescent has infinitely many / zero singleton blocks. Once again the duality ensures the presence / absence of a dust component for the  $\Lambda$  F-V. The last condition on the *No CDI* / *CDI* is due to Bertoin and Le Gall [15] but a former and equivalent condition was stated by Schweinsberg [69] as follows

$$\sum_{n=2}^{\infty} \left( \sum_{k=2}^n (k-1) \binom{n}{k} \lambda_{n,k} \right)^{-1} < \infty.$$

Actually these two equivalent conditions determine whether the  $\Lambda$  coalescent has finitely many blocks at time  $0+$  or infinitely many at any time, almost surely. This explains the denomination coming down from infinity (CDI). By analogy, we say that the  $\Lambda$  Fleming-Viot comes down from infinity when it has finitely many atoms (and no dust) immediately after time 0.

**EXAMPLE 0.6.** *The standard and the Beta( $2 - \alpha, \alpha$ ) Fleming-Viot, with  $\alpha \in (1, 2)$ , belong to REGIME 4. The  $\Lambda$  Fleming-Viot associated to the Bolthausen-Sznitman coalescent is in REGIME 3. Finally the Beta( $2 - \alpha, \alpha$ ) Fleming-Viot, with  $\alpha \in (0, 1)$ , belongs to REGIME 2.*

We turn our attention to the  $\Psi$ -MVBP. The condition for *Finitely many events / Infinitely many events* is proven in Theorem B.2. The condition for the split *Finite variation / Infinite variation* actually determines whether the  $\Psi$ -CSBP has finite or infinite variation paths. That it is a necessary and sufficient condition for the presence of dust is a consequence of Proposition IV.3. Finally the last condition for *No CDI / CDI* is due to Grey [40] and determines whether the  $\Psi$ -CSBP has a positive probability of reaching 0 in finite time. When it does, it is elementary to check that the Lévy measure  $w_t$  of  $u(t, \cdot)$  has a finite mass, and so,  $m_t$  has finitely many atoms.

**EXAMPLE 0.7.** *The Feller diffusion as well as the  $\alpha$ -stable MVBPs, with  $\alpha \in (1, 2)$ , belong to REGIME 4. The Neveu MVBP is in REGIME 3. Finally the  $\alpha$ -stable MVBPs, with  $\alpha \in (0, 1)$ , belong to REGIME 2.*

## 1.4 The Eves

In the previous subsection, we have described the measure  $\rho_t$  (resp.  $m_t$ ) at any given time  $t \in (0, T)$ . In particular, we have paid attention to its number of atoms. It is important to notice that, as time passes, the atomic support does not ‘move’ but only evolves when an atom gets a positive mass or loses its mass. To formalise this idea, we introduce the following notion.

**DEFINITION 0.8.** *A point  $x \in [0, 1]$  is called ancestral type if there exists  $t \in [0, T)$  such that  $\rho_t(\{x\}) > 0$  (resp.  $m_t(\{x\}) > 0$ ).*

An ancestral type  $x \in [0, 1]$  is a genetic type carried by an ancestor with a significant progeny at a given time. We characterize its emergence (the first time it has a strictly positive progeny) and its decay (the last time it has a strictly positive progeny). Specifically, it emerges either immediately after time 0, in REGIMES 3 & 4; or at a strictly positive time, in REGIMES 1 & 2. Concerning the decay, it can either never become extinct; or become extinct at a finite time  $d := \sup\{t \geq 0 : \rho_t(\{x\}) > 0\}$  (resp.  $d := \sup\{t \in [0, T) : m_t(\{x\}) > 0\}$ ), this happens only in REGIME 4. We can now reinterpret the table of the preceding subsection.

- REGIME 1. Finitely many ancestral types emerge on any compact interval of time so that the atomic support at any time is finite. As time passes, the atomic support gets larger and larger since none of the ancestral types become extinct in finite time.
- REGIME 2. The times at which ancestral types emerge are dense in  $(0, \infty)$ . The atomic support "increases" as time passes, since here again, none of the ancestral types become extinct in finite time.
- REGIME 3. The ancestral types emerge immediately after time 0 and do not become extinct in finite time.
- REGIME 4. The ancestral types emerge also immediately after time 0. The difference with the preceding regime is that infinitely many (all but one for the  $\Lambda$  Fleming-Viot / all but a Poisson number with parameter the second root of  $\Psi$  for the  $\Psi$ -MVBP) become extinct in finite time.

So far, we have only considered the *number* of ancestral types. It is natural to ask for a description of the relative sizes of the progenies. We define the *frequency* at time  $t \geq 0$  of an ancestral type  $x$  as the quantity  $\rho_t(\{x\})$  (resp.  $m_t(\{x\})/m_t([0, 1])$ ). Below we present a study of the behaviour of the frequencies when  $t$  goes to  $T$ . This will lead us to a crucial property in this thesis: we say that the Eve property is fulfilled if there exists an ancestral type  $e$  whose frequency converges to 1 as  $t \rightarrow T$ .

The  $\Lambda$  Fleming-Viot case.

**THEOREM** (Bertoin-Le Gall [13]) *There exists a random variable  $e \in [0, 1]$ , uniformly distributed, such that*

$$\rho_t(\{e\}) \xrightarrow[t \rightarrow \infty]{} 1$$

*This point is called the primitive Eve of the population.*

This result ensures that the Eve property is fulfilled by the  $\Lambda$  Fleming-Viot process, without condition on  $\Lambda$ . Notice that  $e$  is an ancestral type according to our definition. To further the study, one would like to have information on those individuals that do not descend from  $e$ . To that end, we propose a generalisation of the primitive Eve of Bertoin and Le Gall.

**DEFINITION 0.9.** *According to the regime of  $\Lambda$ , we introduce the Eves as follows:*

*Eves - persistent case.* In REGIMES 1, 2 & 3, we say that the  $\Lambda$  Fleming-Viot admits an infinite sequence of Eves if there exists a collection  $(e^i)_{i \geq 1}$  of r.v. such that almost surely for all  $i \geq 1$

$$\frac{\rho_t(\{e^i\})}{\rho_t([0, 1] \setminus \{e^1, \dots, e^{i-1}\})} \xrightarrow[t \rightarrow \infty]{} 1$$

*Eves - extinction case.* In REGIME 4, we say that the  $\Lambda$  Fleming-Viot admits an infinite sequence of Eves if one can order the ancestral types by strictly decreasing extinction times, the sequence is then denoted by  $(e^i)_{i \geq 1}$ .

We now present results ensuring the existence / non-existence of the sequence of Eves.

**THEOREM** (cf TH I.1) *Suppose that:*

- $\Lambda$  is in REGIME 1, or
- $\Lambda$  is in REGIME 2 and fulfils  $\int_{(0,1)} \nu(du) u \log \frac{1}{u} < \infty$

*then the  $\Lambda$  Fleming-Viot does not admit an infinite sequence of Eves.*

Let us comment briefly this result. In REGIME 1, we show that after a certain random time all the elementary reproduction events choose a parent with type  $e$ : as a consequence, no other ancestral type gets an overwhelming progeny in the remaining population. In REGIME 2, we show that under the  $u \log \frac{1}{u}$  condition, the dust component is still large (the precise statement strongly relies on the lookdown representation, we refer to Chapter I) so that one cannot identify a recursive sequence of Eves. When the  $u \log \frac{1}{u}$  condition does not hold, we conjecture that, even if the dust component becomes negligible, no ancestral type gets an overwhelming progeny, see Conjecture I.35.

We turn to REGIME 3.

**PROPOSITION** (cf PROP I.4) *When  $\Lambda(du) = du$ , the  $\Lambda$  Fleming-Viot admits an infinite sequence of Eves.*



Unfortunately the existence of this sequence of Eves for other measures in that regime remains an open question, see Conjecture I.36.

In REGIME 4, the definition of the Eves requires distinct extinction times. Consequently we introduce the following event

$$\mathbf{E} := \{\text{There exists } t > 0 \text{ s.t. at least two ancestral types become extinct simultaneously at time } t\}$$

On the complementary event, the ancestral types become extinct at distinct times and therefore, they can be naturally ordered by decreasing extinction times. Conversely on  $\mathbf{E}$ , the ancestral types cannot be ordered by decreasing extinction times and therefore the  $\Lambda$  Fleming-Viot process does not admit an infinite sequence of Eves.

**THEOREM** (cf **TH I.2**) *Consider REGIME 4. The event  $\mathbf{E}$  is trivial, that is,  $\mathbb{P}(\mathbf{E}) \in \{0, 1\}$ . If  $\Lambda(\{0\}) > 0$  or  $\Lambda(du) = f(u)du$  where  $f$  is a regularly varying function at  $0+$  with index  $1 - \alpha$  with  $\alpha \in (1, 2)$  then  $\mathbb{P}(\mathbf{E}) = 0$  so that the  $\Lambda$  Fleming-Viot admits an infinite sequence of Eves.*

We conjecture that  $\mathbb{P}(\mathbf{E}) = 0$  in REGIME 4 without further condition on  $\Lambda$ , see Conjecture I.37.

The  $\Psi$ -MVBP case.

The *Eve property* can be rephrased in the branching setting as follows.

**DEFINITION 0.10.** *We say that the branching mechanism  $\Psi$  verifies the Eve property on the event  $\{Z_T = 0\}$  (resp.  $\{Z_T = +\infty\}$ ) if there exists a random variable  $e$  on  $[0, 1]$  such that*

$$\frac{m_t(\{e\})}{m_t([0, 1])} \xrightarrow{t \rightarrow T} 1$$

*almost surely on  $\{Z_T = 0\}$  (resp.  $\{Z_T = +\infty\}$ ). This point is called the primitive Eve.*

As we will see, there exists a large class of branching mechanisms for which the Eve property is not fulfilled. Therefore, one is naturally interested in the other possible behaviours of the ratio  $M_t(\cdot) := m_t(\cdot)/m_t([0, 1])$  as  $t \rightarrow T$ . In collaboration with Thomas Duquesne, we have obtained a complete classification. First we have proved that  $M_\infty := \lim_{t \rightarrow T} M_t$  exists almost surely, and that the convergence holds for the total variation distance. Then we have a description of  $M_\infty$  according to the branching mechanism  $\Psi$ : we rely on the following definitions. We call settlers the atoms of  $M_\infty$ , and dust its continuous part with respect to Lebesgue measure. When  $M_\infty$  has no dust, and the number of settlers is almost surely 1, we say that  $M_\infty$  verifies the Eve property. Finally we denote by  $q$  the second root of  $\Psi$ .

**THEOREM** (cf **TH III.2**) *We assume that  $\Psi$  is a non-linear branching mechanism. Then the following holds true almost surely*

- (i) *On the event  $A = \{T < \infty\}$ ,  $M_\infty$  verifies the Eve property.*
- (ii) *On the event  $B = \{T = \infty; \lim_{t \rightarrow \infty} Z_t = \infty\}$ :*
  - (ii-a) *If  $\Psi'(0+) = -\infty$ , then  $M_\infty$  verifies the Eve property.*
  - (ii-b) *If  $\Psi'(0+) \in (-\infty, 0)$  and  $q < \infty$ ,  $M_\infty$  has no dust and finitely many settlers whose number, under  $\mathbb{P}(\cdot | B)$ , is distributed as a Poisson r.v. with mean  $q$  conditioned to be non zero.*
  - (ii-c) *If  $\Psi'(0+) \in (-\infty, 0)$  and  $q = \infty$ ,  $M_\infty$  has no dust and infinitely many settlers that form a dense subset of  $[0, 1]$ .*
- (iii) *On the event  $C = \{T = \infty; \lim_{t \rightarrow \infty} Z_t = 0\}$ :*

- (iii-a) If  $\Psi$  is of infinite variation type, then  $M_\infty$  verifies the Eve property.
- (iii-b) If  $\Psi$  is of finite variation type, then the following holds true:
- (iii-b-1) If  $\nu((0, 1)) < \infty$ , then  $M_\infty$  has dust and finitely many settlers whose number, under  $\mathbb{P}(\cdot | C)$ , is distributed as a Poisson r.v. with mean  $\frac{1}{D} \int_{(0, \infty)} e^{-qh} \nu(dh)$ .
  - (iii-b-2) If  $\nu((0, 1)) = \infty$  and  $\int_{(0, 1)} \nu(dh) h \log 1/h < \infty$ , then  $M_\infty$  has dust and infinitely many settlers that form a dense subset of  $[0, 1]$ .
  - (iii-b-3) If  $\int_{(0, 1)} \nu(dh) h \log 1/h = \infty$ , then  $M_\infty$  has no dust and infinitely many settlers that form a dense subset of  $[0, 1]$ .

Case (i) is relatively trivial: the extinction / explosion time has a density w.r.t. the Lebesgue measure so that two independent  $\Psi$ -CSBPs become extinct / explode at distinct times almost surely. Cases (ii-b) and (ii-c) are rather intuitive: when the  $\Psi$ -CSBP is super-critical with a finite mean, two independent copies grow at comparable speeds. Cases (ii-a) and (iii-a) are perhaps the most surprising. Let us also comment briefly the behaviour of the dust. First the dust component of  $M_\infty$  is always the limit of the dust of  $M_t$  when  $t \rightarrow T$ : in other words, it cannot be the result of atoms becoming negligible at the infinite. Second, observe that we have obtained a  $h \log h$  condition for the existence of dust at the limit in case (iii-b): this kind of condition appears regularly in the literature on branching processes [40, 64].

This result yields a necessary and sufficient condition on  $\Psi$  for the Eve property to hold, namely: on  $\{Z_T = 0\}$ ,  $\Psi$  has to be of infinite variation type and on  $\{Z_T = \infty\}$ ,  $\Psi'(0+)$  has to be infinite. Let us further the study of those  $\Psi$ -MVBPs fulfilling the Eve property: we introduce the counterpart of Definition 0.9.

**DEFINITION 0.11.** According to the regime of  $\Psi$ , we introduce the Eves as follows:

*Eves - persistent case.* In REGIMES 1, 2 & 3, we say that the  $\Psi$ -MVBP admits an infinite sequence of Eves if there exists a collection  $(e^i)_{i \geq 1}$  of r.v. such that almost surely for all  $i \geq 1$

$$\frac{\rho_t(\{e^i\})}{\rho_t([0, 1] \setminus \{e^1, \dots, e^{i-1}\})} \xrightarrow{t \rightarrow \infty} 1$$

*Eves - extinction case.* In REGIME 4 when  $\Psi'(0+) \geq 0$ , we say that the  $\Psi$ -MVBP admits an infinite sequence of Eves if one can order the ancestral types by strictly decreasing extinction times, the sequence is then denoted by  $(e^i)_{i \geq 1}$ .

*Eves - hybrid case.* In REGIME 4 when  $\Psi'(0+) \in [-\infty, 0)$ , we say that the  $\Psi$ -MVBP admits an infinite sequence of Eves if one can order the (finitely many) super-critical ancestral types according to the first criterion and then the (infinitely many) sub-critical ancestral types according to the second criterion, the sequence is then denoted by  $(e^i)_{i \geq 1}$ .

Notice that we call super-critical those ancestral types whose frequency goes to infinity.

**THEOREM (cf TH II.5)** Assume that  $\Psi$  is conservative. If the Eve property holds then the  $\Psi$ -MVBP admits an infinite sequence of Eves.

The non-conservative case is different, we refer to Remark II.24.

**Conclusion.** Let us sum up the results exposed in this section. In the  $\Lambda$  Fleming-Viot case, the Eve property is always fulfilled, but the infinite sequence of Eves does not always exist. In the  $\Psi$ -MVBP case, the Eve property is not always fulfilled. But when it is fulfilled (and when  $\Psi$  is conservative), the MVBP always admits an infinite sequence of Eves. Whereas we have a complete classification of the asymptotic behaviours in the branching case, some questions remain open in the  $\Lambda$  Fleming-Viot case (we refer to Chapter I Subsection 4.5 for conjectures): this asymmetry is mainly due to the powerful log-Laplace identity which is missing in the latter case.

Let us also mention that the sequence of Eves, in both cases, verifies some nice properties: for instance they are i.i.d. uniform $[0, 1]$ . We will see later on that they have a profound meaning in the lockdown representation.

## 2 A new formulation of Donnelly and Kurtz's lockdown representation

The purpose of this section is to provide a new formulation of the celebrated lockdown representation of Donnelly and Kurtz [24]. Our presentation does not require previous familiarity with the original article as all the arguments will be provided, but it will also shed a new light on this object. The novelty is primarily due to the introduction of a stochastic flow of partitions encoding the so-called lockdown graph. We also mention that Foucart has developed independently a very similar object for generalised Fleming-Viot processes with immigration [34].

The lockdown construction focuses on neutral population models, that is, population models where the genetic type of an individual does not modify its chances to reproduce or die. This property is fundamental as it allows to construct the genealogical structure regardless of the types carried by the lineages. Although this construction is very general and applies to many objects, we will restrict ourselves to the  $\Lambda$  Fleming-Viot and the  $\Psi$ -MVBP.

We briefly present the objectives of the lockdown construction. Suppose we are given the population size  $t \mapsto Z_t$  (we set  $Z_t = 1$  in the  $\Lambda$  Fleming-Viot case for convenience). We want to define a countable collection of random processes  $(\xi_t(i), t \geq 0)_{i \geq 1}$  such that:

- (i) The process  $t \mapsto (Z_t, \xi_t(1), \xi_t(2), \dots)$  is Markov.
- (ii) At every time  $t \geq 0$ , the sequence  $(\xi_t(i))_{i \geq 1}$  admits a limiting empirical measure

$$\Xi_t(dx) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{\xi_t(i)}(dx)$$

- (iii) The process  $(Z_t \cdot \Xi_t, t \geq 0)$  is a  $\Lambda$  Fleming-Viot process or a  $\Psi$ -MVBP.

The collection  $(\xi_t(i), t \geq 0)_{i \geq 1}$  should be understood as a particle system that discretizes a measure-valued process. To define this particle system, we have to specify:

- The initial types  $(\xi_0(i))_{i \geq 1}$ .
- The dynamics of the particle system.

Our presentation starts with the definition of the dynamics which is given by a stochastic flow of partitions. In a second part, we look at the main properties of this object and derive the exchangeability property of  $(\xi_t(i))_{i \geq 1}$ , which is crucial in this construction. The third part is devoted to the identification of the measure-valued process through a martingale problem.

**Notation.** We denote by  $\mathcal{P}_\infty$  the set of partitions of  $\mathbb{N} := \{1, 2, \dots\}$ . The blocks of a partition  $\pi \in \mathcal{P}_\infty$  are ordered by increasing values of their least elements so that  $\pi(1)$  is the block containing 1,  $\pi(2)$  is the block containing the smallest integer not in  $\pi(1)$  and so on. We set  $0_{[\infty]} := \{1\}, \{2\}, \dots$  and  $1_{[\infty]} := \{1, 2, 3, \dots\}$ . Given  $\pi, \pi' \in \mathcal{P}_\infty$ , we define  $\text{Coag}(\pi, \pi')$  as the partition of  $\mathbb{N}$  whose  $i$ -th block is given by  $\cup_{j \in \pi'(i)} \pi(j)$ . Finally we define  $\mathcal{P}_\infty^*$  as the set of partitions with a unique non-singleton block.

## 2.1 The lockdown graph

Our presentation of the measure-valued processes showed that, although the population is a continuum, there are at most a countable number of ancestors with a positive frequency. Henceforth, we are going to deal with a countable collection of individuals. At any time each individual will be located at a *level* taken to be an element of  $\mathbb{N}$ : two distinct individuals are located at two distinct levels and each level is occupied. The level of an individual is related to the future behaviour of its progeny: low levels tend to carry individuals with large progenies. We set the notation  $(i, t)$  to designate the individual alive at time  $t$  and located at level  $i$ .

**REMARK 0.12.** *We can already notice that this ordering relies on the future of the population, and therefore the lockdown representation is not adapted to the filtration of the population model. This is a fundamental difference with the stochastic flow of Bertoin and Le Gall.*

For any pair of times  $s < t$ , we need to specify the genealogical relationships between the individuals  $(i, s), i \geq 1$  and  $(j, t), j \geq 1$ . In our formalism, these relationships are encoded by a random partition  $\hat{\Pi}_{s,t}$  of  $\mathbb{N}$ : the progeny of  $(i, s)$  is specified by the block  $\hat{\Pi}_{s,t}(i)$  so that the integers  $j$  in this block are in one-to-one correspondence with the individuals  $(j, t)$  that descend from  $(i, s)$ . Let us now explain precisely how one constructs these partitions. First we notice that the population models we are interested in prescribe only two kinds of *elementary reproduction events*:

- binary reproduction events, where a parent gives birth to two individuals before dying.
- multiple reproduction events, where a parent gives birth to an infinity of individuals before dying, the children representing a strictly positive fraction of the population just after the birth.

Suppose that the individual  $(i_0, t-)$  is the parent of a multiple reproduction event at time  $t$ . At this time  $t$ , the population is then divided into the descendants of  $(i_0, t-)$ , we call  $K$  the set of levels they occupy, and the other individuals already alive at time  $t-$  whose levels have been possibly modified by the reproduction event. The lockdown construction prescribes that  $K$  has to be of the form  $\{i_0 < j_1 < j_2 < \dots\}$ : that is, one child is located at the same level as his parent while all the other children are at higher levels. Subsequently the individuals already alive at time  $t-$  are redistributed, keeping the same order, on the remaining levels  $\mathbb{N} \setminus K$ . Let us comment briefly the necessary form of  $K$ . Even if it should be seen as an *ad-hoc* construction, it is possible to give an intuitive explanation in certain cases. Recall that the level is related to the importance of the progeny. In the case where the progenies become extinct in finite time (REGIME 4), the level of an individual is related to the extinction time of its progeny so that, in our example, the extinction time of the progeny of  $(i_0, t-)$  is the supremum of the extinction times of the progenies of its children. Consequently one child has to be located at level  $i_0$  while all the other children are at higher levels. A similar explanation holds when the sizes of the progenies are of distinct asymptotic orders.

**REMARK 0.13.** *Observe that the intuitive explanation we have given for the form taken by  $K$  requires the existence of an infinite sequence of Eves for the measure-valued process. One can already suspect a rigorous connection between our sequence of Eves and the lockdown representation.*

We denote by  $\mathcal{I}_K$  the partition of  $\mathbb{N}$  with a unique non singleton block  $K$ : if we order the blocks by their least elements, this implies that  $\mathcal{I}_K(i_0) = K$  and for any  $j \neq i_0$ ,  $\mathcal{I}_K(j)$  is a singleton. The genealogical transition from time  $t-$  to time  $t$  is completely described by the partition  $\mathcal{I}_K$ : the progeny of  $i_0$  is given by the block  $\mathcal{I}_K(i_0) = K$ , while the progeny of  $j$  is the singleton  $\mathcal{I}_K(j)$  for every  $j \neq i_0$ . Assume we are given the genealogical relationships  $\hat{\Pi}_{s,t-}$  from time  $s$  to time  $t-$ . Then we have the following identity:

$$\hat{\Pi}_{s,t} = \text{Coag}(\mathcal{I}_K, \hat{\Pi}_{s,t-}) \quad (3)$$

To show that (3) holds, observe that the descendants at time  $t$  of  $(i, s)$  are the children at time  $t$  of all the individuals in  $\hat{\Pi}_{s,t-}(i)$ , which can be written formally

$$\hat{\Pi}_{s,t}(i) = \bigcup_{j \in \hat{\Pi}_{s,t-}(i)} \mathcal{I}_K(j)$$

In the particular case of a binary reproduction event,  $K$  has only two elements and (3) still holds. The construction now boils down to generate a suitable (random) collection of partitions of the form  $\mathcal{I}_K$ .

We introduce a random point process  $\mathcal{P}$  on  $[0, T) \times \mathcal{P}_\infty^*$  which collects all the *elementary reproduction events*. This point process is the union of two components:

- A binary part  $\mathcal{N}_{\text{binary}}$ , whose atoms are of the form  $(t, \mathcal{I}_{\{i,j\}})$  where  $1 \leq i < j$ .
- A multiple collisions part  $\mathcal{N}_{\text{multiple}}$ , whose atoms are of the form  $(t, \mathcal{I}_K)$  where  $K \subset \mathbb{N}$  has a strictly positive asymptotic frequency.

The point process  $\mathcal{P}$  has to satisfy an important property: on any interval  $(s, t]$  and for any integer  $n \in \mathbb{N}$ , the intersection of its projection on  $(s, t] \times \mathcal{P}_n$  with  $(s, t] \times \mathcal{P}_n^*$  has a (random) finite number of atoms, let us call them

$$(t_1, \varrho_1), \dots, (t_q, \varrho_q)$$

in the increasing order of their time coordinate. In terms of genealogy, this property ensures that the  $n$  first levels undergo only finitely many elementary reproduction events on compact intervals of time. We set

$$\hat{\Pi}_{s,t}^{[n]} := \text{Coag}(\varrho_q, \text{Coag}(\varrho_{q-1}, \dots, \text{Coag}(\varrho_2, \varrho_1) \dots))$$

which belongs to  $\mathcal{P}_n$ . Obviously, the collection of partitions  $(\hat{\Pi}_{s,t}^{[n]})_{n \in \mathbb{N}}$  is consistent and defines by an inductive limit a partition  $\hat{\Pi}_{s,t} \in \mathcal{P}_\infty$ . In the case where  $s = t$ , we let  $\hat{\Pi}_{s,t} := \mathbf{0}_{[\infty]}$ .

Extending this construction to all intervals  $(s, t]$ , we get a collection  $\hat{\Pi} := (\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  that we call stochastic flow of partitions. The term "flow" is due to the so-called cocycle property:

$$\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s}) \text{ a.s.}$$

for every  $r \leq s \leq t$  in  $[0, T)$ . Actually this property holds for all triplets  $r \leq s \leq t$  simultaneously with probability one, see Subsection 2.2. To end the definition of the lookdown graph, we give the specific intensities of  $\mathcal{N}_{\text{binary}}$  and  $\mathcal{N}_{\text{multiple}}$ .

## Λ Fleming-Viot

$\mathcal{N}_{\text{binary}}$  is a Poisson point process on  $\mathbb{R} \times \mathcal{P}_\infty^*$  with intensity measure

$$dt \otimes \left( \sum_{1 \leq i < j} \Lambda(0) \delta_{\mathcal{I}_{\{i,j\}}}(\cdot) \right)$$

where  $\mathcal{I}_{\{i,j\}}$  is the element of  $\mathcal{P}_\infty^*$  whose unique non-singleton block is  $\{i, j\}$ .  $\mathcal{N}_{\text{multiple}}$  is a Poisson point process on  $\mathbb{R} \times \mathcal{P}_\infty^*$  with intensity measure

$$dt \otimes \int_{(0,1)} \vartheta_x(\cdot) \nu(dx)$$

where  $\vartheta_x$  is the distribution of the random exchangeable partition with mass-partition  $(x, 0, \dots)$  (we refer to [9] or to Chapter II Subsection 2.1 for a reminder on the so-called paint-box distribution). Note that these Poisson point processes are taken on the whole line  $\mathbb{R}$  and not only on  $[0, \infty)$  and that the intensities are stationary in time. The following result is immediate.

**THEOREM** *The collection of partitions  $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$  enjoys the following properties:*

- For every  $r \leq s \leq t$ ,  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$  a.s. (cocycle property).
- $\hat{\Pi}_{s,t}$  is an exchangeable random partition whose law only depends on  $t - s$ . Furthermore, for any  $s_1 < s_2 < \dots < s_n$  the partitions  $\hat{\Pi}_{s_1, s_2}, \hat{\Pi}_{s_2, s_3}, \dots, \hat{\Pi}_{s_{n-1}, s_n}$  are independent.
- $\hat{\Pi}_{0,0} = 0_{[\infty]}$  and  $\hat{\Pi}_{0,t} \rightarrow 0_{[\infty]}$  in probability as  $t \downarrow 0$ , for the distance  $d_\mathcal{P}$ .

Moreover the process  $(\hat{\Pi}_{-t,0}, t \geq 0)$  is a  $\Lambda$  coalescent.

We call this object a  $\Lambda$  flow of partitions.

### $\Psi$ -MVBP

Let  $\mathcal{N}_{\text{binary}}$  be a doubly stochastic Poisson point process on  $[0, T) \times \mathcal{P}_\infty^*$  with random intensity measure

$$dt \otimes \left( \sum_{1 \leq i < j} \frac{\sigma^2}{Z_t} \delta_{\mathcal{I}_{\{i,j\}}}(\cdot) \right)$$

while  $\mathcal{N}_{\text{multiple}}$  is obtained from the point process

$$\bigcup_{\{t \geq 0: \Delta Z_t > 0\}} \left\{ \left( t, \frac{\Delta Z_t}{Z_t} \right) \right\}$$

as follows. For each atom  $(t, \frac{\Delta Z_t}{Z_t})$ , let  $\varrho_t$  be a paint-box based on the mass-partition  $(\frac{\Delta Z_t}{Z_t}, 0, \dots)$ . Then

$$\mathcal{N}_{\text{multiple}} := \bigcup_{\{t \geq 0: \Delta Z_t > 0\}} \{(t, \varrho_t)\}$$

Intuitively, this means that binary reproduction events occur at rate  $\frac{\sigma^2}{Z_t}$ , while each jump  $\Delta Z_t$  corresponds to the birth of a fraction  $\frac{\Delta Z_t}{Z_t}$  of new individuals.

**THEOREM** (cf **TH II.1**) *The collection of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ , together with the  $\Psi$ -CSBP  $Z$ , satisfies*

- For all  $0 \leq s \leq t < T$ ,  $\hat{\Pi}_{s,t}$  is distributed as a paint-box based on an independent subordinator with Laplace exponent  $u(t - s, \cdot)$  and restricted to  $[0, Z_s]$ .
- For all  $0 \leq r < s < t < T$ , a.s.  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$  (cocycle property).

We call  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  a  $\Psi$  flow of partitions with underlying CSBP  $Z$ . This object is very tractable to deal with convergences. We illustrate this fact with the following limit theorem.

**THEOREM** (cf **TH II.4**) *Consider a sequence of branching mechanisms  $(\Psi^m)_{m \geq 1}$  and let  $\hat{\Pi}^m$  be a  $\Psi^m$  flow of partitions. Suppose that*

- i) *For all  $u \in \mathbb{R}_+$ ,  $\Psi^m(u) \rightarrow \Psi(u)$  as  $m \rightarrow \infty$ .*
- ii) *The branching mechanism  $\Psi$  satisfies the Eve property.*
- iii)  *$\Psi$  is not the Laplace exponent of a compound Poisson process.*

then

$$(\hat{\Pi}_{0,t}^m, t \geq 0) \xrightarrow[m \rightarrow \infty]{(d)} (\hat{\Pi}_{0,t}, t \geq 0)$$

in  $\mathbb{D}(\mathbb{R}_+, \mathcal{P}_\infty)$ .

## 2.2 Limiting empirical measure and exchangeability

We consider a flow of partitions  $\hat{\Pi}$  which is either related to  $\Lambda$  or  $\Psi$  as defined in the preceding subsection. A crucial property of this genealogical structure is that for every  $s \leq t$  the random partition  $\hat{\Pi}_{s,t}$  is exchangeable. Indeed for every  $n \in \mathbb{N}$ , the partition  $\hat{\Pi}_{s,t}^{[n]}$  is obtained by coagulating a finite number of exchangeable partitions: exchangeability then follows from classical arguments. A result due to Kingman and de Finetti then ensures that  $\hat{\Pi}_{s,t}$  has asymptotic frequencies almost surely: we denote by  $|\hat{\Pi}_{s,t}(i)|$  the asymptotic frequency of the  $i$ -th block, for every  $i \geq 1$ . Actually we have a stronger result for the flow of partitions.

**THEOREM** (cf **PROP I.15**) *With probability one, the following three properties are fulfilled:*

- i) *(Cocycle property) for all triplets  $r \leq s \leq t$  we have*

$$\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$$

- ii) *(Regularity in  $(\mathcal{P}_\infty, d_\mathcal{P})$ ) for every  $s \leq t$  we have the following convergences in  $(\mathcal{P}_\infty, d_\mathcal{P})$*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \hat{\Pi}_{s,t+\epsilon} &= \lim_{\epsilon \downarrow 0} \hat{\Pi}_{s+\epsilon,t} = \hat{\Pi}_{s,t} \\ \lim_{\epsilon \downarrow 0} \hat{\Pi}_{s,t-\epsilon} &= \hat{\Pi}_{s,t-} \\ \lim_{\epsilon \downarrow 0} \hat{\Pi}_{s-\epsilon,t} &= \hat{\Pi}_{s-,t} \end{aligned}$$

- iii) *(Regularity in frequencies) for every  $s \leq t$  the partitions  $\hat{\Pi}_{s,t}, \hat{\Pi}_{s,t-}, \hat{\Pi}_{s-,t}$  possess asymptotic frequencies and whenever  $s < t$  we have the following convergences for every integer  $i \geq 1$*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s,t+\epsilon}(i)| &= \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s+\epsilon,t}(i)| = |\hat{\Pi}_{s,t}(i)| \\ \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s,t-\epsilon}(i)| &= |\hat{\Pi}_{s,t-}(i)| \\ \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s-\epsilon,t}(i)| &= |\hat{\Pi}_{s-,t}(i)| \end{aligned}$$

- iv) *For every  $s \in \mathbb{R}$ ,  $t \mapsto \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|$  is càdlàg on  $(0, \infty)$ .*

**REMARK 0.14.** *We do not provide the proof of this result in the MVBP case, but it follows as an easy adaptation of the  $\Lambda$  Fleming-Viot case.*

**REMARK 0.15.** *Partitions  $\hat{\Pi}_{s-,t}$  and  $\hat{\Pi}_{s,t-}$  are well defined: they differ from  $\hat{\Pi}_{s,t}$  whenever  $\hat{\Pi}_{s-,s}$  or  $\hat{\Pi}_{t-,t}$  are non trivial, that is, whenever an elementary reproduction event occurs at time  $s$  or  $t$ .*

This illustrates how the formalism of partitions is well-suited to state precise properties on the regularity of the genealogy. From now on, we will work on the event of probability one prescribed by the preceding theorem. Observe that for every  $s \in [0, T)$  the process

$$t \mapsto \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|$$

is non-decreasing and càdlàg on  $(0, \infty)$ . In REGIMES 3 & 4, it jumps from 0 to 1 at time  $0+$ , while in REGIMES 1 & 2 it reaches 1 in infinite time.

Based on this genealogical structure, we define a measure-valued process. We assume that initially  $(i, 0)$  has a given type  $\xi_0(i) \in [0, 1]$ , for every  $i \geq 1$ , and that it transmits its type to all his children. This yields a particle system  $t \mapsto (\xi_t(i))_{i \geq 1}$  that fulfils the following identity:

$$\forall i, j \geq 1, \quad j \in \hat{\Pi}_{0,t}(i) \Rightarrow \xi_t(j) = \xi_0(i).$$

From the regularity of the frequencies of the blocks, we deduce that the following map

$$[0, T) \ni t \mapsto \Xi_t(dx) := \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)| \delta_{\xi_0(i)}(dx) + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)|\right) dx$$

defines an  $\mathcal{M}_1$ -valued process which is càdlàg on  $(0, T)$  for the topology of the total variation distance<sup>1</sup>. Notice that the continuity at  $0+$  does not hold in REGIMES 3 & 4, but we can show that it is almost surely continuous at  $0+$  for the topology of the weak convergence of probability measures. Of course, at any time  $t \geq 0$   $\Xi_t$  is the limiting empirical measure of the exchangeable sequence  $(\xi_t(i))_{i \geq 1}$  and we recover the original definition by Donnelly and Kurtz [24] of the measure-valued process.

Observe that this definition of the measure-valued process allows to identify clearly the respective rôles played by the initial types and the genealogical structure. In particular the evolution of the frequencies does not depend on the initial types.

**REMARK 0.16.** *We have defined the measure-valued process  $(\Xi_t, t \in [0, T))$  from time 0 in the flow of partitions. Similarly we can define a measure-valued process  $(\Xi_{s,t}, t \in [s, T))$  from time  $s$  in the flow of partitions by setting*

$$[s, T) \ni t \mapsto \Xi_{s,t}(dx) := \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)| \delta_{\xi_{s,s}(i)}(dx) + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)|\right) dx$$

if we are given a sequence of initial types  $(\xi_{s,s}(i))_{i \geq 1}$ .

### 2.3 Martingale problem

Given a flow of partitions and a sequence of initial types at time 0, we have defined a measure-valued process  $t \mapsto \Xi_t$ . The goal now is to identify the distribution of this process. An essential ingredient is the following (the proof is postponed to Appendix C).

<sup>1</sup>This regularity for the total variation distance is due to the fact that, in our setting, the individuals inherit the types of their parent and do not undergo a spatial motion. The proof follows from the càdlàg property of each frequency and the monotonicity of the dust.



**THEOREM 0.1.** *Suppose that  $(\xi_0(i))_{i \geq 1}$  is an exchangeable sequence of r.v. on  $[0, 1]$ . Then at any time  $t \geq 0$  the sequence  $(\xi_t(i))_{i \geq 1}$  is exchangeable and for any integer  $n \geq 1$  and any bounded measurable map  $f : [0, 1]^n \rightarrow \mathbb{R}$  we have*

$$\mathbb{E} \left[ f(\xi_t(1), \dots, \xi_t(n)) \mid \mathcal{F}_t^{Z, \Xi} \right] = \int_{[0, 1]^n} f(x_1, \dots, x_n) \Xi_t(dx_1) \cdots \Xi_t(dx_n)$$

where  $\mathcal{F}_t^{Z, \Xi}$  is the  $\sigma$ -field generated by the population size  $Z$  and the process  $\Xi$  up to time  $t$ .

**REMARK 0.17.** *This result is stated in Theorem 4.1 of [24], but the proof does not seem to be complete. Our formalism with partitions turns out to be appropriate to show such properties.*

From now on,  $\xi_0(i), i \geq 1$  is taken i.i.d. uniform $[0, 1]$ . To identify the distribution of  $t \mapsto Z_t \cdot \Xi_t$  we prove that it solves a martingale problem for which uniqueness is known. Below, we provide the specific arguments for the  $\Lambda$  Fleming-Viot as they are relatively easy. Those for the  $\Psi$ -MVBP are postponed to Appendix D. However in both cases, we follow the strategy of proof of Donnelly and Kurtz: first we identify a martingale problem solved by the process  $(Z_t, \xi_t(1), \dots, \xi_t(n))$ . Then we translate this martingale problem in terms of  $(Z_t \cdot \Xi_t)$  thanks to Theorem 0.1 and for appropriate test functions. Finally we remark that it is exactly the martingale problem that characterises the desired measure-valued process.

### $\Lambda$ Fleming-Viot

Let  $\hat{\Pi} = (\hat{\Pi}_{s,t}, s \leq t)$  be a flow of partitions associated with the measure  $\Lambda$  as defined previously. Let  $(\xi_0(i))_{i \geq 1}$  be a sequence of i.i.d. uniform $[0, 1]$  r.v. and  $(\xi_t(i), t \geq 0)_{i \geq 1}, (\Xi_t, t \geq 0)$  be defined as above.

*Step 1.* We define the bounded linear operator  $A_n : \mathcal{C}([0, 1]^n) \rightarrow \mathcal{C}([0, 1]^n)$  as follows. For any function  $f \in \mathcal{C}([0, 1]^n)$  we set (recall the notation from Subsection 1.1)

$$A_n f(x_1, \dots, x_n) := \sum_{\substack{K \subset [n] \\ k = \#K \geq 2}} \lambda_{n,k} (f(R_K(x_1, \dots, x_{n-k+1})) - f(x_1, \dots, x_n))$$

It is a simple matter to check that  $(\xi_t(1), \dots, \xi_t(n); t \geq 0)$  is a continuous time Markov chain with generator  $A_n$ . Hence the process

$$f(\xi_t(1), \dots, \xi_t(n)) - \int_0^t A_n f(\xi_u(1), \dots, \xi_u(n)) du$$

is a martingale in the natural filtration  $\mathcal{F}^\xi$  of the particle system  $\xi$ . Since the sigma-field  $\mathcal{F}_t^\xi$  contains  $\mathcal{F}_t^{\Xi}$ , we deduce that for every  $0 \leq s \leq t$  almost surely

$$\mathbb{E} \left[ f(\xi_t(1), \dots, \xi_t(n)) - f(\xi_s(1), \dots, \xi_s(n)) - \int_s^t A_n f(\xi_u(1), \dots, \xi_u(n)) du \mid \mathcal{F}_s^{\Xi} \right] = 0$$

*Step 2.* We now prove that  $\Xi$  verifies the martingale problem of Definition 0.2: recall the map  $G_f$  and the operator  $L$ . From Theorem 0.1, we deduce that

$$G_f(\Xi_t) = \mathbb{E} \left[ f(\xi_t(1), \dots, \xi_t(n)) \mid \mathcal{F}_t^{\Xi} \right]$$

Also, this theorem combined with a straightforward calculation entails

$$L G_f(\Xi_u) = \mathbb{E} \left[ A_n f(\xi_u(1), \dots, \xi_u(n)) \mid \mathcal{F}_u^{\Xi} \right], \forall u \geq 0$$

This ensures that

$$\begin{aligned}
 & \mathbb{E} \left[ G_f(\Xi_t) - G_f(\Xi_s) - \int_s^t LG_f(\Xi_u) du \mid \mathcal{F}_s^\Xi \right] \\
 = & \mathbb{E} \left[ \mathbb{E} [f(\xi_t(1), \dots, \xi_t(n)) \mid \mathcal{F}_t^\Xi] - f(\xi_s(1), \dots, \xi_s(n)) \right. \\
 & \left. - \int_s^t \mathbb{E} [A_n f(\xi_u(1), \dots, \xi_u(n)) \mid \mathcal{F}_u^\Xi] du \mid \mathcal{F}_s^\Xi \right] \\
 = & \mathbb{E} \left[ f(\xi_t(1), \dots, \xi_t(n)) - f(\xi_s(1), \dots, \xi_s(n)) - \int_s^t A_n f(\xi_u(1), \dots, \xi_u(n)) du \mid \mathcal{F}_s^\Xi \right] \\
 = & 0
 \end{aligned}$$

where we use Fubini theorem at the second equality. Consequently  $(\Xi_t, t \geq 0)$  is a  $\Lambda$  Fleming-Viot process, thanks to the uniqueness of the martingale problem obtained by Bertoin and Le Gall (see Subsection 1.1).

**Conclusion.** The lookdown representation allows to construct the  $\Lambda$  Fleming-Viot (resp. the  $\Psi$ -MVBP) along with its genealogy. Actually our reformulation provides explicitly the genealogy through the flow of partitions. In the next section, we present an alternative representation.

### 3 Bertoin and Le Gall's stochastic flows

Bertoin and Le Gall introduced in a series of four papers [12, 13, 14, 15] a representation of  $\Psi$ -MVBP and  $\Lambda$  Fleming-Viot processes using stochastic flows.

#### 3.1 Stochastic flows of bridges

Recall that a bridge in the sense of Kallenberg [47] is a non-decreasing random process  $F : [0, 1] \rightarrow [0, 1]$  such that  $F(0) = 0$ ,  $F(1) = 1$  and  $F$  has exchangeable increments. Kallenberg showed that there exist a sequence  $(V_i)_{i \geq 1}$  of i.i.d. uniform  $[0, 1]$  r.v. and an independent sequence  $a_1 \geq a_2 \geq \dots \geq 0$  of r.v. verifying  $\sum a_i \leq 1$  such that almost surely for all  $x \in [0, 1]$

$$F(x) = \sum_{i \geq 1} a_i \mathbf{1}_{\{V_i \leq x\}} + x \left( 1 - \sum_{i \geq 1} a_i \right)$$

We have already encountered this object in the definition of the  $\Lambda$  Fleming-Viot process: we then remarked that  $x \mapsto \rho_t([0, x])$  is a bridge. This simple fact is the underlying idea for flows of bridges.

**DEFINITION 0.18.** (Bertoin-Le Gall [13]) A collection of r.v.  $(F_{s,t}, -\infty < s \leq t < \infty)$  that verifies:

- For every  $s \leq t$ ,  $F_{s,t}$  is a bridge whose law only depends on  $t - s$ . Furthermore, if  $t_1 < t_2 < \dots < t_p$ , the bridges  $F_{t_1, t_2}, \dots, F_{t_{p-1}, t_p}$  are independent.
- For every  $r \leq s \leq t$ , we have almost surely

$$F_{r,t} = F_{s,t} \circ F_{r,s} \text{ (cocycle property)}$$

- $F_{0,0} = Id$  and  $F_{0,t} \rightarrow Id$  in probability as  $t \downarrow 0$  in Skorohod's topology.

is called a stochastic flow of bridges. Additionally, if we define  $\rho_{s,t}$  as the probability measure on  $[0, 1]$  whose distribution function is  $F_{s,t}$  and if  $(\rho_{0,t}, t \geq 0)$  is a  $\Lambda$  Fleming-Viot process, then the flow is said associated to the measure  $\Lambda$ .

We postpone the construction of such flows to the next paragraph. We first examine some properties of this object. Consider an independent sequence  $(U_i)_{i \geq 1}$  of i.i.d. uniform $[0, 1]$  r.v. and define for all  $t \geq 0$  the following random partition

$$i \stackrel{\Pi_t}{\sim} j \Leftrightarrow F_{-t,0}^{-1}(U_i) = F_{-t,0}^{-1}(U_j)$$

where  $F_{-t,0}^{-1}$  is the right-continuous inverse of  $F_{-t,0}$  for every  $t \geq 0$ . There is a one-to-one correspondence between the jumps of  $F_{-t,0}$  and the blocks of  $\Pi_t$  with positive asymptotic frequency. Additionally  $\Pi_t$  has dust iff  $F_{-t,0}$  has a drift component. These two properties are consequences of the fact that  $\Pi_t$  is a paint-box based on the mass-partition obtained from the jumps of  $F_{-t,0}$  (see [9] or Chapter II Subsection 2.1 for the definitions). Bertoin and Le Gall proved that the process  $(\Pi_t, t \geq 0)$  is an exchangeable coalescent (or  $\Xi$  coalescent), and in the particular case where the flow of bridges is associated to  $\Lambda$ , it is a  $\Lambda$  coalescent.

To construct a flow of bridges associated to a given measure  $\Lambda$ , Bertoin and Le Gall first considered the case where  $u^{-2}\Lambda(du)$  is a finite measure before passing to the limit. We propose another construction based on Kolmogorov's extension theorem. However in any case, the trajectories of the flow do not have much regularity properties.

**Construction of flows of bridges.** Fix an integer  $n \geq 1$  and a sequence  $t_1 < t_2 < \dots < t_{n+1}$ . Denote by  $\hat{\Pi}$  a flow of partitions associated to the measure  $\Lambda$ . For each  $i \in [n]$ , consider the mass-partition  $|\hat{\Pi}_{t_i, t_{i+1}}|^\downarrow =: (a_{i,j})_{j \geq 1}$  formed by the asymptotic frequencies of the blocks of  $\hat{\Pi}_{t_i, t_{i+1}}$  in the decreasing order. Using an independent sequence  $(V_{i,j})_{j \geq 1}$  of i.i.d. uniform $[0, 1]$  r.v. we define the bridge  $F_{t_i, t_{i+1}}$  as follows. For all  $x \in [0, 1]$ , let

$$F_{t_i, t_{i+1}}(x) := \sum_{j \geq 1} a_{i,j} \mathbf{1}_{\{V_{i,j} \leq x\}} + x \left(1 - \sum_{j \geq 1} a_{i,j}\right)$$

Doing so for each  $i \in [n]$ , we get a collection of independent bridges  $F_{t_i, t_{i+1}}, i \in [n]$  whose distribution is the one expected. We need to verify the consistency of this law when  $n$  varies. To that end, we remark that the ordered jumps of the bridge

$$F_{t_2, t_3} \circ F_{t_1, t_2}$$

are equal in law to the asymptotic frequencies of the partition

$$\text{Coag}(\hat{\Pi}_{t_2, t_3}, \hat{\Pi}_{t_1, t_2})$$

This identity is a consequence of Corollary 1 in [13]. But since this last partition is distributed as  $\hat{\Pi}_{t_1, t_3}$ , we deduce that

$$F_{t_2, t_3} \circ F_{t_1, t_2} \stackrel{(d)}{=} F_{t_1, t_3}$$

The consistency follows. Therefore, one can apply Kolmogorov's extension theorem to deduce the existence of a stochastic flow of bridges associated to  $\Lambda$ .

Notice that a flow of bridges encodes much more than a single  $\Lambda$  Fleming-Viot process: it actually couples an infinite collection  $(\rho_{s,t}, t \in [s, \infty))_{s \in \mathbb{R}}$  of such processes indexed by  $s \in \mathbb{R}$ .

### 3.2 Stochastic flows of subordinators

The idea is very close, but a major difference consists in the varying population size.

**DEFINITION 0.19.** (Bertoin-Le Gall [12]) A collection of r.v.  $(S_{s,t}(a), 0 \leq s \leq t, a \geq 0)$  that verifies:

- For every  $0 \leq s \leq t$ ,  $(S_{s,t}(a), a \geq 0)$  is a subordinator with Laplace exponent  $u(t-s, \cdot)$ .
- For every integer  $p \geq 2$  and  $0 \leq t_1 \leq \dots \leq t_p$ , the subordinators  $S_{t_1, t_2}, \dots, S_{t_{p-1}, t_p}$  are independent and almost surely

$$S_{t_1, t_p}(a) = S_{t_{p-1}, t_p} \circ \dots \circ S_{t_1, t_2}(a), \quad \forall a \geq 0 \text{ (cocycle property)}$$

- For all  $a \geq 0$ ,  $(S_{0,t}(a), t \geq 0)$  is a  $\Psi$ -CSBP started from  $a$ .

is called a stochastic flow of subordinators.

To prove the existence of such an object, Bertoin and Le Gall made use of Kolmogorov's extension theorem, we refer to Proposition 1 in [12] for details.

As we will be interested in an initial population  $[0, 1]$  at time 0, we introduce the  $\Psi$ -MVBP

$$m_{0,t}([0, x]) := S_{0,t}(x), \quad \forall x \in [0, 1]$$

This  $\Psi$ -MVBP admits a càdlàg modification, still denoted  $(m_{0,t}, t \geq 0)$ , that allows to deal with its lifetime  $T$ . Also, we let  $Z_t := m_{0,t}([0, 1])$ . From now on, we will restrict our attention to the flow  $(S_{s,t}(a), 0 \leq s \leq t < T, 0 \leq a \leq Z_s)$ . This object defines a collection of  $\Psi$ -MVBP's  $(m_{s,t}, t \in [s, T])_{s \in [0, T]}$  each starting from a point  $s \in [0, T]$ :

$$m_{s,t}([0, x]) := S_{s,t}(x), \quad \forall x \in [0, Z_s]$$

Note the analogy between the collection of  $\Lambda$  Fleming-Viot obtained from the flow of bridges, and this collection of  $\Psi$ -MVBP's obtained from the flow of subordinators.

**Genealogical interpretation of stochastic flows.** In both cases, the stochastic flows provide a notion of genealogy for the underlying measure-valued process: the progeny of those individuals lying in  $[a, b] \subset [0, 1]$  at time 0 is given by  $t \mapsto F_{0,t}([a, b])$  (and similarly with  $S$ ). Moreover, for any  $0 < s < t$ , the cocycle property expresses the following consistency: the progeny at time  $t$  of  $[a, b]$  is the union of all the progenies at time  $t$  of the descendants at time  $s$  of  $[a, b]$ . This representation through stochastic flows does not provide explicitly a particle system, as the lookdown representation does. The goal of the next section is to explain how one can read the lookdown representation from a stochastic flow.

**A remark on the cocycle property.** One should pay attention to the fact that, for both flows of bridges and subordinators, the cocycle property is verified for each given triplet  $r < s < t$  almost surely, but nothing ensures that it holds for all triplets with probability one. Indeed, the construction from Kolmogorov's extension theorem does not provide such an almost sure result. However, this simultaneous cocycle property is obtained in the particular case where  $\nu(du)$  is a finite measure thanks to a Poissonian construction, see Section 5 in [13]. It is interesting to notice that the cocycle property is verified for all triplets with probability one in the case of flows of partitions, see Subsection 2.2.

## 4 Unification of two representations

Consider a flow of bridges  $F$  (resp. a flow of subordinators  $S$ ) associated to  $\Lambda$  (resp. to  $\Psi$ ). Our goal is to define, from any time  $s \in \mathbb{R}$  (resp.  $s \in [0, T)$ ), a lookdown process whose limiting empirical measure is equal to  $(\rho_{s,t}, t \in [s, \infty))$  (resp. to  $(m_{s,t}, t \in [s, T)$ ) up to the population size  $Z$ ). To that end, we need to identify a sequence of initial types and a flow of partitions.

**Assumption.** From now on,  $\Lambda$  and  $\Psi$  are such that the measure-valued process admits an infinite sequence of Eves (see Subsection 1.4). This requirement will be explained below.

At any time  $s \in \mathbb{R}$ , the  $\Lambda$  Fleming-Viot  $(\rho_{s,t}, t \in [s, \infty))$  admits almost surely a sequence of Eves, say  $(e_s^i)_{i \geq 1}$ . Similarly, when  $s \in [0, T)$ , the  $\Psi$ -MVBP  $(m_{s,t}, t \in [s, T))$  that starts from the Lebesgue measure on  $[0, Z_s]$  admits almost surely a sequence of Eves, but for convenience, we *normalise* this sequence by the total-mass  $Z_s$ . Here again, we denote the Eves by  $(e_s^i)_{i \geq 1}$ . In both cases they are i.i.d. uniform $[0, 1]$ .

Since the flow fulfils a consistency property (the so-called cocycle property), it is natural to look at the genealogical relationships between the Eves taken at different times. For every  $s < t$  we define the random partition  $\hat{\Pi}_{s,t}$  by

$$i \stackrel{\hat{\Pi}_{s,t}}{\sim} j \iff e_t^i \text{ and } e_t^j \text{ descend from a same ancestor at time } s$$

The r.h.s. can be stated formally as  $F_{s,t}^{-1}(e_t^i) = F_{s,t}^{-1}(e_t^j)$  for the  $\Lambda$  Fleming-Viot, and  $S_{s,t}^{-1}(Z_t \cdot e_t^i) = S_{s,t}^{-1}(Z_t \cdot e_t^j)$  for the  $\Psi$ -MVBP. It is a simple matter to check that the ancestor at time  $s$  of an Eve at time  $t$  is necessarily itself an Eve. Additionally for each  $i \geq 1$ , the  $i$ -th block  $\hat{\Pi}_{s,t}(i)$  is the progeny at time  $t$  of the  $i$ -th Eve  $e_s^i$  at time  $s$ .

**THEOREM** (cf **TH I.3 AND TH II.6**) *The collection of partitions  $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$  (respectively  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ ) is a  $\Lambda$  (respectively  $\Psi$ ) flow of partitions.*

Therefore, in both cases we have extracted a genealogical structure from the flow of bridges (resp. subordinators). However, the lack of regularity of the latter (the cocycle property is not verified simultaneously by all triplets) prevents the flow of partitions to be itself regular. More precisely, nothing ensures that with probability one the number of elementary reproduction events in the flow of partitions restricted to  $[n]$  is finite on compact intervals of time and that the cocycle property holds for all triplets. These regularity properties are necessary for the lookdown representation since it is a pathwise construction. Consequently we propose a regularisation procedure to show the existence of a modification of the original flow of partitions that verifies the required properties. We refer to Chapter I Section 2.3 and Proposition II.30 for further details. From now on, we will deal with this regular modification.

We have identified a process of Eves  $s \mapsto (e_s^i)_{i \geq 1}$  and a stochastic flow of partitions  $\hat{\Pi}$ . As explained in Section 2, this is sufficient to define a lookdown process. From every time  $s$ , we introduce:

$$[s, T) \ni t \mapsto \Xi_{s,t}(dx) = \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)| \delta_{e_s^i}(dx) + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)|\right) dx$$

Our main result is the following.

**THEOREM** (cf **TH I.4 AND II.7**) *The lookdown construction based on the flow of partitions  $\hat{\Pi}$  and the process of Eves  $s \mapsto (e_s^i)_{i \geq 1}$  allows to recover the flow of bridges (resp. subordinators). More precisely, in the  $\Lambda$  Fleming-Viot case for every  $s \in \mathbb{R}$  almost surely*

$$(\Xi_{s,t}, t \in [s, \infty)) = (\rho_{s,t}, t \in [s, \infty))$$

while in the  $\Psi$ -MVBP case for every  $s \in [0, T)$  almost surely

$$(Z_t \cdot \Xi_{s,t}, t \in [s, T)) = (m_{s,t}, t \in [s, T))$$

Additionally, this decomposition is unique in the following sense. Consider another flow of partitions  $\hat{\Pi}'$  and another process of initial types  $s \mapsto (\chi_s(i))_{i \geq 1}$  and suppose that the lookdown construction based on these objects allows to recover the flow of bridges (resp. subordinators) then:

- (Initial types) For each  $s$ , almost surely  $\forall i \geq 1 \chi_s(i) = e_s^i$ .
- (Lookdown graph) Almost surely  $\hat{\Pi}' = \hat{\Pi}$ .

**The existence of an infinite sequence of Eves: a necessary condition for uniqueness.** Consider a measure  $\Lambda$  or a branching mechanism  $\Psi$  such that the measure-valued process does not admit an infinite sequence of Eves, and for the moment exclude REGIME 4 for  $\Lambda$  Fleming-Viot. Consider the lookdown representation. We stress that there is no uniqueness of the initial types. Indeed, since the measure-valued process does not admit an infinite sequence of Eves there exist two ancestral types  $x \neq y$  whose progenies at the infinite are of comparable asymptotic. Necessarily  $x$  and  $y$  have to be two initial types in the lookdown representation. Since their progenies are comparable at the infinite,  $x$  has both a positive probability to be located above  $y$  and a positive probability to be located below  $y$ .

We are now concerned with the uniqueness of the lookdown graph associated with the flow of measure-valued processes. We start with the  $\Psi$ -MVBP case. Assume that the Eve property is not fulfilled: from Subsection 1.4 we know that  $T = \infty$  a.s. Necessarily any such  $\Psi$ -flow of partitions  $\hat{\Pi}$  enjoys the following property: the number of elementary reproduction events on the time interval  $[0, T = \infty)$  involving at least two levels among  $[n]$  is finite almost surely (otherwise the Eve property would be fulfilled). The result stated in Theorem 0.1 can be applied to the entire interval  $[0, T = \infty)$  (the proof only requires the finiteness of the number of elementary reproduction events restricted to  $[n]$  on  $[0, t]$ ) and yields another flow of partitions  $\hat{\Pi}'$  which allows to construct the same flow of  $\Psi$ -MVBP. As a consequence there is no uniqueness of the lookdown graph.

We turn to the  $\Lambda$  Fleming-Viot. Consider first REGIME 1. The number of elementary reproduction events involving at least two levels among  $[n]$  is infinite almost surely, therefore the preceding procedure cannot be applied. However one can consider a restriction of this flow of partitions by removing the progeny of the first level at time 0, that is, by removing  $\hat{\Pi}_{0,t}(1)$  for all  $t \geq 0$ . Once we have reindexed the integers in the partitions, we get a new collection of partitions that verifies the cocycle property. For this collection of partitions the number of elementary reproduction events involving at least two levels among  $[n]$  is finite almost surely, and we can apply our procedure to construct another collection of partitions. The non-uniqueness then follows. When the  $\Lambda$  Fleming-Viot is in REGIME 2 OR 3 and does not admits an infinite sequence of Eves, one can adapt the preceding argument to show the non-uniqueness of the flow of partitions.

Finally in REGIME 4 for the  $\Lambda$  Fleming-Viot, there does not exist an infinite sequence of Eves if the process loses two ancestral types simultaneously. The question of uniqueness of the initial types and the lookdown graph remains then open: however we conjecture that there does not exist such  $\Lambda$  Fleming-Viot processes.



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## From flows of Lambda Fleming-Viot processes to lookdown processes via flows of partitions

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*This chapter has been taken from the article [52].*

### 1 Introduction

The  $\Lambda$  coalescent process  $(\Pi_t, t \geq 0)$  has been introduced by Pitman [66] and Sagitov [67]. This is a Markov process with values in the set  $\mathcal{P}_\infty$  of partitions of  $\mathbb{N} := \{1, 2, \dots\}$  whose distribution is characterised by a finite measure  $\Lambda$  on  $[0, 1]$ . Let us describe briefly the dynamics of this process. Using the consistency of the partitions, it suffices to define the dynamics of the restriction  $(\Pi_t^{[n]}, t \geq 0)$  of the process to the set  $\mathcal{P}_n$  of partitions of  $[n] := \{1, \dots, n\}$ , for every  $n \geq 1$ . If  $\Pi_t^{[n]}$  has  $m$  blocks at a given time  $t \geq 0$ , then any  $k$  of them merge at rate

$$\lambda_{m,k} := \int_{[0,1]} x^k (1-x)^{n-k} x^{-2} \Lambda(dx) \quad (\text{I.1})$$

for every integer  $k \in \{2, \dots, m\}$ . If we assume that  $\Pi_0$  is the trivial partition  $0_{[\infty]} := \{\{1\}, \{2\}, \dots\}$  then the process  $(\Pi_t, t \geq 0)$  can be interpreted as the genealogy of an infinite population: each individual is represented by an integer so that the coalescence of a collection of blocks corresponds to groups of individuals finding their most recent common ancestor backward in time.

The  $\Lambda$  coalescent is in duality (see Lemma 5 of Bertoin and Le Gall [13]) with the so-called  $\Lambda$  Fleming-Viot process which, on the contrary, describes the evolution forward in time of an infinite population. This Markov process has been introduced by Bertoin and Le Gall [13], and implicitly by Donnelly and Kurtz [24]. In this population model, each individual possesses a genetic type taken to be a real value in  $[0, 1]$  so that the frequencies of the genetic types among the population are encoded by an element of the set  $\mathcal{M}_1$  of probability measures on  $[0, 1]$ . The  $\Lambda$  Fleming-Viot process is therefore an  $\mathcal{M}_1$ -valued process. It starts from the Lebesgue measure on  $[0, 1]$  and evolves through *reproduction events* that can be informally described as follows. At rate  $\nu(dx) := x^{-2} \Lambda(dx)$  a parent (uniformly chosen among the population) reproduces: a fraction  $x$  of individuals dies out and is replaced by individuals with the same type as the parent. The duality with the  $\Lambda$  coalescent can be thought of as follows: each reproduction event induces a coalescence event backward in time.

Although one can easily construct the  $\Lambda$  coalescent thanks to the consistency of the restrictions of the



partitions, the construction of the  $\Lambda$  Fleming-Viot process is not trivial. The main objective of this paper is to unify two constructions, namely the stochastic flow of bridges introduced by Bertoin and Le Gall [13] and the lockdown representation of Donnelly and Kurtz [24]. This is achieved in three main steps. First we propose a new formulation of the lockdown representation that relies on the introduction of an object called the *stochastic flow of partitions*. Second we investigate the asymptotic behaviour of the  $\Lambda$  Fleming-Viot process, and introduce the notion of *Eves* that generalise the primitive Eve of Bertoin and Le Gall [13]. We stress that this study is interesting in its own right. Third, we use the Eves and the stochastic flow of partitions to define the lockdown representation pathwise from the stochastic flow of bridges.

### 1.1 The lockdown representation via the flow of partitions

In the lockdown representation introduced by Donnelly and Kurtz [24], the  $\Lambda$  Fleming-Viot process is obtained as the process of empirical measures of a countable particle system  $(\xi_t(i), t \geq 0), i \geq 1$  which is called the *lockdown process*. The particle system is constructed in such a way that the  $n$  first particles do not depend on all the other ones, for every  $n \geq 1$ . Let us describe the dynamics of the  $n$  first particles for a given  $n \geq 1$ . Initially the particles start from a sequence of values called *initial types* which are i.i.d.  $\text{uniform}[0, 1]$  r.v. Then to every subset  $K \subset \{1, \dots, n\}$  whose cardinality  $k$  belongs to  $\{2, \dots, n\}$  is associated an exponential clock with parameter  $\lambda_{n,k}$ . When this clock rings, the values of the  $n$  first particles are updated as follows. All the particles indexed by  $i \in K$  take the type of the particle indexed by  $\min K$ , while the remaining particles indexed by  $i \in [n] \setminus K$  take, in the same order, the values of the particles indexed by  $\{1, \dots, \min K - 1, \min K + 1, \dots, n - k + 1\}$ . These transitions are called *reproduction events* and the particle indexed by  $\min K$  is called the parent. The total transition rate of the  $n$  first particles is equal to

$$\lambda_n := \sum_{k=2}^n \binom{n}{k} \lambda_{n,k}. \quad (1.2)$$

The consistency of the particle system comes from the fact that the  $n$ -th particle does never give its type to one of the  $(n-1)$  first particles so that the infinite collection of particles can be defined simultaneously. This particle system admits a process of empirical measures which is a  $\Lambda$  Fleming-Viot process. We refer to Subsection 2.1 for precise definitions.

The collection of reproduction events forms the so-called *lockdown graph*. It is well-known that this object implicitly defines the genealogy of the particle system. However this has never been formalised. One contribution of the present paper is to show that the lockdown graph can be encoded by a collection of partitions of  $\mathbb{N} := \{1, 2, \dots\}$  that we call a stochastic flow of partitions and that gives an explicit description of the genealogy of the lockdown process. To state the definition of this object, we introduce some notation. The restriction of an element  $\pi \in \mathcal{P}_\infty$  to the set  $\mathcal{P}_n$  is denoted  $\pi^{[n]}$ . We endow  $\mathcal{P}_\infty$  with the distance  $d_\mathcal{P}$  defined as follows

$$d_\mathcal{P}(\pi, \pi') = 2^{-i} \text{ where } i = \sup\{n \geq 1 : \pi^{[n]} = \pi'^{[n]}\}. \quad (1.3)$$

We order the blocks of a partition  $\pi$  in the increasing order of their least element. We denote by  $\pi(i)$  the  $i$ -th block according to this ordering, and we define the asymptotic frequency of this block

$$|\pi(i)| := \lim_{n \rightarrow \infty} \frac{1}{n} \#\{\pi(i) \cap [n]\}$$

when this limit exists. For any partitions  $\pi, \pi'$ , we introduce the partition  $\text{Coag}(\pi, \pi')$  as follows. For every  $i \geq 1$ , the  $i$ -th block of  $\text{Coag}(\pi, \pi')$  is the union of the blocks  $\pi(j)$  for  $j \in \pi'(i)$ .

**DEFINITION I.1.** A stochastic flow of partitions is a collection  $\hat{\Pi} = (\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$  of partitions of  $\mathbb{N} := \{1, 2, \dots\}$  such that:

- For every  $r < s < t$ ,  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$  a.s. (cocycle property).
- $\hat{\Pi}_{s,t}$  is an exchangeable random partition whose law only depends on  $t - s$ . Furthermore, for any  $s_1 < s_2 < \dots < s_n$  the partitions  $\hat{\Pi}_{s_1, s_2}, \hat{\Pi}_{s_2, s_3}, \dots, \hat{\Pi}_{s_{n-1}, s_n}$  are independent.
- $\hat{\Pi}_{0,0} = \mathbf{0}_{[\infty]} := \{\{1\}, \{2\}, \dots\}$  and  $\hat{\Pi}_{0,t} \rightarrow \mathbf{0}_{[\infty]}$  in probability as  $t \downarrow 0$  for the distance  $d_{\mathcal{P}}$ .

Furthermore if the process  $(\hat{\Pi}_{-t,0}, t \geq 0)$  is a  $\Lambda$  coalescent then we say that  $\hat{\Pi}$  is a  $\Lambda$  flow of partitions.

In Section 2 we show a correspondence between lockdown graphs and flows of partitions. This correspondence allows us to propose a new formulation of the lockdown representation that we now present. Let  $\xi_0 = (\xi_0(i), i \geq 1)$  be a sequence of i.i.d. uniform $[0, 1]$  r.v., referred to as the *initial types*, and let  $\hat{\Pi}$  be an independent  $\Lambda$  flow of partitions. For every  $t > 0$  and every  $i \geq 1$ , we let  $j$  be the index of the block of  $\hat{\Pi}_{0,t}$  that contains  $i$  and we set  $\xi_t(i) = \xi_0(j)$ . Then  $(\xi_t(i), t \geq 0), i \geq 1$  is a *lookdown process*. Furthermore its process of empirical measures coincides almost surely with the  $\mathcal{M}_1$ -valued process  $(\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0), t \geq 0)$  defined by

$$\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0) := \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)| \delta_{\xi_0(i)} + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)|\right) \ell.$$

where  $\ell$  stands for the Lebesgue measure on  $[0, 1]$ . Consequently  $(\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0), t \geq 0)$  is a  $\Lambda$  Fleming-Viot process. We refer to Subsection 2.3 for the simultaneous existence of the asymptotic frequencies of the blocks at all times  $t \geq 0$ .

Observe that the lockdown representation provides much information on the  $\Lambda$  Fleming-Viot process. In particular, it shows that at any time  $t \geq 0$  the probability measure  $\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$  has an atomic component and a Lebesgue component. When this Lebesgue component is not null, we say that the measure has *dust*. Notice that this terminology comes from the theory of partitions of  $\mathbb{N}$ : if the union of the singleton blocks of  $\pi \in \mathcal{P}_\infty$  admits a positive asymptotic frequency, we say that  $\pi$  has dust (see Section 2.1 in Bertoin [9]).

## 1.2 The Eves

We shall assume that  $\Lambda(\{1\}) = 0$  to avoid trivial reproduction events where all individuals merge at once in the  $\Lambda$  coalescent. For convenience, we introduce the following notation

$$\Psi(u) = \Lambda(\{0\})u^2 + \int_{(0,1)} (e^{-xu} - 1 + xu)\nu(dx), \quad \forall u \geq 0 \quad \text{where } \nu(dx) = x^{-2}\Lambda(dx). \quad (\text{I.4})$$

Let  $\rho = (\rho_t, t \geq 0)$  be a  $\Lambda$  Fleming-Viot process assumed to be càdlàg for the weak convergence in  $\mathcal{M}_1$  (the semigroup is Feller). For the moment we do not assume that  $\rho$  has been obtained from the lockdown representation. From the behaviour (recalled in Section 3) of the dust and the number of non-singleton blocks in the  $\Lambda$  coalescent and thanks to the duality with the  $\Lambda$  coalescent, we are able to identify four regimes for the  $\Lambda$  Fleming-Viot process.

**PROPOSITION I.2.** We identify the following four regimes according to the measure  $\Lambda$ :

1. DISCRETE:  $\nu([0, 1]) < \infty$ . For all  $t > 0$ , almost surely  $\rho_t$  has dust and finitely many atoms.
2. INTENSIVE w. DUST:  $\nu([0, 1]) = \infty$  and  $\int_{[0,1]} x\nu(dx) < \infty$ . For all  $t > 0$ , almost surely  $\rho_t$  has dust and infinitely many atoms.

3. INTENSIVE  $\infty$ :  $\int_{[0,1]} x\nu(dx) = \infty$  and  $\int^\infty \frac{du}{\Psi(u)} = \infty$ . For all  $t > 0$ , almost surely  $\rho_t$  has no dust and infinitely many atoms.
4. CDI:  $\int^\infty \frac{du}{\Psi(u)} < \infty$ . For all  $t > 0$ , almost surely  $\rho_t$  has no dust and finitely many atoms.

This classification will be useful for the study of the asymptotic behaviour of the  $\Lambda$  Fleming-Viot process that we now present. Bertoin and Le Gall [13] showed the existence of a r.v.  $e$  such that almost surely

$$\rho_t(\{e\}) \xrightarrow[t \rightarrow \infty]{} 1.$$

The r.v.  $e$  has a uniform distribution on  $[0, 1]$  and is called the primitive Eve of the population. It can be interpreted as an ancestor whose progeny fixes in the population. In the present paper, we investigate the existence of an infinite sequence of Eves  $e^i, i \geq 1$  that would generalise the primitive Eve in the following sense: for every  $i \geq 1$ , the progeny of  $e^i$  is overwhelming among the population that does not descend from  $e^1, \dots, e^{i-1}$ . The precise definition is given below.

**DEFINITION I.3.** *According to the regime of  $\Lambda$ , we introduce the Eves as follows:*

*Eves - persistent case.* In regimes DISCRETE, INTENSIVE w. DUST and INTENSIVE  $\infty$ , we say that  $\rho$  admits an infinite sequence of Eves if there exists a collection  $(e^i)_{i \geq 1}$  of points in  $[0, 1]$  such that for every  $i \geq 1$

$$\frac{\rho_t(\{e^i\})}{\rho_t([0, 1] \setminus \{e^1, \dots, e^{i-1}\})} \xrightarrow[t \rightarrow \infty]{} 1$$

*Eves - extinction case.* In regime CDI, we say that  $\rho$  admits an infinite sequence of Eves if one can order the atoms by strictly decreasing extinction times, the sequence is then denoted by  $(e^i)_{i \geq 1}$ .

In Proposition I.21 we show that the event where  $\rho$  admits an infinite sequence of Eves is measurable in the sigma-field generated by  $\rho$  and that, on this event, the Eves are measurable. It is simple to check that almost surely  $e^1$  is well-defined and coincides with the primitive Eve. To determine the existence of the whole sequence, we need to study the asymptotic properties of the  $\Lambda$  Fleming-Viot process. The lockdown representation is particularly appropriate to that purpose. Indeed, suppose that  $\rho_t = \mathcal{L}_{0,t}(\hat{\Pi}, \xi_0), t \geq 0$  where  $\hat{\Pi}$  is a  $\Lambda$  flow of partitions and  $\xi_0$  is an independent sequence of i.i.d. uniform $[0, 1]$  r.v., then if the  $\Lambda$  Fleming-Viot process  $(\rho_t, t \geq 0)$  admits an infinite sequence of Eves then necessarily  $e_i = \xi_0(i)$  for every  $i \geq 1$ , see Proposition I.23. We now present our results on the existence of the Eves.

**THEOREM I.1.** *Suppose that  $\Lambda$  belongs to:*

- Regime DISCRETE or
- Regime INTENSIVE w. DUST and fulfils the condition  $\int_{[0,1]} x \log \frac{1}{x} \nu(dx) < \infty$

*then almost surely the  $\Lambda$  Fleming-Viot process does not admit an infinite sequence of Eves.*

Let us comment briefly this result. We rely on the lockdown representation. In regime DISCRETE, we will prove that eventually all the reproduction events choose a parent with type  $\xi_0(1)$  so that only the frequency of the primitive Eve makes positive jumps. It is then easy to derive that the second initial type  $\xi_0(2)$  does not fix in the remaining population that does not descend from  $\xi_0(1)$ . In regime INTENSIVE w. DUST under the condition  $\int_{[0,1]} x \log \frac{1}{x} \nu(dx) < \infty$ , we will prove the almost sure existence of at least one initial type, say  $\xi_0(i)$  for a certain  $i \geq 2$ , whose frequency stays equal to 0 forever. Consequently the  $i$ -th Eve does not exist. Notice that the  $x \log \frac{1}{x}$  condition is fulfilled by all the Beta $(2 - \alpha, \alpha)$  coalescents with  $\alpha \in (0, 1)$ . Let us mention that this  $x \log \frac{1}{x}$  condition does also appear in a very similar context

with continuous state branching processes, see [25].

In regime INTENSIVE  $\infty$ , we have the following result.

**PROPOSITION I.4.** *In regime INTENSIVE  $\infty$ , when  $\Lambda(dx) = dx$  almost surely the  $\Lambda$  Fleming-Viot process admits an infinite sequence of Eves.*

This result strongly relies on the connection between the Bolthausen-Sznitman coalescent [12, 19] and the Neveu continuous state branching process, see Subsection 4.3. The existence of the sequence of Eves for other measures  $\Lambda$  in regime INTENSIVE  $\infty$  remains an open question.

To study regime CDI, we rely again on the lookdown representation. Definition I.3 and the remark that follows the statement imply that the  $\Lambda$  Fleming-Viot process admits an infinite sequence of Eves if and only if the initial types  $\xi_0(2), \xi_0(3), \dots$  become extinct at distinct times almost surely. We call  $E$  the event where at least two initial types become extinct simultaneously.

**THEOREM I.2.** *The event  $E$  has probability 0 or 1. When  $\Lambda(\{0\}) > 0$  or when  $\Lambda(dx) = f(x)dx$  with  $f$  a regularly varying function at 0 with index  $1 - \alpha$  where  $\alpha \in (1, 2)$ , then  $E$  has probability 0 and almost surely the  $\Lambda$  Fleming-Viot process admits an infinite sequence of Eves.*

The regular variation condition on the density of  $\Lambda$  is verified by the well-studied class of Beta( $2 - \alpha, \alpha$ ) coalescents with  $\alpha \in (1, 2)$ . We refer to Subsection 4.5 for conjectures on the existence of the sequence of Eves in the remaining cases.

### 1.3 The flow of bridges and the unification

Let us present another approach to construct the  $\Lambda$  Fleming-Viot process. An exchangeable bridge as defined by Kallenberg [47] is a non-decreasing random process  $F : [0, 1] \rightarrow [0, 1]$  such that  $F(0) = 0$ ,  $F(1) = 1$ , and  $F$  has exchangeable increments. Bertoin and Le Gall observed that the distribution function of the  $\Lambda$  Fleming-Viot process taken at a given time  $t \geq 0$ , say  $F_t$ , is an exchangeable bridge. Moreover for all  $0 < s < t$ ,  $F_t$  has the same distribution as  $F'_{t-s} \circ F_s$  where  $F'_{t-s}$  is an independent copy of  $F_{t-s}$  and  $\circ$  stands for the composition operator of real-valued functions. This observation motivates the following definition due to Bertoin and Le Gall [13].

**DEFINITION I.5.** *A flow of bridges is a collection  $F := (F_{s,t}, -\infty < s \leq t < \infty)$  of bridges such that :*

- For every  $r < s < t$ ,  $F_{r,t} = F_{s,t} \circ F_{r,s}$  a.s. (cocycle property).
- The law of  $F_{s,t}$  only depends on  $t - s$ . Furthermore, if  $s_1 < s_2 < \dots < s_n$  the bridges  $F_{s_1, s_2}, F_{s_2, s_3}, \dots, F_{s_{n-1}, s_n}$  are independent.
- $F_{0,0} = Id$  and  $F_{0,t} \rightarrow Id$  in probability as  $t \downarrow 0$  for the Skorohod's topology.

Furthermore, if one denotes by  $\rho_{s,t}$  the probability measure with distribution function  $F_{s,t}$  and if  $(\rho_{0,t}, t \geq 0)$  is a  $\Lambda$  Fleming-Viot process, then  $F$  is called a  $\Lambda$  flow of bridges.

Observe that the stationarity of the increments of the flow of bridges ensures that the distribution of  $(\rho_{s,s+t}, t \geq 0)$  does not depend on  $s \in \mathbb{R}$ . Consequently a  $\Lambda$  flow of bridges does not only construct one  $\Lambda$  Fleming-Viot process, it actually couples an infinite collection of  $\Lambda$  Fleming-Viot processes indexed by  $s \in \mathbb{R}$ .

We now present the main contribution of the present paper. Consider a measure  $\Lambda$  such that the  $\Lambda$  Fleming-Viot process almost surely admits an infinite sequence of Eves. Notice that we do not restrict

ourselves to the particular cases that we have identified in Proposition I.4 and Theorem I.2 and we only assume that Definition I.3 is verified. Let  $F$  be a  $\Lambda$  flow of bridges. For every  $s \in \mathbb{R}$ , consider a càdlàg modification of the  $\Lambda$  Fleming-Viot process that we still denote by  $(\rho_{s,s+t}, t \geq 0)$  for simplicity. We let  $e_s := (e_s^i, i \geq 1)$  be the sequence of Eves of the  $\Lambda$  Fleming-Viot process  $(\rho_{s,s+t}, t \geq 0)$ . The cocycle property of the flow of bridges expresses the consistency of the collection of  $\Lambda$  Fleming-Viot processes, it is therefore natural to look at the relationships between the Eves taken at different times. We introduce the random partitions  $\hat{\Pi}_{s,t}$  of  $\mathbb{N}$  by setting for every  $i \geq 1$

$$\hat{\Pi}_{s,t}(i) := \{j \geq 1 : F_{s,t}^{-1}(e_t^j) = e_s^i\}.$$

where  $F_{s,t}^{-1}$  is the càdlàg inverse of  $F_{s,t}$ . This formula means that we put in a same block all the Eves at time  $t$  that descend from a same Eve at time  $s$ .

**THEOREM I.3.** *The collection of partitions  $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$  is a  $\Lambda$  flow of partitions.*

From this  $\Lambda$  flow of partitions we can define a collection of measure-valued processes using the lockdown representation. Fix  $s \in \mathbb{R}$  and introduce the  $\mathcal{M}_1$ -valued process  $(\mathcal{E}_{s,s+t}(\hat{\Pi}, e_s), t \geq 0)$  by setting

$$\mathcal{E}_{s,s+t}(\hat{\Pi}, e_s) := \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)| \delta_{e_s^i} + (1 - \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|) \ell, \quad \forall t \geq 0.$$

Simple arguments show that  $e_s$  is a sequence of i.i.d. uniform $[0, 1]$  r.v. which is independent of  $\hat{\Pi}$  so that  $(\mathcal{E}_{s,s+t}(\hat{\Pi}, e_s), t \geq 0)$  is a  $\Lambda$  Fleming-Viot process. This leads us to the statement of our main result.

**THEOREM I.4.** *Suppose that the  $\Lambda$  Fleming-Viot process admits an infinite sequence of Eves. The  $\Lambda$  flow of bridges is uniquely decomposed into two random objects: the flow of partitions and the Eves process. More precisely, for all  $s \in \mathbb{R}$  almost surely*

$$(\mathcal{E}_{s,s+t}(\hat{\Pi}, e_s), t \geq 0) = (\rho_{s,s+t}, t \geq 0)$$

Furthermore if we are given a  $\Lambda$  flow of partitions  $\hat{\Pi}'$  and for each  $s \in \mathbb{R}$  an independent sequence  $\chi_s := (\chi_s(i), i \geq 1)$  i.i.d. uniform $[0, 1]$  such that almost surely  $(\mathcal{E}_{s,s+t}(\hat{\Pi}', \chi_s), t \geq 0) = (\rho_{s,s+t}, t \geq 0)$  then

- (Initial types) For all  $s \in \mathbb{R}$ , almost surely for every  $i \geq 1$   $\chi_s(i) = e_s^i$ .
- (Genealogy) Almost surely  $\hat{\Pi}' = \hat{\Pi}$ .

This theorem shows that the  $\Lambda$  flow of partitions  $\hat{\Pi}$  and the process of Eves  $(e_s, s \in \mathbb{R})$  are sufficient to recover the whole collection of  $\Lambda$  Fleming-Viot processes encoded by the flow of bridges. Furthermore these two ingredients are unique. Notice that our construction actually defines a *flow* indexed by  $s \in \mathbb{R}$  of lockdown processes.

## 2 Flows of partitions

We start this section with the original definition of the lockdown representation. Then we introduce our formalism based on partitions of integers and show a one-to-one correspondence between lockdown graphs and flows of partitions. Let us also mention that stochastic flows of partitions have been independently introduced by Foucart [34] to define generalised Fleming-Viot processes with immigration.

## 2.1 The lockdown representation

In the lockdown representation, the population is composed of a countable collection of individuals: at any time each individual is located at a so-called *level*, taken to be an element of  $\mathbb{N}$ . This induces at any time an ordering of the population. At time 0 the individual located at level  $i \geq 1$  is called the  $i$ -th ancestor, his type is denoted by  $\xi_0(i) \in [0, 1]$ . As time passes, *reproduction events* occur in which an individual, called the parent, gives birth to a collection of children and dies out. Each child is necessarily located at the same or a higher level than the parent and has the same type. All the individuals (except the parent) that were alive before the reproduction event are then redistributed keeping the same order on the levels that are not occupied by a child. The variable  $\xi_t(i)$  is then defined as the type carried at time  $t \geq 0$  by the individual located at level  $i \geq 1$ . We now provide formal definitions.

We define  $\mathcal{S}_\infty$  as the subset of  $\{0, 1\}^\infty$  whose elements have at least two coordinates equal to 1. For every  $n \geq 2$  we also introduce  $\mathcal{S}_n$  as the subset of  $\{0, 1\}^\infty$  whose elements have at least two coordinates equal to 1 among  $[n]$ . For a subset  $\mathfrak{p}$  of  $\mathbb{R} \times \mathcal{S}_\infty$  and an integer  $n \geq 2$ , let  $\mathfrak{p}_{|[s,t] \times \mathcal{S}_n}$  be the intersection of  $\mathfrak{p}$  with  $[s, t] \times \mathcal{S}_n$ .

**DEFINITION I.6.** *A deterministic lockdown graph is a countable subset  $\mathfrak{p}$  of  $\mathbb{R} \times \mathcal{S}_\infty$  such that for all  $n \in \mathbb{N}$  and all  $s \leq t$ ,  $\mathfrak{p}_{|[s,t] \times \mathcal{S}_n}$  has a finite number of points. To every point  $(t, v) \in \mathfrak{p}$  we associate the set  $I_{t,v} := \{i \geq 1 : v(i) = 1\}$ .*

A deterministic lockdown graph should be seen as a collection of *reproduction events*  $(t, v) \in \mathbb{R} \times \mathcal{S}_\infty$ , where  $t$  denotes the reproduction time,  $\min I_{t,v}$  is the level of the parent and  $I_{t,v}$  is the set of levels that participate to the reproduction event.

**DEFINITION I.7.** *Let  $\mathfrak{p}$  be a lockdown graph and  $\xi_0 = (\xi_0(i), i \geq 1)$  be a collection of values in  $[0, 1]$ . The deterministic lockdown process constructed from  $\mathfrak{p}$  and  $\xi_0$  is the particle system  $(\xi_t(i), t \geq 0), i \geq 1$  defined as follows:*

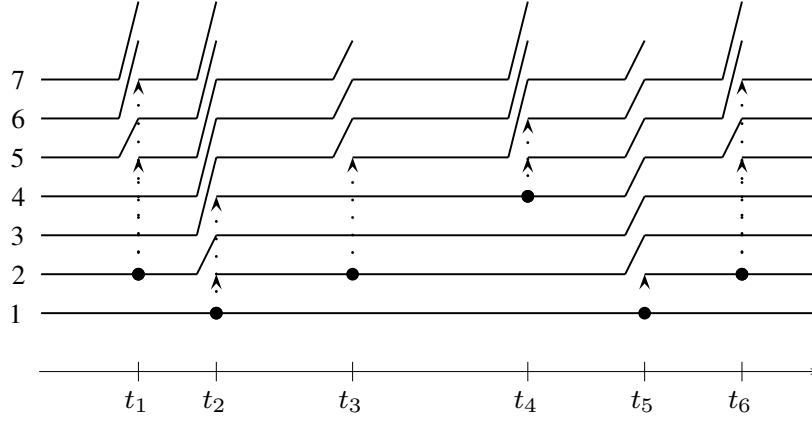
- The initial values are given by  $(\xi_0(i), i \geq 1)$
- At any reproduction event  $(t, v) \in \mathfrak{p}$  with  $t > 0$  we have

$$\begin{cases} \xi_t(i) = \xi_{t-}(\min(I_{t,v})) & \text{for every } i \in I_{t,v} \\ \xi_t(i) = \xi_{t-}(i - (\#\{I_{t,v} \cap [i]\} - 1) \vee 0) & \text{for every } i \notin I_{t,v}. \end{cases} \quad (\text{I.5})$$

The transitions should be interpreted as follows. At level  $\min(I_{t,v})$  is located the parent of the reproduction event, this individual dies at time  $t$ . At any level  $i \in I_{t,v}$  a new individual is born with the type  $\xi_{t-}(\min(I_{t,v}))$  of the parent. All the individuals alive at time  $t-$  except the parent are then redistributed, keeping the same order, on those levels that do not belong to  $I_{t,v}$  (see Figure I.1). When  $v$  does not belong to  $\mathcal{S}_n$ , then  $\xi_t(i) = \xi_{t-}(i)$  for every  $i \in [n]$ . The finiteness of the set  $\mathfrak{p}_{|[s,t] \times \mathcal{S}_n}$  for all  $s < t$  thus ensures that the  $n$  first particles  $(\xi_t(i), t \geq 0), i \in [n]$  make only finitely many jumps on any compact interval of time so that they are well-defined. Observe the consistency of the particle system when  $n$  varies: the  $(n+1)$ -th particle does not affect the  $n$  first. Consequently  $(\xi_t(i), t \geq 0), i \geq 1$  is well-defined.

We now explain how one randomises the previous objects so that the lockdown process admits almost surely a process of empirical measures that forms a  $\Lambda$  Fleming-Viot process. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $\mathcal{P}$  be a Poisson point process on  $\mathbb{R} \times \mathcal{S}_\infty$  with intensity measure  $dt \otimes (\mu_K + \mu_\Lambda)$ . The measures  $\mu_K$  and  $\mu_\Lambda$  are defined by

$$\mu_\Lambda(\cdot) := \int_{(0,1)} \sigma_u(\cdot) \nu(du) \quad , \quad \mu_K(\cdot) := \Lambda(0) \sum_{1 \leq i < j} \delta_{s_{i,j}}(\cdot) \quad (\text{I.6})$$



**Figure I.1:** A lockdown graph. Each arrow corresponds to a reproduction event: the level carrying a dot reproduces on the levels carrying an ending arrow. For example, at time  $t_1$ , level 2 reproduces on levels 5 and 7 while former levels 5, 6 and 7 are pushed up to the next available levels.

The ancestor of the individual alive at time  $t_3$  at level 7 is given by  $A_{t_3}(7) = 4$ .

The corresponding flow of partitions, restricted to  $[7]$ , would be entirely defined by the partitions  $\hat{\Pi}_{t_1-, t_1} = \{\{1\}, \{2, 5, 7\}, \{3\}, \{4\}, \{6\}\}$ ,  $\hat{\Pi}_{t_2-, t_2} = \{\{1, 2, 4\}, \{3\}, \{5\}, \{6\}, \{7\}\}$ ,  $\hat{\Pi}_{t_3-, t_3} = \{\{1\}, \{2, 5\}, \{3\}, \{4\}, \{6\}, \{7\}\}$  and so on.

where for all  $u \in (0, 1)$ ,  $\sigma_u(\cdot)$  is the distribution on  $\mathcal{S}_\infty$  of a sequence of i.i.d. Bernoulli random variables with parameter  $u$ , and for all  $1 \leq i < j$ ,  $s_{i,j}$  is the element of  $\mathcal{S}_\infty$  that has only two coordinates equal to 1:  $i$  and  $j$ . The measure  $\mu_\Lambda$  corresponds to reproduction events involving a positive proportion of individuals while  $\mu_K$  corresponds to reproduction events involving only two individuals at once.

**DEFINITION I.8.** A lockdown graph associated with the measure  $\Lambda$  - or  $\Lambda$  lockdown graph in short - is a Poisson point process  $\mathcal{P}$  on  $\mathbb{R} \times \mathcal{S}_\infty$  with intensity measure  $dt \otimes (\mu_K + \mu_\Lambda)$ .

Using Formula (I.2) we easily get

$$(\mu_K + \mu_\Lambda)(\mathcal{S}_n) = \Lambda(0) \binom{n}{2} + \int_{(0,1)} \sum_{k=2}^n \binom{n}{k} u^k (1-u)^{n-k} \nu(du) = \lambda_n.$$

Basic properties of Poisson point processes ensure that  $\mathcal{P}|_{[s,t] \times \mathcal{S}_n}$  is a Poisson point process on  $[s, t] \times \mathcal{S}_n$  whose intensity has a total mass equal to  $(t-s)\lambda_n < \infty$ . Consequently for  $\mathbb{P}$ -almost all  $\omega$ ,  $\mathcal{P}(\omega)$  is a deterministic lockdown graph. Let  $\xi_0 = (\xi_0(i), i \geq 1)$  be an independent sequence of i.i.d. uniform $[0, 1]$  r.v.

**DEFINITION I.9.** Applying Definition I.7 to  $\mathcal{P}(\omega)$  and  $\xi_0(\omega)$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ , we get a particle system  $(\xi_t(i), t \geq 0), i \geq 1$  that we call the lockdown process associated to  $\mathcal{P}$  and  $\xi_0$ .

We let  $\mathcal{M}_1$  be the set of probability measures on  $[0, 1]$  endowed with the topology of the weak convergence.

**THEOREM I.5.** (Donnelly-Kurtz [24])  $\mathbb{P}$ -almost surely the particle system  $(\xi_t(i), t \geq 0), i \geq 1$  admits a process of empirical measures

$$t \mapsto \Xi_t := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{\xi_t(i)}.$$

This  $\mathcal{M}_1$ -valued process is a càdlàg  $\Lambda$  Fleming-Viot process.

This result is not stated in these terms in the article of Donnelly and Kurtz so that we provide some details on the proof.

**Proof** Let us introduce for every  $n \geq 1$ , the process of empirical measures of the  $n$  first particles:

$$t \mapsto \Xi_t^{[n]} := \frac{1}{n} \sum_{i=1}^n \delta_{\xi_t(i)}.$$

This process takes values in the space  $\mathbb{D}([0, \infty), \mathcal{M}_1)$  of càdlàg  $\mathcal{M}_1$ -valued functions. Let  $f_k, k \geq 1$  be a dense sequence of continuous functions on  $[0, 1]$ . Lemma 3.4 and 3.5 of Donnelly and Kurtz [24] show that for every  $k \geq 1$ , almost surely the sequence of processes  $(\int_{[0,1]} f_k(x) \Xi_t^{[n]}(dx), t \geq 0), n \geq 1$  is a Cauchy sequence in  $\mathbb{D}([0, \infty), \mathbb{R})$  endowed with the norm:

$$d_u(X, Y) = \int_{[0, \infty)} e^{-t} \sup_{s \leq t} 1 \wedge |X_s - Y_s| dt.$$

Although it is not separable, this space is complete. Subsequently using the distance

$$d(m, m') := \sum_{k \geq 1} 2^{-k} \left( \left| \int_{[0,1]} f_k(x) m(dx) - \int_{[0,1]} f_k(x) m'(dx) \right| \wedge 1 \right)$$

that metrises the topology of the weak convergence on  $\mathcal{M}_1$ , it is a simple matter to check that almost surely  $(\Xi_t^{[n]}, t \geq 0), n \geq 1$  is a Cauchy sequence in  $\mathbb{D}([0, \infty), \mathcal{M}_1)$  endowed with the norm:

$$d'_u(M, M') = \int_{[0, \infty)} e^{-t} \sup_{s \leq t} d(M_s, M'_s) dt.$$

The sequence converges almost surely since the latter is a complete space. The identification of the distribution of the limiting process can be carried out by comparing (modulo a simple calculation) the expression of the generator obtained by Donnelly and Kurtz in Section 4 [24] with the expression of the generator of the  $\Lambda$  Fleming-Viot process obtained by Bertoin and Le Gall in Theorem 3 [13]. ■

## 2.2 Deterministic flows of partitions

We start with the definition of the deterministic flow of partitions. Then we prove a one-to-one correspondence between the set of deterministic flows of partitions and the set of deterministic lookdown graphs. Finally we show that the deterministic flow of partitions formalises the implicit genealogy encoded by a deterministic lookdown graph. A key rôle will be played by the coagulation operator. Recall that for all  $\pi, \pi' \in \mathcal{P}_\infty$ ,  $\text{Coag}(\pi, \pi')$  is the element of  $\mathcal{P}_\infty$  whose  $i$ -th block is equal to the union of the  $\pi(j)$ 's for  $j \in \pi'(i)$ . We extend this notation to the case where  $\pi, \pi' \in \mathcal{P}_n$  by taking implicitly  $\text{Coag}(\pi, \pi')$  as an element of  $\mathcal{P}_n$  and using the same definition for its blocks.

**DEFINITION I.10.** *A deterministic flow of partitions is a collection  $\hat{\pi} = (\hat{\pi}_{s,t}, -\infty < s \leq t < \infty)$  of partitions of  $\mathbb{N}$  that verifies:*

- For all  $r < s < t$ , we have  $\hat{\pi}_{r,t} = \text{Coag}(\hat{\pi}_{s,t}, \hat{\pi}_{r,s})$  (cocycle property).
- For all  $s \in \mathbb{R}$ ,  $\hat{\pi}_{s,s} = 0_{[\infty]}$  and the limit  $\lim_{r \uparrow s} \hat{\pi}_{r,s} =: \hat{\pi}_{s-,s}$  exists and is either a partition with a unique non-singleton block or  $0_{[\infty]}$ .
- For all  $t \in \mathbb{R}$ ,  $\lim_{\substack{r < s < t \\ r, s \uparrow t}} \hat{\pi}_{r,s}$  exists and equals  $0_{[\infty]}$  (left regularity).



- For all  $s \in \mathbb{R}$ ,  $\lim_{t \downarrow s} \hat{\pi}_{s,t}$  exists and equals  $0_{[\infty]}$  (right regularity).

It is natural to call  $\{(s, \hat{\pi}_{s-,s}) : \hat{\pi}_{s-,s} \neq 0_{[\infty]}\}$  the collection of the jumps of  $\hat{\pi}$ . For convenience, we write this collection of jumps in a slightly different manner. To every partition  $\hat{\pi}_{s-,s}$  that differs from  $0_{[\infty]}$  we associate the element  $v_s$  of  $\mathcal{S}_\infty$  whose  $i$ -th coordinate is 1 if and only if  $i$  belongs to the non-singleton block of  $\hat{\pi}_{s-,s}$ . We then set  $J(\hat{\pi}) := \{(s, v_s) : \hat{\pi}_{s-,s} \neq 0_{[\infty]}\}$  for the collection of jumps of  $\hat{\pi}$ . Similarly for every  $n \geq 2$ , we set  $J_n(\hat{\pi}) := \{(s, v_s) : \hat{\pi}_{s-,s}^{[n]} \neq 0_{[n]}\}$ . Notice that  $J(\hat{\pi})$  and  $J_n(\hat{\pi})$  are subsets of  $\mathbb{R} \times \mathcal{S}_\infty$  and  $\mathbb{R} \times \mathcal{S}_n$  respectively.

**LEMMA I.11.** *For every  $n \geq 1$ , the intersection of  $J_n(\hat{\pi})$  with any set of the form  $[s, t] \times \mathcal{S}_n$  has finitely many points.*

**Proof** Suppose that the intersection of  $J_n(\hat{\pi})$  with the set  $[s, t] \times \mathcal{S}_n$  has infinitely many points. Necessarily these points accumulate near a certain point in  $[s, t]$ . For simplicity we assume that they accumulate on the right of  $r \in [s, t]$  and we let  $(r_1, v_1), (r_2, v_2), \dots$  be a sequence of these points such that  $r_i \downarrow r$  as  $i \rightarrow \infty$ . Since  $\hat{\pi}_{u, r_i}$  converges to the partition with a unique non-singleton block equal to  $\{j \geq 1 : v_{r_i}(j) = 1\}$  as  $u \uparrow r_i$  and since  $v_{r_i} \in \mathcal{S}_n$ , there exists  $u \in (r_{i+1}, r_i)$  such that  $\hat{\pi}_{u, r_i}^{[n]} \neq 0_{[n]}$ . Therefore the cocycle property  $\hat{\pi}_{r_{i+1}, r_i}^{[n]} = \text{Coag}(\hat{\pi}_{u, r_i}^{[n]}, \hat{\pi}_{r_{i+1}, u}^{[n]})$  ensures that  $\hat{\pi}_{r_{i+1}, r_i}^{[n]} \neq 0_{[n]}$ . Using the cocycle property once again we get that  $\hat{\pi}_{r, r_i}^{[n]} \neq 0_{[n]}$ . Taking the limit as  $i \rightarrow \infty$  we deduce that the right regularity at  $r$  is not verified. If we had assumed that the points accumulate on the left of a given point  $r$  then the left regularity at  $r$  would have failed. ■

We now make the connection between deterministic flows of partitions and deterministic lockdown graphs.

**PROPOSITION I.12.** *The map  $J$  that associates to a deterministic flow of partitions  $\hat{\pi}$  the collection of its jumps  $J(\hat{\pi})$  is a bijection between the set of deterministic flows of partitions and the set of deterministic lockdown graphs.*

**Proof** Let  $\hat{\pi}$  be a deterministic flow of partitions. One can write

$$J(\hat{\pi}) = \bigcup_{n \geq 1} J_n(\hat{\pi}).$$

Hence the set on the left is a countable union of countable sets thanks to Lemma I.11, so that it is itself countable. Moreover Lemma I.11 shows that the intersection of  $J(\hat{\pi})$  with any set of the form  $[s, t] \times \mathcal{S}_n$  has only finitely many points. Consequently  $J(\hat{\pi})$  is a deterministic lockdown graph.

Let  $\mathfrak{p}$  be deterministic lockdown graph. For every  $n \geq 1$ , we construct the restriction of  $\hat{\pi}$  to  $\mathcal{P}_n$  as follows. Fix  $s \leq t$ . If  $s = t$ , set  $\hat{\pi}_{s,t}^{[n]} := 0_{[n]}$ . If  $s < t$ , let  $(t_1, v_1), \dots, (t_k, v_k)$  be the finite collection of points in  $\mathfrak{p}|_{(s,t) \times \mathcal{S}_n}$  ranked by increasing time coordinates. We define the map  $\phi_n : \mathcal{S}_n \rightarrow \mathcal{P}_n$  as follows. For every  $v \in \mathcal{S}_n$ ,  $\phi_n(v)$  is the element of  $\mathcal{P}_n$  with a unique non-singleton block equal to  $\{i \in [n] : v(i) = 1\}$ . We then set

$$\hat{\pi}_{s,t}^{[n]} := \text{Coag} \left( \phi_n(v_k), \text{Coag} \left( \phi_n(v_{k-1}), \dots, \text{Coag}(\phi_n(v_2), \phi_n(v_1)) \dots \right) \right).$$

Our construction is consistent when  $n$  varies so that there exists a unique partition  $\hat{\pi}_{s,t}$  whose restriction to  $[n]$  is given by  $\hat{\pi}_{s,t}^{[n]}$ , for all  $s \leq t$ . Let us now check that  $\hat{\pi}$  is indeed a deterministic flow of partitions. The cocycle property is a direct consequence of our definition with the coagulation operator. The finiteness of  $\mathfrak{p}|_{(s,t) \times \mathcal{S}_n}$  for every  $s \leq t$  and every  $n \geq 1$  ensures the existence of  $\hat{\pi}_{s-,s}$  for all  $s \in \mathbb{R}$  along with

the left and right regularity. The fact that  $\hat{\pi}_{s-,s}$  is either  $0_{[\infty]}$  or a partition with a unique non-singleton block is a consequence of the fact that we have only coagulated partitions with a unique non-singleton block. Finally it is simple to check that  $J(\hat{\pi}) = \mathfrak{p}$ .  $\blacksquare$

Let us now explain why the deterministic flow of partitions is relevant in our context. Let  $\mathfrak{p}$  be a deterministic lookdown graph. For any level  $j \geq 1$  taken at a given time  $t \geq 0$ , one can define its *ancestral line* by tracing its genealogy backward in time until its ancestor at time 0. This ancestral line starts at  $j$  and stays constant between two reproduction events. At a reproduction event, say  $(s, v)$  with  $s \in (0, t]$ , either the ancestral line belongs to  $I_{s,v}$  and then it jumps to  $\min I_{s,v}$ , or the ancestral line does not belong to  $I_{s,v}$ . In the latter case, the ancestral line jumps from its current position, say  $i$ , to  $i - (\#\{I_{t,v} \cap [i]\} - 1) \vee 0$ . The value of the ancestral line at 0 is then the ancestor of level  $j$  at time  $t$ , we denote it by  $A_t(j)$ . We refer to Figure I.1 for an illustration of these ancestral lines.

Denote by  $J^{-1}$  the inverse map of  $J$  and set  $\hat{\pi} := J^{-1}(\mathfrak{p})$ . The following lemma shows that  $\hat{\pi}$  is the genealogical structure implicitly encoded by  $\mathfrak{p}$ .

**LEMMA I.13.** *For all  $i \geq 1$  and all  $t \geq 0$ ,  $\hat{\pi}_{0,t}(i) = \{j \geq 1 : A_t(j) = i\}$ .*

In other words,  $\hat{\pi}_{0,t}(i)$  is the progeny at time  $t$  of the  $i$ -th ancestor. This allows to state an alternative definition of the lookdown process. Consider a sequence of initial types  $\xi_0(i), i \geq 1$ . Then it is simple to check that the lookdown process of Definition I.7 verifies and is characterised by

$$\forall i, j \geq 1, \quad i \in \hat{\pi}_{0,t}(j) \Rightarrow \xi_t(i) = \xi_0(j).$$

**Proof** Fix  $n \geq 1$ . Let  $(s_1, v_1), (s_2, v_2), \dots$  be the elements of  $\mathfrak{p}_{|(0,\infty) \times \mathcal{S}_n}$  ranked by increasing time coordinates. From the definition of the map  $J$ , it is immediate to check that  $t \mapsto \hat{\pi}_{0,t}^{[n]}$  only evolves at times  $s_k, k \geq 1$ . Set  $s_0 = 0$ . We prove by induction on  $k$  that for every  $i \geq 1$ ,  $\hat{\pi}_{0,s_k}^{[n]}(i) = \{j \in [n] : A_{s_k}(j) = i\}$ . At rank 0 this is trivial. Suppose that it holds at a certain rank  $k-1 \geq 0$ . At rank  $k$ , we set  $I_{s_k, v_k}^{[n]} := \{j \in [n] : v_k(j) = 1\}$  and for every  $j \in [n]$  we define  $b(j)$  by setting

$$b(j) = \min I_{s_k, v_k}^{[n]} \quad \text{if } j \in I_{s_k, v_k}^{[n]}, \quad b(j) = j - (\#\{I_{s_k, v_k}^{[n]} \cap [j]\} - 1) \vee 0 \quad \text{otherwise.}$$

Then the definition of the ancestral line immediately yields that  $A_{s_k}(j) = A_{s_{k-1}}(b(j))$ . Let  $\pi$  be the partition of  $[n]$  whose blocks are given by  $\{j \in [n] : A_{s_k}(j) = i\}, i \in [n]$ . From the formula above we check that  $j$  and  $j'$  are in a same block of  $\pi$  if and only if  $A_{s_{k-1}}(b(j)) = A_{s_{k-1}}(b(j'))$  which is equivalent to saying (thanks to the induction hypothesis) that  $b(j)$  and  $b(j')$  are in a same block of  $\hat{\pi}_{0,s_{k-1}}^{[n]}$ . From the very definition of  $\hat{\pi}_{s_{k-1}, s_k}^{[n]}$ , it is elementary to check that  $b(j)$  is the index of the block of  $\hat{\pi}_{s_{k-1}, s_k}^{[n]}$  containing  $j$ . Since  $\hat{\pi}_{0, s_k}^{[n]} = \text{Coag}(\hat{\pi}_{s_{k-1}, s_k}^{[n]}, \hat{\pi}_{0, s_{k-1}}^{[n]})$  we deduce that  $\hat{\pi}_{0, s_k}^{[n]}(i) = \pi$  and the induction is complete. We have checked the asserted equality for the restrictions to  $[n]$  for any given  $n \geq 1$ . The lemma follows.  $\blacksquare$

### 2.3 Stochastic flows of partitions

We now consider the stochastic flow of partitions of Definition I.1. We first propose a Poissonian construction of this object and show that almost surely its trajectories are deterministic flows of partitions. Consequently almost all the trajectories encode a deterministic lookdown graph and can be used to apply the lookdown construction. Second we consider a stochastic flow of partitions not necessarily constructed from a Poisson point process. Nothing ensures that almost all its trajectories are deterministic flows of partitions. However we show the existence of a modification of this stochastic flow of partitions such that almost all its trajectories are deterministic flow of partitions. The lookdown representation can therefore be applied to the modification. We will need this result for the unification.

Poissonian construction

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Fix a finite measure  $\Lambda$  on  $[0, 1)$  and consider a  $\Lambda$  lockdown graph  $\mathcal{P}$  as introduced in Definition I.8. Note that for  $\mathbb{P}$ -a.a.  $\omega \in \Omega$ , for all  $s < t$  and all  $n \geq 1$  the point collection  $\mathcal{P}(\omega)|_{[s,t] \times \mathcal{S}_n}$  has a finite number of points. Therefore we define for  $\mathbb{P}$ -a.a.  $\omega$  the deterministic flow of partitions  $\hat{\Pi}(\omega)$  by setting  $\hat{\Pi}(\omega) := J^{-1}(\mathcal{P}(\omega))$ .

**PROPOSITION I.14.** *The collection of partitions  $\hat{\Pi}$  is a  $\Lambda$  flow of partitions.*

**Proof** The cocycle and continuity properties are verified for  $\mathbb{P}$ -a.a. trajectories by construction. The independence of the increments comes from the independence properties of Poisson point processes. Finally the Poissonian construction of coalescent processes (see Section 4.2.3 of the book of Bertoin [9]) ensures that  $(\hat{\Pi}_{-t,0}, t \geq 0)$  is a  $\Lambda$  coalescent. ■

The trajectories of the stochastic flow of partitions obtained from this Poisson point process are  $\mathbb{P}$ -a.s. deterministic flows of partitions. Actually they enjoy several nice regularity properties as the following result shows (the proof is postponed to Subsection 6.1).

**PROPOSITION I.15.** *On a same event of  $\mathbb{P}$ -probability one, the following holds true:*

- i) *The trajectories of  $\hat{\Pi}$  are deterministic flows of partitions.*
- ii) *(Regularity in frequencies) For every  $s \in \mathbb{R}, t \geq 0$  the partitions  $\hat{\Pi}_{s,s+t}, \hat{\Pi}_{s,s+t-}, \hat{\Pi}_{s-,s+t}$  possess asymptotic frequencies and whenever  $t > 0$  we have the following convergences for every integer  $i \geq 1$*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s,s+t+\epsilon}(i)| &= \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s+\epsilon,s+t}(i)| = |\hat{\Pi}_{s,s+t}(i)| \\ \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s,s+t-\epsilon}(i)| &= |\hat{\Pi}_{s,s+t-}(i)| \\ \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s-\epsilon,s+t}(i)| &= |\hat{\Pi}_{s-,s+t}(i)| \end{aligned}$$

- iii) *For every  $s \in \mathbb{R}, t \mapsto \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|$  is càdlàg on  $(0, \infty)$ .*

Let  $\xi_0 = (\xi_0(i))_{i \geq 1}$  be a sequence of i.i.d. uniform $[0, 1]$  r.v. As remarked after Lemma I.13, one can construct the lockdown process associated with  $\mathcal{P}$  and  $\xi_0$  as follows.

**DEFINITION I.16** (Second definition of the lockdown process.). *The lockdown process associated with  $\hat{\Pi}$  and  $\xi_0$  is the unique collection of processes  $(\xi_t(i), t \geq 0), i \geq 1$  such that for every integers  $i, j \geq 1$  and all  $t \geq 0$*

$$i \in \hat{\Pi}_{0,t}(j) \Rightarrow \xi_t(i) = \xi_0(j).$$

*This process coincides with the lockdown process associated with  $\mathcal{P}$  and  $\xi_0$  of Definition I.9.*

The exchangeability of  $\hat{\Pi}_{0,t}$  and  $\xi_0$  ensures that  $(\xi_t(i))_{i \geq 1}$  is itself exchangeable (see for instance the proof of Theorem 2.1 in [9]). The original formalism of the lockdown representation does not provide such an immediate argument for the exchangeability. Now we set for all  $t \geq 0$

$$\mathcal{E}_{0,t} := \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)| \delta_{\xi_0(i)} + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)|\right) \ell$$

where  $\ell$  is the Lebesgue measure on  $[0, 1]$ . This definition makes sense on the event of  $\mathbb{P}$ -probability one of Proposition I.15.

**PROPOSITION I.17.** *The process  $(\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0), t \geq 0)$  is a càdlàg  $\Lambda$  Fleming-Viot process. It coincides  $\mathbb{P}$ -a.s. with the process of empirical measures of the lockdown process  $(\xi_t(i), t \geq 0), i \geq 1$  introduced in Theorem I.5.*

**REMARK I.18.** *The process  $(\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0), t > 0)$  is even càdlàg for the total variation distance on  $\mathcal{M}_1$ .*

**Proof** From Proposition I.15, we know that  $\mathbb{P}$ -a.s.  $(\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0), t > 0)$  is a càdlàg  $\mathcal{M}_1$ -valued process. Moreover for all  $t \geq 0$ ,  $\mathbb{P}$ -a.s.  $\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$  coincides with the empirical measure of  $(\xi_t(i), i \geq 1)$  (see for instance Lemma 2 of Foucart [34]). From Theorem I.5, we know that the process of empirical measures of the lockdown process is a càdlàg  $\Lambda$  Fleming-Viot process. Since two càdlàg processes that coincide  $\mathbb{P}$ -a.s. for all rational values, are  $\mathbb{P}$ -a.s. equal, the almost sure equality follows. ■

## Regularisation

We now consider a  $\Lambda$  flow of partitions  $\hat{\Pi}$  in the sense of Definition I.1. Observe that its trajectories are not necessarily deterministic flows of partitions. Indeed, the cocycle property does not necessarily hold simultaneously for all triplets  $r < s < t$  on a same event of probability one. Hence the bijection  $J$  cannot be directly applied to obtain lockdown graphs. Below we prove the existence of a modification  $\tilde{\Pi}$  whose trajectories are genuine deterministic flows of partitions. The reason that motivates this technical discussion is that we will identify in Section 5 a stochastic flow of partitions embedded into a flow of bridges from which we will construct a lockdown process.

**PROPOSITION I.19.** *Let  $\hat{\Pi}$  be a  $\Lambda$  flow of partitions. There exists a collection of partitions  $\tilde{\Pi}$  such that for all  $s \leq t$ , almost surely  $\hat{\Pi}_{s,t} = \tilde{\Pi}_{s,t}$  and such that on a same event  $\Omega_{\hat{\Pi}}$  of probability one, all the trajectories  $\tilde{\Pi}(\omega)$  are deterministic flows of partitions.*

Our strategy of proof is to restrict to the rational marginals of the flow of partitions  $\hat{\Pi}$ , and to prove that they verify the properties of a deterministic flow of partitions up to an event of probability zero. Then, we take right and left limits on this object and prove that we recover a modification of the initial flow. The proof is postponed to Subsection 6.2.

**REMARK I.20.** *From a stochastic flow of partitions, we have been able to define a regularised modification. Note that this operation does not seem possible for a stochastic flow of bridges. Indeed, a key argument in our proof relies on the continuity of the coagulation operator whereas this property does not hold with the composition operator for bridges.*

## 3 The Eves

We start with the classification into four regimes of the  $\Lambda$  Fleming-Viot process. In view of the proof of Proposition I.2, we recall some results of the literature on the behaviour of the  $\Lambda$  coalescent, we refer to Pitman [66], Schweinsberg [69], Bertoin and Le Gall [15], Gneden et al. [36] and Freeman [35] for the proofs. Let  $(\Pi_t, t \geq 0)$  be a  $\Lambda$  coalescent. We use the regimes introduced in the statement of Proposition I.2 which are characterised in terms of the measure  $\Lambda$ . In regime DISCRETE, almost surely for all  $t > 0$  the partition  $\Pi_t$  has dust and finitely many non-singleton blocks. In regime INTENSIVE w. DUST, almost surely for all  $t > 0$  the partition  $\Pi_t$  has dust and infinitely many non-singleton blocks. In regime INTENSIVE  $\infty$ , almost surely for all  $t > 0$  the partition  $\Pi_t$  has no dust and infinitely many non-singleton blocks. Finally in regime CDI, almost surely for all  $t > 0$  the partition  $\Pi_t$  has no dust and finitely many non-singleton blocks. In the latter regime, we say that the  $\Lambda$  coalescent Comes Down from Infinity (CDI).

*Proof of Proposition I.2.* Let  $(\rho_t, t \geq 0)$  be a  $\Lambda$  Fleming-Viot process. Fix  $t > 0$ . We know that  $\rho_t$  has the same distribution on  $\mathcal{M}_1$  as the r.v.  $\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$  defined in Subsection 2.3. By definition of the measure  $\mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$ , the dust and the number of atoms of this random probability measure have the same law as the dust and the number of non-singleton blocks of a  $\Lambda$  coalescent taken a time  $t$ . Using the results on the  $\Lambda$  coalescent recalled above, we obtain the asserted classification. ■

To study the existence of the Eves we need to deal with a regular version of the  $\Lambda$  Fleming-Viot process. Let  $\mathbf{P}$  be the distribution of the  $\Lambda$  Fleming-Viot process on the space  $\mathbb{D} := \mathbb{D}([0, \infty), \mathcal{M}_1)$  of càdlàg  $\mathcal{M}_1$ -valued functions endowed with the usual Skorohod's topology. Recall that  $\mathcal{M}_1$  is equipped with the topology of the weak convergence of probability measures. We denote by  $\mathcal{B}(\mathbb{D})$  the Borel sigma-field associated with  $\mathbb{D}$  augmented with the  $\mathbf{P}$ -null sets. Recall also from Proposition I.17 that one can get such a regular version of the  $\Lambda$  Fleming-Viot process using the lockdown representation.

**PROPOSITION I.21.** *The set*

$$\mathcal{O} := \{\rho \in \mathbb{D} : \rho \text{ admits an infinite sequence of Eves}\}$$

belongs to  $\mathcal{B}(\mathbb{D})$ . Moreover the map

$$\begin{aligned} \mathbb{D} \cap \mathcal{O} &\rightarrow [0, 1]^{\mathbb{N}} \\ \rho &\mapsto (e^i)_{i \geq 1} \end{aligned}$$

is measurable when  $[0, 1]^{\mathbb{N}}$  is endowed with the product sigma-field.

**Proof** We start with regimes DISCRETE, INTENSIVE w. DUST and INTENSIVE  $\infty$ . We set for any  $k, l, n \geq 1$  and any  $\epsilon \in (0, 1)$

$$\mathcal{O}(k, \epsilon, l, n) := \bigcup_{i_1, \dots, i_k \in [2^n]} \bigcap_{\substack{j \in [k] \\ t \geq l \\ t \in \mathbb{Q}}} \left\{ \rho_t \left( \left[ \frac{i_j - 1}{2^n}, \frac{i_j}{2^n} \right) \right) > (1 - \epsilon) \left( 1 - \sum_{m=1}^{j-1} \rho_t \left( \left[ \frac{i_m - 1}{2^n}, \frac{i_m}{2^n} \right) \right) \right) \right\}$$

which is clearly an element of  $\mathcal{B}(\mathbb{D})$ . Subsequently we set

$$\mathcal{O}(k) = \bigcap_{\epsilon \in (0, 1) \cap \mathbb{Q}} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{O}(k, \epsilon, l, n).$$

The event  $\mathcal{O}(k)$  is the event where the  $k$  first Eves exist. Then  $\mathcal{O} = \lim \downarrow \mathcal{O}(k)$  so that  $\mathcal{O}$  belongs to  $\mathcal{B}(\mathbb{D})$ . The event  $\{e^1 \leq x\}$  is then obtained by modifying the definition of  $\mathcal{O}(1, \epsilon, l, n)$  by restricting to the  $i_1$ 's which are smaller than  $\inf\{j \in [2^n] : x \leq j2^{-n}\}$  and taking the union on  $n \geq 1$  and the limit as  $l \rightarrow \infty$  and  $\epsilon \downarrow 0$ . This can be easily generalised to prove the measurability of the whole sequence.

We now turn to regime CDI which is more involved. We rely on the following four claims.

**Claim 1.** The set

$$C := \left\{ \rho \in \mathbb{D} : \begin{array}{l} \forall t \in \mathbb{Q}_+, \rho_t \text{ is a weighted sum of finitely many atoms} \\ \forall t, s \in \mathbb{Q}_+ \text{ the atoms as time } t + s \text{ are a subset of the atoms at time } t. \end{array} \right\}$$

belongs to  $\mathcal{B}(\mathbb{D})$ .

**Claim 2.** On the set  $C$ , for all  $t > 0$  the measure  $\rho_t$  is a sum of finitely many atoms. Furthermore, for every  $s \in \mathbb{Q}_+$  and every  $t > s$  the atoms of  $\rho_t$  are necessarily atoms of  $\rho_s$ .

Consequently on  $C$  we can define  $\#\rho_t$  as the number of atoms of the measure  $\rho_t$  for any given time  $t > 0$ . Of course  $\rho \mapsto \#\rho_t$  is measurable from  $C$  to  $\mathbb{N}$ .

**Claim 3.** For every  $i \geq 2$  and every  $\rho \in C$  we define the following quantity  $\tau_i := \inf\{t > 0 : \#\rho_t < i\}$ . Then  $\tau_i$  is a  $\mathcal{H}_t$ -stopping time where  $\mathcal{H}_t := \sigma(\rho_s, s \in [0, t])$  augmented with the  $\mathbf{P}$ -null sets.

**Claim 4.** The set

$$C' := \left\{ \rho \in \mathbb{D} : \begin{array}{l} \forall t \in \mathbb{R}_+^*, \rho_t \text{ is a weighted sum of finitely many atoms} \\ \forall t, s \in \mathbb{R}_+^* \text{ the atoms as time } t + s \text{ are a subset of the atoms at time } t \\ t \mapsto \#\rho_t \text{ is càdlàg on } (0, \infty) \end{array} \right\}$$

belongs to  $\mathcal{B}(\mathbb{D})$ .

The set  $\mathcal{O}$  is the subset of  $C'$  where  $t \mapsto \#\rho_t$  makes only jumps of size  $-1$  so that it belongs to  $\mathcal{B}(\mathbb{D})$ . To end the proof of Proposition I.21 in regime CDI we observe that on the event  $\mathcal{O}$ , we have for every  $x \in [0, 1]$

$$\{e^1 \leq x\} = \mathcal{O} \cap \left( \bigcup_{t \in \mathbb{Q}_+} \{\rho : \#\rho_t = 1, \rho_t([0, x]) = 1\} \right)$$

so that the measurability of  $e^1$  is immediate. This can be generalised easily to prove the measurability of the whole sequence.

*Proof of Claim 1.* For every  $t, s > 0$  and every  $k \geq 1, m \geq 0, n \geq 1$  let

$$\begin{aligned} C(k, t, m, s, n) := & \bigcup_{\substack{i_1, \dots, i_{k+m} \in [2^n] \\ i_1, \dots, i_{k+m} \text{ are distinct}}} \left( \bigcap_{j \in [k]} \left\{ \rho_t \left( \left[ \frac{i_j - 1}{2^n}, \frac{i_j}{2^n} \right) \right) > 0; \rho_{t+s} \left( \left[ \frac{i_j - 1}{2^n}, \frac{i_j}{2^n} \right) \right) > 0 \right\} \right. \\ & \bigcap_{k < j \leq k+m} \left\{ \rho_t \left( \left[ \frac{i_j - 1}{2^n}, \frac{i_j}{2^n} \right) \right) > 0; \rho_{t+s} \left( \left[ \frac{i_j - 1}{2^n}, \frac{i_j}{2^n} \right) \right) = 0 \right\} \\ & \left. \bigcap_{i \in [2^n], i \neq i_1, \dots, i_{k+m}} \left\{ \rho_t \left( \left[ \frac{i - 1}{2^n}, \frac{i}{2^n} \right) \right) = \rho_{t+s} \left( \left[ \frac{i - 1}{2^n}, \frac{i}{2^n} \right) \right) = 0 \right\} \right) \end{aligned}$$

which is obviously a measurable set. Then we have

$$C = \bigcap_{t, s \in \mathbb{Q}_+^*} \bigcup_{k \geq 1} \bigcup_{m \geq 0} \lim_{n \rightarrow \infty} C(k, t, m, s, n).$$

so that it is a Borel set of  $\mathbb{D}$ . Claim 1 is proved.

*Proof of Claim 2.* Fix  $t > 0$  and let  $s \in (0, t) \cap \mathbb{Q}$ . We restrict ourselves to  $\rho$ 's which belong to  $C$ . We know that for every  $\epsilon > 0$  such that  $t + \epsilon \in \mathbb{Q}$  the atoms of  $\rho_{t+\epsilon}$  are atoms of  $\rho_s$ . Let  $S$  be the union of the atoms of  $\rho_{t+\epsilon}$  when  $\epsilon$  varies. Necessarily  $S$  is a finite set so that it is a closed set of  $[0, 1]$ . By the right continuity of  $\rho$  we deduce that

$$\rho_t([0, 1] \setminus S) \leq \lim_{\epsilon \downarrow 0} \rho_{t+\epsilon}([0, 1] \setminus S) = 0$$

where implicitly  $\epsilon$  is taken such that  $t + \epsilon$  belongs to  $\mathbb{Q}$ . This ensures that  $\rho_t$  is a sum of finitely many atoms which are also atoms of  $\rho_s$ . Claim 2 is proved.

*Proof of Claim 3.* If  $\tau_i \leq t$  then there are two cases. Either at time  $t$  the number of atoms is smaller than  $i$ , this is a measurable event. Or at any time  $s \in \{t\} \cup ((0, t) \cap \mathbb{Q})$  the number of atoms is larger than or equal to  $i$  and there exists at least one time  $r \in (0, t) \setminus \mathbb{Q}$  such that  $\#\rho_r < i$ . Let us show that the latter case is measurable. Fix  $m \geq 0$  and consider the interval of time  $(s_-, s_+)$  on which for all rational  $t \in (s_-, s_+) \cap \mathbb{Q}$  the measure  $\rho_t$  has  $m + i$  atoms. We call  $a_1 < \dots < a_{m+i}$  these atoms. By the right

continuity of  $\rho$  we know that  $t \mapsto \rho_t(\{a_1\}), \dots, \rho_t(\{a_{m+i}\})$  are càdlàg processes on  $[s_-, s_+)$ . For each collection  $j_1 < \dots < j_{m+1}$  of  $m+1$  indices among  $[m+i]$ , we introduce the hitting time of  $0 \in \mathbb{R}^{m+1}$  by  $(s_-, s_+) \ni t \mapsto (\rho_t(\{a_{j_1}\}), \dots, \rho_t(\{a_{j_{m+1}}\}))$  which is a stopping time in the augmented filtration  $\mathcal{H}_t$ . Then  $\{\tau_i < t\}$  coincides  $\mathbf{P}$ -a.s. with the union on  $m \geq 0$  and on the  $j_1 < \dots < j_{m+1}$ 's of the event where this hitting time belongs to  $(s_-, s_+)$ . The claim follows.

*Proof of Claim 4.* The set  $C'$  is the subset of  $C$  where the times  $\tau_i$  are the infimum of the rational times at which  $\rho$  has less than  $i$  atoms. Therefore

$$C' := C \bigcap_{i \geq 2} \left\{ \tau_i = \inf \{ t \in \mathbb{Q}_+^* : \#\rho_t < i \} \right\}$$

and the claim follows. This ends the proof of the proposition.  $\blacksquare$

From now on, we rely on the lockdown representation to study the Eves. We consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  on which is defined a  $\Lambda$  flow of partitions  $\hat{\Pi}$  arising from the Poissonian construction of Subsection 2.3 and an independent sequence  $\xi_0$  of i.i.d. uniform $[0, 1]$  r.v. We then set  $\rho_t := \mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$  for all  $t \geq 0$ . Proposition I.17 ensures that  $(\rho_t, t \geq 0)$  is a  $\Lambda$  Fleming-Viot  $\mathbb{P}$ -a.s. càdlàg. The study of the existence of the Eves in regime CDI requires a precise description of the behaviour of the  $\Lambda$  Fleming-Viot process.

**LEMMA I.22.** *In regime CDI we define for every  $i \geq 1$  the r.v.  $d(i) := \inf \{ t > 0 : |\hat{\Pi}_{0,t}(i)| = 0 \}$ . Then  $\mathbb{P}$ -a.s. for every  $i \geq 1$  we have  $d(i) = \inf \{ t > 0 : \hat{\Pi}_{0,t}(i) = \emptyset \}$ . Consequently  $\mathbb{P}$ -a.s.*

- for every  $i \geq 1$ ,  $d(i) \geq d(i+1)$ ,
- for every  $i \geq 1$ , the block  $\hat{\Pi}_{0,t}(i)$  has a strictly positive frequency when  $t \in (0, d(i))$  and is empty when  $t \in [d(i), \infty)$ ,
- $d(i) \downarrow 0$  as  $i \rightarrow \infty$ .

**Proof** First observe the following deterministic fact. For any two partitions  $\pi, \pi' \in \mathcal{P}_\infty$ , if  $\pi$  has  $n$  blocks then  $\text{Coag}(\pi', \pi)$  has at most  $n$  blocks. Recall that in regime CDI at any time  $t > 0$  the exchangeable partition  $\hat{\Pi}_{0,t}$  has  $\mathbb{P}$ -a.s. finitely many blocks that all have strictly positive asymptotic frequency. Using these two observations, we easily get that  $\mathbb{P}(C) = 1$  where  $C$  is the event introduced in Claim 1 of the proof of Proposition I.21.

We now prove that  $\mathbb{P}(C') = 1$  where  $C'$  is the event introduced in Claim 4. For every  $i \geq 2$  we define  $q_i := \inf \{ t \in \mathbb{Q}_+^* : \#\rho_t < i \}$ . As  $\mathbb{P}(C) = 1$  we know that  $\mathbb{P}$ -a.s. the sequence  $q_i, i \geq 2$  is non-increasing. Since  $\hat{\Pi}_{0,0} = 0_{[\infty]}$  and  $\mathbb{P}$ -a.s. for every  $t \in \mathbb{Q}_+^*$  the partition  $\hat{\Pi}_{0,t}$  has a finite number of blocks which have positive asymptotic frequencies we deduce that  $\mathbb{P}$ -a.s. the sequence  $q_i, i \geq 2$  has no positive lower bound and thus converges to 0. For every  $i \geq 2$  and every  $\epsilon \in \mathbb{Q}_+^*$  we define  $\tau_i(\epsilon) := \inf \{ t \geq \epsilon : \#\rho_t < i \}$ . The right continuity of  $\rho$  and the  $\mathbb{P}$ -a.s. finiteness of the number of atoms on  $[\epsilon, \infty)$  entail that  $\mathbb{P}$ -a.s. the measure  $\rho_{\tau_i(\epsilon)}$  has less than  $i$  atoms. Furthermore the arguments in the proof of Claim 3 above ensure that this is an  $\mathcal{H}_t$ -stopping time. Then Proposition 3.1 in Donnelly and Kurtz [24] yields that the sequence  $(\xi_{\tau_i(\epsilon)}(j))_{j \geq 1}$  is exchangeable. Necessarily its empirical measure is  $\mathbb{P}$ -a.s. equal to  $\rho_{\tau_i(\epsilon)}$ . By de Finetti Theorem (see for instance p.103 in Bertoin [9]) conditionally on  $\rho_{\tau_i(\epsilon)}$  the sequence  $(\xi_{\tau_i(\epsilon)}(j))_{j \geq 1}$  is i.i.d. with distribution  $\rho_{\tau_i(\epsilon)}$ . Therefore  $\mathbb{P}$ -a.s. this sequence takes its values in the set of atoms of  $\rho_{\tau_i(\epsilon)}$  and therefore  $\hat{\Pi}_{0,\tau_i(\epsilon)}$  has at most  $i-1$  blocks. This together with the exchangeability of  $\hat{\Pi}_{0,t}$  at any given time  $t$  ensures the existence of an event  $\Omega^*$  of  $\mathbb{P}$ -probability 1 on which  $\hat{\Pi}$  fulfils the regularity properties of Proposition I.15, on which  $\rho \in C$  and on which for all  $t \in \mathbb{Q}_+^* \cup \{\tau_i(\epsilon), i \geq 2, \epsilon \in \mathbb{Q}_+^*\}$  we have

- (a)  $\#\hat{\Pi}_{0,t} = \#\rho_t$ ,
- (b)  $|\hat{\Pi}_{0,t}(j)| = \rho_t(\{\xi_0(j)\})$  is strictly positive i.f.f.  $j \leq \#\hat{\Pi}_{0,t}$ .

We work deterministically on  $\Omega^*$ . Fix  $i \geq 2$ . There exists  $\epsilon \in \mathbb{Q}_+^*$  such that  $\epsilon < q_i$ . Suppose that  $\tau_i(\epsilon) < q_i$ . Then by (a) and (b) above and the deterministic fact at the beginning of the proof, we have  $\#\rho_t < i$  for all  $t \in [\tau_i(\epsilon), \infty)$  which is in contradiction with the definition of  $q_i$ . Therefore  $\tau_i(\epsilon) = q_i$  and for all  $\epsilon' \in \mathbb{Q}_+^*$  such that  $\epsilon' < \epsilon$  we have  $\tau_i(\epsilon') = q_i$ . Since  $\tau_i = \lim_{\epsilon \downarrow 0} \tau_i(\epsilon)$  we deduce that  $\tau_i = q_i$ . Consequently  $\mathbb{P}(C') = 1$ .

Set  $\tau_1 = \infty$ . Notice that the right continuity of the asymptotic frequencies of the blocks ensures that  $|\hat{\Pi}_{0,d(i)}(i)| = 0$ . For every  $i \geq 1$  the atom  $\xi_0(i)$  disappears at time  $d(i)$  and thus necessarily  $d(i) = \tau_k$  for a certain  $k \geq 1$ . Let us that  $d(i) = \tau_i$  for every  $i \geq 1$ . If  $d(1) < \infty$  then properties (a) and (b) above hold true for  $t = d(1)$  and we deduce that  $\#\hat{\Pi}_{0,d(1)}$  has no blocks which is not possible for a partition. Consequently  $d(1) = \infty$ . We now prove by induction that  $d(i) = \tau_i \leq d(i-1)$  for every  $i \geq 2$ . At rank  $i = 2$ , we know that for all  $t \in (0, d(2))$  we have  $\rho_t(\{\xi_0(j)\}) > 0$  when  $j = 1, 2$  so that  $\#\rho_t \geq 2$  and  $\tau_2 \geq d(2)$ . Since  $\rho_{d(2)}(\{\xi_0(2)\}) = 0$  we deduce using property (b) above that  $\tau_2 = d(2)$ . As  $d(1) = \infty$ , the inequality  $d(2) \leq d(1)$  is trivially verified. Suppose now that  $d(j) = \tau_j$  for all  $j \in [i-1]$  with  $i \geq 2$  being given. Necessarily  $d(i) \leq d(i-1)$ . Indeed if  $d(i) > d(i-1)$  then  $\rho_{d(i-1)}(\{\xi_0(i)\}) > 0$  and property (b) would yield that  $\rho_{d(i-1)}(\{\xi_0(i-1)\}) > 0$  which is contradictory. Now observe that for all  $t \in (0, d(i))$  we have  $\rho_t(\{\xi_0(j)\}) > 0$  for  $j \in [i]$  so that  $\#\rho_t \geq i$ . Consequently  $d(i)$  is the first time at which  $\rho$  has less than  $i$  atoms. Hence  $d(i) = \tau_i$ . The induction is complete. We have shown that for every  $i \geq 1$  on  $(0, d(i))$  the asymptotic frequency of  $\hat{\Pi}_{0,t}(i)$  is strictly positive and that  $\hat{\Pi}_{0,t}(i)$  is empty on  $[d(i), \infty)$ . Furthermore since  $d(i) = q_i$  we know that the sequence  $d(i), i \geq 2$  is non-increasing and converges to 0. ■

The following proposition justifies the use of the lookdown representation in the study of the Eves. Recall that we are in the *Eves - extinction case* when the  $\Lambda$  Fleming-Viot process is in regime CDI and admits an infinite sequence of Eves, while we are in the *Eves - persistent case* when the  $\Lambda$  Fleming-Viot process is in any other regime and admits an infinite sequence of Eves.

**PROPOSITION I.23.** *Let  $(\rho_t, t \geq 0)$  be a  $\Lambda$  Fleming-Viot process constructed with the lookdown representation. Then  $\mathbb{P}$ -a.s.  $\xi_0(1)$  coincides with the primitive Eve of Bertoin and Le Gall. If we assume that the  $\Lambda$  Fleming-Viot process  $\mathbb{P}$ -a.s. admits an infinite sequence of Eves ( $e^i, i \geq 1$ ) then we have  $\mathbb{P}$ -a.s. for every  $i \geq 1$ ,  $e^i = \xi_0(i)$ .*

**Proof** Consider first the *Eves - extinction case*. By construction of  $\rho$ , we know that  $\mathbb{P}$ -a.s. for all  $t \geq 0$

$$\rho_t = \sum_{i=1}^{\#\hat{\Pi}_{0,t}} |\hat{\Pi}_{0,t}(i)| \delta_{\xi_0(i)}$$

where  $\#\pi$  denotes the number of blocks of a partition  $\pi$ . The existence of the infinite sequence of Eves ensures that  $\mathbb{P}$ -a.s. for every  $i \geq 1$  we have  $d(i) > d(i+1)$  where  $d(i)$  has been introduced in Lemma I.22. This lemma actually shows that the extinction time of  $\xi_0(i)$  is equal to  $d(i)$ . Consequently  $\mathbb{P}$ -a.s. for every  $i \geq 1$  we have  $\xi_0(i) = e^i$ . Consider now the *Eves - persistent case*. For all  $t \geq 0$ , we denote by  $a_t(1) \geq a_t(2) \geq \dots \geq 0$  the masses of  $\rho_t$  ranked in the decreasing order. We prove by induction on  $i \geq 1$  that

$$\mathbb{P}(\forall j \in [i], \hat{\Pi}_{0,t}(j) = a_t(j)) \xrightarrow{t \rightarrow \infty} 1 \text{ and } \mathbb{P}\text{-a.s. } \forall j \in [i], \frac{|\hat{\Pi}_{0,t}(j)|}{1 - \sum_{k=1}^{j-1} |\hat{\Pi}_{0,t}(k)|} \xrightarrow{t \rightarrow \infty} 1.$$



We start at rank  $i = 1$ . From the definition of  $\rho$ , there is a one-to-one correspondence between the blocks of  $\hat{\Pi}_{0,t}$  with positive frequency and the masses of  $\rho_t$ . The definition of  $e^1$  entails that  $\mathbb{P}$ -a.s.  $|\hat{\Pi}_{0,t}(1)|$  converges to either 0 or 1 as  $t \rightarrow \infty$  and that  $a_t(1)$  converges to 1 as  $t \rightarrow \infty$ . The exchangeability of the partition  $\hat{\Pi}_{0,t}$  implies that  $\mathbb{P}(|\hat{\Pi}_{0,t}(1)| = a_t(1) | (a_t(k))_{k \geq 1}) = a_t(1)$ . Consequently  $\mathbb{P}(|\hat{\Pi}_{0,t}(1)| = a_t(1)) \rightarrow 1$  as  $t \rightarrow \infty$  by the dominated convergence theorem. This implies that  $\mathbb{P}$ -a.s.  $|\hat{\Pi}_{0,t}(1)| \rightarrow 1$  as  $t \rightarrow \infty$ . The claimed property is proved at rank  $i = 1$ . Assume that it holds at a given rank  $i - 1 \geq 1$ . The existence of an infinite sequence of Eves yields that

$$\frac{a_t(i)}{1 - \sum_{j=1}^{i-1} a_t(j)} \xrightarrow{t \rightarrow \infty} 1.$$

The one-to-one correspondence between the atoms of  $\rho_t$  and the blocks of  $\hat{\Pi}_{0,t}$  with positive frequency along with the induction hypothesis and the existence of an infinite sequence of Eves imply that  $\mathbb{P}$ -a.s.

$$\frac{\hat{\Pi}_{0,t}(i)}{1 - \sum_{j=1}^{i-1} \hat{\Pi}_{0,t}(j)}$$

converges to either 0 or 1 as  $t \rightarrow \infty$ . The exchangeability of  $\hat{\Pi}_{0,t}$  entails that

$$\mathbb{P}\left(|\hat{\Pi}_{0,t}(i)| = a_t(i) \mid \{\forall j \in [i-1], |\hat{\Pi}_{0,t}(j)| = a_t(j)\}; (a_t(k))_{k \geq i}\right) = \frac{a_t(i)}{1 - \sum_{j=1}^{i-1} a_t(j)}.$$

Using the dominated convergence theorem and the arguments above we deduce that  $\mathbb{P}(\hat{\Pi}_{0,t}(i) = a_t(i))$  converges to 1. Henceforth  $\mathbb{P}$ -a.s.  $\frac{\hat{\Pi}_{0,t}(i)}{1 - \sum_{j=1}^{i-1} \hat{\Pi}_{0,t}(j)}$  goes to 1 as  $t$  goes to infinity and the induction is complete. Since for all  $i \geq 1$ ,  $\rho_t(\{\xi_0(i)\}) = |\hat{\Pi}_{0,t}(i)|$  we deduce from the convergence obtained by induction and the uniqueness of the Eves that  $\mathbb{P}$ -a.s.  $\xi_0(i) = e^i$ .

In the case where the  $\Lambda$  Fleming-Viot process does not admit an infinite sequence of Eves the arguments above still hold for the primitive Eve.  $\blacksquare$

**COROLLARY I.24.** *Consider a càdlàg  $\Lambda$  Fleming-Viot process that admits  $\mathbb{P}$ -a.s. an infinite sequence of Eves ( $e^i, i \geq 1$ ). Then the Eves are i.i.d. uniform $[0, 1]$  and are independent of the sequence  $(\rho_t(\{e^i\}), t \geq 0), i \geq 1$ .*

**Proof** First assume that the  $\Lambda$  Fleming-Viot process is obtained via the lockdown representation. Since  $(\xi_0(i), i \geq 1)$  is a sequence of i.i.d. uniform $[0, 1]$  r.v. independent of  $\hat{\Pi}$  and since for every  $i \geq 1$ ,  $(\rho_t(\{e^i\}), t \geq 0) = (\hat{\Pi}_{0,t}(i), t \geq 0)$  then Proposition I.23 ensures that the asserted result holds. Now consider any càdlàg  $\Lambda$  Fleming-Viot process  $(\rho_t, t \geq 0)$ . Proposition I.21 ensures that the Eves are  $\sigma(\rho)$ -measurable r.v. and therefore their distribution coincides with that obtained via the lockdown construction.  $\blacksquare$

## 4 Results on the existence of the Eves

The goal of this section is to prove Theorem I.1, Proposition I.4 and Theorem I.2. We consider each of the four regimes separately. Except for regime INTENSIVE  $\infty$  we rely on the lockdown representation of the  $\Lambda$  Fleming-Viot process. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space on which is defined a  $\Lambda$  flow of partitions arising from the Poissonian construction and an independent sequence  $\xi_0$  of i.i.d. uniform $[0, 1]$  r.v. We set  $\rho_t := \mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$  for all  $t \geq 0$ . We also rely on the lockdown process  $(\xi_t(i), t \geq 0), i \geq 1$

defined from  $\hat{\Pi}$  and  $\xi_0$  according to Definition I.16. Let us introduce some technical tools for later use. We define the filtration

$$\mathcal{F}_t := \sigma(\hat{\Pi}_{0,s}, 0 \leq s \leq t), \quad t \geq 0 \quad (\text{I.7})$$

associated to the flow of partitions. We augment this filtration with the  $\mathbb{P}$ -null sets.

**LEMMA I.25.** *The process  $(\hat{\Pi}_{0,t}, t \geq 0)$  is a  $\mathcal{P}_\infty$ -valued Markov process with a Feller semigroup. For any  $\mathcal{F}_t$ -stopping time  $\tau$ , conditionally given  $\{\tau < \infty\}$  the process  $(\hat{\Pi}_{\tau, \tau+t}, t \geq 0)$  is independent of  $\mathcal{F}_\tau$  and has the same distribution as  $(\hat{\Pi}_{0,t}, t \geq 0)$ .*

**Proof** The very definition of stochastic flows of partitions ensures that  $(\hat{\Pi}_{0,t}, t \geq 0)$  is Markov with a semigroup  $Q_t$  defined as follows. For every  $\pi \in \mathcal{P}_\infty$  and every bounded measurable map  $f$  on  $\mathcal{P}_\infty$

$$Q_t f(\pi) = \mathbb{E} \left[ f(\text{Coag}(\hat{\Pi}_{0,t}, \pi)) \right].$$

Consider a bounded continuous map  $f$ . Since Coag is a bi-continuous operator (see Section 4.2 of the book of Bertoin [9]), the dominated convergence theorem ensures that  $\pi \mapsto Q_t f(\pi)$  is a bounded continuous map. Since  $\mathcal{P}_\infty$  is a compact metric space, the map  $f$  is uniformly continuous. For every  $\epsilon > 0$ , there exists  $n \geq 1$  such that  $d_{\mathcal{P}}(\pi, \pi') \leq 2^{-n} \Rightarrow |f(\pi) - f(\pi')| < \epsilon$  where  $d_{\mathcal{P}}$  is the distance introduced in Formula (I.3). Thus we get

$$\sup_{\pi \in \mathcal{P}_\infty} |Q_t f(\pi) - f(\pi)| \leq \epsilon \mathbb{P}(\hat{\Pi}_{0,t}^{[n]} = 0_{[n]}) + 2 \sup_{\pi \in \mathcal{P}_\infty} |f(\pi)| \left( 1 - \mathbb{P}(\hat{\Pi}_{0,t}^{[n]} = 0_{[n]}) \right)$$

Since  $\hat{\Pi}_{0,t} \rightarrow 0_{[\infty]}$  in probability as  $t \downarrow 0$ , the right member goes to  $\epsilon$  as  $t \downarrow 0$ . This implies the Feller property of  $Q$ . Let  $\tau$  be an  $\mathcal{F}_t$ -stopping time. Let  $A \in \mathcal{F}_\tau$ . Let  $f_1, \dots, f_p$  be  $p \geq 1$  bounded continuous maps on  $\mathcal{P}_\infty$ . We want to prove that for every  $0 \leq t_1 \leq \dots \leq t_p$

$$\mathbb{E}[1_A 1_{\{\tau < \infty\}} f_1(\hat{\Pi}_{\tau, \tau+t_1}) \dots f_p(\hat{\Pi}_{\tau, \tau+t_p})] = \mathbb{E}[1_A 1_{\{\tau < \infty\}}] \mathbb{E}[f_1(\hat{\Pi}_{0,t_1}) \dots f_p(\hat{\Pi}_{0,t_p})].$$

We check the result for  $p = 1$ , since the general case can be treated similarly. For every integer  $n \geq 1$ , let  $\tau_n$  be the smallest real number of the form  $k/n$  which is strictly larger than  $\tau$ . The right continuity of the trajectories at the first line, the independence and stationarity of the increments of a flow of partitions at the third and fourth lines ensure that

$$\begin{aligned} \mathbb{E}[1_A 1_{\{\tau < \infty\}} f_1(\hat{\Pi}_{\tau, \tau+t_1})] &= \lim_{n \rightarrow \infty} \mathbb{E}[1_A 1_{\{\tau < \infty\}} f_1(\hat{\Pi}_{\tau_n, \tau_n+t_1})] \\ &= \lim_{n \rightarrow \infty} \sum_{k \geq 0} \mathbb{E}[1_A 1_{\{(k-1)/n \leq \tau < k/n\}} f_1(\hat{\Pi}_{k/n, k/n+t_1})] \\ &= \lim_{n \rightarrow \infty} \sum_{k \geq 0} \mathbb{E}[1_A 1_{\{(k-1)/n \leq \tau < k/n\}}] \mathbb{E}[f_1(\hat{\Pi}_{k/n, k/n+t_1})] \\ &= \mathbb{E}[1_A 1_{\{\tau < \infty\}}] \mathbb{E}[f_1(\hat{\Pi}_{0,t_1})]. \end{aligned}$$

■

We also introduce the lowest level associated to an ancestor.

**DEFINITION I.26.** *For all  $t \geq 0$  and every  $i \geq 1$ , we set  $L_t(i) := \inf\{j \geq 1 : j \in \hat{\Pi}_{0,t}(i)\}$  where  $\inf \emptyset = \infty$  by convention. The r.v.  $L_t(i)$  is the lowest level at time  $t$  that belongs to the progeny of the  $i$ -th ancestor.*

In regime CDI and from Lemma I.22 we know that  $\mathbb{P}$ -a.s. for every  $i \geq 1$  the r.v.  $L_t(i) < \infty$  when  $t \in [0, d(i))$  while  $L_t(i) = \infty$  when  $t \in [d(i), \infty)$ .

**LEMMA I.27.** *Let  $\pi$  be a deterministic partition with a unique non-singleton block whose index is  $k \geq 1$ . Assume that  $\pi(k)$  admits an asymptotic frequency, say  $u \in (0, 1]$ . Then for any exchangeable random partition  $\pi'$  almost surely the partition  $\text{Coag}(\pi, \pi')$  admits asymptotic frequencies and we have*

$$\forall i \geq 1, \quad |\text{Coag}(\pi, \pi')(i)| = |\pi'(i)|(1 - u) + \mathbf{1}_{\{k \in \pi'(i)\}}u.$$

**Proof** Lemma 4.6 in the book of Bertoin [9] ensures that  $\text{Coag}(\pi, \pi')$  admits asymptotic frequencies almost surely, we only need to prove the asserted formula on these frequencies. Let  $\gamma(n) := \min \pi(n)$  be the smallest integer in the  $n$ -th block of  $\pi$ . Since  $\pi$  has an infinite number of blocks,  $\gamma(n)$  goes to infinity as  $n$  tends to infinity. Observe that

$$\forall n \geq 1, \quad \gamma(n) - (\#\{\pi(k) \cap [\gamma(n)]\} - 1) \vee 0 = n.$$

We thus get that  $n/\gamma(n)$  goes to  $1 - u$  as  $n$  goes to infinity. From the definition of the coagulation operator for every  $i \geq 1$  if  $k \in \pi'(i)$  then the block  $\text{Coag}(\pi, \pi')(i)$  contains the elements in  $\pi(k)$  together with the images through  $\gamma$  of the elements in  $\pi'(i)$  while if  $k \notin \pi'(i)$  then the block  $\text{Coag}(\pi, \pi')(i)$  contains only the images through  $\gamma$  of the elements in  $\pi'(i)$ . Therefore we get for every  $n \geq k$

$$\# \left( \text{Coag}(\pi, \pi')(i) \cap [\gamma(n)] \right) = \#(\pi'(i) \cap [n]) + \mathbf{1}_{\{k \in \pi'(i)\}} (\#(\pi(k) \cap [\gamma(n)]) - 1)$$

Since the blocks of  $\pi'$  have asymptotic frequencies almost surely and since  $n/\gamma(n) \rightarrow 1 - u$  we get the asserted equality of the lemma by dividing both members of the above formula by  $\gamma(n)$  and taking the limit as  $n \rightarrow \infty$ . ■

#### 4.1 Regime DISCRETE

Recall from Proposition I.23 that the primitive Eve  $e$  of Bertoin and Le Gall is almost surely equal to  $\xi_0(1)$ . Consider the process  $t \mapsto \rho_t([0, 1] \setminus \{\xi_0(1)\})$ . Theorem 4 of Bertoin and Le Gall [13] shows that this process is Markov with a Feller semigroup. Since we have constructed  $\rho$  with the lockdown representation,  $t \mapsto \rho_t([0, 1] \setminus \{\xi_0(1)\})$  evolves at the reproduction events of the lockdown process. In regime DISCRETE, these reproduction events are finitely many on any compact interval of time. Therefore we can enumerate  $\{(s, \hat{\Pi}_{s-,s}) : s > 0, \hat{\Pi}_{s-,s} \neq 0_{[\infty]}\}$  by increasing time coordinates, say  $(t_i, \pi_i), i \geq 1$ . For each partition  $\pi_i$ , we let  $u_i$  be the asymptotic frequency of its unique non-singleton block (these asymptotic frequencies are well-defined on a same event of probability one, see the proof of Proposition I.15). From Formula (I.6) we deduce that the  $u_i$ 's are i.i.d. with distribution  $\nu(du)/\nu([0, 1])$ . We work with the collection  $(t_i, u_i), i \geq 1$ . We then set  $X_0 := 1$  and for every  $i \geq 1, X_i := \rho_{t_i}([0, 1] \setminus \{\xi_0(1)\})$ . From the above arguments, we deduce that  $(X_i, i \geq 0)$  is a Markov chain. Let us denote by  $\mathbf{Q}_x$  the distribution of this Markov chain when it starts from  $x \in (0, 1]$ . The following lemma provides the transition probabilities of this chain.

**LEMMA I.28.** *Fix  $x \in (0, 1]$  and let  $u$  be a r.v. with law  $\nu(du)/\nu([0, 1])$ . Under  $\mathbf{Q}_x$  the r.v.  $X_1$  is distributed as follows:*

$$X_1 = \begin{cases} (1 - u)x & \text{with probability } u + (1 - x)(1 - u) \\ (1 - u)x + u & \text{with probability } x(1 - u). \end{cases} \quad (\text{I.8})$$

**Proof** We use the lockdown representation and work under measure  $\mathbb{P}$  in the proof. Fix  $i \geq 1$ . We have  $\mathbb{P}$ -a.s.  $X_{t_i} = 1 - |\hat{\Pi}_{0,t_i}(1)|$  and  $\hat{\Pi}_{0,t_i} = \text{Coag}(\pi_i, \hat{\Pi}_{0,t_{i-1}})$ . Let  $K$  be the index of the non-singleton

block of  $\pi_i$ . Lemma I.27 entails that  $\mathbb{P}$ -a.s. if  $K \in \hat{\Pi}_{0,t_{i-1}}(1)$  then  $|\hat{\Pi}_{0,t_i}(1)| = |\hat{\Pi}_{0,t_{i-1}}(1)|(1 - u_i) + u_i$  while if  $K \notin \hat{\Pi}_{0,t_{i-1}}(1)$  then  $|\hat{\Pi}_{0,t_i}(1)| = |\hat{\Pi}_{0,t_{i-1}}(1)|(1 - u_i)$ . Therefore

$$\begin{aligned}\mathbb{P}(X_i = X_{i-1}(1 - u_i) | u_i, X_{i-1}) &= \mathbb{P}(K \in \hat{\Pi}_{0,t_{i-1}}(1) | u_i, X_{i-1}) \\ \mathbb{P}(X_i = X_{i-1}(1 - u_i) + u_i | u_i, X_{i-1}) &= \mathbb{P}(K \notin \hat{\Pi}_{0,t_{i-1}}(1) | u_i, X_{i-1})\end{aligned}$$

Observe that  $\mathbb{P}$ -a.s.  $\{K \in \hat{\Pi}_{0,t_{i-1}}(1)\} = \{\xi_{t_{i-1}}(K) = \xi_0(1)\}$ . Since  $u_i$  has law  $\nu(du)/\nu([0, 1])$  and is independent from  $X_{i-1}$ , the proof boils down to determining  $\mathbb{P}(\xi_{t_{i-1}}(K) = \xi_0(1) | u_i, X_{i-1})$ . Since  $\pi_i$  is an exchangeable random partition independent of the lockdown process up to time  $t_{i-1}$  and since  $u_i$  is the asymptotic frequency of its unique non-singleton block, we have for all  $k \geq 1$

$$\mathbb{P}(K = k | (\xi_{t_{i-1}}(j))_{j \geq 1}, u_i) = (1 - u_i)^{k-1} u_i.$$

On the event where  $K = 1$  the parent of the reproduction event is of type  $\xi_{t_{i-1}}(1)$ . Recall that  $\xi_{t_{i-1}}(1) = \xi_0(1)$  since the first particle of the lockdown process is constant. On the event where  $K = k \geq 2$  the parent of the reproduction event is of type  $\xi_{t_{i-1}}(k)$ . Proposition 3.1 of Donnelly and Kurtz [24] ensures that  $(\xi_{t_{i-1}}(j), j \geq 1)$  is an exchangeable sequence of r.v. Its empirical measure is  $\mathbb{P}$ -a.s. equal to  $\rho_{t_{i-1}}$ . Consequently

$$\forall k \geq 2, \quad \mathbb{P}(\xi_{t_{i-1}}(k) = \xi_{t_{i-1}}(1) | u_i, X_{i-1}) = 1 - X_{i-1}.$$

Therefore we get

$$\begin{aligned}\mathbb{P}(\xi_{t_{i-1}}(K) = \xi_0(1) | u_i, X_{i-1}) &= \sum_{k=1}^{\infty} \mathbb{P}(K = k | (\xi_{t_{i-1}}(j))_{j \geq 1}, u_i) \mathbb{P}(\xi_{t_{i-1}}(k) = \xi_{t_{i-1}}(1) | u_i, X_{i-1}) \\ &= u_i + (1 - X_{i-1})(1 - u_i).\end{aligned}$$

This ends the proof of the lemma. ■

**PROPOSITION I.29.** *There exists a random time  $T > 0$  after which  $\mathbb{P}$ -a.s. the process  $t \mapsto \rho_t(\{\xi_0(1)\})$  makes only positive jumps. In other terms, eventually all the reproduction events choose a parent of type  $\xi_0(1)$ .*

**Proof** We work on  $(\Omega, \mathcal{F}, \mathbb{P})$ . We introduce the random variables  $\tau_0 := \inf\{i \geq 0 : X_i < 1/2\}$  and for each  $n \geq 0$ ,  $r_n := \inf\{i > \tau_n : X_i - X_{i-1} > 0\}$  and  $\tau_{n+1} := \inf\{i > r_n : X_i < 1/2\}$ . We use the convention  $\inf \emptyset = \infty$ . In words,  $\tau_0$  is the first time the process  $X$  hits  $(0, 1/2)$  and  $r_0$  is the first time after  $\tau_0$  the process  $X$  makes a positive jump. Recursively  $\tau_{n+1}$  is the first time after  $r_n$  the process  $X$  hits again  $(0, 1/2)$ , and  $r_{n+1}$  is the time of the next positive jump. Set  $\mathcal{F}_k := \sigma(X_i, 0 \leq i \leq k)$ . Recall that the objective is to prove that  $\mathbb{P}$ -a.s.  $X$  eventually makes only negative jumps. The proof of the proposition therefore boils down to showing that  $\mathbb{P}$ -a.s. the sequence  $(r_n)_{n \geq 0}$  eventually equals  $+\infty$ . The transition probabilities of the chain entail that  $x \mapsto \mathbf{Q}_x(X \text{ makes only negative jumps})$  is decreasing. Thus for all  $n \geq 0$ ,  $\mathbb{P}$ -a.s.

$$\begin{aligned}\mathbb{P}(r_{n+1} = \infty | \mathcal{F}_{\tau_{n+1}}) \mathbf{1}_{\{r_n < \infty\}} &= \mathbf{Q}_{X_{\tau_{n+1}}}(X \text{ makes only negative jumps}) \mathbf{1}_{\{r_n < \infty\}} \\ &\geq \mathbf{Q}_{1/2}(X \text{ makes only negative jumps}) \mathbf{1}_{\{r_n < \infty\}}\end{aligned}$$

Consequently for all  $n \geq 0$

$$\mathbb{P}(r_n < \infty) \leq (1 - \mathbf{Q}_{1/2}(X \text{ makes only negative jumps}))^{n+1}$$

Hence to prove the proposition it is sufficient to show that  $\mathbf{Q}_{1/2}(X \text{ makes only negative jumps})$  is strictly positive. From the transition probabilities of the chain, this quantity is equal to

$$\begin{aligned} \mathbb{E} \left[ \prod_{i \geq 1} \left( u_i + (1 - u_i) \left( 1 - \frac{1}{2} \prod_{j=1}^{i-1} (1 - u_j) \right) \right) \right] &\geq \mathbb{E} \left[ \prod_{i \geq 1} \left( 1 - \frac{1}{2} \prod_{j=1}^{i-1} (1 - u_j) \right) \right] \\ &\geq \mathbb{E} \left[ \exp \left( \sum_{i \geq 1} \log \left( 1 - \frac{1}{2} \prod_{j=1}^{i-1} (1 - u_j) \right) \right) \right] \end{aligned}$$

Let us now prove that the negative r.v. inside the exponential is finite almost surely. Its expectation is given by

$$\begin{aligned} \mathbb{E} \left[ \sum_{i \geq 1} \log \left( 1 - \frac{1}{2} \prod_{j=1}^{i-1} (1 - u_j) \right) \right] &\geq \mathbb{E} \left[ -\frac{c}{2} \sum_{i \geq 1} \prod_{j=1}^{i-1} (1 - u_j) \right] \quad (\text{I.9}) \\ &\geq -\frac{c}{2} \sum_{i \geq 1} \mathbb{E} \left[ \prod_{j=1}^{i-1} (1 - u_j) \right] \end{aligned}$$

where  $c$  is a positive constant such that  $\log(1 - y) \geq -cy$  for all  $y \in [0, 1/2]$ . We have used the monotone convergence theorem to go from the first to the second line. We get

$$\mathbb{E} \left[ \prod_{j=1}^{i-1} (1 - u_j) \right] = \left( \frac{\int_{[0,1]} (1 - u) \nu(du)}{\int_{[0,1]} \nu(du)} \right)^{i-1}.$$

The measure  $\nu$  is supported by  $(0, 1)$  since we are in regime DISCRETE, consequently the right member is strictly smaller than 1. This ensures that the series at the second line of (I.9) is finite, which in turn implies the almost sure finiteness of the negative r.v. inside the exponential above. Therefore  $\mathbf{Q}_{1/2}(X \text{ makes only negative jumps}) > 0$ . ■

*Proof of Theorem I.1 for regime DISCRETE.* We know from Lemma I.27 and Proposition I.29 that  $\mathbb{P}$ -a.s. for every  $i \geq 1$  if  $t_i > T$  then  $|\hat{\Pi}_{0,t_i}(1)| = (1 - u_i)|\hat{\Pi}_{0,t_{i-1}}(1)| + u_i$  and  $|\hat{\Pi}_{0,t_i}(2)| = (1 - u_i)|\hat{\Pi}_{0,t_{i-1}}(2)|$ . Consequently  $\mathbb{P}$ -a.s. for every  $i \geq 1$  if  $t_i > T$  then  $\rho_{t_i}([0, 1] \setminus \{\xi_0(1)\}) = (1 - u_i)\rho_{t_{i-1}}([0, 1] \setminus \{\xi_0(1)\})$  and  $\rho_{t_i}(\{\xi_0(2)\}) = (1 - u_i)\rho_{t_{i-1}}(\{\xi_0(2)\})$ . Moreover Proposition I.2 entails that  $\mathbb{P}$ -a.s. for all  $t \in \mathbb{Q}_+$  the measure  $\rho_t$  has dust and therefore  $\rho_t(\{\xi_0(2)\}) < \rho_t([0, 1] \setminus \{\xi_0(1)\})$ . Consequently  $\mathbb{P}$ -a.s. the process

$$\frac{\rho_t(\{\xi_0(2)\})}{\rho_t([0, 1] \setminus \{\xi_0(1)\})}, t \geq 0 \quad (\text{I.10})$$

is constant after time  $T$  and is strictly lower than 1 so that the  $\Lambda$  Fleming-Viot process does not admit a second Eve. ■

## 4.2 Regime INTENSIVE w. DUST

Recall that for every  $n \geq 2$ ,  $L_t(n)$  is defined as the smallest integer in  $\hat{\Pi}_{0,t}(n)$  - or equivalently as the lowest level with type  $\xi_0(n)$  at time  $t \geq 0$ . For every  $t \geq 0$  we let  $C_t(n)$  be equal to 1 if  $L_s(n)$ ,  $s \in [0, t]$  has been chosen as the parent of a reproduction event, 0 otherwise. In other terms  $C_t(n)$  equals 1 if and only if a reproduction event has chosen a parent of type  $\xi_0(n)$  on the time interval  $[0, t]$ . Let us describe the dynamics of the pair  $(L_t(n), C_t(n), t \geq 0)$ . For every  $k \geq 2$  and every  $l \geq 1$  we set

$$q(k, k + l) := \sum_{i=2}^{(l+1) \wedge k} \binom{k}{i} \binom{l-1}{l+1-i} \lambda_{k+l, l+1}$$

where  $\lambda_{k+l,l+1}$  is defined at Formula (I.1).

**LEMMA I.30.** *The process  $t \mapsto (L_t(n), C_t(n))$  is a continuous time Markov chain with values in  $\mathbb{N} \times \{0, 1\}$ . For every  $k \geq 2$  and every  $l \geq 1$  the transition rates are given by*

$$(k, 0) \rightarrow \begin{cases} (k+l, 0) & \text{at rate } q(k, k+l) \\ (k, 1) & \text{at rate } \lambda_{k,1} \end{cases} \quad \text{and} \quad (k, 1) \rightarrow (k+l, 1) \text{ at rate } q(k, k+l)$$

**Proof** This is a consequence of the transitions of the lookdown process given at Definition (I.7). Indeed suppose that the process  $t \mapsto L_t(n)$  is currently at level  $k$ . It jumps to a higher level if a reproduction event involves at least two levels among  $[k]$ . This higher level equals  $k+l$  if:

- $i$  levels are involved among the  $[k]$  first, with  $i \in \{2, \dots, (l+1) \wedge k\}$ ,
- $l+1-i$  levels are involved among  $\{k+1, \dots, k+l-1\}$ ,
- level  $k+l$  is not involved.

This yields the rate  $q(k, k+l)$ . Concerning the second coordinate, observe that level  $k$  is chosen as the parent of a reproduction event at rate  $\lambda_{k,1}$ . Once the second coordinate reaches 1 it does not evolve any more by definition. ■

This lemma has two consequences. First the process  $(L_t(n), t \geq 0)$  is a continuous time Markov chain. Second conditionally given this process, the probability that  $C_\infty(n) := \lim_{t \rightarrow \infty} C_t(n)$  equals 0 is given by

$$\mathbb{P}(C_\infty(n) = 0 \mid (L_t(n), t \geq 0)) = \exp\left(-\int_0^\infty \lambda_{L_t(n),1} dt\right).$$

Finally let us observe that the map  $k \mapsto \lambda_{k,1}$  is decreasing.

**PROPOSITION I.31.** *Consider regime INTENSIVE w. DUST and assume that  $\int_{[0,1]} x \log \frac{1}{x} \nu(dx) < \infty$ . Almost surely there exists at least one initial type whose frequency remains null.*

**REMARK I.32.** *A simple adaptation of the proof actually shows that there exists an infinity of initial types whose frequencies remain null.*

**Proof** The frequency of  $\xi_0(n)$  remains null if  $L(n)$  is never chosen as the parent of any reproduction event or equivalently if  $C_\infty(n) = 0$ . To prove the proposition, it suffices to show the  $\mathbb{P}$ -a.s. existence of an integer  $n \geq 1$  such that  $C_\infty(n) = 0$ . First we claim the existence of an integer  $n_0 \geq 1$  and of a real value  $w \in (0, 1]$  such that

$$\forall n \geq n_0, \quad \mathbb{P}(C_\infty(n) = 0) = \mathbb{E}\left[\exp\left(-\int_0^\infty \lambda_{L_t(n),1} dt\right)\right] \geq w. \quad (\text{I.11})$$

We postpone the proof of this claim below and complete the proof of the proposition. First observe that  $\mathbb{P}$ -a.s. the set  $\{C_\infty(n) = 0\}$  coincides with  $\{\forall t \geq 0 : \hat{\Pi}_{0,t}(n) \text{ is a singleton}\}$ . We stress that the latter set  $\mathbb{P}$ -a.s. coincides with  $\{\forall t \geq 0, |\hat{\Pi}_{0,t}(n)| = 0\}$ . One inclusion is trivial since a singleton has null frequency. Let us prove the other inclusion.  $\mathbb{P}$ -a.s. if at a given time  $t > 0$  the block  $\hat{\Pi}_{0,t}(n)$  is not a singleton then for all  $s \geq 0$  the block  $\hat{\Pi}_{0,t+s}(n)$  is not a singleton (otherwise the partition  $\hat{\Pi}_{0,t+s}(n)$  would have a finite number of blocks and this is not the case in this regime). Moreover  $\mathbb{P}$ -a.s. for all time  $t \in \mathbb{Q}_+^*$  the block  $\hat{\Pi}_{0,t}(n)$  is either a singleton or admits a strictly positive asymptotic frequencies. The converse inclusion follows. Consequently  $\mathbb{P}$ -a.s. the set  $\{C_\infty(n) = 0\}$  coincides with  $\{\forall t \geq 0, |\hat{\Pi}_{0,t}(n)| = 0\}$ .

Let  $r_0 := \inf\{t \geq 0 : |\hat{\Pi}_{0,t}(n_0)| > 0\}$  be the first time at which the  $n_0$ -th block gets a positive frequency. On the event where  $r_0 < \infty$ , we let  $n_1 := \inf\{n > n_0 : |\hat{\Pi}_{0,r_0}(n)| = 0\}$  be the lowest level  $n$  such that  $\xi_0(n)$  has never reproduced on  $[0, r_0]$ . Observe that  $n_1$  is  $\mathbb{P}$ -a.s. finite on the event  $r_0 < \infty$  since  $\mathbb{P}$ -a.s. the partition  $\hat{\Pi}_{0,r_0}$  has singleton blocks. Then recursively we define for every  $k \geq 1$ ,  $r_k := \inf\{t \geq 0 : |\hat{\Pi}_{0,t}(n_k)| > 0\}$  and on the event where  $r_k < \infty$ ,  $n_{k+1} := \inf\{n > n_k : |\hat{\Pi}_{0,r_k}(n)| = 0\}$ . Here again on the event where  $r_k < \infty$ , the r.v.  $n_{k+1}$  is  $\mathbb{P}$ -a.s. finite thanks to the same argument. To prove the proposition we need to show that  $\mathbb{P}$ -a.s. there exists  $k \geq 1$  such that  $r_k = \infty$ . From (I.11) we have

$$\mathbb{P}(r_0 = \infty) = \mathbb{P}(C_\infty(n_0) = 0) \geq w.$$

For every  $k \geq 0$ ,  $r_k$  is a stopping time in the filtration  $\mathcal{F}_t, t \geq 0$  of the flow of partitions. Lemma I.25 entails that on the event where  $r_k < \infty$  the process  $(\hat{\Pi}_{r_k, r_k+t}, t \geq 0)$  has the same distribution as  $(\hat{\Pi}_{0,t}, t \geq 0)$  so that (I.11) does also hold for this process:

$$\forall n \geq n_0, \quad \mathbb{P}\left(\forall t \geq 0, |\hat{\Pi}_{r_k, r_k+t}(n)| = 0 \mid r_k < \infty\right) \geq w.$$

Thus we get

$$\mathbb{P}(r_{k+1} = \infty \mid r_k < \infty) \geq w.$$

This easily implies that

$$\mathbb{P}(\forall k \geq 0, r_k < \infty) = 0$$

and the proof is complete.

*Proof of (I.11).* The strategy of the proof is to construct on an auxiliary probability space  $(A, \mathcal{A}, \mathbf{Q})$  an integer-valued process  $Y = (Y_t, t \geq 0)$  which is stochastically lower than  $L(n)$  for every  $n \geq n_0$  with  $n_0$  suitably chosen. From the description of the process  $(L(n), C(n))$ , the probability under  $\mathbb{P}$  of  $C_\infty(n) = 0$  is bounded below by the expectation under  $\mathbf{Q}$  of  $\exp\left(-\int_0^\infty \lambda_{Y_t,1} dt\right)$ .

Let  $u^*$  be a real value in  $(0, 1)$  such that  $\Lambda((u^*, 1)) > 0$  and set  $a = \frac{4-u^*}{4-2u^*}$ . Notice that  $a > 1$ . There exists  $n_0 \geq 3$  such that

$$0 < 1 - \frac{1 - \frac{u^*}{4} - \frac{2}{n_0}}{a} < u^* \quad , \quad \frac{4a}{(u^*)^2 n_0} < 1 \quad , \quad an_0 \geq n_0 + 1. \quad (\text{I.12})$$

We set  $u' := 1 - \frac{1 - \frac{u^*}{4} - \frac{2}{n_0}}{a}$ . Then we claim that for every  $n \geq n_0$

$$\sum_{l=0}^{\infty} q(n, \lfloor an \rfloor + l) \geq \int_{(u', 1)} \nu(dx) x^2 \sum_{j=\lfloor an \rfloor - 1 - n}^{\lfloor an \rfloor - 3} \binom{\lfloor an \rfloor - 3}{j} x^j (1-x)^{\lfloor an \rfloor - 3 - j}. \quad (\text{I.13})$$

An analytic proof of this claim is given below but let us make the following comment. On the left, we have the total rate at which  $L(n)$  jumps from  $n$  to a level above (or equal to)  $\lfloor an \rfloor$  while on the right, we have the rate at which occur reproduction events verifying:

- The proportion of individuals involved in the reproduction event belongs to  $(u', 1)$ .
- The two first levels participate to the reproduction event.
- Among levels  $\{3, \dots, \lfloor an \rfloor - 1\}$  the number  $j$  of levels that participate is greater than or equal to  $\lfloor an \rfloor - 1 - n$  so that in such a reproduction event  $L(n)$  jumps from  $n$  to a level above (or equal to)  $\lfloor an \rfloor$ .

Fix  $x \in (u', 1)$ ,  $n \geq n_0$  and let  $B$  be a Binomial r.v. with  $\lfloor an \rfloor - 3$  trials and probability of success  $x$ . We observe that for every given  $x \in (u', 1)$

$$\sum_{j=\lfloor an \rfloor - 1 - n}^{\lfloor an \rfloor - 3} \binom{\lfloor an \rfloor - 3}{j} x^j (1-x)^{\lfloor an \rfloor - 3 - j}$$

is the probability that  $B$  is greater than or equal to  $\lfloor an \rfloor - 1 - n$ . Let  $\mathbf{P}$  be the probability distribution of  $B$ . From (I.12), we easily verify that

$$x(\lfloor an \rfloor - 3) - \lfloor an \rfloor + 1 + n = n - 2 - (1-x)(\lfloor an \rfloor - 3) \geq n - 2 - (1-x)an \geq \frac{u^*}{4}n > 0.$$

Using the Bienaymé-Chebyshev inequality at the third line, we get

$$\begin{aligned} \mathbf{P}(B < \lfloor an \rfloor - 1 - n) &= \mathbf{P}(B - x(\lfloor an \rfloor - 3) < \lfloor an \rfloor - 1 - n - x(\lfloor an \rfloor - 3)) \\ &\leq \mathbf{P}(|B - x(\lfloor an \rfloor - 3)| > n - 2 - (1-x)(\lfloor an \rfloor - 3)) \\ &\leq \frac{x(1-x)(\lfloor an \rfloor - 3)}{(n - 2 - (1-x)(\lfloor an \rfloor - 3))^2} \leq \frac{4a}{(u^*)^2 n} \end{aligned}$$

Using (I.13) we get

$$\begin{aligned} \int_{(u', 1)} \nu(dx) x^2 \sum_{j=\lfloor an \rfloor - 1 - n}^{\lfloor an \rfloor - 3} \binom{\lfloor an \rfloor - 3}{j} x^j (1-x)^{\lfloor an \rfloor - 3 - j} &\geq \int_{(u', 1)} \nu(dx) x^2 \left(1 - \frac{4a}{(u^*)^2 n}\right) \\ &\geq \Lambda((u', 1)) \left(1 - \frac{4a}{(u^*)^2 n_0}\right) =: r. \end{aligned}$$

Together with (I.12) this ensures that the rate at which the process  $L$  jumps from  $n$  to a level above or equal to  $\lfloor an \rfloor$  is greater or equal to  $r > 0$ , uniformly for all  $n \geq n_0$ . Consider an auxiliary probability space  $(A, \mathcal{A}, \mathbf{Q})$  on which is defined a Poisson process with rate  $r$ . Denote by  $0 = t_0 < t_1 < t_2 < \dots$  the jump times of this Poisson process. Still on  $(A, \mathcal{A}, \mathbf{Q})$  we define the process  $Y = (Y_t, t \geq 0)$  as follows. Initially  $Y_0 = n_0$ , and  $Y$  stays constant on every interval  $[t_k, t_{k+1})$  while its transitions are given by  $Y_{t_{k+1}} = \lfloor aY_{t_k} \rfloor$ . We then set  $b := \inf\{\lfloor an \rfloor/n, n \geq n_0\}$ . Thanks to (I.12) we have for every  $n \geq n_0$ ,  $\lfloor an \rfloor/n \geq (n+1)/n$ . Moreover  $\lfloor an \rfloor/n \rightarrow a > 1$  as  $n \rightarrow \infty$ . Consequently  $b > 1$  and thus for every  $k \geq 0$ ,  $Y_{t_k} \geq b^k$ . From the bound obtained on the jump rates, we deduce that the process  $Y$  is stochastically lower than the process  $L(n)$  for every  $n \geq n_0$ . Set  $w(n) := \mathbb{P}(C_\infty(n) = 0)$ . For every  $n \geq n_0$  we have

$$w(n) \geq \mathbf{Q} \left[ \exp \left( - \sum_{k \geq 0} (t_{k+1} - t_k) \lambda_{Y_{t_k}, 1} \right) \right] = \exp \left( \sum_{k=0}^{\infty} \log \left( \frac{r}{r + \lambda_{Y_{t_k}, 1}} \right) \right).$$

The right member is strictly positive if and only if  $\sum_{k \geq 0} \lambda_{Y_{t_k}, 1} < \infty$ . Using the simple inequality

$$\forall m \geq 1, \quad \lambda_{m, 1} \leq \int_{(0, m^{-\frac{1}{2}}]} x \nu(dx) + (1 - m^{-\frac{1}{2}})^{m-1} \int_{(0, 1)} x \nu(dx)$$

we get

$$\sum_{k=0}^{\infty} \lambda_{Y_{t_k}, 1} \leq \sum_{k=0}^{\infty} \int_{(0, b^{-\frac{k}{2}}]} x \nu(dx) + \sum_{k=0}^{\infty} (1 - Y_{t_k}^{-\frac{1}{2}})^{Y_{t_k}-1} \int_{(0, 1)} x \nu(dx)$$



The second sum on the right converges since  $Y_{t_k} \geq b^k > 1$ . Thus we deduce that  $\sum_{k \geq 0} \lambda_{Y_{t_k}, 1} < \infty$  if

$$\sum_{k=0}^{\infty} \int_{(0, b^{-\frac{k}{2}}]} x \nu(dx) < \infty.$$

Observe that this last quantity is equal to

$$\begin{aligned} \sum_{k=0}^{\infty} (k+1) \int_{(b^{-\frac{k+1}{2}}, b^{-\frac{k}{2}}]} x \nu(dx) &= \sum_{k=0}^{\infty} \int_{(b^{-\frac{k+1}{2}}, b^{-\frac{k}{2}}]} x \nu(dx) + \sum_{k=0}^{\infty} k \int_{(b^{-\frac{k+1}{2}}, b^{-\frac{k}{2}}]} x \nu(dx) \\ &= \int_{(0,1)} x \nu(dx) + \frac{2}{\log b} \sum_{k=0}^{\infty} \log \left( b^{\frac{k}{2}} \right) \int_{(b^{-\frac{k+1}{2}}, b^{-\frac{k}{2}}]} x \nu(dx) \end{aligned}$$

which is finite since  $\int_{(0,1)} x \log \frac{1}{x} \nu(dx) < \infty$ . Therefore  $w := \inf_{n \geq n_0} w(n) > 0$ .

*Proof of (I.13).* Consider the set  $S := \{1, \dots, \lfloor an \rfloor + l - 1\}$ . Observe that

$$\sum_{i=2}^{(\lfloor an \rfloor + l + 1 - n) \wedge n} \binom{n}{i} \binom{\lfloor an \rfloor + l - n - 1}{\lfloor an \rfloor + l + 1 - n - i}$$

is the number of combinations with  $\lfloor an \rfloor + l + 1 - n$  elements among  $S$  with at least 2 elements among  $\{1, \dots, n\}$ . This number is greater than the number of combinations with  $\lfloor an \rfloor + l + 1 - n$  elements among  $S$  with the constraint that 1 and 2 are chosen. The latter is equal to  $\binom{\lfloor an \rfloor + l - 3}{\lfloor an \rfloor + l - n - 1}$ . Consequently we can bound the left member of (I.13) as follows.

$$\begin{aligned} &\int_{(0,1)} \nu(dx) \sum_{l=0}^{\infty} \sum_{i=2}^{(\lfloor an \rfloor + l + 1 - n) \wedge n} \binom{n}{i} \binom{\lfloor an \rfloor + l - n - 1}{\lfloor an \rfloor + l + 1 - n - i} x^{\lfloor an \rfloor + l - n + 1} (1-x)^{n-1} \\ &\geq \int_{(w',1)} \nu(dx) \sum_{l=0}^{\infty} x^{\lfloor an \rfloor + l - n + 1} (1-x)^{n-1} \binom{\lfloor an \rfloor + l - 3}{\lfloor an \rfloor + l - n - 1} \end{aligned}$$

The binomial factor in the right member corresponds to the number of combinations with  $\lfloor an \rfloor + l - n - 1$  elements among the set  $\{3, \dots, \lfloor an \rfloor + l - 1\}$ . Splitting this last set into  $\{3, \dots, \lfloor an \rfloor - 1\}$  and  $\{\lfloor an \rfloor, \dots, \lfloor an \rfloor + l - 1\}$  we get that the last quantity is equal to

$$\begin{aligned} &\int_{(w',1)} \nu(dx) \sum_{l=0}^{\infty} x^{\lfloor an \rfloor + l - n + 1} (1-x)^{n-1} \sum_{j=\lfloor an \rfloor - n - 1}^{(\lfloor an \rfloor + l - n - 1) \wedge (\lfloor an \rfloor - 3)} \binom{\lfloor an \rfloor - 3}{j} \binom{l}{\lfloor an \rfloor + l - n - 1 - j} \\ &= \int_{(w',1)} \nu(dx) x^2 \sum_{j=\lfloor an \rfloor - n - 1}^{\lfloor an \rfloor - 3} \binom{\lfloor an \rfloor - 3}{j} x^j (1-x)^{\lfloor an \rfloor - 3 - j} \\ &\quad \times \sum_{l=j+n+1-\lfloor an \rfloor}^{\infty} \binom{l}{\lfloor an \rfloor + l - n - 1 - j} x^{\lfloor an \rfloor - n - 1 - j + l} (1-x)^{n - \lfloor an \rfloor + 2 + j} \end{aligned}$$

Finally using the change of variable  $p = \lfloor an \rfloor + l - n - 1 - j$  and setting  $k := n + 1 + j - \lfloor an \rfloor$ , one gets

$$\sum_{l=j+n+1-\lfloor an \rfloor}^{\infty} \binom{l}{\lfloor an \rfloor + l - n - 1 - j} x^{\lfloor an \rfloor - n - 1 - j + l} (1-x)^{n - \lfloor an \rfloor + 2 + j} = \frac{(1-x)^{1+k}}{k!} \sum_{p=0}^{\infty} \frac{(p+k)!}{p!} x^p$$

A simple induction on  $k$  shows that the last quantity is equal to 1. ■

*Proof of Theorem I.1 in regime INTENSIVE w. DUST.* We know from Proposition I.31 that  $\mathbb{P}$ -a.s. there exists  $n \geq 2$  such that  $\rho_t(\{\xi_0(n)\}) = 0$  for all  $t \geq 0$  so that the  $n$ -th Eve is not defined. ■

### 4.3 Regime INTENSIVE $\infty$

When  $\Lambda$  is the Lebesgue measure on  $[0, 1]$ , one obtains the celebrated Bolthausen-Sznitman coalescent [19]. Its  $\Lambda$  Fleming-Viot counterpart belongs to regime INTENSIVE  $\infty$ . The proof of Proposition I.4 relies strongly on the connection with measure-valued branching processes obtained by Bertoin and Le Gall [12] for the Bolthausen-Sznitman coalescent, and by Birkner et al. [18] for all the Beta( $2 - \alpha, \alpha$ ) coalescents with  $\alpha \in (0, 2)$ . We refer to Dawson [21], Etheridge [30] and Le Gall [60] for further details on measure-valued branching processes. Let us also mention that the existence of an infinite sequence of Eves for the measure-valued branching process with the Neveu branching mechanism can be obtained thanks to the results in [25, 53].

*Proof of Proposition I.4.* One can construct  $\rho$  by rescaling a measure-valued branching process  $(m_t, t \geq 0)$  associated with the Neveu branching mechanism  $\Psi(u) = u \log u$  as follows:

$$\rho_t(dx) := \frac{m_t(dx)}{m_t([0, 1])}, \quad \forall t \geq 0$$

This result was initially stated for the Bolthausen-Sznitman coalescent by Bertoin and Le Gall in [12], and later on by Birkner et al. for the forward-in-time process in [18]. As a consequence of this connection, we deduce that there exists a r.v.  $e$  such that almost surely

$$\frac{m_t(\{e\})}{m_t([0, 1])} \xrightarrow{t \rightarrow \infty} 1$$

The branching property ensures that the restrictions of  $m$  to any two disjoint subintervals of  $[0, 1]$  are independent. For each integer  $n \geq 1$ , we divide  $[0, 1]$  into dyadic subintervals of the form

$$[0, 2^{-n}), [2^{-n}, 2 \times 2^{-n}), \dots, [1 - 2^{-n}, 1]$$

and we consider the corresponding restrictions of  $m$ . Obviously, for each subinterval  $[(i - 1)2^{-n}, i 2^{-n})$  there exists  $e(i, n)$  such that

$$\frac{m_t(\{e(i, n)\})}{m_t([(i - 1)2^{-n}, i 2^{-n})])} \xrightarrow{t \rightarrow \infty} 1$$

Necessarily  $m$  restricted to the union of two subintervals indexed by  $i \neq j \in [2^n]$  admits either  $e(i, n)$  or  $e(j, n)$  as an Eve and therefore

$$\lim_{t \rightarrow \infty} \frac{m_t(\{e(i, n)\})}{m_t(\{e(j, n)\})} \in \{0, +\infty\}$$

Hence one can order the  $e(i, n), i \in [2^n]$  by asymptotic sizes. Using the consistency of the restrictions when  $n$  varies, one gets the existence of a sequence  $(e^i)_{i \geq 1}$  fulfilling the formula of the statement. ■ It is rather unfortunate that this simple argument does not apply to other measures in regime INTENSIVE  $\infty$ .

### 4.4 Regime CDI

In this subsection, we work on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  on which is defined the  $\Lambda$  flow of partitions  $\hat{\Pi}$  and the sequence of initial types  $\xi_0 = (\xi_0(i), i \geq 1)$  i.i.d. uniform $[0, 1]$ . For every  $t \geq 0$  we

set  $\rho_t := \mathcal{E}_{0,t}(\hat{\Pi}, \xi_0)$  and we let  $(\xi_t(i), t \geq 0), i \geq 1$  be the lockdown process constructed from  $\hat{\Pi}$  and  $\xi_0$ . Recall the filtration introduced in (I.7). A classical argument ensures that  $\mathcal{F}_{0+}$  is a trivial sigma-field under  $\mathbb{P}$ , that is, for every  $A \in \mathcal{F}_{0+}$  we have  $\mathbb{P}(A) \in \{0, 1\}$ . We restrict our attention to regime CDI and we define the following event

$$\mathbf{E} := \{\text{There are at least two initial types that become extinct simultaneously}\}.$$

**LEMMA I.33.** *Consider regime CDI. Then  $\mathbb{P}(\mathbf{E})$  is either 0 or 1.*

**Proof** Recall from Lemma I.22 that  $\mathbb{P}$ -a.s. for every  $i \geq 1$ ,  $d(i)$  stands for the extinction time of the initial type  $\xi_0(i)$  and that it coincides with the first time at which  $\hat{\Pi}_{0,t}(i)$  is empty. The r.v.  $d(i)$  is an  $\mathcal{F}_t$ -stopping time. We also know from Lemma I.22 that  $\mathbb{P}$ -a.s.  $d(i) \downarrow 0$  as  $i \rightarrow \infty$ . Then it is easy to check that  $\cap_{i \geq 1} \mathcal{F}_{d(i)} = \mathcal{F}_{0+}$ . The event  $\mathbf{E}$  can be written as follows:

$$\mathbf{E} := \{\exists i \geq 2 : d(i) = d(i+1)\}.$$

We introduce the following collection of nested events

$$\mathbf{E}_i := \{\exists j \geq i : d(j) = d(j+1)\}, \quad i \geq 1$$

and we set  $\mathbf{E}_\infty := \cap_{i \geq 1} \mathbf{E}_i$ . Since  $\mathbf{E}_i \in \mathcal{F}_{d(i)}$ , we deduce that  $\mathbf{E}_\infty \in \mathcal{F}_{0+}$  so that  $\mathbb{P}(\mathbf{E}_\infty) \in \{0, 1\}$ . To end the proof, we show that  $\mathbb{P}(\mathbf{E}_\infty) = \mathbb{P}(\mathbf{E})$ .

Suppose that  $\mathbb{P}(\mathbf{E}_\infty) = 1$ . Since  $\mathbf{E}_\infty \subset \mathbf{E}$ , we deduce that  $\mathbb{P}(\mathbf{E}) = 1$ . Suppose now that  $\mathbb{P}(\mathbf{E}_\infty) = 0$  and assume that there exists  $i \geq 1$  and  $p \in (0, 1]$  such that  $\mathbb{P}(d(i) = d(i+1)) = p$ . Recall Definition I.26. For every  $k \geq i$ , let  $\tau_k(i) := \inf\{t \geq 0 : L_t(i) \geq k\}$ . This is an  $\mathcal{F}_t$ -stopping time. By consistency,  $\mathbb{P}$ -a.s.  $L_{\tau_k(i)}(i) < L_{\tau_k(i)}(i+1)$  and we have

$$\{d(i) = d(i+1)\} = \{(L_{\tau_k(i)+t}(i), t \geq 0) \text{ and } (L_{\tau_k(i)+t}(i+1), t \geq 0) \text{ reach } \infty \text{ simultaneously}\}.$$

Since  $\tau_k(i)$  is a  $\mathbb{P}$ -a.s. finite  $\mathcal{F}_t$ -stopping time, Lemma I.25 ensures that  $(\hat{\Pi}_{\tau_k(i), \tau_k(i)+s}, s \geq 0)$  has the same distribution as  $(\hat{\Pi}_{0,s}, s \geq 0)$  and is independent of  $\mathcal{F}_{\tau_k(i)}$ . We deduce that

$$\begin{aligned} \mathbb{P}(d(i) = d(i+1)) &= \sum_{j' > j \geq k} \mathbb{P}(L_{\tau_k(i)}(i) = j, L_{\tau_k(i)}(i+1) = j') \mathbb{P}(d(j) = d(j')) \\ &\leq \sum_{j' > j \geq k} \mathbb{P}(L_{\tau_k(i)}(i) = j, L_{\tau_k(i)}(i+1) = j') \mathbb{P}(d(j) = d(j+1)) \\ &\leq \mathbb{P}(\mathbf{E}_k) \end{aligned}$$

We have proved that for all  $k \geq i$ ,  $\mathbb{P}(\mathbf{E}_k) \geq p > 0$ . Since the events  $\mathbf{E}_k, k \geq i$  are nested this ensures that  $\mathbb{P}(\mathbf{E}_\infty) \geq p > 0$  which contradicts the hypothesis. Therefore for all  $i \geq 1$ ,  $\mathbb{P}(d(i) = d(i+1)) = 0$  and consequently  $\mathbb{P}(\mathbf{E}) = 0$ .  $\blacksquare$

For every  $n \geq 1$  and every  $c > 0$  we now introduce the event

$$\mathbf{E}_{n,c} := \{\exists m \geq n : d((1+c)m) = d(m)\}.$$

**PROPOSITION I.34.** *Suppose that  $\mathbb{P}(\mathbf{E}) = 1$ . There exists  $c > 0$  such that  $\mathbb{P}(\cap_{n \geq 1} \mathbf{E}_{n,c}) > 0$ .*

**Proof** The equality  $\mathbb{P}(E) = 1$  entails the existence of an integer  $j \geq 2$  such that  $\mathbb{P}(d(j) = d(j+1)) > 0$ . Since the process  $t \mapsto (L_t(2), L_t(3))$  has a positive probability to reach  $(j, j+1)$  at a certain time, a simple argument based on Lemma I.25 entails the existence of  $\eta \in (0, 1]$  such that  $\mathbb{P}(d(2) = d(3)) = \eta$ . We introduce for every  $n \geq 1$  the  $\mathcal{F}_t$ -stopping time

$$\tau_n := \inf \left\{ t \geq 0 : \rho_t([0, 1] \setminus \{\xi_0(1)\}) \leq \frac{1}{n^2} \right\}$$

which is finite almost surely since the primitive Eve  $e = \xi_0(1)$  fixes. We first show that  $L_{\tau_n}(3) - L_{\tau_n}(2)$  becomes large as  $n$  goes to infinity. To that end, we define

$$Z_{\tau_n} := \inf \{k > L_{\tau_n}(2) : k \notin \hat{\Pi}_{0, \tau_n}(1)\} = \inf \{k > L_{\tau_n}(2) : \xi_{\tau_n}(k) \neq \xi_0(1)\}.$$

Hence  $L_{\tau_n}(2)$  and  $Z_{\tau_n}$  are the first and second levels at time  $\tau_n$  which are not of type  $\xi_0(1)$ . Necessarily  $\xi_{\tau_n}(Z_{\tau_n})$  is either equal to  $\xi_0(2)$  or is equal to  $\xi_0(3)$  in which case  $Z_{\tau_n} = L_{\tau_n}(3)$ . In both cases  $Z_{\tau_n}$  is lower than or equal to  $L_{\tau_n}(3)$ . Proposition 3.1 in [24] ensures that the sequence  $(\xi_{\tau'}(i))_{i \geq 1}$  is exchangeable with empirical measure  $\rho_{\tau'}$  as soon as  $\tau'$  is a stopping time in the filtration of  $\rho$ . This result can be extended easily to show that  $(\xi_{\tau'}(i))_{i \geq 2}$  is exchangeable with empirical measure  $\rho_{\tau'}$  whenever  $\tau'$  is a stopping time in the filtration of the pair  $(\rho, \rho(\{\xi_0(1)\}))$ . We deduce that the sequence  $(\xi_{\tau_n}(i))_{i \geq 2}$  is exchangeable with empirical measure  $\rho_{\tau_n}$  so that the  $\xi_{\tau_n}(i), i \geq 2$  are i.i.d. with law  $\rho_{\tau_n}$ . The law of  $(L_{\tau_n}(2), Z_{\tau_n})$  can then be characterised as follows.

Fix a real value  $p \in (0, 1)$  and consider an auxiliary probability space  $(A, \mathcal{A}, \mathbf{Q})$  on which are defined two independent geometric r.v.  $G$  and  $G'$  with parameter  $p$ , that is:

$$\forall k \geq 1, \mathbf{Q}(G = k) = (1-p)^{k-1}p.$$

Then the distribution under  $\mathbb{P}$  of  $(L_{\tau_n}(2), Z_{\tau_n})$  conditionally given  $\rho_{\tau_n}([0, 1] \setminus \{\xi_0(1)\}) = p$  is the distribution under  $\mathbf{Q}$  of  $(1 + G, 1 + G + G')$ . A simple calculation yields for all  $c > 0$

$$\begin{aligned} \mathbf{Q}\left(\frac{G'}{1+G} \geq c; G \geq n\right) &= \sum_{k=n}^{\infty} \sum_{l=c(k+1)}^{\infty} p^2(1-p)^{k+l-2} \\ &= \frac{p(1-p)^{c-2+n(1+c)}}{1 - (1-p)^{1+c}}. \end{aligned} \quad (\text{I.14})$$

By the right continuity of  $\rho$ ,  $\mathbb{P}$ -a.s.  $\rho_{\tau_n}([0, 1] \setminus \{\xi_0(1)\}) \leq 1/n^2$ . From the dominated convergence theorem and (I.14) we get

$$\mathbb{P}\left(\frac{Z_{\tau_n} - L_{\tau_n}(2)}{L_{\tau_n}(2)} \geq c; L_{\tau_n}(2) \geq n\right) \xrightarrow{n \rightarrow \infty} \frac{1}{1+c}.$$

Recall that  $L_{\tau_n}(3) \geq Z_{\tau_n}$  so that

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{L_{\tau_n}(3) - L_{\tau_n}(2)}{L_{\tau_n}(2)} \geq c; L_{\tau_n}(2) \geq n\right) \geq \liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{Z_{\tau_n} - L_{\tau_n}(2)}{L_{\tau_n}(2)} \geq c; L_{\tau_n}(2) \geq n\right) = \frac{1}{1+c}.$$

We now introduce the event

$$B_n := \left\{ d(2) = d(3); \frac{L_{\tau_n}(3) - L_{\tau_n}(2)}{L_{\tau_n}(2)} \geq c; L_{\tau_n}(2) \geq n \right\}.$$

Take  $c > 0$  such that  $\frac{1}{1+c} + \eta > 1$ . There exists  $\epsilon > 0$  such that  $\frac{1-\epsilon}{1+c} + \eta > 1$ . From the bound on the lim above we deduce the existence of  $n_0 \geq 1$  such that for all  $n \geq n_0$

$$\mathbb{P}\left(\frac{L_{\tau_n}(3) - L_{\tau_n}(2)}{L_{\tau_n}(2)} \geq c; L_{\tau_n}(2) \geq n\right) \geq \frac{1-\epsilon}{1+c}.$$

Using the inequality  $\mathbb{P}(D \cap D') \geq \mathbb{P}(D) + \mathbb{P}(D') - 1$  that holds for any two events  $D, D'$  we get

$$\begin{aligned} \mathbb{P}(\mathbf{B}_n) &\geq \mathbb{P}(d(2) = d(3)) + \mathbb{P}\left(\frac{L_{\tau_n}(3) - L_{\tau_n}(2)}{L_{\tau_n}(2)} \geq c; L_{\tau_n}(2) \geq n\right) - 1 \\ &\geq \eta + \frac{1 - \epsilon}{1 + c} - 1 > 0 \end{aligned}$$

Moreover we have

$$\mathbf{B}_n = \bigcup_{m \geq n} \left\{ \begin{array}{l} L_{\tau_n}(3) \geq (1 + c)L_{\tau_n}(2); L_{\tau_n}(2) = m; \\ (L_{\tau_n+t}(3), t \geq 0) \text{ and } (L_{\tau_n+t}(2), t \geq 0) \text{ reach } \infty \text{ simultaneously} \end{array} \right\}. \quad (\text{I.15})$$

Since  $\tau_n$  is a  $\mathbb{P}$ -a.s. finite  $\mathcal{F}_t$ -stopping time, Lemma I.25 ensures that  $(\hat{\Pi}_{\tau_n, \tau_n+s}, s \geq 0)$  has the same distribution as  $(\hat{\Pi}_{0,s}, s \geq 0)$  and is independent of  $\mathcal{F}_{\tau_n}$ . This yields together with (I.15) that for every  $n \geq n_0$

$$\mathbb{P}(\mathbf{E}_{n,c}) \geq \mathbb{P}(\mathbf{B}_n) \geq \eta + \frac{1 - \epsilon}{1 + c} - 1 > 0.$$

Since the events  $\mathbf{E}_{n,c}, n \geq 1$  are nested,  $\mathbb{P}(\bigcap_{n \geq 1} \mathbf{E}_{n,c}) = \lim_{n \rightarrow \infty} \mathbb{P}(\mathbf{E}_{n,c}) > 0$  and the proposition is proved.  $\blacksquare$

*Proof of Theorem I.2.* Recall the function  $\Psi$  from Equation (I.4). Since  $\int^\infty \frac{du}{\Psi(u)} < \infty$ , we can introduce the continuous map  $t \mapsto v(t)$  as the unique solution of

$$\int_{v(t)}^\infty \frac{du}{\Psi(u)} = t, \quad \forall t > 0$$

Proposition 15 in Berestycki et al. [5] ensures that for all  $\epsilon \in (0, 1)$  we have

$$\mathbb{P}\left(\liminf_{t \rightarrow 0} \frac{\#\hat{\Pi}_{0,t}}{v\left(\frac{1+\epsilon}{1-\epsilon}t\right)} \geq \frac{1}{1+\epsilon}; \limsup_{t \rightarrow 0} \frac{\#\hat{\Pi}_{0,t}}{v\left(\frac{1-\epsilon}{1+\epsilon}t\right)} \leq \frac{1}{1-\epsilon}\right) = 1 \quad (\text{I.16})$$

where  $\#\hat{\Pi}_{0,t}$  denotes the number of blocks of  $\hat{\Pi}_{0,t}$ . Assume that  $\Lambda(dx) = f(x)dx$  where  $f$  is regularly varying at  $0+$  with index  $1 - \alpha$  (where  $\alpha \in (1, 2)$ ) or that  $\Lambda(\{0\}) > 0$  (in which case  $\alpha$  is taken equal to 2). Then  $\Psi$  is regularly varying at  $+\infty$  with index  $\alpha$  and  $v$  is itself regularly varying at  $0+$  with index  $-1/(\alpha - 1)$  (see Subsection 6.3 for a proof of this fact). Consequently we have for all  $\epsilon \in (0, 1)$

$$v\left(\frac{1+\epsilon}{1-\epsilon}t\right)\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/(\alpha-1)} \underset{t \downarrow 0}{\sim} v\left(\frac{1-\epsilon}{1+\epsilon}t\right)\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/(\alpha-1)} \underset{t \downarrow 0}{\sim} v(t).$$

Together with (I.16) this yields

$$\mathbb{P}\left(\lim_{t \rightarrow 0} \frac{\#\hat{\Pi}_{0,t}}{v(t)} = 1\right) = 1.$$

This forces the jumps of  $t \mapsto \#\hat{\Pi}_{0,t}$  to be small near  $0+$ . More precisely for any  $c > 0$

$$\mathbb{P}\left(\limsup_{t \rightarrow 0} \frac{\#\hat{\Pi}_{0,t-} - \#\hat{\Pi}_{0,t}}{\#\hat{\Pi}_{0,t}} < c\right) = 1. \quad (\text{I.17})$$

The collection of events  $(\mathbf{E}_{n,c}, n \geq 1)$  is nested. Recall that  $\mathbb{P}$ -a.s.  $d(i) \downarrow 0$  as  $i \rightarrow \infty$ . Using (I.17) we deduce that for any  $c > 0$

$$\mathbb{P}\left(\bigcap_{n \geq 1} \mathbf{E}_{n,c}\right) = 0.$$

This identity combined with Proposition I.34 entails that  $\mathbb{P}(\mathbf{E})$  is not equal to 1. Lemma I.33 in turn ensures that  $\mathbb{P}(\mathbf{E}) = 0$ .  $\blacksquare$

## 4.5 Open questions

Consider regime INTENSIVE w. DUST and recall the function  $\Psi$  from (I.4). In [25], it is shown that the measure-valued branching process with branching mechanism  $\Psi$  (we refer to Dawson [21], Etheridge [30] or Le Gall [60] for a definition of this object) has a residual dust component when  $t$  tends to infinity iff  $\int_{(0,1)} x \log \frac{1}{x} \nu(dx) < \infty$ . However when this  $x \log \frac{1}{x}$  condition is not fulfilled the frequencies in the population when  $t$  goes to infinity are of comparable order and the measure-valued branching process does not admit an infinite sequence of Eves. As branching processes and  $\Lambda$  Fleming-Viot processes present many similarities [6, 15, 18], it is natural to expect the following behaviour.

**CONJECTURE I.35.** *In regime INTENSIVE w. DUST when  $\int_{(0,1)} x \log \frac{1}{x} \nu(dx) = \infty$ , every initial type of the lookdown representation gets a positive frequency at a certain time. However there does not exist an infinite sequence of Eves.*

In regime INTENSIVE  $\infty$ , if we could prove that  $L(n+1)$  goes to infinity much faster than  $L(n)$  then we could deduce the existence of a sequence of Eves. But the main difficulty lies in finding a precise upper bound for  $L(n)$ .

**CONJECTURE I.36.** *In regime INTENSIVE  $\infty$ , the  $\Lambda$  Fleming-Viot admits an infinite sequence of Eves without further condition on  $\Lambda$ .*

Finally in regime CDI, our proof relies strongly on the regular variation of the measure  $\Lambda$ . However the similarity between  $\Lambda$  Fleming-Viot and branching processes (for which the extinction times of two independent copies are distinct almost surely) suggests the following.

**CONJECTURE I.37.** *In regime CDI, without further condition on  $\Lambda$  we have  $\mathbb{P}(\mathbf{E}) = 0$  and thus, the  $\Lambda$  Fleming-Viot always admits an infinite sequence of Eves.*

## 5 Unification

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space on which is defined a  $\Lambda$  flow of bridges  $(F_{s,t}, -\infty < s \leq t < \infty)$  according to Definition I.5. We consider a modification of this flow of bridges that we still denote  $(F_{s,t}, -\infty < s \leq t < \infty)$  and such that for every  $s \in \mathbb{R}$  the process  $(\rho_{s,s+t}, t \geq 0)$  is a càdlàg  $\Lambda$  Fleming-Viot process. This modification does exist since the  $\Lambda$  Fleming-Viot process has a Feller semigroup, see p.278 in [13]. From now on, we assume that the measure  $\Lambda$  is such that the  $\Lambda$  Fleming-Viot process admits  $\mathbb{P}$ -a.s. an infinite sequence of Eves. Let us emphasise the fact that we only rely on Definition I.3 and do not restrict ourselves to the particular examples of measures  $\Lambda$  presented in Proposition I.4 and Theorem I.2.

### 5.1 The evolving sequence of Eves

For each  $s \in \mathbb{R}$ , we define the Eves  $(e_s^i, i \geq 1)$  of the  $\Lambda$  Fleming-Viot process  $(\rho_{s,s+t}, t \geq 0)$ . Notice that this sequence is defined on an event of  $\mathbb{P}$ -probability 1 that depends on  $s$ . For each  $s \in \mathbb{R}$ , outside this event we set an arbitrary value to the sequence  $(e_s^i, i \geq 1)$ . Below we provide a simple description of the connection between the Eves taken at two distinct times. We rely on a key property due to Bertoin and Le Gall [13] that we now recall. Consider an exchangeable bridge  $B$  and an independent sequence  $V = (V_i, i \geq 1)$  of i.i.d. uniform $[0, 1]$  r.v. We define an exchangeable random partition  $\pi = \pi(B, V)$  by setting

$$i \sim j \Leftrightarrow B^{-1}(V_i) = B^{-1}(V_j).$$

For each  $j \geq 1$ , if the  $j$ -th block  $\pi(j)$  is not empty then we define  $V_j' := B^{-1}(V_i)$  for an arbitrary integer  $i \in \pi(j)$ . If the number of blocks of  $\pi$  is finite, we complete the sequence  $V'$  with i.i.d. uniform $[0, 1]$  r.v.

Then Lemma 2 in [13] entails that the  $(V'_j, j \geq 1)$  are i.i.d. uniform $[0, 1]$  and are independent of  $\pi$ .

**PROPOSITION I.38.** *Fix  $s > 0$ . Define the partition  $\pi = \pi(F_{0,s}, (e_s^i)_{i \geq 1})$ . Then the sequence  $(e_0^j, j \geq 1)$  is independent of  $\pi$  and  $\mathbb{P}$ -a.s. for every  $j \geq 1$ , if  $\pi(j)$  is not empty then  $e_0^j = F_{0,s}^{-1}(e_s^i)$  for any  $i \in \pi(j)$ .*

This allows to introduce  $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$  by setting for every  $s \leq t$

$$\hat{\Pi}_{s,t} := \pi(F_{s,t}, (e_t^i)_{i \geq 1}) \text{ if } s < t, \quad \hat{\Pi}_{s,t} := 0_{[\infty]} \text{ if } s = t.$$

**Proof** Let  $\Omega_{0,s}$  be an event of  $\mathbb{P}$ -probability one on which the definitions of the Eves at times 0 and  $s$  hold and on which for all  $t \in \mathbb{Q}_+$  we have  $F_{0,s+t} = F_{s,s+t} \circ F_{0,s}$ . Recall that  $(e_s^i, i \geq 1)$  is a sequence of i.i.d. uniform $[0, 1]$  r.v. which only depends on  $\sigma(F_{s,s+t}, t \geq 0)$  so that  $(e_s^i, i \geq 1)$  is independent of  $F_{0,s}$  from the independence of the increments of the flow of bridges. The sequence  $(e_s^i, i \geq 1)$  plays the rôle of the sequence  $(V_i, i \geq 1)$  above. We introduce  $K$  as the random number of blocks of  $\pi$  which is  $\mathbb{P}$ -a.s. finite in the *Eves - extinction case* (the bridge  $F_{0,s}$  has a finite number of jumps and no drift) while it is  $\mathbb{P}$ -a.s. infinite in the *Eves - persistent case* (the bridge  $F_{0,s}$  has an infinite number of jumps). We define the following sequence of random variables  $V'_j := F_{0,s}^{-1}(e_s^{i_j})$  where  $i_j := \min \pi(j)$  for every  $j \in [K]$ . If  $K$  is finite, we set  $V'_j := e_0^j$  for all  $j > K$ . To prove the proposition, it remains to show that:

- (i)  $\mathbb{P}$ -a.s.  $e_0^j = V'_j$  for every  $j \in [K]$ ,
- (ii)  $(e_0^j)_{j \geq 1}$  is independent of  $\pi$ .

We start with the first assertion. We argue deterministically on the event  $\Omega_{0,s}$  of  $\mathbb{P}$ -probability one. We claim that for every  $j \in [K]$  and all  $t \in \mathbb{Q}_+$

$$\rho_{s,s+t}(\{e_s^{i_j}\}) \leq \rho_{0,s+t}(\{V'_j\}) \leq \rho_{s,s+t}([0, 1] \setminus \{e_s^1, \dots, e_s^{i_j-1}\}). \quad (\text{I.18})$$

Let us prove (I.18). Recall that  $V'_j$  is the pre-image of  $e_s^{i_j}$  through  $F_{0,s}$  so that  $V'_j$  is either the location of a jump of the bridge or it is point of increase. First assume that  $V'_j$  is the location of a jump of  $F_{0,s}$ . This yields that  $F_{0,s}(V'_j) \geq e_s^{i_j} > F_{0,s}(V'_j-)$ . Consequently  $F_{0,s+t}(V'_j) \geq F_{s,s+t}(e_s^{i_j}) \geq F_{s,s+t}(e_s^{i_j}-) \geq F_{0,s+t}(V'_j-)$  and the first inequality of (I.18) follows. To prove the second inequality, observe that for all  $l \in [i_j - 1]$ ,  $e_s^l$  does not belong to  $(F_{0,s}(V'_j-), F_{0,s}(V'_j))$ . Consequently

$$\rho_{0,s+t}(\{V'_j\}) = \rho_{s,s+t}((F_{0,s}(V'_j-), F_{0,s}(V'_j))) \leq 1 - \sum_{l=1}^{i_j-1} \rho_{s,s+t}(\{e_s^l\})$$

and the second inequality follows. Assume now that  $F_{0,s}$  is continuous but increasing at  $V'_j$ . Then  $F_{0,s}(V'_j) = e_s^{i_j} = F_{0,s}(V'_j-)$ . Since  $F_{0,s}$  is increasing at  $V'_j$  we have  $F_{0,s+t}(V'_j-) = F_{s,s+t}(e_s^{i_j}-)$  and we get  $\rho_{0,s+t}(\{V'_j\}) = \rho_{s,s+t}(\{e_s^{i_j}\})$ . The second equality of (I.18) derives from the fact that for every  $l \in [i_j - 1]$ ,  $e_s^l \neq e_s^{i_j}$  so that

$$\rho_{0,s+t}(\{V'_j\}) = \rho_{s,s+t}(\{e_s^{i_j}\}) \leq 1 - \sum_{l=1}^{i_j-1} \rho_{s,s+t}(\{e_s^l\}).$$

Therefore (I.18) is proved. We now carry out the proof of Assertion (i), we treat separately two cases. We start with the *Eves - extinction case*. The bridge  $F_{0,s}$  has no drift and  $K$  jumps. These  $K$  jumps

are located on the points  $(V'_j)_{j \in [K]}$ . Necessarily the  $K$  first Eves are the jump locations of  $F_{0,s}$  so that the sets  $\{V'_j, j \in [K]\}$  and  $\{e_0^j, j \in [K]\}$  coincide. By definition of the Eves, the processes  $t \mapsto \rho_{s,s+t}([0, 1] \setminus \{e_s^1, \dots, e_s^{i_j-1}\})$  and  $t \mapsto \rho_{s,s+t}(\{e_s^{i_j}\})$  reach 0 at the same time. From (I.18) we conclude that  $t \mapsto \rho_{0,s+t}(\{V'_j\})$  reaches 0 also at that time. Hence, the collection  $(V'_j)_{j \in [K]}$  is ordered by decreasing extinction times and we conclude that  $V'_j = e_0^j$  for all  $j \in [K]$ . We turn to the *Eves - persistent case*. The definition of the Eves implies that for all  $j \geq 1$

$$\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{Q}}} \frac{\rho_{s,s+t}(\{e_s^{i_j}\})}{\rho_{s,s+t}([0, 1] \setminus \{e_s^1, \dots, e_s^{i_j-1}\})} = 1$$

The arguments used in the proof of (I.18) can be adapted to show that for all  $t \in \mathbb{Q}_+$

$$\rho_{s,s+t}(\{e_s^1, \dots, e_s^{i_j-1}\}) \leq \rho_{0,s+t}(\{V'_1, \dots, V'_{j-1}\})$$

Together with (I.18) this yields

$$\lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{Q}}} \frac{\rho_{0,s+t}(\{V'_j\})}{\rho_{0,s+t}([0, 1] \setminus \{V'_1, \dots, V'_{j-1}\})} \geq \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{Q}}} \frac{\rho_{s,s+t}(\{e_s^{i_j}\})}{\rho_{s,s+t}([0, 1] \setminus \{e_s^1, \dots, e_s^{i_j-1}\})} = 1$$

By the uniqueness of the Eves we get  $V'_j = e_0^j$  for all  $j \geq 1$ . The first assertion is proved.

We turn to the second assertion. In the *Eves - persistent case*, this assertion is a consequence of the key property of Bertoin and Le Gall and of the first assertion since  $K = \infty$   $\mathbb{P}$ -a.s. We consider the *Eves - extinction case*. Let  $\mathcal{G}_s := \sigma(F_{r,t}, -\infty < r \leq t \leq s)$ , implicitly we augment this sigma-field with the  $\mathbb{P}$ -null sets. Observe that on the event  $\{K = k\}$ , the Eves  $(e_0^j)_{j > k}$  have extinct progenies at time  $s$  so that  $\mathbf{1}_{\{K=k\}}(e_0^j)_{j > k}$  is  $\mathcal{G}_s$ -measurable. For every  $k \geq 1$ , we define a sigma-field on  $\{K = k\}$  by setting

$$\mathcal{B}_k := \{A \cap \{K = k\} : A \in \sigma(F_{0,s}, (e_s^j)_{j \geq 1})\}.$$

We claim that  $(e_0^j)_{j > k}$  is independent of  $\mathcal{B}_k$ . Indeed let  $h$  be a bounded measurable map on  $\mathbb{D}([0, 1], [0, 1])$  and let  $g_0, g_s$  be two bounded measurable maps on  $[0, 1]^{\mathbb{N}}$  (endowed with the product sigma-field).

$$\begin{aligned} \mathbb{E} \left[ g_0((e_0^j)_{j > k}) h(F_{0,s}) g_s((e_s^j)_{j \geq 1}) \mathbf{1}_{\{K=k\}} \right] &= \mathbb{E} \left[ \mathbb{E} \left[ g_0((e_0^j)_{j > k}) h(F_{0,s}) g_s((e_s^j)_{j \geq 1}) \mathbf{1}_{\{K=k\}} \mid \mathcal{G}_s \right] \right] \\ &= \mathbb{E} \left[ g_0((e_0^j)_{j > k}) h(F_{0,s}) \mathbf{1}_{\{K=k\}} \right] \mathbb{E} \left[ g_s((e_s^j)_{j \geq 1}) \right] \\ &= \mathbb{E} \left[ g_0((e_0^j)_{j > k}) \right] \mathbb{E} \left[ h(F_{0,s}) \mathbf{1}_{\{K=k\}} \right] \mathbb{E} \left[ g_s((e_s^j)_{j \geq 1}) \right] \end{aligned}$$

where the second equality comes from the independence of the increments of a flow of bridges and the  $\mathcal{G}_s$ -measurability of  $\mathbf{1}_{\{K=k\}}(e_0^j)_{j > k}$ , while the third equality comes from Corollary I.24 and the fact that on  $\{K = k\}$  the bridge  $F_{0,s}$  only depends on  $(e_0^i)_{i \leq k}$  and  $(\rho_{0,s}(\{e_0^i\}))_{i \leq k}$ . The claimed independence follows. We now prove the independence of  $(e_0^j)_{j \geq 1}$  with  $\pi$ . Let  $m \geq 1$  be an integer,  $f_1, \dots, f_m$  be  $m$  bounded measurable maps on  $[0, 1]$  and  $1 \leq i_1 < i_2 < \dots < i_m$  be  $m$  integers. Let also  $\phi$  be a bounded measurable map on  $\mathcal{P}_\infty$ . Let  $\mathbb{P}_k := \mathbb{P}(\cdot \mid \{K = k\})$ . We have

$$\begin{aligned} \mathbb{E} \left[ \prod_{j=1}^m f_j(e_0^{i_j}) \phi(\pi) \right] &= \sum_{k=1}^{\infty} \mathbb{E}_k \left[ \prod_{j=1}^m f_j(e_0^{i_j}) \phi(\pi) \right] \mathbb{P}(K = k) \\ &= \sum_{k=1}^{\infty} \mathbb{E}_k \left[ \mathbb{E}_k \left[ \prod_{j=1}^m f_j(e_0^{i_j}) \phi(\pi) \mid \mathcal{B}_k \right] \right] \mathbb{P}(K = k) \\ &= \sum_{k=1}^{\infty} \mathbb{E} \left[ \prod_{j:i_j > k} f_j(e_0^{i_j}) \right] \mathbb{E} \left[ \prod_{j:i_j \leq k} f_j(e_0^{i_j}) \phi(\pi) \mathbf{1}_{\{K=k\}} \right] \end{aligned}$$



where we use the claim proved above and the fact that on  $\{K = k\}$  the r.v.  $\pi$  and  $(e_0^j)_{j \leq k}$  are  $\mathcal{B}_k$ -measurable. Then we apply the key property of Bertoin and Le Gall together with Assertion (i) to obtain

$$\begin{aligned} \mathbb{E} \left[ \prod_{j=1}^m f_j(e_0^{i_j}) \phi(\pi) \right] &= \sum_{k=1}^{\infty} \mathbb{E} \left[ \prod_{j:i_j > k} f_j(e_0^{i_j}) \right] \mathbb{E} \left[ \prod_{j:i_j \leq k} f_j(e_0^{i_j}) \right] \mathbb{E} \left[ \phi(\pi) \mathbf{1}_{\{K=k\}} \right] \\ &= \mathbb{E} \left[ \prod_{j=1}^m f_j(e_0^{i_j}) \right] \mathbb{E} \left[ \phi(\pi) \right] \end{aligned}$$

Assertion (ii) follows. This ends the proof of the proposition.  $\blacksquare$

*Proof of Theorem 1.3.* The cocycle property is trivially fulfilled if  $r = s$  or  $s = t$  thus we fix  $r < s < t$  and prove the cocycle property. We argue deterministically on the event of  $\mathbb{P}$ -probability one on which  $F_{r,t} = F_{s,t} \circ F_{r,s}$  and on which the definition of the Eves at times  $r$ ,  $s$  and  $t$  hold. Fix two integers  $i, j$  and let  $k_i, k_j$  be the integers such that  $F_{s,t}^{-1}(e_t^i) = e_s^{k_i}$  and  $F_{s,t}^{-1}(e_t^j) = e_s^{k_j}$ . Observe that

$$F_{r,t}^{-1}(e_t^i) = F_{r,t}^{-1}(e_t^j) \Leftrightarrow F_{r,s}^{-1} \circ F_{s,t}^{-1}(e_t^i) = F_{r,s}^{-1} \circ F_{s,t}^{-1}(e_t^j).$$

The right member is equivalent with

$$k_i = k_j \text{ or } F_{r,s}^{-1}(e_s^{k_i}) = F_{r,s}^{-1}(e_s^{k_j}).$$

Consequently we have proved that  $i$  and  $j$  are in a same block of  $\hat{\Pi}_{r,t}$  if and only if they are in a same block of  $\hat{\Pi}_{s,t}$  or the indices of their respective blocks in  $\hat{\Pi}_{s,t}$ , say  $k_i$  and  $k_j$ , are in a same block of  $\hat{\Pi}_{r,s}$ . The cocycle property follows.

Theorem 1 [13] ensures that for every  $s \in \mathbb{R}$ ,  $(\hat{\Pi}_{s-t,s}, t \geq 0)$  is an exchangeable coalescent. Since the flow of bridges is associated with the measure  $\Lambda$ , we deduce that  $(\hat{\Pi}_{-t,0}, t \geq 0)$  is a  $\Lambda$  coalescent. Another consequence of this fact is that  $\hat{\Pi}_{s,t}$  is an exchangeable random partition whose distribution only depends of  $t - s$ .

Fix  $s_1 < s_2 < \dots < s_n$ . If we prove that  $\hat{\Pi}_{s_{n-1},s_n}$  is independent of  $\hat{\Pi}_{s_1,s_2}, \dots, \hat{\Pi}_{s_{n-2},s_{n-1}}$ , then an easy induction allows to prove the independence of  $\hat{\Pi}_{s_1,s_2}, \dots, \hat{\Pi}_{s_{n-2},s_{n-1}}, \hat{\Pi}_{s_{n-1},s_n}$ . For every  $i \in [n-1]$ ,  $\hat{\Pi}_{s_i,s_{i+1}}$  is a  $\sigma(F_{s_i,s_{i+1}}, (e_{s_{i+1}}^j)_{j \geq 1})$  measurable r.v. and if  $i \in [n-2]$  then  $(e_{s_{i+1}}^j)_{j \geq 1}$  is measurable in the sigma-field

$$\sigma\left((e_{s_{i+2}}^j)_{j \geq 1}, (F_{s_{i+1},t}, s_{i+1} \leq t \leq s_{i+2})\right).$$

Consequently it is sufficient to prove that  $\hat{\Pi}_{s_{n-1},s_n}$  is independent of  $\sigma((e_{s_{n-1}}^j)_{j \geq 1}, (F_{s,t}, -\infty < s \leq t \leq s_{n-1}))$ . Fix  $k \geq 1$ . Let  $\mathcal{G}_{s_{n-1}}$  be the sigma-field generated by  $(F_{s,t}, -\infty < s \leq t \leq s_{n-1})$ . Let  $f_1, \dots, f_k$  be  $k$  bounded measurable maps on  $[0, 1]$  and let  $h$  be a bounded measurable map on  $\mathcal{P}_\infty$ . For all  $j_1, \dots, j_k \geq 1$  and all  $A \in \mathcal{G}_{s_{n-1}}$  we have

$$\begin{aligned} \mathbb{E} \left[ \prod_{l=1}^k f_l(e_{s_{n-1}}^{j_l}) \mathbf{1}_A h(\hat{\Pi}_{s_{n-1},s_n}) \right] &= \mathbb{E} \left[ \mathbb{E} \left[ \prod_{l=1}^k f_l(e_{s_{n-1}}^{j_l}) \mathbf{1}_A h(\hat{\Pi}_{s_{n-1},s_n}) \mid \mathcal{G}_{s_{n-1}} \right] \right] \\ &= \mathbb{E} \left[ \mathbf{1}_A \mathbb{E} \left[ \prod_{l=1}^k f_l(e_{s_{n-1}}^{j_l}) h(\hat{\Pi}_{s_{n-1},s_n}) \right] \right] \\ &= \mathbb{E} \left[ \prod_{l=1}^k f_l(e_{s_{n-1}}^{j_l}) \right] \mathbb{E} [\mathbf{1}_A] \mathbb{E} [h(\hat{\Pi}_{s_{n-1},s_n})] \end{aligned}$$

where we use the independence of the increments of a flow of bridges at the second line, and Proposition I.38 at the third line. Hence  $\hat{\Pi}_{s_{n-1}, s_n}$  is independent of  $\sigma((e_{s_{n-1}}^j)_{j \geq 1}, (F_{s,t}, -\infty < s \leq t \leq s_{n-1}))$ . The asserted independence follows.

Finally the convergence in distribution of  $\hat{\Pi}_{0,t}$  towards  $0_{[\infty]}$  as  $t \downarrow 0$  derives from Lemma 1 in [13]. Since the limit is deterministic and since  $\mathcal{P}_\infty$  is a metric space, the convergence in probability is immediate. ■

## 5.2 Proof of Theorem I.4

From the  $\Lambda$  flow of bridges, we have defined a  $\Lambda$  flow of partitions. The trajectories of the latter are not necessarily deterministic flows of partitions. However using the regularisation procedure of Subsection 2.3, we can consider a modification of this flow of partitions that we still denote  $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$  and whose trajectories are deterministic flows of partitions  $\mathbb{P}$ -a.s. At any given time  $s \in \mathbb{R}$ , we use this flow of partitions and the Eves at time  $s$  to define a measure-valued process thanks to the lockdown construction

$$t \mapsto \mathcal{E}_{s,s+t}(\hat{\Pi}, (e_s^i)_{i \geq 1}) := \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)| \delta_{e_s^i} + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|\right) \ell.$$

Fix  $s \in \mathbb{R}$ . Both  $(\mathcal{E}_{s,s+t}(\hat{\Pi}, (e_s^i)_{i \geq 1}), t \geq 0)$  and  $(\rho_{s,s+t}, t \geq 0)$  are  $\mathbb{P}$ -a.s. càdlàg processes. To prove that they coincide  $\mathbb{P}$ -a.s., it suffices to show that for every  $t \geq 0$ ,  $\mathbb{P}$ -a.s.  $\mathcal{E}_{s,s+t}(\hat{\Pi}, (e_s^i)_{i \geq 1}) = \rho_{s,s+t}$ . Fix  $t \geq 0$ . Since  $F_{s,s+t}$  is an exchangeable bridge, we know that  $\mathbb{P}$ -a.s.  $F_{s,s+t}$  has a collection of jumps and possibly a drift component. Proposition I.38 ensures that  $\mathbb{P}$ -a.s. the jump locations of  $F_{s,s+t}$  are included in the set  $\{e_s^i, i \geq 1\}$ . Moreover since  $(e_{t+s}^i, i \geq 1)$  is independent of  $F_{s,s+t}$   $\mathbb{P}$ -a.s. each block of the partition  $\hat{\Pi}_{s,s+t}$  admits an asymptotic frequency equal to the size of a jump of  $F_{s,s+t}$ . Since the  $i$ -th block is associated with the jump located on the  $i$ -th Eve  $e_s^i$ , we deduce that  $\mathbb{P}$ -a.s. for every  $i \geq 1$ ,  $\rho_{s,s+t}(\{e_s^i\}) = |\hat{\Pi}_{s,s+t}(i)|$ . Consequently  $\mathbb{P}$ -a.s.

$$\rho_{s,s+t} = \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)| \delta_{e_s^i} + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|\right) \ell = \mathcal{E}_{s,s+t}(\hat{\Pi}, (e_s^i)_{i \geq 1}).$$

The first assertion of Theorem I.4 is proved. We turn to the second assertion. Let  $\hat{\Pi}'$  be another  $\Lambda$  flow of partitions and for every  $s \in \mathbb{R}$ , let  $\chi_s = (\chi_s(i), i \geq 1)$  be an independent sequence of i.i.d. uniform $[0, 1]$  r.v. We assume that for each  $s \in \mathbb{R}$ ,  $\mathbb{P}$ -a.s.

$$(\mathcal{E}_{s,s+t}(\hat{\Pi}, (e_s^i)_{i \geq 1}), t \geq 0) = (\mathcal{E}_{s,s+t}(\hat{\Pi}', \chi_s), t \geq 0) = (\rho_{s,s+t}, t \geq 0).$$

From Proposition I.23, we deduce that  $\mathbb{P}$ -a.s. for every  $i \geq 1$ ,  $\chi_s(i) = e_s^i$ .

Thanks to these almost sure equalities we get  $\mathbb{P}$ -a.s. for every  $i \geq 1$  and all  $s \in \mathbb{Q}$  and  $t \in \mathbb{Q}_+$

$$\rho_{s,s+t}(\{e_s^i\}) = |\hat{\Pi}_{s,s+t}(i)| = |\hat{\Pi}'_{s,s+t}(i)|. \quad (\text{I.19})$$

We now work on an event  $\Omega^*$  of  $\mathbb{P}$ -probability one on which (I.19) holds true and on which both  $\hat{\Pi}, \hat{\Pi}'$  verify the regularity properties of Proposition I.15. We argue deterministically on  $\Omega^*$ . Using Proposition I.15, we deduce that for all  $s \in \mathbb{R}, t \geq 0$  and every  $i \geq 1$ ,  $|\hat{\Pi}_{s,s+t}(i)| = |\hat{\Pi}'_{s,s+t}(i)|$ . To end the proof, it suffices to obtain that for all  $s \in \mathbb{R}$ ,  $\hat{\Pi}_{s-,s} = \hat{\Pi}'_{s-,s}$ . This is a consequence of the next lemma which concludes the proof of Theorem I.4.

**LEMMA I.39.** *Let  $\hat{\Pi}^\times$  denote indifferently either  $\hat{\Pi}$  or  $\hat{\Pi}'$ . Let  $I$  be a subset of  $\mathbb{N}$ . The following assertions are equivalent*

- i)  $\hat{\Pi}_{s-,s}^\times$  has a unique non-singleton block  $I$ .

ii) For every  $i \neq \min I$ , let  $b(i)$  be the unique integer such that  $i = b(i) - (\#\{I \cap [b(i)]\} - 1) \vee 0$ . Then  $(|\hat{\Pi}_{s-,s+t}^\times(i)|, t \geq 0) = (|\hat{\Pi}_{s-,s+t}^\times(b(i))|, t \geq 0)$ .

**Proof** Suppose i). Recall that  $\hat{\Pi}_{s-,t}^\times = \text{Coag}(\hat{\Pi}_{s,t}^\times, \hat{\Pi}_{s-,s}^\times)$ . Let  $i$  be an integer distinct from  $\min I$ . The definition of the coagulation operator implies that for all  $t \geq 0$ ,  $\hat{\Pi}_{s-,s+t}^\times(i) = \hat{\Pi}_{s-,s+t}^\times(b(i))$ . Consequently the identity on the asymptotic frequencies follows.

Suppose ii). Since the trajectories of  $\hat{\Pi}^\times$  are deterministic flows of partitions, we know that  $\hat{\Pi}_{s-,s}^\times$  is a partition with at most one non-singleton block. For  $i \neq \min I$  we stress that the equality  $(|\hat{\Pi}_{s-,s+t}^\times(i)|, t \geq 0) = (|\hat{\Pi}_{s-,s+t}^\times(b(i))|, t \geq 0)$  implies that  $\hat{\Pi}_{s-,s}^\times(i) = \{b(i)\}$ . We first prove that  $b(i) \in \hat{\Pi}_{s-,s}^\times(i)$ , let us denote by  $J$  the latter block. In the *Eves - extinction case* for  $j < j'$  the process  $(|\hat{\Pi}_{s-,s+t}^\times(j)|, t \geq 0)$  reaches 0 strictly after  $(|\hat{\Pi}_{s-,s+t}^\times(j')|, t \geq 0)$  so that the process  $(|\hat{\Pi}_{s-,s+t}^\times(i)|, t \geq 0)$  reaches 0 at the same time as  $(|\hat{\Pi}_{s-,s+t}^\times(\min J)|, t \geq 0)$  reaches 0. Consequently  $\min J = b(i)$ . In the *Eves - persistent case* for every  $j$  the ratio  $|\hat{\Pi}_{s-,s+t}^\times(j)| / \sum_{j' \geq j} |\hat{\Pi}_{s-,s+t}^\times(j')|$  goes to 1 as  $t \rightarrow \infty$ . Consequently we have  $|\hat{\Pi}_{s-,s+t}^\times(i)| \sim |\hat{\Pi}_{s-,s+t}^\times(\min J)|$  as  $t \rightarrow \infty$ . If  $\min J < b(i)$  then  $|\hat{\Pi}_{s-,s+t}^\times(b(i))|$  is negligible compared with  $|\hat{\Pi}_{s-,s+t}^\times(i)|$  as  $t \rightarrow \infty$  while if  $\min J > b(i)$  the converse holds true. Therefore  $\min J = b(i)$ . We now prove that  $J = \{b(i)\}$ . Suppose that there is another element  $k \in J$ . Then we have  $|\hat{\Pi}_{s-,s+t}^\times(i)| \geq |\hat{\Pi}_{s-,s+t}^\times(b(i))| + |\hat{\Pi}_{s-,s+t}^\times(k)|$  for all  $t \geq 0$ . The process  $(|\hat{\Pi}_{s-,s+t}^\times(k)|, t \geq 0)$  cannot be null at all times since otherwise the Eve  $e_s^k$  would not be well-defined. We then deduce that the equality  $|\hat{\Pi}_{s-,s+t}^\times(i)| = |\hat{\Pi}_{s-,s+t}^\times(b(i))|$  is not verified at all times, which is contradictory. Hence  $J = \{b(i)\}$ . We have proved that for every  $i \neq \min I$ ,  $\hat{\Pi}_{s-,s}^\times(i) = \{b(i)\}$ . This suffices to recover completely the partition  $\hat{\Pi}_{s-,s}^\times$  and to assert that  $\hat{\Pi}_{s-,s}^\times(\min I) = I$ .  $\blacksquare$

## 6 Appendix

### 6.1 Proof of Proposition I.15

Recall that  $\mathcal{P}$  is a  $\Lambda$  lockdown graph. Without loss of generality, we can assume that for every  $\omega \in \Omega$ , for all  $s < t$  and all  $n \geq 1$  the collection  $\mathcal{P}_{[s,t] \times \mathcal{S}_n}(\omega)$  has a finite number of points. Since  $\hat{\Pi} = J^{-1}(\mathcal{P})$ , we deduce that for every  $\omega \in \Omega$ ,  $\hat{\Pi}(\omega)$  is a deterministic flow of partitions. The difficulty of the proof lies in the regularity of the asymptotic frequencies.

**LEMMA I.40.**  $\mathbb{P}$ -a.s. for all  $s \in \mathbb{R}$ ,  $\hat{\Pi}_{s-,s}$  admit asymptotic frequencies.

**Proof** Fix  $n \geq 1$ . Consider the set  $\{(s, \hat{\Pi}_{s-,s}) : \hat{\Pi}_{s-,s}^{[n]} \neq 0_{[n]}\}$ . A simple argument shows that this is a Poisson point process on  $\mathbb{R} \times \mathcal{P}_\infty$  with intensity  $dt \times \phi(m_n)$  where  $\phi(m_n)$  is the pushforward of the finite measure

$$m_n := (\mu_\Lambda + \mu_K)(\cdot \cap \mathcal{S}_n)$$

through the map  $\phi : \mathcal{S}_\infty \rightarrow \mathcal{P}_\infty$  that associates to an element  $v \in \mathcal{S}_\infty$  the partition with a unique non-singleton block equal to  $\{i \geq 1 : v(i) = 1\}$ . If we enumerate the points in  $\{(s, \hat{\Pi}_{s-,s}) : \hat{\Pi}_{s-,s}^{[n]} \neq 0_{[n]}\}$  by increasing absolute time coordinate, then we get a collection  $(t_i, \pi_i)_{i \geq 1}$  where  $(\pi_i)_{i \geq 1}$  is a sequence of i.i.d. exchangeable  $\mathcal{P}_\infty$ -valued r.v. with distribution  $\phi(m_n)/m_n(\mathcal{S}_n)$ . Consequently almost surely for every  $i \geq 1$  the partition  $\pi_i$  admits asymptotic frequencies. We deduce that with probability one all the  $\hat{\Pi}_{s-,s}$  such that  $\hat{\Pi}_{s-,s}^{[n]} \neq 0_{[n]}$  admit asymptotic frequencies. Taking a countable union on  $n \geq 1$ , we get the asserted result.

**LEMMA I.41.** With probability one for every rational value  $s$  and every integer  $i \geq 1$  the process  $(|\hat{\Pi}_{s-,s+t}^\times(i)|, t \geq 0)$  are well-defined and càdlàg.

**Proof** This follows as an adaptation of Lemmas 3.4 and 3.5 in [24].

To end the proof, we need to distinguish three cases.

1- CDI. The process  $(\hat{\Pi}_{t-r,t}, r \geq 0)$  starts with infinitely many blocks, and immediately after time 0, comes down from infinity. Note that this property holds a priori on an event of probability one that depends on  $t$ , but the cocycle property allows to assert that the coming down from infinity holds for all  $t$  simultaneously on a same event of probability one. The jumps of  $(\hat{\Pi}_{t-r,t}, r \geq 0)$  are finitely many on any compact interval of  $(0, \infty)$  since the jump rate of a coalescent with a finite number of blocks is finite. Thus  $\mathbb{P}$ -a.s. for all  $s \in \mathbb{R}, t > 0$ , there exists a rational value  $q(\omega) = q \in (s, s+t)$  such that  $\hat{\Pi}_{s,s+t} = \hat{\Pi}_{q,s+t}$  and the existence of asymptotic frequencies follows from the rational case. The limit

$$\lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s+\epsilon, s+t}(i)| = |\hat{\Pi}_{s, s+t}(i)|, \forall i \in \mathbb{N}$$

is then obvious. Similarly, there exists a rational value  $p(\omega) = p < s$  such that  $\hat{\Pi}_{p, s+t} = \hat{\Pi}_{s-, s+t}$ . It implies the existence of asymptotic frequencies for the latter along with the limit

$$\lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s-\epsilon, s+t}(i)| = |\hat{\Pi}_{s-, s+t}(i)|, \forall i \in \mathbb{N}$$

The same kind of arguments apply to show the regularity when  $t$  varies.

Finally the process  $(\sum_{i \geq 1} |\hat{\Pi}_{s, s+t}(i)|, t \geq 0)$  is càdlàg on  $(0, \infty)$  since at any time  $t > 0$  only finitely many  $|\hat{\Pi}_{s, s+t}(i)|$ 's are non-null and since each of these are càdlàg processes in  $t$ .

2- DISCRETE and INTENSIVE w. DUST. On any compact interval of time, only a finite number of reproduction events hit any given level (recall that  $\int_{(0,1)} u \nu(du) < \infty$ ), therefore  $r \mapsto \hat{\Pi}_{t-r,t}(i)$  evolves at discrete times for any given  $i \geq 1$ . The arguments of the previous regime can therefore be applied by considering  $\hat{\Pi}_{s,t}(i)$  instead of  $\hat{\Pi}_{s,t}$  in order to show the regularities in frequency.

Showing that  $(\sum_{i \geq 1} |\hat{\Pi}_{s, s+t}(i)|, t > 0)$  is càdlàg on  $(0, \infty)$  is more involved. This will be a consequence of a uniform bound on the block frequencies. More precisely we introduce for every  $t \geq 0$

$$N_{0,t}(n) := \sum_{s \in (0,t]} \#\{i \in [n] : v_s(i) = 1\}.$$

One can show that the sequence of processes  $(N_{0,t}(n)/n, t \geq 0)$  converges  $\mathbb{P}$ -a.s. to a càdlàg process  $(Y_{0,t}, t \geq 0)$  for the uniform norm on  $\mathbb{D}([0, \infty), \mathbb{R})$ . Then a simple argument based on the transitions of the lookdown process ensures that for every  $i, n \geq 1$  and every  $s, t \geq 0$

$$\frac{1}{n} \left| \sum_{j=1}^i \#\{\hat{\Pi}_{s, s+t+\epsilon}(j) \cap [n]\} - \sum_{j=1}^i \#\{\hat{\Pi}_{s, s+t}(j) \cap [n]\} \right| \leq \frac{N_{0, s+t+\epsilon}(n) - N_{0, s+t}(n)}{n}$$

so that for every  $i \geq 1$

$$\left| \sum_{j=1}^i |\hat{\Pi}_{s, s+t+\epsilon}(j)| - \sum_{j=1}^i |\hat{\Pi}_{s, s+t}(j)| \right| \leq Y_{0, s+t+\epsilon} - Y_{0, s+t}.$$

Taking the limit as  $i \rightarrow \infty$ , one gets the asserted right continuity of  $(\sum_{i \geq 1} |\hat{\Pi}_{s, s+t}(i)|, t > 0)$ . A similar argument yields the existence of left limits.

3- INTENSIVE  $\infty$ . In this regime, all the partitions have infinitely many blocks and no singleton. Arguing like in Lemma I.41, we can show that on a same event of probability one for every  $s \in \mathbb{Q}$  the process  $(\sum_{i \geq 1} |\hat{\Pi}_{s, s+t}(i)|, t > 0)$  is constant equal to 1. We now prove the existence of asymptotic frequencies

for  $\hat{\Pi}_{s,s+t}$  when  $s$  is not rational. Fix  $i \geq 1$  and a rational value  $p \in (s, s+t) \cap \mathbb{Q}$ . For every  $n \geq i$ , we set

$$\eta(n) := 1 - \sum_{l=1}^n |\hat{\Pi}_{p,s+t}(l)|$$

From the property proved above  $\eta(n) \rightarrow 0$  as  $n \rightarrow \infty$ . Let us denote by  $j_1, j_2, \dots$  the elements of the block  $\hat{\Pi}_{s,p}(i)$ . From the cocycle property, we have:

$$\sum_{j_i \leq n} |\hat{\Pi}_{p,s+t}(j_i)| \leq \liminf_{m \rightarrow \infty} \frac{\#\{\hat{\Pi}_{s,s+t}(i) \cap [m]\}}{m} \leq \liminf_{m \rightarrow \infty} \frac{\#\{\hat{\Pi}_{s,s+t}(i) \cap [m]\}}{m} \leq \sum_{j_i \leq n} |\hat{\Pi}_{p,s+t}(j_i)| + \eta(n) \quad (\text{I.20})$$

Letting  $n$  go to infinity ensures the existence of  $|\hat{\Pi}_{s,s+t}(i)|$ . The same reasoning applies to  $\hat{\Pi}_{s-,s+t}$  and  $\hat{\Pi}_{s,s+t-}$ . We now prove the regularity properties. Since  $p$  is rational, we know that there exists  $\epsilon_0(\omega) = \epsilon_0 > 0$  small enough so that

$$||\hat{\Pi}_{p,s+t+\epsilon}(j)| - |\hat{\Pi}_{p,s+t}(j)|| \leq \frac{\eta(n)}{n}, \quad \forall \epsilon < \epsilon_0, \forall j \leq n$$

Therefore

$$1 - \sum_{l=1}^n |\hat{\Pi}_{p,s+t+\epsilon}(l)| \leq 2\eta(n), \quad \forall \epsilon < \epsilon_0$$

Combined with Equation (I.20), this ensures the convergence of  $|\hat{\Pi}_{s,s+t+\epsilon}(i)|$  towards  $|\hat{\Pi}_{s,s+t}(i)|$  when  $\epsilon$  goes to 0. We get similarly the convergence when  $t - \epsilon$  goes to  $t-$ . To prove the convergences when  $s - \epsilon$  goes to  $s-$  and  $s + \epsilon$  goes to  $s$ , one remarks that there exists  $\epsilon_0(\omega) = \epsilon_0 > 0$  such that no reproduction events affecting at least two levels among  $[n]$  fall in  $(s - \epsilon_0, s)$  nor in  $(s, s + \epsilon_0)$ . Hence  $\hat{\Pi}_{s-\epsilon,p}(i) \cap [n]$  and  $\hat{\Pi}_{s+\epsilon,p}(i) \cap [n]$  do not vary whenever  $\epsilon$  is in  $(0, \epsilon_0)$ . Similar arguments as above apply.

To end the proof, we need to check that  $(\sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)|, t > 0)$  is constant equal to 1 for all  $s \in \mathbb{R}$ . We have checked it for rational values  $s$ . Fix  $s \in \mathbb{R}$  and  $t > 0$ . Take an arbitrary  $p \in (s, s+t) \cap \mathbb{Q}$ . From the cocycle property and Fatou lemma, we know that for all  $i \geq 1$  we have

$$|\hat{\Pi}_{s,s+t}(i)| \geq \sum_{j \in \hat{\Pi}_{s,p}(i)} |\hat{\Pi}_{p,s+t}(j)|$$

so that

$$\sum_{i \geq 1} |\hat{\Pi}_{s,s+t}(i)| \geq \sum_{i \geq 1} \sum_{j \in \hat{\Pi}_{s,p}(i)} |\hat{\Pi}_{p,s+t}(j)| = \sum_{j \geq 1} |\hat{\Pi}_{p,s+t}(j)| = 1.$$

This concludes the proof. ■

## 6.2 Proof of Proposition I.19

The proof of the proposition relies on three lemmas.

**LEMMA I.42.** *Consider the restriction  $(\hat{\Pi}_{s,t}, s \leq t \in \mathbb{Q})$  of the flow of partitions to its rational marginals. Then there exists an event  $\Omega_{\hat{\Pi}}$  of probability 1 on which:*

- $\forall r < s < t \in \mathbb{Q}, \hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$
- *For every  $n \geq 1$ , every  $a < b$  there exist  $m_0 \geq 1$  and  $k \geq 0$  such that for all  $m \geq m_0$ , every partition  $\hat{\Pi}_{a+(i-1)2^{-m}(b-a), a+i2^{-m}(b-a)}^{[n]}$  has at most one non-singleton block and  $\#\{i \in [2^m] : \hat{\Pi}_{a+(i-1)2^{-m}(b-a), a+i2^{-m}(b-a)}^{[n]} \neq 0_{[m]}\}$  equals  $k$ .*

This lemma implies that almost surely the trajectories of  $(\hat{\Pi}_{s,t}, s \leq t \in \mathbb{Q})$  are deterministic flows of partitions.

**Proof** First, one has

$$\mathbb{P}\left(\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s}), \forall r < s < t \in \mathbb{Q}^3\right) = 1 \quad (\text{I.21})$$

Fix  $a < b$ . The second assertion is trivially verified for a stochastic flow of partitions  $\hat{\Pi}^{\mathcal{P}}$  defined from a  $\Lambda$  lookdown graph  $\mathcal{P}$ . Using Equation (I.21) and the fact that the finite-dimensional distributions of  $\hat{\Pi}$  and  $\hat{\Pi}^{\mathcal{P}}$  are equal, the assertion for these values of  $a, b$  follow. Taking a countable union on  $a < b \in \mathbb{Q}$  the asserted result follows.  $\blacksquare$

We now define for every  $s \leq t \in \mathbb{R}$  the partition  $\tilde{\Pi}_{s,t}$  on the event  $\Omega_{\hat{\Pi}}$  as follows.

**LEMMA I.43.** *On the event  $\Omega_{\hat{\Pi}}$ , the following random partition is well-defined.*

$$\tilde{\Pi}_{s,t} := \begin{cases} \hat{\Pi}_{s,t} & \text{if } s, t \in \mathbb{Q}, s < t \\ 0_{[\infty]} & \text{if } s = t \\ \lim_{v \downarrow t, v \in \mathbb{Q}} \hat{\Pi}_{s,v} & \text{if } s \in \mathbb{Q}, t \notin \mathbb{Q} \\ \lim_{r \downarrow s, r \in \mathbb{Q}} \hat{\Pi}_{r,t} & \text{if } s \notin \mathbb{Q}, t \in \mathbb{Q} \\ \text{Coag}(\tilde{\Pi}_{q,t}, \tilde{\Pi}_{s,q}) & \text{for any arbitrary rational } q \in (s, t) \text{ if } s, t \notin \mathbb{Q} \end{cases} \quad (\text{I.22})$$

Furthermore, for every  $r < s < t$ ,  $\tilde{\Pi}_{r,t} = \text{Coag}(\tilde{\Pi}_{s,t}, \tilde{\Pi}_{r,s})$ .

**Proof** We work on the event  $\Omega_{\hat{\Pi}}$  throughout this proof. Recall the cocycle property together with the left and right regularity properties verified by the flow restricted to its rational marginals. Fix  $s \in \mathbb{Q}$  and let us prove the existence of a limit for  $\tilde{\Pi}_{s,v}$  when  $v$  is rational and goes to a given irrational value  $t \in (s, \infty)$ . Fix  $n \geq 1$ . There exists  $\epsilon = \epsilon(\omega) > 0$  such that for all rational values  $p, q$  in  $(t, t + \epsilon)$ ,  $\hat{\Pi}_{p,q}^{[n]} = 0_{[n]}$ . Combined with the cocycle property on rational marginals, this ensures that  $v \mapsto \hat{\Pi}_{s,v}^{[n]}$  is constant whenever  $v \in (t, t + \epsilon) \cap \mathbb{Q}$ . The existence of the limit follows. A similar argument shows the existence of a limit for  $\tilde{\Pi}_{v,t}$  when  $v$  is rational and goes to an irrational value  $s$  and  $t$  is a given rational value.

Fix  $r < s < t$ . If all three are rational, the corresponding cocycle property holds since we are on  $\Omega_{\hat{\Pi}}$ . Now suppose that either  $s$  is rational or both  $r$  and  $t$  are rational, then we stress that the corresponding cocycle property still holds. Indeed, take a limiting sequence of rational values for which the cocycle property holds and then use the continuity of the coagulation operator (see Lemma 4.2 in Bertoin [9]).

Finally, suppose that  $s, t \notin \mathbb{Q}$ . To verify that our definition of  $\tilde{\Pi}_{s,t}$  makes sense, we need to show that  $\text{Coag}(\tilde{\Pi}_{q,t}, \tilde{\Pi}_{s,q})$  does not depend on the value  $q \in (s, t) \cap \mathbb{Q}$ . Consider two such values  $q, q' \in (s, t) \cap \mathbb{Q}$ , suppose that  $q < q'$  and use the associativity of the coagulation operator (see Lemma 4.2 in [9]) to obtain

$$\begin{aligned} \text{Coag}(\tilde{\Pi}_{q',t}, \tilde{\Pi}_{s,q'}) &= \text{Coag}(\tilde{\Pi}_{q',t}, \text{Coag}(\tilde{\Pi}_{q,q'}, \tilde{\Pi}_{s,q})) \\ &= \text{Coag}(\text{Coag}(\tilde{\Pi}_{q',t}, \tilde{\Pi}_{q,q'}), \tilde{\Pi}_{s,q}) = \text{Coag}(\tilde{\Pi}_{q,t}, \tilde{\Pi}_{s,q}) \end{aligned}$$

Thus, the definition of  $\tilde{\Pi}_{s,t}$  does not depend on  $q \in (s, t)$ .

Finally, consider three irrational  $r < s < t$ , and two rational values  $q, q'$  such that  $q \in (r, s)$  and  $q' \in (s, t)$ .

$$\begin{aligned} \text{Coag}(\tilde{\Pi}_{s,t}, \tilde{\Pi}_{r,s}) &= \text{Coag}(\text{Coag}(\tilde{\Pi}_{q',t}, \tilde{\Pi}_{s,q'}), \text{Coag}(\tilde{\Pi}_{q,s}, \tilde{\Pi}_{r,q})) \\ &= \text{Coag}(\tilde{\Pi}_{q',t}, \text{Coag}(\tilde{\Pi}_{s,q'}, \text{Coag}(\tilde{\Pi}_{q,s}, \tilde{\Pi}_{r,q}))) = \tilde{\Pi}_{r,t} \end{aligned}$$

This concludes the proof. ■

On the complement of  $\Omega_{\hat{\Pi}}$ , set any arbitrary value to  $\tilde{\Pi}_{s,t}$ .

**LEMMA I.44.** *The collection of partitions  $\tilde{\Pi}$  is a modification of  $\hat{\Pi}$ , that is, for every  $s \leq t$ , a.s.  $\tilde{\Pi}_{s,t} = \hat{\Pi}_{s,t}$ . Furthermore, for each  $\omega \in \Omega_{\hat{\Pi}}$ ,  $\tilde{\Pi}(\omega)$  is a deterministic flow of partitions.*

**Proof** Fix  $s \in \mathbb{Q}$ . For every  $t \in \mathbb{Q}_+$ ,  $\tilde{\Pi}_{s,s+t} = \hat{\Pi}_{s,s+t}$  on the event  $\Omega_{\hat{\Pi}}$ , so it holds a.s. We know that the process  $(\hat{\Pi}_{s,s+t}, t \geq 0)$  admits a càdlàg modification from Lemma I.25. Almost surely for all  $t \in \mathbb{Q}$  the  $t$ -marginals of this modification coincide with  $\hat{\Pi}_{s,s+t}$  and also with  $\tilde{\Pi}_{s,s+t}$ . Since the process  $(\tilde{\Pi}_{s,t}, t \in [s, \infty))$  is also càdlàg, we deduce that almost surely it coincides with the modification. Thus it is itself a modification of  $(\hat{\Pi}_{s,s+t}, t \geq 0)$ . Consequently for all  $s \in \mathbb{Q}$  and all  $t \geq 0$  we have almost surely  $\tilde{\Pi}_{s,s+t} = \hat{\Pi}_{s,s+t}$ .

Now suppose  $s$  irrational, take  $t > 0$  and fix  $n \in \mathbb{N}$ . We have for all  $q \in (s, s+t) \cap \mathbb{Q}$

$$\mathbb{P}\left(\tilde{\Pi}_{s,s+t}^{[n]} = \hat{\Pi}_{s,s+t}^{[n]}\right) \geq \mathbb{P}\left(\tilde{\Pi}_{q,s+t}^{[n]} = \hat{\Pi}_{q,s+t}^{[n]}; \hat{\Pi}_{s,s+t}^{[n]} = \text{Coag}(\hat{\Pi}_{q,s+t}^{[n]}, \hat{\Pi}_{s,q}^{[n]}); \hat{\Pi}_{s,q}^{[n]} = 0_{[n]}\right)$$

As  $q \downarrow s$ ,  $\mathbb{P}(\hat{\Pi}_{s,q}^{[n]} = 0_{[n]}) \rightarrow 1$  by definition of a stochastic flow of partitions. The cocycle property of a stochastic flow of partitions together with the almost sure identity  $\tilde{\Pi}_{q,s+t} = \hat{\Pi}_{q,s+t}$  that we have already proved, ensures that the probability of the event on the r.h.s. tends to 1 as  $q \downarrow s$ . Thus  $\tilde{\Pi}_{s,s+t} = \hat{\Pi}_{s,s+t}$  almost surely. Finally, when  $t = 0$  we know that  $\tilde{\Pi}_{s,s} = 0_{[\infty]}$  almost surely by definition. Therefore,  $\tilde{\Pi}$  is a modification of  $\hat{\Pi}$ .

We need to verify that for all  $\omega \in \Omega_{\hat{\Pi}}$ ,  $\tilde{\Pi}(\omega)$  is a deterministic flow of partitions. The cocycle property was proved in the preceding lemma. Let us show the right regularity. Fix  $s \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Recall that there exists  $\epsilon = \epsilon(\omega) > 0$  such that for all rational  $p < q \in (s, s + \epsilon)$ ,  $\tilde{\Pi}_{p,q}^{[n]} = 0_{[n]}$ . Letting  $p \downarrow s$ , we get  $\tilde{\Pi}_{s,q}^{[n]} = 0_{[n]}$  for all  $q \in (s, s + \epsilon) \cap \mathbb{Q}$ . Similarly for all  $r \in (s, s + \epsilon)$ , we have  $\tilde{\Pi}_{q,r}^{[n]} = 0_{[n]}$  as soon as  $q \in (s, r)$ . Using the fact that  $\tilde{\Pi}_{s,r}^{[n]} = \text{Coag}(\tilde{\Pi}_{q,r}^{[n]}, \tilde{\Pi}_{s,q}^{[n]})$  we get that  $\tilde{\Pi}_{s,r}^{[n]} = 0_{[n]}$  for all  $r \in (s, s + \epsilon)$ . This in turn implies that  $\tilde{\Pi}_{s,r}^{[n]} \rightarrow \tilde{\Pi}_{s,s}^{[n]}$  as  $r \downarrow s$  and the right regularity is proved. The left regularity is obtained similarly. ■

### 6.3 Calculations on regular variation

We fix  $\alpha \in (1, 2)$  and assume that  $\Lambda(du) = f(u)du$  with  $f(u) = u^{1-\alpha}L(u)$  for all  $u \in (0, 1)$  and  $L$  is slowly varying at  $0+$  (see [56]). Fix  $\lambda \in (0, \infty)$ . For all  $u > 0$  we have

$$\frac{\Psi(\lambda u)}{\lambda^\alpha} = \Psi(u) + \int_0^\lambda (e^{-xu} - 1 + xu) x^{-1-\alpha} \left( L\left(\frac{x}{\lambda}\right) - L(x) \right) dx - \int_\lambda^1 (e^{-xu} - 1 + xu) x^{-1-\alpha} L(x) dx$$

Since  $L$  is slowly varying at  $0+$ , the ratio  $L(x/\lambda)/L(x)$  goes to 1 as  $x$  tends to  $0+$ . This, together with the fact that  $\Psi(u)/u \rightarrow \infty$  as  $u \rightarrow \infty$ , ensures that

$$\Psi(\lambda u) \underset{u \rightarrow \infty}{\sim} \lambda^\alpha \Psi(u)$$

Let us now prove that the map  $v$  defined in the proof of Theorem I.2 is itself regularly varying at  $0+$  with index  $-1/(\alpha - 1)$ . We have

$$\frac{v(t)}{\Psi(v(t))} = \int_0^t \frac{\Psi(v(s)) - v(s) \Psi'(v(s))}{(\Psi(v(s)))^2} v'(s) ds \underset{t \rightarrow 0}{\sim} (\alpha - 1)t$$

In the above calculations, we use the identity  $v'(s) = -\Psi(v(s))$  and the regular variation of  $\Psi$  at  $\infty$  that ensures the convergence  $u\Psi'(u)/\Psi(u) \rightarrow \alpha$  as  $u \rightarrow \infty$  (see Theorem 2 in [56]). Therefore we have proved that

$$\frac{t v'(t)}{v(t)} \xrightarrow{t \rightarrow 0} \frac{-1}{\alpha - 1}$$

This identity implies, thanks to Theorem 2 in [56], that  $v$  is regularly varying at  $0+$  with index  $-1/(\alpha - 1)$ .

Assume now that  $\Lambda(du) = c \delta_0(du) + f(u)du$  where  $c > 0$  and  $f$  is any positive and measurable function such that  $\int_{(0,1)} f(u)du < \infty$ . It is simple to show that  $\Psi(u) \sim c u^2/2$  as  $u \rightarrow \infty$ . The calculations on  $v$  above then applies with  $\alpha = 2$ . We deduce that  $v$  is regularly varying at  $0+$  with index  $-1$ . ■





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Genealogy of flows of continuous-state branching processes via flows  
of partitions and the Eve property

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*This article [53] has been accepted for publication in the Annales de l'Institut Henri Poincaré.*

## 1 Introduction

A continuous-state branching process (CSBP for short) is a Markov process  $(Z_t, t \geq 0)$  that takes its values in  $[0, +\infty]$  and fulfils the branching property: for all  $z, z' \in [0, +\infty]$ ,  $(Z_t + Z'_t, t \geq 0)$  is a CSBP, where  $(Z_t, t \geq 0)$  and  $(Z'_t, t \geq 0)$  are two independent copies started from  $z$  and  $z'$  respectively. Such a process describes the evolution of an initial population size  $Z_0$ , and the branching property implies that two disjoint subpopulations have independent evolutions. To alleviate notation, we will implicitly consider an initial population size  $Z_0 = 1$ . A CSBP has a Feller semigroup entirely characterized by a convex function  $\Psi$  called its branching mechanism, so we will write  $\Psi$ -CSBP to designate the corresponding distribution. The Feller property entails the existence of a càdlàg modification, still denoted  $(Z_t, t \geq 0)$  and thus allows to define the lifetime of  $Z$  as the stopping time

$$T := \inf\{t \geq 0 : Z_t \notin (0, \infty)\}$$

with the convention  $\inf \emptyset = \infty$ . The denomination *lifetime* is due to the simple fact that both 0 and  $\infty$  are absorbing states.

The process  $Z$  can be seen as the total-mass of a measure-valued process  $(m_t, t \in [0, T))$  on  $[0, 1]$  (or any compact interval), started from the Lebesgue measure on  $[0, 1]$  and such that for all  $x \in [0, 1]$ ,  $(m_t([0, x]), t \geq 0)$  and  $(m_t((x, 1]), t \geq 0)$  are two independent  $\Psi$ -CSBP corresponding to the sizes of the subpopulations started from  $[0, x]$  and  $(x, 1]$  respectively. The process  $m$  is called a measure-valued branching process or  $\Psi$ -MVBP for short. Note that when  $Z_T = \infty$ , the measure is no longer finite and therefore we set  $m_T = \Delta$ , see Subsection 2.3 for further details.

**DEFINITION II.1.** *We say that the branching mechanism  $\Psi$  satisfies the Eve property if and only if there exists a random variable  $e$  in  $[0, 1]$  such that*

$$\frac{m_t(dx)}{m_t([0, 1])} \xrightarrow[t \uparrow T]{} \delta_e(dx) \text{ a.s.} \tag{II.1}$$

in the sense of weak convergence of probability measures. The r.v.  $e$  is called the primitive Eve of the population.

This property means that a fraction asymptotically equal to 1 of the population descends from a single individual located at  $e$  as  $t$  gets close to the lifetime  $T$ . This property seems to have never been studied before, except by Tribe [75] in the case of the Feller diffusion with a spatial motion. From the branching property, we will show that  $e$  is necessarily uniform $[0, 1]$ , when the Eve property is verified. The goal of the present paper is to study this Eve property in connection with the genealogy of the  $\Psi$ -CSBP. Note that the complete classification of the asymptotic behaviour of  $\frac{m_t(dx)}{m_t([0,1])}$  will be provided in a forthcoming work [25].

A CSBP describes the evolution of the population size, but does not provide clear information on the genealogy. In recent years, several approaches have been proposed to study the genealogical structure: one can cite the historical superprocess of Dawson and Perkins [22], the continuum random tree introduced by Aldous in [2], the Lévy trees of Le Gall, Le Jan and Duquesne [26, 61], we also refer to [13, 38, 39] for the genealogy of related population models. This paper presents a new object, called a stochastic flow of partitions associated with a branching mechanism, that unifies two well-known approaches: the flow of subordinators of Bertoin and Le Gall [12] and the lookdown representation of Donnelly and Kurtz [24]. Let us mention that this object focuses on the genealogical structure, and does not pay attention to the genetic types carried by the individuals: hence it would not be appropriate to deal with more elaborate models incorporating mutations or spatial motions. We first introduce this object and its relationships with these two representations, before presenting the connection with the Eve property.

As mentioned above, the population size does not define in itself the genealogy. Therefore we start from a càdlàg  $\Psi$ -CSBP  $(Z_t, t \in [0, T])$  and enlarge the probability space in order to add more information to this process. This is achieved by defining a random point process  $\mathcal{P}$  with values in  $[0, T) \times \mathcal{P}_\infty$ , where  $\mathcal{P}_\infty$  stands for the space of partitions of the integers  $\mathbb{N}$ . To each jump  $(t, \Delta Z_t)$  of the CSBP is associated a point  $(t, \varrho_t)$  in  $\mathcal{P}$  such that the random partition  $\varrho_t$  is distributed according to the paint-box scheme with mass-partition  $(\frac{\Delta Z_t}{Z_t}, 0, 0, \dots)$ , see Subsection 2.1 for a precise definition of the paint-box scheme. The genealogical interpretation is the following:  $(t, \Delta Z_t)$  corresponds to a reproduction event where a parent, chosen uniformly among the population alive at time  $t-$ , gives birth to a subpopulation of size  $\Delta Z_t$ ; therefore a fraction  $\frac{\Delta Z_t}{Z_t}$  of the individuals at time  $t$  descends from this parent. In addition, when  $Z$  has a diffusion part,  $\mathcal{P}$  contains points of the form  $(t, \mathcal{I}_{\{i,j\}})$  that model binary reproduction events, that is, events where an individual  $i$  at time  $t-$  is the parent of two individuals  $i$  and  $j$  at time  $t$ , the partition  $\mathcal{I}_{\{i,j\}}$  having a unique non-singleton block  $\{i, j\}$ , with  $i < j$ . A precise definition of the point process  $\mathcal{P}$  will be given in Subsection 3.2, but it should be seen as an object that collects all the *elementary reproduction events* as time passes.

We then introduce a collection of random partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  by "composing" the partitions contained in  $\mathcal{P}$ . In order not to burden this introduction, we do not provide the precise definition of these partitions but, roughly speaking,  $\hat{\Pi}_{s,t}$  is the result of the composition forward-in-time of all the elementary reproduction events provided by  $\mathcal{P}$  on the interval  $(s, t]$ . Therefore the partitions collect the following information

- Backward-in-time : the process  $s \mapsto \hat{\Pi}_{t-s,t}$  gives the genealogy of the population alive at time  $t$ .
- Forward-in-time : the process  $s \mapsto \hat{\Pi}_{t,t+s}$  gives the descendants of the population alive at time  $t$ .

$(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  is called a  $\Psi$  flow of partitions and  $Z$  its *underlying  $\Psi$ -CSBP*.

*Connection with the lockdown representation*

This object is intimately related to the lockdown representation of Donnelly and Kurtz [24]. A lockdown process is a particle system entirely characterized by a sequence of *initial types*, that provides a sampling of the initial population, and a so-called *lookdown graph*, that stands for the genealogical structure. In a previous work [52], we showed that the flow of partitions formalizes and clarifies the notion of lookdown graph which was implicit in the lockdown construction of Donnelly and Kurtz [24]. To complete the picture of the lockdown construction, note that the limiting empirical measure of the particle system at time  $t$ , say  $\bar{\Xi}_t$ , is a probability measure such that the process  $Z \cdot \bar{\Xi}$  is a  $\Psi$ -MVBP, see Section 3 for further details.

*Connection with the flow of subordinators*

It is well-known that the process  $x \mapsto m_t([0, x])$  is a subordinator whose Laplace exponent  $u_t(\cdot)$  is related to the branching mechanism  $\Psi$  via forthcoming Equation (II.7). In addition, the branching property ensures that  $m_{t+s}$  is obtained by composing the subordinator  $m_t$  with an independent subordinator distributed as  $m_s$ . This is the key observation that allowed Bertoin and Le Gall [12] to describe the genealogy of the  $\Psi$ -MVBP with a collection of subordinators. Formally, a  $\Psi$  flow of subordinators  $(S_{s,t}(a), 0 \leq s \leq t, a \geq 0)$  is a collection of random processes that verify

- For every  $0 \leq s \leq t$ ,  $(S_{s,t}(a), a \geq 0)$  is a subordinator with Laplace exponent  $u_{t-s}$ .
- For every integer  $p \geq 2$  and  $0 \leq t_1 \leq \dots \leq t_p$ , the subordinators  $S_{t_1, t_2}, \dots, S_{t_{p-1}, t_p}$  are independent and

$$S_{t_1, t_p}(a) = S_{t_{p-1}, t_p} \circ \dots \circ S_{t_1, t_2}(a), \quad \forall a \geq 0 \text{ a.s. (cocycle property)}$$

- For all  $a \geq 0$ ,  $(S_{0,t}(a), t \geq 0)$  is a  $\Psi$ -CSBP started from  $a$ .

Each subordinator  $[0, 1] \ni x \mapsto S_{0,t}(x)$  can be seen as the distribution function of a random measure  $m_{0,t}$  on  $[0, 1]$  so that  $(m_{0,t}, t \geq 0)$  forms a  $\Psi$ -MVBP. In particular,  $S_t := S_{0,t}(1)$  is its total-mass process and one can define  $T^S$  as its lifetime. Hence, all the relevant information about this initial population  $[0, 1]$  is contained into the flow  $(S_{s,t}(a), 0 \leq s \leq t, 0 \leq a \leq S_s)$ . Fix  $0 \leq s < t < T^S$  and consider a point  $a \in [0, S_s]$  such that  $S_{s,t}(a) - S_{s,t}(a-) > 0$ . Bertoin and Le Gall interpreted  $a$  as an ancestor alive at time  $s$  and  $S_{s,t}(a) - S_{s,t}(a-)$  as its progeny alive at time  $t$ . We show that our collection of partitions actually formalizes this genealogical structure. To state this result we use the notation  $\mathcal{P}(S_{s,t})$  that stands for the paint-box distribution based on the mass-partition obtained from the rescaled jumps  $\{\frac{S_{s,t}(a) - S_{s,t}(a-)}{S_t}, a \in [0, S_s]\}$  of the subordinator  $S_{s,t}$ , here again we refer to Subsection 2.1 for a precise definition.

**THEOREM II.1.** *The collection of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ , together with its underlying CSBP  $Z$ , satisfies*

- For all  $n \geq 1$  and all  $0 < t_1 < \dots < t_n$ ,

$$(Z_{t_1}, \dots, Z_{t_n}, \hat{\Pi}_{0, t_1}, \dots, \hat{\Pi}_{t_{n-1}, t_n} | t_n < T) \stackrel{(d)}{=} (S_{t_1}, \dots, S_{t_n}, \mathcal{P}(S_{0, t_1}), \dots, \mathcal{P}(S_{t_{n-1}, t_n}) | t_n < T^S)$$

- For all  $0 \leq r < s < t < T$ , a.s.  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$  (cocycle property).

Note that the operator  $\text{Coag}$  is a composition operator for partitions, see Section 4.2 in [9] or Subsection 2.1 of the present paper.

*Main results*

We now study the connection between the Eve property and the genealogy. To alleviate notation, we set  $\hat{\Pi}_t := \hat{\Pi}_{0,t}$  and let  $\sigma$  be the diffusion coefficient appearing in the branching mechanism  $\Psi$ . In addition, we let  $\mathcal{I}_{[\infty]} := \{\{1, 2, 3, \dots\}\}$  denote the partition with a unique block containing all the integers.

**THEOREM II.2.** *There exists an exchangeable partition  $\hat{\Pi}_T$  such that  $\hat{\Pi}_t \rightarrow \hat{\Pi}_T$  almost surely as  $t \uparrow T$ . Moreover, these three assumptions are equivalent*

i)  $\Psi$  satisfies the Eve property.

ii)  $\hat{\Pi}_T = \mathcal{I}_{[\infty]}$  a.s.

iii) 
$$\sum_{\{s < T: \Delta Z_s > 0\}} \left( \frac{\Delta Z_s}{Z_s} \right)^2 + \int_0^T \frac{\sigma^2}{Z_s} ds = \infty \text{ a.s.}$$

This result allows to define  $\hat{\Pi}_t := \hat{\Pi}_T$  for all  $t \geq T$ .

If there are individuals who do not share their ancestors with any other individuals then the partition has singleton blocks: we say that the partition has dust. It is well-known that for coalescent processes with multiple collisions, a dichotomy occurs (except in a very trivial case) between those coalescent processes that have infinitely many singletons at every time  $t > 0$  almost surely and those that have no singletons at every time  $t > 0$  almost surely, see [66]. It is striking that a similar dichotomy holds in the branching process setting.

**THEOREM II.3.** *The following dichotomy holds:*

- If  $\Psi$  is the Laplace exponent of a Lévy process with finite variation paths, then almost surely for all  $t \in (0, T)$ , the partition  $\hat{\Pi}_t$  has singleton blocks.
- Otherwise, almost surely for all  $t \in (0, T)$ , the partition  $\hat{\Pi}_t$  has no singleton blocks.

Furthermore when  $\sigma = 0$ , almost surely for all  $t \in (0, T]$  the asymptotic frequency of the dust component of  $\hat{\Pi}_t$  is equal to  $\prod_{s \leq t} (1 - \frac{\Delta Z_s}{Z_s})$  whereas when  $\sigma > 0$ , almost surely for all  $t \in (0, T]$  there is no dust.

A flow of partitions also appears as an efficient tool to deal with convergences. We illustrate this fact with the following problem. Consider a sequence of branching mechanisms  $(\Psi_m)_{m \geq 1}$  that converges pointwise to another branching mechanism  $\Psi$ . Implicitly,  $Z^m, \hat{\Pi}^m$  will denote  $\Psi_m$ -CSBP and  $\Psi_m$  flow of partitions, for every  $m \geq 1$ . It is easy to deduce from [20] that  $Z^m \rightarrow Z$  in a sense that will be made precise in Subsection 5.2, so that a similar result for the corresponding genealogies is expected.

**THEOREM II.4.** *Suppose that*

i) For all  $u \in \mathbb{R}_+$ ,  $\Psi_m(u) \rightarrow \Psi(u)$  as  $m \rightarrow \infty$ .

ii) The branching mechanism  $\Psi$  satisfies the Eve property.

iii)  $\Psi$  is not the Laplace exponent of a compound Poisson process.

then

$$(\hat{\Pi}_t^m, t \geq 0) \xrightarrow[m \rightarrow \infty]{(d)} (\hat{\Pi}_t, t \geq 0)$$

in  $\mathbb{D}(\mathbb{R}_+, \mathcal{P}_\infty)$ .

The Eve property says that the rescaled  $\Psi$ -MVBP can be approximated by a Dirac mass as  $t$  gets close to  $T$ . It is natural to ask if finer results can be obtained: for instance, does there exist a second Eve that carries a significant part of the remaining population ?

We call *ancestor* a point  $x \in [0, 1]$  for which there exists  $t \in [0, T)$  such that  $m_t(\{x\}) > 0$ ; in that case,  $m_t(\{x\})$  is called the *progeny* of  $x$  at time  $t$ . We will prove in Subsection 4.2 that the collection of ancestors is countable. Roughly speaking, the progeny of a given ancestor is a  $\Psi$ -CSBP started from 0. Therefore, one can naturally compare two ancestors: either by persistence, i.e. according to the extinction times of their progenies (if they become extinct in finite time); or by predominance, i.e. according to the asymptotic behaviours of their progenies (if their lifetimes are infinite). Notice that these two notions (persistence/predominance) are mutually exclusive.

**THEOREM II.5.** *Assume that  $Z$  does not reach  $\infty$  in finite time. If the Eve property holds then one can order the ancestors by persistence/predominance. We denote this ordering  $(e^i)_{i \geq 1}$  and call these points the Eves. In particular,  $e^1$  is the primitive Eve.*

The Eves enjoy several nice properties. For instance, Proposition II.26 shows that the sequence  $(e^i)_{i \geq 1}$  is i.i.d. uniform $[0, 1]$ . Also, the Eves will be of major importance in the last part of this work we now present.

Theorem II.1 shows that flows of subordinators and flows of partitions are related by their finite-dimensional marginals. One could wonder if the connection is deeper: does there exist a flow of partitions embedded into a flow of subordinators ? It turns out that the Eve property plays a crucial rôle in this topic.

Consider a  $\Psi$  flow of subordinators  $(S_{s,t}(a), 0 \leq s \leq t < T^S, 0 \leq a \leq S_s)$ , and for simplicity let  $Z_s := S_s$  denote the total population size and  $T := T^S$  its lifetime. For all  $s \leq t$ , the subordinator  $S_{s,t}$  defines a random measure  $m_{s,t}$  on  $[0, Z_s]$  with total mass  $Z_t$ . Assume that  $Z$  does not reach  $\infty$  in finite time and that the Eve property is verified. Theorem II.5 allows to introduce the *Eves process*  $(e_s^i, s \in [0, T])_{i \geq 1}$  by considering at each time  $s \in [0, T)$ , the sequence of Eves of the  $\Psi$ -MVBP  $(m_{s,t}, t \in [s, T])$  that starts from the Lebesgue measure on  $[0, Z_s]$ . Notice that we actually rescale the Eves  $(e_s^i)_{i \geq 1}$  by the mass  $Z_s$  in order to obtain r.v. in  $[0, 1]$ .

The Eves process is the set of individuals that play a significant rôle in the population as time passes. One is naturally interested in the genealogical relationships between these Eves, so we introduce a collection of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  by setting

$$\hat{\Pi}_{s,t}(i) := \{j \in \mathbb{N} : e_t^j \text{ descends from } e_s^i\}$$

Here " $e_t^j$  descends from  $e_s^i$ " means that  $Z_t \cdot e_t^j \in (S_{s,t}(Z_s \cdot e_s^i -), S_{s,t}(Z_s \cdot e_s^i)]$ .

**THEOREM II.6.** *The collection of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  defined from the flow of subordinators and the Eves process is a  $\Psi$  flow of partitions.*

We end with a decomposition result similar to the main theorem of [52]. For each time  $s \in [0, T)$ , let  $\mathcal{E}_s(\hat{\Pi}, (e_s^i)_{i \geq 1})$  be the measure-valued process defined by

$$[s, T) \ni t \mapsto \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)| \delta_{e_s^i}(dx) + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)|\right) dx$$

and  $r_{s,t}$  the probability measure on  $[0, 1]$  defined by

$$r_{s,t}(dx) := \frac{m_{s,t}(Z_s \cdot dx)}{Z_t}$$

**THEOREM II.7.** *The flow of subordinators can be uniquely decomposed into two random objects: the Eves process  $(e_s^i, s \in [0, T])$  and the flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ .*

i) **Decomposition.** *For each  $s \in \mathbb{R}$ , a.s.  $\mathcal{E}_s(\hat{\Pi}, (e_s^i)_{i \geq 1}) = (r_{s,t}, t \in [s, T])$*

ii) **Uniqueness.** *Let  $(H_{s,t}, 0 \leq s \leq t < T)$  be a  $\Psi$  flow of partitions defined from the  $\Psi$ -CSBP  $Z$ , and for each  $s \in [0, T)$ , consider a sequence  $(\chi_s(i))_{i \geq 1}$  of r.v. taking distinct values in  $[0, 1]$ . If for each  $s \in [0, T)$ , a.s.  $\mathcal{E}_s(H, (\chi_s(i))_{i \geq 1}) = (r_{s,t}, t \in [s, T])$  then*

- *For each  $s \in [0, T)$ , a.s.  $(\chi_s(i))_{i \geq 1} = (e_s^i)_{i \geq 1}$ .*
- *Almost surely  $H = \hat{\Pi}$ .*

This theorem provides an embedding of the lookdown representation into a flow of subordinators and thus, unifies those two representations. Note that the Eve property is actually a necessary condition for the uniqueness. Indeed when the Eve property does not hold, there is no natural order on the ancestors and therefore no uniqueness of the embedding.

## 2 Preliminaries

### 2.1 Partitions of integers

For every  $n \in \mathbb{N} \cup \{\infty\}$ , let  $\mathcal{P}_n$  be the set of partitions of  $[n] := \{1, \dots, n\}$ . We equip  $\mathcal{P}_\infty$  with the distance  $d_{\mathcal{P}}$  defined as follows. For all  $\pi, \pi' \in \mathcal{P}_\infty$

$$d_{\mathcal{P}}(\pi, \pi') = 2^{-i} \Leftrightarrow i = \sup\{j \in \mathbb{N} : \pi^{[j]} = \pi'^{[j]}\} \quad (\text{II.2})$$

where  $\pi^{[j]}$  is the restriction of  $\pi$  to  $[j]$ .  $(\mathcal{P}_\infty, d_{\mathcal{P}})$  is a compact metric space. We also introduce for every  $n \in \mathbb{N} \cup \{\infty\}$ ,  $\mathcal{P}_n^*$  as the subset of  $\mathcal{P}_n$  whose elements have a unique non-singleton block. In particular, for all subsets  $K \subset \mathbb{N}$ , we denote by  $\mathcal{I}_K$  the element of  $\mathcal{P}_\infty^*$  whose unique non-singleton block is  $K$ . Also we denote by  $0_{[\infty]} := \{\{1\}, \{2\}, \dots\}$  the trivial partition of  $\mathbb{N}$  into singletons.

Let  $\pi \in \mathcal{P}_\infty$ , for each  $i \geq 1$  we denote by  $\pi(i)$  the  $i$ -th block of  $\pi$  in the increasing order of their least element. Furthermore, the asymptotic frequency of  $\pi(i)$  when it exists is defined to be

$$|\pi(i)| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{j \in \pi(i)\}}$$

When all the blocks of a partition  $\pi$  admit an asymptotic frequency, we denote by  $|\pi|^\downarrow$  the sequence of its asymptotic frequencies in the decreasing order. We consider the Borel  $\sigma$ -field of  $(\mathcal{P}_\infty, d_{\mathcal{P}})$ , and define an exchangeable random partition  $\pi$  as a random variable on  $\mathcal{P}_\infty$  whose distribution is invariant under the action of any permutation of  $\mathbb{N}$ , see Section 2.3.2 in [9] for further details.

We define the coagulation operator  $\text{Coag} : \mathcal{P}_\infty \times \mathcal{P}_\infty \rightarrow \mathcal{P}_\infty$  as follows. For any elements  $\pi, \pi' \in \mathcal{P}_\infty$ ,  $\text{Coag}(\pi, \pi')$  is the partition whose blocks are given by

$$\text{Coag}(\pi, \pi')(i) = \bigcup_{j \in \pi'(i)} \pi(j) \quad (\text{II.3})$$

for every  $i \in \mathbb{N}$ . This is a Lipschitz-continuous operator and we have

$$\text{Coag}(\pi, \text{Coag}(\pi', \pi'')) = \text{Coag}(\text{Coag}(\pi, \pi'), \pi'') \quad (\text{II.4})$$

for any elements  $\pi, \pi', \pi'' \in \mathcal{P}_\infty$ , see Section 4.2 in [9] for further details.

We call mass-partition a sequence  $s = (s_i)_{i \geq 1}$  such that  $s_1 \geq s_2 \geq \dots \geq 0$ ,  $\sum_{i \geq 1} s_i \leq 1$ . From a mass-partition  $s$  one can define the paint-box based on  $s$ , that is, the distribution  $\mathcal{P}(s)$  of the random exchangeable partition whose sequence of asymptotic frequencies is  $s$ . This can be achieved by considering a sequence  $(U_i)_{i \geq 1}$  i.i.d. uniform $[0, 1]$  and defining the random partition  $\pi$  via the following equivalence relation

$$i \sim j \Leftrightarrow \exists p \geq 1 \text{ s.t. } U_i, U_j \in \left[ \sum_{k=1}^{p-1} s_k, \sum_{k=1}^p s_k \right)$$

In this work, we will consider the mass-partition  $(x, 0, \dots)$  associated to a point  $x \in (0, 1]$  and the corresponding paint-box distribution  $\mathcal{P}(x, 0, \dots)$  in order to define the flow of partitions, see Subsection 3.2.

Finally consider a subordinator  $X$  restricted to  $[0, a]$ , with  $a > 0$ . On the event  $\{X_a > 0\}$ , the sequence  $(\frac{\Delta X_t}{X_a})_{t \in [0, a]}^\downarrow$  will be called the mass-partition induced by the subordinator  $X$ , and the paint-box based on this sequence will be denoted by  $\mathcal{P}(X)$ . This can be achieved by considering an i.i.d. sequence  $(U_i)_{i \geq 1}$  of uniform $[0, 1]$  r.v., and defining on the event  $\{X_a > 0\}$  the exchangeable random partition  $\pi$  by the following equivalence relation

$$i \stackrel{\pi}{\sim} j \Leftrightarrow X^{-1}(X_a U_i) = X^{-1}(X_a U_j) \quad (\text{II.5})$$

where  $X^{-1}$  denotes the right continuous inverse of  $t \mapsto X_t$ . We also complete the definition by setting  $\mathcal{P}(X) := \mathcal{I}_{[\infty]} = \{\{1, 2, 3, \dots\}\}$  on the event  $\{X_a = 0\}$ .

## 2.2 Continuous-state branching processes

We recall the definition of the continuous-state branching processes introduced in the celebrated article of Jirina [46]. A continuous-state branching process (CSBP for short) started from  $a \geq 0$  is a Markov process  $(Z_t^a, t \geq 0)$  with values in  $[0, \infty]$  such that  $(Z_t^{a+b}, t \geq 0)$  has the same distribution as  $(Z_t^a + Z_t^b, t \geq 0)$  where  $Z^a$  and  $Z^b$  are two independent copies started from  $a$  and  $b$  respectively. Such a process is entirely characterized by a convex function  $\Psi : [0, +\infty) \rightarrow (-\infty, +\infty)$ , called its branching mechanism, via the following identity

$$\mathbb{E}[e^{-\lambda Z_t^a}] = e^{-a u_t(\lambda)}, \quad \forall \lambda > 0 \quad (\text{II.6})$$

where the function  $u_t(\lambda)$  solves

$$\frac{\partial u_t(\lambda)}{\partial t} = -\Psi(u_t(\lambda)), \quad u_0(\lambda) = \lambda \quad (\text{II.7})$$

and  $\Psi$  is the Laplace exponent of a spectrally positive Lévy process. Thus  $\Psi$  has the following form

$$\Psi(u) = \gamma u + \frac{\sigma^2}{2} u^2 + \int_0^\infty (e^{-hu} - 1 + hu \mathbf{1}_{\{h \leq 1\}}) \nu(dh) \quad (\text{II.8})$$

where  $\gamma \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\nu$  is a measure on  $(0, \infty)$  such that  $\int_0^\infty (1 \wedge x^2) \nu(dx) < \infty$ . In the sequel, we will omit the symbol  $a$  and consider  $a = 1$  as the results we will expose do not depend on this value. Note that the semigroup is Feller, so a  $\Psi$ -CSBP admits a càdlàg modification. In the rest of this subsection, we consider implicitly a càdlàg modification of  $Z$ .

We say that the  $\Psi$ -CSBP is subcritical, critical or supercritical according as  $\Psi'(0+)$  is positive, null, or



negative. Furthermore since 0 and  $\infty$  are two absorbing states, we introduce the following two stopping times, namely the extinction time and the explosion time by setting

$$T_0 := \inf\{t \geq 0 : Z_t = 0\}, \quad T_\infty := \inf\{t \geq 0 : Z_t = \infty\} \quad (\text{II.9})$$

Let also  $T := T_0 \wedge T_\infty$  denote the *lifetime* of the  $\Psi$ -CSBP  $Z$ . Classical results entail that  $\mathbb{P}(Z_T = 0) = e^{-q}$ , and therefore  $\mathbb{P}(Z_T = \infty) = 1 - e^{-q}$  where  $q := \sup\{u \geq 0 : \Psi(u) \leq 0\}$ . Note that we use the convention  $\sup \mathbb{R}_+ = \infty$ . In [40], Grey provided a complete classification of the possible behaviours of  $Z$  at the end of its lifetime:

**Extinction.** For all  $t > 0$ , we have  $\mathbb{P}(T_0 \leq t) = e^{-u_t(\infty)}$  and

$$u_t(\infty) < \infty \Leftrightarrow \Psi(v) > 0 \text{ for large enough } v \text{ and } \int_v^\infty \frac{du}{\Psi(u)} < \infty$$

If  $u_t(\infty)$  is finite, then  $u_t(\infty) \downarrow q$  as  $t \rightarrow \infty$ . This ensures that on the event  $\{Z_T = 0\}$  either  $T < \infty$  a.s., or  $T = \infty$  a.s.

**Explosion.** For all  $t > 0$ , we have

$$\mathbb{P}(T_\infty > t) = \lim_{\lambda \rightarrow 0^+} \mathbb{E}[e^{-\lambda Z_t}] = e^{-u_t(0^+)} \quad (\text{II.10})$$

Using this last equality, Grey proved that  $T_\infty \stackrel{\text{a.s.}}{=} \infty \Leftrightarrow \int_{0^+} \frac{du}{\Psi(u)} = \infty$ . When this condition holds, we say that the CSBP is conservative. Here again, on the event  $\{Z_T = +\infty\}$  either  $T < \infty$  a.s., or  $T = \infty$  a.s.

The proofs of the following two lemmas are postponed to Section 7.

**LEMMA II.2.** *On the event  $\{T < \infty\}$ ,  $T$  has a distribution absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$ .*

For all  $\epsilon \in (0, 1)$ , we introduce  $T_\epsilon := \inf\{t \geq 0 : Z_t \notin (\epsilon, 1/\epsilon)\}$ , and notice that  $T_\epsilon < T$  a.s. for all  $\epsilon \in (0, 1)$ .

**LEMMA II.3.** *For all  $t \geq 0$  and all  $\epsilon \in (0, 1)$*

$$\mathbb{E} \left[ \sum_{s \leq t \wedge T_\epsilon : \Delta Z_s > 0} \left( \frac{\Delta Z_s}{Z_s} \right)^2 \right] < \infty$$

### 2.3 Measure-valued branching processes and flows of subordinators

In this subsection, we introduce the measure-valued branching processes associated to a branching mechanism  $\Psi$ . For the sake of simplicity, we will consider measures on the interval  $[0, 1]$ , but the definition holds for any other compact interval. Let  $\mathcal{M}_f$  denote the set of finite measures on  $[0, 1]$  and let  $\Delta$  be an extra point that will represent infinite measures. We set  $\overline{\mathcal{M}}_f := \mathcal{M}_f \cup \{\Delta\}$  and equip this space with the largest topology that makes continuous the map

$$\begin{aligned} [0, +\infty] \times \mathcal{M}_1 &\rightarrow \overline{\mathcal{M}}_f \\ (\lambda, \mu) &\mapsto \begin{cases} \lambda \cdot \mu & \text{if } \lambda < \infty \\ \Delta & \text{if } \lambda = \infty \end{cases} \end{aligned}$$

This topology is due to Watanabe [77].

We denote by  $\mathcal{B}^{++}$  the set of bounded Borel functions on  $[0, 1]$  that admit a strictly positive infimum.

We call measure-valued branching process associated with the branching mechanism  $\Psi$ , or  $\Psi$ -MVBP in short, a  $\overline{\mathcal{M}}_f$ -valued Markov process  $(m_t, t \geq 0)$  started from a given measure  $m_0 \in \overline{\mathcal{M}}_f$  that verifies for all  $f \in \mathcal{B}^{++}$

$$\mathbb{E}[\exp(-\langle m_t, f \rangle)] = \exp(-\langle m_0, u_t \circ f \rangle)$$

Note that  $\langle \Delta, f \rangle = +\infty$ , thus  $\Delta$  is an absorbing point. The existence of this process can be obtained using a flow of subordinators as it will be shown below. The uniqueness of the distribution derives from the Markov property and the characterization of the Laplace functional on  $\mathcal{B}^{++}$ .

It is straightforward to check that the total-mass process  $(m_t([0, 1]), t \geq 0)$  is a  $\Psi$ -CSBP, say  $Z$ , started from  $m_0([0, 1])$ . As proved in [28], this process verifies the branching property: for every  $m_0, m'_0 \in \overline{\mathcal{M}}_f$ , the process  $(m_t + m'_t, t \geq 0)$  is a  $\Psi$ -MVBP started from  $m_0 + m'_0$ , where  $(m_t, t \geq 0)$  and  $(m'_t, t \geq 0)$  are two independent  $\Psi$ -MVBP started from  $m_0$  and  $m'_0$  respectively.

Finally, from Lemma 3.5.1 in [21] one can prove that its semigroup verifies the Feller property. This implies that the  $\Psi$ -MVBP admits a càdlàg modification. In the rest of this subsection, we consider implicitly a càdlàg modification of  $m$  and will denote by  $T$  the lifetime of its total-mass process (which is necessarily a càdlàg  $\Psi$ -CSBP).

Suppose that  $(m_t, t \geq 0)$  starts from the Lebesgue measure on  $[0, 1]$ . It is then immediate to deduce that for all  $t \geq 0$ , the process  $x \mapsto m_t([0, x])$  is a (possibly killed) subordinator whose Laplace exponent is given by  $(u_t(\lambda), \lambda > 0)$ . From the Lévy-Khintchine formula, we deduce that there exists a real number  $d_t \geq 0$  and a measure  $w_t$  on  $(0, \infty)$  that verifies  $\int_0^\infty (1 \wedge h) w_t(dh) < \infty$ , such that

$$u_t(\lambda) = u_t(0+) + d_t \lambda + \int_0^\infty (1 - e^{-\lambda h}) w_t(dh), \text{ for all } \lambda > 0 \quad (\text{II.11})$$

Notice that  $u_t(0+)$  is the instantaneous killing rate of the subordinator, which is related to the explosion of the total mass process  $(m_t([0, 1]), t \geq 0)$ . Indeed  $\mathbb{P}(T_\infty \leq t) = \mathbb{P}(m_t([0, 1]) = \infty) = 1 - e^{-u_t(0+)}$ , see Equation (II.10).

From this observation, Bertoin and Le Gall introduced an object called flow of subordinators. Proposition 1 in [12] asserts the existence of a process  $(S_{s,t}(a), 0 \leq s \leq t, a \geq 0)$  such that

- For every  $0 \leq s \leq t$ ,  $(S_{s,t}(a), a \geq 0)$  is a subordinator with Laplace exponent  $u_{t-s}$ .
- For every integer  $p \geq 2$  and  $0 \leq t_1 \leq \dots \leq t_p$ , the subordinators  $S_{t_1, t_2}, \dots, S_{t_{p-1}, t_p}$  are independent and

$$S_{t_1, t_p}(a) = S_{t_{p-1}, t_p} \circ \dots \circ S_{t_1, t_2}(a), \quad \forall a \geq 0 \text{ a.s. (cocycle property)}$$

- For all  $a \geq 0$ ,  $(S_{0,t}(a), t \geq 0)$  is a  $\Psi$ -CSBP started from  $a$ .

Actually in their construction, they excluded the non-conservative branching mechanisms but one can easily adapt their proof to the general case.

Let us now present the connection with the  $\Psi$ -MVBP. Introduce the random Stieltjes measures

$$m_{0,t}(dx) := d_x S_{0,t}(x), \quad \forall x \in [0, 1]$$

From the very definition of the flow of subordinators, one can prove that  $(m_{0,t}, t \geq 0)$  is a  $\Psi$ -MVBP started from the Lebesgue measure on  $[0, 1]$ . One can consider a càdlàg modification still denoted  $(m_{0,t}, t \geq 0)$ , and let  $T^S$  be the lifetime of its total-mass process  $S_t := m_{0,t}([0, 1]), t \geq 0$  (which is a càdlàg  $\Psi$ -CSBP). It is then natural to introduce the random Stieltjes measures

$$m_{s,t}(dx) := d_x S_{s,t}(x), \quad \forall x \in [0, S_s]$$

for every  $0 < s \leq t < T^S$ . Each process  $(m_{s,t}, t \in [s, T^S])$  is a  $\Psi$ -MVBP and admits a càdlàg modification still denoted  $(m_{s,t}, t \in [s, T^S])$ . Then we obtain a flow of  $\Psi$ -MVBP  $(m_{s,t}, 0 \leq s \leq t < T^S)$  that describes the evolution of an initial population  $[0, 1]$ .

### 3 Flows of partitions and the lookdown representation

The goal of this section is to develop the construction of  $\Psi$  flows of partitions presented in the introduction. To that end, we first recall the definition of deterministic flows of partitions as introduced in [52] since the one-to-one correspondence with lookdown graphs is deterministic. Then, we define a random point process  $\mathcal{P}$  pathwise from a càdlàg  $\Psi$ -CSBP  $Z$ , which will allow us to construct a  $\Psi$  flow of partitions. Finally we give a precise characterization of its jump rates which will be necessary in the proof of Theorem II.4, this last subsection can be skipped on first reading.

#### 3.1 Deterministic flows of partitions

Fix  $T \in (0, +\infty]$ . In [52], we introduced deterministic flows of partitions and proved they are in one-to-one correspondence with the so-called lookdown graphs. Lookdown graphs are implicit in the lookdown construction of Donnelly and Kurtz [24], and the upshot of the flows of partitions is to clarify and formalize this notion. In the present paper, we do not recall the definition of the lookdown graph and we refer to [52] for further details.

Below this formal definition of deterministic flows of partitions, the reader should find intuitive comments.

**DEFINITION II.4.** *A deterministic flow of partitions on  $[0, T)$  is a collection  $\hat{\pi} = (\hat{\pi}_{s,t}, 0 \leq s \leq t < T)$  of partitions such that*

- *For every  $r < s < t \in [0, T)$ ,  $\hat{\pi}_{r,t} = \text{Coag}(\hat{\pi}_{s,t}, \hat{\pi}_{r,s})$ .*
- *For every  $s \in (0, T)$ ,  $\lim_{r \uparrow s} \lim_{t \uparrow s} \hat{\pi}_{r,t} =: \hat{\pi}_{s-,s-} = \mathbf{0}_{[\infty]}$ .*
- *For every  $s \in [0, T)$ ,  $\lim_{t \downarrow s} \hat{\pi}_{s,t} = \hat{\pi}_{s,s} = \mathbf{0}_{[\infty]}$ .*

*Furthermore, if for all  $s \in (0, T)$ ,  $\hat{\pi}_{s-,s}$  has at most one unique non-singleton block, then we say that  $\hat{\pi}$  is a deterministic flow of partitions without simultaneous mergers.*

The first property asserts a cocycle property for the collection of partitions: the evolution forward-in-time is obtained by coagulating consecutive partitions. The second and third properties ensure that for all  $n \geq 1$  and every compact interval  $[r, t] \subset [0, T)$ , only a finite number of partitions  $\hat{\pi}_{s-,s}^{[n]}$  differ from the trivial partition  $\mathbf{0}_{[n]}$ . Note that in this paper, we will only consider flows of partitions without simultaneous mergers.

*Construction from a point process*

Let  $p$  be a *deterministic* point process on  $[0, T) \times \mathcal{P}_\infty^*$  whose restriction to any subset of the form  $(s, t] \times \mathcal{P}_n^*$  has finitely many points. Fix an integer  $n \in \mathbb{N}$  and two real numbers  $s \leq t \in [0, T)$ . Let  $(t_i, \varrho_i)_{1 \leq i \leq q}$  be the finitely many points of  $p|_{(s,t] \times \mathcal{P}_n^*}$  in the increasing order of their time coordinate. We introduce

$$\hat{\pi}_{s,t}^{[n]} := \text{Coag}(\varrho_q, \text{Coag}(\varrho_{q-1}, \dots, \text{Coag}(\varrho_2, \varrho_1) \dots)) \quad (\text{II.12})$$

Obviously, the collection of partitions  $(\hat{\pi}_{s,t}^{[n]}, n \in \mathbb{N})$  is compatible and defines by a projective limit a unique partition  $\hat{\pi}_{s,t}$  such that its restriction to  $[n]$  is  $\hat{\pi}_{s,t}^{[n]}$ , for each  $n \in \mathbb{N}$ . Then, one easily verifies that  $(\hat{\pi}_{s,t}, 0 \leq s \leq t < T)$  is a deterministic flow of partitions without simultaneous mergers.

**REMARK II.5.** *This construction gives a hint of the one-to-one correspondence with lockdown graphs. See [52] for further details.*

We can now introduce the lockdown representation using a deterministic flow of partitions. Let  $(\xi_{s,s}(i))_{i \geq 1}$  be a sequence of points in  $\mathbb{R}_+$  and define the particle system  $(\xi_{s,t}(i), t \in [s, T])_{i \geq 1}$  as follows. For all  $t \geq s$  and all  $i, j \geq 1$ ,

$$\xi_{s,t}(j) = \xi_{s,s}(i) \Leftrightarrow j \in \hat{\pi}_{s,t}(i) \quad (\text{II.13})$$

**DEFINITION II.6.** *We use the notation  $\mathcal{L}_s(\hat{\pi}, (\xi_{s,s}(i))_{i \geq 1})$  to denote the deterministic lockdown function  $(\xi_{s,t}(i), t \in [s, \infty))_{i \geq 1}$  defined from the flow of partitions  $\hat{\pi}$  and the initial types  $(\xi_{s,s}(i))_{i \geq 1}$ .*

Moreover, for all  $t \in [s, T)$ , set

$$\Xi_{s,t}(dx) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{\xi_{s,t}(i)}(dx) \quad (\text{II.14})$$

when this is well-defined.

**DEFINITION II.7.** *We denote by  $\mathcal{E}_s(\hat{\pi}, (\xi_{s,s}(i))_{i \geq 1})$  the collection of limiting empirical measures  $(\Xi_{s,t}, t \in [s, \infty))$  defined from the flow of partitions  $\hat{\pi}$  and the initial types  $(\xi_{s,s}(i))_{i \geq 1}$ , when it exists.*

Let us give an intuitive explanation of this particle system. If one considers each point  $\xi_{s,s}(i)$  as some characteristic (type or location for instance) of the  $i$ -th ancestor at time  $s$ , then the underlying idea of the lockdown representation is to give the same characteristic to the descendants of this ancestor at any time  $t > s$ . Therefore, the measure  $\Xi_{s,t}(dx)$  describes the composition of the population at time  $t$ :  $\Xi_{s,t}(\{\xi_{s,s}(i)\})$  is the proportion of individuals at time  $t$  who descend from the  $i$ -th ancestor alive at time  $s$ . In the next subsection, we will see that if one applies this scheme with a random flow of partitions, whose distribution is well chosen, then  $\Xi$  is a MVBP (rescaled by its total-mass).

### 3.2 Stochastic flows of partitions associated with a branching mechanism

We randomize the previous definitions using a point process  $\mathcal{P}$  on  $[0, T) \times \mathcal{P}_\infty$  where  $T$  is a random positive time in order to introduce flows of partitions associated with a branching mechanism  $\Psi$ . As mentioned in the introduction, this point process is obtained as the union of two point processes:  $\mathcal{N}_\sigma$  that stands for the binary reproduction events due to the diffusion of the underlying CSBP, and  $\mathcal{N}_\nu$  that encodes the positive frequency reproduction events due to the jumps of the CSBP. As these objects rely on many definitions, one should refer on first reading to the heuristic definitions given in the introduction.

For every  $z > 0$ , we introduce the map  $\phi_z : \mathbb{R}_+^* \rightarrow [0, 1]$ , that will be used to consider rescaled jumps of a CSBP, by setting

$$\phi_z : h \mapsto \frac{h}{h + z}$$

We define a measure  $\mu^{\text{binary}}$  on  $\mathcal{P}_\infty$  that will encode binary reproduction events often called Kingman reproduction events. Recall that  $\mathcal{I}_{\{i,j\}}$  stands for the element of  $\mathcal{P}_\infty^*$  whose unique non-singleton block is  $\{i, j\}$  for every integers  $1 \leq i < j$ .

$$\mu^{\text{binary}}(d\pi) := \sum_{i < j} \delta_{\mathcal{I}_{\{i,j\}}}(d\pi) \quad (\text{II.15})$$

This ends the introduction of preliminary notation. Fix a branching mechanism  $\Psi$  and consider a  $\Psi$ -CSBP  $(Z_t, t \geq 0)$  started from 1 assumed to be càdlàg. We keep the notation of Section 2.2, in particular  $T$  denotes the lifetime of  $Z$ . We start with the definition of  $\mathcal{N}_\nu$ . Consider the random point process

$$\mathcal{Q} := \bigcup_{\{t \geq 0: \Delta Z_t > 0\}} \left\{ \left( t, \frac{\Delta Z_t}{Z_t} \right) \right\}$$

and define a  $\mathcal{P}$ -randomization  $\mathcal{N}_\nu$  of  $\mathcal{Q}$  in the sense of Chapter 12 in [48], where  $\mathcal{P}$  is the paint-box probability kernel introduced in Subsection 2.1. The point process  $\mathcal{N}_\nu := \cup\{(t, \frac{\Delta Z_t}{Z_t}, \varrho_t)\}$  on  $\mathbb{R}_+ \times [0, 1] \times \mathcal{P}_\infty$  can be described as follows. For all  $t \geq 0$  such that  $\Delta Z_t > 0$ ,  $\varrho_t$  is a r.v. on  $\mathcal{P}_\infty$  distributed according to the paint-box distribution  $\mathcal{P}(\frac{\Delta Z_t}{Z_t}, 0, \dots)$ . It is more convenient to consider the restriction of this point process to  $\mathbb{R}_+ \times \mathcal{P}_\infty$  still denoted by  $\mathcal{N}_\nu = \cup\{(t, \varrho_t)\}$ .

Second, we define a doubly stochastic Poisson point process  $\mathcal{N}_\sigma$  on  $\mathbb{R}_+ \times \mathcal{P}_\infty$ , in the sense of Chapter 12 in [48], with a random intensity measure given by

$$\mathbf{1}_{\{t < T\}} dt \otimes \frac{\sigma^2}{Z_t} \mu^{\text{binary}}(d\pi) \quad (\text{II.16})$$

We finally define the point process  $\mathcal{P}$  on  $\mathbb{R}_+ \times \mathcal{P}_\infty$  as

$$\mathcal{P} := \mathcal{N}_\sigma \cup \mathcal{N}_\nu \quad (\text{II.17})$$

Notice that almost surely this point process takes its values in  $\mathbb{R}_+ \times \mathcal{P}_\infty^*$ , and has finitely many points in any set of the form  $[0, t] \times \mathcal{P}_n^*$  with  $t < T$  and  $n \in \mathbb{N}$ , as we will see in Proposition II.13. Thus for each  $\omega \in \Omega$ , we define a deterministic flow of partitions without simultaneous mergers  $(\hat{\Pi}_{s,t}(\omega), 0 \leq s \leq t < T)$  using the point collection  $\mathcal{P}(\omega)$  and the pathwise construction of Subsection 3.1.

Let us now explain how one defines a lookdown process associated with a  $\Psi$ -MVBP. Fix  $s \geq 0$  and condition on  $\{s < T\}$ . Consider a sequence  $(\xi_{s,s}(i))_{i \geq 1}$  of i.i.d. uniform $[0, 1]$  r.v. and define the lookdown process  $(\xi_{s,t}(i), t \in [s, T])_{i \geq 1} := \mathcal{L}_s(\hat{\Pi}, (\xi_{s,s}(i)))_{i \geq 1}$ . Lemma 3.5 in [24] ensures that almost surely this particle system admits a process of limiting empirical measures  $(\Xi_{s,t}, t \in [s, T]) := \mathcal{E}_s(\hat{\Pi}, (\xi_{s,s}(i))_{i \geq 1})$ , and almost surely for all  $t \in [s, T]$  we have

$$\Xi_{s,t}(dx) = \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)| \delta_{\xi_{s,s}(i)}(dx) + \left( 1 - \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)| \right) dx$$

Moreover, Section 2 in [18] shows that the process  $(Z_t \cdot \Xi_{s,t}(Z_s \cdot dx), t \in [s, T])$  is a càdlàg  $\Psi$ -MVBP started from the Lebesgue measure on  $[0, Z_s]$ , conditionally on  $Z_s$ .

**REMARK II.8.** *The results in [18, 24] are stated with the usual notion of lookdown graph. But they are immediately translated in terms of flows of partitions thanks to our one-to-one correspondence.*

**REMARK II.9.** *We can define from any time  $s \in [0, T)$ , a  $\Psi$ -MVBP with total-mass process  $Z$  using an independent sequence of initial types  $(\xi_{s,s}(i))_{i \geq 1}$  and the flow  $\hat{\Pi}$ . Then, it could seem simple to define a flow of  $\Psi$ -MVBP using this lookdown representation simultaneously for all  $s \in [0, T)$ . However, this is far from being trivial since the initial types  $s \mapsto (\xi_{s,s}(i))_{i \geq 1}$  have to be suitably coupled. In Section 6, we will show that these initial types have to be the Eves.*

An important property of the lookdown process (see [24]) is that for all  $t \geq s$ , conditionally given  $Z_t$  the sequence  $(\xi_{s,t}(i))_{i \geq 1}$  is exchangeable on  $[0, 1]$ . This implies that conditionally given  $Z_t$  the partition  $\hat{\Pi}_{s,t}$  has the paint-box distribution on the subordinator  $x \mapsto Z_t \cdot \Xi_{s,t}([0, x])$ . More generally, we have

**THEOREM II.1** *The collection of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ , together with its underlying CSBP  $Z$ , satisfies*

- For all  $n \geq 1$  and all  $0 < t_1 < \dots < t_n$ ,

$$(Z_{t_1}, \dots, Z_{t_n}, \hat{\Pi}_{0,t_1}, \dots, \hat{\Pi}_{t_{n-1}, t_n} | t_n < T) \stackrel{(d)}{=} (S_{t_1}, \dots, S_{t_n}, \mathcal{P}(S_{0,t_1}), \dots, \mathcal{P}(S_{t_{n-1}, t_n}) | t_n < T^S)$$

- For all  $0 \leq r < s < t < T$ , a.s.  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$  (cocycle property).

**Proof** The cocycle property is a consequence of our construction as we have defined the restrictions of the partitions by coagulating elementary reproduction events. We turn our attention to the finite dimensional distributions. Fix an integer  $n \geq 1$  and a  $n$ -tuple  $0 = t_0 < t_1 < \dots < t_n$ . Let  $(S_{s,t}(a), 0 \leq s \leq t, 0 \leq a \leq S_{0,s}(z))$  be a  $\Psi$  flow of subordinators restricted to an initial population  $[0, z]$  instead of  $[0, 1]$ , and keep the notation  $T^S$  to denote the lifetime of the total mass process  $(S_t := S_{0,t}(z), t \geq 0)$  which is a  $\Psi$ -CSBP started from  $z$ . For every  $i \in [n]$ , let  $H_{t_{i-1}, t_i}$  be distributed according to the paintbox  $\mathcal{P}(S_{t_{i-1}, t_i})$ . Note that  $H_{0,t_1}, \dots, H_{t_{n-1}, t_n}$  are coupled only through their mass-partitions. We use our construction of the beginning of this subsection to define pathwise from the CSBP  $(S_t, t \geq 0)$  a collection  $(\hat{\Pi}_{s,t}^S, 0 \leq s \leq t < T^S)$ . We will use the notation  $\mathbb{P}_z$  to emphasize the dependence on the initial value  $z$ . Implicitly,  $f_i$  will denote a bounded Borel map from  $\mathcal{P}_\infty$  to  $\mathbb{R}$  and  $g_i$  a bounded Borel map from  $\mathbb{R}_+$  to  $\mathbb{R}$ . We now prove by recursion on  $n \geq 1$  that

$$\begin{aligned} & \mathbb{E}_z[f_1(\hat{\Pi}_{0,t_1}^S)g_1(S_{t_1}) \dots f_n(\hat{\Pi}_{t_{n-1}, t_n}^S)g_n(S_{t_n})\mathbf{1}_{\{t_n < T^S\}}] \\ &= \mathbb{E}_z[f_1(H_{0,t_1})g_1(S_{t_1}) \dots f_n(H_{t_{n-1}, t_n})g_n(S_{t_n})\mathbf{1}_{\{t_n < T^S\}}] \end{aligned}$$

The case  $n = 1$  follows from the discussion above the statement of the theorem. Fix  $n \geq 2$  and suppose that for all  $z > 0$  and all  $f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1}$ , we have

$$\begin{aligned} & \mathbb{E}_z[f_1(\hat{\Pi}_{0,t_1}^S)g_1(S_{t_1}) \dots f_{n-1}(\hat{\Pi}_{t_{n-2}, t_{n-1}}^S)g_{n-1}(S_{t_{n-1}})\mathbf{1}_{\{t_{n-1} < T^S\}}] \\ &= \mathbb{E}_z[f_1(H_{0,t_1})g_1(S_{t_1}) \dots f_{n-1}(H_{t_{n-2}, t_{n-1}})g_{n-1}(S_{t_{n-1}})\mathbf{1}_{\{t_{n-1} < T^S\}}] \end{aligned}$$

Then, we obtain at rank  $n$  for any given  $z > 0$  and any  $f_1, \dots, f_n, g_1, \dots, g_n$

$$\begin{aligned} & \mathbb{E}_z[f_1(\hat{\Pi}_{0,t_1}^S)g_1(S_{t_1}) \dots f_n(\hat{\Pi}_{t_{n-1}, t_n}^S)g_n(S_{t_n})\mathbf{1}_{\{t_n < T^S\}}] \\ &= \mathbb{E}_z \left[ f_1(\hat{\Pi}_{0,t_1}^S)g_1(S_{t_1})\mathbf{1}_{\{t_1 < T^S\}} \mathbb{E}_{S_{t_1}} \left[ f_2(\hat{\Pi}_{0,t_2-t_1}^S) \dots f_n(\hat{\Pi}_{t_{n-1}-t_1, t_n-t_1}^S)g_n(S_{t_n-t_1})\mathbf{1}_{\{t_n-t_1 < T^S\}} \right] \right] \\ &= \mathbb{E}_z \left[ f_1(H_{0,t_1})g_1(S_{t_1})\mathbf{1}_{\{t_1 < T^S\}} \mathbb{E}_{S_{t_1}} \left[ f_2(H_{0,t_2-t_1}) \dots f_n(H_{t_{n-1}-t_1, t_n-t_1})g_n(S_{t_n-t_1})\mathbf{1}_{\{t_n-t_1 < T^S\}} \right] \right] \\ &= \mathbb{E}_z[f_1(H_{0,t_1})g_1(S_{t_1}) \dots f_n(H_{t_{n-1}, t_n})g_n(S_{t_n})\mathbf{1}_{\{t_n < T^S\}}] \end{aligned}$$

where the first (resp. last) equality comes from the Markov property applied to the process  $(S_t, t \in [0, T^S])$  (resp. to the homogeneous chain  $(S_{t_i}, H_{t_{i-1}, t_i}, t_{i+1} - t_i)_{1 \leq i \leq n}$ ) while the second equality makes use of the recursion hypothesis and the case  $n = 1$ .  $\blacksquare$

This result motivates the following definition.

**DEFINITION II.10.** *A collection of random partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  defined on a same probability space as a càdlàg  $\Psi$ -CSBP  $(Z_t, t \in [0, T])$  and that verifies*

- For all  $n \geq 1$  and all  $0 < t_1 < \dots < t_n$ ,

$$(Z_{t_1}, \dots, Z_{t_n}, \hat{\Pi}_{0,t_1}, \dots, \hat{\Pi}_{t_{n-1}, t_n} | t_n < T) \stackrel{(d)}{=} (S_{t_1}, \dots, S_{t_n}, \mathcal{P}(S_{0,t_1}), \dots, \mathcal{P}(S_{t_{n-1}, t_n}) | t_n < T^S)$$

- For all  $0 \leq r < s < t < T$ , a.s.  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$  (cocycle property).

is called a  $\Psi$  flow of partitions.  $Z$  is called its underlying CSBP.

**REMARK II.11.** In our construction from a point process, we can verify that the cocycle property is fulfilled almost surely simultaneously for all triplets, that is,

$$\mathbb{P}[\forall 0 \leq r < s < t < T, \hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})] = 1$$

This is not necessarily the case for a general  $\Psi$  flow of partitions: however Proposition II.30 will show that we can define a regularized modification which fulfils that property.

### 3.3 A characterization of the jump rates

The formalism of partitions enables one to restrict to  $n$  individuals sampled uniformly among the population. In this subsection, we give a characterization of the dynamics of this finite-dimensional process. The restriction of  $\mathcal{P}$  to  $\mathbb{R}_+ \times \mathcal{P}_n^*$  is denoted  $\mathcal{P}^{[n]}$ . We introduce, for any integer  $2 \leq k \leq n$  and any subset  $K \subset [n]$  such that  $\#K = k$ , the quantity

$$L_t(n, K) := \#\{r \in (0, t] : (r, \mathbf{1}_K^{[n]}) \in \mathcal{P}^{[n]}\} \quad (\text{II.18})$$

where  $\mathcal{I}_K^{[n]}$  is the restriction of  $\mathcal{I}_K$  to  $[n]$  and  $\mathcal{I}_K$  is the partition whose unique non-singleton block is  $K$ . Moreover, we set

$$L_t(n) := \sum_{\{K \subset [n] : \#K \geq 2\}} L_t(n, K) \quad (\text{II.19})$$

In words,  $L_t(n)$  is the total number of points of  $\mathcal{P}$  restricted to  $(0, t] \times \mathcal{P}_n^*$ . Note that the collection of processes  $\{(L_t(n, K), t \in [0, T]); K \subset [n], \#K \geq 2\}$  is completely equivalent with the restricted flow  $(\hat{\Pi}_{s,t}^{[n]}, 0 \leq s \leq t < T)$ : the knowledge of any one of them is sufficient to recover the other. We denote by  $d_n$  the number of subsets of  $[n]$  with at least 2 elements, that is,  $d_n := \sum_{k=2}^n \binom{n}{k}$  and we introduce the filtration  $(\mathcal{F}_t, t \geq 0)$  by setting for all  $t \geq 0$

$$\mathcal{F}_t := \sigma\{Z_s, s \in [0, t]\} \vee \sigma\{\mathcal{P}_{[0,t] \times \mathcal{P}_\infty}\} \quad (\text{II.20})$$

For every integer  $k$  such that  $2 \leq k \leq n$ , we set

$$\lambda_{n,k}(z, \Psi) := \int_0^1 x^k (1-x)^{n-k} \left( \frac{\sigma^2}{z} x^{-2} \delta_0(dx) + z \nu \circ \phi_z^{-1}(dx) \right)$$

where  $\nu \circ \phi_z^{-1}$  is the pushforward measure of  $\nu$  through the map  $\phi_z$ . Notice that  $\lambda_{n,k}$  can be seen as a map from  $\mathbb{R}_+^* \times \mathcal{M}_f(\mathbb{R}_+)$  to  $\mathbb{R}_+$ . Indeed, any element of  $\mathcal{M}_f(\mathbb{R}_+)$  has the form  $\tilde{\sigma}^2 \delta_0(dh) + (1 \wedge h^2) \tilde{\nu}(dh)$ , where  $\tilde{\sigma} \geq 0$  and  $\tilde{\nu}$  is a measure on  $(0, \infty)$  such that  $\int_0^\infty (1 \wedge h^2) \tilde{\nu}(dh) < \infty$ , so it can be associated to the branching mechanism  $\tilde{\Psi}$  defined by the triplet  $(\tilde{\gamma} = 0, \tilde{\sigma}, \tilde{\nu})$ .

This ends the introduction of notation. We now state two results: the first is a technical continuity statement and the second is the characterization of the jump rates. They will be of main importance for the proof of Theorem II.4.

**PROPOSITION II.12.** Fix  $k, n$  such that  $2 \leq k \leq n$ . The map  $\lambda_{n,k}$  is continuous from  $\mathbb{R}_+^* \times \mathcal{M}_f(\mathbb{R}_+)$ , endowed with the product topology, to  $\mathbb{R}_+$ .

The proof of this first result will be given in Section 7, as it is rather technical. For the next result, we rely on notions of stochastic calculus introduced in Chapters I.3 and II.2 in [45].

**PROPOSITION II.13.** *The collection of counting processes  $\{(L_t(n, \mathbf{K}), t \in [0, T]); \mathbf{K} \subset [n], \#\mathbf{K} \geq 2\}$  is a pure-jump  $d_n$ -dimensional semimartingale on  $[0, T)$  with respect to  $\mathcal{F}$ . Its predictable compensator is the  $d_n$ -dimensional process*

$$\left\{ \left( \int_0^t \lambda_{n, \#\mathbf{K}}(Z_{s-}, \Psi) ds, t \in [0, T) \right); \mathbf{K} \subset [n], \#\mathbf{K} \geq 2 \right\} \quad (\text{II.21})$$

**Proof** Fix  $\mathbf{K} \subset [n]$  such that its cardinality, denoted by  $k := \#\mathbf{K}$ , is greater than 2. It is straightforward to check that  $(L_t(n, \mathbf{K}), t \in [0, T))$  is a counting process adapted to the filtration  $\mathcal{F}$ . Similarly, one can easily verify that the process

$$\left( \int_0^t \lambda_{n, k}(Z_{s-}, \Psi) ds, t \in [0, T) \right)$$

is a predictable increasing process w.r.t. the filtration  $\mathcal{F}$ . Let us prove that the process

$$\left( L_t(n, \mathbf{K}) - \int_0^t \lambda_{n, k}(Z_{s-}, \Psi) ds, t \in [0, T) \right)$$

is a local martingale on  $[0, T)$  w.r.t.  $\mathcal{F}$ . To do so, set for all  $t \in [0, T)$

$$\begin{aligned} M_t^{(1)} &:= L_t(n, \mathbf{K}) - \sum_{s \leq t} \left( \frac{\Delta Z_s}{Z_s} \right)^k \left( 1 - \frac{\Delta Z_s}{Z_s} \right)^{n-k} - \mathbf{1}_{\{k=2\}} \int_0^t \frac{\sigma^2}{Z_{s-}} ds \\ M_t^{(2)} &:= \sum_{s \leq t} \left( \frac{\Delta Z_s}{Z_s} \right)^k \left( 1 - \frac{\Delta Z_s}{Z_s} \right)^{n-k} - \int_0^t \int_0^1 x^k (1-x)^{n-k} Z_{s-} \nu \circ \phi_{Z_{s-}}^{-1}(dx) ds \end{aligned}$$

It is sufficient to show that both  $M^{(1)}$  and  $M^{(2)}$  are local martingales on  $[0, T)$  w.r.t.  $\mathcal{F}$ .

Let us focus on the first one. Fix  $\epsilon \in (0, 1)$  and recall the definition of the stopping time  $T_\epsilon := \inf\{t \geq 0 : Z_t \notin (\epsilon, 1/\epsilon)\}$ . Condition on  $(Z_s, s \in [0, T))$  and consider a time  $s > 0$  such that  $\Delta Z_s > 0$  (note that those times are countably many a.s.). The  $\mathcal{P}$ -randomization procedure implies that the restriction of the random partition  $\varrho_s$  to  $\mathcal{P}_n$  has a probability  $\left(\frac{\Delta Z_s}{Z_s}\right)^k \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{n-k}$  to be equal to  $\mathcal{I}_K^{[n]}$  independently of the other partitions  $(\varrho_t)_{\{t \neq s: \Delta Z_t > 0\}}$ . For all  $t \geq 0$ , the number of occurrences of the partition  $\mathcal{I}_K^{[n]}$  in  $\mathcal{N}_\nu$  restricted to  $[0, t \wedge T_\epsilon] \times \mathcal{P}_n$  is given by the following r.v.

$$\#\{s \in [0, t \wedge T_\epsilon] : (s, \mathcal{I}_K^{[n]}) \in \mathcal{N}_\nu^{[n]}\} \quad (\text{II.22})$$

which is, therefore, distributed as the sum of a sequence, indexed by  $\{s \in [0, t \wedge T_\epsilon] : \Delta Z_s > 0\}$ , of independent Bernoulli r.v. with parameters  $\left(\left(\frac{\Delta Z_s}{Z_s}\right)^k \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{n-k}\right)$ . Since

$$\left(\frac{\Delta Z_s}{Z_s}\right)^k \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{n-k} \leq \left(\frac{\Delta Z_s}{Z_s}\right)^2 \quad (\text{II.23})$$

a simple application of Borel-Cantelli lemma together with Lemma II.3 ensures that the r.v. of Equation (II.22) is finite a.s. One also easily deduces that for all  $t \geq 0$

$$\mathbb{E} \left[ \#\{s \in [0, t \wedge T_\epsilon] : (s, \mathcal{I}_K^{[n]}) \in \mathcal{N}_\nu^{[n]}\} \mid (Z_s, s \in [0, t \wedge T_\epsilon]) \right] = \sum_{s \leq t \wedge T_\epsilon} \left(\frac{\Delta Z_s}{Z_s}\right)^k \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{n-k}$$

Furthermore when  $k = 2$ , we deduce from the definition of  $\mathcal{N}_\sigma$  that the counting process

$$\#\{s \in [0, t] : (s, \mathcal{I}_K^{[n]}) \in \mathcal{N}_\sigma^{[n]}\}, t \in [0, T)$$



is a doubly stochastic Poisson process with intensity  $\mathbf{1}_{\{t < T\}} \frac{\sigma^2}{Z_t} dt$ . Therefore, for all  $t \geq 0$

$$\mathbb{E} \left[ \#\{s \in [0, t \wedge T_\epsilon] : (s, \mathcal{I}_K^{[n]}) \in \mathcal{N}_\sigma^{[n]}\} \mid (Z_s, s \in [0, t \wedge T_\epsilon]) \right] = \int_0^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds$$

Notice that the r.h.s. is finite a.s. Putting together the preceding results, we get that

$$\mathbb{E} \left[ L_{t \wedge T_\epsilon}(n, \mathbf{K}) \mid (Z_s, s \in [0, t \wedge T_\epsilon]) \right] = \sum_{s \leq t \wedge T_\epsilon} \left( \frac{\Delta Z_s}{Z_s} \right)^k \left( 1 - \frac{\Delta Z_s}{Z_s} \right)^{n-k} + \mathbf{1}_{\{k=2\}} \int_0^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds$$

Using Lemma II.3 and Equation (II.23), we deduce that the r.h.s. of the preceding equation is integrable for all  $t \geq 0$ . Therefore

$$\mathbb{E} [M_{t \wedge T_\epsilon}^{(1)}] = 0$$

Note that the integrability is indeed locally uniform since we deal with non-decreasing processes. In addition, we have for all  $0 \leq r \leq t$ ,

$$\begin{aligned} \mathbb{E} [M_{t \wedge T_\epsilon}^{(1)} \mid \mathcal{F}_r] &= M_{r \wedge T_\epsilon}^{(1)} + \mathbb{E} \left[ L_{r \wedge T_\epsilon, t \wedge T_\epsilon}(n, \mathbf{K}) \right. \\ &\quad \left. - \sum_{s \in (r \wedge T_\epsilon, t \wedge T_\epsilon]} \left( \frac{\Delta Z_s}{Z_s} \right)^k \left( 1 - \frac{\Delta Z_s}{Z_s} \right)^{n-k} - \mathbf{1}_{\{k=2\}} \int_{r \wedge T_\epsilon}^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds \mid \mathcal{F}_r \right] \end{aligned}$$

By applying the strong Markov property at time  $r \wedge T_\epsilon$  to the process  $Z$ , one easily gets that the second term in the r.h.s. is zero a.s. using the preceding arguments. Therefore, we have proven that  $(M_{t \wedge T_\epsilon}^{(1)}, t \in [0, T])$  is a locally uniformly integrable martingale. Since  $T_\epsilon \uparrow T$  a.s., it implies that  $M^{(1)}$  is a local martingale on  $[0, T)$ .

We turn our attention to  $M^{(2)}$ . It is well-known that the dual predictable compensator of the random measure

$$\sum_{\{t \geq 0 : \Delta Z_t > 0\}} \delta_{(t, \frac{\Delta Z_t}{Z_t})} \quad (\text{II.24})$$

is the random measure  $\mathbf{1}_{\{t < T\}} Z_{t-} dt \otimes \nu \circ \phi_{Z_{t-}}^{-1}(dx)$  on  $\mathbb{R}_+ \times [0, 1]$ . Thus Th.II.1.8 in [45] ensures that  $M^{(2)}$  is a local martingale on  $[0, T)$ . Indeed, it suffices to take  $W(\omega, t, x) := \mathbf{1}_{\{t < T\}} x^k (1-x)^{n-k}$  and to apply the theorem to the random measure of Equation (II.24).

We have proved that both  $M^{(1)}$  and  $M^{(2)}$  are local martingales w.r.t.  $\mathcal{F}$ , this implies that the process

$$\left( L_t(n, \mathbf{K}) - \int_0^t \lambda_{n,k}(Z_{s-}, \Psi) ds, t \in [0, T) \right)$$

is a local martingale on  $[0, T)$  w.r.t.  $\mathcal{F}$ .

Finally, consider the vector formed by the  $d_n$  counting processes. Since we have identified for each of them their compensator in a same filtration  $\mathcal{F}$ , we have identified the compensator of the vector. The proposition is proved.  $\blacksquare$

## 4 The Eve property

Throughout this section,  $m$  designates a càdlàg  $\Psi$ -MVBP started from the Lebesgue measure on  $[0, 1]$ ,  $Z$  denotes its total-mass process and  $T$  its lifetime. In the first subsection, we define the Eve property and prove Theorem II.2. In the second subsection, we identify a complete sequence of Eves and prove Theorem II.5. Some properties of the Eves are given in the third subsection.

### 4.1 Definition

Recall the definition given in the introduction.

**Definition II.1** We say that the branching mechanism  $\Psi$  satisfies the Eve property if and only if there exists a random variable  $e$  in  $[0, 1]$  such that

$$\frac{m_t(dx)}{m_t([0, 1])} \xrightarrow[t \uparrow T]{} \delta_e(dx) \text{ a.s.} \quad (\text{II.25})$$

in the sense of weak convergence of probability measures. The r.v.  $e$  is called the primitive Eve of the population.

**LEMMA II.14.** Suppose that the Eve property is verified. Then  $e$  is uniform $[0, 1]$ .

**Proof** Let us consider a bijection  $f$  from  $[0, 1]$  to  $[0, 1]$  that preserves the Lebesgue measure. For all  $t \geq 0$ , we denote by  $m_t \circ f^{-1}$  the pushforward measure of  $m_t$  by the function  $f$ . The process  $(m_t \circ f^{-1}, t \in [0, T])$  is still a  $\Psi$ -MVBP whose lifetime is  $T$ . Thus there exists a r.v.  $e' \in [0, 1]$  such that

$$\frac{m_t \circ f^{-1}(dx)}{m_t([0, 1])} \xrightarrow[t \rightarrow T]{} \delta_{e'}(dx) \text{ a.s.} \quad (\text{II.26})$$

Moreover, it is immediate to check that  $e' := f(e)$  and that  $e'$  and  $e$  have the same distribution. We deduce that  $e$  is a r.v. on  $[0, 1]$  whose distribution is invariant under bijections that preserve the Lebesgue measure. Hence it is a uniform $[0, 1]$  r.v. ■

The following proposition specifies an important case where the Eve property is fulfilled.

**PROPOSITION II.15.** If  $T < \infty$  a.s. then  $\Psi$  satisfies the Eve property.

**Proof** Suppose that  $T < \infty$  a.s. The branching property fulfilled by the process  $(m_t, t \geq 0)$  ensures that

$$(m_t([0, 2^{-n}]), t \in [0, T]); (m_t([2^{-n}, 2 \times 2^{-n}]), t \in [0, T]); \dots; (m_t([1 - 2^{-n}, 1]), t \in [0, T])$$

are  $2^n$  i.i.d.  $\Psi$ -CSBP started from  $2^{-n}$  and stopped at the infimum of their lifetimes. Since the lifetimes of these CSBP are independent and finite a.s., we deduce from Lemma II.2 that they are distinct a.s. and that  $T$  is either the first explosion time or the last extinction time of the preceding collection. Therefore, for all  $i \in [2^n]$ ,

$$\lim_{t \rightarrow T} \frac{m_t([(i-1)2^{-n}, i2^{-n}])}{m_t([0, 1])} \in \{0, 1\} \text{ a.s.}$$

This implies that there exists a unique (random) integer  $u_n \in [2^n]$  such that

$$\lim_{t \rightarrow T} \frac{m_t([(u_n-1)2^{-n}, u_n 2^{-n}])}{m_t([0, 1])} = 1 \text{ a.s.}$$

This holds for all  $n \in \mathbb{N}$  and obviously  $[(u_n-1)2^{-n}, u_n 2^{-n}] \supset [(u_{n+1}-1)2^{-(n+1)}, u_{n+1} 2^{-(n+1)}]$ . We can therefore introduce the following random variable

$$e := \inf_{n \in \mathbb{N}} u_n 2^{-n}$$

We have proved that

$$\frac{m_t(dx)}{m_t([0, 1])} \xrightarrow[t \rightarrow T]{} \delta_e(dx) \text{ a.s.} \quad (\text{II.27})$$

in the sense of weak convergence of probability measures. ■

**REMARK II.16.** A complete classification of the asymptotic behaviour of  $\frac{m_t(\cdot)}{m_t([0,1])}$  will be established in a forthcoming work [25]. In particular it will be shown that whenever the CSBP is supercritical, the Eve property is fulfilled if and only if the mean is infinite: an intuitive argument for this result is that two independent copies of a same CSBP have comparable asymptotic sizes iff the mean is finite.

We now present a result that relates the Eve property with the behaviour of the  $\Psi$  flow of partitions at the end of its lifetime  $T$ . In addition, this result provides a necessary and sufficient condition on  $Z$  for the Eve property to hold.

**Theorem II.2** *There exists an exchangeable partition  $\hat{\Pi}_T$  such that  $\hat{\Pi}_t \rightarrow \hat{\Pi}_T$  almost surely as  $t \uparrow T$ . Moreover, these three assumptions are equivalent*

- i)  $\Psi$  satisfies the Eve property.
- ii)  $\hat{\Pi}_T = \mathcal{I}_{[\infty]}$  a.s.
- iii)  $\sum_{\{s < T: \Delta Z_s > 0\}} \left(\frac{\Delta Z_s}{Z_s}\right)^2 + \int_0^T \frac{\sigma^2}{Z_s} ds = \infty$  a.s.

**REMARK II.17.** This theorem should be compared with Theorem 6.1 in [24] where a similar condition on the total-mass process is given but for a much larger class of measure-valued processes. However their result is proved only when  $T < \infty$ , which in our particular case of branching processes, is a trivial case as we already know from Proposition II.15 that the Eve property is fulfilled.

**Proof** To prove the asserted convergence, it suffices to show that for each  $n \in \mathbb{N}$ , the restriction  $\hat{\Pi}_t^{[n]}$  of  $\hat{\Pi}_t$  to  $\mathcal{P}_n$  admits a limit when  $t \uparrow T$  almost surely. We fix  $n \in \mathbb{N}$  until the end of the proof.

*Step 1.* Remark that conditional on  $\int_0^T \frac{\sigma^2}{Z_s} ds$ , the r.v.  $\#\{\mathcal{N}_{\sigma|_{[0,T]} \times \mathcal{P}_n^*}\}$  has a Poisson distribution with parameter  $\binom{n}{2} \int_0^T \frac{\sigma^2}{Z_s} ds$ . Thus we have almost surely

$$\mathbb{P}\left(\#\{\mathcal{N}_{\sigma|_{[0,T]} \times \mathcal{P}_n^*}\} = 0 \mid \int_0^T \frac{\sigma^2}{Z_s} ds\right) = \exp\left(-\binom{n}{2} \int_0^T \frac{\sigma^2}{Z_s} ds\right) \quad (\text{II.28})$$

In addition, thanks to Borel-Cantelli lemma we notice that

$$\begin{aligned} & \mathbb{P}\left(\#\{\mathcal{N}_{\nu|_{[0,T]} \times \mathcal{P}_n^*}\} = 0 \mid \sum_{s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 = \infty\right) \leq \\ & 1 - \mathbb{P}\left(\#\{\mathcal{N}_{\nu|_{[0,T]} \times \mathcal{P}_n^*}\} = \infty \mid \sum_{s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 = \infty\right) = 0 \end{aligned} \quad (\text{II.29})$$

*Step 2.* Introduce  $t_i := \inf\{t \geq 0 : \hat{\Pi}_t^{[i]} = \mathcal{I}_{[i]}\}$  for all  $i \in \mathbb{N}$ , we first prove that  $t_n < T$  conditional on  $\{\sum_{s < T} (\frac{\Delta Z_s}{Z_s})^2 + \int_0^T \frac{\sigma^2}{Z_s} ds = \infty\}$ , thus it will imply that  $\hat{\Pi}_t^{[n]} \rightarrow \mathcal{I}_{[n]}$  as  $t \uparrow T$  on the same event, and also the implication iii)  $\Rightarrow$  ii). We proceed via a recursion. At rank  $i = 2$ , we use Equations (II.28) and (II.29) to obtain

$$\mathbb{P}\left(\#\{\mathcal{P}_{[0,T]} \times \mathcal{P}_2^*\} = 0 \mid \sum_{s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 + \int_0^T \frac{\sigma^2}{Z_s} ds = \infty\right) = 0$$

Hence  $t_2 < T$  a.s. Suppose that  $t_{i-1} < T$  almost surely for a given integer  $i \geq 3$ , then we have  $\hat{\Pi}_{t_{i-1}}^{[i-1]} = \mathcal{I}_{[i-1]}$  a.s. Thus either  $t_i = t_{i-1}$  and the recursion is complete, or  $\hat{\Pi}_{t_{i-1}}^{[i]} = \{1, \dots, i-1\}, \{i\}$ .

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In the latter case, we need to prove that on  $[t_{i-1}, T)$  there will be a reproduction event involving an integer in  $[i-1]$  and the integer  $i$ . We denote by  $A_i$  the subset of  $\mathcal{P}_i^*$  whose elements are partitions with a non-singleton block containing an integer lower than  $i-1$  and the integer  $i$ . Remark that on the event  $\{\sum_{s < T} (\frac{\Delta Z_s}{Z_s})^2 + \int_0^T \frac{\sigma^2}{Z_s} ds = \infty\}$

$$\sum_{t_{i-1} < s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 + \int_{t_{i-1}}^T \frac{\sigma^2}{Z_s} ds = \infty \text{ a.s.}$$

so that almost surely

$$\begin{aligned} \mathbb{P}\left(\#\{\mathcal{N}_{\sigma|(t_{i-1}, T) \times A_i}\} = 0 \mid \int_0^T \frac{\sigma^2}{Z_s} ds = \infty\right) &= 0 \\ \mathbb{P}\left(\#\{\mathcal{N}_{\nu|(t_{i-1}, T) \times A_i}\} = 0 \mid \sum_{s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 = \infty\right) &= 0 \end{aligned}$$

which in turn ensures that  $t_i < T$  a.s. The recursion is complete.

*Step 3.* We now prove that conditional on  $\{\sum_{s < T} (\frac{\Delta Z_s}{Z_s})^2 + \int_0^T \frac{\sigma^2}{Z_s} ds < \infty\}$ , the number of reproduction events  $\#\{\mathcal{P}_{|[0, T) \times \mathcal{P}_n^*}\}$  is finite. This will imply that  $\hat{\Pi}_t^{[n]}$  admits a limit as  $t \uparrow T$  on the same event.

Thanks to the remark preceding (II.28), we deduce that on  $\{\int_0^T \frac{\sigma^2}{Z_s} ds < \infty\}$ , the r.v.  $\#\{\mathcal{N}_{\sigma|[0, T) \times \mathcal{P}_n^*}\}$  is finite. In addition, for each  $s \geq 0$  such that  $\Delta Z_s > 0$ , the probability that the restriction of  $\varrho_s$  to  $\mathcal{P}_n$  differs from  $0_{[n]}$  is equal to

$$\sum_{k=2}^n \binom{n}{k} \left(\frac{\Delta Z_s}{Z_s}\right)^k \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{n-k}$$

independently of the other  $(\varrho_t)_{t \neq s}$ . Since

$$\sum_{s < T} \sum_{k=2}^n \binom{n}{k} \left(\frac{\Delta Z_s}{Z_s}\right)^k \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{n-k} \leq \sum_{s < T} \sum_{k=2}^n \binom{n}{k} \left(\frac{\Delta Z_s}{Z_s}\right)^2$$

an application of Borel-Cantelli lemma implies that

$$\mathbb{P}\left(\#\{\mathcal{N}_{\nu|[0, T) \times \mathcal{P}_n^*}\} < \infty \mid \sum_{s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 < \infty\right) = 1$$

Thus we have proved that  $\#\{\mathcal{P}_{|[0, T) \times \mathcal{P}_n^*}\}$  is finite on the event  $\{\sum_{s < T} (\frac{\Delta Z_s}{Z_s})^2 + \int_0^T \frac{\sigma^2}{Z_s} ds < \infty\}$ .

*Step 4.* We now prove that

$$\mathbb{P}\left(\sum_{s < T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 + \int_0^T \frac{\sigma^2}{Z_s} ds < \infty\right) > 0 \Rightarrow \mathbb{P}(\hat{\Pi}_T \neq \mathcal{I}_{[\infty]}) > 0$$

this will imply ii)  $\Rightarrow$  iii).

Thanks to Equation (II.28), we get

$$\mathbb{P}\left(\#\{\mathcal{N}_{\sigma|[0, T) \times \mathcal{P}_2^*}\} = 0 \mid \int_0^T \frac{\sigma^2}{Z_s} ds < \infty\right) > 0$$

Also, note that

$$\mathbb{P}\left(\#\{\mathcal{N}_{\nu|[0, T) \times \mathcal{P}_2^*}\} = 0 \mid \left\{\left(\frac{\Delta Z_s}{Z_s}\right)^2; s < T\right\}\right) = \prod_{s < T} \left(1 - \left(\frac{\Delta Z_s}{Z_s}\right)^2\right)$$

One can readily prove that the r.h.s. is strictly positive on the event  $\{\sum_{s<T} (\frac{\Delta Z_s}{Z_s})^2 < \infty\}$ . Therefore we have proven that

$$\mathbb{P}\left(\hat{\Pi}_T^{[2]} = 0_{[2]} \mid \sum_{s<T} \left(\frac{\Delta Z_s}{Z_s}\right)^2 + \int_0^T \frac{\sigma^2}{Z_s} ds < \infty\right) > 0$$

This inequality ensures the implication ii)  $\Rightarrow$  iii).

*Step 5.* We turn our attention to the proof of ii)  $\Leftrightarrow$  i). Consider a sequence  $(\xi_0(i))_{i \geq 1}$  of i.i.d. uniform $[0, 1]$  r.v. and let  $(\xi_t(i), t \geq 0)_{i \geq 1} := \mathcal{L}_0(\hat{\Pi}, (\xi_0(i))_{i \geq 1})$  be the lookdown process defined from this last sequence and the flow of partitions  $\hat{\Pi}$ . We know that  $(Z_t \cdot \Xi_t(\cdot), t \geq 0)$  is a  $\Psi$ -MVBP, where  $(\Xi_t, t \geq 0) := \mathcal{E}_0(\hat{\Pi}, (\xi_0(i))_{i \geq 1})$ . Moreover, a.s. for all  $t \in [0, T)$

$$\begin{aligned} \Xi_t(\{\xi_0(1)\}) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \mathbf{1}_{\{\xi_t(i) = \xi_0(1)\}} \\ &= |\hat{\Pi}_t(1)| \end{aligned}$$

It is intuitively easy to see that the Eve property is equivalent with the almost sure convergence

$$\Xi_t(\{\xi_0(1)\}) \xrightarrow{t \uparrow T} 1$$

Roughly speaking, the primitive Eve is necessarily the type  $\xi_0(1)$  in the sequence of initial types of the lookdown representation. For a rigorous proof of this result, see Proposition II.25. Then, it is sufficient to show the following equivalence

$$\Xi_t(\{\xi_0(1)\}) \xrightarrow{t \uparrow T} 1 \text{ a.s.} \iff \hat{\Pi}_t \xrightarrow{t \uparrow T} \mathcal{I}_{[\infty]} \text{ a.s.}$$

Since  $\hat{\Pi}_t$  is, conditionally on  $\{t < T\}$ , an exchangeable random partition, we deduce that for all  $n \geq 1$

$$\mathbb{P}\left(\hat{\Pi}_t^{[n]} = \mathcal{I}_{[n]} \mid |\hat{\Pi}_t(1)|\right) = |\hat{\Pi}_t(1)|^{n-1}$$

Thus

$$\Xi_t(\{\xi_0(1)\}) \xrightarrow{t \uparrow T} 1 \text{ a.s.} \iff |\hat{\Pi}_t(1)| \xrightarrow{t \uparrow T} 1 \text{ a.s.} \iff \forall n \geq 1, \hat{\Pi}_T^{[n]} = \mathcal{I}_{[n]} \text{ a.s.}$$

The proof is complete. ■

Thanks to this theorem, we can set  $\hat{\Pi}_t := \hat{\Pi}_T$  for all  $t \geq T$ .

## 4.2 An ordering of the ancestors

Consider the following definition of an ancestor.

**DEFINITION II.18.** *Fix a point  $x \in [0, 1]$ . If there exists a time  $t > 0$  such that  $m_t(\{x\}) > 0$ , then we say that  $x$  is an ancestor of  $m$  and  $m_t(\{x\})$  is called its progeny at time  $t$ . We say that an ancestor  $x$  becomes extinct at time  $d$ , if  $d := \sup\{t > 0 : m_t(\{x\}) > 0\}$  is finite. If so,  $d$  is called its extinction time.*

Thanks to the lookdown representation the set of ancestors is countable almost surely. Indeed, an ancestor is a point of the atomic support of the MVBP at a given time. Since at any time, the atomic support is included in the set of initial types (of the lookdown representation) and since this last set is countable, the result follows.

**REMARK II.19.** *In the infinite variation case, one can identify an ancestor and its progeny with a Lévy tree among the Lévy forest that represents the genealogy of a CSBP. For further details on Lévy trees see [26, 27, 61].*

The progeny  $m_t(\{x\})$  of an ancestor  $x$  has the same possible long-term behaviours as a  $\Psi$ -CSBP (these behaviours have been recalled in Subsection 2.2). We thus propose a classification of the  $\Psi$ -MVBP according to these possible behaviours; for the moment we do not require the Eve property to be verified. Recall  $d_t, w_t$  from Equation (II.11).

### Classification of the behaviours

- *Extinction.* The total-mass process  $Z$  reaches 0 in finite time. All the ancestors become extinct in finite time but no two of them simultaneously. At any time  $t \in (0, T)$ ,  $m_t$  has finitely many atoms, hence the number of ancestors that have not become extinct is finite, and  $m_t$  has no continuous part, that is,  $d_t = 0$ .
- *Explosion.* The total-mass process  $Z$  reaches  $\infty$  in finite time. All the ancestors, except the primitive Eve, have finite progenies at time  $T$ .
- *Infinite lifetime - no extinction of ancestors.*  $T = \infty$  and  $\Psi$  is either negative or has a second positive root  $q \in [0, \infty)$  and verifies  $\int_0^\infty \frac{du}{\Psi(u)} = \infty$ . Then no ancestor becomes extinct in finite time, and their progenies reach, in infinite time, 0 or  $\infty$ .
- *Infinite lifetime - possible extinction of ancestors.*  $T = \infty$ ,  $\Psi$  has a second positive root  $q > 0$  and  $\int_0^\infty \frac{du}{\Psi(u)} < \infty$ . The set of ancestors can be subdivided into those, infinitely many, which become extinct in finite time (no two of them simultaneously) and those, finitely many, whose progenies reach  $\infty$  in infinite time.

Additionally in the last two cases, the number of ancestors whose progenies reach  $\infty$  is Poisson with parameter  $q$  under the condition that  $\Psi$  is conservative.

**REMARK II.20.** *A  $\Psi$ -MVBP enjoys at most two distinct behaviours: one on the event  $\{Z_T = 0\}$  and another on the event  $\{Z_T = \infty\}$ .*

**EXAMPLE II.21.** *Let us give some examples that illustrate the previous cases*

- $\Psi(u) = u^2$ , the  $\Psi$ -CSBP reaches 0 in finite time almost surely and so we are in the Extinction case almost surely.
- $\Psi(u) = -\sqrt{u}$ , the  $\Psi$ -CSBP reaches  $+\infty$  in finite time almost surely and so we are in the Explosion case almost surely.
- $\Psi(u) = u \ln(u)$ , this is called the Neveu CSBP: it has an infinite lifetime almost surely and so we are in the Infinite lifetime - no extinction of ancestors case almost surely.
- $\Psi(u) = u \ln(u) + u^2$ , this CSBP reaches either 0 in finite time or  $\infty$  in infinite time. On the event  $\{Z_T = \infty\}$  we are in the Infinite lifetime - possible extinction of ancestors case, while on the event  $\{Z_T = 0\}$  we are in the Extinction case.

**Proof (Classification of the behaviours)** It is plain that these four cases cover all the possible combinations of branching mechanisms and asymptotic behaviours of the total-mass processes.

*Extinction case.* If  $d_t > 0$  or  $w_t$  is an infinite measure then we have  $\mathbb{P}(Z_t = 0) = 0$ , therefore necessarily  $w_t$  is a finite measure and  $d_t = 0$ . Since each atom of  $m_t$  is associated with an ancestor with

a positive progeny at time  $t$ , we deduce that at any time  $t > 0$  only finitely many ancestors have not become extinct. Now condition on  $\{t < T\}$  and consider two ancestors  $x_1$  and  $x_2$  in  $[0, 1]$  not yet extinct at time  $t$ . Their progenies after time  $t$  are given by two independent  $\Psi$ -CSBP  $(m_{t+s}(\{x_1\}), s \geq 0)$  and  $(m_{t+s}(\{x_2\}), s \geq 0)$ . The extinction times of these two ancestors are then distinct a.s. thanks to Lemma II.2.

*Explosion case.* Since two independent  $\Psi$ -CSBP cannot explode at the same finite time thanks to Lemma II.2, we deduce that only one ancestor has an infinite progeny at time  $T$ .

*Infinite lifetime cases.* The Poisson distribution of the statement can be derived from Lemma 2 in [11]. Let  $x \in [0, 1]$  be an ancestor. Then, there exists  $t > 0$  s.t.  $m_t(\{x\}) > 0$ . The process  $(m_{t+s}(\{x\}), s \geq 0)$  is a  $\Psi$ -CSBP started from  $m_t(\{x\})$ , and so, either it reaches  $\infty$  in infinite time with probability  $1 - e^{-m_t(\{x\})q}$  or it reaches 0 with the complementary probability. In the latter case, it reaches 0 in finite time if and only if  $\int^\infty \frac{du}{\Psi(u)} < \infty$ .

Now consider the case  $\int^\infty \frac{du}{\Psi(u)} < \infty$ . Remark that in that case  $d_t = 0$  for all  $t > 0$  and that there is no simultaneous extinction (same proof as above). Let us prove that infinitely many ancestors become extinct in finite time. Consider the lookdown representation of the  $\Psi$ -MVBP: we stress that the set of initial types is exactly equal to the set of ancestors. We have already proved one inclusion at the beginning of this subsection: each ancestor is an initial type. The converse is obtained as follows. Observe first that  $\Psi$  is necessarily the Laplace exponent of a Lévy process with infinite variation paths since otherwise  $\int^\infty \frac{du}{\Psi(u)} < \infty$  would not hold. Therefore Theorem II.3 ensures that the partitions  $\hat{\Pi}_{0,t}, t > 0$  have no singleton: each block has a strictly positive asymptotic frequency and therefore each initial types of the lookdown representation has a strictly positive frequency. This implies that each initial type is necessarily an ancestor. As the initial types are infinitely many, so are the ancestors: a Poisson number of them have a progeny that reaches  $\infty$  in infinite time, hence infinitely many become extinct in finite time. ■

**REMARK II.22.** *One should compare the Extinction case with the behaviour of the  $\Lambda$  Fleming-Viot processes that come down from infinity. But in that setting, the question of simultaneous loss of ancestral types remains open, see Section 6 or [52] for further details.*

**THEOREM II.5** *Assume that  $Z$  does not reach  $\infty$  in finite time. If the Eve property holds then one can order the ancestors by persistence/predominance. We denote this ordering  $(e^i)_{i \geq 1}$  and call these points the Eves. In particular,  $e^1$  is the primitive Eve.*

The persistence of an ancestor refers to the extinction time of its progeny (when it reaches 0 in finite time) while the predominance denotes the asymptotic behaviour of its progeny (when it does not become extinct in finite time). The proof of this theorem is thus split into the *Extinction* and the *Infinite lifetime* cases. Note that we have excluded the case where the  $\Psi$ -CSBP is non-conservative for a reason given in Remark II.24.

### Extinction case

One can enumerate the ancestors of  $(m_t, t \in [0, T])$ , say  $(e^i)_{i \geq 1}$ , in the decreasing order of their extinction times  $(d^i)_{i \geq 1}$ , that is,  $T = d^1 > d^2 > d^3 \dots > 0$ . In particular,  $e^1$  is the primitive Eve.

### Infinite lifetime case

We let  $e^1$  be the primitive Eve of the population: necessarily it does not become extinct in finite time. Then we use the following result.

**LEMMA II.23.** *In the Infinite lifetime - no extinction of ancestors case: there exists a sequence*

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$(e^i)_{i \geq 2}$  such that for all  $i \geq 2$

$$\lim_{t \rightarrow \infty} \frac{m_t(\{e^i\})}{m_t([0, 1] \setminus \{e^1, \dots, e^{i-1}\})} = 1$$

*In the Infinite lifetime - possible extinction of ancestors case: let  $K$  be the random number of ancestors which never become extinct. There exists a sequence  $(e^i)_{i \geq 2}$  such that for all  $i \in \{2, \dots, K\}$*

$$\lim_{t \rightarrow \infty} \frac{m_t(\{e^i\})}{m_t([0, 1] \setminus \{e^1, \dots, e^{i-1}\})} = 1$$

*and  $(e^i)_{i > K}$  are the remaining ancestors in the decreasing order of their extinction times  $(d^i)_{i > K}$ , that is,  $T = \infty > d^{K+1} > d^{K+2} > d^{K+3} \dots > 0$ .*

**Proof** We focus on the *Infinite lifetime - no extinction of ancestors* case, the other case is then a mixture of the latter with the *Extinction* case. For every  $n \in \mathbb{N}$ , we subdivide  $[0, 1]$  into

$$[0, 2^{-n}), [2^{-n}, 2 \times 2^{-n}), \dots, [1 - 2^{-n}, 1]$$

Since these intervals are disjoint, the restrictions of  $m_t$  to each of them are independent. Therefore, we define for each  $i \in [2^n]$ , the random point  $e(i, n)$  as the Eve of the process

$$(m_t(\cdot \cap [(i-1)2^{-n}, i2^{-n})), t \geq 0)$$

(note that for  $i = 2^n$  we take  $[1 - 2^{-n}, 1]$ ). In addition, one can define an ordering of the collection  $(e(i, n))_{i \in [2^n]}$  according to the asymptotic behaviours of their progenies. More precisely, for two integers  $i \neq j \in [2^n]$ , thanks to the classification of the behaviours and the Eve property, we have

$$\lim_{t \rightarrow \infty} \frac{m_t(\{e(i, n)\})}{m_t(\{e(j, n)\})} \in \{0, \infty\}$$

Thus, there exist two r.v.  $i_1^n \neq i_2^n \in [2^n]$  such that for all  $i \in [2^n]$

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{m_t(\{e(i_1^n, n)\})}{m_t(\{e(i, n)\})} &= \infty \text{ if } i \neq i_1^n \\ \lim_{t \rightarrow \infty} \frac{m_t(\{e(i_2^n, n)\})}{m_t(\{e(i, n)\})} &= \infty \text{ if } i \neq i_1^n, i_2^n \end{aligned}$$

We set  $e^1(n) := e(i_1^n, n)$  and  $e^2(n) := e(i_2^n, n)$ . We claim that almost surely the sequences  $(e^1(n))_{n \geq 1}$  and  $(e^2(n))_{n \geq 1}$  are eventually constant. This is clear for  $(e^1(n))_{n \geq 1}$  since for each  $n \geq 1$ ,  $e^1(n) = e^1$ , which is the primitive Eve of the entire population  $[0, 1]$ . We turn our attention to the sequence  $(e^2(n))_{n \geq 1}$ , in that case the claim is not so clear. Roughly speaking, the wild behaviour this sequence could have is the following: infinitely often, the second Eve  $e^2(n)$  is "hidden" in the interval  $[(i_1^{n-1} - 1)2^{-(n-1)}, i_1^{n-1}2^{-(n-1)})$  containing the first Eve  $e^1$  at rank  $n - 1$ , but we will see that it cannot occur. Suppose that the claim does not hold. Thus there exists an event  $E$  of positive probability on which there exists a sequence  $(n_k)_{k \geq 1}$  of integers such that  $e^2(n_k - 1) \neq e^2(n_k)$  for every  $k \geq 1$ . From the consistency of the restrictions of the MVBP  $m$  to the subintervals defined at ranks  $n_k - 1$  and  $n_k$ , we deduce that  $e^2(n_k)$  is in  $[(i_1^{n_k-1} - 1)2^{-(n_k-1)}, i_1^{n_k-1}2^{-(n_k-1)})$ , that is, the same interval as  $e^1$  at rank  $n_k - 1$ . Hence on the event  $E$

$$|e^1 - e^2(n)| \rightarrow 0 \text{ as } n \uparrow \infty \tag{II.30}$$



We now exhibit a contradiction. By the exchangeability of the increments of the MVBP  $m$ , we know that  $(i_1^n, i_2^n)$  is distributed uniformly among the pairs of integers in  $[2^n]$ . Therefore, one easily deduces that for all  $n \geq p$

$$\mathbb{P}(|e^1 - e^2(n)| \leq 2^{-p}) \leq \mathbb{P}(|i_1^n - i_2^n| \leq 2^{n-p} + 1) \leq 2^{2-p}$$

This implies that the convergence of Equation (II.30) holds with probability 0, and  $E$  cannot have positive probability. Therefore our initial claim is proved and we can define  $e^2 := \lim_{n \rightarrow \infty} e^2(n)$ .

The property is proved for the first two ancestors  $e^1$  and  $e^2$ . The general case is obtained similarly.  $\blacksquare$

**REMARK II.24.** *In the case where  $Z$  reaches  $\infty$  in finite time (non-conservative case), we cannot obtain a relevant ordering. Indeed, in that case all the progenies  $m_T(\{x\})$  of the ancestors  $x \neq e$  are finite at time  $T$ . Therefore, no natural order appears in that setting.*

### 4.3 The Eves and the lookdown representation

Let us motivate the previous ordering by presenting a striking connection with the lookdown representation. The following proposition implies that, if the process of limiting empirical measures of a lookdown process is equal to a given  $\Psi$ -MVBP, then the initial types are necessarily the sequence of Eves of the  $\Psi$ -MVBP. We denote by  $r_t$  the probability measure obtained by rescaling  $m_t$  by its total-mass  $Z_t$ .

**PROPOSITION II.25.** *Assume that the Eve property is fulfilled and that the  $\Psi$ -CSBP does not explode in finite time. Consider a  $\Psi$  flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  defined from the  $\Psi$ -CSBP  $Z$  and a sequence  $(\xi_0(i))_{i \geq 1}$  of r.v. taking distinct values in  $[0, 1]$ . Let  $(\Xi_t, t \in [0, T]) := \mathcal{E}_0(\hat{\Pi}, (\xi_0(i))_{i \geq 1})$  be the limiting empirical measures of the lookdown process defined from these objects. If  $(\Xi_t, t \in [0, T]) = (r_t, t \in [0, T])$  a.s., then  $(\xi_0(i))_{i \geq 1} = (e^i)_{i \geq 1}$  a.s.*

**Proof** We prove the proposition in the *Extinction* and *Infinite lifetime - no extinction of ancestors* cases, as the *Infinite lifetime - possible extinction of ancestors* case is a combination of these two cases. Consider the lookdown process

$$(\xi_t(i), t \in [0, T])_{i \geq 1} := \mathcal{L}_0(\hat{\Pi}, (\xi_0(i))_{i \geq 1})$$

Suppose this lookdown process verifies the assumptions of the proposition: there exists an event  $\Omega^*$  of probability 1 on which

$$(\Xi_t, t \in [0, T]) = (r_t, t \in [0, T]) \quad (\text{II.31})$$

We have to prove that  $\xi_0(i) = e^i$  for all  $i \geq 1$  a.s. First we notice that each initial type  $\xi_0(i)$  (resp. each ancestor  $e^i$ ) is associated with a process of frequencies  $(|\hat{\Pi}_{0,t}(i)|, t \geq 0)$  (resp.  $(r_t(\{e^i\}), t \geq 0)$ ). In addition, we have

$$\Xi_t(dx) = \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)| \delta_{\xi_0(i)}(dx) + \left(1 - \sum_{i \geq 1} |\hat{\Pi}_{0,t}(i)|\right) dx \quad (\text{II.32})$$

We work on the event  $\Omega^*$  throughout this proof.

*Extinction case.* There is no drift part in Equation (II.32), and the two sets  $\{\xi_0(i); i \geq 1\}$  and  $\{e^i; i \geq 1\}$  are equal. The initial types  $\{\xi_0(i); i \geq 1\}$  of the lookdown process are ordered by decreasing persistence by construction. The Eves of the  $\Psi$ -MVBP  $(m_t, t \in [0, T])$  are also ordered by decreasing persistence. Therefore  $\xi_0(i) = e^i$  for all  $i \geq 1$ .

*Infinite lifetime - no extinction of ancestors case.* From Equations (II.31) and (II.32), we know that

$$(|\hat{\Pi}_{0,t}(i)|)_{i \geq 1}^\downarrow = (r_t(\{e^i\}))_{i \geq 1}^\downarrow \quad (\text{II.33})$$

for all  $t \in [0, T)$ . By definition of the ancestors, we know that for every  $i \geq 1$

$$\mathbb{P}\left[\left(r_t(\{e_0^j\})\right)_{1 \leq j \leq i} = \left(r_t(\{e_0^j\})\right)_{1 \leq j \leq i}^\downarrow\right] \xrightarrow{t \rightarrow \infty} 1$$

Thus, using the exchangeability of the partition  $\hat{\Pi}_{0,t}$ , Equation (II.33) and the last identity, we deduce that for every  $i \geq 1$

$$\mathbb{P}\left[\left(|\hat{\Pi}_{0,t}(j)|\right)_{1 \leq j \leq i} = \left(|\hat{\Pi}_{0,t}(j)|\right)_{1 \leq j \leq i}^\downarrow\right] \xrightarrow{t \rightarrow \infty} 1$$

which entails that

$$\mathbb{P}\left[\left(\xi_0(j)\right)_{1 \leq j \leq i} = \left(e^j\right)_{1 \leq j \leq i}\right] = 1$$

This concludes the proof. ■

We now determine the distribution of the sequence of Eves  $(e^i)_{i \geq 1}$ .

**PROPOSITION II.26.** *The sequence  $(e^i)_{i \geq 1}$  is i.i.d. uniform $[0, 1]$  and is independent of the sequence of processes  $(m_t(\{e^i\}), t \in [0, T))_{i \geq 1}$*

**Proof** Consider a sequence  $(\xi_0(i))_{i \geq 1}$  of i.i.d. uniform $[0, 1]$  r.v. and a  $\Psi$  flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  defined from the  $\Psi$ -CSBP  $(Z_t, t \in [0, T))$ . Let  $(\Xi_t, t \in [0, T)) := \mathcal{E}_0(\hat{\Pi}, (\xi_0(i))_{i \geq 1})$  be the limiting empirical measures of the corresponding lookdown process. Denote by  $\Phi$  the measurable map that associates to a  $\Psi$ -MVBP its sequence of Eves. From Proposition II.25, we deduce that a.s.

$$\Phi((Z_t \cdot \Xi_t(\cdot), t \in [0, T))) = (\xi_0(i))_{i \geq 1}$$

Since  $(Z_t \cdot \Xi_t(\cdot), t \in [0, T)) \stackrel{(d)}{=} (m_t(\cdot), t \in [0, T))$ , we obtain thanks to Proposition II.25 the following identity:

$$\left((m_t(\cdot), t \in [0, T)), (e^i)_{i \geq 1}\right) \stackrel{(d)}{=} \left((Z_t \cdot \Xi_t(\cdot), t \in [0, T)), (\xi_0(i))_{i \geq 1}\right) \quad (\text{II.34})$$

Therefore  $(e^i)_{i \geq 1}$  is a sequence of i.i.d. uniform $[0, 1]$  r.v. In addition the collection of asymptotic frequencies  $(Z_t \cdot \Xi_t(\{\xi_0(i)\}), t \in [0, T))_{i \geq 1}$  only depends on  $\hat{\Pi}$ , thus it is independent of the initial types  $(\xi_0(i))_{i \geq 1}$ . The asserted result follows. ■

## 5 Some properties of the genealogy

Consider a branching mechanism  $\Psi$ , a  $\Psi$ -CSBP  $Z$  started from 1 assumed to be càdlàg and a  $\Psi$  flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  defined from the  $\Psi$ -CSBP  $Z$ . We present some properties of the  $\Psi$  flow of partitions before stating a limit theorem. For the sake of simplicity, let  $\hat{\Pi}_t := \hat{\Pi}_{0,t}$  for all  $t \geq 0$  (recall that Theorem II.2 allows us to extend this process after time  $T$ ).

### 5.1 Dust and modification

**THEOREM II.3** *The following dichotomy holds:*

- *If  $\sigma = 0$  and  $\int_0^\infty (1 \wedge h)\nu(dh) < \infty$ , then almost surely for all  $t \in (0, T)$ , the partition  $\hat{\Pi}_t$  has singleton blocks.*
- *Otherwise, almost surely for all  $t \in (0, T)$ , the partition  $\hat{\Pi}_t$  has no singleton blocks.*

Furthermore when  $\sigma = 0$ , almost surely for all  $t \in (0, T]$  the asymptotic frequency of the dust component of  $\hat{\Pi}_t$  is equal to  $\prod_{s \leq t} (1 - \frac{\Delta Z_s}{Z_s})$  whereas when  $\sigma > 0$ , almost surely for all  $t \in (0, T]$  there is no dust.

**REMARK II.27.** *The condition  $\sigma = 0$  and  $\int_0^\infty (1 \wedge h)\nu(dh) < \infty$  is equivalent to saying that  $\Psi$  is the Laplace exponent of a Lévy process with finite variation paths.*

**Proof** By the definition of a  $\Psi$  flow of partitions, we know that, conditional on  $t < T$ ,  $\hat{\Pi}_t$  is distributed as a paint-box on the subordinator  $S_{0,t}$  therefore it has no singleton blocks iff  $d_t = 0$  (recall that  $d_t$  is the drift term of the Laplace exponent  $u_t(\cdot)$ ). Since for all  $t, s \geq 0$ ,  $u_{t+s}(\cdot) = u_t \circ u_s(\cdot)$ , classical results ensure that  $d_{t+s} = d_t \cdot d_s$ . Therefore

$$\exists t > 0, d_t > 0 \Leftrightarrow \forall t > 0, d_t > 0$$

Also, the equivalence  $d_t > 0 \Leftrightarrow \sigma = 0$  and  $\int_0^\infty (1 \wedge h)\nu(dh) < \infty$  can be found in [72].

Suppose now that  $\sigma = 0$ . Classical results on exchangeable partitions (see [9] for instance) ensure that the asymptotic frequency of the dust is almost surely equal to the probability that the first block is a singleton conditional on the mass partition. If  $t < T$ , then  $\hat{\Pi}_t(1)$  is a singleton iff no elementary reproduction event has involved 1. This occurs with probability  $\prod_{s \leq t} (1 - \frac{\Delta Z_s}{Z_s})$  conditionally on  $Z$ . If  $t = T$ , then either  $\hat{\Pi}_T$  has finitely many blocks and in that case it cannot have dust, or it has infinitely many blocks. In the latter case  $\hat{\Pi}_T(1)$  is a singleton iff for all  $t < T$   $\hat{\Pi}_t(1)$  is a singleton. This occurs with probability  $\prod_{s < T} (1 - \frac{\Delta Z_s}{Z_s})$ .

When  $\sigma > 0$ , the number of blocks in  $\hat{\Pi}_t$  is finite almost surely since we are either in the *Extinction* case or in the *Infinite lifetime - possible extinction of ancestors* case.

Finally, for all  $t \in [0, T)$  we have proved that the asserted properties hold almost surely. Since the process of asymptotic frequencies of  $\hat{\Pi}_t$  is càdlàg, we deduce that these properties hold almost surely for all  $t \in [0, T)$ . ■

Another interesting question about genealogical structures is the following: can we recover the population size from the genealogy ?

**PROPOSITION II.28.** *The process  $(Z_t, t \geq 0)$  is measurable in the filtration  $\mathcal{F}_t^{\hat{\Pi}} := \sigma\{\hat{\Pi}_{r,s}, 0 \leq r \leq s \leq t\}, t \geq 0$ .*

**Proof** We give a sketch of the proof. Suppose that  $\sigma > 0$  then the infinitesimal jumps due to binary coagulation events allow one to recover the jump rates which is  $\sigma^2/Z_t$  at any given time  $t \geq 0$  thus the process  $Z$  is entirely recovered from this only information. Now suppose that  $\sigma = 0$ . The rescaled jumps  $(\frac{\Delta Z_t}{Z_t}, t \geq 0)$  are measurable w.r.t.  $(\mathcal{F}_t^{\hat{\Pi}}, t \geq 0)$ . Conjointly with the knowledge of the deterministic drift  $\gamma$ , we are able to recover the paths of the process. ■

Recall that the trajectories of a stochastic flow of partitions are not necessarily deterministic flows of partitions: the cocycle property does not necessarily hold simultaneously for all triplets  $r < s < t$ . But we have mentioned that this property is actually verified in the particular case of a flow of partitions defined from a càdlàg CSBP as presented in Subsection 3.2. The goal of what follows is to prove that any  $\Psi$  flow of partitions admits a modification whose trajectories are deterministic flows of partitions. The following two results are proved in Section 7.

**LEMMA II.29.** *The process  $(Z_t, \hat{\Pi}_t; t \geq 0)$  is a Markov process in its own filtration with a Feller semigroup.*

**PROPOSITION II.30.** *Consider a  $\Psi$  flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  with underlying  $\Psi$ -CSBP  $(Z_t, 0 \leq t < T)$ . There exists a process  $(\tilde{\Pi}_{s,t}, 0 \leq s \leq t < T)$  such that:*

- For all  $s \leq t$ , almost surely on the event  $\{t < T\}$   $\hat{\Pi}_{s,t} = \tilde{\Pi}_{s,t}$ .
- For  $\mathbb{P}$ -a.a.  $\omega \in \Omega$ ,  $\tilde{\Pi}(\omega)$  is a deterministic flow of partitions without simultaneous mergers.

## 5.2 A limit theorem

We now turn our attention to the continuity properties of the law of  $(\hat{\Pi}_t, t \geq 0)$  according to its branching mechanism  $\Psi$ . To motivate this study we provide a convergence result for sequences of  $\Psi$ -CSBPs, but this requires first to introduce a suitable topology to compare càdlàg functions that possibly reach  $\infty$  in finite time. At first reading, one can replace our topology with the usual Skorohod's topology and skip the next paragraph.

Our topology is the same as the one introduced in [20]. Let  $\bar{d}$  be a metric on  $[0, +\infty]$  that makes this space homeomorphic to  $[0, 1]$ . We denote by  $\mathcal{D}([0, +\infty], [0, +\infty])$  the space of càdlàg functions  $f : [0, +\infty] \rightarrow [0, +\infty]$  such that  $f(t) = 0$  (resp.  $\infty$ ) implies  $f(t+s) = 0$  (resp.  $\infty$ ) and  $\lim_{t \rightarrow \infty} f(t)$  exists in  $[0, +\infty]$  and is equal to  $f(\infty)$ .

We define  $\Lambda_\infty$  as the set of increasing homeomorphisms of  $[0, +\infty]$  into itself. Let  $\bar{d}_\infty$  be the following metric on  $\mathcal{D}([0, +\infty], [0, +\infty])$

$$\bar{d}_\infty(f, g) := 1 \wedge \inf_{\lambda \in \Lambda_\infty} \left( \sup_{s \geq 0} \bar{d}(f(s), g \circ \lambda(s)) \vee \sup_{s \geq 0} |s - \lambda(s)| \right)$$

Let  $(\Psi_m)_{m \in \mathbb{N}}$  be a sequence of branching mechanisms such that Equation (II.8) is fulfilled with the triplet  $(\gamma_m, \sigma_m, \nu_m)_{m \in \mathbb{N}}$  verifying the corresponding assumptions and denote by  $Z^m$  a  $\Psi_m$ -CSBP started from 1. Let  $\Psi$  be another branching mechanism and  $Z$  a  $\Psi$ -CSBP. We consider the following assumption.

**ASSUMPTION 1.** For all  $u \in \mathbb{R}_+$ , we have  $\Psi_m(u) \rightarrow \Psi(u)$  as  $m \rightarrow \infty$ .

**REMARK II.31.** This assumption is equivalent with

$$\gamma_m - \nu_m((1, \infty)) \xrightarrow{m \rightarrow \infty} \gamma - \nu((1, \infty))$$

in  $\mathbb{R}$  and

$$\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh) \xrightarrow{m \rightarrow \infty} \sigma^2 \delta_0(dh) + (1 \wedge h^2) \nu(dh)$$

in the sense of weak convergence in the set  $\mathcal{M}_f(\mathbb{R}_+)$  of finite measures on  $\mathbb{R}_+$ . See Theorem VII.2.9 and Remark VII.2.10 in [45].

The following proposition yields a convergence result on sequences of CSBP, which is a consequence of the work of Caballero, Lambert and Uribe Bravo in [20].

**PROPOSITION II.32.** Under Assumption 1, we have

$$(Z_t^m, t \geq 0) \xrightarrow[m \rightarrow \infty]{(d)} (Z_t, t \geq 0), \text{ in the sense of weak convergence in } \mathcal{D}([0, +\infty], [0, +\infty]) \quad (\text{II.35})$$

**Proof** The proof of Proposition 6 in [20] ensures that there exists a sequence  $(Y_t^m, t \geq 0)$  of  $\Psi_m$ -Lévy processes started from 1 stopped whenever reaching 0 that converges almost surely to a  $\Psi$ -Lévy process  $(Y_t, t \geq 0)$  stopped whenever reaching 0, where the convergence holds in  $\mathcal{D}([0, +\infty], [0, +\infty])$ . Furthermore, Proposition 5 in [20] yields that  $L^{-1}$  is continuous on  $(\mathcal{D}([0, +\infty], [0, +\infty]), \bar{d}_\infty)$  where  $L$  is a time change due to Lamperti, see Subsection 7.1 for the definition. Therefore, we deduce that

$$\bar{d}_\infty(L^{-1}(Y^m), L^{-1}(Y)) \xrightarrow[m \rightarrow \infty]{} 0 \text{ a.s.}$$

Since  $L^{-1}(Y^m)$  (resp.  $L^{-1}(Y)$ ) is a  $\Psi_m$  (resp.  $\Psi$ ) CSBP for all  $m \geq 1$ , this concludes the proof.  $\blacksquare$

Let  $(\hat{\Pi}_{s,t}^m, 0 \leq s \leq t)$  be a  $\Psi_m$  flow of partitions, for each  $m \geq 1$ .

**THEOREM II.4** Suppose that

- i) For all  $u \in \mathbb{R}_+$ ,  $\Psi_m(u) \rightarrow \Psi(u)$  as  $m \rightarrow \infty$ .
- ii) The branching mechanism  $\Psi$  satisfies the Eve property.
- iii)  $\Psi$  is not the Laplace exponent of a compound Poisson process.

then

$$(\hat{\Pi}_t^m, t \geq 0) \xrightarrow[m \rightarrow \infty]{(d)} (\hat{\Pi}_t, t \geq 0)$$

in  $\mathbb{D}(\mathbb{R}_+, \mathcal{P}_\infty)$ .

The proof of this theorem requires a preliminary lemma. Recall that  $T_\epsilon := \inf\{t \geq 0 : Z_t \notin (\epsilon, 1/\epsilon)\}$ .

**LEMMA II.33.** *Under the hypothesis of the theorem, suppose that  $Z^m \rightarrow Z$  almost surely as  $m \uparrow \infty$  in  $\mathcal{D}([0, +\infty], [0, +\infty])$ , then for all  $\epsilon \in (0, 1)$  we have  $T_\epsilon^m \xrightarrow[m \rightarrow \infty]{} T_\epsilon$  a.s. and*

$$(Z_{t \wedge T_\epsilon^m}^m, t \geq 0) \xrightarrow[m \rightarrow \infty]{} (Z_{t \wedge T_\epsilon}, t \geq 0) \text{ a.s.}$$

in  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ .

We postpone the proof of this lemma to Section 7. We are now ready to prove the theorem.

**Proof** (Theorem II.4) The definition of the topology on  $\mathcal{P}_\infty$  entails that it suffices to show that for every  $n \in \mathbb{N}$ ,

$$(\hat{\Pi}_t^{m,[n]}, t \geq 0) \xrightarrow[m \rightarrow \infty]{(d)} (\hat{\Pi}_t^{[n]}, t \geq 0) \quad (\text{II.36})$$

in  $\mathbb{D}(\mathbb{R}_+, \mathcal{P}_n)$ . So we fix  $n \in \mathbb{N}$ . Our proof consists in showing that

- a)  $\mathbb{P}(\hat{\Pi}_{T_\epsilon}^{[n]} = \mathcal{I}_{[n]}) \rightarrow 1$  as  $\epsilon \downarrow 0$ .
- b)  $(\hat{\Pi}_{t \wedge T_\epsilon^m}^{m,[n]}, t \geq 0) \xrightarrow[m \rightarrow \infty]{(d)} (\hat{\Pi}_{t \wedge T_\epsilon}^{[n]}, t \geq 0)$  in  $\mathbb{D}(\mathbb{R}_+, \mathcal{P}_n)$  for every  $\epsilon \in (0, 1)$ .

To see that those two properties imply the asserted convergence, observe that they entail

$$\mathbb{P}(\hat{\Pi}_{T_\epsilon}^{m,[n]} = \mathcal{I}_{[n]}) \rightarrow 1$$

as  $m \rightarrow \infty$  and  $\epsilon \downarrow 0$ . Since  $\mathcal{I}_{[n]}$  is an absorbing state for the processes  $\hat{\Pi}^{[n]}$  and  $\hat{\Pi}^{m,[n]}$ , we deduce that

$$\begin{aligned} \mathbb{P}[(\hat{\Pi}_{t \wedge T_\epsilon}^{[n]}, t \geq 0) = (\hat{\Pi}_t^{[n]}, t \geq 0)] &\rightarrow 1 \\ \mathbb{P}[(\hat{\Pi}_{t \wedge T_\epsilon^m}^{m,[n]}, t \geq 0) = (\hat{\Pi}_t^{m,[n]}, t \geq 0)] &\rightarrow 1 \end{aligned}$$

as  $m \rightarrow \infty$  and  $\epsilon \downarrow 0$ . Hence, the asserted convergence follows. We now prove a) and b).

The first property a) derives from Theorem II.2 and the fact that  $T_\epsilon \rightarrow T$  a.s. as  $\epsilon \downarrow 0$ .

Let us prove the second property b). Note that this is sufficient to show that for every  $\epsilon \in (0, 1)$

$$\left\{ (L_{t \wedge T_\epsilon^m}^m(n, K), t \geq 0); K \subset [n], \#K \geq 2 \right\} \xrightarrow[m \uparrow \infty]{(d)} \left\{ (L_{t \wedge T_\epsilon}(n, K), t \geq 0); K \subset [n], \#K \geq 2 \right\}$$

in  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+^{d_n})$ , where  $\{(L_t(n, K), t \geq 0); K \subset [n], \#K \geq 2\}$  are the counting processes whose jump rates have been characterized in Subsection 3.3. Indeed the knowledge of these processes allows to

determine the elementary reproduction events  $\hat{\Pi}_{t-,t}^{[n]}$ , and so, is sufficient to recover the process  $(\hat{\Pi}_t^{[n]}, t \geq 0)$ . Obviously, this also holds when the processes are stopped at  $T_\epsilon$ .

Let  $Z^m$  (resp.  $Z$ ) be a  $\Psi_m$ -CSBP (resp.  $\Psi$ -CSBP) such that  $Z^m \rightarrow Z$  almost surely as  $m \uparrow \infty$  in  $\mathcal{D}([0, +\infty], [0, +\infty])$ . Fix  $\epsilon \in (0, 1)$ , we know from Lemma II.33 that

$$(Z_{t \wedge T_\epsilon^m}^m, t \geq 0) \xrightarrow{m \rightarrow \infty} (Z_{t \wedge T_\epsilon}, t \geq 0) \text{ a.s.}$$

in  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ . Using Proposition II.12 and the definition of the stopping times  $T_\epsilon^m$ , we deduce that for each  $2 \leq k \leq n$

$$\left( \int_0^{t \wedge T_\epsilon^m} \lambda_{n,k}(Z_{s-}^m, \Psi^m) ds, t \geq 0 \right) \xrightarrow{m \uparrow \infty} \left( \int_0^{t \wedge T_\epsilon} \lambda_{n,k}(Z_{s-}, \Psi) ds, t \geq 0 \right) \text{ a.s.} \quad (\text{II.37})$$

in  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ .

From each  $\Psi_m$ -CSBP  $Z^m$ , we define a flow of partitions and consider the corresponding  $d_n$ -dimensional counting process

$$\left\{ (L_{t \wedge T_\epsilon^m}^m(n, K), t \geq 0); K \subset [n], \#K \geq 2 \right\}$$

We do the same from the  $\Psi$ -CSBP  $Z$  and define

$$\left\{ (L_{t \wedge T_\epsilon}(n, K), t \geq 0); K \subset [n], \#K \geq 2 \right\}$$

We have shown in Proposition II.13 that this process is a  $d_n$ -dimensional pure-jump semimartingale whose predictable compensator is the  $d_n$ -dimensional process

$$\left\{ \left( \int_0^{t \wedge T_\epsilon} \lambda_{n, \#K}(Z_{s-}, \Psi) ds, t \geq 0 \right); K \subset [n], \#K \geq 2 \right\}$$

and similarly for  $\{(L_{t \wedge T_\epsilon^m}^m(n, K), t \geq 0); K \subset [n], \#K \geq 2\}$ , for each  $m \geq 1$ . From Theorem VI.4.18 in [45] we deduce that the collection of  $d_n$ -dimensional processes  $\{(L_{t \wedge T_\epsilon^m}^m(n, K), t \geq 0); K \subset [n], \#K \geq 2\}_{m \geq 1}$  is tight  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+^{d_n})$ . Indeed, in the notation of [45] conditions (i) and (ii) are trivially verified, while condition (iii) is a consequence of Equation (II.37). Furthermore, this last equation ensures that any limit of a subsequence of the collection of  $d_n$ -dimensional semimartingales is a  $d_n$ -dimensional semimartingale whose predictable compensator is

$$\left\{ \left( \int_0^{t \wedge T_\epsilon} \lambda_{n, \#K}(Z_{s-}, \Psi) ds, t \geq 0 \right); K \subset [n], \#K \geq 2 \right\}$$

which characterizes uniquely the semimartingale  $\{(L_{t \wedge T_\epsilon}(n, K), t \geq 0); K \subset [n], \#K \geq 2\}$  (see for instance Theorem IX.2.4 in [45]). This ensures the following convergence

$$\left\{ (L_{t \wedge T_\epsilon^m}^m(n, K), t \geq 0); K \subset [n], \#K \geq 2 \right\} \xrightarrow{m \uparrow \infty} \left\{ (L_{t \wedge T_\epsilon}(n, K), t \geq 0); K \subset [n], \#K \geq 2 \right\} \quad (\text{II.38})$$

in  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+^{d_n})$ . ■

## 6 From a flow of subordinators to the lookdown representation

### 6.1 Connection with generalised Fleming-Viot and motivation

In [52], we considered the class of generalised Fleming-Viot processes: they are Markov processes that take values in the set of probability measures on a set of genetic types, say  $[0, 1]$ , and that describe

the evolution of the asymptotic frequencies of genetic types in a population of constant size 1. A flow of generalised Fleming-Viot processes  $(\rho_{s,t}, -\infty < s \leq t < +\infty)$  is a consistent collection of generalised Fleming-Viot processes and is completely encoded by a stochastic flow of bridges, see the article [13] of Bertoin and Le Gall. This object is similar to the stochastic flow of subordinators (restricted to an initial population  $[0, 1]$ ), but while in the former the population size is constant in the latter it varies as a CSBP. This will be a major difficulty in the present work. We introduced the notion of ancestral type for a generalised Fleming-Viot process, similarly as we have identified the ancestors of a MVBP. Then we restricted our study to the following two subclasses of generalised Fleming-Viot:

- *Eves - extinction*: the ancestral types with a positive frequency are finitely many at any positive time almost surely and any two of them do never get extinct simultaneously. Thus we can order them by decreasing extinction times, hence obtaining a sequence  $(e^i)_{i \geq 1}$  called the Eves.
- *Eves - persistent*: the ancestral types do never become extinct and can be ordered according to the asymptotic behaviours of their progenies as  $t$  tends to  $\infty$ , we called the corresponding sequence  $(e^i)_{i \geq 1}$  the Eves as well.

At each time  $s \in \mathbb{R}$ , we defined the sequence of Eves  $(e_s^i)_{i \geq 1}$  as the ancestral types of the generalised Fleming-Viot process  $(\rho_{s,t}, t \in [s, \infty))$ . Then, expressing the genealogical relationships between those Eves in terms of partitions of integers we obtained a stochastic flow of partitions  $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$ . These two objects catch all the information encoded by the flow of generalised Fleming-Viot processes:  $(e_s^i)_{i \geq 1}$  is the sequence of types carried by the population started at time  $s$  while  $(\hat{\Pi}_{s,t}, t \in [s, \infty))$  tells how the frequencies of these types evolve in time. Additionally, it provides a pathwise connection with the lookdown representation: the main result of [52] asserts that for every time  $s \in \mathbb{R}$ , the process of limiting empirical measures  $\mathcal{E}_s(\hat{\Pi}, (e_s^i)_{i \geq 1})$  is almost surely equal to  $(\rho_{s,t}, t \in [s, \infty))$ .

Many connections exist between generalised Fleming-Viot processes and  $\Psi$ -MVBP: in [12], Bertoin and Le Gall proved that the Bolthausen-Sznitman coalescent is the genealogy of the Neveu branching process, in [18] Birkner et al. exhibited a striking connection between  $\alpha$ -stable branching processes and Beta(2 -  $\alpha$ ,  $\alpha$ ) Fleming-Viot processes and in [15] Bertoin and Le Gall proved that a generalised Fleming-Viot process has a behaviour locally (*i.e.* for a small subpopulation) identical with a branching process. It is thus natural to expect that a result similar to the one stated in [52] holds in the present setting of branching processes.

For our construction to hold, we need the following assumptions:

- $\Psi$  is conservative, that is, the  $\Psi$ -CSBP does not reach  $\infty$  in finite time a.s.
- The branching mechanism  $\Psi$  enjoys the Eve property.

These assumptions ensure the existence of the ordering of ancestors presented in Subsection 4.2 in three different cases: *Extinction*, *Infinite lifetime - no extinction of ancestors* and *Infinite lifetime - possible extinction of ancestors*.

From now on, we consider a flow of  $\Psi$ -MVBP  $(m_{s,t}, 0 \leq s \leq t < T)$  defined from a  $\Psi$  flow of subordinators. As explained in Subsection 2.3, we can assume that each process  $(m_{s,t}, t \in [s, T))$  is càdlàg. Note that in this section, we use  $T$  instead of  $T^S$  for the lifetime of the flow and we set  $Z_t := m_{0,t}([0, 1])$  for all  $t \in [0, T)$  instead of the notation  $S_t$ . Finally, recall the definition of the probability measure  $r_{s,t}$  via the rescaling

$$r_{s,t}(dx) := \frac{m_{s,t}(Z_s \cdot dx)}{Z_t}$$

In the next subsection, we define the Eves process  $(e_s^i, s \in [0, T])_{i \geq 1}$  and a  $\Psi$ -flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  pathwise from the flow of  $\Psi$ -MVBP. In particular, we prove Theorem II.6. In the last subsection, we introduce for all  $s \in [0, T]$ , the lookdown process  $(\xi_{s,t}(i), t \in [s, T])_{i \geq 1} := \mathcal{L}_s(\hat{\Pi}, (e_s^i)_{i \geq 1})$  and define the measure-valued process  $(\Xi_{s,t}, t \in [s, T]) := \mathcal{E}_s(\hat{\Pi}, (e_s^i)_{i \geq 1})$ . The rest of that subsection is devoted to the proof of the following result.

**THEOREM II.7** *The flow of subordinators can be uniquely decomposed into two random objects: the Eves process  $(e_s^i, s \in [0, T])$  and the flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ .*

i) **Decomposition.** *For each  $s \in \mathbb{R}$ , a.s.  $\mathcal{E}_s(\hat{\Pi}, (e_s^i)_{i \geq 1}) = (r_{s,t}, t \in [s, T])$*

ii) **Uniqueness.** *Let  $(H_{s,t}, 0 \leq s \leq t < T)$  be a  $\Psi$  flow of partitions defined from the  $\Psi$ -CSBP  $Z$ , and for each  $s \in [0, T]$ , consider a sequence  $(\chi_s(i))_{i \geq 1}$  of r.v. taking distinct values in  $[0, 1]$ . If for each  $s \in [0, T]$ , a.s.  $\mathcal{E}_s(H, (\chi_s(i))_{i \geq 1}) = (r_{s,t}, t \in [s, T])$  then*

- *For each  $s \in [0, T]$ , a.s.  $(\chi_s(i))_{i \geq 1} = (e_s^i)_{i \geq 1}$ .*
- *Almost surely  $H = \hat{\Pi}$ .*

## 6.2 Eves process and flow of partitions

For each  $s \in [0, T]$ , the process  $(m_{s,t}, t \in [s, T])$  is a  $\Psi$ -MVBP started from the Lebesgue measure on  $[0, Z_s]$ . Therefore, we introduce the sequence  $(e_s^i)_{i \geq 1}$  defined as its sequence of Eves (according to the definition given in Subsection 4.2) but rescaled by the mass  $Z_s$  in order to obtain r.v. in  $[0, 1]$ . The process  $(e_s^i, s \in [0, T])_{i \geq 1}$  is then called the *Eves process*.

A motivation for the rescaling of the Eves by the mass  $Z_s$  is given by the following lemma.

**LEMMA II.34.** *For every  $s \in [0, T]$ , the sequence  $(e_s^i)_{i \geq 1}$  is i.i.d. uniform $[0, 1]$  and independent of the past of the flow until time  $s$ , that is, of  $\sigma\{m_{u,v}, 0 \leq u \leq v \leq s\}$ .*

**Proof** An easy adaptation of Proposition II.26 shows that, conditional on  $\sigma\{m_{u,v}, 0 \leq u \leq v \leq s\}$ , the sequence  $(Z_s e_s^i)_{i \geq 1}$  is i.i.d. uniform $[0, Z_s]$ . Therefore, the sequence  $(e_s^i)_{i \geq 1}$  is i.i.d. uniform $[0, 1]$  and independent of  $\sigma\{m_{u,v}, 0 \leq u \leq v \leq s\}$ . ■

We now express the genealogical relationships between the Eves in terms of partitions. It is convenient to define the process  $F_{s,t}$  as the distribution function of  $r_{s,t}$  for all  $0 \leq s \leq t < T$ . One easily shows that this process is a bridge in the sense of [13]: it is a non-decreasing random process from 0 to 1 with exchangeable increments. We define an exchangeable random partition  $\hat{\Pi}_{s,t}$  for all  $0 \leq s \leq t < T$  thanks to the following equivalence relation

$$i \stackrel{\hat{\Pi}_{s,t}}{\sim} j \Leftrightarrow F_{s,t}^{-1}(e_t^i) = F_{s,t}^{-1}(e_t^j)$$

for all integers  $i, j$ .

**PROPOSITION II.35.** *For all  $0 \leq s \leq t < T$ , almost surely  $(F_{s,t}, (e_t^i)_{i \geq 1}, (e_s^i)_{i \geq 1})$  follows the composition rule, that is:*

- *$(e_t^i)_{i \geq 1}$  is i.i.d. uniform $[0, 1]$ .*
- *$\hat{\Pi}_{s,t}$  is an exchangeable random partition independent of  $(e_s^i)_{i \geq 1}$ . Denote its blocks by  $(A_j)_{j \geq 1}$  in the increasing order of their least elements. Then, for each  $j$  and any  $i \in A_j$ , we have  $e_s^i = F_{s,t}^{-1}(e_t^j)$ .*



- $(e_s^i)_{i \geq 1}$  is i.i.d. uniform $[0, 1]$ .

The proof of this proposition follows from very similar arguments to those developed in Section 5 of [52], where the *Eves - extinction case* corresponds here to the *Extinction case* while the *Eves - persistent case* corresponds to *Infinite lifetime - no extinction of ancestors* here. Once again, the *Infinite lifetime - possible extinction of ancestors* case is obtained as a mixture of the previous two ones.

**THEOREM II.6** *The collection of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  defined from the flow of subordinators and the Eves process is a  $\Psi$  flow of partitions.*

**Proof** Fix  $0 \leq r < s < t < T$ . We know that for all integers  $i, j$

$$i \stackrel{\hat{\Pi}_{r,t}}{\sim} j \Leftrightarrow F_{r,t}^{-1}(e_t^i) = F_{r,t}^{-1}(e_t^j)$$

Recall that  $F_{r,t}^{-1}(e_t^i) = F_{r,s}^{-1} \circ F_{s,t}^{-1}(e_t^i)$  a.s. and similarly for  $j$ . Proposition II.35 shows that there exists an integer  $k_i$  (resp.  $k_j$ ) such that  $F_{s,t}^{-1}(e_t^i) = e_s^{k_i}$  a.s. (resp.  $j$  instead of  $i$ ). Then we obtain that a.s.

$$i \stackrel{\hat{\Pi}_{r,t}}{\sim} j \Leftrightarrow k_i \stackrel{\hat{\Pi}_{r,s}}{\sim} k_j$$

From the definition of the coagulation operator, we deduce that a.s.  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$ . Now we prove the property on the finite dimensional marginals via a recursion on  $n$ . Implicitly  $f_i$  (resp.  $g_i$ ) will denote a bounded Borel map from  $\mathcal{P}_\infty$  (resp.  $\mathbb{R}_+$ ) to  $\mathbb{R}$  while  $\phi$  will be a bounded Borel map from  $[0, 1]^{\mathbb{N}}$  to  $\mathbb{R}$ . For any sequence  $0 = t_0 < t_1 < t_2 < \dots < t_n$ ,  $H_{t_{i-1}, t_i} := \mathcal{P}(S_{t_{i-1}, t_i})$  will denote the random partitions obtained via independent paint-box schemes based on  $S_{t_{i-1}, t_i}$ , with  $i \in [n]$ . In addition, we will consider a more general setting in which the flow of subordinators is taken at time 0 with an initial population  $[0, z]$  for a given  $z > 0$  (whereas in this section we consider only the case  $z = 1$ ). Then we make use of  $\mathbb{P}_z$  to emphasize the dependence on  $z > 0$ . We will prove that for any integer  $n \geq 1$ , for all  $z > 0$ ,  $0 < t_1 \dots < t_n$ , and all  $f_1, \dots, f_n, g_1, \dots, g_n$  we have

$$\begin{aligned} & \mathbb{E}_z [f_1(\hat{\Pi}_{0,t_1})g_1(Z_{t_1}) \dots f_n(\hat{\Pi}_{t_{n-1}, t_n})g_n(Z_{t_n}) \mid (e_0^i)_{i \geq 1}] \\ &= \mathbb{E}_z [f_1(H_{0,t_1})g_1(Z_{t_1}) \dots f_n(H_{t_{n-1}, t_n})g_n(Z_{t_n})] \end{aligned}$$

This identity will ensure the asserted distribution for finite dimensional marginals of  $\hat{\Pi}$ .

At rank  $n = 1$ , we use Lemma II.34 to deduce that the sequence  $(e_{t_1}^i)_{i \geq 1}$  is independent from the subordinator  $S_{0,t_1}$ . Therefore, we can assume that  $H_{0,t_1}$  is defined according to the paint-box scheme with this sequence of i.i.d. uniform $[0, 1]$  and the subordinator  $S_{0,t_1}$ , that is,  $H_{0,t_1} = \hat{\Pi}_{0,t_1}$ . It suffices to prove that  $\hat{\Pi}_{0,t_1}$  and  $Z_{t_1}$  are independent from the sequence  $(e_0^i)_{i \geq 1}$ . The first independence comes from Proposition II.35. The second independence can be obtained from Lemma II.34. The identity follows.

Now suppose that the identity holds at rank  $n - 1$  for all  $z > 0$ , and all  $f_1, g_1, \dots, f_{n-1}, g_{n-1}$ . At rank  $n$ , we get for any  $f_1, g_1, \dots, f_n, g_n, \phi$  and  $z > 0$

$$\begin{aligned} & \mathbb{E}_z [f_1(\hat{\Pi}_{0,t_1})g_1(Z_{t_1}) \dots f_n(\hat{\Pi}_{t_{n-1}, t_n})g_n(Z_{t_n})\phi((e_0^i)_{i \geq 1})] \\ &= \mathbb{E}_z [f_1(\hat{\Pi}_{0,t_1})g_1(Z_{t_1})\phi((e_0^i)_{i \geq 1}) \mathbb{E} [f_2(\hat{\Pi}_{t_1, t_2})g_2(Z_{t_2}) \dots f_n(\hat{\Pi}_{t_{n-1}, t_n})g_n(Z_{t_n}) \mid \mathcal{F}_{t_1}, (e_{t_1}^i)_{i \geq 1}]]] \end{aligned}$$

where  $\mathcal{F}_{t_1}$  is the  $\sigma$ -field generated by the flow of subordinators until time  $t_1$ . Remark that we have used the measurability of  $(e_0^i)_{i \geq 1}$  from  $\mathcal{F}_{t_1}$  and  $(e_{t_1}^i)_{i \geq 1}$ , given by Proposition II.35. We now apply the Markov property to the process  $(Z_t, t \geq 0)$  to obtain

$$= \mathbb{E}_z [f_1(\hat{\Pi}_{0,t_1})g_1(Z_{t_1})\phi((e_0^i)_{i \geq 1}) \mathbb{E}_{Z_{t_1}} [f_2(\hat{\Pi}_{0, t_2 - t_1})g_2(Z_{t_2 - t_1}) \dots g_n(Z_{t_n - t_1}) \mid (e_{t_1}^i)_{i \geq 1}]]]$$

Notice that we use an abusive notation when conditioning on  $(e_{t_1}^i)_{i \geq 1}$ : we mean that the sequence of ancestors at time 0 in the shifted (by  $t_1$ ) process is equal to the sequence  $(e_{t_1}^i)_{i \geq 1}$  of the original flow of subordinators. We believe that an accurate notation would have greatly burdened the preceding equations. We now apply the recursion hypothesis and the case  $n = 1$  to obtain

$$\begin{aligned} &= \mathbb{E}_z \left[ f_1(H_{0,t_1}) g_1(Z_{t_1}) \mathbb{E}_{Z_{t_1}} [f_2(H_{0,t_2-t_1}) g_2(Z_{t_2-t_1}) \dots g_n(Z_{t_n-t_1})] \right] \mathbb{E}_z [\phi((e_0^i)_{i \geq 1})] \\ &= \mathbb{E}_z [f_1(H_{0,t_1}) g_1(Z_{t_1}) f_2(H_{t_1,t_2}) \dots f_n(H_{t_{n-1},t_n}) g_n(Z_{t_n})] \mathbb{E}_z [\phi((e_0^i)_{i \geq 1})] \end{aligned}$$

where the last equality is due to the Markov property applied to the chain  $(Z_{t_i}, H_{t_{i-1},t_i}, (t_{i+1}-t_i))_{1 \leq i \leq n}$ . Note that this discrete chain is homogeneous in time since we include in the state-space the length of the *next* time interval. The recursion is complete.  $\blacksquare$

### 6.3 The pathwise lockdown representation

So far, we have defined pathwise from the flow of  $\Psi$ -MVBP  $(m_{s,t}, 0 \leq s \leq t < T)$  the Eves process  $(e_s^i, s \in [0, T])_{i \geq 1}$  and a  $\Psi$  flow of partitions  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$ . Thanks to Proposition II.30, we can consider a regularized modification of the flow of partitions that we still denote  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  for convenience. We are now able to define a particle system  $(\xi_{s,t}(i), 0 \leq s \leq t < T)_{i \geq 1}$  as follows. For all  $s \in [0, T)$ , let  $(\xi_{s,t}(i), t \in [s, T])_{i \geq 1} := \mathcal{L}_s(\hat{\Pi}, (e_s^i)_{i \geq 1})$  and define the measure-valued process  $(\Xi_{s,t}, t \in [s, T]) := \mathcal{E}_s(\hat{\Pi}, (e_s^i)_{i \geq 1})$ . The rest of this subsection is devoted to the proof of Theorem II.7.

**Proof** (Theorem II.7) Fix  $s \geq 0$  and work conditionally on  $\{s < T\}$ . From the lockdown representation, we know that for all  $t \in [s, T)$ , almost surely

$$\Xi_{s,t}(dx) = \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)| \delta_{e_s^i}(dx) + (1 - \sum_{i \geq 1} |\hat{\Pi}_{s,t}(i)|) dx$$

Thanks to Proposition II.35, for all  $t \in [s, T)$ , almost surely for all  $i \geq 1$ ,  $|\hat{\Pi}_{s,t}(i)| = r_{s,t}(\{e_s^i\})$  and

$$r_{s,t} = \sum_{i \geq 1} r_{s,t}(\{e_s^i\}) \delta_{e_s^i}(dx) + (1 - \sum_{i \geq 1} r_{s,t}(\{e_s^i\})) dx$$

we obtain that almost surely for all  $t \in [s, T) \cap \mathbb{Q}$ ,  $\Xi_{s,t} = r_{s,t}$ . Since both processes are càdlàg, we deduce they are equal almost surely.

We now turn our attention to the proof of the uniqueness property. Let  $(H_{s,t}, 0 \leq s \leq t < T)$  be a  $\Psi$ -flow of partitions defined from  $Z$  and  $(\chi_s(i), s \in [0, T])_{i \geq 1}$  be, at each time  $s \in [0, T)$ , a sequence of r.v. taking distinct values in  $[0, 1]$ . Define for each  $s \in [0, T)$

$$(X_{s,t}, t \in [s, T]) := \mathcal{E}_s(H, (\chi_s(i))_{i \geq 1})$$

and suppose that a.s.  $(X_{s,t}, t \in [s, T]) = (r_{s,t}, t \in [s, T])$ .

From Proposition II.25, we deduce that for each  $s \in [0, T)$ , almost surely  $(\chi_s(i))_{i \geq 1} = (e_s^i)_{i \geq 1}$ . So the first uniqueness property is proved. We now prove the second uniqueness property. There exists an event  $\Omega^*$  of probability 1 such that on this event, for every rational numbers  $s, t$  such that  $0 \leq s \leq t < T$  and every integer  $i \geq 1$  we have

$$r_{s,t}(\{e_s^i\}) = |\hat{\Pi}_{s,t}(i)| = |H_{s,t}(i)| \quad (\text{II.39})$$

In the rest of the proof, we work on the event  $\Omega^*$ . Our proof relies on the following claim.

**Claim** The flow of partitions  $\hat{\Pi}$  is entirely defined by the knowledge of the quantities  $|\hat{\Pi}_{s,t}(i)|$  for every rational values  $0 \leq s \leq t < T$  and every integer  $i \geq 1$ .

Obviously, the same then holds for the flow of partitions  $H$ . Thanks to this result and Equation (II.39), we deduce that  $\hat{\Pi} = H$  almost surely. ■

It remains to prove the **Claim**. This is achieved thanks to the following two lemmas.

**LEMMA II.36.** *Almost surely, for every  $s \leq t$  in  $[0, T)$  such that  $t$  is rational,  $\hat{\Pi}_{s,t}$  admits asymptotic frequencies and the process  $r \mapsto |\hat{\Pi}_{t-r,t}(i)|$  is l'adcàg for every integer  $i$ .*

**LEMMA II.37.** *Let  $I$  be a subset of  $\mathbb{N}$ . The following assertions are equivalent*

- i)  $\hat{\Pi}_{s-,s}$  has a unique non-singleton block  $I$ .
- ii) For every  $i \neq \min I$  let  $b(i)$  be the unique integer such that  $i = b(i) - (\#\{I \cap [b(i), i]\} - 1) \vee 0$ . Then we have  $(|\hat{\Pi}_{s-,t}(i)|, t \in (s, \infty) \cap \mathbb{Q}) = (|\hat{\Pi}_{s,t}(b(i))|, t \in (s, \infty) \cap \mathbb{Q})$ .

and similarly when  $\hat{\Pi}$  is replaced by  $H$ .

**Proof (Claim)** The knowledge of  $|\hat{\Pi}_{s,t}(i)|$  for every rational values  $0 \leq s \leq t < T$  and every integer  $i \geq 1$ , entails, thanks to Lemma II.36, the knowledge of the quantities  $|\hat{\Pi}_{r-,t}(i)|$  and  $|\hat{\Pi}_{r,t}(i)|$  for all  $r \in (0, t)$ . Then, Lemma II.37 ensures that the elementary reproduction events  $\hat{\Pi}_{s-,s}$  are obtained from the preceding quantities. ■

**Proof (Lemma II.36)** From the exchangeability properties of  $\hat{\Pi}$ , we know that almost surely the quantities  $|\hat{\Pi}_{s,t}(i)|$  exist simultaneously for all rational values  $s \leq t$  and integers  $i \geq 1$ . Fix the rational value  $t$ . We differentiate three cases. First if  $w_t$  is a finite measure and  $d_t = 0$ , then the process  $r \mapsto \hat{\Pi}_{t-r,t}$  has finitely many blocks and no dust, and evolves at discrete times by coagulation events. Thus, for all  $s \in (0, T)$ , there exist rational values  $p < s < q$  such that  $\hat{\Pi}_{s-,t} = \hat{\Pi}_{p,t}$  and  $\hat{\Pi}_{s,t} = \hat{\Pi}_{q,t}$ . The result follows. Second if  $d_t > 0$ . Then, one can easily prove that the rate at which the  $i$ -th block is involved in a coalescence event is finite, for every  $i \geq 1$ . Therefore, the same identities, but for the  $i$ -th block, as in the previous case hold.

Finally, consider the case where blocks have infinitely many blocks but no dust. Then, one can adapt the arguments used in Section 7 of [52] to obtain the result. ■

**Proof (Lemma II.37)** The objects are well-defined thanks to Lemma II.36. One can adapt the proof in Section 6 of [52] in this setting to obtain the asserted result. For this, it is enough to remark that the processes  $(|\hat{\Pi}_{s,t}(i)|, t \in (s, T) \cap \mathbb{Q})_{i \geq 1}$  are distinct by pair since either they reach 0 at distinct finite times or their asymptotic behaviours are distinct. ■

## 7 Appendix

### 7.1 The Lamperti representation

The Lamperti representation provides a time change that maps a  $\Psi$ -Lévy process to a  $\Psi$ -CSBP. It relies on the following objects. Define for all  $t \geq 0$  and any  $f \in \mathcal{D}([0, +\infty], [0, +\infty])$ ,

$$I(f)_t := \inf\{s \geq 0 : \int_0^s f(u) du > t\}$$

Then we define  $L : \mathcal{D}([0, +\infty], [0, +\infty]) \rightarrow \mathcal{D}([0, +\infty], [0, +\infty])$  by setting

$$L(f) := f \circ I(f) \text{ for all } f \in \mathcal{D}([0, +\infty], [0, +\infty])$$

Conversely, one can verify that  $L^{-1}(g) = g \circ J(g)$  where, for all  $g \in \mathcal{D}([0, +\infty], [0, +\infty])$

$$J(g)_t := \inf\{s \geq 0 : \int_0^s g^{-1}(u) du > t\}$$

Consider a  $\Psi$ -Lévy process  $Y$  started from 1 stopped whenever reaching 0, the result of Lamperti ensures that  $L^{-1}(Y)$  is a  $\Psi$ -CSBP started from 1, and that  $L(Z)$  is a  $\Psi$ -Lévy started from 1 stopped whenever reaching 0.

## 7.2 Proof of Lemma II.2

A simple calculation ensures that  $t \mapsto u_t(\infty)$  (when  $u_t(\infty) < \infty$ ) and  $t \mapsto u_t(0+)$  are differentiable, with derivatives equal to  $-\Psi(u_t(\infty))$  and  $-\Psi(u_t(0+))$  respectively. Therefore  $t \mapsto \mathbb{P}(T \leq t) = e^{-u_t(\infty)} + 1 - e^{-u_t(0+)}$  is differentiable as well. So the distribution of  $T$  is absolutely continuous with respect to the Lebesgue measure on  $(0, \infty)$  on the event  $\{T < \infty\}$ . ■

## 7.3 Proof of Lemma II.3

Let  $(Y_t, t \geq 0)$  be a  $\Psi$ -Lévy started from 1. Using the Lamperti's result, we define a  $\Psi$ -CSBP  $Z := L^{-1}(Y)$  started from 1. For all  $\epsilon \in (0, 1)$ , we introduce the stopping time

$$T_\epsilon^Y := \inf\{t \geq 0 : Y_t \notin (\epsilon, 1/\epsilon)\}$$

and use the stopping time  $T_\epsilon$  introduced for  $Z$  previously. It is immediate to check that for all  $t \geq 0$ ,  $Y_{J(Y)_t \wedge T_\epsilon^Y} = Z_{t \wedge T_\epsilon}$ . Fix  $t \geq 0$  and notice that

$$J(Y)_t \wedge T_\epsilon^Y \leq \frac{t}{\epsilon}$$

Thus, we get that

$$\begin{aligned} \mathbb{E} \left[ \sum_{s \leq t \wedge T_\epsilon^Y : \Delta Z_s > 0} \left( \frac{\Delta Z_s}{Z_s} \right)^2 \right] &= \mathbb{E} \left[ \sum_{\substack{s \leq J(Y)_t \wedge T_\epsilon^Y : \\ 0 < \Delta Y_s < 1}} \left( \frac{\Delta Y_s}{Y_s} \right)^2 \right] + \mathbb{E} \left[ \sum_{\substack{s \leq J(Y)_t \wedge T_\epsilon^Y : \\ \Delta Y_s \geq 1}} \left( \frac{\Delta Y_s}{Y_s} \right)^2 \right] \\ &\leq \frac{1}{\epsilon^2} \mathbb{E} \left[ \sum_{\substack{s \leq J(Y)_t \wedge T_\epsilon^Y : \\ 0 < \Delta Y_s < 1}} (\Delta Y_s)^2 \right] + \mathbb{E} \left[ \#\{s \leq J(Y)_t \wedge T_\epsilon^Y : \Delta Y_s \geq 1\} \right] \\ &\leq \frac{1}{\epsilon^2} \mathbb{E} \left[ \sum_{s \leq t/\epsilon : 0 < \Delta Y_s < 1} (\Delta Y_s)^2 \right] + \mathbb{E} \left[ \#\{s \leq t/\epsilon : \Delta Y_s \geq 1\} \right] \\ &\leq \frac{t}{\epsilon^3} \int_{(0,1)} h^2 \nu(dh) + \frac{t}{\epsilon} \nu([1, \infty)) < \infty \end{aligned}$$

where the last inequality derives from the very definition of  $\nu$ . ■

## 7.4 Proof of Proposition II.12

Fix  $2 \leq k \leq n$ . Consider a sequence  $(z_m)_{m \geq 1} \in \mathbb{R}_+^*$  that converges to  $z > 0$ , and a sequence of branching mechanisms  $(\Psi_m)_{m \geq 1}$  such that Assumption 1 is verified. We have to prove that

$$\int_0^1 x^k (1-x)^{n-k} \left( \frac{\sigma_m^2}{z_m} x^{-2} \delta_0(dx) + z_m \nu_m \circ \phi_{z_m}^{-1}(dx) \right) \rightarrow \int_0^1 x^k (1-x)^{n-k} \left( \frac{\sigma^2}{z} x^{-2} \delta_0(dx) + z \nu \circ \phi_z^{-1}(dx) \right)$$

Since  $x \mapsto x^{k-2}(1-x)^{n-k}$  is continuous on  $[0, 1]$ , it suffices to prove that

$$\frac{\sigma_m^2}{z_m} \delta_0(dx) + z_m x^2 \nu_m \circ \phi_{z_m}^{-1}(dx) \xrightarrow{(w)} \frac{\sigma^2}{z} \delta_0(dx) + z x^2 \nu \circ \phi_z^{-1}(dx) \quad (\text{II.40})$$

in the sense of weak convergence in  $\mathcal{M}_f([0, 1])$ . Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Set

$$I_m := \int_0^1 f(x) \left( \frac{\sigma_m^2}{z_m} \delta_0(dx) + z_m x^2 \nu_m \circ \phi_{z_m}^{-1}(dx) \right) = \int_0^\infty f\left(\frac{h}{h+z_m}\right) \left( \frac{\sigma_m^2}{z_m} \delta_0(dh) + z_m \left(\frac{h}{h+z_m}\right)^2 \nu_m(dh) \right)$$

We decompose  $I_m = A_m + B_m$  where

$$A_m := \int_0^\infty f\left(\frac{h}{h+z}\right) \left(\frac{h}{h+z}\right)^2 \frac{z}{1 \wedge h^2} (\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh))$$

$$B_m := \int_0^\infty \left( f\left(\frac{h}{h+z_m}\right) \left(\frac{h}{h+z_m}\right)^2 z_m - f\left(\frac{h}{h+z}\right) \left(\frac{h}{h+z}\right)^2 z \right) \frac{1}{1 \wedge h^2} (\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh))$$

Using the remark below Assumption 1, one can check that  $A_m \rightarrow A$  where

$$\begin{aligned} A &:= \int_0^\infty f\left(\frac{h}{h+z}\right) \left(\frac{h}{h+z}\right)^2 \frac{z}{1 \wedge h^2} (\sigma^2 \delta_0(dh) + (1 \wedge h^2) \nu(dh)) \\ &= \int_0^1 f(x) \left( \frac{\sigma^2}{z} \delta_0(dx) + z x^2 \nu \circ \phi_z^{-1}(dx) \right) \end{aligned}$$

Therefore, the proof of Equation (II.40) reduces to show that  $B_m \rightarrow 0$  as  $m \uparrow \infty$ . To that end, we get

$$\begin{aligned} B_m &= \int_0^\infty f\left(\frac{h}{h+z_m}\right) \frac{h}{h+z_m} \left( \frac{h}{h+z_m} \frac{z_m}{1 \wedge h^2} - \frac{h}{h+z} \frac{z}{1 \wedge h^2} \right) [\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh)] \\ &\quad + \int_0^\infty \left( f\left(\frac{h}{h+z_m}\right) \frac{h}{h+z_m} - f\left(\frac{h}{h+z}\right) \frac{h}{h+z} \right) \frac{h}{h+z} \frac{z}{1 \wedge h^2} [\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh)] \end{aligned}$$

Set  $g_m(h) := \frac{h}{h+z_m} \frac{z_m}{1 \wedge h^2} - \frac{h}{h+z} \frac{z}{1 \wedge h^2}$ . One can easily prove that  $g_m(h) \rightarrow 0$  as  $m \uparrow \infty$  uniformly in  $h \in \mathbb{R}_+$ . Therefore we have the following upper bound for the absolute value of the first term on the r.h.s. of the preceding equation

$$\|f\|_\infty \int_0^\infty |g_m(h)| [\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh)] \rightarrow 0 \text{ as } m \uparrow \infty$$

Finally remark that the second term in the r.h.s. of the preceding equation gives

$$\begin{aligned} &\int_0^\infty \left( f\left(\frac{h}{h+z_m}\right) \frac{h}{h+z_m} - f\left(\frac{h}{h+z}\right) \frac{h}{h+z} \right) \frac{h}{h+z} \frac{z}{1 \wedge h^2} [\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh)] \\ &= \int_0^\infty \left( f\left(\frac{h}{h+z_m}\right) - f\left(\frac{h}{h+z}\right) \right) \left(\frac{h}{h+z}\right)^2 \frac{z}{1 \wedge h^2} [\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh)] \\ &\quad + \int_0^\infty f\left(\frac{h}{h+z_m}\right) \left( \frac{1}{h+z_m} - \frac{1}{h+z} \right) \frac{h^2}{h+z} \frac{z}{1 \wedge h^2} [\sigma_m^2 \delta_0(dh) + (1 \wedge h^2) \nu_m(dh)] \end{aligned}$$

Denote by  $C_m$  and  $D_m$  respectively the first and second term on the r.h.s. of the preceding equation. One can easily prove that  $h \mapsto \frac{h}{h+z_m} - \frac{h}{h+z}$  converges to 0 as  $m \uparrow \infty$  uniformly in  $h \in \mathbb{R}_+$ . And since  $f$  is uniformly continuous on  $[0, 1]$ , we deduce that  $h \mapsto f(\frac{h}{h+z_m}) - f(\frac{h}{h+z})$  converges to 0 as  $m \uparrow \infty$  uniformly on  $\mathbb{R}_+$ . Therefore it is immediate to check that  $|C_m| \rightarrow 0$  as  $m \uparrow \infty$ . Moreover,

$$|f(\frac{h}{h+z_m})(\frac{1}{h+z_m} - \frac{1}{h+z})\frac{h^2}{h+z} \frac{z}{1 \wedge h^2}| \leq \|f\|_\infty \frac{|z-z_m|}{z_m} (\frac{1}{z} \vee z)$$

Hence,  $D_m \rightarrow 0$  as  $m \uparrow \infty$ . This ends the proof of the proposition.  $\blacksquare$

## 7.5 Proof of Lemma II.29

The state space of this process is  $(0, \infty) \times \mathcal{P}_\infty$  to which is added formally a cemetery point  $\partial$  that gathers all the states of the form  $(0, \pi)$  and  $(\infty, \pi)$  where  $\pi$  is any partition. The semigroup has been completely defined in Theorem II.1, and it follows that the corresponding process  $(Z_t, \hat{\Pi}_t; t \geq 0)$  is Markov. To prove that this semigroup verifies the Feller property, we have to show first that the map  $(z, \pi) \mapsto \mathbb{E}[f(Z_t, \hat{\Pi}_t)]$  is continuous and vanishes at  $\partial$ , and second that  $\mathbb{E}[f(Z_t, \hat{\Pi}_t)] \rightarrow f(z, \pi)$  as  $t \downarrow 0$ , for any given continuous map  $f : (0, \infty) \times \mathcal{P}_\infty \rightarrow \mathbb{R}$  that vanishes at  $\partial$ , and any initial condition  $(z, \pi) \in (0, \infty) \times \mathcal{P}_\infty$  for the process  $(Z, \hat{\Pi})$ .

To show the first assertion, we consider the map

$$(z, \pi) \mapsto \mathbb{E}[f(S_{0,t}(z), \text{Coag}(\mathcal{P}(S_{0,t}^z), \pi))]$$

where  $(S_{0,t}(a), a \geq 0)$  is a subordinator with Laplace exponent  $u_t(\cdot)$  and  $S_{0,t}^z$  is its restriction to  $[0, z]$ . Let  $(z_m, \pi_m)_{m \geq 1}$  be a sequence converging to  $(z, \pi) \in (0, \infty) \times \mathcal{P}_\infty$ . For all  $\epsilon > 0$ , there exists  $m_0 \geq 1$  such that for all  $m \geq m_0$ , we have

$$\begin{aligned} \mathbb{P}\left(\frac{|S_{0,t}(z) - S_{0,t}(z_m)|}{S_{0,t}(z) \vee S_{0,t}(z_m)} > \epsilon \mid S_{0,t}(z) \notin \{0, \infty\}\right) &< \epsilon \\ \mathbb{P}(S_{0,t}(z_m) \neq S_{0,t}(z) \mid S_{0,t}(z) \in \{0, \infty\}) &< \epsilon \end{aligned}$$

Then, there are two cases: on the event  $\{S_{0,t}(z) \in \{0, \infty\}\}$ , the process starting from  $(z, \pi)$  is in the cemetery point at time  $t$  and with conditional probability greater than  $1 - \epsilon$ , this is also the case at time  $t$  for the process starting from  $(z_m, \pi_m)$ , for every  $m \geq m_0$ . On the complementary event  $\{S_{0,t}(z) \notin \{0, \infty\}\}$ , fix  $m \geq m_0$  and  $n \geq 1$ . Without loss of generality, we can suppose that  $z_m \leq z$ . Let  $(U_i)_{i \geq 1}$  be a sequence of i.i.d. uniform $[0, S_{0,t}(z)]$  r.v. and introduce the partitions  $\mathcal{P}(S_{0,t}^z)$  by applying the paint-box scheme to the subordinator  $S_{0,t}^z$  using the  $(U_i)_{i \geq 1}$ . Let  $(V_i)_{i \geq 1}$  be an independent sequence of i.i.d. uniform $[0, S_{0,t}(z_m)]$  r.v. and define the sequence  $W_i := U_i \mathbf{1}_{\{U_i \leq S_{0,t}(z_m)\}} + V_i \mathbf{1}_{\{U_i > S_{0,t}(z_m)\}}$ . This sequence is also i.i.d. uniform $[0, S_{0,t}(z_m)]$ , and we apply the paint-box scheme to the subordinator  $S_{0,t}^{z_m}$  with that sequence  $(W_i)_{i \geq 1}$ , then obtaining a partition  $\mathcal{P}(S_{0,t}^{z_m})$ . We have (recall the definition of

the metric  $d_{\mathcal{P}}$  given in Equation (II.2))

$$\begin{aligned}
 & \mathbb{P}(d_{\mathcal{P}}(\mathcal{P}(S_{0,t}^{z_m}), \mathcal{P}(S_{0,t}^z)) \leq 2^{-n} \mid S_{0,t}(z) \notin \{0, \infty\}) \\
 & \geq \mathbb{P}\left[\bigcap_{i \leq n} \{U_i = W_i\} \cap \left\{ \frac{|S_{0,t}(z) - S_{0,t}(z_m)|}{S_{0,t}(z) \vee S_{0,t}(z_m)} < \epsilon \right\} \mid S_{0,t}(z) \notin \{0, \infty\}\right] \\
 & \geq \mathbb{P}\left[\bigcap_{i \leq n} \{U_i \leq S_{0,t}(z_m)\} \mid \frac{|S_{0,t}(z) - S_{0,t}(z_m)|}{S_{0,t}(z) \vee S_{0,t}(z_m)} < \epsilon; S_{0,t}(z) \notin \{0, \infty\}\right] \\
 & \quad \times \mathbb{P}\left[\frac{|S_{0,t}(z) - S_{0,t}(z_m)|}{S_{0,t}(z) \vee S_{0,t}(z_m)} < \epsilon \mid S_{0,t}(z) \notin \{0, \infty\}\right] \\
 & \geq \mathbb{P}\left(\frac{U_1}{S_{0,t}(z)} \leq 1 - \epsilon\right)^n (1 - \epsilon) \geq (1 - \epsilon)^{n+1}
 \end{aligned}$$

Putting all these arguments together and using the facts that the coagulation operator is bicontinuous and that  $f$  is continuous and vanishes near  $\partial$ , one deduces that

$$\mathbb{E}[f(S_{0,t}(z), \text{Coag}(\mathcal{P}(S_{0,t}^z), \pi)) - f(S_{0,t}(z_m), \text{Coag}(\mathcal{P}(S_{0,t}^{z_m}), \pi_m))] \xrightarrow{m \rightarrow \infty} 0$$

and the continuity property follows. The fact that it vanishes at  $\partial$  is elementary. Let us now prove that for all  $(z, \pi) \in (0, \infty) \times \mathcal{P}_{\infty}$ , we have

$$\mathbb{E}[f(S_{0,t}(z), \text{Coag}(\mathcal{P}(S_{0,t}^z), \pi))] \xrightarrow[t \downarrow 0]{} f(z, \pi)$$

This convergence follows from the càdlàg property of  $t \mapsto S_{0,t}(z)$  and the fact that  $\mathcal{P}(S_{0,t}^z)$  tends to  $\mathcal{O}_{[\infty]}$  in distribution as  $t \downarrow 0$ . The Feller property follows.  $\blacksquare$

## 7.6 Proof of Proposition II.30

Let  $(\hat{\Pi}_{s,t}, 0 \leq s \leq t < T)$  be a  $\Psi$ -flow of partitions with underlying  $\Psi$ -CSBP  $Z$ . The idea of the proof is the following: we consider the rational marginals of the flow and show that for  $\mathbb{P}$ -a.a.  $\omega$ ,  $(\hat{\Pi}_{s,t}(\omega), 0 \leq s \leq t < T; s, t \in \mathbb{Q})$  is a deterministic flow of partitions. Thus we extend this flow to the entire interval  $[0, T)$  and show that its trajectories are still deterministic flows of partitions, almost surely.

There exists an event  $\Omega_{\hat{\Pi}}$  of probability 1 such that on this event, we have:

- For every  $r < s < t \in [0, T) \cap \mathbb{Q}$ ,  $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$ .
- For every  $s \in (0, T)$ ,  $\forall n \geq 1$ ,  $\exists \epsilon > 0$  s.t.  $\forall p, q \in (s - \epsilon, s) \cap \mathbb{Q}$ ,  $\hat{\Pi}_{p,q}^{[n]} = \mathcal{O}_{[n]}$ .
- For every  $s \in [0, T)$ ,  $\forall n \geq 1$ ,  $\exists \epsilon > 0$  s.t.  $\forall p, q \in (s, s + \epsilon) \cap \mathbb{Q}$ ,  $\hat{\Pi}_{p,q}^{[n]} = \mathcal{O}_{[n]}$ .
- For every  $s \in [0, T) \cap \mathbb{Q}$ ,  $\hat{\Pi}_{s,s} = \mathcal{O}_{[\infty]}$  and the process  $(\hat{\Pi}_{s,t}, t \in [s, T) \cap \mathbb{Q})$  is càdlàg.

The existence of this event follows from the following arguments. First, for each given triplet the coagulation property holds a.s. So it holds simultaneously for all rational triplets, a.s. Second, the probability that  $\hat{\Pi}_{r,t}$  is close to  $\mathcal{O}_{[\infty]}$  increases to 1 as  $t - r \downarrow 0$ . Together with the coagulation property this ensures the second and third properties. Finally, Lemma II.29 shows that the process  $(Z_t, \hat{\Pi}_t; t \geq 0)$  is Markov

with a Feller semigroup, so it admits a càdlàg modification. This ensures the last assertion.

We now define a process  $(\tilde{\Pi}_{s,t}, 0 \leq s \leq t < T)$  as follows. On  $\Omega_{\hat{\Pi}}$ , we set for all  $0 \leq s \leq t < T$

$$\tilde{\Pi}_{s,t} := \begin{cases} \hat{\Pi}_{s,t} & \text{if } s, t \in \mathbb{Q} \\ \lim_{r \downarrow t, r \in \mathbb{Q}} \hat{\Pi}_{s,r} & \text{if } s \in \mathbb{Q}, t \notin \mathbb{Q} \\ \lim_{r \downarrow s, r \in \mathbb{Q}} \hat{\Pi}_{r,t} & \text{if } t \in \mathbb{Q}, s \notin \mathbb{Q} \\ 0_{[\infty]} & \text{if } s = t \\ \text{Coag}(\tilde{\Pi}_{r,t}, \tilde{\Pi}_{s,r}) & \text{if } s, t \notin \mathbb{Q} \text{ with any given } r \in (s, t) \cap \mathbb{Q} \end{cases}$$

On the complementary event  $\Omega \setminus \Omega_{\hat{\Pi}}$ , set any arbitrary values to the flow  $\tilde{\Pi}$ . A long but easy enumeration of all possible cases proves that this defines a modification of  $\hat{\Pi}$  and that almost surely, the trajectories are deterministic flows of partitions. ■

### 7.7 Proof of Lemma II.33

Let  $(f_m)_{m \geq 1}$  and  $f$  be elements of  $\mathcal{D}([0, +\infty], [0, +\infty])$  without negative jumps and introduce for all  $\epsilon \in (0, 1)$

$$T^f(\epsilon) := \inf\{t \geq 0 : f(t) \notin (\epsilon, 1/\epsilon)\} \text{ and } T^{f_m}(\epsilon) := \inf\{t \geq 0 : f_m(t) \notin (\epsilon, 1/\epsilon)\} \text{ for all } m \geq 1$$

We make the following assumptions

- i)  $f_m(0) = f(0) = 1$  and  $f_m \xrightarrow{m \rightarrow \infty} f$  for the distance  $\bar{d}_\infty$ .
- ii)  $T^f(\epsilon) = \inf\{t \geq 0 : f(t) \notin [\epsilon, 1/\epsilon]\}$ .
- iii)  $\Delta f(T^f(\epsilon)) > 0 \Rightarrow f(T^f(\epsilon)) > 1/\epsilon$ .
- iv) For all  $r \in [0, T^f(\epsilon))$ ,  $\inf_{s \in [0, r]} f(s) > \epsilon$ . Moreover when  $\Delta f(T^f(\epsilon)) > 0$ , it remains true with  $r = T^f(\epsilon)$ .

We fix  $\epsilon \in (0, 1)$  until the end of the proof.

*Step 1.* We stress that  $T^{f_m}(\epsilon) \rightarrow T^f(\epsilon)$  as  $m \rightarrow \infty$ . Indeed, suppose that there exists  $\delta > 0$  such that  $T^{f_m}(\epsilon) < T^f(\epsilon) - \delta$  for an infinity of  $m \geq 1$  (for simplicity, say for all  $m \geq 1$ ). Then, for all  $m \geq 1$  we use iv) to deduce

$$\bar{d}_\infty(f, f_m) \geq \frac{\delta}{2} \wedge \inf_{\{s \in [0, T^f(\epsilon) - \delta]\}} \{|1/\epsilon - f(s)| \wedge |f(s) - \epsilon|\} > 0$$

which contradicts the convergence hypothesis. Similarly, suppose that there exists  $\delta > 0$  such that  $T^{f_m}(\epsilon) > T^f(\epsilon) + \delta$  for an infinity of  $m \geq 1$  (here again, say for all  $m \geq 1$ ). For all  $m \geq 1$ , using ii) we have

$$\bar{d}_\infty(f, f_m) \geq \frac{\delta}{2} \wedge \sup_{\{s \in [T^f(\epsilon), T^f(\epsilon) + \delta]\}} \{|f(s) - 1/\epsilon| \wedge |\epsilon - f(s)|\} > 0$$

which also contradicts the convergence hypothesis. Therefore, the asserted convergence  $T^{f_m}(\epsilon) \rightarrow T^f(\epsilon)$  as  $m \rightarrow \infty$  holds. For simplicity, we now write  $T^f$  instead of  $T^f(\epsilon)$  to alleviate the notation.

*Step 2.* We now prove that  $(f^m(t \wedge T^{f_m}), t \geq 0) \rightarrow (f(t \wedge T^f), t \geq 0)$  for the distance  $\bar{d}_\infty$ . We consider



two cases.

*Step 2a.* Suppose that  $T^f$  is a continuity point of  $f$ . Fix  $\eta > 0$ , there exists  $\delta > 0$  such that for all  $t \in [T^f - \delta, T^f + \delta]$ , we have

$$\bar{d}(f(T^f), f(t)) < \eta$$

From i), we know there exists an integer  $m_0 \geq 1$  and a sequence  $(\lambda_m)_{m \geq m_0}$  of homeomorphisms of  $[0, \infty)$  into  $[0, \infty)$  such that for all  $m \geq m_0$ ,  $|T^{f^m} - T^f| < \delta/4$  and

$$\sup_{s \geq 0} |\lambda_m(s) - s| < \delta/4 \text{ and } \sup_{s \geq 0} \bar{d}(f_m(\lambda_m(s)), f(s)) < \eta$$

We consider now any integer  $m \geq m_0$ . For all  $r \in [0, \delta/2]$ , we have

$$\begin{aligned} \bar{d}(f_m(T^{f^m} - r), f(T^f)) &\leq \bar{d}(f^m(T^{f^m} - r), f(\lambda_m^{-1}(T^{f^m} - r))) + \bar{d}(f(\lambda_m^{-1}(T^{f^m} - r)), f(T^f)) \\ &< 2\eta \end{aligned}$$

using the preceding inequalities. In particular we have proven that  $f^m(T^{f^m}) \rightarrow f(T^f)$  as  $m \rightarrow \infty$ . Finally for all  $t \geq 0$  and all  $m \geq m_0$  we have

$$\begin{aligned} \bar{d}(f_m(\lambda_m(t) \wedge T^{f^m}), f(t \wedge T^f)) &\leq \bar{d}(f_m(\lambda_m(t)), f(t)) + \bar{d}(f_m(T^{f^m}), f(t)) \mathbf{1}_{\{\lambda_m(t) > T^{f^m}, t < T^f\}} \\ &\quad + \bar{d}(f_m(\lambda_m(t)), f(T^f)) \mathbf{1}_{\{\lambda_m(t) \leq T^{f^m}, t \geq T^f\}} + \bar{d}(f_m(T^{f^m}), f(T^f)) \end{aligned}$$

The first and the fourth term in the r.h.s are inferior to  $\eta$  and  $2\eta$  thanks to the preceding inequalities. Concerning the second term, one can show that  $|t - T^f| < \delta/2$  when  $\{\lambda_m(t) > T^{f^m}, t < T^f\}$  which ensures that

$$\bar{d}(f_m(T^{f^m}), f(t)) \mathbf{1}_{\{\lambda_m(t) > T^{f^m}, t < T^f\}} \leq \bar{d}(f_m(T^{f^m}), f(T^f)) + \bar{d}(f(T^f), f(t)) \mathbf{1}_{\{|t - T^f| < \delta/2\}} \leq 3\eta$$

Similarly, when  $\{\lambda_m(t) \leq T^{f^m}, t \geq T^f\}$  the quantity  $r := T^{f^m} - \lambda_m(t)$  belongs to  $[0, \delta/2]$  and thus

$$\bar{d}(f_m(\lambda_m(t)), f(T^f)) \mathbf{1}_{\{\lambda_m(t) \leq T^{f^m}, t \geq T^f\}} < 2\eta$$

Hence, we have  $\bar{d}(f_m(\lambda_m(t) \wedge T^{f^m}), f(t \wedge T^f)) \leq 8\eta$ . This proves the asserted convergence when  $T^f$  is a continuity point of  $f$ .

*Step 2b.* Now suppose that  $f$  jumps at time  $T^f$ . Recall that in that case,  $f(T^f) > 1/\epsilon$ . We denote by  $S := \sup\{f(s) : s \in [0, T^f]\}$  the supremum of  $f$  before time  $T^f$ , which is strictly inferior to  $1/\epsilon$ , and similarly  $I := \inf\{f(s) : s \in [0, T^f]\}$  which is strictly superior to  $\epsilon$  thanks to iv). Set

$$\eta := [\bar{d}(1/\epsilon, S) \wedge \bar{d}(1/\epsilon, f(T^f)) \wedge \bar{d}(\epsilon, I)]/2$$

Thanks to i), there exists an integer  $m_0 \geq 1$  and a sequence  $(\lambda_m)_{m \geq m_0}$  of homeomorphisms of  $[0, \infty)$  into itself such that for all  $m \geq m_0$  we have

$$\sup_{s \geq 0} |\lambda_m(s) - s| < \eta \text{ and } \sup_{s \geq 0} \bar{d}(f^m(\lambda_m(s)), f(s)) < \eta$$

Suppose that  $\lambda_m(T^f) > T^{f^m}$ . Then necessarily,

$$\bar{d}(f^m(T^{f^m}), f(\lambda_m^{-1}(T^{f^m}))) > \bar{d}(1/\epsilon, S) \wedge \bar{d}(\epsilon, I) > \eta$$

which contradicts the hypothesis. Similarly, if  $\lambda_m(T^f) < T^{f^m}$ , then

$$\bar{d}(f^m(\lambda_m(T^f)), f(T^f)) > \bar{d}(1/\epsilon, f(T^f)) > \eta$$

which also contradicts the hypothesis. Hence,  $\lambda_m(T^f) = T^{f^m}$ . And we conclude that

$$(f^m(t \wedge T^{f^m}), t \geq 0) \rightarrow (f(t \wedge T^f), t \geq 0)$$

in  $(\mathcal{D}([0, +\infty], [0, +\infty]), \bar{d}_\infty)$ . But these functions are elements of  $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ , so the last convergence also holds in the usual Skorohod's topology (see the remark below Proposition 5 in [20]).

To finish the proof, it suffices to apply these deterministic results to the processes  $Z^m$  and  $Z$  once we have verified that their trajectories fulfil the required assumptions a.s. Recall that a  $\Psi$ -CSBP  $Z$  can be obtained via the Lamperti time change (see Appendix 7.1) of a  $\Psi$  Lévy process  $Y$

$$Z_t = Y_{J(Y)_t}$$

and that the map  $t \mapsto J(Y)_t$  is continuous.

As we have assumed that  $\Psi$  is not the Laplace exponent of a compound Poisson process, we deduce that a.s.  $T_\epsilon = \inf\{s \geq 0 : Z_s \notin [\epsilon, 1/\epsilon]\}$  and that if  $Z$  jumps at time  $T_\epsilon$  then  $Z_{T_\epsilon} > 1/\epsilon$  a.s. Moreover,  $\inf_{s \in [0, r]} Z_s > \epsilon$  a.s. for all  $r \in [0, T_\epsilon)$ , and also for  $r = T_\epsilon$  when  $Z$  jumps at  $T_\epsilon$ . Otherwise, the càdlàg inverse of the infimum of  $Z$  would admit a fixed discontinuity at time  $\epsilon$  with positive probability, but the latter is (the Lamperti time-change of) the opposite of a subordinator, and so, it does not admit any fixed discontinuity. ■



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The Eve property for continuous-state branching processes

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This article [25] has been written in collaboration with Thomas Duquesne.

## 1 Introduction

Continuous State Branching Processes (CSBP for short) have been introduced by Jirina [46] and Lamperti [58, 57, 59]. They are the scaling limits of Galton-Watson processes: see Grimvall [41] and Helland [43] for general functional limit theorems. They represent the random evolution of the size of a *continuous* population. Namely, if  $Z = (Z_t)_{t \in [0, \infty)}$  is a CSBP, the population at time  $t$  can be represented as the interval  $[0, Z_t]$ . In this paper, we focus on the following question: *as  $t \rightarrow \infty$ , does the population concentrate on the progeny of a single ancestor  $e \in [0, Z_0]$ ?* If this holds true, then we say that *the population has an Eve*. More generally, we discuss the asymptotic frequencies of settlers. A more formal definition is given further in the introduction.

The Eve terminology was first introduced by Bertoin and Le Gall [13] for the generalised Fleming-Viot process. Tribe [75] addressed a very similar question for super-Brownian motion with quadratic branching mechanism, while in Theorem 6.1 [24] Donnelly and Kurtz gave a particle system interpretation of the Eve property. In the CSBP setting, the question has been raised for a general branching mechanism in [53]. Let us mention that Grey [40] and Bingham [17] introduced martingales techniques to study the asymptotic behaviours of CSBP under certain assumptions on the branching mechanism: to answer the above question in specific cases, we extend their results using slightly different tools. For related issues, we also refer to Bertoin, Fontbona and Martinez [11], Bertoin [10] and Abraham and Delmas [1].

Before stating the main result of the present paper, we briefly recall basic properties of CSBP, whose proofs can be found in Silverstein [72], Bingham [17], Le Gall [60] or Kyprianou [50]. CSBP are  $[0, \infty]$ -Feller processes whose only two absorbing states are 0 and  $\infty$  and whose transition kernels  $(p_t(x, \cdot); t \in [0, \infty), x \in [0, \infty])$  satisfy the so-called *branching property*:

$$\forall x, x' \in [0, \infty], \forall t \in [0, \infty), \quad p_t(x, \cdot) * p_t(x', \cdot) = p_t(x + x', \cdot). \quad (\text{III.1})$$

Here,  $*$  stands for the convolution product of measures. We do not view  $\infty$  as a cemetery point, and we do not consider killed CSBP. Then, the transition kernels are true probability measures on  $[0, \infty]$  and

they are characterised by their *branching mechanism*  $\Psi : [0, \infty) \rightarrow \mathbb{R}$  as follows: for any  $t, \lambda, x \in [0, \infty)$ ,

$$\int_{[0, \infty)} p_t(x, dy) \exp(-\lambda y) = \exp(-xu(t, \lambda)), \quad (\text{III.2})$$

where  $u(\cdot, \lambda)$  is a  $[0, \infty)$ -valued function that satisfies  $\partial_t u(t, \lambda) = -\Psi(u(t, \lambda))$  and  $u(0, \lambda) = \lambda$ . For short, we write  $\text{CSBP}(\Psi, x)$  for continuous state branching process with branching mechanism  $\Psi$  and initial value  $x$ . The branching mechanism  $\Psi$  is necessarily of the following Lévy-Khintchine form:

$$\forall \lambda \in [0, \infty), \quad \Psi(\lambda) = \gamma\lambda + \beta\lambda^2 + \int_{(0, \infty)} \pi(dr) (e^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r < 1\}}), \quad (\text{III.3})$$

where  $\gamma \in \mathbb{R}$ ,  $\beta \geq 0$  and  $\pi$  is a Borel measure on  $(0, \infty)$  such that  $\int_{(0, \infty)} (1 \wedge r^2) \pi(dr) < \infty$ . We recall that a CSBP with branching mechanism  $\Psi$  is a time-changed spectrally positive Lévy process whose Laplace exponent is  $\Psi$ : see for instance Lamperti [57] and Caballero, Lambert and Uribe Bravo [20]. Consequently, the sample paths of a cadlag CSBP have no negative jump. Moreover, a CSBP has infinite variation sample paths iff the corresponding Lévy process has infinite variation sample paths, which is equivalent to the following assumption:

$$(\text{Infinite variation}) \quad \beta > 0 \quad \text{or} \quad \int_{(0, 1)} r \pi(dr) = \infty. \quad (\text{III.4})$$

Therefore, the finite variation cases correspond to the following assumption:

$$(\text{Finite variation}) \quad \beta = 0 \quad \text{and} \quad \int_{(0, 1)} r \pi(dr) < \infty. \quad (\text{III.5})$$

In the finite variation cases,  $\Psi$  can be rewritten as follows:

$$\forall \lambda \in [0, \infty), \quad \Psi(\lambda) = D\lambda - \int_{(0, \infty)} \pi(dr) (1 - e^{-\lambda r}), \quad \text{where} \quad D := \gamma + \int_{(0, 1)} r \pi(dr). \quad (\text{III.6})$$

In these cases, note that  $D = \lim_{\lambda \rightarrow \infty} \Psi(\lambda)/\lambda$ .

We shall always avoid the cases of deterministic CSBP that correspond to linear branching mechanisms. Namely, *we shall always assume that either  $\beta > 0$  or  $\pi \neq 0$ .*

Since  $\Psi$  is convex, it has a right derivative at 0, that is possibly equal to  $-\infty$ . Furthermore,  $\Psi$  has at most two roots. We introduce the following notation:

$$\Psi'(0+) := \lim_{\lambda \rightarrow 0+} \lambda^{-1} \Psi(\lambda) \in [-\infty, \infty) \quad \text{and} \quad q = \sup \{ \lambda \in [0, \infty) : \Psi(\lambda) \leq 0 \}. \quad (\text{III.7})$$

Note that  $q > 0$  iff  $\Psi'(0+) < 0$ , and that  $q = \infty$  iff  $-\Psi$  is the Laplace exponent of a subordinator.

We next discuss basic properties of the function  $u$  defined by (III.2). The Markov property for CSBP entails

$$\forall t, s, \lambda \in [0, \infty), \quad u(t+s, \lambda) = u(t, u(s, \lambda)) \quad \text{and} \quad \partial_t u(t, \lambda) = -\Psi(u(t, \lambda)), \quad u(0, \lambda) = \lambda. \quad (\text{III.8})$$

If  $\lambda \in (0, \infty)$ , then  $u(\cdot, \lambda)$  is the unique solution of (III.8). If  $\lambda = q$ , then  $u(\cdot, q)$  is constant to  $q$ . An easy argument derived from (III.8) entails the following: if  $\lambda > q$  (resp.  $\lambda < q$ ), then  $u(\cdot, \lambda)$  is decreasing (resp. increasing). Then, by an easy change of variable, (III.8) implies

$$\forall t \in [0, \infty), \quad \forall \lambda \in (0, \infty) \setminus \{q\}, \quad \int_{u(t, \lambda)}^{\lambda} \frac{du}{\Psi(u)} = t. \quad (\text{III.9})$$

For any  $x \in [0, \infty]$ , we denote by  $\mathbb{P}_x$  the canonical law of CSBP( $\Psi, x$ ) on the Skorohod space of cadlag  $[0, \infty]$ -valued functions that is denoted by  $\mathbb{D}([0, \infty), [0, \infty])$ . We denote by  $Z = (Z_t)_{t \in [0, \infty)}$  the canonical process on  $\mathbb{D}([0, \infty), [0, \infty])$ . As  $t \rightarrow \infty$ , a CSBP either converges to  $\infty$  or to 0. More precisely,

$$\forall x \in (0, \infty), \quad e^{-qx} = \mathbb{P}_x\left(\lim_{t \rightarrow \infty} Z_t = 0\right) = 1 - \mathbb{P}_x\left(\lim_{t \rightarrow \infty} Z_t = \infty\right). \quad (\text{III.10})$$

If  $\Psi'(0+) \geq 0$ , then  $q = 0$  and the CSBP gets extinct:  $\Psi$  is said to be *sub-critical*. If  $\Psi'(0+) < 0$ , then  $q > 0$  and the CSBP has a positive probability to tend to  $\infty$ :  $\Psi$  is said to be *super-critical*.

Let us briefly discuss *absorption*: let  $\zeta_0$  and  $\zeta_\infty$  be the times of absorption in resp. 0 and  $\infty$ . Namely:

$$\zeta_0 = \inf\{t > 0 : Z_t \text{ or } Z_{t-} = 0\}, \quad \zeta_\infty = \inf\{t > 0 : Z_t \text{ or } Z_{t-} = \infty\} \quad \text{and} \quad \zeta = \zeta_0 \wedge \zeta_\infty, \quad (\text{III.11})$$

with the usual convention:  $\inf \emptyset = \infty$ . We call  $\zeta$  the time of absorption. The integral equation (III.9) easily implies the following:

$$(\text{Conservative } \Psi) \quad \forall x \in [0, \infty), \quad \mathbb{P}_x(\zeta_\infty < \infty) = 0 \iff \int_{0+} \frac{dr}{(\Psi(r))_-} = \infty. \quad (\text{III.12})$$

Here  $(\cdot)_-$  stands for the negative part function. If  $\Psi$  is *non-conservative*, namely if

$$(\text{Non-conservative } \Psi) \quad \int_{0+} \frac{dr}{(\Psi(r))_-} < \infty, \quad (\text{III.13})$$

then,  $\Psi'(0+) = -\infty$  and for any  $t, x \in (0, \infty)$ ,  $\mathbb{P}_x(\zeta_\infty > t) = \exp(-x\kappa(t))$ , where  $\kappa(t) := \lim_{\lambda \rightarrow 0+} u(t, \lambda)$  satisfies  $\int_0^{\kappa(t)} dr / (\Psi(r))_- = t$ . Note that  $\kappa : (0, \infty) \rightarrow (0, q)$  is one-to-one and increasing. Thus,  $\mathbb{P}_x$ -a.s.  $\lim_{t \rightarrow \infty} Z_t = \infty$  iff  $\zeta_\infty < \infty$  and in this case,  $\lim_{t \rightarrow \zeta_\infty-} Z_t = \infty$ . Namely, the process reaches  $\infty$  continuously.

The integral equation (III.9) also implies the following:

$$(\text{Persistent } \Psi) \quad \forall x \in [0, \infty), \quad \mathbb{P}_x(\zeta_0 < \infty) = 0 \iff \int^\infty \frac{dr}{\Psi(r)} = \infty. \quad (\text{III.14})$$

If  $\Psi$  allows extinction in finite time, namely if

$$(\text{Non-persistent } \Psi) \quad \int^\infty \frac{dr}{\Psi(r)} < \infty, \quad (\text{III.15})$$

it necessarily implies that  $\Psi$  satisfies (III.4), namely that  $\Psi$  is of infinite variation type. In this case, for any  $t, x \in (0, \infty)$ ,  $\mathbb{P}_x(\zeta_0 \leq t) = \exp(-xv(t))$  where  $v(t) := \lim_{\lambda \rightarrow \infty} u(t, \lambda)$  satisfies  $\int_{v(t)}^\infty dr / \Psi(r) = t$ . Note that  $v : (0, \infty) \rightarrow (q, \infty)$  is one-to-one and decreasing. Thus,  $\mathbb{P}_x$ -a.s.  $\lim_{t \rightarrow \infty} Z_t = 0$  iff  $\zeta_0 < \infty$ .

The previous arguments allow to define  $u$  for negative times. Namely, for all  $t \in (0, \infty)$ , set  $\kappa(t) = \lim_{\lambda \rightarrow 0+} u(t, \lambda)$  and  $v(t) = \lim_{\lambda \rightarrow \infty} u(t, \lambda)$ . As already mentioned,  $\kappa(t)$  is positive if  $\Psi$  is non-conservative and null otherwise and  $v(t)$  is finite if  $\Psi$  is non-persistent and infinite otherwise. Then, observe that  $u(t, \cdot) : (0, \infty) \rightarrow (\kappa(t), v(t))$  is increasing and one-to-one. We denote by  $u(-t, \cdot) : (\kappa(t), v(t)) \rightarrow (0, \infty)$  the reciprocal function. It is plain that (III.9) extends to negative times. Then, observe that  $\partial_t u(-t, \lambda) = \Psi(u(-t, \lambda))$  and that (III.8) extends to negative times as soon as it makes sense.

Let us give here the precise definition of the Eve property. To that end, we fix  $x \in (0, \infty)$ , we denote by  $\mathcal{B}([0, x])$  the Borel subsets of  $[0, x]$ . We also denote by  $\mathcal{M}([0, x])$  the set of positive Borel-measures on  $[0, x]$  and by  $\mathcal{M}_1([0, x])$  the set of Borel probability measures. Let us think of  $m_t \in \mathcal{M}_1([0, x])$ ,

$t \in [0, \infty)$ , as the frequency distributions of a continuous population whose set of ancestors is  $[0, x]$  and that evolves through time  $t$ . Namely for any Borel set  $B \subset [0, x]$ ,  $m_t(B)$  is the frequency of the individuals at time  $t$  whose ancestors belong to  $B$ . The relevant convergence mode is the *total variation norm*:

$$\forall \mu, \nu \in \mathcal{M}_1([0, x]), \quad \|\mu - \nu\|_{\text{var}} = \sup \{ |\mu(A) - \nu(A)|; A \in \mathcal{B}([0, x]) \}.$$

Here, it is natural to assume that  $t \mapsto m_t$  is cadlag in total variation norm. The Eve property can be defined as follows.

**DEFINITION III.1.** *We denote by  $\ell$  the Lebesgue measure on  $\mathbb{R}$  (or its restriction to  $[0, x]$  according to the context). Let  $t \in (0, \infty) \mapsto m_t \in \mathcal{M}_1([0, x])$  be cadlag with respect to  $\|\cdot\|_{\text{var}}$  and assume that there exists  $m_\infty \in \mathcal{M}_1([0, x])$  such that  $\lim_{t \rightarrow \infty} \|m_t - m_\infty\|_{\text{var}} = 0$ , where*

$$m_\infty = a\ell + \sum_{y \in S} m_\infty(\{y\})\delta_y. \quad (\text{III.16})$$

Here,  $a$  is called the *dust*,  $S$  is a countable subset of  $[0, x]$  that is the set of *settlers* and for any  $y \in S$ ,  $m_\infty(\{y\})$  is the *asymptotic frequency of the settler*  $y$ .

If  $a = 0$ , then we say that the population  $m := (m_t)_{t \in (0, \infty)}$  has *no dust* (although  $m_t$  may have a diffuse part at any finite time  $t$ ). If  $a = 0$  and if  $S$  reduces to a single point  $\mathbf{e}$ , then  $m_\infty = \delta_{\mathbf{e}}$  and the population  $m$  is said to have an *Eve* that is  $\mathbf{e}$ . Furthermore, if there exists  $t_0 \in (0, \infty)$  such that  $m_t = \delta_{\mathbf{e}}$ , for any  $t > t_0$ , then we say that the population has an *Eve in finite time*.  $\square$

The following theorem asserts the existence of a regular version of the frequency distributions associated with a CSBP.

**THEOREM III.1.** *Let  $x \in (0, \infty)$ . Let  $\Psi$  be a branching mechanism of the form (III.3). We assume that  $\Psi$  is not linear. Then, there exists a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  on which the two following processes are defined.*

(a)  $Z = (Z_t)_{t \in [0, \infty)}$  is a cadlag CSBP  $(\Psi, x)$ .

(b)  $M = (M_t)_{t \in [0, \infty)}$  is a  $\mathcal{M}_1([0, x])$ -valued process that is  $\|\cdot\|_{\text{var}}$ -cadlag on  $(0, \infty)$  such that

$$\forall B \in \mathcal{B}([0, x]), \quad \mathbf{P}\text{-a.s.} \lim_{t \rightarrow 0^+} M_t(B) = x^{-1}\ell(B).$$

The processes  $Z$  and  $M$  satisfy the following property: for any Borel partition  $B_1, \dots, B_n$  of  $[0, x]$  there exist  $n$  independent cadlag CSBP  $(\Psi)$ ,  $Z^{(1)}, \dots, Z^{(n)}$ , with initial values  $\ell(B_1), \dots, \ell(B_n)$ , such that

$$\forall k \in \{1, \dots, n\}, \quad \forall t \in [0, \zeta), \quad M_t(B_k) = Z_t^{(k)} / Z_t, \quad (\text{III.17})$$

where  $\zeta$  stands for the time of absorption of  $Z$ .

We call  $M$  the *frequency distribution process of a CSBP  $(\Psi, x)$* . If  $\Psi$  is of finite variation type, then  $M$  is  $\|\cdot\|_{\text{var}}$ -right continuous at time 0, which is not the case if  $\Psi$  is of infinite variation type as explained in Section 2.3. The strong regularity of  $M$  requires specific arguments: in the infinite variation cases, we need a decomposition of CSBP into Poisson clusters, which is the purpose of Theorem III.3 in Section 2.2. (see this section for more details and comments).

The main result of the paper concerns the asymptotic behaviour of  $M$  on the following three events.

- $A := \{\zeta < \infty\}$  that is the event of absorption. Note that  $\mathbf{P}(A) > 0$  iff  $\Psi$  either satisfies (III.13) or (III.15), namely iff  $\Psi$  is either non-conservative or non-persistent.

- $B := \{\zeta = \infty; \lim_{t \rightarrow \infty} Z_t = \infty\}$  that is the event of explosion in infinite time. Note that  $\mathbf{P}(B) > 0$  iff  $\Psi$  satisfies (III.12) and  $\Psi'(0+) \in [-\infty, 0)$ , namely iff  $\Psi$  is conservative and super-critical.
- $C := \{\zeta = \infty; \lim_{t \rightarrow \infty} Z_t = 0\}$  that is the event of extinction in infinite time. Note that  $\mathbf{P}(C) > 0$  iff  $\Psi$  satisfies (III.14) and  $q < \infty$ .

**THEOREM III.2.** *We assume that  $\Psi$  is a non-linear branching mechanism. Let  $x \in (0, \infty)$  and let  $M$  and  $Z$  be as in Theorem III.1. Then,  $\mathbf{P}$ -a.s.  $\lim_{t \rightarrow \infty} \|M_t - M_\infty\|_{\text{var}} = 0$ , where  $M_\infty$  is of the form (III.16). Moreover, the following holds true  $\mathbf{P}$ -almost surely.*

- (i) *On the event  $A = \{\zeta < \infty\}$ ,  $M$  has an Eve in finite time.*
- (ii) *On the event  $B = \{\zeta = \infty; \lim_{t \rightarrow \infty} Z_t = \infty\}$ :*
  - (ii-a) *If  $\Psi'(0+) = -\infty$ , then  $M$  has an Eve;*
  - (ii-b) *If  $\Psi'(0+) \in (-\infty, 0)$  and  $q < \infty$ , there is no dust and  $M$  has finitely many settlers whose number, under  $\mathbf{P}(\cdot | B)$ , is distributed as a Poisson r.v. with mean  $xq$  conditioned to be non zero;*
  - (ii-c) *If  $\Psi'(0+) \in (-\infty, 0)$  and  $q = \infty$ , there is no dust and  $M$  has infinitely many settlers that form a dense subset of  $[0, x]$ .*
- (iii) *On the event  $C = \{\zeta = \infty; \lim_{t \rightarrow \infty} Z_t = 0\}$ :*
  - (iii-a) *If  $\Psi$  is of infinite variation type, then  $M$  has an Eve;*
  - (iii-b) *If  $\Psi$  is of finite variation type, then the following holds true:*
    - (iii-b-1) *If  $\pi((0, 1)) < \infty$ , then there is dust and  $M$  has finitely many settlers whose number, under  $\mathbf{P}(\cdot | C)$ , is distributed as a Poisson r.v. with mean  $\frac{x}{D} \int_{(0, \infty)} e^{-qr} \pi(dr)$ ;*
    - (iii-b-2) *If  $\pi((0, 1)) = \infty$  and  $\int_{(0, 1)} \pi(dr) r \log 1/r < \infty$ , then there is dust and there are infinitely many settlers that form a dense subset of  $[0, x]$ ;*
    - (iii-b-3) *If  $\int_{(0, 1)} \pi(dr) r \log 1/r = \infty$ , then there is no dust and there are infinitely many settlers that form a dense subset of  $[0, x]$ .*

First observe that the theorem covers all the possible cases, except the deterministic ones that are trivial. On the absorption event  $A = \{\zeta < \infty\}$ , the result is easy to explain: the descendent population of a single ancestor either explodes strictly before the others, or gets extinct strictly after the others, and there is an Eve in finite time.

The cases where there is no Eve – namely, Theorem III.2 (ii-b), (ii-c) and (iii-b) – are simple to explain: the size of the descendent populations of the ancestors grow or decrease in the same (deterministic) scale and the limiting measure is that of a normalised subordinator as specified in Proposition III.12, Lemma III.13, Proposition III.14, Lemma III.15, and also in the proof Section 3.2. Let us mention that in Theorem III.2 (iii-b1) and (iii-b2), the dust of  $M_\infty$  comes only from the dust of the  $M_t$ ,  $t \in (0, \infty)$ : it is not due to limiting aggregations of atoms of the measures  $M_t$  as  $t \rightarrow \infty$ .

Theorem III.2 (ii-a) and (iii-a) are the main motivation of the paper: in these cases, the descendent populations of the ancestors grow or decrease in distinct scales and one dominates the others, which implies the Eve property in infinite time. This is the case of the Neveu branching mechanism  $\Psi(\lambda) = \lambda \log \lambda$ , that is related to the Bolthausen-Sznitman coalescent: see Bolthausen and Sznitman [19], and Bertoin and Le Gall [13].

Let us first make some comments in connection with the Galton-Watson processes. The asymptotic behaviours displayed in Theorem III.2 (ii) find their counterparts at the discrete level: the results of



Seneta [70, 71] and Heyde [44] implicitly entail that the Eve property is verified by a supercritical Galton-Watson process on the event of explosion iff the mean is infinite. However neither the extinction nor the dust find relevant counterparts at the discrete level so that Theorem III.2 (i) and (iii) are specific to the continuous setting.

CSBP present many similarities with generalised Fleming-Viot processes, see for instance the monograph of Etheridge [30]: however for this class of measure-valued processes Bertoin and Le Gall [13] proved that the population has an Eve without assumption on the parameter of the model (the measure  $\Lambda$  which is the counterpart of the branching mechanism  $\Psi$ ). We also mention that when the CSBP has an Eve, one can define a recursive sequence of Eves on which the residual populations concentrate, see [53]. Observe that this property is no longer true for generalised Fleming-Viot processes, see [52].

The paper is organized as follows. In Section 2.1, we gather several basic properties and estimates on CSBP that are needed for the construction of the cluster measure done in Section 2.2. These preliminary results are also used to provide a regular version of  $M$  which is the purpose of Section 2.3. Section 3 is devoted to the proof of Theorem III.2: in Section 3.1 we state specific results on Grey martingales associated with CSBP in the cases where Grey martingales evolve in comparable deterministic scales: these results entail Theorem III.2 (ii-b), (ii-c) and (iii-b), as explained in Section 3.2. Section 3.3 is devoted to the proof of Theorem III.2 (ii-a) and (iii-a): these cases are more difficult to handle and the proof is divided into several steps; in particular it relies on Lemma III.20, whose proof is postponed to Section 3.3.

## 2 Construction of $M$ .

### 2.1 Preliminary estimates on CSBP.

Recall that we assume that  $\Psi$  is not linear: namely, either  $\beta > 0$  or  $\pi \neq 0$ . The branching property (III.1) entails that for any  $t \in (0, \infty)$ ,  $u(t, \cdot)$  is the Laplace exponent of a subordinator. Namely, it is of the following form:

$$u(t, \lambda) = \kappa(t) + d(t)\lambda + \int_{(0, \infty)} \nu_t(dr) (1 - e^{-\lambda r}), \quad \lambda \in [0, \infty), \quad (\text{III.18})$$

where  $\kappa(t) = \lim_{\lambda \rightarrow 0^+} u(t, \lambda)$ ,  $d(t) \in [0, \infty)$  and  $\int_{(0, \infty)} (1 \wedge r) \nu_t(dr) < \infty$ . Since  $\Psi$  is not linear, we easily get  $\nu_t \neq 0$ . As already mentioned in the introduction if  $\Psi$  is conservative,  $\kappa(t) = 0$  for any  $t$  and if  $\Psi$  is non-conservative, then  $\kappa : (0, \infty) \rightarrow (0, q)$  is increasing and one-to-one. To avoid to distinguish these cases, we extend  $\nu_t$  on  $(0, \infty]$  by setting  $\nu_t(\{\infty\}) := \kappa(t)$ . Thus, (III.18) can be rewritten as follows:  $u(t, \lambda) = d(t)\lambda + \int_{(0, \infty]} \nu_t(dr) (1 - e^{-r\lambda})$ , with the usual convention  $\exp(-\infty) = 0$ . Recall from (III.6) the definition of  $D$ .

**LEMMA III.2.** *Let  $t \in (0, \infty)$ . Then  $d(t) > 0$  iff  $\Psi$  is of finite variation type. In this case,  $d(t) = e^{-Dt}$ .*

**Proof** First note that  $d(t) = \lim_{\lambda \rightarrow \infty} \lambda^{-1} u(t, \lambda)$ . An elementary computation implies that

$$\frac{u(t, \lambda)}{\lambda} = \exp \left( \int_0^t \partial_s \log u(s, \lambda) ds \right) = \exp \left( - \int_0^t \frac{\Psi(u(s, \lambda))}{u(s, \lambda)} ds \right). \quad (\text{III.19})$$

If  $\Psi$  satisfies (III.15), then recall that  $\lim_{\lambda \rightarrow \infty} u(t, \lambda) < \infty$ . Thus, in this case,  $d(t) = 0$ . Next assume that  $\Psi$  satisfies (III.14). Then,  $\lim_{\lambda \rightarrow \infty} u(t, \lambda) = \infty$ . Note that  $\Psi(\lambda)/\lambda$  increases to  $\infty$  in the infinite variation cases and that it increases to the finite quantity  $D$  in the finite variation cases, which implies the desired result by monotone convergence in the last member of (III.19).  $\blacksquare$

## 2. Construction of M.

Recall that for any  $x \in [0, \infty]$ ,  $\mathbb{P}_x$  stands for the law on  $\mathbb{D}([0, \infty), [0, \infty])$  of a CSBP( $\Psi, x$ ) and recall that  $Z$  stands for the canonical process. It is easy to deduce from (III.1) the following monotone property:

$$\forall t \in [0, \infty), \forall y \in [0, \infty), \forall x, x' \in [0, \infty] \text{ such that } x \leq x', \quad \mathbb{P}_x(Z_t > y) \leq \mathbb{P}_{x'}(Z_t > y). \quad (\text{III.20})$$

**LEMMA III.3.** *Assume that  $\Psi$  is not linear. Then, for all  $t, x, y \in (0, \infty)$ ,  $\mathbb{P}_x(Z_t > y) > 0$ .*

**Proof** Let  $(S_x)_{x \in [0, \infty)}$  be a subordinator with Laplace exponent  $u(t, \cdot)$  that is defined on an auxiliary probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Thus  $S_x$  under  $\mathbf{P}$  has the same law as  $Z_t$  under  $\mathbb{P}_x$ . Since  $\nu_t \neq 0$ , there is  $r_0 \in (0, \infty)$  such that  $\nu_t((r_0, \infty)) > 0$ . Consequently,  $N := \#\{z \in [0, x] : \Delta S_z > r_0\}$  is a Poisson r.v. with non-zero mean  $x\nu_t((r_0, \infty))$ . Then, for any  $n$  such that  $nr_0 > y$  we get  $\mathbb{P}_x(Z_t > y) = \mathbf{P}(S_x > y) \geq \mathbf{P}(N \geq n) > 0$ , which completes the proof. ■

The following lemmas are used in Section 2.2 for the construction of the cluster measure.

**LEMMA III.4.** *Assume that  $\Psi$  is of infinite variation type. Then, for any  $t, s \in (0, \infty)$ ,*

$$\nu_{t+s}(dr) = \int_{(0, \infty]} \nu_s(dx) \mathbb{P}_x(Z_t \in dr; Z_t > 0). \quad (\text{III.21})$$

**Proof** Let  $\nu$  be the measure in the right member of (III.21). Then, for all  $\lambda \in (0, \infty)$ , (III.2) and (III.8) imply that

$$\int_{(0, \infty]} \nu(dr) (1 - e^{-\lambda r}) = \int_{(0, \infty]} \nu_s(dx) (1 - e^{-xu(t, \lambda)}) = u(s, u(t, \lambda)) = u(s+t, \lambda) = \int_{(0, \infty]} \nu_{t+s}(dr) (1 - e^{-\lambda r}).$$

By letting  $\lambda$  go to 0, this implies that  $\nu(\{\infty\}) = \nu_{t+s}(\{\infty\})$ . By differentiating in  $\lambda$ , we also get  $\int_{(0, \infty)} \nu(dr) r e^{-\lambda r} = \int_{(0, \infty)} \nu_{t+s}(dr) r e^{-\lambda r}$ . Since Laplace transform of finite measures is injective, this entails that  $\nu$  and  $\nu_{t+s}$  coincide on  $(0, \infty)$  which completes the proof. ■

**LEMMA III.5.** *Assume that  $\Psi$  is of infinite variation type. Then, for all  $\varepsilon \in (0, 1)$  and all  $s, t \in (0, \infty)$  such that  $s < t$ ,*

$$\int_{(0, \infty]} \nu_s(dx) \mathbb{P}_x(Z_{t-s} > \varepsilon) = \nu_t((\varepsilon, \infty]) \in (0, \infty). \quad (\text{III.22})$$

**Proof** The equality follows from (III.21). Next observe that  $\nu_t((\varepsilon, \infty]) \leq \frac{1}{\varepsilon} \int_{(0, \infty]} (1 \wedge r) \nu_t(dr) < \infty$ . Since  $\nu_s$  does not vanish on  $(0, \infty)$ , Lemma III.3 entails that the first member is strictly positive. ■

We shall need the following simple result in the construction of  $M$  in Section 2.3.

**LEMMA III.6.** *For all  $a, y \in (0, \infty)$ ,  $\lim_{r \rightarrow 0+} \mathbb{P}_r(Z_a > y) = 0$  and  $\lim_{r \rightarrow 0+} \mathbb{P}_r(\sup_{b \in [0, a]} Z_b > y) = 0$ .*

**Proof** First note that  $\mathbb{P}_r(Z_a > y) \leq (1 - e^{-1})^{-1} \mathbb{E}_r[1 - e^{-Z_a/y}] = (1 - e^{-1})^{-1} (1 - e^{-ru(a, 1/y)}) \rightarrow 0$  as  $r \rightarrow 0$ , which implies the first limit. Let us prove the second limit: if  $q = \infty$ , then  $Z$  is non-decreasing and the second limit is derived from the first one. We next assume that  $q < \infty$ , and we claim that there exist  $\theta, C \in (0, 1)$  that only depend on  $a$  and  $y$  such that

$$\forall z \in [y, \infty), \forall b \in [0, a], \quad p_b(z, [0, \theta y]) \leq C. \quad (\text{III.23})$$

*Proof of (III.23).* We specify  $\theta \in (0, 1)$  further. By (III.20),  $p_b(z, [0, \theta y]) \leq p_b(y, [0, \theta y]) = \mathbb{P}_y(Z_b \leq \theta y)$ . By an elementary inequality, for all  $\lambda \in (0, \infty)$ ,  $\mathbb{P}_y(Z_b \leq \theta y) \leq \exp(y\theta\lambda) \mathbb{E}_y[\exp(-\lambda Z_b)] = \exp(y\theta\lambda -$

$yu(b, \lambda)$ ). We take  $\lambda = q+1$ . Thus,  $u(\cdot, q+1)$  is decreasing and  $p_b(z, [0, \theta y]) \leq \exp(y\theta(q+1) - yu(a, q+1))$ . We choose  $\theta = \frac{u(a, q+1)}{2(q+1)}$ . Then, (III.23) holds true with  $C = \exp(-y\theta(q+1))$ . ■

We next set  $T = \inf\{t \in [0, \infty) : Z_t > y\}$ , with the convention  $\inf \emptyset = \infty$ . Thus  $\{\sup_{b \in [0, a]} Z_b > y\} = \{T \leq a\}$ . Let  $\theta$  and  $C$  as in (III.23). First note that  $\mathbb{P}_r(T \leq a) \leq \mathbb{P}_r(Z_a > \theta y) + \mathbb{P}_r(T \leq a; Z_a \leq \theta y)$ . Then, by the Markov property at  $T$  and (III.23), we get

$$\mathbb{P}_r(T \leq a; Z_a \leq \theta y) = \mathbb{E}_r[\mathbf{1}_{\{T \leq a\}} p_{a-T}(Z_T, [0, \theta y])] \leq C \mathbb{P}_r(T \leq a).$$

Thus,  $\mathbb{P}_r(\sup_{b \in [0, a]} Z_b > y) \leq (1-C)^{-1} \mathbb{P}_r(Z_a > \theta y) \rightarrow 0$  as  $r \rightarrow 0$ , which completes the proof. ■

We next state a more precise inequality that is used in the construction of the cluster measure of CSBP.

**LEMMA III.7.** *We assume that  $\Psi$  is not linear. Then, for any  $\varepsilon, \eta \in (0, 1)$  and for any  $t_0 \in (0, \infty)$ , there exists  $a \in (0, t_0/4)$  such that*

$$\forall x \in [0, \eta], \forall b \in [0, a], \forall c \in [\frac{1}{2}t_0, t_0], \quad \mathbb{P}_x\left(\sup_{t \in [0, b]} Z_t > 2\eta; Z_c > \varepsilon\right) \leq 2 \mathbb{P}_x(Z_b > \eta; Z_c > \varepsilon). \quad (\text{III.24})$$

**Proof** Since  $\Psi$  is not linear,  $\nu_t \neq 0$ . If  $q = \infty$ , the corresponding CSBP has increasing sample paths and the lemma obviously holds true. So we assume that  $q < \infty$ . We first claim the following.

$$\forall x, y, t_0, t_1 \in (0, \infty) \text{ with } t_1 \leq t_0, \quad \inf_{t \in [t_1, t_0]} \mathbb{P}_x(Z_t > y) > 0. \quad (\text{III.25})$$

*Proof of (III.25).* Suppose that there is a sequence  $s_n \in [t_1, t_0]$  such that  $\lim_{n \rightarrow \infty} \mathbb{P}_x(Z_{s_n} > y) = 0$ . Without loss of generality, we can assume that  $\lim_{n \rightarrow \infty} s_n = t$ . Since  $u(\cdot, \lambda)$  is continuous,  $Z_{s_n} \rightarrow Z_t$  in law under  $\mathbb{P}_x$  and the Portmanteau Theorem implies that  $\mathbb{P}_x(Z_t > y) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_x(Z_{s_n} > y) = 0$ , which contradicts Lemma III.3 since  $t > 0$ . ■

We next claim the following: for any  $\eta, \delta \in (0, 1)$ , there exists  $a \in (0, \infty)$  such that

$$\forall x \in [2\eta, \infty), \forall s \in [0, a], \quad \mathbb{P}_x(Z_s \leq \eta) \leq \delta. \quad (\text{III.26})$$

*Proof of (III.26).* We fix  $x \in [2\eta, \infty)$ . Let  $a \in (0, \infty)$  that is specified later. For any  $s \in [0, a]$ , the Markov inequality entails for any  $\lambda \in (0, \infty)$

$$\mathbb{P}_x(Z_s \leq \eta) \leq e^{\lambda\eta} \mathbb{E}_x[e^{-\lambda Z_s}] = e^{\lambda\eta - xu(s, \lambda)} \leq e^{-\lambda\eta + 2\eta(\lambda - u(s, \lambda))}. \quad (\text{III.27})$$

We now take  $\lambda > q$ . Then,  $u(\cdot, \lambda)$  is decreasing and we get

$$-\lambda\eta + 2\eta(\lambda - u(s, \lambda)) \leq -\lambda\eta + 2\eta \int_0^s \Psi(u(b, \lambda)) db \leq -\lambda\eta + 2\eta a \Psi(\lambda). \quad (\text{III.28})$$

Then set  $\lambda = q + 1 - \eta^{-1} \log \delta$  and  $a = (q + 1)/(2\Psi(\lambda))$ , which entails (III.26) by (III.28) and (III.27). ■

We now complete the proof of the lemma. We first fix  $\varepsilon, \eta \in (0, 1)$  and  $t_0 \in (0, \infty)$  and then we set

$$\delta = \frac{1}{2} \frac{\inf_{t \in [\frac{1}{4}t_0, t_0]} \mathbb{P}_{2\eta}(Z_t > \varepsilon)}{\sup_{t \in [\frac{1}{4}t_0, t_0]} \mathbb{P}_\eta(Z_t > \varepsilon)}.$$

## 2. Construction of M.

By (III.25),  $\delta > 0$ . Let  $a \in (0, \frac{1}{4}t_0)$  be such that (III.26) holds true. We then fix  $x \in [0, \eta]$ ,  $b \in [0, a]$  and  $c \in [\frac{1}{2}t_0, t_0]$  and we introduce the stopping time  $T = \inf\{t \in [0, \infty) : Z_t > 2\eta\}$ . Then,

$$A := \mathbb{P}_x \left( \sup_{s \in [0, b]} Z_s > 2\eta; Z_c > \varepsilon \right) = \mathbb{P}_x(T \leq b; Z_c > \varepsilon) \leq \mathbb{P}_x(Z_b > \eta; Z_c > \varepsilon) + B, \quad (\text{III.29})$$

where  $B := \mathbb{P}_x(T \leq b; Z_b \leq \eta; Z_c > \varepsilon)$  is bounded as follows: by the Markov property at time  $b$  and by (III.20), we first get

$$B \leq \mathbb{E}_x [\mathbf{1}_{\{T \leq b; Z_b \leq \eta\}} \mathbb{P}_{Z_b}(Z_{c-b} > \varepsilon)] \leq \mathbb{P}_\eta(Z_{c-b} > \varepsilon) \mathbb{E}_x [\mathbf{1}_{\{T \leq b; Z_b \leq \eta\}}].$$

Recall that  $p_t(x, dy) = \mathbb{P}_x(Z_t \in dy)$  stands for the transition kernels of  $Z$ . The Markov property at time  $T$  then entails

$$B \leq \mathbb{P}_\eta(Z_{c-b} > \varepsilon) \mathbb{E}_x [\mathbf{1}_{\{T \leq b\}} p_{b-T}(Z_T, [0, \eta])].$$

Next observe that  $\mathbb{P}_x$ -a.s.  $b-T \leq a$  and  $Z_T > 2\eta$ , which implies  $p_{b-T}(Z_T, [0, \eta]) \leq \delta$  by (III.26). Thus,

$$B \leq \delta \mathbb{P}_\eta(Z_{c-b} > \varepsilon) \mathbb{E}_x [\mathbf{1}_{\{T \leq b\}}].$$

Since  $c-b \in [\frac{1}{4}t_0, t_0]$ , we get  $\delta \mathbb{P}_\eta(Z_{c-b} > \varepsilon) \leq \frac{1}{2} \inf_{t \in [\frac{1}{4}t_0, t_0]} \mathbb{P}_{2\eta}(Z_t > \varepsilon)$ . Next, observe that

$$\mathbb{P}_x\text{-a.s. on } \{T \leq b\}, \quad \inf_{t \in [\frac{1}{4}t_0, t_0]} \mathbb{P}_{2\eta}(Z_t > \varepsilon) \leq p_{c-T}(2\eta, (\varepsilon, \infty]) \leq p_{c-T}(Z_T, (\varepsilon, \infty]),$$

where we use (III.20) in the last inequality. Thus, by the Markov property at time  $T$  and the previous inequalities, we finally get

$$B \leq \frac{1}{2} \mathbb{E}_x [\mathbf{1}_{\{T \leq b\}} p_{c-T}(Z_T, (\varepsilon, \infty])] = \frac{1}{2} \mathbb{P}_x(T \leq b; Z_c > \varepsilon) = \frac{1}{2} A,$$

which implies the desired result by (III.29). ■

We end the section by a coupling of finite variation CSBP. To that end, let us briefly recall that CSBP are time-changed Lévy processes via Lamperti transform: let  $X = (X_t)_{t \in [0, \infty)}$  be a cadlag Lévy process without negative jump that is defined on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . We assume that  $X_0 = x \in (0, \infty)$  and that  $\mathbf{E}[\exp(-\lambda X_t)] = \exp(-x\lambda + t\Psi(\lambda))$ . We then set

$$\tau = \inf\{t \in [0, \infty) : X_t = 0\}, \quad L_t = \tau \wedge \inf\left\{s \in [0, \tau) : \int_0^s \frac{dr}{X_r} > t\right\} \quad \text{and} \quad Z_t = X_{L_t}, \quad (\text{III.30})$$

with the conventions  $\inf \emptyset = \infty$  and  $X_\infty = \infty$ . Then,  $(Z_t)_{t \in [0, \infty)}$  is a CSBP( $\Psi, x$ ). See [20] for more details. Recall from (III.6) the definition of  $D$ .

**LEMMA III.8.** *Assume that  $\Psi$  is of finite variation type and that  $D$  is strictly positive. Let  $(Z_t)_{t \in [0, \infty)}$  be a CSBP( $\Psi, x$ ) defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . For any  $\lambda \in [0, \infty)$ , set  $\Psi^*(\lambda) := \Psi(\lambda) - D\lambda$ . Then, there exists  $(Z_t^*)_{t \in [0, \infty)}$ , a CSBP( $\Psi^*, x$ ) on  $(\Omega, \mathcal{F}, \mathbf{P})$  such that*

$$\mathbf{P}\text{-a.s.} \quad \forall t \in [0, \infty), \quad \sup_{s \in [0, t]} Z_s \leq Z_t^*.$$

**Proof** Without loss of generality, we assume that there exists a Lévy process  $X$  defined on  $(\Omega, \mathcal{F}, \mathbf{P})$  such that  $Z$  is derived from  $X$  by the Lamperti time-change (III.30). We then set  $X_t^* = X_t + Dt$  that is a subordinator with Laplace exponent  $-\Psi^*$  and with initial value  $x$ . Since  $D$  is positive, we have  $X_t \leq X_t^*$  for all  $t \in [0, \infty)$ . Observe that  $\tau^* = \infty$ . Let  $L^*$  and  $Z^*$  be derived from  $X^*$  as  $L$  and  $Z$  are derived from  $X$  in (III.30). Then,  $Z^*$  is a CSBP( $\Psi^*, x$ ) and observe that  $L_t^* \geq L_t$ . Since  $X^*$  is non-decreasing,  $Z_t^* = X_{L_t^*}^* \geq X_{L_t}^* \geq X_{L_t} = Z_t$ , which easily implies the desired result since  $Z^*$  is non-decreasing. ■

## 2.2 The cluster measure of CSBP with infinite variation.

Recall that  $\mathbb{D}([0, \infty), [0, \infty))$  stands for the space of  $[0, \infty]$ -valued cadlag functions. Recall that  $Z$  stands for the canonical process. For any  $t \in [0, \infty)$ , we denote by  $\mathcal{F}_t$  the canonical filtration. Recall from (III.11) the definition of the times of absorption  $\zeta_0$ ,  $\zeta_\infty$  and  $\zeta$ . Also recall from the beginning of Section 2.1 the definition of the measure  $\nu_t$  on  $(0, \infty]$ .

**THEOREM III.3.** *Let  $\Psi$  be of infinite variation type. Then, there exists a unique sigma-finite measure  $N_\Psi$  on  $\mathbb{D}([0, \infty), [0, \infty))$  that satisfies the following properties.*

- (a)  $N_\Psi$ -a.e.  $Z_0 = 0$  and  $\zeta > 0$ .
- (b)  $\nu_t(dr) = N_\Psi(Z_t \in dr; Z_t > 0)$ , for any  $t \in (0, \infty)$ .
- (c)  $N_\Psi[F(Z_{\cdot \wedge t})G(Z_{t+\cdot}); Z_t > 0] = N_\Psi[F(Z_{\cdot \wedge t})\mathbb{E}_{Z_t}[G]; Z_t > 0]$ , for any nonnegative functionals  $F, G$  and for any  $t \in (0, \infty)$ .

The measure  $N_\Psi$  is called the cluster measure of CSBP( $\Psi$ ).

**Proof** The only technical point to clear is (a): namely, the right-continuity at time 0. For any  $s, t \in (0, \infty)$  such that  $s \leq t$  and for any  $\varepsilon \in (0, 1)$ , we define a measure  $Q_{t,\varepsilon}^s$  on  $\mathbb{D}([0, \infty), [0, \infty))$  by setting

$$Q_{t,\varepsilon}^s[F] = \frac{1}{\nu_t((\varepsilon, \infty])} \int_{(0, \infty]} \nu_s(dx) \mathbb{E}_x[F(Z_{(\cdot-s)_+}); Z_{t-s} > \varepsilon], \quad (\text{III.31})$$

for any functional  $F$ . By Lemma III.5, (III.31) makes sense and it defines a probability measure on the space  $\mathbb{D}([0, \infty), [0, \infty))$ . The Markov property for CSBP and Lemma III.5 easily imply that for any  $s \leq s_0 \leq t$ ,

$$Q_{t,\varepsilon}^s[F(Z_{s_0+\cdot})] = \frac{1}{\nu_t((\varepsilon, \infty])} \int_{(0, \infty]} \nu_{s_0}(dx) \mathbb{E}_x[F(Z); Z_{t-s_0} > \varepsilon], \quad (\text{III.32})$$

We first prove that for  $t$  and  $\varepsilon$  fixed, the laws  $Q_{t,\varepsilon}^s$  are tight as  $s \rightarrow 0$ . By (III.32), it is clear that we only need to control the paths in a neighbourhood of time 0. By a standard criterion for Skorohod topology (see for instance Theorem 16.8 [16] p. 175), the laws  $Q_{t,\varepsilon}^s$  are tight as  $s \rightarrow 0$  if the following claim holds true: for any  $\eta, \delta \in (0, 1)$ , there exists  $a_1 \in (0, \frac{1}{4}t)$  such that

$$\forall s \in (0, a_1], \quad Q_{t,\varepsilon}^s\left(\sup_{[0, a_1]} Z > 2\eta\right) < \delta. \quad (\text{III.33})$$

To prove (III.33), we first prove that for any  $\eta, \delta \in (0, 1)$ , there exists  $a_0 \in (0, t)$  such that

$$\forall s, b \in (0, a_0] \text{ such that } s \leq b, \quad Q_{t,\varepsilon}^s(Z_b > \eta) < \frac{1}{3}\delta. \quad (\text{III.34})$$

*Proof of (III.34).* Recall that  $\mathbf{1}_{[1, \infty)}(y) \leq C(1 - e^{-y})$ , for any  $y \in [0, \infty]$ , where  $C = (1 - e^{-1})^{-1}$ . Fix  $\eta, \delta \in (0, 1)$  and  $s, b \in (0, t)$  such that  $s \leq b$ . Then, (III.32), with  $b = s_0$ , implies that

$$\begin{aligned} Q_{t,\varepsilon}^s(Z_b > \eta) &\leq C Q_{t,\varepsilon}^s(1 - e^{-\frac{1}{\eta}Z_b}) = \frac{C}{\nu_t((\varepsilon, \infty])} \int_{(0, \infty]} \nu_b(dx) (1 - e^{-\frac{1}{\eta}x}) \mathbb{P}_x(Z_{t-b} > \varepsilon) \\ &\leq \frac{C^2}{\nu_t((\varepsilon, \infty])} \int_{(0, \infty]} \nu_b(dx) (1 - e^{-\frac{1}{\eta}x}) \mathbb{E}_x[1 - e^{-\frac{1}{\varepsilon}Z_{t-b}}] \\ &\leq \frac{C^2}{\nu_t((\varepsilon, \infty])} \int_{(0, \infty]} \nu_b(dx) (1 - e^{-\frac{1}{\eta}x}) (1 - e^{-xu(t-b, \frac{1}{\varepsilon})}) =: f(b). \end{aligned}$$

By developing the product in the integral of the last right member of the inequality, we get

$$f(b) = \frac{C^2}{\nu_t((\varepsilon, \infty])} \left( u(b, \frac{1}{\eta}) + u(t, \frac{1}{\varepsilon}) - u(b, \frac{1}{\eta} + u(t-b, \frac{1}{\varepsilon})) \right) \xrightarrow{b \rightarrow 0} 0,$$

We then define  $a_0$  such that  $\sup_{b \in (0, a_0]} f(b) < \frac{1}{3}\delta$ , which implies (III.34).  $\square$

*Proof of (III.33).* We fix  $\eta, \delta \in (0, 1)$ . Let  $a \in (0, \frac{1}{4}t)$  such that (III.24) in Lemma III.7 holds true with  $t_0 = t$ . Let  $a_0$  as in (III.34). We next set  $a_1 = a \wedge a_0$ . We fix  $s \in (0, a_1]$  and we then get the following inequalities:

$$\begin{aligned} Q_{t,\varepsilon}^s \left( \sup_{[0, a_1]} Z > 2\eta \right) &\leq Q_{t,\varepsilon}^s(Z_s > \eta) + Q_{t,\varepsilon}^s \left( \sup_{[0, a_1]} Z > 2\eta; Z_s \leq \eta, \right) \\ &\leq \frac{1}{3}\delta + \frac{1}{\nu_t((\varepsilon, \infty])} \int_{(0, \eta]} \nu_s(dx) \mathbb{P}_x \left( \sup_{[0, a_1-s]} Z > 2\eta; Z_{t-s} > \varepsilon \right) \\ &\leq \frac{1}{3}\delta + \frac{2}{\nu_t((\varepsilon, \infty])} \int_{(0, \eta]} \nu_s(dx) \mathbb{P}_x(Z_{a_1-s} > \eta; Z_{t-s} > \varepsilon) \\ &\leq \frac{1}{3}\delta + 2Q_{t,\varepsilon}^s(Z_{a_1} > \eta) < \delta. \end{aligned}$$

Here we use (III.34) in the second line, (III.24) in the third line and (III.34) in the fourth one.  $\square$

We have proved that for  $t, \varepsilon$  fixed, the laws  $Q_{t,\varepsilon}^s$  are tight as  $s \rightarrow 0$ . Let  $Q_{t,\varepsilon}$  stand for a possible limiting law. By a simple argument,  $Q_{t,\varepsilon}$  has no fixed jump at time  $s_0$  and basic continuity results entail that (III.32) holds true with  $Q_{t,\varepsilon}$  instead of  $Q_{t,\varepsilon}^s$ , which fixes the finite-dimensional marginal laws of  $Q_{t,\varepsilon}$  on  $(0, \infty)$ . Next observe that for  $\eta, \delta \in (0, 1)$  and  $a_1 \in (0, \frac{1}{4}t)$  as in (III.33), the set  $\{\sup_{(0, a_1)} Z > 2\eta\}$  is an open set of  $\mathbb{D}([0, \infty), [0, \infty])$ . Then, by (III.33) and the Portmanteau Theorem,  $Q_{t,\varepsilon}(\sup_{(0, a_1)} Z > 2\eta) \leq \delta$ . This easily implies that  $Q_{t,\varepsilon}$ -a.s.  $Z_0 = 0$ , which completely fixes the finite-dimensional marginal laws of  $Q_{t,\varepsilon}$  on  $[0, \infty)$ . This proves that there is only one limiting distribution and  $Q_{t,\varepsilon}^s \rightarrow Q_{t,\varepsilon}$  in law as  $s \rightarrow 0$ .

We next set  $N_{t,\varepsilon} = \nu_t((\varepsilon, \infty]) Q_{t,\varepsilon}$ . We easily get  $N_{t,\varepsilon} - N_{t,\varepsilon'} = N_{t,\varepsilon}(\cdot; Z_t \in (\varepsilon, \varepsilon'])$ , for any  $0 < \varepsilon < \varepsilon' < 1$ . Fix  $\varepsilon_p \in (0, 1)$ ,  $p \in \mathbb{N}$ , that decreases to 0. We define a measure  $N_t$  by setting

$$N_t = N_{t,\varepsilon_0} + \sum_{p \geq 0} N_{t,\varepsilon_{p+1}}(\cdot; Z_t \in (\varepsilon_{p+1}, \varepsilon_p]) = N_{t,\varepsilon_0} + \sum_{p \geq 0} N_{t,\varepsilon_{p+1}} - N_{t,\varepsilon_p}.$$

By the first equality,  $N_t$  is a well-defined sigma-finite measure; the second equality shows that the definition of  $N_t$  does not depend on the sequence  $(\varepsilon_p)_{p \in \mathbb{N}}$ , which implies  $N_t(\cdot; Z_t > \varepsilon) = \nu_t((\varepsilon, \infty]) Q_{t,\varepsilon}$ , for any  $\varepsilon \in (0, 1)$ . Consequently, we get  $N_t - N_{t'} = N_t(\cdot; Z_{t'} = 0)$ , for any  $t' > t > 0$ . Fix  $t_q \in (0, 1)$ ,  $q \in \mathbb{N}$ , that decreases to 0. We define  $N_\Psi$  by setting

$$N_\Psi = N_{t_0} + \sum_{q \geq 0} N_{t_{q+1}}(\cdot; Z_{t_q} = 0) = N_{t_0} + \sum_{q \geq 0} N_{t_{q+1}} - N_{t_q}.$$

The first equality shows that  $N_\Psi$  is a well-defined measure and the second one that its definition does not depend on the sequence  $(t_q)_{q \in \mathbb{N}}$ , which implies

$$\forall \varepsilon \in (0, 1), \forall t \in (0, \infty), \quad N_\Psi(\cdot; Z_t > \varepsilon) = \nu_t((\varepsilon, \infty]) Q_{t,\varepsilon}. \quad (\text{III.35})$$

This easily entails that for any nonnegative functional  $F$

$$\forall t \in (0, \infty), \quad N_\Psi[F(Z_{t+\cdot}); Z_t > 0] = \int_{(0, \infty)} \nu_t(dx) \mathbb{E}_x[F]. \quad (\text{III.36})$$

Recall that  $\zeta$  is the time of absorption in  $\{0, \infty\}$ . Since  $N_{t_q, \varepsilon_p}(\zeta = 0) = 0$ , we get  $N_\Psi(\zeta = 0) = 0$  and thus,  $N_\Psi(\{0\}) = 0$ , where  $\mathbf{0}$  stands for the null function. Set  $A_{p,q} = \{Z_{t_q} > \varepsilon_p\}$ . Then,  $N_\Psi(A_{p,q}) < \infty$  by (III.35). Since  $\mathbb{D}([0, \infty), [0, \infty)) = \{0\} \cup \bigcup_{p,q \geq 1} A_{p,q}$ ,  $N_\Psi$  is sigma-finite. Properties (b) and (c) are easily derived from (III.36), (III.35) and standard limit-procedures: the details are left to the reader. ■

### 2.3 Proof of Theorem III.1.

#### Poisson decomposition of CSBP.

From now on, we fix  $(\Omega, \mathcal{F}, \mathbf{P})$ , a probability space on which are defined all the random variables that we mention, unless the contrary is explicitly specified. We also fix  $x \in (0, \infty)$  and we recall that  $\ell$  stands for the Lebesgue measure on  $\mathbb{R}$  or on  $[0, x]$ , according to the context.

We first briefly recall Palm formula for Poisson point measures: let  $E$  be a Polish space equipped with its Borel sigma-field  $\mathcal{E}$ . Let  $A_n \in \mathcal{E}$ ,  $n \in \mathbb{N}$ , be a partition of  $E$ . We denote by  $\mathcal{M}_{\text{pt}}(E)$  the set of point measures  $m$  on  $E$  such that  $m(A_n) < \infty$  for any  $n \in \mathbb{N}$ ; we equip  $\mathcal{M}_{\text{pt}}(E)$  with the sigma-field generated by the applications  $m \mapsto m(A)$ , where  $A$  ranges in  $\mathcal{E}$ . Let  $\mathcal{N} = \sum_{i \in I} \delta_{z_i}$ , be a Poisson point measure on  $E$  whose intensity measure  $\mu$  satisfies  $\mu(A_n) < \infty$  for every  $n \in \mathbb{N}$ . We shall refer to the following as to the *Palm formula*: for any measurable  $F : E \times \mathcal{M}_{\text{pt}}(E) \rightarrow [0, \infty)$ ,

$$\mathbf{E} \left[ \sum_{i \in I} F(z_i, \mathcal{N} - \delta_{z_i}) \right] = \int_E \mu(dz) \mathbf{E} [F(z, \mathcal{N})]. \quad (\text{III.37})$$

We next introduce the Poisson point measures that are used to define the population associated with a CSBP.

**Infinite variation cases.** We assume that  $\Psi$  is of infinite variation type. Let

$$\mathcal{P} = \sum_{i \in I} \delta_{(x_i, Z^i)} \quad (\text{III.38})$$

be a Poisson point measure on  $[0, x] \times \mathbb{D}([0, \infty), [0, \infty))$ , with intensity  $\mathbf{1}_{[0,x]}(y) \ell(dy) N_\Psi(dZ)$ , where  $N_\Psi$  is the cluster measure associated with  $\Psi$  as specified in Theorem III.3. Then, for any  $t \in (0, \infty)$ , we define the following random point measures on  $[0, x]$ :

$$\mathcal{Z}_t = \sum_{i \in I} Z_t^i \delta_{x_i} \quad \text{and} \quad \mathcal{Z}_{t-} = \sum_{i \in I} Z_{t-}^i \delta_{x_i}. \quad (\text{III.39})$$

We also set  $\mathcal{Z}_0 = \ell(\cdot \cap [0, x])$ . ■

**Finite variation cases.** We assume that  $\Psi$  is of finite variation type and not linear. Recall from (III.6) the definition of  $D$ . Let

$$\mathcal{Q} = \sum_{j \in J} \delta_{(x_j, t_j, Z^j)} \quad (\text{III.40})$$

be a Poisson point measure on  $[0, x] \times [0, \infty) \times \mathbb{D}([0, \infty), [0, \infty))$ , whose intensity measure is

$$\mathbf{1}_{[0,x]}(y) \ell(dy) e^{-Dt} \ell(dt) \int_{(0, \infty)} \pi(dr) \mathbb{P}_r(dZ),$$

where  $\mathbb{P}_r$  is the canonical law of a CSBP( $\Psi, r$ ) and where  $\pi$  is the Lévy measure of  $\Psi$ . Then, for any  $t \in (0, \infty)$ , we define the following random measures on  $[0, x]$ :

$$\mathcal{Z}_t = e^{-Dt} \ell(\cdot \cap [0, x]) + \sum_{j \in J} \mathbf{1}_{\{t_j \leq t\}} Z_{t-t_j}^j \delta_{x_j}, \quad \mathcal{Z}_{t-} = e^{-Dt} \ell(\cdot \cap [0, x]) + \sum_{j \in J} \mathbf{1}_{\{t_j \leq t\}} Z_{(t-t_j)-}^j \delta_{x_j}. \quad (\text{III.41})$$

We also set  $\mathcal{Z}_0 = \ell(\cdot \cap [0, x])$ . ■

In both cases, for any  $t \in [0, \infty)$  and any  $B \in \mathcal{B}([0, x])$ ,  $\mathcal{Z}_t(B)$  and  $\mathcal{Z}_{t-}(B)$  are  $[0, \infty]$ -valued  $\mathcal{F}$ -measurable random variables. The finite dimensional marginals of  $(\mathcal{Z}_t(B))_{t \in [0, \infty)}$  are those of a CSBP( $\Psi, \ell(B)$ ): in the infinite variation cases, it is a simple consequence of Theorem III.3 (c); in the finite variation cases, it comes from direct computations: we leave the details to the reader. Moreover, if  $B_1, \dots, B_n$  are disjoint Borel subsets of  $[0, x]$ , note that the processes  $(\mathcal{Z}_t(B_k))_{t \in [0, \infty)}$ ,  $1 \leq k \leq n$  are independent. To simplify notation, we also set

$$\forall t \in [0, \infty), \quad \mathcal{Z}_t = \mathcal{Z}_t([0, x]), \quad (\text{III.42})$$

that has the finite dimensional marginals of a CSBP( $\Psi, x$ ).

### Regularity of $\mathcal{Z}$ .

Since we deal with possibly infinite measures, we introduce the following specific notions. We fix a metric  $d$  on  $[0, \infty]$  that generates its topology. For any positive Borel measures  $\mu$  and  $\nu$  on  $[0, x]$ , we define their variation distance by setting

$$d_{\text{var}}(\mu, \nu) := \sup_{B \in \mathcal{B}([0, x])} d(\mu(B), \nu(B)). \quad (\text{III.43})$$

The following proposition deals with the regularity of  $\mathcal{Z}$  on  $(0, \infty)$ , which is sufficient for our purpose. The regularity at time 0 is briefly discussed later.

**PROPOSITION III.9.** *Let  $\mathcal{Z}$  be as in (III.39) or (III.41). Then,*

$$\mathbf{P}\text{-a.s. } \forall t \in (0, \infty), \quad \lim_{h \rightarrow 0+} d_{\text{var}}(\mathcal{Z}_{t+h}, \mathcal{Z}_t) = 0 \quad \text{and} \quad \lim_{h \rightarrow 0+} d_{\text{var}}(\mathcal{Z}_{t-h}, \mathcal{Z}_{t-}) = 0. \quad (\text{III.44})$$

**Proof** We first prove the infinite variation cases. We proceed by approximation. Let us fix  $s_0 \in (0, \infty)$ . For any  $\varepsilon \in (0, 1)$ , we set

$$\forall t \in (0, \infty), \quad \mathcal{Z}_t^\varepsilon = \sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i > \varepsilon\}} Z_t^i \delta_{x_i}.$$

Note that  $\#\{i \in I : Z_{s_0}^i > \varepsilon\}$  is a Poisson r.v. with mean  $xN_\Psi(Z_{s_0} > \varepsilon) = x\nu_{s_0}((\varepsilon, \infty]) < \infty$ . Therefore,  $\mathcal{Z}^\varepsilon$  is a finite sum of weighted Dirac masses whose weights are cadlag  $[0, \infty]$ -valued processes. Then, by an easy argument,  $\mathbf{P}$ -a.s.  $\mathcal{Z}^\varepsilon$  is  $d_{\text{var}}$ -cadlag on  $(0, \infty)$ .

For any  $v \in [0, \infty]$ , then set  $\varphi(v) = \sup\{d(y, z) ; y \leq z \leq y + v\}$ , which is well-defined, bounded, non-decreasing and such that  $\lim_{v \rightarrow 0} \varphi(v) = 0$ . For any  $\varepsilon > \varepsilon' > 0$ , observe that  $\mathcal{Z}_t^{\varepsilon'} = \mathcal{Z}_t^\varepsilon + \sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i \in (\varepsilon', \varepsilon]\}} Z_t^i \delta_{x_i}$ . Then, we fix  $T \in (0, \infty)$ , we set  $Y_t^{\varepsilon', \varepsilon} := \sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i \in (\varepsilon', \varepsilon]\}} Z_{s_0+t}^i$  and we get

$$\sup_{t \in [s_0, s_0+T]} d_{\text{var}}(\mathcal{Z}_t^{\varepsilon'}, \mathcal{Z}_t^\varepsilon) \leq \varphi(V_{\varepsilon', \varepsilon}) \quad \text{where} \quad V_{\varepsilon', \varepsilon} := \sup_{t \in [0, T]} Y_t^{\varepsilon', \varepsilon}.$$

Note that  $Y^{\varepsilon', \varepsilon}$  is a cadlag CSBP( $\Psi$ ). The exponential formula for Poisson point measures and Theorem III.3 (b) imply for any  $\lambda \in (0, \infty)$ ,

$$-\frac{1}{x} \log \mathbf{E}[\exp(-\lambda Y_0^{\varepsilon', \varepsilon})] = \int_{(\varepsilon', \varepsilon]} \nu_{s_0}(dr) (1 - e^{-\lambda r}) \leq \lambda \int_{(0, \varepsilon]} \nu_{s_0}(dr) r \xrightarrow{\varepsilon \rightarrow 0} 0.$$

For any  $\eta \in (0, \infty)$ , it easily implies  $\lim_{\varepsilon \rightarrow 0} \sup_{\varepsilon' \in (0, \varepsilon]} \mathbf{P}(Y_0^{\varepsilon', \varepsilon} > \eta) = 0$ . Next, note that  $r \mapsto \mathbb{P}_r(\sup_{t \in [0, T]} \mathcal{Z}_t > \eta)$  is non-decreasing and recall that  $\lim_{r \rightarrow 0+} \mathbb{P}_r(\sup_{t \in [0, T]} \mathcal{Z}_t > \eta) = 0$ , by Lemma III.6. This limit, combined with the previous argument, entails that  $\lim_{\varepsilon \rightarrow 0} \sup_{\varepsilon' \in (0, \varepsilon]} \mathbf{E}[\varphi(V_{\varepsilon', \varepsilon})] = 0$ .



Therefore, we can find  $\varepsilon_p \in (0, 1)$ ,  $p \in \mathbb{N}$ , that decreases to 0 such that  $\sum_{p \geq 0} \mathbf{E}[\varphi(V_{\varepsilon_{p+1}, \varepsilon_p})] < \infty$ , and there exists  $\Omega_0 \in \mathcal{F}$  such that  $\mathbf{P}(\Omega_0) = 1$  and such that  $R_p := \sum_{q \geq p} \varphi(V_{\varepsilon_{q+1}, \varepsilon_q}) \rightarrow 0$  as  $p \rightarrow \infty$ , on  $\Omega_0$ . We then work deterministically on  $\Omega_0$ : by the previous arguments, for all Borel subsets  $B$  of  $[0, x]$ , for all  $t \in (s_0, s_0 + T)$  and for all  $q > p$ , we get  $d(\mathcal{Z}_t^{\varepsilon_q}(B), \mathcal{Z}_t^{\varepsilon_p}(B)) \leq R_p$  and  $d(\mathcal{Z}_{t-}^{\varepsilon_q}(B), \mathcal{Z}_{t-}^{\varepsilon_p}(B)) \leq R_p$ , since  $d$  is a distance on  $[0, \infty]$ . Since  $t > s_0$ , the monotone convergence for sums entails that  $\lim_{q \rightarrow \infty} \mathcal{Z}_t^{\varepsilon_q}(B) = \mathcal{Z}_t(B)$  and  $\lim_{q \rightarrow \infty} \mathcal{Z}_{t-}^{\varepsilon_q}(B) = \mathcal{Z}_{t-}(B)$ . By the continuity of the distance  $d$ , for all  $B$ , all  $t \in (s_0, \infty)$  and all  $p \in \mathbb{N}$ , we get  $d(\mathcal{Z}_t(B), \mathcal{Z}_t^{\varepsilon_p}(B)) \leq R_p$  and  $d(\mathcal{Z}_{t-}(B), \mathcal{Z}_{t-}^{\varepsilon_p}(B)) \leq R_p$ . This easily implies that  $\mathcal{Z}$  is  $d_{\text{var}}$ -cadlag on  $(s_0, s_0 + T)$  since the processes  $\mathcal{Z}^{\varepsilon_p}$  are also  $d_{\text{var}}$ -cadlag on the same interval. This completes the proof in the infinite variation cases since  $s_0$  can be taken arbitrarily small and  $T$  arbitrarily large.

We next consider the finite variation cases: we fix  $s_0 \in (0, \infty)$  and for any  $\varepsilon \in (0, 1)$ , we set

$$\forall t \in [0, s_0], \quad \mathcal{Z}_t^\varepsilon = \sum_{j \in J} \mathbf{1}_{\{t_j \leq t, Z_0^j > \varepsilon\}} \mathcal{Z}_{t-t_j}^j \delta_{x_j}.$$

Since  $\#\{j \in J : t_j \leq s_0, Z_0^j > \varepsilon\}$  is a Poisson r.v. with mean  $x \pi((\varepsilon, \infty]) \int_0^{s_0} e^{-Dt} dt < \infty$ ,  $\mathcal{Z}^\varepsilon$ , as a process indexed by  $[0, s_0]$ , is a finite sum of weighted Dirac masses whose weights are cadlag  $[0, \infty]$ -valued processes on  $[0, s_0]$ : by an easy argument, it is  $d_{\text{var}}$ -cadlag on  $[0, s_0]$ . Next observe that for any  $\varepsilon > \varepsilon' > 0$ ,  $\mathcal{Z}_t^{\varepsilon'} = \mathcal{Z}_t^\varepsilon + \sum_{j \in J} \mathbf{1}_{\{t_j \leq t, Z_0^j \in (\varepsilon', \varepsilon]\}} \mathcal{Z}_{t-t_j}^j \delta_{x_j}$ . Thus,

$$\sup_{t \in [0, s_0]} d_{\text{var}}(\mathcal{Z}_t^{\varepsilon'}, \mathcal{Z}_t^\varepsilon) \leq \varphi(V_{\varepsilon', \varepsilon}) \quad \text{where} \quad V_{\varepsilon', \varepsilon} := \sum_{j \in J} \mathbf{1}_{\{t_j \leq s_0, Z_0^j \in (\varepsilon', \varepsilon]\}} \sup_{t \in [0, s_0]} \mathcal{Z}_t^j.$$

The exponential formula for Poisson point measures then implies for any  $\lambda \in (0, \infty)$ ,

$$-\frac{1}{x} \log \mathbf{E}[\exp(-\lambda V_{\varepsilon', \varepsilon})] = \int_0^{s_0} e^{-Dt} dt \int_{(\varepsilon', \varepsilon]} \pi(dr) \mathbb{E}_r \left[ 1 - e^{-\lambda \sup_{[0, s_0]} Z} \right].$$

We now use Lemma III.8: if  $D \in (0, \infty)$ , we set  $\Psi^*(\lambda) = \Psi(\lambda) - D\lambda$  and if  $D \in (-\infty, 0]$ , we simply take  $\Psi^* = \Psi$ . Denote by  $u^*$  the function derived from  $\Psi^*$  as  $u$  is derived from  $\Psi$  by (III.9). As a consequence of Lemma III.8, we get  $\mathbb{E}_r[1 - e^{-\lambda \sup_{[0, s_0]} Z}] \leq 1 - e^{-ru^*(s_0, \lambda)}$ . Thus,

$$\begin{aligned} -\frac{1}{x} \log \mathbf{E}[\exp(-\lambda V_{\varepsilon', \varepsilon})] &\leq \int_0^{s_0} e^{-Dt} dt \int_{(\varepsilon', \varepsilon]} \pi(dr) (1 - e^{-ru^*(s_0, \lambda)}) \\ &\leq s_0 e^{|D|s_0} u^*(s_0, \lambda) \int_{(0, \varepsilon]} \pi(dr) r \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

This easily entails  $\lim_{\varepsilon \rightarrow 0} \sup_{\varepsilon' \in (0, \varepsilon]} \mathbf{E}[\varphi(V_{\varepsilon', \varepsilon})] = 0$ . We then argue as in the infinite variation cases: there exists a sequence  $\varepsilon_p \in (0, 1)$ ,  $p \in \mathbb{N}$ , that decreases to 0 and there exists  $\Omega_0 \in \mathcal{F}$  with  $\mathbf{P}(\Omega_0) = 1$ , such that  $R_p := \sum_{q \geq p} \varphi(V_{\varepsilon_{q+1}, \varepsilon_q}) \rightarrow 0$  as  $p \rightarrow \infty$ , on  $\Omega_0$ . We work deterministically on  $\Omega_0$ : we set  $\mathcal{Z}_t^* = \mathcal{Z}_t - e^{-Dt} \ell(\cdot \cap [0, x])$ , that is the purely atomic part of  $\mathcal{Z}_t$ . Then, for all  $B$ , for all  $t \in [0, s_0]$  and for all  $p \in \mathbb{N}$ ,  $d(\mathcal{Z}_t^*(B), \mathcal{Z}_t^{\varepsilon_p}(B)) \leq R_p$  and  $d(\mathcal{Z}_{t-}^*(B), \mathcal{Z}_{t-}^{\varepsilon_p}(B)) \leq R_p$ . This implies that  $\mathbf{P}$ -a.s.  $\mathcal{Z}^*$  is  $d_{\text{var}}$ -cadlag on  $[0, s_0]$ , by the same arguments as in the infinite variation cases. Clearly, a similar result holds true for  $\mathcal{Z}$  on  $[0, s_0]$ , which completes the proof of Proposition III.9, since  $s_0$  can be chosen arbitrarily large.  $\blacksquare$

Note that in the finite variation cases,  $\mathcal{Z}$  is  $d_{\text{var}}$ -right continuous at 0. In the infinite variation cases, this cannot be so: indeed, set  $B = [0, x] \setminus \{x_i; i \in I\}$ , then  $\mathcal{Z}_t(B) = 0$  for any  $t \in (0, \infty)$  but  $\mathcal{Z}_0(B) = \ell(B) = x$ . However, we have the following lemma.

**LEMMA III.10.** *Assume that  $\Psi$  is of infinite variation type. Let  $\mathcal{Z}$  be defined on  $(\Omega, \mathcal{F}, \mathbf{P})$  by (III.39). Then*

$$\forall B \in \mathcal{B}([0, x]), \quad \mathbf{P}\text{-a.s.} \lim_{t \rightarrow 0^+} \mathcal{Z}_t(B) = \ell(B).$$

*This implies that  $\mathbf{P}$ -a.s.  $\mathcal{Z}_t \rightarrow \mathcal{Z}_0$  weakly as  $t \rightarrow 0^+$ .*

**Proof** Since  $(\mathcal{Z}_t(B))_{t \in [0, \infty)}$  has the finite dimensional marginal laws of a CSBP( $\Psi, \ell(B)$ ), it admits a modification  $Y = (Y_t)_{t \in [0, \infty)}$  that is cadlag on  $[0, \infty)$ . By Proposition III.9, observe that  $\mathcal{Z} \cdot (B)$  is cadlag on  $(0, \infty)$ . Therefore,  $\mathbf{P}$ -a.s.  $Y$  and  $\mathcal{Z} \cdot (B)$  coincide on  $(0, \infty)$ , which implies the lemma. ■

Proof of Theorem III.1 and of Theorem III.2 (i).

Recall the notation  $Z_t = \mathcal{Z}_t([0, x])$ . By Proposition III.9,  $Z$  is cadlag on  $(0, \infty)$  and by arguing as in Lemma III.10, without loss of generality, we can assume that  $Z$  is right continuous at time 0: it is therefore a cadlag CSBP( $\Psi, x$ ). Recall from (III.11) the definition of the absorption times  $\zeta_0, \zeta_\infty$  and  $\zeta$  of  $Z$ . We first set

$$\forall t \in [0, \zeta), \quad \forall B \in \mathcal{B}([0, x]), \quad M_t(B) = \frac{\mathcal{Z}_t(B)}{Z_t}. \quad (\text{III.45})$$

Observe that  $M$  has the desired regularity on  $[0, \zeta)$  by Proposition III.9 and Lemma III.10. Moreover  $M$  satisfies property (III.17). It only remains to define  $M$  for the times  $t \geq \zeta$  on the event  $\{\zeta < \infty\}$ .

Let us first assume that  $\mathbf{P}(\zeta_0 < \infty) > 0$ , which can only happen if  $\Psi$  satisfies (III.15). Note that in this case,  $\Psi$  is of infinite variation type. Now recall  $\mathcal{P}$  from (III.38) and  $\mathcal{Z}$  from (III.39). Thus,  $\zeta_0 = \sup_{i \in I} \zeta_0^i$ , where  $\zeta_0^i$  stands for the extinction time of  $Z^i$ . Then,  $\mathbf{P}(\zeta_0 < t) = \exp(-xN_\Psi(\zeta_0 \geq t))$ . Thus,  $N_\Psi(\zeta_0 \geq t) = v(t)$ , that is the function defined right after (III.15) which satisfies  $\int_{v(t)}^\infty dr/\Psi(r) = t$ . Since  $v$  is  $C^1$ , the law (restricted to  $(0, \infty)$ ) of the extinction time  $\zeta_0$  under  $N_\Psi$  is diffuse. This implies that  $\mathbf{P}$ -a.s. on  $\{\zeta_0 < \infty\}$  there exists a unique  $i_0 \in I$  such that  $\zeta_0 = \zeta_0^{i_0}$ . Then, we set  $\xi_0 := \sup\{\zeta_0^i; i \in I \setminus \{i_0\}\}$ ,  $\mathbf{e} = x_{i_0}$  and we get  $M_t = \delta_{\mathbf{e}}$  for any  $t \in (\xi_0, \zeta_0)$ . Thus, on the event  $\{\zeta_0 < \infty\}$  and for any  $t > \zeta_0$ , we set  $M_t = \delta_{\mathbf{e}}$  and  $M$  has the desired regularity on the event  $\{\zeta_0 < \infty\}$ . An easy argument on Poisson point measures entails that conditional on  $\{\zeta_0 < \infty\}$ ,  $\mathbf{e}$  is uniformly distributed on  $[0, x]$ .

Let us next assume that  $\mathbf{P}(\zeta_\infty < \infty) > 0$ , which can only happen if  $\Psi$  satisfies (III.13). We first consider the infinite variation cases: note that  $\zeta_\infty = \inf_{i \in I} \zeta_\infty^i$ , where  $\zeta_\infty^i$  stands for the explosion time of  $Z^i$ . Then,  $\mathbf{P}(\zeta_\infty \geq t) = \exp(-xN_\Psi(\zeta_\infty < t))$ . Thus,  $N_\Psi(\zeta_\infty < t) = \kappa(t)$  that is the function defined right after (III.13) which satisfies  $\int_0^{\kappa(t)} dr/(\Psi(r))_- = t$ . Since  $\kappa$  is  $C^1$ , the law (restricted to  $(0, \infty)$ ) of the explosion time  $\zeta_\infty$  under  $N_\Psi$  is diffuse. This implies that  $\mathbf{P}$ -a.s. on  $\{\zeta_\infty < \infty\}$  there exists a unique  $i_1 \in I$  such that  $\zeta_\infty = \zeta_\infty^{i_1}$ . Then, on  $\{\zeta_\infty < \infty\}$ , we set  $\mathbf{e} = x_{i_1}$  and  $M_t = \delta_{\mathbf{e}}$ , for any  $t \geq \zeta_\infty$ . Then, we get  $\lim_{t \rightarrow \zeta_\infty^-} \|M_t - \delta_{\mathbf{e}}\|_{\text{var}} = 0$  and an easy argument on Poisson point measures entails that conditional on  $\{\zeta_\infty < \infty\}$ ,  $\mathbf{e}$  is uniformly distributed on  $[0, x]$ . This completes the proof when  $\Psi$  is of infinite variation type. In the finite variation cases, we argue in the same way: namely, by simple computations, one shows that for any  $t \in (0, \infty)$ ,  $\#\{j \in J : t_j \leq t, Z_{t-t_j}^j = \infty\}$  is a Poisson r.v. with mean  $x\kappa(t)$ ; it is therefore finite and the times of explosion of the population have diffuse laws: this proves that the descendent population of exactly one ancestor explodes strictly before the others, and it implies the desired result in the finite variation cases: the details are left to the reader. ■

**REMARK III.11.** *Note that the above construction of  $M$  entails Theorem III.2 (i).* ■

### 3 Proof of Theorem III.2.

#### 3.1 Results on Grey martingales.

We briefly discuss the limiting laws of Grey martingales (see [40]) associated with CSBP that are involved in describing the asymptotic frequencies of the settlers. Recall from (III.39) and (III.41) the definition of  $\mathcal{Z}_t$ : for any  $y$  fixed,  $t \mapsto \mathcal{Z}_t([0, y])$  is a CSBP( $\Psi, y$ ) and for any  $t$  fixed,  $y \mapsto \mathcal{Z}_t([0, y])$  is a subordinator. Let  $\theta \in (0, \infty)$  and  $y \in (0, x]$ . We assume that  $u(-t, \theta)$  is well-defined for any  $t \in (0, \infty)$ : namely, we assume that  $\kappa(t) < \theta < v(t)$ , for all  $t \in (0, \infty)$ . Recall that (III.8) extends to negative times. Therefore,  $t \mapsto \exp(-u(-t, \theta)\mathcal{Z}_t([0, y]))$  is a  $[0, 1]$ -valued martingale that a.s. converges to a limit in  $[0, 1]$  denoted by  $\exp(-W_y^\theta)$ , where  $W_y^\theta$  is a  $[0, \infty]$ -valued random variable. Since  $y \mapsto u(-t, \theta)\mathcal{Z}_t([0, y])$  is a subordinator,  $y \mapsto W_y^\theta$  is a (possibly killed) subordinator. We denote by  $\phi_\theta$  its Laplace exponent that has therefore the general Lévy-Khintchine form:

$$\forall \lambda \in [0, \infty), \quad \phi_\theta(\lambda) = \kappa_\theta + d_\theta \lambda + \int_{(0, \infty)} \varrho_\theta(dr) (1 - e^{-\lambda r}),$$

where  $\kappa_\theta, d_\theta \in [0, \infty)$  and  $\int_{(0, \infty)} (1 \wedge r) \varrho_\theta(dr) < \infty$ . Note that  $\phi_\theta(1) = \theta$ , by definition. We first consider the behaviour of CSBP when they tend to  $\infty$ .

**PROPOSITION III.12.** *We assume that  $\Psi$  is not linear and that  $\Psi'(0+) \in (-\infty, 0)$ , which implies that  $\Psi$  is conservative and  $q \in (0, \infty]$ . Let  $\theta \in (0, q)$ . Then,  $u(-t, \theta)$  is well-defined for all  $t \in (0, \infty)$  and  $\lim_{t \rightarrow \infty} u(-t, \theta) = 0$ . For any  $\theta' \in (0, q)$  and any  $y \in (0, x]$ , we then get  $\mathbf{P}$ -a.s.*

$$W_y^\theta = R_{\theta'; \theta} W_y^{\theta'} \quad \text{where} \quad R_{\theta'; \theta} := \exp\left(\Psi'(0+) \int_{\theta'}^\theta \frac{d\lambda}{\Psi(\lambda)}\right). \quad (\text{III.46})$$

$W^\theta$  is a conservative subordinator without drift: namely  $\kappa_\theta = d_\theta = 0$ . Moreover,

$$\forall \lambda \in (0, \infty), \quad \phi_\theta(\lambda) = u\left(\frac{\log \lambda}{-\Psi'(0+)}, \theta\right) \quad \text{and} \quad \varrho_\theta((0, \infty)) = q. \quad (\text{III.47})$$

Thus, if  $q < \infty$ ,  $W^\theta$  is a compound Poisson process with jump-rate  $q$  and jump-law  $\frac{1}{q}\varrho_\theta$  whose Laplace transform is  $\lambda \mapsto 1 - \frac{1}{q}u\left(\frac{\log \lambda}{-\Psi'(0+)}, \theta\right)$ .

**Proof** Let  $\theta \in (0, q)$  and  $t \in (0, \infty)$ . Note that  $v(t) > q$  and since  $\Psi$  is conservative,  $\kappa(t) = 0$ . Thus, for all  $t \in (0, \infty)$ ,  $u(-t, \theta)$  is well-defined. Note that  $\Psi$  is negative on  $(0, q)$ , then, by (III.9),  $\lim_{t \rightarrow \infty} u(-t, \theta) = 0$  and  $\lim_{t \rightarrow \infty} u(t, \theta) = q$ , even if  $q = \infty$ . Next, observe that

$$\frac{u(-t, \theta)}{u(-t, \theta')} = \exp\left(\int_{\theta'}^\theta d\lambda \partial_\lambda \log(u(-t, \lambda))\right) = \exp\left(\int_{\theta'}^\theta \frac{\Psi(u(-t, \lambda))}{u(-t, \lambda)} \frac{d\lambda}{\Psi(\lambda)}\right). \quad (\text{III.48})$$

This entails (III.46) since  $\lim_{\lambda \rightarrow 0} \Psi(\lambda)/\lambda = \Psi'(0+)$ . Thus,  $\phi_\theta(1/R_{\theta'; \theta}) = \phi_{\theta'}(1) = \theta'$ . Then, take  $\theta' = u(t, \theta)$ : by (III.9), it implies that  $\phi_\theta(e^{-\Psi'(0+)t}) = u(t, \theta)$ , for any  $t \in \mathbb{R}$ , which proves the formula for  $\phi_\theta$  in (III.47). Next observe that  $\kappa_\theta = \lim_{\lambda \rightarrow 0} \phi_\theta(\lambda) = \lim_{t \rightarrow \infty} u(-t, \theta) = 0$ . Namely,  $W^\theta$  is conservative. Also note that  $\lim_{\lambda \rightarrow \infty} \phi_\theta(\lambda) = \lim_{t \rightarrow \infty} u(t, \theta) = q$ . Thus, if  $q < \infty$ ,  $d_\theta = 0$  and the last part of the proposition holds true.

We next assume that  $q = \infty$ . Then,  $-\Psi$  is the Laplace exponent of a conservative subordinator and we are in the finite variation cases. We set  $A(t) := \log(e^{\Psi'(0+)t} u(t, \theta))$  and we observe that  $\log d_\theta =$

$\lim_{t \rightarrow \infty} A(t)$ , by taking  $\lambda = e^{-\Psi'(0+)t}$  in (III.47). An easy computation using (III.8) entails

$$\begin{aligned} A(t) - \log \theta &= \int_0^t (\Psi'(0+) + \partial_s \log u(s, \theta)) ds = \int_0^t \left( \Psi'(0+) - \frac{\Psi(u(s, \theta))}{u(s, \theta)} \right) ds \\ &= t \int_0^1 \left( \Psi'(0+) - \frac{\Psi(u(st, \theta))}{u(st, \theta)} \right) ds. \end{aligned}$$

Recall that  $\lim_{\lambda \rightarrow \infty} \Psi(\lambda)/\lambda = D$ . Then, for any  $s \in (0, 1]$ ,

$$\lim_{t \rightarrow \infty} \Psi'(0+) - \frac{\Psi(u(st, \theta))}{u(st, \theta)} = \Psi'(0+) - D = - \int_{(0, \infty)} r \pi(dr) < 0,$$

since  $\pi \neq 0$ . This implies that  $\lim_{t \rightarrow \infty} A(t) = -\infty$  and thus  $d_\theta = 0$ . ■

We complete this result by the following lemma.

**LEMMA III.13.** *We assume that  $\Psi$  is not linear and that  $\Psi'(0+) \in (-\infty, 0)$ , which implies  $\Psi$  is conservative and  $q \in (0, \infty]$ . Let  $\theta \in (0, q)$ . Then,  $u(-t, \theta)$  is well-defined for all  $t \in (0, \infty)$  and  $\lim_{t \rightarrow \infty} u(-t, \theta) = 0$ . Moreover, there exists a cadlag subordinator  $W^\theta$  whose initial value is 0 and whose Laplace exponent is  $\phi_\theta$  as defined by (III.47) such that*

$$\mathbf{P}\text{-a.s.} \quad \forall y \in [0, x], \quad \lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, y]) = W_y^\theta \quad \text{and} \quad \lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t(\{y\}) = \Delta W_y^\theta,$$

where  $\Delta W_y^\theta$  stands for the jump of  $W^\theta$  at  $y$ .

**Proof** We first assume that  $\Psi$  is of finite variation type. Fix  $\varepsilon, s_0 \in (0, \infty)$ . Recall from (III.40) the definition of  $\mathcal{Q}$  and observe that  $\sum_{j \in J} \mathbf{1}_{\{t_j \leq s_0, Z_0^j > \varepsilon\}} \delta_{(x_j, t_j, Z^j)} = \sum_{1 \leq n \leq N} \delta_{(X_n, T_n, Z^{(n)})}$ , where  $N$  is a Poisson r.v. with mean  $C := xD^{-1}(1 - e^{-Ds_0})\pi((\varepsilon, \infty))$  and conditionally given  $N$ , the variables  $X_n, T_n, Z^{(n)}$ ,  $1 \leq n \leq N$  are independent: the  $X_n$  are uniformly distributed on  $[0, x]$ , the law of  $T_n$  is  $(1 - e^{-Ds_0})^{-1} D e^{-Dt} \mathbf{1}_{[0, s_0]}(t) \ell(dt)$  and the processes  $Z^{(n)}$  are distributed as CSBP( $\Psi$ ) whose entrance law is  $\pi((\varepsilon, \infty))^{-1} \mathbf{1}_{(\varepsilon, \infty)}(r) \pi(dr)$ . When  $D = 0$ , one should replace  $(1 - e^{-Dt})D^{-1}$  by  $t$  in the last two expressions. We next observe that  $u(-t, \theta) \mathcal{Z}_{t-T_n}^{(n)} = u(-(t - T_n), u(-T_n, \theta)) \mathcal{Z}_{t-T_n}^{(n)} \rightarrow V_n$  exists as  $t \rightarrow \infty$  and by Proposition III.12,

$$\mathbf{E}[e^{-\lambda V_n}] = \frac{1}{\pi((\varepsilon, \infty))} \int_{(\varepsilon, \infty)} \pi(dr) \mathbf{E}[e^{-r\phi_{u(-T_n, \theta)}(\lambda)}] = xC^{-1} \int_0^{s_0} dt e^{-Dt} \int_{(\varepsilon, \infty)} \pi(dr) e^{-r\phi_{u(-t, \theta)}(\lambda)} \quad (\text{III.49})$$

As  $\varepsilon \rightarrow 0$  and  $s_0 \rightarrow \infty$ , this proves that there exists  $\Omega_0 \in \mathcal{F}$  such that  $\mathbf{P}(\Omega_0) = 1$  and on  $\Omega_0$ , for any  $j \in J$ ,  $\lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t(\{x_j\}) = \lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_{t-t_j}^j =: \Delta_j$  exists in  $[0, \infty)$ . Then, on  $\Omega_0$ , for any  $y \in [0, x]$ , we set  $W_y^\theta = \sum_{j \in J} \mathbf{1}_{[0, y]}(x_j) \Delta_j$  and we take  $W^\theta$  as the null process on  $\Omega \setminus \Omega_0$ . Clearly,  $W^\theta$  is a cadlag subordinator whose initial value is 0. We next prove that its Laplace exponent is  $\phi_\theta$ . To that end fix  $y \in (0, x]$ ; by (III.49)

$$\begin{aligned} \mathbf{E} \left[ \exp \left( -\lambda \sum_{j \in J} \mathbf{1}_{\{x_j \leq y; Z_0^j > \varepsilon; t_j \leq s_0\}} \Delta_j \right) \right] &= \mathbf{E} \left[ \exp \left( -\lambda \sum_{1 \leq n \leq N} \mathbf{1}_{\{X_n \leq y\}} V_n \right) \right] \\ &= \exp \left( -y \int_0^{s_0} dt e^{-Dt} \int_{(\varepsilon, \infty)} \pi(dr) (1 - e^{-r\phi_{u(-t, \theta)}(\lambda)}) \right). \end{aligned}$$

Let  $\varepsilon \rightarrow 0$  and  $s_0 \rightarrow \infty$  to get

$$\begin{aligned} -\frac{1}{y} \log \mathbf{E} \left[ e^{-\lambda W_y^\theta} \right] &= \int_0^\infty dt e^{-Dt} \int_{(0, \infty)} \pi(dr) (1 - e^{-r\phi_{u(-t, \theta)}(\lambda)}) \\ &= \int_0^\infty dt e^{-Dt} (D\phi_{u(-t, \theta)}(\lambda) - \Psi(\phi_{u(-t, \theta)}(\lambda))). \end{aligned} \quad (\text{III.50})$$

Then, we set  $g(t) := e^{-Dt} \phi_{u(-t, \theta)}(\lambda)$ . By (III.47) and (III.8),  $g(t) = e^{-Dt} u(-t, \phi_\theta(\lambda))$ . Thus,  $\partial_t g(t) = e^{-Dt} (\Psi(\phi_{u(-t, \theta)}(\lambda)) - D\phi_{u(-t, \theta)}(\lambda))$  and to compute (III.50), we need to specify the limit of  $g$  as  $t$  tends to  $\infty$ : since  $\lim_{t \rightarrow \infty} u(-t, \phi_\theta(\lambda)) = 0$ ,

$$\partial_t \log g(t) = \frac{\Psi(u(-t, \phi_\theta(\lambda)))}{u(-t, \phi_\theta(\lambda))} - D \xrightarrow{t \rightarrow \infty} \Psi'(0+) - D = - \int_{(0, \infty)} \pi(dr) r < 0$$

which easily implies that  $\lim_{t \rightarrow \infty} g(t) = 0$  and by (III.50), we get  $\mathbf{E}[\exp(-\lambda W_y^\theta)] = \exp(-y\phi_\theta(\lambda))$ . Namely, the Laplace exponent of  $W^\theta$  is  $\phi_\theta$ .

From Proposition III.12, for any  $y \in [0, x]$ , we get  $\mathbf{P}$ -a.s.  $\lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, y]) =: W'_y$ , where the random variable  $W'_y$  has the same law as  $W_y^\theta$ . Next observe that

$$u(-t, \theta) \mathcal{Z}_t([0, y]) = u(-t, \theta) e^{-Dt} y + \sum_{j \in J} \mathbf{1}_{\{x_j \leq y\}} u(-t, \theta) \mathcal{Z}_t(\{x_j\}).$$

Recall from above that  $\lim_{t \rightarrow \infty} e^{-Dt} u(-t, \theta) = 0$ . Thus, by Fatou for sums, we get  $\mathbf{P}$ -a.s.  $W'_y \geq \sum_{j \in J} \mathbf{1}_{\{x_j \leq y\}} \Delta_j = W_y^\theta$ , which implies  $W'_y = W_y^\theta$ . Then, there exists  $\Omega_1 \in \mathcal{F}$  such that  $\mathbf{P}(\Omega_1) = 1$  and on  $\Omega_1$ , for any  $q \in \mathbb{Q} \cap [0, x]$ ,  $\lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, q]) = W_q^\theta$ .

We next work deterministically on  $\Omega_2 = \Omega_0 \cap \Omega_1$ . First observe that if  $y \notin \{x_j; j \in J\}$ ,  $\mathcal{Z}_t(\{y\}) = 0$ . Thus, by definition of  $W^\theta$ , for any  $y \in [0, x]$ , we get  $\lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t(\{y\}) = \Delta W_y^\theta$ . Moreover, for any  $y \in [0, x)$  and any  $q \in \mathbb{Q} \cap [0, x]$  such that  $q > y$ , we get

$$W_y^\theta \leq \liminf_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, y]) \leq \limsup_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, y]) \leq W_q^\theta,$$

the first equality being a consequence of Fatou. Since  $W^\theta$  is right continuous, by letting  $q$  go to  $y$  in the previous inequality we get  $\lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, y]) = W_y^\theta$  for any  $y \in [0, x]$  on  $\Omega_2$ , which completes the proof of the lemma when  $\Psi$  is of finite variation type.

When  $\Psi$  is of infinite variation type the proof follows the same lines. Fix  $\varepsilon, s_0 \in (0, \infty)$ , recall from (III.38) the definition of  $\mathcal{P}$  and recall the Markov property in Theorem III.3 (c). Then observe that  $\sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i > \varepsilon\}} \delta_{(x_i, Z_{s_0+}^i)} = \sum_{1 \leq n \leq N} \delta_{(X_n, Z^{(n)})}$ , where  $N$  is a Poisson random variable with mean  $C := x N_\Psi(Z_{s_0} > \varepsilon)$  and, conditionally on  $N$ , the variables  $X_n, Z^{(n)}$ ,  $1 \leq n \leq N$ , are independent:  $X_n$  is uniformly distributed on  $[0, x]$  and the processes  $Z^{(n)}$  are CSBP( $\Psi$ ) whose entrance law is given by  $N_\Psi(Z_{s_0} \in dr | Z_{s_0} > \varepsilon)$ . Then, note that  $u(-t, \theta) \mathcal{Z}_{t-s_0}^{(n)} = u(-(t-s_0), u(-s_0, \theta)) \mathcal{Z}_{t-s_0}^{(n)} \rightarrow V_n$  exists as  $t \rightarrow \infty$  and by Proposition III.12,  $\mathbf{E}[\exp(-\lambda V_n)] = N_\Psi(\exp(-\phi_{u(-s_0, \theta)}(\lambda) Z_{s_0}) | Z_{s_0} > \varepsilon)$ . By letting  $\varepsilon$  and  $s_0$  go to 0, this proves that there exists  $\Omega_0 \in \mathcal{F}$  such that  $\mathbf{P}(\Omega_0) = 1$  and on  $\Omega_0$ , for any  $i \in I$ ,  $\lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t(\{x_i\}) = \lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t^i =: \Delta_i$  exists in  $[0, \infty)$ . Then, on  $\Omega_0$ , for any  $y \in [0, x]$ , we set  $W_y^\theta = \sum_{i \in I} \mathbf{1}_{[0, y]}(x_i) \Delta_i$  and we take  $W^\theta$  as the null process on  $\Omega \setminus \Omega_0$ . Clearly,  $W^\theta$  is a cadlag subordinator whose initial value is 0 and we prove that its Laplace exponent is  $\phi_\theta$  as follows. First note that

$$\begin{aligned} \mathbf{E} \left[ \exp \left( -\lambda \sum_{i \in I} \mathbf{1}_{\{x_i \leq y; Z_{s_0}^i > \varepsilon\}} \Delta_i \right) \right] &= \mathbf{E} \left[ \exp \left( -\lambda \sum_{1 \leq n \leq N} \mathbf{1}_{\{X_n \leq y\}} V_n \right) \right] \\ &= \exp \left( -y N_\Psi(\mathbf{1}_{\{Z_{s_0} > \varepsilon\}} (1 - e^{-\phi_{u(-s_0, \theta)}(\lambda) Z_{s_0}})) \right) \end{aligned} \quad (\text{III.51})$$

### 3. Proof of Theorem III.2.

By (III.47) and (III.8), we get  $N_\Psi(1 - e^{-\phi_{u(-s_0, \theta)}(\lambda)Z_{s_0}}) = \phi_\theta(\lambda)$ . Then, by letting  $\varepsilon, s_0 \rightarrow 0$  in (III.51), we get  $\mathbf{E}[\exp(-\lambda W_y^\theta)] = \exp(-y\phi_\theta(\lambda))$ . We next proceed exactly as in the finite variation cases to complete the proof of the lemma.  $\blacksquare$

We next consider the behaviour of finite variation sub-critical CSBP.

**PROPOSITION III.14.** *Let  $\Psi$  be a branching mechanism of finite variation type such that  $\Psi'(0+) \in [0, \infty)$ . Then,  $\Psi$  is conservative and persistent,  $D \in (0, \infty)$ , and for all  $\theta, t \in (0, \infty)$ ,  $u(-t, \theta)$  is well-defined and  $\lim_{t \rightarrow \infty} u(-t, \theta) = \infty$ . For any  $\theta, \theta' \in (0, \infty)$ , and any  $y \in (0, x]$ , we also get **P**-a.s.*

$$W_y^\theta = S_{\theta, \theta'} W_y^{\theta'} \quad \text{where} \quad S_{\theta, \theta'} := \exp\left(D \int_{\theta'}^\theta \frac{d\lambda}{\Psi(\lambda)}\right). \quad (\text{III.52})$$

$W^\theta$  is a conservative subordinator. Namely,  $\kappa_\theta = 0$ . Moreover,

$$\forall \lambda \in (0, \infty), \quad \phi_\theta(\lambda) = u\left(-\frac{\log \lambda}{D}, \theta\right) \quad \text{and} \quad \varrho_\theta((0, \infty)) = \pi((0, \infty))/D. \quad (\text{III.53})$$

The subordinator  $W^\theta$  has a positive drift iff  $\int_{(0,1)} \pi(dr) r \log 1/r < \infty$ . In this case,

$$\log d_\theta = \log \theta - \int_\theta^\infty \left(\frac{D}{\Psi(\lambda)} - \frac{1}{\lambda}\right) d\lambda. \quad (\text{III.54})$$

**Proof** Since  $\Psi$  is conservative and persistent,  $\kappa(t) = 0$  and  $v(t) = \infty$  and  $u(-t, \theta)$  is well-defined for any  $\theta \in (0, \infty)$ . Moreover, (III.9) implies  $\lim_{t \rightarrow \infty} u(-t, \theta) = \infty$  and  $\lim_{t \rightarrow \infty} u(t, \theta) = 0$ . Recall that  $\lim_{\lambda \rightarrow \infty} \Psi(\lambda)/\lambda = D$ . Then, (III.48) entails (III.52). We then argue as in the proof of Proposition III.12 to prove that  $\phi_\theta(e^{-Dt}) = u(t, \theta)$  for any  $t \in \mathbb{R}$ , which entails the first part of (III.53). Thus,  $\kappa_\theta = \lim_{\lambda \rightarrow 0} \phi_\theta(\lambda) = \lim_{t \rightarrow \infty} u(t, \theta) = 0$  and  $W^\theta$  is conservative.

We next compute the value of  $d_\theta$ . To that end, we set  $B(t) = \log(e^{-Dt}u(-t, \theta))$  and we observe that  $\log d_\theta = \lim_{t \rightarrow \infty} B(t)$ , by taking  $\lambda = e^{Dt}$  in (III.53). By an easy computation using (III.8), we get

$$\log \theta - B(t) = \int_{-t}^0 ds (D + \partial_s \log u(s, \theta)) = \int_0^t ds \left(D - \frac{\Psi(u(-s, \theta))}{u(-s, \theta)}\right) = \int_\theta^{u(-t, \theta)} d\lambda \left(\frac{D}{\Psi(\lambda)} - \frac{1}{\lambda}\right). \quad (\text{III.55})$$

Now recall that  $D - \lambda^{-1}\Psi(\lambda) = \int_{(0, \infty)} \pi(dr)(1 - e^{-\lambda r})/\lambda$ . Thus,

$$\log \theta - B(t) = \int_{(0, \infty)} \pi(dr) \int_\theta^{u(-t, \theta)} d\lambda \frac{1 - e^{-\lambda r}}{\lambda \Psi(\lambda)} \xrightarrow{t \rightarrow \infty} \int_{(0, \infty)} \pi(dr) \int_\theta^\infty d\lambda \frac{1 - e^{-\lambda r}}{\lambda \Psi(\lambda)} =: I. \quad (\text{III.56})$$

Now observe that  $\lambda \mapsto \lambda^{-1}\Psi(\lambda)$  is increasing and tends to  $D$  as  $\lambda \rightarrow \infty$ . Thus,  $\frac{1}{D}J \leq I \leq \frac{\theta}{\Psi(\theta)}J$  where

$$J := \int_{(0, \infty)} \pi(dr) \int_\theta^\infty d\lambda \frac{1 - e^{-\lambda r}}{\lambda^2} = \int_{(0, \infty)} \pi(dr) r \int_{\theta r}^\infty d\mu \frac{1 - e^{-\mu}}{\mu^2}.$$

Clearly,  $J < \infty$  iff  $\int_{(0,1)} \pi(dr) r \log 1/r < \infty$ , which entails the last point of the proposition. By an easy computation, (III.55) implies (III.54).

It remains to prove the second equality in (III.53). First assume that  $d_\theta = 0$ . This implies that  $\pi((0, \infty)) = \infty$  and the first part of (III.53) entails  $\varrho_\theta((0, \infty)) = \lim_{\lambda \rightarrow \infty} \phi_\theta(\lambda) = \lim_{t \rightarrow \infty} u(-t, \theta) = \infty$ , which proves the second part of (III.53) in this case. We next assume that  $d_\theta > 0$ . We set  $C(t) = u(-t, \theta) - d_\theta e^{Dt}$ . Thus,  $\varrho_\theta((0, \infty)) = \lim_{t \rightarrow \infty} C(t)$ . By (III.55), we get

$$\frac{C(t)}{u(-t, \theta)} = 1 - \frac{d_\theta}{e^{-Dt}u(-t, \theta)} = 1 - \exp\left(-\int_{u(-t, \theta)}^\infty d\lambda \left(\frac{D}{\Psi(\lambda)} - \frac{1}{\lambda}\right)\right) \sim_{t \rightarrow \infty} \int_{u(-t, \theta)}^\infty d\lambda \left(\frac{D}{\Psi(\lambda)} - \frac{1}{\lambda}\right).$$

Then,  $C(t) \sim_{t \rightarrow \infty} F(u(-t, \theta))$  where  $F(x) = x \int_x^\infty (\frac{D}{\Psi(\lambda)} - \frac{1}{\lambda}) d\lambda$ . We then set  $\varphi(\lambda) = D\lambda - \Psi(\lambda)$  and we observe that  $\lim_{\lambda \rightarrow \infty} \varphi(\lambda) = \pi((0, \infty))$ . Thus,

$$F(x) = x \int_x^\infty \frac{\varphi(\lambda)}{\lambda \Psi(\lambda)} d\lambda = \int_1^\infty \frac{\varphi(x\mu)x}{\mu \Psi(x\mu)} d\mu \xrightarrow{x \rightarrow \infty} \frac{\pi((0, \infty))}{D} \int_1^\infty \frac{d\mu}{\mu^2} = \frac{\pi((0, \infty))}{D},$$

which implies the second part of (III.53). ■

We complete this result by the following lemma.

**LEMMA III.15.** *Let  $\Psi$  be a branching mechanism of finite variation type such that  $\Psi'(0+) \in [0, \infty)$ . Then,  $\Psi$  is conservative and persistent,  $D \in (0, \infty)$ , and for all  $\theta, t \in (0, \infty)$ ,  $u(-t, \theta)$  is well-defined and  $\lim_{t \rightarrow \infty} u(-t, \theta) = \infty$ . Moreover, there exists a cadlag subordinator  $W^\theta$  whose initial value is 0 and whose Laplace exponent is  $\phi_\theta$  as defined by (III.53) such that*

$$\mathbf{P}\text{-a.s.} \quad \forall y \in [0, x], \quad \lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t([0, y]) = W_y^\theta \quad \text{and} \quad \lim_{t \rightarrow \infty} u(-t, \theta) \mathcal{Z}_t(\{y\}) = \Delta W_y^\theta,$$

where  $\Delta W_y^\theta$  stands for the jump of  $W^\theta$  at  $y$ .

**Proof** The proof Lemma III.13 works verbatim, except that in (III.50)

$$\int_0^\infty dt e^{-Dt} (D\phi_{u(-t, \theta)}(\lambda) - \Psi(\phi_{u(-t, \theta)}(\lambda))) = \phi_\theta(\lambda) - d_\theta \lambda,$$

which is easy to prove since  $e^{-Dt} \phi_\theta(e^{Dt} \lambda) \rightarrow d_\theta \lambda$  as  $t \rightarrow \infty$ . ■

### 3.2 Proof of Theorem III.2 (ii-b), (ii-c) and (iii-b).

We now consider the cases where there is no Eve property. Recall that  $x \in (0, \infty)$  is fixed and that  $\ell$  stands for Lebesgue measure on  $\mathbb{R}$  or on  $[0, x]$  according to the context. Recall that  $\Psi$  is not linear and recall the notation  $Z_t := \mathcal{Z}_t([0, x])$ . We first need the following elementary lemma.

**LEMMA III.16.** *For any  $t \in (0, \infty]$ , let  $m_t \in \mathcal{M}_1([0, x])$  be of the form  $m_t = a_t \ell + \sum_{y \in S} m_t(\{y\}) \delta_y$ , where  $S$  is a fixed countable subset of  $[0, x]$  and  $a_t \in [0, \infty)$ . We assume that for any  $y \in S$ ,  $\lim_{t \rightarrow \infty} m_t(\{y\}) = m_\infty(\{y\})$  and  $\lim_{t \rightarrow \infty} a_t = a_\infty$ . Then,  $\lim_{t \rightarrow \infty} \|m_t - m_\infty\|_{\text{var}} = 0$ .*

**Proof** For all  $\varepsilon \in (0, \infty)$ , there is  $S_\varepsilon \subset S$ , finite and such that  $\sum_{y \in S \setminus S_\varepsilon} m_\infty(\{y\}) < \varepsilon$ . Then, for any  $A \subset [0, x]$

$$\begin{aligned} |m_t(A) - m_\infty(A)| &\leq x |a_t - a_\infty| + \sum_{y \in S_\varepsilon} |m_t(\{y\}) - m_\infty(\{y\})| + \sum_{y \in S \setminus S_\varepsilon} m_t(\{y\}) + \sum_{y \in S \setminus S_\varepsilon} m_\infty(\{y\}) \\ &\leq x |a_t - a_\infty| + \sum_{y \in S_\varepsilon} |m_t(\{y\}) - m_\infty(\{y\})| + 1 - a_t x - \sum_{y \in S_\varepsilon} m_t(\{y\}) + \varepsilon. \end{aligned}$$

Thus,

$$\limsup_{t \rightarrow \infty} \sup_{A \subset [0, x]} |m_t(A) - m_\infty(A)| \leq 1 - a_\infty x - \sum_{y \in S_\varepsilon} m_\infty(\{y\}) + \varepsilon = \varepsilon + \sum_{y \in S \setminus S_\varepsilon} m_\infty(\{y\}) \leq 2\varepsilon,$$

which implies the desired result. ■

**Proof of Theorem III.2 (ii-b) and (ii-c).** Recall that  $B = \{\zeta = \infty; \lim_{t \rightarrow \infty} Z_t = \infty\}$ . We assume that  $\Psi'(0+) \in (-\infty, 0)$ , which implies  $q \in (0, \infty]$  and that  $\Psi$  is conservative. Let  $\theta \in (0, q)$  and let  $W^\theta$  be a cadlag subordinator as in Lemma III.13. Recall that its Laplace exponent is  $\phi_\theta$  as defined by (III.47). It is easy to prove that  $\mathbf{P}$ -a.s.  $\mathbf{1}_{\{W_x^\theta > 0\}} = \mathbf{1}_B$ . We now work a.s. on  $B$ : it makes sense to set  $M_\infty(dr) = dW_r^\theta/W_x^\theta$  that does not depend on  $\theta$  as proved by (III.46) in Proposition III.12. Note that  $M_t = a_t \ell + \sum_{y \in S} M_t(\{y\}) \delta_y$  either with  $a_t = 0$  and  $S = \{x_i; i \in I\}$  if  $\Psi$  is of infinite variation type, or with  $a_t = e^{-Dt}/Z_t$  and  $S = \{x_j; j \in J\}$  if  $\Psi$  is of finite variation type. Next note that  $\{y \in [0, x] : \Delta W_y^\theta > 0\} \subset S$  and since  $W^\theta$  has no drift, we get  $M_\infty = \sum_{y \in S} M_\infty(\{y\}) \delta_y$ . Then, Lemma III.13 easily entails that a.s. on  $B$ , for any  $y \in S$ ,  $\lim_{t \rightarrow \infty} M_t(\{y\}) = M_\infty(\{y\})$ . Next, recall from the proof of Proposition III.12 that  $\lim_{t \rightarrow \infty} u(-t, \theta) e^{-Dt} = d_\theta = 0$ , which implies that  $\lim_{t \rightarrow \infty} a_t = 0$ . Then, Lemma III.16 entails that a.s. on  $B$ ,  $\lim_{t \rightarrow \infty} \|M_t - M_\infty\|_{\text{var}} = 0$ .

If  $q < \infty$ , then Proposition III.12 entails that  $W^\theta$  is a compound Poisson process: in this case and on  $B$ , there are finitely many settlers and conditionally on  $B$ , the number of settlers is distributed as a Poisson r.v. with parameter  $xq$  conditioned to be non zero, which completes the proof of Theorem III.2 (ii-b). If  $q = \infty$ , then the same proposition shows that  $W^\theta$  has a dense set of jumps. Therefore, a.s. on  $B$  there are a dense countable set of settlers, which completes the proof of Theorem III.2 (ii-c). In both cases, the asymptotic frequencies are described by Proposition III.12 and Lemma III.13  $\blacksquare$

**Proof of Theorem III.2 (iii-b).** Recall that  $C = \{\zeta = \infty; \lim_{t \rightarrow \infty} Z_t = 0\}$ . We assume that  $\Psi$  is of finite variation type, which implies that  $\Psi$  is persistent. Also recall that  $\mathbf{P}(C) = e^{-qx} > 0$ . Thus, we also assume that  $q < \infty$ . Then, observe that  $\mathcal{Z}$  under  $\mathbf{P}(\cdot | C)$  is distributed as the process derived from the finite variation sub-critical branching mechanism  $\Psi(\cdot + q)$ . So, without loss of generality, we can assume that  $\Psi$  is of finite variation and sub-critical, namely  $\Psi'(0+) \in [0, \infty)$ , which implies that  $\Psi$  is conservative and  $D \in (0, \infty)$ .

Let  $\theta \in (0, \infty)$  and let  $W^\theta$  be a cadlag subordinator as in Lemma III.15 whose Laplace exponent  $\phi_\theta$  is defined by (III.53). Since  $\Psi$  is conservative and persistent, it makes sense to set  $M_\infty(dr) = dW_r^\theta/W_x^\theta$  that does not depend on  $\theta$  as proved by (III.52) in Proposition III.14. Note that  $M_t = a_t \ell + \sum_{y \in S} M_t(\{y\})$  where  $a_t = e^{-Dt}/Z_t$  and  $S = \{x_j; j \in J\}$ , and observe that  $\{y \in [0, x] : \Delta W_y^\theta > 0\} \subset S$ . Recall that  $d_\theta$  stands for the (possibly null) drift of  $W^\theta$ . Then, we get  $M_\infty = a_\infty \ell + \sum_{y \in S} M_\infty(\{y\}) \delta_y$ , where  $a_\infty = d_\theta/W_x^\theta$ . By Lemma III.15, a.s. for any  $y \in S$ ,  $\lim_{t \rightarrow \infty} M_t(\{y\}) = M_\infty(\{y\})$  and recall from the proof of Proposition III.12 that  $\lim_{t \rightarrow \infty} u(-t, \theta) e^{-Dt} = d_\theta$ , which implies that  $\lim_{t \rightarrow \infty} a_t = a_\infty$ . Then, Lemma III.16 entails that a.s.  $\lim_{t \rightarrow \infty} \|M_t - M_\infty\|_{\text{var}} = 0$ .

If  $\pi((0, 1)) < \infty$ , then  $\pi((0, \infty)) < \infty$  and  $\int_{(0,1)} \pi(dr) r \log 1/r < \infty$ . Proposition III.14 entails that  $W^\theta$  has a drift part and finitely many jumps in  $[0, x]$ : there is dust and finitely many settlers. More precisely, conditionally given  $C$ , the number of settlers is distributed as a Poisson r.v. with parameter  $\frac{x}{D} \int_{(0,\infty)} e^{-qr} \pi(dr)$  since  $e^{-qr} \pi(dr)$  is the Lévy measure of  $\Psi(\cdot + q)$ . This proves Theorem III.2 (iii-b1). If  $\pi((0, 1)) = \infty$  and  $\int_{(0,1)} \pi(dr) r \log 1/r < \infty$ , Proposition III.14 entails that  $W^\theta$  has a drift part and a dense set of jumps in  $[0, x]$ : thus, a.s. on  $C$ , there is dust and infinitely many settlers. This proves Theorem III.2 (iii-b2). Similarly, if  $\int_{(0,1)} \pi(dr) r \log 1/r = \infty$ , Proposition III.14 entails that a.s. on  $C$ , there is no dust and there are infinitely many settlers, which proves Theorem III.2 (iii-b3). In all cases, conditionally on  $C$ , the asymptotic frequencies are described thanks to Proposition III.14 and Lemma III.15 applied to the branching mechanism  $\Psi(\cdot + q)$ .  $\blacksquare$



### 3.3 Proof of Theorem III.2 (ii-a) and (iii-a).

Preliminary lemmas.

Recall that  $x \in (0, \infty)$  is fixed and recall that  $\mathcal{M}_1([0, x])$  stands for the set of Borel probability measures on  $[0, x]$ . We first recall (without proof) the following result – quite standard – on weak convergence in  $\mathcal{M}_1([0, x])$ .

**LEMMA III.17.** *For any  $t \in [0, \infty)$ , let  $m_t \in \mathcal{M}_1([0, x])$  be such that for all  $q \in \mathbb{Q} \cap [0, x]$ ,  $\lim_{t \rightarrow \infty} m_t([0, q])$  exists. Then, there exists  $m_\infty \in \mathcal{M}_1([0, x])$  such that  $\lim_{t \rightarrow \infty} m_t = m_\infty$  with respect to the topology of the weak convergence.*

Recall the definition of  $(M_t)_{t \in [0, \infty)}$  from Theorem III.1 and Section 2.3.

**LEMMA III.18.** *We assume that  $\Psi$  is not linear and conservative. Then, there exists a random probability measure  $M_\infty$  on  $[0, x]$  such that  $\mathbf{P}$ -a.s.  $\lim_{t \rightarrow \infty} M_t = M_\infty$  with respect to the topology of the weak convergence.*

**Proof** By Lemma III.17, it is sufficient to prove that for any  $q \in \mathbb{Q} \cap [0, x]$ ,  $\mathbf{P}$ -a.s.  $\lim_{t \rightarrow \infty} M_t([0, q])$  exists. To that end, we use a martingale argument: for any  $t \in [0, \infty)$ , we denote by  $\mathcal{G}_t$  the sigma-field generated by the r.v.  $\mathcal{Z}_s([0, q])$  and  $\mathcal{Z}_s((q, x])$ , where  $s$  ranges in  $[0, t]$ . Recall that  $Z_t = \mathcal{Z}_t([0, q]) + \mathcal{Z}_t((q, x])$  and that  $(\mathcal{Z}_t([0, q]))_{t \in [0, \infty)}$  and  $(\mathcal{Z}_t((q, x]))_{t \in [0, \infty)}$  are two independent conservative CSBP( $\Psi$ ). Then, for any  $\lambda, \mu \in (0, \infty)$  and any  $t, s \in [0, \infty)$

$$\mathbf{E}[\exp(-\mu \mathcal{Z}_{t+s}([0, q]) - \lambda \mathcal{Z}_{t+s}((q, x])) \mid \mathcal{G}_t] = \exp(-u(s, \lambda + \mu) \mathcal{Z}_t([0, q]) - u(s, \lambda) \mathcal{Z}_t((q, x]))$$

By differentiating in  $\mu = 0$ , we get

$$\mathbf{E}[\mathbf{1}_{\{Z_{t+s} > 0\}} \mathcal{Z}_{t+s}([0, q]) e^{-\lambda Z_{t+s}} \mid \mathcal{G}_t] = \mathbf{1}_{\{Z_t > 0\}} \mathcal{Z}_t([0, q]) e^{-u(s, \lambda) Z_t} \partial_\lambda u(s, \lambda). \quad (\text{III.57})$$

By continuity in  $\lambda$ , (III.57) holds true  $\mathbf{P}$ -a.s. for all  $\lambda \in [0, \infty)$ . We integrate (III.57) in  $\lambda$ : note that for any  $z \in (0, \infty)$ ,  $I(z) := \int_0^\infty d\lambda e^{-u(s, \lambda) z} \partial_\lambda u(s, \lambda) = z^{-1}(1 - e^{-v(s)z})$  if  $\Psi$  is non-persistent (here  $v$  is the function defined right after (III.15)) and  $I(z) = z^{-1}$  if  $\Psi$  is persistent. In both cases,  $I(z) \leq z^{-1}$  and thus we get

$$\mathbf{E}[\mathbf{1}_{\{Z_{t+s} > 0\}} M_{t+s}([0, q]) \mid \mathcal{G}_t] = \mathbf{E}[\mathbf{1}_{\{Z_{t+s} > 0\}} \frac{\mathcal{Z}_{t+s}([0, q])}{Z_{t+s}} \mid \mathcal{G}_t] \leq \mathbf{1}_{\{Z_t > 0\}} \frac{\mathcal{Z}_t([0, q])}{Z_t} = \mathbf{1}_{\{Z_t > 0\}} M_t([0, q]).$$

Then,  $t \mapsto \mathbf{1}_{\{Z_t > 0\}} M_t([0, q])$  is a nonnegative super-martingale: it almost surely converges and Lemma III.17 applies on the event  $\{\zeta_0 = \infty\}$ . Since we already proved that  $M$  has an Eve on the event  $\{\zeta_0 < \infty\}$ , the proof is complete.  $\blacksquare$

For any  $v \in [0, 1)$  and any  $t \in (0, \infty]$ , we set

$$R_t^{-1}(v) = \inf \{y \in [0, x] : M_t([0, y]) > v\}. \quad (\text{III.58})$$

Let  $U, V : \Omega \rightarrow [0, 1)$  be two independent uniform r.v. that are also independent of the Poisson point measures  $\mathcal{P}$  and  $\mathcal{Q}$ . Then, for any  $t, s \in (0, \infty]$ , the conditional law of  $(R_t^{-1}(U), R_s^{-1}(V))$  given  $\mathcal{P}$  and  $\mathcal{Q}$  is  $M_t \otimes M_s$ . Moreover, Lemma III.18 and standard arguments entail

$$\mathbf{P}\text{-a.s.} \quad \lim_{t \rightarrow \infty} R_t^{-1}(U) = R_\infty^{-1}(U) \quad \text{and} \quad \lim_{t \rightarrow \infty} R_t^{-1}(V) = R_\infty^{-1}(V). \quad (\text{III.59})$$

For any  $t \in (0, \infty)$ , we recall the definition of the function  $v(t) = \lim_{\lambda \rightarrow \infty} u(t, \lambda)$  that is infinite if  $\Psi$  is persistent and finite if  $\Psi$  is non-persistent. Recall that  $u(-t, \cdot) : (\kappa(t), v(t)) \rightarrow (0, \infty)$  is the reciprocal function of  $u(t, \cdot)$ . It is increasing and one-to-one, which implies that  $\lim_{\lambda \rightarrow v(t)} u(-t, \lambda) = \infty$ .

**LEMMA III.19.** *Let us assume that  $\Psi$  is conservative. Then, for all  $t, \theta \in (0, \infty)$  and all  $s \in [0, \infty)$ ,*

$$\mathbf{E}[\mathbf{1}_{\{R_t^{-1}(U) \neq R_{t+s}^{-1}(V)\}} (1 - e^{-\theta Z_{t+s}})] = x^2 \int_0^{v(t)} dw \Psi(w) e^{-xw} \frac{u(-t, w) - (u(-t, w) - u(s, \theta))_+}{\Psi(u(-t, w))}, \quad (\text{III.60})$$

where  $(\cdot)_+$  stands for the positive part function.

**Proof** We first prove the lemma when  $\Psi$  is of infinite variation type: recall from (III.38) the definition of  $\mathcal{P}$  and note that on  $\{Z_{t+s} > 0\}$ ,

$$\mathbf{E}[\mathbf{1}_{\{R_t^{-1}(U) \neq R_{t+s}^{-1}(V)\}} | \mathcal{P}] = \sum_{i,j \in I, i \neq j} \frac{Z_t^i Z_{t+s}^j}{(Z_t^i + Z_t^j + \sum_{k \in I \setminus \{i,j\}} Z_t^k) (Z_{t+s}^i + Z_{t+s}^j + \sum_{k \in I \setminus \{i,j\}} Z_{t+s}^k)},$$

To simplify notation, we denote by  $A$  the left member in (III.60). By Palm's formula (III.37) we then get,

$$A = x^2 \int N_\Psi(dZ) \int N_\Psi(dZ') \mathbf{E} \left[ \frac{\mathbf{1}_{\{Z_{t+s} + Z'_{t+s} + Z_{t+s} > 0\}} Z_t Z'_{t+s} (1 - e^{-\theta(Z_{t+s} + Z'_{t+s} + Z_{t+s})})}{(Z_t + Z'_t + Z_t)(Z_{t+s} + Z'_{t+s} + Z_{t+s})} \right].$$

For any  $\lambda_1, \lambda_2 \in (0, \infty)$ , we then set

$$\begin{aligned} B(\lambda_1, \lambda_2) &= \int N_\Psi(dZ) \int N_\Psi(dZ') \mathbf{E} \left[ Z_t Z'_{t+s} e^{-\lambda_1(Z_t + Z'_t + Z_t)} e^{-\lambda_2(Z_{t+s} + Z'_{t+s} + Z_{t+s})} \right] \\ &= N_\Psi(Z_t e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}}) N_\Psi(Z_{t+s} e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}}) \mathbf{E} \left[ e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}} \right] \end{aligned}$$

Recall that  $N_\Psi(1 - e^{-\lambda Z_t}) = u(t, \lambda)$  and recall Theorem III.3 (c). Then, we first get

$$N_\Psi(Z_t e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}}) = N_\Psi(Z_t e^{-(\lambda_1 + u(s, \lambda_2)) Z_t}) = \partial_\lambda u(t, \lambda_1 + u(s, \lambda_2)).$$

By the same argument we get

$$\begin{aligned} N_\Psi(Z_{t+s} e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}}) &= \partial_{\lambda_2} N_\Psi(1 - e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}}) = \partial_{\lambda_2} N_\Psi(1 - e^{-(\lambda_1 + u(s, \lambda_2)) Z_t}) \\ &= \partial_\lambda u(s, \lambda_2) \partial_\lambda u(t, \lambda_1 + u(s, \lambda_2)). \end{aligned}$$

This implies that

$$B(\lambda_1, \lambda_2) = \partial_\lambda u(s, \lambda_2) (\partial_\lambda u(t, \lambda_1 + u(s, \lambda_2)))^2 e^{-xu(t, \lambda_1 + u(s, \lambda_2))}. \quad (\text{III.61})$$

An easy argument then entails that

$$A = x^2 \int_0^\infty \int_0^\infty d\lambda_1 d\lambda_2 (B(\lambda_1, \lambda_2) - B(\lambda_1, \lambda_2 + \theta)).$$

Set  $C(\theta) := x^2 \int_0^\infty \int_0^\infty B(\lambda_1, \lambda_2 + \theta) d\lambda_1 d\lambda_2$ . The previous equality shows that  $A = C(0) - C(\theta)$ . We recall that  $v(s) = \lim_{\lambda \rightarrow \infty} u(s, \lambda)$  and let us compute  $C(\theta)$ . To that end we use the changes of variable  $y = u(s, \lambda_2 + \theta)$  and  $\lambda = \lambda_1 + y$  to get

$$\begin{aligned} C(\theta) &= x^2 \int_0^\infty d\lambda_1 \int_{u(s, \theta)}^{v(s)} dy (\partial_\lambda u(t, \lambda_1 + y))^2 e^{-xu(t, \lambda_1 + y)}. \\ &= x^2 \int_0^\infty d\lambda_1 \int_{\lambda_1 + u(s, \theta)}^{\lambda_1 + v(s)} d\lambda (\partial_\lambda u(t, \lambda))^2 e^{-xu(t, \lambda)}. \end{aligned}$$

Recall from (III.9) that  $\partial_\lambda u(t, \lambda) = \Psi(u(t, \lambda))/\Psi(\lambda)$  and note that  $\Psi(\lambda) = \Psi(u(-t, u(t, \lambda)))$ . Then, by the change of variable  $w = u(t, \lambda)$ , we get

$$C(\theta) = x^2 \int_0^\infty d\lambda_1 \int_{u(t, \lambda_1 + u(s, \theta))}^{u(t, \lambda_1 + v(s))} \frac{\Psi(w)}{\Psi(u(-t, w))} e^{-xw}.$$

Thus,

$$\begin{aligned} A &= C(0) - C(\theta) = x^2 \int_0^\infty d\lambda_1 \int_{u(t, \lambda_1)}^{u(t, \lambda_1 + u(s, \theta))} \frac{\Psi(w)}{\Psi(u(-t, w))} e^{-xw} \\ &= x^2 \int_0^{v(t)} dw \int_0^\infty d\lambda_1 \mathbf{1}_{\{u(t, \lambda_1) \leq w \leq u(t, \lambda_1 + u(s, \theta))\}} \frac{\Psi(w)}{\Psi(u(-t, w))} e^{-xw} \\ &= x^2 \int_0^{v(t)} dw \Psi(w) e^{-xw} \frac{u(-t, w) - (u(-t, w) - u(s, \theta))_+}{\Psi(u(-t, w))}, \end{aligned}$$

which is the desired result in the infinite variation cases.

The proof in the finite variation cases is similar except that  $\mathcal{Z}$  and  $M$  are derived from the Poisson point measure  $\mathcal{Q}$  defined by (III.40). Note that  $\Psi$  is persistent. We moreover assume it to be conservative: thus,  $Z_t \in (0, \infty)$ , for any  $t \in [0, \infty)$ . Let  $A$  stand for the left member in (III.60). Then,  $A = A_1 + A_2$  where

$$A_1 := \mathbf{E} \left[ \frac{1}{Z_t Z_{t+s}} \sum_{j \in J} \mathbf{1}_{\{t_j \leq t\}} Z_{t-t_j}^j (Z_{t+s} - \mathbf{1}_{\{t_j \leq t+s\}} Z_{t+s-t_j}^j) (1 - e^{-\theta Z_{t+s}}) \right], \quad A_2 := \mathbf{E} \left[ \frac{x e^{-Dt}}{Z_t} (1 - e^{-\theta Z_{t+s}}) \right].$$

$A_1$  corresponds to the event where  $U$  falls on a jump of  $R_t$ , while  $A_2$  deals with the event where it falls on the dust. The latter gives

$$A_2 = x e^{-Dt} \int_0^\infty d\lambda \mathbf{E} [e^{-\lambda Z_t} - e^{-\lambda Z_t - \theta Z_{t+s}}] = x e^{-Dt} \int_0^\infty d\lambda (e^{-x u(t, \lambda)} - e^{-x u(t, \lambda + u(s, \theta))}).$$

We next observe that  $A_1 = \int_0^\infty \int_0^\infty d\lambda_1 d\lambda_2 (\tilde{B}(\lambda_1, \lambda_2) - \tilde{B}(\lambda_1, \lambda_2 + \theta))$ , where for any  $\lambda_1, \lambda_2 \in (0, \infty)$  we have set

$$\begin{aligned} \tilde{B}(\lambda_1, \lambda_2) &= \mathbf{E} \left[ e^{-\lambda_1 Z_t - \lambda_2 Z_{t+s}} \sum_{j \in J} \mathbf{1}_{\{t_j \leq t\}} Z_{t-t_j}^j \left( x e^{-D(t+s)} + \sum_{k \in J \setminus \{j\}} \mathbf{1}_{\{t_k \leq t+s\}} Z_{t+s-t_k}^k \right) \right] \\ &= \mathbf{E} [Z_{t+s} e^{-\lambda_1 Z_t} e^{-\lambda_2 Z_{t+s}}] x \int_0^t e^{-Db} db \int_{(0, \infty)} \pi(dr) \mathbb{E}_r [Z_{t-b} e^{-\lambda_1 Z_{t-b}} e^{-\lambda_2 Z_{t+s-b}}] \quad \text{(III.62)} \end{aligned}$$

Here we apply Palm formula to derive the second line from the first one. The first expectation in (III.62) yields

$$\mathbf{E} [Z_{t+s} e^{-\lambda_1 Z_t} e^{-\lambda_2 Z_{t+s}}] = \partial_\lambda u(s, \lambda_2) \partial_\lambda u(t, \lambda_1 + u(s, \lambda_2)) x e^{-x u(t, \lambda_1 + u(s, \lambda_2))}.$$

The second term of the product in (III.62) gives

$$\begin{aligned} &x \int_0^t e^{-Db} db \int_{(0, \infty)} \pi(dr) \partial_\lambda u(t-b, \lambda_1 + u(s, \lambda_2)) r e^{-ru(t-b, \lambda_1 + u(s, \lambda_2))} \\ &= x \int_0^t e^{-Db} db \partial_\lambda u(t-b, \lambda_1 + u(s, \lambda_2)) (D - \Psi'(u(t-b, \lambda_1 + u(s, \lambda_2)))) \\ &= x (\partial_\lambda u(t, \lambda_1 + u(s, \lambda_2)) - e^{-Dt}) \end{aligned}$$

Here, to derive the second line from the first one, we use  $\int_0^\infty \pi(dr) r e^{-r\lambda} = D - \Psi'(\lambda)$ . To derive the third one from the second one, we use the identity  $\partial_\lambda u(t, \lambda) = -\Psi(\lambda)^{-1} \partial_t u(t, \lambda)$  and we do an integration by part. Recall  $B(\lambda_1, \lambda_2)$  from (III.61). By the previous computations we get

$$\tilde{B}(\lambda_1, \lambda_2) = x^2 B(\lambda_1, \lambda_2) - x^2 \partial_\lambda u(s, \lambda_2) \partial_\lambda u(t, \lambda_1 + u(s, \lambda_2)) e^{-Dt} e^{-xu(t, \lambda_1 + u(s, \lambda_2))}$$

Recall that we already proved that  $x^2 \int_0^\infty \int_0^\infty d\lambda_1 d\lambda_2 (B(\lambda_1, \lambda_2) - B(\lambda_1, \lambda_2 + \theta))$  equals the right member of (III.60). So, to complete the proof, we set

$$F(\theta) := \int_0^\infty \int_0^\infty d\lambda_1 d\lambda_2 \partial_\lambda u(s, \lambda_2 + \theta) \partial_\lambda u(t, \lambda_1 + u(s, \lambda_2 + \theta)) e^{-Dt} e^{-xu(t, \lambda_1 + u(s, \lambda_2 + \theta))}$$

and calculations similar as in the infinite variation case yield  $x^2(F(0) - F(\theta)) = -A_2$ , which entails the desired result in the finite variation cases.  $\blacksquare$

To complete the proof of Theorem III.2, we need the following technical lemma whose proof is postponed.

**LEMMA III.20.** *We assume that  $\Psi$  is not linear. Then,  $\mathbf{P}$ -a.s. for all  $y \in [0, x]$ ,  $\lim_{t \rightarrow \infty} M_t(\{y\})$  exists.*

**Proof of Theorem III.2 (ii-a).**

We temporarily admit Lemma III.20. We assume that  $q > 0$  and that  $\Psi$  is conservative. To simplify notation, we denote by  $\{Z \rightarrow \infty\}$  the event  $\{\lim_{t \rightarrow \infty} Z_t = \infty\}$ . By Lemma III.19 and the Markov property, we first get

$$\begin{aligned} \mathbf{P}(R_t^{-1}(U) \neq R_{t+s}^{-1}(V); Z \rightarrow \infty) &= \mathbf{E}[\mathbf{1}_{\{R_t^{-1}(U) \neq R_{t+s}^{-1}(V)\}} (1 - e^{-qZ_{t+s}})] \\ &= x^2 \int_0^{v(t)} dw \Psi(w) e^{-xw} \frac{u(-t, w) - (u(-t, w) - q)_+}{\Psi(u(-t, w))} =: A(t) \end{aligned}$$

We set  $\mathbf{e} = R_\infty^{-1}(V)$ . Using the Portmanteau theorem as  $s \rightarrow \infty$  on the law of the pair  $(R_t^{-1}(U), R_{t+s}^{-1}(V))$  with the complement of the closed set  $\{(y, y) : y \in [0, x]\}$ , we get  $\mathbf{P}(R_t^{-1}(U) \neq \mathbf{e}; Z \rightarrow \infty) \leq A(t)$ . But now observe that  $\mathbf{E}[\mathbf{1}_{\{R_t^{-1}(U) \neq \mathbf{e}; Z \rightarrow \infty\}} | \mathcal{P}, V] = (1 - M_t(\{\mathbf{e}\})) \mathbf{1}_{\{Z \rightarrow \infty\}}$ . Thus ,

$$\mathbf{E}[(1 - M_t(\{\mathbf{e}\})) \mathbf{1}_{\{Z \rightarrow \infty\}}] \leq A(t) \tag{III.63}$$

We next prove that  $\lim_{t \rightarrow \infty} A(t) = 0$ . First note that for all  $w \in (0, q)$ ,  $w < v(t)$  and  $u(-t, w) < q$ , moreover  $u(-t, w) \downarrow 0$  as  $t \uparrow \infty$ . Since  $\Psi'(0+) = -\infty$ ,  $\lambda/\Psi(\lambda) \uparrow 0$  as  $\lambda \downarrow 0$ . This implies that

$$x^2 \int_0^q dw \Psi(w) e^{-xw} \frac{u(-t, w) - (u(-t, w) - q)_+}{\Psi(u(-t, w))} = x^2 \int_0^q dw \Psi(w) e^{-xw} \frac{u(-t, w)}{\Psi(u(-t, w))} \xrightarrow{t \rightarrow \infty} 0.$$

If  $q = \infty$ , then, this proves  $\lim_{t \rightarrow \infty} A(t) = 0$ . Let us assume that  $q < \infty$ : for all  $w \in (q, v(t))$ ,  $u(-t, w) > q$  and we get

$$x^2 \int_q^{v(t)} dw \Psi(w) e^{-xw} \frac{u(-t, w) - (u(-t, w) - q)_+}{\Psi(u(-t, w))} = x^2 \int_q^{v(t)} dw \Psi(w) e^{-xw} \frac{q}{\Psi(u(-t, w))} . \tag{III.64}$$

There are two cases to consider: if  $\Psi$  is persistent, then  $v(t) = \infty$ . Moreover, for all  $w \in (q, \infty)$ ,  $u(-t, w)$  is well-defined and  $u(-t, w) \uparrow \infty$  as  $t \uparrow \infty$ , which implies that (III.64) tends to 0 as  $t \rightarrow \infty$ . If  $\Psi$  is non-persistent, then  $v(t) < \infty$ . Observe that  $\lim_{t \rightarrow \infty} v(t) = q$  and use (III.19) with  $\lambda = u(-t, w)$  to prove that  $w < u(-t, w)$  for any  $w \in (q, v(t))$ . Since  $\Psi$  increases, we get

$$x^2 \int_q^{v(t)} dw \Psi(w) e^{-xw} \frac{q}{\Psi(u(-t, w))} \leq qx^2 \int_q^{v(t)} dw e^{-xw} \xrightarrow{t \rightarrow \infty} 0.$$

This completes the proof of  $\lim_{t \rightarrow \infty} A(t) = 0$ .

By (III.63) and Lemma III.20, we get  $\mathbf{P}$ -a.s. on  $\{Z \rightarrow \infty\}$ ,  $M_t(\{\mathbf{e}\}) \rightarrow 1$ . Thus, it entails  $\|M_t - \delta_{\mathbf{e}}\|_{\text{var}} \rightarrow 0$  by Lemma III.16, as  $t \rightarrow \infty$ , which implies Theorem III.2 (ii-a).  $\blacksquare$

**Proof of Theorem III.2 (iii-a).**

We assume that  $\Psi$  is persistent, of infinite variation type and such that  $q < \infty$ . Observe that  $\mathcal{P}$  under  $\mathbf{P}(\cdot | \lim_{t \rightarrow \infty} Z_t = 0)$  is a Poisson point measure associated with the branching mechanism  $\Psi(\cdot + q)$  that is sub-critical (and therefore conservative). So the proof of Theorem III.2 (iii-a) reduces to the cases of sub-critical persistent branching mechanisms and without loss of generality, we now assume that  $\Psi$  is so. Thus,  $\lim_{\theta \rightarrow \infty} u(t, \theta) = v(t) = \infty$ . By letting  $\theta$  go to  $\infty$  in Lemma III.19, we get

$$\mathbf{P}(R_t^{-1}(U) \neq R_{t+s}^{-1}(V)) = x^2 \int_0^\infty \frac{u(-t, w)}{\Psi(u(-t, w))} \Psi(w) e^{-xw} dw =: B(t),$$

which does not depend on  $s$ . Then, set  $\mathbf{e} = R_\infty^{-1}(V)$ . By the Portmanteau theorem as  $s \rightarrow \infty$ , we get  $\mathbf{P}(R_t^{-1}(U) \neq \mathbf{e}) \leq B(t)$ . Next observe that  $\mathbf{E}[\mathbf{1}_{\{R_t^{-1}(U) \neq \mathbf{e}\}} | \mathcal{P}, V] = 1 - M_t(\{\mathbf{e}\})$ . Therefore,

$$0 \leq 1 - \mathbf{E}[M_t(\{\mathbf{e}\})] \leq B(t) \quad (\text{III.65})$$

Since  $\Psi$  is sub-critical and persistent for all  $w \in (0, \infty)$ ,  $u(-t, w)$  increases to  $\infty$  as  $t \uparrow \infty$ . Moreover, since  $\Psi$  is of infinite variation type,  $\lambda/\Psi(\lambda)$  decreases to 0 as  $\lambda \uparrow \infty$ , which implies that  $\lim_{t \rightarrow \infty} B(t) = 0$ . By (III.65) and Lemma III.20, we get  $\mathbf{P}$ -a.s.  $M_t(\{\mathbf{e}\}) \rightarrow 1$ , and thus  $\|M_t - \delta_{\mathbf{e}}\|_{\text{var}} \rightarrow 0$  by Lemma III.16, as  $t \rightarrow \infty$ , which completes the proof of Theorem III.2 (iii-a).  $\blacksquare$

**Proof of Lemma III.20.**

To complete the proof of Theorem III.2, it only remains to prove Lemma III.20. We shall proceed by approximation, in several steps. Recall from (III.38) and (III.40) the definition of the Poisson point measures  $\mathcal{P}$  and  $\mathcal{Q}$ . For any  $t \in (0, \infty)$ , we define the following:

$$\mathcal{P}_t = \sum_{i \in I} \delta_{(x_i, Z_{i \wedge t}^i)} \quad \text{and} \quad \mathcal{Q}_t = \sum_{j \in J} \mathbf{1}_{\{t_j \leq t\}} \delta_{(x_j, t_j, Z_{j \wedge (t-t_j)}^j)}. \quad (\text{III.66})$$

We then define  $\mathcal{G}_t$  as the sigma-field generated either by  $\mathcal{P}_t$  if  $\Psi$  is of infinite variation type, or by  $\mathcal{Q}_t$  if  $\Psi$  is of finite variation type.

**LEMMA III.21.** *Assume that  $\Psi$  is conservative and not linear. Then, for all  $s, t, \lambda \in [0, \infty)$*

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E}[e^{-\lambda Z_{t+s}} | \mathcal{G}_t] = e^{-u(s, \lambda) Z_t}.$$

**Proof** We first consider the infinite variation cases. We fix  $s_0, \varepsilon \in (0, \infty)$ . For any  $t \in (s_0, \infty)$ , we set

$$\mathcal{P}_t^{>\varepsilon} = \sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i > \varepsilon\}} \delta_{(x_i, Z_{i \wedge t}^i)} \quad \text{and} \quad \mathcal{Z}_t^\varepsilon = \sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i > \varepsilon\}} Z_t^i \delta_{x_i}. \quad (\text{III.67})$$

### 3. Proof of Theorem III.2.

Since  $t > s_0$ , and by monotone convergence for sums,  $\lim_{\varepsilon \rightarrow 0} \mathcal{Z}_{t+s}^\varepsilon([0, x]) = Z_{t+s}$ . Then, observe that  $\mathcal{Z}_{t+s}^\varepsilon$  is independent from  $\mathcal{P}_t - \mathcal{P}_t^{>\varepsilon}$ . Thus, **P**-a.s.  $\mathbf{E}[e^{-\lambda Z_{t+s}} | \mathcal{G}_t] = \lim_{\varepsilon \rightarrow 0} \mathbf{E}[e^{-\lambda \mathcal{Z}_{t+s}^\varepsilon([0, x])} | \mathcal{P}_t^{>\varepsilon}]$ .

Next, note that  $\mathcal{P}_t^{>\varepsilon}$  is a Poisson point measure whose law is specified as follows. By Theorem III.3 (b) and Lemma III.5, first note that  $N_\Psi(Z_{s_0} > \varepsilon) = \nu_{s_0}((\varepsilon, \infty]) \in (0, \infty)$ . Then,  $Q_{s_0, \varepsilon} = N_\Psi(\cdot | Z_{s_0} > \varepsilon)$  is a well-defined probability on  $\mathbb{D}([0, \infty), [0, \infty])$ . Theorem III.3 (c) easily entails that

$$Q_{s_0, \varepsilon}\text{-a.s.} \quad Q_{s_0, \varepsilon} [e^{-\lambda Z_{t+s}} | Z_{\cdot \wedge t}] = e^{-u(s, \lambda) Z_t}. \quad (\text{III.68})$$

Next, note that  $\mathcal{P}_t^{>\varepsilon}$  can be written as  $\sum_{1 \leq k \leq S} \delta_{(X_k, Y^k_{\wedge t})}$ , where  $(X_k, Y^k)$ ,  $k \geq 1$ , is an i.i.d. sequence of  $[0, x] \times \mathbb{D}([0, \infty), [0, \infty])$ -valued r.v. whose law is  $x^{-1} \mathbf{1}_{[0, x]}(y) \ell(dy) Q_{s_0, \varepsilon}(dZ)$  and where  $S$  is a Poisson r.v. with mean  $x \nu_{s_0}((\varepsilon, \infty])$  that is independent from the  $(X_k, Y^k)_{k \geq 1}$ . By an easy argument, we derive from (III.68) that **P**-a.s.  $\mathbf{E}[e^{-\lambda \mathcal{Z}_{t+s}^\varepsilon([0, x])} | \mathcal{P}_t^{>\varepsilon}] = e^{-u(s, \lambda) \mathcal{Z}_t^\varepsilon([0, x])}$ , which entails the desired result as  $\varepsilon \rightarrow 0$ .

In the finite variation cases, we also proceed by approximation: for any  $\varepsilon \in (0, \infty)$ , we set

$$\mathcal{Q}_t^{>\varepsilon} = \sum_{j \in J} \mathbf{1}_{\{t_j \leq t; Z_0^j > \varepsilon\}} \delta_{(x_j, t_j, Z_{t-t_j}^j)}, \quad \mathcal{Z}_t^\varepsilon = \sum_{j \in J} \mathbf{1}_{\{t_j \leq t; Z_0^j > \varepsilon\}} Z_{t-t_j}^j \delta_{x_j} \quad \text{and} \quad Z_t^* = Z_t - x e^{-Dt}.$$

Then, note that  $\lim_{\varepsilon \rightarrow 0} \mathcal{Z}_{t+s}^\varepsilon([0, x]) = Z_{t+s}^*$  and observe that  $\mathcal{Z}_{t+s}^\varepsilon$  is independent from  $\mathcal{Q}_t - \mathcal{Q}_t^{>\varepsilon}$ . Thus, **P**-a.s.  $\mathbf{E}[e^{-\lambda Z_{t+s}^*} | \mathcal{G}_t] = \lim_{\varepsilon \rightarrow 0} \mathbf{E}[e^{-\lambda \mathcal{Z}_{t+s}^\varepsilon([0, x])} | \mathcal{Q}_t^{>\varepsilon}]$ . Next, note that  $\mathcal{Q}_t^{>\varepsilon}$  is a Poisson point measure that can be written as  $\sum_{1 \leq k \leq S} \delta_{(X_k, T_k, Y^k_{\wedge (t-T_k)})}$  where  $(X_k, T_k, Y^k)_{k \geq 1}$ , is an i.i.d. sequence of  $[0, x] \times [0, t] \times \mathbb{D}([0, \infty), [0, \infty])$ -valued r.v. whose law is  $x^{-1} \mathbf{1}_{[0, x]}(y) \ell(dy) (1 - e^{-Dt})^{-1} D e^{-Ds} \ell(ds) Q_\varepsilon(dZ)$  where

$$Q_\varepsilon(dZ) := \frac{1}{\pi((\varepsilon, \infty))} \int_{(\varepsilon, \infty)} \pi(dr) \mathbb{P}_r(dZ)$$

and  $S$  is an independent Poisson r.v. with mean  $x(1 - e^{-Dt}) D^{-1} \pi((\varepsilon, \infty))$ . When  $D = 0$ , one should replace  $(1 - e^{-Dt}) D^{-1}$  by  $t$  in the last two expressions. Note that the Markov property applies under  $Q_\varepsilon$ . Namely,  $Q_\varepsilon$ -a.s.  $Q_\varepsilon[e^{-\lambda Z_{t+s}} | Z_{\cdot \wedge t}] = e^{-u(s, \lambda) Z_t}$ . This implies **P**-a.s. the following

$$\mathbf{E} \left[ e^{-\lambda \sum_{j \in J} \mathbf{1}_{\{t_j \leq t, Z_0^j > \varepsilon\}} Z_{t+s-t_j}^j} \mid \mathcal{Q}_t^{>\varepsilon} \right] = e^{-u(s, \lambda) \sum_{j \in J} \mathbf{1}_{\{t_j \leq t, Z_0^j > \varepsilon\}} Z_{t-t_j}^j} = e^{-u(s, \lambda) \mathcal{Z}_t^\varepsilon([0, x])}. \quad (\text{III.69})$$

Then, note that  $\sum_{j \in J} \mathbf{1}_{\{t < t_j \leq t+s, Z_0^j > \varepsilon\}} Z_{t+s-t_j}^j$  is independent from  $\mathcal{Q}_t^{>\varepsilon}$ . By the exponential formula for Poisson point measures, we thus **P**-a.s. get

$$-\log \mathbf{E} \left[ e^{-\lambda \sum_{j \in J} \mathbf{1}_{\{t < t_j \leq t+s, Z_0^j > \varepsilon\}} Z_{t+s-t_j}^j} \mid \mathcal{Q}_t^{>\varepsilon} \right] = x e^{-Dt} \int_0^s da e^{-Da} \int_{(\varepsilon, \infty)} \pi(dr) (1 - e^{-ru(s-a, \lambda)}). \quad (\text{III.70})$$

As  $\varepsilon \rightarrow 0$ , the right member of (III.70) tends to  $x e^{-Dt} \int_0^s da e^{-Da} (Du(s-a, \lambda) - \Psi(u(s-a, \lambda)))$  that is equal to  $x e^{-Dt} u(s, \lambda) - x \lambda e^{-D(s+t)}$  by a simple integration by parts. This computation combined with (III.69) and (III.70), implies

$$\lim_{\varepsilon \rightarrow 0} -\log \mathbf{E} [e^{-\lambda \mathcal{Z}_{t+s}^\varepsilon([0, x])} | \mathcal{Q}_t^{>\varepsilon}] = u(s, \lambda) \lim_{\varepsilon \rightarrow 0} \mathcal{Z}_t^\varepsilon([0, x]) + x e^{-Dt} u(s, \lambda) - x \lambda e^{-D(s+t)}$$

Namely,  $-\log \mathbf{E}[e^{-\lambda Z_{t+s}^*} | \mathcal{G}_t] = u(s, \lambda)(Z_t^* + x e^{-Dt}) - x \lambda e^{-D(s+t)} = u(s, \lambda) Z_t - x \lambda e^{-D(s+t)}$ , which implies the desired result.  $\blacksquare$

**LEMMA III.22.** *We assume that  $\Psi$  is conservative and non-linear. We fix  $s_0, \varepsilon \in (0, \infty)$ . For any  $t \in (s_0, \infty)$ , we define  $\mathcal{Z}_t^\varepsilon$  as follows:*

– *If  $\Psi$  is of infinite variation type, then  $\mathcal{Z}_t^\varepsilon = \sum_{i \in I} \mathbf{1}_{\{Z_{s_0}^i > \varepsilon\}} Z_t^i \delta_{x_i}$ .*

– *If  $\Psi$  is of finite variation type, then  $\mathcal{Z}_t^\varepsilon = \sum_{j \in J} \mathbf{1}_{\{t_j \leq s_0; Z_0^j > \varepsilon\}} Z_{t-t_j}^j \delta_{x_j}$ .*

*Recall the definition of the sigma-field  $\mathcal{G}_t$ . Then, for all  $t \in (s_0, \infty)$ , all  $s, \theta \in [0, \infty)$  and all  $y \in [0, x]$ ,*

$$\mathbf{P}\text{-a.s.} \quad \mathbf{E} \left[ \mathbf{1}_{\{Z_{t+s} > 0\}} \frac{\mathcal{Z}_{t+s}^\varepsilon([0, y])}{Z_{t+s}} (1 - e^{-\theta Z_{t+s}}) \mid \mathcal{G}_t \right] = \mathbf{1}_{\{Z_t > 0\}} \frac{\mathcal{Z}_t^\varepsilon([0, y])}{Z_t} (1 - e^{-u(s, \theta) Z_t}). \quad (\text{III.71})$$

**Proof** We first consider the infinite variation cases. Note that in these cases,  $\mathcal{Z}_t^\varepsilon$  is defined as in (III.67). Let  $\lambda \in (0, \infty)$ . Recall the notation  $Q_{s_0, \varepsilon} = \mathbf{N}_\Psi(\cdot \mid Z_{s_0} > \varepsilon)$  from the proof of Lemma III.21: by differentiating (III.68), we get

$$Q_{s_0, \varepsilon}\text{-a.s.} \quad Q_{s_0, \varepsilon} [Z_{t+s} e^{-\lambda Z_{t+s}} \mid Z_{\cdot, \wedge t}] = Z_t e^{-u(s, \lambda) Z_t} \partial_\lambda u(s, \lambda). \quad (\text{III.72})$$

Recall from (III.66), the definition of  $\mathcal{P}_t$ . Let  $F$  be a bounded nonnegative measurable function on the space of point measures on  $[0, x] \times \mathbb{D}([0, \infty), [0, \infty])$ . We then set  $A(\lambda) = \mathbf{E}[\mathcal{Z}_{t+s}^\varepsilon([0, y]) e^{-\lambda Z_{t+s}} F(\mathcal{P}_t)]$ . By Palm formula (III.37), Lemma III.21 and (III.72), we get

$$\begin{aligned} A(\lambda) &= \mathbf{E} \left[ \sum_{i \in I} \mathbf{1}_{\{x_i \in [0, y]; Z_{s_0}^i > \varepsilon\}} Z_{t+s}^i e^{-\lambda Z_{t+s}^i} e^{-\lambda \sum_{k \in I \setminus \{i\}} Z_{t+s}^k} F(\delta_{(x_i, Z_{\cdot, \wedge t}^i)} + \mathcal{P}_t - \delta_{(x_i, Z_{\cdot, \wedge t}^i)}) \right] \\ &= \nu_{s_0}((\varepsilon, \infty]) \int_0^y dr \int Q_{s_0, \varepsilon}(dZ) \mathbf{E} \left[ Z_{t+s} e^{-\lambda Z_{t+s}} e^{-\lambda Z_{t+s}} F(\delta_{(r, Z_{\cdot, \wedge t})} + \mathcal{P}_t) \right] \\ &= (\partial_\lambda u(s, \lambda)) \nu_{s_0}((\varepsilon, \infty]) \int_0^y dr \int Q_{s_0, \varepsilon}(dZ) \mathbf{E} \left[ Z_t e^{-u(s, \lambda) Z_t} e^{-u(s, \lambda) Z_t} F(\delta_{(y, Z_{\cdot, \wedge t})} + \mathcal{P}_t) \right] \\ &= (\partial_\lambda u(s, \lambda)) \mathbf{E} [\mathcal{Z}_t^\varepsilon([0, y]) e^{-u(s, \lambda) Z_t} F(\mathcal{P}_t)]. \end{aligned}$$

By an easy argument, it implies that  $\mathbf{P}$ -a.s. for all  $\lambda \in (0, \infty)$ ,

$$\mathbf{E} [\mathcal{Z}_{t+s}^\varepsilon([0, y]) e^{-\lambda Z_{t+s}} \mid \mathcal{G}_t] = \mathcal{Z}_t^\varepsilon([0, y]) e^{-u(s, \lambda) Z_t} \partial_\lambda u(s, \lambda).$$

Thus,  $\mathbf{P}$ -a.s. for all  $\lambda, \theta \in (0, \infty)$ ,

$$\begin{aligned} \mathbf{E} [\mathbf{1}_{\{Z_{t+s} > 0\}} \mathcal{Z}_{t+s}^\varepsilon([0, y]) e^{-\lambda Z_{t+s}} (1 - e^{-\theta Z_{t+s}}) \mid \mathcal{P}_t] &= \\ \mathbf{1}_{\{Z_t > 0\}} \mathcal{Z}_t^\varepsilon([0, y]) (e^{-u(s, \lambda) Z_t} \partial_\lambda u(s, \lambda) - e^{-u(s, \lambda + \theta) Z_t} \partial_\lambda u(s, \lambda + \theta)). & (\text{III.73}) \end{aligned}$$

When we integrate the first member of (III.73) in  $\lambda$  on  $(0, \infty)$ , we get the first member of (III.71). Then, by an easy change of variable, we get

$$\forall \lambda_0 \in [0, \infty), \forall z \in (0, \infty), \quad \int_{\lambda_0}^{\infty} d\lambda e^{-u(s, \lambda) z} \partial_\lambda u(s, \lambda) = \frac{1}{z} (e^{-u(s, \lambda_0) z} - e^{-v(s) z}), \quad (\text{III.74})$$

where we recall that  $v(s) = \lim_{\lambda \rightarrow \infty} u(s, \lambda)$ , which is infinite if  $\Psi$  is persistent and finite otherwise. Since  $\Psi$  is conservative, recall that  $\kappa(s) = \lim_{\lambda \rightarrow 0^+} u(s, \lambda) = 0$ . Thus, when we integrate the second member of (III.73) in  $\lambda$  on  $(0, \infty)$ , we obtain the second member of (III.71), which completes the proof of the lemma in the infinite variation cases.

We next consider the finite variation cases. Note that the definition of  $\mathcal{Z}^\varepsilon$  is slightly different from the proof of Lemma III.21. Recall from (III.66), the definition of  $\mathcal{Q}_t$ . Let  $F$  be a bounded nonnegative

### 3. Proof of Theorem III.2.

measurable function on the space of point measures on  $[0, x] \times [0, \infty) \times \mathbb{D}([0, \infty), [0, \infty])$ . We set  $A(\lambda) = \mathbf{E}[\mathcal{Z}_{t+s}^\varepsilon([0, y]) e^{-\lambda Z_{t+s}} F(\mathcal{Q}_t)]$ . By Palm formula (III.37) and Lemma III.21 we get

$$\begin{aligned}
A(\lambda) &= \mathbf{E} \left[ \sum_{j \in J} \mathbf{1}_{\{x_j \in [0, y]; t_j \leq s_0; Z_0^j > \varepsilon\}} Z_{t+s-t_j}^j e^{-\lambda Z_{t+s-t_j}^j} e^{-\lambda \sum_{k \in J \setminus \{j\}} \mathbf{1}_{\{t_k \leq t+s\}} Z_{t+s-t_k}^k} e^{-\lambda x e^{-D(t+s)}} \right. \\
&\quad \left. \times F(\delta_{(x_j, t_j, Z_{\cdot \wedge (t-t_j)}^j)} + \mathcal{Q}_t - \delta_{(x_j, t_j, Z_{\cdot \wedge (t-t_j)}^j)}) \right] \\
&= \int_0^y da \int_0^{s_0} db e^{-Db} \int_{(\varepsilon, \infty)} \pi(dr) \mathbb{E}_r \left[ \mathbf{E} \left[ Z_{t+s-b} e^{-\lambda Z_{t+s-b}} e^{-\lambda Z_{t+s}} F(\delta_{(a, b, Z_{\cdot \wedge (t-b)})} + \mathcal{Q}_t) \right] \right] \\
&= \partial_\lambda u(s, \lambda) \int_0^y da \int_0^{s_0} db e^{-Db} \int_{(\varepsilon, \infty)} \pi(dr) \mathbb{E}_r \left[ \mathbf{E} \left[ Z_{t-b} e^{-u(s, \lambda) Z_{t-b}} e^{-u(s, \lambda) Z_t} F(\delta_{(a, b, Z_{\cdot \wedge (t-b)})} + \mathcal{Q}_t) \right] \right] \\
&= \partial_\lambda u(s, \lambda) \mathbf{E}[\mathcal{Z}_t^\varepsilon([0, y]) e^{-u(s, \lambda) Z_t} F(\mathcal{Q}_t)].
\end{aligned}$$

Then, we argue exactly as in the infinite variation cases. ■

We now complete the proof of Lemma III.20. If  $\Psi$  is not conservative, then we have already proved that on  $\{\zeta_\infty < \infty\}$ ,  $M$  has an Eve in finite time. Moreover, conditionally on  $\{\lim_{t \rightarrow \infty} Z_t = 0\}$ ,  $M$  is distributed as the frequency process of a CSBP( $\Psi(\cdot + q)$ ) that is sub-critical, and therefore conservative. Thus, without loss of generality, we assume that  $\Psi$  is conservative. In this case, Lemma III.22 applies: we fix  $s_0, \varepsilon \in (0, \infty)$  and we let  $\theta$  go to  $\infty$  in (III.71); this implies that  $t \mapsto \mathbf{1}_{\{Z_t > 0\}} \frac{\mathcal{Z}_t^\varepsilon([0, y])}{Z_t}$  is a supermartingale. Then,

$$\mathbf{P}\text{-a.s. } \forall q \in \mathbb{Q} \cap [0, x], \quad \lim_{t \rightarrow \infty} \mathbf{1}_{\{Z_t > 0\}} \frac{\mathcal{Z}_t^\varepsilon([0, q])}{Z_t} =: R_q^\varepsilon \text{ exists.}$$

Then observe there exists a finite subset  $S_{s_0, \varepsilon} := \{X_1 < \dots < X_N\} \subset [0, x]$  such that a.s. for all  $t \in (s_0, \infty)$ ,  $\mathcal{Z}_t^\varepsilon([0, x] \setminus S_{s_0, \varepsilon}) = 0$ . Then, for any  $1 \leq k \leq N$ , there exists  $q, q' \in \mathbb{Q} \cap [0, x]$  such that  $q < X_k < q'$  and  $\mathbf{1}_{\{Z_t > 0\}} M_t(\{X_k\}) = \mathbf{1}_{\{Z_t > 0\}} \mathcal{Z}_t^\varepsilon((q, q'])/Z_t \rightarrow R_{q'}^\varepsilon - R_q^\varepsilon$ , as  $t \rightarrow \infty$ .

Now observe that if  $\Psi$  is of infinite variation type,  $\{x_i; i \in I\} = \bigcup_{n, m \in \mathbb{N}} S_{2^{-m}, 2^{-n}}$ . Thus, on the event  $\{\zeta_0 = \infty\}$  (no extinction in finite time), this entails that  $\mathbf{P}$ -a.s. for all  $i \in I$ ,  $\lim_{t \rightarrow \infty} M_t(\{x_i\})$  exists. Moreover, for all  $y \notin \{x_i; i \in I\}$  and all  $t \in (0, \infty)$ ,  $M_t(\{y\}) = 0$ . Finally, on  $\{\zeta_0 < \infty\}$ , we have already proved that  $M$  has an Eve in finite time. This completes the proof of Lemma III.20 when  $\Psi$  is of infinite variation type.

If  $\Psi$  is finite variation type, note that  $\{x_j; j \in J\} = \bigcup_{n, m \in \mathbb{N}} S_{m, 2^{-n}}$ . Since there is no extinction in finite time, we get that  $\mathbf{P}$ -a.s. for all  $j \in J$ ,  $\lim_{t \rightarrow \infty} M_t(\{x_j\})$  exists, which completes the proof since for all  $y \notin \{x_j; j \in J\}$  and all  $t \in (0, \infty)$ , we have  $M_t(\{y\}) = 0$ . ■





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Quasi-stationary distributions associated with explosive CSBP

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This article [54] has been published in *Electronic Communications in Probability*.

## 1 Introduction

Continuous-state branching processes (CSBP) are  $[0, \infty]$ -valued Markov processes that describe the evolution of the size of a continuous population. They have been introduced by Jirina [46] and Lamperti [57]. We recall some basic facts on CSBP and refer to Bingham [17], Grey [40], Kyprianou [51] and Le Gall [60] for details and proofs.

Consider the space  $\mathbb{D}([0, \infty), [0, \infty])$  of càdlàg  $[0, \infty]$ -valued functions endowed with the Skorohod's topology. We denote by  $Z := (Z_t, t \geq 0)$  the canonical process on this space. For all  $x \in [0, \infty]$ , we denote by  $\mathbb{P}_x$  the distribution of the CSBP starting from  $x$  whose semigroup is characterised by

$$\forall t \geq 0, \lambda > 0, \quad \mathbb{E}_x[e^{-\lambda Z_t}] = e^{-x u(t, \lambda)} \tag{IV.1}$$

where for all  $\lambda > 0$ ,  $(u(t, \lambda), t \geq 0)$  is the unique solution of

$$\partial_t u(t, \lambda) = -\Psi(u(t, \lambda)) \quad , \quad u(0, \lambda) = \lambda \tag{IV.2}$$

and  $\Psi$ , the so-called *branching mechanism* of the CSBP, is a convex function of the form

$$\forall u \geq 0, \quad \Psi(u) = \gamma u + \frac{\sigma^2}{2} u^2 + \int_{(0, \infty)} (e^{-uh} - 1 + uh \mathbf{1}_{\{h < 1\}}) \nu(dh) \tag{IV.3}$$

where  $\gamma \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\nu$  is a Borel measure on  $(0, \infty)$  such that  $\int_{(0, \infty)} (1 \wedge h^2) \nu(dh) < \infty$ . The function  $\Psi$  entirely characterises the law of the process. The CSBP fulfils the following branching property: for all  $x, y \in [0, \infty]$  the process starting from  $x + y$  has the same law as the sum of two independent copies starting from  $x$  and  $y$  respectively. Observe that  $\Psi$  is also the Laplace exponent of a spectrally positive Lévy process, we refer to Theorem 1 in [57] for a pathwise correspondence between Lévy processes and CSBP.

The convexity of  $\Psi$  entails that the ratio  $\Psi(u)/u$  is increasing. A direct calculation or Proposition I.2 p.16 [8] shows that it converges to a finite limit as  $u \rightarrow \infty$  iff

$$(Finite\ variation) \quad \sigma = 0 \text{ and } \int_{(0, 1)} h \nu(dh) < \infty \tag{IV.4}$$

When this condition is verified, the limit of the ratio is necessarily equal to  $D := \gamma + \int_{(0,1)} h\nu(dh)$  and  $\Psi$  can be rewritten

$$\forall u \geq 0, \quad \Psi(u) = Du + \int_{(0,\infty)} (e^{-uh} - 1) \nu(dh) \quad (\text{IV.5})$$

As  $t \rightarrow \infty$  the CSBP converges either to 0 or to  $\infty$ , which are absorbing states for the process. Consequently we define the *lifetime* of the CSBP as the stopping time  $T := T_0 \wedge T_\infty$  where

$$(\text{Extinction}) \quad T_0 := \inf\{t \geq 0 : Z_t = 0\}, \quad (\text{Explosion}) \quad T_\infty := \inf\{t \geq 0 : Z_t = \infty\}$$

We denote by  $q := \sup\{u \geq 0 : \Psi(u) \leq 0\} \in [0, \infty]$  the second root of the convex function  $\Psi$ : it is elementary to check from (IV.2) that  $u(t, q) = q$  for all  $t \geq 0$  and that for all  $\lambda > 0$ ,  $u(t, \lambda) \rightarrow q$  as  $t \rightarrow \infty$ . Hence from (IV.1) we get

$$\forall x \in [0, \infty], \quad \mathbb{P}_x\left(\lim_{t \rightarrow \infty} Z_t = 0\right) = 1 - \mathbb{P}_x\left(\lim_{t \rightarrow \infty} Z_t = \infty\right) = e^{-xq}$$

When  $\Psi'(0+) > 0$  (resp.  $\Psi'(0+) = 0$ ) the CSBP is said *subcritical* (resp. *critical*), the convexity of  $\Psi$  then implies  $q = 0$  and the process is almost surely absorbed at 0. Moreover the extinction time  $T_0$  is almost surely finite iff

$$\int^{+\infty} \frac{du}{\Psi(u)} < \infty \quad (\text{IV.6})$$

Otherwise  $T_0$  is almost surely infinite. When  $\Psi'(0+) \in [-\infty, 0)$  the CSBP is said *supercritical* and then  $q \in (0, \infty]$ . The CSBP has a positive probability to be absorbed at 0 iff  $q \in (0, \infty)$ . In that case, on the extinction event  $\{T = T_0\}$  the finiteness of  $T_0$  is governed by the same criterion as above. On the explosion event  $\{T = T_\infty\}$ , the explosion time  $T_\infty$  is almost surely finite iff

$$\int_{0+} \frac{du}{-\Psi(u)} < \infty \quad (\text{IV.7})$$

Observe that  $\Psi'(0+) = -\infty$  is required (but not sufficient) for this inequality to be fulfilled. When (IV.7) does not hold,  $T_\infty$  is almost surely infinite on the explosion event.

By quasi-stationary distribution (QSD for short), we mean a probability measure  $\mu$  on  $(0, \infty)$  such that

$$\mathbb{P}_\mu(Z_t \in \cdot | T > t) = \mu(\cdot)$$

When  $\mu$  is a QSD, it is a simple matter to check that under  $\mathbb{P}_\mu$  the random variable  $T$  has an exponential distribution, the parameter of which is called the *rate of decay* of  $\mu$ . The goal of the present paper is to investigate the QSD associated with a CSBP that explodes in finite time almost surely.

## 1.1 A brief review of the literature: the extinction case

Li [63] and Lambert [55] considered the extinction case  $T = T_0 < \infty$  almost surely, so that  $\Psi'(0+) \geq 0$  and (IV.6) holds, and they studied the CSBP conditioned on non-extinction. We recall some of their results. When  $\Psi$  is subcritical, that is  $\Psi'(0+) > 0$ , there exists a family  $(\mu_\beta; 0 < \beta \leq \Psi'(0+))$  of QSD where  $\beta$  is the rate of decay of  $\mu_\beta$ . These distributions are characterised by their Laplace transforms as follows

$$\forall \lambda \geq 0, \quad \int_{(0,\infty)} \mu_\beta(dr) e^{-r\lambda} = 1 - e^{-\beta\Phi(\lambda)} \quad \text{where} \quad \Phi(\lambda) := \int_\lambda^{+\infty} \frac{du}{\Psi(u)} \quad (\text{IV.8})$$

Notice that  $\Phi$  is well-defined thanks to (IV.6). For any  $\beta > \Psi'(0+)$  they proved that there is no QSD with rate of decay  $\beta$ , and that Equation (IV.8) does not define the Laplace transform of a probability measure on  $(0, \infty)$ . Additionally, the value  $\beta = \Psi'(0+)$  yields the so-called Yaglom limit:

$$\forall x > 0, \mathbb{P}_x(Z_t \in \cdot | T > t) \xrightarrow[t \rightarrow \infty]{} \mu_{\Psi'(0+)}(\cdot)$$

When  $\Psi$  is critical, that is  $\Psi'(0+) = 0$ , the preceding quantity converges to a trivial limit for all  $x > 0$  and Equation (IV.8) does not define the Laplace transform of a probability measure on  $(0, \infty)$ . However, under the condition  $\Psi''(0+) < \infty$ , they proved the following convergence (that extends a result originally due to Yaglom [78] for Galton-Watson processes)

$$\forall x > 0, z \geq 0, \mathbb{P}_x\left(\frac{Z_t}{t} \geq z | T > t\right) \xrightarrow[t \rightarrow \infty]{} \exp\left(-\frac{2z}{\Psi''(0+)}\right) \quad (\text{IV.9})$$

Finally in both critical and subcritical cases, for any given value  $t > 0$  the process  $(Z_r, r \in [0, t])$  conditioned on  $s < T$  admits a limiting distribution as  $s \rightarrow \infty$ , called the  $Q$ -process. The law of the  $Q$ -process is obtained as a  $h$ -transform of  $\mathbb{P}$  as follows

$$\forall x > 0, dQ_{x|\mathcal{F}_t} := \frac{Z_t e^{\Psi'(0)t}}{x} d\mathbb{P}_{x|\mathcal{F}_t}$$

## 1.2 Main results: the explosive case

We now assume that almost surely the CSBP explodes in finite time. From the results recalled above, this is equivalent with (IV.7) and  $q = \infty$  so that  $\Psi$  is convex, decreasing and non-positive. Hence the ratio  $\Psi(u)/u$  cannot converge to  $+\infty$  so that necessarily (IV.4) holds, and  $\Psi$  can be written as in (IV.5). Observe also that in that case the Lévy process with Laplace exponent  $\Psi$  is a subordinator. We set:

$$\Psi(+\infty) := \lim_{u \rightarrow \infty} \Psi(u) \in [-\infty, 0)$$

From (IV.5) we deduce that  $\Psi(+\infty) \in (-\infty, 0)$  iff  $\nu(0, \infty) < \infty$  and  $D = 0$ . When this condition holds, we have  $\Psi(+\infty) = -\nu(0, \infty)$ . Otherwise  $\Psi(+\infty) = -\infty$ .

We start with an elementary remark: conditioning a CSBP on non-explosion does not affect the branching property. Consequently the law of  $Z_t$  conditioned on  $T > t$  is infinitely divisible: if it admits a limit as  $t$  goes to  $\infty$ , the limit has to be infinitely divisible as well. Our result below shows that  $\Psi(+\infty)$  plays a rôle analogue to  $\Psi'(0+)$  in the extinction case.

**THEOREM IV.1.** *Suppose  $T = T_\infty < \infty$  almost surely and set*

$$\forall \lambda \geq 0, \Phi(\lambda) := \int_\lambda^0 \frac{du}{\Psi(u)}$$

*For any  $\beta > 0$  there exists a unique quasi-stationary distribution  $\mu_\beta$  associated to the rate of decay  $\beta$ . This probability measure is infinitely divisible and is characterised by*

$$\forall \lambda \geq 0, \int_{(0, \infty)} \mu_\beta(dr) e^{-r\lambda} = e^{-\beta\Phi(\lambda)} \quad (\text{IV.10})$$

*Additionally, the following dichotomy holds true:*

(i)  $\Psi(+\infty) \in (-\infty, 0)$ . *The limiting conditional distribution is given by*

$$\forall x \in (0, \infty), \lim_{t \rightarrow \infty} \mathbb{P}_x(Z_t \in \cdot | T > t) = \mu_{x\nu(0, \infty)}(\cdot)$$

(ii)  $\Psi(+\infty) = -\infty$ . *The limiting conditional distribution is trivial:*

$$\forall a, x \in (0, \infty), \quad \lim_{t \rightarrow \infty} \mathbb{P}_x(Z_t \leq a \mid T > t) = 0$$

Let us make some comments. Firstly this theorem implies that  $\lambda \mapsto \Phi(\lambda)$  is the Laplace exponent of a subordinator, and so,  $\mu_\beta$  is the distribution of a  $\Phi$ -Lévy process taken at time  $\beta$ . Secondly there is a similarity with the extinction case: the limiting conditional distribution is trivial iff  $\Psi(+\infty) = -\infty$  so that the dichotomy on the value  $\Psi(+\infty)$  is the explosive counterpart of the dichotomy on the value  $\Psi'(0+)$  in the extinction case. Also, note the similarity in the definition of the Laplace transforms (IV.8) and (IV.10). However, there are two major differences with the extinction case: firstly there is no restriction on the rates of decay. Secondly, even if the limiting conditional distribution is trivial when  $\Psi(+\infty) = -\infty$ , there exists a family of QSD.

The following theorem characterises the  $Q$ -process associated with an explosive CSBP. Let  $\mathcal{F}_t$  be the sigma-field generated by  $(Z_r, r \in [0, t])$ , for any  $t \in [0, \infty)$ .

**THEOREM IV.2.** *We assume that  $T = T_\infty < \infty$  almost surely. For each  $x > 0$ , there exists a distribution  $\mathbb{Q}_x$  on  $\mathbb{D}([0, \infty), [0, \infty))$  such that for any  $t \geq 0$*

$$\lim_{s \rightarrow \infty} \mathbb{P}_x(\cdot \mid T > s)_{|\mathcal{F}_t} = \mathbb{Q}_x(\cdot)_{|\mathcal{F}_t}$$

Furthermore,  $\mathbb{Q}_x$  is the law of the  $\Psi^Q$ -CSBP where

$$\Psi^Q(u) = Du$$

The  $Q$ -process appears as the  $\Psi$ -CSBP from which one has removed all the jumps: only the deterministic part remains, see also the forthcoming Proposition IV.3. Notice that the  $Q$ -process cannot be defined through a  $h$ -transform of the CSBP: actually the distribution of the  $Q$ -process on  $\mathbb{D}([0, t], [0, \infty))$  is not even absolutely continuous with respect to that of the  $\Psi$ -CSBP, except when the Lévy measure  $\nu$  is finite.

When  $\Psi(+\infty) = -\infty$ , Theorem IV.1 shows that the process conditioned on non-explosion converges to a trivial limit. In the next theorem, under the assumption that the branching mechanism is regularly varying at 0 we propose a rescaling of the CSBP conditioned on non-explosion such that it converges to a non-trivial limit. Recall that we call slowly varying function at 0 any continuous map  $L : (0, \infty) \rightarrow (0, \infty)$  such that for any  $a \in (0, \infty)$ ,  $L(au)/L(u) \rightarrow 1$  as  $u \downarrow 0$ .

**THEOREM IV.3.** *Suppose that  $\Psi(u) = -u^{1-\alpha}L(u)$  with  $L$  a slowly varying function at 0 and  $\alpha \in (0, 1)$ , and assume that  $\Psi(+\infty) = -\infty$ . Consider any function  $f : [0, \infty) \rightarrow (0, \infty)$  satisfying  $\Psi(f(t)^{-1})f(t) \sim \Psi(u(t, 0+))$  as  $t \rightarrow \infty$ . Then the following convergence holds true:*

$$\forall x, \lambda \in (0, \infty), \quad \mathbb{E}_x \left[ e^{-\lambda Z_t / f(t)} \mid t < T \right] \xrightarrow[t \rightarrow \infty]{} e^{-x \lambda^\alpha / \alpha}$$

Observe that the limit displayed by this theorem is the Laplace transform of the QSD associated with  $\Psi(u) = -u^{1-\alpha}$ .

**EXAMPLE IV.1.** *When  $\Psi(u) = -k u^{1-\alpha}$  with  $k > 0$  and  $\alpha \in (0, 1)$ , we have  $f(t) \sim (\alpha kt)^{(1-\alpha)/\alpha^2}$  as  $t \rightarrow \infty$ . When  $\Psi(u) = -c u - k u^{1-\alpha}$  with  $k, c > 0$  and  $\alpha \in (0, 1)$ , we have  $f(t) \sim (k/c)^{(1-\alpha)/\alpha^2} e^{ct/\alpha}$  as  $t \rightarrow \infty$ .*

The proof of Theorem IV.3 is inspired by calculations of Slack in [73] where it is shown that any critical Galton-Watson process with a regularly varying generating function can be properly rescaled so that, conditioned on non-extinction, it converges towards a non-trivial limit. For completeness we also adapted the result of Slack to critical CSBP conditioned on non-extinction.

**PROPOSITION IV.2.** *Suppose that  $\Psi(u) = u^{1+\alpha}L(u)$  with  $L$  a slowly varying function at 0 and  $\alpha \in (0, 1]$ . Assume that  $T = T_0 < \infty$  almost surely. Fix any function  $f : [0, \infty) \rightarrow (0, \infty)$  verifying  $f(t) \sim u(t, \infty)$  as  $t \rightarrow \infty$ . Then we have the following convergence*

$$\forall x, \lambda \in (0, \infty), \quad \mathbb{E}_x [e^{-\lambda Z_t f(t)} | t < T] \xrightarrow{t \rightarrow \infty} 1 - (1 + \lambda^{-\alpha})^{-1/\alpha}$$

We recover in particular the finite variance case (IV.9) of Lambert and Li. Our result also covers the so-called stable branching mechanisms  $\Psi(u) = u^{1+\alpha}$  with  $\alpha \in (0, 1]$ .

**Organisation of the paper.** We start with a study of continuous-time Galton-Watson processes (which are the discrete-state counterparts of CSBP): we provide a complete description of the QSD when this process explodes in finite time almost surely and compare the results with the continuous-state case. In the third section we prove Theorems IV.1, IV.2 and IV.3. Finally in the fourth section we prove Proposition IV.2.

## 2 The discrete case

A discrete-state branching process  $(Z_t, t \geq 0)$  is a continuous-time Markov process taking values in  $\mathbb{Z}_+ \cup \{+\infty\}$  that verifies the branching property (we refer to Chapter V of Harris [42] for the proofs of the following facts). It can be seen as a Galton-Watson process with offspring distribution  $\xi$  where each individual has an independent exponential lifetime with parameter  $c > 0$ . Let us denote by  $\phi(\lambda) = \sum_{k=0}^{\infty} \lambda^k \xi(k)$ ,  $\forall \lambda \in [0, 1]$  the generating function of the Galton-Watson process. We denote by  $\mathbf{P}_n$  the law on the space  $\mathbb{D}([0, \infty), \mathbb{Z}_+ \cup \{+\infty\})$  of  $\mathcal{Z}$  starting from  $n \in \mathbb{Z}_+ \cup \{+\infty\}$ , and  $\mathbf{E}_n$  the related expectation operator. The semigroup of the DSBP is characterised via the Laplace transform (see Chapter V.4 of [42])

$$\forall r \in (0, 1), \forall t \in [0, \infty), \quad \mathbf{E}_n [r^{Z_t}] = F(t, r)^n \quad \text{where} \quad \int_r^{F(t, r)} \frac{dx}{c(\phi(x) - x)} = t \quad (\text{IV.11})$$

Let  $\tau$  be the lifetime of  $\mathcal{Z}$ , that is, the infimum of the extinction time  $\tau_0$  and the explosion time  $\tau_\infty$ . Taking the limits  $r \downarrow 0$  and  $r \uparrow 1$  in (IV.11) one gets

$$\mathbf{P}_n(\tau_0 \leq t) = F(t, 0+)^n, \quad \mathbf{P}_n(\tau_\infty < t) = 1 - F(t, 1-)^n$$

In this section, we assume that there is explosion in finite time almost surely. Results of Chapters V.9 and V.10 of [42] then entail that the smallest solution of the equation  $\phi(x) = x$  equals 0 (and so  $\xi(0) = 0$ ) and that  $\int_{1-} \frac{dx}{c(\phi(x) - x)}$  is finite. This allows to define

$$\Phi(r) := \int_1^r \frac{dx}{c(\phi(x) - x)}, \quad r \in (0, 1] \quad (\text{IV.12})$$

Clearly  $r \mapsto \Phi(r)$  is the inverse map of  $t \mapsto F(t, 1-)$ , that is for all  $t \geq 0$ ,  $\Phi(F(t, 1-)) = t$ . We say that a measure  $\mu$  on  $\mathbb{N} = \{1, 2, \dots\}$  is a quasi-stationary distribution (QSD) for  $\mathcal{Z}$  if

$$\mathbf{P}_\mu(\mathcal{Z}_t \in \cdot | \tau > t) = \mu(\cdot)$$

From the Markov property, we deduce that  $\tau$  has an exponential distribution under  $\mathbf{P}_\mu$ , the parameter of which is called the rate of decay of  $\mu$ .

**THEOREM IV.4.** *Suppose there is explosion in finite time almost surely. Let  $\beta_0 := c(1 - \xi(1))$ . There is a unique quasi-stationary distribution  $\mu_\beta$  associated with the rate of decay  $\beta$  if and only if  $\beta$  is of the form  $n\beta_0$ , with  $n \in \mathbb{N}$ . It is characterised by its Laplace transform*

$$\sum_k \mu_\beta(\{k\})r^k = e^{-\beta\Phi(r)}, \quad \forall r \in (0, 1] \quad (\text{IV.13})$$

For any initial condition  $n \in \mathbb{N}$  we have

$$\lim_{t \rightarrow \infty} \mathbf{P}_n(\mathcal{Z}_t \in \cdot \mid \tau > t) = \mu_{n\beta_0}(\cdot)$$

Let us make some comments. First there exists only a countable family of QSD. This is due to the restrictive condition that our process takes values in  $\mathbb{Z}_+ \cup \{\infty\}$ . Also, observe the similarity with Theorem IV.1: indeed a DSBP can be seen as a particular CSBP starting from an integer and whose branching mechanism is the Laplace exponent of a compound Poisson process with integer-valued jumps. In particular  $\nu(\{k\}) = c\xi(k+1)$  for all integer  $k \geq 1$ . Hence the quantity  $c(1 - \xi(1))$  in the DSBP case corresponds to  $\nu(0, \infty)$  in the CSBP case. Finally we mention that the  $Q$ -process associated with an explosive DSBP is the constant process, that is, the DSBP with the trivial generating function  $F(t, r) = r$ . This fact can be proved using calculations similar to those in the proof below or it can be deduced from Theorem IV.2 and the remarks above.

**Proof** We start with the proof of the uniqueness of the QSD for a given rate of decay  $\beta > 0$ . Let  $\mu$  be a QSD and let  $\beta > 0$  be its rate of decay. Then we have for all  $t \geq 0$

$$e^{-\beta t} = \mathbb{P}_\mu(\tau > t) = \sum_k \mu(\{k\})\mathbb{P}_k(\tau > t) = \sum_k \mu(\{k\})F(t, 1-)^k$$

Since  $F(\Phi(r), 1-) = r$  we get

$$\forall r \in (0, 1], \quad e^{-\beta\Phi(r)} = \sum_k \mu(\{k\})r^k$$

which ensures the uniqueness of the QSD for a given rate of decay. We now prove that whenever  $\beta = n\beta_0$  with  $n \in \mathbb{N}$ , the last expression is indeed the Laplace transform of a probability measure on  $\mathbb{N}$ .

$$\forall n \in \mathbb{N}, \quad \mathbb{E}_n[r^{\mathcal{Z}_t} \mid \tau > t] = \frac{\mathbb{E}_n[r^{\mathcal{Z}_t}; \tau > t]}{\mathbb{P}_n(\tau > t)} = \left( \frac{F(t, r)}{F(t, 1-)} \right)^n \quad (\text{IV.14})$$

By  $0 \leq F(t, r) \leq F(t, 1-) \rightarrow 0$  as  $t \rightarrow \infty$ ,  $\phi(x) = \xi(1)x + \mathcal{O}(x^2)$  as  $x \downarrow 0$  and (IV.12) we get

$$\Phi(r) = \int_{F(t, 1-)}^{F(t, r)} \frac{dx}{c(\phi(x) - x)} \underset{t \rightarrow \infty}{\sim} \int_{F(t, 1-)}^{F(t, r)} \frac{dx}{cx(\xi(1) - 1)} = -\frac{1}{\beta_0} \log \frac{F(t, r)}{F(t, 1-)}$$

We deduce that the r.h.s. of (IV.14) converges to  $\exp(-\Phi(r)n\beta_0)$  as  $t \rightarrow \infty$ . From this convergence and the fact that  $\Phi(1-) = 0$ , we deduce that  $r \mapsto \exp(-\Phi(r)n\beta_0)$  is the Laplace transform of a probability measure say  $\mu_{n\beta_0}$  on  $\mathbb{Z}_+$ . As  $\Phi(0+) = +\infty$ , we deduce that this probability measure does not charge 0. Also, observe that  $\mu_{\beta_0}(\{1\}) > 0$ . Indeed for all  $r \in (0, 1)$  we have  $\Phi'(r) = -(\beta_0 r)^{-1} - G(r)$  where  $G$  is bounded near 0. Since  $\mu_{\beta_0}(\{1\}) = -\lim_{r \downarrow 0} \beta_0 \Phi'(r) e^{-\beta_0 \Phi(r)}$ , the strict positivity follows.

Fix  $\beta > 0$ . We now assume that  $r \mapsto e^{-\beta\Phi(r)}$  is the Laplace transform of a probability measure on  $\mathbb{N}$  say  $\mu_\beta$ . Denote by  $m \in \mathbb{N}$  the smallest integer such that  $\mu_\beta(\{m\}) > 0$ . Then we have for all  $r \in (0, 1]$

$$\begin{aligned} e^{-\beta\Phi(r)} &= \mu_\beta(\{m\})r^m + \sum_{k>m} \mu_\beta(\{k\})r^k = (e^{-\beta_0\Phi(r)})^{\frac{\beta}{\beta_0}} \\ &= \left( \mu_{\beta_0}(\{1\})r + \sum_{k>1} \mu_{\beta_0}(\{k\})r^k \right)^{\frac{\beta}{\beta_0}} \end{aligned}$$

This implies that  $\mu_\beta(\{m\})r^m \sim (\mu_{\beta_0}(\{1\})r)^{\frac{\beta}{\beta_0}}$  as  $r \downarrow 0$  and so,  $m = \frac{\beta}{\beta_0} \in \mathbb{N}$ . Consequently (IV.13) is the Laplace transform of a probability measure on  $\mathbb{N}$  iff  $\beta$  is of the form  $n\beta_0$ . ■

### 3 Quasi-stationary distributions and Q-process in the explosive case

Consider a branching mechanism  $\Psi$  of the form (IV.3). It is well-known and can be easily checked from (IV.1) that for any  $t \geq 0$  the law of  $Z_t$  under  $\mathbb{P}_x$  is infinitely divisible. Consequently  $u(t, \cdot)$  is the Laplace exponent of a (possibly killed) subordinator (see Chapter 5.1 [51]). Thanks to the Lévy-Khintchine formula, there exist  $a_t, d_t \geq 0$  and a Borel measure  $w_t$  on  $(0, \infty)$  with  $\int_{(0, \infty)} (1 \wedge h) w_t(dh) < \infty$  such that

$$\forall \lambda \geq 0, \quad u(t, \lambda) = a_t + d_t \lambda + \int_{(0, \infty)} (1 - e^{-\lambda h}) w_t(dh) \quad (\text{IV.15})$$

Note that  $a_t = u(t, 0+)$  is positive iff the CSBP has a positive probability to explode in finite time. In the genealogical interpretation, the measure  $w_t$  gives the distribution of the clusters of individuals alive at time  $t$  who share a same ancestor at time 0, while the coefficient  $d_t$  corresponds to the individuals at time  $t$  who do not share their ancestor at time 0 with other individuals. For further use, we write the integral version of (IV.2):

$$\forall t \geq 0, \forall \lambda \in [0, \infty) \setminus \{q\}, \quad \int_{u(t, \lambda)}^\lambda \frac{du}{\Psi(u)} = t \quad (\text{IV.16})$$

The following result shows that the drift  $d_t$  is left unchanged when replacing  $\Psi$  by  $\Psi^Q$  of Theorem IV.2: this means that the  $Q$ -process is obtained by removing all the clusters in the population.

**PROPOSITION IV.3.** *When  $\Psi$  fulfils (IV.4) then  $d_t = e^{-Dt}$  for all  $t \geq 0$ . Otherwise  $d_t = 0$  for all  $t > 0$ .*

**Proof** Corollary p.1049 in [72] entails that  $d_t = 0$  for all  $t > 0$  whenever  $\sigma > 0$  or  $\int_{(0,1)} h\nu(dh) = \infty$ . We now assume the converse, namely that  $\Psi$  fulfils (IV.4) so that  $\Psi(u)/u \rightarrow D$  as  $u \rightarrow \infty$ . A direct computation shows that  $d_t = \lim_{\lambda \rightarrow \infty} u(t, \lambda)/\lambda$ . Then for any  $t \geq 0, \lambda > 0$

$$\log \left( \frac{u(t, \lambda)}{\lambda} \right) = \int_0^t \frac{\partial_s u(s, \lambda)}{u(s, \lambda)} ds = - \int_0^t \frac{\Psi(u(s, \lambda))}{u(s, \lambda)} ds \quad (\text{IV.17})$$

If  $q \in (0, \infty)$ , then for all  $\lambda > q$  and all  $0 \leq s \leq t$  we have  $q < u(t, \lambda) \leq u(s, \lambda) \leq \lambda$  thanks to (IV.2) and by (IV.16) we deduce that  $u(t, \lambda) \uparrow \infty$  as  $\lambda \rightarrow \infty$ . If  $q = \infty$ , then for all  $\lambda > 0$  and all  $0 \leq s \leq t$  we have  $\lambda \leq u(s, \lambda) \leq u(t, \lambda)$  thanks to (IV.2) and obviously  $u(t, \lambda) \uparrow \infty$  as  $\lambda \rightarrow \infty$ . Since  $\Psi(u)/u \uparrow D$  as  $u \rightarrow \infty$  the dominated convergence theorem applied to (IV.17) yields that  $\log(u(t, \lambda)/\lambda) \rightarrow -Dt$  as  $\lambda \rightarrow \infty$ . ■

Until the end of the section, we assume that  $\Psi$  verifies (IV.7) and that  $q = \infty$ . Consequently under  $\mathbb{P}_x$ ,  $Z$  explodes in finite time almost surely and  $a_t = u(t, 0+) > 0$  for all  $t > 0$ . An elementary calculation entails

$$\forall t \geq 0, x > 0, \quad \mathbb{P}_x(T > t) = e^{-x a_t}$$

We introduce for all  $\lambda \geq 0$ ,  $\Phi(\lambda) := \int_\lambda^0 du/\Psi(u)$ . This non-negative, increasing function admits a continuous inverse, namely the function  $t \mapsto a_t$ . Also, thanks to Equation (IV.16) we deduce the identities

$$\forall t, \lambda \geq 0, \quad \Phi(u(t, \lambda)) = t + \Phi(\lambda), \quad u(t, \lambda) = u(t + \Phi(\lambda), 0+) \quad (\text{IV.18})$$



### 3.1 Proof of Theorem IV.1

First we compute the necessary form of the QSD. Fix  $\beta > 0$  and suppose that  $\mu_\beta$  is a QSD with rate of decay  $\beta$ . We get for all  $t \geq 0$

$$e^{-\beta t} = \mathbb{P}_{\mu_\beta}(\mathbf{T} > t) = \int_{(0, \infty)} \mu_\beta(dr) e^{-r a t}$$

Letting  $t = \Phi(\lambda)$  for any  $\lambda \geq 0$  we obtain

$$e^{-\beta \Phi(\lambda)} = \int_{(0, \infty)} \mu_\beta(dr) e^{-r \lambda}$$

Consequently there is at most one QSD corresponding to the rate of decay  $\beta$ . Now suppose that the preceding formula defines a probability distribution on  $(0, \infty)$  then the following calculation ensures that it is quasi-stationary:

$$\begin{aligned} \forall \lambda > 0, \quad \mathbb{E}_{\mu_\beta}[e^{-\lambda Z_t} | \mathbf{T} > t] &= \frac{\mathbb{E}_{\mu_\beta}[e^{-\lambda Z_t}; \mathbf{T} > t]}{\mathbb{P}_{\mu_\beta}(\mathbf{T} > t)} = \frac{\mathbb{E}_{\mu_\beta}[e^{-\lambda Z_t}]}{\mathbb{P}_{\mu_\beta}(\mathbf{T} > t)} = \frac{\int_{(0, \infty)} \mu_\beta(dr) e^{-r u(t, \lambda)}}{e^{-\beta t}} \\ &= e^{-\beta(\Phi(u(t, \lambda)) - t)} = e^{-\beta \Phi(\lambda)} = \mathbb{E}_{\mu_\beta}[e^{-\lambda Z_0}] \end{aligned}$$

We now assume  $\Psi(+\infty) \in (-\infty, 0)$  and we prove that  $\lambda \mapsto e^{-\beta \Phi(\lambda)}$  is indeed the Laplace transform of a probability measure  $\mu_\beta$  on  $(0, \infty)$ . Let  $x := \beta/\nu(0, \infty)$ , for all  $\lambda > 0$  we have

$$\mathbb{E}_x[e^{-\lambda Z_t} | \mathbf{T} > t] = \frac{\mathbb{E}_x[e^{-\lambda Z_t}; \mathbf{T} > t]}{\mathbb{P}_x(\mathbf{T} > t)} = \exp\left(-x(u(t, \lambda) - a_t)\right)$$

From (IV.16) and the definition of  $\Phi$  we get that

$$\int_{u(t, \lambda)}^{a_t} \frac{du}{\Psi(u)} = \Phi(\lambda)$$

Using again (IV.16) and the fact that  $\Psi$  is non-positive, we get that  $a_t \rightarrow \infty$  and  $u(t, \lambda) \rightarrow \infty$  as  $t \rightarrow \infty$ . Since  $\Psi(u) \rightarrow -\nu(0, \infty)$  as  $u \rightarrow \infty$ , one deduces that

$$\int_{u(t, \lambda)}^{a_t} \frac{du}{\Psi(u)} \underset{t \rightarrow \infty}{\sim} \frac{u(t, \lambda) - a_t}{\nu(0, \infty)}$$

and therefore

$$\mathbb{E}_x[e^{-\lambda Z_t} | \mathbf{T} > t] \underset{t \rightarrow \infty}{\longrightarrow} e^{-\Phi(\lambda) x \nu(0, \infty)}$$

Since  $\Phi(\lambda) \rightarrow 0$  as  $\lambda \downarrow 0$ , we deduce that  $\lambda \mapsto e^{-\Phi(\lambda) x \nu(0, \infty)} = e^{-\beta \Phi(\lambda)}$  is the Laplace transform of a probability measure on  $[0, \infty)$ . Moreover, it does not charge 0 since  $\Phi(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ .

We now suppose  $\Psi(+\infty) = -\infty$ . An easy adaptation of the preceding arguments ensures that for any  $x, \lambda > 0$

$$\mathbb{E}_x[e^{-\lambda Z_t} | \mathbf{T} > t] \underset{t \rightarrow \infty}{\longrightarrow} 0$$

Hence the limiting distribution is trivial: it is a Dirac mass at infinity. However, let us prove that  $\lambda \mapsto e^{-\beta \Phi(\lambda)}$  is indeed the Laplace transform of a probability measure  $\mu_\beta$  on  $(0, \infty)$ . For every  $\epsilon > 0$ , define the branching mechanism

$$\Psi_\epsilon(u) := \int_{(0, \infty)} (e^{-hu} - 1)(1_{\{h > \epsilon\}} \nu(dh) + \frac{1}{\epsilon} \delta_{-D\epsilon}(dh)) = \frac{1}{\epsilon} (e^{D\epsilon u} - 1) + \int_{(\epsilon, \infty)} (e^{-hu} - 1) \nu(dh)$$

Observe that for any  $u \geq 0$ ,  $\Psi_\epsilon(u) \downarrow \Psi(u)$  as  $\epsilon \downarrow 0$ . Thus by monotone convergence we deduce that

$$\forall \lambda \geq 0, \int_\lambda^0 \frac{du}{\Psi_\epsilon(u)} \xrightarrow{\epsilon \downarrow 0} \int_\lambda^0 \frac{du}{\Psi(u)}$$

The first part of the proof applies to  $\Psi_\epsilon$ , and therefore the l.h.s. of the preceding equation is the Laplace exponent taken at  $\lambda$  of an infinitely divisible distribution on  $(0, \infty)$ . Since the r.h.s. vanishes at 0 and goes to  $\infty$  at  $\infty$ , it is the Laplace exponent of an infinitely divisible distribution on  $(0, \infty)$ . ■

### 3.2 Proof of Theorem IV.2

Fix  $t \geq 0$ . Since we are dealing with non-decreasing processes and since the asserted limiting process is continuous, the convergence of the finite-dimensional marginals suffices to prove the theorem (see for instance Th VI.3.37 in [45]). By Proposition IV.3, we know that  $u^Q(t, \lambda) = \lambda e^{-Dt}$  is the function related to  $\Psi^Q$  via (IV.2). Hence we only need to prove that for all  $n \geq 1$ , all  $n$ -uplets  $0 \leq t_1 \leq \dots \leq t_n \leq t$  and all coefficients  $\lambda_1, \dots, \lambda_n > 0$  we have

$$\lim_{s \rightarrow \infty} -\frac{1}{x} \log \mathbb{E}_x[e^{-\lambda_1 Z_{t_1} - \dots - \lambda_n Z_{t_n}} | \mathbf{T} > t + s] = \lambda_1 d_{t_1} + \dots + \lambda_n d_{t_n} \quad (\text{IV.19})$$

Thanks to an easy recursion, we get

$$\begin{aligned} & -\frac{1}{x} \log \mathbb{E}_x[e^{-\lambda_1 Z_{t_1} - \dots - \lambda_n Z_{t_n}} | \mathbf{T} > t + s] \\ &= u\left(t_1, \lambda_1 + u\left(t_2 - t_1, \lambda_2 + \dots + u\left(t_n - t_{n-1}, \lambda_n + u(t + s - t_n, 0+)\right) \dots\right)\right) - u(t + s, 0+) \end{aligned}$$

To prove (IV.19), we proceed via a recurrence on  $n$ . We check the case  $n = 1$ . Recall that  $u(t, \lambda)/\lambda \rightarrow d_t$  as  $\lambda \rightarrow \infty$ . Then the concavity of  $\lambda \rightarrow u(t, \lambda)$  (that can be directly checked from (IV.15)) implies that  $\partial_\lambda u(t, \lambda) \rightarrow d_t$  as  $\lambda \rightarrow \infty$ . Writing  $u(t + s, 0+) = u(t_1, u(t + s - t_1, 0+))$ , the preceding arguments and the fact that  $u(t + s - t_1, 0+) = a_{t+s-t_1} \rightarrow \infty$  as  $s \rightarrow \infty$  entail

$$u\left(t_1, \lambda_1 + u(t + s - t_1, 0+)\right) - u(t + s, 0+) \rightarrow \lambda_1 d_{t_1} \text{ as } s \rightarrow \infty$$

Suppose now that the result holds at rank  $n - 1 \geq 1$ , that is, (IV.19) holds true for all  $(n - 1)$ -uplets of times and coefficients. In particular

$$\begin{aligned} & u\left(t_2 - t_1, \lambda_2 + \dots + u\left(t_n - t_{n-1}, \lambda_n + u(t + s - t_n, 0+)\right) \dots\right) - u(t + s - t_1, 0+) \\ & \underset{s \rightarrow \infty}{\sim} \lambda_2 d_{t_2 - t_1} + \dots + \lambda_n d_{t_n - t_1} \end{aligned}$$

Therefore the argument of the case  $n = 1$  applies and shows that

$$\begin{aligned} & u\left(t_1, \lambda_1 + u\left(t_2 - t_1, \lambda_2 + \dots + u\left(t_n - t_{n-1}, \lambda_n + u(t + s - t_n, 0+)\right) \dots\right)\right) - u(t + s, 0+) \\ & \underset{s \rightarrow \infty}{\sim} \lambda_1 d_{t_1} + \lambda_2 d_{t_1} d_{t_2 - t_1} + \dots + \lambda_n d_{t_1} d_{t_n - t_1} \end{aligned}$$

which is the desired result since  $d_{r+r'} = d_r d_{r'}$  for all  $r, r' \geq 0$  by Proposition IV.3. ■

### 3.3 Proof of Theorem IV.3

Recall the notation  $a_t = u(t, 0+)$  and that  $a_t \rightarrow \infty$  as  $t \rightarrow \infty$ . Since  $u \mapsto \Psi(u)/u$  is strictly increasing from  $-\infty$  to  $D$ , there exists a positive function  $f$  such that

$$\Psi(f(t)^{-1})f(t) \sim \Psi(a_t)$$

as  $t \rightarrow \infty$ . Since  $\Psi(a_t) \rightarrow -\infty$  as  $t \rightarrow \infty$ , necessarily  $f(t) \rightarrow \infty$ . Fix  $\lambda, x \in (0, \infty)$ . For any  $t \in (0, \infty)$ , we have

$$-\frac{1}{x} \log \mathbb{E}_x[e^{-\lambda Z_t/f(t)} \mid t < T] = u(t, \lambda f(t)^{-1}) - a_t$$

We rely on two lemmas, whose proofs are postponed to the end of the subsection.

**LEMMA IV.4.** *As  $u \downarrow 0$ , we have  $\Phi(u) \sim u/(-\alpha\Psi(u))$ .*

Since  $f(t) \rightarrow +\infty$  as  $t \rightarrow \infty$  the lemma implies

$$\Psi(a_t)\Phi(\lambda f(t)^{-1}) \underset{t \rightarrow \infty}{\sim} -\frac{\Psi(a_t)\lambda}{\alpha f(t)\Psi(\lambda f(t)^{-1})}$$

Since  $L$  is slowly varying at  $0+$ , we deduce that  $\Psi(\lambda f(t)^{-1}) \sim \lambda^{1-\alpha}\Psi(f(t)^{-1})$  as  $t \rightarrow \infty$ . Thus the very definition of  $f$  entails

$$\Psi(a_t)\Phi(\lambda f(t)^{-1}) \underset{t \rightarrow \infty}{\sim} -\lambda^\alpha \alpha^{-1} \tag{IV.20}$$

**LEMMA IV.5.** *The following holds true as  $t \rightarrow \infty$*

$$\int_{u(t, \lambda f(t)^{-1})}^{a_t} \frac{dv}{\Psi(v)} \sim \int_{u(t, \lambda f(t)^{-1})}^{a_t} \frac{dv}{\Psi(a_t)}$$

From the latter lemma, we deduce

$$\begin{aligned} u(t, \lambda f(t)^{-1}) - a_t &\underset{t \rightarrow \infty}{\sim} -\Psi(a_t) \int_{u(t, \lambda f(t)^{-1})}^{a_t} \frac{dv}{\Psi(v)} = -\Psi(a_t)\Phi(\lambda f(t)^{-1}) \\ &\underset{t \rightarrow \infty}{\sim} \lambda^\alpha \alpha^{-1} \end{aligned}$$

where we use (IV.20) at the second line. The theorem is proved. ■

*Proof of Lemma IV.4.* Recall the definition of  $\Phi$ . An integration by parts yields that for all  $u \in [0, \infty)$

$$\Phi(u) = -\frac{u}{\Psi(u)} + \int_u^0 \frac{1}{\Psi(v)} \frac{v\Psi'(v)}{\Psi(v)} dv$$

Recall from Theorem 2 in [56] that  $v\Psi'(v)/\Psi(v) \rightarrow 1 - \alpha$  as  $v \downarrow 0$ . Therefore an elementary calculation ends the proof. ■

*Proof of Lemma IV.5.* For all  $t \in [0, \infty)$ ,  $a_t \leq u(t, \lambda f(t)^{-1})$ . We write

$$\int_{a_t}^{u(t, \lambda f(t)^{-1})} \frac{dv}{\Psi(v)} - \int_{a_t}^{u(t, \lambda f(t)^{-1})} \frac{dv}{\Psi(a_t)} = \int_{a_t}^{u(t, \lambda f(t)^{-1})} \frac{\Psi(a_t) - \Psi(v)}{\Psi(v)\Psi(a_t)} dv$$

The convexity of  $\Psi$  implies that for all  $v \in [a_t, u(t, \lambda f(t)^{-1})]$  we have

$$0 \leq \Psi(a_t) - \Psi(v) \leq -\Psi'(a_t)(v - a_t) \tag{IV.21}$$

Suppose that  $t \mapsto u(t, \lambda f(t)^{-1}) - a_t$  is bounded for large times. The fact that  $\Psi'(v)/\Psi(v)$  goes to 0 as  $v \rightarrow \infty$  together with (IV.21) then entail

$$\begin{aligned} 0 \leq \int_{a_t}^{u(t, \frac{\lambda}{f(t)})} \frac{\Psi(a_t) - \Psi(v)}{\Psi(v)\Psi(a_t)} dv &\leq -\left(u\left(t, \frac{\lambda}{f(t)}\right) - a_t\right) \frac{\Psi'(a_t)}{\Psi(a_t)} \int_{a_t}^{u(t, \frac{\lambda}{f(t)})} \frac{dv}{\Psi(v)} \\ &\leq_{t \rightarrow \infty} o\left(\int_{a_t}^{u(t, \frac{\lambda}{f(t)})} \frac{dv}{\Psi(v)}\right) \end{aligned}$$

which in turn proves the lemma. We are left with the proof of the boundedness of  $t \mapsto u(t, \lambda f(t)^{-1}) - a_t$  for large times. Fix  $k \in (-D, \infty)$ . Since  $\Psi'(v) \uparrow D$  as  $v \rightarrow \infty$ , for  $t$  large enough we get from (IV.21) that  $\Psi(v) \geq \Psi(a_t) - k(v - a_t)$  for all  $v \in [a_t, u(t, \lambda f(t)^{-1})]$ . A simple calculation then yields

$$0 \leq \frac{1}{k} \log\left(1 - k \frac{u(t, \lambda f(t)^{-1}) - a_t}{\Psi(a_t)}\right) \leq \int_{u(t, \lambda f(t)^{-1})}^{a_t} \frac{dv}{\Psi(v)} = \Phi(\lambda f(t)^{-1})$$

Using  $\log(1 + v) \geq v/2$  for  $v$  small and since  $\Phi(\lambda f(t)^{-1}) \rightarrow 0$ , we get for  $t$  large enough

$$0 \leq -\frac{u(t, \lambda f(t)^{-1}) - a_t}{2\Psi(a_t)} \leq \Phi(\lambda f(t)^{-1})$$

From (IV.20), we deduce that  $t \mapsto u(t, \lambda f(t)^{-1}) - a_t$  is bounded for large times. ■

## 4 Proof of Proposition IV.2

The proof is inspired by that of Theorem 1 in [73] but for completeness we give all the details. Recall that  $\Psi(u) = u^{1+\alpha}L(u)$  with  $L$  slowly varying at 0 and  $\alpha \in (0, 1)$  and that  $T_0 < \infty$  almost surely: consequently  $q = 0$  and (IV.6) holds true. Recall (IV.16). We set for all  $t \geq 0$ ,  $v(t) := u(t, +\infty)$  which is finite by (IV.6). Observe that  $v$  is decreasing from  $+\infty$  to 0. Grey p. 672 [40] proved that

$$\forall t \geq 0, x > 0, \mathbb{P}_x(t \geq T) = e^{-xv(t)} \quad (\text{IV.22})$$

Since  $\Psi(u)/u \rightarrow \infty$  as  $u \rightarrow \infty$  we get for all  $r > 0$

$$\begin{aligned} \frac{v(r)}{r\Psi(v(r))} &= \frac{1}{r} \int_0^r \partial_s \left( \frac{v(s)}{\Psi(v(s))} \right) ds = \frac{1}{r} \int_0^r \partial_s v(s) \frac{\Psi(v(s)) - v(s)\Psi'(v(s))}{\Psi(v(s))^2} ds \\ &= \frac{1}{r} \int_0^r \left( \frac{v(s)\Psi'(v(s))}{\Psi(v(s))} - 1 \right) ds \end{aligned}$$

where we use the identity  $\partial_s v(s) = -\Psi(v(s))$  at the second line. Since  $\Psi$  is regularly varying at 0, Theorem 2 in [56] entails that  $u\Psi'(u)/\Psi(u) \rightarrow 1 + \alpha$  as  $u \downarrow 0$ . Taking the limit  $r \rightarrow \infty$  in the above identity, one gets

$$v(r)^\alpha L(v(r)) \sim \frac{1}{\alpha r} \text{ as } r \rightarrow \infty \quad (\text{IV.23})$$

Since  $v$  is a bijection from  $(0, \infty)$  onto itself, for any  $t \in (0, \infty)$  there exists a unique  $s(t) = s \in (0, \infty)$  such that  $v(s) = \lambda f(t)$ . From the assumption  $f(t) \sim v(t)$  as  $t \rightarrow \infty$ , we deduce that  $s \rightarrow \infty$  as  $t \rightarrow \infty$ . We use (IV.23) and the slowness of the variation of  $L$  to get as  $t \rightarrow \infty$

$$\frac{t}{s} \sim \frac{v(s)^\alpha L(v(s))}{v(t)^\alpha L(v(t))} \sim \frac{\lambda^\alpha f(t)^\alpha L(\lambda f(t))}{f(t)^\alpha L(f(t))} \sim \lambda^\alpha$$

Hence  $\lambda^\alpha s \sim t$  as  $t \rightarrow \infty$ . Using  $\partial_r v(r) = -\Psi(v(r))$  and (IV.23), we obtain for all  $t > 0$

$$\log \left( \frac{v(t+s)}{v(t)} \right) = \int_t^{t+s} \frac{\partial_r v(r)}{v(r)} dr = - \int_t^{t+s} v(r)^\alpha L(v(r)) dr \underset{t \rightarrow \infty}{\sim} -\frac{1}{\alpha} \log(1 + \lambda^{-\alpha})$$

Using the above results, (IV.22) and the identity  $u(t, \lambda f(t)) = u(t, v(s)) = v(t+s)$  we get for all  $t > 0$

$$\begin{aligned} \mathbb{E}_x \left[ e^{-\lambda Z_t f(t)} \mid t < \mathbb{T} \right] &= \frac{\mathbb{E}_x \left[ e^{-\lambda Z_t f(t)} \right] - \mathbb{P}_x(t \geq \mathbb{T})}{\mathbb{P}_x(t < \mathbb{T})} = \frac{e^{-x u(t, \lambda f(t))} - e^{-x v(t)}}{1 - e^{-x v(t)}} \\ &\underset{t \rightarrow \infty}{\sim} 1 - \frac{v(t+s)}{v(t)} \underset{t \rightarrow \infty}{\sim} 1 - (1 + \lambda^{-\alpha})^{-1/\alpha} \end{aligned}$$

This ends the proof. ■

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Alternative construction of the tree length of an evolving Kingman  
coalescent

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In [65], Pfaffelhuber, Wakolbinger and Weisshaupt considered the evolving Kingman coalescent obtained from the lookdown representation of the standard Fleming-Viot process. At any time  $t$ , when tracing backward in time the lineages of all the individuals one gets a Kingman coalescent tree  $\mathcal{T}_t$ . The goal of their work was to study the process of lengths of this coalescent, that is, the sum of the branch lengths of  $\mathcal{T}_t$  when  $t$  varies in  $\mathbb{R}$ . Given that the length of such a tree is infinite almost surely, they defined a compensated tree length process  $(\mathcal{L}_t^n, t \in \mathbb{R})$  by restricting the lookdown process to its  $n$  first levels, for all  $n \geq 1$  and subtracting to the restricted tree length (an equivalent of) its mean. They proved the convergence in probability of  $(\mathcal{L}_t^n, t \in \mathbb{R})_{n \geq 1}$  to a càdlàg process  $(\mathcal{L}_t, t \in \mathbb{R})$ , called the compensated tree length of the evolving Kingman coalescent (see Theorem V.1). In this chapter, we propose an alternative construction based on the erasure of a branch length  $\epsilon > 0$  from each terminal branch of the trees  $\mathcal{T}_t$ , then obtaining a collection  $(\mathcal{T}_t^\epsilon, t \in \mathbb{R})_{\epsilon > 0}$  of càdlàg tree-valued processes. By defining a collection of (suitably) compensated tree length processes based on those trees, we are able to obtain a strong limiting process  $(\mathcal{L}_t, t \in \mathbb{R})$  (see Theorem V.2). Furthermore, this process is almost surely equal to  $(\mathcal{L}_t, t \in \mathbb{R})$  (see Theorem V.3).

In a first section, we recall precisely the results obtained in [65] and expose our results. The proofs are provided in the second section.

## 1 Statement of the result

Let  $\hat{\Pi}$  be a flow of partitions associated to  $\Lambda(du) = \delta_0(du)$ . At any time  $t \in \mathbb{R}$ , we consider the tree  $\mathcal{T}_t$  defined from the coalescent  $s \mapsto \hat{\Pi}_{t-s,t}$ , along with its restriction  $\mathcal{T}_t^n$  to  $[n]$  obtained from the  $n$ -coalescent  $s \mapsto \hat{\Pi}_{t-s,t}^{[n]}$ . We denote by  $\tilde{l}$  the operator that associates to a finite real tree its length, that is, the sum of its branch lengths.

Let us now recall the compensated tree length process as defined in [65]. For all  $n \geq 2, t \in \mathbb{R}$

$$\mathcal{L}_t^n := \tilde{l}(\mathcal{T}_t^n) - 2 \log(n) \tag{V.1}$$

The main result of [65] (see Proposition 2.1, Theorem 1 and Proposition 3.2) can be restated as

**THEOREM V.1.** (Pfaffelhuber, Wakolbinger, Weisshaupt [65]) *There exists a càdlàg process  $\mathcal{L}$*

such that

$$\mathcal{L}^n \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \mathcal{L} \quad (\text{V.2})$$

in the Skorohod's topology. Moreover, for all  $t \in \mathbb{R}$  the r.v.  $\frac{1}{2}\mathcal{L}_t$  follows the Gumbel distribution.

where  $\xrightarrow{(\mathbb{P})}$  denotes convergence in probability. Let us now present our alternative procedure. Fix  $\epsilon > 0$  and define for all  $t \in \mathbb{R}$

$$S_t^\epsilon := \#\hat{\Pi}_{t-\epsilon, t}$$

This is the number of blocks in the coalescent  $r \mapsto \hat{\Pi}_{t-r, t}$  at time  $\epsilon$ . We denote by  $\mathcal{T}_t^\epsilon$  the tree obtained from the coalescent  $s \mapsto \hat{\Pi}_{t-\epsilon-s, t-\epsilon}^{[S_t^\epsilon]}$ , that is, from the coalescent  $s \mapsto \hat{\Pi}_{t-\epsilon-s, t-\epsilon}$  restricted to the  $S_t^\epsilon$  first integers. Also let

$$\mathcal{T}_t := \mathcal{T}_t \quad (\text{V.3})$$

The tree  $\mathcal{T}_t^\epsilon$  can be obtained by erasing a branch length  $\epsilon$  from each leaves of  $\mathcal{T}_t$ . This procedure yields for each  $\epsilon > 0$  a tree-valued process  $(\mathcal{T}_t^\epsilon, t \in \mathbb{R})$ . We then define for all  $t \in \mathbb{R}$

$$\mathcal{L}_t^\epsilon := \tilde{l}(\mathcal{T}_t^\epsilon) + 2 \log\left(\frac{\epsilon}{2}\right) \quad (\text{V.4})$$

We now state our main result

**THEOREM V.2.** *There exists a càdlàg process  $\mathcal{L}$  such that*

$$\mathcal{L}^\epsilon \xrightarrow[\epsilon \downarrow 0]{(\mathbb{P})} \mathcal{L} \quad (\text{V.5})$$

in the Skorohod's topology. Moreover, for all  $t \in \mathbb{R}$  the r.v.  $\frac{1}{2}\mathcal{L}_t$  follows the Gumbel distribution.

A natural question is to compare the two limiting processes  $\mathcal{L}$  and  $\mathcal{L}$ . Since they are related to the same tree-valued process  $(\mathcal{T}_t, t \in \mathbb{R})$ , we expect them to be equal.

**THEOREM V.3.** *Almost surely  $\mathcal{L} = \mathcal{L}$*

## 2 Proofs

### 2.1 Preliminary results

We start with two results of the literature we are going to rely on.

**LEMMA V.1.** *(Pfaffelhuber, Wakolbinger, Weisshaupt - Lemma 4.6 in [65]) Almost surely and in  $L^2$ ,*

$$\epsilon S_t^\epsilon \xrightarrow[\epsilon \downarrow 0]{} 2$$

For  $u, v \rightarrow 0$ ,  $u \leq v$  and  $u/v \rightarrow \Gamma \leq 1$

$$\left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}}, \frac{S_t^v - 2/v}{\sqrt{2/(3v)}} \right) \xrightarrow{(d)} \mathcal{N}(0, C)$$

where  $C = \begin{pmatrix} 1 & \Gamma^{3/2} \\ \Gamma^{3/2} & 1 \end{pmatrix}$

**LEMMA V.2.** *(Depperschmidt, Greven, Pfaffelhuber - Theorem 1 in [23]) Almost surely*

$$\sup_{t \in \mathbb{R}} |\epsilon S_t^\epsilon - 2| \xrightarrow[\epsilon \downarrow 0]{} 0$$

We now state a lemma needed later on.

**LEMMA V.3.** *Fix  $t \in \mathbb{R}$ . For  $u, v \rightarrow 0$ ,  $u \leq v$  and  $u/v \rightarrow \Gamma \leq 1$ , we have the following convergence*

$$\mathbb{E} \left[ \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} \cdot \frac{S_t^v - 2/v}{\sqrt{2/(3v)}} \right] \rightarrow \Gamma^{3/2}$$

**Proof** Since this r.v. converges in distribution towards  $\Gamma^{3/2}$ , uniform integrability would imply  $L^1$  convergence and therefore the asserted convergence. A classical result ensures that if

$$\mathbb{E} \left[ \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} \cdot \frac{S_t^v - 2/v}{\sqrt{2/(3v)}} \right)^{3/2} \right]$$

is uniformly bounded as  $u, v \rightarrow 0$ ,  $u/v \rightarrow \Gamma$  then uniform integrability holds. Using Cauchy-Schwartz inequality, it suffices to show a uniform bound on

$$\mathbb{E} \left[ \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} \right)^3 \right]$$

We proceed as follows. We use the notation  $x_+ := x \vee 0$  for any given real number  $x$  and we bound separately the positive and negative parts of  $\left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} \right)^3$ .

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} \right)_+^3 \right] &\leq \mathbb{P} \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} \in [0, 1] \right) + \int_1^{+\infty} \mathbb{P} \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} > x \right) 3x^2 dx \\ &\leq 1 + \int_1^{+\infty} \mathbb{P} \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} > x \right) 3x^2 dx \end{aligned}$$

Following [65], we define  $T^k$  as the time needed by the Kingman coalescent to come down to  $k$  blocks: it is an independent sum of an infinite collection of exponential r.v. with parameters  $\binom{k+1}{2}, \binom{k+2}{2}, \dots$ . We also introduce the mapping

$$f_u(x) := \left\lfloor \frac{2}{u} + x \sqrt{\frac{2}{3u}} \right\rfloor, \forall u > 0$$

and write

$$\begin{aligned} \mathbb{P} \left( \frac{S_t^u - 2/u}{\sqrt{2/(3u)}} > x \right) &= \mathbb{P}(S_t^u > f_u(x)) = \mathbb{P}(T^{f_u(x)} > u) \\ &= \mathbb{P} \left( \frac{T^{f_u(x)} - 2/f_u(x)}{\sqrt{4/(3f_u(x)^3)}} > \frac{u - 2/f_u(x)}{\sqrt{4/(3f_u(x)^3)}} \right) \\ &\leq \mathbb{E} \left[ \left( \frac{T^{f_u(x)} - 2/f_u(x)}{\sqrt{4/(3f_u(x)^3)}} \right)^4 \right] \times \left( \frac{\sqrt{4/(3f_u(x)^3)}}{u - 2/f_u(x)} \right)^4 \\ &\leq \frac{C}{f_u(x)^6 (u - 2/f_u(x))^4} \end{aligned}$$

where the last inequality comes from the bound  $\mathbb{E}[(T^n - 2/n)^4] \leq C/n^6$  where  $C > 0$ . We need to find a uniform (in  $u$ ) upper bound for

$$\int_1^{+\infty} \frac{3x^2 C dx}{f_u(x)^2 (f_u(x)u - 2)^4}$$



Remark that for all  $u \in (0, 1/4)$  and all  $x \geq 1$ , we have

$$\begin{aligned} f_u(x)^2(f_u(x)u - 2)^4 &\geq \left(\frac{2}{u} + x\sqrt{\frac{2}{3u}} - 1\right)^2 \left(u\left(\frac{2}{u} + x\sqrt{\frac{2}{3u}} - 1\right) - 2\right)^4 \\ &\geq \left(2 + x\sqrt{\frac{2u}{3}} - u\right)^2 \left(x\sqrt{\frac{2}{3}} - \sqrt{u}\right)^4 \geq \left(x\sqrt{\frac{2}{3}} - \frac{1}{2}\right)^4 \end{aligned}$$

and the uniform bound follows. The bound for the negative part of  $\left(\frac{S_t^u - 2/u}{\sqrt{2/(3u)}}\right)^3$  is similar.  $\blacksquare$

## 2.2 Convergence of the sequence of processes

We decompose this proof into three parts. First we prove the strong convergence of the one-dimensional marginals. Then, we prove the tightness in  $\mathbb{D}(\mathbb{R}, \mathbb{R})$  of the collection of càdlàg processes. Finally, we conclude with the convergence in probability.

### One-dimensional marginals

**LEMMA V.4.** *For all  $t \in \mathbb{R}$ ,  $\frac{1}{2}\mathcal{L}_t^\epsilon$  converges in probability as  $\epsilon \downarrow 0$  to a random variable with the Gumbel distribution.*

**Proof** Fix  $t \in \mathbb{R}$ . We start with the convergence in distribution towards a Gumbel r.v. We have

$$\frac{1}{2}\mathcal{L}_t^\epsilon = \frac{1}{2}\tilde{l}(\mathcal{T}_t^\epsilon) - \log(S_t^\epsilon) + \log(S_t^\epsilon) + \log\left(\frac{\epsilon}{2}\right) \quad (\text{V.6})$$

Since  $\log(S_t^\epsilon) + \log(\epsilon/2)$  tends to 0 almost surely as  $\epsilon \downarrow 0$ , it suffices to prove the convergence in distribution of  $\frac{1}{2}\tilde{l}(\mathcal{T}_t^\epsilon) - \log(S_t^\epsilon)$ .

Since the Kingman coalescent is a Markov process, we deduce that for each  $n \geq 2$ , the conditional distribution of  $\mathcal{T}_t^\epsilon$  given  $\{S_t^\epsilon = n\}$  is the same as  $\mathcal{T}_t^n$ . Then, we have for all  $x \geq 0$

$$\begin{aligned} \mathbb{P}\left(\frac{1}{2}\tilde{l}(\mathcal{T}_t^\epsilon) - \log(S_t^\epsilon) < x\right) &= \sum_{n=2}^{\infty} \mathbb{P}\left(\frac{1}{2}\tilde{l}(\mathcal{T}_t^\epsilon) - \log(S_t^\epsilon) < x \mid S_t^\epsilon = n\right) \mathbb{P}(S_t^\epsilon = n) \\ &= \sum_{n=2}^{\infty} \mathbb{P}\left(\frac{1}{2}\tilde{l}(\mathcal{T}_t^n) - \log(n) < x\right) \mathbb{P}(S_t^\epsilon = n) \\ &= \sum_{n=2}^{\infty} (1 - \exp(-x)/n)^{n-1} \mathbb{P}(S_t^\epsilon = n) \end{aligned}$$

where the last equality comes from an easy calculation based on exponential distributions of branch lengths of a Kingman coalescent tree (see for instance the proof of Proposition 2.1 in [65]). Since  $(1 - \exp(-x)/n)^{n-1} \xrightarrow[n \rightarrow \infty]{} \exp(-\exp(-x))$  and  $S_t^\epsilon \xrightarrow[\epsilon \downarrow 0]{} \infty$  a.s., we get

$$\mathbb{P}\left(\frac{1}{2}\tilde{l}(\mathcal{T}_t^\epsilon) - \log(S_t^\epsilon) < x\right) \xrightarrow[\epsilon \downarrow 0]{} \exp(-\exp(-x))$$

Thus, the limiting distribution is Gumbel. Now let us prove the convergence in probability. Fix  $0 < \epsilon < \epsilon'$ , and remark that

$$\tilde{l}(\mathcal{T}_t^\epsilon) + 2\log\left(\frac{\epsilon}{2}\right) - \tilde{l}(\mathcal{T}_t^{\epsilon'}) - 2\log\left(\frac{\epsilon'}{2}\right) = \int_{\epsilon}^{\epsilon'} \left(S_t^u - \frac{2}{u}\right) du \quad (\text{V.7})$$

We have

$$\begin{aligned}
 \mathbb{E}\left[(\mathcal{L}_t^\epsilon - \mathcal{L}_t^{\epsilon'})^2\right] &= \mathbb{E}\left[\left(\int_\epsilon^{\epsilon'} (S_t^u - \frac{2}{u}) du\right)^2\right] \\
 &= 2\mathbb{E}\left[\int_{v \in [\epsilon, \epsilon']} \int_{u \in [\epsilon, v]} (S_t^u - \frac{2}{u})(S_t^v - \frac{2}{v}) du dv\right] \\
 &= 2\int_{v \in [\epsilon, \epsilon']} \int_{u \in [\epsilon, v]} \mathbb{E}\left[\left(S_t^u - \frac{2}{u}\right)\left(S_t^v - \frac{2}{v}\right)\right] du dv \\
 &\stackrel{\epsilon' \downarrow 0}{\sim} 2\int_{v \in [\epsilon, \epsilon']} \int_{u \in [\epsilon, v]} \sqrt{\frac{2}{3u} \frac{2}{3v}} \left(\frac{u}{v}\right)^{3/2} du dv \\
 &\stackrel{\epsilon' \downarrow 0}{\sim} \frac{2}{3}(\epsilon' - 2\epsilon + \epsilon^2/\epsilon')
 \end{aligned}$$

where we use Fubini's theorem at third line and Lemma V.3 at fourth line. Therefore  $(\mathcal{L}_t^\epsilon)_{\epsilon > 0}$  is a Cauchy sequence in  $L^2$  so it converges in  $L^2$  as  $\epsilon \downarrow 0$ , and a fortiori it converges in probability.  $\blacksquare$

### Tightness

We write

$$\mathcal{L}_t^\epsilon = \underbrace{\left(\tilde{l}(\mathcal{T}_t^\epsilon) - \sum_{i=2}^{S_t^\epsilon} \frac{2}{i-1}\right)}_{X_t^\epsilon} + \left(\sum_{i=2}^{S_t^\epsilon} \frac{2}{i-1} + 2\log\left(\frac{\epsilon}{2}\right)\right)$$

It is elementary to check, thanks to Lemma V.2, that  $t \mapsto \mathcal{L}_t^\epsilon - X_t^\epsilon$  converges uniformly towards twice the Euler constant almost surely. Therefore, to obtain tightness of the collection  $(\mathcal{L}_t^\epsilon)_{\epsilon > 0}$  it is sufficient (see Theorems 3.8.6 and 3.8.8 in [31]) to prove that

(i)  $(X_t^\epsilon)_{\epsilon > 0}$  is tight in  $\mathbb{R}$

(ii)  $\exists \beta > 0, \exists \theta > 1, \forall \epsilon > 0, \forall t \in \mathbb{R}, \forall h \in (0, 1], \mathbb{E}[1 \wedge |X_{t+h}^\epsilon - X_t^\epsilon|^\beta \wedge |X_t^\epsilon - X_{t-h}^\epsilon|^\beta] \lesssim h^\theta$

where  $a(h) \lesssim b(h)$  iff  $\exists C > 0$ , independent of  $h$ , such that  $a(h) \leq C b(h)$  for all  $h > 0$ ,  $a$  and  $b$  being two real-valued functions.

From Lemma V.4, we deduce (i). Let us focus on (ii). We fix  $\epsilon > 0$  and follow the proof of [65].

#### Step 1

We prove an inequality useful for the second step. For all  $i \geq 2$  and all  $t \in \mathbb{R}$ , we define  $T_i(t)$  as the time spent by the tree  $\mathcal{T}_t$  with exactly  $i$  branches: it is an exponential r.v. with parameter  $\binom{i}{2}$ . When  $h$  is small, the trees  $\mathcal{T}_t$  and  $\mathcal{T}_{t+h}$  tend to be similar. To quantify this similarity we set

$$\forall h > 0, D_{(t, t+h]} = \min\{i \geq 2 : T_i(t) \neq T_i(t+h)\}$$

If we start from the root of the trees, then  $\mathcal{T}_t$  and  $\mathcal{T}_{t+h}$  do not differ until they have  $D_{(t, t+h]}$  branches. We stress that for all integer  $n \geq 2$ , we have

$$\mathbb{P}(D_{(t-h, t]} \vee D_{(t, t+h]} < n) \lesssim (n^4 h^2) \wedge 1 \tag{V.8}$$

Indeed consider the lookdown graph restricted to the  $n$  first levels. If no elementary reproduction event involving two levels in  $[n]$  occurs on the time interval  $[t, t+h]$  then  $D_{(t, t+h]} \geq n$ . Conversely if  $D_{(t, t+h]} \geq n$  then no coalescence event has involved two branches among the  $n$  first and therefore no elementary reproduction event involving two levels in  $[n]$  occurs on the time interval  $[t, t+h]$ . The probability of this event is  $\exp(-h \binom{n}{2})$ . Finally the independence of the increments of the lookdown

graph ensures the asserted formula.

**Step 2**

Set  $\beta := 10$  and  $\theta := 10/9$ .

$$\begin{aligned} & \mathbb{E}[1 \wedge (X_{t+h}^\epsilon - X_t^\epsilon)^{10} \wedge (X_t^\epsilon - X_{t-h}^\epsilon)^{10}] \\ & \leq \mathbb{E}[1 \wedge (X_{t+h}^\epsilon - X_t^\epsilon)^{10}; D_{(t-h,t]} \leq D_{(t,t+h)}] + \mathbb{E}[1 \wedge (X_t^\epsilon - X_{t-h}^\epsilon)^{10}; D_{(t-h,t]} \geq D_{(t,t+h)}] \end{aligned} \quad (\text{V.9})$$

We restrict ourselves to bounding the first term on the right hand side, since we would proceed in the same manner for the second term.

$$\begin{aligned} & \mathbb{E}[1 \wedge (X_{t+h}^\epsilon - X_t^\epsilon)^{10}; D_{(t-h,t]} \leq D_{(t,t+h)}] \\ & \leq \mathbb{P}[D_{(t-h,t]} \leq D_{(t,t+h)} < h^{-2/9}] + \mathbb{E}[(X_{t+h}^\epsilon - X_t^\epsilon)^{10}; D_{(t-h,t]} \geq h^{-2/9}] \\ & \lesssim h^{10/9} + \mathbb{E}\left[\left(\sum_{i=\lfloor h^{-2/9} \rfloor \vee 2}^{S_{t+h}^\epsilon} i(T_i(t+h) - \frac{1}{\binom{i}{2}}) - \sum_{i=\lfloor h^{-2/9} \rfloor \vee 2}^{S_t^\epsilon} i(T_i(t) - \frac{1}{\binom{i}{2}})\right)^{10}\right] \end{aligned} \quad (\text{V.10})$$

where we use Equation (V.8) for the last inequality. It remains to bound the second term on the right hand side. Let us introduce for all  $n \geq 1$

$$a_n(h) := \sum_{i=\lfloor h^{-2/9} \rfloor \vee 2}^{\infty} \mathbb{E}\left[\left(i(T_i(t) - \frac{1}{\binom{i}{2}})\right)^n\right] \quad (\text{V.11})$$

A simple calculation shows that  $a_1(h) = 0$  for all  $h > 0$  and that for all  $n \geq 2$  there exists  $C_n > 0$  such that for all  $h > 0$ ,  $a_n(h) \leq C_n h^{2(n-1)/9}$ . Now remark that  $(x - y)^n \leq (2x)^n + (2y)^n$  for any even integer  $n$  and all  $x, y \in \mathbb{R}$ . Thus

$$\begin{aligned} & \mathbb{E}\left[\left(\sum_{i=\lfloor h^{-1/9} \rfloor \vee 2}^{S_{t+h}^\epsilon} i(T_i(t+h) - \frac{1}{\binom{i}{2}}) - \sum_{i=\lfloor h^{-1/9} \rfloor \vee 2}^{S_t^\epsilon} i(T_i(t) - \frac{1}{\binom{i}{2}})\right)^{10}\right] \\ & \leq 2^{10} \mathbb{E}\left[\left(\sum_{i=\lfloor h^{-1/9} \rfloor \vee 2}^{S_{t+h}^\epsilon} i(T_i(t+h) - \frac{1}{\binom{i}{2}})\right)^{10}\right] + 2^{10} \mathbb{E}\left[\left(\sum_{i=\lfloor h^{-1/9} \rfloor \vee 2}^{S_t^\epsilon} i(T_i(t) - \frac{1}{\binom{i}{2}})\right)^{10}\right] \\ & \lesssim \sum_{k=2}^{10} \sum_{\substack{n_1, \dots, n_k \\ n_1 + \dots + n_k = 10}} a_{n_1}(h) \dots a_{n_k}(h) \lesssim h^{10/9} \end{aligned} \quad (\text{V.12})$$

By coalescing Equations (V.9), (V.10) and (V.12), we get

$$\mathbb{E}[1 \wedge (X_{t+h}^\epsilon - X_t^\epsilon)^{10} \wedge (X_t^\epsilon - X_{t-h}^\epsilon)^{10}] \lesssim h^{10/9} \quad (\text{V.13})$$

This completes the proof of the tightness.

**Convergence in probability**

From the strong convergence of the one-dimensional marginals, we deduce the convergence in distribution of the finite dimensional marginals. Together with the tightness, we conclude that  $\mathcal{L}^\epsilon$  converge in distribution as  $\epsilon \downarrow 0$ . Using Proposition 6.1 of [65], we conclude to the existence of a càdlàg process  $\mathcal{L}$  such that

$$\mathcal{L}^\epsilon \xrightarrow[\epsilon \downarrow 0]{(\mathbb{P})} \mathcal{L} \quad (\text{V.14})$$

### 2.3 Proof of Theorem V.3

Since both processes are càdlàg, it suffices to prove that the one-dimensional marginals are almost surely equal to prove the theorem. Fix  $t \in \mathbb{R}$  and let  $\epsilon_n := n^{-3}$ ,  $n \in \mathbb{N}$ . We will prove that

$$\mathcal{L}_t^{\epsilon_n} - \mathcal{L}_t^n \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0 \quad (\text{V.15})$$

As our processes are stationary, time plays no rôle in this proof. Therefore we omit the subscript  $t$ . Also to alleviate notation, we write  $\epsilon$  rather than  $\epsilon_n$ . We have

$$\begin{aligned} \mathcal{L}^\epsilon - \mathcal{L}^n &= \tilde{l}(\mathcal{T}^\epsilon) + 2 \log\left(\frac{\epsilon}{2}\right) - \tilde{l}(\mathcal{T}^n) + 2 \log(n) \\ &= \tilde{l}(\mathcal{T}^\epsilon \setminus \mathcal{T}^n) - \tilde{l}(\mathcal{T}^n \setminus \mathcal{T}^\epsilon) + 2 \log\left(\frac{n\epsilon}{2}\right) \end{aligned}$$

Since  $\mathcal{T}^n \setminus \mathcal{T}^\epsilon$  has at most  $n$  branches on a period of time  $\epsilon$ , we have almost surely

$$\tilde{l}(\mathcal{T}^n \setminus \mathcal{T}^\epsilon) \leq n\epsilon \xrightarrow[n \rightarrow \infty]{} 0$$

Therefore we focus on  $Y_n := \tilde{l}(\mathcal{T}^\epsilon \setminus \mathcal{T}^n) + 2 \log(n\epsilon/2)$ , and will prove that it tends in probability to 0 as  $n \rightarrow \infty$ . We introduce some notation. Let  $K_i^n$  be the number of branches in  $\mathcal{T}^n$  at the time where  $\mathcal{T}$  has  $i$  branches (see Section 4.1 of [65]). Since this quantity only depends on the discrete structure of the tree, it is independent of  $S^\epsilon$ . We refer to Lemma 4.1 of [65] for the expression of its distribution. Conditionally given  $S^\epsilon$ , the tree  $\mathcal{T}^\epsilon$  is distributed as a Kingman coalescent started with  $S^\epsilon$  branches. We denote by  $T_i$  the time spent by this tree with  $i$  branches: conditionally given  $S^\epsilon$ , it is an exponential r.v. with parameter  $\binom{i}{2}$  independent of  $K_i^n$ , for every  $i \in \{2, \dots, S^\epsilon\}$ . We write

$$Y_n = \sum_{i=2}^{S^\epsilon} (i - K_i^n) T_i + 2 \log\left(\frac{n\epsilon}{2}\right)$$

and compute

$$\mathbb{E}[(Y_n)^2 | S^\epsilon] = \mathbb{E}\left[\left(\sum_{i=2}^S (i - K_i^n) T_i\right)^2 | S^\epsilon\right] + 4 \log^2\left(\frac{n\epsilon}{2}\right) + 4 \log\left(\frac{n\epsilon}{2}\right) \mathbb{E}\left[\sum_{i=2}^S (i - K_i^n) T_i | S^\epsilon\right]$$

The second conditional expectation gives

$$\mathbb{E}\left[\sum_{i=2}^S (i - K_i^n) T_i | S^\epsilon\right] = \sum_{i=2}^S \frac{2}{n + i - 1} = 2 \log\left(\frac{n + S^\epsilon - 1}{n}\right) + \mathcal{O}(1/n)$$

Concerning the first conditional expectation, we have

$$\begin{aligned}
 & \mathbb{E} \left[ \left( \sum_{i=2}^S (i - K_i^n) T_i \right)^2 \middle| S^\epsilon \right] \\
 = & \sum_{i=2}^S \sum_{j=2}^S \frac{4(1 + \delta_{i,j})}{(n+i-1)(n+j-1)} + \frac{n(n-1)4(1 + \delta_{i,j})}{(n+j-1)(n+i-1)(n+i-2)j(j-1)} \\
 = & 4 \left( \sum_{i=n+1}^{n+S-1} \frac{1}{i} \right)^2 + 4 \sum_{i=n+1}^{n+S-1} \frac{1}{i^2} + 4n(n-1) \sum_{i=n+1}^{n+S-1} \frac{1}{i(i-1)} \sum_{j=2}^S \frac{1}{j(j-1)(n+j-1)} \\
 & + 4n(n-1) \sum_{i=2}^S \frac{1}{i(i-1)(n+i-1)^2(n+i-2)} \\
 = & 4 \log^2 \left( \frac{n + S^\epsilon - 1}{n} \right) + \log \left( \frac{n + S^\epsilon - 1}{n} \right) \mathcal{O}(1/n) \\
 = & 4 \log^2 \left( \frac{n + S^\epsilon - 1}{n} \right) + \mathcal{O}(\log(n)/n)
 \end{aligned}$$

where we use Lemma 4.1 of [65] for the second equality. Finally, we get

$$\begin{aligned}
 \mathbb{E}[(Y_n)^2 | S^\epsilon] &= 4 \left[ \log^2 \left( \frac{n + S^\epsilon - 1}{n} \right) + \log^2 \left( \frac{n\epsilon}{2} \right) + 2 \log \left( \frac{n\epsilon}{2} \right) \log \left( \frac{n + S^\epsilon - 1}{n} \right) \right] + \mathcal{O}(\log(n)/n) \\
 &= 4 \left[ \log \left( \frac{n + S^\epsilon - 1}{n} \right) + \log \left( \frac{n\epsilon}{2} \right) \right]^2 + \mathcal{O}(\log(n)/n) \\
 &= 4 \log^2 \left( \epsilon \frac{n + S^\epsilon - 1}{2} \right) + \mathcal{O}(\log(n)/n)
 \end{aligned}$$

Now we fix  $\eta > 0$  and define the interval  $A_n := [\frac{2}{\epsilon} - \eta n \sqrt{\frac{2}{3\epsilon}}; \frac{2}{\epsilon} + \eta n \sqrt{\frac{2}{3\epsilon}}]$ . It easily follows from Lemma V.1 that  $\mathbb{P}(S^\epsilon \in A_n) \rightarrow 1$  as  $n \uparrow \infty$ . Hence,

$$\begin{aligned}
 \mathbb{P}(|Y_n| > \eta) &\leq \mathbb{P}(S^\epsilon \notin A_n) + \sum_{k \in A_n} \mathbb{P}(|Y_n| > \eta; S^\epsilon = k) \\
 &\leq \mathbb{P}(S^\epsilon \notin A_n) + \eta^{-2} \sum_{k \in A_n} \mathbb{E}[(Y_n)^2 | S^\epsilon = k] \mathbb{P}(S^\epsilon = k) \\
 &\leq \mathbb{P}(S^\epsilon \notin A_n) + \eta^{-2} 4 \log^2 \left( 1 + \eta n \sqrt{\frac{\epsilon}{6}} + \frac{(n-1)\epsilon}{2} \right) + \mathcal{O}(\log(n)/n)
 \end{aligned}$$

Since  $\epsilon = n^{-3}$  the r.h.s. converges to 0 as  $n \uparrow \infty$ . This proves that  $Y_n$  tends to 0 in probability as  $n \uparrow \infty$ , which in turn implies that  $\mathcal{L}_t = \mathcal{L}_t$  a.s.  $\blacksquare$

# Appendices



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Duality Kingman coalescent / standard Fleming-Viot

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**THEOREM A.1.** *Suppose that  $\rho$  is the standard Fleming-Viot process, that is,  $\Lambda(dx) = \delta_0(dx)$  and that  $(\Pi_t, t \geq 0)$  is a Kingman coalescent. The processes  $(\#\rho_t, t \geq 0)$  and  $(\#\Pi_t, t \geq 0)$  have the same distribution.*

**REMARK A.1.** *Simon Tavaré in [74] described the process of the number of lines of descent of a  $i$ -sample from the Moran model with an initial population size  $N$ . By a compatibility argument this could be extended to an infinite population. Here we give a direct proof in the lookdown representation.*

Recall from Chapter 0 Section 3 that  $\#\rho_t$  has the same distribution as  $\#\Pi_t$  for any measure  $\Lambda$ . Our theorem shows that it also holds for the whole processes in the Kingman case. One could wonder what happens in other cases. First, the question makes sense only when the  $\Lambda$  Fleming-Viot /  $\Lambda$  coalescent comes down from infinity. We know that the process  $t \mapsto \#\rho_t$  decreases by jumps of size 1 when  $\mathbb{P}(\mathbf{E}) = 0$ , see Subsection 1.4. While this is clearly not the case for the process  $t \mapsto \#\Pi_t$  as soon as we are not in the Kingman case. Hence, under  $\mathbb{P}(\mathbf{E}) = 0$  the identity holds iff we are in the Kingman case. We now present the proof of the theorem.

We introduce some useful notation. For every  $i \in \mathbb{N}$ , we set

$$Y_t(i) := \min(\hat{\Pi}_{0,t}(i))$$

In other terms,  $Y_t(i)$  is the smallest level at time  $t$  that descends from individual  $(i, 0)$ . Note that for all  $t \geq 0$ ,  $(Y_t(i))_{i \geq 1}$  is an increasing sequence of  $\mathbb{R} \cup \{\infty\}$  and that  $Y_t(1) = 1$ . We then denote by  $d^i := \inf\{t \geq 0 : Y_t(i) = \infty\}$ , that is, the hitting time of  $\infty$  by the process  $(Y_t(i), t \geq 0)$ . This time is finite a.s. for all  $i \geq 2$  and equals  $\infty$  for  $i = 1$ . The interest of the  $Y_t(i)$ 's is that they are related to  $\#\rho$  (resp.  $\#m_t$ ) as follows. For every  $i \geq 2$

$$\#\rho_t = i \Leftrightarrow d^{i+1} \leq t < d^i$$

To prove the theorem, it suffices to show that  $(d^i - d^{i+1})_{i \geq 2}$  is a sequence of independent r.v. such that for each  $i \geq 2$ ,  $d^i - d^{i+1} \stackrel{(d)}{=} \mathcal{E}(\binom{i}{2})$ , where  $\mathcal{E}(x)$  designates the exponential distribution with parameter  $x$ . We introduce a discrete filtration  $(\mathcal{G}_i)_{i \geq 1}$  as follows. For each  $i \geq 1$

$$\mathcal{G}_i := \bigvee_{j \geq i} \sigma\{(Y_t(j), t \geq 0)\} \tag{A.1}$$



From now on, we fix an integer  $i \geq 2$  and we consider the continuous time Markov chain  $(Y_t(i), Y_t(i+1))_{t \geq 0}$ . The dynamics of the lockdown graph implies that whenever the first coordinate of this chain jumps so does the second coordinate, but the converse does not hold true. It is then natural to consider the embedded Markov chain  $(\tilde{Y}_n(i), \tilde{Y}_n(i+1))_{n \geq 0}$  indexed by the jump times of the second coordinate. This chain satisfies  $\tilde{Y}_0(i) = i$  and  $\tilde{Y}_n(i+1) = i+1+n$  for all  $n \in \mathbb{N} \cup \{0\}$ .

**LEMMA A.2.** *The transitions of the Markov chain  $(\tilde{Y}_n(i), \tilde{Y}_n(i+1))_{n \geq 0}$  are given by*

$$\begin{aligned} \mathbb{P}(\tilde{Y}_{n+1}(i) = p+1 | \tilde{Y}_n(i) = p) &= \frac{\binom{p}{2}}{\binom{i+1+n}{2}} \\ \mathbb{P}(\tilde{Y}_{n+1}(i) = p | \tilde{Y}_n(i) = p) &= 1 - \frac{\binom{p}{2}}{\binom{i+1+n}{2}} \end{aligned}$$

Furthermore, the r.v.  $\tilde{Y}_\infty(i) = Y_{d_{i+1}}(i)$  is independent from  $\mathcal{G}_{i+1}$ .

**Proof** Fix an integer  $p$ . If  $p \notin \{i, i+1, \dots, i+n\}$ , then the event  $\{\tilde{Y}_n(i) = p\}$  has a null probability. Otherwise, condition on  $\{\tilde{Y}_n(i) = p\}$ , and notice that the transition  $n \rightarrow n+1$  corresponds to a birth event that affects two of the  $i+n+1$  first levels in the lockdown representation. Among the possible configurations of such birth events, only a proportion  $\frac{\binom{p}{2}}{\binom{i+1+n}{2}}$  of them affect the  $p$  first levels. Thus, the asserted transition rates follow.

Finally, remark that  $\tilde{Y}_\infty(i)$  only depends on the collection of transitions of the chain  $(\tilde{Y}_n(i), \tilde{Y}_n(i+1))_{n \geq 0}$  which are independent of the sigma-field  $\mathcal{G}_{i+1}$ . Indeed, the latter only takes into account the jump times of the lines  $Y_j$  for  $j \geq i+1$ . Therefore, if level  $k$  reproduces on level  $l$  at time  $t$  with  $1 \leq k < l \leq Y_{t-}(i+1)$ , then  $\mathcal{G}_{i+1}$  contains no information about  $k$ , while  $k$  is needed to determine the transition of the chain  $(\tilde{Y}_n(i), \tilde{Y}_n(i+1))_{n \geq 0}$  at this jump time. ■

The following proposition will be needed in the proof of the theorem and will be proven later on.

**PROPOSITION A.3.** *The distribution of  $\tilde{Y}_\infty(i)$  is given by*

$$\mathbb{P}(\tilde{Y}_\infty(i) = i+l) = \frac{(i+l-1)!(i+l-2)!i(i-1)}{l!(2i+l-1)!} \quad (\text{A.2})$$

for every integer  $l \geq 0$ .

We are now ready to derive the proof of the theorem.

*Proof of Theorem A.1.* Let  $S_j^\infty$  be a r.v. distributed as the sum of a sequence (indexed by  $p$ ) of independent  $\mathcal{E}(\binom{p}{2})$  for  $p \geq j$ . Since  $d^i - d^{i+1}$  is the time necessary for the process  $(Y_t(i), t \geq d^{i+1})$  to reach  $\infty$  from  $\tilde{Y}_\infty(i)$ , it is clear that, conditional on  $\{\tilde{Y}_\infty(i) = i+l\}$ , this time is distributed as  $S_{i+l}^\infty$ . Then we get

$$\mathbb{P}(d^i - d^{i+1} \in ds) = \sum_{l=0}^{\infty} \mathbb{P}(\tilde{Y}_\infty(i) = i+l) \mathbb{P}(S_{i+l}^\infty \in ds) \quad (\text{A.3})$$

$$= \sum_{l=0}^{\infty} \frac{(i+l-1)!(i+l-2)!i(i-1)}{l!(2i+l-1)!} \mathbb{P}(S_{i+l}^\infty \in ds) \quad (\text{A.4})$$

We have to prove that the latter distribution is  $\mathcal{E}(\binom{i}{2})$ . Consider a  $i$ -sample embedded in a Kingman coalescent. The first coalescence time  $T$  of this sample is distributed as  $\mathcal{E}(\binom{i}{2})$ . Denote by  $N(i)$  the random

number of blocks in the full coalescent at time  $T$ . Lemma 3 of [68] ensures that  $N(i)$  is distributed as  $\tilde{Y}_\infty(i) - 1$  (take  $l = i + l - 1, j = i, k = i - 1$  and  $i \rightarrow \infty$  in their notation). Moreover, conditional on  $\{N(i) = i + l - 1\}$ ,  $T$  is distributed as  $S_{i+l}^\infty$ . Thanks to the preceding formula, one can easily verify that  $T$  has the same distribution as  $d^i - d^{i+1}$ . Therefore we have proven that  $d^i - d^{i+1} \stackrel{(d)}{=} \mathcal{E}\left(\binom{i}{2}\right)$ .

Now it remains to prove that  $d^i - d^{i+1}$  is independent of  $(d^{i+k} - d^{i+k+1})_{k \geq 1}$ . Let  $\mathcal{F}_t$  be the sigma-field generated by the lookdown representation up to time  $t \geq 0$ . Since  $\tilde{Y}_\infty(i)$  is  $\mathcal{F}_{d^{i+1}}$ -measurable we deduce that conditionally on  $\mathcal{F}_{d^{i+1}}$ , the r.v.  $d^i - d^{i+1}$  is distributed as  $S_{i+l}^\infty$  where  $i + l = \tilde{Y}_\infty(i)$ . Using Lemma A.2 and the fact that  $\mathcal{G}_{i+1} \subset \mathcal{F}_{d^{i+1}}$ , for any event  $A \in \mathcal{G}_{i+1}$  and any Borel function  $f$  we have the following identities

$$\begin{aligned} \mathbb{E}[\mathbf{1}_A f(d^i - d^{i+1})] &= \mathbb{E}[\mathbb{E}[\mathbf{1}_A f(d^i - d^{i+1}) | \mathcal{F}_{d^{i+1}}]] = \mathbb{E}[\mathbf{1}_A \mathbb{E}[f(d^i - d^{i+1}) | \mathcal{F}_{d^{i+1}}]] \\ &= \mathbb{E}[\mathbf{1}_A \Phi(\tilde{Y}_\infty(i))] = \mathbb{P}(A) \mathbb{E}[\Phi(\tilde{Y}_\infty(i))] \end{aligned}$$

where  $\Phi(n) := \mathbb{E}[S_n^\infty]$  for all integer  $n \geq 2$ . We deduce that  $d^i - d^{i+1}$  is independent of  $\mathcal{G}_{i+1}$  and thus is independent of  $(d^{i+k} - d^{i+k+1})_{k \geq 1}$ . The proof of the theorem is thus complete.  $\blacksquare$

*Proof of Proposition A.3* Let us prove the formula in the case  $l = 0$ .

$$\begin{aligned} \mathbb{P}(\tilde{Y}_\infty(i) = i) &= \prod_{n=0}^{\infty} \mathbb{P}(\tilde{Y}_{n+1}(i) = i | \tilde{Y}_n(i) = i) \\ &= \prod_{n=0}^{\infty} \left(1 - \frac{\binom{i}{2}}{\binom{i+1+n}{2}}\right) \\ &= \prod_{n=0}^{\infty} \frac{(n+1)(n+2i)}{(n+i)(n+i+1)} = \frac{(i-1)!i!}{(2i-1)!} \end{aligned}$$

We turn our attention to the case  $l \geq 1$ . We will enumerate all the configurations of jump times of the process  $\tilde{Y}(i)$  and calculate their probabilities. Therefore we introduce  $n_0 := 0 < n_1 < \dots < n_l < n_{l+1} := \infty$ . Define

$$\begin{aligned} A(n_1, \dots, n_l) &:= \{\tilde{Y}_0(i) = \dots = \tilde{Y}_{n_1-1}(i) = i\} \cap \{\tilde{Y}_0(n_1) = \dots = \tilde{Y}_{i_2-1}(i) = i+1\} \dots \\ &\quad \cap \{\tilde{Y}_{i_{l-1}}(i) = \dots = \tilde{Y}_{n_l-1}(i) = i+l-1\} \cap \{\tilde{Y}_{n_l}(i) = \dots = \tilde{Y}_\infty(i) = i+l\} \end{aligned}$$

In words,  $A(n_1, \dots, n_l)$  is the event that the process  $\tilde{Y}(i)$  jumps at times  $n_1, \dots, n_l$  and stays constant otherwise.

We have

$$\begin{aligned}
 \mathbb{P}(A(n_1, \dots, n_l)) &= \left[ \prod_{m=0}^l \prod_{j=i_m}^{i_{m+1}-2} \left(1 - \frac{\binom{i+m}{2}}{\binom{i+j+1}{2}}\right) \right] \left[ \prod_{m=1}^l \frac{\binom{i+m-1}{2}}{\binom{i+i_m}{2}} \right] \\
 &= \frac{(i+l-1)!(i+l-2)!}{(i-1)!(i-2)!} \left[ \prod_{m=0}^l \prod_{j=i_m}^{i_{m+1}-2} \frac{(j-m+1)(j+m+2i)}{(i+1+j)(i+j)} \right] \left[ \prod_{m=1}^l \frac{1}{(i+i_m)(i+i_m-1)} \right] \\
 &= \frac{(i+l-1)!(i+l-2)!}{(i-1)!(i-2)!} \prod_{j=n_l}^{\infty} \frac{(j-l+1)(j+l+2i)}{(j+i+1)(j+i)} \\
 &\times \left[ \prod_{m=0}^{l-1} \frac{(i_m-m+1) \dots (i_m+i-1)}{(i_{m+1}-m) \dots (i_{m+1}+i)} \frac{(i_{m+1}+i) \dots (i_{m+1}+m+2i-2)}{(i_m+1+i) \dots (i_m+m+2i-1)} \right] \\
 &= \frac{(i+l-1)!(i+l-2)!i!(i-1)!}{(i-1)!(i-2)!(2i-1)!} \left[ \prod_{m=1}^{l-1} \frac{1}{(i_m+m+2i-2)(i_m+m+2i-1)} \right] \\
 &\times \frac{1}{(n_l+l+2i-2)(n_l+l+2i-1)} \\
 &= \frac{(i+l-1)!(i+l-2)!i(i-1)}{(2i-1)!} \left[ \prod_{m=1}^l \frac{1}{(i_m+m+2i-2)(i_m+m+2i-1)} \right]
 \end{aligned}$$

We deduce from the preceding calculations that for each  $l \geq 1$

$$\begin{aligned}
 \mathbb{P}(\tilde{Y}_\infty(i) = i+l) &= \sum_{0 < n_1 < \dots < n_l} \mathbb{P}(A(n_1, \dots, n_l)) \\
 &= \frac{(i+l-1)!(i+l-2)!i(i-1)}{(2i-1)!} \sum_{0 < n_1 < \dots < n_l} \prod_{m=1}^l \frac{1}{(i_m+m+2i-2)(i_m+m+2i-1)}
 \end{aligned}$$

The proof of the proposition is now achieved thanks to this formula and using the following lemma.  $\blacksquare$

**LEMMA A.4.** For all  $l \geq 1$ , we have

$$\sum_{0 < n_1 < \dots < n_l} \prod_{m=1}^l \frac{1}{(i_m+m+2i-2)(i_m+m+2i-1)} = \frac{(2i-1)!}{l!(2i+l-1)!} \quad (\text{A.5})$$

**Proof** Using a partial fraction expansion and a simple induction, one can show that for all  $q \geq 1$ , and  $N \geq 1$

$$\sum_{p=q}^{\infty} \frac{1}{p(p+1) \dots (p+N)} = \frac{(q-1)!}{N(N+q-1)!} \quad (\text{A.6})$$

Now let us prove by induction on  $N$  that for all  $l \geq 1$  and  $0 \leq N \leq l$

$$\begin{aligned}
 &\sum_{0 < n_1 < \dots < n_l} \prod_{m=1}^l \frac{1}{(i_m+m+2i-2)(i_m+m+2i-1)} \\
 &= \sum_{0 < n_1 < \dots < i_{l-N}} \left[ \prod_{m=1}^{l-N} \frac{1}{(i_m+m+2i-2)(i_m+m+2i-1)} \right] \frac{(i_{l-N}+l-N+2i-1)!}{N!(i_{l-N}+l+2i-1)!}
 \end{aligned}$$

Case  $N = 0$  is immediate. Suppose that the formula is true at rank  $N - 1$ , for a given  $1 \leq N \leq l$ . From Equation (A.6), we deduce that

$$\begin{aligned} \sum_{i_{l-N+1}=i_{l-N+1}}^{\infty} \frac{(i_{l-N+1} + l - N + 2i - 2)!}{(N - 1)!(i_{l-N+1} + l + 2i - 1)!} &= \sum_{p=i_{l-N}+l-N+2i}^{\infty} \frac{1}{(N - 1)!p \dots (p + N)} \\ &= \frac{(i_{l-N} + l - N + 2i - 1)!}{N!(i_{l-N} + l + 2i - 1)!} \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\sum_{0 < n_1 < \dots < n_l} \prod_{m=1}^l \frac{1}{(i_m + m + 2i - 2)(i_m + m + 2i - 1)} \\ &= \sum_{0 < n_1 < \dots < i_{l-N+1}} \left[ \prod_{m=1}^{l-N+1} \frac{1}{(i_m + m + 2i - 2)(i_m + m + 2i - 1)} \right] \frac{(i_{l-N+1} + l - N + 2i)!}{(N - 1)!(i_{l-N+1} + l + 2i - 1)!} \\ &= \sum_{0 < n_1 < \dots < i_{l-N}} \left[ \prod_{m=1}^{l-N} \frac{1}{(i_m + m + 2i - 2)(i_m + m + 2i - 1)} \right] \\ &\times \sum_{i_{l-N+1}=i_{l-N+1}}^{\infty} \frac{(i_{l-N+1} + l - N + 2i - 2)!}{(N - 1)!(i_{l-N+1} + l + 2i - 1)!} \\ &= \sum_{0 < n_1 < \dots < i_{l-N}} \left[ \prod_{m=1}^{l-N} \frac{1}{(i_m + m + 2i - 2)(i_m + m + 2i - 1)} \right] \frac{(i_{l-N} + l - N + 2i - 1)!}{N!(i_{l-N} + l + 2i - 1)!} \end{aligned}$$

We recover the formula at rank  $N$ . This ends the induction. Finally, at rank  $N = l$ , we get

$$\sum_{0 < n_1 < \dots < n_l} \prod_{m=1}^l \frac{1}{(i_m + m + 2i - 2)(i_m + m + 2i - 1)} = \frac{(2i - 1)!}{l!(2i + l - 1)!}$$

Thus we have proven the lemma. ■



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Number of atoms in REGIMES 1 & 2

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Let us recall the following result due to Freeman.

**THEOREM B.1.** (Freeman [35]) *Suppose that  $\rho$  is a  $\Lambda$  Fleming-Viot in REGIME 1 or REGIME 2. Almost surely for every  $t > 0$ , the number of atoms  $\#\rho_t$  is finite if and only if  $\int_{(0,1)} u^{-2}\Lambda(du) < \infty$ .*

Note that  $\Lambda(0) = 0$  whenever the  $\Lambda$  Fleming-Viot comes down from infinity. Freeman's proof is based on the flow of bridges representation. Here we propose another proof based on the lookdown representation together with a proof of a similar result in the MVBP setting.

**THEOREM B.2.** *Suppose that  $m$  is a  $\Psi$ -MVBP in REGIME 1 or REGIME 2. Almost surely for every  $t \in (0, T)$ , the number of atoms  $\#m_t$  is finite if and only if  $\nu(0, 1) < \infty$ .*

**Proof** We use the lookdown representation of these measure-valued processes, that is, we consider a flow of partitions  $\hat{\Pi}$  associated to  $\Lambda$  or  $\Psi$ , we keep the notation of Chapter 0. Recall that an atom of  $\rho_t$  (resp.  $m_t$ ) corresponds to a block of  $\hat{\Pi}_{0,t}$  with a strictly positive asymptotic frequency at a given time  $t > 0$ , see Chapter 0 Section 2. Since we are not in REGIME 4, no ancestral type becomes extinct. In other terms, a block with a positive frequency at a given time will keep it positive forever.

We begin with the simplest implication: we suppose that  $\int_{(0,1)} u^{-2}\Lambda(du)$  (resp.  $\nu(0, 1)$ ) is finite. To prove that the number of atoms at any time  $t > 0$  is finite, it suffices to prove that the total number of elementary reproduction events on  $(0, t]$  is finite.

In the  $\Lambda$  Fleming-Viot case the intensity measure of the Poisson point process generating these reproduction events is finite on any compact of time, and so the total number of elementary reproduction events on  $(0, t]$  is finite. In the  $\Psi$ -MVBP case, the situation is more complicated. We need to prove that  $Z$  makes finitely many jumps on  $(0, t]$ . To that end, we make use of the Lamperti representation. Recall that  $Z$  is equal in distribution to a time-changed Lévy process with Laplace exponent  $\Psi$  starting from 1 and killed upon reaching 0. From the assumption on  $\nu$ , we easily deduce that this Lévy process makes finitely many jumps on any compact interval of time. Since the time-change sends the hitting time of 0 by the Lévy process onto  $T = \infty$ , we deduce that  $Z$  makes finitely many jumps on any compact interval of time as well.

We now turn our attention to the converse implication. We suppose that  $\int_{(0,1)} u^{-2}\Lambda(du)$  (resp.  $\nu(0, 1)$ ) is infinite. We want to prove that the number of atoms at any time  $t > 0$  (resp. at any time  $t \in$

$(0, T)$  is infinite. Using the preceding arguments, we easily deduce that the total number of elementary reproduction events is infinite on any compact interval of time. Indeed, the intensity measure in the  $\Lambda$  Fleming-Viot case is infinite while the number of jumps of a Lévy process with Laplace exponent  $\Psi$  is infinite on any compact interval of time.

Suppose that  $\hat{\Pi}_{0,t}$  has a finite number of non-singleton blocks and let us exhibit a contradiction. There exists  $i \geq 2$  such that with positive probability, all the blocks with indices greater than  $i$  are singleton blocks. Therefore with positive probability, on  $[0, t]$  no elementary reproduction event has chosen a level  $j > i$  as a parent, and so, on the  $i$  first levels infinitely many elementary reproduction events have fallen. As we are in the finite variation case (REGIME 1 or 2), the number of elementary reproduction events whose parent is a given level  $j \in \mathbb{N}$  is necessarily finite on any compact interval of time since in the  $\Lambda$  Fleming-Viot case it is a Poisson r.v. with parameter

$$t \int_{(0,1)} u(1-u)^{j-1} u^{-2} \Lambda(du) < \infty$$

and in the  $\Psi$ -MVBP case it is a Poisson r.v. with random parameter

$$\sum_{s \in (0,t]: \Delta Z_s > 0} \left(1 - \frac{\Delta Z_s}{Z_s}\right)^{j-1} \frac{\Delta Z_s}{Z_s} < \infty \text{ almost surely}$$

Consequently the  $i$  first levels cannot be the parent of infinitely many reproduction events on  $[0, t]$ . The contradiction is now clear and the theorems are proved. ■

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Proof of Theorem 0.1

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Since  $\hat{\Pi}_{0,t}$  and  $(\xi_0(i))_{i \geq 1}$  are exchangeable, the exchangeability of  $(\xi_t(i))_{i \geq 1}$  follows from standard arguments, see the proof of Theorem 2.1 in [9] for instance. By definition,  $\Xi_t$  is its limiting empirical measure. The hard part is now to prove that it remains exchangeable conditionally given all the information on the asymptotic frequencies and the total-mass up to time  $t$ . Our strategy is the following. Fix  $n \in \mathbb{N}$  and consider a permutation  $\sigma$  of  $[n]$ , that we extend to the whole  $\mathbb{N}$  by setting  $\sigma(i) = i$  whenever  $i > n$ . We are going to construct another lookdown process  $(\xi'_s(i), s \in [0, t])_{i \geq 1}$  such that its limiting empirical measure  $(\Xi'_s, s \in [0, t])$  is almost surely equal to  $(\Xi_s, s \in [0, t])$  and such that the particles at time  $t$  verify  $(\xi'_t(i))_{i \geq 1} = (\xi_t(\sigma(i)))_{i \geq 1}$ . This will ensure that  $(\xi_t(\sigma(i)))_{i \geq 1}$  has the same distribution as  $(\xi_t(i))_{i \geq 1}$ , conditionally on  $(\Xi_s, s \in [0, t])$  and  $(Z_s, s \in [0, t])$ . Therefore the conditional exchangeability will follow. Finally, from the very definition of  $\Xi_t$ , we will immediately deduce that the conditional limiting empirical measure is  $\Xi_t$  and the asserted equality will be proved.

Let us now formalize this idea. We want to construct a flow of partition  $\hat{\Pi}'$  on the interval  $[0, t]$  and a random permutation  $\sigma'$  of  $\mathbb{N}$  such that:

- $\hat{\Pi}'$  and  $\hat{\Pi}$ , restricted to  $[0, t]$ , have the same distribution.
- the sequence  $\xi'_0(i) := \xi_0(\sigma'(i)), i \in \mathbb{N}$  is i.i.d. uniform $[0, 1]$ .
- we have  $\xi'_t(i) = \xi_t(\sigma(i)), \forall i \in \mathbb{N}$ , where  $(\xi'_s(i), s \in [0, t])_{i \geq 1}$  is the particle system defined from  $\hat{\Pi}'$  and the sequence  $\xi'_0(i), i \in \mathbb{N}$ .
- $\Xi'_s = \Xi_s$  for all  $s \in [0, t]$ , where  $\Xi'_s$  is the limiting empirical measure of  $\xi'_s(i), i \in \mathbb{N}$ .

If these assertions hold, then we deduce that  $(\xi_t(\sigma(1)), \dots, \xi_t(\sigma(n)))$  has the same distribution as  $(\xi_t(1), \dots, \xi_t(n))$  conditionally given  $Z$  and  $\Xi$  until time  $t$ . Therefore  $(\xi_t(1), \dots, \xi_t(n))$  is conditionally exchangeable with limiting empirical measure  $\Xi_t$ . It remains to prove the existence of  $\hat{\Pi}'$  and  $\sigma'$  fulfilling the requirements. To that end, we introduce the following:

**DEFINITION C.1.** (Permutation  $\mathfrak{p}$ ) Let  $\pi$  be a partition of  $[n]$  and  $\sigma$  a permutation of  $[n]$ . For each  $i \in [\#\pi]$  define  $\mathfrak{p}(i)$  as the unique integer  $l \in [\#\pi]$  such that

$$\sigma(\pi)(i) = \{j \in [n] : \sigma(j) \in \pi(l)\} = \sigma^{-1}(\pi(l))$$

Then  $\mathfrak{p}$  is a permutation of  $[\#\pi]$ .



Let us enumerate the elementary reproduction events of  $\hat{\Pi}^{[n]}$ : they are finitely many on the interval  $[0, t]$ , say  $(t_1, \varrho_1), \dots, (t_q, \varrho_q)$  in the increasing order of their time coordinates. Using Definition C.1, we define  $p_1, \dots, p_{q+1}$  as follows. First set  $p_{q+1} := \sigma$ . Then  $p_q$  is defined as the permutation of  $[\#\varrho_q]$  obtained from  $p_{q+1}$  and  $\varrho_q$  via Definition C.1. Set  $\pi_{q-1} := \varrho_{q-1}^{[\#\varrho_q]}$ . Define  $p_{q-1}$  as the permutation of  $[\#\pi_{q-1}]$  obtained from  $p_q$  and  $\pi_{q-1}$  via the same definition. Define  $\pi_{q-2} := \varrho_{q-2}^{[\#\pi_{q-1}]}$ . Then  $p_{q-2}$  is the permutation of  $[\#\pi_{q-2}]$  obtained from  $p_{q-1}$  and  $\pi_{q-2}$ . We repeat the procedure recursively. We use the collection of permutations  $p_1, \dots, p_{q+1}$  to define the flow  $\hat{\Pi}'$ . For any  $s \in (0, t]$  such that  $\hat{\Pi}_{s-,s} \neq 0_{[\infty]}$ , there exists a unique integer  $k \in [q+1]$  such that  $s \in [t_{k-1}, t_k)$  with the convention  $t_0 = 0$  and  $t_{q+1} = t$ . We set

$$\hat{\Pi}'_{s-,s} := p_k(\hat{\Pi}_{s-,s})$$

Here again we have implicitly extended  $p_k$  to  $\mathbb{N}$ . It is a simple matter to check that  $\hat{\Pi}'$  has the same distribution as  $\hat{\Pi}$ . Let  $\sigma' := p_1$ , we define the sequence  $\xi'_0(i) := \xi_0(\sigma'(i)), i \in \mathbb{N}$  which is i.i.d. uniform $[0, 1]$  since  $\sigma'$  is independent of  $(\xi_0(i))_{i \geq 1}$ . By construction, we have  $\xi'_t(i) = \xi_t(\sigma(i)), \forall i \in \mathbb{N}$  and  $\Xi'_s = \Xi_s, \forall s \in [0, t]$ . ■

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Martingale problem for the MVBP

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The main difficulty lies in the randomly varying population size that induces unbounded jump rates of our process. To bound the jump rates, we consider the following collection of stopping times

$$T_\epsilon := \inf \left\{ t \geq 0 : Z_t \notin \left( \epsilon, \frac{1}{\epsilon} \right) \right\}, \quad \epsilon \in (0, 1)$$

We rely on the following result to identify the distribution of the MVBP.

**THEOREM D.1.** (El Karoui-Roelly [28]) For any function  $f \in \mathcal{B}^{++}([0, 1])$ , introduce

$$\begin{aligned} G_f(\mu) &:= e^{-\langle \mu, f \rangle} \\ \mathcal{L}G_f(\mu) &:= e^{-\langle \mu, f \rangle} \left( \gamma \langle \mu, f \rangle + \frac{\sigma^2}{2} \langle \mu, f^2 \rangle + \int_{(0, \infty)} \nu(dh) \langle \mu, e^{-hf(\cdot)} - 1 + \mathbf{1}_{(0,1)}(h)hf(\cdot) \rangle \right) \end{aligned}$$

If for any such  $f$  and any  $\epsilon \in (0, 1)$ , the process

$$t \mapsto G_f(Z_{t \wedge T_\epsilon} \cdot \Xi_{t \wedge T_\epsilon}) - \int_0^{t \wedge T_\epsilon} \mathcal{L}G_f(Z_{s \wedge T_\epsilon} \cdot \Xi_{s \wedge T_\epsilon}) ds \quad (\text{D.1})$$

is a martingale, then  $(Z_t \cdot \Xi_t, t \geq 0)$  is a  $\Psi$ -MVBP.  $\square$

**REMARK D.1.** This is derived from Theorem 7 in [28]. Our statement differs slightly from theirs, but is easily justified thanks to the proof in their paper.

We fix  $\epsilon \in (0, 1)$ . To simplify notation, for any integer  $n \geq 1$  and any  $x_1, \dots, x_n \in [0, 1]$ , we write  $\bar{\xi} = (\xi(1), \dots, \xi(n))$  and  $\bar{x} = (x_1, \dots, x_n)$ .

*Step 1.* Fix  $n \in \mathbb{N}$ . We want to identify the generator of the Markov process

$$t \mapsto (Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon})$$

Let  $f$  be any element of  $\mathcal{B}^{++}([0, 1])$  and  $\phi$  be any element of  $\mathcal{C}_c^\infty((\epsilon, 1/\epsilon))$ . We define the map  $H_{\phi, f} : [0, \infty) \times [0, 1]^n \rightarrow \mathbb{R}$  by setting

$$H_{\phi, f}(z, \bar{x}) := \phi(z) f(x_1) \dots f(x_n)$$

and introduce the operator  $A_n$  acting on  $H_{\phi,f}$  as follows. If  $z \notin (\epsilon, 1/\epsilon)$ , then  $A_n H_{\phi,f}(z, \bar{x}) = 0$ . Otherwise

$$\begin{aligned} A_n H_{\phi,f}(z, \bar{x}) := & \left[ -\gamma z \phi'(z) + \frac{\sigma^2}{2} z \phi''(z) + z \int_{(0,\infty)} (\phi(z+h) - \phi(z) - \mathbf{1}_{(0,1)}(h) h \phi'(z)) \nu(dh) \right] \prod_{l \in [n]} f(x_l) \\ & + \sum_{1 \leq i < j \leq n} \frac{\sigma^2}{z} \phi(z) \left[ f(x_i)^2 \prod_{l \in [n] \setminus \{i,j\}} f(x_l) - \prod_{l \in [n]} f(x_l) \right] \\ & + \sum_{K \subset [n]} \int_{(0,\infty)} \nu(dh) \frac{h^k z^{n+1-k}}{(z+h)^n} \phi(z+h) \left[ f(x_{\min K})^k \prod_{l \in [n] \setminus K} f(x_l) - \prod_{l \in [n]} f(x_l) \right] \end{aligned}$$

where implicitly  $k$  stands for  $\#K$  in the last sum. In Subsection 1 below, we provide the arguments showing that

$$H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon}) - H_{\phi,f}(Z_0, \bar{\xi}_0) - \int_0^t A_n H_{\phi,f}(Z_{s \wedge T_\epsilon}, \bar{\xi}_{s \wedge T_\epsilon}) ds$$

is a martingale in the natural filtration  $\mathcal{F}^{Z,\xi}$  of the particle system stopped at  $T_\epsilon$ . Since  $\mathcal{F}_s^{Z,\Xi} \subset \mathcal{F}_s^{Z,\xi}$  for all  $s \geq 0$ , we deduce that for every  $0 \leq s \leq t$

$$\mathbb{E} \left[ H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon}) - H_{\phi,f}(Z_0, \bar{\xi}_0) - \int_s^t A_n H_{\phi,f}(Z_{u \wedge T_\epsilon}, \bar{\xi}_{u \wedge T_\epsilon}) du \mid \mathcal{F}_{s \wedge T_\epsilon}^{Z,\Xi} \right] = 0 \quad (\text{D.2})$$

*Step 2.* The only information we have so far on the process  $(Z_{t \wedge T_\epsilon} \cdot \Xi_{t \wedge T_\epsilon}, t \geq 0)$  comes through the martingale (D.2). What follows aims at showing that "simple" calculations allow to recover the martingale problem of El Karoui and Roelly. We consider the map

$$\begin{aligned} I_f : \mathcal{M}_f & \longrightarrow \mathbb{R} \\ \mu & \longmapsto \phi(\langle \mu, 1 \rangle) \left( \frac{\langle \mu, f \rangle}{\langle \mu, 1 \rangle} \right)^n \end{aligned}$$

and introduce the operator  $\mathcal{A}$  acting on  $I_f$  as follows. Let  $\mu \in \mathcal{M}_f$ . If  $\langle \mu, 1 \rangle \notin (\epsilon, 1/\epsilon)$ , then  $\mathcal{A}I_f(\mu) = 0$ . Otherwise

$$\begin{aligned} \mathcal{A}I_f(\mu) := & \left[ -\gamma \phi'(\langle \mu, 1 \rangle) + \frac{\sigma^2}{2} \phi''(\langle \mu, 1 \rangle) \right. \\ & + \int_{(0,\infty)} \nu(dh) \left( \phi(\langle \mu, 1 \rangle + h) - \phi(\langle \mu, 1 \rangle) - \mathbf{1}_{(0,1)}(h) h \phi'(\langle \mu, 1 \rangle) \right) \left. \right] \frac{\langle \mu, f \rangle^n}{\langle \mu, 1 \rangle^{n-1}} \\ & + \frac{n(n-1)}{2} \frac{\sigma^2}{\langle \mu, 1 \rangle} \phi(\langle \mu, 1 \rangle) \left[ \frac{\langle \mu, f^2 \rangle}{\langle \mu, 1 \rangle} \left( \frac{\langle \mu, f \rangle}{\langle \mu, 1 \rangle} \right)^{n-2} - \left( \frac{\langle \mu, f \rangle}{\langle \mu, 1 \rangle} \right)^n \right] \\ & + \sum_{k=2}^n \binom{n}{k} \int_{(0,\infty)} \nu(dh) \frac{h^k \langle \mu, 1 \rangle^{n+1-k}}{(\langle \mu, 1 \rangle + h)^n} \phi(\langle \mu, 1 \rangle + h) \left[ \frac{\langle \mu, f^k \rangle \langle \mu, f \rangle^{n-k}}{\langle \mu, 1 \rangle^{n+1-k}} - \left( \frac{\langle \mu, f \rangle}{\langle \mu, 1 \rangle} \right)^n \right] \end{aligned}$$

One should notice the similarity between the expression of this operator and  $A_n$ . We use Theorem 0.1 to formalise this similarity. First we have

$$\mathbb{E} \left[ H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon}) \mid \mathcal{F}_{t \wedge T_\epsilon}^{Z,\Xi} \right] = I_f(Z_{t \wedge T_\epsilon} \cdot \Xi_{t \wedge T_\epsilon})$$

Also, Theorem 0.1 and a simple calculation ensure that

$$\mathbb{E}\left[A_n H_{\phi,f}(Z_{t\wedge T_\epsilon}, \bar{\xi}_{t\wedge T_\epsilon}) \mid \mathcal{F}_{t\wedge T_\epsilon}^{Z,\Xi}\right] = \mathcal{A}I_f(Z_{t\wedge T_\epsilon} \cdot \Xi_{t\wedge T_\epsilon})$$

Therefore we get for every  $0 \leq s \leq t$

$$\begin{aligned} & \mathbb{E}\left[I_f(Z_{t\wedge T_\epsilon} \cdot \Xi_{t\wedge T_\epsilon}) - I_f(Z_{s\wedge T_\epsilon} \cdot \Xi_{s\wedge T_\epsilon}) - \int_s^t \mathcal{A}I_f(Z_u \cdot \Xi_{u\wedge T_\epsilon}) du \mid \mathcal{F}_{s\wedge T_\epsilon}^{Z,\Xi}\right] \\ = & \mathbb{E}\left[\mathbb{E}\left[H_{\phi,f}(Z_{t\wedge T_\epsilon}, \bar{\xi}_{t\wedge T_\epsilon}) \mid \mathcal{F}_{t\wedge T_\epsilon}^{Z,\Xi}\right] - H_{\phi,f}(Z_{s\wedge T_\epsilon}, \bar{\xi}_{s\wedge T_\epsilon})\right. \\ & \left. - \int_s^t \mathbb{E}\left[A_n H_{\phi,f}(Z_{u\wedge T_\epsilon}, \bar{\xi}_{u\wedge T_\epsilon}) \mid \mathcal{F}_{u\wedge T_\epsilon}^{Z,\Xi}\right] du \mid \mathcal{F}_{s\wedge T_\epsilon}^{Z,\Xi}\right] \\ = & \mathbb{E}\left[H_{\phi,f}(Z_{t\wedge T_\epsilon}, \bar{\xi}_{t\wedge T_\epsilon}) - H_{\phi,f}(Z_{s\wedge T_\epsilon}, \bar{\xi}_{s\wedge T_\epsilon})\right. \\ & \left. - \int_s^t A_n H_{\phi,f}(Z_{u\wedge T_\epsilon}, \bar{\xi}_{u\wedge T_\epsilon}) du \mid \mathcal{F}_{s\wedge T_\epsilon}^{Z,\Xi}\right] \\ = & 0 \end{aligned}$$

Finally, we use the following fact stated in a different way in [18]: a sketch of proof is provided in Subsection 2.

**Fact.** (Birkner et al. [18]) By taking linear combinations of maps of the form  $I_f$ , one derives that the martingale problem (D.1) is satisfied by  $(Z_{t\wedge T_\epsilon} \cdot \Xi_{t\wedge T_\epsilon}, t \geq 0)$ . ■

## 1 Identification of the generator

The goal of this subsection is to prove that

$$H_{\phi,f}(Z_{t\wedge T_\epsilon}, \bar{\xi}_{t\wedge T_\epsilon}) - H_{\phi,f}(Z_0, \bar{\xi}_0) - \int_0^t A_n H_{\phi,f}(Z_{s\wedge T_\epsilon}, \bar{\xi}_{s\wedge T_\epsilon}) ds \quad (\text{D.3})$$

is a  $\mathcal{F}^{Z,\bar{\xi}}$ -martingale. Let us introduce an appropriate formalism. Let  $\mathcal{E}$  be the space  $[0, \infty] \times [0, 1]^n$  endowed with the usual topology. We consider the set  $\mathcal{F}$  of bounded Borel maps  $F : \mathcal{E} \rightarrow \mathbb{R}$  such that  $\forall x_1, \dots, x_n \in [0, 1]$ ,  $F(\cdot, x_1, \dots, x_n)$  is a continuous map that vanishes outside of  $(\epsilon, 1/\epsilon)$ . Notice that  $H_{\phi,f}$  belongs to  $\mathcal{F}$ . It is elementary to check that  $\mathcal{F}$ , equipped with the supremum norm, is a Banach space. The process

$$t \mapsto (Z_{t\wedge T_\epsilon}, \bar{\xi}_{t\wedge T_\epsilon})$$

is Markov on  $\mathcal{E}$ . One can easily prove that its semigroup  $(P_t^\epsilon, t \geq 0)$  is strongly continuous on  $\mathcal{F}$ , that is,  $P_t^\epsilon(\mathcal{F}) \subset \mathcal{F}$  and, for all  $F \in \mathcal{F}$ ,  $\|P_t^\epsilon F - F\|_\infty \rightarrow 0$  as  $t \downarrow 0$ . Suppose that  $Z_0 = z \in (\epsilon, 1/\epsilon)$  and  $\xi_0(i) = x_i \in [0, 1], \forall i \in [n]$  are all deterministic. If we prove that

$$\sup_{(z,\bar{x}) \in \mathcal{E}} \left| \frac{1}{t} \mathbb{E}\left[H_{\phi,f}(Z_{t\wedge T_\epsilon}, \bar{\xi}_{t\wedge T_\epsilon}) - H_{\phi,f}(z, \bar{x})\right] - A_n H_{\phi,f}(z, \bar{x}) \right| \xrightarrow[t \rightarrow 0]{} 0 \quad (\text{D.4})$$

then Proposition 1.1.5 in [31] ensures that (D.3) is a martingale. We are left with the proof of (D.4). To that end, we split the operator  $A_n$  into three parts as follows:

$$A_n := A_n^{(1)} + A_n^{(2)} + A_n^{(3)}$$

where  $A_n^{(1)}$  controls the infinitesimal evolution of the total mass:

$$A_n^{(1)} H_{\phi,f}(z, \bar{x}) := \left[ -\gamma z \phi'(z) + \frac{\sigma^2}{2} z \phi''(z) + z \int_{(0,\infty)} (\phi(z+h) - \phi(z) - \mathbf{1}_{(0,1)}(h) h \phi'(z)) \nu(dh) \right] \\ \times \prod_{l \in [n]} f(x_l)$$

while  $A_n^{(2)}$  and  $A_n^{(3)}$  deal with the transitions of the  $n$  first particles of the lookdown process:

$$A_n^{(2)} H_{\phi,f}(z, \bar{x}) := \sum_{1 \leq i < j \leq n} \frac{\sigma^2}{z} \phi(z) \left[ f(x_i)^2 \prod_{l \in [n] \setminus \{i,j\}} f(x_l) - \prod_{l \in [n]} f(x_l) \right] \\ A_n^{(3)} H_{\phi,f}(z, \bar{x}) := \sum_{K \subset [n]} \int_{(0,\infty)} \frac{h^k z^{n-k}}{(z+h)^n} \phi(z+h) \left[ f(x_{\min K})^k \prod_{l \in [n] \setminus K} f(x_l) - \prod_{l \in [n]} f(x_l) \right] z \nu(dh)$$

Let us present how this technical proof is organised. The expectation of Equation (D.4) is divided into:

$$\mathbb{E}[H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{x}) - H_{\phi,f}(z, \bar{x})]$$

from which  $A_n^{(1)}$  will arise and

$$\mathbb{E}[H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon}) - H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{x})] \tag{D.5}$$

that will yield  $A_n^{(2)}$  and  $A_n^{(3)}$ .

**Step 1:  $A_n^{(1)}$**

The function  $\phi$  being an element of  $C_c^\infty(\epsilon, 1/\epsilon)$ , we know that

$$\lim_{t \downarrow 0} \sup_{(z, \bar{x}) \in \mathcal{E}} \left| \frac{1}{t} \mathbb{E}[\phi(Z_t) - \phi(z)] - \mathcal{G}\phi(z) \right| = 0$$

where  $\mathcal{G}$  is the generator of the  $\Psi$ -CSBP defined as follows

$$\mathcal{G}\phi(z) = -\gamma z \phi'(z) + \frac{\sigma^2}{2} z \phi''(z) + z \int_{(0,\infty)} (\phi(z+h) - \phi(z) - \mathbf{1}_{(0,1)}(h) h \phi'(z)) \nu(dh)$$

As noticed by Foucart in [33], this result can be obtained by applying Volkonskii's theorem [76] to the generator of the  $\Psi$ -Lévy process, that can be found in Chapter 3 of [3]. Estimates on the supremum and the infimum of the CSBP ensure that

$$\lim_{t \downarrow 0} \sup_{(z, \bar{x}) \in \mathcal{E}} \left| \frac{1}{t} \mathbb{E}[\phi(Z_{t \wedge T_\epsilon}) - \phi(Z_t)] \right| = 0$$

Therefore we have the following convergence

$$\lim_{t \downarrow 0} \sup_{(z, \bar{x}) \in \mathcal{E}} \left| \frac{1}{t} \mathbb{E}[H_{\phi,f}(Z_{t \wedge T_\epsilon}, \bar{x}) - H_{\phi,f}(z, \bar{x})] - A_n^{(1)} H_{\phi,f}(z, \bar{x}) \right| = 0$$

Step 2:  $A_n^{(2)}$  and  $A_n^{(3)}$

Let us denote by  $(\tau_i^{\text{jump}})_{i \geq 1}$  (resp.  $(\tau_i^{\text{bin}})_{i \geq 1}$ ) the increasing sequence of times of the elementary reproduction events involving at least two levels among the  $n$  first and due to the jump part (resp. to the binary part) in  $\hat{\Pi}$ . We start with a simple fact, whose proof is provided at the end of this subsection.

**LEMMA D.2.** *The quantities  $\mathbb{P}(\tau_2^{\text{jump}} \leq t)$ ,  $\mathbb{P}(\tau_2^{\text{bin}} \leq t)$ ,  $\mathbb{P}(\tau_1^{\text{jump}} \vee \tau_1^{\text{bin}} \leq t)$  are of order  $t^2$  as  $t \downarrow 0$  uniformly for  $z \in (\epsilon, 1/\epsilon)$ .*

As we are only considering quantities of order  $t$  (we divide the expectation by  $t$  and let it go to 0), we omit these three events in the sequel so that either there is no event on  $[0, t]$ , and the r.v. in Equation (D.5) vanishes, or there is a single event. Now, we analyse separately the cases  $\tau_1^{\text{jump}} \leq t$  and  $\tau_1^{\text{bin}} \leq t$ . We start with the latter:

$$\begin{aligned} & \mathbb{E} \left[ (H_{\phi, f}(Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon}) - H_{\phi, f}(Z_{t \wedge T_\epsilon}, \bar{x})) ; \tau_1^{\text{bin}} \leq t \right] \\ &= \sum_{1 \leq i < j \leq n} \left( f(x_i)^2 \prod_{l \in [n] \setminus \{i, j\}} f(x_l) - \prod_{l \in [n]} f(x_l) \right) \mathbb{E} \left[ \phi(Z_{t \wedge T_\epsilon}) ; \tau_1^{\text{bin}} \leq t ; \hat{\Pi}_{\tau_1^-, \tau_1}^{[n]} = \mathcal{I}_{\{i, j\}}^{[n]} \right] + \mathcal{O}(t^2) \end{aligned}$$

Notice that

$$\begin{aligned} \mathbb{E} \left[ \phi(Z_{t \wedge T_\epsilon}) ; \tau_1^{\text{bin}} \leq t ; \hat{\Pi}_{\tau_1^-, \tau_1}^{[n]} = \mathcal{I}_{\{i, j\}}^{[n]} \right] &= \mathbb{E} \left[ \phi(Z_{t \wedge T_\epsilon}) \int_0^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds e^{-\int_0^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds} \right] + \mathcal{O}(t^2) \\ &\underset{t \downarrow 0}{\sim} t \frac{\sigma^2}{z} \phi(z) \end{aligned}$$

uniformly for  $z \in (\epsilon, 1/\epsilon)$ . Therefore we deduce that

$$\lim_{t \downarrow 0} \sup_{(z, \bar{x}) \in \mathcal{E}} \frac{1}{t} \left| \mathbb{E} \left[ (H_{\phi, f}(Z_{t \wedge T_\epsilon}, \bar{\xi}_{t \wedge T_\epsilon}) - H_{\phi, f}(Z_{t \wedge T_\epsilon}, \bar{x})) ; \tau_1^{\text{bin}} \leq t \right] - A_n^{(2)} H_{\phi, f}(z, \bar{x}) \right| = 0$$

The case  $\tau_1^{\text{jump}} \leq t$  is similar but more involved: it requires to assume that the jump is larger than a certain threshold  $\delta > 0$  and then to let  $\delta$  go to 0. We do not give the details.

**Proof** (Lemma D.2) For all  $t \geq 0$ , we have

$$\begin{aligned} \mathbb{P}[\tau_1^{\text{jump}} \leq t] &\leq 2^n \mathbb{E} \left[ \sum_{s \leq t \wedge T_\epsilon} \left( \frac{\Delta Z_s}{Z_s} \right)^2 \right] \\ &\leq 2^n t \left( \frac{1}{\epsilon^3} \int_{(0,1)} h^2 \nu(dh) + \frac{1}{\epsilon} \nu([1, \infty)) \right) \end{aligned}$$

thanks to Lemma II.3. So that

$$\begin{aligned} \mathbb{P}[\tau_2^{\text{jump}} \leq t] &\leq 2^{2n} \mathbb{E} \left[ \sum_{s_1 < s_2 \leq t \wedge T_\epsilon} \left( \frac{\Delta Z_{s_1}}{Z_{s_1}} \right)^2 \left( \frac{\Delta Z_{s_2}}{Z_{s_2}} \right)^2 \right] \\ &\leq 2^{2n} t^2 \left( \frac{1}{\epsilon^3} \int_{(0,1)} h^2 \nu(dh) + \frac{1}{\epsilon} \nu([1, \infty)) \right)^2 \end{aligned}$$

Now observe that

$$\begin{aligned} \mathbb{P}[\tau_2^{\text{bin}} \leq t] &= \mathbb{E} \left[ 1 - \frac{1 + \binom{n}{2} \int_0^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds}{\exp\left(\binom{n}{2} \int_0^{t \wedge T_\epsilon} \frac{\sigma^2}{Z_s} ds\right)} \right] \\ &\leq \mathcal{O}(t^2) \end{aligned}$$

uniformly for  $z \in (\epsilon, 1/\epsilon)$ . Finally  $\tau_1^{\text{bin}} \vee \tau_1^{\text{jump}} \leq t$  can be dealt with by combining previous cases.  $\blacksquare$

## 2 Outline of the proof for the linear span

Fix  $f \in \mathcal{B}^{++}([0, 1])$ . We want to show that for  $\epsilon$  small enough, we have

$$\mathbf{E} \left[ G_f(Z_{t \wedge T_\epsilon} \cdot \Xi_{t \wedge T_\epsilon}) - G_f(Z_0 \cdot \Xi_0) - \int_0^{t \wedge T_\epsilon} \mathcal{L}G_f(Z_{s \wedge T_\epsilon} \cdot \Xi_{s \wedge T_\epsilon}) ds \right] = 0 \quad (\text{D.6})$$

Fix  $(a, b) \subset (\epsilon, 1/\epsilon)$  and consider a  $\mathcal{C}^\infty([0, 1])$  function  $\varphi$  such that

1.  $\varphi(0) = 1$  and  $\varphi^{(k)}(0) = 0$  for all  $k \geq 1$ .
2.  $\varphi(x) = 0$  and  $\varphi^{(k)}(x) = 0$  for all  $k \geq 1$  and all  $x \in [1/2, 1]$ .

Notice that  $1/2$  is arbitrary in the definition. Set for any  $p \in \mathbb{N}$  and any  $\mu \in \mathcal{M}_f$

$$J_p(\mu) := \sum_{k=0}^p \frac{(-\langle \mu, f \rangle)^k}{k!} \varphi\left(\frac{(\langle \mu, 1 \rangle - b)_+}{(1/\epsilon - b)}\right) \varphi\left(\frac{(\langle \mu, 1 \rangle - a)_-}{(a - \epsilon)}\right)$$

It is elementary to check that  $J_p$  is a linear combination of maps of the form  $I_f$ . We choose  $p$  such that  $|\sum_{k \leq p} (-x)^k/k! - \exp(-x)|$  is small on a certain compact interval so that  $|J_p(\mu) - G_f(\mu)|$  and  $|\mathcal{A}J_p(\mu) - \mathcal{L}G_f(\mu)|$  are both small whenever  $\langle \mu, 1 \rangle \in (\epsilon, 1/\epsilon)$ . Then using the martingale relation obtained for maps of the form  $I_f$ , we deduce that

$$J_p(Z_{t \wedge T_\epsilon} \cdot \Xi_{t \wedge T_\epsilon}) - J_p(Z_0 \cdot \Xi_0) - \int_0^t \mathcal{A}J_p(Z_{s \wedge T_\epsilon} \cdot \Xi_{s \wedge T_\epsilon}) ds$$

is a martingale. Then we approximate the martingale relation (D.6) with the map  $J_p$  and let  $p$  go to infinity. ■

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