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FINER ANALYSIS OF CHARACTERISTIC CURVE AND ITS APPLICATION TO EXACT, OPTIMAL CONTROLLABILITY, STRUCTURE OF THE ENTROPY SOLUTION OF A SCALAR CONSERVATION LAW WITH CONVEX FLUX

Shyam Ghoshal

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**FINER ANALYSIS OF CHARACTERISTIC CURVE AND ITS APPLICATION TO
EXACT, OPTIMAL CONTROLLABILITY, STRUCTURE OF THE ENTROPY
SOLUTION OF A SCALAR CONSERVATION LAW WITH CONVEX FLUX**

A thesis

submitted to the
Tata Institute of Fundamental Research, Mumbai
for the degree of Doctor of Philosophy
in Mathematics.

by

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to our literature, and acknowledgment of collaborative research and discussions.

This work was done under the guidance of Professor G. D. Veerappa Gowda, at the TIFR Centre for Applicable Mathematics, Bangalore.

Shyam Sundar Ghoshal

In my capacity as supervisor of the candidates thesis, I certify that the above statements are true to the best of my knowledge.

G. D. Veerappa Gowda

Date:

Dedicated To

My father, who first showed me the world of mathematics

My mother, who first showed me the world

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This thesis grew out of countless hours of testing, working and reworking of ideas in front of the blackboard and numerous long sequences of discussions that I have had with Professor Adimurthi and Professor G D V Gowda during the past few years. Not only that this thesis is essentially an outgrowth of this enriching collaboration, their role is so instrumental to it that without them, this thesis would not have ever seen the light of a day.

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ABSTRACT

Goal of this thesis is to study four problems. In chapters 3-5, we consider scalar conservation law in one space dimension with strictly convex flux. First problem is to know the profile of the entropy solution. In spite of the fact that, this was studied extensively in last several decades, the complete profile of the entropy solution is not well understood. Second problem is the exact controllability. This was studied for Burgers equation and some partial results are obtained for large time. It was a challenging problem to know the controllability for all time and also for general convex flux. In a seminal paper [25], Dafermos introduces the characteristic curves and obtain some qualitative properties of a solution of a convex conservation law. In this thesis, we further study the finer properties of these characteristic curves. Here we solve these two problems in complete generality. In view of the explicit formulas of Lax - Oleinik [31], Joseph - Gowda [40], target functions must satisfy some necessary conditions. In this thesis we prove that these are also sufficient. Method of the proof depends highly on the characteristic methods and explicit formula given by Lax - Oleinik and the proof is constructive. Third problem is to solve the optimal controllability problem. In chapter 5 we derive a method to obtain a solution of an optimal control problem for the scalar conservation laws with convex flux. By using the method of descent, this type of problem was considered by Castro-Palacios-Zuazua in [23] for the Burgers equation. Our approach is simple and based on the explicit formulas of Hopf and Lax-Oleinik. Last but not the least is about the problem of total variation bound for solution of scalar conservation laws with discontinuous flux. For the scalar conservation laws with discontinuous flux, an infinite family (A, B) -interface entropies are introduced and each one of them has been shown to form an L^1 -contraction semigroup (see, [8]). One of the main unsettled questions concerning conservation law with discontinuous flux is boundedness of total variation of the solution. Away from the interface, boundedness of total variation of the solution has been proved in a recent paper [16]. In the chapter 6, we discuss this particular issue in detail and produce a counter example to show that the solution, in general, has unbounded total variation near the interface. In fact this example illustrates that smallness of BV norm of the initial data is immaterial. We hereby settled the question of determining for which of the aforementioned (A, B) pairs, the solution will have bounded total variation in case of strictly convex fluxes.

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CHAPTER 1

BASICS OF CONSERVATION LAWS

1.1 Preliminaries

What is conservation laws ?

Let Ω be an open subset of \mathbb{R}^m , and let $f_j, 1 \leq j \leq n$, be n smooth functions from Ω into \mathbb{R}^n . The general form of a system of conservation laws in several space variable is given by

$$\mathbf{u}_t + \sum_{i=1}^n \frac{\partial}{\partial x_i} \mathbf{f}_i(\mathbf{u}) = 0, \quad \mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, t > 0, \quad (1.1.1)$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{pmatrix}$$

is a vector valued function from $\mathbb{R}^n \times [0, \infty)$ into Ω . The set Ω is called the set of states and the functions

$$\mathbf{f}_j = \begin{pmatrix} f_{1j} \\ \cdot \\ \cdot \\ \cdot \\ f_{mj} \end{pmatrix}$$

are called the flux-functions.

Let D be an arbitrary domain of \mathbb{R}^n , and let $\gamma = (\gamma_1, \dots, \gamma_n)^T$ be an outward unit normal to the boundary ∂D of D . Then by Gauss-divergence theorem, (1.1.1) becomes

$$\frac{d}{dt} \int_D \mathbf{u}(\mathbf{x}, t) d\mathbf{x} + \sum_{i=1}^n \int_{\partial D} \mathbf{f}_i(\mathbf{u}) \gamma_i dS = \mathbf{0}.$$

This equation has a very natural meaning : $\int_D \mathbf{u}(\mathbf{x}, t) d\mathbf{x}$ represents the total amount of the densities of various conserved quantities in some physical system within D at time t . Now the above equation typically asserts that the rate of change of $\int_D \mathbf{u}(\mathbf{x}, t) d\mathbf{x}$ within D is governed by flux functions, which controls the rate of loss or increase of \mathbf{u} through ∂D . Hence the system (1.1.1) is called in **conservative form**.

For all $j = 1, \dots, n$, let

$$\mathbf{A}_j(\mathbf{u}) = \left(\frac{\partial \mathbf{f}_{ij}}{\partial \mathbf{u}_k}(\mathbf{u}) \right)_{1 \leq i, k \leq m}$$

be the Jacobian matrix of $\mathbf{f}_j(\mathbf{u})$.

Definition: The system (1.1.1) is called **hyperbolic** if, for any $\mathbf{u} \in \Omega$ and any $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$, $\boldsymbol{\omega} \neq \mathbf{0}$, all the eigenvalues of the matrix $A(\mathbf{u}, \boldsymbol{\omega}) = \sum_{i=1}^n \omega_i \mathbf{A}_i(\mathbf{u})$ are real, say, $\lambda_1(\mathbf{u}, \boldsymbol{\omega}) \leq \dots \leq \lambda_m(\mathbf{u}, \boldsymbol{\omega})$. If, in addition, the eigenvalues $\lambda_k(\mathbf{u}, \boldsymbol{\omega})$ are all distinct, the corresponding m eigenvectors are linearly independent, then the system (1.1.1) is called **strictly hyperbolic**.

If we put $m = 1$ in (1.1.1) then it is called a **scalar conservation laws**. Now our aim is to study initial value problem (IVP) : Find a function \mathbf{u} such that $\mathbf{u}(\mathbf{x}, t) \in D$ which is a solution of (1.1.1) satisfying the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \quad (1.1.2)$$

Where $\mathbf{u}_0 : \mathbb{R}^n \rightarrow D$ is a given function.

It is well known that even if the initial value \mathbf{u}_0 is smooth, the solution to (1.1.1)-(1.1.2) typically develops discontinuities in a finite time. Hence the above problem must be understood in a weak sense.

Concept of weak solutions : Consider the problem (1.1.1)-(1.1.2) and assume $\mathbf{u}_0 \in L_{loc}^\infty(\mathbb{R}^n)^m$, where $L_{loc}^\infty(\mathbb{R}^n)^m$ is the space of locally bounded measurable functions. We say $\mathbf{u} \in L_{loc}^\infty(\mathbb{R}^n \times [0, \infty))^m$ solves (1.1.1)-(1.1.2) weakly if the following holds for all $\varphi \in C_0^1(\mathbb{R}^n \times [0, \infty))^m$

$$\int_0^\infty \int_{\mathbb{R}^n} \left\{ \mathbf{u} \cdot \frac{\partial \varphi}{\partial t} + \sum_{j=1}^n \mathbf{f}_j(\mathbf{u}) \frac{\partial \varphi}{\partial x_j} \right\} dx dt + \int_{\mathbb{R}^n} \mathbf{u}_0 \cdot \varphi(\mathbf{x}, 0) = 0. \quad (1.1.3)$$

We say that a function \mathbf{u} is “piecewise C^1 ” if there exists a finite number of smooth orientable surfaces Γ outside of which \mathbf{u} is a C^1 function and across which \mathbf{u} has jump discontinuity. Given a surface of discontinuity Γ , we denote $\mathbf{n} = (n_t, n_{x_1}, \dots, n_{x_n})^T (\neq 0)$ by a normal vector to Γ and \mathbf{u}_+ and \mathbf{u}_- be the limits of \mathbf{u} on each side of Γ .

$$\lim_{\varepsilon \rightarrow 0} \mathbf{u}_\pm((\mathbf{x}, t) + \varepsilon \mathbf{n}) = \mathbf{u}(\mathbf{x}, t).$$

$$\varepsilon > 0$$

Let $\mathbf{u} : \mathbb{R}^n \times [0, \infty) \rightarrow D$ be a piecewise C^1 function. Then, \mathbf{u} is a solution of (1.1.1)-(1.1.2), in the sense of weak solution on $\mathbb{R}^n \times [0, \infty)$ if and only if the following conditions are satisfied:

- (i) \mathbf{u} is a classical solution of (1.1.1)-(1.1.2) in the domains where \mathbf{u} is C^1 .
- (ii) \mathbf{u} satisfies the jump condition

$$(\mathbf{u}_+ - \mathbf{u}_-)n_t + \sum_{j=1}^n \eta_{x_j} (\mathbf{f}_j(\mathbf{u}_+) - \mathbf{f}_j(\mathbf{u}_-)) = 0 \quad (1.1.4)$$

along the surfaces of discontinuity. The jump relation (1.1.4) is known as the **Rankine-Hugoniot** condition.

In the one-dimensional case ($n = 1$), if we assume Γ is a smooth curve with parametrization $(t, \xi(t))$, and we have

$$\mathbf{n} = (-s, 1)^T, \quad s = \frac{d\xi}{dt}. \quad (1.1.5)$$

So that the Rankine-Hugoniot condition (1.1.4) becomes

$$s[\mathbf{u}] = [\mathbf{f}(\mathbf{u})]. \quad (1.1.6)$$

Example 1.1.1.(Burgers equation) : The scalar parabolic equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}$$

was introduced by Burgers as the simplest differential model for a fluid flow. He studied in particular the limit equation when ϵ tends to zero which we write in conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (u^2/2) = 0. \quad (1.1.7)$$

This is a particular case of (1.1.1) if we put $n = 1, m = 1$ and $f(u) = u^2/2$, trivially it is strictly hyperbolic.

Example 1.1.2.(Euler equations for compressible gas flow in one dimension) :

$$\begin{cases} \rho_t + (\rho v)_x & = 0 & \text{(conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x & = 0 & \text{(conservation of momentum)} \\ (\rho E)_t + (\rho E v + \rho v)_x & = 0 & \text{(conservation of energy).} \end{cases} \quad (1.1.8)$$

Here ρ is the **mass density**, v the **velocity**, and E the **energy density** per unit mass. We assume $E = e + v^2/2$, where e is the **enternal energy** per unit mass and the term $v^2/2$ corresponds to the kinetic energy per unit mass. The p in (1.1.8) denotes the pressure.

Where $p = p(\rho, e)$ is a known function of p and e . Writing $u = (u_1, u_2, u_3) = (\rho, \rho v, \rho E)$, Euler's equation (1.1.8) comprise a stricly hyperbolic system provided we assume $p > 0$ and

$$\frac{\partial p}{\partial \rho} > 0, \frac{\partial p}{\partial e} > 0, \quad (1.1.9)$$

where $p = p(\rho, e)$ is the **constitutive relation** between the **mass density**, the **internal energy density** and the **pressure**. Let us rather change variables and regard the **density** ρ , **velocity** v and **internal energy** e as the unknowns. We can then rewrite Euler's equation (1.1.8) in terms of these quantities, and, so doing, obtain after some calculations the system

$$\begin{cases} \rho_t + v\rho_x + \rho v_x = 0, \\ v_t + vv_x + \frac{1}{\rho}p_x = 0, \\ e_t + ve_x + \frac{p}{\rho}v_x = 0, \end{cases} \quad (1.1.10)$$

provided $\rho > 0$. These equations **are not** in conservative form. Setting now $\mathbf{u} = (u_1, u_2, u_3) = (\rho, v, e)$, we rewrite (1.1.10) as

$$\mathbf{u}_t + \mathbf{B}(\mathbf{u})\mathbf{u}_x = \mathbf{0}, \text{ in } \mathbb{R} \times (0, \infty) \quad (1.1.11)$$

$$\mathbf{B}(z) = z_2 I + \mathbf{D}(z). \quad (1.1.12)$$

The characteristic polynomial of D is $-\lambda(\lambda^2 - \sigma^2)$, for $\sigma^2 = \frac{p}{z_1^2} \frac{\partial p}{\partial e} + \frac{\partial p}{\partial \rho}$.

Recalling (1.1.12) and reverting to physical notation, we see that the eigenvalues of \mathbf{B} are

$$\lambda_1 = v - \sigma, \lambda_2 = v, \lambda_3 = v + \sigma. \quad (1.1.13)$$

Where $\sigma^2 = \frac{p}{z_1^2} \frac{\partial p}{\partial e} + \frac{\partial p}{\partial \rho} > 0$ is the **local sound speed**. We therefore see that the system (1.1.11) is **stricly hyperbolic**, provided the assumption (1.1.9) is valid. At present a good mathematical understanding of the problem (1.1.1) is largely unavailable.

1.1.1 Scalar Conservation Laws

Consider the following scalar conservation laws

$$u_t + \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(t, x, u) = 0, x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \quad (1.1.14)$$

with initial condition

$$u|_{t=0} = u_0(x). \quad (1.1.15)$$

Existence of a solution by Vanishing Viscosity Method : The basic idea for “**Vanishing Viscosity Method**” (see [46]), is to passing to the limit as $\epsilon \rightarrow 0+$ in the

parabolic equation

$$u_t + \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(t, x, u) = \epsilon \Delta u, \quad \epsilon > 0. \quad (1.1.16)$$

Where Δ is the Laplace operator over the space variables x_1, \dots, x_n . This method, which has deep physical meaning, not only allows us to prove the existence of a weak solution of the problem (1.1.14)-(1.1.15) in the sense of the corresponding integral identity, but also makes it possible to show those additional conditions called **entropy condition** on the weak solutions which characterize the uniqueness.

Entropy condition: Kruzkov (see [46]) introduce the following **entropy criteria** to prove the uniqueness :

$$\begin{aligned} & \iint \left\{ |u(t, x) - k| \phi_t + \sum_{i=1}^n \text{sign}(u(t, x) - k) [f_i(t, x, u(t, x)) - f_i(t, x, k)] \phi_{x_i} \right. \\ & \left. - \sum_{i=1}^n \text{sign}(u(t, x) - k) [f_{i x_i}(t, x, k)] \phi \right\} dx dt \geq 0; \end{aligned} \quad (1.1.17)$$

for all $0 \leq \phi \in C_c^\infty(\mathbb{R}^n)$, $k \in \mathbb{R}$.

1.1.2 Scalar conservation laws with strict convex flux

Let us consider the following scalar conservation laws with strict convex flux

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{if } (x, t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x, 0) &= u_0(x) \in L^\infty(\mathbb{R}) \end{aligned} \quad (1.1.18)$$

where the flux $f \in C^2(\mathbb{R})$ be a strict convex function with superlinear growth i.e.,

$$\lim_{u \rightarrow \infty} \frac{f(u)}{|u|} = \infty.$$

The Legendre transform (convex dual) of f is defined by

$$f^*(p) = \sup_{q \in \mathbb{R}} \{pq - f(q)\}. \quad (1.1.19)$$

We say f is uniformly convex if for some constant $\theta > 0$ the following holds

$$f'' \geq \theta > 0.$$

We now try to obtain a formula for an appropriate weak solution of the initial-value problem (1.1.18). Without loss of generality we may as well also take

$$f(0) = 0.$$

As motivation, suppose now $u_0 \in L^\infty(\mathbb{R})$ and define

$$v_0(x) := \int_0^x u_0(y) dy \quad (x \in \mathbb{R}). \quad (1.1.20)$$

Recalling the Hopf-Lax formula (see third chapter of [31]) :

$$w(x, t) := \min_{y \in \mathbb{R}} \left\{ v_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\} \quad (x \in \mathbb{R}, t > 0). \quad (1.1.21)$$

Then w is unique, weak solution for the following **Hamilton-Jacobi** equation with Lipschitz continuous initial data v_0 :

$$\begin{aligned} w_t + f(w_x) &= 0 & \text{in} & \quad \mathbb{R} \times (0, \infty) \\ w &= v_0 & \text{on} & \quad \mathbb{R} \times \{t = 0\}. \end{aligned} \quad (1.1.22)$$

For the moment assume w is smooth. Now by differentiate the PDE and its initial condition with respect to x , we deduce

$$\begin{aligned} w_{xt} + f(w_x)_x &= 0 & \text{in} & \quad \mathbb{R} \times (0, \infty) \\ w_x &= u_0 & \text{on} & \quad \mathbb{R} \times t = 0. \end{aligned} \quad (1.1.23)$$

Hence if we set $u = w_x$, we see u solves problem (1.1.18).

The foregoing computational is only formal, as we know that w defined by (1.1.21) is not in general smooth. But the solution w of (1.1.22) is Lipschitz continuous, hence

$$u(x, t) = \frac{\partial}{\partial x} \left[\min_{y \in \mathbb{R}} \left\{ v_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\} \right] \quad (1.1.24)$$

is defined for a.e.. Lax-Oleinik has given an elegant formula for (1.1.24).

Theorem 1.1.1. (Lax-Oleinik formula). *Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth convex and $u_0 \in L^1(\mathbb{R})$.*

- (i) *For each time $t > 0$, there exists for all but at most countably many values of $x \in \mathbb{R}$, a unique point $y(x, t)$ such that*

$$\min_{y \in \mathbb{R}} \left\{ v_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\} = v_0(y(x, t)) + t f^* \left(\frac{x-y(x, t)}{t} \right).$$

- (ii) *The mapping $x \mapsto y(x, t)$ is nondecreasing.*

- (iii) *For each time $t > 0$, the function u defined by (1.1.24) is*

$$u(x, t) = (f')^{-1} \left(\frac{x-y(x, t)}{t} \right) \quad (1.1.25)$$

for a.e x . In particular, formula (1.1.25) holds for $(x, t) \in \mathbb{R} \times (0, \infty)$.

Theorem 1.1.2. *Under the assumption of Theorem 1.1.1, the function u defined by (1.1.25) is a weak solution of the initial-value problem (1.1.18).*

1.2 Long Time Behavior

a. Decay in sup-norm.

Theorem 1.2.1. *(Asymptotics in L^∞ -norm). Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth uniformly convex and the initial data u_0 has compact support then there exists a constant C such that*

$$|u(x, t)| \leq \frac{C}{t^{1/2}}.$$

b. Decay to N -wave.

Theorem 1.2.2. *(Asymptotics in L^1 -norm). Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth uniformly convex. Assume u_0 has compact support and $p, q > 0$. Then there exists a constant C such that*

$$\int_{-\infty}^{\infty} |u(\cdot, t) - N(\cdot, t)| dx \leq \frac{C}{t^{1/2}}.$$

where the N -wave is given by

$$N(x, t) := \begin{cases} \frac{1}{d} \left(\frac{x}{t} - \sigma \right) & \text{if } -(pdt)^{1/2} < x - \sigma t < (qdt)^{1/2} \\ 0 & \text{otherwise.} \end{cases}$$

and $d := f''(0) > 0$, $((f')^{-1})'(\sigma) = \frac{1}{d}$,

$$p := -2 \min_{y \in \mathbb{R}} \int_{-\infty}^y u_0 dx, \quad q := 2 \max_{y \in \mathbb{R}} \int_y^{\infty} u_0 dx.$$

1.3 Conservation law with discontinuous flux

Introduction. Let us introduce the following single scalar conservation law of the form:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(x, u)) &= 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \mathbb{R} \end{aligned} \tag{1.3.1}$$

where the flux function $F(x, u)$ is given by $F(x, u) = H(x)f(u) + (1 - H(x))g(u)$, which is a **discontinuous function** of x , H is the Heaviside function, f and g are smooth

functions on \mathbb{R} . We are going to consider the equation with the assumption that both f and g are smooth and are of convex type.

It is well known that after a finite time, the IVP (1.3.1), in general, does not admit a continuous solution even if u_0 is sufficiently smooth. Henceforth, by a solution u of (1.3.1) we mean a solution in the weak sense. That is $u \in L_{\text{loc}}^\infty$ such that for all $\varphi \in C_0^1(\overline{\mathbb{R} \times \mathbb{R}_+})$,

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(u \frac{\partial \varphi}{\partial t} + F(x, u) \frac{\partial \varphi}{\partial x} \right) dt dx + \int_{-\infty}^{\infty} u(x, 0) \varphi(x, 0) dx = 0. \quad (1.3.2)$$

It is easy to see that (1.3.2) is the weak formulation of the following problem. Denoting $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, then in the weak sense u satisfies

$$\begin{aligned} u_t + f(u)_x &= 0 & \text{for } x > 0, \quad t > 0, \\ u_t + g(u)_x &= 0 & \text{for } x < 0, \quad t > 0, \\ u(x, 0) &= u_0(x), \end{aligned} \quad (1.3.3)$$

and at $x = 0$, u satisfies the Rankine-Hugoniot condition i.e., for almost all $t > 0$,

$$f(u(0+, t)) = g(u(0-, t)) \quad (1.3.4)$$

where $u(0+, t) = \lim_{x \rightarrow 0+} u(x, t)$ and $u(0-, t) = \lim_{x \rightarrow 0-} u(x, t)$.

Even in the case where the flux is smooth ($f = g$), it is well known that weak solutions are not necessarily unique and have to be augmented by extra admissibility criteria, the so called entropy conditions.

Kruzkov (see [46]) proved that if F is continuous in u and $\frac{\partial F}{\partial x}$ is bounded, then (1.3.1) admits a weak solution. Hence in general the problem (1.3.3) may not admit a solution, even if it admits solution may not be unique.

Non existence of solution: If F is discontinuous in x , Kruzkov's method does not guarantee a solution. For example in (1.3.3) if we take $g(u) = u$, $f(u) = -u$, $u_0(x) = 2$ if $x < 0$ and $u_0(x) = 3$ if $x > 0$, then it is easy to see that $u(x, t) = 2$ if $x < 0$ and $u(x, t) = 3$ if $x > 0$, is a solution for (1.3.3) but do not satisfy (1.3.4). Hence it is not a weak solution of (1.3.1).

Non uniqueness of solution: Now in (1.3.3) if we consider $g(u) = -u$, $f(u) = u$, $u_0 = 0 \quad \forall x$, then for any $\lambda > 0$,

$$u(x, t) = \begin{cases} 0 & \text{if } x < t < 0 \\ -\lambda & \text{if } t < x < 0 \\ \lambda & \text{if } x < t < 0 \\ 0 & \text{if } 0 < t < x \end{cases} \quad (1.3.5)$$

satisfies (1.3.3)-(1.3.4) which shows that weak solutions of (1.3.1) may not be unique.

The discontinuity of the flux function at $x = 0$ causes a discontinuity of a solution which in general not uniquely determined by the initial data.

Adimurthi and Gowda [6] have given an explicit formula for the solution of (1.3.3) satisfying (1.3.4). This agrees with the Lax-Oleinik (see third chapter of [31]) formula when $f = g$. Also they have given an interface entropy condition at $x = 0$ so that the problem (1.3.3)-(1.3.4) admits a unique solution determined by the initial condition like in Kruzkov [46]. This explicit formula satisfies some extra entropy condition called interface entropy condition. Using this entropy condition and Lax-Oleinik entropy condition away from $x = 0$ (Interior Entropy Condition) Adimurthi and Gowda [6] prove the uniqueness like in Kruzkov by showing that the solution forms L^1 contractive semigroup.

Now we state the following results without proof, for details see Adimurthi and Gowda [6].

Entropy conditions and Uniqueness results :

Interior Entropy Condition ($E_{Interior}$): u is said to satisfy the entropy condition ($E_{Interior}$) (Lax-Oleinik entropy conditions) if for all $t > 0$

$$\lim_{0 < z \rightarrow 0} u(x + z, t) \leq \lim_{0 < z \rightarrow 0} u(x - z, t) \quad \text{if } x > 0 \quad (1.3.6)$$

$$\lim_{0 < z \rightarrow 0} u(x + z, t) \leq \lim_{0 < z \rightarrow 0} u(x - z, t) \quad \text{if } x < 0. \quad (1.3.7)$$

Interface Entropy Condition ($E_{Interface}$): At $x = 0$,

$u(0+, t) = \lim_{x \rightarrow 0+} u(x, t)$, $u(0-, t) = \lim_{x \rightarrow 0-} u(x, t)$ exist for almost all $t > 0$. Furthermore for almost all $t > 0$ one of the following conditions must hold:

$$f'(u(0+, t)) \geq 0 \quad \text{and} \quad g'(u(0-, t)) \geq 0, \quad (1.3.8)$$

$$f'(u(0+, t)) \leq 0 \quad \text{and} \quad g'(u(0-, t)) \leq 0, \quad (1.3.9)$$

$$f'(u(0+, t)) \leq 0 \quad \text{and} \quad g'(u(0-, t)) \geq 0. \quad (1.3.10)$$

Entropy pairs: Let φ_1, φ_2 be convex functions. Let $\psi'_1(s) = f'(s)\varphi'_1(s)$, $\psi'_2(s) = g'(s)\varphi'_2(s)$. Then (φ_i, ψ_i) , $i = 1, 2$ are called entropy pairs associated to (1.3.1).

Kruzkov Entropy condition ($E_{Kruzkov}$): A weak solution $u \in L^\infty_{loc}$ of (1.3.3) and (1.3.3) is said to satisfy ($E_{Kruzkov}$) if for every entropy pairs (φ_i, ψ_i) $i = 1, 2$ and for every $\rho \in C^1_0(\mathbb{R} \times \mathbb{R}_+)$, $\rho \geq 0$

$$\int_0^\infty \int_0^\infty \left(\phi_1(u) \frac{\partial \rho}{\partial t} + \psi_1(u) \frac{\partial \rho}{\partial x} \right) dt dx \geq - \int_0^\infty \psi_1(u(0+, t)) \rho(0, t) dt \quad (1.3.11)$$

$$\int_{-\infty}^0 \int_0^\infty \left(\phi_2(u) \frac{\partial \rho}{\partial t} + \psi_2(u) \frac{\partial \rho}{\partial x} \right) dt dx \geq \int_0^\infty \psi_2(u(0-, t)) \rho(0, t) dt. \quad (1.3.12)$$

Then we have the following

Theorem 1.3.1. *Let $u, v \in L^\infty$ be two weak solutions of (1.3.3) and (1.3.4). Assume that*

u, v satisfy the entropy conditions $(E_{Interface})$ and $(E_{Kruzkov})$ and satisfy initial condition in the following sense:

$$\lim_{t \rightarrow 0^+} \|u(\cdot, t) - u_0\|_{L^1} = \lim_{t \rightarrow 0^+} \|v(\cdot, t) - u_0\|_{L^1} = 0$$

Then $u \equiv v$.

Theorem 1.3.2. Let $u, v \in L^\infty \cap BV_{loc}$ be two solutions of (1.3.3) and (1.3.4) satisfying $(E_{Interior})$ and $(E_{Interface})$. Further more assume that the set of points of discontinuity of u and v are discrete set of Lipschitz curves. Then $u \equiv v$.

Explicit formula for the solution

Before going to the explicit formula, let us recall some well known results on convex functions without proof.

$f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a strictly convex and superlinear growth if for $a \neq b, t \in (0, 1)$

$$f(ta + (1-t)b) < tf(a) + (1-t)f(b) \quad \text{and} \quad \lim_{a \rightarrow \infty} \frac{f(a)}{|a|} = \infty.$$

Define convex dual f^* of f by

$$f^*(x) = \sup_{y \in \mathbb{R}} \{xy - f(y)\}.$$

If f is strictly convex and super linear growth then f and f^* satisfies the following:

- (a) $f^*(0) = -\min f$, is finite
- (b) f^* is strictly convex and super linear growth and satisfy

$$f(y) = \sup_{x \in \mathbb{R}} \{xy - f^*(x)\}.$$

Definition 1.3.3. (Admissible curves). Let $0 \leq s < t$ and $\xi \in c([s, t], \mathbb{R})$.

ξ is called an admissible curve if the following hold.

1. ξ consists of atmost three linear curves and each segment lies completely in either $x \geq 0$ or $x \leq 0$.
2. Let $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be such that for $i = 1, 2, 3$, $\xi_i = \xi|_{[t_i, t_{i-1}]}$ be the linear parts of ξ . If ξ consists of three linear curves then $\xi_2 = 0$.

Let

$$\begin{aligned} c(x, t, s) &= \{\xi \in c([s, t], \mathbb{R}); \xi(t) = x, \xi \text{ is an admissible curve}\} \\ c(x, t) &= c(x, t, 0). \end{aligned} \tag{1.3.13}$$

Divide $c(x, s, t)$ into three categories defined as below.

$$\begin{aligned}
c_0(x, t, s) &= \{\xi \in c(x, t, s); \xi \text{ is linear and } x\xi(\theta) \geq 0 \forall \theta \in [s, t]\}. \\
c_r(x, t, s) &= \{\xi \in c(x, t, s); \xi \text{ consists of three pieces and } x\xi(\theta) \geq 0 \\
&\forall \theta \in [s, t]\}. \\
c_b(x, t, s) &= c(x, t, s) - \{c_r(x, t, s) \cup c_0(x, t, s)\}. \\
c_l(x, t) &= c_l(x, t, 0) \text{ for } l = 0, r, b
\end{aligned} \tag{1.3.14}$$

Let f^*, g^* denote the convex duals of f and g respectively. Let w be a function on \mathbb{R} and $\xi \in c(x, t, s)$. Define

$$\begin{aligned}
\rho_{\xi, w}(x, t, s) &= w(\xi(s)) + \int_{\{\theta \in [s, t]; \xi(\theta) > 0\}} f^*\left(\frac{d\xi}{d\theta}\right) d\theta + \int_{\{\theta \in [s, t]; \xi(\theta) < 0\}} g^*\left(\frac{d\xi}{d\theta}\right) d\theta \\
&+ \text{meas}\{\theta \in [s, t]; \xi(\theta) = 0\} \min\{f^*(0), g^*(0)\}.
\end{aligned} \tag{1.3.15}$$

Theorem 1.3.4. *Let v_0 be a uniformly Lipschitz continuous function on \mathbb{R} and $\rho_\xi(x, t) = \rho_{\xi, v_0}(x, t, 0)$. Define*

$$v(x, t) = \inf \{\rho_\xi(x, t), \quad \xi \in c(x, t)\}. \tag{1.3.16}$$

Then v is a uniformly Lipschitz continuous function satisfying the following:

(i)

$$\begin{aligned}
v_t + f(v_x) &= 0 \quad \text{in } x > 0, t > 0 \\
v_t + g(v_x) &= 0 \quad \text{in } x < 0, t > 0.
\end{aligned} \tag{1.3.17}$$

(ii) For almost every t , $v_x(0+, t) = \lim_{x \rightarrow 0+} v_x(x, t)$ and $v_x(0-, t) = \lim_{x \rightarrow 0-} v_x(x, t)$ exist and satisfy $f(v_x(0+, t)) = g(v_x(0-, t))$. Furthermore there exist disjoint sets V, S_1, S_2 such that $(0, \infty) = V \cup S_1 \cup S_2$, V an open set, $\text{meas}(S_2) = 0$ with the property that for almost every $t \in V$, one of the following pairs of inequalities holds:

$$f'(v_x(0+, t)) \geq 0, \quad g'(v_x(0-, t)) \geq 0 \tag{1.3.18}$$

$$f'(v_x(0+, t)) \leq 0, \quad g'(v_x(0-, t)) \leq 0 \tag{1.3.19}$$

and if $t \in S_1$, then

$$f'(v_x(0+, t)) \leq 0, \quad g'(v_x(0-, t)) \geq 0. \tag{1.3.20}$$

(iii) There exist a constant $M > 0$ and Lipschitz continuous functions $R_1(t) \geq 0$, $L_1(t) \geq 0$ on $[0, \infty)$ with $R_1(0) = L_1(0) = 0$ such that for all $z > 0$

$$f'(v_x(x+z, t)) - f'(v_x(x, t)) \leq \begin{cases} \frac{Mz}{z+x} & \text{if } 0 < x < x+z < R_1(t) \\ \frac{z}{t} & \text{if } x > 0 \text{ and not in the above range.} \end{cases} \tag{1.3.21}$$

$$g'(v_x(x, t)) - g'(v_x(x - z, t)) \leq \begin{cases} \frac{Mz}{z-x} & \text{if } L_1(t) < x - z \text{ and } x < 0 \\ \frac{z}{t} & \text{if } x < 0 \text{ and not in the above range.} \end{cases} \quad (1.3.22)$$

Theorem 1.3.5. *Let $u_0 \in L^\infty(\mathbb{R})$ and $v_0(x) = \int_0^x u_0(\theta) d\theta$. Then $u = \frac{\partial v}{\partial x}$ is a weak solution of (1.3.3) and (1.3.4). Also u satisfies the entropy condition (E_i) and (E_b) and the solution can be given explicitly as follows:*

There exist Lipschitz continuous functions $R_1(t) \geq 0$ and $L_1(t) \leq 0$ on $(0, \infty)$ and bounded variation functions $y_+(x, t)$ for $x \geq 0$ (non increasing in $(0, R_1(t))$ and non decreasing in $[R_1(t), \infty)$) and $y_-(x, t)$ for $x \leq 0$ (non decreasing in $(-\infty, 0)$) such that

(i) *For $x > 0$,*

$$u(x, t) = \begin{cases} (f')^{-1} \left(\frac{x}{t - y_+(x, t)} \right) & \text{if } x \leq R_1(t) \\ (f')^{-1} \left(\frac{x - y_+(x, t)}{t} \right) & \text{if } x > R_1(t). \end{cases} \quad (1.3.23)$$

(ii) *For $x < 0$,*

$$u(x, t) = \begin{cases} (g')^{-1} \left(\frac{x}{t - y_-(x, t)} \right) & \text{if } x \leq L_1(t) \\ (g')^{-1} \left(\frac{x - y_-(x, t)}{t} \right) & \text{if } x > L_1(t). \end{cases} \quad (1.3.24)$$

Furthermore u is unique in the class of all solutions for which the set of discontinuities is a discrete set of Lipschitz curves.

CHAPTER 2

INTRODUCTION

2.1 Introduction to the problem of exact, optimal controllability and structure of the solution of scalar conservation laws with strict convex flux

In this chapter we give a brief introduction to the problems which have been tackled in this thesis. Consider the following scalar conservation law in one space dimension.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly convex C^2 function satisfying the super linear growth,

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty. \quad (2.1.1)$$

Let $T > 0$, $I = (A, B)$, $\Omega = I \times (0, T)$, $u_0 \in L^\infty(I)$, $b_0, b_1 \in L^\infty((0, T))$ and consider the problem

$$u_t + f(u)_x = 0 \quad (x, t) \in \Omega, \quad (2.1.2)$$

$$u(x, 0) = u_0(x) \quad x \in I, \quad (2.1.3)$$

$$u(A, t) = b_0(t) \quad t \in (0, T), \quad (2.1.4)$$

$$u(B, t) = b_1(t) \quad t \in (0, T). \quad (2.1.5)$$

This problem was well studied from last several decades starting from the pioneering works of Lax-Oleinik [31], Kruzkov [46], Bardaux-Leraux-Nedeleck [13]. They have studied the existence and uniqueness of weak solutions to (2.1.2)-(2.1.5) satisfying the entropy condition. In spite of being well studied, still there are problems which are open. Notably among them are

1. Profile of a solution, for example how many shocks can a solution exhibit and the nature of the shocks.
2. Exact controllability of initial and initial-boundary value problem.
3. Optimal controllability for initial and initial-boundary value problem.

In the forthcoming chapters we study these problems for the entropy solution of (2.1.2) and we say a solution means a weak solution satisfying the entropy condition. The basic ingredient in studying all these problems comes from the analysis of characteristic curves R_{\pm} . Originally this was introduced by Hopf [33] and later by Dafermos [25], who studied them quite extensively to obtain information on the nature of solutions. Independently this was used in [6] to obtain the explicit formula for solution of discontinuous flux.

First we study the finer properties of the characteristics for the initial value problem, namely

- (i). Comparison properties with respect to the initial data.
- (ii). Failure of the continuity with respect to the initial data.
- (iii). Behavior of the characteristics when one side of the initial data is large.

Before stating the main results for structure of the solution of (2.1.2), we recall some well known results without proof.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 convex function. Assume that

1. f has superlinear growth, that is

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty.$$

2. f is strictly convex. That is for $a, b \in \mathbb{R}$, $a = b$ if and only if

$$f(a) - f(b) - (a - b)f'(b) = 0.$$

3. Let $f^*(p) = \sup_q \{pq - f(q)\}$ be the Legendre transform of f . Then $f^* \in C^1$ and f^* is of superlinear growth and satisfies

$$\begin{aligned} f &= f^{**}, \\ f^{*'}(p) &= (f')^{-1}(p), \\ f^*(f'(p)) &= pf'(p) - f(p), \\ f(f^{*'}(p)) &= pf^{*'}(p) - f^*(p). \end{aligned} \tag{2.1.6}$$

Controlled Curves: Let $x \in \mathbb{R}$, $0 \leq s < t$ and define the class of controlled curves $\Gamma(x, s, t)$ by

$$\Gamma(x, s, t) = \{r : [s, t] \rightarrow \mathbb{R}; r \text{ is a straight line with } r(t) = x\}, \tag{2.1.7}$$

and denote $\Gamma(x, t) = \Gamma(x, 0, t)$.

Value function: Let $u_0 \in L^\infty(\mathbb{R}), b \in \mathbb{R}$, define

$$v_0(x) = \int_b^x u_0(\theta) d\theta, \quad (2.1.8)$$

be its primitive. Define the value function $v(x, t)$ associated with f and u_0 by

$$\begin{aligned} v(x, t) &= \min_{r \in \Gamma(x, t)} \left\{ v_0(r(0)) + t f^* \left(\frac{x - r(0)}{t} \right) \right\} \\ &= \min_{\beta \in \mathbb{R}} \left\{ v_0(\beta) + t f^* \left(\frac{x - \beta}{t} \right) \right\}. \end{aligned} \quad (2.1.9)$$

Then v satisfies the

Dynamic Programming principle: Let $0 \leq s < t$, then

$$v(x, t) = \min_{r \in \Gamma(x, s, t)} \left\{ v(r(s), s) + (t - s) f^* \left(\frac{x - r(s)}{t - s} \right) \right\}. \quad (2.1.10)$$

Characteristics: Define the characteristic set $ch(x, t, s)$ and extreme characteristic points $y_\pm(x, t, s)$ by

$$ch(x, t, s) = \{r \in \Gamma(x, t, s); r \text{ is a minimizer in (2.1.10)}\}, \quad (2.1.11)$$

$$\begin{aligned} y_-(x, t, s) &= \min\{r(s) : r \in ch(x, t, s)\}, \\ y_+(x, t, s) &= \max\{r(s) : r \in ch(x, t, s)\}. \end{aligned} \quad (2.1.12)$$

$$r_\pm(\theta, x, t, s) = x + \frac{x - y_\pm(x, t, s)}{t - s}(\theta - t). \quad (2.1.13)$$

Denote $ch(x, t) = ch(x, t, 0), y_\pm(x, t) = y_\pm(x, t, 0), r_\pm(\cdot, x, t) = r_\pm(\cdot, x, t, 0)$. For $r \in ch(x, t, s)$, we say, $r(s)$ a characteristic point.

Then we have the following theorem due to Hopf, Lax-Oleinik (see [31]).

Theorem 2.1.1. *Let $0 \leq s < t, u_0, v_0, v$ be as above, then*

(1). *v is a uniformly Lipschitz continuous viscosity solution of the Hamilton-Jacobi equation*

$$\begin{aligned} v_t + f(v_x) &= 0 & (x, t) \in \mathbb{R} \times (0, \infty), \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \quad (2.1.14)$$

(2). *There exist $M > 0$, depending only on $\|u_0\|_\infty$ and Lipschitz constant of f, f^* restricted to $[-\|u_0\|_\infty, \|u_0\|_\infty]$ such that for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+, r \in ch(x, t, s)$*

$$\left| \frac{x - r(s)}{t - s} \right| \leq M. \quad (2.1.15)$$

(3). **NIP (Non intersecting property of characteristics)**: Let $x_1 \neq x_2$, $0 \leq s < \min\{t_1, t_2\}$, $r_i \in \text{ch}(x_i, t_i, s)$ and for $i = 1, 2$. Then for all $\theta \in (s, \min\{t_1, t_2\})$,

$$r_1(\theta) \neq r_2(\theta). \quad (2.1.16)$$

From NIP, it follows that

(a). $x \mapsto y_{\pm}(x, t, s)$ are non decreasing functions.

(b). At the points of continuity of y_+ ,

$$y_+(x, t, s) = y_-(x, t, s)$$

and hence $\text{ch}(x, t, s) = \{r\}$ given by

$$r(\theta) = \frac{x - y_+(x, t, s)}{t - s}(\theta - t) + x.$$

(c). Let $r \in \text{ch}(x, t)$, $z = r(s)$. Let $r_1 = r|_{[0, s]}$, $r_2 = r|_{[s, t]}$, then $r_1 \in \text{ch}(z, s)$, $r_2 \in \text{ch}(x, t, s)$.

(4). Let $u(x, t) = \frac{\partial v}{\partial x}(x, t)$. Then u is the unique solution of (2.1.2) with initial data u_0 and satisfying

$$|u(x, t)| \leq \|u_0\|_{\infty}. \quad (2.1.17)$$

For a.e x , $y_-(x, t) = y_+(x, t)$ and u is given by

$$f'(u(x, t)) = \frac{x - y_+(x, t)}{t} = \frac{x - y_-(x, t)}{t}. \quad (2.1.18)$$

Furthermore if x be a point of differentiability of $y_{\pm}(x, t)$ and $y_{\pm}(x, t)$ is a point of differentiability of v_0 , then

$$u(x, t) = u_0(y_{\pm}(x, t)). \quad (2.1.19)$$

(5). If $u_0 \leq v_0$, then $u(x, t) \leq v(x, t)$, for a.e. $t > 0$ and for a.e. $x \in \mathbb{R}$.

In this sequel we will follow the notation of characteristic curves as in [6].

Definition 2.1.2. (Characteristic Curves R_{\pm}) : Let $\alpha \in \mathbb{R}$, $0 \leq s < t$. Define

$$\begin{aligned} R_-(t, s, \alpha) &= \inf\{x; \alpha \leq y_-(x, t, s)\}, \\ R_+(t, s, \alpha) &= \sup\{x; y_+(x, t, s) \leq \alpha\}, \\ R_{\pm}(t, \alpha) &= R_{\pm}(t, 0, \alpha). \end{aligned} \quad (2.1.20)$$

Definition 2.1.3. (Characteristic and regular characteristic line): Let $\alpha, p \in \mathbb{R}$ and $r(t, \alpha, p) = \alpha + tf'(p)$ be a stright line passing through $(\alpha, 0)$ with reciprocal slope $f'(p)$. Then

- (i). r is called a characteristic line if for all $t > 0$, $r(\cdot, \alpha, p) \in ch(r(t, \alpha, p), t)$.
- (ii). Let r be a characteristic line, then r is called regular characteristic line if for all $t > 0$

$$y_+(r(t, \alpha, p), t) = y_-(r(t, \alpha, p), t) = \alpha. \quad (2.1.21)$$

Definition 2.1.4. (Base point and value) : Let $r(\cdot, \alpha, p)$ be a characteristic line. Then α is called the base point and p is called value of the characteristic line $r(\cdot, \alpha, p)$.

Definition 2.1.5. (ASSP-Asymptotically Single Shock Packet) : Let $C_1 < C_2, p \in \mathbb{R}$ and for $i = 1, 2$, $r(\cdot, C_i, p)$ as above. Then define

$$D(C_1, C_2, p) = \{(x, t) : r(t, C_1, p) < x < r(t, C_2, p), t > 0\}. \quad (2.1.22)$$

Then $D(C_1, C_2, p)$ is called ASSP if

- (i). $r(\cdot, C_1, p), r(\cdot, C_2, p)$ are regular characteristic lines.
- (ii). For all $C_1 < \alpha < C_2$, $R_{\pm}(t, \alpha) \in D(C_1, C_2, p)$ for all $t > 0$.
- (iii). For all $C_1 < \alpha \leq \beta < C_2$ there exists $T > 0$ such that for $t \geq T$,

$$R_+(t, \alpha) = R_-(t, \alpha) = R_+(t, \beta) = R_-(t, \beta). \quad (2.1.23)$$

In otherwords all the characteristic curves in D will merge at infinity and represents a single shock at infinity.

With these preliminaries, we can state our main results. Let $A_1 < A_2$, $u_{\pm} \in \mathbb{R}$, $\bar{u}_0 \in L^{\infty}(A_1, A_2)$ and define u_0 by

$$u_0(x) = \begin{cases} u_- & \text{if } x < A_1, \\ \bar{u}_0 & \text{if } A_1 < x < A_2, \\ u_+ & \text{if } x > A_2. \end{cases} \quad (2.1.24)$$

Let u be the solution of (2.1.2) with u_0 as the initial data.

Definition 2.1.6. (Single shock case) : Let u and u_0 be as above, then u is said to be a case of single shock if there exist $T > 0$, $x_0 \in \mathbb{R}$ such that

$$u(x, T) = \begin{cases} u_- & \text{if } x < x_0, \\ u_+ & \text{if } x > x_0. \end{cases} \quad (2.1.25)$$

As a consequence of this, for $t > T$, u is given by

$$u(x, T) = \begin{cases} u_- & \text{if } x < x_0 + (t - T) \left(\frac{f(u_+) - f(u_-)}{u_+ - u_-} \right), \\ u_+ & \text{if } x > x_0 + (t - T) \left(\frac{f(u_+) - f(u_-)}{u_+ - u_-} \right). \end{cases} \quad (2.1.26)$$

Then we have the following structure Theorem.

Main results for structure of the entropy solution

Theorem 2.1.7. (*Structure Theorem*) *Let u and u_0 be as above. Then*

(I). *u represents a case of single shock if and only if $u_- > u_+$.*

(II). *Let $u_- \leq u_+$. Then there exist $A_1 \leq B_1 \leq B_2 \leq A_2$ and a countable number of disjoint ASSP $\{D(C_{1i}, C_{2i}, p_i)\}_{i \in I}$ such that*

(i). *$r(\cdot, B_1, u_-), r(\cdot, B_2, u_+)$, are regular characteristic lines and for $t > 0$,*

$$R_-(t, A_1) \leq r(t, B_1, u_-) \leq r(t, B_2, u_+) \leq R_+(t, A_2) \quad (2.1.27)$$

$$u(x, T) = \begin{cases} u_- & \text{if } x < R_-(t, A_1), \\ u_+ & \text{if } x > R_+(t, A_2) \end{cases} \quad (2.1.28)$$

(ii). *Let*

$$\begin{aligned} E &= \{(x, t) : r(t, B_1, u_-) \leq x \leq r(t, B_2, u_+)\} \\ S &= \cup_{i \in I} D(C_{1i}, C_{2i}, p_i) \\ R &= E \setminus S. \end{aligned}$$

Then R consists of all regular characteristic lines and u is continuous on R .

(iii). *For each $i \in I$, u_0 satisfies*

$$\int_{C_{1i}}^{C_{2i}} u_0(x) dx = (C_{2i} - C_{1i}) p_i \quad (2.1.29)$$

and if u_0 is continuous in a neighbourhood of $\{C_{1i}, C_{2i}\}$, then

$$u_0(C_{1i}) = u_0(C_{2i}) = p_i. \quad (2.1.30)$$

(iv) Let

$$\begin{aligned}
F_- &= \{(x, t) : x < R_-(t, A_1)\} \\
D_- &= \{(x, t) : R_-(t, A_1) < x < r(t, B_1, u_-)\} \\
F_+ &= \{(x, t) : x > R_+(t, A_2)\} \\
D_+ &= \{(x, t) : r(t, B_2, u_+) < x < R_+(t, A_2)\}
\end{aligned}$$

then $\mathbb{R} \times (0, \infty) = F_- \cup F_+ \cup D_- \cup D_+ \cup S \cup R$. Define the N -wave by

$$N(x, t) = \begin{cases} u_- & \text{if } (x, t) \in F_- \\ u_+ & \text{if } (x, t) \in F_+ \\ \frac{x - B_1}{t} & \text{if } (x, t) \in D_- \\ \frac{x - B_2}{t} & \text{if } (x, t) \in D_+ \\ p_i & \text{if } (x, t) \in D(C_{1,i}, C_{2,i}, p_i) \\ p & \text{if } (x, t) \in R, \text{ } p \text{ is the value of the regular} \\ & \text{characteristic line on which } (x, t) \text{ lies.} \end{cases} \quad (2.1.31)$$

Then $x \mapsto N(x, t)$ is a continuous non decreasing function on $D_+ \cup D_- \cup S \cup R$.

(v). Suppose f satisfies the following growth assumptions : There exist $\gamma > 0, \delta > 0, C > 0$ such that for $|h| \leq \delta$,

$$f(u_\pm) - f(u_\pm + h) + hf'(u_\pm + h) \geq C|h|^\gamma \quad (2.1.32)$$

and for any compact interval J , there exists $C(J) > 0, \eta > 0$ such that for all $a, b \in J$

$$|f'(a) - f'(b)| \leq C(J)|a - b|^\eta. \quad (2.1.33)$$

Then for t large we have the following decay estimate

$$\int_{-\infty}^{\infty} |f'(u(x, t)) - f'(N(x, t))| dx = O\left(\frac{1}{t^{\eta/\gamma}}\right). \quad (2.1.34)$$

Furthermore if some $r > 0, a, b \in J, C_1(J) > 0$,

$$|f'(a) - f'(b)| \geq C_1(J)|a - b|^r$$

then,

$$\int_{-\infty}^{\infty} |u(x, t) - N(x, t)|^r dx = O\left(\frac{1}{t^{\eta/\gamma}}\right). \quad (2.1.35)$$

As an immediate consequence we have

Corollary 2.1.1. Let $u_- = u_+ = 0$, u_0 be continuous in $[A_1, A_2]$. Let

$$Z(u_0) = \{x \in (A_1, A_2) : u_0(x) = 0\}.$$

Suppose for any $\alpha, \beta \in Z(u_0)$,

$$\int_{\alpha}^{\beta} u_0(x) dx \neq 0,$$

then ASSP does not exist.

Remark 2.1.1. In general, solution u can have infinitely many discontinuity in ASSP (see example 3.3.10). If u_0 satisfies some smooth assumptions near boundary points in an ASSP, then for each t large, u admits only one discontinuity.

In contrast to Schaffer [56] results which says that for a quite large class of smooth initial data, solution admits finitely many shocks. Then we have the following

Theorem 2.1.8. *There exist a $u_0 \in C_c^\infty(\mathbb{R})$ such that u admits infinitely many ASSP.*

2. Exact Controllability: Normally for the non linear evolution equations, technique of linearization is adopted to study controllability problems. Unfortunately this method does not work (see Horsin [39]) and very few results are available on this subject. Here we consider the following three problems of controllability. Let $u_0 \in L^\infty(\mathbb{R})$,

- (I) **Controllability for pure initial value problem:** Assume that $I = \mathbb{R}, \Omega = \mathbb{R} \times (0, T)$. Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2), g \in L^\infty(J_1)$, a target be given. The question is, does there exist a $\bar{u}_0 \in L^\infty(J_2)$ and u in $L^\infty(\Omega)$ such that u is a solution of (2.1.2) satisfying

$$u(x, T) = g(x) \quad x \in J_1, \tag{2.1.36}$$

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } x \notin J_2, \\ \bar{u}_0(x) & \text{if } x \in J_2. \end{cases} \tag{2.1.37}$$

- (II) **Controllability for one sided initial boundary value problem:** Assume that $I = (0, \infty), \Omega = \mathbb{R} \times (0, T), J = (0, C)$ and a target function $g \in L^\infty(J)$ be given. The question is, does there exist a $u \in L^\infty(\Omega)$ and a $b \in L^\infty((0, T))$ such that u is a solution of (2.1.2) satisfying

$$u(x, T) = g(x) \quad \text{if } x \in J, \tag{2.1.38}$$

$$u(x, 0) = u_0(x) \quad \text{if } x \in (0, \infty), \tag{2.1.39}$$

$$u(0, t) = b(t) \quad \text{if } t \in (0, T). \tag{2.1.40}$$

(III) **Controllability from two sided initial boundary value problem:**

(a). Let $\Omega = \mathbb{R} \times (0, T)$, $I_1 = (B_1, B_2)$, $B_1 \leq C \leq B_2$. Given the target functions $g_1 \in L^\infty(B_1, C)$, $g_2 \in L^\infty(C, B_2)$, does there exist a $\bar{u}_0 \in L^\infty(\mathbb{R} \setminus I_1)$ and $u \in L^\infty(\Omega)$ such that u is a solution of (2.1.2) satisfying

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2. \end{cases} \quad (2.1.41)$$

and

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } B_1 < x < B_2, \\ \bar{u}_0(x) & \text{if } x < B_1 \text{ or } x > B_2. \end{cases} \quad (2.1.42)$$

(b). Here we consider controllability in a strip. Let $I = (B_1, B_2)$, $\Omega = I \times (0, T)$, $B_1 < C < B_2$. Let $g_1 \in L^\infty((B_1, C))$, $g_2 \in L^\infty((C, B_2))$ be given. The question is, does there exist $b_0, b_1 \in L^\infty((0, T))$ and a $u \in L^\infty(\Omega)$ such that u is a solution of (2.1.2) and satisfying

$$u(x, 0) = u_0(x), \quad (2.1.43)$$

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2. \end{cases} \quad (2.1.44)$$

$$u(B_1, t) = b_0(t), \quad (2.1.45)$$

$$u(B_2, t) = b_1(t). \quad (2.1.46)$$

In view of the Lax-Oleinik (Chapter (3) of [31]) explicit formula for solutions of pure initial value problem and by Joseph-Gowda [40] for initial boundary value problem, the targets g or g_1, g_2 cannot be arbitrary. They must satisfy the compatibility condition, for example in the case of problem (I), there exists a non-decreasing function ρ in (B_1, B_2) such that for a.e $x \in (B_1, B_2)$

$$f'(g(x)) = \frac{x - \rho(x)}{T}. \quad (2.1.47)$$

In the case of problem (II), there exists a non-decreasing function ρ in $(0, C)$ such that

$$f'(g(x)) = \frac{x}{T - \rho(x)}. \quad (2.1.48)$$

If the target functions satisfies the compatibility conditions, then the question is **whether the problems (I),(II) and (III) admit a solution?.** In fact, **it is true** and we have the following results. First we describe the class of functions satisfying compatibility conditions.

Definition (Admissible functions): Let $J = (M, N)$ and $T > 0$,

$$S(J) = \{\rho : J \rightarrow \mathbb{R} : \rho \text{ is monotone and left or right continuous function}\}.$$

Then define admissible class of target functions by

(i) Target space for initial value problem (IA):

$$IA(J) = \{g; f'(g(x)) = \frac{x - \rho(x)}{T}, \rho \in S(J), \rho \text{ is a non-decreasing function}\}. \quad (2.1.49)$$

(ii) Target space for left boundary problem (LA):

$$LA(J) = \{g; f'(g(x)) = \frac{x-M}{T-\rho(x)}, \rho \in S(J), \rho \text{ is a non-increasing right continuous function}\}. \quad (2.1.50)$$

(iii) Target space for right boundary problem (RA):

$$RA(J) = \{g; f'(g(x)) = \frac{x-N}{T-\rho(x)}, \rho \in S(J), \rho \text{ is a non-decreasing left continuous function}\}. \quad (2.1.51)$$

Main Theorems for exact controllability :

Theorem 2.1.9. Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2)$. Let $g(x) = (f')^{-1}\left(\frac{x-\rho(x)}{T}\right)$ be in $IA(J_1)$ and $B_1 < A_1 < A_2 < B_2$, satisfying

$$A_1 \leq \rho(x) \leq A_2 \quad \text{if } x \in J_1, \quad (2.1.52)$$

then there exists a $\bar{u}_0 \in L^\infty(J_2), u \in L^\infty(\Omega)$ such that (u, \bar{u}_0) is a solution to problem (I) (see Figure 2.1).

Theorem 2.1.10. Let $\wedge > 0, C > 0, \delta > 0, J = (0, C)$. Let $g \in LA(J)$ given by $f'(g(x)) = \frac{x}{T-\rho(x)}$ for $x \in J$ and satisfying

$$\delta \leq \rho(x) \leq T, \quad (2.1.53)$$

$$\left| \frac{x}{T-\rho(x)} \right| \leq \wedge. \quad (2.1.54)$$

Then there exist a $b \in L^\infty(0, T), u \in L^\infty(\Omega)$ such that (u, b) is a solution to Problem II (see Figure 2.2).

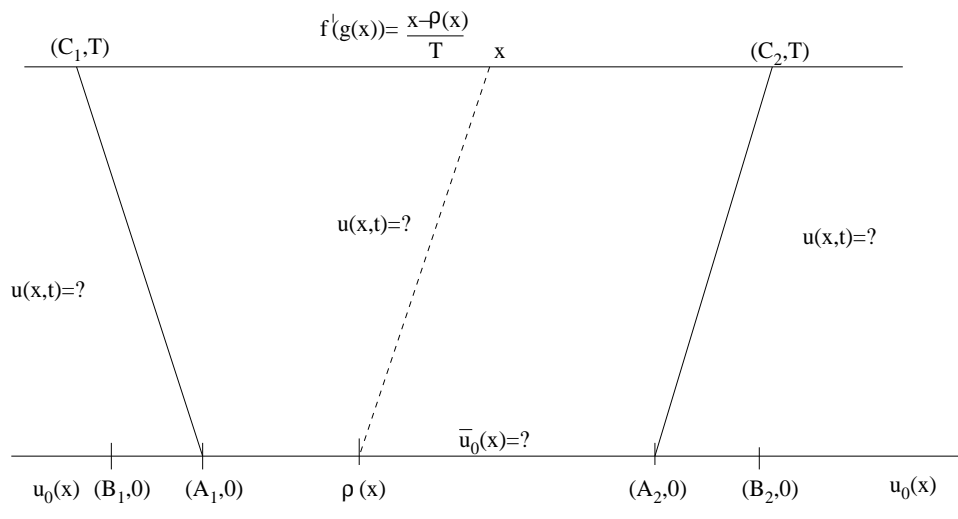


Fig. 2.1:

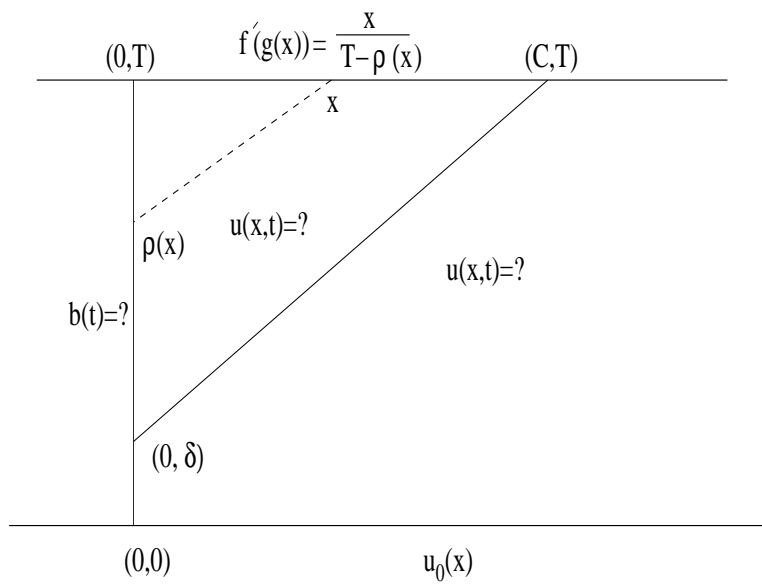


Fig. 2.2:

Theorem 2.1.11. Let $I_1 = (B_1, B_2)$, $B_1 < C < B_2$, $J_1 = (B_1, C)$, $J_2 = (C, B_2)$, then

(a). Let $A_1 < B_1 < B_2 < A_2$ and $g_1 \in IA(J_1)$, $g_2 \in IA(J_2)$ given by $f'(g_1(x)) = \frac{x-\rho_1(x)}{T}$, $f'(g_2(x)) = \frac{x-\rho_2(x)}{T}$, satisfying

$$\rho_1(x) \leq A_1 \quad \text{if } x \in J_1, \quad (2.1.55)$$

$$\rho_2(x) \geq A_2 \quad \text{if } x \in J_2. \quad (2.1.56)$$

Then there exists $\bar{u}_0 \in L^\infty(\mathbb{R} \setminus I_1)$, $u \in L^\infty(\Omega)$ such that (u, \bar{u}_0) is a solution to problem (a) of III (see Figure 2.3).

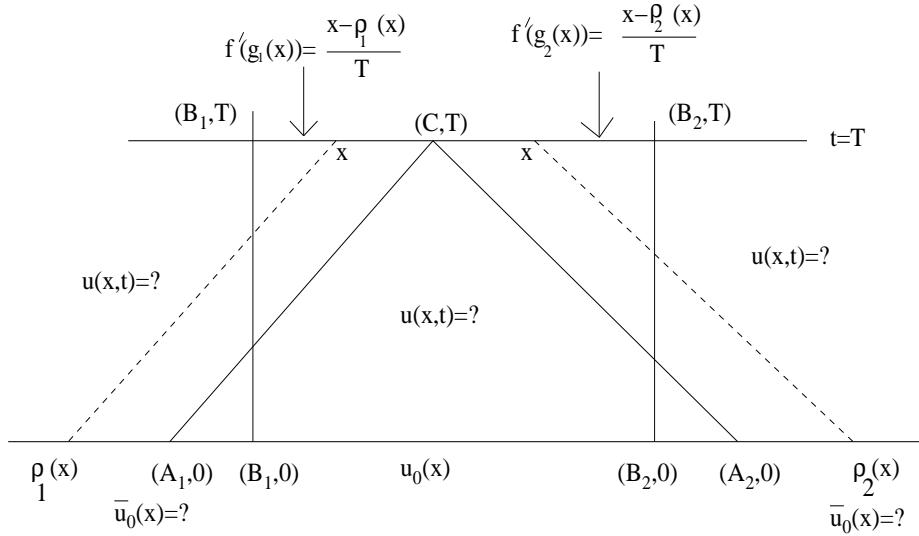


Fig. 2.3:

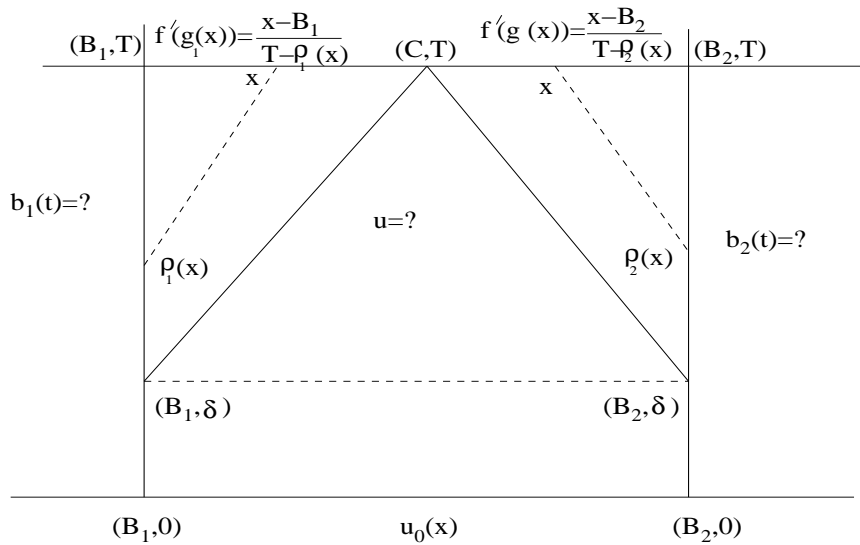


Fig. 2.4:

(b). Let $\wedge > 0$, $0 < \delta < T$, $g_1 \in LA(J_1)$, $g_2 \in RA(J_2)$, given by $f'(g_1(x)) =$

$\frac{x-B_1}{T-\rho_1(x)}, f'(g_2(x)) = \frac{x-B_2}{T-\rho_2(x)}$ satisfying for $i = 1, 2$, $x \in J_i$

$$\delta \leq \rho_i(x) \leq T, \quad (2.1.57)$$

$$\left| \frac{x - B_i}{T - \rho_i(x)} \right| \leq \wedge. \quad (2.1.58)$$

Then there exists $b_0, b_1 \in L^\infty((0, T))$ and $u \in L^\infty(\Omega)$ such that (u, b_0, b_1) is a solution to problem (b) of III (see Figure 2.4).

Before going for further results, let us recall some of the earlier works in this direction.

Problem (a) in III was considered by Horsin [39] for the Burger's equation under similar assumptions on g_1 and g_2 as in (a) of Theorem (4.1.3). He proves that there exists $T_c > 2$, such that (a) of problem III has an approximate controllability solution. That is given $\epsilon > 0$, there exist (u, \bar{u}_0) such that

$$\int_{B_1}^{B_2} |u(x, T_c) - g(x)| dx = O(\epsilon),$$

and $u(x, T_c) = g(x) = \chi_{(B_1, C)}g_1(x) + \chi_{(C, B_2)}g_2(x)$, outside an interval of length ϵ .

For the viscous case Glass-Guerrero [32] proved that $u(x, T) = M \neq 0$ is controllable by using the Cole-Hopf transformation. On the other hand, Guerrero-Imanuvilov [37] proved that $M = 0$ cannot be controllable.

Theorem (4.1.3) is stronger and much more precise result in the non viscous case because

- (i). It removes the condition on time T_c and obtains exact controllability.
- (ii). It deals with general convex flux instead of Burger's equation.
- (iii). Also we give a criterion when the constants are controllable.

In the case of problem (II), Fabio-Ancona and Andrea-Marson [11],[12] studied the problem (II) from the point of view of Hamilton-Jacobi equations. They proved the compactness properties of $\{u(\cdot, T)\}$ when $u(x, 0) = 0$. But they do not address the exact controllability question and Theorem (4.1.2) gives a precise solution for control problem (II).

In our results on controllability, superlinearity of f plays an important role in removing the condition on T_c and obtain a free region. Next using convexity, we explicitly construct solutions in these free regions for particular data which allow us to obtain solutions for control problems.

(3) Optimal controllability: Let $g \in L^\infty(\mathbb{R})$ with compact support be given. For $u_0 \in L^\infty(\mathbb{R})$, let u be the solution of (2.1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$ with initial data $u_0(x)$. Define

the cost functional

$$J(u_0) = \int_{\mathbb{R}} |u(x, T) - g(x)|^2 dx, \quad (2.1.59)$$

and consider the minimization problem

$$C = \inf_{u_0 \in L^\infty(\mathbb{R})} J(u_0).$$

This problem was considered by Castro-Palacios-Zuazua [23] and proved that there exists a minimizer. Since the functional is neither convex nor differentiable, it is quite hard to give a numerical scheme to capture a minimizer.

In chapter 5, we tackle this optimal controllability in a different way. We first reduce the problem to a standard optimization in a Hilbert space by using Lax-Oleinik formula, then we use backward construction to obtain a minimizer. This construction turns out to be far **simpler**.

Let us state our main results of chapter 5 Let

$$f(\theta_f) = \min_{\theta \in \mathbb{R}} f(\theta). \quad (2.1.60)$$

Define

1. **Admissible sets A and B :**

$$A = \{u_0 \in L^\infty(\mathbb{R}) : u_0(x) = \theta_f \text{ outside a compact set}\} \quad (2.1.61)$$

$$B = \{\rho \in L^\infty_{loc}(\mathbb{R}) : \begin{array}{l} (i) \quad \rho \text{ is a non decreasing function} \\ (ii) \quad \rho(x) = x \text{ outside a compact set} \end{array}\}. \quad (2.1.62)$$

For each $N > 0$, define

$$B_N = \{\rho \in B : \rho(x) = x \text{ for } |x| > N\}. \quad (2.1.63)$$

2. **Target function k :** Let $I = (C_1, C_2)$ and k be a measurable function such that

$$k(x) = \theta_f \quad \text{if } x \notin (C_1, C_2), \quad (2.1.64)$$

$$f'(k(x)) \in L^2(I). \quad (2.1.65)$$

3. **Cost function J :** Let k be a target function and $u_0 \in A$. Let $u(x, t)$ be the corresponding solution of (5.1.2) with initial data u_0 . Let $T > 0$ and define the

modified cost function

$$J(u_0) = \int_{\mathbb{R}} |f'(u(x, T)) - f'(k(x))|^2 dx. \quad (2.1.66)$$

Then we have the following proposition.

Proposition 2.1.1. *J is well defined.*

Optimal control problem : Given A, k as above, find a $\tilde{u}_0 \in A$ such that

$$J(\tilde{u}_0) = \min_{u_0 \in A} J(u_0) \quad (2.1.67)$$

and if the minimizer exists, then devise a algorithm to capture it.

Then we have the following main result.

Main Theorem :

Theorem 2.1.12. *There exists a minimizer for (2.1.67) which can be captured by using the standard convex optimization problem in a Hilbert space and backward construction.*

2.2 Introduction to the problem of existence and non-existence of TV bounds for scalar conservation laws with discontinuous flux

Let $I \subset \mathbb{R}$ be an open interval. Let $u_0 : \mathbb{R} \rightarrow I$, $F : \mathbb{R} \times I \rightarrow \mathbb{R}$ be measurable functions. We consider the following equation of conservation law

$$\begin{aligned} u_t + F(x, u)_x &= 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}. \end{aligned} \quad (2.2.1)$$

However smooth F and u_0 , in general (2.2.1) may not admit classical solutions. Even if weak solutions exist, it may not be unique.

This problem has been studied extensively over the last several decades, when the flux F is Lipschitz continuous and u_0 is of bounded variation. One has to impose an extra criteria called ‘‘Entropy condition’’ to obtain the physically relevant solution. In this direction, there are three following methods :

1. **Hamilton-Jacobi method (due to Lax and Olenik [31]).**
2. **Vanishing viscosity method (due to Kruzkov [46], [38]).**
3. **Numerical schemes:** Here $F(x, u) = f(u)$ is Lipschitz. Any monotone, conservative and consistent scheme converges to a unique entropy solution, (for details see [36]).

Next consider the case when the regularity on F fails. Suppose $F(x, u) = f(u)$ is continuous but not Lipschitz continuous. Then the finite domain of dependence fails

and this case was considered by Kruzkov [47]. They defined the concept of generalized entropy condition and proved that the solution is unique.

Now suppose the regularity of F in the space variable fails. Assume that F has a single point of discontinuity at $x = 0$ and is given by

$$F(x, u) = H(x)f(u) + (1 - H(x))g(u) \quad (2.2.2)$$

where f and g are Lipschitz continuous functions and H is the Heaviside function. This equation (2.2.2) decomposes into two equations

$$u_t + f(u)_x = 0 \quad x > 0, t > 0 \quad (2.2.3)$$

$$u_t + g(u)_x = 0 \quad x < 0, t > 0$$

and if $u^+(t) = u(0+, t)$, $u^-(t) = u(0-, t)$ exist, then $u^\pm(t)$ satisfy Rankine-Hugoniot condition at the interface $x = 0$ given by

$$f(u^+(t)) = g(u^-(t)). \quad (2.2.4)$$

A solution to (2.2.1) by definition a weak solution to (2.2.1) satisfying entropy condition of Lax-Olenik, Kruzkov away from the interface $x = 0$.

Under the assumption f and g coincides at least two points Gimse-Risebro [35, 34], Diehl [30] obtained a solution for Riemann data. Main questions are existence of solutions for arbitrary data and unicity.

Assume that f and g are strictly convex C^2 -function and $u_0 \in BV \cap L^\infty$. Now regularize H to a smooth function H^δ and let $F^\delta(x, u) = H^\delta(x)f(u) + (1 - H^\delta(x))g(u)$. Then F^δ is Lipschitz continuous function. Consider two problems

(i). **Vanishing viscosity for Hamilton-Jacobi:**

$$v_t + F^\delta(x, v_x) = \epsilon v_{xx} \quad x \in \mathbb{R}, t > 0 \quad (2.2.5)$$

$$v(x, 0) = v_0(x) = \int_0^x u_0(\theta) d\theta, \quad x \in \mathbb{R}$$

and

(ii). **Vanishing viscosity for conservation law:**

$$\begin{aligned} u_t + F^\delta(x, u)_x &= \epsilon u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}. \end{aligned} \quad (2.2.6)$$

Let $v_{\epsilon, \delta}$ be the unique solution of (2.2.5). Then letting $\epsilon \rightarrow 0$, $\delta \rightarrow 0$ Ostrov [54] showed

that $v_{\epsilon,\delta}$ converges to a unique viscosity solution v of

$$\begin{aligned} v_t + F(x, v_x) &= 0 & x \in \mathbb{R}, t > 0 \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \tag{2.2.7}$$

Independently, (2.2.7) has been studied by Adimurthi-Gowda [6] and obtained an explicit formula for the viscosity solution. Furthermore it was shown that $u = \frac{\partial v}{\partial x}$ is a solution of (2.2.3) satisfying an interface entropy condition

$$\text{meas}\{t : f'(u^+(t)) > 0, g'(u^-(t)) < 0\} = 0 \tag{2.2.8}$$

and Lax-Olenik entropy condition for $x \neq 0$. That is undercompressive waves are not allowed at the interface. The solution forms an L^1 -contractive semi-group. Moreover under some mild regularity, the solution of (2.2.3) satisfying (2.2.8) is unique.

At the conservation law level, this problem was studied by Karlsen, Risebro, Towers [42] and showed that for $\epsilon \rightarrow 0$, $\delta \rightarrow 0$, solution of (2.2.6) converges to a solution of (2.2.3) and it does not satisfy the condition (2.2.8)(see, Example in page 7). In general, these solutions admit undercompressive waves at the interface $x = 0$ which is not allowed in the classical theory of Lax-Olenik and Kruzkov (see, [46]). From the model coming from capillary diffusion, Kaasschieter [41] had studied this problem by using a different diffusion term than the one in (2.2.6) which captured the better physics of that situation. The solution obtained by Kaasschieter also satisfy interface entropy condition (2.2.8). For example, in some cases like clarifier-thickner model, undercompressive waves are allowed at the interface [18, 19, 20, 17, 21, 42, 27, 29, 28, 43]. In view of this discrepancy, a general theory known as (A, B) interface entropy theory was proposed in [8], [9], [22]. It was first shown that (A, B) -entropy solution exists and forms an L^1 -contractive semi-group and is unique. For the strictly convex case, in [9] explicit Lax-Olenik type formulas are established satisfying “ (A, B) interface entropy condition”.

Using this, a numerical scheme of Godunov type is derived in [7], [10]. This scheme is conservative, monotone but not consistent. One cannot expect total variation diminishing property as a constant data gives rise to a non-constant solution. Without total variation bound studying the convergence of the scheme was difficult. This was overcome by singular mapping technique used in [50], [61], [7], [45], [8]. Since the schemes are monotone and conservative and hence by Crandall-Tartar’s Lemma, the solution mapping is L^1 -contractive. From this, it was shown (see, [16]) that the scheme is of total variation bounded away from the interface and this gives an alternative method to prove the convergence of the scheme. These methods do not give any information at the interface $x = 0$. Now the open question was

“does the (A, B) entropy solution admits a total variation bound?”

Here we assume that f and g are strictly convex and of super linear growth. We use

the explicit formula for the (A, B) entropy solution constructed in [9] and show that if (A, B) are away from the critical points, then the solution is of total bounded variation. One of the main ingredients to show this is the study of boundary values $\lambda_{\pm}(t)$. If $\lambda_{\pm}(t)$ are of bounded variation, then it follows easily that the solution itself is of bounded variation. But this is not the case in general. In general $\lambda_{\pm}(t)$ are highly oscillatory and we illustrate this with an example where we show that $\lambda_{\pm}(t)$ are not of bounded variation. The total variation bound exists when A, B are away from the critical points of f and g respectively.

When A is a critical point of g , then for certain data we construct the (A, B) entropy solution which is not of bounded variation at the interface $x = 0$. Idea behind this construction is as follows. At the interface, u satisfies RH condition, namely $f(u^+(t)) = g(u^-(t))$. Hence $u^-(t) = g^{-1}(f(u^+(t)))$. If $f(u^+(t))$ is of bounded variation and the range lies in a neighbourhood of critical point of g , then u^- need not be of bounded variation.

As a consequence of our example, it follows that however small the BV norm of initial data is, BV norm of the solution can blow up at the interface. Hence the BV bounds of the solution need not depend on the smallness of the initial data.

We assume the following assumptions on f, g and u_0 .

- (i). f and g are strictly convex, C^2 and of superlinear growth.
- (ii). $u_0 \in L^\infty$ and let v_0 be its primitive given by

$$v_0(x) = \int_0^x u_0(\theta) d\theta.$$

Let $f(\theta_f) = \min_{\theta \in \mathbb{R}} f(\theta)$, $g(\theta_g) = \min_{\theta \in \mathbb{R}} g(\theta)$ be the points of minima of f and g respectively. Let f^* and g^* be their respective convex duals. Let us recall some of the definitions and notations from [9], [8].

Definition 2.2.1. (*Connection*). Let $(A, B) \in \mathbb{R}^2$. Then (A, B) is called a connection if it satisfies

- (i). $f(B) = g(A)$.
- (ii). $f'(B) \geq 0, g'(A) \leq 0$.

Definition 2.2.2. (*Interphase entropy functional*). Let $u \in L_{\text{loc}}^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm(t) = u(0^\pm, t)$ exist a.e. $t > 0$. Then we define $I_{AB}(t)$, the interface entropy functional by

$$I_{AB}(t) = (g(u^-(t)) - g(A))\text{sign}(u^-(t) - A) - (f(u^+(t)) - f(B))\text{sign}(u^+(t) - B). \quad (2.2.9)$$

Definition 2.2.3. (*Interphase entropy condition*). Let $u \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm(t)$ exist a.e. $t > 0$. Then u is said to satisfy Interphase entropy condition relative to a connection (A, B) if for a.e. $t > 0$

$$I_{AB}(t) \geq 0. \quad (2.2.10)$$

Definition 2.2.4. (*(A, B) entropy solution*). Let $F(x, u) = H(x)f(u) + (1 - H(x))g(u)$ and (A, B) be a connection. Let $u_0 \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$. Then $u \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be a (A, B) entropy solution if

(i). u is a weak solution of

$$\begin{aligned} u_t + F(x, u)_x &= 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}. \end{aligned} \quad (2.2.11)$$

(ii). u satisfies Lax-Olenik-Kruzkov entropy condition away from the interface $x = 0$.

(iii). At the interface $x = 0$, u satisfies (A, B) interface entropy condition (2.2.10).

The (A, B) interface entropy conditions generalizes these two cases. For a particular choice of the (A, B) connection the total variation of the solution of (2.2.11) may increase.

Next we illustrate how to get an explicit (A, B) entropy solution. For this we need to construct boundary values $\lambda_\pm(t)$ and the details are carried out in [9], [6].

Definition 2.2.5. (*Control curves*). Let $t > 0$. $\gamma : [0, t] \rightarrow \mathbb{R}$ be a continuous function. γ is said to be a control curve if there exist $0 \leq t_1 \leq t$ such that

(i). $\gamma|_{[t_1, t]} = 0$,

(ii). $\gamma|_{[0, t_1]}$ is linear and $\gamma|_{(0, t_1)} \neq 0$.

Let

$$\Gamma(t) = \{\gamma : [0, t] \rightarrow \mathbb{R}; \gamma \text{ is control curve}\},$$

$$\Gamma_+(t) = \{\gamma \in \Gamma(t) : \gamma(t) \geq 0\},$$

$$\Gamma_-(t) = \{\gamma \in \Gamma(t) : \gamma(t) \leq 0\}.$$

Define $b_\pm(t)$ by

$$b_+(t) = \inf_{\gamma \in \Gamma_+(t)} \left\{ v_0(\gamma(0)) + \int_0^t f^*(\dot{\gamma}(\theta)) d\theta \right\},$$

$$b_-(t) = \inf_{\gamma \in \Gamma_-(t)} \left\{ v_0(\gamma(0)) + \int_0^t g^*(\dot{\gamma}(\theta)) d\theta \right\},$$

where $v_0(x) = \int_0^x u_0(\theta) d\theta$.

Definition 2.2.6. (*Boundary data*). Let (A, B) be a connection and define $\lambda_{\pm}(t) = \lambda_{\pm}(t, A, B, v_0, f, g)$ by

$$\lambda_+(t) = \begin{cases} f^{-1}(-b'_+(t)) & \text{if } -b'_+(t) > \max(-b'_-(t), f(B)), \\ f_+^{-1}(\max(-b'_-(t), f(B))) & \text{if } -b'_+(t) \leq \max(-b'_-(t), f(B)), \end{cases} \quad (2.2.12)$$

$$\lambda_-(t) = \begin{cases} g^{-1}(-b'_-(t)) & \text{if } -b'_-(t) > \max(-b'_+(t), g(A)), \\ g_+^{-1}(\max(-b'_+(t), g(A))) & \text{if } -b'_-(t) \leq \max(-b'_+(t), g(A)). \end{cases} \quad (2.2.13)$$

Regarding the behavior of $\lambda_{\pm}(t)$, we have the following Theorem.

Theorem 2.2.1. *For a certain choice of f and g there exists u_0 such that $TV(\lambda_{\pm}) = \infty$.*

Now recall from [6], [9] the existence and uniqueness of (A, B) entropy solution.

Theorem 2.2.2. *Let $u_0 \in L^\infty$ and (A, B) be a connection. Let $\lambda_{\pm}(t)$ be as defined earlier. Then there exists an (A, B) entropy solution u with $\|u\|_\infty \leq \|u_0\|_\infty$ and is unique. Furthermore the solution can be described explicitly by Lax-Olenik type formula as follows.*

For each $t > 0$ there exists $R_1(t), R_2(t) \geq 0, L_1(t) \leq 0, L_2(t) \leq 0$ and monotone functions $y_{\pm}(x, t), t_{\pm}(x, t), z_{\pm}(x, t)$ such that

- (i). *for $x \in [R_1(t), \infty)$, $y_+(x, t) \geq 0$ is a non-decreasing function and for $x \in [0, R_1(t))$, $0 \leq t_+(x, t) < t$ is a non-increasing function such that for $x > 0$, (see page 16, equation (44), [9])*

$$u(x, t) = \begin{cases} f^{*'}\left(\frac{x - y_+(x, t)}{t}\right) = u_0(y_+(x, t)) & \text{if } x \geq R_1(t), \\ f^{*'}\left(\frac{x}{t - t_+(x, t)}\right) = \lambda_+(t_+(x, t)) & \text{if } 0 \leq x < R_1(t). \end{cases} \quad (2.2.14)$$

- (ii). *For $x \in (-\infty, L_1(t)]$, $y_-(x, t) \leq 0$, is a non-decreasing function and for $x \in (L_1(t), 0]$, $0 \leq t_-(x, t) < t$, $t_-(x, t)$ is non-increasing function such that for $x < 0$,*

$$u(x, t) = \begin{cases} g^{*'}\left(\frac{x - y_-(x, t)}{t}\right) = u_0(y_-(x, t)) & \text{if } x \leq L_1(t), \\ g^{*'}\left(\frac{x}{t - t_-(x, t)}\right) = \lambda_-(t_-(x, t)) & \text{if } L_1(t) < x < 0. \end{cases} \quad (2.2.15)$$

- (iii). *Without loss of generality we can assume $g(\theta_g) \geq f(\theta_f)$. Let us assume $A = \theta_g$. Then we have the following two cases*

Case 1. $L_1(t) = 0$, $R_1(t) \geq 0$ (see page 53, equation (4.21), (4.22), [6])

$$u(x, t) = \begin{cases} f_+^{-1}(g(u_0(z_+(x, t)))) & \text{if } 0 < x < R_2(t) \\ f_+^{-1}(g(\theta_g)) & \text{if } R_2(t) \leq x < R_1(t). \end{cases} \quad (2.2.16)$$

Case 2. $L_1(t) < 0$, $R_1(t) \geq 0$ (see Lemma 4.8 and page 55, equation (4.30), [6])

$$u(x, t) = \begin{cases} g^{-1}(f(u_0(z_-(x, t)))) & \text{if } 0 > x > L_1(t) = L_2(t) \\ f_+^{-1}(g(\theta_g)) & \text{if } 0 < x < R_1(t) = R_2(t). \end{cases} \quad (2.2.17)$$

See figure 2.5 for clear illustrations.

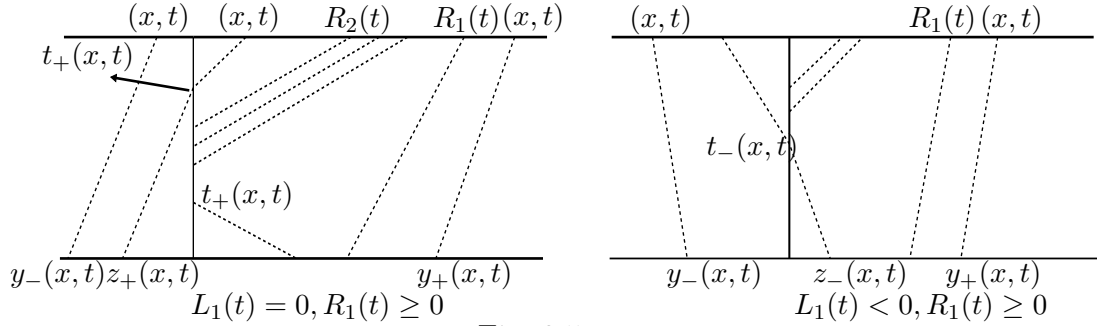


Fig. 2.5:

Now we can state our main results as follows.

Main Theorem :

Theorem 2.2.3. Let $u_0 \in L^\infty(\mathbb{R})$ and u be the solution as above. Let $t > 0$, $\epsilon > 0$, $M > \epsilon$, $I(M, \epsilon) = \{x : \epsilon \leq |x| \leq M\}$. Then

- (1). Suppose there exists an $\alpha > 0$ for which $f'' \geq \alpha$, $g'' \geq \alpha$, then there exist a $C = C(\epsilon, M, \alpha)$ such that

$$TV(u(\cdot, t), I(M, \epsilon)) \leq C(\epsilon, M, t).$$

- (2). Let $T > 0$ and $u_0 \in BV$. Then there exists a $C(\epsilon, T)$ such that for all $0 < t \leq T$

$$TV(u(\cdot, t), |x| > \epsilon) \leq C(\epsilon, t) TV(u_0) + 4\|u_0\|_\infty$$

- (3). Let $u_0 \in BV$, $T > 0$ and $A \neq \theta_g$, $B \neq \theta_f$. Then there exists a $C > 0$ such that for all $0 < t \leq T$,

$$TV(u(\cdot, t)) \leq C TV(u_0) + 6\|u_0\|_\infty.$$

(4). Let $u_0, f_+^{-1}(g(u_0)), g_-^{-1}(f(u_0)) \in BV, T > 0$ and $A = \theta_g$. Then for all $0 < t \leq T$,

$$TV(u(., t))$$

$$\leq TV(u_0) + \max(TV(f_+^{-1}(g(u_0))), TV(g_-^{-1}(f(u_0)))) + 6\|u_0\|_\infty.$$

(5). Let $A = \theta_g$ or $B = \theta_f$, then for certain choice of fluxes f and g there exists a initial data $u_0 \in BV \cap L^\infty$ such that $TV(u(., t)) = \infty$.

CHAPTER 3

STRUCTURE OF AN ENTROPY SOLUTION OF A SCALAR CONSERVATION LAW WITH STRICT CONVEX FLUX

3.1 Introduction

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly convex C^1 function satisfying the super linear growth,

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty. \quad (3.1.1)$$

Let $u_0 \in L^\infty(I)$ and consider the following single scalar conservation law

$$u_t + f(u)_x = 0 \quad x \in \mathbb{R}, t > 0, \quad (3.1.2)$$

$$u(x, 0) = u_0(x) \quad x \in I. \quad (3.1.3)$$

This problem was well studied from last several decades starting from the pioneering works of Lax-Oleinik [31], Kruzkov [36]. They have studied the existence and uniqueness of weak solutions to (3.1.1)-(3.1.2) satisfying the entropy condition. In this chapter we study, profile of a solution, for example how many shocks can a solution exhibit and the nature of the shocks. Some partial results in this direction are obtained by Lax [48] and Dafermos [26].

The basic ingredient in studying the above problem comes from the analysis of characteristic curves R_\pm . Originally this was introduced by Hopf [33] and later in a seminal paper by Dafermos [25], who studied them quite extensively to obtain information on the nature of solutions. Independently, this was used in [6] to obtain the explicit formula for a solution of discontinuous flux.

The plan of the chapter is as follows: In this section with some examples, we illustrate the behaviour of solutions. In section 3.2 we recall the basic theorems of Hopf, Lax-Oleinik and state our main results. Section 3.3 deals with the finer analysis of characteristic and

the proof of main Theorems.

1. Illustration of a profile of a solution : In order to understand the shock profile of a solution, we consider the following two basic examples which form the general pattern. Consider the Burgers equation,

$$u_t + \left(\frac{u^2}{2} \right)_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Let $A_1 < A_2$, $u_-, u_+ \in \mathbb{R}$, $\bar{u}_0 \in L^\infty(A_1, A_2)$ and define the initial data u_0 by

$$u_0(x) = \begin{cases} u_- & \text{if } x < A_1, \\ \bar{u}_0(x) & \text{if } A_1 < x < A_2, \\ u_+ & \text{if } x > A_2. \end{cases} \quad (3.1.4)$$

Example 3.1.1. (Single shock case): (See Figure 3.1) Let $u_- > \alpha > u_+$ and $\bar{u}_0(x) = \alpha$. Define

$$\begin{aligned} \sigma_0 &= \frac{f(u_-) - f(u_+)}{u_- - u_+}, \quad \sigma_1 = \frac{f(u_-) - f(\alpha)}{u_- - \alpha}, \quad \sigma_2 = \frac{f(u_+) - f(\alpha)}{u_+ - \alpha}, \quad T_0 = \frac{A_2 - A_1}{\sigma_1 - \sigma_2}, \\ x_0 &= A_1 + \sigma_1 T_0, \quad s_1(t) = A_1 + \sigma_1 t, \quad s_2(t) = A_2 + \sigma_2 t, \quad s_0(t) = x_0 + (t - T_0)\sigma_0. \end{aligned}$$

Then the solution u is given by,

(i). Let $0 < t < T_0$, then

$$u(x, t) = \begin{cases} u_- & \text{if } x < s_1(t), \\ \alpha & \text{if } s_1(t) < x < s_2(t), \\ u_+ & \text{if } x > s_2(t). \end{cases} \quad (3.1.5)$$

(ii). Let $t > T_0$, then

$$u(x, t) = \begin{cases} u_- & \text{if } x < s_0(t), \\ u_+ & \text{if } x > s_0(t). \end{cases} \quad (3.1.6)$$

Example 3.1.2. (Infinitely many Shocks): (See Figure 3.2 and Figure 3.3) Let $I = (A_1, A_2)$ and define the ASSP (asymtotically single shock packet-see definition 3.2.5) $D(I)$ and single shock solution $u(x, t, I)$ as follows:

$$D(I) = I \times (0, \infty). \quad (3.1.7)$$

$$T(I) = \frac{A_2 - A_1}{2}. \quad (3.1.8)$$

$$\bar{u}_0(x, I) = \begin{cases} 1 & \text{if } A_1 < x < \frac{A_1 + A_2}{2}, \\ -1 & \text{if } \frac{A_1 + A_2}{2} < x < A_2. \end{cases} \quad (3.1.9)$$

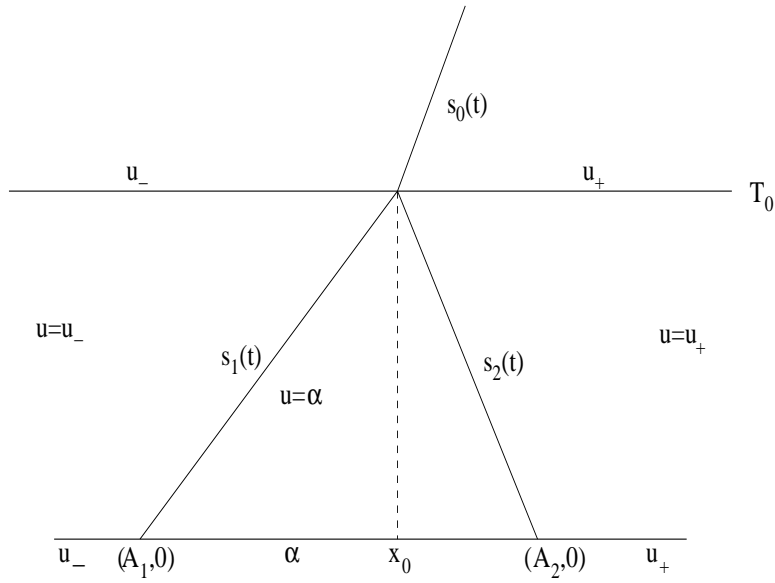


Fig. 3.1:

Let $u(x, t, I)$ be a solution of (3.1.2) in $D(I)$ with initial condition $\bar{u}_0(x, I)$ and satisfying the boundary condition

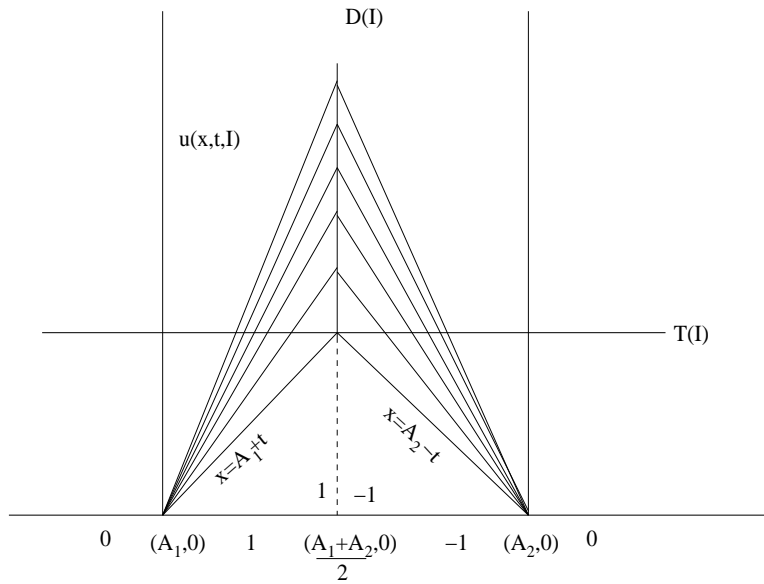


Fig. 3.2:

$$u(A_1+, t, I) = u(A_2-, t, I) = 0, \quad (3.1.10)$$

and it is given by (see Figure 3.2).

(i). Let $0 < t < T(I)$, then

$$u(x, t, I) = \begin{cases} \frac{x-A_1}{t} & \text{if } A_1 < x < A_1 + t, \\ 1 & \text{if } A_1 + t < x < \frac{A_1+A_2}{2}, \\ -1 & \text{if } \frac{A_1+A_2}{2} < x < A_2 - t, \\ \frac{x-A_2}{t} & \text{if } A_2 - t < x < A_2. \end{cases} \quad (3.1.11)$$

(ii). Let $t > T(I)$, then

$$u(x, t, I) = \begin{cases} \frac{x-A_1}{t} & \text{if } A_1 < x < \frac{A_1+A_2}{2}, \\ \frac{x-A_2}{t} & \text{if } \frac{A_1+A_2}{2} < x < A_2. \end{cases} \quad (3.1.12)$$

Next we glue such solutions to produce a single solution having infinitely many shocks for each $t > 0$. Let $n \geq 1$ and define

$$\begin{aligned} I_+ &= (1, \infty), \quad I_- = (-\infty, 0), \quad I_n = \left(\frac{1}{2n}, \frac{1}{2n-1}\right), \quad J_n = \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \quad n \geq 1. \\ D_n &= I_n \times (0, \infty), \quad x_n = \frac{1}{2}\left(\frac{1}{2n} + \frac{1}{2n-1}\right). \end{aligned}$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \in (1, \infty) \cup_{n=1}^{\infty} \left(\frac{1}{2n}, x_n\right), \\ 0 & \text{if } x \in J_n \cup I_-, \\ -1 & \text{if } x \in \cup_{n=1}^{\infty} \left(x_n, \frac{1}{2n-1}\right), \end{cases} \quad (3.1.13)$$

and define the solution by (see Figure 3.3)

$$u(x, t) = \begin{cases} 1 & \text{if } 1 + t < x, \\ \frac{x-1}{t} & \text{if } 1 < x < 1 + t, \\ u(x, t, I_n) & \text{if } (x, t) \in D_n, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1.14)$$

In view of (3.1.12), u satisfies Rankine-Hugoniot condition as well as entropy condition across ∂D_n and hence u is the entropy solution of (3.1.2) with u_0 as its initial condition.

Now the basic question is, under what conditions, solution u admit a single shock (as in example 3.1.1) for t sufficiently large and how it behaves if it does not admits a single shock (as in example 3.1.2) for t large?

Analysis of shocks: If the data satisfies like in example (3.1.1),

$$u_- > u_+, \quad (3.1.15)$$

(v). $\partial D_i \setminus (\mathbb{R} \times \{t = 0\})$ are parallel characteristic lines and any two characteristics curves within D_i intersects after finite time. Asyptotically it represents a single shock packet. In example (3.1.2), $D_i = I_i \times (0, \infty)$.

(vi). N wave (see Figure 3.4) satisfies

$$\int_{-\infty}^{\infty} |u(x, t) - N(x, t)| = O\left(\frac{1}{t^{1/2}}\right)$$

provided $f'' \geq \beta > 0$. In example (3.1.2), rarefaction wave is given by

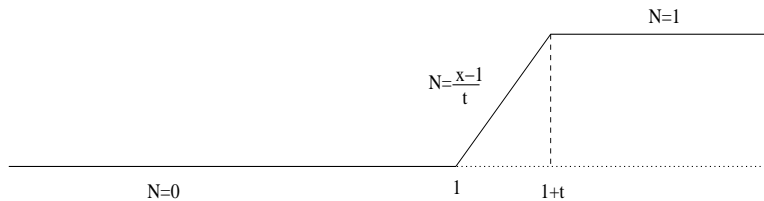


Fig. 3.4:

$$N(x, t) = \begin{cases} 0 & \text{if } x \leq 1, \\ \frac{x-1}{t} & \text{if } 1 \leq x \leq 1 + t, \\ 1 & \text{if } x > 1 + t. \end{cases}$$

The main results of this chapter are : we prove (i) to (iv), for a general convex flux f with L^∞ data. In case of (vi), we have relaxed the condition $f'' \geq \beta > 0$. Earlier this problem was studied by Liu-Pierre [52] (for the power law), Kim [44] (for the algebraic growth rate at zero) and obtained the decay estimates. Here we have generalized these results and obtain the rate of decay of solutions with respect to the N -wave (see Remark 3.3.2).

It has been shown by Schaeffer [56] that for large class of smooth initial data u_0 , the solution can develop atmost finitely many shocks for t large. If we remove the smoothness in the initial data, example (3.1.2) shows the existence of infinitely many asymptotically shock packets D_i and each one represents a single shock. In fact within D_i , one can construct infinitely many shocks by using backward construction [2] and all of them merge at infinity (see example 3.3.10).

In Theorem 3.2.8, we construct a $u_0 \in C_c^\infty(\mathbb{R})$ for which the solution admits infinitely many shocks and hence Schaeffer's result cannot be improved.

3.2 Preliminaries and Main results

Before stating the main results, we recall some well known results without proof.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 convex function. Assume that

1. f has superlinear growth, that is

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty.$$

2. f is strictly convex. That is for $a, b \in \mathbb{R}$, $a = b$ if and only if

$$f(a) - f(b) - (a - b)f'(b) = 0.$$

3. Let $f^*(p) = \sup_q \{pq - f(q)\}$ be the Legendre transform of f . Then $f^* \in C^1$ and f^* is of superlinear growth and satisfies

$$\begin{aligned} f &= f^{**}, \\ f'(p) &= (f')^{-1}(p), \\ f^*(f'(p)) &= pf'(p) - f(p), \\ f(f'(p)) &= pf^*(p) - f^*(p). \end{aligned} \tag{3.2.1}$$

Examples: $f(u) = |u|^p$, $1 < p < \infty$.

Controlled Curves: Let $x \in \mathbb{R}$, $0 \leq s < t$ and define the class of controlled curves $\Gamma(x, s, t)$ by

$$\Gamma(x, s, t) = \{r : [s, t] \rightarrow \mathbb{R}; r \text{ is a straight line with } r(t) = x\}, \tag{3.2.2}$$

and denote $\Gamma(x, t) = \Gamma(x, 0, t)$.

Value function: Let $u_0 \in L^\infty(\mathbb{R})$, $b \in \mathbb{R}$, define

$$v_0(x) = \int_b^x u_0(\theta) d\theta, \tag{3.2.3}$$

be its primitive. Define the value function $v(x, t)$ associated to f and u_0 by

$$\begin{aligned} v(x, t) &= \min_{r \in \Gamma(x, t)} \left\{ v_0(r(0)) + tf^* \left(\frac{x - r(0)}{t} \right) \right\} \\ &= \min_{\beta \in \mathbb{R}} \left\{ v_0(\beta) + tf^* \left(\frac{x - \beta}{t} \right) \right\}. \end{aligned} \tag{3.2.4}$$

Then v satisfies the

Dynamic Programming principle: Let $0 \leq s < t$, then

$$v(x, t) = \min_{r \in \Gamma(x, s, t)} \left\{ v(r(s), s) + (t - s)f^* \left(\frac{x - r(s)}{t - s} \right) \right\}. \tag{3.2.5}$$

Characteristics: Define the characteristic set $ch(x, t, s)$ and extreme characteristic points $y_{\pm}(x, t, s)$ by

$$ch(x, t, s) = \{r \in \Gamma(x, t, s); r \text{ is a minimizer in (3.2.5)}\}, \quad (3.2.6)$$

$$\begin{aligned} y_-(x, t, s) &= \min\{r(s) : r \in ch(x, t, s)\}, \\ y_+(x, t, s) &= \max\{r(s); r \in ch(x, t, s)\}. \end{aligned} \quad (3.2.7)$$

$$r_{\pm}(\theta, x, t, s) = x + \frac{x - y_{\pm}(x, t, s)}{t - s}(\theta - t). \quad (3.2.8)$$

Denote $ch(x, t) = ch(x, t, 0)$, $y_{\pm}(x, t) = y_{\pm}(x, t, 0)$, $r_{\pm}(\cdot, x, t) = r_{\pm}(\cdot, x, t, 0)$. For $r \in ch(x, t, s)$, we say, $r(s)$ a characteristic point.

Then we have the following theorem due to Hopf, Lax-Oleinik (see [31]).

Theorem 3.2.1. *Let $0 \leq s < t$, u_0, v_0, v be as above, then*

(1). *v is a uniformly Lipschitz continuous viscosity solution of the Hamilton-Jacobi equation*

$$\begin{aligned} v_t + f(v_x) &= 0 & (x, t) \in \mathbb{R} \times (0, \infty), \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \quad (3.2.9)$$

(2). *There exist $M > 0$, depending only on $\|u_0\|_{\infty}$ and Lipschitz constant of f, f^* restricted to $[-\|u_0\|_{\infty}, \|u_0\|_{\infty}]$ such that for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $r \in ch(x, t, s)$*

$$\left| \frac{x - r(s)}{t - s} \right| \leq M. \quad (3.2.10)$$

(3). **NIP (Non intersecting property of characteristics):** *Let $x_1 \neq x_2$, $0 \leq s < \min\{t_1, t_2\}$, $r_i \in ch(x_i, t_i, s)$ and for $i = 1, 2$. Then for all $\theta \in (s, \min\{t_1, t_2\})$,*

$$r_1(\theta) \neq r_2(\theta). \quad (3.2.11)$$

From NIP, it follows that

(a). *$x \mapsto y_{\pm}(x, t, s)$ are non decreasing functions.*

(b). *At the points of continuity of y_+ ,*

$$y_+(x, t, s) = y_-(x, t, s)$$

and hence $ch(x, t, s) = \{r\}$ given by

$$r(\theta) = \frac{x - y_+(x, t, s)}{t - s}(\theta - t) + x.$$

(c). Let $r \in ch(x, t)$, $z = r(s)$. Let $r_1 = r|_{[0, s]}$, $r_2 = r|_{[s, t]}$, then $r_1 \in ch(z, s)$,
 $r_2 \in ch(x, t, s)$.

(4). Let $u(x, t) = \frac{\partial v}{\partial x}(x, t)$. Then u is the unique solution of (3.1.2) with initial data u_0 and satisfying

$$|u(x, t)| \leq \|u_0\|_\infty. \quad (3.2.12)$$

For a.e x , $y_-(x, t) = y_+(x, t)$ and u is given by

$$f'(u(x, t)) = \frac{x - y_+(x, t)}{t} = \frac{x - y_-(x, t)}{t}. \quad (3.2.13)$$

Furthermore if x be a point of differentiability of $y_\pm(x, t)$ and $y_\pm(x, t)$ is a point of differentiability of v_0 , then

$$u(x, t) = u_0(y_\pm(x, t)). \quad (3.2.14)$$

(5). If $u_0 \leq v_0$, then $u(x, t) \leq v(x, t)$, for a.e. $t > 0$ and for a.e. $x \in \mathbb{R}$.

In this sequel we will follow the notation of characteristic curves as in [6].

Definition 3.2.2. (Characteristic Curves R_\pm) : Let $\alpha \in \mathbb{R}$, $0 \leq s < t$. Define

$$\begin{aligned} R_-(t, s, \alpha) &= \inf\{x; \alpha \leq y_-(x, t, s)\}, \\ R_+(t, s, \alpha) &= \sup\{x; y_+(x, t, s) \leq \alpha\}, \\ R_\pm(t, \alpha) &= R_\pm(t, 0, \alpha). \end{aligned} \quad (3.2.15)$$

Definition 3.2.3. (Characteristic and regular characteristic line): Let $\alpha, p \in \mathbb{R}$ and $r(t, \alpha, p) = \alpha + tf'(p)$ be a straight line passing through $(\alpha, 0)$ with reciprocal slope $f'(p)$. Then

- (i). r is called a characteristic line if for all $t > 0$, $r(\cdot, \alpha, p) \in ch(r(t, \alpha, p), t)$
- (ii). Let r be a characteristic line, then r is called regular characteristic line if for all $t > 0$

$$y_+(r(t, \alpha, p), t) = y_-(r(t, \alpha, p), t) = \alpha. \quad (3.2.16)$$

Definition 3.2.4. (Base point and value) : Let $r(\cdot, \alpha, p)$ be a characteristic line. Then α is called the base point and p is called value of the characteristic line $r(\cdot, \alpha, p)$.

Definition 3.2.5. (ASSP-Asymptotically single shock packet) : Let $C_1 < C_2, p \in \mathbb{R}$ and for $i = 1, 2$, $r(\cdot, C_i, p)$ as above. Then define

$$D(C_1, C_2, p) = \{(x, t) : r(t, C_1, p) < x < r(t, C_2, p), t > 0\}. \quad (3.2.17)$$

Then $D(C_1, C_2, p)$ is called ASSP if

- (i). $r(\cdot, C_1, p), r(\cdot, C_2, p)$ are regular characteristic lines.
- (ii). For all $C_1 < \alpha < C_2$, $R_{\pm}(t, \alpha) \in D(C_1, C_2, p)$ for all $t > 0$.
- (iii). For all $C_1 < \alpha \leq \beta < C_2$ there exists $T > 0$ such that for $t \geq T$,

$$R_+(t, \alpha) = R_-(t, \alpha) = R_+(t, \beta) = R_-(t, \beta). \quad (3.2.18)$$

In other words all the characteristic curves in D will merge at infinity and represents a single shock at infinity.

With these preliminaries, we can state our main results. Let $A_1 < A_2$, $u_{\pm} \in \mathbb{R}$, $\bar{u}_0 \in L^\infty(A_1, A_2)$ and define u_0 by

$$u_0(x) = \begin{cases} u_- & \text{if } x < A_1, \\ \bar{u}_0 & \text{if } A_1 < x < A_2, \\ u_+ & \text{if } x > A_2. \end{cases} \quad (3.2.19)$$

Let u be the solution of (3.1.2) with u_0 as the initial data.

Definition 3.2.6. (Single shock case) : Let u and u_0 be as above, then u is said to be a case of single shock if there exist $T > 0$, $x_0 \in \mathbb{R}$ such that

$$u(x, T) = \begin{cases} u_- & \text{if } x < x_0, \\ u_+ & \text{if } x > x_0. \end{cases} \quad (3.2.20)$$

As a consequence of this, for $t > T$, u is given by

$$u(x, T) = \begin{cases} u_- & \text{if } x < x_0 + (t - T) \left(\frac{f(u_+) - f(u_-)}{u_+ - u_-} \right), \\ u_+ & \text{if } x > x_0 + (t - T) \left(\frac{f(u_+) - f(u_-)}{u_+ - u_-} \right). \end{cases} \quad (3.2.21)$$

Then we have the following shock profile decomposition.

Main Results :

Theorem 3.2.7. (Structure Theorem) Let u and u_0 be as in (3.2.19). Then

- (I). u represents a case of single shock if and only if $u_- > u_+$.
- (II). Let $u_- \leq u_+$. Then there exist $A_1 \leq B_1 \leq B_2 \leq A_2$ and a countable number of disjoint ASSP $\{D(C_{1i}, C_{2i}, p_i)\}_{i \in I}$ such that
 - (i). $r(\cdot, B_1, u_-), r(\cdot, B_2, u_+)$, are regular characteristic lines and for $t > 0$,

$$R_-(t, A_1) \leq r(t, B_1, u_-) \leq r(t, B_2, u_+) \leq R_+(t, A_2) \quad (3.2.22)$$

$$u(x, T) = \begin{cases} u_- & \text{if } x < R_-(t, A_1), \\ u_+ & \text{if } x > R_+(t, A_2) \end{cases} \quad (3.2.23)$$

(ii). Let

$$\begin{aligned} E &= \{(x, t) : r(t, B_1, u_-) \leq x \leq r(t, B_2, u_+)\} \\ S &= \cup_{i \in I} D(C_{1i}, C_{2i}, p_i) \\ R &= E \setminus S. \end{aligned}$$

Then R consists of all regular characteristic lines and u is continuous on R .

(iii). For each $i \in I$, u_0 satisfies

$$\int_{C_{1i}}^{C_{2i}} u_0(x) dx = (C_{2i} - C_{1i}) p_i \quad (3.2.24)$$

and if u_0 is continuous in a neighbourhood of $\{C_{1i}, C_{2i}\}$, then

$$u_0(C_{1i}) = u_0(C_{2i}) = p_i. \quad (3.2.25)$$

(iv) Let

$$\begin{aligned} F_- &= \{(x, t) : x < R_-(t, A_1)\} \\ D_- &= \{(x, t) : R_-(t, A_1) < x < r(t, B_1, u_-)\} \\ F_+ &= \{(x, t) : x > R_+(t, A_2)\} \\ D_+ &= \{(x, t) : r(t, B_2, u_+) < x < R_+(t, A_2)\} \end{aligned}$$

then $\mathbb{R} \times (0, \infty) = F_- \cup F_+ \cup D_- \cup D_+ \cup S \cup R$. Define the N -wave by

$$N(x, t) = \begin{cases} u_- & \text{if } (x, t) \in F_- \\ u_+ & \text{if } (x, t) \in F_+ \\ \frac{x - B_1}{t} & \text{if } (x, t) \in D_- \\ \frac{x - B_2}{t} & \text{if } (x, t) \in D_+ \\ p_i & \text{if } (x, t) \in D(C_{1,i}, C_{2,i}, p_i) \\ p & \text{if } (x, t) \in R, \text{ } p \text{ is the value of the regular} \\ & \text{characteristic line on which } (x, t) \text{ lies.} \end{cases} \quad (3.2.26)$$

Then $x \mapsto N(x, t)$ is a continuous non decreasing function on $D_+ \cup D_- \cup S \cup R$.

(v). Suppose f satisfies the following growth assumptions : There exist $\gamma > 0, \delta > 0, C >$

0 such that for $|h| \leq \delta$,

$$f(u_{\pm}) - f(u_{\pm} + h) + hf'(u_{\pm} + h) \geq C|h|^{\gamma} \quad (3.2.27)$$

and for any compact interval J , there exists $C(J) > 0, \eta > 0$ such that for all $a, b \in J$

$$|f'(a) - f'(b)| \leq C(J)|a - b|^{\eta}. \quad (3.2.28)$$

Then for t large we have the following decay estimate

$$\int_{-\infty}^{\infty} |f'(u(x, t)) - f'(N(x, t))| dx = O\left(\frac{1}{t^{\eta/\gamma}}\right). \quad (3.2.29)$$

Furthermore if some $r > 0, a, b \in J, C_1(J) > 0$,

$$|f'(a) - f'(b)| \geq C_1(J)|a - b|^r$$

then,

$$\int_{-\infty}^{\infty} |u(x, t) - N(x, t)|^r dx = O\left(\frac{1}{t^{\eta/\gamma}}\right). \quad (3.2.30)$$

As an immediate consequence of (3.2.24), (3.2.25) is

Corollary 3.2.1. Let $u_- = u_+ = 0$, u_0 be continuous in $[A_1, A_2]$. Let

$$Z(u_0) = \{x \in (A_1, A_2) : u_0(x) = 0\}.$$

Suppose for any $\alpha, \beta \in Z(u_0)$,

$$\int_{\alpha}^{\beta} u_0(x) dx \neq 0,$$

then ASSP does not exist.

Remark 3.2.1. In general, solution u can have infinitely many discontinuity in ASSP (see example 3.3.10). If u_0 satisfies some smooth assumptions near boundary points in an ASSP, then for each t large, u admits only one discontinuity.

In contrast to Schaffer [56] results which says that for a quite large class of smooth initial data, solution admits finitly many shocks. Then we have the following

Theorem 3.2.8. *There exist a $u_0 \in C_c^{\infty}(\mathbb{R})$, such that u admits infinitely many ASSP.*

3.3 Proof of the Theorems:

In order to prove the theorems, we need the following lemmas.

Lemma 3.3.1. *Let u_0, u, v_0, v be as in theorem (3.2.1) and $0 \leq s < t$. Then*

1. *Let $x_1 < x_2$ and β be a minimizer in (3.2.4) for $v(x_1, t)$ and $v(x_2, t)$. Then β is a minimizer for all $x_1 < x < x_2$ and satisfies*

$$f'(u(x, t)) = \frac{x - \beta}{t - s}. \quad (3.3.1)$$

2. *Let $x_k \in \mathbb{R}, r_k \in ch(x_k, t, s)$ such that $\lim_{k \rightarrow \infty} (x_k, r_k(0)) = (x, \beta)$. Then $r(\theta) = \beta + \frac{x - \beta}{t - s}(\theta - t)$ is in $ch(x, t, s)$. Furthermore*

$$\lim_{x_k \uparrow x} y_+(x_k, t, s) = y_-(x, t, s), \quad (3.3.2)$$

$$\lim_{x_k \downarrow x} y_-(x_k, t, s) = y_+(x, t, s). \quad (3.3.3)$$

3. *For all $t > 0$,*

$$R_-(t, s, \alpha) \leq R_+(t, s, \alpha), \quad (3.3.4)$$

$$y_-(R_-(t, s, \alpha), t, s) \leq \alpha \leq y_+(R_-(t, s, \alpha), t, s), \quad (3.3.5)$$

$$y_-(R_+(t, s, \alpha), t, s) \leq \alpha \leq y_+(R_+(t, s, \alpha), t, s). \quad (3.3.6)$$

4. *Let $r_{\pm}(\cdot, t, s, \alpha)$ and $\bar{r}_{\pm}(\cdot, t, s, \alpha)$ denote the left and right extreme characteristics at $R_-(t, s, \alpha)$ and $R_+(t, s, \alpha)$ respectively and is given by*

$$\begin{aligned} r_{\pm}(\theta) = r_{\pm}(\theta, t, s, \alpha) &= R_{\pm}(t, s, \alpha) \\ &+ \frac{R_{\pm}(t, s, \alpha) - y_{\pm}(R_{\pm}(t, s, \alpha), t, s)}{t - s}(\theta - t). \end{aligned} \quad (3.3.7)$$

$$\begin{aligned} \bar{r}_{\pm}(\theta) = \bar{r}_{\pm}(\theta, t, s, \alpha) &= R_{\pm}(t, s, \alpha) \\ &+ \frac{R_{\pm}(t, s, \alpha) - y_{\pm}(R_{\pm}(t, s, \alpha), t, s)}{t - s}(\theta - t). \end{aligned} \quad (3.3.8)$$

Then for all $s < \theta < t$

$$r_-(\theta) \leq R_-(\theta, s, \alpha) \leq r_+(\theta). \quad (3.3.9)$$

$$\bar{r}_-(\theta) \leq R_+(\theta, s, \alpha) \leq \bar{r}_+(\theta). \quad (3.3.10)$$

Furthermore, they satisfy the dynamic programming principle

$$R_-(t, \alpha) = R_-(t, s, R_-(s, \alpha)) \quad (3.3.11)$$

$$R_+(t, \alpha) = R_+(t, s, R_-(s, \alpha)) \quad (3.3.12)$$

and $t \mapsto R_{\pm}(t, \alpha)$ are Lipschitz continuous functions with Lipschitz constant independent of α with

$$\lim_{t \rightarrow 0} R_{\pm}(t, \alpha) = \alpha. \quad (3.3.13)$$

5. Let $s > 0$, then

$$R_-(t, s, \alpha) = R_+(t, s, \alpha). \quad (3.3.14)$$

Let $s = 0$ and if $R_-(t, \alpha) < R_+(t, \alpha)$, then for all $R_-(t, \alpha) < x < R_+(t, \alpha)$

$$y_{\pm}(x, t) = \alpha, \quad f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (3.3.15)$$

6. **Collapsing property** : Suppose for $\alpha, \beta \in \mathbb{R}$, there exists a $T > 0$ such that

(i). If $R_-(T, \alpha) = R_-(T, \beta)$, then for $t > T$, $R_-(t, \alpha) = R_+(t, \beta)$.

(ii). If $R_-(T, \alpha) = R_+(T, \beta)$, then for $t > T$, $R_-(t, \alpha) = R_+(t, \beta)$.

(iii). If $R_+(T, \alpha) = R_+(T, \beta)$, then for $t > T$, $R_+(t, \alpha) = R_+(t, \beta)$.

7. Let $u_{1,0} \leq u_{2,0}$ and u_1 and u_2 be the respective solutions of (3.1.2) and (3.1.3). Let $\alpha \in \mathbb{R}$ and $R_{\pm}^{(1)}(t, \alpha), R_{\pm}^{(2)}(t, \alpha)$ be the extreme characteristic curves of u_1 and u_2 respectively. Then for all $t > 0$,

$$R_{\pm}^{(1)}(t, \alpha) \leq R_{\pm}^{(2)}(t, \alpha).$$

Proof :

1. Let $x \in (x_1, x_2)$ and $r \in ch(x, t, s)$. Suppose $r(s) \neq \beta$, then r intersects one of the characteristics $\frac{x_i - \beta}{t - s}(\theta - t) + x_i$, $i = 1, 2$ which contradicts NIP. Hence $\beta = r(s)$ and $y_{\pm}(x, t, s) = \beta$. Furthermore

$$v(x, t) = v(\beta, s) + (t - s)f^* \left(\frac{x - \beta}{t - s} \right)$$

and hence $u(x, t) = \frac{\partial v}{\partial x} = f^* \left(\frac{x - \beta}{t - s} \right) = (f')^{-1} \left(\frac{x - \beta}{t - s} \right)$. This proves (1).

2. From the continuity of v , f^* and from dynamic programming principle

$$\begin{aligned} v(x, t) &= \lim_{k \rightarrow \infty} \left\{ v(r_k(s), s) + (t - s) f^* \left(\frac{x_k - r_k(s)}{t - s} \right) \right\} \\ &= v(\beta, s) + (t - s) f^* \left(\frac{x - \beta}{t - s} \right). \end{aligned}$$

Hence

$$r(\theta) = \beta + \frac{x - \beta}{t - s} (\theta - t) \in ch(x, t, s). \quad (3.3.16)$$

Let $x_1 < x_2$, then from NIP,

$$y_-(x_1, t, s) \leq y_+(x_1, t, s) \leq y_-(x_2, t, s) \leq y_+(x_2, t, s).$$

Let $x_1 = x_k, x_2 = x$ and $\beta = \lim_{x_k \uparrow x} y_+(x_k, t, s)$. Then from (3.3.1), $\beta = r(s)$ is a characteristic point and hence $y_-(x, t, s) \leq \beta$. Therefore

$$y_-(x, t, s) \leq \beta = \lim_{x_k \uparrow x} y_+(x_k, t, s) \leq y_-(x, t, s)$$

This proves (3.3.2) and similarly (3.3.3) holds.

3. Let $x < R_-(t, s, \alpha)$, then $y_-(x, t, s) < \alpha$ and for any $\xi < x$, $y_+(\xi, t, s) \leq y_-(x, t, s) < \alpha$. Hence $\{x < R_-(t, s, \alpha)\} \subset \{x : y_+(x, t, s) < \alpha\}$. This implies that $R_-(t, s, \alpha) \leq \sup_x \{x : y_+(x, t, s) \leq \alpha\} = R_+(t, s, \alpha)$. This proves (3.3.4). Suppose $y_-(R_-(t, s, \alpha), t, s) > \alpha$, then from (3.3.2) there exist a $x < R_-(t, s, \alpha)$ for which $y_-(x, t, s) > \alpha$. Hence $R_-(t, s, \alpha) \leq x$ which is a contradiction. Similarly from (3.3.3), it follows that $\alpha \leq y_+(R_-(t, s, \alpha), t, s)$. This proves (3.3.5) and (3.3.6) follows similarly.

4. Suppose for some $s < \theta_0 < t$, $R_-(\theta_0, s, \alpha) < r_-(\theta_0)$. Hence for $x \in (R_-(\theta_0, t, s, \alpha), r_-(\theta_0))$, $r_-(s) \leq \alpha \leq y_-(x, \theta_0, s)$. If $r_-(s) < \alpha$ or $y_-(x, \theta_0, s) > \alpha$, then from NIP gives a contradiction. Therefore $r_-(s) = y_-(x, \theta_0, s) = \alpha$, for all x and therefore $r(\theta) = x + \frac{x - \alpha}{\theta_0 - s} (\theta - \theta_0) \in ch(x, \theta_0, s)$. From (3.3.2), choose $\xi < R_-(t, s, \alpha)$ such that $y_+(\xi, t, s) < \alpha$ and $\tilde{r}(\theta) = \xi + \frac{\xi - y_+(\xi, t, \alpha)}{t - s} (\theta - s)$ intersect $r(\theta)$ at some $s < \tilde{\theta} < t$. This contradicts NIP and hence $r_-(\theta) \leq R_-(\theta, t, s)$.

Next suppose $r_+(\theta_0) < R_-(\theta_0, s, \alpha)$ for some $\theta_0 \in (s, t)$. From (3.3.5), $r_+(s) \geq \alpha$ and hence for any $x \in (r_+(\theta_0), R_-(\theta_0, t, s, \alpha))$, $y_-(x, t, s) < \alpha$ and therefore the characteristic $r(\theta) = x + \frac{x - y_-(x, \theta_0, s)}{\theta_0 - s} (\theta - s)$ intersect $r_+(\theta)$ at some $\tilde{\theta} \in (s, \theta_0)$ contradicting NIP. This proves (3.3.9) and (3.3.10) follows similarly.

For $s = 0$ and $\theta \in (0, t)$ we have from (3.3.9)

$$r_-(\theta, t, 0, \alpha) \leq R_-(\theta, \alpha) \leq r_+(\theta, t, 0, \alpha),$$

hence from NIP, if $x < R_-(t, \alpha)$, then $y_-(x, t, s) \leq r_-(s, t, 0, \alpha)$ and if $x > R_-(t, \alpha)$, $y_+(x, t, s) \geq r_+(s, t, 0, \alpha)$. This implies (3.3.11). (3.3.12) follows similarly.

From (3.3.7), (3.3.9) and (3.2.10) we have at $s = 0$, $\theta \in (0, t]$

$$\begin{aligned} R_-(t, \alpha) &+ \frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t}(\theta - t) \\ &\leq R_-(\theta, \alpha) \\ &\leq R_-(t, \alpha) + \frac{R_-(t, \alpha) - y_+(R_-(t, \alpha), t)}{t}(\theta - t). \end{aligned}$$

Hence

$$\begin{aligned} |R_-(\theta, \alpha) - R_-(t, \alpha)| &\leq \left| \frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t} \right| |\theta - t| \\ &+ \left| \frac{R_-(t, \alpha) - y_+(R_-(t, \alpha), t)}{t} \right| |\theta - t| \\ &\leq 2M(\theta - t) \end{aligned}$$

and

$$|R_-(t, \alpha) - y_{\pm}(R_-(t, \alpha), t)| \leq Mt.$$

Since $y_-(R_-(t, \alpha), t) \leq \alpha \leq y_+(R_+(t, \alpha), t)$ and hence letting $t \rightarrow 0$ to obtain $R_-(t, \alpha) \rightarrow \alpha$ as $t \rightarrow 0$. Similarly for R_+ and this proves (4).

5. Let $R_-(t, s, \alpha) < R_+(t, s, \alpha)$ and $x \in (R_-(t, s, \alpha), R_+(t, s, \alpha))$. Then $y_+(x, t, s) \geq y_-(x, t, s) \geq \alpha \geq y_+(x, t, s)$ and hence $y_+(x, t, s) = \alpha$. Therefore from (3.3.1), $u(x, t) = \frac{\partial v}{\partial x}$ is given by

$$f'(u(x, t)) = \frac{x - \alpha}{t - s} \tag{3.3.17}$$

and $r_x(\theta) = x + \frac{x - \alpha}{t - s}(\theta - t) \in ch(x, t, s)$. Let $s > 0$, then from NIP, $r_x \in ch(x, t)$ and for $x_1 < x_2$, r_{x_1} , r_{x_2} intersect at $t = s$ which is a contradiction. This proves (3.3.14) and when $s = 0$, (3.3.15) follows from (3.3.17).

6. It is enough to prove (ii) and the rest follows in the same manner. From (3.3.11)

and (3.3.14) for $t > T$,

$$\begin{aligned}
R_-(t, \alpha) &= R_-(t, T, R_-(T, \alpha)) \\
&= R_-(t, T, R_+(T, \beta)) \\
&= R_+(t, T, R_+(T, \beta)) \\
&= R_+(t, \beta).
\end{aligned}$$

This proves (6).

7. From (5) of Theorem 3.2.1, $u_1(x, t) \leq u_2(x, t)$. Hence from Theorem 3.2.1, for a.e. $t > 0$ and for a.e. $x \in \mathbb{R}$,

$$\frac{x - y_{\pm}^{(1)}(x, t)}{t} = f'(u_1(x, t)) \leq f'(u_2(x, t)) = \frac{x - y_{\pm}^{(2)}(x, t)}{t}$$

where $y_{\pm}^{(i)}(x, t)$ are the extreme characteristic points of $u_i, i = 1, 2$. Therefore $y_{\pm}^{(2)}(x, t) \leq y_{\pm}^{(1)}(x, t)$ and hence $R_{\pm}^{(1)}(t, \alpha) \leq R_{\pm}^{(2)}(t, \alpha)$ for a.e. $t > 0$. Now from the continuity in $t > 0$, (7) follows.

Lemma 3.3.2. (1). Let $r(t, \alpha, p) = \alpha + tf'(p)$ and for $T > 0$, $r(\cdot, \alpha, p) \in ch(r(T, \alpha, p), T)$. Let $0 \leq T_1 < T$, then for $T_1 < t < T$, following are equivalent

$$\begin{aligned}
(i). \quad y_-(r(t, \alpha, p), t) &= \alpha. \\
(ii). \quad u(r(t, \alpha, p)-, t) &= p. \\
(iii). \quad u(r(t, \alpha, p)+, t) &= p. \\
(iv). \quad y_+(r(t, \alpha, p), t) &= \alpha.
\end{aligned} \tag{3.3.18}$$

(2). Let $r(t, \alpha, p)$ be a regular characteristic line. In view of (1), u is well defined at $r(t, \alpha, p)$ for all $t > 0$ by

$$u(r(t, \alpha, p), t) = p. \tag{3.3.19}$$

Let $(\alpha_k, p_k) \rightarrow (\alpha, p)$ and $\{r(\cdot, \alpha_k, p_k)\}$ are regular characteristic lines. Then $r(\cdot, \alpha, p)$ is a regular characteristic line. Furthermore if $t_k \rightarrow t > 0$ and $x_k = r(t_k, \alpha_k, p_k)$, $x = r(t, \alpha, p)$, then

$$\lim_{k \rightarrow \infty} u(x_k, t_k) = u(x, t). \tag{3.3.20}$$

(3). Suppose for some $t > 0$, $y_-(R_-(T, \alpha)) = \alpha$, then for all $0 < t < T$,

$$\begin{aligned}
R_-(T, \alpha) &= R_-(T, \alpha) + \frac{R_-(T, \alpha) - \alpha}{T}(t - T) \\
y_-(R_-(t, \alpha), t) &= \alpha.
\end{aligned} \tag{3.3.21}$$

(4). Suppose for some $T > 0$, $y_+(R_+(T, \alpha), T) = \alpha$, then for $0 < t < T$,

$$\begin{aligned} R_+(t, \alpha) &= R_+(T, \alpha) + \frac{R_+(T, \alpha) - \alpha}{T}(t - T) \\ y_+(R_+(t, \alpha), t) &= \alpha. \end{aligned} \tag{3.3.22}$$

Proof. (1). we have the following

claim : Denote $r(t) = r(t, \alpha, p)$, then for $T_1 < t < T$

$$\begin{aligned} f'(u(r(t)-, t)) &= \frac{r(t) - y_-(r(t), t)}{t} \\ f'(u(r(t)+, t)) &= \frac{r(t) - y_+(r(t), t)}{t} \\ u(r(t)-, t) &\geq p \geq u(r(t)+, t). \end{aligned}$$

In order to prove this, observe that that $y_-(r(t), t) \leq \alpha \leq y_+(r(t), t)$ since $r(\cdot)$ is a characteristic. Hence from (3.2.13) and (3.3.2) we have

$$\begin{aligned} f'(u(r(t)-, t)) &= \lim_{x \uparrow r(t)} \frac{x - y_+(x, t)}{t} \\ &= \frac{r(t) - y_-(r(t), t)}{t} \\ &\geq \frac{r(t) - \alpha}{t} = f'(p). \end{aligned}$$

Similarly from (3.2.13) and (3.3.3) we have

$$\begin{aligned} f'(u(r(t)+, t)) &= \frac{r(t) - y_+(r(t), t)}{t} \\ &\leq \frac{r(t) - \alpha}{t} = f'(p). \end{aligned}$$

Hence

$$u(r(t)+, t) \geq p \geq u(r(t)-, t)$$

and this proves the claim.

(i) \implies (ii). Let $y_-(r(t), t) = \alpha$, then from the claim we have $f'(u(r(t)-, t)) = \frac{r(t) - \alpha}{t} = f'(p)$ and hence $u(r(t)-, t) = p$.

(ii) \implies (iii). From the claim and Rankine-Hugoniot condition across $r(\cdot)$, we have for a.e. $T_1 < t < T$

$$\begin{aligned} f'(p) = \frac{dr}{dt} &= \frac{f(u(r(t)-, t)) - f(u(r(t)+, t))}{u(r(t)+, t) - u(r(t)-, t)} \\ &= \frac{f'(p) - f(u(r(t)+, t))}{u(r(t)+, t) - u(r(t)-, t)}. \end{aligned}$$

Since $u(r(t)+, t) \leq p$ and hence by strict convexity of f , we have $u(r(t)+, t) = p$ for a.e. $T_1 < t < T$. Therefore from the claim we have

$$f'(p) = f'(u(r(t)+, t)) = \frac{r(t) - y_+(r(t), t)}{t} \leq f'(p).$$

Hence $y_+(r(t), t) = \alpha$ for a.e. $t > 0$.

Suppose for some $T_1 < t_0 < T$, $y_+(r(t_0), t_0) > \alpha$. Let $T > t_1 > t_0$ such that $y_+(r(t_1), t_1) = \alpha$. Then from (3.3.3), choose a $\xi > r(t_1)$ such that $y_+(\xi, t_1) \in (\alpha, y_+(r(t_0), t_0))$. Then the right extreme characteristics at $(r(t_1), t_1)$ and at $(r(t_0), t_0)$ intersect contradicting NIP. Hence $y_+(r(t), t) = \alpha$ for all $T_1 < t < T$. Therefore from the claim, for $T_1 < t < T$,

$$\begin{aligned} f'(u(r(t)+, t)) &= \frac{r(t) - y_+(r(t), t)}{t} \\ &= \frac{r(t) - \alpha}{t} \\ &= f'(p). \end{aligned}$$

(iii) \implies (iv). From the claim we have

$$f'(p) = f'(u(r(t)+, t)) = \frac{r(t) - y_+(r(t), t)}{t}$$

and hence $y_+(r(t), t) = \alpha$.

Similar analysis proves (iv) \implies (iii) \implies (ii) \implies (i). This proves (1).

(2). From (2) of Lemma 3.3.1, limit of characteristics is a characteristic and hence $r(t, \alpha, p)$ is a characteristic line. If $(\alpha_k, p_k) = (\alpha, p)$ for some k , then there is nothing to prove. Hence assume that $(\alpha_k, p_k) \neq (\alpha, p)$ for all k . Therefore from NIP, given k , for all $t > 0$, either $r(t, \alpha_k, p_k) < r(t, \alpha, p)$ or $r(t, \alpha, p) < r(t, \alpha_k, p_k)$. Suppose for some subsequence still denoted by (α_k, p_k) such that for all k , $r(t, \alpha_k, p_k) < r(t, \alpha, p)$, then $\alpha_k = r(0, \alpha_k, p_k) \leq r(0, \alpha, p) = \alpha$. Suppose for some $t_0 > 0$, $y_-(r(t_0, \alpha, p), t_0) < \alpha$, then choose k_0 large such that

$$y_-(r(t_0, \alpha, p), t_0) < \alpha_k = y_-(r(t, \alpha_{k_0}, p_{k_0}), t) \leq \alpha.$$

Hence the left extreme characteristic at $(r(t_0, \alpha, p), t_0)$ intersect $r(\cdot, \alpha_{k_0}, p_{k_0})$ contradicting NIP. Hence $y_-(r(t, \alpha, p), t) = \alpha$ for all $t > 0$. From the claim it follows that $r(\cdot, \alpha, p)$ is regular. Similarly if for a subsequence still denoted by (α_k, p_k) if $r(t, \alpha_k, p_k) > r(t, \alpha, p)$, it follows that $r(\cdot, \alpha, p)$ is regular.

Let $x_k = r(t_k, \alpha_k, p_k) \rightarrow x = r(t, \alpha, p)$ as $k \rightarrow \infty$, then

$$\lim_{k \rightarrow \infty} u(x_k, t_k) = \lim_{k \rightarrow \infty} p_k = p = u(x, t).$$

This proves (2).

(3). It is enough to prove (3) and (4) follows in a similar way. For $0 \leq \theta$, let $r(\theta) = R_-(T, \alpha) + \frac{R_-(T, \alpha) - \alpha}{T}(\theta - T)$. Then from the hypothesis, $r \in ch(R_-(T, \alpha), T)$ and hence for $0 < t < T$, $R_-(t, \alpha) \leq r(t)$. Suppose for some $t_0 \in (\theta, T)$, $R_-(t_0, \alpha) < r(t_0)$, then from NIP, for all $R_-(t_0, \alpha) < x < r(t_0)$, $\alpha \leq y_-(x, t_0) \leq \alpha$ and hence $y_-(x, t_0) = \alpha$. This implies $r_x(\theta) = \alpha + \frac{x - \alpha}{t_0}(\theta - t_0)$ is in $ch(x, t_0)$. From (3.3.2) we can choose a $\xi < R_-(T, \alpha)$ such that $y_+(\xi, T) \in (R_-(t_0, \alpha), \alpha)$ and hence the characteristic $\xi + \frac{\xi - y_+(\xi, T)}{T}(\theta - T)$ intersect $r_{x_0}(\theta)$ for some $x_0 \in (R_-(t_0, \alpha), r(t_0))$ contradicting NIP. Suppose for some $0 < t_0 < T$, $y_-(R_-(t_0, \alpha), \alpha) < \alpha$, then again choose $\xi < R_-(T, \alpha)$ such that $y_+(\xi, T) \in (y_-(R_-(t_0, \alpha), \alpha), \alpha)$. Hence the right extreme characteristic at (ξ, T) intersect the left extreme characteristic at $(R_-(t_0, \alpha), t_0)$ contradicting NIP. This proves (3).

Lemma 3.3.3. *Let $t > 0$ and suppose $x \mapsto u(x, t)$ is a non decreasing function for $x \in (a, b)$. Then $u(\cdot, t)$ is a continuous function in (a, b) .*

Proof. Let $a < x_1 < x_2 < b$, then $u(x_1, t) \leq u(x_2, t)$ and hence for $x \in (a, b)$, $u(x-, t) \leq u(x+, t)$. Since u is an entropy solution and hence $u(x+, t) \leq u(x-, t)$. This proves $u(x+, t) = u(x-, t)$ and hence the Lemma.

Lemma 3.3.4. *Let $\alpha \in \mathbb{R}$, then*

1. $t \mapsto y_-(R_-(t, \alpha), t)$ is a non increasing function and $t \mapsto y_+(R_-(t, \alpha), t)$ is a non decreasing function. Let

$$(C_{1,\alpha}, C_{2,\alpha}) = \lim_{t \rightarrow \infty} (y_-(R_-(t, \alpha), t), y_+(R_-(t, \alpha), t)) \quad (3.3.23)$$

Then

$$C_{1,\alpha} \leq \alpha \leq C_{2,\alpha}.$$

2. Let $C_{1,\alpha} < \beta < C_{2,\alpha}$, then there exists a $T(\beta) > 0$ such that for $t > T(\beta)$

$$R_{\pm}(t, \beta) = R_-(t, \alpha) \quad (3.3.24)$$

3. There does not exist a regular characteristic line $r(t)$ such that $C_{1,\alpha} < r(0) < C_{2,\alpha}$.

4. Suppose $C_{1,\alpha} > -\infty$, then

$$p_{-, \alpha} = \lim_{t \rightarrow \infty} \frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t} \quad (3.3.25)$$

exists and $r_{1,\alpha}(t) = C_{1,\alpha} + t f'(p_{-, \alpha})$ is a regular characteristic line. Furthermore

(i). if $C_{1,\alpha} = \alpha$, then $r_{1,\alpha}(t) = R_-(t, \alpha)$ for all $t > 0$.

(ii). If $C_{1,\alpha} < \alpha$, then $r_{1,\alpha}(t) < R_-(t, \alpha)$ for all $t > 0$.

5. Suppose u_0 is continuous in a neighbourhood of $C_{1,\alpha}$, then

$$u_0(C_{1,\alpha}) = p_{-,\alpha}. \quad (3.3.26)$$

Furthermore if u_0 be a non decreasing function in $(C_{1,\alpha}, C_{1,\alpha} + \epsilon)$ for some $\epsilon > 0$, then there exists a $T > 0$ such that for all $t > 0$, $u(\cdot, t)$ is continuous in $\{x : r_{1,\alpha}(t) < x < R_-(t, \alpha)\}$.

6. Suppose $C_{2,\alpha} < \infty$, then

$$p_{+,\alpha} = \lim_{t \rightarrow \infty} \frac{R_-(t, \alpha) - y_+(R_-(t, \alpha), t)}{t} \quad (3.3.27)$$

exists and $r_{2,\alpha} = C_{2,\alpha} + tf'(p_{+,\alpha})$ is a regular characteristic line. Furthermore if there exists a $T > 0$ such that $y_-(R_-(T, \alpha), \alpha) < \alpha$, then for all $t > 0$,

$$R_-(t, \alpha) < r_{2,\alpha}(t).$$

7. Suppose u_0 is continuous in neighbourhood of $C_{2,\alpha}$, then

$$u_0(C_{2,\alpha}) = p_{+,\alpha} \quad (3.3.28)$$

Furthermore if u_0 is a non decreasing function in $(C_{2,\alpha} - \epsilon, C_{2,\alpha})$, then there exists $T > 0$ such that for all $t > T$, $u(\cdot, t)$ is continuous in $\{x : R_-(t, \alpha) < x < r_{2,\alpha}(t)\}$.

8. If $-\infty < C_{1,\alpha} < C_{2,\alpha} < \infty$, then for all $t > 0$,

$$\text{def: } p_\alpha = p_{-,\alpha} = p_{+,\alpha} \quad (3.3.29)$$

$$\int_{C_{1,\alpha}}^{C_{2,\alpha}} u_0(x) dx = \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x, t) dx = p_\alpha(C_{2,\alpha} - C_{1,\alpha}). \quad (3.3.30)$$

$$\int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} |f'(u(x, t)) - f'(p_\alpha)| dx \leq \frac{(C_{1,\alpha} - C_{2,\alpha})^2}{t}. \quad (3.3.31)$$

9. $t \mapsto y_-(R_+(t, \alpha), t)$ is a non increasing function and $t \mapsto y_+(R_+(t, \alpha), t)$ is a non decreasing function and let

$$(d_{1,\alpha}, d_{2,\alpha}) = \lim_{t \rightarrow \infty} (y_-(R_+(t, \alpha), \alpha), y_+(R_+(t, \alpha), \alpha)). \quad (3.3.32)$$

10. All the above properties (2) to (8) hold for $R_+(t, \alpha), d_{1,\alpha}, d_{2,\alpha}$, with appropriate changes in (4) and (6).

Proof. It is enough to prove (1) to (8), (9) and (10) follow similarly.

1. Let $t_1 < t_2$ and $r_i(\theta)$ be the left extreme characteristic at $(R_-(t_i, \alpha), t_i)$ for $i = 1, 2$. From (3.3.9) we have $r_2(t_1) \leq R_-(t_1, \alpha) = r_1(t_1)$. Hence from NIP, r_2 and r_1 do not intersect in the interval $(0, t_1)$. Therefore for $\theta \in (0, t_1)$, either $r_2(\theta) \leq r_1(\theta)$ or $r_2(\theta) > r_1(\theta)$. Suppose $r_1(\theta) < r_2(\theta)$, then $y_-(R_-(t_1, \alpha), t_1) = r_1(0) < r_2(0) = y_-(R_-(t_2, \alpha), t_2)$. From (3.3.2), we can choose a $\xi < R_-(t_2, \alpha)$ such that $y_-(R_-(t_1, \alpha), t_1) < y_-(\xi, t_2) < y_-(R_-(t_2, \alpha), t_2)$ and hence the characteristic at (ξ, t_2) intersects r_1 in $(0, t_1)$ which contradicts NIP. Hence

$$y_-(R_-(t_2, \alpha), t_2) = r_2(0) \leq r_1(0) = y_-(R_-(t_1, \alpha), t_1)$$

Similarly for $t \mapsto y_+(R_-(t, \alpha), t)$. Since $y_-(R_-(t, \alpha), t) \leq \alpha \leq y_+(R_-(t, \alpha), t)$ and hence $C_{1,\alpha} \leq \alpha \leq C_{2,\alpha}$. This proves (1).

2. Let $C_{1,\alpha} < \beta < C_{2,\alpha}$. Choose $T > 0$ such that $y_-(R_-(T, \alpha), T) < \beta < y_+(R_-(T, \alpha), T)$. Let $r_1(\theta), r_2(\theta)$ be the left and right characteristic at $(R_-(T, \alpha), T)$. We claim that for $\theta \in (0, T)$, $r_1(\theta) \leq R_\pm(\theta, \beta) \leq r_2(\theta)$. Suppose not, let for some $\theta_0 \in (0, T)$, $R_-(\theta_0, \beta) < r_1(\theta_0)$. Since $y_+(R_-(\theta_0, \beta), \theta_0) \geq \beta > r_1(0)$, hence the right extreme characteristic at $(R_-(\theta_0, \beta), \theta_0)$ intersects r_1 in $(0, \theta_0)$ contradicting NIP. Hence $r_1(\theta) \leq R_-(\theta, \beta)$. Similarly all other cases follow. This proves the claim. Since $r_1(T) = r_2(T) = R_-(T, \alpha)$ and hence there exist $t_1, t_2 \in (0, T]$ such that $R_+(t_1, \beta) = R_-(t_1, \alpha)$ and $R_-(t_2, \beta) = R_-(t_2, \alpha)$. Then from collapsing property, $R_\pm(t, \beta) = R_-(t, \alpha)$ for all $t \geq \max\{t_1, t_2\}$. This proves (2).

3. First observe that $y_-(R_-(t, \alpha), t) \neq \alpha$ for all $t > 0$. Suppose $y_-(R_-(t, \alpha), t) = \alpha$ for all $t > 0$, then from (3) and (1) of Lemma 3.3.2, $y_+(R_-(t, \alpha), t) = \alpha$ for all $t > 0$, contradicting $C_{1,\alpha} < C_{2,\alpha}$. Hence there exists $T > 0$ such that for all $t > T$, $y_-(R_-(t, \alpha), t) < \alpha \leq y_+(R_-(t, \alpha), t)$. Let r be a regular characteristic line with $C_{1,\alpha} < r(0) = \beta < C_{2,\alpha}$. Then we have for all $t > 0$,

$$R_-(t, \beta) \leq r(t) \leq R_+(t, \beta).$$

From (2), there exists $T_1 > 0$ such that for all $t > T_1$, $R_-(t, \beta) = R_+(t, \beta) = R_-(t, \alpha)$. Hence $r(t) = R_-(t, \alpha)$ for all $t > T_1$. Since r is a regular characteristic line, hence for $t > \max\{T, T_1\}$, $\alpha = y_-(r(t), t) = y_-(R_-(t, \alpha), t) < \alpha$, which is a contradiction. This proves (3).

4. Let

$$\begin{aligned} f'(p_t) &= \frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t} \\ r(\theta, t) &= R_-(t, \alpha) + f'(p_t)(\theta - t). \\ &= y_-(R_-(t, \alpha), t) + \theta f'(p_t) \end{aligned}$$

Then from (3.2.10), $\{p_t\}$ is bounded and $r(\cdot, t)$ is the left extreme characteristic at $(R_-(t, \alpha), t)$. Let $0 < t_0 < t$ be fixed and define $x(t_0, t) = r(t_0, t)$. Then for $\theta \in (0, t_0)$, $x(t_0, t)$ and r satisfies

$$x(t_0, t) = y_-(R_-(t, \alpha), t) + t_0 f'(p_t) \quad (3.3.33)$$

$$r(\theta, t) = x(t_0, t) + f'(p_t)(\theta - t_0). \quad (3.3.34)$$

Let $t_0 < t_1 < t_2$. Since $y_-(R_-(t_2, \alpha), t_2) \leq y_-(R_-(t_1, \alpha), t_1)$ and hence from NIP, $x(t_0, t_2) \leq x(t_0, t_1)$. From (3.3.33), $\{x(t_0, t)\}$ is bounded and non increasing function. Let $x(t_0) = \lim_{t \rightarrow \infty} x(t_0, t)$. Therefore from (3.3.33), p_t converges as $t \rightarrow \infty$ and let $p_{-, \alpha} = \lim_{t \rightarrow \infty} p_t$. Then (3.3.33) and (3.3.34) imply

$$\begin{aligned} x(t_0) &= C_{1, \alpha} + t_0 f'(p_{-, \alpha}) \\ \lim_{t \rightarrow \infty} r(\theta, t) &= x(t_0) + f'(p_{-, \alpha})(\theta - t_0) \\ &= C_{1, \alpha} + \theta f'(p_{-, \alpha}) \\ &= r_{1, \alpha}(\theta) \end{aligned}$$

and from (2) of Lemma 3.3.1, $r_{1, \alpha}$ is a characteristic at $(x(t_0), t_0)$. Since t_0 is arbitrary and hence $r_{1, \alpha}$ is a characteristic line with $r_{1, \alpha}(0) = C_{1, \alpha}$. From (3.3.9), we have $x(t_0, t) = r(t_0, t) \leq R_-(t_0, \alpha)$. Hence letting $t \rightarrow \infty$ to obtain

$$r_{1, \alpha}(t_0) \leq R_-(t_0, \alpha).$$

Let $C_{1, \alpha} = \alpha$, then $y_-(R_-(t, \alpha), t) = \alpha$ for all $t > 0$. Hence from (3) and (1) of Lemma 3.3.2, $R_-(t, \alpha)$ is a regular characteristic line and from the definition, $r_{1, \alpha}(t) = R_-(t, \alpha)$.

$C_{1, \alpha} < \alpha$. Suppose for some $T > 0$, $R_-(T, \alpha) = r_{1, \alpha}(T)$. Since $r_{1, \alpha}$ is a characteristic line and $C_{1, \alpha} \leq y_-(R_-(t, \alpha), t)$, hence $y_-(R_-(T, \alpha), T) = C_{1, \alpha}$ and by monotonicity, $y_-(R_-(t, \alpha), t) = C_{1, \alpha}$ for all $t > T$. Next we claim that $R_-(t, \alpha) = r_{1, \alpha}(t)$ for $t > T$. Suppose not, then there exists $T_1 > T$ such that $r_{1, \alpha}(T_1) < R_-(T_1, \alpha)$. Since $y_+(R_-(T, \alpha), T) \geq \alpha > C_{1, \alpha}$ and $y_-(R_-(T, \alpha), T) = C_{1, \alpha}$, hence the right extreme characteristic at $(R_-(T, \alpha), T)$ and left extreme characteristic at $(R_-(T_1, \alpha), T_1)$ intersect in $(0, T)$ contradicting NIP, this proves the claim.

Therefore for $t > T$, $y_-(r_{1, \alpha}(t), t) = y_-(R_-(t, \alpha), t) = C_{1, \alpha}$ and hence from (1) of Lemma 3.3.2, $C_{1, \alpha} = y_+(r_{1, \alpha}(t), t) = y_+(R_-(t, \alpha), t) \geq \alpha > C_{1, \alpha}$ which is a contradiction. This proves that for all $t > 0$, $r_{1, \alpha}(t) < R_-(t, \alpha)$. Hence for $t_0 > 0$ and for any $t > t_0$, $r_{1, \alpha}(t_0) < x(t_0, t) < R_-(t_0, \alpha)$. Since $y_-(x(t_0, t), t_0) = y_-(R_-(t, \alpha), t)$ and therefore

$$\lim_{x \downarrow r_{1, \alpha}(t_0)} y_-(x, t_0) = C_{1, \alpha}.$$

Hence from (3.2.13) and (3.3.3), we have

$$\begin{aligned}
f'(u(r_{1,\alpha}(t_0)+, t_0)) &= \lim_{x \downarrow r_{1,\alpha}(t_0)} \frac{x - y_-(x, t_0)}{t_0} \\
&= \frac{r_{1,\alpha}(t_0) - C_{1,\alpha}}{t_0} \\
&= f'(p_{-, \alpha}).
\end{aligned}$$

Therefore from (1) of Lemma 3.3.1 we have for all $t_0 > 0$, $y_-(r_{1,\alpha}(t_0), t_0) = y_+(r_{1,\alpha}(t_0), t_0) = C_{1,\alpha}$. This proves that $r_{1,\alpha}$ is a regular characteristic and this completes the proof of (4).

5. Let $\epsilon_0 > 0$ be such that u_0 is continuous in $(C_{1,\alpha} - \epsilon_0, C_{1,\alpha} + \epsilon_0)$. Choose $T > 0$ such that for all $t > T$, $y_-(R_-(t, \alpha), t) < C_{1,\alpha} + \epsilon_0$. Let $T < t$, then for all $r_{1,\alpha}(t) < x < R_-(t, \alpha)$, $y_-(x, t_0) \in [C_{1,\alpha}, C_{1,\alpha} + \epsilon_0)$. Since v_0 is differentiable in $(C_{1,\alpha} - \epsilon_0, C_{1,\alpha} + \epsilon_0)$, hence from (3.2.13) and (3.2.14) we have for a.e. $x \in (r_{1,\alpha}(t), R_-(t, \alpha))$,

$$f'(u_0(y_-(x, t))) = f'(u(x, t)) = \frac{x - y_-(x, t)}{t}. \quad (3.3.35)$$

Let $x(t_0, t)$ be as in (3.3.33). Since $x(t_0, t) \rightarrow r_{1,\alpha}(t_0)$ as $t \rightarrow \infty$ and hence

$$\lim_{x \rightarrow r_{1,\alpha}(t_0)} y_-(x, t_0) = C_{1,\alpha}$$

Hence from (3.3.35)

$$\begin{aligned}
f'(u_0(C_{1,\alpha})) &= \lim_{x \rightarrow r_{1,\alpha}(t_0)} f'(u_0(y_-(x, t_0))) \\
&= \frac{r_{1,\alpha}(x_0) - C_{1,\alpha}}{t_0} \\
&= f'(p_{-, \alpha}).
\end{aligned}$$

This proves (3.3.26).

Let u_0 be non decreasing in $[C_{1,\alpha}, C_{1,\alpha} + \epsilon_0)$. Since $y_-(x, t) \in [C_{1,\alpha}, C_{1,\alpha} + \epsilon_0)$, hence $x \mapsto u_0(y_-(x, t))$ is a non decreasing function. Therefore from (3.2.14) and Lemma 3.3.3, $x \mapsto u(x, t) = u_0(y_-(x, t))$ is a continuous function for $x \in (r_{1,\alpha}(t), R_-(t, \alpha))$. This proves (5).

(6). (3.3.27) follows exactly as in (4) and $r_{2,\alpha}$ is a characteristic line.

Claim: Let for some $T > 0$, $R_-(T, \alpha) = r_{2,\alpha}(T)$, then $y_-(R_-(T, \alpha), \alpha) = \alpha$.

Suppose $y_-(R_-(T, \alpha), \alpha) < \alpha$, then we show that $R_-(t, \alpha) = r_{2,\alpha}(t)$ for all $t > T$. If not, then there exists a $t_0 > T$ such that $R_-(t_0, \alpha) < r_{2,\alpha}(t_0)$. Since $y_-(R_-(T, \alpha), \alpha) < \alpha \leq y_+(R_-(t_0, \alpha), t_0)$, the left extreme characteristic at $(R_-(T, \alpha), T)$ and the right extreme characteristic at $(R_-(t_0, \alpha), t)$ intersect in $(0, T)$ contradicting

NIP. Hence $R_-(t, \alpha) = r_{2,\alpha}(t)$ for all $t > T$. Since $y_+(R_-(t, \alpha), t) \leq C_{2,\alpha}$ and hence $y_+(R_-(t, \alpha), t) = C_{2,\alpha}$ for $t > T$. Therefore from (1) of Lemma 3.3.2, for $t > T$, $\alpha \leq C_{2,\alpha} = y_+(R_-(t, \alpha), t) = y_-(R_-(t, \alpha), t) < \alpha$ which is a contradiction. This proves the claim.

From the hypothesis, there exists $T > 0$ such that $y_-(R_-(T, \alpha), T) < \alpha$ and hence from the above claim, there exists $0 \leq T_0 < \infty$ such that

$$T_0 = \max\{t \geq 0 : R_-(t, \alpha) = r_{2,\alpha}(t)\}.$$

Since $R_-(t, \alpha) \leq r_{2,\alpha}(t)$ for all $t > 0$, hence we have for $t > T_0$, $R_-(t, \alpha) < r_{2,\alpha}(t)$ and from the claim $y_-(R_-(T_0, \alpha), T_0) = \alpha$. Therefore from (3),(1) of Lemma 3.3.2, for all $0 < t \leq T_0$,

$$\alpha = y_-(R_-(t, \alpha), t) = y_+(R_-(t, \alpha), t) = C_{2,\alpha},$$

and for all $t > T_0$

$$y_-(R_-(t, \alpha), t) < \alpha \leq y_+(R_-(t, \alpha), t) \leq C_{2,\alpha}.$$

Hence if $\alpha < C_{2,\alpha}$, then the left extreme characteristic at $(R_-(T_0, \alpha), T_0)$ and the right extreme characteristic at $(R_-(t, \alpha), t)$, for $t > T_0$ intersect in $(0, T)$ contradicting NIP. Therefore $\alpha = C_{2,\alpha}$ and $y_+(R_-(t, \alpha), t) = C_{2,\alpha}$ for all $t > T$. Let $t_0 > T_0$ and $r_+(\cdot)$ be the right extreme characteristic at $(R_-(t_0, \alpha), t_0)$. Let $T_0 < t \leq t_0$ be such that $r_+(t_1) \leq R_-(t_1, \alpha)$ and t_1 exists because $R_-(T_0, \alpha) = r_{2,\alpha}(T_0)$. Since $y_-(R_-(t, \alpha), \alpha) < \alpha = C_{2,\alpha}$ and $r_+(0) = y_+(R_-(t_0, \alpha), \alpha) = C_{2,\alpha}$, therefore r_+ and left extreme characteristic at $(R_-(t_1, \alpha), t_1)$ intersect in $(0, t_1)$ contradicting NIP. Hence $T_0 = 0$ and therefore for all $t > 0$, $R_-(t, \alpha) < r_{2,\alpha}(t)$. Following the similar proof as in (4) to yield $r_{2,\alpha}(\cdot)$ is a regular characteristic line. This proves (6).

7. Proof of this assertion follows by similar arguments as in (5).

8. Let $-\infty < C_{1,\alpha} < C_{2,\alpha} < \infty$, then $C_{1,\alpha} \leq y_-(R_-(t, \alpha), t) \leq \alpha \leq y_+(R_-(t, \alpha), t) \leq C_{2,\alpha}$ and hence

$$\begin{aligned} f'(p_{-, \alpha}) &= \lim_{t \rightarrow \infty} \frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t} \\ &\geq \lim_{t \rightarrow \infty} \frac{R_-(t, \alpha) - y_+(R_-(t, \alpha), t)}{t} \\ &= f'(p_{+, \alpha}). \end{aligned}$$

Hence $p_{-, \alpha} \geq p_{+, \alpha}$. Suppose $p_{-, \alpha} > p_{+, \alpha}$, then the characteristic lines $r_{1,\alpha}$ and $r_{2,\alpha}$ intersect at $t_0 = \frac{C_{2,\alpha} - C_{1,\alpha}}{f'(p_{-, \alpha}) - f'(p_{+, \alpha})} > 0$ contradicting NIP. Hence $p_{-, \alpha} = p_{+, \alpha}$. This proves (3.3.29).

Let $\Omega(t) = \{(x, \theta) : 0 < \theta < t, r_{1,\alpha}(\theta) < x < r_{2,\alpha}(\theta)\}$. Then integrating by parts

yields

$$\begin{aligned}
0 = \int_{\Omega(t)} (u_\theta + f(u)_x) dx d\theta &= - \int_{C_{1,\alpha}}^{C_{2,\alpha}} u_0(x) dx + \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x, t) dx \\
&+ \int_0^t \frac{dr_{1,\alpha}}{d\theta} u(r_{1,\alpha}(\theta)_+, \theta) d\theta \\
&- \int_0^t \frac{dr_{2,\alpha}}{d\theta} u(r_{2,\alpha}(\theta)_-, \theta) d\theta \\
&- \int_0^t f(u(r_{1,\alpha}(\theta)_+, \theta)) d\theta \\
&+ \int_0^t f(u(r_{2,\alpha}(\theta)_-, \theta)) d\theta.
\end{aligned}$$

Since $\frac{dr_{1,\alpha}}{d\theta} = \frac{dr_{2,\alpha}}{d\theta} = f'(p_\alpha)$, $r_{1,\alpha}$ and $r_{2,\alpha}$ are regular characteristic lines and hence from (1) of Lemma 3.3.2, we have $u(r_{1,\alpha}(\theta)_+, \theta) = u(r_{2,\alpha}(\theta)_-, \theta) = p_\alpha$. Therefore by change of variable $\xi = x - tf'(p_\alpha)$ to obtain

$$\begin{aligned}
\int_{C_{1,\alpha}}^{C_{2,\alpha}} u_0(x) dx &= \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x, t) dx \\
&= \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} (f')^{-1} \left(\frac{x - y_-(x, t)}{t} \right) dx \\
&= \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} (f')^{-1} \left(\frac{x - tf'(p_\alpha) - y_-(x, t)}{t} + f'(p_\alpha) \right) dx \\
&= \int_{C_{1,\alpha}}^{C_{2,\alpha}} (f')^{-1} \left(\frac{\xi - y_-(x, t)}{t} + f'(p_\alpha) \right) d\xi \\
&\rightarrow p_\alpha(C_{1,\alpha} - C_{2,\alpha}) \text{ as } t \rightarrow \infty.
\end{aligned}$$

This proves (3.3.30).

$$\begin{aligned}
\int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} |f'(u(x,t)) - f'(p_\alpha)| dx &= \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} \left| \frac{x - y_-(x,t)}{t} - f'(p_\alpha) \right| dx \\
&= \int_{C_{1,\alpha}}^{C_{2,\alpha}} \left| \frac{\xi - y_-(x,t)}{t} \right| d\xi \\
&\leq \frac{(C_{2,\alpha} - C_{1,\alpha})^2}{t}.
\end{aligned}$$

This proves (3.3.31) and hence (8). (9) and (10) follow exactly as in the previous cases and this proves the Lemma.

Let u_0 be as in (3.2.19) and $I = (A_1, A_2)$. Define

$$m = \inf_{x \in I} \{\bar{u}_0(x), u_+\}, \quad M = \sup_{x \in I} \{\bar{u}_0(x), u_-\}.$$

$$f'(k_1) = \begin{cases} f'(u_-) & \text{if } u_- < m \\ \frac{f(u_-) - f(m)}{u_- - m} & \text{if } u_- \geq m. \end{cases} \quad (3.3.36)$$

$$f'(k_2) = \begin{cases} f'(u_+) & \text{if } u_+ > M \\ \frac{f(u_+) - f(M)}{u_+ - M} & \text{if } u_+ \leq M. \end{cases} \quad (3.3.37)$$

$$r_1(t) = A_1 + tf'(k_1), \quad r_2(t) = A_2 + tf'(k_2). \quad (3.3.38)$$

Let u_m, u_M be the solutions of (3.1.2) with respective initial datas u_0^m and u_0^M given by

$$u_0^m = \begin{cases} u_- & \text{if } x < A_1, \\ m & \text{if } x > A_1. \end{cases}$$

$$u_0^M = \begin{cases} u_+ & \text{if } x > A_2, \\ M & \text{if } x < A_2. \end{cases}$$

Then u_m and u_M are given by

$$u_m(x,t) = \begin{cases} u_- & \text{if } x < r_1(t), \\ (f')^{-1} \left(\frac{x - A_1}{t} \right) & \text{if } r_1(t) < x < \text{Max}\{r_1(t), A_1 + tf'(m)\}, \\ m & \text{if } x > \text{Max}\{r_1(t), A_1 + tf'(m)\}. \end{cases}$$

$$u_m(x, t) = \begin{cases} u_+ & \text{if } x > r_2(t), \\ (f')^{-1} \left(\frac{x - A_2}{t} \right) & \text{if } \text{Min}\{r_2(t), A_2 + tf'(M)\} < x < r_2(t), \\ M & \text{if } x < \text{Min}\{r_2(t), A_2 + tf'(M)\}. \end{cases}$$

Then clearly the left extreme characteristic curve R_- at A_1 for u_m is r_1 and right extreme characteristic R_+ at A_2 for u_M is r_2 . Since $u_0^m \leq u_0 \leq u_0^M$, then from (7) of Lemma 3.3.1 we have for $t > 0$,

$$r_1(t) \leq R_-(t, A_1) \leq R_+(t, A_2) \leq r_2(t). \quad (3.3.39)$$

Furthermore, for $x < R_-(t, A_1)$ we have $y_-(x, t) < A_1$ and for $x > R_+(t, A_2)$ we have $y_+(x, t) > A_2$. Since v_0 is differentiable in $(-\infty, A_1) \cup (A_2, \infty)$, hence from (3.2.14)

$$u(x, t) = \begin{cases} u_- & \text{if } x < R_-(t, A_1), \\ u_+ & \text{if } x > R_+(t, A_2). \end{cases} \quad (3.3.40)$$

$$\begin{aligned} f'(u_-) &= \lim_{x \uparrow R_-(t, A_1)} \frac{x - y_+(x, t)}{t} \\ &= \frac{R_-(t, A_1) - y_-(R_-(t, A_1), t)}{t}. \end{aligned} \quad (3.3.41)$$

$$\begin{aligned} f'(u_+) &= \lim_{x \downarrow R_+(t, A_2)} \frac{x - y_-(x, t)}{t} \\ &= \frac{R_+(t, A_2) - y_+(R_+(t, A_2), t)}{t}. \end{aligned} \quad (3.3.42)$$

Let v_0 be given by

$$v_0(x) = \int_{A_1}^x u_0(\theta) d\theta. \quad (3.3.43)$$

Then from (3.2.4), for any $x \in \mathbb{R}, t > 0$,

$$\begin{aligned} v(x, t) &= v_0(y_-(x, t)) + tf^* \left(\frac{x - y_-(x, t)}{t} \right) \\ v(x, t) &= v_0(y_+(x, t)) + tf^* \left(\frac{x - y_+(x, t)}{t} \right), \end{aligned}$$

therefore if $y_-(x, t) \leq A_1$, then

$$\begin{aligned} (y_-(x, t) - A_1)u_- + tf^* \left(\frac{x - y_-(x, t)}{t} \right) &= \int_{A_1}^{y_+(x, t)} u_0(\theta) d\theta \\ &+ tf^* \left(\frac{x - y_+(x, t)}{t} \right). \end{aligned} \quad (3.3.44)$$

and if $y_+(x, t) > A_2$,

$$\begin{aligned} \int_{A_1}^{y_-(x,t)} u_0(\theta) d\theta + t f^* \left(\frac{x - y_-(x, t)}{t} \right) &= \int_{A_1}^{A_2} \bar{u}_0(\theta) d\theta + (y_+(x, t) - A_2) u_+ \\ &+ t f^* \left(\frac{x - y_+(x, t)}{t} \right). \end{aligned} \quad (3.3.45)$$

Then we have the following

Lemma 3.3.5. *Assume that for all $t > 0$,*

$$R_-(t, A_1) < R_+(t, A_2) \quad (3.3.46)$$

and denote

$$y(t) = y_+(R_-(t, A_1), t), \quad Y(t) = y_-(R_+(t, A_2), t) \quad (3.3.47)$$

$$(B_1, B_2) = \lim_{t \rightarrow \infty} (y(t), Y(t)) \quad (3.3.48)$$

then,

1. $A_1 \leq B_1 \leq B_2 \leq A_2$,

$$f'(u_-) = \lim_{t \rightarrow \infty} \frac{R_-(t, A_1) - y(t)}{t} \quad (3.3.49)$$

$$f'(u_+) = \lim_{t \rightarrow \infty} \frac{R_+(t, A_2) - Y(t)}{t} \quad (3.3.50)$$

2. Let $\Gamma_1(t) = R_-(t, B_1)$, $\Gamma_2(t) = R_+(t, B_2)$, then Γ_1 and Γ_2 are regular characteristic lines given by

$$R_-(t, A_1) \leq \Gamma_1(t) = B_1 + t f'(u_-) \quad (3.3.51)$$

$$R_+(t, A_2) \geq \Gamma_2(t) = B_2 + t f'(u_+). \quad (3.3.52)$$

Furthermore,

$$u_- \leq u_+. \quad (3.3.53)$$

3. Assume that there exist $\gamma > 0, \delta > 0, C > 0$ such that for all $|h| \leq \delta$,

$$f(u_{\pm}) - f(u_{\pm} + h) + h f'(u_{\pm} + h) \geq C|h|^\gamma. \quad (3.3.54)$$

Then

$$u_- - (f')^{-1} \left(\frac{R_-(t, A_1) - y(t)}{t} \right) = O \left(\frac{1}{t^{1/\gamma}} \right) \quad (3.3.55)$$

$$u_+ - (f')^{-1} \left(\frac{R_+(t, A_2) - Y(t)}{t} \right) = O \left(\frac{1}{t^{1/\gamma}} \right). \quad (3.3.56)$$

Furthermore for some $\eta > 0$, for all $a, b \in [-\|u_0\|, \|u_0\|]$, f satisfies

$$|f'(a) - f'(b)| \leq C|b - a|^\eta. \quad (3.3.57)$$

Then

$$f'(u_-) - \frac{R_-(t, A_1) - y(t)}{t} = O \left(\frac{1}{t^{\eta/\gamma}} \right) \quad (3.3.58)$$

$$f'(u_+) - \frac{R_+(t, A_2) - Y(t)}{t} = O \left(\frac{1}{t^{\eta/\gamma}} \right). \quad (3.3.59)$$

Proof : From (1) and (9) of Lemma 3.3.4, (B_1, B_2) exist and satisfies $A_1 \leq B_1, B_2 \leq A_2$. Suppose $B_2 < B_1$, then there exists $t_0 > 0$ such that $B_2 < Y(t_0) < y(t_0) < B_1$. Since by hypothesis $R_-(t_0, A_1) < R_+(t_0, A_2)$ and hence the left extreme characteristic at $(R_+(t_0, A_2), t_0)$ and right extreme characteristic at $(R_-(t_0, A_1), t_0)$ intersect which contradicts NIP. Hence $A_1 \leq B_1 \leq B_2 \leq A_2$.

Let $f'(p_t) = \frac{R_-(t, A_1) - y(t)}{t}$, then from (3.3.39), we have

$$A_1 + tf'(k_1) \leq R_-(t, A_1) \leq A_2 + tf'(k_2)$$

and hence

$$\frac{A_1 - y(t)}{t} + f'(k_1) \leq f'(p_t) \leq \frac{A_2 - y(t)}{t} + f'(k_2)$$

Since $y(t) \in [A_1, B_1]$ and hence $\{p_t\}$ is bounded as $t \rightarrow \infty$. From (3.3.41) and (3.3.44) we have

$$\begin{aligned} & (y_-(R_-(t, A_1), t) - A_1)u_- + tf^* \left(\frac{R_-(t, A_1) - y_-(R_-(t, A_1), t)}{t} \right) \\ &= \int_{A_1}^{y(t)} u_0(\theta) d\theta + tf^* \left(\frac{R_-(t, A_1) - y(t)}{t} \right). \end{aligned}$$

That is

$$\begin{aligned}
-u_- f'(u_-) + u_- \left(\frac{R_-(t, A_1) - A_1}{t} \right) + f^*(f'(u_-)) \\
&= \frac{1}{t} \int_{A_1}^{y(t)} u_0(\theta) d\theta + f^*(f'(p_t)) \\
-u_- f'(u_-) + f^*(f'(u_-)) &= \frac{1}{t} \int_{A_1}^{y(t)} u_0(\theta) d\theta - u_- \left(\frac{y(t) - A_1}{t} \right) \\
&\quad - u_- \left(\frac{R_-(t, A_1) - y(t)}{t} \right) + f^*(f'(p_t)) \\
&= O\left(\frac{1}{t}\right) + f^*(f'(p_t)) - u_- f'(p_t)
\end{aligned}$$

Since $f^*(f'(p)) = pf'(p) - f(p)$ we have

$$f(p_t) - f(u_-) - (p_t - u_-)f'(p_t) = O\left(\frac{1}{t}\right). \quad (3.3.60)$$

Let for a sequence $t_k \rightarrow \infty$, $p_{t_k} \rightarrow p$, then from (3.3.60) we obtain

$$f(p) - f(u_-) - (p - u_-)f'(p) = 0$$

and hence by strict convexity, $p = u_-$. This proves (3.3.49) and similarly (3.3.50). This proves (1).

2. From (6) and (9) of Lemma 3.3.3, $\Gamma_1 = R_-(t, B_1)$, $\Gamma_2 = R_+(t, B_2)$ are regular characteristic lines given by (3.3.51) and (3.3.52). Since

$$\frac{R_-(t, A_1) - y(t)}{t} < \frac{R_+(t, A_2) - Y(t)}{t} + \frac{Y(t) - y(t)}{t}$$

and hence letting $t \rightarrow \infty$ to obtain $f'(u_-) \leq f'(u_+)$. This implies that $u_- \leq u_+$. This proves (3.3.53) and hence (2).

From (3.3.54) and (3.3.60) we have

$$|u_- - p_t| = O\left(\frac{1}{t^{1/\gamma}}\right).$$

$$R_-(t, A_1) = y(t) + tf' \left(u_- + O\left(\frac{1}{t^{1/\gamma}}\right) \right).$$

Therefore from (3.3.57) we have

$$\begin{aligned} \left| f'(u_-) - \frac{R_-(t, A_1) - y(t)}{t} \right| &= \left| f'(u_-) - f' \left(u_- + O \left(\frac{1}{t^{1/\gamma}} \right) \right) \right| \\ &= O \left(\frac{1}{t^{1/\gamma}} \right). \end{aligned} \quad (3.3.61)$$

This proves (3.3.55) and (3.3.58). Similarly (3.3.56) and (3.3.59) follow. This proves (3) and hence the Lemma.

Proof of Theorem 3.2.7 : We have to consider the following two cases:

(I). There exists $T > 0$ such that

$$R_-(T, A_1) = R_+(T, A_2). \quad (3.3.62)$$

(II). For all $t > 0$,

$$R_-(t, A_1) < R_+(t, A_2). \quad (3.3.63)$$

(I). In this case, from (6) of Lemma 3.3.1 and (3.3.40) we have for all $t > T$,

$$\begin{aligned} S(t) &= R_-(t, A_1) = R_+(t, A_2) \\ &= S(T) + \frac{f(u_+) - f(u_-)}{u_+ - u_-} (t - T) \\ u(x, t) &= \begin{cases} u_+ & \text{if } x > S(t), \\ u_- & \text{if } x < S(t). \end{cases} \end{aligned}$$

Since u is an entropy solution and hence $u_- \geq u_+$. From (3.3.41) and (3.3.42), for $t > T$, we have

$$\begin{aligned} f'(u_-) &= \lim_{x \uparrow S(t)} \frac{x - y_+(x, t)}{t} = \frac{S(t) - y_-(S(t), t)}{t} \\ f'(u_+) &= \lim_{x \downarrow S(t)} \frac{x - y_-(x, t)}{t} = \frac{S(t) - y_+(S(t), t)}{t} \end{aligned}$$

and $y_-(S(t), t) \leq A_1 < A_2 \leq y_+(S(t), t)$. This implies that $f'(u_-) > f'(u_+)$ and hence $u_- > u_+$. This proves (I).

(II). From Lemma 3.3.5, there exists $A_1 \leq B_1 \leq B_2 \leq A_2$ such that $\Gamma_1(t) = r_1(t, B_1, u_-)$, $\Gamma_2(t) = r_2(t, B_2, u_+)$ are regular characteristic lines. Hence from (3.3.52), $u_- \leq u_+$ and (3.2.22) follows from (3.3.51), (3.3.52). (3.2.23) follows from (3.3.40). This proves (i).

Define the regular set \tilde{R} and singular set \tilde{D} by

$$\tilde{R} = \{ \alpha \in [B_1, B_2] : \alpha \text{ is the base point of a regular characteristic line} \}. \quad (3.3.64)$$

Since then sequence of regular characteristics converges to a regular characteristic (see

(2) of Lemma 3.3.2) and hence \tilde{R} is a closed set. Let

$$\tilde{D} = [B_1, B_2] \setminus \tilde{R}. \quad (3.3.65)$$

Then \tilde{D} is open set. Let $\alpha \in \tilde{D}$. Then from (1) and (3) of Lemma 3.3.2 there exists a $T > 0$ such that for $t > T$,

$$y_-(R_-(t, \alpha), \alpha) < \alpha \leq y_+(R_-(t, \alpha), \alpha).$$

Since Γ_1, Γ_2 are characteristic lines, hence from NIP,

$$B_1 \leq y_-(R_-(t, \alpha), \alpha) \leq y_+(R_-(t, \alpha), \alpha) \leq B_2.$$

Then from Lemma 3.3.4 there exist $C_{1,\alpha} < C_{2,\alpha}, p_\alpha$ such that

$$\begin{aligned} (C_{1,\alpha}, C_{2,\alpha}) &= \lim_{t \rightarrow \infty} (y_-(R_-(t, \alpha), \alpha), y_+(R_-(t, \alpha), \alpha)) \\ r_{1,\alpha} &= C_{1,\alpha} + tf'(p_\alpha), \quad r_{2,\alpha}(t) = C_{2,\alpha} + tf'(p_\alpha) \end{aligned}$$

are regular characteristic lines and $\{C_{1,\alpha}, C_{2,\alpha}\} \cap \tilde{D} = \emptyset$. Let

$$D_\alpha = \{(x, t) : r_{1,\alpha}(t) < x < r_{2,\alpha}(t)\},$$

then from Lemma 3.3.4 D_α is an ASSP and $D_\alpha \subset E$, where

$$E = \{(x, t) : \Gamma_1(t) \leq x \leq \Gamma_2(t)\}.$$

Let $\alpha, \beta \in \tilde{D}$ such that $D_\alpha \cap D_\beta \neq \emptyset$. Suppose $D_\alpha \neq D_\beta$, since $\partial D_\alpha \setminus [C_{1,\alpha}, C_{2,\alpha}], \partial D_\beta \setminus [C_{1,\beta}, C_{2,\beta}]$ are regular characteristic lines and hence from NIP, for some $i \in \{1, 2\}$, for all $t > 0$, the regular characteristic line $r_{i,\alpha}(t) \in D_\beta$, contradicting the fact that D_β does not contain regular characteristic line. Hence $D_\alpha = D_\beta$. Therefore we can define an equivalence relation in \tilde{D} by $\alpha \sim \beta$ if and only if $D_\alpha = D_\beta$. From ASSP, it follows that the equivalence class of α , denoted by $(\alpha) = (C_{1,\alpha}, C_{2,\alpha})$. Hence there exists a countable set I , disjoint intervals $\{(C_{1,i}, C_{2,i})\}_{i \in I}$, real numbers $\{p_i\}$, regular characteristic lines $r_{1,i}(t) = C_{1,i} + tf'(p_i)$, $r_{2,i}(t) = C_{2,i} + tf'(p_i)$ and disjoint ASSP $\{D_i = D(C_{1,i}, C_{2,i}, p_i)\}_{i \in I}$ given by

$$\begin{aligned} \tilde{D} &= \cup_{i \in I} (C_{1,i}, C_{2,i}) \\ D_i &= \{(x, t) : r_{1,i}(t) < x < r_{2,i}(t)\}. \end{aligned}$$

Claim 1 : Let R be the union of all regular characteristic lines in E . Then

$$E = \cup_{i \in I} D_i \cup R.$$

Let $(x_0, t_0) \in E$ and $r \in ch(x_0, t_0)$. Let $\alpha = r(0)$, then from the definition we have for $0 \leq t \leq t_0$,

$$R_-(t, \alpha) \leq r(t) \leq R_+(t, \alpha).$$

Suppose $\alpha \in (C_{1,i}, C_{2,i})$ for some i , since D_i is an ASSP and hence for all $R_-(t, \alpha) \leq x \leq R_+(t, \alpha)$, $(x, t) \in D_i$. This implies $(x_0, t_0) \in D_i$.

Suppose $\alpha \notin \tilde{D}$, then $\alpha \in \tilde{R}$ and hence there exist a regular characteristic line $\eta(t) = \alpha + tf'(p)$. Since $R_-(t, \alpha) \leq \eta(t) \leq R_+(t, \alpha)$ and hence either $R_-(t_0, \alpha) \leq x_0 \leq \eta(t_0)$ or $\eta(t_0) \leq x_0 \leq R_+(t_0, \alpha)$. Without loss of generality we can assume that $R_-(t_0, \alpha) \leq x_0 \leq \eta(t_0)$. Since η is a regular characteristic and hence $y_+(R_-(t, \alpha)) = \alpha$ for all $t > 0$. We have to consider two cases.

Case (i) : $y_-(R_-(t, \alpha), t) = \alpha$ for all $t > 0$.

Then from (3) of Lemma 3.3.2, $R_-(t, \alpha) \leq x \leq \eta(t)$, $f'(u(x, t)) = \frac{x-\alpha}{t}$. Hence (x_0, t_0) lies on the line $L(t) = \alpha + \frac{x_0-\alpha}{t_0}t$ and $R_-(t, \alpha) \leq L(t) \leq \eta(t)$ for all $t > 0$. Hence from NIP, $y_-(L(t), t) = y_+(L(t), t) = \alpha$ and therefore L is a regular characteristic line.

Case (ii) : There exists $T > 0$ such that $y_-(R_-(T, \alpha), T) < \alpha$. Since $R_-(t, \alpha) \leq \eta(t)$ and hence $y_+(R_-(t, \alpha), t) = \alpha$ for all $t > 0$. Therefore from (2),(4),(6),(7) of Lemma 3.3.3

$$D_\alpha = \{(x, t) : r_{1,\alpha}(t) < x < r_{2,\alpha}(t)\}$$

is an ASSP. Hence $D_\alpha = D_i$ for some i . From (6) of Lemma 3.3.3, we have for all $t > 0$, $R_-(t, \alpha) < r_{2,\alpha}(t) \leq \eta(t)$. Hence if $R_-(t_0, \alpha) \leq x_0 < r_{2,\alpha}(t_0)$, then $(x_0, t_0) \in D_i$. If $r_{2,\alpha}(t_0) \leq x_0 \leq \eta(t_0)$, then by the same argument as earlier, $x_0 = L(t_0)$ for some regular characteristic line. Hence $(x_0, t_0) \in R$. This proves the claim.

From the above analysis and claim, (i) and (ii) follow. (3.2.24) follows from (3.3.30) and (3.2.25) follows from (3.3.28),(3.3.29).

(iv). From (2) of Lemma 3.3.2, u is continuous on R and hence $N = u$ on R is continuous. Since Γ_1 and Γ_2 are characteristic lines with values u_- and u_+ and hence N is continuous on $R \cup D_- \cup D_+$. By definition, N is continuous on $\cup_{i \in I} D_i$. Therefore it is enough to show that if $(x_k, t_k) \rightarrow (x_0, t_0)$, $(x_k, t_k) \in R$, $(x_0, t_0) \in \partial D_i \setminus \tilde{D}$ for some i , then $N(x_k, t_k) \rightarrow N(x_0, t_0)$. Let $r_k(t) = \alpha_k + tf'(p_k)$ be the regular characteristic line such that $r_k(t_k) = x_k$. Without loss of generality we can assume that $r_{1,i}(t_0) = x_0$. Since $\alpha_k \in [B_1, B_2]$ and hence $f'(p_k) = \frac{x_k - \alpha_k}{t_k}$ are bounded. Then for a subsequence still denoted by $\{\alpha_k, p_k\}$ such that $(\alpha_k, p_k) \rightarrow (\alpha_0, p_0)$. Therefore for all t , $r_k(t) \rightarrow r(t) = \alpha_0 + tf'(p_0)$ a regular characteristic line. Since $r_{1,i}(t_0) = x_0 = \lim_{k \rightarrow \infty} r_k(t_k) = r(t_0)$, r and $r_{1,i}$ intersect at (x_0, t_0) and by NIP it follows that $r = r_{1,i}$ and $\alpha_0 = C_{1,i}, p_0 = p_i$. Hence $N(x_k, t_k) = p_k \rightarrow p_i = N(x_0, t_0)$.

This proves that N is continuous.

Let $(x_1, t), (x_2, t) \in R$ and $x_1 < x_2$. Let the corresponding regular characteristic lines be r_1, r_2 and for $i = 1, 2$, $x_i = r_i(t) = y_-(x_i, t) + tf'(p_i)$, for some p_1, p_2 . Since $y_-(x_1, t) \leq y_-(x_2, t)$ and hence from NIP, $f'(p_1) \leq f'(p_2)$. This implies that $N(x_1, t) = p_1 \leq p_2 = N(x_2, t)$. Since N is constant on D_i and hence $x \mapsto N(x, t)$ is a non decreasing function for $(x, t) \notin F_- \cup F_+$.

For $A \subset \mathbb{R} \times \mathbb{R}_+, t > 0$, define t section A_t of A by

$$A_t = \{x : (x, t) \in A\}.$$

Let f satisfies (3.2.27) and (3.2.28). Then

$$\begin{aligned} J &= \int_{-\infty}^{\infty} |f'(u(x, t)) - f'(N(x, t))| dx = \int_{R_-(t, A_1)}^{\Gamma_1(t)} |f'(u(x, t)) - f'(u_-)| dx \\ &+ \int_{R_t} |f'(u(x, t)) - f'(N(x, t))| dx + \sum_{i \in I} \int_{D_{it}} |f'(u(x, t)) - f'(N(x, t))| dx \\ &+ \int_{\Gamma_2(t)}^{R_+(t, A_2)} |f'(u(x, t)) - f'(u_+)| dx \\ &= \int_{R_-(t, A_1)}^{\Gamma_1(t)} \left| \frac{x - y_-(x, t)}{t} - \frac{x - B_1}{t} \right| dx + \int_{\Gamma_2(t)}^{R_+(t, A_2)} \left| \frac{x - y_-(x, t)}{t} - \frac{x - B_2}{t} \right| dx \\ &+ \sum_{i \in I} \int_{r_{1,i}(t)}^{r_{2,i}(t)} \left| \frac{x - y_-(x, t)}{t} - f'(p_i) \right| dx \end{aligned}$$

since on R_t , $N(x, t) = u(x, t)$. Let $\xi = x - tf'(p_i)$, then

$$\begin{aligned} \int_{r_{1,i}(t)}^{r_{2,i}(t)} \left| \frac{x - y_-(x, t)}{t} - f'(p_i) \right| dx &= \int_{C_{1,i}}^{C_{2,i}} \left| \frac{\xi - y_-(x, t)}{t} \right| d\xi \\ &\leq \frac{(C_{2,i} - C_{1,i})^2}{t} \end{aligned}$$

From (3.3.61) we have for $\xi = x - tf'(u_-)$

$$\int_{R_-(t, A_1)}^{\Gamma_1(t)} \left| \frac{x - y_-(x, t)}{t} - \frac{x - B_1}{t} \right| dx = \int_{R_-(t, A_1)}^{\Gamma_1(t)} \left| \frac{B_1 - y_-(x, t)}{t} \right| dx$$

$$\begin{aligned}
&\leq \frac{B_1 - A_1}{t} (\Gamma_1(t) - R_-(t, A_1)) \\
&= \frac{B_1 - A_1}{t} \left[B_1 - y(t) - t \left(f'(u_-) + O\left(\frac{1}{t^{1/\gamma}}\right) \right) - f'(u_-) \right] \\
&\leq \frac{B_1 - A_1}{t} \left[(B_1 - A_1) + tO\left(\frac{1}{t^{\eta/\gamma}}\right) \right] \\
&= \frac{(B_1 - A_1)^2}{t} + O\left(\frac{1}{t^{\eta/\gamma}}\right).
\end{aligned}$$

Similarly

$$\int_{\Gamma_2(t)}^{R_-(t, A_2)} \left| \frac{x - y_-(x, t)}{t} - \frac{x - B_2}{t} \right| dx \leq \frac{(A_2 - B_2)^2}{t} + O\left(\frac{1}{t^{\eta/\gamma}}\right).$$

Hence

$$\begin{aligned}
J &\leq \frac{(B_1 - A_1)^2 + (B_2 - A_2)^2 + \sum_{i \in I} (C_{2,i} - C_{1,i})^2}{t} + O\left(\frac{1}{t^{\eta/\gamma}}\right) \\
&= \frac{(B_1 - A_1)^2 + (B_2 - A_2)^2 + (B_2 - B_1)^2}{t} + O\left(\frac{1}{t^{\eta/\gamma}}\right) \\
&= O\left(\frac{1}{t} + \frac{1}{t^{\eta/\gamma}}\right) \\
&= O\left(\frac{1}{t^{\eta/\gamma}}\right),
\end{aligned}$$

since $\eta/\gamma \leq 1$. This proves (3.2.29). Now (3.2.30) follows from (3.2.29). This proves the theorem.

Before proving Theorem 3.2.8, we need some preliminary results.

Definition 3.3.6. u_0 is said to be mildly oscillating at a if there exist $\theta_1 < a < \theta_2$ such that either a is a local maxima or a local minima of u_0 or u_0 is monotone in (θ_1, θ_2) .

Definition 3.3.7. Let $\alpha < \beta$ and u_0 is said to have same parity at α, β if there exist $\alpha < \theta_1 < \theta_2 < \beta$ such that

$$u_0(\alpha) = u_0(\beta)$$

and u_0 is a non decreasing function in $(\alpha, \theta_1) \cup (\theta_2, \beta)$.

Lemma 3.3.8. Let $C_1 < C_2$, $p \in \mathbb{R}$ and $r_1(t) = C_1 + tf'(p)$, $r_2(t) = C_2 + tf'(p)$. Let

$$D = \{(x, t) : r_1(t) < x < r_2(t)\}$$

be an ASSP. Assume that u_0 is continuous in a neighbourhood of $\{C_1, C_2\}$ and mildly oscillating at C_1 and C_2 . Then u_0 have same parity at C_1 and C_2 .

Proof : Let $C_1 < \alpha < C_2$, then α is not a regular point since D is an ASSP. Hence from Lemma 3.3.4, $D = D_\alpha$ is the ASSP with $u_0(C_1) = u_0(C_2) = p$. Suppose they are not of same parity. For simplicity we assume that there exists $C_1 < \theta_1 < \theta_2 < C_2$ such that u_0 is non decreasing continuous function in (C_1, θ_1) and u_0 is non increasing continuous function in (θ_2, C_2) . Let $T > 0$ be such that for all $t > T$, $r_1(t) < x < R_-(t, \alpha) < \xi < r_2(t)$,

$$R_-(t, \theta_1) = R_+(t, \theta_2) = R_-(t, \alpha) \quad (3.3.66)$$

$$C_1 \leq y_-(x, t) \leq y_+(x, t) < \theta_1 < \theta_2 < y_-(\xi, t) \leq y_+(\xi, t) \leq C_2. \quad (3.3.67)$$

By choosing θ_1 close to C_1 and θ_2 close to C_2 , we can assume that u_0 is continuous in $[C_1, \theta_1] \cup [\theta_2, C_2]$. Hence from (3.2.14), for a.e. x and ξ

$$u(x, t) = u_0(y_+(x, t)), \quad u(\xi, t) = u_0(y_+(\xi, t)). \quad (3.3.68)$$

Therefore

$$p = u_0(C_1) \leq u_0(y_+(x, t)) = u(x, t) \quad (3.3.69)$$

$$p = u_0(C_2) \leq u_0(y_+(\xi, t)) = u(\xi, t). \quad (3.3.70)$$

Claim : For a.e. $x \in (r_1(t), R_-(t, \alpha))$, a.e. $\xi \in (R_-(t, \alpha), r_2(t))$

$$u(x, t) = u(\xi, t) = p$$

does not hold.

Suppose not, then choose a sequence $x_k \uparrow R_-(t, \alpha)$, $\xi_k \downarrow R_-(t, \alpha)$ such that $f'(p) = f'(u(x_k, t)) = \frac{x_k - y_+(x_k, t)}{t}$ and $f'(p) = f'(u(\xi_k, t)) = \frac{\xi_k - y_-(\xi_k, t)}{t}$. Then from (3.3.2) and (3.3.3)

$$\begin{aligned} f'(p) &= \lim_{x_k \uparrow R_-(t, \alpha)} \frac{x_k - y_+(x_k, t)}{t} \\ &= \frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t} \\ f'(p) &= \lim_{\xi_k \downarrow R_-(t, \alpha)} \frac{\xi_k - y_-(\xi_k, t)}{t} \\ &= \frac{R_-(t, \alpha) - y_+(R_-(t, \alpha), t)}{t}. \end{aligned}$$

Comparing both the expression and from (3.3.67) we have $y_-(R_-(t, \alpha), t) = y_+(R_-(t, \alpha), t)$ and $y_-(R_-(t, \alpha), t) < \theta_1 < \theta_2 < y_+(R_-(t, \alpha), t)$ which is a contradiction. This proves the claim.

From (3.3.30),(3.3.69),(3.3.70) and the claim we have

$$\begin{aligned}
p(C_2 - C_1) &= \int_{r_1(t)}^{r_2(t)} u(x, t) dx = \int_{r_1(t)}^{R_-(t, \alpha)} u(x, t) dx + \int_{R_-(t, \alpha)}^{r_2(t)} u(x, t) dx \\
&> p(r_2(t) - r_1(t)) \\
&= p(C_2 - C_1)
\end{aligned}$$

which is a contradiction.

Similar proof follows if u_0 is non increasing continuous function in (C_1, θ_1) and non decreasing continuous function in (θ_2, C_2) . Next assume that u_0 is non increasing continuous function in $(C_1, \theta_1) \cup (\theta_2, C_2)$. Choose $T > 0$ such that (3.3.66), (3.3.67) and (3.3.68) hold for all $t > T$. Therefore for a.e $r_1(t) < x < R_-(t, \alpha) < \xi < r_2(t)$,

$$\begin{aligned}
f'(u_0(y_+(x, t))) &\leq f'(u_0(C_1)) = f'(p) \\
f'(u_0(y_-(\xi, t))) &\geq f'(u_0(C_2)) = f'(p).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{R_-(t, \alpha) - y_-(R_-(t, \alpha), t)}{t} &= \lim_{x \uparrow R_-(t, \alpha)} \frac{x - y_+(x, t)}{t} \\
&= \lim_{x \uparrow R_-(t, \alpha)} f'(u(x, t)) \\
&= \lim_{x \uparrow R_-(t, \alpha)} f'(u_0(y_+(x, t))) \\
&\leq f'(p).
\end{aligned}$$

Hence $R_-(t, \alpha) \leq y_-(R_-(t, \alpha), \alpha) + t f'(p)$. Similarly we get $R_-(t, \alpha) \geq y_+(R_-(t, \alpha), \alpha) + t f'(p)$ and hence $\theta_2 \leq y_+(R_-(t, \alpha), \alpha) \leq y_-(R_-(t, \alpha), \alpha) \leq \theta_1$, which is a contradiction. This proves the Lemma.

Existence of $u_0 \in C_c^\infty$ for which the solution of (3.1.2) admits infinitely many ASSP : Let $u_0 \in C_c^\infty(\mathbb{R})$ and u be the solution of (3.1.2) with initial data u_0 . First observe that if an ASSP exists, then u_0 necessarily satisfies the compatibility conditions (3.3.29), (3.3.30) and having the same parity at the end points. Hence we look for data satisfying these conditions.

First we construct a solution having single ASSP. From this we construct a data having infinitely many ASSP.

Let $A_1 < \alpha_1 < \alpha_2 < A_2$ and $u_0 \in C_c^\infty(\mathbb{R})$ such that

1. Support of $u_0 \subset [A_1, A_2]$.
2. $\int_{A_1}^{A_2} u_0(x) dx = 0$.

$$3. \quad 0 < u(\alpha_1) = \max u, \quad 0 > u(\alpha_2) = \min u,$$

and u_0 is strictly increasing in $(A_1, \alpha_1) \cup (\alpha_2, A_2)$ and strictly decreasing in (α_1, α_2) , then

Lemma 3.3.9. *Assume that $f(0) = f'(0) = 0$ and u be the solution of (3.1.2) with initial data u_0 . Then u admits a single ASSP in $D = (A_1, A_2) \times (0, \infty)$ and satisfies for all $x \notin (A_1, A_2), t > 0$*

$$u(x, t) = u(A_1+, t) = u(A_2-, t) = 0. \quad (3.3.71)$$

Proof. Let $\alpha_1 < \beta < \alpha_2$ be such that $u_0(\beta) = 0$. Then $u_0 > 0$ in (A_1, β) and $u_0 < 0$ in (β, A_2) . Let

$$\begin{aligned} R_1(t) &= R_-(t, \alpha_1), R_2(t) = R_+(t, \alpha_2), \\ y_{\pm}(t) &= y_{\pm}(R_1(t), t), Y_{\pm}(t) = y_{\pm}(R_2(t), t). \end{aligned}$$

Claim 1: Let $t_0 > 0$ such that for $0 < t < t_0, A_1 < R_1(t)$. Then for all $z < A_1 < x < R_1(t)$

$$A_1 \leq y_-(x, t) \leq y_+(x, t) \leq \alpha_1 \quad (3.3.72)$$

$$u(A_1+, t) = u(z, t) = 0. \quad (3.3.73)$$

Suppose for some $x_1 \in (A_1, R_1(t)), y_-(x_1, t) < A_1$, then for all $x < x_1, y_-(x, t) < A_1$. Since u_0 is differentiable and hence from (3.2.13) and (3.2.14) for a.e. $x \in (A_1, x_1)$

$$0 = f'(0) = f'(u_0(y_-(x, t))) = f'(u(x, t)) = \frac{x - y_-(x, t)}{t}$$

and hence $y_-(x, t) < A_1 < x = y_-(x, t)$ which is a contradiction. Since $y_+(x, t) \leq y_-(x, t) \leq \alpha_1$, hence (3.3.72).

Let $z_0 < A_1$ be such that $y_+(z_0, t) \geq A_1$, then for all $z \in (z_0, A_1), y_+(z, t) \geq A_1$. Then from NIP, it follows that $y_+(z, t) \leq y_-(z, t) \leq \alpha_1$ and hence $f'(u_0(y_+(z, t))) \geq 0$. Hence from (3.2.13) and (3.2.14), for a.e. $z \in [z_0, A_1]$

$$0 \leq f'(u_0(y_+(z, t))) = f'(u(z, t)) = \frac{z - y_+(z, t)}{t}.$$

Therefore $A_1 > z \geq y_+(z, t) \geq A_1$ which is a contradiction. Hence $y_+(z, t) < A_1$ for all $z < A_1$ and $u(z, t) = u_0(y_+(z, t)) = 0$. Therefore $u(A_1-, t) = \lim_{z \uparrow A_1} u(z, t) = 0$. Hence from Rankine-Hugoniot condition across $x = A_1, 0 < t < t_0$ gives $u(A_1+, t) = 0$. This proves (3.3.73).

Claim 2: Let $t_1 > 0$ be such that for $0 < t < t_1$, $R_2(t) < A_2$, then for $R_2(t) < x < A_2 < z$

$$\alpha_2 \leq y_-(x, t) \leq y_+(x, t) \leq A_2 \quad (3.3.74)$$

$$u(A_2-, t) = u(z, t) = 0. \quad (3.3.75)$$

Proof follows exactly as in claim 1.

Claim 3: Let $t_0 > 0$ such that for all $0 < t < t_0$,

$$A_1 < R_1(t) \leq R_2(t) < A_2 \quad (3.3.76)$$

then

$$A_1 < R_1(t_0) \leq R_2(t_0) < A_2. \quad (3.3.77)$$

Suppose $R_1(t_0) = A_1$. Then $A_1 = R_1(t_0) \leq R_2(t_0) \leq A_2$. At $R_1(t_0)$, $y_+(t_0) \geq \alpha_1$, hence for $R_1(t_0) < x$, $y_+(x, t_0) \geq \alpha_1$ and from (3.2.13) and (3.2.14) for a.e. $x \in (R_1(t_0), \alpha_1)$, we have

$$f'(u_0(y_+(x, t_0))) = f'(u(x, t_0)) = \frac{x - y_+(x, t_0)}{t_0} < \frac{\alpha_1 - \alpha_1}{t_0} = 0.$$

Therefore $u_0(y_+(x, t_0)) < 0$ and hence $y_+(x, t_0) > \beta$. By monotonicity, for all $x > R_1(t_0) = A_1$, $y_+(x, t_0) > \beta$. This implies from (3.3.74), for all $x \in (A_1, A_2)$.

$$\beta < y_+(x, t_0) \leq A_2.$$

Therefore from the hypothesis on u_0 and from (3.2.14) we have for a.e. $x \in (A_1, A_2)$

$$u(x, t_0) = u_0(y_+(x, t_0)) < 0. \quad (3.3.78)$$

From (3.3.73) and (3.3.75) $u(A_1+, t) = u(A_2-, t) = 0$. Therefore we have

$$\begin{aligned} 0 &= \int_{A_1}^{A_2} \int_0^{t_0} (u_t + f(u)_x) dx dt = \int_{A_1}^{A_2} u(x, t_0) dx - \int_{A_1}^{A_2} u_0(x) dx \\ &= \int_{A_1}^{A_2} u(x, t_0) dx < 0, \end{aligned}$$

which is a contradiction. Similarly $R_2(t_0) \in (A_1, A_2)$ and this proves the claim.

Since for t small, $A_1 < R_1(t) < R_2(t) < A_2$ and hence from claim (3), $A_1 < R_1(t) \leq R_2(t) < A_2$ for all $t > 0$. Therefore from claim 1 and claim 2 $A_1 \leq y_{\pm}(t) \leq A_2$. Hence

from Lemma 3.3.4 there exists an ASSP $D = (\tilde{A}_1, \tilde{A}_2, p)$ such that

$$\begin{aligned} (\tilde{A}_1, \tilde{A}_2) &= \lim_{t \rightarrow \infty} (y_-(t), y_+(t)) \\ f'(p) &= \lim_{t \rightarrow \infty} \frac{R_1(t) - y_-(t)}{t} = 0 \\ u_0(\tilde{A}_1) &= u_0(\tilde{A}_2) = p = 0. \end{aligned}$$

From Lemma 3.3.8 \tilde{A}_1 and \tilde{A}_2 have the same parity. Since u_0 has only three zeros $\{A_1, \beta, A_2\}$ and A_1 and A_2 are the only pair of points having same parity. Hence $\tilde{A}_1 = A_1, \tilde{A}_2 = A_2$ and this proves the Lemma.

Proof of Theorem 3.2.8

Proof. Let u_0 be as in Lemma 3.3.9 with $A_1 = -1, A_2 = 1$. For $n \geq 2$. define $\{\psi_{0,n}, u_n\}$ as follows

$$\psi_{0,n}(x) = e^{-n} u_0(2n^2(x - \frac{1}{n})).$$

Then $\psi_{0,n} \in C_c^\infty(\mathbb{R})$ and support of $\psi_{0,n}$ is contained in $(\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2})$ and for each k , there exists $c(k) > 0$ such that

$$\left\| \frac{d^k \psi_{0,n}}{dx^k} \right\|_\infty \leq c(k) n^{2k} e^{-n} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.3.79)$$

$\psi_{0,n}$ satisfies all the hypothesis of Lemma 3.3.9 and let u_n be the corresponding solution. Define \bar{u}_0 and u by

$$\bar{u}_0(x) = \begin{cases} \psi_{0,n}(x) & \text{if } x \in (\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}), \\ 0 & \text{otherwise.} \end{cases} \quad (3.3.80)$$

$$u(x, t) = \begin{cases} u_n(x, t) & \text{if } x \in (\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}), \\ 0 & \text{otherwise,} \end{cases} \quad (3.3.81)$$

then from (3.3.71), u is a solution of (3.1.2) with initial condition \bar{u}_0 . From (3.3.79), $\bar{u}_0 \in C_c^\infty(\mathbb{R})$ and u is an ASSP in $(\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2})$ for all $n \geq 2$. This proves the theorem.

Example 3.3.10. (See Figure 3.5) Next we construct an ASSP which contains infinitely many shocks. Let $f(u) = \frac{u^2}{2}$ and $0 = x_0 < x_1 < x_2 < \dots < 1$ and $0 = t_0 < t_1 < t_2 \dots < \infty$ be such that

$$\lim_{n \rightarrow \infty} x_n = 1, \lim_{n \rightarrow \infty} \frac{x_n}{t_n} = 0.$$

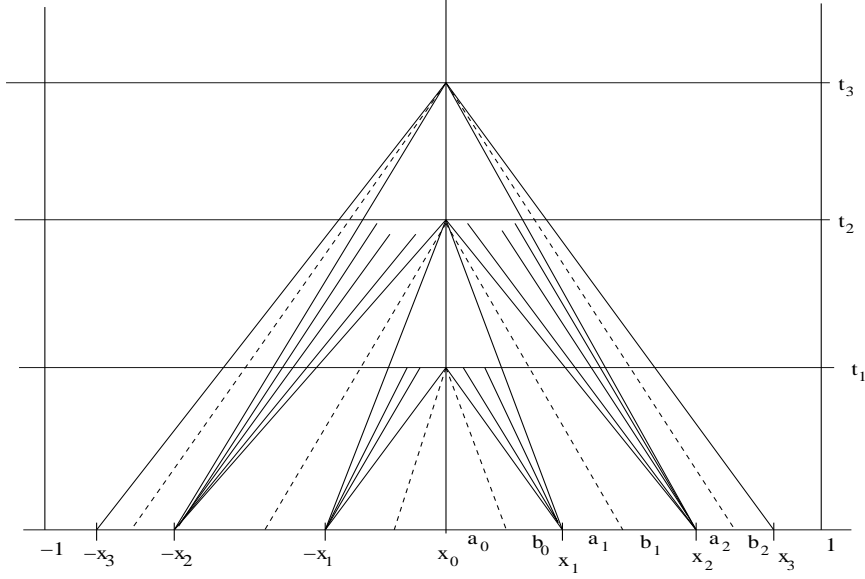


Fig. 3.5:

Define for $n \geq 1$, $a_n, b_n, l_n, m_n, \tilde{a}_n, \tilde{b}_n, \tilde{l}_n, \tilde{m}_n$ as follows:

$$\begin{aligned}
 f'(a_n) &= -f'(\tilde{a}_n) = \frac{-x_{n-1}}{t_n} \\
 f'(b_n) &= -f'(\tilde{b}_n) = \frac{-x_n}{t_n} \\
 l_n(t) &= -\tilde{l}_n(t) = x_{n-1} + t f'(a_n) \\
 m_n(t) &= -\tilde{m}_n(t) = x_n + t f'(b_n). \\
 s_n(t) &= (t - t_n) \frac{f(a_n) - f(b_n)}{a_n - b_i} \\
 \tilde{s}_n(t) &= -s_n(t) \\
 -\tilde{\alpha}_n &= \alpha_n = s_n(0).
 \end{aligned}$$

Since $x_{n-1} < x_n$ and hence $f'(a_n) = -\frac{x_{n-1}}{t_n} > -\frac{x_n}{t_n} = f'(b_n)$, therefore $b_n < a_n$ and

$$\begin{aligned}
 f'(b_n) &< \frac{f(a_n) - f(b_n)}{a_n - b_n} < f'(a_n), \\
 x_{n-1} = -t_n f'(a_n) &< -t_n \frac{f(a_n) - f(b_n)}{a_n - b_n} < -t_n f'(b_n) = x_n, \\
 l_n(0) &< s_n(0) < m_n(0).
 \end{aligned}$$

Define $u_+(x, t)$ for $x > 0, t > 0$ by

$$u_+(x, t) = \begin{cases} a_n & \text{if } 0 < t < t_n, l_n(t) < x < s_n(t), \\ b_n & \text{if } 0 < t < t_n, s_n(t) < x < m_n(t), \\ (f')^{-1}\left(\frac{x-x_{n+1}}{t}\right) & \text{if } 0 < t < t_n, m_n(t) < x < l_{n+1}(t), \\ 0 & \text{otherwise,} \end{cases}$$

and u and u_0 by

$$u(x, t) = \begin{cases} u_+(x, t) & \text{if } x > 0, t > 0, \\ -u_+(-x, t) & \text{if } x < 0, t > 0, \end{cases}$$

$$u_0(x) = u(x, 0).$$

On the line $x = 0$, $u(0+, t) = -u(0-, t)$ and hence $\frac{1}{2}(u(0+, t) + u(0-, t)) = 0$. This implies that u satisfies RH condition across $x = 0$ and hence u is the solution of (1.2) with initial data u_0 . Furthermore $u = 0$ for $|x| \geq 1$ and $D = (-1, 1) \cup (0, \infty)$ is an ASSP and each line $s_n(t)$ and $\tilde{s}_n(t)$ for $0 < t < t_n$ is a shock curve.

Remark 3.3.1. In Lemma 3.3.9, one can relax the conditions on u_0 as follows Let u_0 be a bounded measurable function with support of u_0 is in $[A_1, A_2]$. Let

$$z(u_0) = \{x \in [A_1, A_2] : u_0(x) = 0\}$$

be its zero set and satisfies the following conditions

- (1). u_0 is monotone in a neighbourhood of $z(u_0)$.
- (2). $\int_{A_1}^{A_2} u_0(x) dx = 0$ and whenever $\alpha < \beta, \alpha, \beta \in z(u_0)$, $\int_{\alpha}^{\beta} u_0(x) dx = 0$, then $\alpha = A_1, \beta = A_2$.
- (3). u_0 is a non decreasing continuous function in a neighbourhood of A_1, A_2 , then

Lemma 3.3.11. *Let u_0 satisfies the above conditions and u be the solution of (3.1.2) with initial data u_0 . Then $D = (A_1, A_2) \times (0, \infty)$ is the only ASSP for u and $u(x, t) = 0$ for $x \notin (A_1, A_2)$.*

Proof follows exactly as in Lemma 3.3.9.

Remark 3.3.2. In Theorem 3.2.7, the decay estimates (3.2.29) and (3.2.30) are obtained under the assumption $f(u) = |u|^p$ for $1 < p < 2$, by Liu-Pierre [52] and generalised by Kim [44] when $u_- = u_+ = 0$.

CHAPTER 4

EXACT CONTROLLABILITY OF SCALAR CONSERVATION LAWS WITH STRICT CONVEX FLUX

4.1 Introduction:

In this chapter we consider the following scalar conservation law in one space dimension. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly convex C^1 function satisfying the super linear growth,

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty. \quad (4.1.1)$$

Let $T > 0$, $0 \leq \delta < T$, $A < B$, $I = (A, B)$, $\Omega = I \times (\delta, T)$, $u_0 \in L^\infty(I)$, $b_0, b_1 \in L^\infty((0, T))$ and consider the problem

$$u_t + f(u)_x = 0 \quad (x, t) \in \Omega, \quad (4.1.2)$$

$$u(x, \delta) = u_0(x) \quad x \in I, \quad (4.1.3)$$

$$u(A, t) = b_0(t) \quad t \in (\delta, T), \quad (4.1.4)$$

$$u(B, t) = b_1(t) \quad t \in (\delta, T). \quad (4.1.5)$$

This problem has been well studied over the last several decades starting from the pioneering works of Lax-Oleinik [31], Kruzkov [46], Bardaux-Leraux-Nedeleck [13]. They have studied the existence and uniqueness of weak solutions to (4.1.2)-(4.1.5) satisfying the entropy condition. In spite of being well studied, still there are problems which are open. Notably among them are

1. Profile of a solution, for example how many shocks can a solution exhibit and the nature of the shocks.
2. Optimal controllability for initial and initial-boundary value problem.
3. Exact controllability of initial and initial-boundary value problem.

Problems (1) and (2) have been dealt in [4] and [3] respectively. In this chapter we investigate problem (3) for the entropy solution of (4.1.2). Through out the chapter solution of (4.1.2) always means a weak solution satisfying the entropy condition. The basic ingredient in studying all these problems comes from the analysis of characteristic curves R_{\pm} . Originally this was introduced by Hopf [33] and later by Dafermos [25], who studied them quite extensively to obtain information on the nature of solutions. Independently this was used in [6] to obtain the explicit formula for solution of discontinuous flux.

The plan of the chapter is as follows:

In section (4.1) we state the main results. In section (4.2) we prove these results assuming four Lemmas without proof. The First two Lemmas deals with the backward construction which will be proved in section (4.3). The remaining two Lemmas deals with free regions. In order to prove these Lemmas, one has to study the finer properties of the generalized characteristics namely

- (i). Comparison properties with respect to the initial data.
- (ii). Failure of the continuity with respect to the initial data.
- (iii). Behavior of the characteristics when one side of the initial data is large.

This has been carried out in section (4.4). Main tool to study all these properties are the Hopf [33], Lax-Oleinik [31] explicit formulas and we recall them without proof.

Main results, Exact Controllability: Normally for the non linear evolution equations, technique of linearization is adopted to study controllability problems. Unfortunately this method does not work (see Horsin [39]) and very few results are available on this subject. Here we consider the following three problems of controllability. Let $u_0 \in L^{\infty}(\mathbb{R})$ and

- (I) **Controllability for pure initial value problem:** Assume that $I = \mathbb{R}, \Omega = \mathbb{R} \times (0, T)$. Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2), g \in L^{\infty}(J_1)$, a target be given. The question is, does there exist a $\bar{u}_0 \in L^{\infty}(J_2)$ and u in $L^{\infty}(\Omega)$ such that u is a solution of (4.1.2) satisfying

$$u(x, T) = g(x) \quad x \in J_1, \quad (4.1.6)$$

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } x \notin J_2, \\ \bar{u}_0(x) & \text{if } x \in J_2. \end{cases} \quad (4.1.7)$$

- (II) **Controllability for one sided initial boundary value problem:** Assume that $I = (0, \infty), \Omega = \mathbb{R} \times (0, T), J = (0, C)$ and a target function $g \in L^{\infty}(J)$ be given. The question is, does there exist a $u \in L^{\infty}(\Omega)$ and $b \in L^{\infty}((0, T))$ such that u is a

solution of (4.1.2) satisfying

$$u(x, T) = g(x) \quad \text{if } x \in J, \quad (4.1.8)$$

$$u(x, 0) = u_0(x) \quad \text{if } x \in (0, \infty), \quad (4.1.9)$$

$$u(0, t) = b(t) \quad \text{if } t \in (0, T). \quad (4.1.10)$$

(III) **Controllability from two sided initial boundary value problem:**

(a). Let $\Omega = \mathbb{R} \times (0, T)$, $I_1 = (B_1, B_2)$, $B_1 \leq C \leq B_2$. Given the target functions $g_1 \in L^\infty(B_1, C)$, $g_2 \in L^\infty(C, B_2)$, does there exist a $\bar{u}_0 \in L^\infty(\mathbb{R} \setminus I_1)$ and $u \in L^\infty(\Omega)$ such that u is a solution of (4.1.2) satisfying

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2. \end{cases} \quad (4.1.11)$$

and

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } B_1 < x < B_2, \\ \bar{u}_0(x) & \text{if } x < B_1 \text{ or } x > B_2. \end{cases} \quad (4.1.12)$$

(b). Here we consider controllability in a strip. Let $I = (B_1, B_2)$, $\Omega = I \times (0, T)$, $B_1 < C < B_2$. Let $g_1 \in L^\infty((B_1, C))$, $g_2 \in L^\infty((C, B_2))$ be given. Then the question is, does there exist $b_0, b_1 \in L^\infty((0, T))$ and a $u \in L^\infty(\Omega)$ such that u is a solution of (4.1.2) and satisfying

$$u(x, 0) = u_0(x), \quad (4.1.13)$$

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2. \end{cases} \quad (4.1.14)$$

$$u(B_1, t) = b_0(t), \quad (4.1.15)$$

$$u(B_2, t) = b_1(t). \quad (4.1.16)$$

In view of the Lax-Oleinik (Chapter (3) of [31]) explicit formula for solutions of pure initial value problem and by Joseph-Gowda [40] for initial boundary value problem, the targets g or g_1, g_2 cannot be arbitrary. They must satisfy the compatibility condition, for example in the case of problem (I), there exists a non-decreasing function ρ in (C_1, C_2) such that for a.e $x \in (C_1, C_2)$

$$f'(g(x)) = \frac{x - \rho(x)}{T}. \quad (4.1.17)$$

In the case of problem (II), there exists a non-decreasing function ρ in $(0, C)$ such

that

$$f'(g(x)) = \frac{x}{T - \rho(x)}. \quad (4.1.18)$$

Assuming that the target functions satisfy the compatibility conditions, then the question is

whether the problems (I),(II) and (III) admit a solution? In fact, **it is true** and we have the following results. First we describe the class of functions satisfying compatibility conditions.

Definition (Admissible functions): Let $J = (M, N)$ and $T > 0$,

$$S(J) = \{\rho : J \rightarrow \mathbb{R} : \rho \text{ is monotone and left or right continuous function}\}.$$

Then define admissible class of target functions by

(i) Target space for initial value problem (IA):

$$IA(J) = \{g \in L^\infty(J) : f'(g(x)) = \frac{x - \rho(x)}{T}, \rho \in S(J), \\ \rho \text{ is a non-decreasing function}\}. \quad (4.1.19)$$

(ii) Target space for left boundary problem (LA):

$$LA(J) = \{g \in L^\infty(J) : f'(g(x)) = \frac{x - M}{T - \rho(x)}, \rho \in S(J), \\ \rho \text{ is a non-increasing right continuous function}\}. \quad (4.1.20)$$

(iii) Target space for right boundary problem (RA):

$$RA(J) = \{g \in L^\infty(J) : f'(g(x)) = \frac{x - N}{T - \rho(x)}, \rho \in S(J), \\ \rho \text{ is a non-decreasing left continuous function}\}. \quad (4.1.21)$$

Theorem 4.1.1. Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2)$. Let $g(x) = (f')^{-1} \left(\frac{x - \rho(x)}{T} \right)$ be in $IA(J_1)$ and $B_1 < A_1 < A_2 < B_2$, satisfying

$$A_1 \leq \rho(x) \leq A_2 \quad \text{if } x \in J_1, \quad (4.1.22)$$

then there exists a $\bar{u}_0 \in L^\infty(J_2)$, $u \in L^\infty(\Omega)$ such that (u, \bar{u}_0) is a solution to problem (I) (see Figure 4.1).

Theorem 4.1.2. Let $\wedge > 0, C > 0, \delta > 0, J = (0, C)$. Let $g \in LA(J)$ given by $f'(g(x)) =$

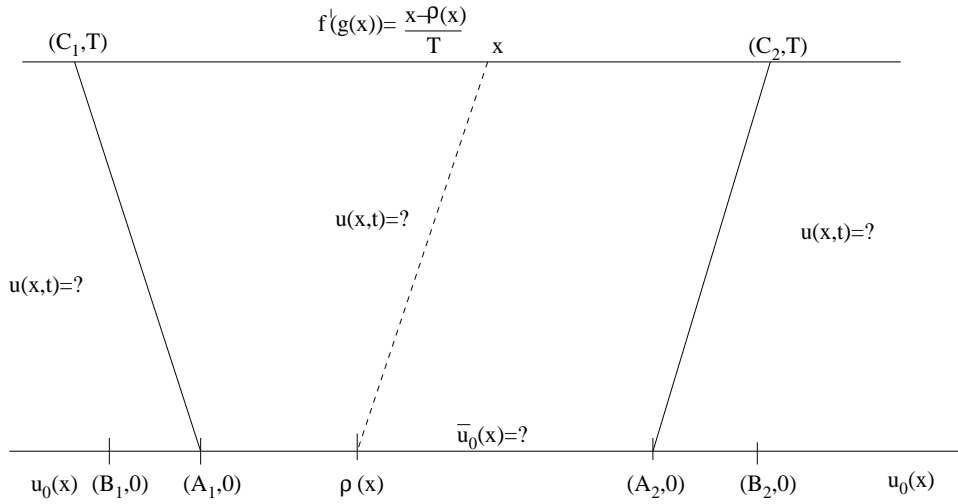


Fig. 4.1:

$\frac{x}{T - \rho(x)}$ for $x \in J$ and satisfying

$$\delta \leq \rho(x) \leq T, \quad (4.1.23)$$

$$\left| \frac{x}{T - \rho(x)} \right| \leq \wedge. \quad (4.1.24)$$

Then there exist a $b \in L^\infty(0, T)$, $u \in L^\infty(\Omega)$ such that (u, b) is a solution to Problem II (see Figure 4.2).

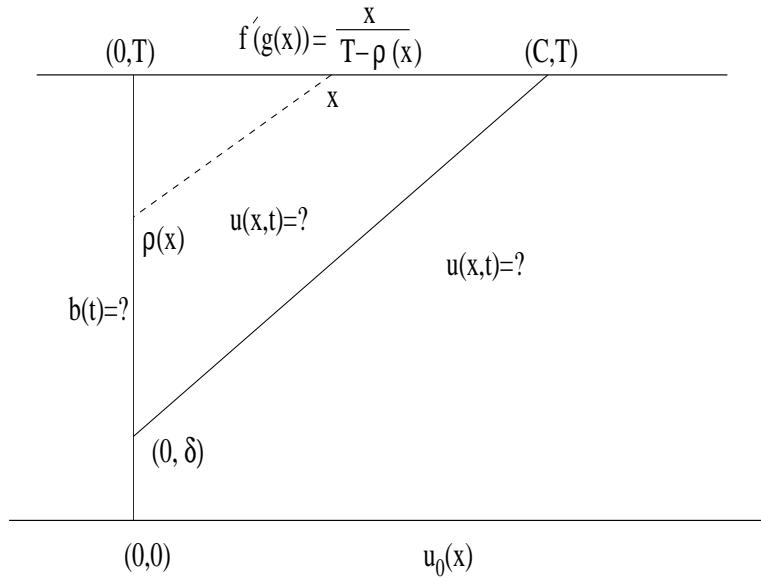


Fig. 4.2:

Theorem 4.1.3. Let $I_1 = (B_1, B_2)$, $B_1 < C < B_2$, $J_1 = (B_1, C)$, $J_2 = (C, B_2)$, then

(a). Let $A_1 < B_1 < B_2 < A_2$ and $g_1 \in IA(J_1), g_2 \in IA(J_2)$ given by $f'(g_1(x)) = \frac{x-\rho_1(x)}{T}$, $f'(g_2(x)) = \frac{x-\rho_2(x)}{T}$, satisfying

$$\rho_1(x) \leq A_1 \quad \text{if } x \in J_1, \quad (4.1.25)$$

$$\rho_2(x) \geq A_2 \quad \text{if } x \in J_2. \quad (4.1.26)$$

Then there exists $\bar{u}_0 \in L^\infty((\mathbb{R} \setminus I_1)), u \in L^\infty(\Omega)$ such that (u, \bar{u}_0) is a solution to problem (a) of III (see Figure 4.3).

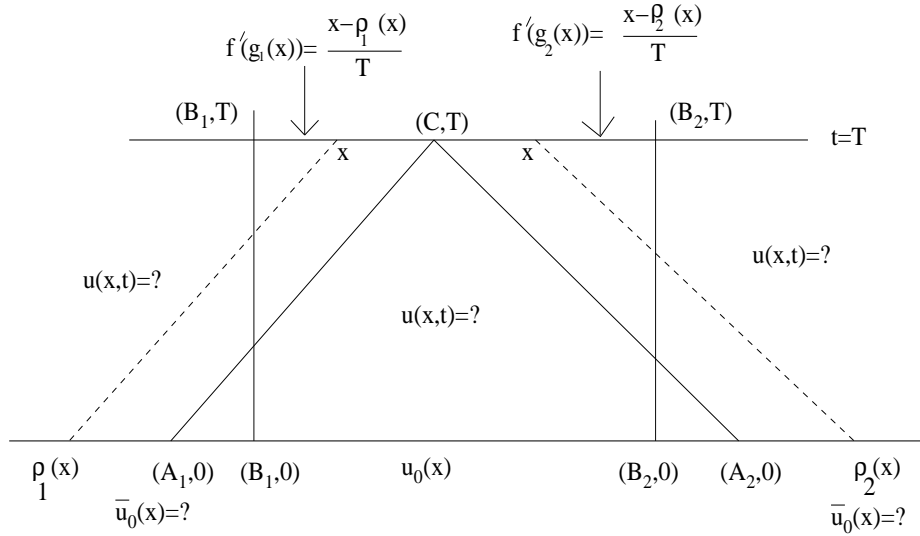


Fig. 4.3:

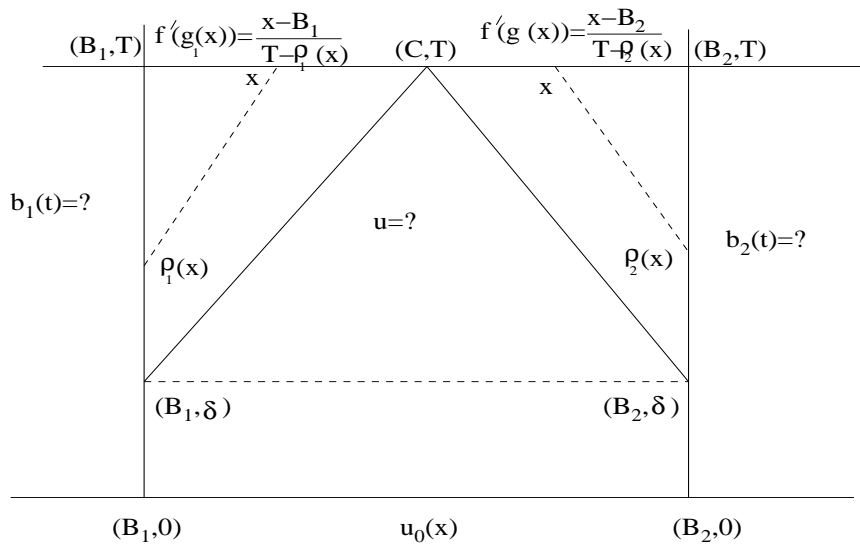


Fig. 4.4:

(b). Let $\wedge > 0, 0 < \delta < T, g_1 \in LA(J_1), g_2 \in RA(J_2)$, given by $f'(g_1(x)) =$

$\frac{x-B_1}{T-\rho_1(x)}, f'(g_2(x)) = \frac{x-B_2}{T-\rho_2(x)}$ satisfying for $i = 1, 2$, $x \in J_i$

$$\delta \leq \rho_i(x) \leq T, \quad (4.1.27)$$

$$\left| \frac{x - B_i}{T - \rho_i(x)} \right| \leq \wedge. \quad (4.1.28)$$

Then there exists $b_0, b_1 \in L^\infty((0, T))$ and $u \in L^\infty(\Omega)$ such that (u, b_0, b_1) is a solution to problem (b) of III (see Figure 4.4).

Before going for further results, let us recall some of the earlier works in this direction and compare them with these results.

Problem (a) in III was considered by Horsin [39] for the Burger's equation under similar assumptions on g_1 and g_2 as in (a) of Theorem 4.1.3. He proves that given any $T > 2$ there exists a $T_c \geq T$, such that (a) of problem III has an approximate controllability solution. That is given $\epsilon > 0$, there exist (u, \bar{u}_0) such that

$$\int_{B_1}^{B_2} |u(x, T_c) - g(x)| dx = O(\epsilon),$$

and $u(x, T_c) = g(x) = \chi_{(B_1, C)}(x)g_1(x) + \chi_{(C, B_2)}(x)g_2(x)$, outside an interval of length ϵ .

In the viscous case the same problem was considered by Glass-Guerrero [32] for the control $u(x, T) = M$ is constant. Using the Cole-Hopf transformation, they show that there exist $T_0 > 0$ such that for all time $T > T_0$ and small viscosity, they prove the exact controllability. Also Guerrero-Imanuvilov [37] proves a negative result by showing that $M = 0$ cannot be controllable.

Theorem 4.1.3 is stronger and much more precise result in the non viscous case because

- (i). It removes the condition on time T_c and obtains exact controllability.
- (ii). It deals with general convex flux instead of Burger's equation.
- (iii). In section (4.5) we give a criterion when the constants are controllable.

In the case of problem (II), Fabio-Ancona and Andrea-Marson [11],[12] studied the problem from the point of view of Hamilton-Jacobi equations and studies the compactness properties of $\{u(\cdot, T)\}$ when $u(x, 0) = 0$ and $u(\cdot, 0) \in \mathcal{U}$, here \mathcal{U} is a set of controls satisfying some properties.

In our results on controllability, superlinearity of f plays an important role in removing the condition on T_c and by creating free regions (see Lemmas 4.2.3 and 4.2.4). Next using convexity and backward construction, we explicitly construct solutions in these free regions for particular data which allow to obtain solutions for control problems (see Lemmas 4.2.1 and 4.2.2).

Remark 4.1.1. The conditions in Theorems 4.1.1-4.1.3 are optimal. That is, in general, we cannot take $A_1 = B_1$, $A_2 = B_2$ in Theorem 4.1.1 and $\delta = 0$ in Theorems 4.1.2 and 4.1.3. This can be illustrated by a simple counter example (see counter example 4.4.14).

4.2 Exact controllability and main results

In this section we give the proof of Theorems 4.1.1 to 4.1.3. Basically following two main ideas are used to prove these results

- (a) Free regions : By suitable variations of parameters in the initial data, one can obtain a sub region in Ω , where the solution is prescribed as a constant. These sub regions are called free regions. This is achieved in Lemmas 4.4.8 and 4.4.9. For example from (4.4.95), the region

$$\{(x, t) : x < L_1(t)\} \quad (4.2.1)$$

is a free region since u^λ is constant.

- (b) Backward construction : By using this method, one can construct a solution which achieves the given target at any time $t = T$.

We use this backward construction to prove the main theorems in the free regions. Then we glue the different solutions by using Rankine-Hugoniot condition.

We state the following Lemmas which deal with this construction.

Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq \delta < T$, $A, C \in \mathbb{R}$. Let $l(\cdot, \delta, A, C)$ be the line joining between (C, T) and (A, δ) with slope $1/f'(a(\delta, A, C))$, intersecting $t = 0$ axis at $D(\delta, A, C)$ and is given by

$$f'(a(\delta, A, C)) = \frac{C - A}{T - \delta}, \quad (4.2.2)$$

$$l(t, \delta, A, C) = A + f'(a(\delta, A, C))(t - \delta), \quad (4.2.3)$$

$$D(\delta, A, C) = A - \delta f'(a(\delta, A, C)), \quad (4.2.4)$$

$$= A - \frac{\delta(C - A)}{T - \delta}. \quad (4.2.5)$$

Lemma 4.2.1. 1. Let $\wedge > 0$, $A < C$ and $\rho \in LA((A, C))$ satisfying

$$0 \leq \delta \leq \rho(x) \leq T, \quad (4.2.6)$$

$$\left| \frac{x - A}{T - \rho(x)} \right| \leq \wedge. \quad (4.2.7)$$

Let $\Omega = (A, \infty) \times (\delta, T)$. Then there exists a $\tilde{b}_1(t, \delta, A, C) \in L^\infty((\delta, T))$ and a solution

$\tilde{u}_1(x, t, \delta, A, C)$ of (4.1.2) satisfying

$$f'(\tilde{u}_1(x, T, \delta, A, c)) = \frac{x - A}{T - \rho(x)}, \quad x \in (A, C), \quad (4.2.8)$$

with initial and boundary conditions

$$\tilde{u}_1(A, t, \delta, a, C) = \tilde{b}_1(t, \delta, A, C), \quad t \in (\delta, T), \quad (4.2.9)$$

$$\tilde{u}_1(x, t, \delta, a, C) = a(\delta, A, C), \quad x > l(t, \delta, A, C), \quad (4.2.10)$$

$$\tilde{u}_1(l(t, \delta, A, C)-, t, \delta, a, C) = a(\delta, A, C). \quad (4.2.11)$$

2. Let $C < A$ and $\rho \in RA((C, A))$ satisfying (4.2.6) and (4.2.7) for $x \in (C, A)$. Let $\Omega = (-\infty, A) \times (\delta, T)$. Then there exist $\tilde{b}_2(t, \delta, A, C) \in L^\infty((\delta, T))$ and a solution $\tilde{u}_2(x, t, \delta, a, C)$ of (4.1.2) satisfying

$$f'(\tilde{u}_2(x, T, \delta, A, C)) = \frac{x - A}{T - \rho(x)}, \quad x \in (C, A), \quad (4.2.12)$$

with initial and boundary conditions

$$\tilde{u}_2(A, t, \delta, A, C) = \tilde{b}_2(t, \delta, A, C), \quad t \in (\delta, T), \quad (4.2.13)$$

$$\tilde{u}_2(x, t, \delta, A, C) = a(\delta, a, C), \quad x < l(t, \delta, A, C), \quad (4.2.14)$$

$$\tilde{u}_2(l(t, \delta, A, C)+, t, \delta, A, C) = a(\delta, A, C), \quad t \in (\delta, T). \quad (4.2.15)$$

Lemma 4.2.2. Let $A_1 < A_2$, $C_1 < C_2$, $\rho \in IA((C_1, C_2))$ such that for all $x \in (C_1, C_2)$,

$$A_1 \leq \rho(x) \leq A_2. \quad (4.2.16)$$

Let $\Omega = \mathbb{R} \times \mathbb{R}_+$, for $i = 1, 2$, $l_i(t) = l(t, 0, A_i, C_i)$, $a_i = a_i(0, A_i, C_i)$, then there exist $\tilde{u}_0 \in L^\infty((A_1, A_2))$ and a solution \tilde{u} of (4.1.2) such that for $0 < t < T$,

$$f'(\tilde{u}(x, T)) = \frac{x - \rho(x)}{T}, \quad \text{for } x \in (C_1, C_2), \quad (4.2.17)$$

$$\tilde{u}(l_1(t)+, t) = a_1, \quad (4.2.18)$$

$$\tilde{u}(l_2(t)-, t) = a_2, \quad (4.2.19)$$

with initial conditions

$$\tilde{u}(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \tilde{u}_0(x) & \text{if } A_1 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (4.2.20)$$

Let $T > 0, \mu, \lambda \in \mathbb{R}, A < B, l_1(t) = l(t, 0, A, C), l_2(t) = l(t, 0, B, C), a_1 = a_1(0, A, C), a_2 = a_2(0, B, C)$. Define u_0^λ and u_0^μ by

$$u_0^\lambda(x) = \begin{cases} a_1 & \text{if } x < A, \\ \lambda & \text{if } A < x < B, \\ u_0(x) & \text{if } x > B \end{cases} \quad (4.2.21)$$

and

$$u_0^\mu(x) = \begin{cases} a_2 & \text{if } x > B, \\ \mu & \text{if } A < x < B, \\ u_0(x) & \text{if } x < A. \end{cases} \quad (4.2.22)$$

Let $u_\lambda(x, t)$ and $u_\mu(x, t)$ be the solutions of (4.1.2) with initial data u_0^λ and u_0^μ respectively. Then we have the following

Lemma 4.2.3. *There exists $\mu_0 < \lambda_0$ such that for all $\mu \leq \mu_0, \lambda \geq \lambda_0, 0 < t < T, x \in \mathbb{R}, u_\lambda$ and u_μ satisfy*

$$u_\lambda(x, t) = a_1, \quad \text{if } x < l_1(t), \quad u_\lambda(l_1(t)+, t) = a_1 \quad (4.2.23)$$

$$u_\mu(x, t) = a_2, \quad \text{if } x > l_2(t), \quad u_\mu(l_2(t)-, t) = a_2. \quad (4.2.24)$$

Let $\delta > 0, T > 0, B_1 \leq B_2, l_1(t) = l(t, \delta, B_1, C), l_2(t) = l(t, \delta, B_2, C), A_1 = l_1(0) < B_1, A_2 = l_2(0) > B_2, a_1 = a(\delta, B_1, C), a_2 = a(\delta, B_2, C)$. For $\lambda, \mu \in \mathbb{R}$, define $u_0^{\lambda, \mu}$ by

$$u_0^{\lambda, \mu}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2 \end{cases} \quad (4.2.25)$$

and $u_{\lambda, \mu}$ be the solution of (4.1.2) with initial data $u_0^{\lambda, \mu}$.

Then we have the following

Lemma 4.2.4. *Given any λ_0, μ_0 , there exist $\lambda_2 \geq \lambda_0, \mu_2 \leq \mu_0$ such that for $0 \leq t \leq T, u_{\lambda_2, \mu_2}$ satisfies*

$$u_{\lambda_2, \mu_2}(x, t) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t), \end{cases} \quad (4.2.26)$$

$$u_{\lambda_2, \mu_2}(l_1(t)+, t) = a_1, \quad u_{\lambda_2, \mu_2}(l_2(t)-, t) = a_2. \quad (4.2.27)$$

Proof of Theorem 4.1.1 Let $\Omega = \mathbb{R} \times (0, T)$, A_i, B_i, C_i, g and ρ be as in Theorem 4.1.1
Let

$$f'(a_1) = \frac{C_1 - A_1}{T}, \quad f'(a_2) = \frac{C_2 - A_2}{T}, \quad l_1(t) = A_1 + tf'(a_1), \quad l_2(t) = A_2 + tf'(a_2).$$

Then from Lemma 4.2.3 choose λ, μ and solutions u_λ and u_μ of (4.1.2) such that

$$u_\lambda(x, t) = a_2 \quad \text{if } x < l_2(t), \quad u_\lambda(l_2(t)+, t) = a_2 \quad (4.2.28)$$

$$u_\mu(x, t) = a_1 \quad \text{if } x > l_1(t), \quad u_\mu(l_1(t)-, t) = a_1 \quad (4.2.29)$$

with

$$u_\lambda(x, 0) = \begin{cases} a_2 & \text{if } x < A_2, \\ \lambda & \text{if } A_2 < x < B_2, \\ u_0(x) & \text{if } x > B_2 \end{cases} \quad (4.2.30)$$

and

$$u_\mu(x, 0) = \begin{cases} a_1 & \text{if } x > A_1, \\ \mu & \text{if } B_1 < x < A_1, \\ u_0(x) & \text{if } x < B_1 \end{cases} \quad (4.2.31)$$

From (4.1.22) and Lemma 4.2.1 there exist a solution u_1 of (4.1.2) and $\tilde{u}_0 \in L^\infty(A_1, A_2)$ satisfying

$$u_1(x, T) = g(x), \quad \text{if } x \in (C_1, C_2) \quad (4.2.32)$$

$$u_1(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \tilde{u}_0(x) & \text{if } A_1 < x < A_2, \\ a_2 & \text{if } x > A_2, \end{cases} \quad (4.2.33)$$

and

$$u_1(x, t) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t). \end{cases} \quad (4.2.34)$$

$$u_1(l_1(t)+, t) = a_1, \quad u_1(l_2(t)-, t) = a_2. \quad (4.2.35)$$

Let

$$\bar{u}_0(x) = \begin{cases} u_0(x) & \text{if } x \notin (B_1, B_2) \\ \lambda & \text{if } A_2 < x < B_2 \\ \tilde{u}_0(x) & \text{if } A_1 < x < A_2 \\ \mu & \text{if } B_1 < x < A_1. \end{cases}$$

From (4.2.28), (4.2.29) , (4.2.34) and RH condition, glue u_λ, u_μ, u_1 to form a single solution u of (1.2) for $0 < t < T$ by

$$u(x, t) = \begin{cases} u_\mu(x, t) & \text{if } x < l_1(t), \\ u_1(x, t) & \text{if } l_1(t) < x < l_2(t), \\ u_\lambda(x, t) & \text{if } l_2(t) < x. \end{cases} \quad (4.2.36)$$

Then from (4.2.30), (4.2.31) and (4.2.33), (u, \bar{u}_0) is the required solution. This proves the Theorem.

Proof of Theorem 4.1.2 Let $f'(a) = \frac{C}{T-\delta}$ and $l(t)$ be the line joining (C, T) and $(0, \delta)$ given by $l(t) = (t - \delta)f'(a)$. Let $A = l(0) = -\delta f'(a) < 0$. From Lemma 4.2.3 by choosing λ large, we can find a solution u_λ of (4.1.2) in $\Omega = \mathbb{R} \times (0, T)$ satisfying

$$u_\lambda(x, 0) = \begin{cases} a & \text{if } x < A, \\ \lambda & \text{if } A < x < 0, \\ u_0(x) & \text{if } x > 0. \end{cases} \quad (4.2.37)$$

$$u_\lambda(x, t) = a \text{ if } x < l(t), \quad (4.2.38)$$

$$u_\lambda(l(t)+, t) = a. \quad (4.2.39)$$

From (4.1.23) , (4.1.24) and (1) of Lemma 4.2.1, choose a solution u_1 of (4.1.2) and $b_1 \in L^\infty(\delta, T)$ such that

$$u_1(x, T) = g(x) \quad (4.2.40)$$

$$u_1(0, t) = b_1(t) \text{ if } \delta < t < T, \quad (4.2.41)$$

$$u_1(x, t) = a \text{ if } x > l(t), t > \delta, \quad (4.2.42)$$

$$u_1(l(t)-, t) = a \text{ if } t > \delta. \quad (4.2.43)$$

From (4.2.39), (4.2.43) and RH conditions we glue the solutions u_λ and u_1 to obtain a solution u of (4.1.2) by

$$u(x, t) = \begin{cases} u_\lambda(x, t) & \text{if } x > l(t), 0 < t < T, \\ u_1(x, t) & \text{if } 0 < x < l(t), \delta < t < T. \end{cases} \quad (4.2.44)$$

Define $b \in L^\infty(0, T)$ by

$$b(t) = \begin{cases} u_\lambda(0+, t) & \text{if } 0 < t < \delta, \\ b_1(t) & \text{if } \delta < t < T. \end{cases} \quad (4.2.45)$$

Then from (4.2.37) , (4.2.40), (u, b) is the required solution. This proves the theorem.

Proof of Theorem 4.1.3 Let $f'(a_1) = \frac{C-A_1}{T}$, $f'(a_2) = \frac{C-A_2}{T}$, $l_1(t) = A_1 + tf'(a_1) = A_2 + tf'(a_2)$ be the respective lines joining (C, T) , $(A_1, 0)$ and (C, T) , $(A_2, 0)$.

From Lemma 4.2.4 choose (λ, μ) and a solution $u_{\lambda, \mu}$ of (4.1.2) in $\mathbb{R} \times (0, T)$ satisfying

$$u_{\lambda, \mu}(x, t) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t), \end{cases} \quad (4.2.46)$$

with initial condition

$$u_{\lambda, \mu}(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (4.2.47)$$

(a). Since g_i is a non decreasing function for $i = 1, 2$ satisfying (4.1.23) , (4.1.26) and hence

$$D_1 = \rho_1(B_1) \leq A_1, A_2 \leq \rho_2(B_2) = D_2.$$

Let η_i be the line joining (B_i, T) and $(D_i, 0)$ with $f'(m_i) = \frac{B_i - D_i}{T}$ for $i = 1, 2$. Then from Lemma 4.2.2, there exist solutions u_i of (4.1.2) in $\mathbb{R} \times (0, T)$ with initial condition $u_0^i \in L^\infty(D_i, A_i)$ for $i = 1, 2$ such that

$$u_1(x, T) = g_1(x, T) \quad \text{if } x \in (B_1, C), \quad (4.2.48)$$

$$u_2(x, T) = g_2(x, T) \quad \text{if } x \in (C, B_2), \quad (4.2.49)$$

$$u_1(x, t) = m_1 \quad \text{if } x < \eta_1(t), \quad (4.2.50)$$

$$u_1(l_1(t)-, t) = u_1(l_1(t)+, t) = a_1, \quad (4.2.51)$$

$$u_2(x, t) = m_2 \quad \text{if } x > \eta_2(t), \quad (4.2.52)$$

$$u_2(l_2(t)-, t) = u_2(l_2(t)+, t) = a_2, \quad (4.2.53)$$

and

$$u_1(x, 0) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ a_1 & \text{if } x > A_1. \end{cases} \quad (4.2.54)$$

$$u_2(x, 0) = \begin{cases} m_2 & \text{if } x > D_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ a_2 & \text{if } x < A_2. \end{cases} \quad (4.2.55)$$

From (4.2.46) , (4.2.47) ,(4.2.51), (4.2.53) and from RH conditions, we can glue $u_1, u_2, u_{\lambda,\mu}$ to a solution u of (4.1.2) with initial data $u(x, 0)$ given by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } x < l_1(t), \\ u_{\lambda,\mu}(x, t) & \text{if } l_1(t) < x < l_2(t), \\ u_2(x, t) & \text{if } x > l_2(t), \end{cases} \quad (4.2.56)$$

$$u(x, 0) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ m_2 & \text{if } x > D_2. \end{cases}$$

Define \bar{u}_0 by

$$\bar{u}_0(x) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ \mu & \text{if } B_2 < x < A_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ m_2 & \text{if } x > D_2. \end{cases} \quad (4.2.57)$$

From (4.2.52) , (4.2.53) u satisfies

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2, \end{cases} \quad (4.2.58)$$

and (u, \bar{u}_0) is the required solution. This proves (a).

(b). Given $\delta > 0$ choose $A_1 < B_1 < B_2 < A_2$ such that $\max(l_1(B_1), l_2(B_2)) = \delta$ and $u_{\lambda,\mu}$ be the solution of (4.1.2) as in (4.2.46). From (4.1.27),(4.1.28) and from Lemma 4.2.1, there exist solutions u_1 of (4.1.2) in $(B_1, \infty) \times (\delta, T)$ and boundary data \tilde{b}_1, u_2 of (4.1.2) in $(-\infty, B_2) \times (\delta, T)$ and boundary data \tilde{b}_2 such that

$$\begin{aligned} u_1(x, T) &= g_1(x) & \text{if } x \in (B_1, C), \\ u_2(x, T) &= g_2(x) & \text{if } x \in (C, B_2), \end{aligned}$$

and for $\delta < t < T$,

$$\begin{aligned} u_1(B_1, t) &= \tilde{b}_1(t), & u_1(l_1(t)-, t) &= a_1, \\ u_2(B_2, t) &= \tilde{b}_2(t), & u_2(l_2(t)+, t) &= a_2. \end{aligned}$$

Then from RH condition glue $u_1, u_2, u_{\lambda, \mu}$ in $\Omega = (B_1, B_2) \times (0, T)$ by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } 0 < t < \delta, B_1 < l_1(x) < t, \\ u_2(x, t) & \text{if } 0 < t < \delta, t < l_1(x) < B_2, \\ u_{\lambda, \mu}(x, t) & \text{otherwise.} \end{cases}$$

Then u is a solution of (4.1.2) satisfying the boundary conditions (b_1, b_2) given by

$$b_1(t) = \begin{cases} \tilde{b}_1(t) & \text{if } \delta < t < T, \\ u_{\lambda, \mu}(B_1+, t) & \text{if } 0 < t < \delta, \end{cases}$$

$$b_2(t) = \begin{cases} \tilde{b}_2(t) & \text{if } \delta < t < T, \\ u_{\lambda, \mu}(B_2-, t) & \text{if } 0 < t < \delta. \end{cases}$$

Then (u, b_1, b_2) is the solution for problem (4.1.3). This proves the Theorem.

4.3 Proof of Lemmas :

Initial Value problem partitions: (See Figure 4.5) Let $0 \leq \delta < T$, $I = (A_1, A_2)$, $J = (C_1, C_2)$. Let $P = \{y_0, y_1 \dots y_n, x_0, x_1 \dots x_n\}$ be a partition of (I, J) if

$$A_1 = y_0 < y_1 < \dots < y_n = A_2, \quad C_1 = x_0 \leq x_1 \leq \dots \leq x_n = C_2.$$

Let $P(I, J) = \{P : P \text{ is a partition of } (I, J)\}$. For a partition P denote $a_i(P), s_i(P), b_i(P), a_i(t, P), s_i(t, P), b_i(t, P)$ by

$$\begin{aligned} f'(a_i(P)) &= \frac{x_i - y_i}{T - \delta}, \\ f'(b_i(P)) &= \frac{x_i - y_{i+1}}{T - \delta}, \\ s_i(P) &= \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)}, \\ a_i(t, P) &= x_i + f'(a_i(P))(t - T), \\ b_i(t, P) &= x_i + f'(b_i(P))(t - T), \\ s_i(t, P) &= x_i + s_i(P)(t - T). \end{aligned}$$

Clearly $a_i(\delta, P) = y_i$, $b_i(\delta, P) = y_{i+1}$.

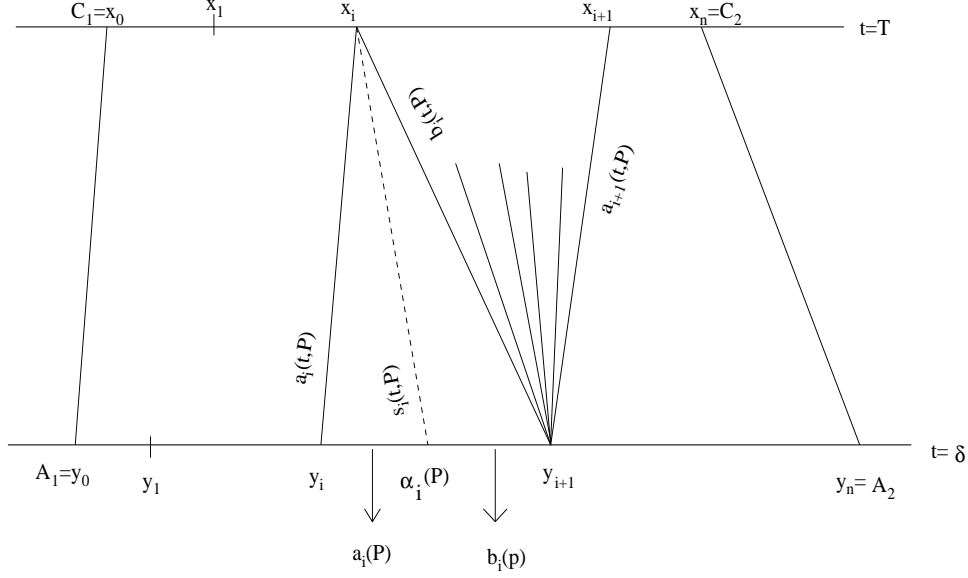


Fig. 4.5:

Lemma 4.3.1. *Let $\alpha_i(P) = s_i(\delta, P)$, then for $\delta \leq t \leq T$,*

$$b_i(P) < a_i(p), \quad b_i(P) \leq a_{i+1}(P), \quad (4.3.1)$$

$$y_i < \alpha_i(P) < y_{i+1}, \quad (4.3.2)$$

$$a_i(t, P) < s_i(t, P) < b_i(t, P) \quad \text{for } \delta < t < T. \quad (4.3.3)$$

Proof. Since $y_i < y_{i+1}$, hence

$$f'(a_i(P)) = \frac{x_i - y_i}{T - \delta} > \frac{x_i - y_{i+1}}{T - \delta} = f'(b_i(P)),$$

therefore $a_i(P) > b_i(P)$. By convexity of f

$$f'(a_i(p)) > \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)} > f'(b_i(P)),$$

$$\frac{x_i - y_i}{T - \delta} > \frac{x_i - \alpha_i(P)}{T - \delta} > \frac{x_i - y_{i+1}}{t - \delta},$$

and hence $y_i < \alpha_i(P) < y_{i+1}$. Since $x_i \leq x_{i+1}$ and hence $f'(b_i(P)) = \frac{x_i - y_{i+1}}{T - \delta} \leq \frac{x_{i+1} - y_{i+1}}{T - \delta} = f'(a_{i+1}(P))$. This implies $b_i(P) \leq a_{i+1}(P)$. This proves (4.3.1) to (4.3.3) and hence the Lemma.

Let $\Omega_i(P) = \{(x, t) : \delta < t < T, a_i(t, P) < x < a_{i+1}(t, P)\}$. In view of Lemma 4.3.1,

let $u_i(x, t, P)$ be a solution of (4.1.2) in $\Omega_i(P)$ defined by

$$u_i(x, t, P) = \begin{cases} a_i(P) & \text{if } a_i(t, P) < x < s_i(t, P), \\ b_i(P) & \text{if } s_i(t, P) < x < b_i(t, P), \\ (f')^{-1}\left(\frac{x-y_{i+1}}{t-\delta}\right) & \text{if } b_i(t, P) \leq x < a_{i+1}(t, P). \end{cases} \quad (4.3.4)$$

Hence $u_i(a_{i+1}(t, P)-, t, P) = a_{i+1}(P) = u_{i+1}(a_{i+1}(t, P)+, t, P)$. Therefore define the solution $u(x, t, P)$ of (4.1.2) in $\mathbb{R} \times (\delta, T)$ by

$$u(x, t, P) = \begin{cases} u_i(x, t, P) & \text{if } (x, t) \in \Omega_i(P), \\ a_0(P) & \text{if } x < a_0(t, P), \\ a_n(P) & \text{if } x > a_n(t, P), \end{cases} \quad (4.3.5)$$

satisfying the initial condition

$$u(x, \delta, P) = \begin{cases} u_0(x, P) & \text{if } x \in (A_1, A_2), \\ a_0(P) & \text{if } x < A_1, \\ a_n(P) & \text{if } x > A_2, \end{cases} \quad (4.3.6)$$

where u_0 is given by

$$u_0(x, P) = \begin{cases} a_i(P) & \text{if } y_i < x < \alpha_i(P), \\ b_i(P) & \text{if } \alpha_i(P) < x < y_{i+1}. \end{cases} \quad (4.3.7)$$

Furthermore at $t = T, x \in (C_1, C_2)$, u satisfies

$$f'(u(x, T, P)) = \sum_{i=0}^{n-1} \chi_{[x_i, x_{i+1})}(x) \left(\frac{x - y_{i+1}}{T - \delta} \right). \quad (4.3.8)$$

Next we calculate the L^∞ and TV bounds of u_0 . First observe that $f'(a_0(P)) = \frac{C_1 - A_1}{T - \delta}$ and $f'(a_n(P)) = \frac{C_2 - A_2}{T - \delta}$, hence $a_0(P), a_n(P)$, are independent of P and denote

$$a_0 = a_0(P), a_n = a_n(P),$$

$$a_0(t) = a_0(t, P), a_n(t) = a_n(t, P).$$

Let $M = \frac{\max(C_2, A_2) - \min(C_1, A_1)}{T - \delta}$, then $|f'(a_i(P))| \leq \left| \frac{x_i - y_i}{T - \delta} \right| \leq M$, $|f'(b_i(P))| \leq M$ and hence

$$|f'(u_0(x, p))| \leq M \quad (4.3.9)$$

$$\begin{aligned}
TV(f'(u_0(\cdot, P))) &= \sum_{i=0}^n |f'(a_i(P)) - f'(b_i(P))| \\
&= + \sum_{i=0}^{n-1} |f'(b_i(P)) - f'(a_{i+1}(P))| \\
&= \sum_{i=0}^n \left| \frac{x_i - y_i}{T - \delta} - \frac{x_i - y_{i+1}}{T - \delta} \right| + \sum_{i=0}^{n-1} \left| \frac{x_i - y_{i+1}}{T - \delta} - \frac{x_{i+1} - y_{i+1}}{T - \delta} \right| \\
&= \frac{1}{T - \delta} (A_2 - A_1) + \frac{1}{T - \delta} (C_2 - C_1).
\end{aligned} \tag{4.3.10}$$

Since $|f'(\theta)| \rightarrow \infty$ as $|\theta| \rightarrow \infty$ and hence we have

Lemma 4.3.2. *There exists a constant M_1 independent of P such that*

$$\|u_0(\cdot, P)\|_\infty + TV(f'(u_0(\cdot, P))) \leq M_1. \tag{4.3.11}$$

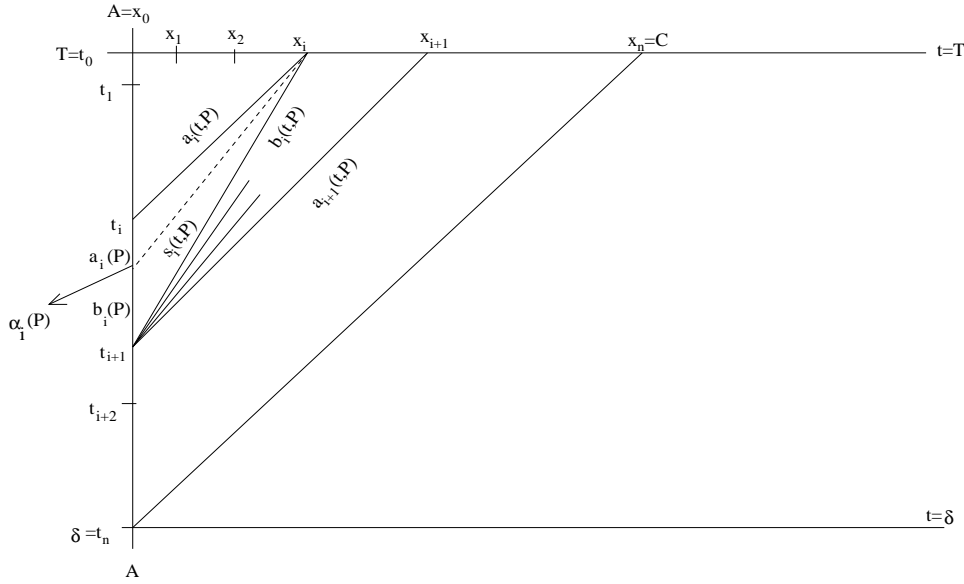


Fig. 4.6:

Boundary value partition: (See Figure 4.6) Let $0 \leq \delta < T$, $A < C$, $I = (A, C)$, $J = (\delta, T)$. Let $P = \{t_0, t_1 \dots t_n, x_0, x_1 \dots x_n\}$ is called a boundary value partition if

$$T = t_0 > t_1 > t_2 \dots > t_n = \delta, \quad A = x_0 \leq x_1 \leq x_2 < \dots \leq x_n = C.$$

$$\text{Let } P(I, J) = \{P : P \text{ is a boundary value partition of } I, J\}. \tag{4.3.12}$$

For a $P \in P(I, J)$ denote $a_i(P), s_i(P), b_i(P), a_i(t, P), s_i(t, P), b_i(t, P)$ by

$$\begin{aligned} f'(a_i(P)) &= \frac{x_i - A}{T - t_i}, \\ f'(b_i(P)) &= \frac{x_i - A}{T - t_{i+1}}, \\ s_i(P) &= \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)}, \\ a_i(t, P) &= x_i + f'(a_i(P))(t - T), \\ b_i(t, P) &= x_i + f'(b_i(P))(t - T), \\ s_i(t, P) &= x_i + s_i(P)(t - T), \end{aligned}$$

Clearly $a_i(t_i, P) = t_i$, $b_i(t_{i+1}, P) = t_{i+1}$.

Lemma 4.3.3. *Define $\alpha_i(P)$ such that $s_i(\alpha_i(P), P) = A$. Then for $t \leq T$*

$$a_i(P) > b_i(P), \quad a_{i+1}(P) \geq b_i(P), \quad (4.3.13)$$

$$t_i > \alpha_i(P) > t_{i+1}, \quad a_i(t, P) \leq s_i(t, P) \leq b_i(t, P), \quad (4.3.14)$$

Proof. Since $t_i > t_{i+1}, x_i \leq x_{i+1}$, hence

$$\frac{x_i - A}{T - t_i} > \frac{x_i - A}{T - t_{i+1}}, \quad \frac{x_i - A}{T - t_{i+1}} \leq \frac{x_{i+1} - A}{T - t_{i+1}}.$$

This implies (4.3.13). From strict convexity of f and (4.3.13), we have

$$f'(a_i(P)) > \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)} > f'(b_i(P)),$$

hence $t_i > \alpha_i(P) > t_{i+1}$ and for all $t < T$, $a_i(t, P) \leq s_i(t, P) \leq b_i(t, P)$. This proves the Lemma.

Let $\Omega_i(P) = \{(x, t) : a_i(t, P) < x < a_{i+1}(t, P), t_{i+1} < t < T\}$. In view of Lemma 4.3.3, let $u_i(x, t, P)$ be a solution of (4.1.2) in $\Omega_i(P)$ defined by

$$u_i(x, t, P) = \begin{cases} a_i(P) & \text{if } a_i(t, P) < x < s_i(t, P), \\ b_i(P) & \text{if } s_i(t, P) < x < b_i(t, P), \\ (f')^{-1}\left(\frac{x-A}{T-t_{i+1}}\right) & \text{if } b_i(t, P) < x < a_{i+1}(t, P). \end{cases} \quad (4.3.15)$$

Then

$$u_{i+1}(a_{i+1}(t, P)+, t, P) = a_{i+1}(P) = u_i(a_{i+1}(t, P)-, t, P). \quad (4.3.16)$$

Also $a_n(P)$ and $a_n(t, P)$ are independent of P and denote by $a_n, a_n(t)$. Then from (4.3.16) it follows that $u_{n-1}(a_n(t)-, t, P) = a_n$. Therefore define the solution $u(x, t, P)$ of (4.1.2)

in $\Omega = (A, \infty) \times (\delta, T)$ by

$$u(x, t, P) = \begin{cases} u_i(x, t, P) & \text{if } (x, t) \in \Omega_i(P), 0 < i \leq n-1, \\ a_n & \text{if } x > a_n(t), \delta < t < T, \end{cases} \quad (4.3.17)$$

and $u(x, t, P)$ takes the boundary value $b(t, P)$ and initial value a_n given by

$$u(A, t, P) = b(t, P) = \begin{cases} \theta_f & \text{if } t_1 < t < T, \\ a_i(P) & \text{if } \alpha_i(P) < t < t_i, \\ b_i(P) & \text{if } t_{i+1} < t < \alpha_i(P). \end{cases} \quad (4.3.18)$$

$$u(x, \delta, P) = u_0(x, P) = a_n = (f)^{-1} \left(\frac{C - A}{T - \delta} \right) \quad \text{if } x \in (A, \infty). \quad (4.3.19)$$

Further more at $t = T$, and $x \in (A, C)$, u satisfies

$$f'(u(x, T, P)) = \sum_{i=1}^n \chi_{[x_i, x_{i+1})}(x) \left(\frac{x - A}{T - t_i} \right). \quad (4.3.20)$$

Next we calculate the L^∞ and TV bounds of the boundary value $b(\cdot, P)$.

$$\begin{aligned} |f'(b(t, P))| &= \max_{1 \leq i \leq n} (|f'(a_i(P))|, |f'(b_i(P))|) \\ &= \max_{1 \leq i \leq n} \left(\left| \frac{x_i - A}{T - t_i} \right|, \left| \frac{x_i - A}{T - t_{i+1}} \right| \right) \\ &= \max_{1 \leq i \leq n} \left(\left| \frac{x_i - A}{T - t_i} \right| \right). \end{aligned} \quad (4.3.21)$$

$$\begin{aligned} TV(f'(b(\cdot, P))) &= \sum_{i=1}^{n-1} |f'(a_i(P)) - f'(b_i(P))| + \sum_{i=0}^{n-1} |f'(b_i(P)) - f'(a_{i+1}(P))| \\ &= \sum_{i=1}^{n-1} \left| \frac{x_i - A}{T - t_i} - \frac{x_i - A}{T - t_{i+1}} \right| + \sum_{i=1}^{n-1} \left| \frac{x_i - A}{T - t_{i+1}} - \frac{x_{i+1} - A}{T - t_{i+1}} \right| + \left| \frac{x_1 - A}{T - t_1} \right| \\ &= \sum_{i=1}^{n-1} \frac{(x_i - A)(t_i - t_{i+1})}{(T - t_i)(T - t_{i+1})} + \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)}{(T - t_{i+1})} + \left| \frac{x_1 - A}{T - t_1} \right| \\ &\leq \left(\frac{T - \delta}{T - t_1} \right) \max_{1 \leq i \leq n} \left| \frac{x_i - A}{T - t_i} \right| + \left(\frac{C - A}{T - t_1} \right). \end{aligned} \quad (4.3.22)$$

Analysis of Discretization and Convergence: Let $\rho : [A, C] \rightarrow [\delta, T]$ be a non increasing right continuous function. Then it follows that $\{x : \rho(x) \leq t\}$ is a closed interval for any t . Let $0 < \epsilon < C - A$, define

$$\rho_\epsilon(x) = \min\{\rho(x), \rho(A + \epsilon)\}.$$

Then ρ_ϵ is a non-increasing right continuous function. Let m, n be non negative integers and let $T = t_0 > t_1 = \rho(A + \epsilon) > t_2 > \dots > t_n = \delta$ be such that $|t_i - t_{i+1}| \leq \frac{1}{m}$ for all $i \geq 1$. Let $k \leq n - 1$ such that $\{x : \rho_\epsilon(x) \leq t_{k+1}\} = \phi, \{x : \rho(x) \leq t_k\} \neq \phi$ and define

$\{x_i\}$ by $x_i = C$ if $i \geq k + 1$ and for $1 \leq i \leq k$,

$$\{x : \rho_\varepsilon(x) > t_i\} = (x_i, C).$$

Denote $P_{n,m,\varepsilon}$ by $P_{m,n,\varepsilon} = \{t_0, t_1, \dots, t_n, x_0, x_1, \dots, x_n\}$ the partition depending on n, m and ε . Associate to $P_{m,n,\varepsilon}$ define

$$\rho(x, P_{m,n,\varepsilon}) = \sum_{i=1}^{n-1} t_i \chi_{[x_{i-1}, x_i)}(x) + t_n \chi_{[x_{n-1}, x_n]}(x). \quad (4.3.23)$$

Then it follows from definition,

$$\sup_n |\rho_\varepsilon(x) - \rho(x)| \leq \sup_{A < x < \varepsilon} |\rho(x) - \rho(A + \varepsilon)| \quad (4.3.24)$$

$$\sup_n |\rho_\varepsilon(x) - \rho(x, P_{m,n,\varepsilon})| \leq \frac{1}{m}. \quad (4.3.25)$$

Definition: Let $\varepsilon_2 < \varepsilon_1, n_2 \geq n_1$. For $i = 1, 2$, let $P_{m,n_i,\varepsilon_i} = \{t_0, t_{1,i}, \dots, t_{n_i,i}, x_0, x_{1,i}, \dots, x_{n_i,i}\}$ be the partitions. Then we say P_{m,n_2,ε_2} dominates P_{m,n_1,ε_1} and is denoted by $P_{m,n_2,\varepsilon_2} \geq P_{m,n_1,\varepsilon_1}$ if for $1 \leq j \leq n_1$

$$\begin{aligned} t_{j,1} &= t_{n_2-n_1+j,2}, \\ x_{j,1} &= x_{n_2-n_1+j,2}. \end{aligned} \quad (4.3.26)$$

For a partition $P_{m,n,\varepsilon}$, define $\Omega(P_{m,n,\varepsilon})$ by

$$\Omega(P_{m,n,\varepsilon}) = \{(x, t) : a_1(t, P_{m,n,\varepsilon}) < x, \delta < t < T\}. \quad (4.3.27)$$

Properties of the domination: Let $\varepsilon_2 < \varepsilon_1, n_2 \geq n_1$ and let for $i = 1, 2, u_i(x, t) = u(x, t, P_{m,n_i,\varepsilon_i}), b_i(t) = b(t, P_{m,n_i,\varepsilon_i})$ as in (4.3.17) and (4.3.18) respectively. Let $P_{m,n_2,\varepsilon_2} \geq P_{m,n_1,\varepsilon_1}$, then from the construction it follows

$$\rho_{\varepsilon_1}(x) = \rho_{\varepsilon_2}(x) \quad \text{if } x \geq \varepsilon_1 + A, \quad (4.3.28)$$

$$u_1(x, t) = u_2(x, t) \quad \text{if } (x, t) \in \Omega(P_{m,n,\varepsilon_1}), \quad (4.3.29)$$

$$b_1(t) = b_2(t) \quad \text{if } \delta < t \leq \rho(A + \varepsilon_1), \quad (4.3.30)$$

$$f'(u_i(x, T)) = \frac{x - A}{T - \rho(x, P_{m,n_i,\varepsilon_i})}, \quad i = 1, 2. \quad (4.3.31)$$

$$\begin{aligned}
\int_{\delta}^T |f'(b_1(t)) - f'(b_2(t))| dt &= \int_{\rho(\epsilon_1+A)}^{\rho(\epsilon_2+A)} |f'(b_2(t)) - f'(b_1(t))| dt \\
&\leq |\rho(A + \epsilon_1) - \rho(A + \epsilon_2)| \max_j \left\{ \left| \frac{x_{j,2}-A}{T-t_{j,2}} \right| \right\} \\
&= |\rho(A + \epsilon_1) - \rho(A + \epsilon_2)| \max_j \left\{ \left| \frac{x_{j,2}-A}{T-\rho_{\epsilon_2}(x_{j,2})} \right| \right\}.
\end{aligned} \tag{4.3.32}$$

Construction of dominations: Let $\epsilon_2 < \epsilon_1$ and $P_{m,n_1,\epsilon_1} = \{t_0, t_{1,1}, \dots, t_{n_1,1}, x_0, x_{1,1}, \dots, x_{n_1,1}\}$. Now choose $\rho(\epsilon_2 + A) = t_{1,2} > t_{2,2} > \dots, t_{r_2,2} = t_{11} = \rho(\epsilon_1 + A)$ such that $|t_{i,2} - t_{i+1,2}| \leq \frac{1}{m}$ for $1 \leq i \leq r_2 - 1$. Let $n_2 = n_1 + n_2$ and define $t_{i,2}$ for $i \geq r_2$ by

$$t_{i,2} = t_{i-r_2+1,1},$$

and $\{x_{i,2}\}$ be associated to $\{t_{i,2}\}$. Let $n_2 = r_2 + n_1 - 1$ and $P_{m,n_2,\epsilon_2} = \{t_0, t_{1,2}, \dots, t_{n_2,2}, x_0, x_{1,2}, \dots, x_{n_2,2}\}$, then $P_{m,n_2,\epsilon_2} \geq P_{m,n_1,\epsilon_1}$.

Let $0 < \epsilon_{i+1} < \epsilon_i < C - A$, $\lim_{i \rightarrow \infty} \epsilon_i = 0$. Let $m \geq 1$ and $\{P_{m,n_1,\epsilon_1}\}_m$ be a partition corresponding to ρ_{ϵ_1} . From the above construction, extend this partition to $\{P_{m,n_2,\epsilon_2}\}_m$ to ρ_{ϵ_2} such that $P_{m,n_2,\epsilon_2} \geq P_{m,n_1,\epsilon_1}$. By induction there exist partitions $\{P_{m,n_j,\epsilon_j}\}_m$ of ρ_{ϵ_j} such that

$$P_{m,n_j,\epsilon_j} \geq P_{m,n_{j-1},\epsilon_{j-1}}. \tag{4.3.33}$$

Denote $P_{m,n_j,\epsilon_j} = \{t_0, t_{1,m,j}, \dots, t_{n_j,m,j}, x_0, x_{1,m,j}, \dots, x_{n_j,m,j}\}$. Since $\rho_{\epsilon_j} \leq \rho$ and hence

$$\left| \frac{x - A}{T - \rho_{\epsilon_j}(x)} \right| \leq \left| \frac{x - A}{T - \rho(x)} \right|,$$

and

$$\left| \frac{x_{k,m,j} - A}{T - t_{k,m,j}} \right| = \left| \frac{x_{k,m,j} - A}{T - \rho_{\epsilon_j}(x_{k,m,j})} \right| \leq \max_x \left| \frac{x - A}{T - \rho(x)} \right|. \tag{4.3.34}$$

Assume that ρ satisfies (4.2.7). Then from (4.3.34)

$$\max_{k \leq n_j} \left\{ \left| \frac{x_{k,m,j} - A}{T - t_{j,m,j}} \right| \right\} \leq \wedge. \tag{4.3.35}$$

For each m, j , let

$$u_{m,j}(x, t) = u(x, t, P_{m,n_j,\epsilon_j}),$$

$$b_{m,j}(t) = b(t, P_{m,n_j,\epsilon_j}),$$

where u and b are given in (4.3.17) and (4.3.18) respectively. From (4.3.21), (4.3.22) and (4.3.35) we have for all m, j

$$\left| f'(b_{m,j}(t)) \right| \leq \wedge. \tag{4.3.36}$$

$$TV(f'(b_{m,j})) \leq \left(\frac{T - \delta}{T - \rho(\epsilon_j)} \right) \wedge + \frac{C - A}{T - \rho(\epsilon_j + A)}. \tag{4.3.37}$$

Let $j > k$, then from (4.3.32)

$$\int_{\delta}^T \left| f'(b_{m,j}(t)) - f'(b_{m,k}(t)) \right| dt \leq \wedge |\rho(\epsilon_j + A) - \rho(\epsilon_k + A)|. \quad (4.3.38)$$

Under the above notations we have

Proof of Lemma 4.2.1 Let ρ satisfies (4.2.7), then for $\rho(\epsilon_j + A) < T$ and from (4.3.36), (4.3.37), for each j , $\{f'(b_{m,j})\}_{m \in \mathbb{N}}$ is bounded in total variation norm. Therefore from super linearity of f , $\{b_{m,j}\}_{m \in \mathbb{N}}$ is uniformly bounded in L^∞ for all j, m . Hence from Helly's theorem and Cantors diagonalization, we can extract a subsequence still denoted by $\{b_{m,j}\}$ such that for every j , $f'(b_{m,j}) \rightarrow f'(b_j)$ as $m \rightarrow \infty$ in L^1 and for *a.e.* t . Since $(f')^{-1}$ exists and hence $b_{m,j} \rightarrow b_j$ *a.e.* t and by dominated convergence Theorem, $b_{m,j} \rightarrow b_j$ in L^1 . Let $\rho_{m,j}(x) = \rho(x, P_{m,n_j,\epsilon_j})$, then from (4.3.25) $\rho_{m,j}(x) \rightarrow \rho_{\epsilon_j}(x)$ uniformly. Since $f'(u_{m,j}(x, \delta)) = \frac{C-A}{T-\delta}$, hence by L^1_{loc} contraction, $u_{m,j}$ converges in L^1_{loc} and for *a.e.* (x, t) to a solution u_j of (4.1.2) with initial boundary condition

$$u_j(A, t) = b_j(t) \quad (4.3.39)$$

$$f'(u_j(x, \delta)) = \frac{C - A}{T - \delta}. \quad (4.3.40)$$

From (4.3.20) , (4.3.23) and (4.3.25), for *a.e.* $x \in (A, C)$

$$f'(u_j(x, T)) = \frac{x - A}{T - \rho_{\epsilon_j}(x)}. \quad (4.3.41)$$

Letting $m \rightarrow \infty$ in (4.3.38) to obtain

$$\int_{\delta}^T \left| f'(b_j(t)) - f'(b_k(t)) \right| \leq \wedge |\rho(A + \epsilon_j) - \rho(A + \epsilon_k)|. \quad (4.3.42)$$

Since ρ is right continuous and hence $|\rho(A + \epsilon_j) - \rho(A + \epsilon_k)| \rightarrow 0$ as $j, k \rightarrow \infty$. Therefore from L^1_{loc} contractivity, there exists a subsequence still denoted by j such that $u_j \rightarrow \tilde{u}_1$, a solution of (4.1.2), $b_j \rightarrow \tilde{b}_1$ in L^1_{loc} and *a.e.* Letting $j \rightarrow \infty$ in (4.3.39) to (4.3.41), then $(\tilde{u}_1, \tilde{b}_1)$ satisfies (4.2.7) to (4.2.10). From Rankine-Hugoniot condition across $a_n(t, \delta)$, \tilde{u}_1 satisfies (4.2.10). This proves (1). Similarly (2) follows and hence the Lemma.

Proof of Lemma 4.3.2 is much simpler than that of Lemma 4.3.1 because of (4.3.9) and (4.3.10), TV bounds exist for discretization of ρ . Let $\rho : [C_1, C_2] \rightarrow [A_1, A_2]$ be in $IA([C_1, C_2])$. First assume that ρ is a strictly increasing continuous function. Let $n \geq 1$ and $A_1 = y_0 < y_1 < \dots < y_n = A_2$ such that $|y_i - y_{i+1}| \leq \frac{1}{n}$. Let k be such that

$\{x : \rho(x) < y_{k-1}\} = \phi$ and $\{x : \rho(x) < y_k\} \neq \phi$. Define $x_i = A_1$ if $i \leq k-1$ and for $i \geq k$,

$$\{x : \rho(x) < y_i\} = [A_1, x_i).$$

Let $P_n = \{y_0, y_1 \dots y_n, x_0, x_1 \dots x_n\}$ be the corresponding partition and define

$$\rho(x, P_n) = \sum_{i=0}^{n-2} y_i \chi_{[x_i, x_{i+1})}(x) + y_{n-1} \chi_{[x_{n-1}, C_2]}(x).$$

Clearly

$$|\rho(x) - \rho(x, P_n)| \leq \frac{1}{n},$$

and hence $\rho(x, P_n) \rightarrow \rho(x)$ uniformly as $n \rightarrow \infty$.

Proof of Lemma 4.2.2 First assume that ρ is a strictly increasing continuous function. For $n \geq 1$, let P_n and $\rho_n(x) = \rho(x, P_n)$ be constructed as above. Let $u_n(x, t) = u(x, t, P_n)$ as in (4.3.5) a solution of (4.1.2) with initial data $u_{0,n}(x) = u_0(x, \delta, P_n)$ as in (4.3.6). From (4.3.9), $\{f'(u_{0,n})\}$ is bounded in $BV_{loc}(\mathbb{R})$ and hence by Helly's theorem, there exists a subsequence still denoted by $\{f'(u_{0,n})\}$ converges to $f'(u_0)$ in L^1_{loc} and *a.e.* Since $u_{0,n}$ is uniformly bounded and f' is strictly increasing function, therefore $u_{0,n} \rightarrow u_0$ in L^1_{loc} . Hence from L^1_{loc} contractivity, u_n converges to \tilde{u} a solution of (4.1.2) *a.e.* $(x, t) \in \mathbb{R} \times (\delta, T)$ with initial data u_0 . Since $f'(a_0(P)) = \frac{C_1 - A_1}{T - \delta}$, $f'(a_n(P)) = \frac{C_2 - A_2}{T - \delta}$, hence if $\tilde{u}_0 = \tilde{u}_0|_{[A_1, A_2]}$, then from Rankine-Hugoniot condition across $a_0(t), a_n(t), (\tilde{u}, \tilde{u}_0)$ satisfies (4.2.18) to (4.2.20). At $t = T$,

$$f'(u_n(x, T)) = \frac{x - \rho_n(x)}{T - \delta}, \quad \text{if } x \in [C_1, C_2],$$

and hence letting $n \rightarrow \infty$, for *a.e.* $x \in (C_1, C_2)$,

$$f'(u(x, T)) = \frac{x - \rho(x)}{T - \delta}.$$

Then (\tilde{u}, \tilde{u}_0) is the required solution satisfying (4.2.17). Let $\rho \in IA((C_1, C_2))$ and ρ_n be a strictly increasing continuous function with values in (A_1, A_2) and converging to ρ in L^1 and *a.e.* Let $(\tilde{u}_n, \tilde{u}_{n,0})$ be the corresponding solutions satisfying (4.2.17) to (4.2.20). Hence from Helly's theorem, there exists a subsequence still denoted by $\tilde{u}_{n,0}$ converging to \tilde{u}_0 in L^1_{loc} and *a.e.* Therefore from L^1_{loc} contractivity, for a subsequence still denoted by \tilde{u}_n converging to \tilde{u} *a.e.* to a solution of (4.1.2) satisfying (4.2.18) to (4.2.20).

For *a.e.* $x \in (C_1, C_2)$ we have

$$f'(\tilde{u}(x, t)) = \lim_{n \rightarrow \infty} f'(\tilde{u}_n(x, T)) = \lim_{n \rightarrow \infty} \frac{x - \rho_n(x)}{T - \delta} = \frac{x - \rho(x)}{T - \delta}.$$

This proves the Lemma.

Remark 4.3.1. Given ρ , we have exhibited a method to construct an initial data \bar{u}_0 and the solution u such that at $t = T$

$$f'(u(x, T)) = \frac{x - \rho(x)}{T}. \quad (4.3.43)$$

This method is not unique. In fact we can construct infinitely many initial datas and all the solutions to these initial data satisfy (4.3.43). Here we illustrate this method with an example.

Example 4.3.4. Let $T > 0$ and $x_1 < x_2$, $y_1 < y_2$. Define

$$\rho(x) = \begin{cases} x - x_1 + Ty_1 & \text{if } x < x_1, \\ y_2 & \text{if } x_1 < x < x_2, \\ x - x_2 + Ty_2 & \text{if } x > x_2. \end{cases}$$

Let $f'(a_1) = \frac{x_1 - y_1}{T}$, $f'(b_1) = \frac{x_1 - y_2}{T}$, $f'(a_2) = \frac{x_2 - y_2}{T}$. By strict convexity, it follows that $b_1 < \min\{a_1, a_2\}$. Let $y_1 = \xi_1 < \xi_2 < \dots < \xi_n = y_2$ be a sequence and define $a_1 = c_1 < c_2 < \dots < c_n = b_1$ and $\{d_i\}$ by

$$f'(c_i) = \frac{x_1 - \xi_i}{T}, \quad f'(d_i) = \frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}.$$

By strict convexity $c_i < d_i < c_{i+1}$. For $0 \leq t \leq T$, let

$$\begin{aligned} \alpha_i(t) &= x_1 + f'(c_i)(t - T) \\ s_i(t) &= x_1 + f'(d_i)(t - T) \\ \beta(t) &= x_2 + f'(a_2)(t - T), \end{aligned}$$

then $\alpha_i(t) < s_i(t) < \alpha_{i+1}(t) < \beta(t)$ for $1 \leq i \leq n - 1$, $t \in (0, T)$. Let $s_i = s_i(0) = x_i - Tf'(d_i)$, then $\xi_i < s_i < \xi_{i+1}$. Now define u and \bar{u}_0 by (see figure 4.7)

$$\bar{u}_0 = \begin{cases} c_1 = a_1 & \text{if } x < s_1, \\ c_i & \text{if } \xi_i < x < s_i, \\ c_{i+1} & \text{if } s_i < x < \xi_{i+1}, \\ c_n = a_2 & \text{if } x > s_{n-1}, \end{cases}$$

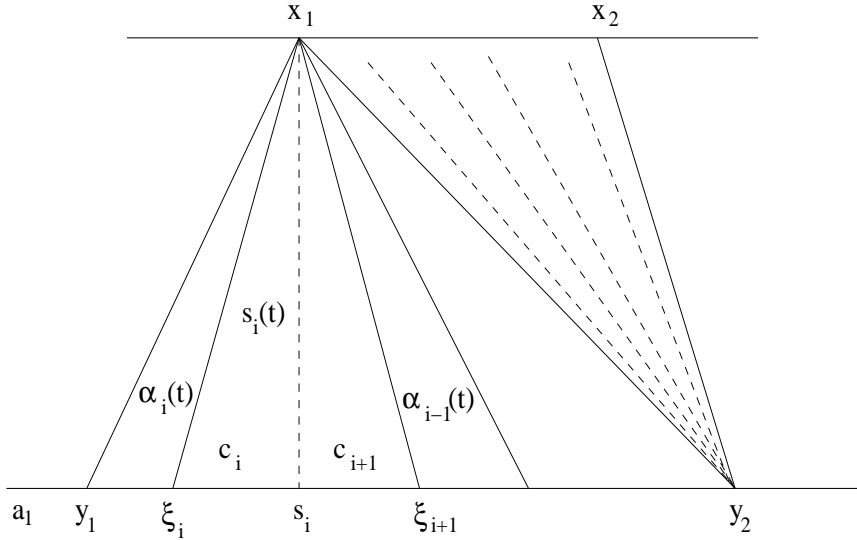


Fig. 4.7:

then the solution u with initial data \bar{u}_0 in $(0, T)$ is given by

$$u(x, t) = \begin{cases} c_1 & \text{if } x < s_1, \\ c_i & \text{if } \alpha_i(t) < x < s_i(t) \\ c_{i+1} & \text{if } s_i(t) < x < s_{i+1}(t) \\ c_n & \text{if } s_{n-1}(t) < x < \alpha_n \\ (f')^{-1} \left(\frac{x-y_2}{t} \right) & \text{if } \alpha_n(t) < x < \beta(t) \\ a_2 & \text{if } x > \beta(t). \end{cases}$$

Clearly u satisfies (4.3.43).

Since $\{\xi_i\}$ are arbitrary and hence there exist infinitely many solutions satisfying (4.3.43). In the above example, $s_i(t)$ are shock curves. In fact one can also introduce the backward rarefaction in the region $\alpha_i(t) < x < \alpha_{i+1}(t)$ by

$$u(x, t) = (f')^{-1} \left(\frac{x - x_1}{t - T} \right) \quad \text{for } \alpha_i(t) < x < \alpha_{i+1}(t).$$

4.4 Finer Analysis of Characteristics

In a beautiful paper, Dafermos [25] had extensively studied the properties of characteristic curves. Here we make a finer analysis of these characteristic curves and then use them to obtain our results. In order to do this, first we recollect the results of Lax-Olenik explicit formula and a good reference for this, is third chapter in [31].

Let $f^*(p) = \sup_q \{pq - f(q)\}$ denote the Legendre transform of f .

Then f^* is in C^1 , strictly convex, super linear growth and satisfies

$$\begin{aligned} f &= f^{**}, \\ f^{*'}(p) &= (f')^{-1}(p), \\ f^*(f'(p)) &= pf'(p) - f(p), \\ f(f^{*'}(p)) &= pf^{*'}(p) - f^*(p). \end{aligned} \tag{4.4.1}$$

Controlled Curves: Let $x \in \mathbb{R}, 0 \leq s < t$ and define the controlled curves $\Gamma(x, s, t)$ by

$$\Gamma(x, s, t) = \{r : [s, t] \rightarrow \mathbb{R}; r \text{ is linear and } r(t) = x\}, \tag{4.4.2}$$

and denote $\Gamma(x, t) = \Gamma(x, 0, t)$.

Value function: Let $u_0 \in L^\infty(\mathbb{R}), x_0 \in \mathbb{R}$, define

$$v_0(x) = \int_{x_0}^x u_0(\theta) d\theta, \tag{4.4.3}$$

be its primitive. Define the value function $v(x, t)$ by

$$\begin{aligned} v(x, t) &= \min_{r \in \Gamma(x, t)} \left\{ v_0(r(0)) + tf^* \left(\frac{x-r(0)}{t} \right) \right\} \\ &= \min_{\beta \in \mathbb{R}} \left\{ v_0(\beta) + tf^* \left(\frac{x-\beta}{t} \right) \right\}. \end{aligned} \tag{4.4.4}$$

Then v satisfies the

Dynamic Programming principle: For $0 \leq s < t$,

$$v(x, t) = \min_{r \in \Gamma(x, s, t)} \left\{ v(r(s), s) + (t-s)f^* \left(\frac{x-r(s)}{t-s} \right) \right\}. \tag{4.4.5}$$

Define the characteristic set $ch(x, s, t, u_0)$ and extreme characteristics $y_\pm(x, s, t, u_0)$ by

$$ch(x, s, t, u_0) = \{r \in \Gamma(x, s, t); r \text{ is a minimizer in (4.4.5)}\}, \tag{4.4.6}$$

$$y_-(x, s, t, u_0) = \min\{r(s) : r \in ch(x, s, t, u_0)\}, \tag{4.4.7}$$

$$y_+(x, s, t, u_0) = \max\{r(s); r \in ch(x, s, t, u_0)\}, \tag{4.4.8}$$

Denote $ch(x, t, u_0) = ch(x, 0, t, u_0)$, $y_\pm(x, t, u_0) = y_\pm(x, 0, t, u_0)$. Then we have the following result due to Hopf, Lax -Oleinik:

Theorem 4.4.1. *Let $0 \leq s < t, u_0, v_0, v$ be as above, then*

1. v is a uniformly Lipschitz continuous function and is a unique viscosity solution of

the Hamilton-Jacobi equation

$$\begin{aligned} v_t + f(v_x) &= 0 & (x, t) \in \mathbb{R} \times (0, \infty), \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \quad (4.4.9)$$

2. There exists $M > 0$, depending only on $\|u_0\|_\infty$ and Lipschitz constant of f restricted to the interval $[-\|u_0\|_\infty, \|u_0\|_\infty]$ such that for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $ch(x, s, t, u_0) \neq \emptyset$ and for $r \in ch(x, s, t, u_0)$

$$\left| \frac{x - r(s)}{t - s} \right| \leq M. \quad (4.4.10)$$

3. NIP (Non intersecting property of characteristics): Let $x_1 \neq x_2$, $t_1 > 0, t_2 > 0$ and for $i = 1, 2, r_i \in ch(x_i, s, t_i, u_0)$. Then $r_1(\theta) \neq r_2(\theta)$ for all $\theta \in (s, \min\{t_1, t_2\})$.

From NIP, it follows that for $0 \leq s < t$,

(a). $x \mapsto y_\pm(x, s, t, u_0)$ are non decreasing functions,

(b). At the points of continuity of y_+ ,

$$y_+(x, s, t, u_0) = y_-(x, s, t, u_0),$$

and hence $ch(x, s, t, u_0) = \{r\}$, where r is given by

$$r(\theta) = \frac{x - y_+(x, s, t, u_0)}{t - s}(\theta - t) + x.$$

- (c). Let $r \in ch(x, t, u_0)$, $z = r(s)$. Let $r_1(\theta) = r(\theta)$ for $0 \leq \theta \leq s$, $r_2(\theta) = r(\theta)$ for $s \leq \theta \leq t$. Then $r_1 \in ch(z, s, u_0)$, $r_2 \in ch(x, s, t, u_0)$.

4. Let $u(x, t) = \frac{\partial v}{\partial x}(x, t)$. Then u is the unique solution of (4.1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$ with initial data u_0 and satisfying

$$|u(x, t)| \leq \|u_0\|_\infty. \quad (4.4.11)$$

For a.e x , $y_-(x, t) = y_+(x, t)$ and u is given by

$$f'(u(x, t)) = \frac{x - y_+(x, t, u_0)}{t} = \frac{x - y_-(x, t, u_0)}{t}. \quad (4.4.12)$$

Let x be a point of differentiability of $y_\pm(x, t, u_0)$ and $y_\pm(x, t, u_0)$ is a point of differentiability of v_0 , then

$$u(x, t) = u_0(y_\pm(x, t, u_0)). \quad (4.4.13)$$

5. Let $u_0, w_0 \in L^\infty(\mathbb{R})$ and u, w be the solutions given in (4) with initial data u_0, w_0 respectively. Then

(a). *Monotonicity:* Let $u_0(x) \leq w_0(x)$ for $x \in \mathbb{R}$, there exists a set $N \subset \mathbb{R}$ of measure zero such that for each $t \notin N$, for a.e $x \in \mathbb{R}$,

$$u(x, t) \leq w(x, t). \quad (4.4.14)$$

(b). *L^1_{loc} contractivity:* Let $c = \max(\|u_0\|_\infty, \|w_0\|_\infty)$ and $I = [-c, c]$. Then there exists a $M > 0$, depending on Lipschitz constant f restricted to I such that for all $t > 0$, $a < b$,

$$\int_a^b |u(x, t) - w(x, t)| dx \leq \int_{a-Mt}^{b+Mt} |u_0(x) - w_0(x)| dx. \quad (4.4.15)$$

For the proofs of (1) to (4) see chapter (3) of [31] and for (5), see chapter (3) of [36].

In this sequel we follow the notations of characteristic curves as in [6]. From now onwards, we assume that $\Omega = \mathbb{R} \times (0, \infty)$, $u_0 \in L^\infty(\mathbb{R})$.

Left and right characteristic curves: Let $0 \leq s < t$, u be a solution of (4.1.2) with initial data u_0 and $\alpha \in \mathbb{R}$. Define the left characteristic curve $R_-(t, s, \alpha, u_0)$ and right characteristic curve $R_+(t, s, \alpha, u_0)$ and denote $R_\pm(t, \alpha, u_0) = R_\pm(t, 0, \alpha, u_0)$ by

$$R_-(t, s, \alpha, u_0) = \inf\{x; \alpha \leq y_-(x, s, t, u_0)\}, \quad (4.4.16)$$

$$R_+(t, s, \alpha, u_0) = \sup\{x : y_+(x, s, t, u_0) \leq \alpha\}. \quad (4.4.17)$$

In view of (4.4.10), $y_-(x, s, t, u_0) \rightarrow -\infty$ as $x \rightarrow -\infty$, $y_+(x, s, t, u_0) \rightarrow +\infty$ as $x \rightarrow +\infty$. Hence (4.4.16) and (4.4.17) are well defined. Our aim is to study the continuous dependence of R_\pm on their arguments (t, α, u_0) .

For $x, y, \in \mathbb{R}, t > 0$, let $r(\theta, t, x, y) \in \Gamma(x, t)$ be the line joining $(x, t), (y, 0)$ given by

$$r(\theta, t, x, y) = \left(\frac{x-y}{t} \right) (\theta - t) + x. \quad (4.4.18)$$

Observe that $r(0, t, x, y) = y$ and hence $r \in ch(x, t, u_0)$ if and only if y is a minimizer in (4.4.4). Hence define the extreme characteristic lines by

$$r_\pm(\theta, t, x) = r(\theta, t, y_\pm(x, t, u_0)). \quad (4.4.19)$$

Since $r_\pm(0, t, x) = y_\pm(x, t, u_0)$ and $y_-(x, t, u_0) \leq y_+(x, t, u_0)$, hence for all $\theta \in [0, t]$,

$$r_-(\theta, t, x) \leq r_+(\theta, t, x). \quad (4.4.20)$$

Then we have the following

Lemma 4.4.2. *Let $u_0, w_0, \{u_0^k\}$ are in $L^\infty(\mathbb{R})$ and $\alpha, \{\alpha_k\}$ are in \mathbb{R} . Let $v, W, \{v_k\}$ be the value functions defined in (4.4.4) with respect to the data $u_0, w_0, \{u_0^k\}$ respectively. Let $u = \frac{\partial v}{\partial x}, w = \frac{\partial W}{\partial x}, u_k = \frac{\partial v_k}{\partial x}$ be the solutions of (4.1.2). Then*

1. *Let $x_1 < x_2, 0 \leq s < t$ and $\beta \in \mathbb{R}$ be a minimizer for $v(x_1, t)$ and $v(x_2, t)$ in (4.4.5). Then for $x_1 < x < x_2$, β is the unique minimizer for $v(x, t)$ and satisfies*

$$f'(u(x, t)) = \frac{x - \beta}{t - s}. \quad (4.4.21)$$

2. *Let $x_k \in \mathbb{R}, r_k \in ch(x_k, t, u_0)$ such that $\lim_{k \rightarrow \infty} (x_k, r_k(0)) = (x, \beta)$. Then $r(\cdot, t, x, \beta) \in ch(x, t, u_0)$. Furthermore*

$$\lim_{x_k \uparrow x} y_+(x_k, t, u_0) = y_-(x, t, u_0), \quad (4.4.22)$$

$$\lim_{x_k \downarrow x} y_-(x_k, t, u_0) = y_+(x, t, u_0). \quad (4.4.23)$$

In particular, y_- is left continuous and y_+ is right continuous.

3. (i). *For all $t > 0$,*

$$R_-(t, \alpha, u_0) \leq R_+(t, \alpha, u_0), \quad (4.4.24)$$

$$\begin{cases} y_-(R_-(t, \alpha, u_0), t, u_0) \leq \alpha \leq y_+(R_-(t, \alpha, u_0), t, u_0), \\ y_-(R_+(t, \alpha, u_0), t, u_0) \leq \alpha \leq y_+(R_+(t, \alpha, u_0), t, u_0). \end{cases} \quad (4.4.25)$$

Further more if $R_-(t, \alpha, u_0) < R_+(t, \alpha, u_0)$, then for all $x \in (R_-(t, \alpha, u_0), R_+(t, \alpha, u_0))$ (see Figure 4.8)

$$y_\pm(x, t, \alpha) = \alpha, f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (4.4.26)$$

- (ii). *Let $0 < s < t$, then*

$$R_-(t, s, \alpha, u_0) = R_+(t, s, \alpha, u_0). \quad (4.4.27)$$

4. *Let $0 \leq s < t$. Then $t \mapsto R_\pm(t, \alpha, u_0)$ are Lipschitz continuous function with Lipschitz norm depends only on α and $\|u_0\|_\infty$ and satisfying*

$$\lim_{t \rightarrow 0} R_\pm(t, \alpha, u_0) = \alpha, \quad (4.4.28)$$

$$R_\pm(t, \alpha, u_0) = R_\pm(t, s, R_\pm(s, \alpha, u_0), u_0). \quad (4.4.29)$$

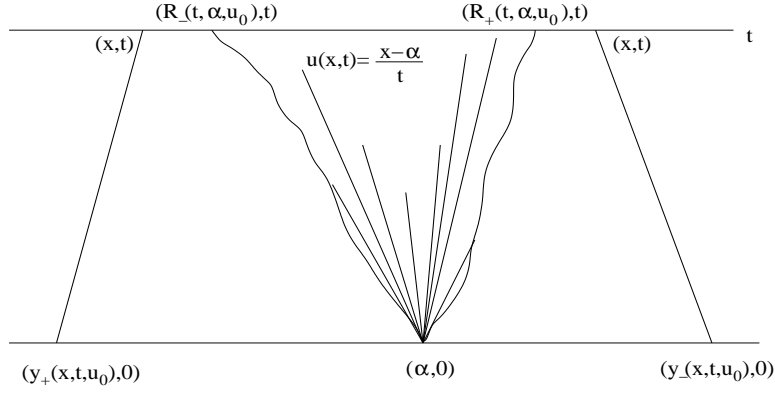


Fig. 4.8:

5. *Monotonicity: Let $u_0 \leq w_0, \alpha \leq \beta$, then*

$$R_{\pm}(t, \alpha, u_0) \leq R_{\pm}(t, \alpha, w_0), \quad (4.4.30)$$

$$R_{\pm}(t, \alpha, u_0) \leq R_{\pm}(t, \beta, u_0). \quad (4.4.31)$$

6. *Continuity with respect to data: Let $\{u_0^k\}$ be bounded in $L^\infty(\mathbb{R})$. Let $\alpha_k \rightarrow \alpha, u_0^k \rightarrow u_0$ in $L^1_{loc}(\mathbb{R})$. Then for $t > 0$.*

(a). *Suppose for all $k, R_-(t, \alpha_k, u_0^k) \leq R_-(t, \alpha, u_0)$, then*

$$\lim_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k) = R_-(t, \alpha, u_0). \quad (4.4.32)$$

(b). *Suppose for all $k, R_+(t, \alpha_k, u_0^k) \geq R_+(t, \alpha, u_0)$, then*

$$\lim_{k \rightarrow \infty} R_+(t, \alpha_k, u_0^k) = R_+(t, \alpha, u_0). \quad (4.4.33)$$

(c). *Suppose $R_-(t, \alpha, u_0) < \bar{R} = \bar{\lim}_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k)$, then for all $x \in (R_-(t, \alpha, u_0), \bar{R}), y_{\pm}(x, t, u_0) = \alpha$ and*

$$f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (4.4.34)$$

(d). *Suppose $\lim_{k \rightarrow \infty} R_+(t, \alpha_k, u_0^k) = \bar{R} < R_+(t, \alpha, u_0)$, then for all $x \in (\bar{R}, R_+(t, \alpha, u_0)), y_{\pm}(x, t, u_0) = \alpha$ and*

$$f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (4.4.35)$$

As an immediate consequence of this, if $R_-(t, \alpha, u_0) = R_+(t, \alpha, u_0)$ for $t > 0$, then $R_{\pm}(t, \alpha, u_0)$ is continuous at (α, u_0) .

Proof. (1). Let $x \in (x_1, x_2)$ and $r \in ch(x, s, t, u_0)$. Suppose $r(s) \neq \beta$. then r inter-sets one of the characteristics $\left(\frac{x_i - \beta}{t - s}\right) (\theta - t) + x_i$, $i = 1, 2$, which contradicts NIP of Theorem 4.2.1 Hence $\beta = r(s) = y_{\pm}(x, s, t, u_0)$. Furthermore

$$v(x, t) = v(\beta, s) + (t - s)f^* \left(\frac{x - \beta}{t - s} \right),$$

and for a.e x ,

$$u(x, t) = \frac{\partial v}{\partial x} = f^{*'} \left(\frac{x - \beta}{t - s} \right) = (f')^{-1} \left(\frac{x - \beta}{t - s} \right).$$

This proves (1).

(2). From the continuity of v and f^* , we have

$$\begin{aligned} v(x, t) &= \lim_{k \rightarrow \infty} v(x_k, t) \\ &= \lim_{k \rightarrow \infty} \left\{ v_0(r_k(0)) + t f^* \left(\frac{x_k - r_k(0)}{t} \right) \right\} \\ &= v_0(\beta) + t f^* \left(\frac{x - \beta}{t} \right), \end{aligned}$$

and hence $r(\cdot, t, x, \beta) \in ch(x, t, u_0)$. Let $x_1 < x_2$, then from NIP, $y_+(x_1, t, u_0) \leq y_-(x_2, t, u_0)$. From monotonicity of y_{\pm} , we have

$$y_-(x_1, t, u_0) \leq y_+(x_1, t, u_0) \leq y_-(x_2, t, u_0) \leq y_+(x_2, t, u_0).$$

Let $x_k \uparrow x$, then from above inequality,

$$\beta = \lim_{k \rightarrow \infty} y_+(x_k, t, u_0) \leq y_-(x, t, u_0).$$

Since a subsequence of $y_+(x_k, t, u_0)$ converges to β , hence $r(\cdot, t, x, \beta) \in ch(x, t, u_0)$. Therefore $\beta \leq y_-(x, t, u_0) \leq r(0, t, x, \beta) = \beta$. This proves (4.4.22). Similarly (4.4.23) follows. This proves (2).

(3). (i). Suppose $y_-(R_-(t, \alpha, u_0), t, u_0) > \alpha$. Then from (4.4.22) there exists $x_0 < R_-(t, \alpha, u_0)$. such that for all $x \in (x_0, R_-(t, \alpha, u_0))$, $y_+(x, t, u_0) > \alpha$. Let x be a point of continuity of y_+ , then from (3) of theorem 4.2.1, $y_-(x, t, u_0) = y_+(x, t, u_0) > \alpha$ and hence $R_-(t, \alpha, u_0) \leq x < R_-(t, \alpha, u_0)$ which is a contradiction. Suppose $y_+(R_-(t, \alpha, u_0), t, u_0) < \alpha$, again from (4.4.23) there exists $x_0 > R_-(t, \alpha, u_0)$ such that for all $x \in (R_-(t, \alpha, u_0), x_0)$, $y_-(x, t, u_0) < \alpha$. Therefore at points x of continuity, $\alpha \leq y_+(x, t, u_0) = y_-(x, t, u_0) < \alpha$, which is a contradiction. This proves (4.4.25) and (4.4.26) follows similarly.

Suppose $R_+(t, \alpha, u_0) < R_-(t, \alpha, u_0)$, then from (4.4.25), $y_-(R_-(t, \alpha, u_0), t, u_0) > \alpha$, therefore from NIP, $y_-(R_-(t, \alpha, u_0), t, u_0) = \alpha = y_+(R_+(t, \alpha, u_0), t, u_0)$. Hence from (4.4.21), for all $x \in (R_+(t, \alpha, u_0), R_-(t, \alpha, u_0))$, α is a minimizer for $v(x, t)$ which implies that $R_-(t, \alpha, u_0) \leq x < R_-(t, \alpha, u_0)$ which is a contradiction. This proves (4.4.24).

Suppose $R_-(t, \alpha, u_0) < R_+(t, \alpha, u_0)$. then from (4.4.24), (4.4.25), we have

$$\alpha \leq y_+(R_-(t, \alpha, u_0), t, u_0) \leq y_-(R_+(t, \alpha, u_0), t, u_0) \leq \alpha.$$

Therefore from (1), for all $x \in (R_-(t, \alpha, u_0), R_+(t, \alpha, u_0))$, $y_{\pm}(x, t, u_0) = \alpha$ and $f'(u(x, t)) = \frac{x-\alpha}{t}$. This proves (4.4.26).

(3). (ii). Let $0 < s < t$, then as in (4.4.24) we have $R_-(t, s, \alpha, u_0) \leq R_+(t, s, \alpha, u_0)$. Suppose $R_-(t, s, \alpha, u_0) < R_+(t, s, \alpha, u_0)$, then as in (4.4.26), we have for all $x \in (R_-(t, s, \alpha, u_0), R_+(t, s, \alpha, u_0))$, $f'(u(x, t)) = \frac{x-\alpha}{t-s}$. Let $R_-(t, s, \alpha, u_0) < x_1 < x_2 < R_+(t, s, \alpha, u_0)$ and $r_{\pm}(\cdot, t, x_1), r_{\pm}(\cdot, t, x_2)$ be the extreme characteristics at x_1, x_2 . Since $r_{\pm}(s, t, x_1) = r_{\pm}(s, t, x_2) = \alpha$, which contradicts NIP. This proves (ii) and hence (3).

(4). Let $0 \leq s < t, R_- = R_-(t, \alpha, u_0), y_{\pm} = y_{\pm}(R_-, t, u_0)$ and $r_{\pm}(\theta) = r(\theta, t, R_-, y_{\pm}) \in ch(R_-, t, u_0)$. Then from (3) of theorem 4.2.1, $r_{\pm}|_{(0,s)} \in ch(r_{\pm}(s), s, u_0)$.

Claim : $r_-(s) \leq R_-(s, \alpha, u_0) \leq r_+(s)$.

Suppose $R_-(s, \alpha, u_0) < r_-(s)$. For $x \in (R_-(s, \alpha, u_0), r_-(s))$, $y_-(x, s, \alpha) \geq \alpha$. Hence if $y_- < \alpha$ or $y_-(x, s, \alpha) > \alpha$, then the characteristics $r_-(\theta), r_-(\theta, s, x)$ intersect for some $\theta \in (0, s)$ which contradicts NIP. Therefore $\alpha = y_- = y_-(x, s, \alpha)$ and from (2) $\tilde{r}(\theta) = \tilde{r}(\theta, s, R_-(s, \alpha, u_0), \alpha) \in ch(R_-(s, \alpha, u_0), s, u_0)$. From (4.4.22) choose a $\xi < R_-$, $y_-(\xi, t, u_0) < \alpha$ such that the characteristic $\tilde{r}(\theta)$ and $r(\theta, t, \xi, y_+(\xi, t, u_0))$ intersect for some $\theta \in (0, s)$ which contradicts NIP.

Suppose $r_+(s) < R_-(s, \alpha, u_0)$, then for $x \in (r_+(s), R_-(s, \alpha, u_0))$, $y_-(x, s, u_0) < \alpha \leq r_+(0) = y_+$ and therefore the characteristic at (x, s) with end point $(y_-(x, s, u_0), 0)$ intersects $r_+(\theta)$ for some $\theta \in (0, s)$ contradicting NIP. This proves the claim.

From (4.4.10) and the claim, we have

$$R_- + \left(\frac{R_- - y_-}{t} \right) (s - t) \leq R_-(s, \alpha, u_0) \leq R_- + \left(\frac{R_- - y_+}{t} \right) (s - t)$$

that is

$$\begin{aligned} |R_- - R_-(s, \alpha, u_0)| &\leq \left(\left| \frac{R_- - y_-}{t} \right| + \left| \frac{R_- - y_+}{t} \right| \right) |s - t| \\ &\leq 2M|s - t|. \end{aligned}$$

Also from (4.4.10), we have $|R_- - y_{\pm}| = |R_- - r_{\pm}(0)| \leq Mt$, hence

$\lim_{t \rightarrow 0} R_-(t, \alpha, u_0) = \alpha$. Similarly for $R_+(t, \alpha, u_0)$.

From (c) of (3) in Theorem 4.2.1, we have $r_{\pm}|_{[s,t]} \in ch(R_-(t, \alpha, u_0), s, t, u_0)$, hence from NIP and from the above claim we have for any $x < R_-(t, \alpha, u_0) < z$, $y_+(x, s, t, u_0) < r_-(s) \leq R_-(s, \alpha, u_0) \leq r_+(s) < y_-(z, s, t, u_0)$. Therefore from the definitions it follows that $R_-(t, \alpha, u_0) = R_-(t, s, R_-(t, s, u_0), u_0)$. Similarly for R_+ and this proves (4).

(5). From (5) of Theorem 4.2.1, for $t \in N$, a.e. x , $u(x, t) \leq w(x, t)$. Let $y_{1,\pm}(x) =$

$y_{\pm}(x, t, u_0)$, $y_{2,\pm}(x) = y_{\pm}(x, t, w_0)$. Choose a dense set $D \subset \mathbb{R}$ such that for $i = 1, 2, x \in D$, $u(x, t) \leq w(x, t)$, $y_{i,+}(x) = y_{i,-}(x)$. Hence from (4.4.12) we have for $x \in D$,

$$\frac{x - y_{1,\pm}(x)}{t} = f'(u(x, t)) \leq f'(w(x, t)) = \frac{x - y_{2,\pm}(x)}{t}.$$

This implies $y_{2,\pm}(x) \leq y_{1,\pm}(x)$. Therefore from (4.4.22) and (4.4.23),

$$\begin{aligned} R_-(t, \alpha, u_0) &= \inf\{x \in D : y_{1,-}(x) \geq \alpha\} \\ &\leq \inf\{x \in D : y_{2,-}(x) \geq \alpha\} \\ &= R_-(t, \alpha, w_0). \end{aligned}$$

$$\begin{aligned} R_+(t, \alpha, u_0) &= \sup\{x \in D, y_{1,+}(x) \leq \alpha\} \\ &\leq \sup\{x \in D : y_{2,+}(x) \leq \alpha\} \\ &= R_+(t, \alpha, w_0). \end{aligned}$$

From (4), $t \mapsto (R_{\pm}(t, \alpha, u_0), R_{\pm}(t, \alpha, w_0))$ are continuous and hence (4.4.30) holds for all $t > 0$.

$$\begin{aligned} R_-(t, \alpha, u_0) &= \inf\{x : y_-(x, t, u_0) \geq \alpha\} \\ &\leq \inf\{x : y_-(x, t, u_0) \geq \beta\} \\ &= R_-(t, \beta, u_0), \end{aligned}$$

and similarly for R_+ . This proves (5).

(6). From L^1_{loc} contractivity, $u_k \rightarrow u$ in L^1_{loc} and hence for a.e. s , $u_k(\cdot, s) \rightarrow u(\cdot, s)$ in L^1_{loc} . Let $t > 0$ be such that for a subsequence still denoted by k such that for a.e. x

$$\lim_{k \rightarrow \infty} u_k(x, t) = u(x, t). \quad (4.4.36)$$

Let $y_{\pm}^k(x) = y_{\pm}(x, t, u_0^k)$, $R_{\pm}^k = R_{\pm}(t, \alpha_k, u_0^k)$. Since $\{y_{\pm}^k\}$ are monotone functions and $\{R_{\pm}^k\}$ are bounded. Hence from Helly's theorem, there exists a subsequence still denoted by k such that for a.e. x ,

$$\lim_{k \rightarrow \infty} y_{\pm}^k(x) = y_{\pm}(x) \quad (4.4.37)$$

$$\left(\lim_{k \rightarrow \infty} R_{\pm}^k, \lim_{k \rightarrow \infty} R_{\pm}^k \right) = \left(\bar{R}_{\pm}, \tilde{R}_{\pm} \right), \quad (4.4.38)$$

where u is the solution of (4.1.2) with $u(x, 0) = u_0(x)$. Let $D \subset \mathbb{R}$ be a dense set such

that for all $x \in D$, (4.4.36) to (4.4.38) holds and further for all k ,

$$y_+^k(x) = y_-^k(x) \quad (4.4.39)$$

$$y_+(x, t, u_0) = y_-(x, t, u_0) \quad (4.4.40)$$

$$f'(u_k(x, t)) = \frac{x - y_{\pm}^k(x)}{t} \quad (4.4.41)$$

$$f'(u(x, t)) = \frac{x - y_{\pm}(x, t, u_0)}{t}. \quad (4.4.42)$$

Hence from (4.4.37), (4.4.41) and (4.4.42), for $x \in D$,

$$y_{\pm}(x) = \lim_{k \rightarrow \infty} y_{\pm}^k(x) = y_{\pm}(x, t, u_0). \quad (4.4.43)$$

Case (i): Let for all k , $R_-^k \leq R_-(t, \alpha, u_0)$, then $\bar{R}_- \leq R_-(t, \alpha, u_0)$. Suppose $\bar{R}_- < R_-(t, \alpha, u_0)$. Let $I = (\bar{R}_-, R_-(t, \alpha, u_0))$, $x \in D \cap I$ and choose $k_0 = k_0(x) > 0$ such that for all $k \geq k_0$, $R_-^k < x$, then

$$\alpha = \lim_{k \rightarrow \infty} \alpha_k \leq \lim_{k \rightarrow \infty} y_-^k(x) = y_-(x, t, u_0) < \alpha,$$

which is a contradiction. Hence $\bar{R}_- = R_-(t, \alpha, u_0)$.

Case (ii) : Let for all k , $R_-(t, \alpha, u_0) \leq R_-^k$, then $R_-(t, \alpha, u_0) \leq \tilde{R}_-$. Suppose $R_-(t, \alpha, u_0) < \tilde{R}_-$, then for $x \in D \cap (R_-(t, \alpha, u_0), \tilde{R}_-)$ choose $k_0 = k_0(x)$ such that for a subsequence $k > k_0$, $x < R_-^k$. Hence $\alpha \leq y_-(x, t, u_0) = \lim_{k \rightarrow \infty} y_-^k(x) \leq \alpha$ and therefore $y_-(x, t, u_0) = \alpha$. Therefore from (4.4.12), $f'(u(x, t)) = \frac{x - \alpha}{t}$.

Since $\{u_0^k\}$ are bounded in L^∞ and hence from (4), there exists a $C > 0$ independent of k such that for all s_1, s_2 we have

$$\begin{aligned} |R_{\pm}(s_1, \alpha_k, u_0^k) - R_{\pm}(s_2, \alpha_k, u_0^k)| &\leq |s_1 - s_2| \\ |R_{\pm}(s_1, \alpha, u_0) - R_{\pm}(s_2, \alpha, u_0)| &\leq |s_1 - s_2| \end{aligned}$$

Now suppose for $t > 0$ and for a subsequence still denoted by k such that

$$R_- = \lim_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k) < R_-(t, \alpha, u_0).$$

Therefore choose $\epsilon > 0$, $k_0 > 0$ such that for all $k \geq k_0$

$$R_-(t, \alpha_k, u_0^k) < R_-(t, \alpha, u_0) - 2\epsilon.$$

Let $|s - t| \leq \frac{\epsilon}{2C}$, then from the above uniform estimates we have for $k \geq k_0$

$$\begin{aligned} R_-(s, \alpha_k, u_0^k) &\leq \frac{\epsilon}{2} + R_-(t, \alpha_k, u_0^k) \\ &\leq \frac{\epsilon}{2} + R_-(t, \alpha, u_0) - 2\epsilon \\ &\leq \frac{\epsilon}{2} + R_-(s, \alpha_k, u_0^k) + \frac{\epsilon}{2} - 2\epsilon \\ &\leq R_-(s, \alpha, u_0) - \epsilon < R_-(s, \alpha, u_0). \end{aligned}$$

Now choose an $|s_0 - t| < \frac{\epsilon}{2}$ such that the previous analysis holds. Then at s_0 , we have

$$R_-(s, \alpha, u_0) - \epsilon \geq \liminf_{k \rightarrow \infty} R_-(s, \alpha_k, u_0^k) = R_-(s, \alpha, u_0) < R_-$$

which is a contradiction. This proves (2.32) and similarly (2.33) holds.

Let $R_-(t, \alpha, u_0) < \bar{R} = \overline{\lim}_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k)$ and $R_-(t, \alpha, u_0) < x < \bar{R}$. Then as earlier choose an $\epsilon > 0$, a subsequence still denoted by k such that for $|s - t| < \frac{\epsilon}{2C}$ and $k \geq k_0(\epsilon)$, following holds :

$$R_-(s, \alpha, u_0) + \epsilon < x < \bar{R} - \epsilon \leq R_-(s, \alpha_k, u_0^k).$$

Now choose an $s > t$ such that $u_k(\xi, s) \rightarrow u(\xi, s)$ a.e. ξ . Hence from the previous analysis we have for all $\xi \in (R_-(s, \alpha, u_0), \bar{R} - \epsilon)$, $f'(u(\xi, s)) = \frac{\xi - \alpha}{s}$. Since $s > t$ and hence we have $f'(u(x, t)) = \frac{x - \alpha}{t}$. This proves (2.34) and similarly (2.35) follows. This proves (6) and hence the Lemma.

Next we study the characterization of R_{\pm} and some comparison properties. For this we need some well known results which will be proved in the following Lemma.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $B(1)$ denote the unit ball in \mathbb{R}^n . Let $0 \leq \chi \in C_c^\infty(B(1))$ with $\int_{\mathbb{R}^n} \chi(x) dx = 1$. Let $\epsilon > 0$ and denote $\chi_\epsilon(x) = \frac{1}{\epsilon^n} \chi(\frac{x}{\epsilon})$ be the usual mollifiers. Let $u_0 \in L_{loc}^1(\mathbb{R}^n)$ and define

$$u_0^\epsilon(x) = (\chi_\epsilon * u_0)(x) = \int_{B(1)} \chi(y) u_0(x - \epsilon y) dy ,$$

then

Lemma 4.4.3. *Denote ess inf and ess sup by inf and sup. Then*

1. *With the above notation, for $x \in \Omega$, there exists a $\epsilon_0 = \epsilon_0(x) > 0$ such that for all $0 < \epsilon < \epsilon_0$,*

$$\inf_{y \in \Omega} u_0(y) \leq u_0^\epsilon(x) \leq \sup_{y \in \Omega} u_0(y). \quad (4.4.44)$$

2. *Let $t_0, \epsilon_0, \alpha \in \mathbb{R}$ and $\omega \in L^\infty((0, t_0))$. Let $R : (0, t_0] \rightarrow \mathbb{R}$ be a locally Lipschitz*

continuous function such that for a.e $t \in (0, t_0)$,

$$\omega(t) \geq (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \varepsilon_0 \quad (4.4.45)$$

$$\frac{dR}{dt} = \frac{f(\omega(t)) - f((f')^{-1}(\frac{R(t)-\alpha}{t}))}{\omega(t) - (f')^{-1}(\frac{R(t)-\alpha}{t})}, \quad (4.4.46)$$

then

$$\overline{\lim}_{t \rightarrow 0} \left| \frac{R(t) - \alpha}{t} \right| = \infty. \quad (4.4.47)$$

Proof. (1). Let $\Omega_\varepsilon = \{x; d(x, \Omega^c) > \varepsilon\}$. Then for $x \in \Omega$, there exists an $\varepsilon_0 > 0$, such that $x \in \Omega_\varepsilon$, for all $\varepsilon < \varepsilon_0$. Hence $x - \varepsilon y \in \Omega$, for a.e $y \in B(1)$ a.e

$$\inf_{\xi \in \Omega} u_0(\xi) \leq u_0(x - \varepsilon y) \leq \sup_{\xi \in \Omega} u_0(\xi).$$

Multiply this identity by χ and integrate over $B(1)$ gives (4.4.44).

(2). Suppose (4.4.47) is not true. That is

$$\sup_{t > 0} \left| \frac{R(t) - \alpha}{t} \right| < \infty. \quad (4.4.48)$$

Let m be defined by

$$\begin{aligned} m &= \inf_{t \in (0, t_0)} \int_0^1 f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \right) \\ &\quad - \left(\frac{R(t) - \alpha}{t} \right) d\theta. \end{aligned} \quad (4.4.49)$$

Claim : $m > 0$.

From (4.4.45), $w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) > 0$ and hence by convexity we have

$$\begin{aligned} f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \right) &\geq f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \\ &= \frac{R(t) - \alpha}{t}. \end{aligned}$$

Hence $m \geq 0$. Suppose $m = 0$, then there exists a sequence $t_k \rightarrow \tilde{t}$ in $[0, 1]$ such that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \int_0^1 f' \left((f')^{-1} \left(\frac{R(t_k) - \alpha}{t_k} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t_k) - \alpha}{t_k} \right) \right) \right) \\ &\quad - \left(\frac{R(t_k) - \alpha}{t_k} \right) d\theta. \end{aligned}$$

Then from (4.4.48), we can choose a subsequence such that

$$\frac{R(t_k) - \alpha}{t_k} \rightarrow a, \quad w(t_k) \rightarrow b \quad \text{as } k \rightarrow \infty.$$

Then from (4.4.45) we have $b \geq (f')^{-1}(a) + \epsilon_0$ and

$$0 = \int_0^1 [f'((f')^{-1}(a) + \theta(b - (f')^{-1}(a))) - a] d\theta$$

and hence by strict convexity

$$0 < f'((f')^{-1}(a) + \theta(b - (f')^{-1}(a))) - a = 0$$

which is a contradiction. This proves the claim. From Taylor series and the claim we have

$$\begin{aligned} \frac{dR}{dt} &= \frac{R(t) - \alpha}{t} + \int_0^1 f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \right) \\ &\quad - \left(\frac{R(t) - \alpha}{t} \right) d\theta. \\ &\geq \frac{R(t) - \alpha}{t} + m\epsilon_0 \end{aligned}$$

or

$$t \frac{d}{dt} \left(\frac{R(t) - \alpha}{t} \right) \geq m\epsilon_0.$$

For $0 < t_1 < t_0$, integrating t to t_1 to obtain

$$\frac{R(t) - \alpha}{t} \leq \frac{R(t_1) - \alpha}{t_1} - m\epsilon_0 \log \frac{t_1}{t} \rightarrow -\infty \quad \text{as } t \rightarrow 0.$$

Lemma 4.4.4. *Let $T > 0, \alpha, \beta \in \mathbb{R}, u_0, v_0$ and v be as in (4.4.3) and (4.4.4). Then*

(1). *Let $x_0 \in \mathbb{R}, t > 0$ such that*

$$y_-(x_0, t, u_0) \leq \alpha \leq y_+(x_0, t, u_0), \quad (4.4.50)$$

then

- (i). *if $x_0 \leq R_-(t, \alpha, u_0)$, then $x_0 = R_-(t, \alpha, u_0)$. If $R_-(t, \alpha, u_0) < x_0$, then for all $x \in (R_-(t, \alpha, u_0), x_0)$, $f'(u(x, t)) = \frac{x - \alpha}{t}$.*
- (ii). *if $x_0 \geq R_+(t, \alpha, u_0)$, then $x_0 = R_+(t, \alpha, u_0)$. If $x_0 < R_+(t, \alpha, u_0)$, then for all $x \in (x_0, R_+(t, \alpha, u_0))$, $f'(u(x, t)) = \frac{x - \alpha}{t}$.*

(2). (i). Let $x \geq R_-(t, \alpha, u_0)$, then

$$v(x, t) = \inf_{y \geq \alpha} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}. \quad (4.4.51)$$

(ii). Let $x \leq R_+(t, \alpha, u_0)$, then

$$v(x, t) = \inf_{y \leq \alpha} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}. \quad (4.4.52)$$

(iii). Let $\alpha < \beta$ and for $0 < t < T$ assume that

$$R_+(t, \alpha, u_0) < R_-(t, \beta, u_0),$$

then for $R_+(t, \alpha, u_0) < x < R_-(t, \beta, u_0)$,

$$v(x, t) = \inf_{\alpha \leq y \leq \beta} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}, \quad (4.4.53)$$

$$m = \inf_{y \in [\alpha, \beta]} u_0(y) \leq u(x, t) \leq \sup_{y \in [\alpha, \beta]} u_0(y) = M. \quad (4.4.54)$$

$$f'(m) \leq \frac{x - y_+(x, t, u_0)}{t} \leq f'(M). \quad (4.4.55)$$

(3). Let $L(t, \alpha, u_0) \in \{R_\pm(t, \alpha, u_0)\}$, $R(t, \beta, u_0) \in \{R_\pm(t, \beta, u_0)\}$. Suppose at $t = T$,

$$L(T, \alpha, u_0) = R(T, \beta, u_0), \quad (4.4.56)$$

then for all $t \geq T$, (see Figure 9).

$$L(t, \alpha, u_0) = R(t, \beta, u_0). \quad (4.4.57)$$

Furthermore, let $\{u_0^k\}$ and u_0 are in $L^\infty(\mathbb{R})$ with $\sup_k \|u_0^k\|_\infty < \infty$. Let $(\alpha_k, \beta_k, u_0^k) \rightarrow (\alpha, \beta, u_0)$ as $k \rightarrow \infty$ in $\mathbb{R}^2 \times L_{loc}^1(\mathbb{R})$ and $T_k \rightarrow T$ in \mathbb{R} such that

$$\begin{aligned} R_-(T, \alpha, u_0) &= R_+(T, \beta, u_0) \\ R_-(T_k, \alpha_k, u_0^k) &= R_+(T_k, \beta_k, u_0^k). \end{aligned} \quad (4.4.58)$$

Then for $t > T$,

$$\begin{aligned}
\lim_{k \rightarrow \infty} R_+(t, \alpha_k, u_0^k) &= \lim_{k \rightarrow \infty} R_-(t, \beta_k, u_0^k) \\
&= R_+(t, \alpha, u_0) \\
&= R_-(t, \alpha, u_0).
\end{aligned} \tag{4.4.59}$$

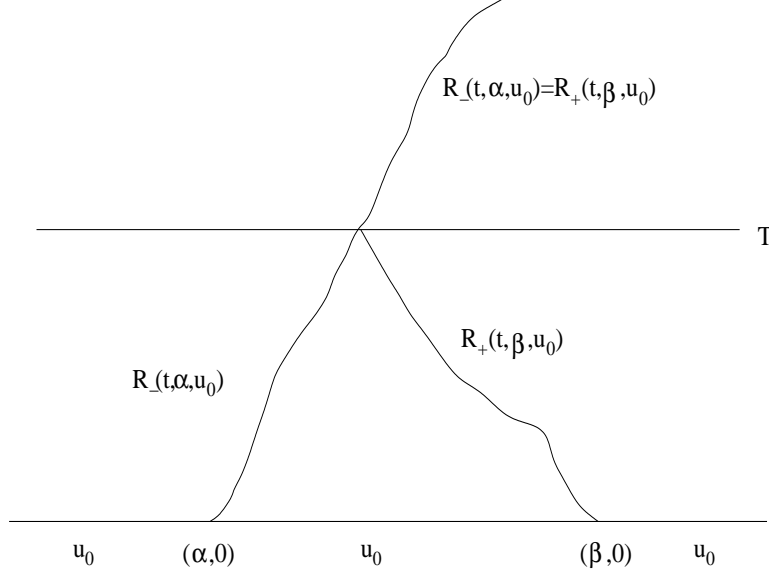


Fig. 4.9:

Proof. (1). It is enough to prove (i) and (ii) follows similarly. Let $C = R_-(t, \alpha, u_0)$, then from (4.4.22) $y_-(C, t, u_0) \leq \alpha$. Suppose $x_0 < C$, then from (4.4.50), the characteristic line joining $(C, t), (y_-(C, t, u_0), 0)$ and $(x_0, t), (y_+(x_0, t, u_0), 0)$ intersects if $y_+(x_0, t, u_0) > \alpha$ or $y_-(C, t, u_0) < \alpha$, which contradicts NIP. Hence $y_+(x_0, t, u_0) = y_-(C, t, u_0) = \alpha$. Therefore from (4.4.21), for $x_0 < x < C$, $f'(u(x, t)) = \frac{x-\alpha}{t}$. This implies that $C = R_-(t, \alpha, u_0) < x < C$, which is a contradiction. Hence $x_0 = R_-(t, \alpha, u_0)$. Suppose $C < x_0$, then from the definition and (4.4.50), we have $y_-(x_0, t, u_0) \leq \alpha \leq y_-(x_0, t, u_0)$ and hence $y_-(x_0, t, u_0) = \alpha$ and from (4.4.21), $f'(u(x, t)) = \frac{x-\alpha}{t}$ for all $C < x < x_0$. This proves (1).

(2). It is enough to prove (i) and (ii) follows similarly. Let $x \geq R_-(t, \alpha, u_0)$, then from (4.4.25), $y_+(x, t, u_0) \geq \alpha$. Therefore

$$\begin{aligned}
&\inf \left\{ \inf_{y \geq \alpha} \{v_0(y) + tf^* \left(\frac{x-y}{t} \right)\}, \inf_{y < \alpha} \{v_0(y) + tf^* \left(\frac{x-y}{t} \right)\} \right\} \\
&= v(x, t) \\
&= v_0(y_+(x, t, u_0)) + tf^* \left(\frac{x - y_+(x, t, u_0)}{t} \right).
\end{aligned}$$

Hence

$$v(x, t) = \inf_{y \geq \alpha} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}.$$

(iii). (4.4.53) follows from (4.4.51) and (4.4.52). Let $\varepsilon > 0$, $u_0^\varepsilon = \chi_\varepsilon * u_0$ and $v_0^\varepsilon, v_\varepsilon$ be as in (4.4.3), (4.4.4) respectively. Let $u^\varepsilon = \frac{\partial v^\varepsilon}{\partial x}$ be the solution of (4.1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$. Since v_0^ε is differentiable and hence for a.e x and from (4.4.13), $u^\varepsilon(x, t) = u_0^\varepsilon(y_+(x, t, u_0))$. Since $u_0^\varepsilon \rightarrow u_0$ in L^1_{loc} and hence $u^\varepsilon \rightarrow u$ in L^1_{loc} . Therefore from (4.4.32) to (4.4.35), we have for $0 < t < T$,

$$\overline{\lim}_{\varepsilon \rightarrow 0} R_+(t, \alpha, u_0^\varepsilon) \leq R_+(t, \alpha, u_0) < R_-(t, \beta, u_0) \leq \underline{\lim}_{\varepsilon \rightarrow 0} R_-(t, \beta, u_0^\varepsilon).$$

Let $\varepsilon_k \rightarrow 0$ and choose a dense set $D \subset (R_+(t, \alpha, u_0), R_-(t, \beta, u_0))$ such that for all $x \in D$,

$$\begin{aligned} \lim_{k \rightarrow \infty} u^{\varepsilon_k}(x, t) &= u(x, t) \\ y(x) &= y_+(x, t, u_0) = y_-(x, t, u_0) \\ y_k(x) &= y_+(x, t, u_0^{\varepsilon_k}) = y_-(x, t, u_0^{\varepsilon_k}). \end{aligned}$$

For $x \in D$, choose $k_0(x)$ such that for all $k \geq k_0(x)$, $x \in (R_+(t, \alpha, u_0^{\varepsilon_k}), R_-(t, \beta, u_0^{\varepsilon_k}))$. Then from ((4.4.53)), $y_k \in [\alpha, \beta]$. Since $u^{\varepsilon_k}(x, t) = u_0^{\varepsilon_k}(y_k(x))$, hence from (4.4.44),

$$m \leq u_0^{\varepsilon_k}(y_k(x)) = u^{\varepsilon_k}(x, t) \leq M.$$

Letting $k \rightarrow \infty$ to obtain (4.4.54). From (4.4.12),

$$f'(u^{\varepsilon_k}(x, t)) = \frac{x - y_k(x)}{t},$$

letting $k \rightarrow \infty$ to obtain

$$\begin{aligned} \frac{x - y(x)}{t} &= f'(u(x, t)) = \lim_{k \rightarrow \infty} f'(u^{\varepsilon_k}(x, t)) \\ &= \lim_{k \rightarrow \infty} \frac{x - y_k(x)}{t}. \end{aligned}$$

Hence $\lim_{k \rightarrow \infty} y_k(x) = y(x)$,

$$f'(m) \leq f'(u_0^{\varepsilon_k}(y_k(x))) = \frac{x - y_k(x)}{t} \leq f'(M),$$

Now letting $k \rightarrow \infty$ to obtain

$$f'(m) \leq \frac{x - y(x)}{t} \leq f'(M),$$

For $x \notin D$, choose $x_k \uparrow x$, $y_+(x_k, t, u_0) = y_-(x_k, t, u_0)$. Then from (4.4.22), $y_+(x_k, t, u_0)$

$\rightarrow y_-(x, t, u_0)$. Now apply the inequalities for x_k and let $k \rightarrow \infty$ to obtain (4.4.54), (4.4.55). This proves (2).

(3). Without loss of generality we can take $L(t, \alpha, u_0) = R_-(t, \alpha, u_0)$ and $R(t, \beta, u_0) = R_+(t, \beta, u_0)$. Similar proof follows in all other cases. Let $C = R_-(T, \alpha, u_0) = R_+(T, \beta, u_0)$ and $t > T$. Then from (4.4.27) and (4.4.29) we have

$$\begin{aligned} R_-(t, \alpha, u_0) &= R_-(t, T, C, u_0) \\ &= R_+(t, T, C, u_0) \\ &= R_+(t, \beta, u_0). \end{aligned} \tag{4.4.60}$$

This proves (4.4.57).

Let $t > T$, then choose $k_0 = k_0(t)$ such that $t > T_k$, for all $k > k_0$. Then from (4.4.57) we have

$$\begin{aligned} R_k(t) &= R_-(t, \alpha_k, u_0^k) = R_+(t, \alpha_k, u_0^k), \\ R(t) &= R_-(t, \alpha, u_0) = R_+(t, \alpha, u_0). \end{aligned}$$

Hence from (6) of Lemma 4.2,

$$\overline{\lim}_{k \rightarrow \infty} R_k(t) \leq R_+(t, \alpha, u_0) = R_-(t, \alpha, u_0) \leq \underline{\lim}_{k \rightarrow \infty} R_k(t). \tag{4.4.61}$$

This proves (4.4.59) and hence the Lemma.

Next we give a criteria under which $R_+ = R_-$. Let $\beta < \gamma$ and $I_1 = [\beta, \gamma]$, Define

$$m = \inf_{y \in I_1} u_0(y), \quad M = \sup_{y \in I_1} u_0(y), \quad I_2 = [f'(m), f'(M)].$$

Let

$$a_0 = \max\{f^*(q) - Mq; q \in I_2\}, \quad f'(q_0) = \max\{q; f^*(q) - Mq \leq a_0\}.$$

Then we have the following.

Lemma 4.4.5. *Let $\alpha < \beta < \gamma, \varepsilon_0 > 0$. Let $u_0 \in L^\infty(\mathbb{R})$, a_0 and q_0 as above. Suppose*

$$\inf_{[\alpha, \beta]} u_0(y) \geq q_0 + \varepsilon_0, \tag{4.4.62}$$

then for all $t > 0$,

$$R_+(t, \beta, u_0) = R_-(t, \beta, u_0).$$

Proof. Suppose for some $T > 0$, $R_+(T, \beta, u_0) > R_-(T, \beta, u_0)$, then from (4.4.57), for $0 < t < T$,

$$R_-(t, \beta, u_0) < R_+(t, \beta, u_0)$$

and from (4.4.26) for $R_-(t, \beta, u_0) < x < R_+(t, \beta, u_0)$.

$$f'(u(x, t)) = \frac{x - \beta}{t}. \quad (4.4.63)$$

From (4.4.28) we can choose T sufficiently small such that for all $0 < t \leq T$,

$$R_+(t, \alpha, u_0) < R_-(t, \beta, u_0) < R_+(t, \beta, u_0) < R_-(t, \gamma, u_0). \quad (4.4.64)$$

Claim: Let $L(t) = R_-(t, \beta, u_0)$, then for $0 < t \leq T$

$$f'(u(L(t)+, t)) \leq \frac{R_+(t, \beta, u_0) - \beta}{t} \leq f'(q_0). \quad (4.4.65)$$

Let $x_k > R_+(t, \beta, u_0)$ be such that $y_+(x_k, t, u_0) = y_-(x_k, t, u_0)$ and $\lim_{k \rightarrow \infty} x_k = R_+(t, \beta, u_0)$. Then from (4.4.55)

$$f'(m) \leq \frac{x - y_-(x_k, t, u_0)}{t} \leq f'(M).$$

Letting $k \rightarrow \infty$ and from (4.4.23) we have

$$f'(m) \leq \frac{R_+(t, \beta, u_0) - y_+(R_+(t, \beta, u_0), t, u_0)}{t} \leq f'(M). \quad (4.4.66)$$

Let $v_0(y) = \int_{\beta}^y u_0(\theta) d\theta$, hence $v_0(\beta) = 0$. Denote $R(t) = R_+(t, \beta, u_0)$, $y_{\pm}(t) = y_{\pm}(R_+(t, \beta, u_0), t, u_0)$, then from (4.4.63), $y_-(t) = \beta$ and from (4.4.4) we have

$$\begin{aligned} t f^* \left(\frac{R(t) - \beta}{t} \right) &= v_0(y_-(t)) + t f^* \left(\frac{R(t) - y_-(t)}{t} \right) \\ &= v_0(y_+(t)) + t f^* \left(\frac{R(t) - y_+(t)}{t} \right) \\ &\leq M(y_+(t) - \beta) + t f^* \left(\frac{R(t) - y_+(t)}{t} \right) \\ &\leq M(y_+(t) - R(t)) + M(R(t) - \beta) + t f^* \left(\frac{R(t) - y_+(t)}{t} \right), \end{aligned}$$

and hence

$$f^* \left(\frac{R(t) - \beta}{t} \right) - M \left(\frac{R(t) - \beta}{t} \right) \leq f^* \left(\frac{R(t) - y_+(t)}{t} \right) - M \left(\frac{R(t) - y_+(t)}{t} \right).$$

From (4.4.66) it follows that

$$f^* \left(\frac{R(t) - \beta}{t} \right) - M \left(\frac{R(t) - \beta}{t} \right) \leq a_0,$$

Letting x tends to $L(t)$ in (4.4.63) to obtain

$$f'(u(L(t)+, t)) = \frac{L(t) - \beta}{t} \leq \frac{R_+(t, \beta, u_0) - \beta}{t} \leq f'(q_0). \quad (4.4.67)$$

This proves (4.4.65) and hence the claim.

From (4.4.54), for $R_+(t, \alpha, u_0) < x < R_-(t, \beta, u_0) = L(t)$, $u(x, t) \geq \inf_{y \in [\alpha, \beta]} u_0(y)$, hence from (4.4.62) and (4.4.67), we have

$$\begin{aligned} u(L(t)-, t) &\geq \inf_{y \in [\alpha, \beta]} u_0(y) \geq q_0 + \varepsilon_0 \\ &\geq u(L(t)+, t) + \varepsilon_0 \\ &= f^{*'}\left(\frac{L(t) - \beta}{t}\right) + \varepsilon_0. \end{aligned} \quad (4.4.68)$$

From RH condition across $L(t)$ gives

$$\frac{dL}{dt} = \frac{f(u(L(t)-, t)) - f\left(f^{*'}\left(\frac{L(t) - \beta}{t}\right)\right)}{u(L(t)-, t) - f^{*'}\left(\frac{L(t) - \beta}{t}\right)}. \quad (4.4.69)$$

Therefore $L(t)$ satisfies the hypothesis (2) of Lemma 4.2.3 and hence from (4.4.47)

$$\lim_{t \rightarrow 0}^- \left| \frac{L(t) - \beta}{t} \right| = \infty,$$

which contradicts the uniform Lipschitz continuity of L from (4) of Lemma 4.2.2. Hence $R_-(t, \beta, u_0) = R_+(t, \beta, u_0)$, for all t , and this proves the Lemma.

Remark 4.4.1. Observe that q_0 entirely depends on the bounds of u_0 in $[\beta, \gamma]$.

Lemma 4.4.6. Let u be the solution of (4.1.2) with

$$\bar{u}_0(x) = u(x, 0) = \begin{cases} a & \text{if } x < \alpha, \\ u_0(x) & \text{if } x > \alpha. \end{cases}$$

Then for $x < R_-(t, \alpha, \bar{u}_0)$,

$$u(x, t) = a, \quad (4.4.70)$$

$$f'(a) = \frac{R_-(t, \alpha, \bar{u}_0) - y_-(R_-(t, \alpha, \bar{u}_0), t, \bar{u}_0)}{t}. \quad (4.4.71)$$

Proof. Since $\bar{v}_0(x) = \int_{\alpha}^x \bar{u}_0(\theta) d\theta$ is differentiable for $x < \alpha$ and hence from (4.4.13), for

a.e, $x < \alpha$, $u(x, t) = \bar{u}_0(y_+(x, t, \bar{u}_0)) = a$ and

$$f'(a) = \frac{x - y_+(x, t, \bar{u}_0)}{t}.$$

From (4.4.22) and letting $x \uparrow R_-(t, \alpha, \bar{u}_0)$ to obtain (4.4.71). This proves the Lemma.

Analysis of initial value problem with data taking three values : Consider the following initial value problem taking three values. Let $a, \lambda, m \in \mathbb{R}$, $\alpha < \beta$ and consider

$$u_0^\lambda(x) = \begin{cases} a & \text{if } x < \alpha, \\ \lambda & \text{if } \alpha < x < \beta, \\ m & \text{if } x > \beta. \end{cases} \quad (4.4.72)$$

and denote

$$v_0^\lambda(x) = \int_{\beta}^x u_0^\lambda(\theta) d\theta, \quad (4.4.73)$$

and v^λ be as in (4.4.4). Let $u^\lambda = \frac{\partial v^\lambda}{\partial x}$ be the entropy solution of (4.1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$ with initial data u_0^λ . Assume that

$$\lambda > \max(a, m), \quad (4.4.74)$$

then α is a point of rarefaction and β is the shock point.

Let

$$\begin{aligned} L_1(t) &= \alpha + f'(a)t, \\ L_2^\lambda(t) &= \alpha + f'(\lambda)t, \\ S^\lambda(t) &= \beta + \left(\frac{f(\lambda) - f(m)}{\lambda - m} \right) t. \end{aligned}$$

Let $(x_0(\lambda), T_0(\lambda))$ be the point of intersection of L_2^λ and S^λ given by

$$\begin{aligned} T_0(\lambda) &= \frac{\beta - \alpha}{f'(\lambda) - \left(\frac{f(\lambda) - f(m)}{\lambda - m} \right)}, \\ x_0(\lambda) &= \alpha + \frac{(\beta - \alpha)f'(\lambda)}{f'(\lambda) - \left(\frac{f(\lambda) - f(m)}{\lambda - m} \right)}. \end{aligned}$$

Since β is the point of shock and hence from (4.4.26) we have

$$R_+(t, \beta, u_0^\lambda) = R_-(t, \beta, u_0^\lambda) = R^\lambda(t)(\text{def}). \quad (4.4.75)$$

Then the solution u^λ for $t \leq T_0(\lambda)$ is given by

$$R^\lambda(t) = S^\lambda(t). \quad (4.4.76)$$

$$u^\lambda(x, t) = \begin{cases} m & \text{if } x > S^\lambda(t), \\ \lambda & \text{if } L_2^\lambda(t) < x < S^\lambda(t), \\ (f')^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } L_1(t) < x < L_2^\lambda(t), \\ a & \text{if } x < L_1(t). \end{cases} \quad (4.4.77)$$

Define $T_1(\lambda) > T_0(\lambda)$ be the first point of intersection of L_2^λ and R^λ . If they do not meet, then define $T_1(\lambda) = \infty$. Next Lemma describes the behavior of u^λ for $t > T_0(\lambda)$.

Lemma 4.4.7. *Let λ satisfy (4.4.74). Then u^λ is given by (see Figure 4.10).*

(i). For $T_0(\lambda) < t < T_1(\lambda)$, $y_\pm(L_1(t), t, u_0^\lambda) = a$ and

$$u^\lambda(x, t) = \begin{cases} m & \text{if } x > R^\lambda(t), \\ f'^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } L_1(t) < x < R^\lambda(t), \\ a & \text{if } x < L_1(t). \end{cases} \quad (4.4.78)$$

(ii). $t > T_1(\lambda)$, then u^λ is the solution of (4.1.2) with initial data

$$u^\lambda(x, T_1(\lambda)) = \begin{cases} a & \text{if } x < R^\lambda(T_1(\lambda)), \\ m & \text{if } x > R^\lambda(T_1(\lambda)). \end{cases} \quad (4.4.79)$$

Furthermore for any compact sets K_1 and K_2 of \mathbb{R} with $K = K_1 \times K_2$, $\eta > 0$, $T \leq T_1(\lambda)$ be bounded, then

$$\lim_{\lambda \rightarrow \infty} \inf_{(a,m) \in K} T_1(\lambda) = \infty, \quad (4.4.80)$$

$$f'(u^\lambda(R^\lambda(t)-, t)) = \begin{cases} f'(\lambda) & \text{if } 0 < t < T_0(\lambda), \\ \frac{R^\lambda(t)-\alpha}{t} & \text{if } T_0(\lambda) < t < T_1(\lambda). \end{cases} \quad (4.4.81)$$

$$\lim_{\lambda \rightarrow \infty} \inf_{\substack{(a,m) \in K \\ T_0(\lambda) \leq t \leq T}} u^\lambda(R^\lambda(t)-, t) = \infty. \quad (4.4.82)$$

$$\lim_{\lambda \rightarrow \infty} \inf_{\eta \leq t \leq T} R^\lambda(t) = \infty, \quad (4.4.83)$$

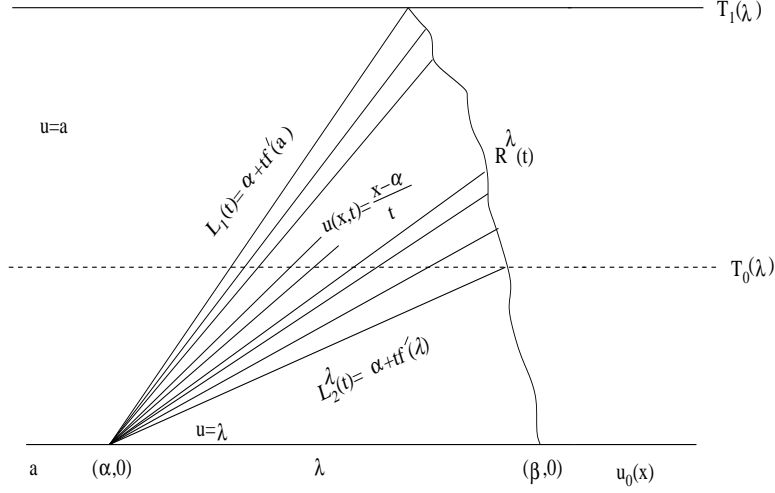


Fig. 4.10:

Proof. Let $T_0(\lambda) < t \leq T_1(\lambda)$. Since $u_0^\lambda(x)$ is differentiable for $x > \beta$ and hence from (4.4.13) and (4.4.51), $u^\lambda(x, t) = u_0(y_+(x, t, u_0^\lambda)) = m$ if $x > R_+(t, \beta, u_0^\lambda) = R^\lambda(t)$. Next we show that for $L^1(t) < x < R^\lambda(t)$, $y_\pm(x, t, u_0^\lambda) = \alpha$.

$L_1(t) < x < R^\lambda(t)$. Then $y_+(x, t, u_0^\lambda) < \beta$. Suppose for some $x_0 \in (L_1(t), R^\lambda(t))$, $y_-(x_0, t, u_0^\lambda) < \alpha$, then for all $x \in (L_1(t), x_0)$, $y_-(x, t, u_0^\lambda) < \alpha$, $u^\lambda(x, t) = u_0(y_-(x, t, u_0^\lambda)) = a$ and

$$\frac{L_1(t) - \alpha}{t} = f'(a) = f'(u^\lambda(x, t)) = \frac{x - y_-(x, t, u_0^\lambda)}{t} > \frac{L_1(t) - \alpha}{t},$$

which is a contradiction. Suppose $y_+(x_0, t, u_0^\lambda) > \alpha$, then for all $x_0 < x < R^\lambda(t)$, $\alpha < y_+(x, t, u_0^\lambda) < \beta$. Since u_0^λ is differentiable in (α, β) and hence from (4.4.13), for a.e. $x \in (x_0, R^\lambda(t))$,

$$u^\lambda(x, t) = u_0^\lambda(y_+(x, t, u_0^\lambda)) = \lambda, f'(\lambda) = f'(u^\lambda(x, t)) = \frac{x - y_+(x, t, u_0^\lambda)}{t}.$$

Suppose $x_0 < L_2^\lambda(t)$, then for $x_0 < x < \min(L_2^\lambda(t), R^\lambda(t))$,

$$f'(\lambda) = \frac{x - y_+(x, t, u_0^\lambda)}{t} < \frac{L_2^\lambda(t) - \alpha}{t} = f'(\lambda),$$

which is a contradiction. Suppose $L_2^\lambda(t) < x_0 < R^\lambda(t)$, then for $x \in (x_0, R^\lambda(t))$, characteristic γ at (x, t) given by $\gamma(\theta) = y_+(x, t, u_0^\lambda) + f'(\lambda)\theta$ intersects S^λ at t_0 , where

$$t_0 = \frac{\beta - y_+(x, t, u_0^\lambda)}{f'(\lambda) - \frac{f(\lambda) - f(m)}{\lambda - m}} < \frac{\beta - \alpha}{f'(\lambda) - \frac{f(\lambda) - f(m)}{\lambda - m}} = T_0(\lambda),$$

which contradicts NIP, since $S^\lambda(t)$ is a characteristic for $0 < t < T_0(\lambda)$. Hence for

$L_1(t) < x < R^\lambda(t)$, $y_+(x, t, u_0^\lambda) = y_-(x, t, u_0^\lambda) = \alpha$ and from (4.4.21), we have

$$f'(u^\lambda(x, t)) = \frac{x - \alpha}{t}. \quad (4.4.84)$$

Now letting $x \downarrow L_1(t)$ and from (4.4.22) to obtain $y_+(L_1(t), t, u_0^\lambda) = \alpha$ and $f'(u^\lambda(L_1(t) +, t)) = \frac{L_1(t) - \alpha}{t} = f'(a)$. This implies $u^\lambda(L_1(t) +, t) = a$. From RH condition across $L_1(t)$ implies that $u^\lambda(L_1(t) -, t) = a$. Therefore from (4.4.12), (4.4.22), (4.4.23) $y_\pm(L_1(t), t, u_0^\lambda) = a$. This implies for $x < L_1(t)$, $y_+(x, t, u_0^\lambda) < \alpha$ and hence from (4.4.13), $u^\lambda(x, t) = u_0(y_+(x, t, u_0^\lambda)) = a$. This proves (4.4.78) and hence (4.4.79).

Let

$$\begin{aligned} y_\pm(t, \lambda) &= y_\pm(R^\lambda(t), t, u_0^\lambda), \\ y_\pm(\lambda) &= y_\pm(R^\lambda(T_1(\lambda)), T_1(\lambda), u_0^\lambda), \\ R^\lambda &= R^\lambda(T_1(\lambda)). \end{aligned}$$

Let $T_0(\lambda) < t \leq T_1(\lambda)$ and letting $x \uparrow R^\lambda(t)$ in (4.4.84) to obtain

$$\frac{R^\lambda(t) - y_-(t, \lambda)}{t} = f'(u^\lambda(R^\lambda(t) -, t)) = \frac{R^\lambda(t) - \alpha}{t}. \quad (4.4.85)$$

Hence $y_-(t, \lambda) = \alpha$. Also at $t = T_1(\lambda)$,

$$f'(a) = \frac{R^\lambda - \alpha}{T_1(\lambda)} = \frac{R^\lambda - y_-}{T_1(\lambda)}. \quad (4.4.86)$$

$$\frac{R^\lambda(t) - y_+(t, \lambda)}{t} = \lim_{x \downarrow R^\lambda(t)} \frac{x - y_-(x, t, u_0^\lambda)}{t} \quad (4.4.87)$$

$$= \lim_{x \downarrow R^\lambda(t)} f'(u^\lambda(x, t)) = f'(m). \quad (4.4.88)$$

From (4.4.85) to (4.4.88) we can evaluate $v^\lambda(R^\lambda(t), t)$ by

$$\begin{aligned} -(\beta - \alpha)\lambda + tf^* \left(\frac{R^\lambda(t) - \alpha}{t} \right) &= (y_+(t, \lambda) - \beta)m + tf^* \left(\frac{R^\lambda(t) - y_+(t, \lambda)}{t} \right) \\ &= m \left(\frac{y_+(t, \lambda) - R^\lambda(t)}{t} t + R^\lambda(t) - \beta \right) \\ &+ tf^*(f'(m)) \\ &= -tmf'(m) + m(R^\lambda(t) - \alpha) + m(\alpha - \beta) \\ &+ tf^*(f'(m)). \end{aligned}$$

$$\frac{(\beta - \alpha)(\lambda - m)}{t} = f^* \left(\frac{R^\lambda(t) - \alpha}{t} \right) - \frac{R^\lambda(t) - \alpha}{t} - f^*(f'(m) + mf'(m)). \quad (4.4.89)$$

Let $t = T_1(\lambda)$ then $\frac{R^\lambda - \alpha}{T_1(\lambda)} = f'(a)$ and hence the right hand side of (4.4.89) is bounded uniformly for $(a, m) \in K$ and hence as $\lambda \rightarrow \infty, T_1(\lambda) \rightarrow \infty$. This proves (4.4.80).

Observe that $R_+(t, \alpha, u_0^\lambda) = L_2^\lambda(t)$ and $L_2^\lambda(t) < R^\lambda(t)$ for $0 < t < T_0(\lambda)$. Hence for a.e $x \in (L_2^\lambda(t), R^\lambda(t)), y_+(x, t, u_0^\lambda) = y_-(x, t, u_0^\lambda) \in (\alpha, \beta)$ and from (4.4.13),

$$u^\lambda(x, t) = u_0^\lambda(y_+(x, t, u_0^\lambda)) = \lambda.$$

From this and (4.4.85), (4.4.81) follows. Let $T_0(\lambda) < t \leq T$, then from superlinearity of f^* , (4.4.82) follows from (4.4.85), (4.4.89). Suppose $\lim_{\lambda \rightarrow \infty} T_0(\lambda) = 0$, and then (4.4.83) follows from (4.4.81), (4.4.82). Hence assume that $\lim_{\lambda \rightarrow \infty} T_0(\lambda) > 0$, then if $\eta < T_0(\lambda)$, then (4.4.83) follows from (4.4.89). This proves the Lemma.

Next we generalize the above Lemma by replacing m by u_0 . More precisely let

$$u_0^\lambda(x) = \begin{cases} a & \text{if } x < \alpha, \\ \lambda & \text{if } \alpha < x < \beta, \\ u_0(x) & \text{if } x > \beta, \end{cases} \quad (4.4.90)$$

and u^λ be the solution of (4.1.2) with initial data u_0^λ . Let

$$m_1 = \inf_{x \geq \alpha} u_0(x), m_2 = \sup_{x \geq \alpha} u_0(x). \quad (4.4.91)$$

For $i = 1, 2$, define $u_0^{i,\lambda}$ by

$$u_0^{i,\lambda}(x) = \begin{cases} a & \text{if } x < \alpha, \\ \lambda & \text{if } \alpha < x < \beta, \\ m_i & \text{if } \beta < x, \end{cases} \quad (4.4.92)$$

and let u_i^λ be the solution of (4.1.2) with initial data $u_0^{i,\lambda}$. Let $L_1(t), L_2^\lambda(t)$ be as defined earlier, then

Lemma 4.4.8. *Let $T > 0$ be fixed, then there exist $\lambda_0 = \lambda_0(m_1, m_2, a, t)$ such that for $\lambda \geq \lambda_0, 0 < t \leq T$,*

$$R_-(t, \beta, u_0^\lambda) = R_+(t, \beta, u_0^\lambda). \quad (4.4.93)$$

and denote $R(\lambda, t) = R_-(t, \beta, u_0^\lambda)$, then

(i). $t \rightarrow R(\lambda, t)$ is a strictly increasing function.

(ii). $\lambda \rightarrow R(\lambda, t)$ is a strictly increasing function. Let $T_1(\lambda)$ be the first point of inter-

section of $L_1(t)$ and $R(t, \lambda)$. Then for any fixed $T > 0$

$$\lim_{t \rightarrow \infty} T_1(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \infty} R(\lambda, T) = \infty. \quad (4.4.94)$$

Let $T_0(\lambda)$ be the first point of intersection of $L_2^\lambda(t)$ and $R(\lambda, t)$. Then

$$u^\lambda(x, t) = \begin{cases} a & \text{if } x < L_1(t) \\ \lambda & \text{if } L_2^\lambda(t) < x < R(\lambda, t), \quad 0 < t < T_0(\lambda). \end{cases} \quad (4.4.95)$$

Proof. Let q_0 be as in (4.4.62), then for $\lambda > q_0$, from Lemma 4.4.5 we have for $i = 1, 2$,

$$R_-(t, \beta, u_0^\lambda) = R_+(t, \beta, u_0^\lambda), \quad (4.4.96)$$

$$R_-(t, \beta, u_0^{i,\lambda}) = R_+(t, \beta, u_0^{i,\lambda}) \quad (4.4.97)$$

and denote $R_i(\lambda, t) = R_-(t, \beta, u_0^{i,\lambda})$, $T_{11}(\lambda), T_1(\lambda), T_{21}(\lambda)$ the first points of intersection of $L_1(t)$ with $R_1(\lambda, t), R(\lambda, t), R_2(\lambda, t)$ respectively. Since $u_0^{1,\lambda} \leq u_0^\lambda \leq u_0^{2,\lambda}$, hence from (4.4.30)

$$R_1(\lambda, t) \leq R(\lambda, t) \leq R_2(\lambda, t), \quad T_{11}(\lambda) \leq T_1(\lambda) \leq T_{21}(\lambda). \quad (4.4.98)$$

Then from (4.4.80), it follows that

$$\lim_{\lambda \rightarrow \infty} T_1(\lambda) = \infty.$$

Next we obtain a bound on $u^\lambda(R(\lambda, t)+, t)$. For this let $\bar{u}(x, t)$ be the solution of (4.1.2) with initial data $\bar{u}_0(x)$ defined by

$$\bar{u}_0(x) = \begin{cases} \min(a, m_1) & \text{if } x < \beta, \\ u_0(x) & \text{if } x > \beta, \end{cases}$$

then for $\lambda > m$, $\bar{u}_0(x) \leq u_0^{1,\lambda}(x) \leq u_0^\lambda(x)$ and hence $\bar{u}(x, t) \leq u^\lambda(x, t)$ and $R_+(t, \beta, \bar{u}_0) \leq R(\lambda, t)$. Since for $y > \beta$, $\int_\beta^y \bar{u}_0(\theta) d\theta = \int_\beta^y u_0^\lambda(\theta) d\theta$ and hence from (4.4.51) we have for

$x > R(\lambda, t)$,

$$\begin{aligned}
V^\lambda(x, t) &= \inf_{y \geq \beta} \left\{ \int_{\beta}^y u_0^\lambda(\theta) d\theta + t f^* \left(\frac{x-y}{t} \right) \right\} \\
&= \inf_{y \geq \beta} \left\{ \int_{\beta}^y \bar{u}_0(\theta) d\theta + t f^* \left(\frac{x-y}{t} \right) \right\} \\
&= V(x, t),
\end{aligned}$$

where $u^\lambda = \frac{\partial V^\lambda}{\partial x}$ and $\bar{u} = \frac{\partial V}{\partial x}$. Hence for $x > R(\lambda, t)$,

$$u^\lambda(x, t) = \bar{u}(x, t). \quad (4.4.99)$$

Therefore

$$\begin{aligned}
|u^\lambda(R(\lambda, t)+, t)| &\leq \|\bar{u}\|_\infty \\
&\leq \max(m_2, a).
\end{aligned} \quad (4.4.100)$$

For $i = 1, 2$, let $T_{i,0}(\lambda)$ be the first intersection point of $L_2^\lambda(t)$ and $R_i(\lambda, t)$ and $T_{i,1}(\lambda) > T_{i,0}(\lambda)$ be the points of intersections of $L_1(t)$ and $R_i(\lambda, t)$. Then from Lemma 4.8, we can choose $\lambda_0 \geq q_0 + \|\bar{u}\|_\infty$ such that for all $\lambda \geq \lambda_0$, $f'(\lambda) > 0$, $f(\lambda) > f(\|\bar{u}\|_\infty)$ and

$$T_{1,1}(\lambda) > T, \quad R_1(\lambda, T) > L_1(t). \quad (4.4.101)$$

$$\inf_{T_{1,0}(\lambda) \leq t \leq T} f^{*'} \left(\frac{R_1(\lambda, t) - \alpha}{t} \right) = \inf_{T_{1,0}(\lambda) \leq t \leq T} u^{1,\lambda}(R_1(\lambda, t)-, t) > \lambda_0. \quad (4.4.102)$$

From (4.4.83) and (4.4.98) we have

$$\lim_{\lambda \rightarrow \infty} R(\lambda, T) \geq \lim_{\lambda \rightarrow \infty} R_1(\lambda, T) = \infty.$$

This proves (4.4.94).

Next imitating the proof as in Lemma 4.4.7 and from (4.4.99) we have for $0 < t < T$,

$$u^\lambda(x, t) = \begin{cases} \bar{u}(x, t) & \text{if } x > R(\lambda, t), \\ (f')^{-1} \left(\frac{x-\alpha}{t} \right) & \text{if } t > T_0(\lambda), L_1(t) < x < R(\lambda, t), \\ \lambda & \text{if } 0 < t < T_0(\lambda), L_2^\lambda(t) < x < R(\lambda), \\ a & \text{if } x < L_1(t). \end{cases} \quad (4.4.103)$$

Let $0 < t < T_0(\lambda)$ then from (4.4.100) and the choice of λ_0 , we have for a.e. t ,

$$\begin{aligned}\frac{d}{dt}R(\lambda, t) &= \frac{f(u^\lambda(R(\lambda, t)-, t)) - f(u^\lambda(R(\lambda, t)+, t))}{u^\lambda(R(\lambda, t)-, t) - u^\lambda(R(\lambda, t)+, t)} \\ &= \frac{f(\lambda) - f(u^\lambda(R(\lambda, t)+, t))}{\lambda - u^\lambda(R(\lambda, t)+, t)} > 0.\end{aligned}$$

Let $T_0(\lambda) < t \leq T$, then from (4.4.98), $T_{1,0}(\lambda) \leq T_0(\lambda)$. Hence from (4.4.103), (4.4.102)

$$\begin{aligned}u^\lambda(R(\lambda, t)-, t) &= f^{*'}\left(\frac{R(\lambda, t) - \alpha}{t}\right) \\ &\geq f^{*'}\left(\frac{R_1(\lambda, t) - \alpha}{t}\right) \\ &= u^{1,\lambda}(R_1(\lambda, t)-, t) \\ &> \lambda_0.\end{aligned}$$

Since $f'(\lambda) > 0$ for $\lambda \geq \lambda_0$, hence

$$f(u^\lambda(R(\lambda, t)-, t)) \geq f(\lambda_0) > f(\|\bar{u}\|_\infty).$$

Therefore from (4.4.99), (4.4.100) we have for $T_0(\lambda) < t \leq T$.

$$\begin{aligned}\frac{d}{dt}R(\lambda, t) &= \frac{f(u^\lambda(R(\lambda, t)-, t)) - f(u^\lambda(R(\lambda, t)+, t))}{u^\lambda(R(\lambda, t)-, t) - u^\lambda(R(\lambda, t)+, t)} \\ &= \frac{f(u^\lambda(R(\lambda, t)-, t)) - f(\bar{u}(R(\lambda, t)+, t))}{u^\lambda(R(\lambda, t)-, t) - \bar{u}(R(\lambda, t)+, t)} \\ &> 0.\end{aligned}$$

This proves that $t \rightarrow R(\lambda, t)$ is a strictly increasing function.

Claim: $R(\lambda, t) \leq L_2^\lambda(t)$ for $t > T_0(\lambda)$.

Suppose for some $t_0 > T_0(\lambda)$, $R(\lambda, t_0) > L_2^\lambda(t_0)$, then for a.e. $x \in (L_2^\lambda(t_0), R(\lambda, t_0))$, $y_+(x, t, u_0^\lambda) \in (\alpha, \beta)$ and hence from (4.4.13) and differentiability of u_0^λ in (α, β) gives $u(x, t_0) = \lambda$ and $f'(\lambda) = \frac{x - y_+(x, t_0, u_0^\lambda)}{t_0}$. Hence the characteristic line $r(\theta)$ at (x, t_0) is parallel to L_2^λ and $r(\theta) \geq L_2^\lambda(\theta)$ for $\theta \in [0, t_0]$. Since $t \rightarrow R(\lambda, t)$ is an increasing function for $t \in (0, T_1(\lambda))$ and $T_0(\lambda) < t_0$, hence $R(\lambda, T_0(\lambda)) < x$. Furthermore $y_+(R(\lambda, T_0(\lambda)), T_0(\lambda), u_0^\lambda) \geq \beta$. Hence the characteristic line at $(R(\lambda, T_0(\lambda)), T_0(\lambda))$ intersect r which contradicts NIP. This proves the claim.

Hence for $t \geq T_0(\lambda)$,

$$\frac{R(\lambda, t) - \alpha}{t} \leq \frac{L_2^\lambda(t) - \alpha}{t} = f'(\lambda). \quad (4.4.104)$$

Let $\lambda_0 \leq \lambda_1 < \lambda_2$, then $u_0^{\lambda_1} \leq u_0^{\lambda_2}$ and hence $R(\lambda_1, t) \leq R(\lambda_2, t)$ and for a.e. x ,

$y_{\pm}(x, t, u_0^{\lambda_1}) \geq y_{\pm}(x, t, u_0^{\lambda_2})$. Suppose for some $0 < t_0 < T, R = R(\lambda_1, t_0) = R(\lambda_2, t_0)$. From (4.4.99) at $x = R$, we have $\alpha \leq y_-(R, t_0, u_0^{\lambda_2}) \leq y_-(R, t_0, u_0^{\lambda_1}) < \beta$, and $u^{\lambda_1}(R+, t_0) = \bar{u}(R+, t_0) = u^{\lambda_2}(R+, t_0)$. Hence from (4.4.23) $y_+(R, t_0, u_0^{\lambda_1}) = y_+(R, t_0, u_0^{\lambda_2})$.

Let for $i = 1, 2$, $y = y_+(R, t_0, u_0^{\lambda_i})$, $y_i = y_-(R, t_0, u_0^{\lambda_i})$ and $V_0^{\lambda_i}(y) = \int_{\beta}^y u_0^{\lambda_i}(\theta) d\theta$, then $V_0^{\lambda_1}(y) = V_0^{\lambda_2}(y)$ for $y \geq \beta$. Hence from (4.4.4) we have

$$\begin{aligned} \lambda_2(y_2 - \beta) + t_0 f^* \left(\frac{R - y_2}{t_0} \right) &= V_0^{\lambda_2}(y_2) + t f^* \left(\frac{R - y_2}{t_0} \right) \\ &= V_0^{\lambda_2}(y) + t f^* \left(\frac{R - y}{t_0} \right) \\ &= V_0^{\lambda_1}(y) + t f^* \left(\frac{R - y}{t_0} \right) \\ &= V_0^{\lambda_1}(y_1) + t f^* \left(\frac{R - y_1}{t_0} \right) \\ &= \lambda_1(y_1 - \beta) + t_0 f^* \left(\frac{R - y_1}{t_0} \right). \end{aligned}$$

Let $f'(\theta_i) = \frac{R - y_i}{t_0}$, then $R = f'(\theta_i)t_0 + y_i$ and since $y_2 \leq y_1$ implies that $f'(\theta_2) \geq f'(\theta_1)$, hence $\theta_2 \geq \theta_1$. Substituting this in the above expression and using $f^*(f'(p)) = pf'(p) - f(p)$ to obtain

$$(R - t_0 f'(\theta_2))\lambda_2 + t_0 f^*(f'(\theta_2)) = (R - t_0 f'(\theta_1))\lambda_1 + t_0 f^*(f'(\theta_1)) + \beta(\lambda_2 - \lambda_1)$$

$$R = \beta + \frac{t_0}{(\lambda_2 - \lambda_1)} [(\lambda_2 - \theta_2)f'(\theta_2) - (\lambda_1 - \theta_1)f'(\theta_1)] + \left(\frac{f(\theta_2) - f(\theta_1)}{\lambda_2 - \lambda_1} \right) t_0.$$

That is for $i = 1, 2$,

$$y_i = \beta + \frac{t_0}{(\lambda_2 - \lambda_1)} [(\lambda_2 - \theta_2)f'(\theta_2) - (\lambda_1 - \theta_1)f'(\theta_1)] + t_0 \left[\frac{f(\theta_2) - f(\theta_1)}{\lambda_2 - \lambda_1} - f'(\theta_i) \right]. \quad (4.4.105)$$

Case (i) : Let $y_2 = y_1$. Then $\theta_2 = \theta_1$ and hence from (4.4.105), $\beta = y_1 < \beta$ which is a contradiction.

Case (ii): Let $\alpha < y_2 < y_1$.

Since $V_0^{\lambda_i}$ is differentiable for $y \in (\alpha, \beta)$ and hence from (4.4.13) , (4.4.23), we have $f'(\lambda_i) = \frac{R - y_i}{t_0}$. Therefore from (4.4.105) and from strict convexity of f we have

$$y_1 = \beta + t_0 \left[\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} - f'(\lambda_1) \right] > \beta.$$

which is a contradiction.

Case(iii): Let $\alpha = y_2 < y_1$.

Since $y_1 > \alpha$, hence $f'(\theta_1) = \frac{R - y_1}{t_0} = f'(\lambda_1)$ and $\frac{R - \alpha}{t_0} = f'(\theta_2)$. From (4.4.104),

$f'(\theta_2) \leq f'(\lambda_2)$ and hence $\lambda_2 \geq \theta_2$. Since $\lambda_1 \geq \lambda_0$ and hence $f'(\theta_2) \geq f'(\lambda_1) > 0$. From (4.4.105), $\theta_1 = \lambda_1$ and convexity of f we have

$$\begin{aligned} 0 > \frac{(\lambda_2 - \lambda_1)(y_1 - \beta)}{t_0} &= (\lambda_2 - \theta_2)(f'(\theta_2) - f'(\lambda_1)) + f(\theta_2) - f(\lambda_1) \\ &\geq (\theta_2 - \lambda_1)f'(\lambda_1) \\ &> 0, \end{aligned}$$

which is a contradiction. This proves $\lambda \rightarrow R(\lambda, t)$ is a strictly increasing function for $\lambda \geq \lambda_0$ and $0 < t \leq T$. This proves the Lemma.

Next we consider the variation from the right, Let u^μ be the solution of (4.1.2) with initial data u_0^μ given by

$$u_0^\mu = \begin{cases} u_0(x) & \text{if } x < \alpha, \\ \mu & \text{if } \alpha < x < \beta, \\ a & \text{if } x > \beta. \end{cases}$$

We state the following Lemma without proof since the proof follows exactly as that of Lemma 4.4.8.

Define

$$L_1(t) = \beta + f'(a)t, L_2^\mu(t) = \beta + f'(\mu)t.$$

Lemma 4.4.9. *There exist $\mu_1 = \mu_1(m_1, m_2, a)$ such that for $\mu < \mu_1, t > 0$,*

$$R_-(t, \alpha, u_0^\mu) = R_+(t, \alpha, u_0^\mu), \quad (4.4.106)$$

and denote $R(\mu, t) = R_-(t, \alpha, u_0^\mu)$. Let $T_0(\mu) > 0$ be the first point of intersection of $R(\mu, t)$ and $L_2^\mu(t)$ and $T_1(\mu) > T_0(\mu)$ be the first point of intersection of $R(\mu, t)$ and $L_1(t)$. Then

$$\lim_{\mu \rightarrow -\infty} T_1(\mu) = \infty. \quad (4.4.107)$$

For $0 < t < T_1(\mu)$,

$$u^\mu(x, t) = \begin{cases} a & \text{if } x > L_1(t), \\ (f')^{-1}\left(\frac{x-\beta}{t}\right) & \text{if } T_0(\mu) < t < T_1(\mu), \\ & R(\mu, t) < x < L_1(t), \\ \mu & \text{if } 0 < t < T_0(\mu), \\ & R(\mu, t) < x < L_2^\mu(t). \end{cases} \quad (4.4.108)$$

Furthermore let $T > 0$ be fixed, then there exist $\mu_0 = \mu_0(T, \mu_1) < \mu_1$ such that

(i). $\mu \rightarrow R(\mu, t)$ is a strictly increasing function for $0 < t \leq T$ and

$$\lim_{\mu \rightarrow -\infty} R(\mu, t) = -\infty. \quad (4.4.109)$$

(ii). For $0 < t < T_1(\mu)$, $t \rightarrow R(\mu, t)$ is a strictly decreasing function of t .

Interaction of R_{\pm} with initial data: We study the interaction of R_{\pm} with varying parameters in the data. For this first we need the following elementary results.

Let $B_1, B_2, \mu_0 < \lambda_0$, $L \in C(\mathbb{R}_+ \times [\lambda_0, \infty))$, $R \in C(\mathbb{R}_+ \times (-\infty, \mu_0])$ be given and for $\lambda \geq \lambda_0, \mu \leq \mu_0$, L and R satisfy the following hypothesis,

(H₁). $\lambda \mapsto L(t, \lambda)$, $\mu \mapsto R(t, \mu)$ are strictly increasing functions such that for all $\lambda \geq \lambda_0, \mu \leq \mu_0$,

$$L(0, \lambda) = B_1, \quad R(0, \mu) = B_2, \quad (4.4.110)$$

and for any $0 < \alpha < \beta$,

$$\lim_{\lambda \rightarrow \infty} \inf_{t \in [\alpha, \beta]} L(t, \lambda) = \infty, \quad \lim_{\mu \rightarrow -\infty} \sup_{t \in [\alpha, \beta]} R(t, \mu) = -\infty. \quad (4.4.111)$$

(H₂). For $\lambda \geq \lambda_0, \mu \leq \mu_0$, $t \mapsto L(t, \lambda)$, is a strictly increasing function and $t \mapsto R(t, \mu)$ is a strictly decreasing function.

Let $I = [\lambda_0, \infty) \times (-\infty, \mu_0]$ and define $x_0(t), y_0(t), \lambda(x, t), \mu(y, t), \delta(\lambda, \mu), c(\lambda, \mu)$ as follows:

$$x_0(t) = L(t, \lambda_0), \quad y_0(t) = R(t, \mu_0) \quad (4.4.112)$$

$$L(t, \lambda(x, t)) = x, \quad R(t, \mu(y, t)) = y \quad (4.4.113)$$

$$L(\delta(\lambda, \mu), \lambda) = R(\delta(\lambda, \mu), \mu) = c(\lambda, \mu), \quad (4.4.114)$$

then we have the following

Lemma 4.4.10. 1. x_0 is a strictly increasing continuous and y_0 is a strictly decreasing function satisfying

$$(x_0(0), y_0(0)) = (B_1, B_2). \quad (4.4.115)$$

2. For $x \geq x_0(t), y \leq y_0(t), (\lambda(x, t), \mu(y, t)) \in I, x \mapsto \lambda(x, t), t \rightarrow \mu(y, t)$ are strictly increasing functions and $t \mapsto \lambda(x, t), y \mapsto \mu(y, t)$ are strictly decreasing continuous functions in $(0, \infty)$. Also for $x > B_1, y < B_2$

$$\lim_{t \rightarrow 0} (\lambda(x, t), \mu(y, t)) = (\infty, -\infty). \quad (4.4.116)$$

3. Let $B_1 < B_2$ and $(\lambda, \mu) \in I$. Then $\delta(\lambda, \mu)$ exists and is a continuous function. Furthermore $\lambda \rightarrow \delta(\lambda, \mu)$ is a decreasing function and $\mu \mapsto \delta(\lambda, \mu)$ is an increasing function

and

$$\lim_{\lambda \rightarrow \infty} \delta(\lambda, \mu) = \lim_{\mu \rightarrow -\infty} \delta(\lambda, \mu) = 0 \quad (4.4.117)$$

$$\lim_{\mu \rightarrow -\infty} c(\lambda, \mu) = B_1, \quad \lim_{\lambda \rightarrow \infty} c(\lambda, \mu) = B_2. \quad (4.4.118)$$

Proof.

1. Follows from (H_1) .
2. From (4.4.111) for $t > 0$, $L(t, \cdot) : [\lambda_0, \infty) \rightarrow [x_0(t), \infty)$ is a homeomorphism and hence $\lambda(x, t)$ exists and $x \mapsto \lambda(x, t)$ is a strictly increasing function. Let $t_1 < t_2$ and suppose $\lambda(x, t_1) \leq \lambda(x, t_2)$, then

$$x = L(t_1, \lambda(x, t_1)) \leq L(t_1, \lambda(x, t_2)) < L(t_2, \lambda(x, t_2)) = x,$$

which is a contradiction. Hence $t \mapsto \lambda(x, t)$ is a strictly decreasing function. Let $(x_n, t_n) \rightarrow (x, t)$, $\lambda(x_n, t_n) \rightarrow \lambda$, then

$$x = \lim_{n \rightarrow \infty} L(t_n, \lambda(x_n, t_n)) = L(t, \lambda),$$

and hence $\lambda = \lambda(x, t)$. This proves the continuity of $\lambda(x, t)$. Suppose as $t_n \rightarrow 0$, $\{\lambda(x, t_n)\}$ is bounded. Then for a subsequence still denote by n such that $\lambda(x, t_n) \rightarrow \lambda$ as $n \rightarrow \infty$. Therefore by continuity of L and (4.4.110)

$$B_1 < x = \lim_{n \rightarrow \infty} L(t_n, \lambda(x, t_n)) = L(0, \lambda) = B_1,$$

which is a contradiction. Hence $\lambda(x, t) \rightarrow \infty$ as $t \rightarrow 0$. Similarly for $\mu(y, t)$ and this proves (2).

3. For $(\lambda, \mu) \in I$, $t \mapsto L(t, \lambda) \geq B_1$ and is a strictly increasing function and $t \mapsto R(t, \mu) \leq B_2$ is a strictly decreasing function. Hence there exists a unique $\delta(\lambda, \mu)$ satisfying (4.4.114) and $B_1 \leq c(\lambda, \mu) \leq B_2$ and continuity follows from the uniqueness of $\delta(\lambda, \mu)$.

Let $\lambda_1 < \lambda_2$ and $\delta(\lambda_1, \mu) \leq \delta(\lambda_2, \mu)$. Then

$$\begin{aligned} R(\delta(\lambda_1, \mu), \mu) &= L(\delta(\lambda_1, \mu), \lambda_1) \leq L(\delta(\lambda_2, \mu), \lambda_1) \\ &< L(\delta(\lambda_2, \mu), \lambda_2) \\ &= R(\delta(\lambda_2, \mu), \mu) \end{aligned}$$

and hence $\delta(\lambda_2, \mu) < \delta(\lambda_1, \mu)$ which is a contradiction. Suppose $\lim_{\lambda \rightarrow \infty} \delta(\lambda, \mu) = \delta_0 > 0$, then from (4.4.111),

$$\infty = \lim_{\lambda \rightarrow \infty} L(\delta(\lambda, \mu), \lambda) = \lim_{\delta(\lambda, \mu) \rightarrow \delta_0} R(\delta(\lambda, \mu), \mu) = R(\delta_0, \mu) < \infty,$$

which is a contradiction hence $\delta_0 = 0$ and

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} c(\lambda, \mu) &= \lim_{\lambda \rightarrow \infty} L(\delta(\lambda, \mu), \lambda) \\ &= \lim_{\delta(\lambda, \mu) \rightarrow 0} R(\delta(\lambda, \mu), \mu) \\ &= B_2,\end{aligned}$$

similarly for $\mu \rightarrow \delta(\lambda, \mu)$. This proves (3) and hence the Lemma.

Corollary 4.4.1. Let $\delta_0 > 0$, then there exist $\lambda_1 \geq \lambda_0, \mu_1 \leq \mu_0$ such that for all $\lambda \geq \lambda_1, \mu \leq \mu_1$,

$$\delta(\lambda, \mu) \leq \delta_0. \quad (4.4.119)$$

Proof. Since $\delta(\lambda, \mu_0) \rightarrow 0$ as $\lambda \rightarrow \infty$, hence choose $\lambda_1 \geq \lambda_0$ such that $\delta(\lambda_1, \mu_0) \leq \delta_0$. Let $\mu_1 = \mu_0$, then for $\lambda \geq \lambda_1, \mu \leq \mu_1$, we have,

$$\delta(\lambda, \mu) \leq \delta(\lambda_1, \mu) \leq \delta(\lambda_1, \mu_1) \leq \delta_0.$$

This proves the Corollary.

Let $T > 0$ and $A_1 < B_1 \leq C \leq B_2 < A_2$ and for $i = 1, 2$, define $a_i, l_i, 0 < \delta_0 < T$ by

$$\begin{aligned}f'(a_i) &= \frac{C - A_i}{T} \\ l_i(t) &= A_i + t f'(a_i) \\ \delta_0 &= \min\{l_1(B_1), l_2(B_2)\}.\end{aligned}$$

Let u_1^λ and u_2^μ be solutions of (4.1.2) with respective initial data u_0^λ, u_0^μ given by

$$u_0^{1,\lambda}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \theta_f & \text{if } x > B_2. \end{cases} \quad (4.4.120)$$

$$u_0^{2,\mu}(x) = \begin{cases} \theta_f & \text{if } x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (4.4.121)$$

From Lemma 4.4.8 and 4.4.9 we can choose $\lambda_0 = \lambda_0(\|u_0\|_\infty), \mu_0 = \mu_0(\|u_0\|_\infty)$ such that

for all $\lambda \geq \lambda_0$, $\mu \leq \mu_0$, $t > 0$,

$$L(t, \lambda) = R_-(t, B_1, u_0^{1,\lambda}) = R_+(t, B_1, u_0^{1,\lambda}) \quad (4.4.122)$$

$$R(t, \mu) = R_-(t, B_2, u_0^{2,\mu}) = R_+(t, B_2, u_0^{2,\mu}), \quad (4.4.123)$$

and for $0 < t \leq T$, L and R satisfies the hypothesis (H_1) , (H_2) of Lemma 4.4.10. Let $(c(\lambda, \mu), \delta(\lambda, \mu))$ be the point of intersection of $L(t, \lambda)$ and $R(t, \mu)$ as defined in (4.4.114). From Corollary 4.4.1, choose $\lambda_1 \geq \lambda_0, \mu_1 \leq \mu_0$ such that for all $\lambda \geq \lambda_1, \mu \leq \mu_1$

$$\delta(\lambda, \mu) < \delta_0. \quad (4.4.124)$$

Lemma 4.4.11. *With the above notation and let $u(x, t, \lambda, \mu)$ be the solution of (4.1.2) with initial condition $u_0^{\lambda, \mu}$ given by*

$$u_0^{\lambda, \mu}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0 & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (4.4.125)$$

then for $0 < t < \delta(\lambda, \mu)$,

$$u_1^\lambda(x, t) = u_2^\mu(x, t) \quad \text{if } L(t, \lambda) < x < R(t, \mu), \quad (4.4.126)$$

$$u(x, t, \lambda, \mu) = \begin{cases} u_1^\lambda(x, t) & \text{if } x < L(t, \lambda), \\ u_1^\lambda(x, t) & \text{if } L(t, \lambda) < x < R(t, \mu), \\ u_2^\mu(x, t) & \text{if } x > R(t, \mu). \end{cases} \quad (4.4.127)$$

Proof. Let $\gamma = \frac{B_1+B_2}{2}$ and define $v_0^{1,\lambda}(x) = \int_\gamma^x u_0^{1,\lambda}(\theta) d\theta$, $v_0^{2,\mu}(x) = \int_\gamma^x u_0^{2,\mu}(\theta) d\theta$, $v_0^{\lambda,\mu}(x) = \int_\gamma^x u_0^{\lambda,\mu}(\theta) d\theta$. Then for $x \in [B_1, B_2]$,

$$v_0^{1,\lambda}(x) = v_0^{2,\mu}(x) = \int_{\frac{B_1+B_2}{2}}^x u_0(\theta) d\theta. \quad (4.4.128)$$

$$v_0^{\lambda,\mu}(x) = \begin{cases} v_0^{1,\lambda}(x) & \text{if } x < B_1, \\ v_0^{2,\mu}(x) & \text{if } x > B_2. \end{cases} \quad (4.4.129)$$

Claim: Let $v^{1,\lambda}, v^{2,\mu}$ be the corresponding value functions associated by $v_0^{1,\lambda}, v_0^{2,\mu}$ defined

in (4.4.4). Then

$$v^{1,\lambda}(x, t) = \inf_{y \in [B_1, B_2]} \left\{ v_0^{1,\lambda}(y) + t f^* \left(\frac{x-y}{t} \right) \right\}, \quad \text{if } L(t, \lambda) < x < B_2 \quad (4.4.130)$$

$$v^{2,\mu}(x, t) = \inf_{y \in [B_1, B_2]} \left\{ v_0^{2,\mu}(y) + t f^* \left(\frac{x-y}{t} \right) \right\}, \quad \text{if } B_1 < x < R(t, \lambda). \quad (4.4.131)$$

Let $L(t, \lambda) < x < B_2$, then from (4.4.122) $y_{\pm}(x, t, u_0^{1,\lambda}) > B_1$. Suppose for some $x_0 \in (L(t, \lambda), B_2)$, $y_+(x_0, t, u_0^{1,\lambda}) > B_2$. Since $v_0^{1,\lambda}$ is differentiable in (B_2, ∞) and hence from (4.4.13) for a.e. $x \in (x_0, B_2)$, $u^{1,\lambda}(x, t) = \frac{\partial v^{1,\lambda}}{\partial x}(x, t) = \theta_f$ and $0 = f'(\theta_f) = \frac{x - y_+(x, t, u_0^{1,\lambda})}{t}$. Hence $B_2 > x = y_+(x, t, u_0^{1,\lambda}) > B_2$, which is a contradiction. Therefore $y_{\pm}(x, t, u_0^{1,\lambda}) \in [B_1, B_2]$ and hence (4.4.130) follows. Similarly (4.4.131) holds. This proves the claim.

From (4.4.128), (4.4.130), (4.4.131), for $L(t, \lambda) < x < R(t, \mu)$, $v^{1,\lambda}(x, t) = v^{2,\mu}(x, t)$ and hence for a.e. x , $u^{1,\lambda}(x, t) = \frac{\partial v^{1,\lambda}}{\partial x}(x, t) = \frac{\partial v^{2,\mu}}{\partial x}(x, t) = u^{2,\mu}(x, t)$. This proves (4.4.126). In view of (4.4.126), RHS of (4.4.127) is a solution of (4.1.2) with initial data $u_0^{\lambda,\mu}$. Hence from uniqueness of solutions (4.4.127) follows. This proves the Lemma.

As an immediate consequence of Lemma 4.4.11 and (4.4.27), (4.4.122), (4.4.123) we have

Corollary 4.4.2. Let $\lambda \geq \lambda_1, \mu \leq \mu_1$, then

$$\begin{aligned} R_{\pm}(t, B_1, u_0^{\lambda,\mu}) &= L(t, \lambda) \quad 0 < t < \delta(\lambda, \mu), \\ R_{\pm}(t, B_2, u_0^{\lambda,\mu}) &= R(t, \mu) \quad 0 < t < \delta(\lambda, \mu), \\ R_{\pm}(t, B_1, u_0^{\lambda,\mu}) &= R_{\pm}(t, B_2, u_0^{\lambda,\mu}), \quad t \geq \delta(\lambda, \mu). \end{aligned}$$

Furthermore, denote $S(t, \lambda, \mu) = R_+(t, B_1, u_0^{\lambda,\mu})$ for $t > \delta(\lambda, \mu)$, then $(t, \lambda, \mu) \mapsto S(t, \lambda, \mu)$ is continuous and

$$u(x, t, \lambda, \mu) = \begin{cases} u^{1,\lambda}(x, t) & \text{if } x < S(t, \lambda, \mu), \\ u^{2,\mu}(x, t) & \text{if } x > S(t, \lambda, \mu), \end{cases} \quad (4.4.132)$$

Proof. Let $(t_k, \lambda_k, \mu_k) \rightarrow (t, \lambda, \mu)$. From Lemma 4.4.10, $\delta(\lambda_k, \mu_k) \rightarrow \delta(\lambda, \mu)$ and hence for $t > \delta(\lambda, \mu)$

$$\begin{aligned} |S(t_k, \lambda_k, \mu_k) - S(t, \lambda, \mu)| &\leq |S(t_k, \lambda_k, \mu_k) - S(t, \lambda_k, \mu_k)| \\ &\quad + |S(t, \lambda_k, \mu_k) - S(t, \lambda, \mu)|. \end{aligned}$$

From (4) of Lemma 4.4.2 and from (3) of Lemma 4.4.4, the right hand side tends to zero as $k \rightarrow \infty$. Let $v^{\lambda,\mu}$ be the cost function associated to $v_0^{\lambda,\mu}$ defined in (4.4.4). For $x < S(t, \lambda, \mu)$, $y_{\pm}(x, t, u_0^{\lambda,\mu}) < B_1$ and hence from (4.4.129), $v^{\lambda,\mu}(x, t) = v^{1,\lambda}(x, t)$. Hence $u(x, t, \lambda, \mu) = \frac{\partial v^{1,\lambda}}{\partial x}(x, t) = u^{1,\lambda}(x, t)$. Similarly for $x > S(t, \lambda, \mu)$, $u(x, t, \lambda, \mu) = u^{2,\mu}(x, t)$,

this proves (4.4.127) and hence the Lemma.

Lemma 4.4.12. *Let $\lambda \geq \lambda_1, \mu \leq \mu_1$ and $\delta(\lambda, \mu) < t_0 \leq T$, then*

(i). *Suppose $l_1(t_0) = S(t_0, \lambda, \mu)$. Then for all $t_0 < t < T$,*

$$S(t, \lambda, \mu) < l_1(t). \quad (4.4.133)$$

$$u(x, t, \lambda, \mu) = \begin{cases} a_2 & \text{if } 0 < t < T, x > l_2(t) \\ a_1 & \text{if } x < \min(l_1(t), S(t, \lambda, \mu)). \end{cases} \quad (4.4.134)$$

(ii). *Suppose $l_2(t_0) = S(t_0, \lambda, \mu)$. Then for all $t_0 < t < T$,*

$$S(t, \lambda, \mu) > l_2(t). \quad (4.4.135)$$

$$u(x, t, \lambda, \mu) = \begin{cases} a_1 & \text{if } 0 < t < T, x < l_1(t), \\ a_2 & \text{if } x > \max(l_2(t), S(t, \lambda, \mu)). \end{cases} \quad (4.4.136)$$

Furthermore there exist λ_2 and μ_2 such that $S(T, \lambda_2, \mu_2) = C$ and for $0 < t < T$, u satisfies

$$u(x, t, \lambda_2, \mu_2) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t). \end{cases} \quad (4.4.137)$$

Proof. (See Figure 4.11) Let $g(t) = \min(l_1(t), S(t, \lambda, \mu))$. Then we claim that for all $x < g(t)$,

$$u(x, t, \lambda, \mu) = a_1. \quad (4.4.138)$$

Suppose $x < l_1(t) \leq S(t, \lambda, \mu)$, then from (4.4.132), (4.4.95) we have $u(x, t, \lambda, \mu) = u^{1, \lambda}(x, t) = a_1$. Hence assume that $S(t, \lambda, \mu) < l_1(t)$. Suppose there exists $x_0 < S(t, \lambda, \mu)$ such that $y_+(x_0, t, u_0^{\lambda, \mu}) > A_1$, then for all $x \in (x_0, S(t, \lambda, \mu))$, $A_1 < y_+(x, t, u_0^{\lambda, \mu}) < B_1$. Since $u_0^{\lambda, \mu}$ is differentiable in (A_1, B_1) and hence from (4.4.13), for a.e. $x \in (x_0, S(t, \lambda, \mu))$

$$\begin{aligned} f'(\lambda) = f'(u(x, t, \lambda, \mu)) &= \frac{x - y_+(x, t, u_0^{\lambda, \mu})}{t} \\ &< \frac{l_1(t) - A_1}{t} = f'(a_1), \end{aligned}$$

which is a contradiction since $\lambda > a_1$. Hence $y_+(x, t, u_0^{\lambda, \mu}) \leq A_1$ for all $x \in (x_0, S(t, \lambda, \mu))$. Suppose $y_+(x_0, t, u_0^{\lambda, \mu}) = A_1$. Then from (4.4.21) $f'(u(x, t)) = \frac{x - A_1}{t}$ for $x \in (x_0, S(t,$

λ, μ). Let $\gamma_x(\theta) = A_1 + \theta \left(\frac{x-A_1}{t} \right) < l_1(\theta)$ be the characteristic at (x, t) , then from (c) of (3) in Theorem 4.2.1, γ_x is also a characteristic at $(\gamma_x(s), s)$ for $0 < s < t$ and $f'(u(\gamma_x(s), s, \lambda, \mu)) = \frac{\gamma_x(s)-A_1}{s} < \frac{l_1(s)-A_1}{s} = f'(a_1)$. Let $s < \delta(\lambda, \mu)$, then $l_1(s) < L(s, \lambda)$ and hence $f'(a_1) > f'(u(\gamma_x(s), s, \lambda, \mu)) = f'(a_1)$ which is a contradiction. This proves the claim.

Let $t_0 > \delta(\lambda, \mu)$ such that $l(t_0) = S(t_0, \lambda, \mu)$. From (4.4.132) and Lemma 4.4.9 for $x > S(t_0, \lambda, \mu)$,

$$a_2 = \max(\mu, a_2) \geq u(x, t_0, \lambda, \mu). \quad (4.4.139)$$

Let $t_0 < t < T$ and w be the solution of (4.1.2) with initial data w_0 at t_0 is given by

$$w_0(x) = \begin{cases} a_1 & \text{if } x < S(t_0, \lambda, \mu) = l_1(t_0), \\ a_2 & \text{if } x > S(t_0, \lambda, \mu) = l_1(t_0). \end{cases}$$

Then w admits a shock at $l_1(t_0)$ and for $t > t_0$ is given by

$$\begin{aligned} \eta(t) &= l_1(t_0) + \frac{f(a_1)-f(a_2)}{a_1-a_2}(t-t_0) \\ &< l_1(t_0) + f'(a_1)(t-t_0) \\ &= A_1 + f'(a_1)t \\ &= l_1(t), \end{aligned} \quad (4.4.140)$$

since f is strictly convex and $f'(a_1) > 0 > f'(a_2)$. From (4.4.138) and (4.4.139), $w_0(x) \geq$

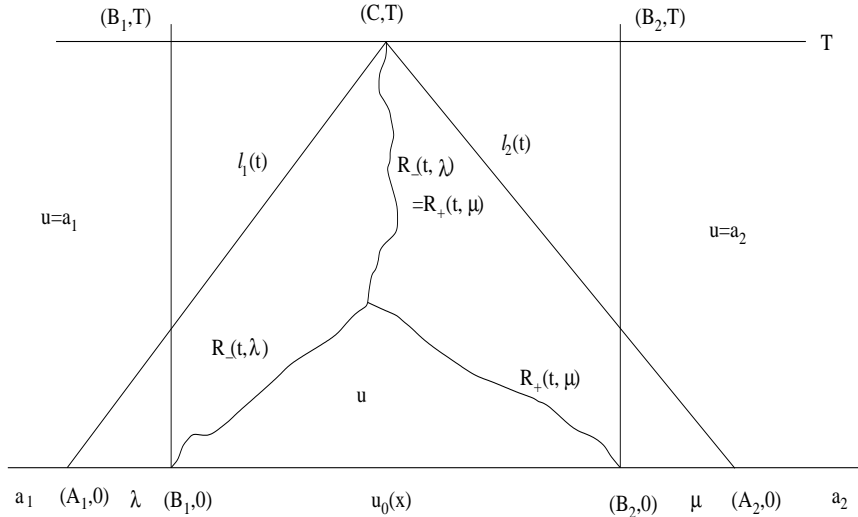


Fig. 4.11:

$u(x, t_0, \lambda, \mu)$ and therefore from (4.4.29) and (4.4.30) we have for $t > t_0$, $l_1(t) > \eta(t) \geq S(t, \lambda, \mu)$. This proves (4.4.133).

From (3) of Lemma 4.2.4, $(\lambda, \mu) \rightarrow S(T, \lambda, \mu)$ is a continuous function for $\lambda \geq \lambda_1$ and $\mu \leq \mu_1$. From (4.4.94), choose a $\tilde{\lambda}_1 > \lambda_1$ such that $S(T, \tilde{\lambda}_1, \mu_1) > T$ and from (4.4.109)

choose $\tilde{\mu}_1 < \mu_1$ such that $S(T, \lambda_1, \tilde{\mu}_1) < T$. From Corollary 4.4.1, S is continuous in $[\lambda_1, \tilde{\lambda}_1] \times [\mu_1, \tilde{\mu}_1]$ and therefore there exists a $(\lambda_2, \mu_2) \in [\lambda_1, \tilde{\lambda}_1] \times [\mu_1, \tilde{\mu}_1]$ such that $S(T, \lambda_2, \mu_2) = C$. Hence (4.4.137) follows from (4.4.136). This proves the Lemma.

Proof of Lemma 4.2.3. In Lemma 4.4.8, take $A = \alpha$, $B = \beta$, $l(t) = L_1(t)$. Then from (4.4.94), choose a λ_0 large such that for all $0 < t \leq T$ and for all $\lambda \geq \lambda_0$, $l(t) < R(\lambda, t)$. Then (4.2.23) follows from (4.4.95) and from Rankine-Hugoniot condition across $l(t)$. Similarly (4.2.24) follows from Lemma 4.4.9 and (4.4.107) and (4.4.108).

Proof of Lemma 4.2.4 This follows from Lemma 4.4.12 and (4.4.136) and Rankine-Hugoniot conditions across $l_1(t)$ and $l_2(t)$.

Example 4.4.13. (Counter Example) : Let $\alpha = 0, x_k < 0, \lim_{k \rightarrow \infty} x_k = 0, \lambda > \theta_f$ and define u_0, u_0^k by

$$u_0(x) = \begin{cases} \theta_f & \text{if } x < 0, \\ \lambda & \text{if } x > 0. \end{cases}$$

$$u_0^k(x) = \begin{cases} \theta_f & \text{if } x < x_k, \\ \lambda & \text{if } x > x_k. \end{cases}$$

Then the solution u and u_k with respective initial datas u_0 and u_0^k are given by

$$u(x, t) = \begin{cases} \theta_f & \text{if } x < 0, t > 0, \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } 0 < x \leq f'(\lambda)t, \\ \lambda & \text{if } x > f'(\lambda)t, \end{cases}$$

then

$$R_-(t, 0, u_0) = 0.$$

$$u_k(x, t) = \begin{cases} \theta_f & \text{if } x < x_k, t > 0, \\ (f')^{-1}\left(\frac{x-x_k}{t}\right) & \text{if } x_k < x < f'(\lambda)t + x_k, \\ \lambda & \text{if } x > f'(\lambda)t + x_k, \end{cases}$$

then

$$R_-(t, 0, u_0^k) = f'(\lambda)t,$$

$$\int_{\mathbb{R}} |u_0(x) - u_0^k(x)| dx = \int_{x_k}^0 (\lambda - \theta_f) = (\lambda - \theta_f)|x_k| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

But

$$\lim_{k \rightarrow \infty} R_-(t, 0, u_0^k) = f'(\lambda)t > 0 = R_-(t, 0, u_0).$$

Example 4.4.14. (Counter Example) : Let $A_1 = B_1 = C_1$, $A_2 = B_2 = C_2$, $\rho(x) = x$

for $x \in (C_1, C_2)$ and

$$u_0(x) = \begin{cases} a_2 & \text{if } x > B_2 \\ a_1 & \text{if } x < B_1, \end{cases}$$

where $a_2 < \theta_f$ and θ_f is the point of minima of f .

Suppose there exists a solution (u, \bar{u}_0) to problem (I), then by Lax-Oleinik formula we have

$$\begin{aligned} \bar{u}_0(x) &= \theta_f & \text{if } x \in (B_1, B_2) \\ u(x, t) &= \theta_f & \text{if } (x, t) \in (B_1, B_2) \times (0, T). \end{aligned}$$

On the otherhand, since $a_2 < \theta_f$ there is a shock wave entering the region $(B_1, B_2) \times (0, T)$ at $(B_2, 0)$ which is a contradiction because the solution $u = \theta_f$ in this region.

4.5 Extensions:

Proposition 4.5.1. (Controllability of constant states):

1. In theorem 4.1.1, $g(x) = m$ a constant if and only if m satisfies

$$\frac{C_2 - A_2}{T} \leq f'(m) \leq \frac{C_1 - A_1}{T}. \quad (4.5.1)$$

2. In theorem 4.1.2, $g(x_0) = m$ a constant if and only m satisfies

$$f'(m) \geq \frac{C}{T - \delta}. \quad (4.5.2)$$

3. In theorem 4.1.3, $g_1(x) = m_1, g_2(x) = m_2$ are constants. then g_1, g_2 is controllable if and only if m_1, m_2 satisfies

$$f'(m_1) \geq \frac{C - A_1}{T - \delta}, f'(m_2) \leq \frac{A_2 - C}{T - \delta}. \quad (4.5.3)$$

Proof. (1). $g(x) = m$ if and only if $\rho(x) = x - Tf'(m)$ for all $x \in (C_1, C_2)$. Hence from (4.1.22) we have $A_1 \leq \rho(x) \leq A_2$ implies that $\frac{x-A_2}{T} \leq f'(m) \leq \frac{x-A_1}{T}$ and hence (4.5.1) holds.

(2). From (4.1.23), $g(x) = m$ if and only if $\delta \leq \rho(x) \leq T$ and hence $\delta \leq x - Tf'(m) \leq T$. This implies (4.5.2). Similarly (4.5.3) follows from (4.1.25) and (4.1.26). This proves the theorem.

(3). Follows similarly.

4.5.1 Controllability on the boundary

As mentioned in the introduction problems (I) and (III) deal with the controllability at time $t = T$. What about the controllability at $x = A_2$. More precisely

Problem (IV): Let $T > 0$ and $A_1 < A_2$. Given $u_0 \in L^\infty(\mathbb{R})$, $g \in L^\infty(0, T)$ find $\bar{u}_0 \in L^\infty((A_1, A_2))$ and u a solution of (4.1.2) in $\Omega = (-\infty, A_2) \times (0, T)$ such that

$$f'(u(A_2, t)) = g(t) \quad \text{if } 0 < t \leq T, \quad (4.5.4)$$

and

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } x < A_1 \\ \bar{u}_0(x) & \text{if } A_1 < x < A_2. \end{cases} \quad (4.5.5)$$

Then we have the following

Theorem 4.5.1. : Let $A_1 < B < A_2$, $\wedge > 0$ and $\rho : [0, T] \rightarrow [B, A_2]$ be a non increasing left continuous function such that for all $t \in [0, T]$,

$$\left| \frac{A_2 - \rho(t)}{t} \right| \leq \wedge, \quad (4.5.6)$$

and $f'(g(t)) = \frac{A_2 - \rho(t)}{t}$. Then there exist (u, \bar{u}_0) satisfying (4.5.4) and (4.5.5).

Proof. Proof follows on the same lines as in theorem (4.1.2) and hence only sketch the main idea of the proof.

Step 1. This step is analogous to Lemma 4.2.1. Frist assume that ρ is discrete. That is there exist a partition $0 = t_n \leq t_{n-1} \leq \dots \leq t_0 = T$ and $B = x_0 < x_1 < \dots < x_n = A_2$. Define a_i and b_i by

$$\begin{aligned} f'(a_i) &= \frac{A_2 - x_i}{t_i}, \quad f'(b_i) = \frac{A_2 - x_{i-1}}{t_i} \\ s_i(t) &= A_2 + (t - t_i) \frac{f(a_i) - f(b_i)}{a_i - b_i} \\ f'(a_i) &= \frac{A_2 - x_i}{t_i} > \frac{A_2 - x_{i-1}}{t_i} = f'(b_i). \end{aligned}$$

Then

$$f'(b_i) = \frac{A_2 - x_{i-1}}{t_i} < \frac{A_2 - x_{i-1}}{t_{i-1}} = f'(a_{i-1}).$$

Hence $a_i > b_i$, $a_{i+1} > b_i$ and from convexity. $f'(a_i) > \frac{f(a_i) - f(b_i)}{a_i - b_i} > f'(b_i)$. Therefore

$$\begin{aligned} x_i = A_2 - t f'(a_i) &< A_2 - t \frac{f(a_i) - f(b_i)}{a_i - b_i} \\ &= s_i(0) < A_2 - t f'(b_i) \\ &= x_{i+1}. \end{aligned}$$

Hence for $0 \leq t \leq T$,

$$l_i(t) \leq s_i(t) \leq m_i(t),$$

where $l_i(t) = x_i + f'(a_i)t$, $m_i(t) = x_{i-1} + f'(b_i)t$. Define ρ_n and g_n by

$$\begin{aligned} \rho_n(t) &= x_0\chi_{[T,t_1]} + \sum_{i=1}^n x_i\chi_{(t_i,t_{i+1}]}(t) \\ f'(g_n(t)) &= \frac{A_2 - \rho_n(t)}{t}. \end{aligned}$$

Define u_n in $\Omega = (-\infty, A_2) \times (0, T)$ by

$$f'(u_n(x, t)) = \begin{cases} a_n & \text{if } x \leq l_n(t), \\ a_i & \text{if } l_i(t) \leq x < s_i(t), \\ b_i & \text{if } s_i(t) < x \leq m_i(t), \\ (f')^{-1}\left(\frac{A_2 - x_i}{t}\right) & \text{if } m_i(t) \leq x \leq l_{i-1}(t), \end{cases}$$

then u_n is a solution of (4.1.2) in Ω satisfying

$$\begin{aligned} f'(u_n(x, t)) &= a_n \quad \text{if } x \leq l_n(t) = B + t\left(\frac{A_2 - B}{T}\right) \\ f'(u_n(A_2, t)) &= \frac{x - \rho_n(t)}{t}. \end{aligned}$$

Let $\bar{u}_{n,0}(x) = u_n(x, 0)$ for $B \leq x \leq A_2$, then as in the proof of Lemma 4.2.1 and from (4.5.6), for a subsequence $u_n \rightarrow \tilde{u}$ in $L^1_{loc}(\Omega)$, $u_n(\cdot, 0) \rightarrow \tilde{u}_0$ in $L^1((B, A_2))$, $\rho_n \rightarrow \rho$ a. e. such that u satisfies (4.1.2) and for a.e. t ,

$$f'(\tilde{u}(A_2, t)) = \frac{A_2 - \rho(t)}{t} \quad \text{if } t \in (0, T), \quad (4.5.7)$$

$$\tilde{u}(x, 0) = \tilde{u}_0(x) \quad \text{if } x \in (B, A_2), \quad (4.5.8)$$

$$f'(\tilde{u}(x, t)) = \frac{A_2 - B}{T} \quad \text{if } x \leq l_0(t). \quad (4.5.9)$$

Step 2. From Lemma 4.4.9 there exists a μ and a solution u_1 of (4.1.2) in Ω satisfying

$$f'(u_1(x, t)) = a_0 \quad \text{if } x > l_0(t), \quad 0 \leq t < T \quad (4.5.10)$$

$$u_1(x, 0) = \begin{cases} a_0 & \text{if } x > B, \\ \mu & \text{if } A_1 < x < B, \\ u_0(x) & \text{if } x < A_1. \end{cases}$$

Now define (u, \bar{u}_0) in Ω by

$$u(x, t) = \begin{cases} \tilde{u}(x, t) & \text{if } x > l_1(t), \\ u_1(x, t) & \text{if } x < l_0(t), \end{cases}$$

$$\bar{u}_0(x) = \begin{cases} u_0(x) & \text{if } x < A_1, \\ \mu & \text{if } A_1 < x < B, \\ \tilde{u}_0(x) & \text{if } x \in (B, A_2). \end{cases}$$

Then (u, \bar{u}_0) is the required solution to problem (IV).

4.5.2 Controllability of initial and boundary values:

All three problems deals with finding either initial data or purely boundary data. In fact one can combine both and is as follows.

Problem V: Let $u_0 \in L^\infty, T > 0, 0 < C_1 < C_2, 0 < A$, Let $\rho_1 : [0, C_1] \rightarrow [0, T], \rho_2 : [C_1, C_2] \rightarrow [0, A]$ be such that

(i) ρ_1 is a non increasing right continuous function.

(ii) ρ_2 is a non decreasing function.

Define g_1 and g_2 by

$$f'(g_1(x)) = \frac{x}{T - \rho_1(x)} \quad \text{if } x \in [0, C_1]$$

$$f'(g_2(x)) = \frac{x - \rho_2(x)}{T} \quad \text{if } x \in [C_1, C_2].$$

Then the problem is to find $b \in L^\infty(0, T)$ and $\bar{u}_0 \in L^\infty(0, A)$ such that a solution u of (4.1.2) in $\mathbb{R} \times (0, T)$ satisfying the following initial boundary data

$$u(0, t) = b(t) \quad \text{if } 0 < t < T. \quad (4.5.11)$$

$$u(x, 0) = \begin{cases} \bar{u}_0(x) & \text{if } x \in (0, A), \\ u_0(x) & \text{if } x \in (A, 0), \end{cases} \quad (4.5.12)$$

and

$$f'(u(x, t)) = \begin{cases} g_1(x) & \text{if } x \in (0, C_1), \\ g_2(x) & \text{if } x \in (C_1, C_2). \end{cases} \quad (4.5.13)$$

Theorem 4.5.2. Let $\lambda > 0, 0 < A_1 < A$ be given. Let ρ_1 and ρ_2, g_1 and g_2 be as above and satisfying

$$0 \leq \rho_2(x) \leq A_1, \quad \left| \frac{x}{T - \rho_1(x)} \right| \leq \wedge \quad (4.5.14)$$

then problem (V) admits a solution.

Idea of the proof. First get a free region by choosing λ large such that the solution u_λ of (4.1.2) in $\mathbb{R} \times (0, \infty)$ satisfying for $0 < t < T$,

$$u_\lambda(x, t) = a_1 = \frac{C_2 - A_1}{T}, \quad \text{if } x < A_1 + tf'(a_1) = l_1(t),$$

$$u_\lambda(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < A, \\ u_0(x) & \text{if } x > A. \end{cases}$$

Existence of u_λ is guaranteed from Lemma 4.4.8. Let $f'(a_0) = \frac{C_1}{T}$ and for $0 < t < T$ define the free regions F_1 and F_2 by

$$F_1 = \{(x, t) : 0 < x < l_0(t) = tf'(a_0)\}, F_2 = \{(x, t) : l_0(t) < x < l_1(t) = A_1 + tf'(a_1)\}.$$

Since $0 \leq \rho_1(x) \leq T$ for $x \in (0, C_1)$ and satisfying (4.5.14), therefore from Lemma 4.4.1, there exists a solution u_1 of (4.1.2) in F_1 and $b \in L^\infty(0, T)$ such that

$$\begin{aligned} u_1(0, t) &= b(t) \\ u_1(x, T) &= g_1(x) \quad \text{if } x \in (0, C_1) \\ u_1(l_0(t)-, t) &= a_0. \end{aligned}$$

From Lemma 4.4.2, there exists a solution u_2 of (4.1.2) in F_2 and $\tilde{u}_0 \in L^\infty(0, A_1)$ such that

$$\begin{aligned} u_2(x, T) &= g_2(x) \quad \text{if } x \in (C_1, C_2) \\ u_2(x, 0) &= \tilde{u}_0(x) \quad \text{if } x \in (0, A_1) \\ u_2(l_0(t)+, t) &= a_0, \quad u_2(l_1(t)-, t) = a_1. \end{aligned}$$

From RH conditions, glue u_1, u_2, u_λ to a single solution u of (4.1.2) in $(0, \infty) \times (0, T)$ by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } (x, t) \in F_1, \\ u_2(x, t) & \text{if } (x, t) \in F_2, \\ u_\lambda(x, t) & \text{if } x > l_1(t), \end{cases}$$

and

$$u(x, 0) = \begin{cases} \tilde{u}_0(x) & \text{if } x \in (0, A_1), \\ \lambda & \text{if } x \in (A_1, A), \\ u_0(x) & \text{if } x \in (A, \infty). \end{cases}$$

Then $(u, u(x, 0), b)$ is the required solution to problem (V) The same method allows to generalize problem III also.

CHAPTER 5

OPTIMAL CONTROLLABILITY FOR SCALAR CONSERVATION LAWS WITH CONVEX FLUX

5.1 Introduction

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be C^1 , strictly convex function having super linear growth. That is

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty. \quad (5.1.1)$$

Let $u_0 \in L^\infty(\mathbb{R})$ and consider the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0, & \text{if } x \in \mathbb{R}, t > 0, \\ u(x, 0) &= u_0(x). \end{aligned} \quad (5.1.2)$$

In general, (5.1.2) does not admit classical solutions and hence look for weak solutions. This problem was well studied and showed that (5.1.2) admits a unique weak solution satisfying Lax-Olenik-Kruzkov entropy conditions. In the sequel we mean u a solution of (5.1.2) if it is a weak solution satisfying entropy condition.

In [23] following optimal control problem associated to (5.1.2) had been considered. Let $k \in L^2_{loc}(\mathbb{R})$ be a target function and $A \subset L^\infty(\mathbb{R})$, a set of admissible controls. Let u_0, u be the associated solution of (5.1.2) and $T > 0$. Define the cost functional \tilde{J} on A by

$$\tilde{J}(u_0) = \int_{-\infty}^{\infty} |u(x, T) - k(x)|^2 dx. \quad (5.1.3)$$

Then the optimal control problem is to find an $u_0 \in A$ such that

$$\tilde{J}(u_0) = \min_{w_0 \in A} \tilde{J}(w_0). \quad (5.1.4)$$

Under a suitable conditions on k and A , they prove that u_0 exists. In general u_0 is not unique. The basic problem related to this is “to capture a minimizer”. It is a very hard problem because the cost functional \tilde{J} is highly nonlinear, non differentiable and non convex. For the Burger’s equation, in [23], they have proposed a numerical scheme called “alternating descent direction” by using the linearization technique developed in [15],[13]. In that work, convergence analysis is completely open.

In this chapter, we tackle this problem in a completely different way. In view of the non linearity, we modify the cost function \tilde{J} to J so that the optimal control problem reduces to the standard convex optimization problem via Lax-Oleinik explicit formula. Then we use the backward construction introduced in [5] to obtain an optimal solution. The novelty of this method is that it is constructive and easy to derive a numerical scheme to capture an optimal solution.

5.2 Preliminaries

Before stating the main results, let us recall Hopf, Lax-Oleinik explicit formulas (see [31]).

Let $u_0 \in L^\infty(\mathbb{R})$ and f be as in the introduction. let f^* denote its convex dual defined by

$$f^*(p) = \sup_q \{pq - f^*(q)\} \quad (5.2.1)$$

then f^* is C^1 , superlinear growth and satisfies

$$\begin{aligned} f(p) &= \sup_q \{pq - f^*(q)\}, \\ f^*(f'(p)) &= pf'(p) - f(p), \\ f^{**} &= f'^{-1}. \end{aligned} \quad (5.2.2)$$

Let $b \in \mathbb{R}$ and define v_0 by

$$v_0(x) = \int_b^x u_0(y)dy, \quad (5.2.3)$$

and the associated value function v is given by

$$v(x, t) = \inf_y \left\{ v_0(y) + tf^* \left(\frac{x-y}{t} \right) \right\}. \quad (5.2.4)$$

Definition 5.2.1. (Characteristic): Define the characteristic set $ch(x, t)$, extreme char-

acteristics $y_{\pm}(x, t)$ by

$$ch(x, t) = \{y : y \text{ is a minimizer in (5.2.4)}\} \quad (5.2.5)$$

$$y_-(x, t) = \inf\{y : y \text{ is in } ch(x, t)\} \quad (5.2.6)$$

$$y_+(x, t) = \sup\{y : y \text{ is in } ch(x, t)\}. \quad (5.2.7)$$

Then we have the following theorem due to Hopf, Lax-Oleinik (see [31]).

Theorem 5.2.2. (1). v is a uniformly Lipschitz continuous function with Lipschitz constant depending only on $\|u_0\|_{\infty}$, f^* and satisfies the Hamilton-Jacobi equation

$$v_t + f(v_x) = 0 \quad x \in \mathbb{R}, t > 0, \quad (5.2.8)$$

$$v(x, 0) = v_0(x) \quad (5.2.9)$$

in the sense of viscosity.

(2). $ch(x, t) \neq \emptyset$ and there exists $M > 0$ depending only on $\|u_0\|_{\infty}$ and f^* such that for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $y \in ch(x, t)$

$$\left| \frac{x - y}{t} \right| \leq M. \quad (5.2.10)$$

(3). For each time t , $x \mapsto y_{\pm}(x, t)$ are non decreasing functions and for a.e. x ,

$$y_+(x, t) = y_-(x, t).$$

(4). Let $u = \frac{\partial v}{\partial x}$, then u is the solution of (5.1.2) such that for a.e. x , $t > 0$,

$$f'(u(x, t)) = \frac{x - y_+(x, t)}{t} = \frac{x - y_-(x, t)}{t}. \quad (5.2.11)$$

Furthermore

$$u(x, t) = u_0(y_{\pm}(x, t)) \quad (5.2.12)$$

if x is a point of differentiability of $y_{\pm}(x, t)$ and $y_{\pm}(x, t)$ is a point of differentiability for v_0 .

As an immediate consequence of this theorem, we have the following Lemma on finite speed of propagation.

Lemma 5.2.3. Let $A_1 < A_2$, $u_-, u_+ \in \mathbb{R}$, $\bar{u}_0 \in L^{\infty}(A_1, A_2)$. Define u_0 by

$$u_0(x) = \begin{cases} u_- & \text{if } x < A_1, \\ \bar{u}_0(x) & \text{if } A_1 < x < A_2, \\ u_+ & \text{if } x > A_2. \end{cases} \quad (5.2.13)$$

Then for $t > 0$, the solution u satisfies

$$u(x, t) = \begin{cases} u_- & \text{if } x < A_1 - Mt, \\ u_+ & \text{if } x > A_2 + Mt. \end{cases} \quad (5.2.14)$$

Proof. Let $t > 0$, then from (5.2.10), if $x < A_1 - Mt$, then

$$-M \leq \frac{x - y_{\pm}(x, t)}{t} < \frac{A_1 - Mt - y_{\pm}(x, t)}{t}.$$

Hence $y_{\pm}(x, t) < A_1$. Since v_0 is differentiable in $(-\infty, A_1)$ and hence from (5.2.12), at the differentiable point x of $y_{\pm}(x, t)$, we have

$$u(x, t) = u_0(y_{\pm}(x, t)) = u_-.$$

Similarly if $x > A_2 + Mt$, then for a.e. x , $u(x, t) = u_+$. This proves (5.2.14) and hence the Lemma.

5.3 Main results

Let

$$f(\theta_f) = \min_{\theta \in \mathbb{R}} f(\theta). \quad (5.3.1)$$

Define

1. Admissible sets A and B :

$$A = \{u_0 \in L^\infty(\mathbb{R}) : u_0(x) = \theta_f \text{ outside a compact set}\} \quad (5.3.2)$$

$$B = \{\rho \in L^\infty_{loc}(\mathbb{R}) : \begin{array}{l} (i) \quad \rho \text{ is a non decreasing function} \\ (ii) \quad \rho(x) = x \text{ outside a compact set} \end{array}\}. \quad (5.3.3)$$

For each $N > 0$, define

$$B_N = \{\rho \in B : \rho(x) = x \text{ for } |x| > N\}. \quad (5.3.4)$$

2. Target function k :

Let $I = (C_1, C_2)$ and k be a measurable function such that

$$k(x) = \theta_f \quad \text{if } x \notin (C_1, C_2), \quad (5.3.5)$$

$$f'(k(x)) \in L^2(I). \quad (5.3.6)$$

3. Cost function J :

Let k be a target function and $u_0 \in A$. Let $u(x, t)$ be the

corresponding solution of (5.1.2) with initial data u_0 . Let $T > 0$ and define the modified cost function

$$J(u_0) = \int_{\mathbb{R}} |f'(u(x, T)) - f'(k(x))|^2 dx. \quad (5.3.7)$$

Then we have the following proposition.

Proposition 5.3.1. *J is well defined.*

Proof. Choose $A_1 < C_1 < C_2 < A_2$ such that $u_0(x) = \theta_f$ for $x \notin (A_1, A_2)$. Then from Lemma 5.2.3, there exists an $M > 0$ depending only on $\|u_0\|_\infty$ and f^* such that

$$u(x, T) = \theta_f \quad \text{for } x \notin [A_1 - MT, A_2 + MT] \quad (5.3.8)$$

and hence $f'(u(x, T)) = f'(\theta_f) = 0$. Hypothesis on k implies for $x \notin [C_1, C_2]$

$$f'(k(x)) = f'(\theta_f) = 0.$$

Hence

$$J(u_0) = \int_{A_1 - MT}^{A_2 + MT} |f'(u(x, T)) - f'(k(x))|^2 dx. \quad (5.3.9)$$

This proves the proposition.

Optimal control problem : Given A, k as above, find a $\tilde{u}_0 \in A$ such that

$$J(\tilde{u}_0) = \min_{u_0 \in A} J(u_0) \quad (5.3.10)$$

and if the minimizer exists, then devise a scheme to capture it.

Then we have the following main result.

Theorem 5.3.1. *There exists a minimizer for (5.3.10) which can be captured by using the standard convex optimization problem in a Hilbert space and backward construction.*

In order to prove this Theorem, first we reduce the problem to a standard projection Theorem in a Hilbert space and then use the backward construction.

Reduction to a projection on a convex set : Let $y_+(x, t)$ be the right extreme characteristic corresponding to the solution u . Let

$$\rho(x) = y_+(x, T), \quad x \in \mathbb{R},$$

then ρ is a non decreasing function and from (5.2.11) for a.e. x ,

$$f'(u(x, t)) = \frac{x - \rho(x)}{T}. \quad (5.3.11)$$

Since $u(x, T) = \theta_f$ for $x \notin [A_1 - MT, A_2 + MT]$, hence $\rho(x) = x$ for $x \notin [A_1 - MT, A_2 + MT]$ and

$$\begin{aligned} J(u_0) &= \int_{-\infty}^{\infty} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^2 dx \\ &= \int_{A_1 - MT}^{A_2 + MT} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^2 dx. \end{aligned} \quad (5.3.12)$$

Now define \bar{J} on B by

$$\bar{J}(\rho) = \int_{-\infty}^{\infty} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^2 dx. \quad (5.3.13)$$

Then from (5.3.11) to (5.3.13) we have

$$\inf_{\rho \in B} \bar{J}(\rho) \leq \inf_{u_0 \in A} J(u_0). \quad (5.3.14)$$

Lemma 5.3.2. *Let*

$$\tilde{B} = \{\rho \in B : \rho(x) = x \text{ if } x \notin [\min(C_1, \rho(C_1)), \max(C_2, \rho(C_2))]\}.$$

Then

$$\inf_{\rho \in \tilde{B}} \bar{J}(\rho) = \inf_{\tilde{\rho} \in \tilde{B}} \bar{J}(\tilde{\rho}). \quad (5.3.15)$$

Proof. Since $\tilde{B} \subset B$, hence

$$\inf_{\tilde{\rho} \in \tilde{B}} \bar{J}(\tilde{\rho}) \geq \inf_{\rho \in B} \bar{J}(\rho). \quad (5.3.16)$$

Let $\rho \in B$ and define $\tilde{\rho}$ by

$$\tilde{\rho}(x) = \begin{cases} x & \text{if } x \notin [\min(C_1, \rho(C_1)), \max(C_2, \rho(C_2))], \\ \rho(C_1) & \text{if } \min(C_1, \rho(C_1)) \leq x < C_1 \\ \rho(x) & \text{if } x \in [C_1, C_2] \\ \rho(C_2) & \text{if } C_2 < x < \max(C_2, \rho(C_2)). \end{cases}$$

Then

$$\begin{aligned}
\bar{J}(\tilde{\rho}) &= \int_{\min(C_1, \rho(C_1))}^{\max(C_2, \rho(C_2))} \left| \frac{x - \tilde{\rho}(x)}{T} - f'(k(x)) \right|^2 dx \\
&= \int_{C_1}^{C_2} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^2 dx + \int_{\min(C_1, \rho(C_1))}^{C_1} \left| \frac{x - \rho(C_1)}{T} \right|^2 dx \\
&\quad + \int_{C_2}^{\max(C_2, \rho(C_2))} \left| \frac{x - \rho(C_2)}{T} \right|^2 dx.
\end{aligned}$$

Since $\rho(x) \leq \rho(C_1)$ for $x \in (\min(C_1, \rho(C_1)), C_1)$ and $\rho(x) \geq \rho(C_2)$ for $x \in (C_2, \max(C_2, \rho(C_2)))$, hence

$$\begin{aligned}
\bar{J}(\tilde{\rho}) &\leq \int_{C_1}^{C_2} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^2 dx \\
&\quad + \int_{\min(C_1, \rho(C_1))}^{C_1} \left| \frac{x - \rho(x)}{T} \right|^2 dx + \int_{C_2}^{\max(C_2, \rho(C_2))} \left| \frac{x - \rho(x)}{T} \right|^2 dx \\
&\leq \int_{-\infty}^{\infty} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^2 dx \\
&= \bar{J}(\rho).
\end{aligned}$$

Hence

$$\inf_{\tilde{\rho} \in \tilde{B}} \bar{J}(\tilde{\rho}) \leq \inf_{\rho \in B} J(\rho).$$

This proves the Lemma.

Lemma 5.3.3. *Let*

$$\begin{aligned}
\tilde{C}_1 &= C_1 - (3T^2 (\|f'(k)\|_2^2 + 1))^{1/3} \\
\tilde{C}_2 &= C_2 + (3T^2 (\|f'(k)\|_2^2 + 1))^{1/3} \\
\tilde{B}_0 &= \{\rho \in \tilde{B} : \rho(x) = x \text{ for } x \notin [\tilde{C}_1, \tilde{C}_2]\},
\end{aligned}$$

then there exists a unique $\tilde{\rho}_0 \in \tilde{B}_0$ such that

$$\bar{J}(\tilde{\rho}_0) = \min_{\tilde{\rho} \in \tilde{B}_0} \bar{J}(\tilde{\rho}) = \min_{\rho \in B} J(\rho).$$

Proof. Let $\rho_k \in \tilde{B}$ be such that

$$\lim_{k \rightarrow \infty} \bar{J}(\tilde{\rho}_k) = \inf_{\tilde{\rho} \in \tilde{B}} \bar{J}(\tilde{\rho}).$$

Let $\eta(x) = x$ for all $x \in \mathbb{R}$, then $\eta \in \tilde{B}$ and

$$\begin{aligned} \|f'(k)\|_2^2 &= \bar{J}(\eta) \geq \lim_{k \rightarrow \infty} \bar{J}(\tilde{\rho}_k) \\ &\geq \lim_{k \rightarrow \infty} \left\{ \int_{\min(C_1, \rho_k(C_1))}^{C_1} \left| \frac{x - \rho_k(x)}{T} \right|^2 dx + \int_{C_2}^{\max(C_2, \rho_k(C_2))} \left| \frac{x - \rho_k(x)}{T} \right|^2 dx \right\}. \end{aligned}$$

Since

$$\rho_k(x) = \begin{cases} \rho_k(C_1) & \text{if } x \in (\min(C_1, \rho_k(C_1)), C_1), \\ \rho_k(C_2) & \text{if } x \in (C_2, \max(C_2, \rho_k(C_2))) \end{cases}$$

and hence there exists $k_0 > 0$ such that for all $k \geq k_0$,

$$\begin{aligned} 3T^2 (\|f'(k)\|_2^2 + 1) &\geq 3T^2 \int_{\min(C_1, \rho_k(C_1))}^{C_1} \left(\frac{x - \rho_k(C_1)}{T} \right)^2 dx \\ &\quad + 3T^2 \int_{C_2}^{\max(C_2, \rho_k(C_2))} \left(\frac{x - \rho_k(C_2)}{T} \right)^2 dx \\ &= (C_1 - \min(C_1, \rho_k(C_1)))^3 + (\max(C_2, \rho_k(C_2)) - C_2)^3. \end{aligned}$$

Hence

$$\begin{aligned} \min(C_1, \rho_k(C_1)) &\geq \tilde{C}_1 \\ \max(C_2, \rho_k(C_2)) &\leq \tilde{C}_2. \end{aligned}$$

This implies that $\rho_k(C_1), \rho_k(C_2) \in [\tilde{C}_1, \tilde{C}_2]$ and hence

$$\inf_{\tilde{\rho} \in \tilde{B}} \bar{J}(\tilde{\rho}) = \inf_{\tilde{\rho} \in \tilde{B}_0} \bar{J}(\tilde{\rho}).$$

Let

$$H = L^2(\tilde{C}_1, \tilde{C}_2), \quad K = \{\rho \in \tilde{B} : \tilde{C}_1 \leq \min(C_1, \rho(C_1)) \leq \max(C_2, \rho(C_2)) \leq \tilde{C}_2\}.$$

Then K is a closed convex set in H and hence from the projection Theorem in a Hilbert space, there exists a unique $\tilde{\rho}_0 \in K$ such that

$$\bar{J}(\tilde{\rho}_0) = \min_{\tilde{\rho} \in K} \bar{J}(\tilde{\rho}).$$

This proves the Lemma.

In order to prove the main results, we need the following backward construction Lemma which is proved in [5],[2]. For the sake of completeness we prove this Lemma after completing the proof of the theorem.

Lemma 5.3.4. (*Backward construction*): Let $D_1 < D_2$, $A_1 < A_2$ and $\rho : [D_1, D_2] \rightarrow [A_1, A_2]$ be a non decreasing function. Define a_1 and a_2 by

$$f'(a_1) = \frac{D_1 - A_1}{T}, \quad f'(a_2) = \frac{D_2 - A_2}{T}.$$

Then there exists a $\bar{u}_0 \in L^\infty(A_1, A_2)$ and the solution u of (5.1.2) with initial data u_0 given by

$$u_0(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \bar{u}_0(x) & \text{if } A_1 < x < A_2, \\ a_2 & \text{if } x > A_2 \end{cases} \quad (5.3.17)$$

satisfies

$$f'(u(x, T)) = \frac{x - \rho(x)}{T} \quad \text{if } x \in [D_1, D_2] \quad (5.3.18)$$

$$u(x, t) = \begin{cases} a_1 & \text{if } x < A_1 + tf'(a_1), \quad t < T \\ a_2 & \text{if } x > A_2 + tf'(a_2), \quad t < T. \end{cases} \quad (5.3.19)$$

Proof of Theorem 5.3.1 From Lemma 5.3.3, there exists a unique minimizer $\tilde{\rho}_0 \in \tilde{B}_0$ for \bar{J} . Take $D_1 = A_1 = \tilde{C}_1$, $D_2 = A_2 = \tilde{C}_2$, $\rho = \tilde{\rho}_0$, then $\tilde{\rho}_0$ satisfies the hypothesis of Lemma 5.3.4. Hence there exists an initial data u_0 and the corresponding solution u such that

$$\begin{aligned} f'(u(x, T)) &= \frac{x - \tilde{\rho}_0(x)}{T} \quad \text{if } x \in [\tilde{C}_1, \tilde{C}_2], \\ u_0(x) &= \theta_f \quad \text{if } x \notin [\tilde{C}_1, \tilde{C}_2]. \end{aligned}$$

Since $f'(a_1) = f'(a_2) = 0$. Therefore $u_0 \in A$ and hence from (5.3.14)

$$\inf_{w_0 \in A} J(w_0) \leq J(u_0) = \bar{J}(\tilde{\rho}_0) = \inf_{\rho \in B} \bar{J}(\rho) \leq \inf_{w_0 \in A} J(w_0).$$

Hence u_0 is an optimal solution to the problem and this proves the Theorem.

Initial Value problem partitions: (See Figure 5.1) Let $I = (A_1, A_2)$, $J = (D_1, D_2)$. Let $P = \{y_0, y_1 \dots y_n, x_0, x_1 \dots x_n\}$ is called a partition of (I, J) if

$$A_1 = y_0 < y_1 < \dots < y_n = A_2, \quad D_1 = x_0 \leq x_1 \leq \dots \leq x_n = D_2.$$

Let $P(I, J) = \{P : P \text{ is a partition of } (I, J)\}$. For a partition P denote $a_i(P)$, $s_i(P)$, $b_i(P)$, $a_i(t, P)$, $s_i(t, P)$, $b_i(t, P)$ by

$$\begin{aligned} f'(a_i(P)) &= \frac{x_i - y_i}{T}, \\ f'(b_i(P)) &= \frac{x_i - y_{i+1}}{T}, \\ s_i(P) &= \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)}, \\ a_i(t, P) &= x_i + f'(a_i(P))(t - T), \\ b_i(t, P) &= x_i + f'(b_i(P))(t - T), \\ s_i(t, P) &= x_i + s_i(P)(t - T). \end{aligned}$$

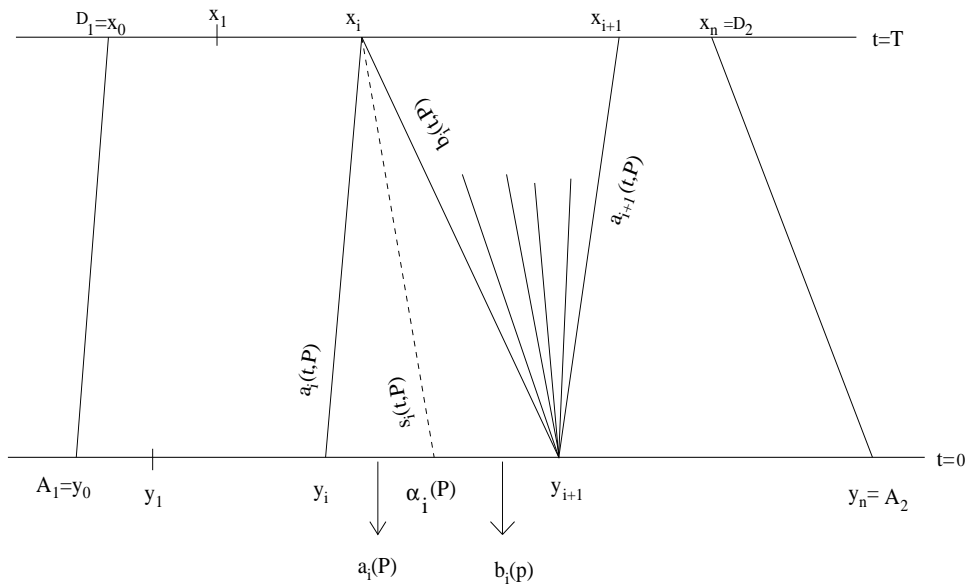


Fig. 5.1:

Clearly $a_i(0, P) = y_i$, $b_i(0, P) = y_{i+1}$.

Lemma 5.3.5. Let $\alpha_i(P) = s_i(0, P)$, then for $0 \leq t \leq T$,

$$b_i(P) < a_i(P), \quad b_i(P) \leq a_{i+1}(P), \quad (5.3.20)$$

$$y_i < \alpha_i(P) < y_{i+1}, \quad (5.3.21)$$

$$a_i(t, P) < s_i(t, P) < b_i(t, P) \quad \text{for } 0 < t < T. \quad (5.3.22)$$

Proof. Since $y_i < y_{i+1}$, hence

$$f'(a_i(P)) = \frac{x_i - y_i}{T} > \frac{x_i - y_{i+1}}{T} = f'(b_i(P)),$$

therefore $a_i(P) > b_i(P)$. By convexity of f

$$f'(a_i(P)) > \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)} > f'(b_i(P)),$$

$$\frac{x_i - y_i}{T} > \frac{x_i - \alpha_i(P)}{T} > \frac{x_i - y_{i+1}}{t},$$

and hence $y_i < \alpha_i(P) < y_{i+1}$. Since $x_i \leq x_{i+1}$ and hence $f'(b_i(P)) = \frac{x_i - y_{i+1}}{T} \leq \frac{x_{i+1} - y_{i+1}}{T} = f'(a_{i+1}(P))$. This implies $b_i(P) \leq a_{i+1}(P)$. This proves (5.3.20) to (5.3.22) and hence the Lemma.

Let $\Omega_i(P) = \{(x, t) : 0 < t < T, a_i(t, P) < x < a_{i+1}(t, P)\}$. In view of Lemma 5.3.5, let $u_i(x, t, P)$ be a solution of (5.1.2) in $\Omega_i(P)$ defined by

$$u_i(x, t, P) = \begin{cases} a_i(P) & \text{if } a_i(t, P) < x < s_i(t, P), \\ b_i(P) & \text{if } s_i(t, P) < x < b_i(t, P), \\ (f')^{-1}\left(\frac{x - y_{i+1}}{t}\right) & \text{if } b_i(t, P) \leq x < a_{i+1}(t, P). \end{cases} \quad (5.3.23)$$

Hence $u_i(a_{i+1}(t, P)-, t, P) = a_{i+1}(P) = u_{i+1}(a_{i+1}(t, P)+, t, P)$. Therefore define the solution $u(x, t, P)$ of (5.1.2) in $\mathbb{R} \times (0, T)$ by

$$u(x, t, P) = \begin{cases} u_i(x, t, P) & \text{if } (x, t) \in \Omega_i(P), \\ a_0(P) & \text{if } x < a_0(t, P), \\ a_n(P) & \text{if } x > a_n(t, P), \end{cases} \quad (5.3.24)$$

satisfying the initial condition

$$u(x, P) = \begin{cases} u_0(x, P) & \text{if } x \in (A_1, A_2), \\ a_0(P) & \text{if } x < A_1, \\ a_n(P) & \text{if } x > A_2, \end{cases} \quad (5.3.25)$$

where u_0 is given by

$$u_0(x, P) = \begin{cases} a_i(P) & \text{if } y_i < x < \alpha_i(P), \\ b_i(P) & \text{if } \alpha_i(P) < x < y_{i+1}. \end{cases} \quad (5.3.26)$$

Furthermore at $t = T, x \in (D_1, D_2), u$ satisfies

$$f'(u(x, T, P)) = \sum_{i=0}^{n-1} \chi_{[x_i, x_{i+1})}(x) \left(\frac{x - y_{i+1}}{T} \right). \quad (5.3.27)$$

Next we calculate the L^∞ and TV bounds of u_0 . First observe that $f'(a_0(P)) = \frac{D_1 - A_1}{T}$ and $f'(a_n(P)) = \frac{D_2 - A_2}{T}$, hence $a_0(P), a_n(P)$, are independent of P and denote

$$a_0 = a_0(P), a_n = a_n(P),$$

$$a_0(t) = a_0(t, P), a_n(t) = a_n(t, P).$$

Let $M = \frac{\max(D_2, A_2) - \min(D_1, A_1)}{T}$, then $|f'(a_i(P))| \leq \left| \frac{x_i - y_i}{T - \delta} \right| \leq M, |f'(b_i(P))| \leq M$ and hence

$$|f'(u_0(x, p))| \leq M. \quad (5.3.28)$$

$$\begin{aligned} TV(f'(u_0(\cdot, P))) &= \sum_{i=0}^n |f'(a_i(P)) - f'(b_i(P))| \\ &= + \sum_{i=0}^{n-1} |f'(b_i(P)) - f'(a_{i+1}(P))| \\ &= \sum_{i=0}^n \left| \frac{x_i - y_i}{T} - \frac{x_i - y_{i+1}}{T} \right| + \sum_{i=0}^{n-1} \left| \frac{x_i - y_{i+1}}{T} - \frac{x_{i+1} - y_{i+1}}{T} \right| \\ &= \frac{1}{T}(A_2 - A_1) + \frac{1}{T}(D_2 - D_1). \end{aligned} \quad (5.3.29)$$

Since $|f'(\theta)| \rightarrow \infty$ as $|\theta| \rightarrow \infty$ and hence we have

Lemma 5.3.6. *There exists a constant M_1 independent of P such that*

$$\|u_0(\cdot, P)\|_\infty + TV(f'(u_0(\cdot, P))) \leq M_1. \quad (5.3.30)$$

Furthermore, for $\phi \in C_c^\infty(\mathbb{R} \times [0, T])$ we have

$$\begin{aligned} \int_{-f}^f \int_0^T (u(x, t, P)\phi_t + f(u(x, t, P))\phi_x) dx dt + \int_{-\infty}^{\infty} u_0(x, P)\phi(x, 0) dx \\ = \int_{-\infty}^{\infty} u(x, T, P)\phi(x, T) dx. \end{aligned} \quad (5.3.31)$$

Let ρ be a strictly increasing continuous function. Let $n \geq 1$ and $A_1 = y_0 < y_1 < \dots < y_n = A_2$ such that $|y_i - y_{i+1}| \leq \frac{1}{n}$. Let k be such that $\{x : \rho(x) < y_{k-1}\} = \phi$ and $\{x : \rho(x) < y_k\} \neq \phi$. Define $x_i = A_1$ if $i \leq k - 1$ and for $i \geq k$,

$$\{x : \rho(x) < y_i\} = [A_1, x_i).$$

Let $P_n = \{y_0, y_1 \dots y_n, x_0, x_1 \dots x_n\}$ be the corresponding partition and define

$$\rho(x, P_n) = \sum_{i=0}^{n-2} y_i \chi_{[x_i, x_{i+1})}(x) + y_{n-1} \chi_{[x_{n-1}, D_2]}(x).$$

Clearly

$$|\rho(x) - \rho(x, P_n)| \leq \frac{1}{n},$$

and hence $\rho(x, P_n) \rightarrow \rho(x)$ uniformly as $n \rightarrow \infty$.

Proof of Lemma 5.3.4 First assume that ρ is a strictly increasing continuous function. For $n \geq 1$, let P_n and $\rho_n(x) = \rho(x, P_n)$ be constructed as above. Let $u_n(x, t) = u(x, t, P_n)$ as in (5.3.21) a solution of (5.1.2) with initial data $u_{0,n}(x) = u_0(x, \delta, P_n)$ as in (5.3.22). From (5.3.27), $\{f'(u_{0,n})\}$ is bounded in $BV_{loc}(\mathbb{R})$ and hence by Helly's theorem, there exists a subsequence still denoted by $\{f'(u_{0,n})\}$ converges to $f'(u_0)$ in L^1_{loc} and *a.e.* Since $u_{0,n}$ is uniformly bounded and f' is strictly increasing function, therefore $u_{0,n} \rightarrow u_0$ in L^1_{loc} . Hence from L^1_{loc} contractivity, u_n converges to \tilde{u} a solution of (5.1.2) *a.e.* $(x, t) \in \mathbb{R} \times (0, T)$ with initial data u_0 . Since $f'(a_0(P)) = \frac{D_1 - A_1}{T}$, $f'(a_n(P)) = \frac{D_2 - A_2}{T}$, hence if $\tilde{u}_0 = \tilde{u}_0|_{[A_1, A_2]}$, then from Rankine-Hugoniot condition across $a_0(t), a_n(t), (\tilde{u}, \tilde{u}_0)$ satisfies (5.3.17) to (5.3.19). At $t = T$,

$$f'(u_n(x, T)) = \frac{x - \rho_n(x)}{T}, \quad \text{if } x \in [D_1, D_2],$$

and hence letting $n \rightarrow \infty$ and from (5.3.31) for *a.e.* $x \in (D_1, D_2)$,

$$f'(u(x, T)) = \frac{x - \rho(x)}{T}.$$

Then (\tilde{u}, \tilde{u}_0) is a required solution. For general ρ , let ρ_n be a strictly increasing continuous function with values in (A_1, A_2) and converging to ρ in L^1 and *a.e.* Let $(\tilde{u}_n, \tilde{u}_{n,0})$ be the corresponding solutions satisfying (5.3.17) to (5.3.19). Hence from Helly's theorem, there exists a subsequence still denoted by $\tilde{u}_{n,0}$ converging to \tilde{u}_0 in L^1_{loc} and *a.e.* Therefore from L^1_{loc} contractivity, for a subsequence still denoted by \tilde{u}_n converging to \tilde{u} *a.e.* to a solution of (5.1.2) satisfying (5.3.17) to (5.3.19).

For *a.e.* $x \in (D_1, D_2)$ we have

$$f'(\tilde{u}(x, t)) = \lim_{n \rightarrow \infty} f'(\tilde{u}_n(x, T)) = \lim_{n \rightarrow \infty} \frac{x - \rho_n(x)}{T} = \frac{x - \rho(x)}{T}.$$

This proves the Lemma.

5.4 Generalization

Remark 5.4.1. In the optimal Control Problem: condition on k can be relaxed and is as follows. We can assume that for $x \notin (C_1, C_2)$

$$k(x) = \begin{cases} \alpha_1 & \text{if } x < C_1, \\ \alpha_2 & \text{if } x > C_2. \end{cases}$$

Then the class B is defined as follows. Let $\rho \in L_{loc}^\infty(\mathbb{R})$ be such that

- (i). ρ is non decreasing function
- (ii). There exist $A_1 < A_2$ such that

$$\rho(x) = \begin{cases} x - \alpha_1 T & \text{if } x < A_1, \\ x - \alpha_2 T & \text{if } x > A_2. \end{cases}$$

Let

$$B = \{\rho; \rho \text{ satisfying (i) and (ii)}\}.$$

Then by the similar arguments one can show that there exists a unique $\tilde{\rho} \in B$ such that

$$J(\tilde{\rho}) = \inf_{\rho \in B} J(\rho). \quad (5.4.1)$$

In view of the controllability of initial and boundary value problems, we can extend the optimal controllability for the boundary value problem. To illustrate this, let us consider one sided initial boundary value problem. Let $0 < T, 0 < C$, and $k \in L_{loc}^2(\mathbb{R})$ such that $k(x) = \theta_f$ for x large. Let $u_0 \in L^\infty, b \in L^\infty(0, T)$ and u be the solution of (1.2) in $\Omega = (0, \infty) \times (0, T)$ with

$$u(t, 0) = b(t) \quad 0 < t < T,$$

$$u(x, 0) = u_0(x) \quad x > 0,$$

$$J(u_0, b) = \int_{\mathbb{R}} |f'(u(x, T)) - f'(k(x))|^2 dx.$$

In order to make integral finite assume $u_0(x) = \theta_f$ for x large. Hence define

$$A = \{(u_0, b) \in L^\infty(\mathbb{R}) \times L^\infty(0, T); u_0(x) = \theta_f, \text{ for large } x\}.$$

Then optimal control problem is to find (\tilde{u}_0, \tilde{b}) such that

$$J(\tilde{u}_0, \tilde{b}) = \inf_{(u_0, b) \in A} J(u_0, b). \quad (5.4.2)$$

From Joseph-Gowda [40] and Lax - Olienik [31] formulas for any $(u_0, b) \in A$, there exist $0 \leq C_1 \leq C$ and $\rho_1 : [0, C_1] \rightarrow [0, T]$ a non increasing function and $\rho_2 : [C_1, C] \rightarrow \mathbb{R}$ a non decreasing function such that

(i). $\rho_2(x) = x$ for x large and

$$f'(u(x, T)) = \frac{x - \rho_2(x)}{T} \quad \text{if } x \in (C_1, C).$$

(ii). $f'(u(x, t)) = \frac{x}{T - \rho_1(x)}$, $0 < x < C_1$ and

$$\left| \frac{x}{T - \rho(x)} \right| \leq \wedge,$$

where \wedge is a constant depending on the Lipschitz constant of f on $[-\|b\|_\infty, \|b\|_\infty]$.

Therefore

$$\begin{aligned} J(u_0, b) &= \int_0^{C_1} \left| \frac{x}{T - \rho_1(x)} - f'(k(x)) \right|^2 dx + \int_{C_1}^C \left| \frac{x - \rho_2(x)}{T} - f'(k(x)) \right|^2 dx \\ &= J_1(\rho_1, C_1) + J_2(\rho_2, C_1) \\ &= J(\rho_1, \rho_2, C_1) \end{aligned}$$

$$m = \inf_{u_0, b} J(u_0, b) = \inf_{\rho_1, \rho_2, C_1} \{J_1(\rho_1, C_1) + J_2(\rho_2, C_1)\}.$$

Let $(\rho_{1,k}, \rho_{2,k}, C_{1,k})$ be a minimizing sequence. As in the previous case, it follows that there exists $A > 0$, such that $\rho_{2,k}(x) = x$ for all k and $x \geq A$. Hence $\{\rho_{i,k}\}$ are uniformly bounded monotone functions, therefore from Helly's theorem there exists a subsequence still denoted by $\{\rho_{1,k}, \rho_{2,k}, C_k\}$ converges $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{C})$ for all $x \in [0, C]$. Hence by Fataou's Lemma $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{C})$ is an optimal solution and from [2] we can construct the solution (\tilde{u}_0, \tilde{b}) .

If \tilde{C} is known, then $\tilde{\rho}_1$ and $\tilde{\rho}_2$ can be obtained from the L^2 - projection as follows: Let $\eta_1(x) = \frac{x}{T - \rho_1(x)}$ then $\frac{x}{\eta_1} \leq T$, $\frac{\eta_1(x)}{x}$ is a non increasing right continuous function. Let $\eta : [0, C] \rightarrow \mathbb{R}$ be function such that

1. $\frac{\eta(x)}{x}$ is non increasing right, continuous function in $[0, \tilde{C}]$ and $\frac{x}{\eta(x)} \leq T$ in $[0, \tilde{C}]$.
2. $\eta|_{(\tilde{C}, C)}$ is non decreasing function.

Let

$$K_N = \{\eta; \eta \text{ satisfying (1) , (2) and } 0 \leq \eta \leq N\}.$$

Then K_N is a closed convex set in $L^2((0, C))$ and let $\tilde{\eta}_k \in K_N$ such that

$$J(\tilde{\eta}) = \inf_{\eta \in K_N} J(\eta) = \inf_0^{\tilde{C}} \left\{ \int_0^{\tilde{C}} |\eta(x) - f'(k(x))|^2 dx + \int_{\tilde{C}}^C \left| \frac{x - \eta(x)}{T} - f'(k(x)) \right|^2 dx \right\}.$$

Then for large η , if we define

$$\begin{aligned} \tilde{\rho}_1(x) &= T - \frac{x}{\tilde{\eta}(x)} \quad \text{for } x \in (0, \tilde{C}) \\ \tilde{\rho}_2(x) &= \tilde{\eta}(x) \quad \text{for } x \in (\tilde{C}, C) \end{aligned}$$

then $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{C})$ is the optimal solution.

Let $1 < p < \infty$ and k be a measurable function satisfying (5.3.5) and $f'(k) \in L^p(\mathbb{R})$. Let $u_0 \in A$ and define the cost functional

$$J_p(u_0) = \int_{\mathbb{R}} |f'(u(x, T)) - f'(k(x))|^p dx.$$

Then from Lax-Oleinik formula, there exists a $\rho \in B$ such that

$$J_p(u_0) = \bar{J}_p(\rho) = \int_{\mathbb{R}} \left| \frac{x - \rho(x)}{T} - f'(k(x)) \right|^p dx$$

and

$$\inf_{u_0 \in A} J_p(u_0) = \inf_{\rho \in B} \bar{J}_p(\rho).$$

Then as in Lemma 5.3.3

$$\inf_{\rho \in B} \bar{J}_p(\rho) = \inf_{\rho \in \tilde{B}_0} \bar{J}_p(\rho).$$

and $\tilde{B}_0 \subset L^p([\tilde{C}_1, \tilde{C}_2])$ is a closed convex set. Hence from uniform convexity, there exists a unique $\tilde{\rho}_0 \in \tilde{B}_0$ such that

$$\bar{J}_p(\tilde{\rho}_0) = \inf_{u_0 \in \tilde{B}_0} \bar{J}_p(\rho).$$

CHAPTER 6

EXISTENCE AND NON-EXISTENCE OF TV BOUNDS FOR SCALAR CONSERVATION LAWS WITH DISCONTINUOUS FLUX

6.1 Introduction

¹ Let $I \subset \mathbb{R}$ be an open interval. Let $u_0 : \mathbb{R} \rightarrow I$, $F : \mathbb{R} \times I \rightarrow \mathbb{R}$ be measurable functions. We consider the following equation of conservation law

$$\begin{aligned} u_t + F(x, u)_x &= 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}. \end{aligned} \tag{6.1.1}$$

In general (6.1.1) may not admit classical solutions even for smooth F and u_0 and it is well known that even if weak solutions exist, it may not be unique.

This problem has been studied extensively over the last several decades, when the flux F is Lipschitz continuous and u_0 is of bounded variation. One has to impose an extra criteria called “Entropy condition” to obtain the physically relevant solution. For achieving this goal, the following three methods are used:

1. **Hamilton-Jacobi method (due to Lax and Olenik [31]):** Basic assumption is that $F(x, u) = f(u)$ which is strictly convex C^2 -functions. Let $v_0(x)$ be the primitive of u_0 and consider the following Hamilton-Jacobi equation

$$\begin{aligned} v_t + f(v_x) &= 0 & x \in \mathbb{R}, t > 0 \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \tag{6.1.2}$$

Then this problem admits a unique viscosity solution v which can be explicitly calculated

¹This chapter is taken from the work [1] and the contents are also added in the thesis of Rajib Dutta as he is one of the coauthor of [1]

from the Hopf formula or from the parabolic regularization

$$\begin{aligned} v_t + f(v_x) &= \epsilon v_{xx} & x \in \mathbb{R}, t > 0 \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}, \end{aligned} \tag{6.1.3}$$

and then letting $\epsilon \rightarrow 0$. This v turns out to be Lipschitz continuous function and $u = \frac{\partial v}{\partial x}$ is the entropy solution of (6.1.1) with $\text{TV}(u(\cdot, t)) \leq \text{TV}(u_0)$ for all $t > 0$. In the case of initial-boundary value problem in the quarter plane explicit formula obtained in [40].

2. Vanishing viscosity method (due to Kruzkov [46], [38]): Here F is assumed to be Lipschitz continuous function in x and u and use directly the parabolic regularizations to (6.1.1), i.e.

$$\begin{aligned} u_t + F(x, u)_x &= \epsilon u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}, \end{aligned} \tag{6.1.4}$$

and let $\epsilon \rightarrow 0$ to obtain a unique entropy solution with $\text{TV}(u(\cdot, t))$ is bounded for all $t > 0$ whenever $\text{TV}(u_0)$ is finite.

3. Numerical schemes: Here $F(x, u) = f(u)$ is Lipschitz. Any monotone, conservative and consistent scheme converges to a unique entropy solution, (for details see [36]), having $\text{TV}(u(\cdot, t)) \leq \text{TV}(u_0) < \infty$ for all $t > 0$.

Next consider the case when the regularity on F fails. Suppose $F(x, u) = f(u)$ is continuous but not Lipschitz continuous. Then the finite domain of dependence fails and this case was considered by Kruzkov [47]. They defined the concept of generalized entropy condition and proved that the solution is unique.

Now suppose the regularity of F in the space variable fails. Assume that F has a single point of discontinuity at $x = 0$ and is given by

$$F(x, u) = H(x)f(u) + (1 - H(x))g(u) \tag{6.1.5}$$

where f and g are Lipschitz continuous functions and H is the Heaviside function. This equation (6.1.1) decomposes into two equations

$$\begin{aligned} u_t + f(u)_x &= 0 & x > 0, t > 0 \\ u_t + g(u)_x &= 0 & x < 0, t > 0 \end{aligned} \tag{6.1.6}$$

and if $u^+(t) = u(0+, t)$, $u^-(t) = u(0-, t)$ exist, then $u^\pm(t)$ satisfies Rankine-Hugoniot condition at the interface $x = 0$ given by

$$f(u^+(t)) = g(u^-(t)). \tag{6.1.7}$$

A solution to (6.1.1) by definition a weak solution to (6.1.1) satisfying entropy condition

of Lax-Olenik, Kruzkov away from the interface $x = 0$.

Equations of the above type arise while dealing with fluid flows in heterogeneous media like in two phase flow in a porous medium with changing rock types that arise in the petroleum industry. In this case, the discontinuities can be on account of the change in permeabilities due to changing rock types. Equations of the type (6.1.1) with discontinuous flux also arise while modeling gravity settling in an ideal clarifier thickener unit used in waste water treatment plants. See [17] for details. Here, discontinuities are on account of the separation of the flow into bulk upward and downward flows at the feed inlet and also due to the presence of singular source terms. Other examples are in the modeling of traffic on highways with changing surface conditions (see [53]) and in ion etching in the semiconductor industry (see [55]).

A simple example by taking $f = 0$, $g = 1$, shows that (6.1.6)-(6.1.7), does not admit any solutions. Hence for existence, one need to put an extra assumption, namely that f and g coincide at least two points. Under this assumption Gimse-Risebro [35, 34], Diehl [30] obtained a solution for Riemann data. Main questions are existence of solutions for arbitrary data and unicity.

In order to understand the problem clearly, assume that f and g are strictly convex C^2 -function. Now regularize H to H^δ , a smooth function and let $F^\delta(x, u) = H^\delta(x)f(u) + (1 - H^\delta(x))g(u)$. Then F^δ is Lipschitz continuous function. Let $u_0 \in BV \cap L^\infty$ and consider two problems

(i). **Vanishing viscosity for Hamilton-Jacobi:**

$$\begin{aligned} v_t + F^\delta(x, v_x) &= \epsilon v_{xx} & x \in \mathbb{R}, t > 0 \\ v(x, 0) &= v_0(x) = \int_0^x u_0(\theta) d\theta, & x \in \mathbb{R} \end{aligned} \tag{6.1.8}$$

and

(ii). **Vanishing viscosity for conservation law:**

$$\begin{aligned} u_t + F^\delta(x, u)_x &= \epsilon u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}. \end{aligned} \tag{6.1.9}$$

Let $v_{\epsilon, \delta}$ be the unique solution of (6.1.8). Then letting $\epsilon \rightarrow 0$, $\delta \rightarrow 0$ Ostrov [54] showed that $v_{\epsilon, \delta}$ converges to a unique viscosity solution v of

$$\begin{aligned} v_t + F(x, v_x) &= 0 & x \in \mathbb{R}, t > 0 \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \tag{6.1.10}$$

Independently, (6.1.10) has been studied by Adimurthi-Gowda [6] and obtained an explicit formula for the viscosity solution. Furthermore it was shown that $u = \frac{\partial v}{\partial x}$ is a solution of

(6.1.1) satisfying an interface entropy condition

$$\text{meas}\{t : f'(u^+(t)) > 0, g'(u^-(t)) < 0\} = 0 \quad (6.1.11)$$

and Lax-Olenik entropy condition for $x \neq 0$. That is undercompressive waves at the interface are not allowed. The solution so obtained form an L^1 -contractive semi-group. Moreover under some mild regularity (see Remark 6.2.1), the solutions of (6.1.1) satisfying (6.1.11) is unique.

At the conservation law level, this problem was studied by Karlsen, Risebro, Towers [42] and showed that for $\epsilon \rightarrow 0$, $\delta \rightarrow 0$, solution of (6.1.9) converges to a solution of (6.1.1) and it does not satisfy the condition (6.1.11). In general, these solutions admit undercompressive waves at the interface $x = 0$ which is not allowed in the classical theory of Lax-Olenik and Kruzkov (see, [46]). From the model coming from capillary diffusion, Kaasschieter [41] had studied this problem by using a different diffusion term than the one in (6.1.9) which captured the better physics of that situation. He used travelling wave method and also arrived to the conclusion that the solution satisfy interface entropy condition (6.1.11). For example, in some cases like clarifier-thickner model, undercompressive waves are allowed at the interface [18, 19, 20, 17, 21, 42, 27, 29, 28, 43]. In view of this discrepancy, a general theory known as (A, B) interface entropy theory was proposed in [8], [9], [22]. It was first shown that (A, B) -entropy solution exists and forms an L^1 -contractive semi-group and is unique. For the strictly convex case, in [9] explicit Lax-Olenik type formulas are established satisfying “ (A, B) interface entropy condition”.

Using this, a numerical scheme of Godunov type is derived in [7], [10]. This scheme is conservative, monotone but not consistent. One cannot expect total variation diminishing property as a constant data gives rise to a non-constant solution. Without total variation bound studying the convergence of the scheme was difficult. This was overcome by singular mapping technique used in [50], [61], [7], [45], [8]. Since the schemes are monotone and conservative and hence by Crandall-Tartar’s Lemma, the solution mapping is L^1 -contractive. From this, it was shown (see, [16]) that the scheme is of total variation bounded away from the interface and this gives an alternative method to prove the convergence of the scheme. Both the methods does not give any information at the interface $x = 0$. Now the open question was

“Does the (A, B) entropy solution admit a total variation bound?”

It is easy to see that if the fluxes f and g are Lipschitz continuous and for almost everywhere modulus of their derivatives are bounded below by a positive constant then the associated singular mappings are invertible and Lipschitz continuous, this implies that (A, B) entropy solutions are of bounded variation. In view of this observation, it is expected that the total variation bounds fail if one of A or B is a critical point of f or g respectively. Aim of this chapter is to show indeed this is the case.

Basically we assume that f and g are strictly convex and of super linear growth. We use the explicit formula for the (A, B) entropy solution constructed in [9] and show that if (A, B) are away from the critical points, then the solution is of total bounded variation (see (1), (2) and (3) of Main Theorem 6.2.5). One of the main ingredients to show this is the study of boundary values $\lambda_{\pm}(t)$ ((6.2.5), (6.2.6)). If $\lambda_{\pm}(t)$ are of bounded variation, then it follows easily that the solution itself is of bounded variation. But this is not the case in general. In general $\lambda_{\pm}(t)$ are highly oscillatory and we illustrate this with an example (see, Theorem 6.2.2) where we show that $\lambda_{\pm}(t)$ are not of bounded variation. In spite of this anomaly, total variation bound exists when A, B are away from the critical points of f and g respectively.

In (5) of Main Theorem 6.2.5, when A is a critical point of g , then for certain data we construct the (A, B) entropy solution which is not of bounded variation at the interface $x = 0$. Idea behind this construction is as follows. At the interface, u satisfies RH condition, namely $f(u^+(t)) = g(u^-(t))$. Hence $u^-(t) = g^{-1}(f(u^+(t)))$. If $f(u^+(t))$ is of bounded variation and the range lies in a neighbourhood of critical point of g , then u^- need not be of bounded variation.

Plan of this chapter is organized as follows. In Section 6.2, we recall from [6], [9], the properties of (A, B) entropy solutions. Basically (A, B) entropy solution depends heavily on construction of left $\lambda_-(t)$ and right $\lambda_+(t)$ boundary values at the interface $x = 0$. In general they need not be of bounded variation. In Section 6.2, we give an example to show that $\lambda_+(t)$ is not of bounded variation whose proof is carried out in Section 6.3. In spite of this anomaly, next we prove that

- (1). (A, B) entropy solutions are of bounded variation if A, B are not critical points of g and f respectively.
- (2). construct an explicit example to show that total variation is unbounded under the assumption that one of A or B is a critical point.

As a consequence of our example, it follows that however small the BV norm of initial data is, BV norm of the solution can blow up at the interface. Hence the BV bounds of the solution need not depend on the smallness of the initial data (see Remark 6.2.2).

Conservation laws with discontinuous flux can also be written as 2×2 system of hyperbolic equations. This system is non strictly hyperbolic near the critical points of f and g . Hence the TV bound may blow up due to occurrence of the critical points of f and g . This type of TV bounds failure usually is called resonance (see [60], [59]).

6.2 Preliminaries

We assume the following assumptions on f, g and u_0 .

- (i). f and g are strictly convex, C^2 and of superlinear growth.

(ii). $u_0 \in L^\infty$ and let v_0 be its primitive given by

$$v_0(x) = \int_0^x u_0(\theta) d\theta.$$

Let $f(\theta_f) = \min_{\theta \in \mathbb{R}} f(\theta)$, $g(\theta_g) = \min_{\theta \in \mathbb{R}} g(\theta)$ be the points of minima of f and g respectively. Let f^* and g^* be their respective convex duals. Let us recall some of the definitions and notations from [9], [8].

Definition 6.2.1. (*Connection*). Let $(A, B) \in \mathbb{R}^2$. Then (A, B) is called a connection if it satisfies

(i). $f(B) = g(A)$.

(ii). $f'(B) \geq 0$, $g'(A) \leq 0$.

Definition 6.2.2. (*Interphase entropy functional*). Let $u \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm(t) = u(0^\pm, t)$ exist a.e. $t > 0$. Then we define $I_{AB}(t)$, the interface entropy functional by

$$I_{AB}(t) = (g(u^-(t)) - g(A))\text{sign}(u^-(t) - A) - (f(u^+(t)) - f(B))\text{sign}(u^+(t) - B). \quad (6.2.1)$$

Definition 6.2.3. (*Interphase entropy condition*). Let $u \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm(t)$ exist a.e. $t > 0$. Then u is said to satisfy Interphase entropy condition relative to a connection (A, B) if for a.e. $t > 0$

$$I_{AB}(t) \geq 0. \quad (6.2.2)$$

Definition 6.2.4. (*(A, B) entropy solution*). Let $F(x, u) = H(x)f(u) + (1 - H(x))g(u)$ and (A, B) be a connection. Let $u_0 \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$. Then $u \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be a (A, B) entropy solution if

(i). u is a weak solution of

$$\begin{aligned} u_t + F(x, u)_x &= 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) &= u_0(x) & x \in \mathbb{R}. \end{aligned} \quad (6.2.3)$$

(ii). u satisfies Lax-Olenik-Kruzkov entropy condition away from the interface $x = 0$.

(iii). At the interface $x = 0$, u satisfies (A, B) interface entropy condition (6.2.2).

The following examples illustrate the different approaches give rise to different solutions for the same initial data.

Example. Assume that f and g intersect at a point p where $f'(p) > 0$, $g'(p) < 0$. Let $g(\theta_g) \geq f(\theta_f)$ and $u_0(x) = p$. Then

1. Parabolic regularization of (6.2.3) gives $u \equiv p$ as a solution. For every $t > 0$,

$$f'(u^+(t)) = f'(p) > 0, \quad g'(u^-(t)) = g'(p) < 0$$

and hence this solution produces an undercompressive wave at the interface $x = 0$. In this case, total variation of $u(\cdot, t)$ does not increase.

2. From the explicit formula [6], the solution u of (6.2.3) is given by

$$u(x, t) = \begin{cases} p & \text{if } x < g'(p)t, \\ g'^{-1}\left(\frac{x}{t}\right) & \text{if } g'(p)t < x < 0, \\ B & \text{if } 0 < x < f'(B)t, \\ f'^{-1}\left(\frac{x}{t}\right) & \text{if } f'(B)t < x < f'(p)t, \\ p & \text{if } x > f'(p)t \end{cases} \quad (6.2.4)$$

where $f(B) = g(\theta_g)$, $f'(B) \geq 0$. In this case $u^+(t) = B$, $u^-(t) = \theta_g$, hence $f'(u^+(t)) \geq 0$, $g'(u^-(t)) = 0$ for all t . Here total variation of $u(\cdot, t)$ increases.

(A, B) interface entropy conditions generalize these two cases and for particular choice of (A, B) connection total variation of the solution of (6.2.3) may increase.

Next we illustrate how to get an explicit (A, B) entropy solution. For this we need to construct boundary values $\lambda_{\pm}(t)$ and the details are carried in [9], [6].

Definition 6.2.5. (*Control curves*). Let $t > 0$. $\gamma : [0, t] \rightarrow \mathbb{R}$ be a continuous function. γ is said to be a control curve if there exists $0 \leq t_1 \leq t$ such that

- (i). $\gamma|_{[t_1, t]} = 0$,
- (ii). $\gamma|_{[0, t_1]}$ is linear and $\gamma|_{(0, t_1)} \neq 0$.

Let

$$\Gamma(t) = \{\gamma : [0, t] \rightarrow \mathbb{R}; \gamma \text{ is control curve}\},$$

$$\Gamma_+(t) = \{\gamma \in \Gamma(t) : \gamma(t) \geq 0\},$$

$$\Gamma_-(t) = \{\gamma \in \Gamma(t) : \gamma(t) \leq 0\}.$$

Define $b_{\pm}(t)$ by

$$b_+(t) = \inf_{\gamma \in \Gamma_+(t)} \left\{ v_0(\gamma(0)) + \int_0^t f^*(\dot{\gamma}(\theta)) d\theta \right\},$$

$$b_-(t) = \inf_{\gamma \in \Gamma_-(t)} \left\{ v_0(\gamma(0)) + \int_0^t g^*(\dot{\gamma}(\theta)) d\theta \right\},$$

where $v_0(x) = \int_0^x u_0(\theta) d\theta$.

Lemma 6.2.1. Let $u_0 \in L^\infty$, then $b_{\pm}(t) \in L^\infty$ and satisfies

(i). $b_{\pm}(t)$ are Lipschitz continuous functions.

(ii). Let $u_0 \in BV$, then $b'_{\pm}(t)$ are of bounded variation on any interval $[0, T]$ for $T > 0$ with $TV(b_{\pm}, [0, T]) \leq CTV(u_0)$, for some constant $C > 0$.

Proof of this Lemma follows from Lemma 6.2.1, (24) and (27) of [9].

In order to construct an (A, B) entropy solution, we split the problem into two boundary value problems with appropriate boundary conditions $\lambda_{\pm}(t)$. For this, we denote f_+^{-1} the inverse of f restricted to the increasing part of f and f_-^{-1} the inverse of f restricted to the decreasing part of f . Similarly define for g_+^{-1}, g_-^{-1} .

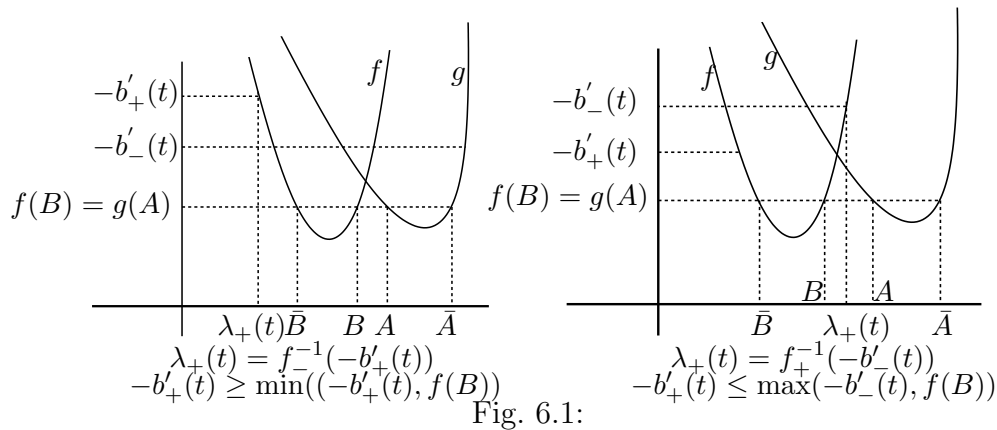
Definition 6.2.6. (*Boundary data*). Let (A, B) be a connection and define $\lambda_{\pm}(t) = \lambda_{\pm}(t, A, B, v_0, f, g)$ (see Figure 1, Figure 2) by

$$\lambda_+(t) = \begin{cases} f_-^{-1}(-b'_+(t)) & \text{if } -b'_+(t) > \max(-b'_-(t), f(B)), \\ f_+^{-1}(\max(-b'_-(t), f(B))) & \text{if } -b'_+(t) \leq \max(-b'_-(t), f(B)), \end{cases} \quad (6.2.5)$$

$$\lambda_-(t) = \begin{cases} g_-^{-1}(-b'_-(t)) & \text{if } -b'_-(t) > \max(-b'_+(t), g(A)), \\ g_+^{-1}(\max(-b'_+(t), g(A))) & \text{if } -b'_-(t) \leq \max(-b'_+(t), g(A)). \end{cases} \quad (6.2.6)$$

Regarding the behavior of $\lambda_{\pm}(t)$, we have the following theorem whose proof will be given in Section 6.3.

Theorem 6.2.2. *For a certain choice of f and g there exists u_0 such that $TV(\lambda_{\pm}) = \infty$.*



Let $\bar{B} < B, \bar{A} > A$ be such that $f(B) = f(\bar{B}) = g(A) = g(\bar{A})$. Now from (6.2.5) and (6.2.6), it follows immediately the following

Lemma 6.2.3. *Let $u_0 \in BV \cap L^\infty$, then λ_{\pm} satisfies*

(i). $f(\lambda_+(t)) \geq f(B)$. If $f'(\lambda_+(t)) \geq 0$, then $\lambda_+(t) \geq B$. Furthermore if $B \neq \theta_f$ then there exists an $\alpha > 0$, such that $\forall 0 \leq t_1 < t_2 < \dots < t_{k+1}$ and satisfies $\lambda_+(t_i) \geq B$, then

$$\sum_{i=1}^k |\lambda_+(t_i) - \lambda_+(t_{i+1})| \leq \alpha TV(u_0)$$

(ii). $g(\lambda_-(t)) \geq g(A)$. If $g'(\lambda_-(t)) \leq 0$, then $\lambda_-(t) \leq A$. Furthermore if $A \neq \theta_g$, then there exists a $\beta > 0$, such that $\forall 0 \leq t_1 < t_2 < \dots < t_{k+1}$ and satisfies $\lambda_-(t_i) \leq A$, then

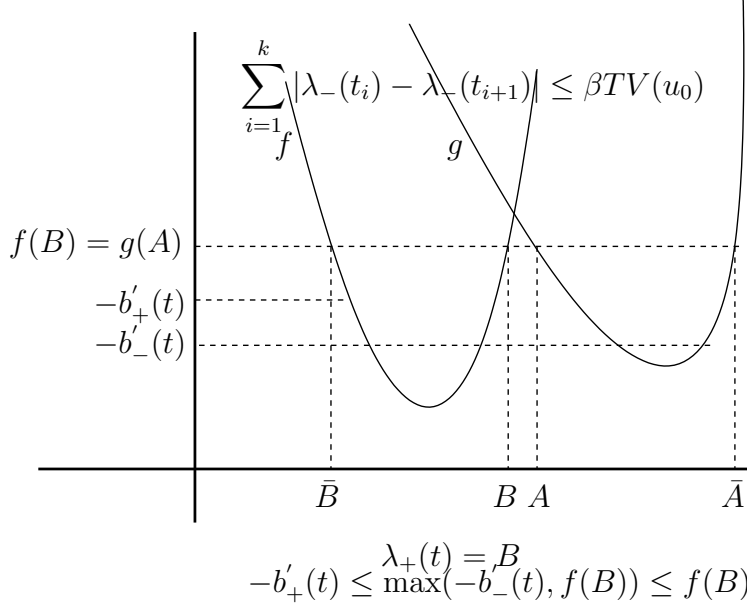


Fig. 6.2:

Proof. If $-b'_+(t) > \text{Max}(-b'_-(t), f(B))$. Then $f(\lambda_+(t)) = -b'_+(t) \geq f(B)$. If $-b'_+(t) \leq \text{Max}(-b'_-(t), f(B))$, then $f(\lambda_+(t)) = \text{Max}(-b'_-(t), f(B)) \geq f(B)$. This immediately gives that if $f'(\lambda_+) \geq 0$, then necessarily $\lambda_+(t) \geq B$. Let $B > \theta_f$, then f_+^{-1} restricted to $[f(B), \infty]$ is locally Lipschitz continuous function. From Lemma 6.2.1, b'_\pm are in L^∞ and hence $\lambda_+ \in L^\infty$. Let α_1 be the Lipschitz constant of f_+^{-1} in $[f(B), \|\lambda_+\|_\infty]$. Hence from Lemma 6.2.1, there exists $C > 0$ such that

$$\begin{aligned} & \sum_{i=1}^k |\lambda_+(t_i) - \lambda_+(t_{i+1})| \\ &= \sum_{i=1}^k |f_+^{-1}(\text{Max}(-b'_-(t_i), f(B))) - f_+^{-1}(\text{Max}(-b'_-(t_{i+1}), f(B)))| \\ &\leq \alpha_1 \sum_{i=1}^k |(\text{Max}(-b'_-(t_i), f(B))) - (\text{Max}(-b'_-(t_{i+1}), f(B)))| \\ &\leq \alpha_1 \sum_{i=1}^k |b'_-(t_i) - b'_-(t_{i+1})| \\ &\leq \alpha_1 C TV(u_0) \\ &= \alpha TV(u_0). \end{aligned}$$

Similarly for λ_- and this proves the corollary.

Now recall from [6], [9] the existence and uniqueness of (A, B) entropy solution.

Theorem 6.2.4. *Let $u_0 \in L^\infty$ and (A, B) be a connection . Let $\lambda_\pm(t)$ be as defined earlier. Then there exists an (A, B) entropy solution u with $\|u\|_\infty \leq \|u_0\|_\infty$ and is unique under mild regularity assumption (see Remark 6.2.1). Furthermore the solution can be described explicitly by Lax-Olenik type formula as follows.*

For each $t > 0$ there exists $R_1(t), R_2(t) \geq 0, L_1(t) \leq 0, L_2(t) \leq 0$ and monotone functions $y_\pm(x, t), t_\pm(x, t), z_\pm(x, t)$ such that

- (i). *for $x \in [R_1(t), \infty)$, $y_+(x, t) \geq 0$ is a non-decreasing function and for $x \in [0, R_1(t))$, $0 \leq t_+(x, t) < t$ is a non-increasing function such that for $x > 0$, (see page 16, equation (44), [9])*

$$u(x, t) = \begin{cases} f^{*'}\left(\frac{x - y_+(x, t)}{t}\right) = u_0(y_+(x, t)) & \text{if } x \geq R_1(t), \\ f^{*'}\left(\frac{x}{t - t_+(x, t)}\right) = \lambda_+(t_+(x, t)) & \text{if } 0 \leq x < R_1(t). \end{cases} \quad (6.2.7)$$

- (ii). *For $x \in (-\infty, L_1(t)]$, $y_-(x, t) \leq 0$, is a non-decreasing function and for $x \in (L_1(t), 0]$, $0 \leq t_-(x, t) < t$, $t_-(x, t)$ is non-increasing function such that for $x < 0$,*

$$u(x, t) = \begin{cases} g^{*'}\left(\frac{x - y_-(x, t)}{t}\right) = u_0(y_-(x, t)) & \text{if } x \leq L_1(t), \\ g^{*'}\left(\frac{x}{t - t_-(x, t)}\right) = \lambda_-(t_-(x, t)) & \text{if } L_1(t) < x < 0. \end{cases} \quad (6.2.8)$$

- (iii). *Without loss of generality we can assume $g(\theta_g) \geq f(\theta_f)$. Let us assume $A = \theta_g$. Then we have the following two cases*

Case 1. $L_1(t) = 0, R_1(t) \geq 0$ (see page 53, equation (4.21), (4.22), [6])

$$u(x, t) = \begin{cases} f_+^{-1}(g(u_0(z_+(x, t)))) & \text{if } 0 < x < R_2(t) \\ f_+^{-1}(g(\theta_g)) & \text{if } R_2(t) \leq x < R_1(t). \end{cases} \quad (6.2.9)$$

Case 2. $L_1(t) < 0, R_1(t) \geq 0$ (see Lemma 4.8 and page 55, equation (4.30), [6])

$$u(x, t) = \begin{cases} g_+^{-1}(f(u_0(z_-(x, t)))) & \text{if } 0 > x > L_1(t) = L_2(t) \\ f_+^{-1}(g(\theta_g)) & \text{if } 0 < x < R_1(t) = R_2(t). \end{cases} \quad (6.2.10)$$

See figure (6.3) for clear illustrations.

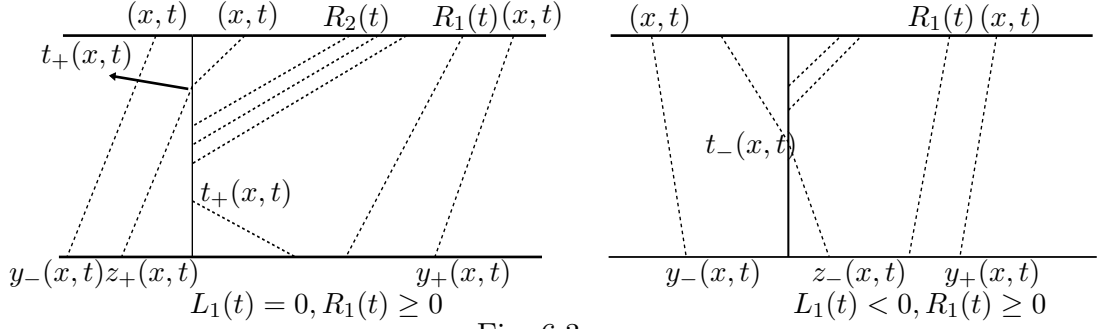


Fig. 6.3:

Remark 6.2.1. In the above theorem by “mild regularity” we mean the set of points of discontinuity of the solution of (6.1.6) is discrete set of Lipschitz curves. In [22], uniqueness was proven without the “mild regularity assumption”.

Now we can state our main result of this chapter as follows.

Theorem 6.2.5. (Main Theorem) Let $u_0 \in L^\infty(\mathbb{R})$ and u be the solution obtained in Theorem 6.2.4 Let $t > 0$, $\epsilon > 0$, $M > \epsilon$, $I(M, \epsilon) = \{x : \epsilon \leq |x| \leq M\}$. Then

- (1). Suppose there exists an $\alpha > 0$ such that $f'' \geq \alpha$, $g'' \geq \alpha$, then there exist $C = C(\epsilon, M, \alpha)$ such that

$$TV(u(\cdot, t), I(M, \epsilon)) \leq C(\epsilon, M, t).$$

- (2). Suppose $u_0 \in BV$, and $T > 0$. Then there exists $C(\epsilon, T)$ such that for all $0 < t \leq T$

$$TV(u(\cdot, t), |x| > \epsilon) \leq C(\epsilon, t) TV(u_0) + 4\|u_0\|_\infty$$

- (3). Let $u_0 \in BV$, $T > 0$ and $A \neq \theta_g$ and $B \neq \theta_f$. Then there exists $C > 0$ such that for all $0 < t \leq T$,

$$TV(u(\cdot, t)) \leq C TV(u_0) + 6\|u_0\|_\infty.$$

- (4). Let $u_0, f_+^{-1}(g(u_0)), g_-^{-1}(f(u_0)) \in BV$, $T > 0$ and $A = \theta_g$. Then for all $0 < t \leq T$,

$$\begin{aligned} & TV(u(\cdot, t)) \\ & \leq TV(u_0) + \max(TV(f_+^{-1}(g(u_0))), TV(g_-^{-1}(f(u_0)))) + 6\|u_0\|_\infty. \end{aligned}$$

- (5). For a certain choice of fluxes f and g there exists $u_0 \in BV \cap L^\infty$ such that $TV(u(\cdot, t)) = \infty$ if $A = \theta_g$ or $B = \theta_f$. For general f and g see Remark 6.2.4.

Proof.(1) Let $\epsilon \leq x_0 < x_1 < \dots < x_l \leq R_1(t) \leq x_{l+1} < \dots < x_N \leq M$ be a partition. Then by monotonicity of y_+ , t_+ , we have

$$\begin{aligned} \frac{1}{t} \sum_{i=l}^{N-1} |x_i - y_+(x_i, t) - x_{i+1} + y_+(x_{i+1}, t)| &\leq \frac{1}{t} \{ |M - R_1(t)| \\ &\quad + |y_+(M, t) - y_+(R_1(t), t)| \}, \\ \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right| &\leq \frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} \\ &\quad + \frac{R_1(t) |t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2}. \end{aligned}$$

$f^{*'} = f'^{-1}$ is Lipschitz continuous with Lipschitz constant bounded by $\frac{1}{\alpha}$ since $f'' \geq \alpha$. Hence from (6.2.7) and above inequalities we obtain

$$\begin{aligned} \sum_{i=0}^{N-1} |u(x_i, t) - u(x_{i+1}, t)| &\leq \frac{1}{\alpha t} \{ |M - R_1(t)| + |y_+(M, t) - y_+(R_1(t), t)| \} \\ &\quad + \frac{1}{\alpha} \left\{ \frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + R_1(t) \frac{|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2} \right\} \\ &\quad + |u(x_l, t) - u(x_{l+1}, t)| \\ &= C(M, \epsilon, t). \end{aligned}$$

Hence $\text{TV}(u(\cdot, t), [\epsilon, M]) \leq C(M, \epsilon, t)$. Similarly for $x \leq -\epsilon$. Now let $u_0 \in \text{BV}$. Let $z_i = y_+(x_i, t)$, $i \geq l + 1$, then $z_i \leq z_{i+1}$ and hence from (6.2.7)

$$\sum_{i=l}^{N-1} |u(x_i, t) - u(x_{i+1}, t)| \leq \sum_{i=l}^{N-1} |u_0(z_i) - u_0(z_{i+1})| \leq \text{TV}(u_0, [0, \infty)).$$

Let $0 < t \leq T$, define

$$\begin{aligned} a(T) &= \sup\{R_1(t) : 0 < t \leq T\} \\ \rho(T, \epsilon) &= \inf\{t - t_+(x, t) : 0 < t \leq T, \epsilon \leq x \leq R_1(t)\}. \end{aligned}$$

Then for $\epsilon \leq x \leq R_1(t)$, we have

$$\frac{\epsilon}{T} \leq \frac{x}{t - t_+(x, t)} \leq \frac{a(T)}{\rho(T, \epsilon)}.$$

Now $f^{*'} = f'^{-1}$ is a Lipschitz continuous function in $[\frac{\epsilon}{T}, \frac{a(T)}{\rho(T, \epsilon)}]$. Let $C_1(T)$ be its Lipschitz constant. Then from (6.2.6)

$$\begin{aligned}
& \sum_{i=0}^{l-1} |u(x_i, t) - u(x_{i+1}, t)| \\
= & \sum_{i=0}^{l-1} \left| f^{*'}\left(\frac{x_i}{t - t_+(x_i, t)}\right) - f^{*'}\left(\frac{x_{i+1}}{t - t_+(x_{i+1}, t)}\right) \right| \\
\leq & C_1(T) \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right| \\
\leq & C_1(T) \left\{ \frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + R_1(t) \frac{|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2} \right\} \\
\leq & C_1(T) \sup_{0 \leq t \leq T} \left\{ \frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + R_1(t) \frac{|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2}, \right. \\
& \left. \frac{L_1(t) - \epsilon}{|t - t_-(\epsilon, t)|} + L_1(t) \frac{|t_-(\epsilon, t) - t_-(L_1(t), t)|}{|t - t_-(\epsilon, t)|^2} \right\} \\
= & C(T).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\sum_{i=0}^{N-1} |u(x_i, t) - u(x_{i+1}, t)| &= \sum_{i=0}^{l-1} |u(x_i, t) - u(x_{i+1}, t)| \\
&\quad + |u(x_l, t) - u(x_{l+1}, t)| \\
&\quad + \sum_{i=l+1}^{N-1} |u(x_i, t) - u(x_{i+1}, t)| \\
&\leq C(T) + 2\|u_0\|_\infty + \text{TV}(u_0, (0, \infty)).
\end{aligned}$$

Similarly for $x \leq -\epsilon$. Hence we obtain

$$\text{TV}(u(\cdot, t), |x| \geq \epsilon) \leq \text{TV}(u_0) + C(T) + 4\|u_0\|_\infty.$$

This proves (1) and (2).

(3). Let $A \neq \theta_f, B \neq \theta_g$. From (2.7) we have for $x_0 < x_1 < \dots < x_l \leq R_1(t), t_i = t_+(x_i, t), f'_+(\lambda_+(t_i)) \geq 0$. Hence from (i) of Lemma 6.2.3 we have

$$\begin{aligned}
& \sum_{i=0}^{N-1} |u(x_i, t) - u(x_{i+1}, t)| \\
&= \sum_{i=0}^{l-1} |\lambda_+(t_+(x_i, t)) - \lambda_+(t_+(x_{i+1}, t))| + |\lambda_+(x_l, t) - u_0(y_+(x_{l+1}, t))| \\
&+ \sum_{i=l}^{N-1} |u_0(y_+(x_i, t)) - u_0(y_+(x_{i+1}, t))| \\
&\leq \alpha TV(u_0) + 2\|u_0\|_\infty + TV(u_0) \\
&= (\alpha + 1)TV(u_0) + 2\|u_0\|_\infty.
\end{aligned}$$

This is true for all $t > 0, \epsilon, M$ and hence

$$TV(u(\cdot, t), [0, \infty)) \leq (\alpha + 1)TV(u_0) + 2\|u_0\|_\infty$$

Similarly in $(-\infty, 0]$. Hence we have

$$\begin{aligned}
TV(u(\cdot, t)) &\leq TV(u(\cdot, t), [0, \infty)) + TV(u(\cdot, t), [0, \infty)) \\
&+ |u(0+, t) - u(0-, t)| \\
&\leq (\alpha + \beta + 2)TV(u(\cdot, t)) + 6\|u_0\|_\infty.
\end{aligned}$$

This proves (3).

(4). Since $A = \theta_g$ and hence $f^*(0) \geq g^*(0)$. It is enough if we consider the following two cases.

Case 1. $L_1(t) = 0, R_1(t) \geq 0$.

Let $\dots < x_{-1} < x_0 \leq 0 < x_1 < \dots < x_{N_1} \leq R_2(t) < x_{N_1+1} < \dots < x_{N_2} \leq R_1(t) < x_{N_2+1} < \dots$ be a partition. Then from (6.2.7), (6.2.8), (6.2.9) we obtain

$$\begin{aligned}
& \sum_{i=-\infty}^{\infty} |u(x_i, t) - u(x_{i+1}, t)| \\
&= \sum_{i=-\infty}^{-1} |u_0(y_-(x_i, t)) - u_0(y_-(x_{i+1}, t))| + \sum_{i=N_2}^{\infty} |u_0(y_+(x_i, t)) - u_0(y_+(x_{i+1}, t))| \\
&+ \sum_{i=1}^{N_1} |f_+^{-1}(g(u_0(z_+(x_i, t)))) - f_+^{-1}(g(u_0(z_+(x_{i+1}, t))))| \\
&\quad + |u(x_0, t) - u(x_1, t)| + |u(x_{N_1}, t) - u(x_{N_1+1}, t)| \\
&\quad + |u(x_{N_2}, t) - u(x_{N_2+1}, t)| \\
&\leq TV(u_0) + TV(f_+^{-1}g(u_0)) + 6\|u_0\|_\infty.
\end{aligned}$$

Case 2. $L_1(t) < 0, R_1(t) \geq 0$.

Let $\dots < x_{-N_1} \leq L_2(t) = L_1(t) < x_{-N_1+1} < \dots < x_0 \leq 0 < x_1 < \dots < x_{N_2} \leq R_2(t) =$

$R_1(t) < x_{N_2+1} < \dots$ be a partition. Then from (6.2.7), (6.2.8), (6.2.10) we obtain

$$\begin{aligned}
& \sum_{-\infty}^{\infty} |u(x_i, t) - u(x_{i+1}, t)| \\
&= \sum_{-\infty}^{-N_1-1} |u_0(y_-(x_i, t)) - u_0(y_-(x_{i+1}, t))| + \sum_{N_2}^{\infty} |u_0(y_+(x_i, t)) - u_0(y_+(x_{i+1}, t))| \\
&\quad + \sum_{-N_1}^{-1} |g_-^{-1}(f(u_0(z_-(x_i, t)))) - g_-^{-1}(f(u_0(z_-(x_{i+1}, t))))| \\
&\quad\quad + |u(x_{-N_1}, t) - u(x_{-N_1+1}, t)| + |u(x_0, t) - u(x_1, t)| \\
&\quad\quad\quad + |u(x_{N_2}, t) - u(x_{N_2+1}, t)| \\
&\leq TV(u_0) + TV(f_+^{-1}g(u_0)) + 6\|u_0\|_{\infty}.
\end{aligned}$$

This proves (4).

(5). Assume

$$f|_{[-1,0]} = -u, \quad g = u^2. \quad (6.2.11)$$

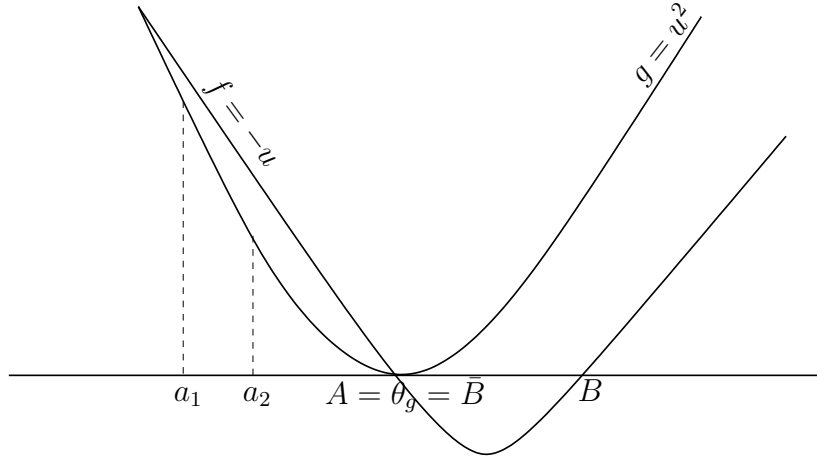


Fig. 6.4:

Here we take $A = \theta_g = 0 = \bar{B}$, where $f(\bar{B}) = f(B)$, $f'(\bar{B}) < 0$ (see Figure 6.4).

Step 1. In order to construct a counter example first we study a Riemann boundary value problem

$$\begin{aligned}
u_t + (u^2)_x &= 0 & x < 0, \quad t > t_1, \\
u(x, t_1) &= a_1 & x < 0, \\
u(0, t) &= a_2 & t > t_1
\end{aligned} \quad (6.2.12)$$

where $a_i \leq 0$, $i = 1, 2$.

Shock case: $a_1 > a_2$. Then the solution is given by (see Figure 6.5)

$$u_S(x, t) = \begin{cases} a_1 & \text{if } x < (a_1 + a_2)(t - t_1), \quad t > t_1, \\ a_2 & \text{if } (a_1 + a_2)(t - t_1) < x < 0, \quad t > t_1 \end{cases} \quad (6.2.13)$$

and denote the shock curve by

$$S(a_1, a_2, t_1) : x = (a_1 + a_2)(t - t_1), \quad t > t_1. \quad (6.2.14)$$

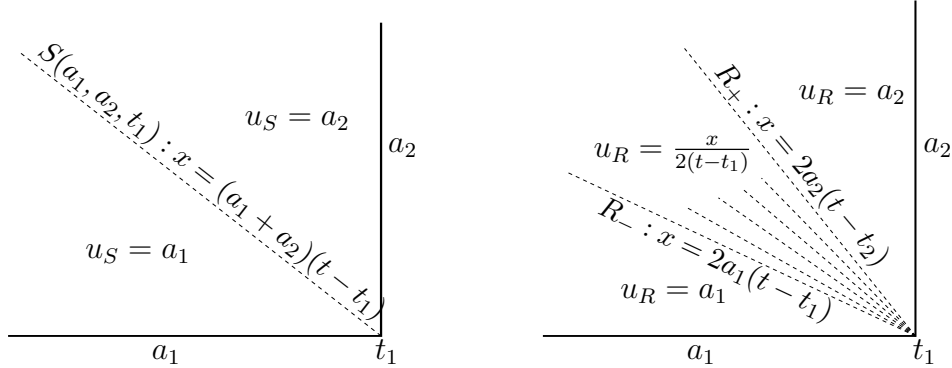


Fig. 6.5: Shock and rarefaction wave

Rarefaction case: $a_1 < a_2$. Then the solution is given by (see Figure 6.5)

$$u_R(x, t) = \begin{cases} a_1 & \text{if } x < 2a_1(t - t_1), \quad t > t_1, \\ \frac{x}{2(t-t_1)} & \text{if } 2a_1(t - t_1) < x < 2a_2(t - t_1), \quad t > t_1, \\ a_2 & \text{if } 2a_2(t - t_1) < x < 0, \quad t > t_1 \end{cases}$$

and denote the boundary of the rarefaction by

$$\begin{aligned} R_-(a_1, a_2, t_1) &: x = 2a_1(t - t_1), \\ R_+(a_1, a_2, t_1) &: x = 2a_2(t - t_1). \end{aligned}$$

Step 2. Next consider the following data. Let $0 \leq t_1 < t_2 < t_3$, $a_2 < a_1 < 0$, $a_2 < a_3 < 0$, $a_4 < a_3$,

$$u_0(x) = a_1 \quad \text{if } t = t_1,$$

$$u(0, t) = \begin{cases} a_2 & \text{if } t_1 < t < t_2, \\ a_3 & \text{if } t_2 < t < t_3, \\ a_4 & \text{if } t_3 < t. \end{cases}$$

Since $a_2 < a_1$, $a_4 < a_3$, at t_1 and t_3 we get shocks and the shock curves are given by

$$\begin{aligned} S(a_1, a_2, t_1) &: x = (a_1 + a_2)(t - t_1), \\ S(a_3, a_4, t_3) &: x = (a_3 + a_4)(t - t_3). \end{aligned}$$

Since $a_2 < a_3$, hence at t_2 we get rarefaction and the boundaries of the rarefaction wave is given by

$$\begin{aligned} R_-(a_2, a_3, t_2) &: x = 2a_2(t - t_2), \\ R_+(a_2, a_3, t_2) &: x = 2a_3(t - t_2). \end{aligned}$$

Let $T \geq t_1$ such that these lines do not meet, then the solution u (see Figure 6.6) for $t \leq T$ is given by

$$u(x, t) = \begin{cases} a_1 & \text{if } x < (a_1 + a_2)(t - t_1), \\ a_2 & \text{if } (a_1 + a_2)(t - t_1) < x < 2a_2(t - t_2), \\ \frac{x}{2(t-t_2)} & \text{if } 2a_2(t - t_2) < x < 2a_3(t - t_2), \\ a_3 & \text{if } 2a_3(t - t_2) < x < (a_3 + a_4)(t - t_3), \\ a_4 & \text{if } x > (a_3 + a_4)(t - t_3). \end{cases}$$

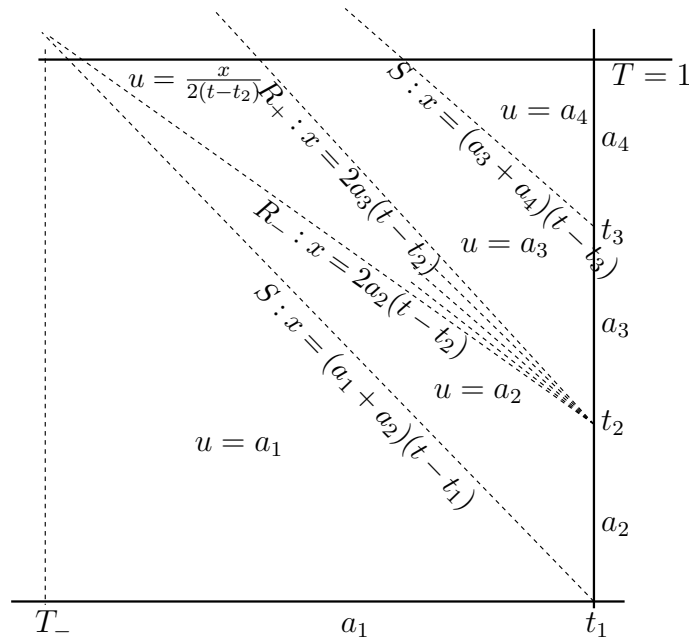


Fig. 6.6:

Next we find the conditions for intersections. $S(a_1, a_2, t_1)$ and $R_-(a_2, a_3, t_2)$ meet at T_- if and only if

$$(a_1 + a_2)(T_- - t_1) = 2a_2(T_- - t_2),$$

that is,

$$T_- = t_1 - \frac{2a_2}{a_1 - a_2}(t_2 - t_1) \quad (6.2.15)$$

and $R_+(a_2, a_3, t_2)$ and $S(a_3, a_4, t_3)$ meet at T_+ if and only if

$$2a_3(T_+ - t_2) = (a_3 + a_4)(T_+ - t_3)$$

that is,

$$T_+ = t_2 - \frac{a_3 + a_4}{a_3 - a_4}(t_3 - t_2). \quad (6.2.16)$$

Step 3. Let $0 = t_0 < t_1 < t_2 < \dots$ be a sequence to be determined. Let $a_0 = 0$ and $\{a_i\}_{i \geq 1}$ be defined by

$$a_{2i-1} = -\frac{1}{(i+1)}, \quad a_{2i} = -\frac{1}{(i+1)^2}$$

and consider the initial boundary condition

$$u_0(x) = 0 \quad \text{for } x \leq 0$$

$$u(0, t) = \begin{cases} a_{2i+1} & \text{if } t_{2i} < t < t_{2i+1}, \\ a_{2i+2} & \text{if } t_{2i+1} < t < t_{2i+2}. \end{cases}$$

From Step 1 it follows that the shock line $S_{2i} = S(a_{2i}, a_{2i+1}, t_{2i})$ emerge from t_{2i} and boundary of rarefaction $R_-(2i+1) = R_-(a_{2i+1}, a_{2i+2}, t_{2i+1})$, $R_+(2i+1) = R_+(a_{2i+1}, a_{2i+2}, t_{2i+1})$ emerge from t_{2i+1} . From (6.2.15) and (6.2.16) S_{2i} and $R_-(2i+1)$ meet at $T_-(i)$ and $R_+(2i+1)$ and S_{2i+2} meet at $T_+(i)$.

Claim 1. We can choose a sequence $\{t_i\}$ such that

- (i). $T_{\pm}(i) \geq 1$ for all i .
- (ii). $t_1 = \frac{1}{2}$, $t_i < t_{i+1}$ for all i , $\lim_{i \rightarrow \infty} t_i = 1$

Proof. Proof is by induction. At t_1 , $a_0 = 0$, $a_1 = -\frac{1}{2}$ and hence from (6.2.15)

$$1 \leq T_-(0) = 2t_1 \Rightarrow t_1 \geq \frac{1}{2}.$$

Hence we can choose $t_1 = \frac{1}{2}$.

Now from (6.2.15) and (6.2.16) we have

$$1 \leq T_-(i) = \frac{a_{2i} + a_{2i+1}}{a_{2i} - a_{2i+1}} t_{2i} - \frac{2a_{2i+1}}{a_{2i} - a_{2i+1}} t_{2i+1},$$

$$1 \leq T_+(i) = \frac{2a_{2i+2}}{a_{2i+2} - a_{2i+3}} t_{2i+1} - \frac{a_{2i+2} + a_{2i+3}}{a_{2i+2} - a_{2i+3}} t_{2i+2}.$$

Hence,

$$t_{2i+1} - t_{2i} \geq \frac{a_{2i} - a_{2i+1}}{2|a_{2i+1}|} (1 - t_{2i}), \quad (6.2.17)$$

$$t_{2i+2} - t_{2i+1} \geq \left| \frac{a_{2i+2} - a_{2i+3}}{a_{2i+2} + a_{2i+3}} \right| (1 - t_{2i+1}). \quad (6.2.18)$$

Now choose for $i \geq 1$

$$t_{2i+1} = t_{2i} + \frac{1}{2} \left| \frac{a_{2i} - a_{2i+1}}{2a_{2i+1}} \right| (1 - t_{2i}),$$

$$t_{2i+2} = t_{2i+1} + \left| \frac{a_{2i+2} - a_{2i+3}}{a_{2i+2} + a_{2i+3}} \right| (1 - t_{2i+1}).$$

Now substituting values of a_i to obtain

$$t_{2i+1} = t_{2i} + \frac{1}{2} \left(\frac{i^2 + i - 1}{2(i+1)^2} \right) (1 - t_{2i}),$$

$$t_{2i+2} = t_{2i+1} + \left(\frac{i^2 + i + 1}{i^2 + 3i + 7} \right) (1 - t_{2i+1}).$$

Therefore if $t_{2i} < 1$, $t_{2i+1} < 1$, then

$$(1 - t_{2i+1}) = (1 - t_{2i}) \left(1 - \frac{i^2 + i - 1}{4(i+1)^2} \right) > 0,$$

$$(1 - t_{2i+2}) = (1 - t_{2i+1}) \left(1 - \frac{1}{2} \frac{i^2 + i + 1}{i^2 + 3i + 7} \right) > 0.$$

Hence by induction $t_i \leq 1 \forall i$ and $t_{2i} < t_{2i+1} < t_{2i+2}$. This implies that $T_0 = \lim_{i \rightarrow \infty} t_i$ exists and from the above definition, $T_0 = 1$. This proves the claim.

As an immediate consequence of Claim 1 and Step 2, solution u (see Figure 6.7) of the boundary value problem in $\mathbb{R} \times [0, 1]$ is given by

$$u(x, t) = \begin{cases} 0 & \text{if } x \leq a_1 t, \\ a_{2i+1} & \text{if } (a_{2i} + a_{2i+1})(t - t_{2i}) < x < 2a_{2i+1}(t - t_{2i+1}), \\ \frac{x}{2(t-t_{2i})} & \text{if } 2a_{2i+1}(t - t_{2i+1}) < x < 2a_{2i+2}(t - t_{2i+1}), \\ a_{2i+2} & \text{if } 2a_{2i+2}(t - t_{2i+1}) < x < (a_{2i+1} + a_{2i+2})(1 - t_{2i+2}). \end{cases} \quad (6.2.19)$$

Then we have the following.

Claim 2. $\text{TV}(u(\cdot, 1)) = \infty$.

From the construction, shocks and rarefaction of u do not intersect in $\mathbb{R} \times [0, 1]$ and hence from (6.2.19), we can find a sequence $x_1 < x_2 < \dots < x_n < 0$ such that $u(x_i, 1) = a_i$ and $\lim_{n \rightarrow \infty} x_n = 0$. Hence

$$\begin{aligned} \text{TV}(u(\cdot, 1)) &\geq \sum_{n=1}^{\infty} |u(x_n, 1) - u(x_{n+1}, 1)| \\ &= \sum_{n=1}^{\infty} |a_n - a_{n+1}| \\ &\geq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n^2} = \infty. \end{aligned}$$

This proves the claim.

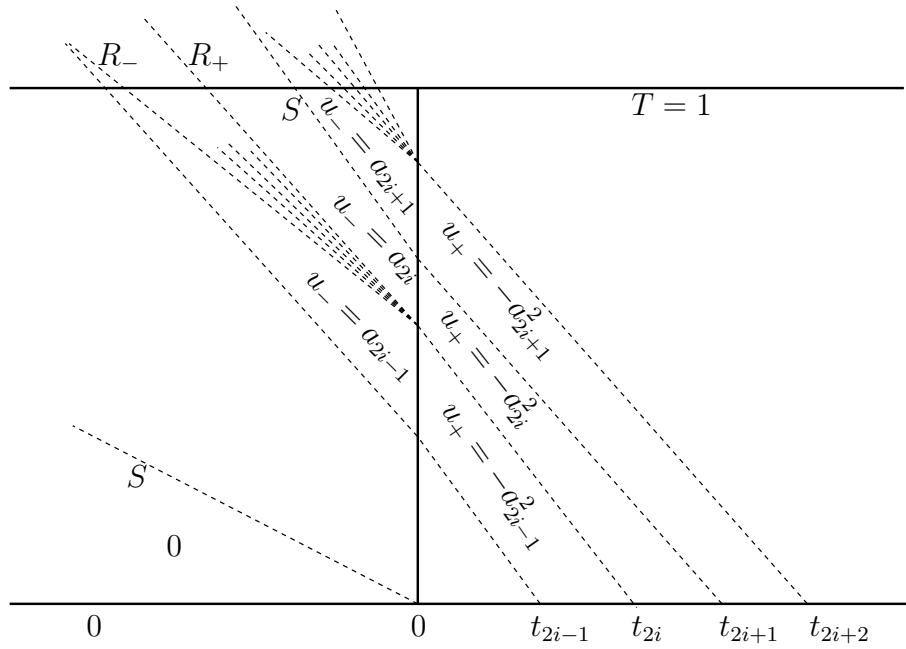


Fig. 6.7:

Step 4. Here we complete the proof of the theorem. Let $\{a_i\}$ and $\{t_i\}$ as in Step 3 and define the initial condition

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ -a_{2i+1}^2 & \text{if } t_{2i} < x < t_{2i+1}, \\ -a_{2i+2}^2 & \text{if } t_{2i+1} < x < t_{2i+2}, \\ -1 & \text{if } x \geq 1. \end{cases}$$

Now for $x > 0$, the solution is given by

$$u(x, t) = \begin{cases} -1 & \text{if } 1 - t \leq x, \\ -a_n^2 & \text{if } t_{n-1} - t < x < t_n - t. \end{cases}$$

At the interface, $x = 0$, R-H condition $f(u^+(t)) = g(u^-(t))$ implies that $-u^+(t) = u^-(t)^2$ and hence $u^-(t)$ is given by,

$$u^-(t) = \begin{cases} -1 & \text{if } t \geq 1, \\ a_n & \text{if } t_{n-1} \leq t \leq t_n. \end{cases}$$

Step 3 gives the solution to the above data in $\mathbb{R} \times [0, 1]$ and hence $\text{TV}(u(\cdot, 1)) = \infty$. This proves the theorem.

Remark 6.2.2. Let $\epsilon > 0$ and $v_{0,\epsilon} = \epsilon^2 u_0$ where u_0 is as in Step 4. Then $\text{TV}(v_{0,\epsilon}) = \epsilon^2 \text{TV}(u_0)$. Let v_ϵ be the corresponding solution. Observe that from (6.2.15) and (6.2.16), $T_\pm^\epsilon(i)$ is independent of ϵ . Hence $\text{TV}(v_\epsilon(\cdot, 1)) = \infty$. This shows that smallness of the

total variation of initial data does not guarantee the boundedness of the total variation at the interface.

Remark 6.2.3. In general, if f and g are not strictly convex Theorem 6.2.5 (3) fails. Here we have an example where total variation bound fails for any (A, B) connection provided one of the points is a critical point though $A \neq \theta_g$ and $B \neq \theta_f$. Let $f(u) = |u - 1| - 1$ and g is given by

$$g(u) = \begin{cases} -u^3 & \text{if } u \leq 0, \\ (u - 1)^2 - 1 & \text{if } u \geq 0 \end{cases}$$

and let $A = 2, B = 0$, for the initial data like in the main theorem, if u is an (A, B) entropy solution, then $\text{TV}(u(\cdot, 1)) = \infty$. The proof follows exactly as in the theorem.

Remark 6.2.4. For general f and g strictly convex C^2 functions and not multiple of each other, the following construction gives the unbounded total variation for (θ_g, B) entropy solution.

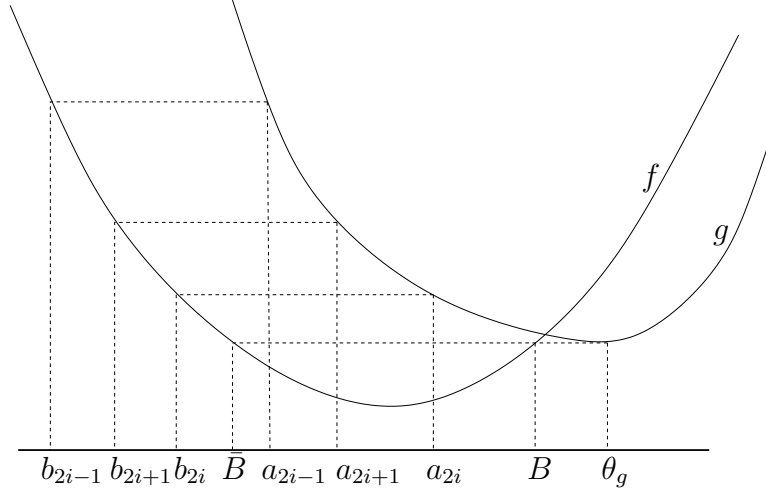


Fig. 6.8:

Assume that $g(\theta_g) > f(\theta_f)$ and let $\bar{B} < B$ such that $f(\bar{B}) = f(B) = g(\theta_g)$, $f'(\bar{B}) < 0$, $f'(B) > 0$.

Step 1. We can choose a sequence $b_n < \bar{B}$ and a sequence $a_n < \theta_g$ with the following property:

- (i). $\lim b_n = \bar{B}$,
- (ii). $b_{2i-1} < b_{2i}, b_{2i+1} < b_{2i}, b_{2i-1} < b_{2i+1}, b_{2i} < b_{2i+2}$,
- (iii). $g(a_n) = f(b_n)$ with $g'(a_n) < 0$ such that $\text{TV}(b_n) < \infty, \text{TV}(a_n) = \infty$. This can be done since $g'(\theta_g) = 0$ and $f'(\bar{B}) > 0$.

Step 2. Let $0 = \xi_0 < \xi_1 < \dots < \xi_n < \dots < \bar{\xi}$, where $\xi_i, \bar{\xi}$ is going to be chosen

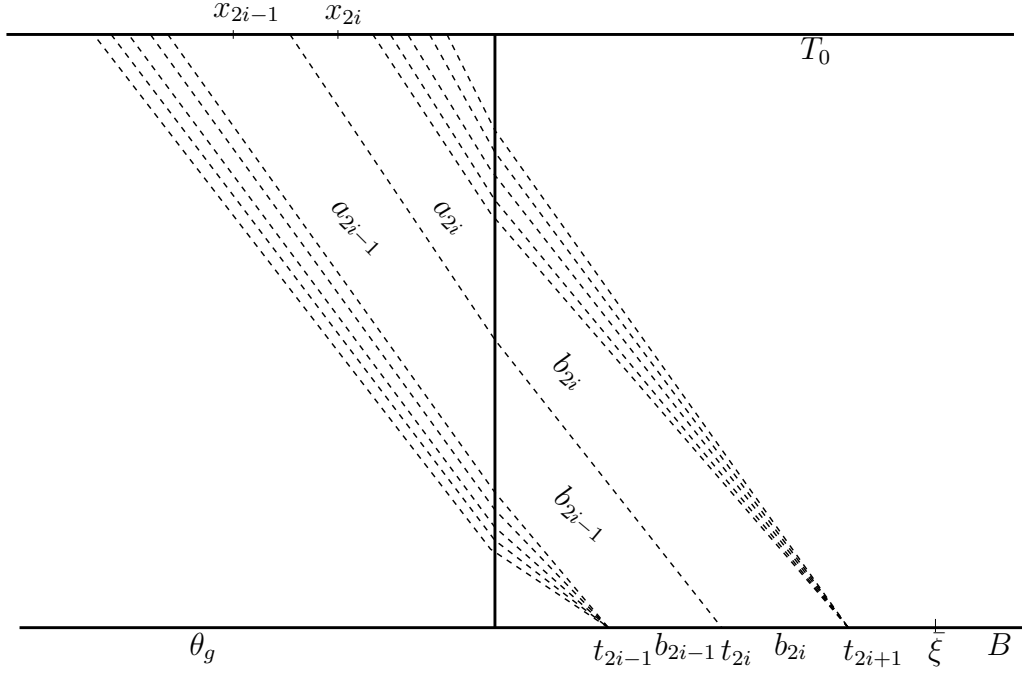


Fig. 6.9:

later. Then we define initial data (see Figure 6.8)

$$u_0(x) = \begin{cases} \theta_g & \text{if } x < 0 = \xi_0, \\ b_n & \text{if } \xi_{n-1} < x < \xi_n, \\ \bar{B} & \text{if } x \geq \bar{\xi}. \end{cases}$$

Step 3. Since $b_{2i-1} < b_{2i}$ and $b_{2i+1} < b_{2i}$, it creates rarefaction and shocks alternatively. Now we can choose $\xi_n, \bar{\xi}, t_0$ such a way that rarefaction and shocks do not meet in the region $(0, t_0) \times \mathbb{R}$ (see Figure 6.9).

Step 4. Now at time t_0 , we can find the points $x_n < 0$ such that $\lim x_n = 0$ and $u(x_n, t_0) = a_n$, which implies $\text{TV}(u(\cdot, t_0)) = \infty$, where u is (θ_g, B) entropy solution.

6.3 Proof of Theorem 6.2.2

In this section we will construct an example showing $\lambda_+(t)$ is not of bounded variation. Basic idea is to choose the data appropriately so that $\lambda_+(t)$ has lot of oscillations. In order to do this we define the following:

Let $f(u) = \frac{u^2}{2}$, $g(u) = \frac{(u-1)^2}{2} = f(u-1)$ then $f^*(u) = \frac{u^2}{2}$, $g^*(u) = u + \frac{u^2}{2}$. For $n \geq 0$,

define

$$\begin{aligned}
x_n &= \left(1 - \frac{1}{2^n}\right), \quad y_n = -\left(1 - \frac{1}{2^n}\right), \\
b_n &= -\left(1 - \frac{1}{2^{n+1}}\right), \quad c_n = 1 - b_n + \frac{(-1)^n}{2^{4(n+2)}}, \\
\tau_n &= -\frac{(x_n + x_{n+1})}{2b_n}, \quad \eta_n = -\frac{(y_n + y_{n+1})}{2(c_n - 1)}, \\
\alpha_n &= -\frac{x_n}{b_n}, \quad \beta_n = -\frac{y_n}{c_n - 1}, \\
\gamma_n &= -\frac{2x_n}{b_{n-1} + b_n}, \quad \delta_n = -\frac{2y_n}{c_{n-1} + c_n - 2}.
\end{aligned}$$

Then the following proposition are needed to prove the theorem. Proposition 6.3.1 is a direct verification. In Proposition 6.3.2, we calculate explicitly $b'_+(t)$ and $b'_-(t)$ for some range of t . Then from these two propositions, we can choose a sequence $\{t_n\}$ such that $f'(\lambda_+(t_{2n+1})) < 0$, $f'(\lambda_+(t_{2n})) > 0$ and $\{\lambda_+(t_n)\}$ has unbounded oscillations.

Proposition 6.3.1. (1) $\tau_{n-1} < \alpha_n < \gamma_n < \tau_n$,

(2) $\eta_{n-1} < \beta_n < \delta_n < \eta_n$,

(3) $\eta_{2n-2} < \alpha_{2n-1} < \beta_{2n-1} < \delta_{2n-1} < \gamma_{2n-1} < \tau_{2n-1} < \eta_{2n-1} < \beta_{2n} < \alpha_{2n} < \gamma_{2n} < \delta_{2n} < \eta_{2n} < \tau_{2n}$,

(4) $c_{2n} - 1 - b_{2n-1} \geq \frac{1}{2}$.

Proof. It follows from direct calculations. Since $x_n < x_{n+1}$, $y_{n+1} < y_n < 0$, $b_n < 0$, $c_n - 1 > 0$, it follows that

$$\begin{aligned}
\alpha_n &= -\frac{x_n}{b_n} < -\frac{(x_n + x_{n+1})}{2b_n} = \tau_n, \\
\beta_n &= -\frac{y_n}{c_n - 1} < -\frac{(y_n + y_{n+1})}{2(c_n - 1)} = \eta_n.
\end{aligned}$$

Now $\tau_{n-1} < \alpha_n$ if and only if $-\frac{(x_n + x_{n-1})}{2b_{n-1}} < -\frac{x_n}{b_n}$ and hence

$$\begin{aligned}
&\left(1 - \frac{1}{2^{n+1}}\right)\left(2 - \frac{1}{2^n} - \frac{1}{2^{n-1}}\right) < 2\left(1 - \frac{1}{2^n}\right)^2, \\
&\text{i.e., } 2 - \frac{4}{2^n} + \frac{3}{2^{2n+1}} < 2 - \frac{4}{2^n} + \frac{2}{2^{2n}},
\end{aligned}$$

i.e., $3 < 4$.

Now $-b_{n-1} < -b_n$ implies that $\alpha_n = \frac{x_n}{-b_n} < \frac{2x_n}{-b_n - b_{n-1}} = \gamma_n$. Also $\gamma_n < \tau_n$ if and only if

$$\frac{2x_n}{-b_n - b_{n-1}} < \frac{x_n + x_{n+1}}{-2b_n},$$

$$\text{i.e., } -4b_n x_n < (x_n + x_{n+1})(-b_n - b_{n-1}),$$

$$\text{i.e., } 4\left(1 - \frac{1}{2^{n+1}}\right)\left(1 - \frac{1}{2^n}\right) < \left(2 - \frac{1}{2^n} - \frac{1}{2^{n+1}}\right)^2,$$

$$\text{i.e., } 4\left(1 - \frac{3}{2^{n+1}} + \frac{1}{2^{2n+1}}\right) < \left(4 - \frac{12}{2^{n+1}} + \frac{9}{2^{2n+1}}\right),$$

and hence $8 < 9$. This proves (1).

Now $\eta_{n-1} < \beta_n$ if and only if $-\frac{(y_n + y_{n-1})}{2(c_{n-1} - 1)} < -\frac{y_n}{c_n - 1}$ and hence

$$\begin{aligned} & \left(1 - \frac{1}{2^{n+1}} + \frac{(-1)^n}{2^{4(n+1)}}\right)\left(2 - \frac{1}{2^n} - \frac{1}{2^{n-1}}\right) < 2\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{2^n} + \frac{(-1)^{n-1}}{2^{4n}}\right), \\ \text{i.e., } & \frac{(-1)^{n+1}}{2^{2n}}\left(4 - \frac{1}{2^{n-1}}\right) < 4. \end{aligned}$$

Since $c_{n-1} < c_n$ implies that $\beta_n < \delta_n$. Also $\delta_n < \eta_n$ if and only if for $n \geq 1$

$$\frac{-2y_n}{c_n + c_{n-1} - 2} < -\frac{y_n + y_{n+1}}{2(c_n - 1)},$$

$$\text{i.e., } 4x_n\left(-b_n + \frac{(-1)^n}{2^{4(n+2)}}\right) < (x_n + x_{n+1})\left(-b_n - b_{n-1} + \frac{(-1)^n}{2^{4(n+2)}} + \frac{(-1)^{n-1}}{2^{4(n+1)}}\right),$$

$$\text{i.e., } \frac{8}{2^{2n+2}} + \frac{4x_n(-1)^n}{2^{4(n+2)}} < \frac{9}{2^{2n+2}} + \frac{(-1)^{n-1}}{2^{4(2n+2)}}\left(2 - \frac{3}{2^{n+1}}\right)(2^4 - 1),$$

$$\text{i.e., } \frac{(-1)^n}{2^{2n+6}}\left(2^5 + 2 - \frac{59}{2^{n+1}}\right) < 1.$$

This proves (2).

Since $c_{2n-1} - 1 + b_{2n-1} < 0$, hence $\alpha_{2n-1} = -\frac{x_{2n-1}}{b_{2n-1}} < \frac{x_{2n-1}}{(c_{2n-1} - 1)} = \beta_{2n-1}$ and $\tau_{2n-1} = -\frac{(x_{2n-1} + x_{2n})}{2b_{2n-1}} < \frac{(x_{2n-1} + x_{2n})}{2(c_{2n-1} - 1)} = \eta_{2n-1}$. Since $c_{2n} - 1 + b_{2n} > 0$, hence $\alpha_{2n} = -\frac{x_{2n}}{b_{2n}} >$

$\frac{x_{2n}}{(c_{2n} - 1)} = -\frac{y_{2n}}{c_{2n} - 1} = \beta_{2n}$. Now $\tau_{2n-1} > \beta_{2n-1}$ if and only if for $n \geq 1$

$$\begin{aligned} & -\frac{x_{2n-1} + x_{2n}}{-2b_{2n-1}} > \frac{x_{2n-1}}{(c_{2n-1} - 1)}, \\ \text{i.e., } & (c_{2n-1} - 1 + 2b_{2n-1})x_{2n-1} + x_{2n} > 0, \\ \text{i.e., } & -\left(1 - \frac{1}{2^{2n}} + \frac{1}{2^{4(2n+1)}}\right)\left(1 - \frac{1}{2^{2n-1}}\right) + \left(1 - \frac{1}{2^{2n}}\right) > 0, \\ \text{i.e., } & 1 - \frac{1}{2^{2n-1}}\left(1 - \frac{1}{2^{2n}}\right) - \frac{1}{2^{4(2n+1)}}\left(1 - \frac{1}{2^{2n-1}}\right) > 0. \end{aligned}$$

Next $\alpha_{2n} < \eta_{2n}$ if and only if

$$\begin{aligned} & -\frac{x_{2n}}{b_{2n}} < -\frac{(y_{2n} + y_{2n+1})}{2(c_{2n} - 1)} = \frac{(x_{2n} + x_{2n+1})}{2(c_{2n} - 1)}, \\ \text{i.e., } & x_{2n}(2(c_{2n} - 1) + b_{2n}) < -b_{2n}x_{2n+1}, \\ \text{i.e., } & \left(1 - \frac{1}{2^{2n}}\right) + \left(1 - \frac{1}{2^{2(2n+1)}}\right) < \left(1 - \frac{1}{2^{2n+1}}\right)\left(1 - \frac{1}{2^{2n+1}}\right), \\ \text{i.e., } & \left(1 - \frac{1}{2^{2n+1}}\right)\left(1 - \frac{1}{2^{2n}} - 1 + \frac{1}{2^{2n+1}}\right) + \frac{2}{2^{4(2n+1)}}\left(1 - \frac{1}{2^{2n}}\right) < 0, \\ \text{i.e., } & -1 + \frac{1}{2^{2n+1}} + \frac{2}{2^{3(2n+1)}}\left(1 - \frac{1}{2^{2n}}\right) < 0. \end{aligned}$$

Now $\eta_{2n-2} < \alpha_{2n-1}$ if and only if for $n \geq 1$

$$\begin{aligned} & \frac{x_{2n-2} + x_{2n-1}}{2(c_{2n-1} - 1)} < -\frac{x_{2n-1}}{b_{2n-1}}, \\ \text{i.e., } & -b_{2n-1}x_{2n-2} < x_{2n-1}(2(c_{2n-1} - 1) + b_{2n-1}), \\ \text{i.e., } & \left(1 - \frac{1}{2^{2n}}\right)\left(1 - \frac{1}{2^{2n-2}}\right) < \left(1 - \frac{1}{2^{2n-1}}\right)\left(1 - \frac{1}{2^{2n}} - \frac{2}{2^{4n}}\right), \\ \text{i.e., } & \left(1 - \frac{1}{2^{2n}}\right)\left(1 - \frac{1}{2^{2n-2}} - 1 + \frac{1}{2^{2n-1}}\right) < -\frac{1}{2^{4n-2}}\left(1 - \frac{1}{2^{2n-1}}\right), \\ \text{i.e., } & -1 + \frac{1}{2^{2n}} < -\frac{1}{2^{3n-1}}\left(1 - \frac{1}{2^{2n-1}}\right). \end{aligned}$$

Now

$$\begin{aligned} c_{2n} + c_{2n-1} - 2 &= -b_{2n} - b_{2n-1} + \frac{1}{2^{4(2n+2)}} - \frac{1}{2^{4(2n+1)}} \\ &< -b_{2n} - b_{2n-1} \end{aligned}$$

and $c_{2n} + c_{2n-1} - 2 > -b_{2n} - b_{2n+1}$. Hence

$$\begin{aligned} \gamma_{2n} &= -\frac{2x_{2n}}{b_{2n-1} + b_{2n}} < \frac{-2y_{2n}}{c_{2n} + c_{2n-1} - 2} = \delta_{2n}, \\ \gamma_{2n-1} &= -\frac{2x_{2n-1}}{b_{2n-2} + b_{2n-1}} > \frac{-2y_{2n-1}}{c_{2n-1} + c_{2n-2} - 2} = \delta_{2n-1}. \end{aligned}$$

This proves (3).

Now we have

$$\begin{aligned}
c_{2n} - 1 - b_{2n-1} &= -b_{2n} + \frac{1}{2^{4(2n+1)}} - b_{2n-1} \\
&= 2 - \frac{1}{2^{2n+1}} - \frac{1}{2^{2n}} - \frac{1}{2^{4(2n+1)}} \\
&\geq \frac{1}{2}.
\end{aligned}$$

This proves (4) and hence the proposition.

Define

$$u_0(x) = \begin{cases} -1 & \text{if } x \geq 1, \\ b_n & \text{if } x_n \leq x < x_{n+1}, \\ c_n & \text{if } y_{n+1} < x \leq y_n, \\ -1 & \text{if } x \leq -1. \end{cases}$$

For $y \geq 0$,

$$J_+(t, y) = v_0(y) + tf^*\left(-\frac{y}{t}\right) = v_0(y) + \frac{y^2}{2t}.$$

For $y \leq 0$,

$$\begin{aligned}
J_-(t, y) &= v_0(y) + tg^*\left(-\frac{y}{t}\right) = v_0(y) - y + \frac{y^2}{2t}. \\
b_+(t) &= \inf_{y \geq 0} J_+(t, y), \\
b_-(t) &= \inf_{y \leq 0} J_-(t, y).
\end{aligned}$$

Then we have the following

Proposition 6.3.2. *With the above definitions, we have*

- (1). $0 < t < \tau_i$ if and only if $J_+(t, x_i) < J_+(t, x_{i+1})$.
- (2). $t \in (\tau_{i-1}, \tau_i)$ if and only if $J_+(t, x_i) < J_+(t, x_j)$ for $i \neq j$.
- (3). $t \in (\gamma_i, \tau_i)$ then $y = -tb_i \in (x_i, x_{i+1})$ such that

$$b_+(t) = J_+(t, -tb_i) = v_0(x_i) - b_i x_i - \frac{tb_i^2}{2} \tag{6.3.1}$$

$$-b'_+(t) = \frac{b_i^2}{2} = f(b_i). \tag{6.3.2}$$

- (4). $0 < t < \eta_i$ if and only if $J_-(t, y_i) < J_-(t, y_{i+1})$.
- (5). $t \in (\beta_i, \eta_i)$ if and only if $J_-(t, y_i) < J_-(t, y_j)$ for $i \neq j$.
- (6). $t \in (\delta_i, \eta_i)$ then $y = -t(c_i - 1) \in (y_{i+1}, y_i)$ such that

$$b_-(t) = J_-(t, -t(c_i - 1)) = v_0(y_i) - c_{i-1}y_i - \frac{t(c_i - 1)^2}{2} \tag{6.3.3}$$

$$-b'_-(t) = \frac{(c_i - 1)^2}{2} = g(c_i). \tag{6.3.4}$$

Proof. By symmetry, it is enough to prove for J_+ . Now $J_+(t, x_i) < J_+(t, x_{i+1})$ if and only if $v_0(x_i) + \frac{x_i^2}{2t} < v_0(x_i) + b_i(x_{i+1} - x_i) + \frac{x_{i+1}^2}{2t}$ and hence $t < -\frac{x_{i+1} + x_i}{2b_i} = \tau_i$. This proves (1).

From (1) it follows that if $t \in (\tau_{i-1}, \tau_i)$, then $J_+(t, x_i) < J_+(t, x_{i+1}) < J_+(t, x_{i+2}) \dots < J_+(t, x_k)$ for all $k \geq i + 1$ and $J_+(t, x_i) < J_+(t, x_{i-1}) < J_+(t, x_{i-2}) \dots < J_+(t, x_k)$ for all $k \leq i - 1$. This proves (2). Converse follows immediately from (1).

Let $t \in (\gamma_i, \tau_i)$ and $J_+(t, z_0) = \min J_+(t, y)$. Then

Claim. $z_0 = -tb_i \in (x_i, x_{i+1})$

$$J_+(t, -tb_i) = v_0(x_i) - b_i x_i - \frac{tb_i^2}{2}.$$

Proof of the claim: From (1), $\{x_j\}_{j \neq i}$ are not the minimizers. Let $z_0 \in [x_{j_0}, x_{j_0+1}]$ for some j_0 . Then we have to consider two cases

Case 1. $j_0 \geq i + 1$. Since $z_0 \notin \{x_{j_0}, x_{j_0+1}\}$ and hence $J_+(t, z_0) < J_+(t, x_{j_0})$. This gives

$$v_0(x_{j_0}) + b_j(z_0 - x_{j_0}) + \frac{y_j^2}{2t} < v_0(x_{j_0}) + \frac{x_{j_0}^2}{2t}.$$

That is, $t \geq \frac{z_0 + x_{j_0}}{-2b_{j_0}} \geq \frac{-x_{j_0}}{b_{j_0}} = \alpha_{j_0}$ which contradicts the fact that $t < \tau_i < \alpha_{j_0}$.

Case 2. Let $j_0 + 1 \leq i$. Let $j \leq i$, then $-b_j = x_{j+1}$. Since $t < 1$ and hence $-tb_j < x_{j+1}$. Also $\alpha_j \leq \alpha_i < t$ implies that $x_j < -tb_j$. This implies that $z_j = -tb_j \in (x_j, x_{j+1})$ is the unique critical point of $J_+(t, y)$ for $y \in (x_j, x_{j+1})$ and hence $J_+(t, z_j) = \min\{J_+(t, y); y \in (x_j, x_{j+1})\}$ and $J_+(t, z_j) = v_0(x_j) - b_j x_j - \frac{tb_j^2}{2}$.

Now $J_+(t, z_j) > J_+(t, z_{j+1})$ if and only if

$$v_0(x_j) - b_j x_j - \frac{tb_j^2}{2} > v_0(x_j) - b_j(x_{j+1} - x_j) - b_j x_j - \frac{tb_{j+1}^2}{2}.$$

This implies that for $j \leq i - 1$,

$$\gamma_{j+1} = -\frac{2x_{j+1}}{b_j + b_{j+1}} < t.$$

Since $t > \gamma_i \geq \gamma_{j+1} \quad \forall j \leq i - 1$, hence $J_+(t, z_j) > J_+(t, z_{j+1}) > \dots > J_+(t, z_i)$. Therefore $z_0 = z_i = -tb_i$ is the unique minimizer for $J_+(t, y)$ and $b_+(t) = J_+(t, -tb_i) = v_0(x_i) - b_i x_i - \frac{tb_i^2}{2}$. This proves the claim.

From this (6.3.1) and (6.3.2) follows and hence the proposition.

With this preliminary, we will construct the following example which concludes the Theorem 6.2.2.

Proposition 6.3.3. *Let (A, B) be a connection such that $f(B) = g(A) \leq \frac{1}{4}$. Then with the above notation, total variation of λ_+ is infinity.*

Proof: From (3) of proposition (6.3.1) and proposition (6.3.2), we can choose a sequence $t_n < t_{n+1}$ (see Figure 6.9) such that

- (1) $t_{2n} \in (\delta_{2n}, \eta_{2n}), t_{2n-1} \in (\gamma_{2n-1}, \tau_{2n-1})$.
- (2) $-b'_+(t_{2n}) = f(b_{2n}), -b'_-(t_{2n}) = g(c_{2n})$.
- (3) $-b'_+(t_{2n-1}) = f(b_{2n-1}), -b'_-(t_{2n-1}) = g(c_{2n-1})$.

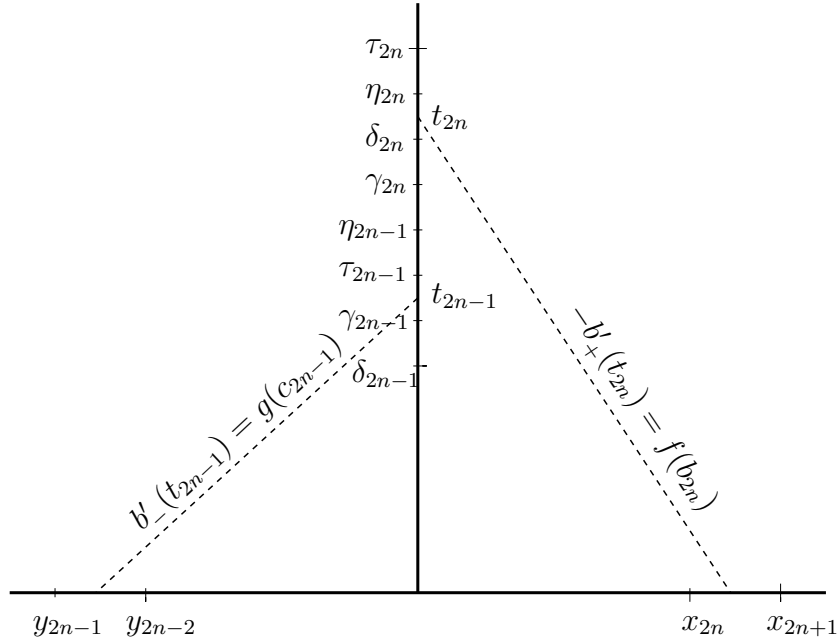


Fig. 6.10:

Since $c_{2n} - 1 > b_{2n}$ and hence $g(c_{2n}) > f(b_{2n})$ and this implies $-b'_-(t_{2n}) > -b'_+(t_{2n})$. Hence from the definition of λ_+ ,

$$\lambda_+(t_{2n}) = f_+^{-1}(g(c_{2n})) = c_{2n} - 1.$$

Also $c_{2n-1} - 1 < b_{2n-1}$ and hence $-b'_-(t_{2n-1}) < -b'_+(t_{2n-1})$. Therefore

$$\lambda_+(t_{2n-1}) = f_-^{-1}(f(b_{2n-1})) = b_{2n-1}.$$

From (4) of proposition (6.3.1)

$$\lambda_+(t_{2n}) - \lambda_+(t_{2n-1}) = c_{2n} - 1 - b_{2n-1} \geq \frac{1}{2}$$

and hence

$$\sum_{n=1}^{\infty} |\lambda_+(t_n) - \lambda_+(t_{n-1})| = \infty.$$

This completes the proof of the example.

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