



HAL
open science

Foncteurs polynomiaux et homologie stable à coefficients polynomiaux

Christine Vespa

► **To cite this version:**

Christine Vespa. Foncteurs polynomiaux et homologie stable à coefficients polynomiaux. Topologie algébrique [math.AT]. Université de Strasbourg, 2013. tel-00903321

HAL Id: tel-00903321

<https://theses.hal.science/tel-00903321>

Submitted on 11 Nov 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INSTITUT DE
RECHERCHE
MATHÉMATIQUE
AVANCÉE

UMR 7501

Strasbourg

Habilitation à diriger des recherches

Université de Strasbourg
Spécialité MATHÉMATIQUES

Christine Vespa

**Foncteurs polynomiaux et homologie stable à
coefficients polynomiaux**

Soutenue le 21 novembre 2013
devant la commission d'examen

William Dwyer, examinateur
Vincent Franjou, rapporteur
Hans-Werner Henn, garant
Nick Kuhn, rapporteur
Birgit Richter, rapporteur
Lionel Schwartz, examinateur

www-irma.u-strasbg.fr



Contents

1	Polynomial functors	13
1.1	What are polynomial functors?	13
1.2	The results	14
1.2.1	Polynomial functors from abelian groups	15
1.2.2	Polynomial functors from groups	18
1.2.3	Polynomial functors from pointed sets	21
1.2.4	Polynomial functors from algebras over a set-operad	21
1.2.5	Polynomial functors from a pointed theory	25
1.3	The methods developped in [HV11] and [HPV]	28
1.3.1	The methods developped in [HV11]	28
1.3.2	The methods developped in [HPV]	29
1.4	Work in progress and open questions	30
1.4.1	Polynomial functors with values in a semi-abelian category [HV]	30
1.4.2	Polynomial functors on a symmetric monoidal category with an initial object [DVa]	32
2	Stable homology with twisted coefficients and functor homology	35
2.0.3	What is stable homology?	35
2.1	The results	36
2.1.1	Stable homology with trivial coefficients	37
2.1.2	Stable homology with twisted coefficients	39
2.2	The methods developped for twisted coefficients in [DV10] [Dja12] and [DVb]	41
2.2.1	Step 1: A general theorem	42
2.2.2	Step 2: Comparison of functor homologies	45
2.3	Work in progress and open questions	49
2.3.1	Automorphisms of free groups $Aut(\mathbb{Z}^{*n})$	49
2.3.2	Other families of groups	49

Introduction

Mon domaine de recherche se situe à l'intersection entre l'algèbre et la topologie algébrique. Mes résultats ont tous pour point commun d'utiliser de manière fondamentale les catégories de foncteurs si bien que mon travail illustre la devise suivante: utiliser des catégories de foncteurs dans leur diversité, transiter de l'une à l'autre via des équivalences, et les comparer s'avère extrêmement fructueux.

Mes résultats se divisent en deux parties correspondant aux deux chapitres de cette habilitation. Le premier chapitre concerne l'étude de la structure des foncteurs polynomiaux et présente les articles [HV11], [HPV] et [DVa]. Le second concerne le calcul de l'homologie stable d'une famille de groupes à coefficients donnés par un foncteur polynomial et décrit les articles [DV10] et [DVb]. Ces deux parties ne sont pas indépendantes. D'une part, les résultats obtenus dans la première partie nourrissent et motivent ceux obtenus dans la seconde et vice versa. D'autre part, l'utilisation de catégories de foncteurs de Mackey généralisés, qui est une des spécificités de mon travail, intervient de manière cruciale dans chacune de ces deux parties. Nous précisons maintenant ces deux points.

Le calcul de l'homologie stable de familles de groupes à coefficients donnés par un foncteur repose sur une interprétation de cette homologie en termes d'homologie des foncteurs (i.e. d'algèbre homologique dans des catégories de foncteurs). Notons que l'homologie des foncteurs a de nombreuses autres applications. Par exemple, plusieurs théories classiques d'homologie peuvent être interprétées en termes d'homologie des foncteurs (l'homologie cyclique et de Hochschild des algèbres commutatives dans [Pir00b] ou des algèbres associatives dans [PR02] ou la E_n -homologie des algèbres commutatives non-unitaires dans [LR11]). Une connaissance fine de la structure des foncteurs polynomiaux est indispensable pour obtenir des calculs explicites d'homologie des foncteurs. Ainsi, l'étude des foncteurs polynomiaux menée dans [DVb] et s'inspirant du résultat obtenu dans [HV11] est cruciale pour montrer un des résultats principaux de [DVb] donnant la trivialité de l'homologie stable des groupes d'automorphismes des groupes libres tordue par un foncteur covariant. Lorsqu'on tord cette homologie par un foncteur contravariant, un résultat partiel de [DVb] montre que cette homologie n'est pas triviale. On peut s'attendre à ce que cette homologie s'interprète en termes d'homologie des foncteurs sur les groupes. Dans un travail en cours avec Pirashvili [PV] nous étudions plus précisément l'homologie des foncteurs sur les groupes en nous inspirant de l'étude des foncteurs polynomiaux sur les groupes menée dans [DVb] et [HPV]. Dans un travail en cours avec Djament [DVa], motivé par le calcul de l'homologie stable des sous-groupes de congruences à coefficients tordus, nous étendons la définition de foncteur polynomial à un cadre plus général et étudions la structure de ces foncteurs.

Un foncteur de Mackey est la donnée d'un couple de foncteurs, un foncteur covariant et un foncteur contravariant, satisfaisant des relations de compatibilité. Originellement, les foncteurs

de Mackey apparaissent en théorie des groupes. Pour un groupe fini G , un foncteur de Mackey classique est un couple de foncteurs additifs ayant pour source la catégorie des G -ensembles et satisfaisant les relations classiques entre les applications d'induction, de restriction et de conjugaison (voir [Web00]). Cette définition peut être généralisée dans plusieurs directions: on peut considérer d'autres catégories sources, relâcher la condition d'additivité des foncteurs et avoir des relations de compatibilité plus subtiles entre les deux foncteurs. Les catégories de foncteurs de Mackey généralisés ont une structure extrêmement riche permettant d'encoder des structures complexes et variées et sont au cœur de mes travaux de recherche bien qu'étant parfois absents des énoncés principaux. Ainsi, dans mes travaux de thèse, j'ai introduit et étudié des foncteurs de Mackey généralisés sur les espaces quadratiques sur le corps \mathbb{F}_2 (voir [Ves08b], [Ves07], [Ves08a]). Dans [DV10], une des étapes de la preuve de notre théorème principal utilise des équivalences de catégories impliquant des foncteurs de Mackey généralisés sur les espaces vectoriels avec injections. Dans [HPV], nous obtenons une description de foncteurs polynomiaux en termes de foncteurs de Mackey généralisés sur le PROP associé à une opérade ensembliste. Une des étapes intermédiaires d'un travail en cours avec Djament [DVa] consiste à étendre à un cadre très général la description des foncteurs polynomiaux de la catégorie des foncteurs de Mackey généralisés sur les espaces quadratiques sur \mathbb{F}_2 que j'avais obtenue dans [Ves08a].

Dans la suite, nous détaillons le contenu des deux chapitres de ce mémoire.

Le premier chapitre concerne l'étude de la structure des foncteurs polynomiaux. Dans les catégories de foncteurs entre catégories abéliennes, les foncteurs additifs jouent un rôle privilégié dans plusieurs domaines de l'algèbre. Cependant il existe de nombreux foncteurs très intéressants qui ne sont pas additifs. Par exemple, le produit tensoriel de groupes abéliens définit un foncteur $T^2 : Ab \rightarrow Ab$ donné par $T^2(G) = G \otimes G$ qui n'est pas additif mais polynomial de degré deux. Plus généralement, le produit tensoriel d'ordre n d'un groupe abélien G définit un foncteur polynomial de degré n entre groupes abéliens. Les foncteurs polynomiaux ont été introduits par Eilenberg et MacLane dans [EML54] pour les foncteurs entre catégories de modules. La définition d'Eilenberg et MacLane peut être étendue à des catégories plus générales que des catégories de modules (voir la section 1.1 pour un cadre général convenable pour cette définition et la section 1.4.2 pour une généralisation très récente de cette définition). De nombreux exemples de foncteurs polynomiaux apparaissent naturellement en topologie algébrique. Par exemple, l'homologie et l'homotopie stable des espaces d'Eilenberg-MacLane sont des foncteurs polynomiaux sur les groupes abéliens (voir [EML54]). Les foncteurs polynomiaux apparaissent également en théorie des représentations des groupes algébriques, en K-théorie algébrique ainsi que pour la théorie des modules sur l'algèbre de Steenrod (voir [FFPS03]). La catégorie des modules instables sur l'algèbre de Steenrod est étroitement reliée à la catégorie des foncteurs polynomiaux entre les espaces vectoriels sur un corps fini (voir [HLS93]) et cette relation est le point de départ de l'étude approfondie de cette catégorie de foncteurs (voir [Kuh94a], [Kuh94b], [Kuh95]).

Il n'est pas difficile de voir que les valeurs d'un foncteur polynomial de degré n d'une catégorie générale ayant de bonnes propriétés dans la catégorie des groupes abéliens sont entièrement déterminées par la donnée de n groupes abéliens. Inversement, on aimerait savoir comment recoller n groupes abéliens pour obtenir un foncteur polynomial de degré n . Cette question est en général difficile et nous sommes loin de comprendre la description combinatoire des foncteurs polynomiaux dans le cas général. Cependant, une telle description a été obtenue

par Baues, Pirashvili *et al.* dans plusieurs cas (voir le tableau à la section 1.2). Ma recherche sur la structure des foncteurs polynomiaux est dans la continuité de ces résultats, dans des contextes plus généraux.

Dans un article en collaboration avec Manfred Hartl [HV11] nous donnons une description explicite des foncteurs quadratiques allant d’une théorie algébrique pointée \mathcal{T} (i.e. une catégorie \mathcal{T} ayant un objet nul et où tous les objets sont des coproduits finis d’un générateur E) dans la catégorie des groupes abéliens. Plus précisément, nous définissons dans [HV11] les \mathcal{T} -modules quadratiques par rapport à E qui sont des diagrammes

$$M = (T_{11}(cr_2\bar{P}_E)(E, E) \otimes_{\bar{P}_E(E)} M_e \xrightarrow{\hat{H}} M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e)$$

où $T_{11}(cr_2\bar{P}_E)$ est la bilinéarisation du second effet croisé cr_2 du foncteur projectif standard réduit $\bar{P}_E : \mathcal{T} \rightarrow Ab$, M_e (resp. M_{ee}) est un module sur $\bar{P}_E(E)$ (resp. $\bar{P}_E(E) \otimes \bar{P}_E(E)$), l’application T est une involution de M_{ee} et \hat{H} et P sont des homomorphismes satisfaisant certaines conditions. Nous obtenons:

Théorème 1. [HV11] *La catégorie des foncteurs quadratiques de \mathcal{T} dans Ab est équivalente à la catégorie des \mathcal{T} -modules quadratiques par rapport à E .*

En appliquant notre résultat à la catégorie des groupes libres de type fini nous retrouvons un cas particulier d’un résultat obtenu par Baues et Pirashvili dans [BP99] et en l’appliquant à la catégorie des ensembles finis pointés nous retrouvons un cas particulier d’un résultat obtenu par Pirashvili dans [Pir00a]. En appliquant notre résultat au cas où E est un cogroupe nous obtenons une description assez simple de ces foncteurs quadratiques, qui complète le papier [BP99] où ce cas était considéré sans être résolu. Pour démontrer le Théorème 1, nous transitons à travers la catégorie des foncteurs additifs sur une certaine catégorie préadditive à deux objets, via le théorème de Gabriel-Popescu, puis nous réinterprétons cette catégorie en termes de \mathcal{T} -modules quadratiques.

En s’inspirant du résultat précédent, Djament et moi obtenons dans [DVb] un résultat pour les foncteurs polynomiaux de degré arbitraire, sur une théorie pointée engendrée par E , dans les groupes abéliens. Plus précisément, nous obtenons le diagramme de recollement entre les foncteurs polynomiaux de degré $n-1$ et ceux de degré n . Pour les foncteurs quadratiques, ce résultat est moins précis que celui obtenu dans [HV11]. Cependant, en appliquant ce résultat à la catégorie des groupes libres de type fini gr nous obtenons que les foncteurs polynomiaux allant de gr dans la catégorie des groupes abéliens Ab s’obtiennent par extensions successives de foncteurs polynomiaux se factorisant par le foncteur d’abélianisation. Plus précisément, nous obtenons le résultat suivant:

Proposition 2. [DVb] *Soit F un foncteur polynomial de degré n allant de gr dans Ab . Le foncteur F admet une filtration*

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \dots \subset F_{n-1} \subset F_n = F$$

telle que chaque sous-quotient F_i/F_{i-1} est polynomial de degré inférieur ou égal à $n-i$ et est dans l’image essentielle du foncteur $Fonc(ab, Ab) \rightarrow Fonc(gr, Ab)$ obtenu par précomposition avec le foncteur d’abélianisation.

Ce résultat sur la structure des foncteurs polynomiaux de gr dans Ab est essentiel dans la preuve du résultat concernant l'homologie stable des groupes d'automorphismes des groupes libres obtenu dans [DVb].

Dans un papier de Baues, Dreckmann, Franjou et Pirashvili [BDFP01], les auteurs donnent une description des foncteurs polynomiaux de degré arbitraire sur les groupes abéliens en termes de foncteurs de Mackey généralisés. Dans [HPV] nous généralisons ce résultat. D'une part nous obtenons une description des foncteurs allant des \mathcal{P} -algèbres dans les groupes abéliens, pour \mathcal{P} une opérade ensembliste (i.e. une opérade dans la catégorie monoïdale des ensembles) en combinant plusieurs équivalences de catégories. D'autre part, nous obtenons une description des foncteurs polynomiaux allant des groupes dans les groupes abéliens à partir du résultat précédent et d'un isomorphisme entre la catégorie des foncteurs polynomiaux allant des monoïdes dans les groupes abéliens et celle des foncteurs polynomiaux allant des groupes dans les groupes abéliens. Plus précisément nous obtenons le résultat suivant:

Théorème 3. [HPV] *Soit $n \in \mathbb{N}$, il existe une équivalence de catégories:*

$$Pol_n(gr, Ab) \simeq (PMack(\Omega(\mathcal{A}s), Ab))_{\leq n}$$

où $(PMack(\Omega(\mathcal{A}s), Ab))_{\leq n}$ est la sous-catégorie pleine de la catégorie de foncteurs de Mackey généralisés $PMack(\Omega(\mathcal{A}s), Ab)$ ayant pour objets les foncteurs qui s'annulent sur $\underline{0}$ et sur les ensembles \underline{m} tels que $m > n$ et $\Omega(\mathcal{A}s)$ est le PROP associé à \mathcal{P} (voir Définition 1.2.24).

L'intérêt du théorème 3 réside dans le fait que la description explicite d'un objet de $PMack(\Omega(\mathcal{A}s), Ab)$ est aisément accessible. Ainsi, ce théorème fournit une description des objets de $Pol_n(gr, Ab)$ en termes de n groupes abéliens, de morphismes de groupes les reliant et de relations satisfaites par ces morphismes de groupes. Nous nous contenterons ici de dire que la structure de foncteur de Mackey généralisé permet d'encoder facilement toutes ces informations et nous renvoyons le lecteur à [HPV] pour la définition précise de la catégorie $PMack(\Omega(\mathcal{A}s), Ab)$. L'équivalence de catégorie du théorème 3 explique donc comment recoller n groupes abéliens pour obtenir un foncteur polynomial de degré n allant des groupes dans les groupes abéliens. A titre d'exemple, nous donnons à la définition 1.2.15 la description combinatoire d'un foncteur cubique allant des groupes dans les groupes abéliens obtenue à partir de l'équivalence de catégorie du théorème 3.

Le second chapitre concerne l'homologie stable de familles de groupes à coefficients donnés par des foncteurs polynomiaux. Soit \mathcal{C} une catégorie monoïdale symétrique et A un objet dans \mathcal{C} . Le groupe d'automorphismes $Aut_{\mathcal{C}}(A^{\oplus n})$ est noté $G(n)$. En évaluant un foncteur $F : \mathcal{C} \rightarrow Ab$ sur $A^{\oplus n}$ nous obtenons, de manière naturelle, un module sur $Aut_{\mathcal{C}}(A^{\oplus n})$, ce qui nous permet de considérer $H_*(G(n), F(A^{\oplus n}))$. De plus, remarquons que l'injection des n premières variables $A^{\oplus n} \hookrightarrow A^{\oplus n+1}$ est compatible avec l'action des groupes d'automorphismes $Aut_{\mathcal{C}}(A^{\oplus n})$ via le morphisme $Aut(A^{\oplus n}) \xrightarrow{g \mapsto g \oplus Id_A} Aut(A^{\oplus n+1})$. Nous obtenons ainsi une suite naturelle de morphismes:

$$\dots \rightarrow H_k(G(n), F(A^{\oplus n})) \rightarrow H_k(G(n+1), F(A^{\oplus n+1})) \rightarrow \dots \quad (1)$$

qui donne lieu à la question naturelle suivante: *Peut-on calculer la colimite de cette suite?*

Cette colimite est appelée l'homologie stable de la famille de groupes $(G(n))$ à coefficients tordus par F et sera notée par $H_k(G_{\infty}, F_{\infty})$. Bien que l'homologie stable de groupes classiques à coefficients tordus a un comportement plus régulier que l'homologie instable, elle

est inaccessible par un calcul direct. Les résultats obtenus dans ce domaine consistent à interpréter l'homologie stable à coefficients tordus en termes d'homologie des foncteurs. La force de ces résultats réside dans le fait que nous avons des outils puissants pour calculer des groupes Tor dans les catégories de foncteurs polynomiaux (voir [FLS94] [FFSS99]) ce qui nous permet d'obtenir des calculs explicites d'homologie stable. En 1999, Betley [Bet99] et Suslin [FFSS99] montrent indépendamment que l'homologie stable des groupes linéaires peuvent être interprétés en termes d'homologie des foncteurs: ils établissent un isomorphisme entre les groupes Tor de foncteurs polynomiaux entre espaces vectoriels sur un corps fini et l'homologie stable des groupes linéaires à coefficients tordus convenables.

Dans [DV10] nous développons une stratégie générale permettant de calculer l'homologie stable de la suite de groupes $(G(n))$ à coefficients donnés par un foncteur covariant polynomial réduit. D'une part, on obtient un théorème général qui exprime l'homologie stable de la suite de groupes $(G(n))$ à coefficients tordus, à partir de l'homologie de la catégorie \mathcal{C} et de l'homologie stable de la suite de groupes $(G(n))$ à coefficients triviaux. D'autre part, on calcule l'homologie de la catégorie \mathcal{C} , au cas par cas, en reliant cette catégorie à une autre catégorie dans laquelle l'algèbre homologique est accessible. Cette seconde étape est la plus difficile. Elle consiste à obtenir un isomorphisme entre l'homologie de chacune de ces deux catégories en utilisant une suite spectrale de Grothendieck. Non seulement nous pouvons obtenir de nouveaux résultats à l'aide de cette méthode mais nous retrouvons également, grâce à elle, tous les autres résultats connus précédemment sur ce sujet. Ainsi, en prenant pour \mathcal{C} des catégories convenables, nous sommes capables de traiter le cas des groupes linéaires, des groupes orthogonaux, des groupes symplectiques, des groupes unitaires ... Par exemple, pour les groupes orthogonaux (ou symplectiques), nous montrons qu'il existe un isomorphisme naturel entre l'homologie stable des groupes orthogonaux à coefficients tordus par un foncteur polynomial et des groupes Tor entre des endofoncteurs sur les espaces vectoriels. Plus précisément, pour les groupes orthogonaux, nous obtenons:

Théorème 4. [DV10] *Soit k un corps fini de caractéristique p et F un foncteur polynomial entre k -espaces vectoriels. Il existe un isomorphisme naturel:*

$$\operatorname{colim}_{n \in \mathbb{N}} H_*(O_{n,n}(k); F(k^{2n})) \simeq \operatorname{Tor}_*^{\mathcal{E}_k^f}(V \mapsto k[S^2(V^*)], F) \quad \text{si } p \text{ est impair}$$

$$\operatorname{colim}_{n \in \mathbb{N}} H_*(O_{n,n}(k); F(k^{2n})) \simeq \operatorname{Tor}_*^{\mathcal{E}_k^f}(V \mapsto k[S^2(V^*)], F) \otimes H_*(\mathbb{Z}/2; k) \quad \text{si } p = 2$$

où \mathcal{E}_k^f est la catégorie des k -espaces vectoriels de dimension finie et S^2 est la deuxième puissance symétrique.

Comme conséquence du théorème 4 nous obtenons des calculs explicites d'homologie stable des groupes orthogonaux à coefficients tordus. Par exemple, pour les puissances symétriques, nous obtenons:

Théorème 5. [DV10] *Soit k un corps fini de cardinal q impair. Pour les puissances symétriques S^\bullet , l'algèbre bigraduée*

$$\operatorname{colim}_{n \in \mathbb{N}} H_*(O_{n,n}(k); S^\bullet(k^{2n}))$$

est polynomiale sur des générateurs $\alpha_{m,s}$ de bidegré $(2q^s m, q^s + 1)$ où l'entier m est le degré homologique.

Dans [DVb] nous appliquons notre méthode générale à l'étude de l'homologie stable des groupes d'automorphismes des groupes libres. Plus précisément, nous obtenons le théorème suivant:

Théorème 6. [DVb] *Soit F un foncteur polynomial réduit allant de la catégorie gr des groupes libres de type fini dans la catégorie Ab des groupes abéliens. Alors*

$$\operatorname{colim}_{n \in \mathbb{N}} H_* \left(\operatorname{Aut}(\mathbb{Z}^{*n}); F(\mathbb{Z}^{*n}) \right) = 0.$$

Un des ingrédients de la preuve de ce théorème est le résultat sur la structure des foncteurs polynomiaux sur les groupes donné à la proposition 2.

Contrairement au cas des groupes linéaires, l'homologie stable des groupes d'automorphismes des groupes libres à coefficients donnés par un foncteur contravariant ne peut pas être obtenue à partir de celle à coefficients donnés par un foncteur covariant. Pour le moment, nous ne savons pas calculer l'homologie stable des groupes d'automorphismes des groupes libres à coefficients donnés par un foncteur contravariant. Cependant, nous obtenons dans [DVb] le résultat suivant qui montre que la situation est complètement différente pour l'homologie stable des groupes d'automorphismes des groupes libres à coefficients donnés par un foncteur contravariant que celle à coefficients donnés par un foncteur covariant.

Proposition 7. [DVb] *Soit F un foncteur polynomial contravariant et réduit allant de la catégorie ab des groupes abéliens libres de type fini dans la catégorie Ab . Il existe un isomorphisme naturel*

$$\operatorname{colim}_{n \in \mathbb{N}} H_1 \left(\operatorname{Aut}(\mathbb{Z}^{*n}); F(\mathbb{Z}^n) \right) \simeq F \otimes_{\mathbf{ab}} \operatorname{Id}$$

où $\operatorname{Aut}(\mathbb{Z}^{*n})$ agit sur $F(\mathbb{Z}^n)$ via la projection naturelle sur $GL_n(\mathbb{Z})$ et $\operatorname{Id} : ab \rightarrow Ab$ est le foncteur d'inclusion.

Bibliography of the author:

- [Ves08b] Generic representations of orthogonal groups: the functor category \mathcal{F}_{quad}
Journal of Pure and Applied Algebra 212/6 (2008), 1472–1499
- [Ves07] Generic representations of orthogonal groups: the mixed functors
Algebraic & Geometric Topology 7 (2007), 379–410.
- [Ves08a] Generic representations of orthogonal groups: projective functors in the category \mathcal{F}_{quad}
Fund. Math. 200 (2008), no. 3, 243–278.
- [DV10] Sur l’homologie des groupes orthogonaux et symplectiques à coefficients tordus
(with Aurélien Djament)
Ann. Scient. Éc. Norm. Sup., (4) 43 (2010), no. 3, 395–459.
- [HV11] Quadratic functors on pointed categories
(with Manfred Hartl)
Advances in Mathematics 226 (2011), 3927-4010
- [HPV] Polynomial functors from algebras over a set-operad and non-linear Mackey functors
(with Manfred Hartl et Teimuraz Pirashvili)
(Accepted for publication in Int. Math. Res. Not. IMRN) (73 pages)
- [DVb] Sur l’homologie des groupes d’automorphismes des groupes libres à coefficients polynomiaux
(with Aurélien Djament)
(Accepted for publication in Commentarii Mathematici Helvetici) (23 pages)
- [DVa] De la structure des foncteurs polynomiaux sur les espaces hermitiens
(with Aurélien Djament)
(Prepublication available on ArXiv)(36 pages)
- [PV] Cohomologie des foncteurs sur les groupes libres
(with Teimuraz Pirashvili) (In preparation)
- [HV] Quadratic functors to semi-abelian categories
(with Manfred Hartl) (In preparation)

Notations:

We denote by:

- Gr the category of groups;
- gr the full subcategory of Gr having as objects finitely generated free objects;
- Ab the category of abelian groups;
- ab the full subcategory of Ab having as objects finitely generated free objects;
- Γ is the category of finite pointed sets;

I take advantage of the writing of this report to include some partial results that I obtained and which appear, for the moment, nowhere in the literature. Here is a list of these results:

- The description of the cubical \mathbb{Z} -module associated to the third Lie functor L^3 in Example 1.2.9;
- the description of cubical functors from gr to Ab in Corollary 1.2.16;
- the description of the quadratic abelian group associated to the Passi functor P_2 in Example 1.2.17;
- a projective resolution of $\Gamma^2 \circ \mathfrak{a}$ in $Quad(gr, Ab)$ in Example 1.2.18;
- the values of $Ext_{Func(gr, Ab)}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$ in section 2.3.1.

Chapter 1

Polynomial functors

In their fundamental work on homology of spaces thereafter linked to their names [EML54] Eilenberg and Mac Lane introduced cross effects and polynomial functors for functors between categories of modules.

1.1 What are polynomial functors?

Let $(\mathcal{C}, \oplus, 0)$ be a small strict symmetric monoidal category where the unit 0 is the null object of \mathcal{C} .

Example 1.1.1. *Any pointed category with finite coproducts is an example of such category. In particular, the categories Gr and Ab are pointed categories with finite coproducts.*

In the sequel, we need the following notations: for $X_1, X_2, \dots, X_n \in \mathcal{C}$, let

$$i_k^n : X_1 \oplus \dots \oplus \hat{X}_k \oplus \dots \oplus X_n \rightarrow X_1 \oplus \dots \oplus X_k \oplus \dots \oplus X_n$$

be the composition:

$$X_1 \oplus \dots \oplus \hat{X}_k \oplus \dots \oplus X_n \simeq X_1 \oplus \dots \oplus 0 \oplus \dots \oplus X_n \rightarrow X_1 \oplus \dots \oplus X_k \oplus \dots \oplus X_n$$

where the second map is induced by the unique map $0 \rightarrow X_k$. Similarly one obtains morphisms:

$$r_k^n = (1_{X_1}, \dots, 1_{X_1}, 0, 1_{X_1}, \dots, 1_{X_n}) : X_1 \oplus \dots \oplus X_k \oplus \dots \oplus X_n \rightarrow X_1 \oplus \dots \oplus \hat{X}_k \oplus \dots \oplus X_n$$

using the existence of the unique map $X_k \rightarrow 0$. We have the relation:

$$r_k^n i_k^n = 1_{X_1 \oplus \dots \oplus \hat{X}_k \oplus \dots \oplus X_n}.$$

In the sequel, \mathcal{D} denotes one of the categories Ab or Gr . Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

Definition 1.1.2. *The n -th cross-effect of F is a functor $cr_n F : \mathcal{C}^{\times n} \rightarrow \mathcal{D}$ (or a multi-functor) defined inductively by*

$$cr_1 F(X) = \ker(F(0) : F(X) \rightarrow F(0))$$

$$cr_2 F(X_1, X_2) = \ker((F(r_2^2), F(r_1^2))^t : F(X_1 \oplus X_2) \rightarrow F(X_1) \times F(X_2))$$

and, for $n \geq 3$, by

$$cr_n F(X_1, \dots, X_n) = cr_2(cr_{n-1} F(-, X_3, \dots, X_n))(X_1, X_2).$$

In other words, to define the n -th cross-effect of F we consider the $(n - 1)$ -st cross-effect, we fix the $n - 2$ last variables and we consider the second cross-effect of this functor. One often writes $F(X_1 | \dots | X_n) = cr_n F(X_1, \dots, X_n)$. Note that $F(X) \simeq cr_1 F(X) \rtimes F(0)$ as $F(0) : F(0) \rightarrow F(X)$ is a natural section of $F(0) : F(X) \rightarrow F(0)$. Moreover, one easily checks by induction that $cr_n F(X_1, \dots, X_n)$ is a subgroup of $F(X_1 \oplus \dots \oplus X_n)$. So, we are mainly interested in *reduced* functors $F : \mathcal{C} \rightarrow \mathcal{D}$, that is satisfying $F(0) = 0$. We denote by $Func_*(\mathcal{C}, \mathcal{D})$ the category of reduced functors $F : \mathcal{C} \rightarrow \mathcal{D}$.

There is an alternative description of cross effects for reduced functors.

Proposition 1.1.3. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then the n -th cross-effect $cr_n F(X_1, \dots, X_n)$ is equal to the kernel of the natural homomorphism*

$$\hat{r}^F : F(X_1 \oplus \dots \oplus X_n) \longrightarrow \bigoplus_{k=1}^n F(X_1 \oplus \dots \oplus \hat{X}_k \oplus \dots \oplus X_n).$$

where \hat{r}^F is the map $(F(r_1^n), \dots, F(r_n^n))^t$.

Definition 1.1.4. *A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be polynomial of degree lower or equal to n if $cr_{n+1} F = 0$. Such a functor is called linear if $n = 1$ and is called quadratic if $n = 2$.*

We denote by $\mathcal{P}ol_n(\mathcal{C}, \mathcal{D})$ (resp. $\mathcal{L}in(\mathcal{C}, \mathcal{D})$, resp. $\mathcal{Q}uad(\mathcal{C}, \mathcal{D})$) the full subcategory of $Func(\mathcal{C}, \mathcal{D})$ consisting of polynomial functors of degree lower or equal to n (resp. linear functors, resp. quadratic functors).

It is well known that additive functors from ab to Ab is equivalent to the category of abelian groups Ab . The following question arrives naturally:

- **Could we extend the previous equivalence of categories to polynomial functors in order to obtain a combinatorial description of polynomial functors?**

In general, to give a combinatorial description of polynomial functors is a difficult question. However, such a description has been obtained by Baues, Pirashvili *et al.* in several cases. **My research on polynomial functors can be considered as a continuation of these results in more general contexts.**

1.2 The results

In this section we summarize the known results concerning the combinatorial description of polynomial functors for $\mathcal{D} = Ab$.

First, we recall the definition of a pointed algebraic theory used here and in many papers by Baues, Jibladze and Pirashvili.

Definition 1.2.1. *A pointed (algebraic) theory \mathcal{T} is a pointed category \mathcal{T} with an object E such that any object of \mathcal{T} is isomorphic to a finite sum of copies of E . In particular, for any object E of \mathcal{C} we denote by $\langle E \rangle_{\mathcal{C}}$ the theory generated by E , i.e. the full subcategory of \mathcal{C} consisting of the objects $E^{\oplus n} = E \oplus \dots \oplus E$ (n times), $n \geq 0$, with $E^{\oplus 0} = 0$.*

Example 1.2.2. *1. The category of finitely generated free abelian groups ab is a pointed theory. We have $ab = \langle \mathbb{Z} \rangle_{Ab}$.*

2. The category of finitely generated free groups gr is a pointed theory. We have $gr = \langle \mathbb{Z} \rangle_{Gr}$.
3. The category of finite pointed sets Γ is a pointed theory. We have $\Gamma = \langle [1] \rangle_{Sets_*}$.
4. Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. The category of finitely generated free algebras over an operad \mathcal{P} in \mathcal{C} , $Free(\mathcal{P})$ is a pointed theory. We have $Free(\mathcal{P}) = \langle \mathcal{F}_{\mathcal{P}}(1) \rangle_{Free(\mathcal{P})}$ where $\mathcal{F}_{\mathcal{P}}$ is the left adjoint of the forgetful functor.

The values of a polynomial functor of degree n , $F : \langle E \rangle_{\mathcal{C}} \rightarrow Ab$ are determined by the n abelian groups $F(E)$, $cr_2F(E, E)$, $cr_3F(E, E, E)$, \dots , $cr_nF(E, \dots, E)$ (see Proposition 1.17 in [HV11]). The difficulty is to know how to glue together n abelian groups A_1, A_2, \dots, A_n in order to obtain a polynomial functor of degree n . The known results are summarized in the following table.

\mathcal{C}	linear functors $Lin(\mathcal{C}, Ab)$	quadratic functors $Quad(\mathcal{C}, Ab)$	higher degree $Pol_n(\mathcal{C}, Ab)$
ab	$\simeq Ab$	[Bau94]	[BDFP01]
gr	$\simeq Ab$	[BP99]	[HPV]
Γ	$\simeq Ab$	[Pir00a]	[Pir00a]
$\mathcal{F}_{Free(\mathcal{P})}$ \mathcal{P} set-operad	[HPV]	[HPV]	[HPV]
$\langle E \rangle_{\mathcal{C}}$	[HV11]	[HV11]	? but partial result in [DVb]

1.2.1 Polynomial functors from abelian groups

The category of linear functors from ab to Ab is equivalent to Ab . The description of quadratic functors is given by Baues in 1994.

Definition 1.2.3. [Bau94] A quadratic \mathbb{Z} -module \mathbf{A} is a diagram:

$$A_1 \begin{array}{c} \xleftarrow{P_1^2} \\ \xrightarrow{H_1^2} \end{array} A_2$$

where

- A_i for $i \in \{1, 2\}$ is an abelian group;
- H_1^2 and P_1^2 are group morphisms;

satisfying the following relations:

1. $P_1^2 H_1^2 P_1^2 = 2P_1^2$;
2. $H_1^2 P_1^2 H_1^2 = 2H_1^2$.

A morphism $\phi : \mathbf{A} \rightarrow \mathbf{B}$ of quadratic \mathbb{Z} -modules is a pair (f_1, f_2) where $f_i : A_i \rightarrow B_i$ for $i \in \{1, 2\}$ which commute with the structure maps H_1^2 and P_1^2 . Composition of morphisms of quadratic \mathbb{Z} -modules is defined in the obvious way. This allows to give the following definition.

Definition 1.2.4. *The category $\mathbf{QM}(\mathbb{Z})$ is the category having as objects quadratic \mathbb{Z} -modules and as maps morphisms of quadratic \mathbb{Z} -module.*

Theorem 1.2.5. *[Bau94] There is an equivalence of categories $\sigma_2 : \text{Quad}(ab, Ab) \rightarrow \mathbf{QM}(\mathbb{Z})$ given by:*

$$\sigma_2(F) = F(\mathbb{Z}) \begin{array}{c} \xleftarrow{P_1^2(F)} \\ \xrightarrow{H_1^2(F)} \end{array} cr_2 F(\mathbb{Z}, \mathbb{Z}).$$

In [BDFP01], the authors extend this theorem by proving that the combinatoric of polynomial functors from ab to Ab is encapsulated in the structure of pseudo-Mackey functors. Roughly speaking a pseudo-Mackey functor is a pair of functors (M_*, M^*) where M_* is covariant and M^* is contravariant, satisfying a certain compatibility property (see section 1 in [HPV] for a precise definition of Mackey and pseudo-Mackey functors). More precisely they obtain the following result:

Theorem 1.2.6 ([BDFP01] Theorem 6.1). *Let $n \in \mathbb{N}$, there is an equivalence of categories*

$$\text{Pol}_n(ab, Ab) \simeq (\text{PMack}(\Omega, Ab))_{\leq n}$$

where $(\text{PMack}(\Omega, Ab))_{\leq n}$ is the full subcategory of $\text{PMack}(\Omega, Ab)$ having as objects the functors which vanish on the sets X such that $|X| > n$ and on \emptyset .

The condition on Mackey functors can easily be translated in terms of diagram of abelian groups but the number of relations between the structure maps rapidly increases with the degree of polynomiality. For example, they obtain the following description of cubical functors (i.e. polynomial functors of degree 3).

Definition 1.2.7. *[BDFP01] A cubical \mathbb{Z} -module \mathbf{A} is a diagram:*

$$\begin{array}{ccccc} & & & \xleftarrow{P_2^3} & \\ & & & \xleftarrow{P_1^3} & \\ A_1 & \xleftarrow{P_1^2} & A_2 & \xleftarrow{P_1^3} & A_3 \\ & \xrightarrow{H_1^2} & & \xrightarrow{H_1^3} & \\ & & & \xrightarrow{H_2^3} & \end{array}$$

where

- A_i for $i \in \{1, 2, 3\}$ is an abelian group;
- H_j^i and P_j^i for $i \in \{2, 3\}$ and $j \in \{1, 2\}$ are group morphisms;

satisfying the following relations:

1. $H_1^3 H_1^2 = H_2^3 H_1^2$
2. $P_1^2 P_1^3 = P_1^2 P_2^3$

3. $H_2^3 P_1^3 = 0$
4. $H_1^3 P_2^3 = 0$
5. $H_1^3 P_1^3 H_1^3 = 2H_1^3$
6. $P_1^3 H_1^3 P_1^3 = 2P_1^3$
7. $H_2^3 P_2^3 H_2^3 = 2H_2^3$
8. $P_2^3 H_2^3 P_2^3 = 2P_2^3$
9. $H_1^2 P_1^2 H_1^2 = 2(P_1^3 + P_2^3)H_1^3 H_1^2 + 2H_1^2$
10. $P_1^2 H_1^2 P_1^2 = 2P_1^2 P_1^3 (H_1^3 + H_2^3) + 2P_1^2$
11. $H_1^3 H_1^2 P_1^2 + H_1^3 + H_2^3 = H_2^3 P_2^3 H_1^3 P_1^3 H_2^3 + H_1^3 P_1^3 H_2^3 P_2^3 H_1^3$
12. $H_1^2 P_1^2 P_1^3 + P_1^3 + P_2^3 = P_2^3 H_1^3 P_1^3 H_2^3 P_2^3 + P_1^3 H_2^3 P_2^3 H_1^3 P_1^3$.

Corollary 1.2.8. [BDFP01] *There is an equivalence of categories $\sigma_3 : \text{Cub}(ab, Ab) \rightarrow \mathbf{CM}(\mathbb{Z})$ where $\mathbf{CM}(\mathbb{Z})$ is the category of cubical \mathbb{Z} -modules.*

Example 1.2.9. *The cubical \mathbb{Z} -module associated to the third Lie functor L^3 is*

$$\begin{array}{ccc}
 & & \begin{array}{c} \xleftarrow{P_2^3} \\ \xleftarrow{P_1^3} \end{array} \\
 0 & \xleftarrow{\quad} \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{\quad} \mathbb{Z} \oplus \mathbb{Z} \\
 & & \begin{array}{c} \xrightarrow{H_1^3} \\ \xrightarrow{H_2^3} \end{array}
 \end{array}$$

where

$$H_1^3 = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix} \quad H_2^3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad P_1^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_2^3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Remark 1.2.10. 1. In [Mik] Mikhailov uses the description of polynomial functors given in [BDFP01] in order to compute Hom -groups and Ext^1 -groups between polynomial functors. In this paper he gives also the combinatorial description of several concrete examples of polynomial functors coming from algebraic topology (i.e. third and fourth stable homotopy groups of the classifying spaces of free abelian groups). We expect to use the same methods in order to compute Hom and Ext^1 groups of polynomial functors from groups to abelian groups (typical examples of such functors are the Passi functors (cf Example 1.2.17 for the definition of Passi functors)). Examples are given in Example 1.2.18. Computations of extensions groups between polynomial functors from groups to abelian groups could be interesting in order to have a conjecture for the stable homology of automorphisms of free groups with coefficients given by a contravariant functor (see section 2.3.1).

2. The work of Baues is the starting point of the representation theoretical work of Drozd [Dro01] [Dro03] on quadratic and cubical functors. At the end of [Dro03] Drozd posed a conjecture concerning polynomial functors of higher degrees and proved by Zimmermann in [Zim].

1.2.2 Polynomial functors from groups

The category of linear functors from gr to Ab is equivalent to Ab . In 1999, Baues and Pirashvili generalize the notion of quadratic \mathbb{Z} -module and obtain, in particular, a description of quadratic functors from gr to Ab .

Definition 1.2.11. [BP99] *An abelian square group \mathbf{A} is a diagram:*

$$A_1 \begin{array}{c} \xleftarrow{P_1^2} \\ \xrightarrow{H_1^2} \end{array} A_2$$

where

- A_i for $i \in \{1, 2\}$ is an abelian group;
- H_1^2 and P_1^2 are group morphisms;

satisfying the following relation:

$$1. \quad P_1^2 H_1^2 P_1^2 = 2P_1^2.$$

As for quadratic \mathbb{Z} -modules, we can define morphisms of abelian square groups and composition of such morphisms. This allows to give the following definition.

Definition 1.2.12. *The category \mathbf{AbSqGr} is the category having as objects abelian square groups and as maps morphisms of abelian square groups.*

Theorem 1.2.13. [BP99] *There is an equivalence of categories $\sigma_2 : \mathbf{Quad}(gr, Ab) \rightarrow \mathbf{AbSqGr}$ given by:*

$$\sigma_2(F) = F(\mathbb{Z}) \begin{array}{c} \xleftarrow{P_1^2(F)} \\ \xrightarrow{H_1^2(F)} \end{array} cr_2 F(\mathbb{Z}, \mathbb{Z}).$$

The starting point of my joint work with Hartl and Pirashvili [HPV] was to understand polynomial functors from gr to Ab and to obtain a description of these functors in the same spirit that the description given by Baues, Dreckmann, Franjou and Pirashvili in [BDFP01] for polynomial functors from ab to Ab . We obtain the following result:

Theorem 1.2.14. [HPV] *Let $n \in \mathbb{N}$, there is an equivalence of categories*

$$Pol_n(gr, Ab) \simeq (PMack(\Omega(\mathcal{A}s), Ab))_{\leq n}$$

where $(PMack(\Omega(\mathcal{A}s), Ab))_{\leq n}$ is the full subcategory of $PMack(\Omega(\mathcal{A}s), Ab)$ having as objects the functors which vanish on $\underline{0}$ and on the sets \underline{m} such that $m > n$ and $\Omega(\mathcal{A}s)$ is the PROP associated to $\mathcal{A}s$ (see Definition 1.2.24).

For example, I obtained the following description of cubical functors from gr to Ab

Definition 1.2.15. An abelian cubical group \mathbf{A} is a diagram:

$$\begin{array}{ccccc} & & & \xleftarrow{P_2^3} & \\ & & & \xleftarrow{P_1^3} & \\ A_1 & \xleftarrow{P_1^2} & A_2 & \xleftarrow{P_1^3} & A_3 \\ & \xrightarrow{H_1^2} & & \xrightarrow{H_1^3} & \\ & & & \xrightarrow{H_2^3} & \end{array}$$

where

- A_i for $i \in \{1, 2, 3\}$ is an abelian group;
- H_j^i and P_j^i for $i \in \{2, 3\}$ and $j \in \{1, 2\}$ are group morphisms;

satisfying the following relations:

1. $H_1^3 H_1^2 = H_2^3 H_1^2$
2. $P_1^2 P_1^3 = P_1^2 P_2^3$
3. $H_2^3 P_1^3 = 0$
4. $H_1^3 P_2^3 = 0$
5. $P_1^3 H_1^3 P_1^3 = 2P_1^3$
6. $P_2^3 H_2^3 P_2^3 = 2P_2^3$
7. $P_1^2 H_1^2 P_1^2 = 2P_1^2 P_1^3 (H_1^3 + H_2^3) + 2P_1^2$
8. $H_1^3 H_1^2 P_1^2 = H_2^3 P_2^3 H_1^3 P_1^3 H_2^3 + H_1^3 P_1^3 H_2^3 P_2^3 H_1^3 + H_1^3 - H_1^3 P_1^3 H_1^3 + H_2^3 - H_2^3 P_2^3 H_2^3$
9. $H_1^2 P_1^2 P_1^3 = P_2^3 H_1^3 P_1^3 H_2^3 P_2^3 + P_1^3 H_2^3 P_2^3 H_1^3 P_1^3 - P_1^3 - P_2^3$
10. $2H_1^2 P_1^2 P_1^3 H_1^3 + 2H_1^2 P_1^2 P_1^3 H_2^3 = P_2^3 H_1^3 P_1^3 H_2^3 H_1^2 P_1^2 + P_1^3 H_2^3 P_2^3 H_1^3 H_1^2 P_1^2 - 2P_1^3 H_2^3 + P_1^3 H_2^3 P_2^3 H_2^3 + P_1^3 H_2^3 P_2^3 H_1^3 P_1^3 H_1^3 - 2P_1^3 H_2^3 P_2^3 H_1^3 + P_2^3 H_1^3 P_1^3 H_2^3 P_2^3 H_2^3 - 2P_2^3 H_1^3 P_1^3 H_2^3 - 2P_2^3 H_1^3 + P_2^3 H_1^3 P_1^3 H_1^3$.

Corollary 1.2.16. There is an equivalence of categories $\sigma_3 : \text{Cub}(gr, Ab) \rightarrow \mathbf{AbCubGr}$ where $\mathbf{AbCubGr}$ is the category of abelian cubical groups.

Example 1.2.17. Passi functors Let M be a monoid with a unit element e and $I(M)$ be the augmentation ideal of the monoid algebra $\mathbb{Z}[M]$ (i.e. the kernel of the map $\epsilon : \mathbb{Z}[M] \rightarrow \mathbb{Z}$ defined by $\epsilon(\sum_{m \in M} \alpha_m m) = \sum_{m \in M} \alpha_m$). We know that $I(M)$ is a free abelian group having as basis the elements $m - 1$ where $m \in M$ and $m \neq e$. For $k \geq 0$, we denote by $I^k(M)$ the k -th power of $I(M)$ with the convention $I^0(M) = \mathbb{Z}[M]$. Let A be an abelian group, for a map $f : M \rightarrow A$ let $\bar{f} : \mathbb{Z}[M] \rightarrow A$ denote the extension of f to a \mathbb{Z} -linear homomorphism.

The Passi functor of order n $P_n : gr \rightarrow Ab$ is defined by $P_n(G) = I(G)/I^{n+1}(G)$. By the following short exact sequence of abelian groups:

$$0 \longrightarrow I^n G / I^{n+1} G \longrightarrow IG / I^{n+1} G \longrightarrow IG / I^n G \longrightarrow 0. \quad (1.1)$$

we obtain a short exact sequence of functors:

$$0 \longrightarrow Q_n \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow 0 \quad (1.2)$$

where $Q_n = \otimes^n \circ \mathbf{a}$ and $\mathbf{a} : gr \rightarrow ab$ is the abelianization functor.

In particular the functor P_n is polynomial of degree n .

We have:

$$\sigma_2(P_2) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{H=\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{P=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

Example 1.2.18. Let $a \in \mathbb{Z}$, consider the quadratic functor $K_a : gr \rightarrow Ab$: corresponding to the following abelian square group

$$\mathbb{Z} \begin{array}{c} \xleftarrow{P=0} \\ \xrightarrow{H=a} \end{array} \mathbb{Z}$$

- **If $a = 0$.** We have $HPH = 2H$ so the abelian square group comes from a quadratic \mathbb{Z} -module. So K_0 factorizes through the abelianization functor $\mathbf{a} : gr \rightarrow ab$ and we have:

$$K_0 = \Lambda^2 \circ \mathbf{a} \oplus \mathbf{a}.$$

- **If $a \neq 0$.** The square group is not a quadratic \mathbb{Z} -module so K_a does not factorize through the abelianization but by Corollary 1.2.40, K_a has a filtration

$$0 \subset F_0 \subset F_1 \subset F_2 = K_a$$

such that each sub-quotient F_i/F_{i-1} is polynomial of degree less than or equal to $2 - i$ and is in the essential image of the functor $\text{Func}(ab, Ab) \rightarrow \text{Func}(gr, Ab)$ obtained by the precomposition with the abelianization functor \mathbf{a} . Such a filtration is:

$$0 \subset \Lambda^2 \circ \mathbf{a} \subset K_a \subset F_2 = K_a$$

and $K_a/\Lambda^2 \circ \mathbf{a} \simeq \mathbf{a}$. So K_a is in $\text{Ext}_{\text{Func}(gr, Ab)}^1(\mathbf{a}, \Lambda^2 \circ \mathbf{a})$.

Using the description in terms of abelian square groups we can prove that the map:

$$\sigma : \mathbb{Z} \rightarrow \text{Ext}_{\text{Func}(gr, Ab)}^1(\mathbf{a}, \Lambda^2 \circ \mathbf{a})$$

defined by $\sigma(a) = K_a$, is a group isomorphism.

This result must be compared with the following: $\text{Ext}_{\text{Func}(ab, Ab)}^1(\text{Id}, \Lambda^2) = 0$.

(We have $\text{Ext}_{\text{Func}(gr, Ab)}^1(\Lambda^2 \circ \mathbf{a}, \mathbf{a}) = 0$.)

The functor K_1 seems to be of particular interest. In fact, using the description of quadratic functors in terms of abelian square groups, we can prove that we have short exact sequences:

$$0 \rightarrow K_1 \rightarrow P_2 \rightarrow \Gamma^2 \circ \mathbf{a} \rightarrow 0$$

and

$$0 \rightarrow \Gamma^2 \circ \mathbf{a} \rightarrow P_2 \rightarrow K_1 \rightarrow 0$$

where Γ^2 is the second exterior power functor and P_2 is the second Passi functor. Using these two short exact sequences we obtain a projective resolution of $\Gamma^2 \circ \mathbf{a}$ in $\text{Quad}(gr, Ab)$ (the functor P_2 is projective in $\text{Quad}(gr, Ab)$). Using this resolution we can compute Ext-groups of the form $\text{Ext}_{\text{Quad}(gr, Ab)}^*(\Gamma^2 \circ \mathbf{a}, -)$.

1.2.3 Polynomial functors from pointed sets

Using the notion of cross-effect of functors Pirashvili constructs in [Pir00a] a functor $cr : Func(\Gamma, Ab) \rightarrow Func(\Omega, Ab)$ where Ω is the category of nonempty finite sets and surjections and proves the following result:

Theorem 1.2.19 (Dold-Kan type theorem for Γ [Pir00a]). *The functor*

$$cr : Func(\Gamma, Ab) \rightarrow Func(\Omega, Ab)$$

is an equivalence of categories.

Remark 1.2.20. In [Pir00a] Pirashvili constructs, more generally, a functor $cr : Func(\Gamma, Gr) \rightarrow Func(\Omega, Gr)$.

Applying this equivalence of categories to polynomial functors we obtain combinatorial description of polynomial functors from Γ to Ab . To be more precise, for $F \in Func(\Gamma, Ab)$ we have $cr(F)(\underline{n}) = cr_n F([1], \dots, [1])$ and the previous equivalence induces an equivalence

$$Pol_n(\Gamma, Ab) \simeq (Func(\Omega, Ab))_{\leq n}$$

where $(Func(\Omega, Ab))_{\leq n}$ is the full subcategory of $Func(\Omega, Ab)$ having as objects the functors which vanish on the sets X such that $card(X) > n$ and on $\underline{0}$.

1.2.4 Polynomial functors from algebras over a set-operad

Let \mathcal{P} be a set-operad and $Free(\mathcal{P})$ be the category of (finitely generated free) \mathcal{P} -algebras. In [HPV] we obtain a description of functors from finitely generated free \mathcal{P} -algebras to abelian groups. Before giving the precise statements we need to introduce some definitions.

First, we recall the construction of the PROP associated to an operad due to May and Thomason in [MT78].

Definition 1.2.21. *Let \mathcal{P} be a unitary set-operad. The **May-Thomason category** associated to \mathcal{P} (or the category of operators or the PROP associated to the operad \mathcal{P}): $\mathcal{S}(\mathcal{P})$, has as objects the finite sets $\underline{n} = \{1, \dots, n\}$, and morphisms from \underline{n} to \underline{m} in $\mathcal{S}(\mathcal{P})$ are the pairs (f, ω^f) where $f \in \mathcal{S}(\underline{n}, \underline{m})$, $\omega^f \in \prod_{y \in \underline{m}} \mathcal{P}(|f^{-1}(y)|)$. For $(f, \omega^f) : \underline{n} \rightarrow \underline{m}$ and $(g, \omega^g) : \underline{m} \rightarrow \underline{l}$ morphisms in $\mathcal{S}(\mathcal{P})$, the composition $(g, \omega^g) \circ (f, \omega^f)$ in $\mathcal{S}(\mathcal{P})$ is the pair $(g \circ f, \omega^{g \circ f})$ where, for $i \in \{1, \dots, l\}$ we have:*

$$\omega_i^{g \circ f} = \gamma(\omega_i^g; \omega_{j_1}^f, \dots, \omega_{j_s}^f) \cdot \sigma_i$$

where $g^{-1}(i) = \{j_1, \dots, j_s\}$ with $j_1 < \dots < j_s$, for $h : \underline{p} \rightarrow \underline{q}$ and $\alpha \in \underline{q}$, $\omega_\alpha^h \in \mathcal{P}(|h^{-1}(\alpha)|)$ is the component of ω^h and σ_i is the permutation of $|(g \circ f)^{-1}(i)|$ defined in the following way: let $A_i = (x_1, \dots, x_{n_j})$ be the natural ordering of $(g \circ f)^{-1}(i)$ and let $B_i = (y_1, \dots, y_{n_j})$ be its ordering obtained by regarding it as $\prod_{g(j)=i} f^{-1}(j)$ so ordered that:

- if $j < j'$, all elements of $f^{-1}(j)$ precede all elements of $f^{-1}(j')$;
- each $f^{-1}(j)$ has its natural ordering as a subset of \underline{n} ;

σ_i then is defined by: $\forall k \in \{1, \dots, n_j\}$, $\sigma_i(k) = l$ such that $y_l = x_k$.

Remark 1.2.22. For $\underline{n} \in \mathcal{S}(\mathcal{P})$ the identity morphism of \underline{n} is $(Id_{\underline{n}}, (1_{\mathcal{P}}, \dots, 1_{\mathcal{P}}) \in \mathcal{P}(1)^n)$ and the associativity of composition follows from the definition of operads.

Example 1.2.23 (Example of composition in $\mathcal{S}(\mathcal{P})$). *Let $(f, \omega_1 \in \mathcal{P}(1), \omega_2 \in \mathcal{P}(2)) : \underline{3} \rightarrow \underline{2}$ be the morphism in $\mathcal{S}(\mathcal{P})$ defined by $f(1) = f(3) = 2$ and $f(2) = 1$ and $(g, \alpha_1 \in \mathcal{P}(2), \alpha_2 \in \mathcal{P}(1), \alpha_3 \in \mathcal{P}(3)) : \underline{6} \rightarrow \underline{3}$ defined by $g(1) = g(5) = 1$, $g(2) = 2$ and $g(3) = g(4) = g(6) = 3$. We have:*

$$(f, \omega_1, \omega_2)(g, \alpha_1, \alpha_2, \alpha_3) = (fg, \gamma(\omega_1, \alpha_2) \in \mathcal{P}(1), \gamma(\omega_2, \alpha_1, \alpha_3) \cdot \sigma \in \mathcal{P}(5))$$

where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$ (in this example $A_2 = (1, 3, 4, 5, 6)$ and $B_2 = (1, 5, 3, 4, 6)$).

Definition 1.2.24. *Let \mathcal{P} be a unitary set-operad. Let $\Omega(\mathcal{P})$ be the subcategory of $\mathcal{S}(\mathcal{P})$ having as morphisms the morphisms of $\mathcal{S}(\mathcal{P}) : (f, \omega^f)$ such that f is a surjective map.*

Remark 1.2.25. 1. Let As be the unitary associative set operad. The categories $\Gamma(As)$ and $\mathcal{S}(As)$ appear in previous work. The category $\mathcal{S}(As)$ is the category of non-commutative sets considered in [PR02] and [Pir02] and this category is isomorphic to the category ΔS introduced in [FL91].

2. Let \mathcal{P} be an operad in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ that admits finite co-products. The construction given in Definition 1.2.21 can be generalized in this setting: we can associate to \mathcal{P} a category enriched over \mathcal{C} whose set of objects is \mathbb{N} .

Theorem 1.2.26. [*HPV*] *There is a natural equivalence of categories:*

$$\text{Func}(\text{Free}(\mathcal{P}), \text{Ab}) \simeq \text{PMack}(\Omega(\mathcal{P}), \text{Ab})$$

where $\text{PMack}(\Omega(\mathcal{P}), \text{Ab})$ is the category of pseudo-Mackey functors.

This theorem is the generalization of Theorem 0.2 in [BDFP01] giving an equivalence of categories between the functors from (finitely generated free) commutative monoids and pseudo-Mackey functors. In fact, applying Theorem 1.2.26 to the operad Com we recover Theorem 0.2 in [BDFP01] since $\text{Free}(\text{Com})$ is the category of (finitely generated free) commutative monoids.

Applying this equivalence of categories to polynomial functors we obtain combinatorial description of polynomial functors from $\text{Free}(\mathcal{P})$ to Ab .

For example, we obtain in [*HPV*] the following presentations of linear and quadratic functors

Proposition 1.2.27. [*HPV*] *There is an equivalence of categories between $\text{Lin}(\text{Free}(\mathcal{P}), \text{Ab})$ and the category of diagrams:*

$$\begin{array}{c} \mathcal{P}(1) \times M_e \\ \downarrow I_e \\ M_e \end{array}$$

where

- M_e is an abelian group;

- $\forall \omega \in \mathcal{P}(1); I_e^\omega := I_e(\omega, -) : M_e \rightarrow M_e$ is a group morphism;

satisfying the following relations $\forall \omega \in \mathcal{P}(1), \forall \omega' \in \mathcal{P}(1)$

1. $I_e^{1\mathcal{P}} = Id_{M_e}$;
2. $I_e^{\omega'} I_e^\omega = I_e^{\gamma(\omega; \omega')}$.

Remark 1.2.28. Using the description of linear functors recalled in [HV11] we have: $Lin(Free(\mathcal{P}), Ab) \simeq T_1 \bar{P}_{\mathcal{F}_{\mathcal{P}(1)}}(\mathcal{F}_{\mathcal{P}(1)})\text{-mod} \simeq \mathbb{Z}[\mathcal{P}(1)]\text{-mod}$ where $\bar{P}_{\mathcal{F}_{\mathcal{P}(1)}}$ is the reduced projective functor associated to $\mathcal{F}_{\mathcal{P}(1)}$. We verify easily that this description coincide with that given in the last proposition.

Proposition 1.2.29. [HPV] *There is an equivalence of categories between $Quad(Free(\mathcal{P}), Ab)$ and the category of diagrams:*

$$\begin{array}{ccc}
 \mathcal{P}(1) \times \mathcal{P}(1) \times M_{ee} & & \mathcal{P}(1) \times M_e \\
 \downarrow I_{ee} & & \downarrow I_e \\
 T \curvearrowright M_{ee} & \xrightarrow{P} & M_e \\
 & \nwarrow H & \\
 & & \mathcal{P}(2) \times M_e
 \end{array}$$

where

- M_e and M_{ee} are abelian groups;
- $\forall \omega \in \mathcal{P}(1); I_e^\omega := I_e(\omega, -) : M_e \rightarrow M_e$ is a group morphism;
- $\forall \omega \in \mathcal{P}(1), \omega' \in \mathcal{P}(1); I_{ee}^{\omega, \omega'} := I_{ee}(\omega, \omega', -) : M_{ee} \rightarrow M_{ee}$ is a group morphism;
- $\forall \omega \in \mathcal{P}(2); H^\omega := H(\omega, -) : M_e \rightarrow M_{ee}$ is a group morphism;
- P and T are group morphisms;

satisfying the following relations $\forall (\omega, \omega', \omega_1, \omega'_1) \in \mathcal{P}(1)^{\times 4}, \forall \alpha \in \mathcal{P}(2), \forall \alpha' \in \mathcal{P}(2)$

1. $T.T = Id_{M_{ee}}$;
2. $PT = P$;
3. $I_e^{\omega'} I_e^\omega = I_e^{\gamma(\omega; \omega')}$;
4. $I_{ee}^{\omega, \omega'} I_{ee}^{\omega_1, \omega'_1} = I_{ee}^{\gamma(\omega_1, \omega), \gamma(\omega'_1, \omega')}$;
5. $I_{ee}^{\omega, \omega'} T = T I_{ee}^{\omega', \omega}$;
6. $H^\alpha I_e^\omega = H^{\gamma(\omega, \alpha)}$;
7. $I_{ee}^{\omega, \omega'} H^\alpha = H^{\gamma(\alpha; \omega, \omega')}$;
8. $TH^\alpha = H^{\alpha \cdot \tau}$ where τ is the transposition of \mathfrak{S}_2 exchanging 1 and 2;

$$9. T = (M^*(\tau, (1_{\mathcal{P}}, 1_{\mathcal{P}})))^{-1}$$

$$10. I_e^\omega P = P I_{ee}^{\omega, \omega}$$

$$11. H^\alpha P = I_{ee}^{\gamma(\alpha; 1_{\mathcal{P}}, 0_{\mathcal{P}}), \gamma(\alpha; 0_{\mathcal{P}}, 1_{\mathcal{P}})} + T I_{ee}^{\gamma(\alpha; 0_{\mathcal{P}}, 1_{\mathcal{P}}), \gamma(\alpha; 1_{\mathcal{P}}, 0_{\mathcal{P}})}.$$

In the particular case $\mathcal{P}(1) = \{1_{\mathcal{P}}\}$ we obtain the following simplification of the previous proposition:

Corollary 1.2.30. *[HPV] For \mathcal{P} an unitary set operad such that $\mathcal{P}(1) = \{1_{\mathcal{P}}\}$, there is an equivalence of categories between $Quad(\text{Free}(\mathcal{P}), \text{Ab})$ and the category of diagrams:*

$$\begin{array}{ccc} M_{ee} & \xrightarrow{P} & M_e \\ & \swarrow H & \\ & & \mathcal{P}(2) \times M_e \end{array}$$

where

- M_e and M_{ee} are abelian groups;
- $\forall \omega \in \mathcal{P}(2)$; $H^\omega := H(\omega, -) : M_e \rightarrow M_{ee}$ is a group morphism;
- P is a group morphism;

satisfying the following relations $\forall \alpha \in \mathcal{P}(2), \forall \alpha' \in \mathcal{P}(2)$

$$1. PH^\alpha P = 2P;$$

$$2. H^\alpha P H^{\alpha'} = H^{\alpha'} + H^{\alpha' \cdot \tau} \text{ where } \tau \text{ is the transposition of } \mathfrak{S}_2 \text{ exchanging 1 and 2.}$$

For $\mathcal{P} = \text{Com}$, since $\text{Com}(2) = \underline{1}$ we recover the equivalence between $Quad(\text{ab}, \text{Ab})$ and quadratic \mathbb{Z} -module obtained by Baues in [Bau94] (see Theorem 1.2.5).

For $\mathcal{P} = \text{As}$, since $\text{As}(2) = \mathfrak{S}_2 = \{1, \tau\}$, we have two group morphisms from M_e to M_{ee} : H^1 and H^τ . Relation (1) in the corollary becomes:

$$(1-1) \quad PH^1 P = 2P$$

$$(1-2) \quad PH^\tau P = 2P$$

and relation (2) becomes:

$$(2-1) \quad H^1 P H^1 = H^1 + H^\tau$$

$$(2-2) \quad H^\tau P H^1 = H^1 + H^\tau$$

$$(2-3) \quad H^1 P H^\tau = H^1 + H^\tau$$

$$(2-4) \quad H^\tau P H^\tau = H^1 + H^\tau.$$

We deduce from relation (2-1) that H^τ is determined by the other data. Then conditions (1-2), (2-2), (2-3) and (2-4) are consequences of relation (1-1)

So, we recover the equivalence between $Quad(\text{gr}, \text{Ab})$ and abelian square groups obtained by Baues and Pirashvili in [BP99] (see Theorem 1.2.13).

1.2.5 Polynomial functors from a pointed theory

In [HV11] we obtain a combinatorial description of linear and quadratic functors from a pointed theory to Ab . Before giving the precise statements we need to introduce some notation and intermediate results.

Let \mathcal{C} be a pointed category with finite coproducts. For E a fixed object of \mathcal{C} , we consider the reduced standard projective functor associated with E , $\bar{P}_E : \mathcal{C} \rightarrow Ab$. Recall that \bar{P}_E is the quotient of $\mathbb{Z}[\mathcal{C}(E, -)]$ by the subfunctor $\mathbb{Z}[0]$ where, for a set S , $\mathbb{Z}[S]$ denotes the free abelian group with basis S and $\mathbb{Z}[0]$ is the subfunctor of $\mathbb{Z}[\mathcal{C}(E, -)]$ defined by $\mathbb{Z}[0](X) = \mathbb{Z}\{E \xrightarrow{0} X\}$ for $X \in \mathcal{C}$.

Let $U_n : Pol_n(\mathcal{C}, Ab) \rightarrow Func(\mathcal{C}, Ab)$ denote the forgetful functor. The n -Taylorisation functor $T_n : Func(\mathcal{C}, Ab) \rightarrow Pol_n(\mathcal{C}, Ab)$ is the left adjoint to U_n .

For linear functors we obtain the following result:

Theorem 1.2.31. [HV11] *The category of linear functors from $\langle E \rangle_{\mathcal{C}}$ to Ab is equivalent to the category of $(T_1 \bar{P}_E)(E)$ -modules.*

Example 1.2.32. 1. For $\langle \mathbb{Z} \rangle_{Ab}$ we obtain $(T_1 \bar{P}_{\mathbb{Z}})(\mathbb{Z}) \simeq \mathbb{Z}$. So we recover the classical result $Lin(ab, Ab) \simeq Ab$.

2. For $\langle \mathbb{Z} \rangle_{Gr}$ we obtain $(T_1 \bar{P}_{\mathbb{Z}})(\mathbb{Z}) \simeq \mathbb{Z}$. So we recover the classical result $Lin(gr, Ab) \simeq Ab$.

For quadratic functors, we need to introduce the following definition:

Definition 1.2.33. [HV11] *A quadratic \mathcal{C} -module relative to E is a diagram of group homomorphisms:*

$$M = (T_{11}(cr_2 \bar{P}_E)(E, E) \otimes_{\bar{P}_E(E)} M_e \xrightarrow{\hat{H}} M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e)$$

where

- M_e is a left $\bar{P}_E(E)$ -module;
- M_{ee} is a symmetric $(T_1 \bar{P}_E)(E) \otimes (T_1 \bar{P}_E)(E)$ -module with involution T ;
- $P : M_{ee} \rightarrow M_e$ is a homomorphism of $\bar{P}_E(E)$ -modules with respect to the diagonal action of $\bar{P}_E(E)$ on M_{ee} , i.e. for $\alpha \in \mathcal{C}(E, E)$ and $m \in M_{ee}$:

$$P((\bar{\alpha} \otimes \bar{\alpha})m) = \alpha P(m),$$

and satisfies $PT = P$;

- \hat{H} is a homomorphism of symmetric $(T_1 \bar{P}_E)(E) \otimes (T_1 \bar{P}_E)(E)$ -modules such that for $\xi \in \mathcal{C}(E, E \oplus E)$, $a \in M_e$ and $m \in M_{ee}$ the following relations hold:

$$(QM1) \quad (\nabla^2 \xi)a = (r_1^2 \xi)a + (r_2^2 \xi)a + P(\hat{H}(\overline{\rho_{12}^2(\xi)} \otimes a));$$

$$(QM2) \quad \hat{H}(\overline{\rho_{12}^2(\xi)} \otimes Pm) = (\overline{r_1^2 \xi} \otimes \overline{r_2^2 \xi})(m + Tm).$$

One easily defines morphisms of quadratic \mathcal{C} -modules relative to E and their composition. This gives rise to the category of quadratic \mathcal{C} -modules relative to E . The principal result of [HV11] is the following theorem:

Theorem 1.2.34. [HV11] *The category of quadratic functors from $\langle E \rangle_{\mathcal{C}}$ to Ab is equivalent to the category of quadratic \mathcal{C} -modules relative to E .*

Applying this result to the case where E is a co-group we obtain a simple description of such functors which completes the paper [BP99] where this case was considered but unsolved. We obtain the following description:

Theorem 1.2.35. [HV11] *Let $\mathcal{T} = \langle E \rangle_{\mathcal{C}}$ be a theory such that E has a co-group structure. The category of quadratic \mathcal{T} -modules relative to E is equivalent to the category of diagrams of group homomorphisms:*

$$M = \left(\begin{array}{ccc} & M_e & \\ & \searrow^{H_1} & \\ & & M_{ee} \xrightarrow{P} M_e \\ & \nearrow_{H_2} & \\ T_{11}cr_2(\mathcal{T}(E, -))(E, E) \otimes_{\bar{P}_E(E)} coker(P) & & \end{array} \right)$$

where

- M_e is a left $\bar{P}_E(E)$ -module;
- M_{ee} is a left $(T_1\bar{P}_E)(E) \otimes (T_1\bar{P}_E)(E)$ -module;
- $P : M_{ee} \rightarrow M_e$ is a homomorphism of $\bar{P}_E(E)$ -modules with respect to the diagonal action of Λ on M_{ee} ;
- H_1 is a homomorphism of abelian groups;
- H_2 is a homomorphism of $(T_1\bar{P}_E)(E) \otimes (T_1\bar{P}_E)(E)$ -modules,

satisfying the following relations for $a \in M_e$, $m \in M_{ee}$, $\gamma \in cr_2(\mathcal{T}(E, -))(E, E)$, $\alpha, \beta \in \mathcal{T}(E, E)$:

$$\begin{aligned} (T1) \quad & PH_1P = 2P \\ (T2) \quad & H_1P((\bar{\alpha} \otimes \bar{\beta})m) - (\bar{\beta} \otimes \bar{\alpha})H_1P(m) = (\bar{\alpha} \otimes \bar{\beta})m - (\bar{\beta} \otimes \bar{\alpha})m \\ (T3) \quad & H_1PH_1(a) = 2H_1(a) + H_2(t_{11}([i_2, i_1]) \otimes \bar{a}) \\ (T4) \quad & H_1PH_2(t_{11}\gamma \otimes \bar{a}) = H_2(t_{11}(\gamma \bullet \tau\gamma) \otimes \bar{a}) \\ (T5) \quad & (\nabla^2\gamma \bullet \alpha \bullet \beta)a = \alpha a + \beta a + P((\bar{\alpha} \otimes \bar{\beta})H_1(a)) + PH_2(t_{11}\gamma \otimes \bar{a}) \\ (T6) \quad & H_1(\alpha a) = H_2(t_{11}h(\alpha) \otimes \bar{a}) + (\bar{\alpha} \otimes \bar{\alpha})H_1(a) \end{aligned}$$

where $[i_2, i_1] = i_2 \bullet i_1 \bullet (i_2)^{-1} \bullet (i_1)^{-1}$, τ is the canonical switch of $E \oplus E$ and $h : \mathcal{T}(E, E) \rightarrow \mathcal{T}(E, E | E)$ is given by

$$h(\alpha) = (i_{(1,2)}^2)^{-1}(((i_1 \bullet i_2)\alpha) \bullet (i_2\alpha)^{-1} \bullet (i_1\alpha)^{-1}).$$

Inspired by Theorem 1.2.34, Djament and I give in [DVb] a general result on polynomial functors of arbitrary degree from a pointed theory generated by E to abelian groups.

Recall the following definition:

Definition 1.2.36. Let R be a ring equipped with an action (on the left) by a group G . The twist algebra of G over R is the R -module $R[G]$ equipped with the multiplication given by $r[g].s[h] = (r^g s).[gh]$ for all $(r, s, g, h) \in R \times R \times G \times G$ (where $^g s$ is the action of g over s). Let A be a ring. For $R = A^{\otimes n}$ and $G = \mathfrak{S}_n$ (the action of the symmetric group on $A^{\otimes n}$ is given by the permutation of the factors), we denote by $\mathfrak{S}_n \wr A$ the twisted algebra.

Theorem 1.2.37. [DVb] There exists a recollement diagram of abelian categories:

$$\mathcal{P}ol_{n-1}(\langle E \rangle_{\mathcal{C}}) \begin{array}{c} \xleftarrow{l} \\ \xleftarrow{\text{incl}} \\ \xleftarrow{r} \end{array} \mathcal{P}ol_n(\langle E \rangle_{\mathcal{C}}) \begin{array}{c} \xleftarrow{\alpha_n} \\ \xleftarrow{\text{cr}_n} \\ \xleftarrow{\beta_n} \end{array} (\mathfrak{S}_n \wr \Lambda_{\mathcal{C}}(E))\text{-Mod}$$

where incl is the inclusion and

$$\alpha_n(M) = T_1(\bar{P}_E)^{\otimes n} \otimes_{\mathfrak{S}_n \wr \Lambda_{\mathcal{C}}(E)} M \simeq (T_1(\bar{P}_E)^{\otimes n} \otimes_{\Lambda_{\mathcal{C}}(E)^{\otimes n}} M)_{\mathfrak{S}_n}$$

where $\Lambda_{\mathcal{C}}(E) = T_1(\bar{P}_E)(E)$.

Recall the definition of recollement diagram of abelian categories in this example (see [Kuh94b] and [FP04] for further information on recollement diagrams)

1. the functor incl is the right adjoint (resp. left adjoint) of the functor l (resp. r) and the unit $Id \rightarrow r.\text{incl}$ is an isomorphism;
2. the functor cr_n is the right adjoint (resp. left adjoint) of the functor α_n (resp. β_n) and the unit $Id \rightarrow \text{cr}_n.\alpha_n$ is an isomorphism;
3. the functor incl is fully faithful and its essential image is the kernel of the functor cr_n .

(Remark that the definition of recollement diagram given in [Kuh94b] contains a redundancy. In fact, by Proposition 3.4.1 and 3.4.2 in [ML98], the co-unit of the adjunction between cr_n and β_n is an isomorphism if and only if the unit of the adjunction between cr_n and α_n is an isomorphism.)

Recall that these properties imply that the functor cr_n induces an equivalence of categories:

$$\mathcal{P}ol_n(\langle E \rangle_{\mathcal{C}}) / \mathcal{P}ol_{n-1}(\langle E \rangle_{\mathcal{C}}) \xrightarrow{\simeq} (\mathfrak{S}_n \wr \Lambda_{\mathcal{C}}(E))\text{-Mod}.$$

Remark 1.2.38. When the category \mathcal{C} is additive, the right adjoint β_n of the functor cr_n is isomorphic to

$$M \mapsto (\bar{T}_1(P_E)^{\otimes n} \otimes_{\Lambda_{\mathcal{C}}(E)^{\otimes n}} M)_{\mathfrak{S}_n}.$$

But this result is no more true in general. In particular, for $\mathcal{C} = gr$ this functor is a quite mysterious functor. However, we prove in [PV] that this functor is exact.

For quadratic functors, Theorem 1.2.37 is less precise than the one obtained in [HV11]. However applying Theorem 1.2.37 to finitely generated free groups we obtain the following result:

Proposition 1.2.39. [DVb] The abelianization functor $\mathfrak{a} : gr \rightarrow ab$ induces equivalences of categories:

$$\mathcal{P}ol_n(gr) / \mathcal{P}ol_{n-1}(gr) \simeq \mathcal{P}ol_n(ab) / \mathcal{P}ol_{n-1}(ab) \simeq \mathbb{Z}[\mathfrak{S}_n]\text{-Mod}.$$

As a corollary we prove that polynomial functors from finitely generated free groups to abelian groups can be understood in terms of functors which factor through the abelianization functor. More precisely we prove the following result:

Corollary 1.2.40. *[DVb] Let F be a polynomial functor of degree n from gr to Ab . The functor F has a filtration*

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that each sub-quotient F_i/F_{i-1} is polynomial of degree less than or equal to $n - i$ and is in the essential image of the functor $Func(ab, Ab) \rightarrow Func(gr, Ab)$ obtained by the precomposition with the abelianization functor.

1.3 The methods developed in [HV11] and [HPV]

1.3.1 The methods developed in [HV11]

The proof of Theorem 1.2.34 in [HV11] illustrates the fruitfulness of the method consisting to compare categories via several equivalences. This proof can be decomposed in two steps:

- we generalize an approach of Pirashvili in [Pir88] to obtain an equivalence between the category $Quad(\langle E \rangle_{\mathcal{C}}, Ab)$ and the category of modules over a particular pre-additive category with two objects: we determine a pair of projective generators of the abelian category $Quad(\langle E \rangle_{\mathcal{C}}, Ab)$ of quadratic functors from $\langle E \rangle_{\mathcal{C}}$ to Ab , compute the maps between them by using a Yoneda lemma for polynomial functors, thus providing the pre-additive category \mathcal{R} , and deduce an equivalence of categories between $Quad(\langle E \rangle_{\mathcal{C}}, Ab)$ and the category of \mathcal{R} -modules from the Gabriel-Popescu theorem (see section 4 in [HV11] for details);
- we obtain an equivalence of categories between \mathcal{R} -modules and quadratic \mathcal{C} -modules relative to E (see section 5 in [HV11]).

We then extend the correspondance between quadratic functors and quadratic \mathcal{C} -modules relative to E to categories \mathcal{C} with sums and a small regular projective generator, in particular any algebraic variety \mathcal{V} (i.e. the category of models (or algebras) of a pointed theory \mathcal{T}); here \mathcal{T} identifies with the full subcategory of free objects of finite rank in \mathcal{V} .

Such an extension of quadratic functor theory from an algebraic theory to its category of models was established in [BP99], in the case where $Model(\mathcal{T})$ is the category Gr of groups. This is achieved by making explicit the functors in the equivalence given in Theorem 1.2.34. Thus, we introduce the *quadratic tensor product* $G \otimes M \in Gr$ for a group G and a square group M . We generalize this device to arbitrary categories \mathcal{C} as above, by constructing a quadratic tensor product $- \otimes M : \mathcal{C} \rightarrow Ab$ for any quadratic \mathcal{C} -module M , and by studying its properties: we compute its effect on E (in fact, $E \otimes M \cong M_e$) and its cross-effect (Theorem 6.27 in [HV11]), and show that it preserves filtered colimits and suitable coequalizers (Theorem 6.24 in [HV11]).

1.3.2 The methods developed in [HPV]

The proof of the principal theorem of [BDFP01] (see Theorem 1.2.6) is based on the two following results:

1. There is an equivalence of categories: $Func(c\text{-}mon, Ab) \simeq PMack(\Omega, Ab)$ where $c\text{-}mon$ is the category of finitely generated free commutative monoids.
2. There is an isomorphism of categories between the reduced polynomial functors of degree n from $c\text{-}mon$ to Ab and those from ab to Ab .

The proof of Theorem 1.2.14 follows the same scheme than that of Theorem 1.2.6 but the results are stronger.

- The first point is replaced by the more general result Theorem 1.2.26.

In [BDFP01], in order to prove Theorem 1.2.6 the authors consider the categories Γ and Ω . An essential tool of the proof is the Dold-Kan type theorem proved in [Pir00a], giving a Morita equivalence between the categories Γ and Ω . In [HPV], we replace the categories Γ and Ω by categories associated to the operad \mathcal{P} (see Definition 1.2.21) denoted by $\Gamma(\mathcal{P})$ and $\Omega(\mathcal{P})$. The extension of the Dold-Kan type theorem to this context is an application of the general result of Słomińska in [Sł04] describing general conditions which imply Morita equivalences of functor categories. In [HPV], however, we give another proof of this result which has the advantage to provide an explicit description of the functors constituting the equivalence. Theorem 1.2.26 is then obtained by combining the following equivalences of categories:

- the category $Free(\mathcal{P})$ is equivalent to the category of Spans on the double category $\mathcal{S}(\mathcal{P})_2$ having as objects finite sets and where horizontal maps are set maps and vertical maps are set maps decorated by certain elements depending on \mathcal{P} (see Proposition 4.2 in [HPV]);
- in Proposition 4.6 in [HPV], we obtain an equivalence of categories between the functors on this category of Spans and another category whose objects are called the \mathcal{M} -functors on $\Gamma(\mathcal{P})$ (see Definition 1.16 in [HPV]);
- we prove that the Morita equivalence between $\Gamma(\mathcal{P})$ and $\Omega(\mathcal{P})$ provides an equivalence of categories between the \mathcal{M} -functors on $\Gamma(\mathcal{P})$ and the pseudo-Mackey functors on $\Omega(\mathcal{P})$ to Ab (see Proposition 4.8 in [HPV]).

Applying Theorem 1.2.26 to the operad Com , since $Free(Com)$ is the category of (finitely generated free) commutative monoids $c\text{-}mon$, we recover the first result obtained in [BDFP01].

Applying Theorem 1.2.26 to the operad $\mathcal{A}s$, since $Free(\mathcal{A}s)$ is the category of (finitely generated free) monoids, we obtain a description of functors from (finitely generated free) monoids to abelian groups in terms of pseudo-Mackey functors.

- In order to obtain a description of polynomial functors from gr to Ab we would like to obtain a relationship between polynomial functors from (finitely generated free) monoids to abelian groups and those from gr to Ab generalizing the second result obtained in [BDFP01]. This relation is the second main result of [HPV]:

Theorem 1.3.1. *[HPV] There is an isomorphism of categories between the reduced polynomial functors of degree n from (finitely generated free) monoids to Ab and those from gr to Ab .*

In [BDFP01] the sketched proof of Proposition 8.3 uses, in an essential way, the fact that the composition:

$$Hom_{c-mon}(B, C) \times Hom_{c-mon}(A, B) \rightarrow Hom_{c-mon}(A, C)$$

is a bilinear map. In the non-commutative case, this map is linear in the second variable (i.e. $f(g+h) = fg + fh$) but no more linear in the first variable. So we can't adapt the proof proposed in [BDFP01] to the non-commutative setting. Consequently, to prove Theorem 1.3.1 we have to introduce several new tools of independent interest. In particular, we extend the notion of polynomial maps in the sense of Passi (from monoids to abelian groups) to maps from a set of morphisms in a (suitable) category to abelian groups. Using this notion we can characterize polynomial functors in terms of their effect on morphism sets, like in the classical case of polynomial functors between abelian categories. This leads us to introduce the categories $T_n\overline{\mathbb{Z}}[\mathcal{C}]$ extending the categories $P_n(\mathcal{A})$ for \mathcal{A} an additive category introduced in [Pir93]. These categories have the following important property: polynomial functors of degree n from \mathcal{C} to Ab are equivalent to additive functors from $T_n\overline{\mathbb{Z}}[\mathcal{C}]$ to Ab . We then compute the category $T_n\overline{\mathbb{Z}}[\mathcal{C}]$ for \mathcal{C} being the category of \mathcal{P} -algebras (for \mathcal{P} a set-operad having a binary operation for which $0_{\mathcal{P}} \in \mathcal{P}(0)$ is a unit). The desired equivalence between polynomial functors from monoids and those from groups then follows from a canonical isomorphism $T_n\overline{\mathbb{Z}}[mon] \simeq T_n\overline{\mathbb{Z}}[gr]$ obtained from the computation of these categories.

We point out that these methods also allow to establish a similar isomorphism between polynomial functors from (finitely generated free) loops (i.e. quasigroups with unit) and polynomial functors from (finitely generated free) Mag -algebras where Mag is the set-operad encoding magmas with unit; the latter can be computed using the results of this paper.

1.4 Work in progress and open questions

In this section I present work in progress on generalizations of polynomial functors.

1.4.1 Polynomial functors with values in a semi-abelian category [HV]

In section 1.2, we restrict ourselves to the case $\mathcal{D} = Ab$. However there are some results on the description of polynomial functors with values in Gr : in [BP99] the authors describe quadratic functors from gr to Gr and in [Pir00a] Pirashvili describes polynomial functors from Γ to Gr . In the first case, abelian square groups are replaced by square groups whose definition is similar to the abelian square groups but the group morphism H is replaced by a quadratic map of groups.

In a work in progress with Hartl [HV] we intend to generalize the results obtained in [HV11] in two directions:

1. We allow the *source category* to be generated (in the sense either of finite sums or of small regular projective generators) by a family of objects instead of just one. Examples of such categories are
 - algebras over coloured operads [Lei04]
 - certain homotopy categories, e.g. the homotopy category of Moore spaces with finitely generated homology which is generated by the suspended pseudoprojective planes;
 - modules over square ringoids which generalize modules over square rings studied in [BHP97].
2. We allow the *target category* to be an arbitrary cocomplete semi-abelian category. In fact, the notion of semi-abelian category is kind of an outmost generalization of the notion of abelian category, forcing a maximal number of good algebraic properties to hold; in particular, exact sequences satisfy all classical diagram lemmas. For example the category of groups is semi-abelian, and so are the categories of algebras over an algebraic operad, crossed modules, compact Hausdorff topological groups, C^* -algebras, and, most important in quadratic algebra, modules over square rings or square ringoids. We refer the reader to [BB04] for further information on semi-abelian category.

These two generalizations oblige us to consider and explicitly analyze Kan extensions of functors. This substantially complicates the relations satisfied by the algebraic objects equivalent to quadratic functors.

The difficulties created by the transition from the category of groups to a semi-abelian category are of the same order that those created by the transition from the category of abelian groups to an abelian category: simple arguments based on calculations with elements must be replaced by more intricate arguments based on maps. For example, we need to carry out a subtle analysis of the coequalizer defining the quadratic tensor product, exploiting the fact that both maps involved are central (in the sense of semi-abelian category theory).

Here is a description of the work done so far.

- We determined the quadratic functors from a many-sorted algebraic theory to a cocomplete semi-abelian category. Here a many-sorted (or S -sorted) algebraic theory means a pointed category with chosen objects $E_s, s \in S$ whose objects are all finite sums $E_{s_1} \oplus \dots \oplus E_{s_n} (s_i \in S)$.

We obtained a functorial equivalence between these functors and certain minimal algebraic data, generalizing the quadratic \mathcal{C} -modules relative to a generator E of \mathcal{C} defined in [HV11]; we call this new data *square objects in \mathcal{D} on \mathcal{C} relative to the generating subcategory \mathcal{E} of \mathcal{C}* (which is the full subcategory of \mathcal{C} with objects $E_s, s \in S$, as above). The square object corresponding to a quadratic functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the first and second cross-effect of F restricted to \mathcal{E} , together with certain natural transformations \mathcal{H}, P and T connecting them. Conversely, the quadratic functor corresponding to a square object is explicitly constructed as kind of a non-linear and non-abelian tensor product generalizing the quadratic tensor products constructed in [Bau94], [BP99] and [HV11]. Recall that in [HV11], where the functors take values in abelian groups, the quadratic tensor product is defined by a pushout diagram of abelian groups where tensor products of certain functors are involved. In the case of a semi-abelian target category

the push-out diagram must be replaced by a coequalizer diagram and the tensor product is replaced by suitable left Kan extensions of functors.

- We discover that in the particular case of \mathcal{C} having a cogroup structure, the natural action of the category of abelian groups on any cocomplete abelian category allows us to decompose the square objects in this case, as a direct consequence of the corresponding decomposition in the special situation considered in our previous paper [HV11].
- In the present form our description of the square objects mentioned above is not yet satisfactory, as the source of the transformation \mathcal{H} is a fairly complicated construction invoking the bilinearized second cross-effect of a Kan-extension. We want to obtain a more explicit computation of the source functor of \mathcal{H} : recall that for quadratic functors between groups the map corresponding to \mathcal{H} is a quadratic map between groups [BP99]; from our viewpoint, this is essentially due to the key fact that the canonical retraction of the defining inclusion of the cross-effect, $F(X|Y) \hookrightarrow F(X \oplus Y)$, $X, Y \in \mathcal{C}$, is a quadratic map if the functor F is quadratic. So, in order to understand \mathcal{H} in our situation, we introduced and began to study a notion of quadratic maps in a semi-abelian category generalizing the notion of quadratic maps between groups.
- Applying our results in [HV11] we obtained a description of the good quadratic functors from $\mathcal{P}\text{-Alg}$ to $\mathcal{Q}\text{-Alg}$. We intend to recover this description from our general result for \mathcal{D} semi-abelian.

1.4.2 Polynomial functors on a symmetric monoidal category with an initial object [DVa]

Several natural functors are not defined on a symmetric monoidal category having a null object but have similar polynomial properties as classical polynomial functors. For example, the cohomology of configuration spaces are not defined on the category of finite sets but on the category of finite sets with injections denoted by FI. The FI-modules are functors from FI to abelian groups or modules over a fixed ring. In [CEF12], Church, Ellenberg and Farb study FI-modules and give a lot of examples of them. In particular, they prove that a finitely generated FI-module has polynomial properties (see Theorem 1.3 in [CEF12]).

In fact, several FI-modules considered in [CEF12] have a richer structure. For example, the homology groups of congruence subgroups are not just abelian groups but we have also an action of $GL_n(\mathbb{Z})$ on them. This observation is crucial in the work of Putman in [Put12] on the stability in the homology of congruence subgroups. So the homology of congruence subgroups is not just a functor on FI but also a functor on the category $\mathbf{S}(\mathbf{ab})$ of finitely generated free abelian groups with split monomorphisms.

The categories FI and $\mathbf{S}(\mathbf{ab})$ are monoidal symmetric but 0 is just an initial object in these categories. Thus, Djamant needs an appropriate notion of polynomial functor for symmetric monoidal categories having an initial object in order to study the homology of congruence subgroups with twisted coefficients given by a polynomial functor [Dja] (see chapter 2 for further information on stable homology). In [DVa] we define such polynomial functors. For example, with our terminology, FI-modules of finite type are polynomial functors. For \mathcal{C} a small additive category with duality, the category of hermitian objects on \mathcal{C} , $H(\mathcal{C})$, is a symmetric monoidal category having an initial object. In [DVa] we study the quotient categories

$Pol_n(H(\mathcal{C}), \mathcal{A})/Pol_{n-1}(H(\mathcal{C}), \mathcal{A})$ where \mathcal{A} is an abelian category and $Pol_n(H(\mathcal{C}), \mathcal{A})$ is the category of polynomial functors of degree n (in our generalized sense). The principal result of [DVa] is:

Theorem 1.4.1. *The forgetful functor $H(\mathcal{C}) \rightarrow \mathcal{C}$ induces equivalences of categories*

$$Pol_n(\mathcal{C}, \mathcal{A})/Pol_{n-1}(\mathcal{C}, \mathcal{A}) \xrightarrow{\cong} Pol_n(H(\mathcal{C}), \mathcal{A})/Pol_{n-1}(H(\mathcal{C}), \mathcal{A}).$$

One of the steps in the proof of this result is a generalization of the classification of polynomial functors in \mathcal{F}_{quad} that I obtained in [Ves08a]. In fact, for \mathcal{M} a symmetric monoidal category having an initial object we construct a symmetric monoidal category having a null object $\widetilde{\mathcal{M}}$ and a monoidal functor $\mathcal{M} \rightarrow \widetilde{\mathcal{M}}$ universal for this property. We prove that this functor induces an equivalence:

$$Pol_n(\widetilde{\mathcal{M}}, \mathcal{A})/Pol_{n-1}(\widetilde{\mathcal{M}}, \mathcal{A}) \xrightarrow{\cong} Pol_n(\mathcal{M}, \mathcal{A})/Pol_{n-1}(\mathcal{M}, \mathcal{A}).$$

When \mathcal{M} is the category \mathcal{E}_q of quadratic spaces over \mathbb{F}_2 , the category $\widetilde{\mathcal{M}}$ is equivalent to the category \mathcal{T}_q introduced in [Ves08b] where I studied the category $\mathcal{F}_{quad} = Func(\mathcal{T}_q, \mathbb{F}_2\text{-mod})$.

Chapter 2

Stable homology with twisted coefficients and functor homology

This chapter concerns homology of discrete groups. We are interested in stable phenomena, that is we are not interested by a particular group G but by families of groups $\{G_n\}_{n \geq 0}$ indexed by the natural numbers and related by canonical inclusions $G_n \hookrightarrow G_{n+1}$.

In this part, we fix a commutative ring \mathbb{k} .

2.0.3 What is stable homology?

Let $(\mathcal{C}, \oplus, 0)$ be a symmetric monoidal category. We suppose that the unit 0 is the null object of \mathcal{C} (i.e. is initial and terminal). Let A be an object in \mathcal{C} , we denote by $G(n)$ the automorphism group $\text{Aut}_{\mathcal{C}}(A^{\oplus n})$. Evaluating a functor $F : \mathcal{C} \rightarrow \mathbb{k}\text{-Mod}$ on $A^{\oplus n}$ we obtain, in a natural way a module over $\text{Aut}_{\mathcal{C}}(A^{\oplus n})$, so we can consider $H_*(G(n), F(A^{\oplus n}))$.

Moreover note that the canonical map $A^{\oplus n} \simeq A^{\oplus n} \oplus 0 \xrightarrow{\text{Id} \oplus (0 \rightarrow A)} A^{\oplus n} \oplus A \simeq A^{\oplus n+1}$ is compatible with the action of the automorphism groups $\text{Aut}_{\mathcal{C}}(A^{\oplus n})$ via the morphism $\text{Aut}(A^{\oplus n}) \xrightarrow{g \mapsto g \oplus \text{Id}_A} \text{Aut}(A^{\oplus n+1})$. So we obtain a natural sequence of morphisms:

$$\dots \rightarrow H_k(G(n), F(A^{\oplus n})) \rightarrow H_k(G(n+1), F(A^{\oplus n+1})) \rightarrow \dots \quad (2.1)$$

Most of the time, we are unable to compute completely this sequence of homology groups. But for several cases we know the answers to the following questions which appear naturally:

- (Q1): **Does this sequence stabilize? (i.e. for all $k \in \mathbb{N}$ does it exist an integer $N(k)$ such that**

$$\forall n \geq N(k), H_k(G(n), F(A^{\oplus n})) \simeq H_k(G(n+1), F(A^{\oplus n+1})).$$

If the answer is yes, what is the smaller integer $N(k)$ having this property? When the previous sequence stabilizes we say that the groups $\{G(n)\}$ satisfy homology stability (with coefficients given by F).

- (Q2): **Can we compute the colimit of this sequence? This colimit is called the stable homology of the family of groups $\{G(n)\}$ with coefficients twisted by F and will be denoted by $H_k(G_{\infty}, F_{\infty})$.**

Remark that these two questions are independent but if we know the answer to the second question and if we know the value of $N(k)$ in the first question, we obtain information on $H_k(G(n), F(A^{\oplus n}))$ for n big enough.

As we can see in the next example, this general setting of a symmetric monoidal category let us to cover several families of groups.

Example 2.0.2. • *The category of finite sets equipped with the disjoint union \amalg is a symmetric monoidal category (the unit is the empty set). Let A be the set having one element. In this case $\text{Aut}(A^{\oplus n}) = \mathfrak{S}_n$ (the symmetric group on n letters).*

- *Let R be a ring. The category of projective left R -modules of finite type $\mathbf{P}(R)$ equipped with the direct sum \oplus is a symmetric monoidal category (the unit is the null module). Let A be the R -module R . In this case $\text{Aut}(A^{\oplus n}) = GL_n(R)$.*
- *Let k be a field. The category of finite dimensional quadratic space over k with the orthogonal sum \perp is a symmetric monoidal category (the unit is the null space). Let A be the hyperbolic plane (i.e. the vector space k^2 with the quadratic form given by $(x, y) \mapsto xy$). In this case $\text{Aut}(A^{\oplus n}) = O_{n,n}(k)$.*
- *The categories of finitely generated free groups \mathbf{gr} equipped with the free product $*$ is a symmetric monoidal category (the unit is the null group). Let A be the group \mathbb{Z} . In this case $\text{Aut}(A^{\oplus n}) = \text{Aut}(\mathbb{Z}^{*n})$ (the automorphism group of the free groups in n generators).*

When the action of the groups on the coefficients is non trivial we say that the homology is with twisted coefficients. However, we say that the homology is with trivial coefficients.

My research mainly concerns (Q2) so, in the sequel, I will focus on the answers of this second question. However, note that functor categories methods can also be used to prove stability results (see [CDG]).

2.1 The results

In this section we will give the known results concerning the question (Q2). These results are summarized in the following table.

	constant functor			reduced covariant polynomial functor F	reduced contravariant polynomial functor F	bireduced polynomial bifunctor B
	\mathbb{Q}	\mathbb{F}_p	\mathbb{Z}			
\mathfrak{S}_n	0	[Nak61]		[Bet02]	?	?
$GL_n(k)$	[Bor74]	[Qui72]		[Bet92]	consequence of [Bet92]	[FFSS99] [Bet99] [Sco]
$O_n(k)$		[FP78]		[DV10] [Dja12]	[DV10] [Dja12]	[DV10] [Dja12]
$Sp_n(k)$		[FP78]		[DV10] [Dja12]	[DV10] [Dja12]	[DV10] [Dja12]
$Aut(\mathbb{Z}^{*n})$	[Gal11]	[Gal11]	[Gal11]	[DVb]	? partial result in [DVb]	?

The methods used to compute the stable homology with trivial coefficients (i.e. for F the constant functor) and with twisted coefficients (i.e. F a non constant functor) are completely different. **My research concerns the case of twisted coefficients.** However, to compute the stable homology with twisted coefficients we need to know the value of stable homology with trivial coefficients. In fact we obtain in [DV10] a general result computing stable homology with twisted coefficients from stable homology with trivial coefficients and functor homology (see Theorem 2.2.3). That is why we recall also the values of stable homology with trivial coefficients in this document.

2.1.1 Stable homology with trivial coefficients

The symmetric groups \mathfrak{S}_n Since the symmetric group is a finite group, rational homology of \mathfrak{S}_n is trivial and so is also the rational stable homology.

For coefficients in a finite field, the result is given by Nakaoka in 1961.

Theorem 2.1.1 (Nakaoka [Nak61]). *1. For $p = 2$ we have:*

$$H_*(\mathfrak{S}_\infty; \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2, \dots, x_I, \dots]$$

where I runs over all admissible sequences, x_I has dimension $d(I)$ and bidegree $b(I)$.

2. For p an odd prime, $H_(\mathfrak{S}_\infty; \mathbb{F}_p)$ is a tensor product of exterior algebras on odd dimensional generators and polynomial algebras on even generators where each generator has a known bidegree.*

The proof of Nakaoka is based on topological methods. A good reference for a more algebraic proof of stable homology of symmetric groups with trivial coefficients is [AM04] Chapter VI.

Groups of matrices: $GL_n(k), O_n(k)$ and $Sp_n(k)$

General linear groups $GL_n(k)$ The computation of the homology of $GL_\infty(k)$ is out of reach for any k . However, for particular rings k this computation is accessible.

In [Qui72], Quillen computes the (co)homology of $GL_n(k)$ when k is a finite field of characteristic p . He computes it completely for coefficients in \mathbb{F}_m if $m \neq p$. For $m = p$ he obtains the following result:

Theorem 2.1.2 (Quillen [Qui72]). *For k a finite field of characteristic p we have:*

$$H_0(GL_\infty(k); \mathbb{F}_p) = \mathbb{F}_p \quad \text{and} \quad H_j(GL_\infty(k); \mathbb{F}_p) = 0 \quad \text{for } j > 0.$$

Remark 2.1.3. Quillen proves also in [Qui72] that

$$H_0(GL_n(\bar{\mathbb{F}}_p); \mathbb{F}_p) = \mathbb{F}_p \quad \text{and} \quad H_j(GL_n(\bar{\mathbb{F}}_p); \mathbb{F}_p) = 0 \quad \text{for } j > 0.$$

In [Sus83], Suslin shows that the natural map $H_*(GL_\infty(k); \mathbb{Z}/n) \rightarrow H_*(GL_\infty(K); \mathbb{Z}/n)$ is an isomorphism for any extension $k \subset K$ of algebraically closed fields. Combining this with Quillen's result the calculation of $H_*(GL_\infty(K), \mathbb{F}_p)$ is obtained for any algebraically closed field K of characteristic p .

Remark 2.1.4. Results of Borel in [Bor74] let us to compute $H_*(GL_\infty(\mathbb{R}); \mathbb{Q})$.

For more details on the homology of linear groups see [Knu01].

Orthogonal and symplectic groups Generalizing the methods of Quillen, Fiedorowicz and Priddy obtain in [FP78] the following results:

Theorem 2.1.5 (Fiedorowicz-Priddy [FP78]). *For k a field with $q = p^r$ elements we have:*

$$H_0(O_\infty(k); \mathbb{F}_p) = \mathbb{F}_p \quad \text{and} \quad H_j(O_\infty(k); \mathbb{F}_p) = 0 \quad \text{for } j > 0.$$

$$H_0(Sp_\infty(k); \mathbb{F}_p) = \mathbb{F}_p \quad \text{and} \quad H_j(Sp_\infty(k); \mathbb{F}_p) = 0 \quad \text{for } j > 0.$$

Automorphism groups of free groups $Aut(\mathbb{Z}^{*n})$ In a recent work, Galatius obtains the following beautiful theorem:

Theorem 2.1.6 (Galatius [Gal11]). *Let \mathbb{Z}^{*n} be the free group freely generated on $\{x_1, \dots, x_n\}$ and $\phi_n : \mathfrak{S}_n \rightarrow Aut(\mathbb{Z}^{*n})$ the homomorphism given by $\phi_n(\sigma) : x_i \mapsto x_{\sigma(i)}$ for $\sigma \in \mathfrak{S}_n$. The induced homomorphism:*

$$(\phi_n)_* : H_k(\mathfrak{S}_n; \mathbb{Z}) \rightarrow H_k(Aut(\mathbb{Z}^{*n}); \mathbb{Z})$$

is an isomorphism for $2k + 1 \leq n$.

By the computations of Nakaoka for the symmetric groups, we can deduce the computation of the stable homology of automorphism groups of free groups $Aut(\mathbb{Z}^{*n})$ with integral coefficients. For k a field, using the universal coefficient theorem, the following isomorphism is deduced:

$$(\phi_n)_* : H_k(\mathfrak{S}_n; k) \simeq H_k(Aut(\mathbb{Z}^{*n}); k)$$

for $2k + 1 \leq n$. In particular, since \mathfrak{S}_n is a finite group, the homology groups of \mathfrak{S}_n with rational coefficients vanish so the previous theorem has the following corollary.

Corollary 2.1.7 (Galatius [Gal11]). *Let \mathbb{Z}^{*n} be the free group, then:*

$$H_k(Aut(\mathbb{Z}^{*n}); \mathbb{Q}) = 0 \quad \text{for } 2k + 1 \leq n.$$

This result had been conjectured by Hatcher and Vogtmann in [HV98].

2.1.2 Stable homology with twisted coefficients

The results in this area are based on an interpretation of stable homology with twisted coefficients as functor homology (i.e. as *Tor* groups in functor categories). The power of these results is related to the fact that we have powerful tools to compute such *Tor* groups in order to obtain explicit computations.

The symmetric groups \mathfrak{S}_n In 2002 Betley proves that stable homology of symmetric groups with twisted coefficients is highly non trivial even for functors of finite degree. He obtains the following result

Theorem 2.1.8 (Betley [Bet02]). *For all functors F from Γ to Ab we have a non natural isomorphism:*

$$H_n(\mathfrak{S}_\infty; F_\infty) \simeq \bigoplus_{\substack{p+q=n \\ i \in \mathbb{N}}} H_p(\mathfrak{S}_\infty; H_q(\mathfrak{S}_i; cr(F)(\{1, \dots, i\}))).$$

General linear groups $GL_n(k)$

Theorem 2.1.9 (Betley [Bet92]). *Let R be a commutative ring and F be a reduced polynomial functor from $\mathbf{P}(R)$ to $\mathbb{k}\text{-mod}$ then:*

$$H_*(GL_\infty(R); F_\infty) = 0.$$

In 1999, Betley and Suslin independently obtained a generalization of the previous results for coefficients given by a bipolynomial bifunctor.

Theorem 2.1.10 (Betley [Bet99], Suslin [FFSS99] (appendix)). *Let k be a finite field and B be a bireduced bipolynomial bifunctor from $\mathbf{P}(k)^{op} \times \mathbf{P}(k)$ to $\mathbb{k}\text{-mod}$ then there is a natural isomorphism of graded k -vector-spaces:*

$$H_*(GL_\infty(k); B_\infty) \simeq Tor_*^{\mathbf{P}(k)^{op} \times \mathbf{P}(k)}(\mathbb{k}[Hom_{\mathbf{P}(k)^{op}}(-, -)], B).$$

Concrete examples of *Tor* groups in $\mathbf{P}(k)^{op} \times \mathbf{P}(k)$ are given in [FFSS99] for classical functors, see also the new approach of Touzé in [Tou].

This theorem has been extended by Scorichenko in 2000 in his PhD-thesis. Recall that, for $B \in (\mathcal{C}^{op} \times \mathcal{C})\text{-mod}$, the Hochschild homology of \mathcal{C} with coefficients in B is:

$$HH_*(\mathcal{C}; B) = Tor_*^{(\mathcal{C}^{op} \times \mathcal{C})\text{-mod}}(\mathbb{k}[\mathcal{C}^{op}], B)$$

where $\mathbb{k}[\mathcal{C}^{op}] : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathbb{k}\text{-mod}$ is given by: $\mathbb{k}[\mathcal{C}^{op}](a, b) = \mathbb{k}[\mathcal{C}(b, a)]$.

Theorem 2.1.11 (Scorichenko [Sco]). *Let R be a ring and B be a bipolynomial bifunctor from $\mathbf{P}(R)^{op} \times \mathbf{P}(R)$ to $\mathbb{k}\text{-mod}$. Then there is a natural isomorphism of graded k -vector-spaces:*

$$H_*(GL_\infty(R); B_\infty) \simeq HH_*(GL_\infty(R) \times \mathbf{P}(R); B)$$

(where the action of $GL_\infty(R)$ in the right hand side is trivial).

Remark 2.1.12. 1. When $R = k$ (a finite field) the homology of $GL_\infty(k)$ with coefficients in k is trivial (see Theorem 2.1.2). So the term $GL_\infty(R)$ in the right hand side of the isomorphism in Theorem 2.1.11 disappears and we recover Theorem 2.1.10.

2. Theorem 2.1.9 can be obtained as a corollary of Theorem 2.1.11.

Orthogonal and symplectic groups

Theorem 2.1.13. [DV10] *Let k be a finite field of odd characteristic and F be a polynomial functor from $\mathbf{P}(k)$ to $k\text{-Mod}$. There exists a natural isomorphism:*

$$H_*(O_\infty(k); F_\infty) \simeq \text{Tor}_*^{\mathcal{E}_k^f}(V \mapsto k[S^2(V^*)], F)$$

where \mathcal{E}_k^f is the category of finite k -vector spaces and S^2 is the second symmetric power.

As a consequence of Theorem 2.1.13 we obtain explicit computations of stable homology of orthogonal groups with twisted coefficients. For example, for the symmetric powers, we obtain:

Theorem 2.1.14. [DV10] *Let k be a finite field of odd cardinality q . For the symmetric powers S^\bullet , the bigraduate algebra*

$$H_*(O_\infty(k); S_\infty^\bullet)$$

is polynomial on the generators $\alpha_{m,s}$ of bidegree $(2q^s m, q^s + 1)$ (where the first degree is the homological degree and the second degree is the internal degree).

In [DV10] we obtain also the explicit computation of $H_*(O_\infty(k); \Lambda_\infty^\bullet)$ (see Theorem 4.19 in [DV10]).

Theorem 2.1.15. [DV10] *Let k be a finite field and F be a polynomial functor from $\mathbf{P}(k)$ to $k\text{-Mod}$. There exists a natural isomorphism:*

$$H_*(Sp_\infty(k); F_\infty) \simeq \text{Tor}_*^{\mathcal{E}_k^f}(V \mapsto k[\Lambda^2(V^*)], F)$$

where \mathcal{E}_k^f is the category of finite k -vector spaces and Λ^2 is the second exterior power.

As a consequence of Theorem 2.1.15 we obtain explicit computations of stable homology of symplectic groups with twisted coefficients.

Remark 2.1.16. Touzé obtained in [Tou10] similar results for orthogonal and symplectic groups over a finite field *viewed as algebraic groups*. In this setting polynomial functors are replaced by *strict* polynomial functors. Recall that strict polynomial functors were introduced by Friedlander and Suslin in [FS97]. Roughly speaking they are the good analogue of polynomial functors in the algebro-geometric setting.

In [Dja12] Djament generalizes Theorem 2.1.13 for a ring. More precisely, he proves the following result:

Theorem 2.1.17. [Dja12] *Let R be a ring with involution $a \mapsto \bar{a}$ and F be a polynomial functor from $\mathbf{P}(R)$ to Ab . There exists a isomorphism of abelian groups:*

$$H_{p+q}(U_{\infty, \infty}(R); F_\infty) \simeq \bigoplus_{p+q=n} H_p(U_{\infty, \infty}(R); \text{Tor}_q^{\mathbf{P}(R)}(\mathbb{Z}[\mathcal{S}^2], F))$$

where $\mathcal{S}^2 : P(R)^{op} \rightarrow \text{Ens}$ is the functor given by $\mathcal{S}^2(M) = \{\text{hermitian forms on } M\}$.

See [Dja12] for explicit computations.

Automorphism groups of free groups $Aut(\mathbb{Z}^{*n})$

Theorem 2.1.18. [DVb] *Let F be a reduced polynomial functor from the category \mathbf{gr} of finitely generated free groups to the category Ab of abelian groups. Then*

$$\operatorname{colim}_{n \in \mathbb{N}} H_* \left(\operatorname{Aut}(\mathbb{Z}^{*n}); F(\mathbb{Z}^{*n}) \right) = 0.$$

Remark 2.1.19. If F is the abelianization functor $\mathbf{a} : \mathbf{gr} \rightarrow Ab$, Theorem 2.1.18 was known. In fact, Hatcher and Wahl proved that the stable homology groups of $Aut(\mathbb{Z}^{*n})$ with coefficients in \mathbb{Z}^n are trivial using the stability of the homology groups of the mapping class groups of certain 3-manifolds (see [HW05], [HW08] and [HW10]). Furthermore, by an observation of Randal-Williams in [RW], for $F = T^n \circ \mathbf{a}$ Theorem 2.1.18 can be deduced from the paper of Hatcher and Vogtmann [HV04] (see also its erratum [HVW] with Wahl). By completely different and purely algebraic methods, Satoh studies in [Sat06] and [Sat07] first and second homology groups of the automorphism group of a free group with coefficients given by the abelianization functor and obtains some computations in the non stable range.

Remark 2.1.20. Theorem 2.1.18 should be compare with Theorem 2.1.6 due to Galatius. In the stable range, by Galatius' Theorem, the stable integral homology groups of $Aut(\mathbb{Z}^{*n})$ are isomorphic to those of the symmetric group \mathfrak{S}_n . Theorem 2.1.18 tells us that it is no more the case for twisted coefficients. In fact, by Theorem 2.1.8 due to Betley, stable homology of symmetric groups with twisted coefficients is highly non trivial.

Contrary to the case of general linear groups, the stable homology of automorphism groups of free groups with coefficients given by a contravariant functor can not be obtained from the one with coefficients given by of a covariant functor. For the moment we do not know how to compute stable homology of automorphisms groups of free groups with coefficients given by contravariant functors. But we prove in [DVb] the following result which shows that the situation is completely different for stable homology of automorphisms groups of free groups with coefficients given by covariant functors and those with coefficients given by contravariant functors.

Proposition 2.1.21. [DVb] *Let F be a reduced contravariant polynomial functor from the category \mathbf{ab} of finitely generated free abelian groups to Ab . There exists a natural isomorphism*

$$\operatorname{colim}_{n \in \mathbb{N}} H_1 \left(\operatorname{Aut}(\mathbb{Z}^{*n}); F(\mathbb{Z}^n) \right) \simeq F \otimes_{\mathbf{ab}} \operatorname{Id}$$

where $\operatorname{Aut}(\mathbb{Z}^{*n})$ acts on $F(\mathbb{Z}^n)$ via the natural projection to $GL_n(\mathbb{Z})$ and $\operatorname{Id} : \mathbf{ab} \rightarrow Ab$ is the inclusion functor.

2.2 The methods developed for twisted coefficients in [DV10] [Dja12] and [DVb]

In [DV10] we develop a general method to compute stable homology of families of groups with twisted coefficients given by a reduced covariant polynomial functor. **Not only can we prove new results by this method, but also can we recover all the older known results in the table.** For example, in [DV10] (Appendice E) we explain how to recover Betley's theorem for symmetric groups 2.1.8 from our general setting and in [Dja12] Djament

explains how to recover Scorischenko's theorem for general linear groups from our general setting (this is based on the fact that general linear groups over a ring R are unitary groups on $R^{op} \times R$ (with the canonical anti-involution)).

Remark 2.2.1. In my PhD-thesis and subsequent work, I tried to compute the stable homology for orthogonal groups over the field \mathbb{F}_2 with twisted coefficients by trying to construct a category of functors which plays the role of the category of functors from $\mathbf{P}(\mathbb{F}_2)$ to \mathbb{F}_2 -vector spaces in the case of orthogonal groups. Thus in [Ves08b] I construct such a category of functors denoted \mathcal{F}_{quad} . In [Ves07], I study a very special family of objects of \mathcal{F}_{quad} called "mixed functors" and in [Ves08a] I study projective functors of the category \mathcal{F}_{quad} . Surprisingly, the results in [DV10] show that stable homology of orthogonal and symplectic groups are given by *Tor* groups of endofunctors of the category of vector spaces. Notably, the categories of quadratic or symplectic spaces does not involve. However variations around the category \mathcal{F}_{quad} appear in different parts of my work. First, the definition of the category \mathcal{F}_{quad} is very close to the definition of the category of pseudo-Mackey functors appearing in the first chapter of this report. A similar category appears as an intermediate ingredient in [DV10] (but Djament proved in [Dja12] that this part of the proof can be solve differently.) Finally, a key ingredient in the description of polynomial functors on hermitian spaces given in [DVa] is a generalization of the description of polynomial functors obtained in [Ves08a].

In the sequel we explain the ideas behind this general method. We refer the reader to [DV10] for details.

This method can be decomposed into two steps:

1. We obtain a general theorem computing stable homology with twisted coefficients from stable homology with trivial coefficients and the homology of a certain category \mathcal{D} (see Theorem 2.2.3). Unfortunately, in most of the cases we are unable to compute directly the homology of this category \mathcal{D} .
2. This step consists of computing the homology of the category \mathcal{D} on a case by case basis. The idea is to relate \mathcal{D} to another category for which we are able to do homological computations and to obtain an isomorphism between the homology of the two categories using a Grothendieck spectral sequence. This step is the most difficult.

In the sequel, we develop these two steps.

2.2.1 Step 1: A general theorem

The first important theorem in [DV10] computes stable homology of a family of groups with twisted coefficients from stable homology of the same family of groups with trivial coefficients and the homology of a certain category.

Let $(\mathcal{D}, \oplus, 0)$ be a symmetric monoidal category. We suppose that the unit 0 is the **initial** object of \mathcal{D} . Let A be an object in \mathcal{D} , we denote by $G(n)$ the automorphism group $Aut_{\mathcal{D}}(A^{\oplus n})$.

We have a natural map:

$$H_*(G_{\infty}, F_{\infty}) \rightarrow H_*(\mathcal{D} \times G_{\infty}, \pi^*F)$$

where $\pi : \mathcal{D} \times G_{\infty} \rightarrow \mathcal{D}$. This natural map is an isomorphism for $F = \mathbb{k}$ (the constant functor) but it is no more the case in general.

The aim of this first theorem is to give conditions on \mathcal{D} such that this natural map is an isomorphism for each functor F .

In the sequel, $H_*(\mathcal{D} \times G_\infty, \pi^*F)$ will be denoted by $H_*(\mathcal{D} \times G_\infty, F)$ for simplicity.

Remark 2.2.2. We chose to present here a simplified version of the hypothesis on \mathcal{D} . For a more general setting we refer the reader to the first section of [DV10].

We consider the following hypothesis on the source category \mathcal{D} :

- **(H1): Transitivity:**

Strong version: Let X and Y be two objects of \mathcal{D} , $\text{Hom}_{\mathcal{D}}(X, Y)$ is a transitive $\text{Aut}_{\mathcal{D}}(Y)$ -set.

Weak version or stable version: For all morphisms $f : X \rightarrow Y$ in \mathcal{D} , there exists an automorphism $\alpha : X \oplus Y \rightarrow X \oplus Y$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & X \oplus Y \\ & \searrow & & & \downarrow \alpha \\ & & & & X \oplus Y \\ & & & & \uparrow i' \\ & & & & X \end{array}$$

where i and i' are the canonical inclusions.

- **(H2): Stabilizers:** Let X and Y be two objects of \mathcal{D} . The canonical group morphism $\text{Aut}_{\mathcal{D}}(Y) \rightarrow \text{Aut}_{\mathcal{D}}(X \oplus Y)$ is an injection and its image is the subgroup of automorphisms ϕ making the following diagram commutative:

$$\begin{array}{ccc} X & \longrightarrow & X \oplus Y \\ & \searrow & \downarrow \phi \\ & & X \oplus Y \end{array}$$

- **(H3): Generator:**

Strong version: There exists $A \in \mathcal{D}$ such that for all objects X in \mathcal{D} there exists $n \in \mathbb{N}$ such that $X \simeq A^{\oplus n}$.

Weak version There exists $A \in \mathcal{D}$ such that for all objects X in \mathcal{D} , there exists an object Y in \mathcal{D} and $n \in \mathbb{N}$ such that $X \oplus Y \simeq A^{\oplus n}$.

The fundamental result in this first step is the following.

Theorem 2.2.3. [DV10] *Let $(\mathcal{D}, \oplus, 0)$ be a symmetric monoidal category such that the unit 0 is the **initial** object of \mathcal{D} and satisfying the three previous hypothesis. Then the natural morphism of graded -modules:*

$$H_*(G_\infty; F_\infty) \rightarrow H_*(G_\infty \times \mathcal{D}; F)$$

is an isomorphism for all functors $F : \mathcal{D} \rightarrow \mathbb{k}\text{-Mod}$.

Note that in this theorem we need no assumption on the functor F .

The following proposition is a consequence of the Künneth theorem.

Proposition 2.2.4. *If \mathbb{k} is a field then we have a natural isomorphism of graded \mathbb{k} -modules:*

$$H_*(G_\infty \times \mathcal{D}; F) \simeq H_*(G_\infty; \mathbb{k}) \otimes_{\mathbb{k}} H_*(\mathcal{D}; F).$$

The categories introduced in Example 2.0.2 do not satisfy these hypothesis. So, we introduce in the next examples the suitable categories for the different families of groups.

Example 2.2.5. • **Symmetric groups:** *Let Θ be the category having as objects finite sets and injections as morphisms. The category Θ is a symmetric monoidal category for the disjoint union (the unit \emptyset is the initial object) and satisfies the three hypothesis (H1) (strong), (H2) and (H3) (strong). (The category Θ is strongly generated by $A = \underline{1}$ (the set having one element)).*

By Theorem 2.2.3 we deduce that: for $G : \Theta \rightarrow \mathbb{k}\text{-mod}$,

$$H_*(S_\infty; F_\infty) \simeq H_*(S_\infty \times \Theta; F).$$

- **Linear groups:** *Let R be a ring. The category $\mathbf{P}(R)$ satisfies the hypothesis (H3) (weak version: for $A = R$) but does not satisfy the hypothesis (H1) and (H2). Let $\mathbf{M}(R)$ be the subcategory of $\mathbf{P}(R)$ having as morphisms split monomorphisms. The category $\mathbf{M}(R)$ satisfies the hypothesis (H3) and (H1) but does not satisfy (H2).*

*Let $\mathbf{S}(R)$ be the category having as objects projective left R -modules of finite type and as morphisms $M \rightarrow N$ ordered pairs (u, v) where $v : N \rightarrow M$ and $u : M \rightarrow N$ are R -linear maps such that $v \circ u = \text{Id}_M$. (In particular, u is a monomorphism.) The category $\mathbf{S}(R)$ is a symmetric monoidal category for the direct sum (the unit 0 is the **initial** object) and satisfies the three hypothesis (H1) (weak), (H2) and (H3) (weak).*

By Theorem 2.2.3 we deduce that: for $F : \mathbf{S}(R) \rightarrow \mathbb{k}\text{-mod}$,

$$H_*(GL_\infty(R); F_\infty) \simeq H_*(GL_\infty(R) \times \mathbf{S}(R); F).$$

- **Orthogonal and symplectic groups:** *Let $\mathcal{E}_q(k)$ be the category having as objects finite dimensional non degenerated quadratic spaces and as morphisms linear maps compatible with the quadratic forms. The category $\mathcal{E}_q(k)$ is a symmetric monoidal category for the orthogonal sum \perp (the unit 0 is the initial object) and satisfies the three hypothesis (H1) (strong), (H2) (by the Witt's theorem) and (H3) (weak version): let H be the hyperbolic plane (i.e. the vector space k^2 with the quadratic form $(x, y) \mapsto xy$) for all $V \in \mathcal{E}_q(k)$ there exists $W \in \mathcal{E}_q(k)$ such that $V \perp W \simeq H^{\perp n}$.*

By Theorem 2.2.3 we deduce that: for $F : \mathcal{E}_q(k) \rightarrow \mathbb{k}\text{-mod}$,

$$H_*(O_\infty(k); F_\infty) \simeq H_*(O_\infty(k) \times \mathcal{E}_q(k); F).$$

- **Automorphism groups of free groups.** *Let \mathcal{G} be the category having as objects finitely generated free groups and as morphisms $A \rightarrow B$ ordered pairs (u, H) where $u : A \rightarrow B$ is a group monomorphism and H is a subgroup of B such that $B = u(A) * H$. The composition $A \xrightarrow{(u, H)} B \xrightarrow{(v, K)} C$ is given by $(v \circ u, v(H) * K)$. The category \mathcal{G} is a symmetric monoidal category for the free product $*$ (the unit 0 is the **initial** object)*

and satisfies the three hypothesis (H1) (strong), (H2) and (H3) (strong for $A = \mathbb{Z}$). By Theorem 2.2.3 and Proposition 2.2.4 we deduce that: If \mathbb{k} is a field then we have a natural isomorphism of graded \mathbb{k} -modules:

$$H_*(\text{Aut}(\mathbb{Z}^{*\infty}); F_\infty) \simeq H_*(\text{Aut}(\mathbb{Z}^{*\infty}); \mathbb{k}) \otimes_{\mathbb{k}} H_*(\mathcal{G}; F).$$

- **Unitary groups.** Let S be a ring with an involution $\sigma : S \rightarrow S$ and $H^{\text{deg}}(S)$ be the category having as objects finitely generated free S -modules equipped with an hermitian form. Let $H(S)$ be the full subcategory of $H^{\text{deg}}(S)$ having as objects finitely generated free S -modules equipped with a non degenerate hermitian form. The category $H(S)$ is a symmetric monoidal category for the orthogonal sum \perp (the unit 0 is the **initial** object). Let X be the S -module S^2 equipped with the hyperbolic form represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have $\text{Aut}_{H(S)}(X^{\perp n}) = U_{n,n}(S)$.

The category $H(S)$ satisfies the three hypothesis (H1) (weak), (H2) and (H3) (weak) ((V, q) is non degenerate, $(V, q) \perp (V, -q)$ is hyperbolic).

By Theorem 2.2.3 we deduce that: for $F : H(S) \rightarrow \mathbb{k}\text{-mod}$,

$$H_*(U_{\infty, \infty}(S); F_\infty) \simeq H_*(U_{\infty, \infty}(S) \times H(S); F).$$

Two sub-examples:

- If S is a commutative ring with the trivial involution, an hermitian form is a quadratic form and we have $H(S) = \mathcal{E}_q(S)$. (This extends the example of orthogonal groups given previously).
- Let R be a ring. We consider the ring $S = R^{\text{op}} \times R$ with the involution $\sigma : R^{\text{op}} \times R \rightarrow R^{\text{op}} \times R$ given by $\sigma(a, b) = (b, a)$. We have $H(S) = S(R)$ and $U_{n,n}(R^{\text{op}} \times R) \simeq GL_{2n}(R)$. (We recover the example of linear groups).

Applying Theorem 2.2.3 to the category Θ and applying Pirashvili's Dold-Kan type theorem (see [Pir00a]) we recover Betley's Theorem 2.1.8. (See Appendice E in [DV10] for more details).

2.2.2 Step 2: Comparison of functor homologies

Unfortunately, apart from the case of symmetric groups, the computation of $H_*(\mathcal{D}; F)$ is inaccessible directly. The idea of this second step is to find on a case by case basis a category \mathcal{C} related to \mathcal{D} by a functor $\phi : \mathcal{D} \rightarrow \mathcal{C}$ such that homological algebra in \mathcal{C} is accessible and to deduce homological algebra computations in \mathcal{D} from computations in \mathcal{C} . The hypothesis concerning the polynomiality of the functor F is crucial in this step.

More precisely, this step is based on the following general construction. Recall that for \mathcal{C} a small category, the functor category $\text{Func}(\mathcal{C}, \mathbb{k}\text{-Mod})$ is an abelian category having a set of projective generators given by $P_A^{\mathcal{C}} := \mathbb{k}[\text{Hom}_{\mathcal{C}}(A, -)]$. Let $\phi : \mathcal{D} \rightarrow \mathcal{C}$ be a functor, consider the following composition of functors:

$$\text{Func}(\mathcal{D}^{\text{op}}, \mathbb{k}\text{-Mod}) \xrightarrow{\phi_!} \text{Func}(\mathcal{C}^{\text{op}}, \mathbb{k}\text{-Mod}) \xrightarrow{- \otimes_{\mathcal{C}}^L F} \mathbb{k}\text{-Mod} \quad (2.2)$$

where $\phi_! : \text{Func}(\mathcal{D}^{op}, \mathbb{k}\text{-Mod}) \rightarrow \text{Func}(\mathcal{C}^{op}, \mathbb{k}\text{-Mod})$ is the left Kan extension of $\phi : \mathcal{D} \rightarrow \mathcal{C}$ i.e. $\phi_!(G)(c) := G \otimes_{\mathcal{D}} \phi^*(P_c^{\mathcal{C}})$ (where ϕ^* is the precomposition by ϕ). The functor $\phi_!$ is right exact and preserves projective objects ($\phi_!(P_D^{\mathcal{D}^{op}}) \simeq P_{\phi(D)}^{\mathcal{C}^{op}}$) and the functor $- \otimes_{\mathcal{C}} F$ is right exact. So, we can consider the Grothendieck spectral sequence associated to the composition (2.2):

$$E_{pq}^2 = (\mathbb{L}_p(- \otimes_{\mathcal{C}} F))(\mathbb{L}_q(\phi_!))(X) \Rightarrow \mathbb{L}_{p+q}(\phi_!(-) \otimes_{\mathcal{C}} F)(X)$$

where \mathbb{L}_q are the left derived functors. Using the natural isomorphism $\phi_!(G) \otimes_{\mathcal{C}} F \simeq G \otimes_{\mathcal{D}} \phi^*(F)$ (for $F \in \text{Func}(\mathcal{C}, \mathbb{k}\text{-Mod})$ and $G \in \text{Func}(\mathcal{D}^{op}, \mathbb{k}\text{-Mod})$) and the definition of *Tor* we obtain:

$$E_{pq}^2 = \text{Tor}_p^{\mathcal{C}}(\mathbb{L}_q(\phi_!)(X), F) \Rightarrow \text{Tor}_{p+q}^{\mathcal{D}}(X, \phi^*(F)) \quad (2.3)$$

where $\mathbb{L}_q(\phi_!)$ are given by

$$\mathbb{L}_q(\phi_!)(X)(c) := \text{Tor}_*^{\mathcal{D}}(X, \phi^*(P_c^{\mathcal{C}})).$$

The principle of this second step is to study, for each family of groups, this spectral sequence.

In the sequel we give a criterion to obtain an isomorphism between *Tor*-groups in \mathcal{C} and in \mathcal{D} . Let $\mathcal{L}_*^{\phi}(Y) = \text{Ker}(\mathbb{L}_*(\phi_!)(\phi^*Y) \rightarrow Y)$ for $Y \in \text{Func}(\mathcal{C}^{op}, \mathbb{k}\text{-Mod})$ the kernel of the unit. We have:

Proposition 2.2.6. *Let $Y \in \text{Func}(\mathcal{C}^{op}, \mathbb{k}\text{-Mod})$ and $F \in \text{Func}(\mathcal{C}, \mathbb{k}\text{-Mod})$ such that*

$$\forall i, j \quad \text{Tor}_i^{\mathcal{C}}(\mathcal{L}_j^{\phi}(Y), F) = 0.$$

Then the morphism:

$$\text{Tor}_*^{\mathcal{D}}(\phi^*Y, \phi^*F) \rightarrow \text{Tor}_*^{\mathcal{C}}(Y, F)$$

induced by ϕ is an isomorphism.

For $X = \mathbb{k}$ (the constant functor) the spectral sequence (2.3) becomes:

$$E_{pq}^2 = \text{Tor}_p^{\mathcal{C}}(L_q, F) \Rightarrow H_{p+q}(\mathcal{D}, \phi^*(F)) \quad (2.4)$$

where $L_q(C) = H_q(\mathcal{D}/C; \mathbb{k})$ (\mathcal{D}/C is the category having as objects ordered pairs (D, d) where D is an object of \mathcal{D} and $d : C \rightarrow \phi(D)$ is a morphism in \mathcal{C} and as morphisms those of \mathcal{D} making the obvious diagram commute.) We have $\mathcal{L}_i^{\phi}(\mathbb{k}) = \tilde{H}_i(\mathcal{D}/C; \mathbb{k})$. Proposition 2.2.6 becomes in this particular case:

Proposition 2.2.7. *Let $F \in \text{Func}(\mathcal{C}, \mathbb{k}\text{-Mod})$ such that*

$$\forall i, j \quad \text{Tor}_i^{\mathcal{C}}(\mathcal{L}_j^{\phi}(\mathbb{k}), F) = 0.$$

Then the morphism:

$$H_*(\mathcal{D}, \phi^*F) \rightarrow H_*(\mathcal{C}, F)$$

induced by ϕ is an isomorphism.

This step consists of finding, for each family of groups, a suitable category \mathcal{C} , a functor ϕ and conditions on F such that $\forall i, j \quad \text{Tor}_i^{\mathcal{C}}(\mathcal{L}_j^\phi(\mathbb{k}), F) = 0$. This kind of result is called "cancellation results".

Example 2.2.8. • *Automorphism groups of free groups* $\text{Aut}(\mathbb{Z}^{*n})$

Let $i : \mathcal{G} \rightarrow gr$ be the functor which is equal to the identity on objects and which associates to a morphism $(u, H) : A \rightarrow B$ in \mathcal{G} the group morphism $u : A \rightarrow B$.

The difficult part of [DVb] is the following proposition.

Proposition 2.2.9. (Cancellation result) For $K : gr \rightarrow \mathbb{k}\text{-Mod}$ **polynomial** we have:

$$\text{Tor}^{gr}(\mathcal{L}_i^\phi(\mathbb{k}), K) = 0.$$

By Proposition 2.2.7 we obtain:

$$H_*(\mathcal{G}; \phi^*(K)) \xrightarrow{\cong} H_*(gr; K) = K(0)$$

(where the last equality is a consequence of the fact that 0 is the final object of gr (see Lecture 2)).

Combining Step 1 and Step 2 we obtain:

Theorem 2.2.10. [DVb] Let F be a reduced polynomial functor from the category gr of finitely generated free groups to the category Ab of abelian groups. Then

$$\text{colim}_{n \in \mathbb{N}} H_*(\text{Aut}(\mathbb{Z}^{*n}); F(\mathbb{Z}^{*n})) = 0.$$

Sketch of the proof of Proposition 2.2.9 Let $\mathbf{a} : gr \rightarrow \mathbb{k}\text{-Mod}$ be the abelianization functor tensorised by \mathbb{k} .

1. The bar resolution of the free group gives rise to an explicit projective resolution of \mathbf{a} . (This result is already present in a previous work of Jibladze and Pirashvili). (See Proposition 4.1 in [DVb]).
2. Taking the tensor product of this resolution with $P_r = P_{\mathbb{Z}^{*r}}^{gr}$, we obtain a complex whose homology is isomorphic to $\text{Tor}_*^{gr}(X, \mathbf{a} \otimes P_r)$ for $X : gr^{op} \rightarrow \mathbb{k}\text{-Mod}$. (See Corollary 4.2 in [DVb]).
3. We give a general cancellation criteria (i.e. conditions on X such that $\text{Tor}_*^{gr}(X, \mathbf{a} \otimes P_r) = 0$). (See Proposition 4.4 in [DVb]).
4. The functors $\mathcal{L}_i^\phi(\mathbb{k})$ satisfy the conditions of the previous criteria. (See Lemma 5.2 in [DVb]).
5. If $\text{Tor}_*^{gr}(X, \mathbf{a} \otimes P_r) = 0$ then $\text{Tor}_*^{gr}(X, F \otimes G) = 0$ (for $F : gr \rightarrow \mathbb{k}\text{-Mod}$ a **reduced polynomial functor**). (This result is a consequence of a general result on the structure of polynomial functors on gr). (See Proposition 5.3 in [DVb]).

• **Orthogonal and symplectic groups**

Let $i : \mathcal{E}_q(\mathbb{k}) \rightarrow \mathcal{E}_q^{deg}(\mathbb{k})$ be the inclusion functor where $\mathcal{E}_q^{deg}(\mathbb{k})$ is the category having as objects finite dimensional quadratic spaces and morphisms injective linear maps. The difficult part of [DV10] is the following proposition.

Proposition 2.2.11. (Cancellation result) For $K : \mathcal{E}_q^{\text{deg}}(\mathbb{k}) \rightarrow \mathbb{k}\text{-Mod}$ **polynomial** we have:

$$\text{Tor}^{\mathcal{E}_q^{\text{deg}}(\mathbb{k})}(\mathcal{L}_i^\phi(\mathbb{k}), K) = 0.$$

By Proposition 2.2.7 we obtain:

$$H_*(\mathcal{E}_q; \phi^*(K)) \xrightarrow{\cong} H_*(\mathcal{E}_q^{\text{deg}}(\mathbb{k}), K).$$

A quadratic form on V is an homogeneous polynomial functor of degree 2 on V so it is an element of $S^2(V^*)$. So we have:

$$H_*(\mathcal{E}_q^{\text{deg}}(\mathbb{k}), K) \simeq \text{Tor}^{\mathcal{E}_{\text{inj}}^f}(V \mapsto \mathbb{k}[S^2(V^*)], K).$$

By a result of Suslin:

$$\text{Tor}^{\mathcal{E}_{\text{inj}}^f}(A, K) \simeq \text{Tor}_*^{\mathcal{E}^f}(A, K)$$

so, $H_*(\mathcal{E}_q(\mathbb{k}), \phi^*(K)) \simeq \text{Tor}^{\mathcal{E}_{\text{inj}}^f}(V \mapsto \mathbb{k}[S^2(V^*)], K)$.

Sketch of the proof of Proposition 2.2.11 The functors $\mathcal{L}_i^\phi(\mathbb{k})$ transform inclusions of V into $V \perp H$ (where H is non-degenerate) into isomorphisms. By this remark we can define $\mathcal{L}_i^\phi(\mathbb{k})$ over the category of fractions where we invert the inclusions. We prove that this category of fractions is equivalent to the category of Spans $\text{Sp}(\mathcal{E}_{\text{inj}}^f)$. (So $\mathcal{L}_i^\phi(\mathbb{k})$ are non-additive Mackey functors). By a previous cancellation result due to Djament we deduce:

$$\text{Tor}_p^{\mathcal{E}_q^{\text{deg}}}(\mathcal{L}_i^\phi(\mathbb{k}), K) \simeq \text{Tor}_p^{\mathcal{E}_q^{\text{deg}}}(\mathcal{L}_i^\phi(\mathbb{k})(0), K).$$

We have $\mathcal{L}_i^\phi(\mathbb{k})(0) \simeq H_q(O_\infty(k), k) = 0$ for $q > 0$ by the result of Fiedorowicz-Priddy given in Theorem 2.1.5.

- **Unitary groups**

Let $H(S) \xrightarrow{i} H^{\text{deg}}(S) \xrightarrow{j} P(S)$ where i is the inclusion functor and j is the functor which forget the hermitian form. The difficult part of [Dja12] is the following proposition.

Proposition 2.2.12. (Cancellation result) For $K : P(S) \rightarrow \mathbb{k}\text{-Mod}$ **polynomial** we have:

$$\text{Tor}^{P(S)}(\mathcal{L}_q^{ji}(\mathbb{k}), j^*K) = 0.$$

By Proposition 2.2.7 we obtain:

$$H_*(H(S); j^*i^*(K)) \xrightarrow{\cong} H_*(H^{\text{deg}}(S), j^*K).$$

Let $\mathcal{S}^2 : P(S)^{\text{op}} \rightarrow \text{Ens}$ be the functor given by $\mathcal{S}^2(M) = \{\text{hermitian forms on } M\}$. So we have:

$$H_*(H^{\text{deg}}(S), j^*K) \simeq \text{Tor}^{P(S)}(\mathbb{k}[\mathcal{S}^2], K).$$

Two sub-examples:

- If S is a finite field we recover the result obtained in [DV10]. Djament extends this result to a commutative ring R (with the trivial involution).
- Let R be a ring. We consider the ring $S = R^{\text{op}} \times R$ with the involution $\sigma : R^{\text{op}} \times R \rightarrow R^{\text{op}} \times R$ given by $\sigma(a, b) = (b, a)$. We saw that $H(S) = S(R)$ and $U_{n,n}(R^{\text{op}} \times R) \simeq GL_{2n}(R)$. We have $\mathcal{S}^2 = \text{Hom}_{P(R)^{\text{op}}}$ so we recover Scorichenko's theorem.

2.3 Work in progress and open questions

2.3.1 Automorphisms of free groups $Aut(\mathbb{Z}^{*n})$

The computation of stable homology of automorphisms of free groups with coefficients given by a contravariant functor is still an open question. We expect that this stable homology is connected to *Tor* groups between polynomial functors from finitely generated free groups to abelian groups. So computations of such *Tor* groups can be interesting. One of the crucial key step in [DVb] is the explicit projective resolution in $Func(gr, Ab)$ of the abelianization functor. This fact is remarkable and explains why homological algebra in $Func(gr, Ab)$ is more accessible than homological algebra in $Func(ab, Ab)$. (Note that homological algebra in $Func(ab, Ab)$ and $Func(gr, Ab)$ have different behaviors. For example $Ext_{Func(ab, Ab)}^1(Id, T^2) = 0$ but $P_2 \in Ext_{Func(gr, Ab)}^1(\mathfrak{a}, T^2 \circ \mathfrak{a})$ by Example 1.2.17.) In work in progress we compute *Ext*-groups in $Func(gr, Ab)$. For example, we have:

$$Ext_{Func(gr, Ab)}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) = \begin{cases} \mathbb{Z}[Ens(m, n)] & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases}$$

In work in progress with Pirashvili [PV] we compare homological algebra in $Func(gr, Ab)$ and in $Pol_n(gr, Ab)$. For $F, G \in Pol_n(gr, Ab)$ we conjecture:

$$Ext_{Func(gr, Ab)}^*(F, G) \simeq Ext_{Pol_n(gr, Ab)}^*(F, G).$$

This result is quite surprising (remark that the analogous result for $Func(ab, Ab)$ is not true).

2.3.2 Other families of groups

We would like to apply our general method to other families of groups. For example, we are interested by the stable homology of braid groups with twisted coefficients. A typical example of twisted coefficients is given by the Burau representation. Unfortunately, the Burau representation does not enter in the usual setting of polynomial functors from a symmetric monoidal category even if this representation has clearly a polynomial behaviour. So, a first step in this work consists to introduce a suitable notion of polynomial functors for braid groups.

We are also interested in the stable homology of mapping class groups with twisted coefficients. Note that homological stability with twisted coefficients for the mapping class groups is proved in [Bol12]. In this paper the coefficients are given by functors having polynomial behavior.

Bibliography

- [AM04] Alejandro Adem and R. James Milgram. *Cohomology of finite groups*, volume 309 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.
- [Bau94] Hans Joachim Baues. Quadratic functors and metastable homotopy. *J. Pure Appl. Algebra*, 91(1-3):49–107, 1994.
- [BB04] Francis Borceux and Dominique Bourn. *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2004.
- [BDFP01] Hans-Joachim Baues, Winfried Dreckmann, Vincent Franjou, and Teimuraz Pirashvili. Foncteurs polynomiaux et foncteurs de Mackey non linéaires. *Bull. Soc. Math. France*, 129(2):237–257, 2001.
- [Bet92] Stanislaw Betley. Homology of $GL(R)$ with coefficients in a functor of finite degree. *J. Algebra*, 150(1):73–86, 1992.
- [Bet99] Stanislaw Betley. Stable K -theory of finite fields. *K-Theory*, 17(2):103–111, 1999.
- [Bet02] Stanislaw Betley. Twisted homology of symmetric groups. *Proc. Amer. Math. Soc.*, 130(12):3439–3445 (electronic), 2002.
- [BHP97] Hans Joachim Baues, Manfred Hartl, and Teimuraz Pirashvili. Quadratic categories and square rings. *J. Pure Appl. Algebra*, 122(1-2):1–40, 1997.
- [Bol12] Søren K. Boldsen. Improved homological stability for the mapping class group with integral or twisted coefficients. *Math. Z.*, 270(1-2):297–329, 2012.
- [Bor74] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.
- [BP99] Hans-Joachim Baues and Teimuraz Pirashvili. Quadratic endofunctors of the category of groups. *Adv. Math.*, 141(1):167–206, 1999.
- [CDG] Gaël Collinet, Aurélien Djament, and James Griffin. Stabilité homologique pour les groupes d'automorphismes des produits libres. International Mathematics research Notices, 2012, doi: 10.1093/imrn/rns181.
- [CEF12] Thomas Church, Jordan S. Ellenberg, and Benson Farb. FI-modules: a new approach to stability for S_n -representations. arXiv:1204.4533, 2012.

- [Dja] Aurélien Djament. Sur l'homologie des groupes de congruences à coefficients polynomiaux. En préparation.
- [Dja12] Aurélien Djament. Sur l'homologie des groupes unitaires à coefficients polynomiaux. *J. K-Theory*, 10(1):87–139, 2012.
- [Dro01] Yuriy A. Drozd. Finitely generated quadratic modules. *Manuscripta Math.*, 104(2):239–256, 2001.
- [Dro03] Yuriy A. Drozd. On cubic functors. *Comm. Algebra*, 31(3):1147–1173, 2003.
- [DVa] Aurélien Djament and Christine Vespa. De la structure des foncteurs polynomiaux sur les espaces hermitiens. Prepublication available on ArXiv (36 pages).
- [DVb] Aurélien Djament and Christine Vespa. Sur l'homologie des groupes d'automorphismes des groupes libres à coefficients polynomiaux. Accepted for publication in *Commentarii Mathematici Helvetici* (23 pages).
- [DV10] Aurélien Djament and Christine Vespa. Sur l'homologie des groupes orthogonaux et symplectiques à coefficients tordus. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(3):395–459, 2010.
- [EML54] Samuel Eilenberg and Saunders Mac Lane. On the groups $H(\Pi, n)$. II. Methods of computation. *Ann. of Math. (2)*, 60:49–139, 1954.
- [FFPS03] Vincent Franjou, Eric M. Friedlander, Teimuraz Pirashvili, and Lionel Schwartz. *Rational representations, the Steenrod algebra and functor homology*, volume 16 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2003.
- [FFSS99] Vincent Franjou, Eric M. Friedlander, Alexander Scorichenko, and Andrei Suslin. General linear and functor cohomology over finite fields. *Ann. of Math. (2)*, 150(2):663–728, 1999.
- [FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. *Trans. Amer. Math. Soc.*, 326(1):57–87, 1991.
- [FLS94] Vincent Franjou, Jean Lannes, and Lionel Schwartz. Autour de la cohomologie de Mac Lane des corps finis. *Invent. Math.*, 115(3):513–538, 1994.
- [FP78] Zbigniew Fiedorowicz and Stewart Priddy. *Homology of classical groups over finite fields and their associated infinite loop spaces*, volume 674 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.
- [FP04] Vincent Franjou and Teimuraz Pirashvili. Comparison of abelian categories recollèments. *Doc. Math.*, 9:41–56 (electronic), 2004.
- [FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.
- [Gal11] Søren Galatius. Stable homology of automorphism groups of free groups. *Ann. of Math. (2)*, 173(2):705–768, 2011.

- [HLS93] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects. *Amer. J. Math.*, 115(5):1053–1106, 1993.
- [HPV] Manfred Hartl, Teimuraz Pirashvili, and Christine Vespa. Polynomial functors from algebras over a set-operad and non-linear Mackey functors. Accepted for publication in *Int. Math. Res. Not. IMRN*.
- [HV] Manfred Hartl and Christine Vespa. Quadratic functors to semi-abelian categories. In preparation.
- [HV98] Allen Hatcher and Karen Vogtmann. Rational homology of $\text{Aut}(F_n)$. *Math. Res. Lett.*, 5(6):759–780, 1998.
- [HV04] Allen Hatcher and Karen Vogtmann. Homology stability for outer automorphism groups of free groups. *Algebr. Geom. Topol.*, 4:1253–1272, 2004.
- [HV11] Manfred Hartl and Christine Vespa. Quadratic functors on pointed categories. *Adv. Math.*, 226(5):3927–4010, 2011.
- [HVW] Allan Hatcher, Karen Vogtmann, and Natalie Wahl. Erratum to: “Homology stability for outer automorphism groups of free groups [Algebr. Geom. Topol. **4** (2004), 1253–1272 (electronic)].
- [HW05] Allen Hatcher and Nathalie Wahl. Stabilization for the automorphisms of free groups with boundaries. *Geom. Topol.*, 9:1295–1336 (electronic), 2005.
- [HW08] Allen Hatcher and Nathalie Wahl. Erratum to: “Stabilization for the automorphisms of free groups with boundaries” [Geom. Topol. **9** (2005), 1295–1336; 2174267]. *Geom. Topol.*, 12(2):639–641, 2008.
- [HW10] Allen Hatcher and Nathalie Wahl. Stabilization for mapping class groups of 3-manifolds. *Duke Math. J.*, 155(2):205–269, 2010.
- [Knu01] Kevin P. Knudson. *Homology of linear groups*, volume 193 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001.
- [Kuh94a] Nicholas J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. I. *Amer. J. Math.*, 116(2):327–360, 1994.
- [Kuh94b] Nicholas J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. II. *K-Theory*, 8(4):395–428, 1994.
- [Kuh95] Nicholas J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. III. *K-Theory*, 9(3):273–303, 1995.
- [Lei04] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.
- [LR11] Muriel Livernet and Birgit Richter. An interpretation of E_n -homology as functor homology. *Math. Z.*, 269(1-2):193–219, 2011.

- [Mik] Roman Mikhailov. On the splitting of polynomial functors. ArXiv: 1202.0586.
- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [MT78] J. Peter May and Robert Thomason. The uniqueness of infinite loop space machines. *Topology*, 17(3):205–224, 1978.
- [Nak61] Minoru Nakaoka. Homology of the infinite symmetric group. *Ann. of Math. (2)*, 73:229–257, 1961.
- [Pir88] Teimuraz Pirashvili. Polynomial functors. *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR*, 91:55–66, 1988.
- [Pir93] Teimuraz Pirashvili. Polynomial approximation of Ext and Tor groups in functor categories. *Comm. Algebra*, 21(5):1705–1719, 1993.
- [Pir00a] Teimuraz Pirashvili. Dold-Kan type theorem for Γ -groups. *Math. Ann.*, 318(2):277–298, 2000.
- [Pir00b] Teimuraz Pirashvili. Hodge decomposition for higher order Hochschild homology. *Ann. Sci. École Norm. Sup. (4)*, 33(2):151–179, 2000.
- [Pir02] Teimuraz Pirashvili. On the PROP corresponding to bialgebras. *Cah. Topol. Géom. Différ. Catég.*, 43(3):221–239, 2002.
- [PR02] Teimuraz Pirashvili and Birgit Richter. Hochschild and cyclic homology via functor homology. *K-Theory*, 25(1):39–49, 2002.
- [Put12] Andrew Putman. Stability in the homology of congruence subgroups. arXiv:1201.4876, 2012.
- [PV] Teimuraz Pirashvili and Christine Vespa. Cohomologie des foncteurs sur les groupes libres. In preparation.
- [Qui72] Daniel Quillen. On the cohomology and K -theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.
- [RW] Oscar Randal-Williams. The stable cohomology of automorphisms of free groups with coefficients in the homology representation. arXiv:1012.1433.
- [Sat06] Takao Satoh. Twisted first homology groups of the automorphism group of a free group. *J. Pure Appl. Algebra*, 204(2):334–348, 2006.
- [Sat07] Takao Satoh. Twisted second homology groups of the automorphism group of a free group. *J. Pure Appl. Algebra*, 211(2):547–565, 2007.
- [Sco] Alexander Scorichenko. Stable k -theory and functor homology over a ring. PhD-thesis, Evanston, 2000.
- [Sło04] Jolanta Słomińska. Dold-Kan type theorems and Morita equivalences of functor categories. *J. Algebra*, 274(1):118–137, 2004.

-
- [Sus83] A. Suslin. On the K -theory of algebraically closed fields. *Invent. Math.*, 73(2):241–245, 1983.
- [Tou] Antoine Touzé. Bar complexes and extensions of classical exponential functors. arXiv:1012.2724 To appear in the Annales de l’Institut Fourier.
- [Tou10] Antoine Touzé. Cohomology of classical algebraic groups from the functorial viewpoint. *Adv. Math.*, 225(1):33–68, 2010.
- [Ves07] Christine Vespa. Generic representations of orthogonal groups: the mixed functors. *Algebr. Geom. Topol.*, 7:379–410, 2007.
- [Ves08a] Christine Vespa. Generic representations of orthogonal groups: projective functors in the category $\mathcal{F}_{\text{quad}}$. *Fund. Math.*, 200(3):243–278, 2008.
- [Ves08b] Christine Vespa. Generic representations of orthogonal groups: the functor category $\mathcal{F}_{\text{quad}}$. *J. Pure Appl. Algebra*, 212(6):1472–1499, 2008.
- [Web00] Peter Webb. A guide to Mackey functors. In *Handbook of algebra, Vol. 2*, pages 805–836. North-Holland, Amsterdam, 2000.
- [Zim] Alexander Zimmermann. The ring of polynomial functors of prime degree. ArXiv: 1304.4850.

Dans les catégories de foncteurs entre catégories abéliennes, les foncteurs additifs jouent un rôle privilégié dans plusieurs domaines de l'algèbre. Cependant il existe de nombreux foncteurs très intéressants qui ne sont pas additifs. Par exemple, le produit tensoriel de groupes abéliens définit un foncteur $T^2 : Ab \rightarrow Ab$ donné par $T^2(G) = G \otimes G$ qui n'est pas additif mais polynomial de degré deux. Les foncteurs polynomiaux ont été introduits par Eilenberg et MacLane pour les foncteurs entre catégories de modules. De nombreux exemples de foncteurs polynomiaux apparaissent naturellement en topologie algébrique. En particulier, l'homologie stable de familles de groupes à coefficients donnés par des foncteurs polynomiaux peut être interprétée en termes d'homologie des foncteurs. Dans les cas favorables, cette homologie des foncteurs est accessible et fournit ainsi des calculs explicites des valeurs stables des homologies à coefficients tordus. Ce mémoire comporte deux parties. La première concerne l'étude de la structure des foncteurs polynomiaux et la seconde concerne le calcul de l'homologie stable d'une famille de groupes à coefficients donnés par un foncteur polynomial.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE
UMR 7501
 Université de Strasbourg et CNRS
 7 Rue René Descartes
 67 084 STRASBOURG CEDEX

Tél. 03 68 85 01 29
 Fax 03 68 85 03 28

www-irma.u-strasbg.fr
irma@math.unistra.fr

IRMA
 Institut de Recherche
 Mathématique Avancée

IRMA 2013/015
<http://tel.archives-ouvertes.fr/tel->



UNIVERSITÉ DE STRASBOURG

ISSN 0755-3390