Resolution of singularities in foliated spaces

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Resolution of singularities in foliated spaces

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Résumé

Considérons une variété regulière analytique M sur $\mathbb{R}$ ou $\mathbb{C}$, un faisceau d'idéaux $\mathcal{I}$ défini sur M, un diviseur à croisement normaux simples E et une distribution singulière involutive θ tangent à E. L'objectif principal de ce travail est d'obtenir une résolution des singularités du faisceau d'idéaux $\mathcal{I}$ qui préserve certaines "bonnes" propriétés de la distribution singulière θ. Ce problème est naturel dans le contexte où on doit étudier "l'interaction" d'une variété et d'un feuilletage.

Pour être plus explicite, considérons un éclatement admissible

$$\sigma : (M^{'}, E^{'}) \longrightarrow (M, E).$$

Il existe plusieurs notions de transformée de la distribution singulière θ par l'éclatement σ. On travaillera avec une transformée algébriquement bien adaptée, qu'on appelle *transformée analytique stricte* et qu'on note $\theta^{'}$. On peut désormais écrire l'éclatement de la façon suivante:

$$\sigma : (M^{'}, \theta^{'}, E^{'}) \longrightarrow (M, \theta, E)$$

Dans ce contexte, on cherche une "bonne" résolution de $\mathcal{I}$, i.e., une suite d'éclatements admissible $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$:

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M, \theta, E)$$

telle que

- Le faisceau d'idéaux $\mathcal{I}.\mathcal{O}_{M_r}$ est principal et son support est contenu dans E_r;
- La distribution singulière θ_r a les mêmes "bonnes" propriétés que θ.

La bonne propriété considérée dans ce travail s'appelle *R-monomialité*: elle est l'une des plus simples propriétés qu'on peut exiger d'une distribution singulière, après la régularité. On remarque que cette propriété est liée à l'existence d'intégrales premières monomiales pour la distribution singulière θ et, donc, est aussi reliée au problème de la monomilisation des applications et de résolution "quasi-lisse" des familles d'idéaux.

Dans la première partie, on introduit la notion de centre *θ-admissible*, qui est bien adaptée à la distribution singulière. On donne une (bonne) description géométrique de ces centres pour une distribution singulière quelconque et on démontre que les éclatements avec de tels centres préservent la propriété de R-monomialité.

Dans la deuxième partie, on démontre l'existence d'une résolution des singularités de l'idéal $\mathcal{I}$ en utilisant seulement des éclatements dont les centres sont θ-admissibles:

- Le premier résultat donne une résolution globale si le faisceau d'idéaux $\mathcal{I}$ est invariant par la distribution singulière θ (et aucune hypothèse sur la distribution singulière θ n'est demandée);
- Le deuxième résultat donne une résolution globale si la distribution singulière θ est de dimension 1 (et aucune hypothèse sur le faisceau d'idéaux $\mathcal{I}$ n'est demandée);
- Le troisième résultat donne une uniformisation locale si la distribution singulière θ est de dimension 2 (et aucune hypothèse sur le faisceau d'idéaux $\mathcal{I}$ n'est demandée).

Dans la troisième partie, on présente deux utilisations des résultats précédents. La première application concerne la résolution des singularités en famille analytique, soit pour une famille d'idéaux, soit pour une famille de champs de vecteurs. Pour la deuxième, on applique les résultats ´à un problème de système dynamique, motivé par une question de Mattei.

Avant de finir, on remarque que d'autres applications sont aussi possibles. En particulier, nous pensons que les outils développés dans cette thèse seront utiles pour traiter l'équi-résolution de Zariski et la monomialisation des applications analytiques.

Abstract

Let M be an analytic manifold over $\mathbb{R}$ or $\mathbb{C}$, $\mathcal{I}$ a coherent and everywhere non-zero ideal sheaf over M, E a reduced SNC divisor and θ an involutive singular distribution everywhere tangent to E. The main objective of this work is to obtain a resolution of singularities for the ideal sheaf $\mathcal{I}$ that preserves some "good" properties of the singular distribution θ. This problem arises naturally when we study the "interaction" between a variety and a foliation.

More precisely, let:

$$\sigma : (M^{'}, E^{'}) \longrightarrow (M, E).$$

be an admissible blowing-up. There exists several notions of transforms for a singular distribution θ. We work with an "algebraically well-adapted" one called *adapted strict analytic transform*, denoted by $\theta^{'}$. The blowing-up can now be written as:

$$\sigma : (M^{'}, \theta^{'}, E^{'}) \longrightarrow (M, \theta, E)$$

In this context, we search a "good" resolution of $\mathcal{I}$, i.e. a sequence of admissible blowings-up $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$:

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M, \theta, E)$$

such that

- The ideal sheaf $\mathcal{I}.\mathcal{O}_{M_r}$ is principal and its support is contained in E_r;
- The singular distribution θ_r have the same "good" properties of θ.

The "good" property we are mainly interested is called R-monomiallity: it is one of the simplest properties after regularity. We remark that this property is related to the existence of monomial first integrals for the singular distribution θ and, thus, is also related with the monomialisation of analytic maps and the "quasi-smooth" resolution of a family of ideal sheaves.

In a first part, we introduce the notion of θ-*admissible* center, which, intuitively speaking, is "well-adapted" to the leafs of the foliation associated to the singular distribution θ (the analog to require that a center has SNC with a divisor). We give a (good) geometrical description of these centers for general involutive singular distributions and prove that blowings-up with such centers preserve the R-monomiality property.

In a second part, we prove the existence of a resolution of singularities for the ideal sheaf $\mathcal{I}$ by blowings-up with θ-admissible centers in the following cases:

- In the first result, we give a global resolution if the ideal sheaf $\mathcal{I}$ is invariant by the singular distribution θ (and no extra hypotheses on the singular distribution θ);
- In the second result, we give a global resolution if the singular distribution θ is one dimensional (and no extra hypotheses on the ideal sheaf $\mathcal{I}$);
- In the third result, we give a local uniformization if the singular distribution θ is two dimensional (and no extra hypotheses on the ideal sheaf $\mathcal{I}$).

In a third part, we present two applications of these results. The first application deals with a resolution for analytic families, either for a family of ideal sheaves or vector fields. The second applications deals with a dynamical system problem, motivated by a question of Mattei.

To conclude, we remark that other applications seem possible. In particular, we believe that the tools developed in this thesis may be useful for dealing with the Zariski equiresolution and the monomialisation of analytic maps.

Contents

Chapter 1

Introduction

1.1 Historical overview and motivation

The interest in *resolution of singularities* dates back to 1860, when the problem of "re-solving" an algebraic curve C over the complex plane was brought to the attention of the mathematical community (see [Ha]). By "resolving" an algebraic curve, we mean some kind of process where the input is a singular curve C and the output is a regular curve C'. This process is motivated by the desire to give a local description of the curve C in the vicinity of its singularities.

Since then, many different resolution processes for an algebraic curve were proposed. We refer to Kollar's book [Ko] for a nice exposition of different methods. But the problem was destinated to have a much wider generality: it naturally motivated the same problem for surfaces and for general varieties. The resolution of algebraic varieties was obtained in 1964 by Hironaka (see the original article [Hi]; see section 3.6 below), and is a landmark on the subject. We refer to an article of Hauser [Ha] for a more complete historical overview.

A connected field of interest is the resolution of foliations. Here the goal is to give a local description of the leaves of a foliation in the vicinity of a singularity. The first result on the subject dates back to Bendixson in 1902, where he states that a resolution of a foliation by curves on the plane is possible (see [Ben]). A complete proof of this result was firstly given by Seidenberg in 1968 (see [Se]). The extension of this result for dimension three ambient spaces is much more recent: in 2004, Cano proves the result for codimension one foliations (see [Ca]); in 2007, Panazzolo proves the result for foliations by curves (see [P2]). No general result is known for arbitrary dimensions.

In applications, sometimes we are lead to combine both subjects. For example, suppose that we have an ambient space containing a variety and a foliation, but the object of study is the "interaction" between them (see chapter 8 for an example of this kind of problem). In this case, either the resolution of the variety or the resolution of the foliation should take into account the other object. Since we still do not have a general result of resolution for foliations, we may try to resolve the variety in a way that does not make the foliation "worse". This leads to the (informal) formulation of the main problem of this thesis:

Problem: Can we obtain a resolution of singularities for a variety that preserves good conditions of an ambient foliation?

We give a rigorous formulation of this problem at the end of the next section. This problem is not only natural, but also establishes a bridge between algebraic/analytic-geometry and dynamical system. Our ambition is that not only the results of this work, but also the techniques here developed, will be of interest to everyone that works with the interaction between varieties and foliations.

1.2 The main problem

A *foliated manifold* is a triple (M, θ, E) where:

- M is a smooth analytic manifold of dimension n over a field $\mathbb{K}$, where the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$;
- E is an ordered collection $E = (E^{(1)}, ..., E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on M such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings;

- θ is an involutive singular distribution defined over M and everywhere tangent to E.

We recall the basic notions of singular distributions (we follow closely [BB]). Let Der_M denote the sheaf of analytic vector fields over M, i.e. the sheaf of analytic sections of TM. A *singular distribution* is a coherent sub-sheaf θ of Der_M. A singular distribution is *involutive* if for each point p in M, the stalk $\theta_p := \theta.\mathcal{O}_p$ is closed under the Lie bracket operation. All singular distributions of this thesis are involutive unless stated otherwise.

Consider the quotient sheaf $Q = Der_M/\theta$. The *singular set* of θ is defined by the closed analytic subset $S(\theta) = \{p \in M : Q_p \text{ is not a free } \mathcal{O}_p \text{ module}\}$. A singular distribution θ is called *regular* if $S(\theta) = \emptyset$. On $M \setminus S(\theta)$ there exists a unique analytic subbundle L of $TM|_{M\setminus S(\theta)}$ such that θ is the sheaf of analytic sections of L. We assume that the dimension of the $\mathbb{K}$ vector space L_p is the same for all $p \in M \setminus S$ (this always holds if M is connected). It will be called the *leaf dimension* of θ and denoted by d. In this case θ is called an involutive *d-singular distribution* and (M, θ, E) a *d-foliated manifold*.

Given a point p in M, a *coherent set of generators* of θ_p is a set $\{X_1, ..., X_{d_p}\}$ of $d_p \geq d$ vector fields germs with representatives defined in a neighborhood U_p of p such that $\{X_1, ..., X_{d_p}\}.\mathcal{O}_q$ generates θ_q for every $q \in U_p$.

We recall that a blowing-up $\sigma : (M', E') \longrightarrow (M, E)$ is *admissible* if the center $\mathcal{C}$ is a closed and regular submanifold of M that has simple normal crossings with E (see section 3.1 or pages 137-138 of [Ko] for details).

We introduce a natural transform of θ under admissible blowing-up called *adapted analytic strict transform*. It is an involutive singular distribution θ', everywhere tangent to E', obtained as a suitable extension of the pull-back of θ from $M \setminus \mathcal{C}$ to $M' \setminus \sigma^{-1}(\mathcal{C})$. The precise definition is given in section 3.2: we stress that, in general, it is neither the strict nor the total transform of θ (see section 3.2). We denote an admissible blowing-up by:

$$\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$$

A *foliated ideal sheaf* is a quadruple $(M, \theta, \mathcal{I}, E)$ where:

- (M, θ, E) is a foliated manifold;
- $\mathcal{I}$ is a coherent and everywhere non-zero ideal sheaf of M.

The *support* of $\mathcal{I}$ is the subset:

$$V(\mathcal{I}) := \{p \in M; \mathcal{I}.\mathcal{O}_p \subset m_p\}$$

where m_p is the maximal ideal of the structural ring $\mathcal{O}_p$.

An ideal sheaf $\mathcal{I}$ is *invariant* by a singular distribution θ if $\theta[\mathcal{I}] \subset \mathcal{I}$, where θ is regarded as a set of derivations taking action over $\mathcal{I}$. An analytic sub-manifold N is *invariant* by a singular distribution θ if the reduced ideal sheaf $\mathcal{I}_N$ that generates N (i.e. $V(\mathcal{I}_N) = N$) is invariant by θ (see section 2.4 for details).

We say that an admissible blowing-up $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ is *of order one* on $(M, \theta, \mathcal{I}, E)$ if the center $\mathcal{C}$ is contained in the variety $V(\mathcal{I})$ (see section 3.3 or definition 3.65 of [Ko] for details). In this case, the *controlled transform* of the ideal sheaf $\mathcal{I}$ is the coherent and everywhere non-zero ideal sheaf $\mathcal{I}^c := \mathcal{O}(-F)(\mathcal{I}.\mathcal{O}_{M'})$, where F stands for the exceptional divisor of the blowing-up (see section 3.3 or subsection 3.58 of [Ko] for details). Finally, an admissible blowing-up of order one of the foliated ideal sheaf is the mapping:

$$\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$$

where the ideal sheaf $\mathcal{I}'$ is the controlled transform of $\mathcal{I}$.

A *resolution* of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ (see section 3.5 for details) is a sequence of admissible blowing-ups of order one:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

such that $\mathcal{I}_r = \mathcal{O}_{M_r}$. In particular, $\mathcal{I}.\mathcal{O}_{M_r}$ is the ideal sheaf of a SNC divisor on M_r with support contained in E_r.

Our main objective is to find a resolution algorithm that preserves as much as possible "good" properties that the singular distribution θ might possess. For example, one could ask if, assuming that the singular distribution θ is regular (i.e. $S(\theta) = \emptyset$), there exists a resolution of the foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ such that the singular distribution θ_r is regular. Unfortunately, it is easy to get examples of foliated ideal sheaves whose resolution necessarily breaks the regularity of a regular distribution:

Example 1.2.1. *Let $(M, \theta, \mathcal{I}, E) = (\mathbb{C}^2, \frac{\partial}{\partial x}, (x, y), \emptyset)$: the only possible strategy for a resolution is to blow up the origin, which breaks the regularity of the distribution.*

The next best thing is a (locally) *monomial* singular distribution: given a ring R such that $\mathbb{Z} \subset R \subset \mathbb{K}$, a d-singular distribution θ is *R-monomial* at $p \in M$ if there exists a local coordinate system $x = (x_1, ..., x_n)$ and a coherent set of generators $\{X_1, ..., X_d\}$ of θ_p such that:

- Either $X_i = \frac{\partial}{\partial x_i}$, or;
- $X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,j} \in R$.

A singular distribution is *R-monomial* if it is R-monomial at every point $p \in M$ (see section 2.2 for details).

The main problem of this work can now be enunciated rigorously:

Problem: Given a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ such that the singular distribution θ is R-monomial, is there a resolution of $(M, \theta, \mathcal{I}, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

such that the singular distribution θ_r is also R-monomial?

In this thesis we prove the following:

- If the ideal sheaf $\mathcal{I}$ is invariant by the singular distribution θ (i.e. $\theta[\mathcal{I}] \subset \mathcal{I}$), then there exists a resolution that preserves regularity and R-monomiality (see Theorem 5.1.1);
- If the leaf dimension of the singular distribution θ is one, then there exists a resolution that preserves R-monomiality (see Theorem 5.3.1);
- If the leaf dimension of the singular distribution θ is two, then there exists a local uniformization that preserves R-monomiality (see Theorem 6.1.1).

We are more precise in the formulation of these results in section 1.4.

1.3 Example

We give a simple example in order to illustrate the difficulty of the problem. We work over the $\mathbb{Z}$-monomial foliated ideal sheaf $(M, \theta, \mathcal{I}, E) = (\mathbb{C}^3, \theta, \mathcal{I}, \emptyset)$, where θ is a $\mathbb{Z}$-monomial singular distribution generated by the regular vector field $X = \frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$ and $\mathcal{I}$ is an ideal generated by (x, y).

On one hand if we consider the admissible blowing-up of order one $\sigma : (M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'}) \longrightarrow (M, \theta, \mathcal{I}, E)$ with center $\mathcal{C} = V(x, y)$ we obtain a resolution of $\mathcal{I}$. On the other hand, the transform of the singular distribution θ (in this case, the adapted analytic strict transform and the strict transform coincide) restricted to the x-chart is generated by the vector field:

$$X^{'} = x\frac{\partial}{\partial z} + z(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$$

which is not $\mathbb{Z}$-monomial (indeed the linear part is nilpotent). So, this naive attempt breaks $\mathbb{Z}$-monomiality. Intuitively, this happens because the center $\mathcal{C}$ is tangent to the orbit of the vector field X at the origin.

So, let $\sigma : (M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'}) \longrightarrow (M, \theta, \mathcal{I}, E)$ be the admissible blowing-up of order one with center $\mathcal{C} = V(x, y, z)$. The only interesting chart is the z-chart, where we obtain:

$$\begin{array}{ll} I^{*} = (x^{'}z^{'}, y^{'}z^{'}) & X^{*} = \frac{1}{z^{'}}(z^{'}\frac{\partial}{\partial z^{'}} - x^{'}\frac{\partial}{\partial x^{'}} - y^{'}\frac{\partial}{\partial y^{'}}) + \frac{\partial}{\partial x^{'}} \\ I^{'} = (x^{'}, y^{'}) & X^{'} = z^{'}\frac{\partial}{\partial z^{'}} + (z^{'} - x^{'})\frac{\partial}{\partial x^{'}} - y^{'}\frac{\partial}{\partial y^{'}} \end{array}$$

where $\mathcal{I}^*$ and X^* stand for the pull-back of the ideal sheaf and the vector field respectively. We claim that θ' is $\mathbb{Z}$-monomial. Indeed, if we consider the change of coordinates:

$$(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (2x' - z', y', z')$$

we obtain that the vector field X^* in these new coordinates is given by:

$$X' = \widetilde{z}\frac{\partial}{\partial \widetilde{z}} - \widetilde{x}\frac{\partial}{\partial \widetilde{x}} - \widetilde{y}\frac{\partial}{\partial \widetilde{y}}$$

Now, let $\sigma : (M'', \theta'', \mathcal{I}'', E'') \longrightarrow (M', \theta', \mathcal{I}', E')$ be the admissible blowing-up of order one with center $\mathcal{C}' = V(x', y')$. Once again, we obtain a resolution of $\mathcal{I}'$ that breaks $\mathbb{Z}$-monomiality. Indeed, the transform of the singular distribution θ' restricted to the x'-chart is generated by the vector field:

$$X'' = x''z''\frac{\partial}{\partial z''} + (z'' - x'')x''\frac{\partial}{\partial x''} - z''y''\frac{\partial}{\partial y''}$$

which is not $\mathbb{Z}$-monomial (indeed the linear part is nilpotent). So, this naive attempt breaks $\mathbb{Z}$-monomiality. Intuitively, this happens because the vector field X' is singular in the origin and transverse to the center $\mathcal{C}$ everywhere else.

So, let $\sigma : (M'', \theta'', \mathcal{I}'', E'') \longrightarrow (M', \theta', \mathcal{I}', E')$ be the admissible blowing-up of order one with center $\mathcal{C}' = V(x', y', z')$. The only interesting chart is the z'-chart, where we obtain:

$$I'' = (x'', y'') \quad X'' = z''\tfrac{\partial}{\partial z''} + (1 - 2x'')\tfrac{\partial}{\partial x''} - 2y''\tfrac{\partial}{\partial y''}$$

We leave to the reader the verification that X'' is $\mathbb{Z}$-monomial.

We finally claim that a third blowing-up with center $\mathcal{C}'' = V(x'', y'')$ gives a resolution of I'' such that θ''' is $\mathbb{Z}$-monomial. The crucial intuitive reason is that the vector field X'' is everywhere transverse to the center $\mathcal{C}''$. We leave the details to the interested reader.

1.4 Ideas and results

In Chapter 2 we define some tools to study the interaction between a singular distribution (or a foliation) and an ideal sheaves (or varieties). It is well-known that a good strategy to do so when the singular distribution is regular are *Fitting ideals* (see definition in [Te]). In section 2.3 we introduce a notion of k-generalized Fitting ideals, which coincides with the definition of Fitting ideals when the singular distribution θ is regular.

This tool allows us to tackle one of the first difficulties of the main problem: how to control the transforms of a singular distribution θ under blowing-up. We deal with this difficulty restricting the possible centers of blowing-up to *θ-admissible centers* (see section 4.1 for the precise definition).

Intuitively, a center $\mathcal{C}$ is θ-admissible at a point p in $\mathcal{C}$ if there exists a local decomposition $\theta_p = \theta_{tr} + \theta_{inv}$ (as $\mathcal{O}_p$-modules) of the singular distribution θ_p into two singular distributions $\{\theta_{tr}, \theta_{inv}\}$ such that:

- The singular distribution θ_{tr} is totally transversal to $\mathcal{C}$, i.e. no vector of $T_p\mathcal{C}$ is contained in the subspace of T_pM generated by θ_{tr};
- The singular distribution θ_{inv} is everywhere tangent to $\mathcal{C}$, i.e. $\mathcal{C}$ is invariant by θ_{inv}.

Later, we formalize this intuitive interpretation (see Proposition 4.4.1). An admissible blowing-up with a θ-admissible center is called a θ-admissible blowing-up. This notion is defined for arbitrary singular distributions, but is particularly important for R-monomial singular distributions because of the following result:

Theorem 1.4.1. *Let (M, θ, E) be a R-monomial d-foliated manifold and:*

$$\sigma : (M^{'}, \theta^{'}, E^{'}) \longrightarrow (M, \theta, E)$$

a θ-admissible blowing-up. Then $\theta^{'}$ is R-monomial.

This is proved in chapter 4, Theorem 4.1.1.

Nevertheless, the definition of θ-admissible center seems to have a much wider range of application. We believe that this kind of blowing-ups could preserve other interesting conditions of a singular distribution θ (actually, it may even be a necessary condition). For example, if θ has leaf dimension one and has only canonical singularities (see I.1.2 of [Mc] for the definition), then θ' has only canonical singularities if, and only if, the blowing-up is θ-admissible (this follows from fact I.2.8 of [Mc] and Proposition 4.4.1 below). More generally, it seems that a similar statement holds for an arbitrary d-foliated manifold.

Based on Theorem 1.4.1, the next step is to look for a resolution of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ with only θ-admissible blowings-up. To achieve this goal, we introduce a new invariant called the tg-order (abbreviation for tangency order) attached to each point p in M and denoted by $\nu_p(\theta, \mathcal{I})$ (see section 2.5 for the precise definition). This invariant gives a measure of the order of tangency between an ideal sheaf $\mathcal{I}$ and a singular distribution θ, even if one of the objects is singular.

The idea behind the invariant is a notion of *tangency chain of ideal sheaves* (see section 2.5 for the precise definition). This "chain" can be seen as a sequence of ideal sheaves $(\mathcal{I}_n)$, where the n-ideal sheaf $\mathcal{I}_n$ contains all the analytic informations about order n tangency points between the ideal sheaf $\mathcal{I}$ and the singular distribution θ. This chain gives a well-known stratification of the variety $V(\mathcal{I})$ into locally closed sub-varieties where θ is n-tangent to $V(\mathcal{I})$. But the fact that the information is analytic and not only geometric is crucial.

This invariant allows us to prove the two main results of this work, which give a θ-admissible resolution if:

- Either the ideal sheaf $\mathcal{I}$ is θ-invariant, or;
- The singular distribution θ has leaf dimension equal to one.

In order to be precise, we define the notion of *local foliated ideal sheaves* as quintuples $(M, M_0, \theta, \mathcal{I}, E)$:

- $(M, \theta, \mathcal{I}, E)$ is a foliated ideal sheaf;
- M_0 is an open relatively compact subset of M.

A *resolution* of a local foliated ideal sheaf $(M, M_0, \theta, \mathcal{I}, E)$ is a resolution of the foliated ideal sheaf $(M_0, \mathcal{I}_0, \theta_0, E_0) := (M_0, \mathcal{I}.\mathcal{O}_{M_0}, \theta.\mathcal{O}_{M_0}, E \cap M_0)$. With this notation, we present the main Theorems of this work in their simplest forms:

Theorem 1.4.2. *Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local d-foliated ideal sheaf and suppose that $\mathcal{I}_0$ is θ_0-invariant. There exists a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:*

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ is a sequence of θ-admissible blowings-up;

ii) The composition $\sigma = \sigma_1 \circ ... \circ \sigma_r$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;

iii) If θ_0 is R-monomial, then so is θ_r;

iv) If θ_0 is regular, then so is θ_r.

Theorem 1.4.3. *Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local foliated ideal sheaf and suppose that θ has leaf dimension equal to* 1*. There exists a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:*

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ is a sequence of θ-admissible blowings-up;

ii) The composition $\sigma = \sigma_1 \circ ... \circ \sigma_1$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;

iii) If θ_0 is R-monomial, then so is θ_r.

In fact Theorems 1.4.2 and 1.4.3 are corollaries of the more general Theorems 5.1.1 and 5.3.1, where we also prove the functoriality of the resolution for a certain kind of morphisms called *chain-preserving smooth morphisms* (see section 2.6 for the definition).

The study of a θ-admissible resolution when the singular distribution θ has leaf dimension bigger then one has some extra difficulties that we describe in section 5.7. Nevertheless, we can present a slightly weaker result for a singular distribution with leaf dimension equals to two. A *local uniformization* of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ at a point p of M is a finite collection of pairs $\{\tau_\alpha : M_\alpha \longrightarrow M, \theta_\alpha\}$ where:

- $\tau_\alpha : M_\alpha \longrightarrow M$ is a proper analytic morphisms;
- θ_α is a singular distributions M_α.

such that:

- The union of the images $\bigcup \tau_\alpha(M_\alpha)$ is an open neighborhood of p.
- For each morphism $\tau_\alpha : M_\alpha \longrightarrow M$ there exists a sequence of admissible local blowings-up of order one:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\tau_{r,\alpha}} \cdots \xrightarrow{\tau_{2,\alpha}} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\tau_{1,\alpha}} (M, \theta, \mathcal{I}, E)$$

  such that $\mathcal{I}_r = \mathcal{O}_{M_r}$, $\theta_\alpha = \theta_r$ and the morphism τ_α is the composition of this local blowings-up: $\tau_\alpha = \tau_{1,\alpha} \circ \ldots \circ \tau_{r,\alpha}$.

where a local blowing-up is the composition of a blowing-up with a injective local isomorphism (see section 3.4 for the precise definition on local blowings-up and section 3.5 for more details about local uniformizations). A local uniformization is θ-admissible if all local blowings-up are θ-admissible.

Accepting this weaker “resolution”, we are able to obtain the following result:

Theorem 1.4.4. *Let $(M, \theta, \mathcal{I}, E)$ be a 2-foliated ideal sheaf and p a point of M. Then, there exists a θ-admissible local uniformization of $(M, \theta, \mathcal{I}, E)$ at p. In particular, if θ is R-monomial, then θ_α is R-monomial for every α.*

This is proved in chapter 6, Theorem 6.1.1.

1.5 Applications and Open problems

1.5.1 Application 1: Resolution in Families - Chapter 7

Resolution of singularities in families (or simultaneous resolution of singularities) is a natural problem which has been considered by several authors. For instance, we could mention the following two motivations:

- ZP) The Zarisky search for a good notion of “equiresolution” (see [Z, ENV, V3] for some results on the subject);
- RP) The study of bifurcations of vector fields and the 16^o Hilbert problem (see [R] for more details on the subject).

We start by being more precise about ZP. Following the results in [ENV], we can change focus from an “equiresolution” to a resolution of a *smooth family of ideal sheaves*. In this work, we define *smooth family of ideal sheaves* to be a quadruple $(B, \Lambda, \pi, \mathcal{I})$ where:

- The ambient space B and the parameter space Λ are two smooth analytic manifolds;
- The morphism $\pi : B \longrightarrow \Lambda$ is smooth;
- The ideal sheaf $\mathcal{I}$ is coherent and everywhere non-zero over B.

Given $\lambda \in \Lambda$, the set $\pi^{-1}(\lambda)$ is a regular sub-manifold of B called *fiber*. A point $\lambda_0 \in \Lambda$ is called an *exceptional value* of a smooth family of ideal sheaves $(B, \Lambda, \pi, \mathcal{I})$ if the fiber $\pi^{-1}(\lambda_0)$ is contained in $V(\mathcal{I})$.

Strictly saying, we would like to find a *family of resolutions with respect to* π, i.e. a resolution $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ of $(B, \mathcal{I}, \emptyset)$ such that $\pi \circ \sigma$ is smooth, where $\sigma = \sigma_1 \circ ... \circ \sigma_r$. This would give the notion of equiresolution desired by Zariski (but we remark that we have fixed a family structure already).

Since this is not always possible (see example 1.2.1), one may try to find a weaker notion of resolution for a family of ideal sheaves. We work with a new one called *uniform* *resolution in families of ideal sheaves* (see section 7.1 for the definition) which was first introduced (in a different context) in [DR]. As a first step to obtain an uniform resolution, we present the following result on elimination of exceptional values:

Theorem 1.5.2. *Let $(B, \Lambda, \pi, \mathcal{I})$ be a smooth family of ideal sheaves such that all fibers are connected. Then, there exists a smooth family of ideal sheaves $(B', \Lambda', \pi', \mathcal{I}')$ and two proper analytic maps $\sigma : B' \longrightarrow B$ and $\tau : \Lambda' \longrightarrow \Lambda$ such that:*

i) The smooth family of ideal sheaves $(B', \Lambda', \pi', \mathcal{I}')$ has no exceptional value;

ii) The following diagram:

$$\begin{array}{ccc} B' & \xrightarrow{\pi'} & \Lambda' \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

commutes;

iii) For any relatively compact open subset B_0 of B, there exists a sequence of admissible blowings-up of order one for $(B_0, \mathcal{I}_0, E_0) = (B_0, \mathcal{I}.\mathcal{O}_{B_0}, \emptyset)$:

$$(B_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B_0, \mathcal{I}_0, E_0)$$

such that $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ ... \circ \sigma_r$ and $\mathcal{I}'.\mathcal{O}_{B_r} = \mathcal{I}_r$;

iv) For any relatively compact open subset Λ_0 of Λ, there exists a sequence of admissible blowings-up by $(\Lambda_0, E_0) = (\Lambda_0, \emptyset)$:

$$(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)$$

such that $\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ ... \circ \tau_r$.

This is proved in section 7.1 Theorem 7.1.1.

Remark 1.5.3. *To make the statements of the Theorem more clear, suppose that the analytic manifolds B and Λ are compact. In this case, σ is the composition of a sequence of blowing-ups $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ and τ is the composition of a sequence of blowing-ups $\vec{\tau} = (\tau_r, ..., \tau_1)$ that commutes at each step, i.e. the following diagram:*

$$\begin{array}{ccccccc}
B_r & \xrightarrow{\sigma_r} & \cdots & \xrightarrow{\sigma_2} & B_1 & \xrightarrow{\sigma_1} & B \\
\downarrow{\scriptstyle \pi_r} & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
\Lambda_r & \xrightarrow{\tau_r} & \cdots & \xrightarrow{\tau_2} & \Lambda_1 & \xrightarrow{\tau_1} & \Lambda
\end{array}$$

commutes, where $B_r = B^{'}$, $\Lambda^{'} = \Lambda_r$ and the morphisms $\pi_i : B_i \longrightarrow \Lambda_i$ are all smooth.

Now, we are more precise about RP. A *smooth family of foliations by curves* is given by a quadruple $(B, \Lambda, \pi, \mathcal{X})$ where:

- The ambient space B and the parameter space Λ are two smooth analytic manifolds;
- The morphism $\pi : B \longrightarrow \Lambda$ is smooth;
- The singular distribution $\mathcal{X}$ is:
    - Everywhere non-zero over B and $d\pi(\mathcal{X}) \equiv 0$;
    - At each point p in B, there exists a vector field X_p that generates the singular distribution $\mathcal{X}_p$.

We recall that the set $S(\mathcal{X}) := V(\mathcal{X}[\mathcal{O}_B])$ is the *singular set* of the vector field $\mathcal{X}$. A point $\lambda_0 \in \Lambda$ is called an *exceptional value* of a smooth family of vector field $(B, \Lambda, \pi, \mathcal{X})$ if the fiber $\pi^{-1}(\lambda_0)$ is contained in $S(\mathcal{X})$.

Finding a general resolution of vector fields is a very difficult problem, yet to be solved. Nevertheless, based on the Bendixson-Seidenberg result for planar vector fields, one could hope to find a resolution when $dim\Lambda = dimB - 2$. In this case, we say that the family of foliations by curves is *planar*. This problem is an essential step in the so-called

Roussarie's program (see [DR]) to prove the existential part of the 16^{th} Hilbert problem and has been solved in some cases. In particular, we mention the following:

- Denkowska and Roussarie [DR]: The authors propose a different meaning of "resolving families of foliation by curves". Their idea is to change focus from a family to the foliation associated to it. It is worth remarking that this idea motivated this thesis;

- Panazzolo [P1]: The author presents a resolution (in the sense of Denkowska and Roussarie) of a smooth family of foliation by curves $(B, \Lambda, \pi, \mathcal{X})$ when the restriction of the linear part of $\mathcal{X}$ to the leafs are non-zero;

- Trifonov [Tr]: The author presents a reduction of a smooth family of foliation by curves $(B, \Lambda, \pi, \mathcal{X})$ into another smooth family of vector field $(B', \Lambda', \pi', \mathcal{X}')$ where no "persistent" singularity exists. It is worth remarking that $(B', \Lambda', \pi', \mathcal{X}')$ may still be complicated since singular perturbations phenomenas are persistent through this reduction. Nevertheless, this is the best known reduction that preserves smoothness.

In this work, we prove a generalization of Proposition $IV.3$ of [DR], concerning elimination of exceptional values. This can be seen as a first step to find the resolution proposed in [DR]:

Theorem 1.5.4. *Let $(B, \Lambda, \pi, \mathcal{X})$ be a smooth family of foliations by curves such that all fibers are connected. Then, there exists a smooth family of foliations by curves $(B', \Lambda', \pi', \mathcal{X}')$ and two proper analytic maps $\sigma : B' \longrightarrow B$ and $\tau : \Lambda' \longrightarrow \Lambda$ such that:*

i) $(B', \Lambda', \pi', \mathcal{X}')$ has no exceptional value;

ii) The following diagram:

$$\begin{array}{ccc} B' & \xrightarrow{\pi'} & \Lambda' \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

commutes;

iii) For any relatively compact open subset B_0 *of* B*, there exists a sequence of admissible blowings-up by* $(B_0, \mathcal{X}_0, E_0) = (B_0, \mathcal{X}.\mathcal{O}_{B_0}, \emptyset)$*:*

$$(B_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (B_0, \mathcal{X}_0, E_0)$$

where $\mathcal{X}_i = \sigma_i^* \mathcal{X}_{i-1}.\mathcal{O}(-F_i)$*, such that* $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ ... \circ \sigma_r$ *and* $\mathcal{X}'.\mathcal{O}_{B_r} = \mathcal{X}_r$*;*

iv) For any relatively compact open subset Λ_0 *of* Λ*, there exists a sequence of admissible blowings-up by* $(\Lambda_0, E_0) = (\Lambda_0, \emptyset)$*:*

$$(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)$$

such that $\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ ... \circ \tau_r$.

This is proved in Section 7.1, Theorem 7.3.1.

1.5.5 Application 2: Generalized Flow-Box and a problem proposed by Mattei - Chapter 8

Mattei's Problem

The problem proposed by Mattei concerns the action of an specific algebraic Lie Group action. Consider a general Lie group G acting on an analytic manifold M:

$$\begin{array}{rccc} A: & G \times M & \longrightarrow & M \\ & (g,m) & \mapsto & g(m) \end{array}$$

and an analytic sub-variety N of M. We say that the triple (M, N, A) satisfies the *G-FB* property if, for each point p in N, there exists an open neighborhood $U_p \subset N$ of p and open neighborhood $V \subset G$ of the neutral element e of G such that:

T) For all point q in U_p, and for all g in V, the point $g(q)$ is contained in N if, and only if, $g(q) = q$.

It is clear that such a property is not always satisfied. So, Mattei suggests to introduce an extra hypothesis: For every point p of N, let $\mathcal{L}_p$ be the analytic sub-variety of M given by the orbit of G through p. We say that the triple (M, N, A) is *geometrically quasi-transverse* if:

H) The tangent space of N and $\mathcal{L}_p$ at p have trivial intersection.

Geometric intuition leads one to ask the following natural question (see figure 1.1):

- General Problem: If the triple (M, N, A) is geometrically quasi-transverse, does it satisfy the $G - FB$ property?

The original problem of Mattei deals with a more specific case where $A : G \times M \to M$ is an algebraic group action. More precisely:

- M is the space of the k-jets of 1-form germs in $(\mathbb{C}^2, 0)$, singular at the origin;
- The group G is the product $G_1 \times G_2$, where G_1 is the group of the k-jets germs of unities u of $(\mathbb{C}^2, 0)$, and G_2 is the group of k-jets of bi-holomorphic germs $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$;
- Given a point $j^k(\omega)$ of M and (u, F) of G, the action $(u, F) * (j^k(\omega))$ is given by $j^k(u(F^*\omega))$ (where j^k is the function that maps an analytic germ to its k-jet).

And then, the question can finally be formulated as follows:

- Mattei Problem: Suppose M, G and A are as above and let $N \subset M$ be an analytic regular sub-variety (not necessarily algebraic) such that the triple (M, N, A) is geometric quasi-transverse. Does the triple (M, N, A) satisfy the $G - FB$ property?

The original motivation of this question is to prove that the semi-universal equisingular unfolding of one-forms constructed by Mattei is actually universal (see Theorem 3.2.1 of [Ma]).

In this work, we are going to give a positive answer to the general question under additional hypotheses, and a counter-example for $dimM = 4$ (see section 8.5.3 and notice that the vector field is complete). Nevertheless, we stress that the Mattei problem is still open.

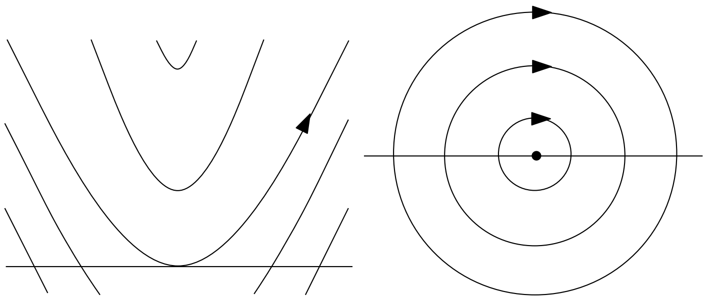

Figure 1.1: In the left, the phase space of Example 1. In the right the phase space of Example 2.

Generalized Flow-Box problem

In order to deal with the above problem, we change focus from the Lie group to the Lie algebra and we reformulate the problem in a dynamical system language. To simplify the discussion, we presently consider a one-dimensional Lie algebra. The general case is studied in chapter 8.

Let M be an analytic manifold, N a regular sub-manifold of M and X an analytic vector field over M. We say that the triple (M, N, X) satisfies the *G-FB* property (Generalized Flow-Box) if: for each point p of N, there exists a pair (U_p, δ_p), where U_p is an open neighborhood of p and $\delta_p > 0$ is a positive real number, such that the orbit $\gamma_q(t)$ of the vector field X passing through the point q of $(N \cap U_p) \setminus Sing(X)$ does not intersect N for $0 < \|t\| < \delta_p$. In section 8.3 we give a more general definition, in the context of a fixed Sub-Riemannian metric on M.

The problem is, given a triple (M, N, X), to establish a local criterium depending on the sub-variety N and the vector field X which guarantees that the $G - FB$ property holds. We give two preliminary examples to motivate:

Example 1: Consider $(M, N, X) = (\mathbb{R}^2, V(y), \frac{\partial}{\partial x} + x\frac{\partial}{\partial y})$ (see figure 1.1, left). A simple calculation shows that the $G - FB$ property is **not** satisfied at the origin. This happens because there is a tangent point between the vector field and the variety.

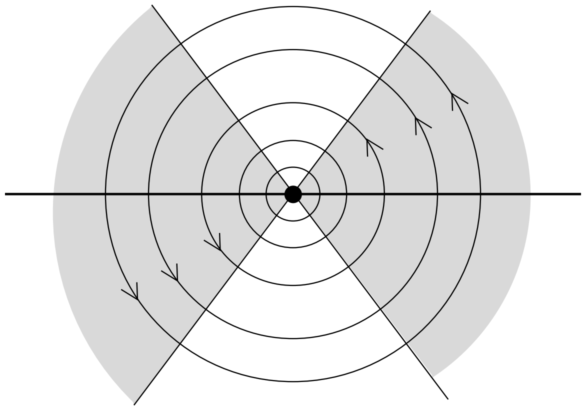

Figure 1.2: An illustration of the $G - FB$ property in Example 2 for $\delta = \frac{\pi}{4}$.

Example 2: Consider $(M, N, X) = (\mathbb{R}^2, V(y), y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$ (see figure 1.1, right). Notice that the $G - FB$ property is satisfied even at the origin, which is singular. For example, figure 1.2 shows the case of $\delta = \frac{\pi}{4}$.

So, one may conjecture that the difficulty of the problem lies in the tangency points between the variety N and the vector field X (just as for the group actions).

We say that a triple (M, N, X) is *geometrically quasi-transverse* if, at each point p in N:

$$dim_{\mathbb{K}} T_pN + dim_{\mathbb{K}} X(p) = dim_{\mathbb{K}}(T_pN + X(p))$$

where $X(p)$ is the subspace of T_pM generated by X. In section 8.1 we give a more general definition, and Lemma 8.1.2 provides the link between these two definitions. In other words, geometrically quasi-transverse triples (M, N, X) don't have points of tangency between the variety and the vector field. Following the intuition of these examples, we may ask the following question:

Question: Does geometrical quasi-transversality implies G-FB?

We answer this question with two results:

Theorem 1.5.6. *If (M, N, X) is geometrically quasi-transverse and one of the following* *conditions is satisfied:*

- *The dimension of N is one;*
- *The codimension of N is one;*
- *(M, N, X) is algebraically quasi-transverse (see definition in section 8.1).*

Then, the $G - FB$ property holds.

Remark 1.5.7. *In particular if the dimension of M is smaller or equal to 3, then geometrical quasi-transversality always implies G-FB.*

The next result shows that the additional condition of algebraic quasi-transversality cannot be dropped for $dimM \geq 4$:

Theorem 1.5.8. *For $dimM \geq 4$, there exists a geometrical quasi-transverse triple (M, N, X) that does not satisfy the $G - FB$ property.*

These Theorems are a reinterpretation of the results contained in section 8.5.

1.5.9 Further applications and Open Problems

We start presenting two objects of research where the techniques here developed could be useful:

- **Monomialization of maps**: An analytic map $\Phi : M \longrightarrow N$ is *monomial* if at every point p in M, there exists a system of coordinates $(x) = (x_1, ..., x_m)$ over $\mathcal{O}_p$ and $(y) = (y_1, ..., y_n)$ of $\mathcal{O}_{\Phi(p)}$ such that:

  $$\Phi(x) = (\Phi_1(x), ..., \Phi_n(x)) = (\prod_{j=1}^{n} x_j^{q_{1,j}}, ..., \prod_{j=1}^{n} x_j^{q_{1,j}})$$

  where the exponents $q_{i,j}$ are natural numbers such that the matrix:

  $$\begin{bmatrix} q_{1,1} & \cdots & q_{1,n} \\ \vdots & \ddots & \vdots \\ q_{n-d,1} & \cdots & q_{n-d,n} \end{bmatrix}$$

is of maximal rank. The problem is the following: given an analytic map $\Phi : M \longrightarrow N$ such that $d\Phi$ is generically of maximal rank then, up to a sequence of blowings-up in M and N, can we assume that the map $\Phi : M \longrightarrow N$ is monomial?

In other words, can we find two analytic proper morphisms $\sigma : M^{'} \longrightarrow M$ and $\tau : N^{'} \longrightarrow N$, which are compositions of blowings-up, and a monomial analytic map $\Phi^{'} : M^{'} \longrightarrow N^{'}$ such that the following diagram:

$$\begin{array}{ccc} M^{'} & \xrightarrow{\Phi^{'}} & N^{'} \\ \sigma \downarrow & & \downarrow \tau \\ M & \xrightarrow{\Phi} & N \end{array}$$

commutes?

This problem is stated by King in [Ki]. The best results, up to our knowledge, are given by:

- Cutkosky in a series of papers [Cu1, Cu2, Cu3], where he (mainly) proves two results: the monomializtion of mapping exists along a valuation (a local uniformization result) and a global monomialization of maps exists if $dimM = 3$ and $dimN = 2$;
- Dan Abramovich, Jan Denef and Kalle Karu in [ADK], where they prove that a monomialization process by "modifications", instead of blowings-up, always exists.

We stress that these results are stated in the algebraic category and for a more general class of fields of characteristic zero.

A possible strategy for tackling the problem is to find a resolution of all Fitting ideals related to the map $\Phi : M \longrightarrow N$. This does not seem to be possible, at least for the notion of resolution we gave in this work (it seems that, in general, Fitting ideals can not be monomialized). Nevertheless, we remark that Fitting ideals are related with singular distributions, and this might be a key idea for applying the present work to the problem of monomialization of maps.

- **Equiresolution**: In this part we allow ourself to be less precise. We follow the ideas from [V2, V3], even though we work in the analytic category. We refer to these two articles for details. An *idealistic ideal* is a triple $(M, \mathcal{I}, E)$ where:

  - M is a smooth analytic manifold of dimension n over a field $\mathbb{K}$, where the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$;
  - $\mathcal{I}$ is a coherent and everywhere non-zero ideal sheaf over M;
  - E is an ordered collection $E = (E^{(1)}, ..., E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on M such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings.

  Consider a smooth subvariety N of $V(\mathcal{I})$ and fix a point p in N. We say that the idealistic triple $(M, \mathcal{I}, E)$ is *equiresolvable* along N locally at p if there exists a triple $(U, \pi : U \to Y, \sigma : U^{'} \to U)$, where:

  - U is an open neighborhood of p;
  - The morphism $\pi : U \longrightarrow Y$ is smooth;
  - The morphism $\sigma : (U^{'}, E^{'}) \longrightarrow (U, E)$ is the composition of admissible blowings-up that gives a resolution of $(U, \mathcal{I}.\mathcal{O}_U, E \cap U)$.

  such that:

  - The morphism $\pi \circ \sigma : U^{'} \longrightarrow Y$ is smooth;
  - For each point q in Y, the morphism σ restricted to the fiber $U(q) = \pi^{-1}(\{q\})$ over q: $\sigma_q : U^{'}(q) \longrightarrow U(q)$ is the composition of blowings-up that resolve $(U(q), \mathcal{I}.\mathcal{O}_{U(q)}, E \cap U(q))$;
  - For any subset $\{E^{(i_1)}, ..., E^{(i_s)}\} \subset E^{'}$, if we define:

$$F(i_1, ..., i_s) = E^{(i_1)} \cap ... \cap E^{(i_s)}$$

Then either $F(i_1, ..., i_s)$ is empty or the induced morphism $F(i_1, ..., i_s) \longrightarrow Y$ is also smooth.

In [V3] the author answers an old question of Zariski about the equiresolution for the case of hypersurfaces. In particular, it gives a characterization of equiresolvable parameter spaces based on a purely geometrical notion called "equisingularity" (see [V3] for details).

In [ENV], the authors shift the focus from equiresolutions to resolution of families. Once fixed a family of idealistic ideals, they describe the necessary conditions that a resolution algorithm needs to verify, so that an equiresolution may be obtained. They finish giving a stratification of the parameter space into locally closed subsets over which equiresolution may be obtained.

We believe that using the ideas of this thesis, one could look for a principalization of ideal sheaves satisfying the condition (AE) of [ENV] (at least for some cases). This belief is motivated by the intuitive interpretation of the condition (AE) of the same article: the centers of the resolution sequence should "spread evenly" over the parameter space N. In the context of this work, consider a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ such that $d\pi(\theta) = Der_N$. If there exists a resolution of $(M, \theta, \mathcal{I}, E)$ by θ-invariant centers (which is a particular kind of θ-admissible centers - see Theorem 5.1.1 and Proposition 5.4.1 for results in this direction), then there exists a equiresolution of the family.

And now, we present four open problems that seem to be natural follow-ups of this thesis:

- **The general resolution of foliated spaces**: The main problem of this work still does not have a complete solution. Can we obtain a global resolution that preserves R-monomiality for a general d-singular distribution? If not, can we at least get a local uniformization?

- **Blowing-up foliations**: The "informal" problem presented in section 1.1 have different possible interpretations. In general:

- What kind of property of a singular distribution θ can be preserved through a resolution of an ideal sheaf?

Since we already have good results when using θ-admissible blowing-ups, we are lead to consider the properties that a θ-admissible blowing-up might preserve. For example:

- Does θ-admissible blowing-up preserves canonicity? And log-canonicity?
- Is the property of being θ-admissible necessary to preserve R-monomiality?

- **Marked ideals**: A *foliated marked ideal sheaf* is a quintuple $(M, \theta, \mathcal{I}, s, E)$ where:

  - $(M, \theta, \mathcal{I}, E)$ is a foliated ideal sheaf;
  - s is a positive integer.

  The *support* of the $(\mathcal{I}, s)$ is the subset:

$$V(\mathcal{I}, s) = \{p \in M; \mathcal{I}.\mathcal{O}_p \subset m_p^s\}$$

  where m_p is the maximal ideal of the structural ideal $\mathcal{O}_p$.

  An admissible blowing-up $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ is *of order* s by $(M, \theta, \mathcal{I}, s, E)$ if the center $\mathcal{C}$ is contained in the variety $V(\mathcal{I}, s)$.

  There exists a natural transform of $\mathcal{I}$ over admissible blowing-ups of order s called *s-controlled transform*. It is the coherent and everywhere non-zero ideal sheaf $\mathcal{I}^{c,s}$ defined as $\mathcal{I}^{c,s} = \mathcal{O}(-sF)(\mathcal{I}.\mathcal{O}_{M'})$, where F stands for the exceptional divisor of the blowing-up. We denote an admissible blowing-up of order s by:

$$\sigma : (M', \theta', \mathcal{I}', s, E') \longrightarrow (M, \theta, \mathcal{I}, s, E)$$

  where the ideal sheaf $\mathcal{I}'$ is the s-controlled transform of $\mathcal{I}$.

  A *resolution* of a foliated marked ideal sheaf $(M, \theta, \mathcal{I}, s, E)$ is a sequence of admissible blowing-ups of order s:

$$(M_r, \theta_r, \mathcal{I}_r, s, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, s, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, s, E_0)$$

such that $V(\mathcal{I}_r, s) = \emptyset$. In this case, we can formulate an analogous version of the main problem of this work for marked ideals:

Open Problem - Marked ideals: Given a foliated marked ideal sheaf $(M, \theta, \mathcal{I}, s, E)$ such that the singular distribution θ is R-monomial, is there a resolution of $(M, \theta, \mathcal{I}, s, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, s, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, s, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, s, E_0)$$

such that the singular distribution θ_r is also R-monomial?

- **Generalized Flow-Box problem for higher dimensions**: In section 8.6 we prove that a class of foliated ideal sheaves called d-algebraically transverse (see Theorem 8.6.4) satisfies the $G-FB$ property. But this property is far from being easy to verify. Either a better definition, or a better characterization of the definition is necessary, and remains an open problem.

Chapter 2

Relations between Foliations and Varieties

2.1 Main Objects

We stress that all objects of this work are analytic. We start with a list of the main objects of this work:

A *manifold with divisor* is a pair (M, E):

- M is a smooth analytic manifold of dimension n over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$);
- E is an ordered collection $E = (E^{(1)}, ..., E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on M such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings.

A *foliated manifold* is a triple (M, θ, E):

- (M, E) is an analytic manifold with divisor;
- θ is an involutive singular distribution defined over M and everywhere tangent to E.

A *foliated ideal sheaf* is a quadruple $(M, \theta, \mathcal{I}, E)$:

- (M, θ, E) is a foliated manifold;

- $\mathcal{I}$ is a coherent and everywhere non-zero ideal sheaf over M.

A *local foliated manifold* is a quadruple (M, M_0, θ, E):

- (M, θ, E) is a foliated manifold;
- M_0 is an open relatively compact subset of M.

We recall the basic notions of singular distributions (we follow closely [BB]). Let Der_M denote the sheaf of analytic vector fields over M, i.e. the sheaf of analytic sections of TM. A *singular distribution* is a coherent sub-sheaf θ of Der_M. A singular distribution is *involutive* if for each point p in M, the stalk $\theta_p := \theta.\mathcal{O}_p$ is closed under the Lie bracket operation.

Consider the quotient sheaf $Q = Der_M/\theta$. The *singular set* of θ is defined by the closed analytic subset $S = \{p \in M : Q_p \text{ is not a free } \mathcal{O}_p \text{ module}\}$. A singular distribution θ is called *regular* if $S = \emptyset$. On $M \setminus S$ there exists a unique analytic subbundle L of $TM|_{M\setminus S}$ such that θ is the sheaf of analytic sections of L. We assume that the dimension of the $\mathbb{K}$ vector space L_p is the same for all points p in $M \setminus S$ (this always holds if M is connected). It will be called the *leaf dimension* of θ and denoted by d. In this case θ is called an involutive *d-singular distribution* and (M, θ, E) a *d-foliated manifold.*

A *coherent set of generators* of θ_p is a set $\{X_1, ..., X_{d_p}\}$ of $d_p \geq d$ vector fields germs with representatives defined in a neighborhood U_p of p such that $\{X_1, ..., X_{d_p}\}.\mathcal{O}_q$ generates θ_q for every $q \in U_p$.

According to the Stefan-Sussmann Theorem (see [St, Su]) an involutive singular distribution θ is *integrable*, i.e. for all point p in M, there exists an immersed locally closed submanifold (N, ϕ) passing through p such that:

- $D_q\phi(T_qN) = L_{\phi(q)}$ for all $q \in N$

where $L_q \subset TM_q$ is the linear subspace generated by θ_q. The maximal connected submanifolds with respect to this property are called *leaves* and denoted by $\mathcal{L}$. The partition of M into leaves is called the *singular foliation* generated by θ (not necessarily saturated).

A singular foliation should be seen as a geometrical counterpart of a singular distribution, just as a variety is a geometrical counterpart of an ideal sheaf. We remark that two different singular distributions may generate the same singular foliation. We say that θ is a *full* involutive singular distribution if, for every involutive singular distribution ω that generates the same singular foliation as θ, the singular distribution ω is a sub-sheaf of θ.

2.2 The R-monomial singular distribution

Given a ring R such that $\mathbb{Z} \subset R \subset \mathbb{K}$ and a point p in M, we say that a d-singular distribution θ is *R-monomial* at p if there exists a local coordinate system $x = (x_1, ..., x_n)$ and a coherent set of generators $\{X_1, ..., X_d\}$ of θ_p such that:

- Either $X_i = \frac{\partial}{\partial x_i}$, or;
- $X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,j} \in R$.

In this case, we say that $x = (x_1, ..., x_n)$ is a *R-monomial coordinate system* and $\{X_1, ..., X_d\}$ is a *R-monomial basis* of θ_p. A singular distribution is *R-monomial* if it is R-monomial in all points. A foliated manifold (M, θ, E) (respectively a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$) is R-monomial if θ is R-monomial.

Lemma 2.2.1. *The R-monomiality is an open condition i.e. if θ is R-monomial at p in M, then there exists an open neighborhood U of p such that θ is R-monomial at every point q in U.*

Examples:

- Any regular distribution is a $\mathbb{Z}$-monomial singular distribution;
- We say that a d-singular distribution θ is *R-monomially integrable* at p if there exists a local coordinate system $x = (x_1, ..., x_n)$ and $n - d$ monomial functions $\lambda_i = \prod_{j=1}^{n} x_j^{q_{i,j}}$ for $1 \leq i \leq n - d$ with exponents $q_{i,j} \in R$ such that:

- Each λ_i is a first integral for all vector fields contained in θ_p, and;
- The matrix:

$$(q_{i,j}) := \begin{bmatrix} q_{1,1} & \dots & q_{1,n} \\ \vdots & \ddots & \vdots \\ q_{n-d,1} & \dots & q_{n-d,n} \end{bmatrix}$$

is of maximal rank.

Lemma 2.2.2. *Given a singular distribution θ:*

I) If it is full and R-monomially integrable, then it is R-monomial;

II) If it is R-monomial, then it is R-monomially integrable.

In particular, we say that θ_p is *meromorphically* (respectively *Darboux*) *monomially integrable* if $R = \mathbb{Z}$ (respectively $R = \mathbb{R}$).

Now, we prove the above Lemmas:

Proof. (Lemma 2.2.1): Let θ be a R-monomial d-singular distribution over $p \in M$. There exists an open set $U \subset M$ containing p, a R-monomial coordinate system $x = (x_1, ..., x_n)$ defined over U and a R-monomial basis $\{X_1, ..., X_d\}$ such that X_i is defined over U for all $i \leq d$. We claim that θ is R-monomial at every point $q \in U$.

Fix $q \in U$. There exists $\xi = (\xi_1, ..., \xi_n) \in \mathbb{K}^n$ such that $q = \xi$ in the coordinate system $x = (x_1, ..., x_n)$.

First, suppose that all vector fields:

$$X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$$

are singular at p. Without loss of generality, suppose that $\xi = (\xi_1, ..., \xi_t, 0, ..., 0)$, where $\xi_i \neq 0$ for all $i \leq t$. Consider the matrix:

$$A = \begin{bmatrix} \alpha_{1,1} & \dots & \alpha_{1,t} \\ \vdots & \ddots & \vdots \\ \alpha_{d,1} & \dots & \alpha_{d,t} \end{bmatrix}$$

and let s be its rank. Without loss of generality, we assume that:

$$A = \begin{bmatrix} D & B \\ 0 & 0 \end{bmatrix}$$

where D is a $s \times s$-diagonal matrix, B is a $s \times (d-s)$-matrix and both matrices have only elements in R. This implies that:

- $X_i = \alpha_{i,i} x_i \frac{\partial}{\partial x_i} + \sum_{j=s}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,i} \neq 0$ for all $i \leq s$;
- $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ for all $i > s$.

and all $\alpha_{i,j} \in R$. Now, taking the change of coordinates $(y_1, ..., y_n) = (x_1 - \xi_1, ..., x_n - \xi_n)$ we obtain:

- $X_i = \alpha_{i,i}(y_i + \xi_i) \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j}(y_j + \xi_j) \frac{\partial}{\partial y_j} + \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$ for all $i \leq s$;
- $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$ for all $i > s$.

And $q = (0, ..., 0)$ at this coordinate system. We proceed with three coordinate changes:

- First change: let $y_i = \xi_i(-1 + \exp(\alpha_{i,i}\bar{y}_i))$ for all $i \leq s$ and $y_i = \bar{y}_i$ otherwise. One can easily check that this is bi-analytic in an open neighborhood of the origin and that:

  $$\frac{\partial}{\partial \bar{y}_i} = \alpha_{i,i}(y_i + \xi_i) \frac{\partial}{\partial y_i}$$

  for all $i < s$. This implies that:

  - For $i \leq s$, we have that $X_i = \frac{\partial}{\partial \bar{y}_i} + \sum_{j=s}^{t} \alpha_{i,j}(\bar{y}_j + \xi_j) \frac{\partial}{\partial \bar{y}_j} + \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}$;
  - For $i > s$, we have that $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial \bar{y}_j}$.

  In what follows, we drop the bars;

- Second change: let $y_i = -\xi_i + (\bar{y}_i + \xi_i) \exp(\sum_{j=1}^{s} \alpha_{j,i} \bar{y}_j)$ if $s < i \leq t$ and $\bar{y}_i = y_i$ otherwise. One can easily check that this is bi-analytic in an open neighborhood of the origin and that:

  $$\frac{\partial}{\partial \bar{y}_i} = \frac{\partial}{\partial y_i} + \sum_{j=s}^{t} \alpha_{i,j}(y_j + \xi_j) \frac{\partial}{\partial y_j}$$

  for all $i < s$. This implies that:

- For $i \leq s$, we have that $X_i = \frac{\partial}{\partial \bar{y}_i} + \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y} \frac{\partial}{\partial \bar{y}_j}$;
- For $i > s$, we have that $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial y_j}$.

In what follows, we drop the bars again;

- Third change: let $y_i = \bar{y}_i \exp(\sum_{j=1}^{s} \alpha_{j,i} \bar{y}_j)$ if $i > t$ and $\bar{y}_i = y_i$ otherwise. One can easily check that this is bi-analytic in an open neighborhood of the origin and that:

$$\frac{\partial}{\partial \bar{y}_i} = \frac{\partial}{\partial y_i} + \sum_{j=t+1}^{n} \alpha_{i,j} y_j \frac{\partial}{\partial y_j}$$

for all $i < s$ and

$$\bar{y}_i \frac{\partial}{\partial \bar{y}_i} = y_1 \frac{\partial}{\partial y_i}$$

for $i > t$. This implies that:

- For $i \leq s$, we have that $X_i = \frac{\partial}{\partial \bar{y}_i}$;
- For $i > s$, we have that $X_i = \sum_{j=t+1}^{n} \alpha_{i,j} \bar{y}_j \frac{\partial}{\partial y_j}$.

which forms a R-monomial basis.

Now, suppose that for $i \leq r$, the vector field X_i is non-singular at p. Without loss of generality, $X_i = \frac{\partial}{\partial x_i}$ and $X_j(x_i) \equiv 0$ whenever $i \leq r$ and $j > r$. In particular, when we make the translation $(y_1, ..., y_n) = (x_1 - \xi_1, ..., x_n - \xi_n)$, we have that $X_i = \frac{\partial}{\partial y_i}$ for $i \leq r$.

Consider the quotient $\mathcal{O}_U/(x_1, ..., x_r)$. It is another regular ring with a R-monomial singular distribution $\{\bar{X}_{r+1}, ..., \bar{X}_t\}$ that is all singular over the origin. Using the first part of the proof, there exists a change of coordinates in $\mathcal{O}_q/(x_1, ..., x_r)$ that turns $\{\bar{X}_{r+1}, ..., \bar{X}_t\}$ into a R-monomial basis. Moreover, this coordinate change is invariant by the first r-coordinates. Taking the equivalent change in $\mathcal{O}_q$, we conclude the Lemma. □

Proof. (Lemma 2.2.2)

I) Consider a vector field $X = \sum_{i=1}^{n} a_i x_i \frac{\partial}{\partial x_i}$ locally defined in p. Since θ_p is full and R-monomially integrable, we have that $X \in \theta_p$ if, and only if, it satisfies the following system of equations:

$$\left\{\begin{array}{c} X(\lambda_1)=0 \\ \vdots \\ X(\lambda_{n-d})=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} \sum_{i=1}^n a_i q_{1,i}=0 \\ \vdots \\ \sum_{i=1}^n a_i q_{n-d,i}=0 \end{array}\right.$$

Since the matrix of the exponents $(q_{i,j})$ is of maximal rank, the solutions of this system forms a subspace $L \subset \mathbb{K}^n$ of dimension d. Take a generator set $(A_1, ..., A_d)$ of L, where $A_k = (a_{k,1}, ..., a_{k,n}) \in R^n$, and consider the associated vector fields $X_k = \sum_{i=1}^n a_{k,i} x_i \frac{\partial}{\partial x_i}$. Clearly X_k is contained in θ_p. Apart from a Gram-Schmidt process and a change in the coordinate systems, we can assume that:

$$X_k = a_{k,k} x_k \frac{\partial}{\partial x_k} + \sum_{i=d+1}^{n} a_{k,i} x_i \frac{\partial}{\partial x_i}$$

where $a_{k,k} \neq 0$, for all $k \leq d$. If $a_{k,i} = 0$ for all $d+1 \leq i \leq n$, then instead of $X_k = \alpha_{k,k} x_k \frac{\partial}{\partial x_k}$, consider the vector field $X_k = \frac{\partial}{\partial x_k}$. After this process, we claim that $\{X_1, ..., X_d\}$ generates θ_p.

Indeed, let $X = \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i}$ be an arbitrary vector field locally defined in p such that:

$$\left\{\begin{array}{c} X(\lambda_1)=0 \\ \vdots \\ X(\lambda_{n-d})=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} \sum_{i=1}^n \frac{\alpha_i(x)}{x_i} q_{1,i}=0 \\ \vdots \\ \sum_{i=1}^n \frac{\alpha_i(x)}{x_i} q_{n-d,i}=0 \end{array}\right.$$

This implies that, either $\frac{\alpha_i(x)}{x_i}$ is analytic or $q_{j,i} = 0$ for all $1 \leq j \leq n-d$. We remark that, by construction, if $q_{j,i} = 0$ for all $1 \leq j \leq n-d$, then $\frac{\partial}{\partial x_i} \in \{X_1, ..., X_d\}$. So, without loss of generality, we assume that $\bar{\alpha}_i(x) := \frac{\alpha_i(x)}{x_i}$ is analytic for all i, which implies that $X = \sum_{i=1}^n \bar{\alpha}_i(x) x_i \frac{\partial}{\partial x_i}$. Thus, clearly X must be a $\mathcal{O}_p$-linear combination of $\{X_1, ..., X_d\}$.

II) Take a coherent set of generators $\{X_1, ..., X_d\}$ of θ_p such that:

- Either $X_i = \frac{\partial}{\partial x_i}$ and we set $a_{i,i} = 1$, $a_{i,j} = 0$ otherwise, or;
- $X_i = \sum_{j=1}^n \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$ with $\alpha_{i,j} \in R$.

and consider an arbitrary monomial $\lambda = \prod_{k=1}^{n} x_k^{q_k}$. This monomial is a first integral of θ_p if, and only if, it satisfies the following system of equations:

$$\begin{cases} X_1(\lambda) = 0 \\ \vdots \\ X_d(\lambda) = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^{n} a_{1,i} q_i = 0 \\ \vdots \\ \sum_{i=1}^{n} a_{d,i} q_i = 0 \end{cases}$$

Since the matrix $(a_{i,j})$ is of maximal rank, the solutions of this system is a subspace $L \subset \mathbb{K}^n$ of dimension $n - d$. So, take a generator set $(Q_1, ..., Q_{n-d})$ of L, where $Q_k = (q_{k,1}, ..., q_{k,n}) \in R^n$, and consider the associated monomial $\lambda_k = \prod_{i=1}^{n} x_i^{q_{k,i}}$. By construction, λ_k are the searched first integrals.

□

2.3 Generalized k-Fitting Opperation

Let (M, θ, E) be a foliated manifold. The *generalized k-Fitting operation* (for $k \leq d$) is a mapping $\Gamma_{\theta,k}$ that associates to each coherent ideal sheaf $\mathcal{I}$ over M the ideal sheaf $\Gamma_{\theta,k}(\mathcal{I})$ whose stalk at each point p in M is given by:

$$\Gamma_{\theta,k}(\mathcal{I}).\mathcal{O}_p = < \{det[X_i(f_j)]_{i,j \leq k};\ X_i \in \theta_p, f_j \in \mathcal{I}.\mathcal{O}_p\} >$$

where $< S >$ stands for the ideal generated by the subset $S \subset \mathcal{O}_p$. The operation $\Gamma_{\theta,1}$ will play an important role in this work and, for simplifying the notation, we denote it by $\theta[\mathcal{I}]$.

Remark 2.3.1. *If $\mathcal{I}$ is a coherent ideal sheaf, then $\Gamma_{\theta,k}(\mathcal{I})$ is also coherent for every $k \leq d$. This follows from the coherence of the singular distribution θ.*

Remark 2.3.2. *In this work, we mainly use the ideal sheaf $< \Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I} >$. In particular, we notice that if $\theta = Der_M$ then the ideal sheaf $< \Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I} >$ coincides with the usual k-Fitting ideal sheaf (see [Te]).*

Remark 2.3.3. *If $\theta = Der_M$, the generalized 1-Fitting ideal sheaf coincides with the derivative ideal (see chapter 3.7 of [Ko] for details on derivative ideal sheaves).*

Lemma 2.3.4. *A d-singular distribution θ is regular at a point p in M if, and only if, $< \Gamma_{\theta,d}(m_p) + m_p >= \mathcal{O}_M$, where m_p stands for the maximal ideal of the structural ideal $\mathcal{O}_p$.*

Proof. First suppose that θ is a regular distribution in a point p of M. In this case, there exists a coordinate system $x = (x_1, ..., x_n)$ of $\mathcal{O}_p$ and a coherent set of generators $\{X_1, ..., X_d\}$ of θ_p which, by the flow-box Theorem, can be assumed to be equal to $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d}\}$. Now, it is clear that the determinant of the matrix:

$$\left\| \begin{matrix} X_1(x_1) & ... & X_1(x_d) \\ \vdots & \ddots & \vdots \\ X_d(x_1) & ... & X_d(x_d) \end{matrix} \right\|$$

is one. Thus, $\Gamma_{\theta,d}(m_p).\mathcal{O}_p$ is equal to $\mathcal{O}_p$, which implies that $< \Gamma_{\theta,d}(m_p) + m_p >$ is equal to $\mathcal{O}_M$.

Now, suppose that $< \Gamma_{\theta,d}(m_p) + m_p >$ is equal to $\mathcal{O}_M$. This implies that $\Gamma_{\theta,d}(m_p).\mathcal{O}_p$ is equal to $\mathcal{O}_p$. So, there exists a coherent set of generators $\{X_1, ..., X_{d_p}\}$ of θ_p and a collection of functions $\{f_1, ..., f_d\} \subset m_p$ such that the determinant of the matrix:

$$\left\| \begin{matrix} X_1(f_1) & ... & X_1(f_d) \\ \vdots & \ddots & \vdots \\ X_d(f_1) & ... & X_d(f_d) \end{matrix} \right\|$$

is an unity of $\mathcal{O}_p$. In particular, this implies that the vector fields $\{X_1, ..., X_d\}$ are regular and generates linearly independent vectors of T_pM. Since the leaf-dimension of θ is d, we conclude that d_p may be taken equal to d and the singular distribution θ is regular. □

Given a coherent ideal sheaf $\mathcal{I}$, we say that:

- $\mathcal{I}$ is *invariant* by θ or *θ-invariant* if $\theta[\mathcal{I}] \subset \mathcal{I}$;
- $\mathcal{I}$ is *totally transverse* to θ or *θ-totally transverse* if $\Gamma_{\theta,d}(\mathcal{I}) = \mathcal{O}_M$.

The *θ-differential closure* of $\mathcal{I}$ is the smallest θ-invariant ideal sheaf $\mathcal{I}_\#$ containing $\mathcal{I}$.

Remark 2.3.5. *The existence of the θ-differential closure $\mathcal{I}_\#$ is a consequence of the Zorn Lemma.*

2.4 Geometric invariance

Consider (M, θ, E) a foliated manifold and $\mathcal{I}$ a coherent ideal sheaf. We say that $\mathcal{I}$ is *geometrically invariant* by θ if every leaf of θ that intersects $V(\mathcal{I})$ is totally contained in $V(\mathcal{I})$.

This definition corresponds to the geometrical intuition of what invariance by a foliation means. But it does not corresponds to the notion of θ-invariance that we have defined:

Example: Consider $(M, \theta, E) = (\mathbb{K}^2, \frac{\partial}{\partial x}, \emptyset)$ and $\mathcal{I} = (yx, y^2)$. Notice that $\mathcal{I}$ is **not** invariant by θ, since $\theta[\mathcal{I}] = (y)$. But $\mathcal{I}$ is geometrically invariant by θ because $V(\mathcal{I}) = \{y = 0\}$ is a leaf of θ.

The following result gives the relation between these two notions of invariance:

Lemma 2.4.1. *Let θ be an involutive d-singular distribution and $\mathcal{I}$ a coherent ideal sheaf.*

- *I) If $\mathcal{I}$ is an ideal sheaf θ-invariant, then $\mathcal{I}$ is geometrically invariant by θ;*
- *II) If $\mathcal{I}$ is a reduced ideal sheaf geometrically invariant by θ, then $\mathcal{I}$ is θ-invariant.*

Now, consider N a sub-variety of M. We denote by $\mathcal{I}_N$ the reduced ideal sheaf over M such that $V(\mathcal{I}_N) = N$. We say that:

- N is *invariant* by θ if $\mathcal{I}_N$ is invariant by θ;
- N is *geometrically invariant* by θ if $\mathcal{I}_N$ is geometrically invariant by θ.

Remark 2.4.2. *Since $\mathcal{I}_N$ is a reduced ideal sheaf, by Lemma* 2.4.1*, the two definitions always coincide for sub-varieties.*

Now we prove the Lemma of this section:

Proof. (Lemma 2.4.1): We start supposing that θ is a 1-singular distribution. Take a point p in $V(\mathcal{I})$ and let $\mathcal{L}$ be the leaf of θ through p (recall that $\mathcal{L}$ is a sub-manifold of M).

- I): If $\mathcal{L}$ is zero dimensional then it is clear that $\mathcal{L} \subset V(\mathcal{I})$, so we assume that $\mathcal{L}$ is one dimensional. In this case, for each point q in $\mathcal{L} \cap V(\mathcal{I})$, the singular distrbution θ_q is generated by a regular vector field X_q and, by Lemma 4.2.4, there exists a system of generators $\{f_1, ..., f_s\}$ of $\mathcal{I}.\mathcal{O}_q$ such that $X_q(f_i) \equiv 0$. This implies the existence of an open neighborhood U_q of q and a local coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ over U_q, such that $X_q = \frac{\partial}{\partial x}$ and $\mathcal{I} = (f_1(y), ..., f_r(y))$. Thus $(\mathcal{L} \cap U_q) \cap V(\mathcal{I}) = \mathcal{L} \cap U_q$, and, since the choice of q in $\mathcal{L}$ was arbitrary, $\mathcal{L} \cap V(\mathcal{I})$ is an open subset of $\mathcal{L}$. Furthermore, since $\mathcal{L}$ is locally closed and $V(\mathcal{I})$ is closed, $\mathcal{L} \cap V(\mathcal{I})$ is a closed subset of $\mathcal{L}$. Thus $\mathcal{L} \subset V(\mathcal{I})$.

- II): We claim that $V(\mathcal{I}) \subset V(\theta[\mathcal{I}])$. The claim implies the result because:

$$\theta[\mathcal{I}] \subset \sqrt{\theta[\mathcal{I}]} \subset \sqrt{\mathcal{I}} = \mathcal{I}$$

  So, take $p \in V(\mathcal{I})$ and let $\mathcal{L}$ be the leaf of θ passing through p. If $\mathcal{L}$ is zero dimensional, then all vector fields germs of θ_p are singular and it is clear that $p \in V(\theta[\mathcal{I}])$, so we assume that $\mathcal{L}$ is one dimensional. In this case θ_p is generated by a regular vector field X_p. Consider $f \in \mathcal{I}.\mathcal{O}_p$: by hypotheses $f|_{\mathcal{L}} \equiv 0$, which implies that $X_p(f)|_{\mathcal{L}} = X_p(f|_{\mathcal{L}}) \equiv 0$. Since the choice of $f \in \mathcal{I}.\mathcal{O}_p$ is arbitrarily, $p \in V(\theta[\mathcal{I}])$.

Now, we prove the result for θ an involutive d-singular distribution. Take a point p in $V(\mathcal{I})$ and let $\mathcal{L}$ be the leaf of θ through p and $\{X_1, ..., X_{d_p}\}$ be a set of coherent generators of θ in a small neighborhood U_p of p.

- I): For a sufficiently small neighborhood U_p of p, every point q in $U_p \cap \mathcal{L}$ is the image of the flow $(Fl^{X_1}_{t_1} \circ ... \circ Fl^{X_{d_p}}_{t_{d_p}})(p) = q$ for some $(t_1, ..., t_{d_p}) \in \mathbb{K}^{d_p}$, where $Fl^X_t(p)$ is the flow of the vector field X at time t and with initial point p (see Lemma 3.24 of [Mi]). Since $X_i(\mathcal{I}.\mathcal{O}_{U_p}) \subset \mathcal{I}.\mathcal{O}_{U_p}$ by hypotheses, by the first part of the proof $Fl^{X_i}_t(p) \in V(\mathcal{I})$ for any t. A recursive use of this argument implies that $q \in V(\mathcal{I})$. Thus, $V(\mathcal{I}) \cap \mathcal{L}$ is open in $\mathcal{L}$. Furthermore, since $\mathcal{L}$ is locally closed and $V(\mathcal{I})$ is closed, $\mathcal{L} \cap V(\mathcal{I})$ is a closed subset of $\mathcal{L}$. Thus $\mathcal{L} \subset V(\mathcal{I})$;

- II): Take any vector field X in θ_p and let γ be the orbit of X at p. Since $\mathcal{L} \subset V(\mathcal{I})$, it is clear that $\gamma \subset V(\mathcal{I})$ and, by the first part of the proof, $X(\mathcal{I}.\mathcal{O}_p) \subset \mathcal{I}.\mathcal{O}_p$. Since the choice of the point and vector field is arbitrarily, we conclude that $\theta[\mathcal{I}] \subset \mathcal{I}$.

□

2.5 Chain of Ideal sheaves

A chain of ideal sheaves consists of a sequence $(\mathcal{I}_i)_{i\in\mathbb{N}}$ such that:

- $\mathcal{I}_i$ is an ideal sheaf over $\mathcal{O}_M$;
- $\mathcal{I}_i \subset \mathcal{I}_j$ if $i \leq j$.

The *length* of a chain of ideal sheaves at a point p of M is the minimal number $\nu_p \in \mathbb{N}$ such that $\mathcal{I}_i.\mathcal{O}_p = \mathcal{I}_{\nu_p}.\mathcal{O}_p$ for all $i \geq \nu_p$. We distinguish two cases:

- if $\mathcal{I}_{\nu_p}.\mathcal{O}_p = \mathcal{O}_p$, then the chain is said to be of type 1 at p;
- if $\mathcal{I}_{\nu_p}.\mathcal{O}_p \neq \mathcal{O}_p$, then the chain is said to be of type 2 at p.

Given a chain of ideal sheaf $(\mathcal{I}_n)$, it is not difficult to see that the functions:

$$\begin{array}{rccc} \nu: & M & \longrightarrow & \mathbb{N} \\ & p & \mapsto & \nu_p \end{array} \quad , \quad \begin{array}{rccl} type: & M & \longrightarrow & \{1,2\} \\ & p & \mapsto & type_p = \text{ type of } (\mathcal{I}_n) \text{ at } p \end{array}$$

are upper semi-continuous. So, given a subset U of M, the definition of length and type naturally extends to U as follows:

- The length of $(\mathcal{I}_n)$ at U is $\nu_U := sup\{\nu_p; p \in U\}$;
- The type of $(\mathcal{I}_n)$ at U is $type_U := sup\{type_p; p \in U\}$.

Notice that ν_U may be infinity. Nevertheless, if U is a relatively compact open subset of M, ν_U is necessarily finite.

Given a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$, the *tangency chain* of the pair $(\theta, \mathcal{I})$ is defined as the following chain of ideal sheaves:

$$\mathcal{T}g(\theta, \mathcal{I}) = \{H(\theta, \mathcal{I}, i); i \in \mathbb{N}\}$$

where the ideal sheaves $H(\theta, \mathcal{I}, i)$ are given by;

$$\begin{cases} H(\theta, \mathcal{I}, 0) := \mathcal{I} \\ H(\theta, \mathcal{I}, i+1) := H(\theta, \mathcal{I}, i) + \theta[H(\theta, \mathcal{I}, i)] \end{cases}$$

At each $p \in M$, the length of this chain is called the tangent order (or shortly, the *tg-order*) at p, and is denoted by $\nu_p(\theta, \mathcal{I})$. The type of the chain is denoted by $type_p(\theta, \mathcal{I})$.

Remark 2.5.1. *Suppose that θ is generated by a regular vector field X and let γ_p be the orbit of X passing through a point p of $V(\mathcal{I})$. In this simple case, we can interpret these invariants as follow:*

- *If the orbit γ_p is contained in the variety $V(\mathcal{I})$, then the type of $(\theta, \mathcal{I})$ at p is two;*
- *If the orbit γ_p is **not** contained in $V(\mathcal{I})$, then the type of $(\theta, \mathcal{I})$ at p is one. Furthermore, the tg-order of $(\theta, \mathcal{I})$ is equal to the order of tangency between the orbit γ_p and the variety $V(\mathcal{I})$ at p.*

In other words, the type identifies the presence of invariant leaves and the tg-order measures the order of tangency between the a leaves and the variety $V(\mathcal{I})$.

2.6 Smooth morphism and Chain-preserving smooth morphism

A morphism $\phi : M \longrightarrow N$ between regular analytic manifold is *smooth* if, and only if, it is a local submersion. In particular, a projection is smooth.

Remark 2.6.1. *In the algebraic category, a morphism $\phi : X \longrightarrow Y$ between two schemes is said to be smooth if:*

- *it is locally of finite type;*
- *it is flat;*
- *for every geometric point $\bar{y} \longrightarrow Y$ the fiber $X_{\bar{y}} = X \times_Y \bar{y}$ is regular.*

Given two foliated ideal sheaves $(M, \theta, \mathcal{I}, E_M)$ and $(N, \omega, \mathcal{J}, E_N)$, a morphism $\phi : M \longrightarrow N$ is *smooth with respect to* $(M, \theta, \mathcal{I}, E_M)$ *and* $(N, \omega, \mathcal{J}, E_N)$ if:

- The morphism $\phi : M \longrightarrow N$ is smooth;
- The set $\phi^{-1}(E_N)$ is equal to E_M;
- The ideal sheaf $\mathcal{J}.\mathcal{O}_M$ is equal to $\mathcal{I}$.

In this case, we abuse notation and denote the morphism as:

$$\phi : (M, \theta, \mathcal{I}, E_M) \longrightarrow (N, \omega, \mathcal{J}, E_N)$$

Notice that this definition is independent of the singular distributions θ and ω. We say that a smooth morphism $\phi : (M, \theta, \mathcal{I}, E_M) \longrightarrow (N, \omega, \mathcal{J}, E_N)$ is *chain-preserving* if:

$$\mathcal{T}g(\omega, \mathcal{J}).\mathcal{O}_M = \mathcal{T}g(\theta, \mathcal{I})$$

i.e $H(\omega, \mathcal{J}, i), \mathcal{O}_M = H(\theta, \mathcal{I}, i)$ for all $i \in \mathbb{N}$.

Remark 2.6.2. *A morphism may be chain preserving even if θ and ω are very "different". This notion depends on the interaction between the singular distributions and the ideal sheaves. This implies, for example, that outside the support of the ideal sheaves, the singular distributions don't need to satisfy any relation.*

We will further say that a smooth morphism $\phi : (M, \theta, \mathcal{I}, E_M) \longrightarrow (N, \omega, \mathcal{J}, E_N)$ is *k-chain-preserving* if the morphism is chain preserving and θ and ω have leaf dimension equal to k.

Whenever we work with local foliated ideal sheaf, a morphism $\phi : (M, M_0, \theta, \mathcal{I}, E_M) \longrightarrow (N, N_0, \omega, \mathcal{J}, E_N)$ satisfies a property P if:

$$\phi|_{M_0} : (M_0, \theta.\mathcal{O}_{M_0}, \mathcal{I}.\mathcal{O}_{M_0}, E_M \cap M_0) \longrightarrow (N_0, \omega.\mathcal{O}_{N_0}, \mathcal{J}.\mathcal{O}_{N_0}, E_N \cap N_0)$$

satisfies property P, where P may be: smoothness, chain-preserving smoothness and k-chain-preserving smoothness.

Chapter 3

Blowings-up

3.1 Admissible blowings-up

Let $\sigma : M^{'} \longrightarrow M$ be a blowing-up with center $\mathcal{C}$, F be the exceptional divisor of the blowing-up and S a subset of M:

- The total transform of S is $S^{*} = \sigma^{*}S = \sigma^{-1}S$;
- The strict transform of S is $S^{s} := \sigma^{s}S = \overline{\sigma^{-1}(S \setminus \mathcal{C})}$ (where $\overline{S}$ stands for the topological closure of S).

Given (M, E) an analytic manifold with divisor, a blowing-up $\sigma : M^{'} \longrightarrow M$ is said to be *admissible* by (M, E) if:

- The center $\mathcal{C}$ is a closed and regular sub-manifold of M;
- The center $\mathcal{C}$ has SNC with E.

If $\sigma : M^{'} \longrightarrow M$ is an admissible blowing-up, there is a natural structure of analytic manifold with divisor in $M^{'}$ given by the pair $(M^{'}, E^{'})$, where $E^{'} = ((E^{(1)})^{s}, ..., (E^{(l)})^{s}, F)$. We denote this blowing-up by:

$$\sigma : (M^{'}, E^{'}) \longrightarrow (M, E)$$

A *sequence* $\vec{\sigma}$ *of admissible blowings-up* is a sequence $(\sigma_r, ..., \sigma_1)$ such that:

$$(M_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, E_1) \xrightarrow{\sigma_1} (M_0, E_0)$$

where $\sigma_i : (M_i, E_i) \longrightarrow (M_{i-1}, E_{i-1})$ is an admissible blowing-up for (M_{i-1}, E_{i-1}) and $E_i = E'_{i-1}$. We establish the following notations:

- The exceptional divisor of σ_i is denoted by F_i;
- $\sigma := \sigma_1 \circ ... \circ \sigma_r$;
- $[i\sigma] := \sigma_{i+1} \circ ... \circ \sigma_r$;
- $[\sigma i] := \sigma_1 \circ ... \circ \sigma_i$.

3.2 Transforms of a singular distribution θ

Let (M, θ, E) be a d-foliated manifold and $\sigma : (M', E') \longrightarrow (M, E)$ an admissible blowing-up with exceptional divisor F. At this subsection we define a classical transforms of θ and we introduce a new one.

Consider the sheaf of $\mathcal{O}_{M'}$-modules $\mathcal{O}(-F) \otimes_{\mathcal{O}_{M'}} Der_{M'}$ which we denote by $\mathcal{B}lDer_{M'}$ (from *blowed-up* derivations). There exists a mapping from $Der_{M'}$ to $\mathcal{B}lDer_{M'}$:

$$\zeta : \ Der_{M'} \ \longrightarrow \ \mathcal{B}lDer_{M'}$$

which, given an open subset U of M', associates to a vector field $X \in Der_{M'}(U)$ the element $\zeta(X) = 1 \otimes X \in \mathcal{B}lDer_{M'}(U)$. Notice that this mapping is injective.

Given a sub-sheaf ω of $Der_{M'}$, we abuse notation and denote by $\zeta(\omega)$ the sub-sheaf of $\mathcal{B}lDer_{M'}$, with the structure of a $\mathcal{O}_{M'}$-module, generated by the image of ω. Reciprocally, given a sub-sheaf ω of $\mathcal{B}lDer_{M'}$ we denote by $\zeta^{-1}(\omega)$ the sub-sheaf of $Der_{M'}$ defined in each open set U of M' by the following elements:

$$\zeta^{-1}(\omega)_U = \{X \in Der_U;\ \zeta(X) \in \omega_U\}$$

Since the blowing-up $\sigma : M' \longrightarrow M$ is a morphism, it gives rise to a mapping on the structural sheaves $\sigma^* : \mathcal{O}_M \longrightarrow \mathcal{O}_{M'}$. Abusing notation, this morphism also gives rise to an application:

$$\sigma^* : Der_M \longrightarrow \mathcal{B}lDer_{M'}$$

which, given an open subset U of M, associates to a vector field X of Der_U the element $\sigma^*(X) = (\frac{1}{f} \otimes fX^*)$, where the principal ideal (f) is equal to $\mathcal{O}(F).\mathcal{O}_{\sigma^{-1}(U)}$ and X^* is the pull-back of the derivation (i.e. $X^*(\sigma^* f) = \sigma^* X(f)$).

The necessity to consider meromorphic functions is illustrated by the following example:

Example: $M = \mathbb{C}^2$, $X = \frac{\partial}{\partial x}$ and let $V(x, y)$ be the center of blowing-up. Then:

- In the x-chart $X^* = \frac{1}{x}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$;
- In the y-chart $X^* = \frac{1}{y}\frac{\partial}{\partial x}$.

In particular, even though θ is analytic, we cannot guarantee that $\sigma^*\theta$ is analytic.

Remark 3.2.1. *The blowing-up of an analytic vector field has at most poles of order one (as in the previous example). This implies that $\sigma^* : Der_M \longrightarrow \mathcal{B}lDer_{M'}$ is well-defined.*

The image $\sigma^*(\theta)$ is a coherent sub-sheaf of the sheaf of $\mathcal{O}_{M'}$-modules $\mathcal{B}lDer_{M'}$. We remark that θ^* is also a morphism of Lie-algebras.

We now define two possible transforms of θ:

- The *total transform* of θ is given by $\theta^* := \sigma^*(\theta)$;
- The *analytic strict transform* of θ is given by $\theta^a := \zeta^{-1}(\theta^*)$.

Whenever $\zeta^{-1}\sigma^*\theta$ is isomorphic to $\sigma^*\theta$, we will abuse notation and write $\theta^* = \zeta^{-1}(\theta^*)$.

We claim that the analytic strict transform is an involutive d-singular distributions (not necessarily tangent to E'). The following Lemma proves the claim:

Lemma 3.2.2. *The sub-sheaf θ^a is an involutive d-singular distribution. Moreover, consider a point q of M' and let $p = \sigma(q)$ and $\{X_1, ..., X_{d_p}\}$ be a coherent set of generators of θ_p. Then θ^a_q has a coherent set of generators $\{Y_i, Z_j, W_k\}$ with $i = 1, ..., r$, $j = 1, ..., s$ ($r + s = d_p$) and $k = 1, ..., t$, where:*

- $Y_i = (\mathcal{O}(F)X_i^*).\mathcal{O}_q$ *whenever* $X_i^*.\mathcal{O}_q$ *is not analytic;*
- $Z_j = X_j^*.\mathcal{O}_q$ *whenever* $X_j^*.\mathcal{O}_q$ *is analytic;*
- $W_k = \mathcal{O}(-F)\sum \gamma_{i,k} Y_i$ *for some* $\Gamma_{\theta,k} \in \mathcal{O}_U^r$ *such that* $W_k \notin < Y_i, Z_j >$.

Consider the involutive n-singular distribution $Der_{M'}(-logF)$ of $Der_{M'}$ composed by all the derivations leaving the exceptional divisor F invariant. The *adapted analytic strict* transform of θ is defined as $\theta^{a,ad} = \theta^a \cap Der_{M'}(-logF)$. It follows from Oka's Theorem that $\theta^{a,ad}$ is an involutive d-singular distribution.

Now, we prove the result stated on this subsection:

Proof. (Lemma 3.2.2)

- Coherence: If q is a point outside the exceptional divisor F, the result is clear because σ is a local isomorphism and, thus, $\zeta : \theta^a_q \longrightarrow \theta^*_q$ is a local isomorphism. So, consider the point q contained in F and let $p = \sigma(q)$. If $\{X_1, ..., X_{d_p}\}$ is a coherent set of generators of θ_p, then it is clear that:

  $$\theta^*_q =< \sigma^*(\zeta(X_1)), ..., \sigma^*(\zeta(X_{d_p})) > .\mathcal{O}_q =< (\frac{1}{f} \otimes fX_1^*), ..., (\frac{1}{f} \otimes fX_{d_p}^*)) > .\mathcal{O}_q$$

  Take U a sufficiently small neighborhood of q and $(x, y) = (x, y_1, ..., y_{n-1})$ a coordinate system such that $f = x$ and $\theta^a_U =< (\frac{1}{x} \otimes xX_1^*), ..., (\frac{1}{x} \otimes xX_{d_p}^*) > .\mathcal{O}_U$. Notice that whenever $X_i^*.\mathcal{O}_U$ is an analytic vector field: $(\frac{1}{x} \otimes xX_i^*.\mathcal{O}_U) = (1 \otimes X_i^*.\mathcal{O}_U)$. Reorganizing the set of generators, we can suppose that $\sigma_U^* =< (\frac{1}{x} \otimes Y_1), ..., (\frac{1}{x} \otimes Y_r), (1 \otimes Z_1), ..., (1 \otimes Z_s) >$ where $r + s = d_p$, $Y_i = xX_i^*.\mathcal{O}_U$ (such that $Y_\zeta(0, y) \not\equiv 0$) and $Z_i = X_i^*.\mathcal{O}_U$.

  Let $\mathcal{R}$ be the sub-module of relations of $\{Y_i|_{x=0}\}$, i.e. the r-tuples $(f_1, ..., f_r) \in \mathcal{O}_U^r$ such that $(\sum_{i=1}^r f_i Y_i)|_{x=0} \equiv 0$. It is easy to see that this is the same sub-module of relations of $\{Y_\zeta(x)|_{x=0}, Y_\zeta(y_j)|_{x=0}\}_{i\leq r, j\leq n-1}$. Thus, by the Oka's Theorem (see Theorem 6.4.1 of [Ho]), $\mathcal{R}$ is finitely generated: $\mathcal{R} = (F_1, ..., F_t)$ where $F_i = (f_{1,i}, ..., f_{r,i})$.

In particular, for every $j \leq t$, $\sum f_{i,j} Y_i$ is divisible by x. So, for each F_j, we have that:

$$\sum_{i=1}^{r} (\frac{f_{i,j}}{x} \otimes Y_i) = (\frac{1}{x} \otimes \sum_{i=1}^{r} f_{i,j} Y_i) =: (1 \otimes W_j)$$

We claim that $\{Y_i, Z_j, W_k\}_{i\leq r, j\leq s, k\leq t}$ generates θ_U^a, which implies the coherence. Indeed, consider $X \in \theta_U^a$: we only need to check that $\zeta(X) \in \{\zeta(Y_i), \zeta(Z_j), \zeta(W_k)\}_{i\leq r, j\leq s, k\leq t}$. We know there exists $\alpha \in \mathcal{O}_U^r$ and $\beta \in \mathcal{O}_U^s$ such that:

$$\zeta(X) = (1 \otimes X) = \sum \alpha_i (\frac{1}{x} \otimes Y_i) + \sum \beta_j (1 \otimes Z_j)$$

Now, $\alpha_i = x \widetilde{\alpha}_\zeta(x, y) + \bar{\alpha}_\zeta(y)$ and thus:

$$\zeta(X) = \sum \widetilde{\alpha}_i(x, y)(1 \otimes Y_i) + \sum \beta_j (1 \otimes Z_j) + \sum \bar{\alpha}_i(y)(\frac{1}{x} \otimes Y_i)$$

It is clear that $\sum \bar{\alpha}_i(y) Y_i$ is divisible by x. This implies that $(\bar{\alpha}_i) \subset \mathcal{R}$. So, there exists $\gamma \in \mathcal{O}_U^t$ such that $(\bar{\alpha}) = \sum \gamma_k F_k$. This finally implies that:

$$\zeta(X) = \sum \widetilde{\alpha}_\zeta(x, y)(1 \otimes Y_i) + \sum \beta_j (1 \otimes Z_j) + \sum \gamma_k (1 \otimes W_k)$$

- Involutiviness: For any point q of M', consider vector fields X and Y contained in θ_q^a. Then the elements $\zeta(X)$ and $\zeta(Y)$ are contained in θ_q^*. Since θ_q^* is closed under Lie brackets, necessarily $[\zeta(X), \zeta(Y)] \in \theta_q^*$ and since the Lie bracket of two analytic derivations is still an analytic derivation, we deduce that $[X, Y] \in \theta_q^a$.

- Leaf dimension: Since the blowing-up $\sigma : M' \longrightarrow M$ and the morphism $\zeta : \theta^a \longrightarrow \theta^*$ are local isomorphisms in an open and dense set, θ^a has also leaf dimension d.

□

3.3 Transforms of foliated manifolds and foliated ideal sheaves

Given (M,θ,E) a foliated manifold and $\sigma:(M^{'},E^{'})\longrightarrow(M,E)$ an admissible blowing-up, there is a natural structure of foliated manifold associated to $(M^{'},E^{'})$ given by $(M^{'},\theta^{'},E^{'})$ where $\theta^{'}$ is the adapted analytic strict transform of θ. We denote the blowing-up by:

$$\sigma:(M^{'},\theta^{'},E^{'})\longrightarrow(M,\theta,E)$$

A sequence $\vec{\sigma}$ of admissible blowings-up gives rise to a sequence:

$$(M_r,\theta_r,E_r)\xrightarrow{\sigma_r}\cdots\xrightarrow{\sigma_2}(M_1,\theta_1,E_1)\xrightarrow{\sigma_1}(M_0,\theta_0,E_0)$$

where $\sigma_i:(M_i,\theta_i,E_i)\longrightarrow(M_{i-1},\theta_{i-1},E_{i-1})$ is an admissible blowing-up and $\theta_i=\theta^{'}_{i-1}$.

Given an admissible blowing-up $\sigma:(M^{'},\theta^{'},E^{'})\longrightarrow(M,\theta,E)$, let F be the exceptional divisor of the blowing-up and consider a coherent everywhere non-zero ideal sheaf $\mathcal{I}$ over M. We define two transforms of $\mathcal{I}$:

- The total transform of $\mathcal{I}$ is $\mathcal{I}^*:=\sigma^*\mathcal{I}=\mathcal{I}.\mathcal{O}_{M'}$;
- The strict transform of $\mathcal{I}$ is $\mathcal{I}^s:=\cup_{i\in\mathbb{N}}(\sigma^*\mathcal{I}:\mathcal{O}(iF))$, where $\mathcal{O}(iF)$ is the ideal sheaf $\mathcal{O}(F)^i$.

Furthermore, if $\mathcal{C}\subset V(\mathcal{I})$, then we also define:

- The controlled transform of $\mathcal{I}$ is $\mathcal{I}^c:=\mathcal{I}^*.\mathcal{O}(-F)$.

The following Lemma gives a crucial algebraic relation between the interactions of θ and $\mathcal{I}$ under blowing-up based on the k-generalized Fitting opperations:

Lemma 3.3.1. *Let $\sigma:(M^{'},\theta^{'},E^{'})\to(M,\theta,E)$ be an admissible blowing-up over a foliated ideal sheaf $(M,\theta,\mathcal{I},E)$. Then:*

- $[\Gamma_{\theta,s}(\mathcal{I})]^*\subset\Gamma_{\theta^*,s}(\mathcal{I}^*)$;

- $[\Gamma_{\theta,s}(\mathcal{I})+\mathcal{I}]^* = \Gamma_{\theta^*,s}(\mathcal{I}^*)+\mathcal{I}^*$.

for all $s \leq d$.

Remark 3.3.2. *In the above Lemma, if* θ^* *is a meromorphic singular distribution, there is a natural way to extend the definition of the operation* $\Gamma_{\theta^*,s}$ *to the sheaf of meromorphic functions over* M.

Proof. Notice that, since $\sigma^* : Der_M \to BlDer_{M'}$ is a morphism, it is clear that:

$$[\Gamma_{\theta,s}(\mathcal{I})]^* \subset \Gamma_{\theta^*,s}(\mathcal{I}^*)$$

And, in particular $[\Gamma_{\theta,s}(\mathcal{I})+\mathcal{I}]^* \subset \Gamma_{\theta^*,s}(\mathcal{I}^*)+\mathcal{I}^*$. To prove the other inclusion, fix a point q of $M^{'}$, let $p=\sigma(q)$ and consider a coherent set of generators $\{g_1,...,g_t\}$ of $\Gamma_{\theta^*,s}(\mathcal{I}^*).\mathcal{O}_q$. For simplicity, we assume that $s=1$ (the other cases follows from analogous reasons). We can chose the generators g_i's of the following form:

$$g_i = \sum_j X^*_{i,j}(\sum_k a_{i,j,k} f^*_{i,j,k})$$

where $X_{i,j}$ are vector fields of θ_p, $a_{i,j,k}$ are functions in $\mathcal{O}_q$ and $f_{i,j,k}$ are functions in $\mathcal{I}.\mathcal{O}_p$. This clearly implies that g_i is contained in the ideal $([\Gamma_{1,\theta}(\mathcal{I})]^*+\mathcal{I}^*).\mathcal{O}_q$, which proves the other inclusion. This finally gives the desired result. □

Given $(M,\theta,\mathcal{I},E)$ a foliated ideal sheaf, we say that an admissible blowing-up $\sigma : (M^{'},\theta^{'},E^{'}) \longrightarrow (M,\theta,E)$ is of *order one* for $(M,\theta,\mathcal{I},E)$ if $\mathcal{C} \subset V(\mathcal{I})$. In this case, there is a natural structure of foliated ideal sheaf associated to $(M^{'},\theta^{'},E^{'})$ given by $(M^{'},\theta^{'},\mathcal{I}^{'},E^{'})$, where $\mathcal{I}^{'}$ is the controlled transform of $\mathcal{I}$. We denote the blowing-up by:

$$\sigma : (M^{'},\theta^{'},\mathcal{I}^{'},E^{'}) \longrightarrow (M,\theta,\mathcal{I},E)$$

A *sequence* $\vec{\sigma}$ *of admissible blowings-up of order one* is a sequence $(\sigma_r,...,\sigma_1)$ of admissible blowings-up such that:

$$(M_r,\theta_r,\mathcal{I}_r,E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1,\theta_1,\mathcal{I}_1,E_1) \xrightarrow{\sigma_1} (M_0,\theta_0,\mathcal{I}_0,E_0)$$

where $\sigma_i : (M_i,\theta_o,\mathcal{I}_i,E_i) \longrightarrow (M_{i-1},\theta_{i-1},\mathcal{I}_{i-1},E_{i-1})$ is an admissible blowing-up of order one for $(M_{i-1},\theta_{i-1},\mathcal{I}_{i-1},E_{i-1})$ and $\mathcal{I}_i = \mathcal{I}^{'}_{i-1}$.

3.4 Local blowings-up

Following (section 2.4 of) [BM3], a *local blowing-up* is a morphism $\tau : M' \longrightarrow M$ that is equal to the composition of a blowing-up $\sigma : M' \longrightarrow \widetilde{M}$ and an injective local isomorphism $\pi : \widetilde{M} \longrightarrow M$, i.e $\tau = \pi \circ \sigma$. Furthermore:

- If the blowing-up $\sigma : (M', E') \longrightarrow (\widetilde{M}, \widetilde{E})$ is admissible, we say that $\tau : (M', E') \longrightarrow (M, E)$ is an *admissible local blowing-up*, where $\widetilde{E} = \pi^{-1}(E)$;
- If the blowing-up $\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (\widetilde{M}, \widetilde{\theta}, \widetilde{\mathcal{I}}, \widetilde{E})$ is an admissible blowing-up of order one, we say that $\tau : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$ is an *admissible local blowing-up of order one*, where $\widetilde{\theta} = \theta.\mathcal{O}_{\widetilde{M}}$ and $\widetilde{\mathcal{I}} = \mathcal{I}.\mathcal{O}_{\widetilde{M}}$.

A *sequence $\vec{\tau}$ of admissible local blowings-up of order one* is a sequence $(\tau_r, ..., \tau_1)$ such that:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\tau_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

where $\tau_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \longrightarrow (M_{i-1}, \theta_{i-1}, \mathcal{I}_{i-1}, E_{i-1})$ is an admissible local blowing-up of order one.

3.5 Resolution and local uniformization of an ideal sheaf

A *resolution* of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is a sequence $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of admissible blowings-up of order one:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

such that $\mathcal{I}_r = \mathcal{O}_{M_r}$. In particular, $\mathcal{I}.\mathcal{O}_{M_r}$ is the ideal sheaf of a SNC divisor on M_r contained in E_r. A *resolution of a local foliated ideal sheaf* $(M, M_0, \theta, \mathcal{I}, E)$ is a resolution of $(M_0, \theta.\mathcal{O}_{M_0}, \mathcal{I}.\mathcal{O}_{M_0}, E \cap M_0)$. A *weak-resolution* of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is a proper and analytic morphism:

$$\sigma : M' \longrightarrow M$$

such that, for every relatively compact open subset M_0 of M, there exist a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that $\sigma|_{\sigma^{-1}M_0} = \sigma_1 \circ ... \circ \sigma_r$.

A "good" resolution will also respect a functorial property. More precisely, following [Ko] (see definition 3.31), we look for a functor $\mathcal{R}$ that has:

- input: The category whose objects are foliated ideal sheaves $(M, \theta, \mathcal{I}, E_M)$ and whose morphisms are smooth morphisms;
- output: The category whose objects are admissible blowing-up sequences:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

  with specified admissible centers $\mathcal{C}_i$ and whose morphisms are given by the Cartesian product.

The functor $\mathcal{R}$ is said to be a *resolution functor* if for all $(M, \theta, \mathcal{I}, E_M)$, it associates a resolution of $(M, \theta, \mathcal{I}, E_M)$ that commutes with smooth morphisms. One can define in the same manner the notion of resolution functor for local ideal sheaves with divisor and for *weak-resolution functors.*

Remark 3.5.1. *For such a functor to be well defined, we will accept blowings-up with empty centers (isomorphisms).*

Following (the ennunciate of Theorem 1.1 of) [BM3], a *local uniformization* of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ at a point p of M is a finite collection of pairs $\{\tau_\alpha : M_\alpha \longrightarrow M, \theta_\alpha\}$ where:

- $\tau_\alpha : M_\alpha \longrightarrow M$ is a proper analytic morphism;
- θ_α is a singular distribution over M_α.

such that:

- The union of the images $\bigcup \tau_\alpha(M_\alpha)$ is an open neighborhood of p.
- For each morphism $\tau_\alpha : M_\alpha \longrightarrow M$ there exists a sequence of admissible local blowings-up of order one:

$$(M_r, \theta_{r,\alpha}, \mathcal{I}_r, E_r) \xrightarrow{\tau_{r,\alpha}} \cdots \xrightarrow{\tau_{2,\alpha}} (M_1, \theta_{r,\alpha}, \mathcal{I}_1, E_1) \xrightarrow{\tau_{1,\alpha}} (M, \theta, \mathcal{I}, E)$$

  such that $\mathcal{I}_r = \mathcal{O}_{M_r}$, $\theta_\alpha = \theta_{r,\alpha}$ and the morphism τ_α is the composition of this local blowings-up: $\tau_\alpha = \tau_{1,\alpha} \circ ... \circ \tau_{r,\alpha}$.

To simplify notation, we abuse notation and denote a local uniformization $\{\tau_\alpha : M_\alpha \longrightarrow M, \theta_\alpha\}$ simply as $\{\tau_\alpha : (M_\alpha, \theta_\alpha) \longrightarrow (M, \theta)\}$.

3.6 The Hironaka's Theorem

Let us state the version of Hironaka's Theorem that we are going to use:

Theorem 3.6.1. *(Hironaka): Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local foliated ideal sheaf. Then there exists a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:*

$$\mathcal{R}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

- *The composition $\sigma = \sigma_1 \circ ... \circ \sigma_r$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;*
- *$\mathcal{R}$ is a resolution functor that commutes with smooth morphisms.*

Remark 3.6.2. *The above Theorem is an interpretation of Theorem 1.3 of* [*BM2*] *or Theorems 2.0.3 and 6.0.6 of* [*W*] *in the following sense:*

- *Neither of the Theorems need the notion of singular distribution;*

- *Theorem* 1.3 *of* [*BM2*] *is enunciated in algebraic category. But the paragraph before Theorem* 1.1 *of* [*BM2*] *justifies the analytic statement;*

- *In* [*BM2*] *and* [*W*]*, the authors work with marked ideal sheaves. We specialize their result to marked ideal sheaves with weight one. The reader may verify that the definition of Support and (weak) transform give rise to the interpretations formulated in this work;*

- *In order to stress the functorial property of the resolution, we follow Kollor's presentation (see [Ko]).*

Remark 3.6.3. *The functorial property implies an intuitive sense of "unicity". For example, let* $\mathcal{C}_i$ *be the centers of* $\mathcal{R}(M, M_0, \theta, \mathcal{I}, E)$ *and* N *a compact analytic manifold. Then* $\mathcal{C}_i \times N$ *are the centers of* $\mathcal{R}(M \times N, M_0 \times N, \omega, \mathcal{I}.\mathcal{O}_{M\times N}, E \times N)$ *for any singular distribution* ω*.*

An important consequence of the functoriality is the following global version of Theorem 3.6.1:

Theorem 3.6.4. *Let* $(M, \theta, \mathcal{I}, E)$ *be a foliated ideal sheaf. Then there exists a proper analytic morphism:*

$$\mathcal{RG}(M, \theta, \mathcal{I}, E) = \sigma : \widetilde{M} \longrightarrow M$$

such that:

- *for every* $M_0 \subset M$ *relatively compact open set,* $\sigma|_{\sigma^{-1}M_0}$ *is the composition of the sequence of blowings-up* $\mathcal{R}(M, M_0, \theta, \mathcal{I}, E)$ *given on Theorem* 3.6.1*;*

- σ *is an isomorphism over* $M_0 \setminus V(\mathcal{I})$*;*

- $\mathcal{RG}(M, \theta, \mathcal{I}, E)$ *is a weak-resolution functor that commutes with smooth morphisms.*

The proof of Theorem 3.6.4 follows the same steps of Theorem 13.3 of [BM1]. We present the proof because the idea will be useful for us.

Proof. (Theorem 3.6.4): Let $(U_i)_{i\in\mathbb{N}}$ be an open cover of M by relatively compact subsets U_i of M such that $U_i \subset U_{i+1}$. Theorem 3.6.1 guarantees the existence of a resolution $\vec{\sigma_i} = (\sigma_{i,1}, ..., \sigma_{i,r_i})$ for each $(M, U_i, \theta, \mathcal{I}, E)$. Consider the morphism, $\sigma_i := \sigma_{i,1} \circ ... \circ \sigma_{i,r_i}$.

The inclusion $\epsilon_i : U_i \longrightarrow U_{i+1}$ is a smooth morphism and, by the functoriality of Theorem 3.6.1, there exists a smooth morphism $\delta_i : U_i^{'} \longrightarrow U_{i+1}^{'}$ such that the following diagram:

$$\begin{array}{ccc} U_i^{'} & \xrightarrow{\delta_i} & U_{i+1}^{'} \\ \sigma_i \downarrow & & \downarrow \sigma_{i+1} \\ U_i & \xrightarrow{\epsilon_i} & U_{i+1} \end{array}$$

commutes. It is clear that M is isomorphic to the direct limit of the U_i, i.e. the disjoint union $\sqcup U_i$ identified by the morphisms ϵ_i. Let $M^{'}$ be the direct limit of $U_i^{'}$ (identified by the morphisms δ_i) and $\sigma : M^{'} \longrightarrow M$ be the direct limit of σ. By construction, $\sigma|_{U_i^{'}}$ coincides with σ_i.

The functorial statement follows from the functoriality of each σ_i. $\square$

Chapter 4

The θ-admissible blowing-up

4.1 Definition and Main result

Let (M, θ, E) be a d-foliated manifold and let $\mathcal{C}$ be an analytic sub-manifold of M. Consider the reduced ideal sheaf $\mathcal{I}_{\mathcal{C}}$ that generates $\mathcal{C}$, i.e. $V(\mathcal{I}_{\mathcal{C}}) = \mathcal{C}$. We say that $\mathcal{C}$ is a θ-*admissible* center if:

- $\mathcal{C}$ is a regular closed sub-variety;
- $\mathcal{C}$ has SNC with E;
- There exists $0 \leq d_0 \leq d$ such that the k-generalized Fitting-ideal $\Gamma_{\theta,k}(\mathcal{I}_{\mathcal{C}})$ is equal to the structural ideal $\mathcal{O}_M$ for all $k \leq d_0$ and is contained in the ideal sheaf $\mathcal{I}_{\mathcal{C}}$ otherwise.

We give a geometrical interpretation of θ-admissible centers in Remark 4.3.2.

Examples:

- If $\mathcal{C}$ is an admissible and θ-invariant center, it is θ-admissible;
- If $\mathcal{C}$ is an admissible and θ-totally transverse center, it is θ-admissible;
- Let $(M, \theta, E) = (\mathbb{C}^3, \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}, \emptyset)$ and $\mathcal{C} = \{x = 0\}$. Then $\mathcal{C}$ is a θ-admissible center, but it is neither invariant nor totally transverse. Indeed, $\Gamma_{\theta,1}(\mathcal{I}_{\mathcal{C}}) = \mathcal{O}_M$ and $\Gamma_{\theta,2}(\mathcal{I}_{\mathcal{C}}) \subset \mathcal{I}_{\mathcal{C}}$.

- Let $(M,\theta,E) = (\mathbb{C}^3, \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}, \emptyset)$ and $\mathcal{C} = \{x^2 - z = 0\}$. Then $\mathcal{C}$ is **not** a θ-admissible center. Indeed, $\Gamma_{\theta,1}(\mathcal{I}_\mathcal{C}) = (x,z)$.

An admissible blowing-up $\sigma : (M^{'},\theta^{'},E^{'}) \longrightarrow (M,\theta,E)$ is *θ-admissible* if the center $\mathcal{C}$ is θ-admissible. We emphasize two particular cases of θ-admissible blowings-up:

- An admissible blowing-up $\sigma : (M^{'},\theta^{'},E^{'}) \longrightarrow (M,\theta,E)$ is *θ-invariant* if the center $\mathcal{C}$ is θ-invariant (i.e $\theta[\mathcal{I}_\mathcal{C}] \subset \mathcal{I}_\mathcal{C}$);

- An admissible blowing-up $\sigma : (M^{'},\theta^{'},E^{'}) \longrightarrow (M,\theta,E)$ is *θ-totally transverse* if the center $\mathcal{C}$ is totally transverse to θ (i.e $\Gamma_{\theta,d}(\mathcal{I}_\mathcal{C} = \mathcal{O}_M)$.

A *sequence $\vec{\sigma} = (\sigma_1,...,\sigma_r)$ of θ-admissible blowings-up* is a sequence of admissible blowings-up:

$$(M_r,\theta_r,E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1,\theta_1,E_1) \xrightarrow{\sigma_1} (M_0,\theta_0,E_0)$$

such that $\sigma_i : (M_{i+1},\theta_{i+1},E_{i+1}) \longrightarrow (M_i,\theta_i,E_i)$ is a θ_i-admissible blowing-up. A *sequence $\vec{\sigma} = (\sigma_1,...,\sigma_r)$ of θ-invariant blowings-up* and of *θ-totally transverse blowings-up* are defined analogously. The following Theorem enlightens the interest of θ-admissible blowings-up:

Theorem 4.1.1. *Let (M,θ,E) be a R-monomial d-foliated manifold and:*

$$\sigma : (M^{'},\theta^{'},E^{'}) \longrightarrow (M,\theta,E)$$

a θ-admissible blowing-up. Then $\theta^{'}$ is R-monomial.

The proof is divided in three parts. The two first subsections prove the existence of a "good" coordinate systems. The proof of the Theorem is given in subsection 4.4. An important corollary of the proof of this Theorem is the following:

Corollary 4.1.2. *Let (M,θ,E) be a d-foliated manifold such that θ is regular and:*

$$\sigma : (M^{'},\theta^{'},E^{'}) \longrightarrow (M,\theta,E)$$

a θ-invariant blowing-up. Then, $\theta^{'}$ is regular.

Which is proved in the end of this chapter.

4.2 Local coordinates for a θ-invariant center

The main result of this subsection is the following:

Proposition 4.2.1. *Let (M, θ, E) be a R-monomial d-foliated manifold and $\mathcal{C}$ an invariant θ-admissible center. Then, at each point $p \in \mathcal{C}$, there exists a R-monomial coordinate system $x = (x_1, ..., x_n)$ such that $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (x_1, ..., x_t)$.*

In what follows, $\mathcal{C}$ is always a θ-invariant admissible center and, given a point p of M, we denote by $I_{\mathcal{C}}$ the ideal $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p$ when there is no risk of confusion on the point p.

The fundamental step for proving proposition 4.2.1 is the following result:

Lemma 4.2.2. *Let (M, θ, E) be a R-monomial d-foliated manifold and $\mathcal{I}$ a θ-invariant regular coherent ideal sheaf. Given a point p of M and a R-monomial coordinate system $x = (x_1, ..., x_n)$ with a R-monomial basis $\{X_1, ..., X_d\}$, there exists a set of generators $\{f_1, ..., f_t\}$ of $I := \mathcal{I}.\mathcal{O}_p$ such that:*

- *$X_i(f_j) \equiv 0$ if X_i is regular;*
- *$X_i(f_j) = K_{i,j} f_j$ for some $K_{i,j} \in R$, if X_i is singular.*

Let us see how this result proves proposition 4.2.1:

Proof. (Proposition 4.2.1) Take $p \in \mathcal{C}$. Our proof is by induction on the pair (d, n), where d is the leaf dimension of θ_p and n is the dimension of the ring $\mathcal{O}_p$.

Notice that for $d = 0$ or $n = 1$ the result is trivial (if $n = 1$, the support of the ideal is a point). By induction, suppose that for all $(d', n') < (d, n)$, where $<$ is the lexicografical order, there is always a R-monomial coordinate system $x = (x_1, ..., x_{n'})$ such that $I_{\mathcal{C}} = (x_1, ..., x_t)$. We prove it to (d, n).

Fix a R-monomial coordinate system $x = (x_1, ..., x_n)$ and $\{X_1, ..., X_d\}$ a R-monomial basis. By lemma 4.2.2, there exists a set of generators $\{f_1, ..., f_t\}$ of the ideal $I_{\mathcal{C}}$ such that:

- $X_i(f_j) \equiv 0$ if X_i is regular;
- $X_i(f_j) = K_{i,j} f_j$ for some $K_{i,j} \in R$, if X_i is singular.

We have two cases to consider:

- Case I: Without loss of generality, suppose $X_1 = \frac{\partial}{\partial x_1}$ and that $X_j(x_1) = 0$ for all $j \neq 1$. Since $X_1(f_i) \equiv 0$ for all i, the set of generators is independent of x_1.

  Let U_p be an open neighborhood of p such that the coordinate system $x = (x_1, ..., x_n)$ is well defined over U_p and the vector fields X_i have representatives over U_p. Consider the quotient:

  $$\Pi : \mathcal{O}_{U_p} \longrightarrow \mathcal{O}_{U_p}/(x_1)$$

  The image of the distribution θ by Π is a R-monomial involutive $(d-1)$-singular distribution $\bar{\theta}$ given by the image of X_i, for $i > 1$. We denote the image of the coordinate system $x = (x_1, ..., x_n)$ by Π as $\bar{x} = (\bar{x}_2, ..., \bar{x}_n)$. By induction, there exists a change of coordinates over $\mathcal{O}_{U_p}/(x_1)$ such that $\bar{I}_{\mathcal{C}} = (\bar{x}_2, ..., \bar{x}_t)$. Doing the equivalent change of coordinates in $\mathcal{O}_{U_p}$, since the change is invariant by x_1, we get $I_{\mathcal{C}} = (x_2, ..., x_t)$.

- Case II: All vector fields of the R-monomial basis $\{X_1, ..., X_d\}$ are singular:

  $$X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$$

  Since $I_{\mathcal{C}}$ is regular, we can suppose that f_1 is regular and, without loss of generality, that $\frac{\partial}{\partial x_1} f_1(p) \neq 0$. Take the change of coordinates $\bar{x}_1 = f_1$ and $\bar{x}_i = x_i$ otherwise. In the new coordinates, we get:

  $$X_i = \sum_{j=2}^{n} \alpha_{i,j} \bar{x}_j \frac{\partial}{\partial \bar{x}_j} + K_{1,i} \bar{x}_1 \frac{\partial}{\partial \bar{x}_1}$$

  because $X_i(f_1) = K_{1,i} f_1$ for $K_{1,j} \in R$. Notice that $\{X_1, ..., X_d\}$ is also a R-monomial basis at this coordinate system. We drop the bars of this coordinate system in order to have simpler notation.

Let U_p be an open neighborhood of p such that the coordinate system $x = (x_1, ..., x_n)$ is well defined over U_p and the vector fields X_i have representatives over U_p. Consider the quotient:

$$\Pi : \mathcal{O}_{U_p} \longrightarrow \mathcal{O}_{U_p}/(x_1)$$

Notice that $\mathcal{O}_{U_p}/(x_1)$ is an analytic manifold of dimension $n-1$. The image of the distribution θ by Π is a R-monomial involutive singular distribution $\bar{\theta}$ given by the image of all X_i. Furthermore, $\bar{\theta}$ satisfies one of the following conditions:

- Either $\bar{\theta}$ is a R-monomial singular distribution of dimension d, or;
- $\bar{\theta}$ is a R-monomial singular distribution of dimension $d-1$ and we can assume $X_1 = x_1 \frac{\partial}{\partial x_1}$.

Either way, by induction, there exists a R-monomial coordinate system $\bar{x} = (\bar{x}_2, ..., \bar{x}_n)$ at $\mathcal{O}_{U_p}/(x_1)$ such that $\bar{I}_{\mathcal{C}} = (\bar{x}_2, ..., \bar{x}_{t+1})$. Doing the equivalent change of coordinates in $\mathcal{O}_p$, since the change is invariant by x_1, we deduce the result.

□

In order to prove Lemma 4.2.2, we will need some preliminary definitions:

- Let $\widehat{\mathcal{O}}_p$ denote the completion of $\mathcal{O}_p$ and fix a coordinate system $x = (x_1, ..., x_n)$. We introduce the topology of simple convergence in $\widehat{\mathcal{O}}_p$, defined by a countably many semi norms:

$$f = \sum a_\alpha x^\alpha \longrightarrow |a_\alpha|$$

  Thus $f_i \longrightarrow f$ means that the coefficients of x^α in f_i converges to the coefficient of x^α in f;

- Fixed a coordinate system $x = (x_1, ..., x_n)$, and given $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, let δ^α be the derivation $\frac{\partial^{\alpha_1}}{\partial x_1} ... \frac{\partial^{\alpha_n}}{\partial x_n}$. Given two functions $f, g \in \mathcal{O}_p$ we say that g *is contained in the Taylor expansion of* f at p if, for all α, either $\delta^\alpha g(p) = \delta^\alpha f(p)$ or $\delta^\alpha g(p) = 0$.

We also recall the following result (see section 6.3 and Theorems 6.3.4 and 6.3.5 of [Ho]):

Proposition 4.2.3. *Let I be an ideal of $\mathcal{O}_p$ and $(f_n)_{n\in\mathbb{N}} \subset I$ be a sequence of analytic function germs which converges simply to an analytic function germ f. Then, $f \in I$.*

We start the proof of Lemma 4.2.2 supposing that the distribution θ has leaf dimension 1. In the next Lemma, the coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ is fixed:

Lemma 4.2.4. *In the notation of Lemma 4.2.2, if θ_p has leaf dimension 1 and $\theta_p =< \frac{\partial}{\partial x} >$, then there exists a set of generators $(h_1, ..., h_t)$ of I such that $X(h_i) \equiv 0$. Moreover, if $(f_1, ..., f_r)$ is any set of generators of I, we can choose $(h_1, ..., h_t)$ such that each h_j is contained in the Taylor expansion of a f_i at p.*

Proof. Take $(f_1, ..., f_r)$ any set of generators of I and let $f := f_1$. Consider its Taylor expansion in x:

$$f = \sum_{i=0}^{\infty} h_i(y)x^i$$

Since I is invariant by X, we have that $(f)_\# \subset I$ (we recall that $(f)_\#$ is the θ-differential closure of the ideal (f)). We claim that $(h_i(y))_{i\in\mathbb{N}} = (f)_\#$.

Indeed, let us prove that $h_0(y) \in (f)_\#$ (the proof for the other coefficients is analogous). We set $g_0 = f$ and define recursively the expressions:

$$g_{i+1} := g_i - xX(g_i)\frac{1}{i}$$

It is easy to see that:

$$g_i = h_0(y) + \sum_{j=i}^{\infty} \beta_{i,j} h_j(y)x^j$$

for some $\beta_{i,j} \in \mathbb{K}$. It is clear that the sequence $(g_n)_n$ is contained in I and converges simply to $h_0(y)$. By Proposition 4.2.3, this implies that $h_0(y) \in (f)_\# \subset I$. Repeating the process for every $i \in \mathbb{N}$, we conclude that $h_i(y) \in I$ for all i. Thus $(h_i(y))_{i\in\mathbb{N}} \subset (f)_\#$.

Using again Proposition 4.2.3, it is clear that the ideal generated by $(h_i(y))_{i\in\mathbb{N}}$ contains $(f)_\#$. Moreover, since the structural ring is noetherian, we have that $(h_i(y))_{i\leq N} = (f)_\#$ for some $N \in \mathbb{N}$. Doing this for all the generators of I, we get the desired result. □

In the next Lemma, the R-monomial coordinate system $x = (x_1, ..., x_n)$ is fixed:

Lemma 4.2.5. *In the notation of Proposition 4.2.2, if θ_p has leaf dimension 1 and $\theta_p =< X >$ where X is a singular R-monomial vector field, then there exists a set of generators $(h_1, ..., h_t)$ of I such that $X(h_i) = K_i h_i$, for $K_i \in R$. Moreover, if $(f_1, ..., f_r)$ is any set of generators of I, we can choose $(h_1, ..., h_t)$ such that each h_j is contained in the Taylor expansion of a f_i at p.*

Proof. Let $(f_1, ..., f_r)$ be a set of generators of I and set $f = f_1$. Since the coordinate system is R-monomial we have that $X = \sum_{i=1}^n K_i x_i \frac{\partial}{\partial x_i}$ for $K_i \in R$. Taking any monomial $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ we get:

$$X(x^\alpha) = \sum_{i=1}^{n} K_i \alpha_i x^\alpha = K_\alpha x^\alpha$$

For some $K_\alpha \in R$ (because $\alpha_i \in \mathbb{Z}$ and $K_i \in R$). Since the number of different monomials is countable, there exists a countable set $R' \subset R$ such that $K_\alpha \in R'$, for all $\alpha \in \mathbb{Z}^n$. This allow us to rewrite the Taylor expansion of $f = f_1$ in the following form:

$$f(x) = \sum_{i \in \mathbb{N}} h_i(x)$$

with $h_i(x)$ such that $X h_i(x) = K_i h_i(x)$, $K_i \in R'$ and $K_i \neq K_j$ whenever $i \neq j$. Moreover, since there exists a representative of f convergent in a open neighborhood of p (thus absolutely convergent), $h_i(x) \in \mathcal{O}_p$. We claim that $(h_i(x))_{i \in \mathbb{N}} = (f)_\#$. Indeed, we show that $h_0 \in (f)_\#$ (the others are analogous). Define $g_0 = f$ and:

$$g_1 := \frac{1}{K_0 - K_1}(K_1 f - X(f)) = \frac{1}{K_0 - K_1}[\sum_{i \in \mathbb{N}} K_i h_i(x) - K_1 \sum_{i \in \mathbb{N}} K_i h_i(x)] = h_0 + \sum_{i \geq 2} \beta_{i,1} h_i \in (f)_\#$$

where $\beta_{i,1} = \frac{K_i - K_1}{K_0 - K_1}$. We define recursively:

$$g_n = \frac{1}{K_0 - K_n}(K_n g_{n-1} - X(g_{n-1})) = h_0 + \sum_{i \geq n+1} \beta_{i,n} h_i \in (f)_\#$$

for non-zero constants $\beta_{i,n}$. It is clear that $(g_n) \subset I$ converges simply to $h_0(x)$. By the proposition 4.2.3, this implies that $h_0(x) \in (f)_\#$. Repeating the process for every $i \in \mathbb{N}$, we conclude that $(h_i(y)) \subset (f)_\#$ for all i.

Using again proposition 4.2.3, it is clear that $(h_i(x))_{i\in\mathbb{N}}$ contains $(f)_{\#}$. Moreover, since the structural ring is noetherian, we have that $(h_i(x))_{i\leq N}$ is equal to $(f)_{\#}$ for some $N \in \mathbb{N}$. Doing this for all f_i in the set of generators of I, we get the desired result. □

We are ready to prove Lemma 4.2.2:

Proof. (Lemma 4.2.2): We prove the result by induction on the leaf dimension of θ. Fix a R-monomial coordinate system $x = (x_1, ..., x_n)$ and a R-monomial base $\{X_1, ..., X_d\}$. Let $(f_1, ..., f_t)$ be a set of generators of I and assume by induction that the lemma is true for $d' < d$.

By the induction hypotheses, we can assume without loss of generality that:

- $X_i(f_j) \equiv 0$ if X_i is regular;
- $X_i(f_j) = K_{i,j} f_j$ for some $K_{i,j} \in R$, if X_i is singular.

for all $i < d$. Now, by lemma 4.2.4 or 4.2.5 there exists another set of generators $(h_1, ..., h_l)$ such that:

- Either $X_d(h_j) \equiv 0$ if X_d is regular, or;
- $X_d(h_j) = K_{d,j} h_j$ for some $K_{i,j} \in R$, if X_d is singular.

Furthermore, as each h_i is a part of the Taylor expansion of some f_j, we have that:

- $X_i(h_j) \equiv 0$ if X_i is regular;
- $X_i(h_j) = K_{i,j} h_j$ for some $K_{i,j} \in R$, if X_i is singular.

for all $i \leq d$. □

4.3 Local coordinates for a θ-admissible center

The main result of this section is the following:

Proposition 4.3.1. *Let (M, θ, E) be a d-foliated manifold and $\mathcal{C}$ a θ-admissible center. Then, at each point $p \in \mathcal{C}$, there exists a coherent set of generators $\{Y_i, Z_j\}$ of θ_p with $i = 1, ..., r$ and $j = 1, ..., s$ such that:*

- *$\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p$ is totally transverse to $\{Y_i\}$;*
- *$\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p$ is invariant by $\{Z_j\}$;*
- *There exists a coordinate system $x = (x_1, ..., x_n)$ of $\mathcal{O}_p$ such that: $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (x_1, ..., x_t)$, $Y_i = \frac{\partial}{\partial x_i}$ and $Z_j(x_i) = 0$ for $i = 1, ..., r$;*
- *If θ is R-monomial, then there exists a R-monomial coordinate system $x = (x_1, ..., x_n)$ such that $\{Y_i, Z_j\}$ is a R-monomial basis. Moreover, this coordinate system can be chosen so that $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (x_1, ..., x_t)$, $Y_i = \frac{\partial}{\partial x_i}$ and $Z_j(x_i) = 0$ for $i = 1, ..., r$.*

In what follows, $\mathcal{C}$ is always a θ-admissible center and, given a point p of M, we denote by $I_{\mathcal{C}}$ the ideal $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p$ when there is no risk of confusion on the point p.

Proof. (Proposition 4.3.1): We prove this Proposition for θ a R-monomial singular distribution. In the general case, we only have to prove the first three statements, and it is not necessary to be careful with coordinate changes.

Fix a point $p \in \mathcal{C}$ and take a R-monomial coordinate system $x = (x_1, ..., x_n)$ of $\mathcal{O}_p$ and a R-monomial basis $\{X_1, ..., X_d\}$ of θ_p. If the center $\mathcal{C}$ is invariant by θ, the Proposition trivially follows from Proposition 4.2.2. So, suppose that the center $\mathcal{C}$ is not invariant by θ. There exists a maximal integer $d_0 > 0$ such that $\Gamma_{\theta,d_0}(\mathcal{I}_{\mathcal{C}}) = \mathcal{O}_M$. This implies that there exists $(f_1, ..., f_{d_0}) \subset I_{\mathcal{C}}$ such that the determinant of the matrix:

$$A = \left\| \begin{matrix} X_1(f_1) & ... & X_1(f_{d_0}) \\ \vdots & \ddots & \vdots \\ X_{d_0}(f_1) & ... & X_{d_0}(f_{d_0}) \end{matrix} \right\|$$

is an unity of $\mathcal{O}_p$. Without loss of generality, we assume that $X_i = \frac{\partial}{\partial x_i}$ for $i \leq d_0$ and $X_j(x_i) = 0$ for $i \leq d_0$ and $j > d_0$.

The next step is a change of coordinate system and R-monomial basis that diagonalizes the matrix A in $\mathcal{O}_p$. But we need to be careful with this process, so to not destroy the R-monomial structure.

Without loss of generality, we assume that $X_i(f_i)$ is an unity for $i \leq d_0$. Consider the change of coordinates $\bar{x}_1 = f_1$ and $\bar{x}_i = x_i$ otherwise. After the change we get:

$$X_1 = U\frac{\partial}{\partial \bar{x}_1}$$
$$X_i(\bar{x}_1) = g_i(\bar{x})$$

for some unit U of $\mathcal{O}_p$. Notice that X_1 is equivalent to $\frac{\partial}{\partial \bar{x}_1}$ and that $\{X_1, X_i - \frac{g_i}{U}X_1\}$ is a R-monomial basis of this new coordinate system.

Repeating this process for all the others f_i, with $i \leq d_0$ we can assume that $x = (x_1, ..., x_n)$ is a R-monomial coordinate system of $\mathcal{O}_p$ such that $f_i = x_i$ and $X_i = \frac{\partial}{\partial x_i}$ for $i \leq d_0$.

Let $Y_i := X_i$ for $i \leq d_0$ and $Z_j := X_{j+d_0}$ for $j \leq d_p - d_0$. It is clear that $\{Y_i\}$ is totally transverse to $I_{\mathcal{C}}$ and that $\{Y_i, Z_j\}$ is a R-monomial basis. Let us prove that $I_{\mathcal{C}}$ is invariant by $\{Z_j\}$: Since $\mathcal{I}_{\mathcal{C}}$ is θ-admissible, we conclude that $\Gamma_{d_0+1}(I_{\mathcal{C}}) \subset I_{\mathcal{C}}$. In particular, taking $Z = \sum h_j Z_j$ a $\mathcal{O}_p$-linear combination of the $\{Z_j\}$, we get:

$$det \left\| \begin{array}{cccc} Y_1(f_1) & ... & Y_1(f_{d_0}) & Y_1(g) \\ \vdots & \ddots & \vdots & \vdots \\ Y_{d_0}(f_1) & ... & Y_{d_0}(f_{d_0}) & Y_{d_0}(g) \\ Z(f_1) & ... & Z(f_{d_0}) & Z(g) \end{array} \right\| \in I_{\mathcal{C}} \longrightarrow det \left\| \begin{array}{cc} Id & Y_i(g) \\ 0 & Z(g) \end{array} \right\| \in I_{\mathcal{C}}$$

So $Z(g) \in I_{\mathcal{C}}$ for every $g \in I_{\mathcal{C}}$ and we conclude that $I_{\mathcal{C}}$ is invariant by $\{Z_j\}$.

In this coordinate system, we have that $I_{\mathcal{C}} = (x_1, ..., x_{d_0}, h_1, ..., h_s)$ where h_i does not depend on $(x_1, ..., x_{d_0})$.

Let U_p be an open neighborhood of p such that the coordinate system $x = (x_1, ..., x_n)$ is well defined over U_p and the vector fields X_i have representatives over U_p. Consider the map:

$$\Pi : \mathcal{O}_{U_p} \longrightarrow \mathcal{O}_{U_p}/(x_1, ..., x_{d_0})$$

We denote the image of the coordinate system $x = (x_1, ..., x_n)$ under Π by $\bar{x} = (\bar{x}_{d_0+1}, ..., \bar{x}_n)$. At this coordinate system the image $\bar{I}_{\mathcal{C}}$ of $I_{\mathcal{C}}$ is generated by $(\bar{h}_1, ..., \bar{h}_s)$, and the image $\bar{\theta}$ of the singular distribution θ is generated by $\{\bar{Z}_j\}$. This implies that $\bar{I}_{\mathcal{C}}$ is invariant by $\bar{\theta}$ and, by Proposition 4.2.1, there exists a change of coordinates such that $\bar{I}_{\mathcal{C}} = (\bar{x}_{d_0+1}, ..., \bar{x}_t)$ and $\{\bar{Z}_j\}$ is a R-monomial basis of $\bar{\theta}$. Since neither Z_j nor h_i depends on $(x_1, ..., x_{d_0})$, using the equivalent change of coordinates in $\mathcal{O}_p$ we get $I_{\mathcal{C}} = (x_1, ..., x_t)$ and $\{Y_i, Z_j\}$ a R-monomial basis such that $Z_j(x_i) = 0$ for $i < d_0$. □

Remark 4.3.2. *If a center $\mathcal{C}$ is θ-admissible, for each point p in $\mathcal{C}$, there exists two singular distributions germs θ_{inv} and θ_{tr} such that:*

- *The singular distribution θ_p is generated by $\{\theta_{inv}, \theta_{tr}\}$;*
- *The ideal $I_{\mathcal{C}}$ is invariant by θ_{inv};*
- *The ideal $I_{\mathcal{C}}$ is totally transverse by θ_{tr}.*

4.4 Proof of Theorem 4.1.1

We present a Proposition that trivially implies Theorem 4.1.1:

Proposition 4.4.1. *Let (M, θ, E) be a d-foliated manifold, $\mathcal{C}$ a θ-admissible center and $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ the blowing-up with center $\mathcal{C}$. For a point q in the exceptional divisor F, let $p = \sigma(q)$. Then there exists a coherent set of generators $\{Y_i, Z_j\}$ of θ_p with $i = 1, ..., r$ and $j = 1, ..., s$ (the same of Proposition 4.3.1) such that:*

- *The singular distribution $\theta'.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F)Y_i^*, Z_j^*\}.\mathcal{O}_q$.*
- *If the singular distribution θ is R-monomial, so is θ'.*

Proof. In the notation of the enunciate, consider the coordinate system $x = (x_1, ..., x_n)$ of $\mathcal{O}_p$ and the coherent set of generators $\{Y_i, Z_j\}$ of θ_p given by Proposition 4.3.1. In this case, we have that $I_{\mathcal{C}} := \mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (x_1, ..., x_t)$ is totally transverse to $\{Y_i\}$ and invariant by $\{Z_j\}$.

Consider a vector field X contained in θ_p:

$$X = \sum A_i \frac{\partial}{\partial x_i}$$

such that $I_{\mathcal{C}}$ is invariant by X. This implies that $(A_i)_{i\leq t} \subset I_{\mathcal{C}}$. After the blowing-up, without loss of generality, we can assume that q is the origin of the x_1 chart:

$$(x_1, y_2, ..., y_t, x_{t+1}, ..., x_n) = (x_1, x_1x_2, ..., x_1x_t, x_{t+1}, ..., x_n)$$

In this chart, we get:

$$X^* = A_1^* \frac{\partial}{\partial x_1} + \sum_{i=2}^{t} \frac{1}{x_1}(A_i^* - A_1^* y_i)\frac{\partial}{\partial y_i} + \sum_{i=t+1}^{n} A_i^* \frac{\partial}{\partial x_i}$$

Since $(A_i)_{i\leq t} \subset I_{\mathcal{C}}$, the function $\frac{1}{x_1}A_i^*$ is analytic for $i \leq t$. Thus, X^* is analytic. In particular, this implies that Z_j^* are all analytic.

In the other hand, the expressions of the blowing-up of the Y_i are given by the following expressions:

- If $t = r$, we can always assume that q is the origin of the x_1 chart:

$$\begin{array}{c} Y_1^* = \frac{\partial}{\partial x_1}^* = \frac{1}{x_1}(x_1\frac{\partial}{\partial x_1} - \sum_{i=t}^{m} y_i \frac{\partial}{\partial y_i}) \\ Y_i^* = \frac{\partial}{\partial x_i}^* = \frac{1}{x_1}\frac{\partial}{\partial y_i} \end{array} \tag{4.1}$$

- If $t > r$, then:
  - The point q can be assumed to be the origin of the x_1 chart and the transform expressions are the same as in (4.1);
  - The point q can be assumed to be the origin of the x_t chart:

$$Y_i^* = \frac{\partial}{\partial x_i}^* = \frac{1}{x_t}\frac{\partial}{\partial y_i} \tag{4.2}$$

  for all $i \leq r$.

Thus, they are all meromorphic and we must multiply by $\mathcal{O}(F)$ exactly one time to get analytic vector fields. Furthermore, we claim that $\{\mathcal{O}(F).Y_i^*, Z_j^*\}.\mathcal{O}_q$ is contained in $Der_{\mathcal{O}_q}(-logF)$. Indeed:

- It is clear by the expressions (4.1) and (4.2) that $\mathcal{O}(F)Y_i^*.\mathcal{O}_q$ leaves $F = \{x_1 = 0\}$ invariant.

- Consider a vector field X contained in θ_p such that $I_{\mathcal{C}}$ is invariant by X. Then:

$$[X^*(\mathcal{O}(F)) + \mathcal{O}(F)].\mathcal{O}_q = [X^*(I^*_{\mathcal{C}}) + I^*_{\mathcal{C}}].\mathcal{O}_q = (X(I_{\mathcal{C}}) + I_{\mathcal{C}})^*.\mathcal{O}_q = I^*_{\mathcal{C}}.\mathcal{O}_q = \mathcal{O}(F).\mathcal{O}_q$$

Thus $Z^*_j.\mathcal{O}_q$ is contained in $Der_{\mathcal{O}_q}(-logF)$.

By Lemma 3.2.2, the singular distribution $\theta^a.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F).Y^*_i, Z^*_j, W_k\}.\mathcal{O}_q$ where W_k is a combination of $Y^*_i.\mathcal{O}_q$ that is analytic and not generated by $\{\mathcal{O}(F).Y^*_i, Z^*_j\}.\mathcal{O}_q$. We have two cases to consider:

i) If $t = r$, then there exists a linear combination that generates $W_1 = \frac{\partial}{\partial x_1}$. But remark that W_1 is not contained in $Der_{\mathcal{O}_q}(-logF)$ and $\mathcal{O}(F)Y^*_1.\mathcal{O}_q = x_1\frac{\partial}{\partial x_1}$ is the minimal multiple of W_1 contained in $Der_{\mathcal{O}_q}(-logF)$. Thus: $\theta'.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F).Y^*_i, Z^*_j\}.\mathcal{O}_q$;

ii) If $t > r$, then it is clear by the expressions (4.1) and (4.2) that there is no possible W_k. This implies that $\theta'.\mathcal{O}_q$ is generated by $\{\mathcal{O}(F).Y^*_i, Z^*_j\}.\mathcal{O}_q$.

Furthermore, if the θ is R-monomial, we can write Z_j in one of the following forms:

$$Z_j = \sum_{i=1}^n \alpha_{i,j} x_i \frac{\partial}{\partial x_i}$$
$$Z_j = \frac{\partial}{\partial x_{k_j}}$$

with $\alpha_{i,j} \in R$ and $k_j > t$. Without loss of generality, we assume that q is in the x_1-chart so to get:

$$Z^*_j = \sum_{i=j}^t (\alpha_{i,j} - \alpha_{1,j}) y_i \frac{\partial}{\partial y_1} + \sum_{i=t+1}^n \alpha_{i,j} x_i \frac{\partial}{\partial x_i}$$
$$Z^*_j = \frac{\partial}{\partial x_{k_j}}$$

which are R-monomial at the origin. Moreover, using the expressions (4.1) and (4.2), it is clear that $\{\mathcal{O}(F)Y^*_i, Z^*_j\}$ is a R-monomial basis at the origin. Now, the the same proof of Lemma 2.2.1 is enough to show that θ' is also R-monomial at q. □

And we are finally ready to prove corollary 4.1.2:

Proof. (Corollary 4.1.2) By Lemma 2.3.4, a d-singular distribution θ is regular at a point p of M if, and only if, $\Gamma_{d,\theta}(m_p) + m_p = \mathcal{O}_M$, where m_p is the maximal ideal of the structural ideal $\mathcal{O}_p$.

Now, consider a point q of $M^{'}$ and let $p = \sigma(q)$. Since $\mathcal{C}$ is θ-invariant, by Proposition 4.4.1, the singular distribution $\theta^{'}$ is equal to the total transform θ^{*}. Since $m_p^{*} \subset m_q$, by Lemma 3.3.1:

$$\Gamma_{d,\theta'}(m_q) = \Gamma_{d,\theta^*}(m_q) \supset \Gamma_{d,\theta^*}(m_p^*) \supset [\Gamma_{d,\theta}(m_p)]^* = \mathcal{O}_{M'}$$

which proves the result. $\square$

Chapter 5

Two Resolutions subordinated to a foliation

5.1 A resolution Theorem for an invariant ideal sheaf

A resolution of $(M,\theta,\mathcal{I},E)$:

$$(M_r,\theta_r,\mathcal{I}_r,E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1,\theta_1,\mathcal{I}_1,E_1) \xrightarrow{\sigma_1} (M,\theta,\mathcal{I},E)$$

is said to be θ-*admissible* (resp. θ-*invariant*) if $\sigma_i : (M_i,\theta_i,\mathcal{I}_i,E_i) \longrightarrow (M_{i-1},\theta_{i-1},\mathcal{I}_{i-1},E_{i-1})$ is θ_{i-1}-admissible (resp. θ_{i-1}-invariant).

In this first section we consider d-foliated ideal sheaves $(M,\theta,\mathcal{I},E)$ such that $\mathcal{I}$ is invariant by θ. In this case, we obtain a resolution:

Theorem 5.1.1. *Let $(M,M_0,\theta,\mathcal{I},E)$ be a local d-foliated ideal sheaf. Suppose that $\mathcal{I}_0$ is invariant by θ_0, i.e. $\theta[\mathcal{I}].\mathcal{O}_{M_0} \subset \mathcal{I}.\mathcal{O}_{M_0}$. Then, there exists a resolution of $(M,M_0,\theta,\mathcal{I},E)$:*

$$\mathcal{R}_{inv}(M,M_0,\theta,\mathcal{I},E) : (M_r,\theta_r,\mathcal{I}_r,E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0,\theta_0,\mathcal{I}_0,E_0)$$

such that:

i) $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ *is a sequence of invariant blowings-up (in particular, a sequence of* θ*-admissible blowings-up);*

ii) *The composition* $\sigma = \sigma_1 \circ ... \circ \sigma_r$ *is an isomorphism over* $M_0 \setminus V(\mathcal{I}_0)$*;*

iii) *If* θ_0 *is R-monomial, then so is* θ_r*;*

iv) *If* θ_0 *is regular, then so is* θ_r*;*

v) $\mathcal{R}_{inv}$ *is a resolution functor that commutes with chain-preserving smooth morphisms.*

This functoriality property allows us to prove a global result just as in the Hironaka's Theorem:

Theorem 5.1.2. *Let* $(M, \theta, \mathcal{I}, E)$ *be a d-foliated ideal sheaf. Suppose that* $\mathcal{I}$ *is invariant by* θ*. Then there exists a proper analytic morphism:*

$$\mathcal{RG}_{inv}(M, \theta, \mathcal{I}, E) = \sigma : (\widetilde{M}, \widetilde{\theta}) \longrightarrow (M, \theta)$$

such that:

i) *for every* $M_0 \subset M$ *a relatively compact open set of* M*,* $\sigma|_{\sigma^{-1}M_0}$ *is the composition of the sequence of blowings-up* $\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E)$ *given on Theorem* 5.1.1*;*

ii) *If* θ *is R-monomial, so is* $\widetilde{\theta}$*;*

iii) *If* θ *is regular, so is* $\widetilde{\theta}$*;*

iv) σ *is an isomorphism over* $M \setminus V(\mathcal{I})$*;*

v) $\mathcal{RG}_{inv}(M, \theta, \mathcal{I}, E)$ *is a weak-resolution functor that commutes with chain-preserving smooth morphisms.*

The proof follows, mutatis mutandis, the same proof of Theorem 3.6.4.

5.2 Proof of Theorem 5.1.1

By the Hironaka's Theorem 3.6.1, there exists a resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of $(M, M_0, \theta, \mathcal{I}, E)$:

$$\mathcal{R}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

where $\sigma_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \longrightarrow (M_{i-1}, \theta_{i-1}, \mathcal{I}_{i-1}, E_{i-1})$ has center $\mathcal{C}_i$. **Claim:** The admissible sequence of blowings-up $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ is θ-invariant.

Proof. Suppose by induction that the centers $\mathcal{C}_i$ are θ_{i-1}-invariant for $i < k$. We need to verify that $\mathcal{C}_k$ is also θ_{k-1}-invariant (including for $k = 1$).

First, notice that $\mathcal{I}_{k-1}$ is invariant by θ_{k-1}. This follows from the induction hypotheses and a recursive use of the following lemma:

Lemma 5.2.1. *Consider an admissible blowing-up of order one* $\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$ *with a center* $\mathcal{C}$ *invariant by* θ*. Then* $\mathcal{I}'$ *is invariant by* θ'*.*

This Lemma is proved in the end of this section. We continue with the proof of the Claim: Since $\mathcal{C}_k$ is regular, by Lemma 2.4.1, we only need to verify that $\mathcal{C}_k$ is geometrically invariant by θ_{k-1}. We divide in two cases:

- First case: θ_{k-1} has leaf dimension one. Let $\mathcal{L}$ be a connected leaf of θ_{k-1} with non-empty intersection with $\mathcal{C}_k$. We need to verify that $\mathcal{L} \subset \mathcal{C}_k$, which is clear if $\mathcal{L}$ is zero-dimensional. So, assume that the leaf $\mathcal{L}$ is one-dimensional and take a point p in $\mathcal{C}_k \cap \mathcal{L}$.

  Locally, the singular distribution $\theta_{k-1}.\mathcal{O}_p$ is generated by a unique non-singular vector field germ X_p with a representative in an open neighborhood U_p of p. By the flow-box Theorem there exists a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ in U_p such that $X_p = \frac{\partial}{\partial x}$.

  Furthermore, without loss of generality, $U_p = V \times W$ where V is a domain of $\mathbb{K}^{n-1}$ and W a domain of $\mathbb{K}$ such that:

- The leaves of $\theta.\mathcal{O}_{U_p}$ are given by $\{q\} \times W$, for every $q \in V$;
- The divisor $E_{k-1} \cap U_p$ is equal to $E_V \times W$, where E_V is a SNC divisor over V;
- There exist a natural smooth morphism $\pi : V \times W \longrightarrow V$.

By the coherence of $\mathcal{I}_{k-1}$ and Proposition 4.2.2, without loss of generality, the ideal sheaf $\mathcal{I}_{U_p} := \mathcal{I}_{k-1}.\mathcal{O}_{U_p}$ has a finite set of generators $\{f_1(y), ..., f_k(y)\}$ independent of x.

Let $g_i \in \mathcal{O}_V$ be functions such that $g_i(y) = f_i(0, y)$ and $\mathcal{J}$ be the ideal sheaf over $\mathcal{O}_V$ generated by the $(g_1(y), ..., g_t(y))$: this clearly implies that $\mathcal{J}.\mathcal{O}_{V\times W} = \mathcal{I}_{U_p}$. Furthemore, the functorial statement of Hironaka's Theorem 3.6.1 guarantees that the resolution of $(U_p, \mathcal{I}.\mathcal{O}_{U_p}, E_{k-1} \cap U_p)$ and $(V, \mathcal{J}, E_V)$ commutes. This finally implies that $\mathcal{C}_k = \pi(\mathcal{C}_k) \times W$ (see remark 3.6.3) and the intersection $\mathcal{L} \cap \mathcal{C}_k$ must be open over $\mathcal{L}$. By analyticity it is also closed and $\mathcal{L} \subset \mathcal{C}_k$;

- Second case: θ_{k-1} has leaf dimensional d. Let $\mathcal{L}$ be a connected leaf of θ_{k-1} with non-empty intersection with $\mathcal{C}_k$. Take a point $p \in \mathcal{C}_k \cap \mathcal{L}$ and a coherent set of generators $\{X_1, ..., X_{d_p}\}$ of $\theta.\mathcal{O}_p$ with representatives defined in an open neighborhood U_p of p. Without loss of generality, every point q of $U_p \cap \mathcal{L}$ is contained in the image of the flux $(Fl^{X_1}_{t_1} \circ ... \circ Fl^{X_{d_p}}_{t_{d_p}})(p)$ for some $(t_1, ..., t_{d_p})$.

  If $\mathcal{L}_i$ is the leaf of X_i passing through p, by the first part of the proof $\mathcal{L}_i \subset \mathcal{C}_k$. A recursive use of this argument implies that $(Fl^{X_1}_{t_1} \circ ... \circ Fl^{X_{d_p}}_{t_{d_p}})(p) \in \mathcal{C}_k$ for small enough $(t_1, ..., t_{d_p})$ which implies that $\mathcal{L} \cap \mathcal{C}_k$ is open over $\mathcal{L}$. By analyticity it is also closed, which implies that $\mathcal{L} \subset \mathcal{C}_k$.

Thus, by induction, $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ is a sequence of θ-invariant admissible blowings-up of order one. □

The functoriality statement of Theorem 5.1.1 is a direct consequence of the functoriality of Theorem 3.6.1. The R-monomiality statement is a direct consequence of Theorem 4.1.1 and the regularity statement is a direct consequence of Corollary 4.1.2.

To finish, we only need to prove Lemma 5.2.1:

Proof. (Proof of Lemma 5.2.1): Since $\mathcal{C}$ is a regular sub-manifold geometrically invariant by θ, by Lemma 2.4.1 it is also invariant by θ. Furthermore, by Lemma 3.3.1 we have that:

$$\theta[\mathcal{I}_{\mathcal{C}}] \subset \mathcal{I}_{\mathcal{C}} \quad \longrightarrow \quad \theta^*[\mathcal{O}(F)] \subset \mathcal{O}(F)$$

Moreover, σ is a θ-admissible blowing-up and, by Proposition 4.4.1, $\theta' = \theta^*$. Thus, again by Lemma 3.3.1:

$$\theta'[\mathcal{I}'] + \mathcal{I}' = \theta^*[\mathcal{I}^*.\mathcal{O}(-F)] + \mathcal{I}' \subset$$
$$\theta^*[\mathcal{I}^*]\mathcal{O}(-F) + \mathcal{I}^*\theta^*[\mathcal{O}(-F)] + \mathcal{I}' = \mathcal{I}'$$

□

5.3 A resolution Theorem subordinated to a 1-foliation

In this section we consider foliated ideal sheaves $(M, \theta, \mathcal{I}, E)$ such that θ has leaf dimension one. In this case, our main result is the following:

Theorem 5.3.1. *Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local 1-foliated ideal sheaf. Then, there exists a resolution of $(M, M_0, \theta, \mathcal{I}, E)$:*

$$\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ is a sequence of θ-admissible blowings-up;

ii) The composition $\sigma = \sigma_1 \circ ... \circ \sigma_1$ is an isomorphism over $M_0 \setminus V(\mathcal{I}_0)$;

iii) If θ_0 is R-monomial, then so is θ_r;

iv) $\mathcal{R}_1$ is a resolution functor that commutes with 1-chain-preserving smooth morphisms.

v) If ω is a d-involutive distribution such that $\mathcal{I}$ is ω-invariant and $\{\omega, \theta\}$ is an involutive $d+1$-singular distribution, the sequence of blowings-up $\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E)$ is ω-invariant;

Remark 5.3.2. *The functorial property* [v] *of Theorem* 5.3.1 *will be used in the proof of Proposition* 6.2.4 *below.*

The functorial property [iv] of Theorem 5.3.1 allows us to prove a global result just as in the Hironaka's Theorem:

Theorem 5.3.3. *Let* $(M, \theta, \mathcal{I}, E)$ *be a* 1*-foliated ideal sheaf. Then there exists a proper analytic morphism:*

$$\mathcal{RG}_1(M, \theta, \mathcal{I}, E) = \sigma : (\widetilde{M}, \widetilde{\theta}) \longrightarrow (M, \theta)$$

such that:

i) for every $M_0 \subset M$ *relatively compact open set of* M*,* $\sigma|_{\sigma^{-1}M_0}$ *is the composition of the sequence of blowings-up* $\mathcal{R}_1(M, M_0, \theta, \mathcal{I}, E)$ *given on Theorem* 5.3.1*;*

ii) If θ *is R-monomial, so is* $\widetilde{\theta}$*;*

iii) σ *is an isomorphism over* $M \setminus V(\mathcal{I})$*;*

iv) $\mathcal{RG}_1(M, \theta, \mathcal{I}, E)$ *is a weak-resolution functor that commutes with* 1*-chain-preserving smooth morphisms.*

The proof follows, mutatis mutandis, the same proof of Theorem 3.6.4.

5.4 Proof of Theorem 5.3.1

Let us start giving the intuitive idea of the proof. Given a local 1-foliated ideal sheaf $(M, M_0, \theta, \mathcal{I}, E)$ the main invariant we consider is the pair:

$$(\nu, t) := (\nu_{M_0}(\theta, \mathcal{I}), type_{M_0}(\theta, \mathcal{I}))$$

where we recall that the tg-order $\nu_{M_0}(\theta, \mathcal{I})$ stands for the length of the tangency chain $\mathcal{T}g(\theta, \mathcal{I})$ over M_0 and the $type_{M_0}(\theta, \mathcal{I})$ stands for the type of this chain at M_0 (see section 2.5).

The proof of the Theorem relies on two steps:

- First step: $(\nu, 2) \longrightarrow (\nu, 1)$;
- Second step: $(\nu, 1) \longrightarrow (\nu - 1, 2)$.

which shows that this invariant drops. The following Propositions formalize the above steps:

Proposition 5.4.1. *Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local d-foliated ideal sheaf and suppose that $type_{M_0}(\theta, \mathcal{I}) = 2$. Then, there exists a sequence of θ-invariant admissible blowings-up of order one:*

$$\mathcal{S}_1(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) $\nu_{M_r}(\theta_r, \mathcal{I}_r) \leq \nu_{M_0}(\theta, \mathcal{I})$ and $type_{M_r}(\theta_r, \mathcal{I}_r) = 1$;

ii) If ω is a d'-involutive distribution such that $\mathcal{I}$ is ω-invariant and $\{\omega, \theta\}$ generates an involutive $d + d'$-singular distribution, the sequence of blowings-up is ω-invariant;

iii) If $\phi : (M, M_0, \theta, \mathcal{I}, E_M) \longrightarrow (N, N_0, \omega, \mathcal{J}, E_N)$ is a chain-preserving smooth morphism, then there exists a chain-preserving smooth morphism $\psi : (M_r, \theta_r, \mathcal{I}_r, E_{M,r}) \longrightarrow (N_r, \omega_r, \mathcal{J}_r, E_{N,r})$.

Proposition 5.4.2. *Let $(M, M_0, \theta, \mathcal{I}, E)$ be a local 1-foliated ideal sheaf and suppose that $type_{M_0}(\theta, \mathcal{I}) = 1$. Then, there exists a sequence of θ-admissible blowings-up of order one:*

$$\mathcal{S}_2(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) $\nu_{M_r}(\theta_r, \mathcal{I}_r) < \nu_{M_0}(\theta, \mathcal{I})$;

ii) If ω is a d-involutive distribution such that $\mathcal{I}$ is ω-invariant and $\{\omega, \theta\}$ generates an involutive $d + 1$-singular distribution, the sequence of blowings-up is ω-invariant;

iii) If $\phi : (M, M_0, \theta, \mathcal{I}, E_M) \longrightarrow (N, N_0, \omega, \mathcal{J}, E_N)$ *is a* 1*-chain-preserving smooth morphism, then there exists a* 1*-chain-preserving smooth morphism* $\psi : (M_r, \theta_r, \mathcal{I}_r, E_{M,r}) \longrightarrow (N_r, \omega_r, \mathcal{J}_r, E_{N,r})$*.*

These two Propositions will be proved in the next two sections. For now, we assume them so to prove Theorem 5.3.1:

Proof. (Theorem 5.3.1): Let N be a relatively compact open subset of M. The tg-order and type $(\nu(N), t(N)) := (\nu_N(\theta, \mathcal{I}), type_N(\theta, \mathcal{I}))$ are well-defined.

In particular, if N_1 and N_2 are two relatively open subsets of M such that $N_1 \subset N_2$, then $(\nu(N_1), t(N_1)) \leq (\nu_{N_2}(\theta, \mathcal{I}), type_{N_2}(\theta, \mathcal{I}))$ (where the order is lexicographically).

Fix N a relatively compact open subset of M such that $\overline{M}_0 \subset N$. We claim that there exists $\overline{M}_0 \subset N_0 \subset N$ a relatively compact open subset N_0 that satisfies $\overline{M}_0 \subset N_0 \subset \overline{N}_0 \subset N$ and a sequence of θ-admissible blowings-up:

$$(N_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (N_1, \theta_1, E_1) \xrightarrow{\sigma_1} (N_0, \theta_0, E_0)$$

such that $(\nu(N_r), t(N_r)) < (\nu(N), t(N))$.

We prove the claim: Take any relatively compact open subset N_0 satisfying $\overline{M}_0 \subset N_0 \subset \overline{N}_0 \subset N$. If $(\nu(N_0), t(N_0)) < (\nu(N), t(N))$, the claim is obvious, so assume that $(\nu(N_0), t(N_0)) = (\nu(N), t(N))$. By Propositions 5.4.1 or 5.4.2 applied to $(N, N_0, \theta, \mathcal{I}, E)$, there exists a sequence of θ-admissible blowings-up:

$$(N_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (N_1, \theta_1, E_1) \xrightarrow{\sigma_1} (N_0, \theta_0, E_0)$$

such that $(\nu(N_r), t(N_r)) < (\nu(N_0), t(N_0)) = (\nu(N), t(N))$, which proves the claim.

As a mater of fact, the recursive use of this claim will prove the Theorem: since the pair (ν, t) is bounded below by $(0, 1)$ one cannot recursively apply the claim an infinite number of times. Once the process stops, we restrict all blowings-up to M_0 and its transforms, which is well-defined because of the functoriality statements of Propositions 5.4.1 and 5.4.2.

The functoriality statements [*iv*] and [*v*] of the Theorem follows directly from the functoriality statements [*ii*] and [*iii*] of Propositions 5.4.1 and 5.4.2. Furthermore, as all blowings-up are θ-admissible, by Theorem 4.1.1 if $\theta.\mathcal{O}_{M_0}$ is R-monomial, so will be its transforms. □

5.5 Proof of Proposition 5.4.1

Consider a d-foliated ideal sheaf $(M, M_0, \theta, \mathcal{I}, E)$ such that $type_{M_0}(\theta, \mathcal{I}) = 2$. Let $\nu = \nu_{M_0}(\theta, \mathcal{I})$ and $\mathcal{C}l(\mathcal{I}) := H(\theta, \mathcal{I}, \nu)$ (see section 2.5). By Theorem 5.1.1, there exists a θ-invariant resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of $(M, M_0, \theta, \mathcal{C}l(\mathcal{I}), E)$:

$$(M_r, \theta_r, (\mathcal{C}l(\mathcal{I}))_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, (\mathcal{C}l(\mathcal{I}))_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, (\mathcal{C}l(\mathcal{I}))_0, E_0)$$

Claim 1: The sequence of blowings-up $\vec{\sigma}$ is θ-admissible of order one for $(M, M_0, \theta, \mathcal{I}, E)$. Furthermore:

$$(\mathcal{C}l(\mathcal{I}))_j = \mathcal{C}l(\mathcal{I}_j)$$

for all $j \leq r$.

The main step for proving the claim is the following Lemma:

Lemma 5.5.1. *Let $\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$ be an invariant θ-admissible blowing-up of order one for $(M, \theta, \mathcal{C}l(\mathcal{I}), E)$. Then:*

$$H(\theta', \mathcal{I}', i) = H(\theta, \mathcal{I}, i)' = H(\theta, \mathcal{I}, i)^* \mathcal{O}(-F)$$

for every $i \leq \nu$. In particular: $(\mathcal{C}l(\mathcal{I}))' = \mathcal{C}l(\mathcal{I}')$.

Which we prove in the end of this section. We now proceed with the proof of the Claim 1:

Proof. (Claim 1) Suppose by induction that, for $i < k$, the sequence $(\sigma_1, ..., \sigma_i)$ is admissible of order one for $(M, M_0, \theta, \mathcal{I}, E)$ and:

$$(\mathcal{C}l(\mathcal{I}))_j = \mathcal{C}l(\mathcal{I}_j)$$

for $j \leq i$. We prove the result for $i = k$ (including $k = 1$). Since σ_k is a blowing-up of order one for $(M_{k-1}, \theta_{k-1}, (\mathcal{C}l(\mathcal{I}))_{k-1}, E_{k-1})$, by the induction hypotheses, it is also of order one for $(M_{k-1}, \theta_{k-1}, \mathcal{C}l(\mathcal{I}_{k-1}), E_{k-1})$. Finally, since $\mathcal{I}_{k-1} \subset \mathcal{C}l(\mathcal{I}_{k-1})$, the blowing-up σ_k is of order one for $(M_{k-1}, \theta_{k-1}, \mathcal{I}_{k-1}, E_{k-1})$, which implies that $(\sigma_1, ..., \sigma_i)$ is of order one for $(M, M_0, \theta, \mathcal{I}, E)$.

Now, by Lemma 5.5.1 and the induction hypotheses:

$$(\mathcal{C}l(\mathcal{I}))_k = (\mathcal{C}l(\mathcal{I}))'_{k-1} = (\mathcal{C}l(\mathcal{I})_{k-1})' = \mathcal{C}l(\mathcal{I}'_{k-1}) = \mathcal{C}l(\mathcal{I}_k)$$

□

This implies that $\vec{\sigma}$ gives rise to an invariant θ-admissible sequence of blowings-up of order one for $(M, M_0, \theta, \mathcal{I}, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

$$H(\theta, \mathcal{I}_r, \nu) = \mathcal{C}l(\mathcal{I}_r) = (\mathcal{C}l(\mathcal{I}))_r = \mathcal{O}_{M_r}$$

which implies that $\nu_{M_r}(\theta_r, \mathcal{I}_r) \leq \nu_{M_0}(\theta, \mathcal{I})$ and $type_{M_r}(\theta_r, \mathcal{I}_r) = 1$.

We now prove the functorial statement $[ii]$ of the Proposition:

Claim 2: The ideal sheaves $H(\mathcal{I}, \theta, i)$ are ω-invariant for all $i \in \mathbb{N}$.

Proof. We prove the result by induction on i. For $i = 0$, the result follows by hypotheses, so assume the result proved for $i = k$. Since $\{\theta, \omega\}$ is an involutive singular distribution, the following calculation shows that the Claim 2 is valid for $k + 1$:

$$\omega[H(\mathcal{I}, \theta, k+1)] = \omega[\theta[H(\mathcal{I}, \theta, k)] + H(\mathcal{I}, \theta, k)] \subset$$
$$\theta[\omega[H(\mathcal{I}, \theta, k)]] + \theta[H(\mathcal{I}, \theta, k)] + \omega[H(\mathcal{I}, \theta, k)] \subset H(\mathcal{I}, \theta, k+1)$$

□

So, by part [*iv*] of Theorem 5.1.1, the resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ is also ω-invariant, because the identity is a chain-preserving smooth morphism between $(M, M_0, \theta, \mathcal{C}l(\mathcal{I}), E)$ and $(M, M_0, \{\theta, \omega\}, \mathcal{C}l(\mathcal{I}), E)$. This proves the functorial statement [*ii*] of the Proposition.

We now prove the functorial statement [*iii*] of the Proposition:

Let $\phi : (M, M_0, \theta, \mathcal{I}, E_M) \longrightarrow (N, N_0, \omega, \mathcal{J}, E_N)$ be a chain-preserving smooth morphism. Let $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ and $\vec{\tau} = (\tau_1, ..., \tau_r)$ be the sequences of blowings-up given in the Proposition (the length of the sequence may be chosen to be the same because of the functoriality of Theorem 5.1.1). Furthermore, for any ideal sheaf $\mathcal{K}$ over N_{i-1}, because of the functoriality of Theorem 5.1.1, we deduce that:

$$(\sigma_i)^*(\mathcal{K}.\mathcal{O}_{M_{i-1}}) = (\tau_i^*\mathcal{K}).\mathcal{O}_{M_i}$$

In particular, if $F_{M,i}$ is the exceptional divisor of the blowing-up $\sigma_i : M_i \longrightarrow M_{i-1}$ and $F_{N,i}$ is the exceptional divisor of the blowing-up $\tau_i : N_i \longrightarrow N_{i-1}$, we have that:

$$\mathcal{O}(-F_{N,i}).\mathcal{O}_{M_i} = \mathcal{O}(-F_{M,i})$$

Claim 3: The following equality holds:

$$H(\mathcal{J}_i, \omega_i, j).\mathcal{O}_{M_i} = H(\mathcal{I}_i, \theta_i, j)$$

for $i \leq r$ and $j \in \mathbb{N}$.

Proof. Suppose by induction that $H(\omega_i, \mathcal{J}_i, j).\mathcal{O}_{M_i} = H(\theta_i, \mathcal{I}_i, j)$ for $i < k$ and any $j \in \mathbb{N}$. Then:

$$H(\omega_k, \mathcal{J}_k, j).\mathcal{O}_{M_k} = (\mathcal{O}(-F_{N,k})\tau_k^* H(\omega_{k-1}, \mathcal{J}_{k-1}, j)).\mathcal{O}_{M_k} =$$
$$= \mathcal{O}(-F_{M,k})\sigma_k^* H(\theta_{k-1}, \mathcal{I}_{k-1}, j) = H(\theta_k, \mathcal{I}_k, j)$$

for any $j \in \mathbb{N}$, which proves Claim 3. □

It is clear that Claim 3 implies the functoriality statement [*iii*] of the Proposition.

To finish, we only need to prove Lemma 5.5.1:

Proof. (Lemma 5.5.1) First, notice that, since $H(\theta,\mathcal{I},i) \subset \mathcal{C}l(\mathcal{I})$ for $i \leq \nu$, the blowing-up is also of order one for $(M,\theta,H(\theta,\mathcal{I},i),E)$.

By hypotheses, the center $\mathcal{C}$ is invariant by θ and, by Proposition 4.3.1, the adapted analytic strict transform $\theta^{'}$ coincides with the total transform θ^{*}. Thus, if F is the exceptional divisor and $\mathcal{J}$ is a coherent ideal sheaf, by Lemma 3.3.1:

$$\theta^{'}[\mathcal{O}(F)] \subset \mathcal{O}(F) \Rightarrow \mathcal{J}\theta^{'}[(\mathcal{O}(-F))] \subset \mathcal{J}\mathcal{O}(-F)$$

In particular, this implies that:

$$\theta^{'}[\mathcal{J}\mathcal{O}(-F)] + \mathcal{J}\mathcal{O}(-F) = \mathcal{O}(-F)(\theta^{'}[\mathcal{J}] + \mathcal{J})$$

Now, it rests to prove that the following equality:

$$H(\theta^{'},\mathcal{I}^{'},i) = H(\theta,\mathcal{I},i)^{*}\mathcal{O}(-F)$$

is valid for all $i \leq \nu$. Indeed, suppose by induction that the equality is valid for $i < k$ (notice that for $k = 0$, the equality is trivial). Since the blowing-up is of order one for $(M,\theta,H(\theta,\mathcal{I},k),E)$, we have that:

$$\begin{gathered} H(\theta^{'},\mathcal{I}^{'},k) = H(\theta^{'},\mathcal{I}^{'},k-1) + \theta^{'}[H(\theta^{'},\mathcal{I}^{'},k-1)] = \\ H(\theta^{'},\mathcal{I}^{'},k-1) + \theta^{*}[H(\theta,\mathcal{I},k-1)^{*}\mathcal{O}(-F)] = \\ \mathcal{O}(-F)\{H(\theta,\mathcal{I},k-1) + \theta[H(\theta,\mathcal{I},k-1)]\}^{*} = \\ \mathcal{O}(-F)H(\theta,\mathcal{I},k)^{*} = H(\theta,\mathcal{I},k)^{'} \end{gathered}$$

which proves the equality and the Lemma. □

5.6 Proof of Proposition 5.4.2

Consider a 1-foliated ideal sheaf $(M,M_0,\theta,\mathcal{I},E)$ such that $type_{M_0}(\theta,\mathcal{I}) = 1$. Let $\nu = \nu_{M_0}(\theta,\mathcal{I})$ and $\mathcal{M}tg(\mathcal{I}) := H(\theta,\mathcal{I},\nu-1)$. By Theorem 3.6.1, there exists a resolution $\vec{\sigma} = (\sigma_1,...,\sigma_r)$ of $(M,M_0,\theta,\mathcal{M}tg(\mathcal{I}),E)$:

$$(M_r,\theta_r,(\mathcal{M}tg(\mathcal{I}))_r,E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0,\theta_0,(\mathcal{M}tg(\mathcal{I}))_0,E_0)$$

Claim 1: the sequence of blowing-up $\vec{\sigma}$ is θ-admissible. Furthermore, the center of blowing-up $\mathcal{C}_k$ are totally transverse to θ_{k-1}.

Proof. Suppose by induction that, for $i < k$:

a) The sequence $(\sigma_1, ..., \sigma_i)$ of blowing-up is θ-admissible;

b) For $p \in V((\mathcal{M}tg(\mathcal{I}))_i)$, there exists a coherent coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ of $\mathcal{O}_p$ such that $x \in \mathcal{M}tg(\mathcal{I})_i.\mathcal{O}_p$ and $\frac{\partial}{\partial x}$ generates $\theta_{i,p} := \theta_i.\mathcal{O}_p$.

We prove the result for k:

- Step $k = 1$. In this case, since the type of the tangency chain $\mathcal{T}g(\theta, \mathcal{I})$ is one, for every $p \in V((\mathcal{M}tg(\mathcal{I}))_0)$ the distribution θ_p is generated by a non-singular vector field. By the flow-box Theorem, there exists a coherent coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ of $\mathcal{O}_p$ such that θ_p is generated by $X := \frac{\partial}{\partial x}$.

  Furthermore, there exists $g \in \mathcal{T}g(\theta, \mathcal{I}).\mathcal{O}_p$ such that $X(g)$ is an unity of $\mathcal{O}_p$. This implies that $g = xU(x, y) + h(y)$ where $U(x, y)$ is an unity. Making the change of coordinates $\bar{x} = g(x, y)$ and $\bar{y} = y$ we get a coordinate system such that $\bar{x} \in \mathcal{T}g(\theta, \mathcal{I}).\mathcal{O}_p$ and θ_p is generated by $X = V\frac{\partial}{\partial \bar{x}}$, where $V = U(x, y) + xU_x(x, y)$ is an unity.

- Step $k > 1$. Take any $p \in V((\mathcal{M}tg(\mathcal{I}))_{k-1})$. Since the center $\mathcal{C}_k$ of the blowing-up $\sigma_k : M_k \longrightarrow M_{k-1}$ is contained in $V((\mathcal{M}tg(\mathcal{I}))_{k-1})$, by the induction hypotheses [*b*] it is also totally transverse to θ at p. This implies that the sequence $(\sigma_1, ..., \sigma_k)$ of blowing-up is θ-admissible.

  Consider $q \in V((\mathcal{M}tg(\mathcal{I}))_k)$ and $p = \sigma_k(q)$. If σ_k is a local isomorphism over q, the result is trivial, so we assume that $q \in F_k$. By the induction hypotheses [*ii*], there exists a coherent coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ of $\mathcal{O}_p$ such that $x \in \mathcal{M}tg(\mathcal{I})_{k-1}.\mathcal{O}_p$ and $\frac{\partial}{\partial x}$ generates $\theta_{k-1,p}$. Since $\mathcal{C} \subset V(\mathcal{M}tg(\mathcal{I})_{k-1})$, without loss of generality $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (x, y_1, ..., y_t)$ and q is the origin of the y_1-chart. It is now easy to compute the transforms of the blowing-up at q and see that the induction hypotheses $[b]$ is valid for $i = k$.

□

Using claim 1 together and Proposition 4.3.1, we deduce that:

$$\theta_{k+1} = \mathcal{O}(F_k)\sigma_k^*(\theta_k) \tag{5.1}$$

and, since the center is totally transverse:

$$\theta_{k+1}[\mathcal{O}(F_k)] \subset \mathcal{O}(F_k) \tag{5.2}$$

to simplify notation, define $(i\sigma_k) = \sigma_{i+1} \circ ... \circ \sigma_k$ for $i < k$, $(k\sigma_k) = id$ and $\bar{\sigma}_k = \sigma_1 \circ ... \circ \sigma_k$. We also introduce:

$$\mathcal{K}_k(\alpha) = \prod_{i=1}^{k-1}[(i\sigma_{k-1})^*\mathcal{O}(\alpha F_i)]$$

Using the equation (5.1) recursively, we get that:

$$\theta_k = \mathcal{K}_k(1)\bar{\sigma}_k^*\theta \tag{5.3}$$

Using the equation (5.2) recursively, we get that:

$$\theta_k(\mathcal{K}_k(\alpha)) \subset \mathcal{K}_k(\alpha) \tag{5.4}$$

Furthermore, given an ideal sheaf $\mathcal{J}$, equation (5.4) implies that:

$$\theta_k[\mathcal{K}_k(\alpha)\mathcal{J}] + \mathcal{K}_k(\alpha)\mathcal{J} = \mathcal{K}_k(\alpha)(\mathcal{J} + \theta_k[\mathcal{J}]) \tag{5.5}$$

Claim 2: the sequence of blowing-up $\vec{\sigma}$ is of order one for $(M, M_0, \theta, \mathcal{I}, E)$ and:

$$H(\theta_k, \mathcal{I}_k, j) = \mathcal{K}_k(-1).\sum_{i=0}^{j}\mathcal{K}_k(i)\bar{\sigma}_k^* H(\theta_0, \mathcal{I}_0, i) \tag{5.6}$$

for all $j \leq \nu$.

Proof. Suppose by induction that, for $k < k_0$:

a) The sequence $(\sigma_1, ..., \sigma_i)$ of blowing-up is of order one for $(M, M_0, \theta, \mathcal{I}, E)$;

b) Equation (5.6) is valid for $k < k_0$.

We prove the result for k_0. Notice that the step $k_0 = 0$ is trivial, so we can treat only the case $k_0 > 0$:

- Step $k_0 > 0$. Using the induction hypotheses $[ii]$ we deduce that:

$$H(\theta_{k_0-1}, \mathcal{I}_{k_0-1}, j) = \mathcal{K}_{k_0-1}(-1). \sum_{i=0}^{j} \mathcal{K}_{k_0-1}(i) \bar{\sigma}^*_{k_0-1} H(\theta_0, \mathcal{I}_0, i) \subset$$
$$\subset \mathcal{K}_{k_0-1}(-1). \sum_{i=0}^{j} \bar{\sigma}^*_{k_0-1} H(\theta_0, \mathcal{I}_0, i) = \mathcal{K}_{k_0-1}(-1) \bar{\sigma}^*_{k_0-1} H(\theta_0, \mathcal{I}_0, j)$$

In particular:

$$\mathcal{M}tg(\mathcal{I}_{k_0-1}) \subset (\mathcal{M}tg(\mathcal{I}_0))_{k_0-1} \tag{5.7}$$

Which implies that $\mathcal{C}_{k_0} \subset V(\mathcal{M}tg(\mathcal{I}_{k_0-1}))$. So the sequence of blowings-up $(\sigma_1, ..., \sigma_{k_0})$ is of order one for $(M, M_0, \theta, \mathcal{I}, E)$.

We now verify the induction hypotheses $[b]$ for $k = k_0$ by induction on j. Indeed, the formula is clearly true for $j = 0$, so consider it proved for $j < j_0$. We prove it for $j = j_0$. Indeed, by equation (5.5):

$$H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0) = H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \theta_{k_0}[H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1)] =$$
$$= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \theta_{k_0}[\mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i) \bar{\sigma}^*_{k_0} H(\theta_0, \mathcal{I}_0, i)] =$$
$$= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i) \theta_{k_0}[\bar{\sigma}^*_{k} H(\theta_0, \mathcal{I}_0, i)]$$

Now, using equation (5.3) and Lemma 3.3.1, we can continue the deduction:

$$= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i+1) \bar{\sigma}^*_{k_0}(\theta[H(\theta_0, \mathcal{I}_0, i)]) =$$
$$= H(\theta_{k_0}, \mathcal{I}_{k_0}, j_0 - 1) + \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0-1} \mathcal{K}_{k_0}(i+1) \bar{\sigma}^*_{k_0}(H(\theta_0, \mathcal{I}_0, i+1)) =$$
$$= \mathcal{K}_{k_0}(-1). \sum_{i=0}^{j_0} \mathcal{K}_{k_0}(i) \bar{\sigma}^*_{k_0} H(\theta_0, \mathcal{I}_0, i)$$

So the formula is proved.

□

Claim 2 implies that the sequence of θ-admissible blowings-up $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of order one for $(M, M_0, \theta, \mathcal{M}tg(\mathcal{I}), E)$ is also of order one for $(M, M_0, \theta, \mathcal{I}, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that: **Claim 3:** The tg-order $\nu(\theta_r, \mathcal{I}_r)$ is strictly smaller then ν.

Proof. Let $\bar{\sigma} = \sigma_1 \circ ... \circ \sigma_r$ and we recall that $(\mathcal{M}tg(\mathcal{I}))_r = \mathcal{O}_{M_r}$, which implies that $\bar{\sigma}^* H(\theta_0, \mathcal{I}_0, \nu - 1) = \mathcal{K}_r(1)$. By claim 2, we deduce that:

$$H(\theta_r, \mathcal{I}_r, \nu - 1) = \mathcal{K}_r(-1).\sum_{i=0}^{\nu-1} \mathcal{K}_r(i)\bar{\sigma}^* H(\theta_0, \mathcal{I}_0, i) =$$
$$= \mathcal{K}_r(-1).\sum_{i=0}^{\nu-2} \mathcal{K}_r(i)\bar{\sigma}^* H(\theta_0, \mathcal{I}_0, i) + \mathcal{K}_r(\nu - 2) =$$
$$= H(\theta_r, \mathcal{I}_r, \nu - 2) + \mathcal{K}_r(\nu - 2)$$

which implies that:

$$\theta_r[H(\theta_r, \mathcal{I}_r, \nu - 1)] + H(\theta_r, \mathcal{I}_r, \nu - 1) = H(\theta_r, \mathcal{I}_r, \nu - 1) + \theta_r[\mathcal{K}_r(\nu - 2)] \subset$$
$$\subset H(\theta_r, \mathcal{I}_r, \nu - 1) + \mathcal{K}_r(\nu - 2) = H(\theta_r, \mathcal{I}_r, \nu - 1)$$

Which proves that the chain is stabilizing in at most $\nu - 1$ steps. □

We now prove the functorial statement [*ii*] of the proposition:

Claim 4: The ideal sheaves $H(\mathcal{I}, \theta, i)$ are ω-invariant for all $i \in \mathbb{N}$.

Proof. The proof follows, mutantis mutatis, the same proof of Claim 2 contained in the proof of Proposition 5.4.1. □

So, by part [*iv*] of Theorem 5.1.1, the resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ is also ω-invariant, because the identity is a chain-preserving smooth morphism between $(M, M_0, 0, \mathcal{M}tg(\mathcal{I}), E)$ and $(M, M_0, \omega, \mathcal{M}tg(\mathcal{I}), E)$.

We now prove the functorial statement [*iii*] of the Proposition.

Let $\phi : (M, M_0, \theta, \mathcal{I}, E_M) \longrightarrow (N, N_0, \omega, \mathcal{J}, E_N)$ be a 1-chain-preserving smooth morphism. Consider $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ and $\vec{\tau} = (\tau_1, ..., \tau_r)$ the sequences of blowings-up given in the Proposition (the length of the sequence may be chosen to be the same because of the functoriality of Theorem 3.6.1). Furthermore, for any ideal sheaf $\mathcal{K}$ over N_{i-1}, because of the functoriality of Theorem 3.6.1, we deduce that:

$$(\sigma_i)^*(\mathcal{K}.\mathcal{O}_{M_{i-1}}) = (\tau_i^* \mathcal{K}).\mathcal{O}_{M_i}$$

In particular, if $F_{M,i}$ is the exceptional divisor of the blowing-up $\sigma_i : M_i \longrightarrow M_{i-1}$ and $F_{N,i}$ is the exceptional divisor of the blowing-up $\tau_i : N_i \longrightarrow N_{i-1}$, we have that:

$$\mathcal{O}(-F_{N,i}).\mathcal{O}_{M_i} = \mathcal{O}(-F_{M,i})$$

Furthermore, define $\mathcal{K}_{M,k}(\alpha)$ and $\mathcal{K}_{N,k}(\alpha)$ in the obvious way. We have that:

$$\mathcal{K}_{N,i}(\alpha).\mathcal{O}_{M_i} = \mathcal{K}_{M,i}(\alpha)$$

Claim 5: The following equality holds:

$$H(\mathcal{J}_i, \omega_i, j).\mathcal{O}_{M_i} = H(\mathcal{I}_i, \theta_i, j)$$

for $i \leq r$ and $j \in \mathbb{N}$.

Proof. Suppose by induction that $H(\omega_i, \mathcal{J}_i, j).\mathcal{O}_{M_i} = H(\theta_i, \mathcal{I}_i, j)$ for $i < k_0$ and any $j \in \mathbb{N}$. Then:

$$H(\omega_{k_0}, \mathcal{J}_{k_0}, j).\mathcal{O}_{M_{k_0}} = (\mathcal{K}_{N,k_0}(-1) \sum_{i=0}^{j} \mathcal{K}_{N,k_0}(i)\bar{\tau}_{k_0}^{*} H(\omega_0, \mathcal{J}_0, i)).\mathcal{O}_{M_{k_0}} =$$
$$\mathcal{K}_{M,k_0}(-1) \sum_{i=0}^{j} \mathcal{K}_{M,k_0}(i)\bar{\sigma}_{k_0}^{*} H(\theta_0, \mathcal{I}_0, i) = H(\theta_{k_0}, \mathcal{I}_{k_0}, j)$$

for any $j \in \mathbb{N}$, which proves the claim. □

It is clear that Claim 5 implies the functoriality statement [*iii*] of the Proposition.

5.7 Appendix: Considerations about the general case

In the general case of a local d-foliated ideal sheaf $(M, M_0, \theta, \mathcal{I}, E)$, obtaining a global resolution seems to be a challenging problem that may need new ideas. To discuss the difficulty of this problem, we follow a more intuitive presentation in this section.

We start giving two intuitive reasons of why the one dimensional case is technically simpler:

i) The only non-trivial generalized Fitting operation is $\Gamma_{\theta,1}$. Intuitively, this implies that the tangency chain $\mathcal{T}g(\theta, \mathcal{I})$ completely describes the intersection between the variety $V(\mathcal{I})$ and the singular distribution θ.

ii) There exists a dichotomy of θ-admissible centers: either they are θ-invariant, or they are θ-totally transverse.

These two statements don't hold when θ is a d-singular distribution. There exists two naive ideas (maybe complementary) to continue the search of a resolution:

i) An induction over the leaf dimension d of θ;

ii) A refinement of the invariants, using the other k-generalized Fitting operations $\Gamma_{\theta,k}$.

At first, we tried to follow a strictly [I] approach, believing that most of the difficulty could disappear under an induction machinery. This is also the main idea behind the next chapter, where a local uniformization is presented for 2-singular distributions. But, for the general problem, the fact [ii] turned out to be a serious difficulty to this naive idea. A generalization of Proposition 5.4.2 may follow from the exact same arguments of the proof of that result. But this seems technically difficult to prove, because the tg-order $\nu(\theta,\mathcal{I})$ is not stable by θ-admissible blowings-up when $d>1$. This is illustrated in the following example:

Example 1: Let $(M,\theta,\mathcal{I},E)=(\mathbb{C}^3,\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\},(x^2+zy^3),\emptyset)$. Notice that:

$$\nu(\theta,\mathcal{I})=2$$

because:

$$H(\theta,\mathcal{I},0)=(x^2+zy^3)$$
$$H(\theta,\mathcal{I},1)=(x,zy^2)$$
$$H(\theta,\mathcal{I},2)=\mathcal{O}_{\mathbb{C}^3}$$

Let $\sigma:(M',\theta',\mathcal{I}',E')\longrightarrow(M,\theta,\mathcal{I},E)$ be the blow with center $\mathcal{C}=V(x,z)$. Notice that σ is a θ-admissible blowing-up of order one, but that $\mathcal{C}$ is neither invariant, nor totally transverse to θ. In the z-chart we have:

$$(M',\theta',\mathcal{I}',E')=(\mathbb{C}^3,\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\},(zx^2+y^3),\{z=0\})$$

which implies that:

$$\nu(\theta',\mathcal{I}')=3$$

because:

$$H(\theta, \mathcal{I}, 0) = (zx^2 + y^3)$$
$$H(\theta, \mathcal{I}, 1) = (zx, y^2)$$
$$H(\theta, \mathcal{I}, 2) = (z, y)$$
$$H(\theta, \mathcal{I}, 2) = \mathcal{O}_{\mathbb{C}^3}$$

Which implies that $\nu(\theta', \mathcal{I}') > \nu(\theta, \mathcal{I})$.

We then turned out for the approach [II]. It seemed reasonable to first answer the following question: what invariants can completely describe the "worst" intersection between a variety and a foliation? We have not been able to give a satisfactory answer so far, although we strongly believe that the generalized Fitting operations are the key for an answer. A naive possibility, which we here present for 2-singular foliations, is the following:

- If $type(\theta, \mathcal{I}) = 2$, as invariant take $(\nu, type)$;
- If $type(\theta, \mathcal{I}) = 1$, then consider a "second chain of tangency's" defined by the operation $\Gamma_{\theta,2}$ applied to $H(\theta, \mathcal{I}, \nu - 1)$. More specifically, consider $\mathcal{J} = H(\theta, \mathcal{I}, \nu - 1)$ and the *second chain of tangency's*:

$$H_2(\theta, \mathcal{J}, 0) = \mathcal{J}$$
$$H_2(\theta, \mathcal{J}, i) = \Gamma_{\theta,2}(H_2(\theta, \mathcal{J}, i-1)) + H_2(\theta, \mathcal{J}, i-1)$$

  This chain also has a length, that we call $\nu_2(\theta, \mathcal{I})$, and type, that we call $type_2(\theta, \mathcal{I})$. As invariant, take $(\nu, type, \nu_2, type_2)$.

We exemplify this idea, even though it does not seem to work in this naive form: one can not be too "picky" with the choice of the centers:

Example 2: Let $(M, \theta, \mathcal{I}, E) = (\mathbb{C}^3, \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}, (x^2 + zy^2), \{z = 0\})$. Notice that:

$$H(\theta, \mathcal{I}, 0) = (x^2 + zy^2)$$
$$H(\theta, \mathcal{I}, 1) = (x, zy)$$
$$H(\theta, \mathcal{I}, 2) = \mathcal{O}_{\mathbb{C}^3}$$

Furthermore, let $\mathcal{J} = H(\theta, \mathcal{I}, \nu - 1) = H(\theta, \mathcal{I}, 1)$, then the second chain of tangency's is given by:

$$H_2(\theta, \mathcal{J}, 0) = (x, zy)$$
$$H_2(\theta, \mathcal{J}, i) = (x, z)$$

for all $i > 0$. So, the natural choice of a blowing-up center is $\mathcal{C} = V(H_2(\theta, \mathcal{J}, 1)) = V(x, z)$. Let $\sigma : (M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'}) \longrightarrow (M, \theta, \mathcal{I}, E)$ be the blow with center $\mathcal{C}$. In the z-chart we have:

$$(M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'}) = (\mathbb{C}^3, \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}, (zx^2 + y^2), \{z = 0\})$$

Notice that, exchanging the coordinates x and y, we are in the same situation of the beginning.

Chapter 6

A local uniformization subordinated to a 2-foliation

6.1 Presentation of the result

A local uniformization (see section 3.5) of $(M, \theta, \mathcal{I}, E)$ at a point p of M:

$$\{\tau_\alpha : (M_\alpha, \theta_\alpha) \longrightarrow (M, \theta)\}$$

is said to be *θ-admissible* if the morphisms τ_α are the composition of θ-admissible local blowings-up.

The main result of this chapter is:

Theorem 6.1.1. *Let $(M, \theta, \mathcal{I}, E)$ be a 2-foliated ideal sheaf and p a point of M. Then, there exists a θ-admissible local uniformization of $(M, \theta, \mathcal{I}, E)$ at p. In particular, if θ is R-monomial, then θ_α is R-monomial for every α.*

In the following remarks, we briefly discuss the reasons behind the conclusions and hypotheses of the Theorem:

Remark 6.1.2. *Theorem 6.1.1 is not global because the proof here presented depends on a choice of a particular vector field X contained in the singular distribution $\theta.\mathcal{O}_p$. This choice is not uniquely defined and, thus, no functorial property is obtained.*

Remark 6.1.3. *The reason why Theorem 6.1.1 demands the leaf dimension of θ to be two is "hidden" in the Propositions* 6.2.4 *and* 6.2.5 *below. All other proofs can be adapted for leaf dimension equals to d. Furthermore, if we assume that a θ-admissible local uniformization is proven for leaf dimension $d-1$, it is worth remaking that:*

- *A Proposition* 6.2.5 *for leaf dimension d seems to follow the same exact steps of the proof for leaf dimension two;*
- *A Proposition* 6.2.4 *for leaf dimension d is more delicate (if it is true at all). This is the main technical difficulty for getting a local uniformization for any leaf dimension. We explain this difficulty in remark* 6.3.1.

6.2 Proof of Theorem 6.1.1

In order to prove Theorem 6.1.1 we introduce a new invariant. The λ-*order* of a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ at a point p of M is given by:

$$\lambda_p(\theta, \mathcal{I}) = min\{\nu_p(X, \mathcal{I});\ X \in \theta_p \text{ is regular and } type_p(X, \mathcal{I}) = 1\}$$

If there is no such vector field $X \in \theta_p$, we define $\lambda_p(\theta, \mathcal{I}) = \infty$.

Given an open relatively compact open subset M_0 of M, we define the λ-order on M_0 as:

$$\lambda_{M_0}(\theta, \mathcal{I}) = sup\{\lambda_q(\theta, \mathcal{I}); q \in M_0\}$$

Notice that this invariant is valid for any leaf dimension of θ and it is clearly upper semi-continuous (because the tg-order is upper semi-continuous). We start given a good condition for the λ-order to be well-behaved:

Lemma 6.2.1. *If the $type_p(\theta, \mathcal{I})$ is one, then the λ-order $\lambda_p(\theta, \mathcal{I})$ is finite. Furthermore:*

$$\lambda_p(\theta, \mathcal{I}) = \nu_q(\theta, \mathcal{I})$$

Remark 6.2.2. *The converse is clearly true by a contra-positive argument: if the* $type_p(\theta, \mathcal{I})$ *is two, then the* λ*-order* $\lambda_p(\theta, \mathcal{I})$ *is infinite.*

Proof. Fix a point p of M and let $\nu := \nu_p(\theta, \mathcal{I})$. Since $type_M(\theta, \mathcal{I}) = 1$, there exists a finite set of vector fields $\{X_1, ..., X_\nu\}$ contained in θ_p and a function $f \in \mathcal{I}$ such that:

$$X_\nu(X_{\nu-1}(...(X_1(f))...))$$

is an unity of $\mathcal{O}_p$. Furthermore, it is clear that all vector fields X_i are regular (otherwise there would exist a smaller set of vector fields with this property, which contradicts the definition of ν). Consider a ν-tuple $\alpha = (\alpha_1, ..., \alpha_\nu) \in \mathbb{K}^\nu$ and the vector field:

$$Y_\alpha = \sum_{i=1}^{\nu} \alpha_i X_i$$

It is clear that, for a generic α, if we apply the vector field Y_α ν-times on f, we get an unity of $\mathcal{O}_p$. Since the vector field Y_α is contained in θ_p and is generically regular, we conclude that $\lambda_p(\theta, \mathcal{I}) = \nu_p(\theta, \mathcal{I})$. □

And now we give the result that motivates the introduction of this invariant:

Proposition 6.2.3. *Suppose that the invariant* $\lambda_p(\theta, \mathcal{I})$ *is finite and* θ *has leaf dimension two. Then there exists a* θ*-admissible local uniformization of* $(M, \theta, \mathcal{I}, E)$ *at* p*.*

The proof of this Proposition depends on the following two Propositions:

Proposition 6.2.4. *Let* $(M, M_0, \theta, \mathcal{I}, E)$ *be a local* 2*-foliated ideal sheaf and* $\mathcal{X}$ *a* 1*-singular distribution defined in an open neighborhood of* $\overline{M}_0$*. Suppose that there exists an involutive* 1*-singular distribution* ω *defined in an open neighborhood of* $\overline{M}_0$ *such that* $\{\mathcal{X}, \omega\}.\mathcal{O}_{M_0}$ *generates* $\theta.\mathcal{O}_{M_0}$*. Then, there exists a sequence of* θ*-admissible and* $\mathcal{X}$*-admissible blowings-up of order one:*

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) The tg-order $\nu_{M_r}(\mathcal{X}_r, \mathcal{I}_r)$ *is smaller or equal to* $\nu_{M_0}(\mathcal{X}_0, \mathcal{I}_0)$*;*

ii) The type $type_{M_r}(\mathcal{X}_r, \mathcal{I}_r)$ *is equal to* 1.

Proposition 6.2.5. *Let* $(M, M_0, \theta, \mathcal{I}, E)$ *be a local* 2*-foliated ideal sheaf and* $\mathcal{X}$ *a* 1*-singular distribution defined in an open neighborhood of* $\overline{M}_0$*. Suppose that:*

- *The singular distribution* $\mathcal{X}.\mathcal{O}_{M_0}$ *is contained in the singular distribution* $\theta.\mathcal{O}_{M_0}$*;*
- *The type* $type_{M_0}(\mathcal{X}, \mathcal{I})$ *is* 1*;*
- *The tg-order* $\nu_{M_0}(\mathcal{X}, \mathcal{I})$ *is equal to* $\nu_{M_0}(\theta, \mathcal{I})$*;*
- *There exists a coordinate system* $(x, y) = (x, y_1, ..., y_{n-1})$ *defined in an open neighborhood* U *of* $\overline{M_0}$ *such that the singular distribution* $\mathcal{X}.\mathcal{O}_{M_0}$ *is generated by the vector field* $X = \frac{\partial}{\partial x}$.

Then, there exists a sequence of θ*-admissible and* $\mathcal{X}$*-admissible blowings-up of order one:*

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

i) The tg-order $\nu_{M_r}(\mathcal{X}_r, \mathcal{I}_r)$ *is strictly smaller then* $\nu_{M_0}(\mathcal{X}_0, \mathcal{I}_0)$*;*

ii) There exists an involutive 1*-singular distribution* ω_r *such that* $\{\mathcal{X}_r, \omega_r\}$ *generates* θ_r.

These two propositions will be proved in the next two sections. For now, we assume them in order to prove Proposition 6.2.3:

Proof. (Proposition 6.2.3) We proceed by induction in the invariant $\lambda_p(\theta, \mathcal{I})$. Suppose that we have already proved the existence of a θ-admissible local uniformization for a 2-foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ at any point p such that $\lambda_p(\theta, \mathcal{I}) < k$. We prove the result for k:

- Step $k = 0$: If $\lambda_p(\theta, \mathcal{I}) = 0$, then $\mathcal{I} = \mathcal{O}_p$ and the result is clear;

- Step $k > 0$: Since the invariant $\lambda_p(\theta, \mathcal{I})$ is finite, we conclude that there exists a regular vector field X in θ_p. By the Flow-box Theorem, there exists a relatively compact open neighborhood M_0 of p, a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ defined in an open neighborhood of $\overline{M_0}$ and a 1-singular distribution $\mathcal{X}$ defined in an open neighborhood of $\overline{M_0}$ such that:

  - The singular distribution $\mathcal{X}.\mathcal{O}_{M_0}$ is contained in the singular distribution $\theta.\mathcal{O}_{M_0}$;
  - The type $type_{M_0}(\mathcal{X}, \mathcal{I})$ is 1;
  - The singular distribution $\mathcal{X}.\mathcal{O}_{M_0}$ is generated by the vector field $X = \frac{\partial}{\partial x}$.

  Furthermore, since the λ-order is upper semi-continuous, we can further suppose that:

  - The tg-order $\nu_{M_0}(\mathcal{X}, \mathcal{I})$ is equal to $\nu_{M_0}(\theta, \mathcal{I})$ and $\lambda_p(\theta, \mathcal{I})$.

  Now, by Proposition 6.2.5, there exists a sequence $\vec{\sigma}_1$ of θ-admissible and $\mathcal{X}$-admissible blowings-up:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

  such that:

  i) $\nu_{M_r}(\mathcal{X}_r, \mathcal{I}_r) < \nu_{M_0}(\mathcal{X}_0, \mathcal{I}_0)$;

  ii) There exists an involutive 1-singular distribution ω_r such that $\{\mathcal{X}_r, \omega_r\}$ generates θ_r.

  Let K_1 be the compact set which is the pre-image of p by the sequence $\vec{\sigma}_1$ of blowings-up. Consider N_0 a relatively compact open neighborhood of K, strictly contained in M_r, and let the morphism $\pi : N_0 \longrightarrow M_r$ be the inclusion. Consider the 2-local foliated manifold $(M_r, N_0, \theta_r, \mathcal{I}_r, E_r)$ and notice that it satisfies the hypotheses of Proposition 6.2.4. So, there exists a sequence $\vec{\sigma}_2$ of θ_r-admissible and $\mathcal{X}_r$-admissible blowings-up:

$$(N_s, \theta_{r+s}, \mathcal{I}_{r+s}, E_{r+s}) \xrightarrow{\sigma_{r+s}} \cdots \xrightarrow{\sigma_{r+1}} (N_0, \theta_r, \mathcal{I}_r, E_r)$$

such that $\nu_{N_s}(\mathcal{X}_{r+s}, \mathcal{I}_{r+s}) \leq \nu_{N_0}(\mathcal{X}_r, \mathcal{I}_r)$ and $type_{N_s}(\mathcal{X}_{r+s}, \mathcal{I}_{r+s}) = 1$.

In particular, we have that:

$$\nu_{N_s}(\mathcal{X}_{r+s}, \mathcal{I}_{r+s}) \leq \nu_{N_0}(\mathcal{X}_r, \mathcal{I}_r) \leq \nu_{M_r}(\mathcal{X}_r, \mathcal{I}_r) < \nu_{M_0}(\mathcal{X}_0, \mathcal{I}_0)$$

which implies that:

$$\lambda_{N_s}(\theta_{r+s}, \mathcal{I}_{r+s}) < \lambda_{M_0}(\theta, \mathcal{I}) = k$$

Let K_2 be the compact set which is the pre-image of K_1 by the sequence $\vec{\sigma}_2$ of blowings-up. Since the λ-order is strictly smaller then k in every point of K_2, by induction and the compacity of K_2, there exists a θ-admissible local uniformization of $(N_s, \theta_{r+s}, \mathcal{I}_{r+s}, E_{r+s})$ over all K_2. Composing this local uniformization with the θ-admissible sequence of blowings-up $\vec{\sigma}_2$, with the morphism π and finally with the sequence of θ-admissible blowings-up $\vec{\sigma}_1$, we finally obtain a θ-admissible local uniformization of $(M, \theta, \mathcal{I}, E)$ at p.

□

Now, we are ready to prove the Theorem:

Proof. (Theorem 6.1.1) If at the point p of M, the λ-order $\lambda_p(\theta, \mathcal{I})$ is finite, the result follows from Lemma 6.2.3. So, we can assume that λ-order is infinite at p, which implies that the type $type_p(\theta, \mathcal{I})$ is two.

Let M_0 be any relatively compact open neighborhood of p and $\pi : M_0 \longrightarrow M$ the inclusion morphism. By Proposition 5.4.1 there exists a sequence $\vec{\sigma}$ of invariant θ-admissible blowings-up of order one:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

$$type_{M_r}(\theta_r, \mathcal{I}_r) = 1$$

Let K be the compact pre-image of the point p by the sequence of blowings-up. Notice that the sequence $\vec{\sigma}$ of blowings-up can be regarded as a sequence of local blowings-up $\vec{\tau} = (\tau_r, ..., \tau_1)$ where $\tau_i = id \circ \sigma_i$ for $i > 1$ and $\tau_1 = \pi_1 \circ \sigma_1$.

Furthermore, by Lemma 6.2.1, notice that the λ-order is finite over all points of K. By the compacity of K and Lemma 6.2.3, there exists a local uniformization that covers all K. Composing this local uniformization with the sequence of local blowings-up $\vec{\tau}$, gives a local uniformization of $(M, \theta, \mathcal{I}, E)$ at p. □

6.3 Proof of Proposition 6.2.4

If the type $type_{M_0}(\mathcal{X}, \mathcal{I})$ is 1, the result is trivial. So suppose that $type_{M_0}(\mathcal{X}, \mathcal{I})$ is 2.

Let $\nu = \nu_{M_0}(\mathcal{X}, \mathcal{I})$ and $\mathcal{C}l_{\mathcal{X}}(\mathcal{I}) := H(\mathcal{X}, \mathcal{I}, \nu)$. By Theorem 5.3.1, there exists a ω-admissible resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of the local foliated ideal sheaf $(M, M_0, \omega, \mathcal{C}l_{\mathcal{X}}(\mathcal{I}), E)$:

$$(M_r, \theta_r, (\mathcal{C}l_{\mathcal{X}}(\mathcal{I}))_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, (\mathcal{C}l_{\mathcal{X}}(\mathcal{I}))_0, E_0)$$

Furthermore, since $\mathcal{C}l_{\mathcal{X}}(\mathcal{I})$ is $\mathcal{X}$-invariant and $\{\mathcal{X}, \theta\}$ is an involutive singular distribution, by the part $[v]$ of Theorem 5.3.1, we conclude that the resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ is also $\mathcal{X}$-invariant ($*$ - see Remark 6.3.1 below). Moreover:

Claim 1: The sequence of blowings-up $\vec{\sigma}$ is θ-admissible.

Proof. We prove the result by induction. Suppose that, for $i < k$:

- The sequence of blowings-up $(\sigma_i, ..., \sigma_1)$ is θ-admissible;
- $\{\mathcal{X}_i, \omega_i\}$ generates θ_i.

We prove the result for $i = k$:

- Step $k = 0$: It trivially follows from the hypotheses;

- Step $k>0$. Consider $\mathcal{C}_k$ the center of the blowing-up σ_k. Since $\{\mathcal{X}_{k-1},\omega_{k-1}\}$ generates θ_{k-1} and $\mathcal{C}_k$ is $\mathcal{X}_{k-1}$-invariant, we have that the 2-generalized Fitting ideal sheaf $\Gamma_{\theta_{k-1},2}(\mathcal{I}_{\mathcal{C}_k})$ is contained in $\mathcal{I}_{\mathcal{C}_k}$. Furthermore:

  - If $\omega[\mathcal{I}_{\mathcal{C}_k}]\subset\mathcal{I}_{\mathcal{C}_k}$ then $\theta[\mathcal{I}_{\mathcal{C}_k}]\subset\mathcal{I}_{\mathcal{C}_k}$;
  - If $\omega[\mathcal{I}_{\mathcal{C}_k}]=\mathcal{O}_{M_{k-1}}$ then $\theta[\mathcal{I}_{\mathcal{C}_k}]=\mathcal{O}_{M_{k-1}}$.

  Which implies that σ_k is θ_{k-1} admissible and $(\sigma_k,...,\sigma_1)$ is θ-admissible.

  It is now easy to see (using Lemma 3.2.2 and Proposition 4.4.1) that $\{X_k,\omega_k\}$ generates θ_k.

□

Claim 2: The sequence of blowings-up $\vec{\sigma}$ is θ-admissible of order one for $(M,M_0,\theta,\mathcal{I},E)$. Furthermore:

$$(\mathcal{C}l_{\mathcal{X}}(\mathcal{I}))_j=\mathcal{C}l_{\mathcal{X}_j}(\mathcal{I}_j)$$

for all $j\leq r$.

Proof. This claim has, mutantis mutatis, the same proof of the Claim 1 contained in the proof of Proposition 5.4.1. □

This implies that $\vec{\sigma}$ gives rise to an invariant θ-admissible sequence of blowings-up of order one for $(M,M_0,\theta,\mathcal{I},E)$:

$$(M_r,\theta_r,\mathcal{I}_r,E_r)\xrightarrow{\sigma_r}\cdots\xrightarrow{\sigma_2}(M_1,\theta_1,\mathcal{I}_1,E_1)\xrightarrow{\sigma_1}(M_0,\theta_0,\mathcal{I}_0,E_0)$$

such that:

$$H(\mathcal{X}_r,\mathcal{I}_r,\nu)=\mathcal{C}l_{\mathcal{X}_r}(\mathcal{I}_r)=(\mathcal{C}l_{\mathcal{X}}(\mathcal{I}))_r=\mathcal{O}_{M_r}$$

which implies that $\nu_{M_r}(\mathcal{X}_r,\mathcal{I}_r)\leq\nu_{M_0}(\mathcal{X},\mathcal{I})$ and $type_{M_r}(\mathcal{X}_r,\mathcal{I}_r)=1$.

Remark 6.3.1. *See* $(*)$ *above: This is the technical point where we strongly use the fact that the leaf dimension of* θ *is two. When we apply Theorem* 5.3.1*, we use the statement* $[v]$ *of the Theorem (which is a functorial property of the resolution) to conclude that the* *sequence of ω-admissible blowings-up are also $\mathcal{X}$-invariant.*

A straight proof of Proposition 6.2.4 *for general leaf dimension d, would need a local uniformization for leaf dimension $d-1$ that satisfies some analogous functorial property. But even for leaf dimension two, obtaining this result is not clear.*

6.4 Proof of Proposition 6.2.5

Without loss of generality, we can suppose that there exists a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ defined on an open neighborhood U of $\overline{M_0}$ such that the 1-singular distribution $\mathcal{X}.\mathcal{O}_U$ is generated by the vector field $X = \frac{\partial}{\partial x}$. Let ω be the 1-singular distribution defined as follow:

$$\omega = \{Y \in \theta.\mathcal{O}_U ; Y(x) \equiv 0\}$$

It is clear that:

- The 2-singular distribution $\theta.\mathcal{O}_{M_0}$ is generated by $\{\mathcal{X}, \omega\}.\mathcal{O}_{M_0}$;
- The ideal (x) is ω-invariant.

Let $\nu := \nu_{M_0}(\mathcal{X}, \mathcal{I})$ and $\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}) := H(\mathcal{X}, \mathcal{I}_0, \nu - 1)$. By Theorem 5.3.1 there exists a ω-admissible resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of the local foliated ideal sheaf $(M, M_0, \omega, \mathcal{M}tg_{\mathcal{X}}(\mathcal{I}), E)$:

$$(M_r, \omega_r, (\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \omega_0, (\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_0, E_0)$$

such that:

Claim 1: The sequence of blowings-up $\vec{\sigma}$ is $\mathcal{X}$-totally transverse.

Proof. Suppose by induction that, for $i < k$:

i) the sequence $(\sigma_1, ..., \sigma_i)$ of blowing-up is $\mathcal{X}$-totally transverse;

ii) for each point p contained in the variety $V((\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_i)$, there exists a coherent coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ of $\mathcal{O}_p$ such that:

- The function x is contained in the ideal $(\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_i.\mathcal{O}_p$;
- The vector field $\frac{\partial}{\partial x}$ generates the singular distribution $\mathcal{X}_{i,p} := \mathcal{X}_i.\mathcal{O}_p$;
- The ideal (x) is ω_i-invariant.

We prove the result for k:

- Step $k = 0$. This trivially follows from the choice of the coordinate systems before blowing-up.
- Step $k > 0$. Consider a point q on the variety $V((\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_k)$ and let $p = \sigma_k(q)$. If σ_k is a local isomorphism over q, the result is trivial, so we assume that the point q is contained in the exceptional divisor F_k. By the induction hypotheses $[ii]$, there exists a coherent coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ of $\mathcal{O}_p$ such that:
  - The function x is contained in the ideal $(\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_{k-1}.\mathcal{O}_p$;
  - The vector field $\frac{\partial}{\partial x}$ generates the singular distribution $\mathcal{X}_{k-1,p}$;

  Since the center $\mathcal{C}_k$ of the blowing-up $\sigma_k : M_k \longrightarrow M_{k-1}$ is contained in $V((\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_{k-1})$ we conclude that:

  - The center $\mathcal{C}_k$ is $\mathcal{X}_{k-1}$-totally transverse at p. This implies that the sequence $(\sigma_1, ..., \sigma_k)$ of blowings-up is $\mathcal{X}$-totally transverse;
  - Without loss of generality $\mathcal{I}_{\mathcal{C}_k}.\mathcal{O}_p = (x, y_1, ..., y_t)$ and the point q is the origin of the y_1-chart. It is now easy to compute the transforms of the blowing-up at q and see that the induction hypotheses $[ii]$ is valid for $i = k$.

□

Claim 2: The sequence of blowings-up $\vec{\sigma}$ is θ-admissible and θ_r is generated by $\{\mathcal{X}_r, \omega_r\}$.

Proof. Suppose by induction that, for $i < k$:

i) The sequence $(\sigma_1, ..., \sigma_i)$ of blowing-up is θ-admissible;

ii) for each point p contained in the variety $V((\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_i)$, there exists a coherent coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ of $\mathcal{O}_p$ such that:

- The function x is contained in the ideal $(\mathcal{M}tg_{\mathcal{X}}(\mathcal{I}))_i.\mathcal{O}_p$;
- The vector field $\frac{\partial}{\partial x}$ generates the singular distribution $\mathcal{X}_{i,p} := \mathcal{X}_i.\mathcal{O}_p$;
- The ideal (x) is ω_i-invariant.

iii) The singular distribution θ_i is generated by $\{\mathcal{X}_i, \omega_i\}$.

We remark that hypotheses $[ii]$ of the induction was already proved in Claim 1 for any i. We prove the result for k:

- Step $k = 0$. This trivially follows from the hypotheses.
- Step $k > 0$. Since $\mathcal{C}_k$ is totally transverse to $\mathcal{X}_{k-1}$, we have that $\theta_{k-1}(\mathcal{I}_{\mathcal{C}_k}) = \mathcal{O}_{M_{k-1}}$.

  So, to show the induction hypotheses $[i]$ for the step k, we only have to prove that the 2-generalized Fitting ideal $\Gamma_{\theta,2}(\mathcal{I}_{\mathcal{C}_k})$ is either the structural sheaf $\mathcal{O}_p$ or is contained in $\mathcal{I}_{\mathcal{C}_k}$. Since σ_k is a ω_{k-1}-admissible blowing-up, we have that:

  - If $\omega_{k-1}[\mathcal{I}_{\mathcal{C}_k}] \subset \mathcal{I}_{\mathcal{C}_k}$ then $\Gamma_{\theta_{k-1},2}(\mathcal{I}_{\mathcal{C}_k}) \subset \mathcal{I}_{\mathcal{C}_k}$;
  - If $\omega_{k-1}[\mathcal{I}_{\mathcal{C}_k}] = \mathcal{O}_{M_{k-1}}$, we claim that $\Gamma_{\theta_{k-1},2}(\mathcal{I}_{\mathcal{C}_k}) = \mathcal{O}_{M_{k-1}}$. Indeed, there exists a function h on the ideal $\mathcal{I}_{\mathcal{C}_k}.\mathcal{O}_p$ and a vector field Y on the singular distribution $\omega_{k-1}.\mathcal{O}_p$ such that $Y(h)$ is an unity. Furthermore, since there exists a coordinate system such that $\omega_{k-1}[(x)] \subset (x)$, we have that $Y(x) = xg$. Thus:

    $$det \begin{Vmatrix} X(x) & Y(x) \\ X(h) & Y(h) \end{Vmatrix} = det \begin{Vmatrix} 1 & xg \\ X(h) & U \end{Vmatrix}$$

    is an unity.

  To finish, it is now easy to see (using Lemma 3.2.2 and Proposition 4.4.1) that $\{X_k, \omega_k\}$ generates θ_k.

□

Claim 3: The sequence of blowings-up $\vec{\sigma}$ give rise to a θ-admissible sequence:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

$$\nu_{M_r}(\mathcal{X}_r, \mathcal{I}_r) < \nu_{M_0}(\mathcal{X}_0, \mathcal{I}_0)$$

Proof. This claim has, mutantis mutatis, the same proof of the Claims 2 and 3 contained in the proof of Proposition 5.4.2. □

These three Claims are enough to prove the Proposition.

Chapter 7

Application 1: Resolution in Families

7.1 Families of ideal sheaves

A *smooth family of ideal sheaves* is given by a quadruple $(B, \Lambda, \pi, \mathcal{I})$ where:

- The ambient space B and the parameter space Λ are two smooth analytic manifolds;
- The morphism $\pi : B \longrightarrow \Lambda$ is smooth;
- The ideal sheaf $\mathcal{I}$ is coherent and everywhere non-zero over B.

Given $\lambda \in \Lambda$, the set $\pi^{-1}(\lambda)$ is a regular sub-manifold of B called *fiber*. A point $\lambda_0 \in \Lambda$ is called an *exceptional value* of a smooth family of ideal sheaf $(B, \Lambda, \pi, \mathcal{I})$ if the fiber $\pi^{-1}(\lambda_0)$ is contained in $V(\mathcal{I})$.

Many works have addressed resolution process for families of ideal sheaves. By this, we intuitively mean a resolution of $(B, \mathcal{I}, \emptyset)$ that preserves, in some way, the structure of family. The precise meaning of resolution in families is not unique in the literature (see e.g [ENV, V2]). For example, in [ENV], one defined a stratification Σ in the parameter space Λ in such a way that the resolution algorithm of the parametrized ideal sheaf behaves uniformly along each strata of Σ.

In the context of this work, a smooth family of ideal sheaves $(B, \Lambda, \pi, \mathcal{I})$ gives rise to a foliated ideal sheaf $(B, \theta, \mathcal{I}, \emptyset)$, where θ is the maximal regular distribution such that $(D\pi)\theta = 0$. This motivates another possible definition of resolution in families:

Uniform Resolution in Families of Ideal sheaves: An *uniform resolution* of a smooth family of ideal sheaves $(B, \Lambda, \pi, \mathcal{I})$ is a sequence $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of admissible blowings-up of order one:

$$(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B, \theta, \mathcal{I}, \emptyset)$$

such that $\mathcal{I}_r = \mathcal{O}_{B_r}$ and θ_r is $\mathbb{Z}$-monomial.

This kind of resolution in families has originally been introduced at [DR] in the context of smooth families of planar foliations by curves, where it is an essential step in Roussarie's program for the existential part of Hilbert 16^{th} Problem. This approach is also similar to the one adopted at [V2], where it is proved the existence of an uniform resolution in families for the case $dim\Lambda = 1$, under the hypotheses that the morphism π is flat over $V(\mathcal{I})$.

The existence of an uniform resolution in families would give rise to a resolution (in some sense) "uniform" in the parameter space. In particular, the study of the fibers of the resolution (i.e. of the morphism $\sigma_r \circ ... \circ \sigma_1 \circ \pi$) may be useful for equiresolution and bifurcation theory. In particular, it might give rise to a stratification of the parameter space in the same sense given in [ENV].

With the results of this work, we can prove the existence of an uniform resolution for a smooth family of ideal sheaves when $dim\Lambda = dimB - 1$ (it is a trivial consequence of Theorem 5.3.1). Furthermore, we can eliminate exceptional values of a smooth family of ideal sheaves (see the Theorem 7.1.1 below). This can be seen as a first step in the solution of the problem of uniform resolution in families.

Theorem 7.1.1. *Let* $(B, \Lambda, \pi, \mathcal{I})$ *be a smooth family of ideal sheaves such that all fibers are connected. Then, there exists a smooth family of ideal sheaves* $(B', \Lambda', \pi', \mathcal{I}')$ *and two proper analytic maps* $\sigma : B' \longrightarrow B$ *and* $\tau : \Lambda' \longrightarrow \Lambda$ *such that:*

i) The smooth family of ideal sheaves $(B', \Lambda', \pi', \mathcal{I}')$ *has no exceptional value;*

ii) The following diagram:

$$\begin{array}{ccc} B' & \xrightarrow{\pi'} & \Lambda' \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

commutes;

iii) For any relatively compact open subset B_0 *of* B*, there exists a sequence of invariant admissible blowings-up of order one for* $(B, B_0, \theta, \mathcal{I}, \emptyset)$*:*

$$(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B_0, \theta_0, \mathcal{I}_0, E_0)$$

such that $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ ... \circ \sigma_r$ *and* $\mathcal{I}'.\mathcal{O}_{B_r} = \mathcal{I}_r$*;*

iv) For any relatively compact open subset Λ_0 *of* Λ*, there exists a sequence of admissible blowings-up:*

$$(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)$$

such that $\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ ... \circ \tau_r$*.*

Proof. Consider the two foliated manifolds $(B, \theta, \emptyset)$ and $(\Lambda, \omega, \emptyset)$, where $\omega = 0$, and let $\mathcal{I}_{\#}$ be the smaller θ-invariant ideal sheaf containing $\mathcal{I}$.

Claim: There exists an ideal sheaf $\mathcal{J}$ over Λ such that $\mathcal{J}.\mathcal{O}_B = \mathcal{I}_{\#}$.

Proof. Consider a point λ in Λ and let p be a point contained in the fiber $\pi^{-1}(\lambda)$. Since θ is regular, there exists a coordinate system $(x, y) = (x_1, ..., x_d, y_1, ..., y_{n-d})$ of $\mathcal{O}_p$ such that $\pi(x, y) = y$ and $\{\frac{\partial}{\partial x_1,}, ..., \frac{\partial}{\partial x_d}\}$ is a coherent set of generators of θ_p.

Since $\mathcal{I}_{\#}$ is θ-invariant, by Proposition 4.2.2 there exists a set of generators $\{f_1(y), ..., f_s(y)\}$ of $\mathcal{I}_{\#}.\mathcal{O}_p$. Let J_p be the ideal of $\mathcal{O}_\lambda$ generated by $\{f_1(y), ..., f_s(y)\}$. Notice that this construction can be done for any point q contained in the fiber $\pi^{-1}(\lambda)$, and generates an ideal J_q.

By the construction of J_p, there exists an open neighborhood U of p such that, for every q in $U \cap \pi^{-1}(\lambda)$, $J_q = J_p$. Furthermore, by analyticity, if (q_i) is a sequence of points in the fiber $\sigma^{-1}(\lambda)$ that are converging to a point q such that $J_{q_1} = J_{q_i}$ for all $i \in \mathbb{N}$, then $J_q = J_{q_1}$. Because of this two properties and the fact that the fiber $\sigma^{-1}(\lambda)$ is connected, we conclude that the ideal J_p is independent of the point p in the fiber $\pi^{-1}(\lambda)$.

Now, we only need to define $\mathcal{J}$ as the ideal sheaf locally given by $\mathcal{J}.\mathcal{O}_\lambda = J_p$ for some p in the fiber $\pi^{-1}(\lambda)$. □

Notice that $\pi : B \longrightarrow \Lambda$ is a chain-preserving smooth morphism between $(B, \theta, \mathcal{I}_{\#}, \emptyset)$ and $(\Lambda, \omega, \mathcal{J}, \emptyset)$. By Theorem 5.1.2 there exists two proper analytic maps $\sigma : B' \longrightarrow B$ and $\tau : \Lambda' \longrightarrow \Lambda$ and a smooth map $\pi' : B' \longrightarrow \Lambda'$ such that:

- The morphism $\sigma : B' \longrightarrow B$ is a weak-resolution of $(B, \theta, \mathcal{I}_{\#}, \emptyset)$;
- The morphism $\tau : \Lambda' \longrightarrow \Lambda$ is a weak-resolution of $(\Lambda, \omega, \mathcal{J}, \emptyset)$;
- The following diagram:

$$\begin{array}{ccc} B' & \xrightarrow{\pi'} & \Lambda' \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

  commutes.

Furthermore, given a relatively compact open subset B_0 of B, the proof of Proposition 5.4.1 guarantees that the sequence of invariant blowings-up $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$, where $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ ... \circ \sigma_r$:

$$(B_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

is of order one for $(B, B_0, \theta, \mathcal{I}, \emptyset)$ and $\mathcal{I}_r$ is of type 1.

Define the ideal sheaf $\mathcal{I}'$ of B' given as the direct limit of the controlled transforms $\mathcal{I}_r$ over all relatively compact open subsets B_0 of B. By construction $(B', \Lambda', \pi', \mathcal{I}')$ has no exceptional value and satisfies all hypotheses of the Theorem. □

7.2 Resolution of foliations

A *nested foliation* is a quadruple (M, θ, ω, E):

- (M, θ, E) is a foliated manifold;
- $\mathcal{X}$ is an everywhere non-zero involutive singular distribution that is a sub-sheaf of θ.

A d-singular distribution θ is said to have *complete intersection* if at each point p in M, there exists a coherent set of generators $\{X_1, ..., X_{d_p}\}$ of θ_p such that d_p is equal to d.

A nested foliation (M, θ, ω, E) is said to be a *nested foliation by curves* if $\mathcal{X}$ has leaf dimension one and complete intersection. In other words, at each point p in M, there exists a vector field X_p in Der_p such that $\{X_p\}$ generates $\mathcal{X}_p$.

In this work, a *resolution* of a nested foliation by curves $(M, \omega, \mathcal{X}, E)$ is a sequence of admissible blowings-up:

$$(M_r, \theta_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{X}_0, E_0)$$

such that $\mathcal{X}_r$ is an *elementary* singular distribution contained in θ_r, i.e. at each point p in M, if X_p is a vector field generating $\mathcal{X}_p$, then the linear part of X_p has a non-zero eigenvalue.

Remark 7.2.1. *In this definition, we don't restrict the transforms of $\mathcal{X}$ to be the analytic strict transform. To obtain a resolution of a nested foliation by curves, one may need to work if other kinds of transforms.*

7.3 Families of foliations by curves

A *smooth family of foliations* is given by a quadruple $(B, \Lambda, \pi, \mathcal{X})$ where:

- The ambient space B and the parameter space Λ are two smooth analytic manifolds;
- The morphism $\pi : B \longrightarrow \Lambda$ is smooth;
- The singular distribution $\mathcal{X}$ is everywhere non-zero and $d\pi(\mathcal{X}) \equiv 0$.

Furthermore, we a smooth family of foliations *by curves* is a smooth family of foliations $(B, \Lambda, \pi, \mathcal{X})$ such that $\mathcal{X}$ has leaf dimension one and complete intersection. A smooth *planar* family of foliations by curves is a smooth family of foliations by curves $(B, \Lambda, \pi, \mathcal{X})$ such that $dim\Lambda = dimB - 2$.

Given $\lambda \in \Lambda$, the set $\pi^{-1}(\lambda)$ is a regular sub-manifold of B called *fiber*. A point $\lambda_0 \in \Lambda$ is called an *exceptional value* of a smooth family of foliations by curves $(B, \Lambda, \pi, \mathcal{X})$ if the fiber $\pi^{-1}(\lambda_0)$ is contained in the singular set $S(\mathcal{X})$.

A notion of resolution process for families of foliations by curves is not unique because the notions of resolution for families of ideal sheaves may be adapted to the case of families of foliations by curves. As an example, we refer to the work [Tr], where a notion of resolution of smooth planar families of foliations by curves is presented. Although the process presented is not a resolution in the sense of this work (because it ends with non-elementary singularities - in particular singular perturbation problems are persistent through this resolution), it is the best known result that preserves the structure of smooth family.

In the context of this work, a smooth family of foliations $(B, \Lambda, \pi, \mathcal{X})$ gives rise to a nested foliation $(B, \theta, \mathcal{X}, E)$, where θ is the maximal regular distribution such that $D\pi(\theta) = 0$. This motivates another possible definition of resolution in families wich was introduced at [DR] in the context of smooth planar families of foliations by curves. We present a generalization of the idea:

Uniform Resolution in Families of Foliations by curves: An *uniform resolution* of a smooth family of foliations by curves $(B, \Lambda, \pi, \mathcal{X})$ is a sequence $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of admissible blowings-up:

$$(B_r, \theta_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (B, \theta, \mathcal{X}, \emptyset)$$

such that $\mathcal{X}_r$ is elementary and θ_r is $\mathbb{Z}$-monomial.

In particular, an uniform resolution for smooth planar families of foliations by curves is an essential step in Roussarie's program for proving the existential part of Hilbert 16^{th} Problem.

The best result in this context is given by Panazzolo in [P1], where an uniform resolution for smooth planar families of foliations by curves is presented, under the hypotheses that at each point p in B, if X_p is a vector field generating $\mathcal{X}_p$, then X_p has non-zero linear part.

With the results of this work we can eliminate exceptional values of smooth families of foliations by curves (see the Theorem 7.3.1 below). This result is a generalization of Proposition $IV.3$ of [DR] and can be seen as a first step towards the solution of the problem of uniform resolution in families of foliations by curves.

Theorem 7.3.1. *Let $(B, \Lambda, \pi, \mathcal{X})$ be a smooth family of foliations by curves such that all fibers are connected. Then, there exists a smooth family of foliations by curves $(B', \Lambda', \pi', \mathcal{X}')$ and two proper analytic maps $\sigma : B' \longrightarrow B$ and $\tau : \Lambda' \longrightarrow \Lambda$ such that:*

i) $(B', \Lambda', \pi', \mathcal{X}')$ has no exceptional value;

ii) The following diagram:

$$\begin{array}{ccc} B^{'} & \xrightarrow{\pi^{'}} & \Lambda^{'} \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

commutes;

iii) For any relatively compact open subset B_0 of B, there exists a sequence of invariant admissible blowings-up:

$$(B_r, \theta_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (B_0, \theta_0, \mathcal{X}_0, E_0)$$

where $\mathcal{X}_i = \mathcal{O}(-F_i).\sigma_i^* \mathcal{X}_{i-1}$, such that $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ ... \circ \sigma_r$ and $\mathcal{X}^{'}.\mathcal{O}_{B_r} = \mathcal{X}_r$;

iv) For any relatively compact open subset Λ_0 of Λ, there exists a sequence of admissible blowings-up:

$$(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)$$

such that $\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ ... \circ \tau_r$.

Proof. Consider the ideal sheaf $\mathcal{S}(\mathcal{X}) := \mathcal{X}[\mathcal{O}_B]$. By Theorem 7.1.1, there exists two proper analytic maps $\sigma : B^{'} \longrightarrow B$ and $\tau : \Lambda^{'} \longrightarrow \Lambda$ respecting [*ii*] and [*iv*] such that, for every open relatively compact subset B_0 of B, there exists a sequence of invariant admissible blowings-up of order one for $(B, B_0, \mathcal{S}(\mathcal{X}), \emptyset)$:

$$(B_r, \theta_r, (\mathcal{S}(\mathcal{X}))_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (B_0, \theta_0, (\mathcal{S}(\mathcal{X}))_0, E_0)$$

such that $(\mathcal{S}(\mathcal{X}))_r$ is of type 1.

Claim: If $\sigma : (M^{'}, \theta^{'}, (\mathcal{S}(\mathcal{X}))^{'}, E^{'}) \longrightarrow (M, \theta, \mathcal{S}(\mathcal{X}), E)$ is an invariant admissible blowing-up of order one between regular singular distributions, then the transform $\mathcal{X}' := \mathcal{O}(-F).\mathcal{X}^*$ is well-defined and:

$$(\mathcal{S}(\mathcal{X}))' = \mathcal{S}(\mathcal{X}')$$

Proof. At each point $p$in M there exists a local system of coordinates $(x, y) = (x_1, ..., x_d, y_1, ..., y_{n-d})$ such that the singular distribution $\theta.\mathcal{O}_p$ is generated by $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d}\}$ and the center of blowing-up $\mathcal{C}$ is such that $\mathcal{I}_{\mathcal{C}}.\mathcal{O}_p = (y_1, ..., y_t)$. In particular, this implies that the singular distribution $\mathcal{X}_p$ is generated by a vector field X of the form:

$$X = \sum_{i=1}^{d} A_i(x, y) \frac{\partial}{\partial x_i}$$

and, for any q in $\sigma^{-1}(p)$, the singular distribution $\mathcal{X}'.\mathcal{O}_q$ is generated by the vector field X' of the form:

$$X' = \sum_{i=1}^{d} A_i(x, y)^* \mathcal{O}(-F) \frac{\partial}{\partial x_i} = \sum_{i=1}^{d} A_i(x, y)' \frac{\partial}{\partial x_i}$$

which implies that:

$$(\mathcal{S}(\mathcal{X}))' = \mathcal{S}(\mathcal{X}')$$

In particular, $\mathcal{X}'$ is analytic and well-defined. □

The Claim implies that $(\sigma_1, ..., \sigma_r)$ is also a sequence of invariant admissible blowings-up:

$$(B_r, \theta_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (B_0, \theta_0, \mathcal{X}_0, E_0)$$

such that $\mathcal{S}(\mathcal{X}_r)$ is of type one. This implies that $\mathcal{X}_r$ has no exceptional values (otherwise $\mathcal{S}(\mathcal{X}_r)$ would be of type 2).

Define the foliation by curves $\mathcal{X}'$ of B' as the direct limit of the transforms $\mathcal{X}_r$ over all relatively compact open subsets B_0 of B. By construction $(B', \Lambda', \pi', \mathcal{X}')$ has no exceptional value and satisfies all hypotheses of the Theorem. □

7.4 Dim 1 Nested foliation by curves

A dim 1 nested foliation by curves is a nested foliation by curve $(M, \theta, \mathcal{X}, E)$ such that θ has leaf dimension one and complete intersection. A *reduction* of $(M, \theta, \mathcal{X}, E)$ is a sequence of θ-admissible blowings-up $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$:

$$(M_r, \theta_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{X}_0, E_0)$$

where:

- The singular distribution $\mathcal{X}_i$ is given by $\mathcal{X}'_{i-1}.\mathcal{O}(-F_i)$ if the blowing-up is θ_{i-1}-invariant;
- The singular distribution $\mathcal{X}_i$ is given by $\mathcal{X}'_{i-1}$ if the blowing-up is θ_{i-1}-totally transverse.

such that:

- The singular distribution $\mathcal{X}_r$ is equal to θ_r.

Remark that the singular distribution $\mathcal{X}_r$ will possess all the "good" properties of θ that are preserved by θ-admissible blowings-up. In particular, if θ is R-monomial, so will be $\mathcal{X}_r$, which also implies that $\mathcal{X}_r$ is an elementary vector field and the reduction is actually a resolution of $(M, \theta, \mathcal{X}, E)$.

For example, consider a vector field:

$$X = A(x, \lambda)\frac{\partial}{\partial x}$$

and let $\theta =< \frac{\partial}{\partial x}$. Then a modification of $(M, \theta, \mathcal{X}, E)$ gives rise to a resolution of $(M, \theta, \mathcal{X}, E)$ and, consequently, for the vector field X.

The following result proves that a reduction is always possible:

Theorem 7.4.1. *Let $(M, M_0, \theta, \mathcal{X}, E)$ be a dim 1 local nested foliation by curves. Then, there exists a reduction of $(M, M_0, \theta, \mathcal{X}, E)$.*

Proof. By hypotheses, for each point p in M, let Y be the vector field that generates θ_p and X the vector field that generates $\mathcal{X}_p$. Since $\mathcal{X}$ is a sub-sheaf of θ, there exists a germ $f \in \mathcal{O}_p$ such that $X = fY$.

Let $\mathcal{I}(\theta, \mathcal{X})$ be the ideal sheaf such that $\mathcal{I}(\theta, \mathcal{X}).\mathcal{O}_p = (f)$. This ideal sheaf is clearly well-defined, coherent and everywhere non-zero.

By Theorem 5.3.1 there exists a θ-admissible resolution $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ of $(M, M_0, \theta, \mathcal{I}(\theta, \mathcal{X}), E)$:

$$(M_r, \theta_r, [\mathcal{I}(\theta, \mathcal{X})]_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, [\mathcal{I}(\theta, \mathcal{X})]_0, E_0)$$

Claim: If $\sigma : (M', \theta', [\mathcal{I}(\theta, \mathcal{X})]', E') \longrightarrow (M, \theta, \mathcal{I}(\theta, \mathcal{X}), E)$ is a θ admissible blowing-up of order one, then the transform $\mathcal{X}'$ is well-defined and:

$$[\mathcal{I}(\theta, \mathcal{X})]' = \mathcal{I}(\theta', \mathcal{I}')$$

Proof.

- Suppose that $\sigma : (M', \theta', [\mathcal{I}(\theta, \mathcal{X})]', E') \longrightarrow (M, \theta, \mathcal{I}(\theta, \mathcal{X}), E)$ is an θ-invariant blowing-up. Fixed a point q in M' and $p = \sigma(q)$, the vector field X' that generated $\mathcal{X}'.\mathcal{O}_q$ is given by:

  $$X' = f^* Y^* \mathcal{O}(-F).\mathcal{O}_q = [f^* \mathcal{O}(-F)] Y'.\mathcal{O}_q$$

  which implies that X' is well-defined and that:

  $$[\mathcal{I}(\theta, \mathcal{X})]' = \mathcal{I}(\theta', \mathcal{I}')$$

- Suppose that $\sigma : (M', \theta', [\mathcal{I}(\theta, \mathcal{X})]', E') \longrightarrow (M, \theta, \mathcal{I}(\theta, \mathcal{X}), E)$ is a θ-totally transverse blowing-up. Fixed a point q in M' and $p = \sigma(q)$, the vector field X' that generated $\mathcal{X}'.\mathcal{O}_q$ is given by:

  $$X' = f^*[\mathcal{O}(-F)Y'].\mathcal{O}_q = [f^* \mathcal{O}(-F)] Y'.\mathcal{O}_q$$

  which implies that X' is well-defined and that:

  $$[\mathcal{I}(\theta, \mathcal{X})]' = \mathcal{I}(\theta', \mathcal{I}')$$

□

The Claim implies that $(\sigma_1, ..., \sigma_r)$ is also a sequence of invariant admissible blowings-up:

$$(M_r, \theta_r, \mathcal{X}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{X}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{X}_0, E_0)$$

such that $\mathcal{I}(\theta_r, \mathcal{X}_r) = \mathcal{O}_{M_r}$, which clearly implies the Theorem. □

Chapter 8

Application 2: Generalized Flow-Box and a problem proposed by Mattei

8.1 Quasi-transversality

A foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is said to be:

- *geometrically quasi-transverse* if:

$$\mathcal{O}(E)\sqrt{\Gamma_{\theta,k}(\mathcal{O}_M) + \mathcal{I}} \subset \sqrt{\Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I}}$$

  for all $k \leq d$;

- 1-*algebraically quasi-transverse* if it is geometrically quasi-transverse, the singular distribution θ has leaf dimension 1 and $\nu_p(\theta, \mathcal{I}) \leq 1$ for all points p in M.

Example 1: Let $(M, \mathcal{I}, E) = (\mathbb{R}^2, (y), \emptyset)$:

i) Consider θ generated by $\{\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\}$. Then $(M, \theta, \mathcal{I}, E)$ is **not** geometrically quasi-transverse. Remark that there exists a leaf of θ finitely tangent to $V(\mathcal{I})$;

ii) Consider θ generated by $\{y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\}$. Then$(M, \theta, \mathcal{I}, E)$ is 1-algebraically quasi-transverse. Remark that the leafs of θ are either transverse to $V(\mathcal{I})$ or singular at it.

Example 2: Let $(M, \mathcal{I}, E) = (\mathbb{R}^4, (z, w), E)$:

i) Let $E = \{y = 0\}$ and θ be generated by $\{(x - xz)\frac{\partial}{\partial z} + (y - xw)\frac{\partial}{\partial w} + (1 - x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\}$. Then $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse but it is **not** 1-algebraically quasi-transverse. Remark that there exists a leaf of θ finitely tangent to $V(\mathcal{I})$ **contained** in the exceptional divisor E;

ii) Let $E = \emptyset$ and θ be generated by $\{x\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial w} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\}$. Then $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse but it is **not** 1-algebraically quasi-transverse. Remark that after the blowing-up of the origin, in the y-chart, we are precisely in the situation of example 2[i].

With these examples, we expect to make clear two major intuitive properties of these definitions:

- First intuitive property (from Example 1): If $E = \emptyset$ and $\mathcal{I}$ is regular, $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse if there is no leaf of θ that is finitely tangent to $V(\mathcal{I})$;
- Second intuitive property (from Example 2): If $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse, after a sequence of invariant blowings-up there may appear leafs of θ finitely tangent to $V(\mathcal{I})$. On the other hand, if $(M, \theta, \mathcal{I}, E)$ is 1-algebraically quasi-transverse, no sequence of invariant blowings-up will create this phenomena.

Remark 8.1.1. *In section* 8.6*, we will discuss what would be a good notion of d-algebraically quasi-transversality for $d > 1$. In particular, the second intuitive property will hold in the generalized context.*

Now, we formalize these intuitions. We start with the first one:

Lemma 8.1.2. *Suppose that the ideal sheaf $\mathcal{I}$ is regular. Then a d-foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse if, and only if, for all point p in $V(\mathcal{I}) \setminus E$:*

$$dim_{\mathbb{K}} L_p + dim_{\mathbb{K}} T_p V(\mathcal{I}) = dim_{\mathbb{K}}(< L_p + T_p V(\mathcal{I}) >)$$

where L_p is the linear sub-space of $T_p M$ generated by θ_p and $< S >$ stands for the smallest linear sub-space of $T_p M$ containing S.

Proof. Let $r = dim_{\mathbb{K}} L_p$. There exists a coherent set of generators $\{X_1, ..., X_r, Y_1, ..., Y_s\}$ of θ_p such that: the vector fields X_i are regular and generates L_p, the vector fields Y_i are singular and $r + s = d_p$. In particular, this implies that the ideal sheaf $\Gamma_{\theta,k}(\mathcal{O}_p)$ is equal to $\mathcal{O}_p$ for $k \leq r$ and the ideal sheaf $\Gamma_{\theta,k}(\mathcal{O}_p)$ contain the maximal ideal sheaf m_p for $k > r$.

Now, let $t = codim_{\mathbb{K}} T_p V(\mathcal{I})$. There exists a local coordinate system $x = (x_1, ..., x_n)$ where $p = (0, ..., 0)$ and $\mathcal{I} = (x_1, ..., x_t)$. We are ready to prove the result:

- ($\Rightarrow$) By the hypotheses, we have that:

$$(\sqrt{\Gamma_{\theta,k}(\mathcal{O}_p) + \mathcal{I}}).\mathcal{O}_p \subset (\sqrt{\Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I}}).\mathcal{O}_p$$

  for all $k \leq d$. In particular, $(\sqrt{\Gamma_{\theta,r}(\mathcal{I}) + \mathcal{I}}).\mathcal{O}_p = \mathcal{O}_p$ for $k = r$. So, there exists a set of analytic germs $(f_1, ..., f_r)$ contained in the ideal $\mathcal{I}.\mathcal{O}_p$ such that:

$$det \begin{Vmatrix} X_1(f_1) & ... & X_1(f_r) \\ \vdots & \ddots & \vdots \\ X_r(f_1) & ... & X_r(f_r) \end{Vmatrix}$$

  is an unity of $\mathcal{O}_p$. Without loss of generality, after a change of coordinates, we may assume that $f_i = x_i$ and $X_i = \frac{\partial}{\partial x_i}$ (see the proof of 4.3.1 for details on the coordinate change). It is now clear that every vector of $T_p V(\mathcal{I})$ must be in a complementary sub-space to $\{X_1, ..., X_r\}$, which proves the result.

- ($\Leftarrow$) The hypotheses implies that the linear sub-spaces L_p and $T_p V(\mathcal{I})$ are transverse at each point p of the set $V(\mathcal{I}) \setminus E$. So, without loss of generality, we can suppose that $X_i = \frac{\partial}{\partial x_i}$ for $i = 1, ..., r \leq t$. In particular, the ideal $\Gamma_{\theta,r}(\mathcal{I}).\mathcal{O}_p$ is equal to $\mathcal{O}_p$ and the ideal $\Gamma_{\theta,r+1}(\mathcal{I}).\mathcal{O}_p$ is contained in the maximal ideal m_p (because the Y_j are all singular at p). Since this is true for all points outside the exceptional divisor, we have that:

$$V(\Gamma_{\theta,k}(\mathcal{O}_M) + \mathcal{I}) \setminus E = V(\Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I}) \setminus E$$

  Which clearly implies that $V(\Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I}) \subset V(\Gamma_{\theta,k}(\mathcal{O}_M) + \mathcal{I}) \cup E$. Thus:

$$\mathcal{O}(E)\sqrt{\Gamma_{\theta,k}(\mathcal{O}_M) + \mathcal{I}} \subset \sqrt{\Gamma_{\theta,k}(\mathcal{I}) + \mathcal{I}}$$

  for all $k \leq d$.

□

Now, we present two Lemmas that formalizes the second intuitive property:

Lemma 8.1.3. *Let $(M, \theta, \mathcal{I}, E)$ be an 1-algebraically quasi-transverse foliated ideal sheaf. For all point p contained in $E \cap V(\mathcal{I})$ the leaf $\mathcal{L}$ of the singular distribution θ passing through p respects one of the following conditions:*

i) Either the leaf $\mathcal{L}$ is transverse to $V(\mathcal{I})$, or;

ii) The leaf $\mathcal{L}$ is contained in $V(\mathcal{I})$.

Proof. If the leaf $\mathcal{L}$ is zero-dimensional, the lemma is trivial. So, we can suppose that $\mathcal{L}$ is one-dimensional.

If the ideal $H(\theta, \mathcal{I}, 1).\mathcal{O}_p = (\mathcal{I} + \theta[\mathcal{I}]).\mathcal{O}_p$ is equal to $\mathcal{O}_p$ then it is clear that the leaf $\mathcal{L}$ is transverse to $V(\mathcal{I})$.

So, we can suppose that the ideal $H(\theta, \mathcal{I}, 1).\mathcal{O}_p$ is different from the structural ideal $\mathcal{O}_p$. By the flow-box Theorem, there exists a coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ such that the vector field $X = \frac{\partial}{\partial x}$ generates $\theta.\mathcal{O}_p$. Furthermore, by Lemma 4.2.4, there exists a set of generators $\{f_1(y), ..., f_s(y)\}$ of $H(\theta, \mathcal{I}, 1).\mathcal{O}_p$ independent of the coordinate x.

This implies that the intersection $\mathcal{L} \cap V(\mathcal{I})$ is an open subset of $\mathcal{L}$. Since $\mathcal{L}$ is locally closed and $V(\mathcal{I})$ is closed, the intersection is a closed subset of $\mathcal{L}$. By connexity, we conclude that $\mathcal{L} \subset V(\mathcal{I})$. □

Lemma 8.1.4. *Let $\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$ be an invariant blowing-up of order one and suppose that $(M, \theta, \mathcal{I}, E)$ is 1-algebraically quasi-transverse. Then $(M', \theta', \mathcal{I}', E')$ is algebraically quasi-transverse.*

Proof. It is clear that $(M', \theta', \mathcal{I}', E')$ is geometrically quasi-transverse, because outside the exceptional divisor F, the blowing-up σ is an isomorphism.

So, we only need to show that $H(\theta',\mathcal{I}',1)$ is θ'-invariant. But this clearly follows from Lemma 5.5.1, which states that:

$$H(\theta',\mathcal{I}',i) = H(\theta,\mathcal{I},i)'$$

and the fact that $H(\theta,\mathcal{I},1)$ is θ-invariant. □

8.2 Sub-Riemannian Geometry

In this section, we introduce some basic concept of sub-Riemannian geometry. We follow closely [Bell].

8.2.1 Basic Definitions

Consider $\mathbb{K} = \mathbb{R}$, W a regular analytic manifold and $\{X_1, ..., X_m\}$ be globally defined analytic vector fields. For each point p in W, we denote by L_p the subspace of T_pW generated by $\{X_1(p), ..., X_m(p)\}$. Given any vector v of L_p, there always exists $(u_1, ..., u_m) \in \mathbb{K}$ (not necessarily unique) such that:

$$v = \sum_{i=1}^{m} u_i X_i(p)$$

So, for each point p of W, consider the mapping:

$$\begin{array}{cccc} \Phi_p : & \mathbb{R}^m & \longrightarrow & T_pW \\ & (u_1, ..., u_m) & \mapsto & \sum_{i=1}^{m} u_i X_i(p) \end{array}$$

Notice that Φ_p restricted to the linear subspace $(ker\Phi_p)^{\perp}$ is a linear isomorphism onto L_p. Let $\Psi_p : L_p \longrightarrow (ker\Phi_p)^{\perp}$ be the inverse mapping. Then, if v and w are vectors contained in L_p, we define the *sub-Riemannian metric* $g_p(v,w)$ associated to $\{X_1, ..., X_m\}$ by:

$$g_p(v,w) = <\Psi_p(v), \Psi_p(w)>$$

where $<,>$ is the euclidean norm of $\mathbb{R}^m$. Based on this metric, we define the notion of *sub-Riemannian norm* $\|.\|_p$ associated with $\{X_1, ..., X_m\}$:

$$\|v\|_p = g_p(v,v)^{\frac{1}{2}}$$

We also define a notion of ∞-*sub-Riemannian norm* $\|.\|_{\infty,p}$ associated to $\{X_1, ..., X_m\}$:

$$\|v\|_{\infty,p} = \|\Psi_p(v)\|_\infty$$

where $\|.\|_\infty$ is the ∞-norm of $\mathbb{R}^m$.

We extend both norms for every vector v of T_pW by setting $\|v\|_p = \|v\|_{\infty,p} = \infty$ if v is not contained in L_p.

With this metric, we can define a notion of length of a path. Given an absolutely continuous path $c(t)$ contained in W, with $t \in [a, b]$, we define:

$$length(c(t)) = \int_a^b \|\dot{c(t)}\|_{c(t)} dt$$
$$length_\infty(c(t)) = \int_a^b \|\dot{c(t)}\|_{\infty,c(t)} dt$$

and the distance associated to the metric g is given by:

$$d(p, q) = inf\{length(c(t));\ \text{where } c(t) \text{ is absolutely continuous and } c(a) = p;\ c(b) = q\}$$
$$d_\infty(p, q) = inf\{length_\infty(c(t));\ \text{where } c(t) \text{ is absolutely continuous and } c(a) = p;\ c(b) = q\}$$

If there is no curve absolutely continuous $c(t)$ such that $c(a) = p$ and $c(b) = q$, we set $d(p, q) = +\infty$ (this happens, for example, if M is not connected). It is clear that we have the two following relations:

$$\begin{aligned} \|v\|_{\infty,p} \leq \|v\|_p \leq \sqrt{m}.\|v\|_{\infty,p} \\ d_\infty(p, q) \leq d(p, q) \leq \sqrt{m}.d_\infty(p, q) \end{aligned} \tag{8.1}$$

8.2.2 The complex definition

We can extend the precious definitions to the complex setting as follows: let $\{X_1, ..., X_m\}$ be globally defined analytic vector fields. We remark that W can be seen as a real variety. Let $\{Y_1, ..., Y_{2m}\}$ be globally defined real-analytic vector fields over W such that $\{Y_i, Y_{m+i}\}$ generates X_i. Then, we can define a notion of sub-Riemannian metric and sub-Riemannian norm just as in the real case for $\{Y_1, ..., Y_{2m}\}$.

We remark that, for each set $\{X_1, ..., X_m\}$, the definition of this metric and norms are not unique, because they will actually depend upon the vector fields $\{Y_1, ..., Y_{2m}\}$.

8.2.3 Global Definitions

From now on, we work with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An analytic *sub-Riemannian metric* on M is a function $g : TM \otimes TM \longrightarrow \mathbb{R} \cup \{\infty\}$ which, locally, may be defined as the metric associated to some system of analytic vector fields. In other words, for each point p in M, there exists a neighborhood U_p of p and a set of vector fields $\{X_1, ..., X_m\}$ defined over U_p such that $g|_{U_p} = g_{\{X_1,...,X_m\}}$ where $g_{\{X_1,...,X_m\}}$ is the sub-Riemmanin metric on U_p associated with $\{X_1, ..., X_m\}$. A sub-Riemannian distance on M is a distance that can be defined, via the length paths, from such a metric.

For the sake of simplicity, we shall always suppose in the sequel that a sub-Riemannian metric $g : TM \otimes TM \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined by a given system of global vector fields $\{X_1, ..., X_m\}$. This assumption is perfectly legitimate in all purely local questions (in particular, the G-FB property defined below is local).

Let ω_g be the singular distribution generated by the given set of global vector fields $\{X_1, ..., X_m\}$. Notice that this singular distribution is not necessarily involutive and it depends on the fixed vector fields $\{X_1, ..., X_m\}$. Nevertheless, since we are always supposing that g is generated by a fixed system of global vector fields $\{X_1, ..., X_m\}$, the singular distribution ω_g is well-defined. Let θ_g be the smaller (in the sense of sub-sheaves) involutive singular distribution containing ω_g. The involutive singular distribution θ_g is well-defined because the intersection of involutive singular distributions is an involutive singular distribution and ω_g is a sub-sheaf of Der_M.

We are now ready to define the main objects of this Chapter and a notion of quasi-transversality:

A *sub-Riemannian manifold* is a triple (M, g, E) such that:

- (M, E) is an analytic manifold with divisor;
- g is a sub-Riemmanin metric over M totally tangent to E i.e. the divisor E is θ_g-

invariant.

A *sub-Riemannian ideal sheaf* is a quadruple $(M, g, \mathcal{I}, E)$ such that:

- The triple (M, g, E) is a sub-Riemannian manifold;
- The ideal sheaf $\mathcal{I}$ is coherent and everywhere non-zero.

A sub-Riemannian ideal sheaf $(M, g, \mathcal{I}, E)$ is said to be:

- *geometrically quasi-transverse* if $(M, \theta_g, \mathcal{I}, E)$ is geometrically quasi-transverse;
- 1-*algebraically quasi-transverse* if $(M, \theta_g, \mathcal{I}, E)$ is 1-algebraically quasi-transverse.

8.2.4 Blowing-up

An admissible blowing-up $\sigma : (M^{'}, E^{'}) \longrightarrow (M, E)$ is *invariant by* (M, g, E) if $\sigma : (M^{'}, \theta_g^{'}, E^{'}) \longrightarrow (M, \theta_g, E)$ is invariant.

The *total transform of* g under an invariant blowing-up is the metric $g^{'}$ over $M^{'}$ defined by the pull-back of the vector fields $\{X_1, ..., X_m\}$. This process is well-defined by Proposition 4.4.1 and the fact that the blowing-up is invariant. Furthermore, it clearly implies the equality $\theta_{g'} = (\theta_g)^{'}$.

8.3 The $G - FB$ property

Let $(M, g, \mathcal{I}, E)$ be a sub-Riemannian ideal sheaf. Given p in M and $\delta > 0$, the set:

$$B^g_\delta(p) = \{q \in M; d_g(p, q) < \delta\}$$

is called the *g-ball* at p with radius δ.

We say that a sub-Riemannian ideal sheaf $(M, g, \mathcal{I}, E)$ satisfies the *Generalized Flow-Box* property, or simply the *G-FB* property, if for each point p in the variety $V(\mathcal{I})$ there exists a pair (U_p, δ_p) where:

- U_p is an open neighborhood of p (in the usual topology of M);
- δ_p is a positive real number.

such that, for all points q in $(V(\mathcal{I}) \cap U_p) \setminus E$ and all positive real number $\delta < \delta_p$, the g-ball $B^g_\delta(q)$ with center q with radius δ:

- intersects the variety $V(\mathcal{I})$ only at q;
- is homeomorphic to a k_q-euclidean ball, where k_q is the dimension of the leaf of θ_g passing through q.

Remark 8.3.1. *If g is locally generated by a single vector field and $E = \emptyset$, then the problem described in the introduction is equivalent to asking whether $(M, g, \mathcal{I}, E)$ satisfies the $G - FB$ property.*

We say that a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ satisfies the *Generalized Flow-Box* property, or simply the *G-FB* property, if there exists a sub-Riemannian metric g on M such that:

- For each point p in M, there exists a choice of vector fields $\{X_1, ..., X_m\}$ that generates g on an open neighborhood U_p of p, such that the singular distribution ω_g (generated by $\{X_1, ..., X_m\}$) is involutive and equal to $\theta.\mathcal{O}_{U_p}$;
- The sub-Riemannian ideal sheaf $(M, g, \mathcal{I}, E)$ satisfies the $G - FB$ property.

Lemma 8.3.2. *If a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ satisfies the $G - FB$ property, then a sub-Riemannian ideal sheaf $(M, g, \mathcal{I}, E)$ such that the involutive singular distribution θ_g coincides with θ also satisfies the $G - FB$ property.*

Remark 8.3.3. *In particular, the $G - FB$ property for a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is independent of the choice of the sub-Riemannian metric g such that $\omega_g = \theta$.*

Proof. Since the problem is local, without loss of generality we can suppose that $M = U_p$ and let $\{X_1, ..., X_m\}$ be globally defined vector fields defining g.

By hypotheses, there exists a sub-Riemannian metric h over M such that $\omega_h = \theta$

and $(M, h, \mathcal{I}, E)$ satisfies the $G - FB$ property. Without loss of generality, this implies that there exists globally defined vector fields $\{Y_1, ..., Y_s\}$ generating θ and h.

Since $\{Y_1, ..., Y_s\}$ is a set of generators of θ, there exists an analytic matrix A such that $(X_1, ..., X_r) = A(Y_1, ..., Y_s)$. Let $\|A\|$ denote the ∞-norm of the matrix A, i.e. $\|A\|_p = max\{\|a_{i,j}(p)\|; i, j\}$.

If v is a vector of T_pM that can be written as $\sum_{i=1}^{r} u_i X_i$, we have that:

$$v = \sum_{i}^{r} u_i \sum_{j=1}^{s} a_{i,j} Y_j = \sum_{j=1}^{s} Y_j [\sum_{i=1}^{r} a_{i,j} u_i]$$

This implies that $\max\{\| \sum_{i=1}^{r} a_{i,j} u_i \|\} \leq \|A\| \max\{\|u_i\|\}$. Thus:

$$\|v\|_{h,\infty,p} \leq \|A\| \|v\|_{g,\infty,p}$$

Since $\|v\|_{h,p} \leq \sqrt{s} \|v\|_{h,\infty,p}$ and $\|v\|_{g,\infty,p} \leq \|v\|_{g,p}$, apart from taking a smaller open set U_p, we have that:

$$\|v\|_{h,p} \leq M \|v\|_{g,p}$$

where $M > 0$ is a constant that depends on $max\{\|A\|_q; q \in U_p\}$. This is enough to prove that $length_h(c(t)) \leq M length_g(c(t))$ and that $d_h(p, q) \leq M d_g(p, q)$, which implies the Lemma. □

8.4 Setting the Problems 1 and 2

The two problems we want to address are the following:

- **Problem 1:** Given a sub-Riemmaninan ideal sheaf $(M, g, \mathcal{I}, E)$ such that $\mathcal{I}$ is regular, is it true that geometrically quasi-transversality implies the $G - FB$ property?

- **Problem 2:** How can we characterize a sub-Riemmaninan ideal sheaf $(M, g, \mathcal{I}, E)$ that satisfies the $G - FB$ property?

The first question is based on the problem proposed by Mattei (see subsection 1.5.5). The main idea is to find a characterization of the $G-FB$ property that only depends on geometrical conditions. The hypotheses on the regularity of $\mathcal{I}$ is actually strong enough for the Problem 1 to have a positive answer when $dimM \leq 3$. But it is not true when $dimM = 4$, situation where we will present a counter-example.

We also remark that the second problem is more general since the variety $V(\mathcal{I})$ may have singularities.

In what follows we will divide the study in two parts: one when the leaf dimension of the singular distribution is one, and another when the leaf dimension is bigger than one. Nevertheless, these two studies are based on the same techniques, and we now present two Lemmas that are useful in both of the studies:

Lemma 8.4.1. *(Local Triviality) Let $(M, \theta, \mathcal{I}, E)$ be a geometrically quasi-transverse foliated ideal sheaf such that θ is totally transverse to $\mathcal{I}$. Then, $(M, \theta, \mathcal{I}, E)$ satisfies the $G-FB$ property.*

Proof. This result follows trivially from the Flow-Box Theorem. □

Lemma 8.4.2. *(Blowing-up reduction) Let $\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$ be an invariant blowing-up of order one and suppose that $(M', \theta', \mathcal{I}', E')$ satisfies the $G-FB$ property. Then $(M, \theta, \mathcal{I}, E)$ satisfies the $G-FB$ property.*

Proof. If a point p in M is outside the center $\mathcal{C}$, then σ is an isomorphism close to $\sigma^{-1}(p)$ and the result follows trivially. So, consider p a point over the center $\mathcal{C}$.

Consider a sub-Riemannian metric g such that $\omega_g = \theta$. Since the blowing-up is invariant, by Proposition 4.4.1, we have that $\theta' = \theta^*$. This implies that the pull-back g' is such that $\omega_{g'} = \theta'$.

Take a relatively compact open neighborhood U of p. Since the blowing-up σ is a proper morphism, $V := \sigma^{-1}(U)$ is a relatively compact open set of $M^{'}$. So, without loss of generality, V is equal to $\cup_{i \leq N} U_{q_i}$, where:

- The points q_i are contained on the exceptional divisor F and $\sigma(q_i) = p$;
- The pair (U_{q_i}, δ_i) satisfies the $G - FB$ property for the sub-Riemannian ideal sheaf $(M^{'}, g^{'}, \mathcal{I}^{'}, E^{'})$ at q_i.

Let $\delta_p := min\{\delta_i; i \leq N\}$. We claim that the pair (U, δ_p) satisfies the $G - FB$ property for the sub-Riemannian ideal sheaf $(M, g, \mathcal{I}, E)$ at p.

Indeed, take a point $q \in (V(\mathcal{I}) \cap U) \setminus E$:

- $(*)$ If q is contained in the center of blowing-up $\mathcal{C}$ then q is contained in the singular set $\Gamma_{\theta,d}(\mathcal{O}_M)$ (because the blowing-up is of order one). So the leaf of θ passing through q is just $\{q\}$, which implies that the g-ball $B^g_\delta(q)$ trivially satisfies the $G-FB$ properties for any δ;
- If q is outside the blowing-up center $\mathcal{C}$ then, since $g^{'}$ is given by the pull-back of g and at this point σ is a local isomorphism, the g-ball $B^g_\delta(q)$ satisfies the $G-FB$ properties for any $\delta < \delta_p$.

□

Remark 8.4.3. *The only point where we need the blowing-up to be of order one is in the argument $(*)$. If we can obtain the same conclusion under a different hypotheses, the Lemma is also valid.*

8.5 The 1-dimensional case

8.5.1 Main result

The main result of this section is the following:

Theorem 8.5.2. *Let $(M, \theta, \mathcal{I}, E)$ be a geometrically quasi-transverse 1-foliated ideal sheaf. If one of the following conditions is verified:*

i) *The* 1*-foliated ideal sheaf* $(M, \theta, \mathcal{I}, E)$ *is* 1*-algebraically quasi-transverse;*

ii) *The ideal sheaf* $\mathcal{I}$ *is regular and the variety* $V(\mathcal{I})$ *has dimension one;*

iii) *The ideal sheaf* $\mathcal{I}$ *is regular and the variety* $V(\mathcal{I})$ *has co-dimension one.*

Then $(M, \theta, \mathcal{I}, E)$ *satisfies the* $G - FB$ *property.*

Proof. Since the problem is local, we can fix a point p in M and a relatively compact open subset M_0 of M containing p. By Proposition 5.4.1 there exists a sequence of θ-invariant blowings-up of order one:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

such that:

I) $type_{M_r}(\theta_r, \mathcal{I}_r) = 1$;

II) $\nu_{M_r}(\theta_r, \mathcal{I}_r) \leq \nu_{M_0}(\theta, \mathcal{I})$;

Conclusion $[I]$ implies that θ_r is non-singular in every point of $V(\mathcal{I}_r)$ and no leaf of θ_r is contained in $V(\mathcal{I})$. Furthermore, for each point p in $V(\mathcal{I}_r)$, the leaf $\mathcal{L}$ of θ_r through p is either transverse or finitely tangent to $V(\mathcal{I}_r)$. We also remark that if $\mathcal{I}_0$ is regular and a leaf $\mathcal{L}$ is tangent to $V(\mathcal{I}_r)$, then it is contained in E_r.

Now, we consider separately the various cases of the Theorem:

i) If $(M_0, \theta_0, \mathcal{I}_0, E_0)$ is 1-algebraically quasi-transverse, then conclusion $[II]$ implies that θ_r is totally transverse to $\mathcal{I}_r$. So, by Lemma 8.4.1, $(M_r, \theta_r, \mathcal{I}_r, E_r)$ satisfies the $G - FB$ property;

ii) If $\mathcal{I}$ is regular and $V(\mathcal{I})$ has dimension one, we remark that $V(\mathcal{I}_r) \cap E_r$ has dimension zero and $V(\mathcal{I}_r)$ is transverse to E_r. Since E_r is invariant by θ_r, we conclude that θ_r must be transverse to every point in $V(\mathcal{I}_r) \cap E_r$, which implies that θ_r is totally transverse to $\mathcal{I}_r$. So, by Lemma 8.4.1, $(M_r, \theta_r, \mathcal{I}_r, E_r)$ satisfies the $G - FB$ property;

iii) If $\mathcal{I}$ is regular and $V(\mathcal{I})$ has co-dimension one, we divide in two cases:

a If $\mathbb{K} = \mathbb{C}$, then the variety of tangencies $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$ is of codimension one in $V(\mathcal{I}_r)$. On the other hand, the variety of tangencies $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$ must be contained in the exceptional divisor E_r, and $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$ must be equal to an union of connected components of $E_r \cap V(\mathcal{I}_r)$. Since the singular distribution has only finite tangencies with the variety $V(\mathcal{I}_r)$, we conclude that $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$ must be empty. So, θ_r is totally transverse to $\mathcal{I}_r$ and, by Lemma 8.4.1, $(M_r, \theta_r, \mathcal{I}_r, E_r)$ satisfies the $G - FB$ property;

b If $\mathbb{K} = \mathbb{R}$, then we can assume that the variety of tangencies $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$ has codimension at least two (by the same argument of item $[iii][a]$). In this case, notice that the variety $V(\mathcal{I}_r)$ is locally orientable. So, for each point p in $V(\mathcal{I}_r)$, let N be a (normal) vector field defined in an open neighborhood U_p of p that gives a local orientation for $V(\mathcal{I}_r)$ in U_p. Furthermore, let X be a vector field defined in U_p that generates $\theta.\mathcal{U}_p$. Then the following function:

$$\begin{array}{cccc} \phi: & U_p \cap V(\mathcal{I}_r) & \longrightarrow & \mathbb{R} \\ & q & \mapsto & <N, X> (q) \end{array}$$

is continuous and equal to zero, if and only if, q is contained in the tangency variety $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$. Since $V(\theta[\mathcal{I}_r]) \cap V(\mathcal{I}_r)$ has codimension at least two in $V(\mathcal{I}_r)$, we can assume that $\phi(q) \geq 0$ in all points q in $U_p \cap V(\mathcal{I}_r)$ because $V(\mathcal{I}_r) \setminus V(\theta[\mathcal{I}_r])$ is connected. Furthermore, by the continuity of X, we conclude that all orbits of X are cutting $V(\mathcal{I}_r) \cap U_p$ with the same orientation. So, by the flow-box Theorem, we conclude that $(U_p, \theta_r.\mathcal{O}_{U_p}, \mathcal{I}_r.\mathcal{O}_{U_p}, E_r \cap U_p)$ satisfies the $G - FB$ property. Since the choice of p in $\mathcal{I}_r$ is arbitrary, we conclude that $(M_r, \theta_r, \mathcal{I}_r, E_r)$ satisfies the $G - FB$ property;

In all the cases, by Lemma 8.4.2, $(M_0, \theta_0, \mathcal{I}_0, E_0)$ satisfies the $G - FB$ property. Since the choice of p was arbitrary, $(M, \theta, \mathcal{I}, E)$ satisfies the $G - FB$ property. □

8.5.3 Counter-example to Problem 1

In this section we present an example of a geometrically quasi-transverse 1-foliated ideal sheaf $(M, \mathcal{X}, \mathcal{I}, E)$, that does not satisfy the $G - FB$ property, even though $\mathcal{I}$ is regular. This example is valid both in $\mathbb{R}$ and $\mathbb{C}$.

We start the construction by stating a Lemma:

Lemma 8.5.4. *The equation:*

$$(1+s^2)\cos(\theta)\sin(\theta) - \theta = 0 \tag{8.2}$$

has an analytic solution $(\theta, s) = (h(s), s) = (sU(s), s)$, where $U(s)$ is an unity defined in an open neighborhood of the origin such that $U(0) = \frac{\sqrt{6}}{2}$.

Now, consider 1-foliated ideal sheaf $(M, \mathcal{X}, \mathcal{I}, E)$, where:

- The variety M is a small open neighborhood of the origin of $\mathbb{R}^4$;
- The singular distribution $\mathcal{X}$ is generated by a unique vector field X:

$$X = y^2 \frac{\partial}{\partial w} + x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

- The ideal sheaf $\mathcal{I}$ is generated by two functions $(z - f(x,y), 2w - g(x,y))$ where:

$$f(x,y) = y^2 \cos(h(x^2+y^2)) - xy \sin(h(x^2+y^2))$$
$$g(x,y) = xy(x^2+y^2)^2$$

  where $h(s)$ is given by Lemma 8.5.4;

- The divisor E is empty.

Claim 1: This sub-Riemannian ideal sheaf is geometrically quasi-transverse.

Proof. Notice that:

$$H(\mathcal{X}, \mathcal{I}, 1) = (z - f(x,y), 2w - g(x,y), X[z - f(x,y)], X[2w - g(x,y)])$$

Furthermore, we have that:

$$X[z - f(x,y)] = 2xy\cos(h(x^2+y^2)) + (x^2-y^2)\sin(h(x^2+y^2))$$
$$X[2w - g(x,y)] = 2y^2 - (x^2-y^2)(x^2+y^2)^2$$

Now, since the points of tangency between the singular distribution $\mathcal{X}$ and the variety $(\mathcal{I})$ are contained in the variety $V(H(\mathcal{X},\mathcal{I},1))$, if the variety $V(H(\mathcal{X},\mathcal{I},1))$ is just the origin (for a sufficiently small ambient space M), then the claim 1 is proved. So, we only need to prove that the variety $V(X[z-f(x,y)], X[2w-g(x,y)])$ is contained in $V(x,y)$. Indeed, from the equation:

$$2y^2 - (x^2-y^2)(x^2+y^2)^2 = 0$$

we get two solutions over y close to the origin:

$$y_1 := x^3V(x)$$
$$y_2 := -x^3V(x)$$

where $V(x)$ is an analytic unity such that $V(0) = \frac{1}{2}\sqrt{2}$. Now, making the substitution on the equation $X[z-f(x,y)] = 0$, we get:

$$\pm 2x^4V(x)\cos(h(x^2+x^6V(x)^2)) + (x^2-x^6V(x)^2)\sin(h(x^2+x^6V(x)^2)) = 0$$

now, taking the Taylor expansion in x on the origin, we get:

$$(\pm 2V(0) + U(0))x^4 + O(x,5) = 0$$

where we recall that $h(s) = sU(s)$. Since $\pm 2V(0) + U(0) = \pm\sqrt{2} + \frac{\sqrt{6}}{2} \neq 0$, we conclude that the equation is equivalent, close to the origin to $x^4W(x) = 0$, where $W(x)$ is an unity. Thus $x^4 = 0$, which implies that $y = 0$ and we are done. □

Claim 2: This sub-Riemannian ideal sheaf does not satisfy the $G-FB$ property.

Proof. We prove this claim when $\mathbb{K} = \mathbb{R}$. We remark that the complex case trivially follows from the real case.

Notice that the variety $V(x,y)$ is invariant by X, since X is singular at each point of it. So, if we consider the blowing-up $\sigma : M' \longrightarrow M$ with center $V(x,y)$, by Lemma 8.4.2 and remark 8.4.3 the the $G-FB$ property is preserved. Since we are in a real field, we may work with a polar blowing-up:

$$\begin{array}{cccc} \sigma : & \mathbb{R}^+ \times S^1 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^4 \\ & (r,\theta,z,w) & \mapsto & (r\cos(\theta), r\sin(\theta), z, w) \end{array}$$

So, after the blowing-up, we get:

$$X' = r^2 \sin(\theta)^2 \frac{\partial}{\partial w} + \frac{\partial}{\partial \theta}$$

and $\mathcal{I}' = (z - f'(r,\theta), 2w - g'(r,\theta))$, where:

$$f'(r,\theta) = r^2 \sin(\theta)[\sin(\theta)\cos(h(r^2)) - \cos(\theta)\sin(h(r^2))]$$
$$g'(r,\theta) = r^6 \sin(\theta)\cos(\theta)$$

Fix $r_0 > 0$. Notice that the orbit of X' passing thought $(r,\theta,z,w) = (r_0,0,0,0)$ at $t=0$ is given by:

$$\gamma(r_0,t) = (r_0, t, 0, \frac{r_0^2}{4}[2t - \sin(2t)])$$

In particular, notice that:

$$\gamma(r_0, h(r_0^2)) = (r_0, h(r_0^2), 0, \frac{r_0^2}{4}[2h(r_0^2) - \sin(2h(r_0^2))])$$

Now, we claim that $\gamma(r_0,0)$ and $\gamma(r_0,h(r_0^2))$ are contained in the variety $V(\mathcal{I}')$. Indeed, we recall that $\mathcal{I}' = (z - f'(r,\theta), 2w - g'(r,\theta))$, where:

$$f'(r,\theta) = r^2 \sin(\theta)[\sin(\theta)\cos(h(r^2)) - \cos(\theta)\sin(h(r^2))]$$
$$g'(r,\theta) = r^6 \sin(\theta)\cos(\theta)$$

Which allow us to show that:

$$[z - f'(r,\theta)](r_0,0,0,0) = r_0^2 \sin(0)[\sin(0 - h(r^2))] = 0$$
$$\left[2w - g'(r,\theta)\right](r_0,0,0,0) = r_0^6 \sin(0)\cos(0) = 0$$

and, thus, $\gamma(r_0,0)$ is contained in the variety $V(\mathcal{I}')$. And that:

$$[z - f'(r,\theta)](\gamma(r_0,h(r_0^2))) = r_0^2 \sin(h(r_0^2))[\sin(h(r_0^2) - h(r^2))] = 0$$
$$\left[2w - g'(r,\theta)\right](\gamma(r_0,h(r_0^2))) = 2\tfrac{r_0^2}{4}[2h(r_0^2) - \sin(2h(r_0^2))] - r_0^6 \sin(h(r_0^2))\cos(h(r_0^2))$$

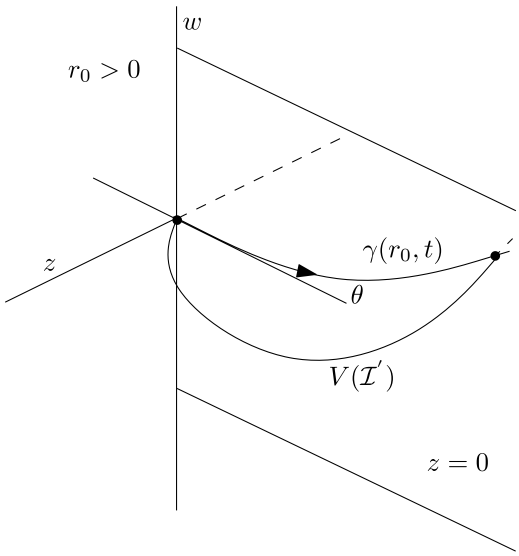

Figure 8.1: In the picture $r_0 > 0$ is fixed. The orbit $\gamma(r_0, t)$ is contained in the plane $\{z = 0\}$ and the variety $V(\mathcal{I}')$ is a curve that cuts the orbit $\gamma(r_0, t)$ two times.

Now, since:

$$2\frac{r_0^2}{4}[2h(r_0^2) - \sin(2h(r_0^2))] - r_0^6 \sin(h(r_0^2))\cos(h(r_0^2)) = r_0^2[h(r_0^2) - \sin(h(r_0^2))\cos(h(r_0^2))(1 + r_0^4)]$$

and $h(s)$ is a solution of:

$$(1 + s^2)cos(\theta)sin(\theta) - \theta$$

we conclude that:

$$[2w - g'(r, \theta)](\gamma(r_0, h(r_0^2))) = 0$$

and, thus, $\gamma(r_0, h(r_0^2))$ is contained in the variety $V(\mathcal{I}')$. This implies that for each $r_0 > 0$ fixed, there exists an orbit of X' that cuts $V(\mathcal{I}')$ two times. Furthermore, the time between each cut is equal to $h(r_0^2)$, which goes to zero when r_0 goes to zero. This implies that $(M', X', \mathcal{I}', E')$ does not satisfy the $G - FB$ property, which proves the claim. $\square$

We also remark that the tg-order $\nu_0(\mathcal{X}, \mathcal{I})$ equals two. Indeed we have that $H(X, \mathcal{I}, 2) = (x^2, xy, y^2, z, w)$ which is invariant by the vector field, thus $\nu_0(\mathcal{X}, \mathcal{I}) \leq 2$. Furthermore, since this example does not satisfy the $G - FB$ property, even though it is geometrically quasi-transverse, we conclude that it can not stabilize at $\nu \leq 1$ (otherwise it would contradict Theorem 8.5.2).

The only thing left to prove is Lemma 8.5.4:

Proof. (Lemma 8.5.4): We start noticing that $(\theta, s) = (0, s)$ is a solution of equation (8.2). So, the equation:

$$(1 + s^2)cos(\theta)\frac{sin(\theta)}{\theta} - 1 = 0 \tag{8.3}$$

has the same solutions of equation (8.2) apart from $(\theta, s) = (0, s)$. Now, taking the Taylor expansion of equation (8.3) in relation with the variable θ at $(\theta, s) = (0, 0)$, we get:

$$s^2 - \frac{2}{3}(1 + s^2)\theta^2 + O(\theta, 3) = 0 \tag{8.4}$$

So, by the Weierstrass Preparation Theorem and the symmetry of equation (8.3) in respect with the transform $\theta \longrightarrow -\theta$, the equation (8.3) can be written as:

$$(f(s)^2 + \theta^2)u(\theta, s) = 0 \tag{8.5}$$

where $u(\theta, s)$ is an unity and $f(s)$ is an analytic function. Furthermore, we have that:

$$f(s)^2u(0, s) = s^2$$

which implies that:

$$f(s) = \pm sU(s)$$

where $U(s) = \sqrt{\frac{1}{u(0,s)}}$ is an analytic unity in a neighborhood of zero. Taking $h(s) = sU(s)$ gives the desired result. To finish, making the substitution of $h(s)$ into the equation (8.2) and taking the Taylor expansion of this expressions in terms of s, we obtain:

$$(U(0) - \frac{2}{3}U(0)^3)s^3 + O(s, 4)$$

which implies that $U(0)$ can be taken equal to $\frac{\sqrt{6}}{2}$. □

8.6 The d-dimensional case

In this section we partially extend the results of the previous section to the case of higher leaf dimension.

8.6.1 d-algebraically quasi-transversality

A a foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is *d-algebraically quasi-transverse* if:

i) The foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse and the tg-order $\nu_p(\theta, \mathcal{I}) = 1$ at all point p in the variety $V(\mathcal{I})$;

ii) Given an invariant blowing-up of order one $\sigma : (M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'}) \longrightarrow (M, \theta, \mathcal{I}, E)$, the foliated ideal sheaf $(M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'})$ is also d-algebraically quasi-transverse;

iii) If $\theta[\mathcal{I}] = \mathcal{O}_M$, then at each point $p \in V(\mathcal{I})$, if $\mathcal{X}$ and ω are two singular distributions defined over an open neighborhood U_p of p such that:

- The distributions $\mathcal{X}$ and ω generates $\theta.\mathcal{O}_{U_p}$;
- The 1-singular distribution $\mathcal{X}$ is generated by a regular vector field X and is totally transverse to $\mathcal{I}$;
- The $(d-1)$-singular distribution ω has a generator set $\{Y_1, ..., Y_s\}$ such that $[X, Y_i] \equiv 0$.

Then $(U_p, \omega, \mathcal{I}.\mathcal{O}_{U_p}, E \cap U_p)$ is $d-1$-algebraically quasi-transverse.

This condition depends on a sequence of blowing-ups and it is quite difficult to define à priori for a general ideal sheaf $\mathcal{I}$. But there is one geometrical interesting case:

Lemma 8.6.2. *Let $(M, \theta, \mathcal{I}, E)$ be a geometrically quasi-transverse d-foliated ideal sheaf and suppose that $\mathcal{I}$ is regular and $V(\mathcal{I})$ has dimension one, i.e. is is a regular analytic curve. Then $(M, \theta, \mathcal{I}, E)$ is d-algebraically quasi-transverse.*

Proof. We claim that the hypotheses (H):

- The foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ is geometrically quasi-transverse;
- The ideal sheaf $\mathcal{I}$ is regular;
- The variety $V(\mathcal{I})$ has dimension one.

is preserved by invariant blowing-ups of order one. Indeed, given an invariant blowing-up of order one $\sigma : (M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'}) \longrightarrow (M, \theta, \mathcal{I}, E)$, we have that:

- The foliated ideal sheaf $(M^{'}, \theta^{'}, \mathcal{I}^{'}, E^{'})$ is geometrically quasi-transverse
- The ideal sheaf $\mathcal{I}^{'}$ is regular;
- The variety $V(\mathcal{I}^{'})$ has dimension one;

Furthermore, if $\theta[\mathcal{I}] = \mathcal{O}_M$, then at each point $p \in V(\mathcal{I})$, if $\mathcal{X}$ and ω are two singular distributions defined over an open neighborhood U_p of p such that:

- The distributions $\mathcal{X}$ and ω generates $\theta.\mathcal{O}_{U_p}$;
- The 1-singular distribution $\mathcal{X}$ is generated by a regular vector field X and is totally transverse to $\mathcal{I}$;
- The $d-1$-singular distribution ω has a generator set $\{Y_1, ..., Y_s\}$ such that $[X, Y_i] \equiv 0$.

Then:

- The foliated ideal sheaf $(U_p, \omega, \mathcal{I}.\mathcal{O}_{U_p}, E \cap U_p)$ is geometrically quasi-transverse
- The ideal sheaf $\mathcal{I}.\mathcal{O}_{U_p}$ is regular;
- The variety $V(\mathcal{I}.\mathcal{O}_{U_p})$ has dimension one;

So, we only need to verify that hypotheses (H) implies that the tg-order $\nu_p(\theta, \mathcal{I})$ is one for every point p in the variety $V(\mathcal{I})$. Fix a point p of the variety $V(\mathcal{I})$, since $\mathcal{I}$ is regular there exists a local coordinate system $(x, y) = (x_1, ..., x_{n-1}, y)$ such that the ideal $\mathcal{I}.\mathcal{O}_p$ is equal to $(x) = (x_1, ..., x_{n-1})$.

The hypotheses of geometrically quasi-transverse, implies that:

- Either $H(\theta, \mathcal{I}, 1).\mathcal{O}_p = \mathcal{O}_p$ and the tg-order $\nu_p(\theta, \mathcal{I}) = 1$, or;
- $H(\theta, \mathcal{I}, 1).\mathcal{O}_p = (x, y^r)$ for some $r \in \mathbb{N}$. In this case, we claim that $H(\theta, \mathcal{I}, 1).\mathcal{O}_p$ is θ_p invariant, which implies that the tg-order $\nu_p(\theta, \mathcal{I}) = 1$. Indeed, if $H(\theta, \mathcal{I}, 1).\mathcal{O}_p$ is not invariant, this implies that the ideal $H(\theta, \mathcal{I}, 1).\mathcal{O}_p$ must be equal to (x, y^{r-1}). Thus, $H(\theta, \mathcal{I}, r).\mathcal{O}_p = \mathcal{O}_p$, which contradicts the hypotheses of geometrical quasi-transversality.

So condition $[I]$ of the definition is verified. This implies that hypotheses (H) satisfies all hypotheses of the d-algebraically quasi-transversality and we are done. □

8.6.3 Main result

A natural extension of Theorem 8.5.2 is the following:

Theorem 8.6.4. *If $(M, \theta, \mathcal{I}, E)$ is a d-algebraically quasi-transverse foliated ideal sheaf, then $(M, \theta, \mathcal{I}, E)$ satisfies the $G - FB$ property.*

Remark 8.6.5. *In particular, if $(M, \theta, \mathcal{I}, E)$ is a geometrically quasi-transverse d-foliated ideal sheaf, $\mathcal{I}$ is regular and $V(\mathcal{I})$ has dimension one, by Lemma* 8.6.2*, the foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ satisfies the $G - FB$ property.*

The idea to prove such a result is an argument by induction on the leaf dimension of θ. It relies in two Propositions:

Proposition 8.6.6. *Let $(M, \theta, \mathcal{I}, E)$ be a geometrically quasi-transverse foliated ideal sheaf and suppose that $\theta[\mathcal{I}] = \mathcal{O}_M$. Then at each point p of $V(\mathcal{I})$, there exists an open neighborhood U of p, a regular vector field X over U and a collection of vector fields $\{Y_1, ..., Y_s\}$ over U such that:*

i) The singular distribution $\mathcal{X}$ generated by X is totally transverse to $\mathcal{I}$;

ii) The singular distribution ω generated by $\{Y_1, ..., Y_s\}$ is involutive;

iii) The singular distribution $\{\mathcal{X}, \omega\}$ is equal to $\theta.\mathcal{O}_U$;

iv) $[X, Y_i] \equiv 0$ for all $i \leq s$.

Remark 8.6.7. *This result can be find in the literature when θ is a reduced singular distribution (see e.g. [MY]). Here we prove a slightly more general result.*

Proposition 8.6.8. *Let $(M, \theta, \mathcal{I}, E)$ be a geometrically quasi-transverse d-foliated ideal sheaf such that the singular distribution θ is locally generated by vector fields $\{X, Y_1, ..., Y_s\}$ satisfying conditions [i], [ii], [iii], [iv] of Proposition 8.6.6. Then, if $(M, \omega, \mathcal{I}, E)$ satisfies the $G - FB$ property, so does $(M, \theta, \mathcal{I}, E)$.*

We now show how these Propositions are enough to prove the Theorem:

Proof. (Theorem 8.6.4) The case of 1-algebraically quasi-transverse is done by Theorem 8.5.2, so suppose the Theorem proved for $d-1$-algebraically quasi-transverse foliated ideal sheaves and take $(M,\theta,\mathcal{I},E)$ a d-algebraically quasi-transverse foliated ideal sheaf.

Since the problem is local, we can fix a point p in $V(\mathcal{I})$. Take a relatively compact open subset M_0 of M containing p. By Proposition 5.4.1 there exists a sequence of θ-invariant blowings-up of order one:

$$(M_r,\theta_r,\mathcal{I}_r,E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1,\theta_1,\mathcal{I}_1,E_1) \xrightarrow{\sigma_1} (M_0,\theta_0,\mathcal{I}_0,E_0)$$

such that:

i) $type_{M_r}(\theta_r,\mathcal{I}_r)=1$;

ii) $\nu_{M_r}(\theta_r,\mathcal{I}_r)\leq \nu_{M_0}(\theta,\mathcal{I})$;

By hypotheses $[II]$ of the d-algebraically quasi-transverse definition, we have that $(M_r,\theta_r,\mathcal{I}_r,E_r)$ is d-algebraically quasi-transverse. Furthermore, by hypotheses $[I]$ of the d-algebraically quasi-transverse definition, we have that $\nu_{M_r}(\theta_r,\mathcal{I}_r)=1$. This implies that $\theta_r[\mathcal{I}]=\mathcal{O}_{M_r}$. By Proposition 8.6.6, without loss of generality, there exists a regular vector field X and an involutive singular distribution ω generated by vector fields $\{Y_1,...,Y_s\}$ such that:

- The vector fields $\{X,Y_1,...,Y_s\}$ generates θ_r;
- $[X,Y_i]\equiv 0$ for all $i\leq s$.

By hypotheses $[III]$ of the d-algebraically quasi-transverse definition, $(M_r,\omega,\mathcal{I}_r,E_r)$ is a $d-1$-algebraically quasi-transverse foliated ideal sheaf. By the induction hypotheses, the foliated ideal sheaf $(M_r,\omega,\mathcal{I}_r,E_r)$ satisfies the $G-FB$ property. So, by Proposition 8.6.8, the foliated ideal sheaf $(M_r,\theta_r,\mathcal{I}_r,E_r)$ also satisfies the $G-FB$ property. Finally, by Lemma 8.4.1, the foliated ideal sheaf $(M_0,\theta_0,\mathcal{I}_0,E_0)$ satisfies the $G-FB$ property.

Since the choice of the point p of $V(\mathcal{I})$ was arbitrary, the foliated ideal sheaf $(M, \theta, \mathcal{I}, E)$ satisfies the $G - FB$ property. □

8.6.9 Proof of Proposition 8.6.6

Since the problem is local, without loss of generality we can suppose that there exists a global coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ such that the vector field $X = \frac{\partial}{\partial x}$ is contained in θ and the analytic function x is contained in the ideal $\mathcal{I}$.

There always exists vector fields $\{Y_1, ..., Y_s\}$ such that $\{X, Y_1, ..., Y_s\}$ generates θ. Furthermore, we can suppose that $Y_i(x) \equiv 0$, which implies that:

$$[X, Y_j] = \sum_{j=1}^{s} A_{i,j}(x, y) Y_j$$

Now, consider a vector field of the form $Y = \sum_{i=1}^{s} \mu_i Y_i$, where $\mu_i \in \mathcal{O}_M$. We have that:

$$[X, Y] = \sum_{j=1}^{s} X(\mu_j) Y_j + \sum_{i=1}^{s} \mu_i \sum_{j=1}^{s} A_{i,j}(x, y) Y_j = \\ \sum_{j=1}^{s} Y_j [X(\mu_j) + \sum_{i=1}^{s} \mu_i A_{i,j}(x, y)]$$

Since X is a regular vector field, the equations:

$$X(\mu_j) + \sum_{i=1}^{s} \mu_i A_{i,j}(x, y) = 0$$

for $j = 1, ..., s$ give rise to an analytic system of ODE's. Since the system is analytic, there exists s locally defined analytic solutions $\vec{\mu}_i = (\mu_{i,1}, ..., \mu_{i,s})$ such that $\vec{\mu}_i(0) = e_i = (0, ..., 0, 1, 0, ..., 0)$, where the 1 is on the i position. Without loss of generality, we suppose that these solutions are globally defined.

Let $Z_i = \sum_{j=1}^{s} \mu_{i,j} Y_j$, then it is clear that:

- The vector fields $\{Z_1, ..., Z_s\}$ generates an involutive $d - 1$-singular distribution ω;
- The vector fields $\{X, Z_1, ..., Z_s\}$ generates θ;
- $[X, Z_i] \equiv 0$ for all $i \leq s$.

8.6.10 Proof of Proposition 8.6.8

Since $\mathcal{X}$ is totally transverse to $\mathcal{I}$, by Lemma 8.4.1, the foliated ideal sheaf $(M, \mathcal{X}, \mathcal{I}, E)$ satisfies the $G - FB$ property.

Furthermore, since the problem is local, without loss of generality we can suppose that there exists a global coordinate system $(x, y) = (x, y_1, ..., y_{n-1})$ such that the vector field X is equal to $\frac{\partial}{\partial x}$ and the analytic function x is contained in the ideal $\mathcal{I}$. We can further suppose that $Y_i(x) \equiv 0$ and each Y_i is independent of x (because $[X, Y_i] \equiv 0$). Let g be the sub-Riemannian metric generated by $\{X, Y_1, ..., Y_s\}$, g_X be the sub-Riemannian metric generated by $\{X\}$ and g_Y be the sub-Riemannian metric generated by $\{Y_1, ..., Y_s\}$.

Without loss of generality, the hypotheses implies that there exists $\delta_p > 0$ such that:

$$d_{g_X}(q, V(\mathcal{I}) \setminus \{q\}) > \delta_p$$
$$d_{g_Y}(q, V(\mathcal{I}) \setminus \{q\}) > \delta_p$$

for all point $q \in V(\mathcal{I}) \setminus E$. The next two Claims proves the Proposition:

Claim 1: For all point q in $V(\mathcal{I}) \setminus E$, the g-ball $B^g_\delta(q)$ intersects $V(\mathcal{I})$ only in q.

Proof. Claim 1 will follow if, for all point q in $V(\mathcal{I}) \setminus E$:

$$d_g(q, V(\mathcal{I}) \setminus \{q\}) > \delta$$

So, suppose by absurd that there exists a point q in $V(\mathcal{I}) \setminus E$ such that:

$$d_g(q, V(\mathcal{I}) \setminus \{q\}) < \delta$$

This implies the existence of a absolutely continuous curve $c : [a, b] \longrightarrow M$ such that $c(a) = q$, $c(b) \in V(\mathcal{I}) \setminus \{q\}$ and $length_g(c(t)) < \delta$. We remark that $c(t) = (x(t), y(t))$ and we define the absolutely continuous curve $\gamma(t) = (0, y(t))$. Notice that, since $\mathcal{I} \supset (x)$, we have that $c(a) = (0, y(a))$ and $c(b) = (0, y(b))$. This implies that $\gamma(a) = q$ and $\gamma(b) \in V(\mathcal{I}) \setminus \{q\}$.

Notice that, because of the chosen decomposition of X and ω:

$$\|\dot{c(t)}\|_{g,c(t)} = \|\dot{x(t)}\|_{g_X,c(t)} + \|\dot{y(t)}\|_{g_Y,c(t)} \geq \|\dot{y(t)}\|_{g_Y,c(t)}$$

Furthermore, since $(0, \dot{y(t)}) = \dot{\gamma(t)}$ and ω is independent of the coordinate x:

$$\|\dot{y(t)}\|_{g_Y,c(t)} = \|\dot{\gamma(t)}\|_{g_Y,c(t)} = \|\dot{\gamma(t)}\|_{g_Y,\gamma(t)}$$

Thus:

$$\|\dot{c(t)}\|_{g,c(t)} \geq \|\dot{\gamma(t)}\|_{g_Y,\gamma(t)}$$

Which implies that:

$$length_{g_Y}(\gamma(t)) \leq length_g(c(t)) < \delta$$

which contradicts the absurd hypotheses and Claim 1 is proved. □

Claim 2: For all point q in $V(\mathcal{I})\backslash E$, the g-ball $B^g_\delta(q)$ is homeomorphic to a k_q-euclidean ball, where k_q is the dimension of the leaf of θ_g passing through q.

Proof. Notice that the g_Y-ball $B^{g_Y}_\delta(q)$ is homeomorphic to a $(k_q - 1)$-euclidean ball for any q in $V(\mathcal{I})$ and $\delta < \delta_p$. From the explicit expression $g_X = (dx)^2$ and the fact that the vector fields $\{Y_1, ..., Y_s\}$ are independent of the x-coordinate, we conclude that the g-ball $B^g_\delta(q)$ is homeomorphic to a k_q-euclidean ball. □

Chapter 9

Les résultats de la thèse en Français

Dans ce chapitre notre objectif est d'énoncer tous les résultats importants de la thèse. Pour ce faire, on va aussi présenter les définitions et notations nécessaires.

9.1 Relations entre un feuilletage et une variété - Chapitre 2

On commence avec une liste d'objets (voir la section 2.1):

- Une *variété feuilletée* est un triplet (M, θ, E), où :
  - M est une variété analytique régulière de dimension n sur $\mathbb{K}$ (où $\mathbb{K}$ est $\mathbb{R}$ ou $\mathbb{C}$);
  - E est une collection ordonnée $E = (E^{(1)}, ..., E^{(l)})$, où $E^{(i)}$ est un diviseur régulier de M tel que $\sum_i E^{(i)}$ est un diviseur réduit à croisements normaux simples;
  - θ est une distribution singulière involutive sur M, tangente à E.
- Un *faisceau d'idéaux feuilleté* est un quadruplet $(M, \theta, \mathcal{I}, E)$, où :
  - (M, θ, E) est une variété feuilletée;
  - $\mathcal{I}$ est un faisceau d'idéaux cohérent et ne s'annulent nulle part sur M.
- Une *variété feuilletée locale* est un quadruplet (M, M_0, θ, E), où :

- (M, θ, E) est une variété feuilletée;
- M_0 est un ouvert relativement compacte de M.

- Un *faisceau d'idéaux feuilleté local* est un quintuplet $(M, M_0, \theta, \mathcal{I}, E)$, où :
  - $(M, \theta, \mathcal{I}, E)$ est un faisceau d'idéaux feuilleté;
  - M_0 est un ouvert relativement compact de M.

Maintenant, on définit quelques outils pour étudier "l'interaction" entre une variété et un feuilletage:

- On considère un anneau R tel que $\mathbb{Z} \subset R \subset \mathbb{K}$. On dit qu'une distribution singulière est *R-monomiale* dans un point p dans M s'il existe un système des coordonnées $x = (x_1, ..., x_n)$ de $\mathcal{O}_p$ et un système générateur cohérent $\{X_1, ..., X_d\}$ de θ_p tels que:
  - ou bien $X_i = \frac{\partial}{\partial x_i}$;
  - ou bien $X_i = \sum_{j=1}^{n} \alpha_{i,j} x_j \frac{\partial}{\partial x_j}$, où $\alpha_{i,j} \in R$.

  La distribution singulière est *R-monomiale* si elle est R-monomiale dans tous les points p dans M.

- L'opération *k-Fitting généralisée* est une application $\Gamma_{\theta,k}$ qui associe à chaque faisceau d'idéaux cohérent $\mathcal{I}$, un faisceau d'idéaux cohérent localement défini par:

$$\Gamma_{\theta,k}(\mathcal{I}).\mathcal{O}_p = \langle \{det[X_i(f_j)]_{i,j \leq k};\ X_i \in \theta_p, f_j \in \mathcal{I}.\mathcal{O}_p\} \rangle$$

  où $\langle S \rangle$ est l'idéal engendre par le sous-espace $S \subset \mathcal{O}_p$.

- La *chaîne de tangence* d'un faisceau d'idéaux feuilleté $(M, \theta, \mathcal{I}, E)$ est la suite:

$$\mathcal{T}g(\theta, \mathcal{I}) = \{H(\theta, \mathcal{I}, i); i \in \mathbb{N}\}$$

  où $H(\theta, \mathcal{I}, i)$ est le faisceau d'idéaux:

$$\begin{cases} H(\theta, \mathcal{I}, 0) := \mathcal{I} \\ H(\theta, \mathcal{I}, i+1) := H(\theta, \mathcal{I}, i) + \theta[H(\theta, \mathcal{I}, i)] \end{cases}$$

La *tg-ordre* de $(M, \theta, \mathcal{I}, E)$ dans un point p de M est le nombre minimal ν_p (qu'on note par $\nu_p(\theta, \mathcal{I})$) tel que:

$$H(\theta, \mathcal{I}, \nu_p).\mathcal{O}_p = H(\theta, \mathcal{I}, i).\mathcal{O}_p$$

pour tout i supérieur à ν_p. Nous distinguons deux cas:

- si $H(\theta, \mathcal{I}\nu_p).\mathcal{O}_p = \mathcal{O}_p$, la chaîne de tangence est de *type* 1 dans p;
- si $H(\theta, \mathcal{I}\nu_p).\mathcal{O}_p \neq \mathcal{O}_p$, la chaîne de tangence est de *type* 2 dans p.

- Un morphisme $\phi : M \longrightarrow N$ est lisse en rapport avec deux faisceaux d'idéaux feuilletés $(M, \theta, \mathcal{I}, E_M)$ et $(N, \omega, \mathcal{J}, E_N)$ si:

  - le morphisme $\phi : M \longrightarrow N$ est lisse;
  - l'ensemble $\phi^{-1}(E_N)$ est égal à E_M;
  - le faisceau d'idéaux $\mathcal{J}.\mathcal{O}_M$ est égal à $\mathcal{I}$.

  On dit qu'un morhisme lisse $\phi : (M, \theta, \mathcal{I}, E_M) \longrightarrow (N, \omega, \mathcal{J}, E_N)$ est préserve-chaîne si:

$$\mathcal{T}g(\omega, \mathcal{J}).\mathcal{O}_M = \mathcal{T}g(\theta, \mathcal{I})$$

  i.e $H(\omega, \mathcal{J}, i), \mathcal{O}_M = H(\theta, \mathcal{I}, i)$ pour tout $i \in \mathbb{N}$. Si la dimension de θ et ω est égale à k, on dit que le morphisme $\phi : M \longrightarrow N$ est *k-préserve-chaîne*.

9.2 Éclatements - Chapitre 3

On présente les outils et les notations plus importants concernant un éclatement:

- Soit (M, θ, E) une variété d-feuilletée et $\sigma : (M', E') \longrightarrow (M, E)$ un éclatement admissible avec diviseur exceptionnel F. On considère le faisceau de $\mathcal{O}_{M'}$-module $\mathcal{B}lDer_{M'} := \mathcal{O}(-F) \otimes_{\mathcal{O}_{M'}} Der_{M'}$. Il existe une application de $Der_{M'}$ sur $\mathcal{B}lDer_{M'}$:

$$\zeta : \ Der_{M'} \ \longrightarrow \ \mathcal{B}lDer_{M'}$$

  telle que, donné un ouvert U de M', l'application associe à un champ de vecteur $X \in Der_{M'}(U)$ l'élément $\zeta(X) = 1 \otimes X \in \mathcal{B}lDer_{M'}(U)$.

Soit ω un sous-faisceau de $Der_{M'}$. On note $\zeta(\omega)$ le sous-faisceau de $\mathcal{B}lDer_{M'}$ engendré par l'image de ω. Réciproquement, si ω est un sous-faisceau de $\mathcal{B}lDer_{M'}$, on note $\zeta^{-1}(\omega)$ le sous-faisceau de $Der_{M'}$ defini dans chaque ouvert U de M' pour les éléments:

$$\zeta^{-1}(\omega)_U = \{X \in Der_U;\ \zeta(X) \in \omega_U\}$$

Comme l'éclatement $\sigma : M' \longrightarrow M$ est un morphisme, il engendre une application:

$$\sigma^* : Der_M \longrightarrow \mathcal{B}lDer_{M'}$$

telle que, étant donné un ouvert U de M, l'application associe à un champ de vecteurs X de Der_U l'élément $\sigma^*(X) = (\frac{1}{f} \otimes fX^*)$, où l'idéal principal (f) engendre $\mathcal{O}(F).\mathcal{O}_{\sigma^{-1}(U)}$ et X^* est le tire-en-arrière de la dérivation (i.e. $X^*(\sigma^* f) = \sigma^* X(f)$).

La *transformée analytique stricte* de θ est la distribution singulière $\theta^a := \zeta^{-1}(\theta^*)$.

On considère la distribution singulière involutive $Der_{M'}(-logF)$ engendrée par toutes les dérivations tangentes à F. La *transformée analytique stricte adaptée* est la distribution singulière $\theta^{a,ad} = \theta^a \cap Der_{M'}(-logF)$.

- Un *éclatement admissible d'ordre un* est un éclatement admissible:

$$\sigma : (M', \theta', \mathcal{I}', E') \longrightarrow (M, \theta, \mathcal{I}, E)$$

tel que le centre $\mathcal{C}$ est contenu dans la variété $V(\mathcal{I})$ et:

  - la distribution singulière θ' est la transformée analytique stricte adaptée de θ;
  - le faisceau d'idéaux $\mathcal{I}'$ est la transformée contrôlée de $\mathcal{I}$ i.e. $\mathcal{I}' = \mathcal{O}(-F)\mathcal{I}^*$ où F est le diviseur exceptionnel.

- Une *résolution* d'un faisceau d'idéaux feuilleté $(M, \theta, \mathcal{I}, E)$ est une suite $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ d'éclatements admissibles d'ordre un:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

telle que $\mathcal{I}_r = \mathcal{O}_{M_r}$.

Un *foncteur résolution* est un foncteur $\mathcal{R}$ avec:

- en entrée: La catégorie des faisceaux d'idéaux feuilleté $(M, \theta, \mathcal{I}, E_M)$ où les morphismees sont morphismees lisses;
- en sortie: La catégorie des suites d'éclatements admissible:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

  où les morphismees sont donnés par le produit cartésien.

tel que $\mathcal{R}(M, \theta, \mathcal{I}, E_M)$ est une résolution de $(M, \theta, \mathcal{I}, E)$.

- Un *éclatement local* est un morphisme $\tau : M^{'} \longrightarrow M$ qui est égal à la composition d'un éclatement $\sigma : M^{'} \longrightarrow \widetilde{M}$ et d'un isomorphisme local injective $\pi : \widetilde{M} \longrightarrow M$, i.e $\tau = \pi \circ \sigma$.
- Une *uniformisation locale* d'un faisceau d'idéaux feuilleté $(M, \theta, \mathcal{I}, E)$ sur un point p de M est une collection finie $\{\tau_\alpha : M_\alpha \longrightarrow M, \theta_\alpha\}$ où:

  - θ_α est une distribution singulière involutive sur M_α;
  - $\tau_\alpha : M_\alpha \longrightarrow M$ est un morphisme propre.

  telle que :

  - l'union des images $\bigcup \tau_\alpha(M_\alpha)$ est une voisinage ouverte de p;
  - pour chaque morphisme $\tau_\alpha : M_\alpha \longrightarrow M$, il existe une suite d'éclatements locaux admissible d'ordre un :

$$(M_r, \theta_{r,\alpha}, \mathcal{I}_r, E_r) \xrightarrow{\tau_{r,\alpha}} \cdots \xrightarrow{\tau_{2,\alpha}} (M_1, \theta_{1,\alpha}, \mathcal{I}_1, E_1) \xrightarrow{\tau_{1,\alpha}} (M, \theta, \mathcal{I}, E)$$

    telle que $\mathcal{I}_r = \mathcal{O}_{M_r}$, $\theta_\alpha = \theta_{r,\alpha}$ et le morphisme τ_α est égal à la composition d'éclatements locaux: $\tau_\alpha = \tau_{1,\alpha} \circ ... \circ \tau_{r,\alpha}$.

9.3 Éclatement θ-admissible - Chapitre 4

Soit (M, θ, E) une d-variété feuilleté et $\mathcal{C}$ une sous-variété analytique de M. Considérons le faisceau d'idéal réduit $\mathcal{I}_{\mathcal{C}}$ qui engendre $\mathcal{C}$, i.e. $V(\mathcal{I}_{\mathcal{C}}) = \mathcal{C}$. On dit que $\mathcal{C}$ est un *centre θ-admissible* si:

- $\mathcal{C}$ est une sous-variété régulière fermé;
- $\mathcal{C}$ est à croisement normal avec E;
- Il existe $0 \leq d_0 \leq d$ tel que l'idéal k-Fitting généralisée $\Gamma_{\theta,k}(\mathcal{I}_{\mathcal{C}})$ est égal à $\mathcal{O}_M$ pour tout $k \leq d_0$ et $\Gamma_{\theta,k}(\mathcal{I}_{\mathcal{C}})$ est contenu dans $\mathcal{I}_{\mathcal{C}}$ pour tout $k > d_0$.

Un éclatement admissible $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ est *θ-admissible* si le centre $\mathcal{C}$ est θ-admissible. Nous soulignons deux cas particuliers:

- Un éclatement admissible $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ est *θ-invariant* si le centre $\mathcal{C}$ est θ-invariant;
- Un éclatement admissible $\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$ est *θ-totalement transversal* si le centre $\mathcal{C}$ est θ-totalement transversal.

Une suite d'éclatements $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ θ-admissible est une suite:

$$(M_r, \theta_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, E_1) \xrightarrow{\sigma_1} (M_0, \theta_0, E_0)$$

telle que $\sigma_i : (M_{i+1}, \theta_{i+1}, E_{i+1}) \longrightarrow (M_i, \theta_i, E_i)$ est θ_i-admissible. Le théorème suivant explique l'intérêt porté à ces éclatements:

Théorème 4.1.1 *Soit (M, θ, E) une d-variété feuilletée R-monomiale et:*

$$\sigma : (M', \theta', E') \longrightarrow (M, \theta, E)$$

un éclatement θ-admissible. Alors θ' est R-monomiale.

Et, comme corollaire de sa preuve, nous obtenons le résultat suivant:

Corollaire 4.1.2 *Soit (M, θ, E) une d-variété feuilletée telle que la distribution singulière θ est régulière et:*

$$\sigma : (M^{'}, \theta^{'}, E^{'}) \longrightarrow (M, \theta, E)$$

un éclatement θ-invariant. Alors $\theta^{'}$ est régulière.

9.4 Trois résolutions subordonnées à un feuilletage - Chapitre 5 et 6

Une résolution de $(M, \theta, \mathcal{I}, E)$:

$$(M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (M, \theta, \mathcal{I}, E)$$

est dite *θ-admissible* (resp. *θ-invariante*) si $\sigma_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \longrightarrow (M_{i-1}, \theta_{i-1}, \mathcal{I}_{i-1}, E_{i-1})$ est θ_{i-1}-admissible (resp. θ_{i-1}-invariante).

Théorème 5.1.1 *Soit $(M, M_0, \theta, \mathcal{I}, E)$ un faisceau d'idéaux feuilleté local. On suppose que $\mathcal{I}_0$ est θ-invariant. Alors il existe une résolution de $(M, M_0, \theta, \mathcal{I}, E)$:*

$$\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

telle que:

i) $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ *est une suite d'éclatements θ-invariants;*

ii) *La composition $\sigma = \sigma_1 \circ ... \circ \sigma_r$ est un isomorphisme sur $M_0 \setminus V(\mathcal{I}_0)$;*

iii) *Si θ_0 est R-monomial, alors θ_r est aussi R-monomial;*

iv) *Si θ_0 est régulière, alors θ_r est aussi régulière;*

v) *$\mathcal{R}_{inv}$ est un foncteur résolution qui commute avec les morphismes lisses préserve-chaîne.*

Théorème 5.3.1 *Soit $(M, M_0, \theta, \mathcal{I}, E)$ un faisceau d'idéaux feuilleté local, où θ est de dimension 1. Alors, il existe une résolution de $(M, M_0, \theta, \mathcal{I}, E)$:*

$$\mathcal{R}_{inv}(M, M_0, \theta, \mathcal{I}, E) : (M_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_0, \theta_0, \mathcal{I}_0, E_0)$$

telle que:

i) $\vec{\sigma} = (\sigma_r, ..., \sigma_1)$ *est une suite d'éclatements θ-admissible;*

ii) *la composition $\sigma = \sigma_1 \circ ... \circ \sigma_r$ est un isomorphisme sur $M_0 \setminus V(\mathcal{I}_0)$;*

iii) *si θ_0 est R-monomial, alors θ_r est aussi R-monomial*;

iv) $\mathcal{R}_1$ *est un foncteur résolution qui commute avec morphismes lisses* 1*-préserve-chaîne.*

v) si ω est une distribution d-singulière involutive telle que $\mathcal{I}$ est ω-invariant et $\{\omega, \theta\}$ engendre une distribution $(d + 1)$-singulière involutive, la suite des éclatements $\vec{\sigma} = (\sigma_1, ..., \sigma_r)$ est ω-invariant;

Une uniformisation locale de $(M, \theta, \mathcal{I}, E)$ dans un point p de M:

$$\tau_\alpha : (M_\alpha, \theta_\alpha) \longrightarrow (M, \theta)$$

est dite *θ-admissible* si les morphismes τ_α sont des composés d'éclatements locaux θ-admissibles.

Théorème 6.1.1 *Soit $(M, \theta, \mathcal{I}, E)$ un faisceau d'idéaux feuilleté où θ est de dimension* 1 *et p un point dans M. Alors il existe une uniformisation locale θ-admissible de $(M, \theta, \mathcal{I}, E)$ dans p. En particulier, si θ est R-monomial, alors θ_α est R-monomial.*

9.5 Application 1 - Résolution dans les familles - Chapitre 7

Une *famille lisse des faisceaux d'idéaux* est un quadruplet $(B, \Lambda, \pi, \mathcal{I})$, où :

- L'espace ambiant B et l'espace des paramètres Λ sont deux variétés analytiques lisses;
- Le morphismee $\pi : B \longrightarrow \Lambda$ est lisse;
- Le faisceau d'idéaux $\mathcal{I}$ est cohérent et partout non nul sur B.

Une *famille lisse de feuilletages par courbes* est un quadruplet $(B, \Lambda, \pi, \mathcal{X})$, où :

- l'espace ambiant B et l'espace des paramètres Λ sont deux variétés analytiques lisses;
- le morphismee $\pi : B \longrightarrow \Lambda$ est lisse;
- la distribution singulière $\mathcal{X}$ est:
  - partout non nulle sur B;
  - $d\pi(\mathcal{X}) \equiv 0$;
  - localement engendrée par un champ de vecteurs.

On considère un point λ dans Λ: l'ensemble $\pi^{-1}(\lambda)$ est une sous-variété régulière de B qu'on appelle *fibre*. Un point λ_0 dans Λ est appelé un *valeur exceptionnelle* d'une famille lisse de faisceaux d'idéaux (respectivement, d'une famille lisse de feuilletages par courbes) si la fibre $\pi^{-1}(\lambda_0)$ est contenu dans la variété $V(\mathcal{I})$ (respectivement, dans la variété $S(\mathcal{X})$).

Théorème 7.1.1 *Soit* $(B, \Lambda, \pi, \mathcal{I})$ *une famille lisse des faisceaux d'idéaux telle que toutes les fibres sont connexes. Alors, il existe une famille lisse des faisceaux d'idéaux* $(B', \Lambda', \pi', \mathcal{I}')$ *et deux applications analytiques propres* $\sigma : B' \longrightarrow B$ *et* $\tau : \Lambda' \longrightarrow \Lambda$ *telles que:*

i) *la famille lisse de faisceaux d'idéaux* $(B', \Lambda', \pi', \mathcal{I}')$ *n'a pas de valeurs exceptionnelles;*

ii) *le diagramme suivant:*

$$\begin{array}{ccc} B' & \xrightarrow{\pi'} & \Lambda' \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

commute;

iii) *pour tout sous-ensemble ouvert relativement compact* B_0 *de* B*, il existe une suite d'éclatements admissibles d'ordre un par* $(B, B_0, \theta, \mathcal{I}, \emptyset)$*:*

$$(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B_0, \theta_0, \mathcal{I}_0, E_0)$$

telle que $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ ... \circ \sigma_r$ *et* $\mathcal{I}^{'}.\mathcal{O}_{B_r} = \mathcal{I}_r$*;*

iv) *pour tout sous-ensemble ouvert relativement compact* Λ_0 *de* Λ*, il existe une suite d'éclatements admissibles:*

$$(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)$$

telle que $\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ ... \circ \tau_r$.

Théorème 7.3.1 *Soit* $(B, \Lambda, \pi, \mathcal{X})$ *une famille lisse des feuilletages par courbes telle que toutes les fibres sont connexes. Alors il existe une famille lisse des feuilletages par courbes* $(B^{'}, \Lambda^{'}, \pi^{'}, \mathcal{X}^{'})$ *et deux applications analytiques propres* $\sigma : B^{'} \longrightarrow B$ *et* $\tau : \Lambda^{'} \longrightarrow \Lambda$ *telles que:*

i) *la famille lisse des feuilletages par courbes* $(B^{'}, \Lambda^{'}, \pi^{'}, \mathcal{X}^{'})$ *n'a pas de valeurs exceptionnelles;*

ii) *le diagramme suivant:*

$$\begin{array}{ccc} B^{'} & \xrightarrow{\pi^{'}} & \Lambda^{'} \\ \sigma \downarrow & & \downarrow \tau \\ B & \xrightarrow{\pi} & \Lambda \end{array}$$

commute;

iii) *pour tout sous-ensemble ouvert relativement compact* B_0 *de* B*, il existe une suite d'éclatements admissible d'ordre un par* $(B, B_0, \theta, \mathcal{X}, \emptyset)$*:*

$$(B_r, \theta_r, \mathcal{I}_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (B_1, \theta_1, \mathcal{I}_1, E_1) \xrightarrow{\sigma_1} (B_0, \theta_0, \mathcal{I}_0, E_0)$$

où $\mathcal{X}_i = \mathcal{O}(-F_i).\sigma_i^*\mathcal{X}_{i-1}$*, telle que* $\sigma|_{\sigma^{-1}B_0} = \sigma_1 \circ \ldots \circ \sigma_r$ *et* $\mathcal{X}'.\mathcal{O}_{B_r} = \mathcal{X}_r$*;*

iv) *pour tout sous-ensemble ouvert relativement compact* Λ_0 *de* Λ*, il existe une suite d'éclatements admissible:*

$$(\Lambda_r, E_r) \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_2} (\Lambda_1, E_1) \xrightarrow{\tau_1} (\Lambda_0, E_0)$$

telle que $\tau|_{\tau^{-1}\Lambda_0} = \tau_1 \circ \ldots \circ \tau_r$.

9.6 Application 2 - Le temps de retour et un problème proposé par Mattei - Chapitre 8

On considère ici un problème proposé par Mattei à propos d'une action d'un groupe de Lie, qu'on transformera en une question sur l'algèbre de Lie. Pour simplifier, dans cette section, on considère une algèbre de Lie unidimensionnelle. Le cas général est étudié dans le chapitre 8. Néanmoins, nous tenons à souligner que le problème proposé par Mattei est toujours ouvert.

Soit M une variété analytique, N une sous-variété régulière de M et X un champ de vecteurs analytique sur M. On dit qu'un triplet (M, N, X) satisfait la *G-FB propriété* si: pour chaque point p dans N, il existe une paire (U_p, δ_p), où U_p est un voisinage ouvert de p et $\delta_p > 0$ est un nombre réel positif, de sorte que l'orbite $\gamma_q(t)$ du champ de vecteurs X passant par un point q dans $(N \cap U_p) \setminus Sing(X)$ n'intersecte pas N pour $0 < \|t\| < \delta_p$.

Le problème est le suivant: Étant donné un triplet (M, N, X), établir des critères locaux en fonction de la sous-variété N et le champ de vecteurs X qui garantissent que la propriété $G - FB$ est satisfaite.

On peut conjecturer que la difficulté du problème réside dans les points de tangence entre la variété N et le champ de vecteurs X. On dit qu'un triplet (M, N, X) est *géométriquement quasi-traverse* si, à chaque point p dans N, nous avons l'égalité:

$$dim_{\mathbb{K}} T_p N + dim_{\mathbb{K}} X(p) = dim_{\mathbb{K}}(T_p N + X(p))$$

où $X(p)$ est le sous-espace de $T_p M$ engendré par X. On peux poser la question suivante:

Question: Est-ce que la quasi-transversalité géométrique implique la propriété G-FB?

On répond à cette question avec deux résultats:

Theorem 9.6.1. *Si* (M, N, X) *est géométriquement quasi-transversal et l'une des conditions suivantes est remplie:*

- *la dimension de* N *est un;*
- *la codimension de* N *est un;*
- (M, N, X) *est algébriquement quasi-transversal (voir la définition dans la section* 8.1*);*

alors la propriété $G - FB$ *est satisfaite.*

Remark 9.6.2. *En particulier, si la dimension de* M *est inférieure ou égale à* 3*, alors quasi-transversalité géométrique implique toujours la propriété G-FB.*

Theorem 9.6.3. *Pour* $dimM \geq 4$*, il existe un triplet* (M, N, X) *géométriquement quasi-transversal qui ne satisfait pas la propriété G-FB.*

Ces théorèmes sont une conséquence immédiate des résultats figurant dans la section 8.5.

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