Etude théorique d'indicateurs d'analyse technique

Dalia Ibrahim

UNIVERSITÉ NICE SOPHIA ANTIPOLIS – UFR Sciences

École Doctorale Sciences Fondamentales et Appliquées

THÈSE

pour obtenir le titre de
Docteur En Sciences
Spécialité MATHÉMATIQUES

présentée et soutenue par
Dalia IBRAHIM

Étude théorique d'indicateurs d'analyse technique

Soutenue le 8 Février 2013
devant le jury composé des professeurs

Huyên Pham	*rapporteur*
Mathieu Rosenbaum	*rapporteur*
François Delarue	*président du jury*
Frédéric Abergel	*examinateur*
Rajna Gibson	*examinateur*
Denis Talay	*directeur de thèse*
Etienne Tanré	*co-directeur de thèse*

Laboratoire Jean-Alexandre Dieudonné, Université de Nice, Parc Valrose, 06108 NICE

Équipe TOSCA, INRIA Sophia Antipolis Méditerranée, 2004 route des lucioles BP 93, 06902 SOPHIA ANTIPOLIS

Remerciements

Cette recherche a été financée par FINRISK (Financial Risk) forum lancé par la fondation nationale suisse et géré par l'université de Zurich. Je remercie vivement pour ce financement, sans ce dernier, cette thèse n'aurait jamais vu le jour.

Je souhaiterais ensuite pouvoir remercier mon directeur de thèse, Denis Talay, pour m'avoir fait découvrir le monde de la recherche en mathématiques financières et contrôle stochastique. Je tiens également à lui exprimer toute ma gratitude et mon profond respect pour sa rigueur scientifique, son dynamisme et ses précieux conseils dont j'ai tiré un grand profit.

Je tiens à remercier mon co-encadreur Etienne Tanré. Je lui suis particulièrement reconnaissant pour sa grande disponibilité lors de ces trois années, ses nombreux conseils m'ont beaucoup apporté.

Mes remerciements les plus respectueux vont également à Huyên Pham et Mathieu Rosenbaum qui m'ont fait l'honneur de bien vouloir rapporter la thèse en y prêtant toute l'attention ncessaire.

Je présente mes sincères remerciements à François Delarue qui m'a fait l'honneur d'accepter de présider mon jury de thèse.

Je remercie chaleureusement Rajna Gibson, et Frédéric Abergel qui m'ont fait l'honneur de participer 'a mon jury de thèse.

Je pense également à mes amis thésards, Julien, Nicolas, Sébastian et Laurent avec qui j'ai passé des moments agréables et très sympathiques pendant les déjeuners et les pauses café.

Merci à notre assistante Christine Faber, pour sa gentillesse, sa bonne humeur et les nombreux services qu'elle a rendu toujours avec le sourire.

En ce moment où je passe par une étape majeure dans mon parcours universitaire, mes pensées vont vers mes chers et adorables parents : Hassan et Fatina qui n'ont ménagé aucun effort pour me soutenir et me réconforter tout le long de mes études.

Enfin, un affectueux remerciement à mon chèri Ibrahim pour son soutien et sa présence à mes cotés, surtout dans les moments difficiles.

Contents

Chapter 1

Introduction

When you're making the attempt to analyze securities and make investment decisions, the strategies you will use most likely find themselves in three very broad categories: fundamental analysis, technical analysis and mathematical analysis. Fundamental analysis involves analyzing the characteristics of a company in order to estimate its value. The technical analysis (sometimes called the chartist analysis) looks at the past price movement and uses this data to predict its future price movements. The mathematical analysis is based on mathematical models.

In our study, we focus on the chartist analysis and the mathematical analysis. The main hypothesis of technical analysis is that all the information is contained inside the records of prices. They do not worry about the value of the stock, but strictly past prices or volumes. The analysis of the charts is sufficient to predict the future price movements, this hypothesis contracts most of mathematical models, which are essentially Markov. So, technical analysis seems to have limited theoretical justifications and their efficiency is questionable. For that we aim to analyse mathematically a chartist indicator widely used by the practitioners in the trading market, then we study the performance of this indicator in a universe that is governed by a stochastic differential equations, for a practitioner seeking to maximize an objective function (for instance, the expected utility of the wealth at a certain maturity). We compare the performance of trader who uses a chartist analysis technique with a trader who uses a portfolio allocation strategy which is optimal when the mathematical model is perfectly specified and calibrated.

To compare the performance of chartist strategies and mathematical strategies, we will be able to provide a conceptual framework where their performance can be compared. If one considers a non-stationary financial economy. The problem that it is impossible to specify and calibrate models which can capture all the sources of instability during a long time interval. For that one can only pretend to divide a long investment period into sub-periods such that, in each one of these sub-periods, the market can reasonably be supposed to follow some particular model, that is a stochastic differential system.

Therefore, one can only use small amounts of data during each sub-period to calibrate the model, and the calibration errors can be substantial. However, any investment strategy's performance depends on the underlying model for the market evolution, and also on the values of the parameters involved in the model. Thus, in a non-stationary economy, one can use strategies which have been optimally designed under the assumption that the market is well described by a prescribed model, but these strategies are extremely misleading in practice because the prescribed model does not fit the actual evolution of the market. In such a situation, some technicians propose that, in a non-stationary economy, technical analysis may be a better indicator to capture some basic trends of the market without assuming model dependency.

In order to understand this problem, recently some mathematicians are interested by this subject, like Talay and al [5]. In this article, they have compared the performance of the trading strategies based on the simple moving average rule with the trading strategies designed from mathematical models, like a modified Black-Scholes such that the instantaneous expected rate of return of the stock changes once at an unknown random time. They have made explicit the trader's expected logarithmic utility of wealth, but unfortunately, the explicit formulae are not propitious to mathematical comparisons. Therefore by Monte Carlo numerical experiments, they have observed from these experiments that technical analysis techniques may overperform mathematical techniques in the case of parameter misspecifications.

In my thesis, we examine and model the performance for another chartist technique designed to detect changes in the "volatility term". We study the performance of the Bollinger Bands technique in a modified Black-Scholes model such that the volatility changes at an unknown and unobservable random time τ (which is independent of the Brownian motion governing the price). My thesis in divided into two parts: The first part is devoted to a theoretical study of the Bollinger Bands indicator with numerical results. In the second part, we deal with a optimal portfolio allocation problem with random time change to provide analytical formulae for portfolios managed by means of mathematical model.

In chapter 2, we introduce the Bollinger Bands technique and we present the mathematical framework to study this indicator. We deal with a continuous financial market with two asset. A risk free asset with dynamics:

$$\begin{cases} dS_t^0 = S_t^0 r dt, \\ S_0^0 = 1. \end{cases}$$

and a risky asset with dynamics:

$$\begin{cases} dS_t = \mu S_t dt + S_t \left(\sigma_1 + (\sigma_0 - \sigma_1)\, \mathbb{1}_{t \leqslant \tau}\right) dB_t, \\ S_0 = S^0. \end{cases} \tag{1.0.1}$$

where $(B_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion. The random time of the volatility change τ is independent of B and at this time τ , which is neither known, nor directly observable, the instantaneous volatility rate changes from σ_0 to σ_1.

We are interested in this chapter to the Bollinger Bandwidth indicator (BWI) which is derived from the Bollinger indicator and it is given by:

$$BWI_t = 4\sqrt{\delta\frac{\int_{t-\delta}^{t} S_u^2 du}{\left(\int_{t-\delta}^{t} S_u du\right)^2} - 1}.$$

The parameter δ denotes the size of the time window used to compute the moving average for the Bollinger Bands.

We prove some identities in law which will be useful in next chapters in order to show that the Bandwidth indicator can be useful to detect the change time at witch the volatility changes his value.

In chapter 3, we are interested by an asymptotic analysis of the Bollinger Bands in the case of small volatilities. We show that the density behavior of the Bandwidth indicator (BWI) depends on the value of the volatility, this implies that (BWI) can detect the volatility change.

In chapter 4, we show that the Bandwidth indicator can be also used to detect a change in the volatility term in the case of large value of volatilities. Of course this situation of large value is not realistic in the finance context but can be arise in the physics context, like statistical mechanics of disordered systems.

We are interested in this chapter by the Normalized indicator (NI_t) defined in function of the Bandwidth indicator as follows:

$$NI_t := \frac{1}{\delta}\left(\frac{(BWI_t)^2}{16} + 1\right) \tag{1.0.2}$$

We aim to show that the normalized indicator can detect a change in the volatility term, that is we show that the behavior of its probability density depends on the value of the volatility. For simplifying the mathematical study, we can assume that the price process (S_t) evolves according to:

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right).$$

We first show that the law of NI_t does not depend of time t and it equals in law to $\sigma^2 k^{(\tilde{\mu})}_{\sigma^2\delta}$ where $\tilde{\mu} = \frac{\mu}{\sigma^2} - \frac{1}{2}$ and $k_t^{(\tilde{\mu})}$ is given in function of exponential functionals of

Brownian motion with drift as follows:

$$\frac{\int_0^t \exp(2\tilde{\mu}s + 2B_s)ds}{\left(\int_0^t \exp(\tilde{\mu}s + B_s)ds\right)^2}.$$

The important result in this chapter is given in proposition 4.1.1 which proves that from a certain value of t, the variable $k_t^{(\tilde{\mu})}$ converges to that of $k_\infty^{(-|\tilde{\mu}|)}$, which is well defined. This result have been showed by a technical method which consists in finding an upper and lower bounds for the cumulative distribution function of the variable $\log(k_t^{(\tilde{\mu})})$ and to prove that for large t, these bounds converges to the cumulative distribution function of the variable $\log(k_\infty^{(-|\tilde{\mu}|)})$. Thanks to the equality in law $NI_t \overset{\mathcal{L}}{=} \sigma^2 k_{\sigma^2\delta}^{(\tilde{\mu})}$, we show that the distribution function of the normalized variable NI_δ converges to that of the variable $\sigma^2 k_\infty^{(-|\tilde{\mu}|)}$ for large value of $\sigma^2\delta$.

So we are interested by studying the law of the variable $k_\infty^{(-|\tilde{\mu}|)}$ in order to deduce that the behavior of the density of the variable NI_δ depends on the value of σ.

From Yor [34], the law of the variable $k_\infty^{(-|\tilde{\mu}|)}$ has characterized by its Laplace transform. But unfortunately, it is not easy to invert this formula in order to obtain the density of $k_\infty^{(-|\tilde{\mu}|)}$. Thanks to Tauberian theory, which enables us to find a connection between the behavior properties of the Laplace transform at infinity (resp.at zero) and the behavior properties of the corresponding cumulative distribution function near zero (resp.near infinity), we deduce the behavior of the cumulative distribution function $\mathbb{P}[k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$ for small x and the behavior of $\mathbb{P}[k_\infty^{(-|\tilde{\mu}|)} \geqslant x]$ for large x. Therefore the behavior of $\mathbb{P}[NI_\delta \leqslant x]$ for small x and the behavior of $\mathbb{P}[NI_\delta \geqslant x]$ for large x are straightforward consequence and we show that these behaviors are dependent of the value of σ.

In chapter 5, we aim to compare two indicators designed to detect a volatility change: the Bollinger bands and the quadratic variation indicators. It is well known in the literature that the quadratic variation indicator is a optimal estimator of the volatility if we select the time increment as small as possible. So our aim in this chapter is to compare the performance of these two indicators in the case of large value for the time increment. We consider a trader who does not perfectly detect τ but, at least, uses one of these two indicators to detect τ and then to decide when he reinvests his portfolio. We assume that τ has a exponential law with parameter λ. We show, in the case of large value of time increment, that for large value of λ, Bollinger bands can overperform and detect the change time τ faster than quadratic variation indicator.

In chapter 6, we examine the performance of a trader whose strategy is based on mathematical models. We study the optimal portfolio allocation strategy in the case where the model has a change in the volatility term at a random variable τ.

We want to exhibit a mathematical optimal strategy from the stochastic control theory, but it appears that we have some difficulties which come from the fact that the trader's strategy needs to be adapted to the filtration generated by the price process, which is different from the Brownian filtration due to the time change τ and it is right-discontinuity. So we will resolve the optimization problem by using a specific feature of the change time τ.

We first use the density approach which have been introduced by Pham and Jiao in [17]. We show with this approach how we can separate the initial optimisation problem into a problem after the change time and a problem before the change time, by relying on the density hypothesis on the change time τ. We show that the pair value function is a unique viscosity solution of a Hamilton-Jacobi Bellman system.

We second show by verification theorem that under smoothness assumptions on the value function, the pair value function is a solution of a Hamilton-Jacobi Bellman system and the optimal strategy of the trader is to follow the optimal strategy which characterizes the value function before the change time and then to switch when the change occurs to the optimal strategy which characterizes the value function after the change time.

Part I

Technical Analysis

Chapter 2

The Bollinger Bands indicator in technical analysis

We present in this chapter the Bollinger Bands indicator, designed to detect the time at which the volatility of the stock switches. We describe the basic setting underlying our mathematical modeling. Finally we prove some identities in law which will be useful to prove in next chapters that the Bollinger Bands can be used to detect the time change of the volatility.

2.1 Technical analysis

Technical analysis is a method of predicting price movements and future market trends by studying charts of past market action. This is done by comparing current prices with historical prices to predict a reasonable outcome. The basis of modern-day technical analysis can be traced back to the Dow Theory, developed around 1900 by Charles Dow. It includes principles such as the trending nature of prices, confirmation and divergence, support and resistance. Technical analysts, or chartists, use a number of tools to help them identify potential trades.

The technical analyst (or the Dow Theory) believes that all the relevant market information is reflected (or discounted) in the averages, hence no other information is needed to make trading decisions. Watching financial markets, it becomes obvious that there are trends, momentum and patterns that repeat over time, not exactly the same way but similar.

A very large number of technical indicators have been developed over the years, including the widely used overbought/oversold indicators such as the Relative Strength Index, and the trend following indicators such as Moving Averages. While technical analysis can be a great help in trading the market, no technical indicator is infallible. Further, technical analysis is only as good as its interpreter. A significant of time

must be spent in learning the principles of technical analysis, and in how to properly interpret the various charts and other technical indicators.

Technical Analysis is based on three Principles:

The first principle is that the market discounts everything. At any time, anything that can or could affect a company will be reflected in the stock price including fundamental, political and psychological factors. Price action should reflect shifts in supply and demand. If demand exceeds supply, prices should rise. If supply exceeds demand, prices should fall.

Secondly, technical analysis is based on the premise that prices move in trends. This means that once a trend in the direction of a share price has been established, the next move in share prices is more likely to be in the same direction as that trend rather than in a different direction. In other words, if a share price is firmly established in a upward trend, the share price is more likely to continue increasing rather than decrease in the next trading period. Most technical trading strategies are based on this assumption.

The last principle is that history repeats itself. Chartists believe that the historical data will show repetitive patterns in price movement. Since these patterns have worked well in the past, it is assumed that they will continue to work well in the future.

A very large number of technical analysis indicators are used by pracitioners. For example:

- Moving average indicators (MA): The MA indicators are precisely calculated according to specific mathematical formulae. This makes moving averages an objective way to determine the current trend direction of a market, and anticipate its most likely future direction. Mathematically, moving averages filter out the random "noise" in market data by smoothing out fluctuations and short-term volatility in price movement. Graphically superimposing a moving average on a price chart makes it easy to visualize the underlying trend within the data.

- Moving average convergence/divergence (MACD): The MACD is a trend following momentum indicator that shows the relationship between two moving averages of prices. The MACD is the difference between a 26-day and 12-day exponential moving average. A 9-day exponential moving average, called the "signal" line is plotted on top of the MACD to show buy/sell opportunities. The basic MACD trading rule is to sell when the MACD falls below its signal line. Similarly, a buy signal occurs when the MACD rises above its signal line. It is also popular to buy/sell when the MACD goes above/below zero.

There is also the relative strength index, momentun, Bollinger Bands, etc.... Here we limit ourselves to the Bollinger Bands technique because it is often used to detect changes in the volatility.

2.1.1 Bollinger Bands

Bollinger Bands is a technical trading tool created by John Bollinger in the early 1980s and it's one of the most popular trading band technique. Trading bands became even more attractive when Bollinger suggested concentrating on volatility. Standard deviation was selected as the best measure for volatility because of its sensitivity to extreme deviations.

Denote by (S_t) the price process. The Bollinger Bands are fully described as follows:

- A δ-period simple moving average MA_t^δ
$$MA_t^\delta = \frac{1}{\delta}\sum_{k=0}^{\delta-1} S_{t-k}.$$
- A standard deviation Y_t^δ
$$(Y_t^\delta)^2 = \frac{1}{\delta}\sum_{k=0}^{\delta-1}(S_{t-k} - MA_t^\delta)^2.$$
- An upper band UB_t^δ at 2 times a δ-period standard deviation above the moving average
$$UB_t^\delta = MA_t^\delta + 2Y_t^\delta.$$
- A lower band LB_t^δ at 2 times a δ-period standard deviation below the moving average
$$LB_t^\delta = MA_t^\delta - 2Y_t^\delta.$$

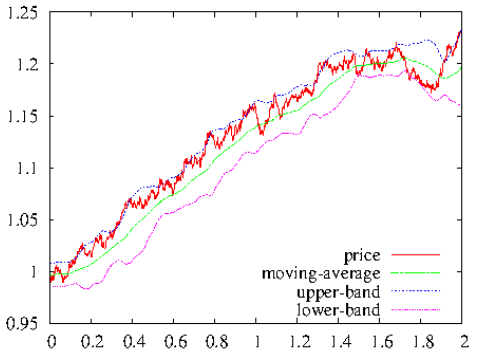
In continuous time we have the following extension:

$$MA_t^\delta = \frac{1}{\delta}\int_{t-\delta}^{t} S_u du$$

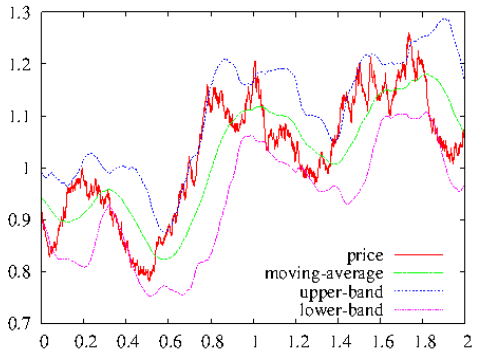
$$(Y_t^\delta)^2 = \frac{1}{\delta}\int_{t-\delta}^{t}(S_u - MA_t^\delta)^2 du$$

$$UB_t^\delta = MA_t^\delta + 2Y_t^\delta$$

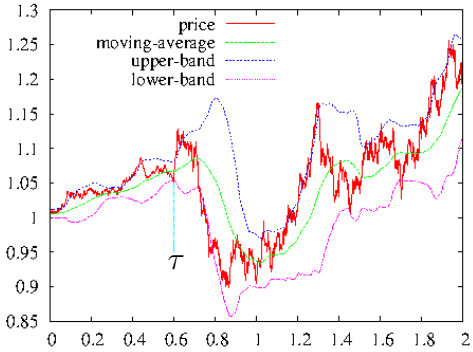
$$LB_t^\delta = MA_t^\delta - 2Y_t^\delta$$



(a) $\sigma = 0.05, \mu = 0.2$



(b) $\sigma = 0.25, \mu = 0.2$



(c) σ_0=0.05, σ_1=0.25, μ=0.2 and τ=0.6

Figure 2.1: Typical trajectories of the price (red), a moving average of size $\delta = 0.2$ (green) and the corresponding lower (pink) and upper (blue) Bollinger bands.

Numerical illustrations:

In Fig.2.1(a) and Fig.2.1(b) we present a typical Bollinger bands where the price S has no change in his volatility term and satisfies:

$$dS_t = S_t \left(\mu dt + \sigma dB_t\right).$$

In Fig.2.1(c), we present a Bollinger Bands where the volatility of the price process changes its value at time τ from a small volatility $\sigma_0 = 0.05$ to a large volatility $\sigma_1 = 0.25$. In this example, the change of volatility occurs at $\tau = 0.6$.

Remark 2.1.1. *Bollinger bands are plotted above and below the moving average price at standard deviation level. Since standard deviation is a measure of volatility, the bands adjust according the market volatility. Their band width expands during volatile markets and contracts during less active periods.*

2.1.2 Bandwidth indicator

An important indicator derived from the Bollinger Bands is the Bandwidth indicator BWI. This indicator is a relative measure of the width of the Bollinger Bands.

$$BWI_t = \frac{UB_t^\delta - LB_t^\delta}{MA_t^\delta} = \frac{4Y_t^{(\delta)}}{MA_t^\delta} = 4\sqrt{\delta \frac{\int_{t-\delta}^t S_u^2 du}{\left(\int_{t-\delta}^t S_u du\right)^2} - 1}.$$

In next section we will give a theoretical study for this indicator in order to show in next chapter that this indicator can be used to detect a change in the volatility term in two cases: the case of small volatilities and the case of large volatilities. We show that we have different behaviors for the density of the Bandwidth indicator for different values of volatilities.

We describe in the next section the basic setting underlying our mathematical modeling.

2.2 A mathematical framework to study the Bandwidth indicator

We deal with the following model for a financial market, in which two assets are traded continuously. The first one is a risk free asset, typically a bond (or bank account), whose price evolves according to the following equation:

$$\begin{aligned} dS_t^0 &= S_t^0 r dt, \\ S_0^0 &= 1. \end{aligned} \tag{2.2.1}$$

The second is a risky asset with price described by the linear stochastic differential equation:

$$\begin{aligned} dS_t &= \mu S_t dt + S_t \left(\sigma_1 + (\sigma_0 - \sigma_1)\, \mathbb{1}_{t \leqslant \tau}\right) dB_t, \\ S_0 &= S^0. \end{aligned} \tag{2.2.2}$$

where $(B_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$. The random time τ is independent of B and at this time τ , which is neither known, nor directly observable, the instantaneous volatility rate changes from σ_0 to σ_1.

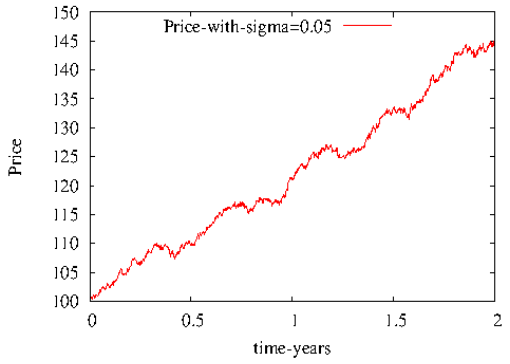
A simple computation shows that:

$$S_t = S^0 \exp\left(\left(\mu - \frac{1}{2}\sigma_0^2\right) t + \sigma_0 B_t + (\sigma_1 - \sigma_0)\left(B_t - B_\tau\right) \mathbb{1}_{\tau<t} - \frac{1}{2}\left(\sigma_1^2 - \sigma_0^2\right)(t-\tau)^+\right). \tag{2.2.3}$$

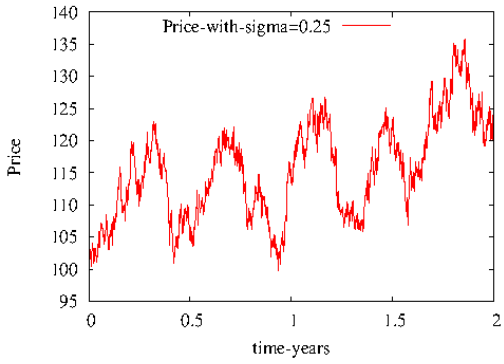
In Fig.2.2(a) and 2.2(b), we illustrate a typical trajectories of the price without a change in the diffusion term, that is the process (S) satisfies:

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right). \tag{2.2.4}$$

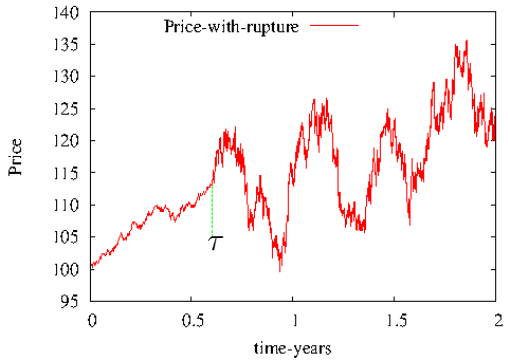
In Fig. 2.2(c), we illustrate the trajectory of S which satisfies (2.2.2) with $\sigma_0 = 0.05$, $\sigma_1 = 0.25$, $\mu = 0.2$ and the change time τ is equal to 0.6.



(a) $\sigma = 0.05$,$\mu = 0.2$



(b) $\sigma = 0.25$,$\mu = 0.2$



(c) $\sigma_0 = 0.05$,$\sigma_1 = 0.25$,$\mu = 0.2$ and $\tau = 0.6$

Figure 2.2: Typical trajectories of the stock price

2.3 Some identities in law

In this section we show a collection of identities in law which will be useful in the next chapter to analyze asymptotic behaviors of the Bandwidth Bands BWI. We aim in the next chapter to prove that the behavior of the probability density function of BWI depends on the value of the volatility. So for more understanding this result, we can first assume that the price process (S_t) hasn't a change in its volatility term and it evolves according to (2.2.4).

Lemma 2.3.1. *Assume (S_t) evolves according to the classical Black and Scholes equation (2.2.4) with a constant volatility, then the law of BWI_t does not depend on time t and*

$$BWI_t \overset{\mathcal{L}}{=} BWI_\delta.$$

Proof :

We have:

$$BWI_t = 4\sqrt{\delta \frac{\int_{t-\delta}^{t} S_u^2 du}{\left(\int_{t-\delta}^{t} S_u du\right)^2} - 1}$$

Use the change of variable $s \to u - (t-\delta)$ and the identity in law

$$(B_{s+(t-\delta)} - B_{t-\delta}\ ; s \geqslant 0) \overset{\mathcal{L}}{=} (B_s\ ; s \geqslant 0)$$

it follows that the law of BWI is independent of the time t and the initial condition S_0. $\forall t > 0$,

$$BWI_t \overset{\mathcal{L}}{=} 4\sqrt{\delta \frac{\int_0^{\delta} S_u^2 du}{\left(\int_0^{\delta} S_u du\right)^2} - 1} \quad := BWI_\delta. \tag{2.3.1}$$

□

Let us now define the process $(k_t^\nu)_{t\geqslant 0}$ by

$$k_t^{(\nu)} := \frac{A_{2,t}^{(\nu)}}{\left(A_{1,t}^{(\nu)}\right)^2} \tag{2.3.2}$$

where $A_{1,t}^{(\nu)}$ and $A_{2,t}^{(\nu)}$ are the exponential functional of Brownian motion defined by:

$$A_{1,t}^{(\nu)} := \int_0^t \exp(B_s + \nu s) ds \qquad \text{and} \qquad A_{2,t}^{(\nu)} := \int_0^t \exp(2B_s + 2\nu s) ds.$$

We now show that the law of BWI_δ can be expressed in terms of the exponential functionals of Brownian motion.

Proposition 2.3.2. *Assume (S_t) evolves according to equation (2.2.4), then we have:*

$$BWI_\delta \overset{\mathcal{L}}{=} 4\sqrt{(\sigma^2\delta)k^{(\tilde{\mu})}_{\sigma^2\delta} - 1}, \tag{2.3.3}$$

where $\tilde{\mu} = \frac{\mu}{\sigma^2} - \frac{1}{2}$.

Proof :

We have:

$$BWI_\delta = 4\sqrt{\delta\frac{\int_0^\delta S_u^2 du}{\left(\int_0^\delta S_u du\right)^2} - 1}$$

$$= 4\sqrt{\delta\frac{\int_0^\delta \exp\left(2\left(\mu - \frac{1}{2}\sigma^2\right)u + 2\sigma B_u\right)du}{\left(\int_0^\delta \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)u + \sigma B_u\right)du\right)^2} - 1}$$

Using the following change of variable

$$s \to \sigma^2 u \qquad \text{and} \qquad \tilde{\mu} = \frac{\mu}{\sigma^2} - \frac{1}{2}$$

and by the scaling property of Brownian motion we obtain:

$$BWI_\delta \overset{\mathcal{L}}{=} 4\sqrt{\sigma^2\delta\frac{\int_0^{\sigma^2\delta} \exp\left(2\tilde{\mu}s + 2B_s\right)ds}{\left(\int_0^{\sigma^2\delta} \exp\left(\tilde{\mu}s + B_s\right)ds\right)^2} - 1} = 4\sqrt{(\sigma^2\delta)k^{(\tilde{\mu})}_{\sigma^2\delta} - 1}. \tag{2.3.4}$$

□

2.3.1 Properties of integral functionals of Brownian motion with drift

The above process $A^{(\nu)}_{1,t}$ and $A^{(\nu)}_{2,t}$ are called exponential functional of Brownian motion and have been studied by a number of authors, e.g, Yor [21], Dufresne [13] and Donati-Martin et al [11]. In the literature, there are several studies concern the law of $A^{(\nu)}_{1,t}$

or $A^{(\nu)}_{2,t}$ for fixed t and for $t \to \infty$. In our next chapter, we will be interested by limit distributions of these processes.

Notice that from the change of variable $u = \frac{t}{4}$ and the scaling property of the Brownian motion, we have

$$A^{(\nu)}_{1,t} \overset{\mathcal{L}}{=} 4A^{(2\nu)}_{2,\frac{t}{4}}. \tag{2.3.5}$$

We introduce the Brownian motion with constant drift $\gamma \in \mathbb{R}$, $(B^{(\gamma)}_t := B_t + \gamma t)_{t \geqslant 0}$.

For $\gamma > 0$ and f a non-negative Borel measurable function f, we set:

$$A^{(\gamma)}_\infty(f) = \int_0^\infty f(B^{(\gamma)}_s)ds$$

We show that for $\nu < 0$, $A^{(\nu)}_{2,\infty}$ is finite. For that, we need the following theorem given in [21], which provides a necessary and sufficient condition in order that $A^{\gamma}_\infty(f)$ defined below is almost surely finite.

Theorem 2.3.3. *Let $\gamma > 0$ and f be a non-negative and locally integrable Borel function on $\mathbb{R}$. Then $A^{(\gamma)}_\infty(f)$ is almost surely finite if and only if $\int_0^\infty f(y)dy < \infty$.*

So for $\nu < 0$, we have:

$$A^{(\nu)}_{2,\infty} = \int_0^\infty \exp(2B^{(\nu)}_s)ds \overset{\mathcal{L}}{=} \int_0^\infty \exp(-2B^{(-\nu)}_s)ds = \int_0^\infty f(B^{(-\nu)}_s)ds$$

where $f(x) = \exp(-2x)$ verifies theorem 2.3.3. Thus for $\nu < 0$, $A^{(\nu)}_{2,\infty}$ is almost sure finite.

The following result shows that the integral (at infinity) of the exponential of Brownian motion with negative drift is distributed as the inverse of a gamma variable.

Proposition 2.3.4. *For $\nu < 0$, $A^{(\nu)}_{2,\infty}$ is almost surely finite and is distributed as $\frac{1}{2\gamma_{-\nu}}$, where $\gamma_{-\nu}$ denotes a gamma random variable with parameter $-\nu$.*

Remark 2.3.5. *From (2.3.5) and the above proposition, we have that for $\nu < 0$, $A^{(\nu)}_{1,\infty}$ is almost surely finite and is distributed as $\frac{2}{\gamma_{-2\nu}}$.*

In next chapter we will be interested by two important results concerning the variable $k^{(\nu)}_t$. The first one is the following equality in law and the second one is the law of $k^{(\nu)}_\infty$ for $\nu < 0$.

Proposition 2.3.6. *For $\nu \in \mathbb{R}$, we have:*

$$k_t^{(\nu)} \overset{\mathcal{L}}{=} k_t^{(-\nu)}$$

Proof :

Using the identity in law $(B_t - B_{t-s}\ ; 0 \leqslant s \leqslant t) \overset{\mathcal{L}}{=} (B_s\ ; 0 \leqslant s \leqslant t)$, we obtain:

$$\begin{aligned}
k_t^{(\nu)} &= \frac{\displaystyle\int_0^t \exp(2B_s + 2\nu s)ds}{\displaystyle\left(\int_0^t \exp(B_s + \nu s)ds\right)^2} \\
&\overset{\mathcal{L}}{=} \frac{\displaystyle\int_0^t \exp\Big(2\left(B_t - B_{t-s}\right) + 2\nu s\Big)ds}{\displaystyle\left(\int_0^t \exp\Big(\left(B_t - B_{t-s}\right) + \nu s\Big)ds\right)^2} \\
&= \frac{\exp\left(2B_t + 2\nu t\right)\displaystyle\int_0^t \exp\Big(-2B_{t-s} - 2\nu(t-s)\Big)ds}{\exp\left(2B_t + 2\nu t\right)\displaystyle\left(\int_0^t \exp\Big(-B_{t-s} - \nu(t-s)\Big)ds\right)^2} \\
&= \frac{\displaystyle\int_0^t \exp\left(-2B_u - 2\nu u\right)du}{\displaystyle\left(\int_0^t \exp\left(-B_u - \nu u\right)du\right)^2} \\
&\overset{\mathcal{L}}{=} \frac{\displaystyle\int_0^t \exp\left(2B_u - 2\nu u\right)du}{\displaystyle\left(\int_0^t \exp\left(B_u - \nu u\right)du\right)^2} = k_t^{(-\nu)}.
\end{aligned}$$

The last equality comes from the symmetric property of the Brownian motion.

□

In [34], Yor has characterized the law of $k_\infty^{(\nu)}$ by its Laplace transform as following:

Proposition 2.3.7. *Let $\nu < 0$; the Laplace transform of $k_\infty^{(\nu)} = \frac{A_{2,\infty}^{(\nu)}}{(A_{1,\infty}^{(\nu)})^2}$ is given by:*

$$\forall \lambda > 0, \qquad \mathbb{E}\left[e^{-\lambda k_\infty^{(\nu)}}\right] = \frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})(\cosh(\sqrt{\lambda/2}))^{-2\nu}}. \tag{2.3.6}$$

Chapter 3

Asymptotic analysis of the Bandwidth indicator in the case of small volatilities

We aim in this chapter to prove that the Bollinger Bandwidth indicator can detect the time change in the case of small volatilities. We show that the density behavior of the Bandwidth indicator depends on the value of the volatility, and the difference between the behaviors becomes more significant when the quotient between the volatilities is greater than a fixed level.

3.1 Asymptotic convergence in law for the Bandwidth indicator

In order to understand in the sequel how the density behavior of the Bandwidth indicator depends on the volatility, we can assume that the price process evolves according to:

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

where μ and σ are constants.

We show in this section that for small value of $\sigma^2\delta$, the Bandwidth indicator normalized by $\sqrt{\sigma^2\delta}$ converges in law to a quadratic additive functionals of Brownian motion.

Proposition 3.1.1. *For small $\sigma^2\delta$, we have the following:*

$$\frac{BWI_\delta}{\sqrt{\sigma^2\delta}} \underset{\sigma^2\delta \to 0}{\xrightarrow{\mathcal{L}}} 4 \times \sqrt{\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2}. \tag{3.1.1}$$

Proof :
To simplify the notation, let $\epsilon = \sigma^2\delta$. From (2.3.3), we have:

$$BWI_\delta \stackrel{\mathcal{L}}{=} 4\sqrt{\epsilon k_\epsilon^{(\tilde{\mu})} - 1}$$

We have:

$$\begin{aligned}
\frac{1}{\epsilon}\left(\epsilon\, k_\epsilon^{(\tilde{\mu})} - 1\right) &= \frac{1}{\epsilon}\left(\epsilon\frac{\displaystyle\int_0^\epsilon \exp\left(2\tilde{\mu}s + 2B_s\right)ds}{\displaystyle\left(\int_0^\epsilon \exp\left(\tilde{\mu}s + B_s\right)ds\right)^2} - 1\right) \\
&= \frac{1}{\epsilon}\frac{\displaystyle\int_0^1 \exp(2\tilde{\mu}\theta\epsilon + 2B_{\theta\epsilon})d\theta}{\displaystyle\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + B_{\theta\epsilon})d\theta\right)^2} - \frac{1}{\epsilon} \qquad \text{(change of variable } \theta = \tfrac{s}{\epsilon}\text{)} \\
&\stackrel{\mathcal{L}}{=} \frac{1}{\epsilon}\frac{\displaystyle\int_0^1 \exp(2\tilde{\mu}\theta\epsilon + 2\sqrt{\epsilon}B_\theta)d\theta}{\displaystyle\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta\right)^2} - \frac{1}{\epsilon} \qquad \text{(from scaling property).}
\end{aligned} \tag{3.1.2}$$

It obviously suffices to prove that for small ϵ:

$$\frac{1}{\epsilon}\frac{\displaystyle\int_0^1 \exp(2\tilde{\mu}\theta\epsilon + 2\sqrt{\epsilon}B_\theta)d\theta}{\displaystyle\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta\right)^2} - \frac{1}{\epsilon} \text{ converges almost surely to } \int_0^1 B_\theta^2 d\theta - \left(\int_0^1 B_\theta d\theta\right)^2. \tag{3.1.3}$$

Indeed, we have:

$$\begin{aligned}
&\frac{1}{\epsilon}\frac{\displaystyle\int_0^1 \exp(2\tilde{\mu}\theta\epsilon + 2\sqrt{\epsilon}B_\theta)d\theta}{\displaystyle\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta\right)^2} - \frac{1}{\epsilon} \\
&= \frac{\frac{1}{\epsilon}\displaystyle\int_0^1 \exp(2\tilde{\mu}\theta\epsilon + 2\sqrt{\epsilon}B_\theta)d\theta - \frac{1}{\epsilon}\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta\right)^2}{\displaystyle\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta\right)^2}.
\end{aligned}$$

The denominator converges almost surely to 1, and

$$\frac{1}{\epsilon}\int_0^1 \exp(2\tilde{\mu}\theta\epsilon + 2\sqrt{\epsilon}B_\theta)d\theta - \frac{1}{\epsilon}\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta\right)^2$$
$$= \int_0^1 \left(\frac{\exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta) - 1}{\sqrt{\epsilon}}\right)^2 d\theta - \left(\frac{1}{\sqrt{\epsilon}}\left(\int_0^1 \exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta)d\theta - 1\right)\right)^2.$$

As

$$\frac{1}{\sqrt{\epsilon}}\Big(\exp(\tilde{\mu}\theta\epsilon + \sqrt{\epsilon}B_\theta) - 1\Big) \underset{\epsilon \to 0}{\xrightarrow{\text{a.s}}} B_\theta,$$

Then the result follows.

□

Numerical illustration

We now illustrate numerically the above convergence in law. For that we are interested by the kernel density estimator which is a non-parametric estimation of the probability density of a random variable:

Let $X_1, X_2, ...X_n$ be a sample drawn of size n from a random variable with density f. A kernel density estimation of f at the point x is

$$\tilde{f}_h(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right) \tag{3.1.4}$$

where the kernel K satisfies $\int K(x)dx = 1$ and $h > 0$ is a smoothing parameter.

A popular choice of K is the Gaussian kernel, namely,

$$K(y) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-y^2}{2}\right).$$

For simplifying the notation, let Q be the random variable defined by:

$$Q = \int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2.$$

In figure 3.1, we illustrate in blue the estimated density of the variable $4\sqrt{Q}$. We first estimate the sum of integrals $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ by discretization and then we estimate the density of $4\sqrt{Q}$ as in (3.1.4). In red and green, we illustrate respectively as in (3.1.4) the estimated density of $\frac{BWI_\delta}{\sigma^2\delta}$ for different value of σ. In red, $\sigma = 0.08$ and in green $\sigma = 0.2$, where we have fixed $\mu = 0.1$ and $\delta = 0.1$.

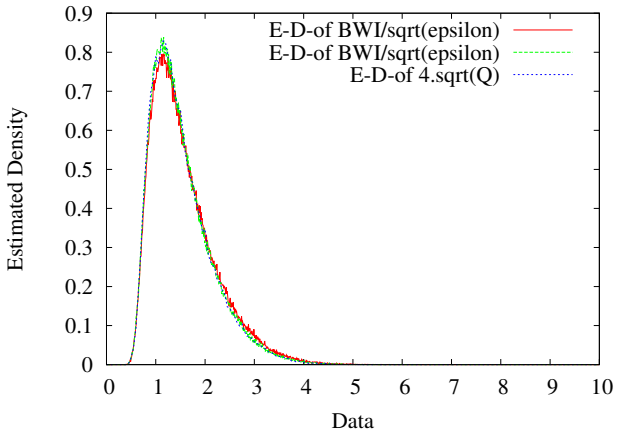


Figure 3.1: Estimated Densities of $4\sqrt{Q}$ (blue), $BWI/\sqrt{\sigma^2\delta}$ (green) with $\sigma = 0.2, \delta = 0.1$ and $BWI/\sqrt{\sigma^2\delta}$ (red) with $\sigma = 0.08, \delta = 0.1$.

3.2 Theoretical study of the law of Q.

In Donati-Martin and Yor [12], Rogers et al [7], the law of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ is characterized by its Laplace transform. Then the density of this variable can be obtained by inverting its Laplace transform, but unfortunately, the explicit formula is not propitious to study the behavior of this density. On the other hand, we show that $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ has an unimodal distribution. We first show that this variable is a generalized gamma convolution and a self-decomposable random variable. Therefore from Sato [28], we can deduce that this random variable has a unimodal distribution.

We now give some definitions and results which will be useful in the sequel. We will be interested by two families of random variables defined as follows:

Definition 3.2.1. *A random variable X is said to be self-decomposable if for each $0 < u < 1$ there is the equality in distribution*

$$X \stackrel{d}{=} uX + Y$$

for some random variable Y independent of X. We denote by S the set of self-decomposable random variables, taking values in $\mathbb{R}_+$.

Definition 3.2.2. *A positive random variable X is called a generalized gamma convolution GGC if it is a limit distribution for sums of independent gamma distributed random variables and the gamma distributions may have different shape and scale parameters. We denote by G the set of positive generalized gamma convolution (GGC) random variables.*

A important result about these families is the following strict inclusions which are mentioned in Yor et al [15] and Sato [28]:

$$G \subset S. \tag{3.2.1}$$

We now present a result concerning the unimodality of a self-decomposable distribution on $\mathbb{R}$. This study has been an open problem for many years, since the 1940s. The affirmative answer was given in the symmetric case by Wintner [29] in 1956, in the one-sided case Wolfe [30] in 1971, and finally in the general case by Yamazato [31] in 1978. This result of Yamazato is given in Sato [28, p.404,Th 53.1] as follows:

Theorem 3.2.3. *If X is a self-decomposable random variable on $\mathbb{R}$, then its distribution is unimodal.*

Application: study of the law of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$

An important studies have been done by Donati-Martin and Yor [12], Rogers et al [7] about the law of a quadratic functional of Brownian motion by computing its Laplace transform $\Phi(\lambda)$ given below. In [12], they have used Fubini's theorem for double Wiener integrals in order to compute the Laplace transform $\Phi(\lambda)$. In [7], Rogers and al have used another technique which is based on Ray-Knight theorem for Brownian motion. The Laplace transform $\Phi(\lambda)$ of the variable $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ is given as follows:

$$\Phi(\lambda) = \left(\frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})}\right)^{1/2}. \tag{3.2.2}$$

Let us introduce the random variable S_h defined in [3] by

$$S_h := \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{\Gamma_{h,n}}{n^2} \tag{3.2.3}$$

for independent random variables $\Gamma_{h,n}$ with gamma law of parameter h.

This variable is also used in next chapter where we present more properties and results for this law.

We now show that $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ is equal in law to a limit of sums of independent gamma variables with different parameters.

Proposition 3.2.4. *Let (B_t) be a one dimensional Brownian motion, we have the identity in law*

$$\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2 \stackrel{\mathcal{L}}{=} S_{1/2}. \tag{3.2.4}$$

where the random variable $S_{1/2}$ is defined in (3.2.3).

Proof :
Remind that the distribution of the gamma variables $\Gamma_{h,n}$ is characterized by the Laplace transform

$$\mathbb{E}\left[e^{-\lambda\Gamma_{h,n}}\right] = (1+\lambda)^{-h} \qquad (\lambda > -1) \tag{3.2.5}$$

and from [8] we have the following Euler's formula:

$$\sinh(z) = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2\pi^2}\right). \tag{3.2.6}$$

From (3.2.3), we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda S_{1/2}}\right] &= \mathbb{E}\left[\prod_{n=1}^{\infty} e^{-\frac{\lambda\Gamma_{1/2,n}}{2n^2\pi^2}}\right] \\
&= \prod_{n=1}^{\infty}\left(\mathbb{E}\left[e^{-\frac{\lambda\Gamma_{1/2,n}}{2n^2\pi^2}}\right]\right) \qquad \text{(because } \Gamma_{1/2,n} \text{ are independents)} \\
&= \prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2\pi^2n^2}\right)^{-1/2} \qquad \text{(from (3.2.5))} \\
&= \left(\frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})}\right)^{1/2} \qquad \text{(from (3.2.6)).}
\end{aligned}$$

we then have that the Laplace transform of $S_{1/2}$ is equal to (3.2.2). Therefore the result follows.

□

Remark 3.2.5. *Notice that the density of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ can be obtained as the sum of two infinite series by inverting its Laplace transform and using Lévy's formula (4.2.4) given in next chapter. Unfortunately, it is not easy to study its behavior by relying on this complicated formula, but relying on the above equality in law, we show that the distribution of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ is unimodal and we give an approximation for its mode.*

Proposition 3.2.6. *The distribution of* $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ *is unimodal.*

Proof :
From the equality in law (3.2.4) and the definition of $S_{1/2}$ as a limit of sums of independent gamma distribution, we can deduce from definition 3.2.2 that $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ belongs to the set G. Thus from (3.2.1) it also belongs to the set S, therefore the result follows from theorem 3.2.3.

□

Let X be a random variable with distribution function μ and finite mean m. The central absolute moment γ_p of order p is defined by:

$$\gamma_p := \mathbb{E}\left[(X-m)^p\right].$$

We now present a result given by Sato in [27] about a relation for a unimodal distribution between its mode and the central absolute moment γ_p of order p.

Proposition 3.2.7 (Sato [27])**.** *For* $p \geqslant 1$*, if* μ *is a unimodal distribution with mode* a *and has finite mean* m*, then*

$$\mid a - m \mid \leqslant (p+1)^{1/p} \gamma_p^{1/p}. \tag{3.2.7}$$

Proposition 3.2.8. *The mode* a *of* $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ *satisfies:*

$$\mid a - \frac{1}{6} \mid \leqslant 0.25. \tag{3.2.8}$$

Proof :
Let us denote by a and m respectively the mode and the mean of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$.
From 3.2.7, we have for even order $2p$

$$\mid a - m \mid \leqslant (2p+1)^{1/2p} \gamma_{2p}^{1/2p}.$$

The mean m is calculated as follows

$$\begin{aligned} m :&= \mathbb{E}\left[\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2\right] \\ &= \frac{1}{2} - \mathbb{E}\left[\left(\int_0^1 B_s ds\right)^2\right] \\ &= \frac{1}{2} - \mathbb{E}\left[\left(B_1 - \int_0^1 s dB_s\right)^2\right] \\ &= \frac{1}{2} - \mathbb{E}\left[\left(\int_0^1 (1-s) dB_s\right)^2\right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

We now compute the central absolute moment

$$\gamma_{2p} := \mathbb{E}\left[\left(\left(\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2\right) - m\right)^{2p}\right].$$

Observe,

$$\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2 - m = \int_0^1 \left(B_s^2 - s\right) ds - \left(\int_0^1 B_s ds\right)^2 + \frac{1}{3}$$

Applying Itô's formula to (tB_t^2) between 0 and 1, we have

$$\begin{aligned} \int_0^1 \left(B_s^2 - s\right) ds &= B_1^2 - 2\int_0^1 s ds - 2\int_0^1 s B_s dB_s \\ &= 2\int_0^1 B_s dB_s - 2\int_0^1 s B_s dB_s \\ &= 2\int_0^1 B_s (1-s) dB_s. \end{aligned}$$

and

$$\begin{aligned} \left(\int_0^1 B_s ds\right)^2 &= \left(\int_0^1 (1-s) dB_s\right)^2 \qquad \text{(Applying Itô to } (tB_t) \text{ between 0 and 1)} \\ &= 2\int_0^1 \int_0^s (1-\theta)\, dB_\theta\, (1-s)\, dB_s + \int_0^1 (1-s)^2 ds \\ &= 2\int_0^1 \int_0^s (1-\theta)\, dB_\theta\, (1-s)\, dB_s + \frac{1}{3}. \end{aligned}$$

Therefore

$$\left(\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2\right) - m = 2\int_0^1 (1-s)\left(B_s - \int_0^s (1-\theta)\, dB_\theta\right) dB_s$$
$$= 2\int_0^1 (1-s)\int_0^s \theta dB_\theta dB_s.$$

Finally for $p = 1$, one obtains:

$$\mathbb{E}\left[\left(\left(\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2\right) - m\right)^2\right] = 4\mathbb{E}\left[\int_0^1 (1-s)^2 \left(\int_0^s \theta dB_\theta\right)^2 ds\right]$$
$$= \frac{4}{3}\int_0^1 (1-s)^2 s^3 ds \approx 0.022.$$

Then for $p = 1$, the result (3.2.8) follows. □

Remark 3.2.9. *The result in (3.2.8) can be refined by choosing another order p. We will use in this case the Burkholder-Davis-Gundy (BDG) inequalities (current calculations will be inserted in the final version of the manuscript).*

In figure 3.2 we use (3.1.4) to estimate the density of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$ where $h = 10^{-3}$. We observe that the mode is into the interval calculated in (3.2.8).

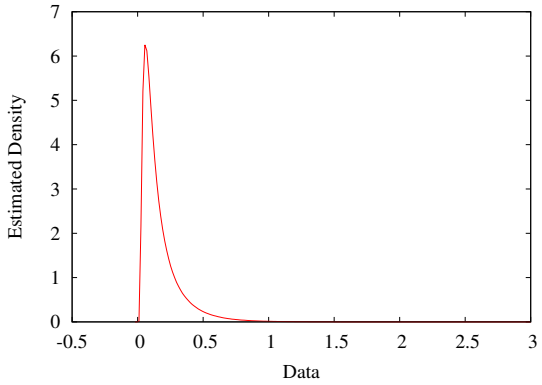


Figure 3.2: Estimated Density of $\int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$.

3.3 Detection change in the case of small volatilities

We now aim to show how the Bandwidth indicator can detect a change of volatilities. This result is confirmed as long as the quotient between σ_0 and σ_1 is greater than or equal to 1.8.

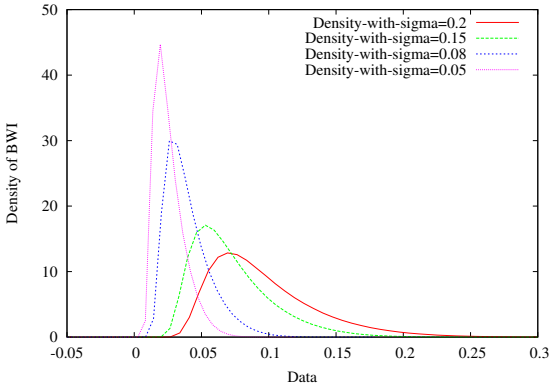


Figure 3.3: Estimated Density of BWI_δ
for different values of σ, $\delta = 0.1$.

In figure 3.3 we use (3.1.4) to estimate the density of the Bandwidth indicator for different value of σ. For each value of σ, we have simulated 10^7 trajectories of BWI_δ, with $\delta = 0.1$, $\mu = 0.1$ and $h = 10^{-3}$. It is clear from this figure that we have different behaviors of the tail for different values of the volatility, and this difference becomes

more significant when the quotient between σ_0 and σ_1 is greater than or equal to 1.8. To better understand this result, we now comment it.

In order to show that the Bandwidth indicator can detect a change of volatilities, we show that for:

$$\begin{cases} \sigma_{min} & \text{fixed} \\ \sigma_{max} & \text{fixed} \end{cases}$$

we can find α and β such that, for example,

i) $\forall \sigma_0 \leqslant \sigma_{min}$, we have:

$$\mathbb{P}\left[BWI_\delta(\sigma_0) \leqslant \alpha\right] \text{is large} \quad \text{and} \quad \mathbb{P}\left[BWI_\delta(\sigma_0) \geqslant \beta\right] \text{is small,}$$

and

ii) $\forall \sigma_1 \geqslant \sigma_{max}$, we have:

$$\mathbb{P}\left[BWI_\delta(\sigma_1) \leqslant \alpha\right] \text{is small} \quad \text{and} \quad \mathbb{P}\left[BWI_\delta(\sigma_1) \geqslant \beta\right] \text{is large.}$$

We remind the notation $Q = \int_0^1 B_s^2 ds - \left(\int_0^1 B_s ds\right)^2$. Let us first find q_1, q_2, q_3 and q_4 such that:

$$\mathbb{P}\left[Q \leqslant q_1\right] \text{is large} \quad \text{and} \quad \mathbb{P}\left[Q \geqslant q_2\right] \text{is small,} \tag{3.3.1}$$

and

$$\mathbb{P}\left[Q \leqslant q_3\right] \text{is small} \quad \text{and} \quad \mathbb{P}\left[Q \geqslant q_4\right] \text{is large.} \tag{3.3.2}$$

So from figure 3.2, we can deduce that:

$$q_1 \geqslant 0.3 \quad , \quad q_2 \geqslant 0.35 \quad , \quad q_3 \leqslant 0.1 \quad , \quad q_4 \leqslant 0.15.$$

Then from proposition (3.1.1), the inequalities in $i)$ and $ii)$ become:

$$\mathbb{P}\left[Q \leqslant \frac{\alpha^2}{16\sigma_0^2\delta}\right] \text{is large} \quad \text{and} \quad \mathbb{P}\left[Q \geqslant \frac{\beta^2}{16\sigma_0^2\delta}\right] \text{is small,}$$

and

$$\mathbb{P}\left[Q \leqslant \frac{\alpha^2}{16\sigma_1^2\delta}\right] \text{ is small} \quad \text{and} \quad \mathbb{P}\left[Q \geqslant \frac{\beta^2}{16\sigma_1^2\delta}\right] \text{ is large.}$$

Suppose for example that $\sigma_{min} = 0.08$ and $\sigma_{max} = 0.15$, then if we take $\sigma_0 = \sigma_{min}$ and $\sigma_1 = \sigma_{max}$ in the above inequalities, we can deduce from (3.3.1) and (3.3.2) that α and β verify:

$$\frac{\alpha^2}{16\sigma_{min}^2\delta} \geqslant q_1 \quad , \quad \frac{\beta^2}{16\sigma_{min}^2\delta} \geqslant q_2 \quad , \quad \frac{\alpha^2}{16\sigma_{max}^2\delta} \leqslant q_3 \quad , \quad \frac{\beta^2}{16\sigma_{max}^2\delta} \leqslant q_4. \tag{3.3.3}$$

By consequent

$$0.055 \leqslant \alpha \leqslant 0.06,$$

and

$$0.059 \leqslant \beta \leqslant 0.073.$$

Then if we observe an event with a large probability in a case and with small probability in another case, then we can deduce in which regime of volatility we are. This observation becomes more significant, that is the difference between the probabilities in the two cases becomes more large, if the quotient between the two volatilities is greater than or equal to 1.8. This result can be deduced from (3.3.3).

Chapter 4

Asymptotic analysis of the Bandwidth indicator in the case of large volatilities

In the previous chapter, we have done an asymptotic analysis of the Bollinger Bandwidth indicator in the case of small value of $\sigma^2\delta$. We have showed that the Bandwidth indicator has the capacity to detect a regime change in volatility in this case.

In this chapter, we aim to show that the Bandwidth indicator can be also used to detect a regime change in the diffusion term in the case of large value of $\sigma^2\delta$.

We are interested in this chapter to the normalized indicator (NI_t) defined as:

$$NI_t := \frac{1}{\delta}\left(\frac{(BWI_t)^2}{16} + 1\right) \tag{4.0.1}$$

where (BWI_t) is the Bandwidth indicator defined in chapter 1.

We first show that the law of NI_t does not depend on time t and it satisfies:

$$NI_t \overset{\mathcal{L}}{=} NI_\delta \overset{\mathcal{L}}{=} \sigma^2 k^{(\tilde{\mu})}_{\sigma^2\delta}.$$

where the variable k is defined in (2.3.2) and $\tilde{\mu} = \frac{\mu}{\sigma^2} - \frac{1}{2}$.

Then we prove that for large value of $\sigma^2\delta$, the cumulative distribution function of NI_δ converges to that of $\sigma^2 k_\infty^{-|\tilde{\mu}|}$. We study the behavior of the cumulative distribution function of $k_\infty^{-|\tilde{\mu}|}$, which can be useful to study and compare the tails-behavior of NI_δ for different value of σ. We make explicit the density and the cumulative distribution function of $k_\infty^{-|\tilde{\mu}|}$. Due to the complexity of these formulae, we have not yet succeeded to use them in order to study the behavior of the cumulative distribution function of $k_\infty^{-|\tilde{\mu}|}$. We therefore use Tauberian theory which allows us to deduce the behavior of the cumulative distribution function of $k_\infty^{-|\tilde{\mu}|}$ from the behavior of its Laplace transform.

4.1 Asymptotic convergence of the cumulative distribution function of the Bollinger normalized indicator

To simplify the mathematical study, we suppose that the price process (S_t) evolves according to:

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right).$$

Then from the change of variables $s \to \sigma^2 u$, $\tilde{\mu} = \frac{\mu}{\sigma^2} - \frac{1}{2}$ and the scaling property of Brownian motion, we prove that the law of the normalized indicator NI does not depend on time t and it satisfies:

$$NI_t \overset{\mathcal{L}}{=} NI_\delta \overset{\mathcal{L}}{=} \sigma^2 k^{(\tilde{\mu})}_{\sigma^2\delta}.$$

We desire to show that the normalized Bandwidth indicator can detect a change in the constant coefficient σ for the case of large value for $\sigma^2\delta$. So we need to compare the tail behaviors of the distribution of NI for different values of σ.

In next section, we show the convergence of the cumulative distribution function of NI_δ to that of $\sigma^2 k_\infty^{-|\tilde{\mu}|}$.

Let us start with exhibiting an upper and lower bounds for the cumulative distribution function of $\log(k_t^{(\tilde{\mu})})$ for large t. Then thanks to the above equality in law, the upper and lower bounds for the cumulative distribution function of NI_δ are straightforward consequence.

4.1.1 Upper and lower bounds for the cumulative distribution function of $\log(k_t^{(\nu)})$ for large t

Proposition 4.1.1. *For $x \in \mathbb{R}$,ϵ,ϵ' and $\vartheta \in \mathbb{R}^+$, $\exists t_0 > 0$ such that for $t > t_0$ one has:*

- *For $|\nu| > 2$,*

$$\mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x - 2\epsilon'\right] - \frac{(1+\vartheta)}{\epsilon'}\left(\frac{-2|\nu|+2}{-2|\nu|+1}\right)e^{\frac{1}{2}(-2|\nu|+1)t} \leqslant \mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right] \leqslant$$
$$\mathbb{P}\Big[\log(k_\infty^{(-|\nu|)}) \leqslant x + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon}\left(\frac{-|\nu|+2}{-|\nu|+1}\right)e^{2(-|\nu|+1)t}$$

- *For* $|\nu| < 1$,

$$\mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x - 2\epsilon^{'}\right] - \frac{(1+\vartheta)}{\epsilon^{'}} \frac{e^{-|\nu|^2 t/2}\left(\frac{2\pi}{t}\right)^{3/2}}{4|\nu|^2(1-\cos(2\pi|\nu|))} \leqslant \mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right] \leqslant \mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x + \epsilon\right] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-|\nu|^2 t/2}\left(\frac{\pi}{2t}\right)^{3/2}}{|\nu|^2(1-\cos(\pi|\nu|))}$$

- *For* $1 < |\nu| < 2$,

$$\mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x - 2\epsilon^{'}\right] - \frac{(1+\vartheta)}{\epsilon^{'}} \left(\frac{-2|\nu|+2}{-2|\nu|+1}\right) e^{\frac{1}{2}(-2|\nu|+1)t} \leqslant \mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right] \leqslant \mathbb{P}\Big[\log(k_\infty^{(-|\nu|)}) \leqslant x + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-|\nu|^2 t/2}\left(\frac{\pi}{2t}\right)^{3/2}}{|\nu|^2(1-\cos(\pi|\nu|))}$$

- *For* $|\nu| = 1$,

$$\mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x - 2\epsilon^{'}\right] - \frac{(1+\vartheta)}{\epsilon^{'}} \frac{1}{\sqrt{(\frac{1}{2}\pi t)}} e^{\frac{-1}{2}t} \leqslant \mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right] \leqslant \mathbb{P}\Big[\log(k_\infty^{(-|\nu|)}) \leqslant x + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-|\nu|^2 t/2}\left(\frac{\pi}{2t}\right)^{3/2}}{\nu^2(1-\cos(\pi|\nu|))}$$

- *For* $|\nu| = 2$,

$$\mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x - 2\epsilon^{'}\right] - \frac{(1+\vartheta)}{\epsilon^{'}} \left(\frac{-2|\nu|+2}{-2|\nu|+1}\right) e^{\frac{1}{2}(-2|\nu|+1)t} \leqslant \mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right] \leqslant \mathbb{P}\Big[\log(k_\infty^{(-|\nu|)}) \leqslant x + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon} \frac{1}{\sqrt{(2\pi t)}} e^{-2t}$$

Proof :

i) Suppose that $\nu < 0$:

We aim to find an upper and a lower bound for the cumulative distribution function of $\log(k_t^{(\nu)})$ in terms of the cumulative distribution function of the limit distribution $\log(k_\infty^{(\nu)})$.

- First step: the upper bound for $\mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right]$.

$\forall \epsilon > 0$, we have:

$$\begin{aligned}
&\mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right] = \mathbb{P}\left[\log\left(A_{2,t}^{(\nu)}\right) \leqslant x + \log\left(\left(A_{1,t}^{(\nu)}\right)^2\right)\right] \\
&\leqslant \mathbb{P}\left[\log\left(A_{2,t}^{(\nu)}\right) \leqslant x + \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right)\right] \qquad \text{(As } A_{1,t}^{(\nu)} \text{ is an increasing processes)} \\
&\leqslant \mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) \leqslant x + \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right) + \left(\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(A_{2,t}^{(\nu)}\right)\right)\right] \\
&\leqslant \mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) \leqslant x + \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right) + \epsilon\right] + \mathbb{P}\left[\left(\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(A_{2,t}^{(\nu)}\right)\right) > \epsilon\right] \\
&\leqslant \mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) \leqslant x + \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right) + \epsilon\right] + \frac{1}{\epsilon}\left(\mathbb{E}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(A_{2,t}^{(\nu)}\right)\right]\right)
\end{aligned}$$

The last inequality comes from the Markov inequality.

Therefore:

$$\mathbb{P}\left[\log\left(k_t^{(\nu)}\right) \leqslant x\right] \leqslant \mathbb{P}\left[\log\left(k_\infty^{(\nu)}\right) \leqslant x + \epsilon\right] + \frac{1}{\epsilon}\left(\mathbb{E}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(A_{2,t}^{(\nu)}\right)\right]\right) \quad (4.1.1)$$

In [9], the last term $\left(\mathbb{E}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(A_{2,t}^{(\nu)}\right)\right]\right)$ is well described for large t:

$$\mathbb{E}\left[\log\left(A_{2,\infty}^{(\nu)}\right)\right] - \mathbb{E}\left[\log\left(A_{2,t}^{(\nu)}\right)\right] \underset{t\to\infty}{\sim} \begin{cases} \left(\frac{\nu+2}{\nu+1}\right) e^{2(\nu+1)t} & \text{if } \nu < -2 \\ \frac{1}{\sqrt{(2\pi t)}} e^{-2t} & \text{if } \nu = -2 \\ \frac{1}{\nu^2(1-\cos(-\pi\nu))}\left(\frac{\pi}{2t}\right)^{3/2} e^{-\nu^2 t/2} & \text{if } -2 < \nu < 0 \end{cases}$$

- Second step: the lower bound for $\mathbb{P}\left[\log\left(k_t^{(\nu)}\right) \leqslant x\right]$

Let $\xi \in \mathbb{R}$, we have:

$$\begin{aligned}
&\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(A_{1,\infty}^{(\nu)}\right)^2 \leqslant x + \xi\right] = \\
&\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right) \leqslant x + \xi; \log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right) < \epsilon^{'}\right] \\
&\quad + \mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right) \leqslant x + \xi; \log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right) \geqslant \epsilon^{'}\right] \\
\leqslant &\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) \leqslant 2\log\left(\left(A_{1,t}^{(\nu)}\right)\right) + x + \xi + 2\epsilon^{'}\right] + \mathbb{P}\left[\log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right) \geqslant \epsilon^{'}\right] \\
\leqslant &\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(\left(A_{1,t}^{(\nu)}\right)^2\right) \leqslant x\right] + \frac{1}{\epsilon^{'}}\mathbb{E}\left[\log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right)\right]
\end{aligned}$$

where the last inequality is obtained by choosing $\xi = -2\epsilon^{'}$ and using Markov inequality. Therefore:

$$\begin{aligned}
\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(\left(A_{1,t}^{(\nu)}\right)^2\right) \leqslant x\right] \geqslant &\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(\left(A_{1,\infty}^{(\nu)}\right)^2\right) \leqslant x - 2\epsilon^{'}\right] \\
&- \frac{1}{\epsilon^{'}}\mathbb{E}\Big[\log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right)\Big]
\end{aligned}$$

As $A_{2,t}^{(\nu)}$ is an increasing process, we deduce:

$$\begin{aligned}
\mathbb{P}\left[\log\left(A_{2,t}^{(\nu)}\right) - \log\left(\left(A_{1,t}^{(\nu)}\right)^2\right) \leqslant x\right] \geqslant &\mathbb{P}\left[\log\left(A_{2,\infty}^{(\nu)}\right) - \log\left(\left(A_{1,t}^{(\nu)}\right)^2\right) \leqslant x - 2\epsilon^{'}\right] \\
&- \frac{1}{\epsilon^{'}}\left[\log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right)\right]
\end{aligned}$$

Finally we have the following inequality

$$\mathbb{P}\left[\log\left(k_t^{(\nu)}\right) \leqslant x\right] \geqslant \mathbb{P}\left[\log\left(k_\infty^{(\nu)}\right) \leqslant x - \epsilon^{'}\right] - \frac{2}{\epsilon^{'}}\mathbb{E}\left[\log\left(A_{1,\infty}^{(\nu)}\right) - \log\left(A_{1,t}^{(\nu)}\right)\right] \quad (4.1.2)$$

As above, from [9], the term $\mathbb{E}\Big[\log(A_{1,\infty}^{(\nu)}) - \log(A_{1,t}^{(\nu)})\Big]$ satisfies

$$\mathbb{E}[\log(A_{1,\infty}^{(\nu)})] - \mathbb{E}[\log(A_{1,t}^{(\nu)})] \underset{t\to\infty}{\sim} \begin{cases} \left(\frac{2\nu+2}{2\nu+1}\right) e^{\frac{1}{2}(2\nu+1)t} & \text{if } \nu < -1 \\ \frac{1}{\sqrt{(\frac{1}{2}\pi t)}} e^{\frac{-1}{2}t} & \text{if } \nu = -1 \\ \frac{1}{4\nu^2(1-\cos(-2\pi\nu))}\left(\frac{2\pi}{t}\right)^{3/2} e^{-\nu^2 t/2} & \text{if } -1 < \nu < 0 \end{cases}$$

This completes the proof of proposition 4.1.1 for $\nu < 0$.

ii) In view of proposition 2.3.6, we can repeat the above two steps for $\nu > 0$ by replacing ν by $-\nu$. This ends the proof of proposition 4.1.1.

□

Numerical examples:

We illustrate in figure 4.1 and 4.2 the above result about the convergence of $\mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right]$ to $\mathbb{P}\left[\log(k_\infty^{(\nu)}) \leqslant x\right]$ in the case where $|\nu| > 2$. We plot in green the cumulative distribution function $\mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right]$ for different values of t. We plot in red the cumulative distribution function $\mathbb{P}\left[\log(k_\infty^{(-|\nu|)}) \leqslant x\right]$, we plot in blue the lower bound and in pink the upper bound.

In figure 4.1, we suppose that $\nu = -3$ and we illustrate the cumulative distribution function of $\mathbb{P}\left[\log(k_t^{(\nu)}) \leqslant x\right]$ for different values of t, when we assume that the variable $k_t^{(\nu)}$ with the terminal time $t = 10$ approximates well the the variable $k_\infty^{(\nu)}$. We observe satisfying results from $t \geqslant 2$ as it is showed in figure c) and d).

In figure 4.2, we suppose that $\nu = -5$. We observe satisfying results from $t \geqslant 0.8$ as it is showed in figure a),b),c) and d).

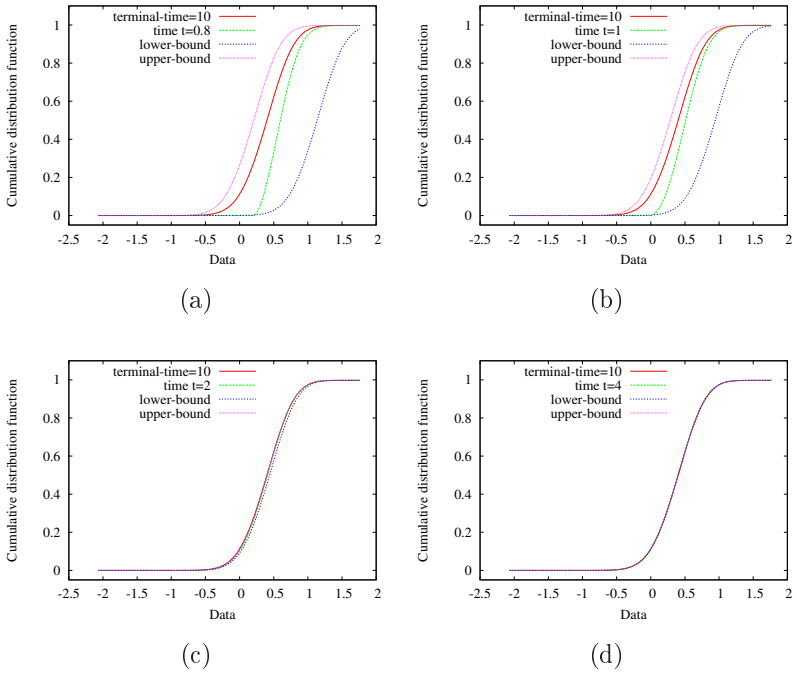


Figure 4.1: Convergence of the cumulative distribution function for $\nu = -3$

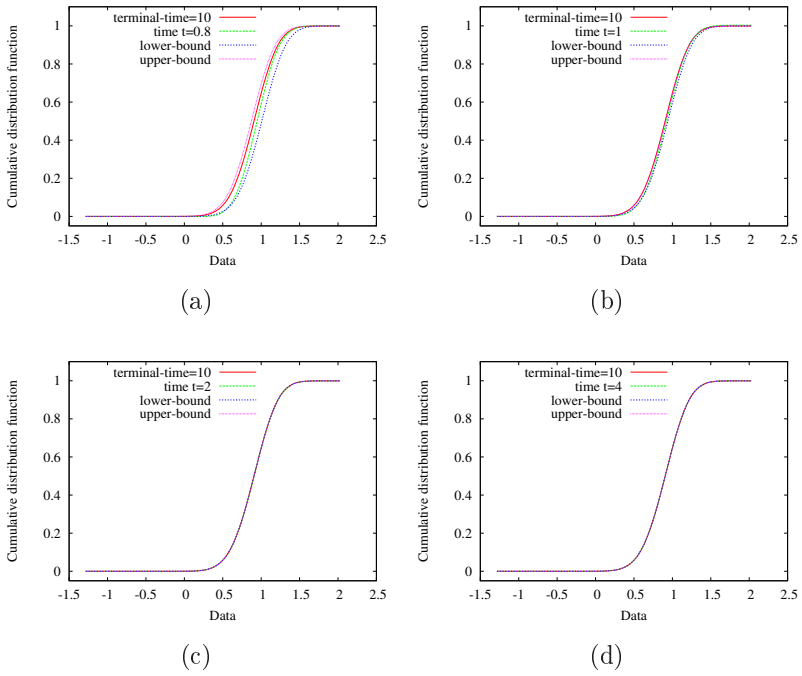


Figure 4.2: Convergence of the cumulative distribution function for $\nu = -5$

We now deduce an upper and lower bounds for the cumulative distribution function of NI_δ.

4.1.2 Upper and lower bounds for the cumulative distribution function of NI_δ

As already noticed,

$$NI_\delta \overset{\mathcal{L}}{=} \sigma^2 k^{(\tilde{\mu})}_{\sigma^2\delta}. \tag{4.1.3}$$

Therefore, from proposition 4.1.1, we can deduce that for a large value of $\sigma^2\delta$ we have:

Corollary 4.1.2. *For $\theta, \epsilon\ \epsilon'$ and $\vartheta \in \mathbb{R}^+$, $\exists t_0 > 0$ such that for $t > t_0$ one has:*

- *For $|\tilde{\mu}| > 2$,*

$$\mathbb{P}\Big[\log(k^{(-|\tilde{\mu}|)}_\infty) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) - 2\epsilon'\Big] - \frac{(1+\vartheta)}{\epsilon'}\left(\frac{-2|\tilde{\mu}|+2}{-2|\tilde{\mu}|+1}\right) e^{\frac{1}{2}(-2|\tilde{\mu}|+1)\sigma^2\delta} \leqslant \mathbb{P}\left[NI_\delta \leqslant \theta\right] \leqslant$$
$$\mathbb{P}\Big[\log(k^{(-|\tilde{\mu}|)}_\infty) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon}\left(\frac{-|\tilde{\mu}|+2}{-|\tilde{\mu}|+1}\right) e^{2(-|\tilde{\mu}|+1)\sigma^2\delta}$$

- *For* $|\tilde{\mu}| < 1$,

$$\mathbb{P}\Big[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) - 2\epsilon^{'}\Big] - \frac{(1+\vartheta)}{\epsilon^{'}} \frac{e^{-|\tilde{\mu}|^2(\sigma^2\delta)/2}\left(\frac{2\pi}{\sigma^2\delta}\right)^{3/2}}{4|\tilde{\mu}|^2(1-\cos(2\pi|\tilde{\mu}|))} \leqslant \mathbb{P}\left[NI_\delta \leqslant \theta\right] \leqslant \mathbb{P}\Big[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-|\tilde{\mu}|^2\sigma^2\delta/2}\left(\frac{\pi}{2\sigma^2\delta}\right)^{3/2}}{|\tilde{\mu}|^2(1-\cos(\pi|\tilde{\mu}|))}$$

- *For* $1 < |\tilde{\mu}| < 2$,

$$\mathbb{P}\left[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) - 2\epsilon^{'}\right] - \frac{(1+\vartheta)}{\epsilon^{'}} \left(\frac{-2|\tilde{\mu}|+2}{-2|\tilde{\mu}|+1}\right) e^{\frac{1}{2}(-2|\tilde{\mu}|+1)\sigma^2\delta} \leqslant \mathbb{P}\left[NI_\delta \leqslant \theta\right] \leqslant \mathbb{P}\Big[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-|\tilde{\mu}|^2\sigma^2\delta/2}\left(\frac{\pi}{2\sigma^2\delta}\right)^{3/2}}{|\tilde{\mu}|^2(1-\cos(\pi|\tilde{\mu}|))}$$

- *For* $|\tilde{\mu}| = 1$,

$$\mathbb{P}\Big[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) - 2\epsilon^{'}\Big] - \frac{(1+\vartheta)}{\epsilon^{'}} \frac{1}{\sqrt{(\frac{1}{2}\pi\sigma^2\delta)}} e^{\frac{-1}{2}\sigma^2\delta} \leqslant \mathbb{P}\left[NI_\delta \leqslant \theta\right] \leqslant \mathbb{P}\left[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) + \epsilon\right] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-|\tilde{\mu}|^2\sigma^2\delta/2}\left(\frac{\pi}{2\sigma^2\delta}\right)^{3/2}}{|\tilde{\mu}|^2(1-\cos(\pi|\tilde{\mu}|))}$$

- *For* $|\tilde{\mu}| = 2$,

$$\mathbb{P}\left[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) - 2\epsilon^{'}\right] - \frac{(1+\vartheta)}{\epsilon^{'}} \left(\frac{-2|\tilde{\mu}|+2}{-2|\tilde{\mu}|+1}\right) e^{\frac{1}{2}(-2|\tilde{\mu}|+1)\sigma^2\delta} \leqslant \mathbb{P}\left[NI_\delta \leqslant \theta\right] \leqslant \mathbb{P}\Big[\log(k_\infty^{(-|\tilde{\mu}|)}) \leqslant \log\left(\frac{\theta}{\sigma^2}\right) + \epsilon\Big] + \frac{(1+\vartheta)}{\epsilon} \frac{e^{-2\sigma^2\delta}}{\sqrt{(2\pi\sigma^2\delta)}}$$

Observation 4.1.3. *First from (4.1.3) we have*

$$NI_\delta \overset{\mathcal{L}}{=} \sigma^2 k_{\sigma^2\delta}^{(\tilde{\mu})}.$$

also from the above numerical examples, we can deduce for example that for $\tilde{\mu} = -3$, *the cumulative distribution function of the indicator* NI *which is equal to* $\mathbb{P}[\sigma^2 k_{\sigma^2\delta}^{(\tilde{\mu})} \leqslant x]$, *converges to* $\mathbb{P}[\sigma^2 k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$ *and the limit is quasi-reached when* $\sigma^2\delta \geqslant 2$.

Similarly, in the case where $\tilde{\mu} = -5$ *we deduce that the cumulative distribution function* $\mathbb{P}[NI_\delta \leqslant x]$ *can be well approximated by* $\mathbb{P}[\sigma^2 k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$ *when* $\sigma^2\delta \geqslant 0.8$.

We are now interested by the variable $k_\infty^{(-|\tilde{\mu}|)}$. We wish to make explicit the density of this variable to deduce the behavior of its cumulative distribution function $\mathbb{P}[k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$ for small x and the behavior of $1 - \mathbb{P}[k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$ for large x.

4.2 The density of the random variable $k_\infty^{(-|\tilde{\mu}|)}$

In this section, we aim to study the probability density function of the random variable $k_\infty^{(-|\tilde{\mu}|)}$. From proposition 2.3.7, the law of this variable has been characterized by its Laplace transform as follows:

$$\forall \lambda > 0, \qquad \mathbb{E}\left[e^{-\lambda k_\infty^{(-|\tilde{\mu}|)}}\right] = \frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})(\cosh(\sqrt{\lambda/2}))^{2|\tilde{\mu}|}} \tag{4.2.1}$$

Then the density of the variable $k_\infty^{(-|\tilde{\mu}|)}$ can be obtained by inverting its Laplace transform. Unfortunately, the formula (4.2.1) is not easy to invert. So we will make explicit the density as the convolution of two densities. Before proving this result in proposition 4.2.6, we need to study the laws of the random variables S_h and C_h defined in [3] by:

$$S_h := \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{h,n}}{n^2} \quad \text{and} \quad C_h := \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{h,n}}{(n-\frac{1}{2})^2},$$

where $(\Gamma_{h,n})_{n\geqslant 0}$ are independent random variables with gamma law of parameter h, that is, the density of $\Gamma_{h,n}$ is given by

$$g(x) = \Gamma(h)^{-1}x^{h-1}e^{-x} \qquad (h>0, x>0).$$

where $\Gamma(h)$ is a normalizing constant.

Remind that the distribution of the gamma variables $\Gamma_{h,n}$ is characterized by the Laplace transform

$$\mathbb{E}\left[e^{-\lambda\Gamma_{h,n}}\right] = (1+\lambda)^{-h} \qquad (\lambda > -1) \tag{4.2.2}$$

and from [8] we have the following Euler's formulae:

$$\sinh(z) = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2\pi^2}\right) \text{ and } \cosh(z) = \prod_{n=1}^{\infty}\left(1+\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right) \tag{4.2.3}$$

Proposition 4.2.1. *The Laplace transforms of S_h and C_h are given for $\lambda > 0$ by*

$$\mathbb{E}\left[e^{-\lambda S_h}\right] = \left(\frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})}\right)^h \quad \textit{and} \quad \mathbb{E}\left[e^{-\lambda C_h}\right] = \frac{1}{\left(\cosh\left(\sqrt{\lambda/2}\right)\right)^h}.$$

Proof :
From the definition of the variable S_h, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda S_h}\right] &= \mathbb{E}\left[\prod_{n=1}^{\infty} e^{-\frac{\lambda\Gamma_{h,n}}{2n^2\pi^2}}\right] \\
&= \prod_{n=1}^{\infty}\left(\mathbb{E}\left[e^{-\frac{\lambda\Gamma_{h,n}}{2n^2\pi^2}}\right]\right) \qquad \left(\text{because } \Gamma_{h,n} \text{ are independent }\right) \\
&= \prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2\pi^2 n^2}\right)^{-h} \qquad \text{(from (4.2.2))} \\
&= \left(\frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})}\right)^{h} \qquad \text{(from (4.2.3))}.
\end{aligned}$$

Similarly for the random variable C_h, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda C_h}\right] &= \mathbb{E}\left[\prod_{n=1}^{\infty} e^{-\frac{\lambda\Gamma_{h,n}}{2\pi^2(n-\frac{1}{2})^2}}\right] \\
&= \prod_{n=1}^{\infty}\left(\mathbb{E}\left[e^{-\frac{\lambda\Gamma_{h,n}}{2\pi^2(n-\frac{1}{2})^2}}\right]\right) \\
&= \prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2\pi^2(n-\frac{1}{2})^2}\right)^{-h} \qquad \text{(from (4.2.2))} \\
&= \frac{1}{\left(\cosh(\sqrt{\lambda/2})\right)^{h}} \qquad \text{(from (4.2.3))}.
\end{aligned}$$

□

The following lemma can be proven from the formula (4.2.1) and using the independence between the random variables S_1 and $C_{2|\tilde{\mu}|}$.

Lemma 4.2.2. *As S_1 and $C_{2|\tilde{\mu}|}$ are independent, then*

$$k_{\infty}^{(-|\tilde{\mu}|)} \overset{\mathcal{L}}{=} S_1 + C_{2|\tilde{\mu}|}$$

Thus from the above lemma, the density function of $k_{\infty}^{(-|\tilde{\mu}|)}$ can be obtained as the convolution between the density of the variable S_1 and the density of $C_{2|\tilde{\mu}|}$.

4.2.1 Probability densities of S_1 and $C_{2|\tilde{\mu}|}$

We denote by f_{S_1} (resp.$f_{C_{2|\tilde{\mu}|}}$) the probability density function of the random variable S_1 (resp.$C_{2|\tilde{\mu}|}$). We aim in this paragraph to make explicit these two densities.

We recall the Lévy's formula in [19]

$$\int_0^\infty e^{-\frac{\lambda^2}{2}t} e^{-\frac{y^2}{2t}} \frac{y}{\sqrt{2\pi t^3}} dt = e^{-\lambda y} \quad (y \geqslant 0), \tag{4.2.4}$$

and the Binomial expansion

$$\frac{1}{(1-x)^h} = \frac{1}{\Gamma(h)} \sum_{n=0}^{\infty} \frac{\Gamma(n+h)}{\Gamma(n+1)} x^n \qquad (h > 0, |x| < 1), \tag{4.2.5}$$

Therefore we can deduce the following expansion formulae which will be useful to compute the densities f_{S_1} and $f_{C_{2|\tilde{\mu}|}}$.

$$\begin{aligned}\left(\frac{x}{\sinh(x)}\right)^h = \frac{2^h x^h}{\left(e^x - e^{-x}\right)^h} = \frac{2^h x^h e^{-hx}}{\left(1 - e^{-2x}\right)^h} &= 2^h x^h e^{-xh} \frac{1}{\Gamma(h)} \sum_{n=0}^{\infty} \frac{\Gamma(n+h)}{\Gamma(n+1)} e^{-2nx} \\ &= \frac{2^h x^h}{\Gamma(h)} \sum_{n=0}^{\infty} \frac{\Gamma(n+h)}{\Gamma(n+1)} e^{-(2n+h)x}\end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned}\left(\frac{1}{\cosh(x)}\right)^h = \frac{2^h}{\left(e^x + e^{-x}\right)^h} = \frac{2^h e^{-hx}}{\left(1 + e^{-2x}\right)^h} &= 2^h e^{-xh} \frac{1}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} e^{-2nx} \\ &= \frac{2^h}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} e^{-(2n+h)x}\end{aligned} \tag{4.2.7}$$

Remark that from the formula (4.2.4), we obtain:

$$\int_0^\infty p_t(y) e^{-\frac{\lambda^2}{2}t} dt = \lambda e^{-\lambda y} \qquad (y \geqslant 0), \tag{4.2.8}$$

where $p_t(y)$ is given by:

$$p_t(y) = \frac{\partial^2}{\partial y^2} \left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \right). \tag{4.2.9}$$

Now we are in a position to invert the Laplace transforms of $C_{2|\tilde{\mu}|}$ and S_1.

Proposition 4.2.3. *The density $f_{C_{2|\tilde{\mu}|}}$ is given by*

$$f_{C_{2|\tilde{\mu}|}}(t) = \frac{2^{2|\tilde{\mu}|}}{\Gamma(2|\tilde{\mu}|)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2|\tilde{\mu}|)}{\Gamma(n+1)} \frac{(2n+2|\tilde{\mu}|)}{2\sqrt{2\pi t^3}} e^{-\frac{((2n+2|\tilde{\mu}|)/2)^2}{2t}}$$

Proof :

From proposition 4.2.1 we have

$$\mathbb{E}\left[e^{-\lambda C_{2|\tilde{\mu}|}}\right] = \frac{1}{\left(\cosh\left(\sqrt{\lambda/2}\right)\right)^{2|\tilde{\mu}|}}.$$

So by applying the expansion formula (4.2.7), we obtain that

$$\mathbb{E}\left[e^{-\lambda C_{2|\tilde{\mu}|}}\right] = \frac{2^{2|\tilde{\mu}|}}{\Gamma(2|\tilde{\mu}|)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2|\tilde{\mu}|)}{\Gamma(n+1)} e^{-(2n+2|\tilde{\mu}|)\sqrt{\lambda/2}}. \tag{4.2.10}$$

Finally the density $f_{C_{2|\tilde{\mu}|}}$ can be obtained by inverting its Laplace transform in (4.2.10) using Lévy formula (4.2.4):

$$\begin{aligned}
f_{C_{2|\tilde{\mu}|}}(t) &= \Phi_\lambda^{-1}\left(\frac{1}{\left(\cosh\left(\sqrt{\lambda/2}\right)\right)^{2|\tilde{\mu}|}}\right)(t) \\
&= \Phi_\lambda^{-1}\left(\frac{2^{2|\tilde{\mu}|}}{\Gamma(2|\tilde{\mu}|)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2|\tilde{\mu}|)}{\Gamma(n+1)} e^{-(2n+2|\tilde{\mu}|)\sqrt{\lambda/2}}\right)(t) \\
&= \frac{2^{2|\tilde{\mu}|}}{\Gamma(2|\tilde{\mu}|)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2|\tilde{\mu}|)}{\Gamma(n+1)} \Phi_\lambda^{-1}\left(e^{-(2n+2|\tilde{\mu}|)\sqrt{\lambda/2}}\right)(t) \\
&= \frac{2^{2|\tilde{\mu}|}}{\Gamma(2|\tilde{\mu}|)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2|\tilde{\mu}|)}{\Gamma(n+1)} \frac{(2n+2|\tilde{\mu}|)}{2\sqrt{2\pi t^3}} e^{-\frac{((2n+2|\tilde{\mu}|)/2)^2}{2t}}
\end{aligned}$$

where Φ_λ^{-1} denotes the inverse Laplace transform defined by:

$$\Phi_\lambda^{-1}(f(\lambda))(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\chi-iT}^{\chi+iT} e^{\lambda t} f(\lambda) d\lambda.$$

□

We now aim to make explicit the density of f_{S_1}. Similarly to the above argument, the density f_{S_1} can be obtained by inverting its Laplace transform as follows:

From proposition 4.2.1 we have

$$\mathbb{E}\left[e^{-\lambda S_1}\right] = \left(\frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})}\right).$$

By applying the expansion formula (4.2.6), we obtain that

$$\mathbb{E}\left[e^{-\lambda S_1}\right] = \left(\frac{\sqrt{\lambda/2}}{\sinh(\sqrt{\lambda/2})}\right) = 2\sum_{n=0}^{\infty}\sqrt{\lambda/2}e^{-(2n+1)\sqrt{\lambda/2}} \tag{4.2.11}$$

Then the density f_{S_1} can be obtained as follows:

$$\begin{aligned} f_{S_1}(t) &= \Phi_\lambda^{-1}\left(2\sum_{n=0}^{\infty}\sqrt{\lambda/2}e^{-(2n+1)\sqrt{\lambda/2}}\right)(t) \\ &= \sum_{n=0}^{\infty}\Phi_\lambda^{-1}\left(\sqrt{2\lambda}e^{-\frac{(2n+1)}{2}\sqrt{2\lambda}}\right)(t) \\ &= \sum_{n=0}^{\infty} p_t\left(\frac{2n+1}{2}\right) \end{aligned} \tag{4.2.12}$$

The last equality is deduced from (4.2.8), where p_t is as in (4.2.9).

Remark that $H(t,y) := \frac{1}{\sqrt{2\pi t}}\exp(-(y^2/2t))$ satisfies the heat equation

$$\frac{\partial}{\partial t}H(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}H(t,y).$$

Then

$$f_{S_1}(t) = 2\sum_{n=0}^{\infty}\frac{\partial}{\partial t}H(t,(2n+1)/2)$$

$$= -\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=0}^{\infty}e^{-\frac{(n+\frac{1}{2})^2}{2t}} + \left(\frac{1}{2\pi t^3}\right)^{1/2}\frac{(n+\frac{1}{2})^2}{t}\sum_{n=0}^{\infty}e^{-\frac{(n+\frac{1}{2})^2}{2t}}$$

$$= -\frac{1}{2}\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=-\infty}^{\infty}e^{-\frac{(n+\frac{1}{2})^2}{2t}} + \frac{1}{2t}\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=-\infty}^{\infty}\left(n+\frac{1}{2}\right)^2 e^{-\frac{(n+\frac{1}{2})^2}{2t}} \tag{4.2.13}$$

Unfortunately, this formula is not propitious to mathematical analysis. We therefore obtain a simple formula for f_{S_1} by using a result obtained in [3], which proves a relation between the densities f_{S_1} and f_{C_2}. The proof of this result relies on the fact that the densities of C_h and S_h for $h = 1; 2$ can be also given by infinite series related to the derivatives of Jacobi's theta function.

Let us start with the Jacobi's theta functions

$$\theta(t) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi t} \quad , \quad \theta_0(t) := \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi n^2 t} \tag{4.2.14}$$

and

$$\theta_1(t) := \sum_{n=-\infty}^{\infty} e^{-\pi(n+1/2)^2 t} \tag{4.2.15}$$

We recall the Jacobi's theta function identity (Poisson summation formula):

$$\frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} e^{-(n+x)^2/t} = \sum_{n=-\infty}^{\infty} \cos(2n\pi x) e^{-n^2\pi^2 t} \qquad (x \in \mathbb{R}, t > 0). \tag{4.2.16}$$

Remark that by replacing x by $1/2$ and t *by* $= t/\pi$ in equation (4.2.16) we have the identity

$$\theta_1(1/t) = \sqrt{t}\ \theta_0(t) \tag{4.2.17}$$

Now we have the infinite series formulas for the densities f_{S_1} and f_{C_2}.

Proposition 4.2.4. *The densities f_{S_1} and f_{C_2} are given by*

$$f_{S_1}(t) = \frac{d}{dt} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi^2 n^2 2t} = 2\pi\ \theta_0'(2t\pi) \tag{4.2.18}$$

and

$$f_{C_2}(t) = \sum_{n=-\infty}^{\infty} \left(4\pi^2 \left(n + \frac{1}{2} \right)^2 t - 1 \right) e^{-2\left(n+\frac{1}{2}\right)^2 \pi^2 t} \tag{4.2.19}$$

Proof :

From (4.2.13), we have:

$$f_{S_1}(t) = -\frac{1}{2}\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=-\infty}^{\infty} e^{-\frac{(n+\frac{1}{2})^2}{2t}} + \frac{1}{2t}\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=-\infty}^{\infty}\left(n+\frac{1}{2}\right)^2 e^{-\frac{(n+\frac{1}{2})^2}{2t}}$$

On the other hand we have:

$$\begin{aligned}
2\pi\theta_0'(2t\pi) &= \frac{d}{dt}\left(\theta_0(2t\pi)\right)\\
&= \frac{d}{dt}\left(\left(\frac{1}{2t\pi}\right)^{1/2}\theta_1\left(\frac{1}{2t\pi}\right)\right) \qquad \text{(from 4.2.17)}\\
&= -\frac{1}{2}\left(\frac{1}{2\pi t^3}\right)^{1/2}\theta_1\left(\frac{1}{2t\pi}\right) - \frac{1}{2\pi t}\left(\frac{1}{2\pi t^3}\right)^{1/2}\theta_1'\left(\frac{1}{2t\pi}\right)\\
&= -\frac{1}{2}\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=-\infty}^{\infty} e^{-\frac{(n+\frac{1}{2})^2}{2t}} + \frac{1}{2t}\left(\frac{1}{2\pi t^3}\right)^{1/2}\sum_{n=-\infty}^{\infty}\left(n+\frac{1}{2}\right)^2 e^{-\frac{(n+\frac{1}{2})^2}{2t}}
\end{aligned}$$

Therefore (4.2.18) is deduced.

Similarly from proposition 4.2.3 and by using the same calculus as above, we can deduce that:

$$f_{C_2}(t) = \sum_{n=-\infty}^{\infty}\left(4\pi^2\left(n+\frac{1}{2}\right)^2 t - 1\right)e^{-2\left(n+\frac{1}{2}\right)^2\pi^2 t}$$

□

Finally we have the relation between f_{S_1} and f_{C_2}.

Corollary 4.2.5. *The densities f_{S_1} and f_{C_2} satisfy the following relation:*

$$f_{S_1}(t) = \left(\frac{1}{8\pi t^3}\right)^{1/2} f_{C_2}\left(\frac{1}{4\pi^2 t}\right)$$

Proof :
From proposition 4.2.4 and using the Jacobi's theta identity (4.2.16), $f_{S_1}(t)$ can be written as:

$$f_{S_1}(t) = \frac{d}{dt}\left(\left(\frac{1}{2\pi t}\right)^{1/2}\sum_{n=-\infty}^{\infty} e^{-\frac{\left(n+\frac{1}{2}\right)^2}{2t}}\right) \tag{4.2.20}$$

Therefore from (4.2.20) and (4.2.19), we can deduce the above relation. □

Now the density f of the random variable $k_\infty^{(-|\tilde{\mu}|)}$ can be obtained explicitly as:

Proposition 4.2.6. *The density f of the random variable $k_\infty^{(\tilde{\mu})}$ is given by*

$$f(t) = \left(f_{S_1} * f_{C_{2|\tilde{\mu}|}}\right)(t) \tag{4.2.21}$$

where

$$f_{S_1}(t) = 4\pi^2 \sum_{n=0}^{\infty} (-1)^n (n+1)^2 e^{-2(n+1)^2\pi^2 t} \tag{4.2.22}$$

and

$$f_{C_{2|\tilde{\mu}|}}(t) = \frac{2^{2|\tilde{\mu}|}}{\Gamma(2|\tilde{\mu}|)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2|\tilde{\mu}|)}{\Gamma(n+1)} \frac{(2n+2|\tilde{\mu}|)}{2\sqrt{2\pi t^3}} e^{-\frac{(n+|\tilde{\mu}|)^2}{2t}} \tag{4.2.23}$$

Proof :

The result follows from the identity in law in lemma 4.2.2.

The formula (4.2.22) is deduced from corollary 4.2.5 and proposition 4.2.3.

□

The formula (4.2.21) is not propitious to mathematical study of the behavior of the cumulative distribution function of $k_\infty^{(-|\tilde{\mu}|)}$. We thus now use a result due to Yor [34, p.370] on the joint law of $\left(\frac{1}{4}A_{2,\infty}^{(\nu)}, A_{1,\infty}^{(\nu)}\right)$ for $\nu < 0$.

4.3 The cumulative distribution function of $k_\infty^{(-|\tilde{\mu}|)}$

We use in this section two important results proved by Yor, in order to prove our result in theorem 4.3.9. The first one concerns the identity in law in theorem 4.3.3, and the second one is the extension of the classical Ray-Knight formula to the perturbed Brownian motion X^γ (see theorem 4.3.4).

4.3.1 Reflecting Brownian motion perturbed by its local time at zero

Let (B_t) denote a one-dimensional Brownian motion started from 0 and $(l_t(B), t \geqslant 0)$ its local time process at level 0. For fixed $\gamma > 0$, the perturbed reflecting Brownian motion (X_t^γ) is defined for all $t \geqslant 0$ by

$$X_t^\gamma = \mid B_t \mid -\gamma l_t(B)$$

where we have the following notations:

- $(l_t(B), t \geqslant 0)$ is the local time process of B at level 0,
- $(l_t^x(X^\gamma), t \geqslant 0)$ is the local time process of X^γ at level x,
- $(\tau_u^\gamma,\ u \geqslant 0)$ is the right continuous inverse of the local time at 0, $(l_t^0(X^\gamma),\ t \geqslant 0)$ of X^γ.

Definition 4.3.1. *(Bessel processes)*
For every $\rho \geqslant 0$ and $x \geqslant 0$, the unique strong solution to the equation

$$Y_t = x + \rho t + 2\int_0^t \sqrt{Y_s} dB_s$$

is called the square of a ρ-dimensional Bessel process started at x and is denoted by $BESQ^\rho(x)$

Remark 4.3.2. *Denote the law of $BESQ^\rho(x)$ by Q_x^ρ. We call the number ρ the dimension of $BESQ^\rho$, and $\nu = \rho/2 - 1$ is called the index of the process $BESQ^\rho$. Also we denote by $BESQ_t^\rho(x,y)$ the Bessel square bridge of dimension $\rho > 0$ from x to y on $[0,t]$. The law of $BESQ_t^\rho(x,y)$ denoted by $Q_t^\rho(x,y)$ is viewed as the Q_x^ρ conditional distribution of $(Y_u, 0 \leqslant u \leqslant t)$ given $Y_t = y$, i.e:*

$$Q_t^\rho(x,y) = Q_x^\rho\left[Y | L_y = t\right] \tag{4.3.1}$$

where $L_y = \sup\{u : Y_u = y\}$.

We now state the following result due to Yor [34, p.370] on the joint law of $\left(\frac{1}{4}A_{2,\infty}^{(\nu)}, A_{1,\infty}^{(\nu)}\right)$ for $\nu < 0$.

Theorem 4.3.3. *Let $\nu < 0$ and define $\gamma = -\frac{1}{2\nu}$.*
Then the following identity in law holds:

$$\left(\tfrac{1}{4}A_{2,\infty}^{(\nu)}, A_{1,\infty}^{(\nu)}\right) \overset{\mathcal{L}}{=} \left(\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_u^\gamma \leqslant 0)} du, \gamma l_{\tau_1^\gamma}(B)\right)$$

A second important result due to Yor [33, p.118] concerns the extended Ray-Knight theorem for the case of perturbed Brownian motion X^γ. This result shows that the local time of X^γ in the space variable up to time τ_s^γ is a Bessel square process of dimension that depends on γ.

Theorem 4.3.4. ***(Ray-Knight Theorem)***

Fix $s > 0$. The processes $(l^x_{\tau_s^\gamma}(X^\gamma),\ x \geqslant 0)$ and $(l^{-x}_{\tau_s^\gamma}(X^\gamma),\ x \geqslant 0)$ are independent, and their respective laws are Q_s^0 and $Q_s^{2-\frac{2}{\gamma}}$, where $Q_s^{2-\frac{2}{\gamma}}$ denotes the law of the square of the Bessel process starting from s, with dimension $2-\frac{2}{\gamma}$, and absorbed at 0.

We now state some lemmas which will be important to prove our result in proposition 4.3.9.

Lemma 4.3.5.

$$-\gamma l_{\tau_s^\gamma}(B) = \inf\{X_u^\gamma,\ u \leqslant \tau_s^\gamma\} \tag{4.3.2}$$

Proof :
Let t be fixed and $g_t = \sup\{s \leqslant t / B_s = 0\}$. Then $B_s \neq 0$ for $s \in (g_t, t)$.

As the local time of B at level 0 is constant until (B_t) hits 0, we can deduce that for all t we have:

$$l_{g_t}(B) = l_t(B)$$

As the above equality is true for all t, so we have $l_{g_{\tau_s^\gamma}}(B) = l_{\tau_s^\gamma}(B)$. Then

$$B_{g_{\tau_s^\gamma}} = 0 \Rightarrow X_{g_{\tau_s^\gamma}} = -\gamma l_{\tau_s^\gamma}(B).$$

For $u \in [0, \tau_s^\gamma]$, we have $X_u^\gamma \geqslant -\gamma l_u(B) \geqslant -\gamma l_{\tau_s^\gamma}(B)$. Then

$$\inf\{X_u^\gamma,\ u \leqslant \tau_s^\gamma\} = -\gamma l_{\tau_s^\gamma}(B)$$

□

The following two lemmas are given in [24]:

Lemma 4.3.6. *Assume $\rho < 2$. Let Z be $BESQ^\rho(x)$ for $x > 0$ and let T_0 be the first hitting time of 0 : $T_0 = \inf\{s | Z_s = 0\}$. Conditionally on $\{T_0 = t\}$, the process $(Z_s, 0 \leqslant s \leqslant t)$ is a $BESQ_t^{4-\rho}(x, 0)$.*

Lemma 4.3.7. *The random variables $M = -\inf\{X_t^\gamma | 0 \leqslant t \leqslant \tau_x^\gamma\}$ and $T_0 = \inf\{t | Y_t = 0\}$ where Y is $BESQ^{2-2/\gamma}(x)$ are equal in law.*

Let T_0 be the first hitting time of 0 by $(l_{\tau_s^\gamma}^{-y}(X^\gamma), y \geqslant 0)$.

Proposition 4.3.8. *Conditionally on* $\{T_0 = t\}$ *the law of* $\left(l_{\tau_s^\gamma}^{y-t};\ 0 \leqslant y \leqslant t\right)$ *is* $Q_t^{2+\frac{2}{\gamma}}(0,s)$.

Proof :
From theorem 4.3.4, $(l_{\tau_s^\gamma}^{-y}(X^\gamma), y \geqslant 0)$ is $BESQ_s^{2-\frac{2}{\gamma}}$. Then from lemma 4.3.6, conditionally on $\{T_0 = t\}$ the process $(l_{\tau_s^\gamma}^{-y}(X^\gamma), y \geqslant 0)$ is a $BESQ_t^{2+\frac{2}{\gamma}}(s,0)$. To conclude the proof we observe that the process $(l_{\tau_s^\gamma}^{-(t-y)}(X^\gamma), 0 \leqslant y \leqslant t)$ is also Bessel square bridge but s and 0 interchanged. Therefore $(l_{\tau_s^\gamma}^{y-t}(X^\gamma), 0 \leqslant y \leqslant t) := (l_{\tau_s^\gamma}^{-(t-y)}(X^\gamma), 0 \leqslant y \leqslant t)$ is $BESQ_t^{2+\frac{2}{\gamma}}(0,s)$ with law $Q_t^{2+\frac{2}{\gamma}}(0,s)$. □

Theorem 4.3.9. *Define* $\gamma = \frac{1}{2|\tilde{\mu}|}$*; then one has*

$$\mathbb{P}\left[k_\infty^{(-|\tilde{\mu}|)} \leqslant u\right] = \int_0^\infty \int_0^{\gamma^2 b^2 u/4} h(z)dz.q(b)db \tag{4.3.3}$$

where

$$h(x) = 2(\gamma b)^{1+\frac{1}{\gamma}} e^{\frac{1}{2\gamma b}} \Phi_\lambda^{-1}\left[\frac{(2\lambda)^{(\frac{1}{2}+\frac{1}{2\gamma})}}{\left(\sinh(\gamma b\sqrt{2\lambda})\right)^{1+\frac{1}{\gamma}}} \exp\left(-\frac{1}{2}\frac{\sqrt{2\lambda}\ \cosh(\gamma b\sqrt{2\lambda})}{\sinh(\gamma b\sqrt{2\lambda})}\right)\right](x)\,,$$

and

$$q(x) = \frac{1}{2\gamma\Gamma(1/\gamma)} x^{-\frac{1}{\gamma}-1} e^{-\frac{1}{x}}.$$

Φ_λ^{-1} *denotes the inverse Laplace transform.*

Proof :
In the following, we denote by q the density function of $l_{\tau_1^\gamma}$.

$$\begin{aligned}
\mathbb{P}\left[k_\infty^{(-|\tilde{\mu}|)} \leqslant u\right] &= \mathbb{P}\left[\frac{A_{2,\infty}^{(-|\tilde{\mu}|)}}{(A_{1,\infty}^{(-|\tilde{\mu}|)})^2} \leqslant u\right] \\
&= \mathbb{P}\left[\frac{4\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_v^\gamma \leqslant 0)} dv}{(\gamma l_{\tau_1^\gamma})^2} \leqslant u\right] \qquad \text{(from theorem 4.3.3)} \\
&= \int_0^\infty \mathbb{P}\left[\frac{4}{\gamma^2 b^2}\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_v^\gamma \leqslant 0)} dv \leqslant u \mid l_{\tau_1^\gamma}(B) = b\right].q(b)db \\
&= \int_0^\infty \mathbb{P}\left[\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_v^\gamma \leqslant 0)} dv \leqslant \frac{\gamma^2 b^2 u}{4} \mid l_{\tau_1^\gamma}(B) = b\right].q(b)db \qquad (4.3.4)
\end{aligned}$$

- First step: We compute $\mathbb{P}\left[\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_v^\gamma \leqslant 0)} dv \leqslant \frac{\gamma^2 b^2 u}{4} \mid l_{\tau_1^\gamma}(B) = b\right]$:

  By the occupation times formula, one finds that

$$\begin{aligned}
&\mathbb{P}\left[\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_v^\gamma \leqslant 0)} dv \leqslant \frac{\gamma^2 b^2 u}{4} \mid l_{\tau_1^\gamma}(B) = b\right] = \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_{-\infty}^0 l^y_{\tau_1^\gamma}(X^\gamma) dy \leqslant u \mid l_{\tau_1^\gamma}(B) = b\right] \\
&= \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_{\inf\limits_{0\leqslant v\leqslant \tau_1^\gamma}(X_v^\gamma)}^0 l^y_{\tau_1^\gamma}(X^\gamma) dy \leqslant u \mid l_{\tau_1^\gamma}(B) = b\right] \\
&= \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_{-\gamma b}^0 l^y_{\tau_1^\gamma}(X^\gamma) dy \leqslant u \mid l_{\tau_1^\gamma}(B) = b\right] \quad \text{(from lemma.4.3.2)} \\
&= \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_0^{\gamma b} l^{(-y)}_{\tau_1^\gamma}(X^\gamma) dy \leqslant u \mid l_{\tau_1^\gamma}(B) = b\right] \\
&= \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_0^{\gamma b} l^{(-y)}_{\tau_1^\gamma}(X^\gamma) dy \leqslant u \mid T_0 := \inf\{x \geqslant 0, l^{-x}_{\tau_1^\gamma}(X^\gamma) = 0\} = \gamma b\right] \\
&= \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_0^{\gamma b} l^{(z-\gamma b)}_{\tau_1^\gamma}(X^\gamma) dz \leqslant u \mid T_0 = \gamma b\right] \quad \text{(change of variable } z = \gamma b - y \text{)} \\
&= \mathbb{P}\left[\frac{4}{\gamma^2 b^2} \int_0^{\gamma b} Y_z dz \leqslant u \mid Y_{\gamma b} = 1\right], \qquad (4.3.5)
\end{aligned}$$

  where Y is the square of $(2 + \frac{2}{\gamma})$-dimensional Bessel process started at 0. From proposition 4.3.8, conditionally on $T_0 = \gamma b$, the process $(l^{(z-\gamma b)}_{\tau_1^\gamma}(X^\gamma), 0 \leqslant z \leqslant \gamma b)$ is a Bessel bridge $Q^{2+\frac{2}{\gamma}}_{\gamma b}(0, 1)$. Then the last equality (4.3.5) is deduced from remark 4.3.2.

  So finally, from (4.3.5) we obtain that

$$\begin{aligned}
\mathbb{P}\left[\int_0^{\tau_1^\gamma} \mathbb{1}_{(X_v^\gamma \leqslant 0)} dv \leqslant \frac{\gamma^2 b^2 u}{4} \mid l_{\tau_1^\gamma}(B) = b\right] &= \int_0^{\gamma^2 b^2 u/4} f_{\int_0^{\gamma b} Y_v dv}(z \mid Y_{\gamma b} = 1) dz \\
&= \int_0^{\gamma^2 b^2 u/4} \frac{f_{(\int_0^{\gamma b} Y_v dv, Y_{\gamma b})}(z, 1)}{f_{Y_{\gamma b}}(1)} dz \qquad (4.3.6)
\end{aligned}$$

  where

– $f_{\int_0^{\gamma b} Y_v dv}(z \mid Y_{\gamma b} = 1)$ is the conditional probability density function of $\int_0^{\gamma b} Y_v dv$ given the value 1 of $Y_{\gamma b}$.

– $f_{(\int_0^{\gamma b} Y_v dv, Y_{\gamma b})}(z, 1)$ gives the joint density of $\int_0^{\gamma b} Y_v dv$ and $Y_{\gamma b}$, and $f_{Y_{\gamma b}}(1)$ gives the marginal density for $Y_{\gamma b}$.

□

From lemma 4.6.15 and equation (4.6.12) in Appendix B, we obtain

$$f_{Y_{\gamma b}}(1) = \frac{(\gamma b)^{-1-\frac{1}{\gamma}} e^{-\frac{1}{2\gamma b}}}{2\Gamma(\frac{1}{\gamma}+1)}. \tag{4.3.7}$$

In addition, from lemma 4.6.15, we obtain

$$\begin{aligned} f_{(\int_0^{\gamma b} Y_z dz, Y_{\gamma b})}(z,1) &= \lim_{x\to 0} \frac{1}{x^{\frac{1}{\gamma}}} i s_z \left(\frac{1}{\gamma}, \gamma b, 0, (x^2+1)/2, x/2\right) \\ &= \frac{1}{\Gamma(\frac{1}{\gamma}+1)} \Phi_\lambda^{-1}\left[\frac{(2\lambda)^{(\frac{1}{2}+\frac{1}{2\gamma})}}{\left(\sinh(\gamma b\sqrt{2\lambda})\right)^{1+\frac{1}{\gamma}}} \exp\left(-\frac{1}{2}\frac{\sqrt{2\lambda}\,\cosh(\gamma b\sqrt{2\lambda})}{\sinh(\gamma b\sqrt{2\lambda})}\right)\right](z) \end{aligned} \tag{4.3.8}$$

where Φ_λ^{-1} denotes the inverse Laplace transform. The equation (4.3.8) is deduced from the linearity of the inverse Laplace transform and from the dominated convergence theorem.

Finally, from (4.3.4) and (4.3.6) we obtain equality (4.3.3).

Remark 4.3.10. *From proposition 4.2.6 and theorem 4.3.4, the density and the cumulative distribution function of the random variable $k_\infty^{(-|\tilde{\mu}|)}$ are given explicitly. But unfortunately, the explicit formulae are not propitious to study the behavior of the cumulative distribution function $\mathbb{P}[k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$. In next section, we aim to use Tauberian theory which enables us to find connections between the behavior properties of the Laplace transform at infinity (resp.at zero) and the behavior properties of the corresponding cumulative distribution function near zero (resp.near infinity).*

4.4 Tails behavior of the distribution of $k_\infty^{(-|\tilde{\mu}|)}$

We are now interested to use Tauberian theory to study the behavior of the Laplace transform of $k_\infty^{(-|\tilde{\mu}|)}$ for large λ (respectively for small λ), in order to deduce the behavior of $\mathbb{P}[k_\infty^{-|\tilde{\mu}|} \leqslant x]$ for small x (respectively the behavior of $1 - \mathbb{P}[k_\infty^{-|\tilde{\mu}|} \leqslant x]$ for large x).

Before proving our main result, let us remind some definitions and results concerning Tauberian theory. We refer to [4].

4.4.1 Laplace transform and Tauberian theorems

Definition 4.4.1. *Let μ be a measure on $[0, \infty)$ and finite on bounded sets. The Laplace transform $\hat{\mu}(\lambda)$ of μ is the real-valued function defined for $\lambda \geqslant c$ by*

$$\hat{\mu}(\lambda) := \int_0^\infty e^{-\lambda x} \mu(dx),$$

where $c = \inf \left\{ \lambda \in \mathbb{R} : \int_0^\infty e^{-\lambda x} \mu(dx) < \infty \right\}$

Notice that in the case where μ is a finite measure, then $\hat{\mu}$ is defined at least for all $\lambda \geqslant 0$.

This section is decomposed in two parts: In the first part I), we recall Bruijn's Tauberian theorem, which proves that the behavior of the Laplace transform $\hat{\mu}(\lambda)$ for large λ is limited to the behavior of $\mu[0, x]$ for small x. In the second part II), we state two Tauberian results: Karamata's Tauberian theorem 4.4.4 and analytic Tauberian theorem 4.4.7. These two results can be used to obtain the behavior of $\mu[x, \infty)$ for large x.

I) **Bruijn's Tauberian Theorem**

In this section, $R_\alpha(0+)$ denotes the class of regularly varying functions at origin with index α (see definition 4.6.1 in Appendix B). And $\overleftarrow{f}$ denotes the generalized inverse of f given by

$$\overleftarrow{f}(x) := \sup\{t : f(t) > x\}$$

The following theorem is given in [4, p.254].

Theorem 4.4.2. *(Bruijn's Tauberian theorem).*

Let μ be a measure on $(0,\infty)$ and finite on bounded sets, whose Laplace transform

$$\tilde{\mu}(\lambda) := \int_0^\infty e^{-\lambda x} d\mu(x)$$

is well defined for all $\lambda > 0$. If $\alpha < 0$, $\phi \in R_\alpha(0+)$, put $\psi(\lambda) = \frac{\phi(\lambda)}{\lambda} \in R_{\alpha-1}(0+)$; then for $B > 0$,

$$-\log \mu(0,x] \underset{x\to 0+}{\sim} \frac{B}{\overleftarrow{\phi}(1/x)} \tag{4.4.1}$$

if and only if

$$-\log \tilde{\mu}(\lambda) \underset{\lambda\to\infty}{\sim} \frac{(1-\alpha)(\frac{B}{-\alpha})^{\frac{\alpha}{\alpha-1}}}{\overleftarrow{\psi}(\lambda)} \tag{4.4.2}$$

Remark 4.4.3. *Notice that in the case where $\mu = \mathbb{P} \circ X^{-1}$ is the distribution of a non-negative random variable X, then $\hat{\mu}(\lambda)$ is also referred as the Laplace transform of X,*

$$\hat{\mu}(\lambda) = \mathbb{E}\left[e^{-\lambda X}\right],$$

and

$$\mu(0,x] = \mathbb{P}[X \leqslant x].$$

II) Karamata's and analytic Tauberian Theorems

Details of this part are provided in paragraph 4.6, appendix B.

Let X be a positive random variable with distribution function F and denote by $\hat{F}$ its Laplace transform. Then we have the following result:

Theorem 4.4.4. *(Karamata's Tauberian theorem)*

Let l be a slowly varying at infinity (see definition 4.6.2 in Appendix B), then

a) For $0 \leqslant \alpha < 1$, the following are equivalent

$$i)\ 1 - \hat{F}(\lambda) \sim \lambda^\alpha l(1/\lambda) \qquad (\lambda \to 0)$$

$$ii)\ 1 - F(x) \sim \frac{l(x)}{x^\alpha \Gamma(1-\alpha)} \qquad (x \to \infty)$$

b) For $\alpha = 1$, the following are equivalent

$$\begin{aligned} &i)\ 1 - \hat{F}(\lambda) \sim \lambda^{\alpha} l(1/\lambda) && (\lambda \to 0) \\ &ii)\ \int_0^x (1 - F(t))dt \sim l(x) && (x \to \infty) \end{aligned}$$

Proof :
The proof is given in Appendix B, section 4.6. □

Remark 4.4.5. ***Interpretation*** *The interpretation of this theorem lies directly to the argument if "the first moment is finite", that is the Laplace transform is differentiable at the origin (see remark 4.6.11). So in the case when the Laplace transform is differentiable e at the origin, $\hat{F}(s)$ may be expanded in a Taylor series and then near to the origin, $1 - \hat{F}(\lambda)$ is equivalent to $A\lambda + o(\lambda)$ (A is a constant). Then α is equal to 1 in the above theorem, therefore we can deduce from b) an information concerning the integrated tail of F but we don't have an information concerning the tail behavior of F.*

In appendix B (paragraph 4.6), we consider the case of the exponential law in order to understand the limitation of this theorem.

We now present the analytic Tauberian theorem which describes the asymptotic behavior of the tail probability based on analytic properties of the Laplace-Stieltjes transform. In [23] and [22], Nakagawa has proved that if the abscissa of convergence of the Laplace-Stieltjes transform is negative and the singularities of this transform on the axis of convergence are only a finite number of poles, then the tail probability decays exponentially fast with a constant related to the abscissa of convergence.

Let X be a non-negative random variable with probability distribution function

$$F(x) = \mathbb{P}\left[X \leqslant x\right].$$

The Laplace-Stieltjes transform Φ of the distribution function F is defined by:

$$\Phi(s) := \int_0^{\infty} e^{-sx} dF(x), \tag{4.4.3}$$

where $s \in \mathbb{C}$.

Definition 4.4.6. *The abscissa of convergence of $\Phi(s)$ is defined as the real number a_0 such that the integral (4.4.3) converges for $Re(s) > a_0$ and diverges for $Re(s) < a_0$. The line $Re(s) = a_0$ is called the axis of convergence of $\Phi(s)$.*

Now we present the main theorem in Nakagawa [22]:

Theorem 4.4.7. *(Analytic Tauberian theorem)*
If $-\infty < a_0 < 0$ and the singularities of $\Phi(s)$ on the axis of convergence $Re(s) = a_0$ are only a finite number of poles, then we have:

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}[X > x] = a_0$$

Remark 4.4.8. *The assumption of theorem 4.4.7 implies that there exists an open neighborhood U of $Re(s) = a_0$ such that $\Phi(s)$ is analytic on U except for the finite number of poles on $Re(s) = a_0$.*

4.4.2 The behavior of $\tilde{F}(x)$ for small x

Let us denote by $\tilde{F}$ the cumulative distribution function of $k_\infty^{(-|\tilde{\mu}|)}$:

$$\tilde{F}(x) = \mathbb{P}\left[k_\infty^{(-|\tilde{\mu}|)} \leqslant x\right],$$

and

$$1 - \tilde{F}(x) = \mathbb{P}\left[k_\infty^{(-|\tilde{\mu}|)} > x\right].$$

Proposition 4.4.9. *The behavior of the cumulative distribution function $\tilde{F}$ for small x is given by:*

$$-\log \tilde{F}(x) \underset{x\to 0+}{\sim} \frac{B}{x}, \tag{4.4.4}$$

where $B = \frac{1}{8}(1 + 2|\tilde{\mu}|)^2$ is a positive constant.

Proof :

Let

$$\phi(x) = \frac{1}{x} \quad , \psi(x) = \frac{1}{x^2} \quad \text{and} \qquad B = \frac{1}{8}(1 + 2|\tilde{\mu}|)^2$$

We have $\phi \in R_{-1}(0+)$ and $\psi(x) = \frac{\phi(x)}{x} \in R_{-2}(0+)$.

From (4.2.1), we have:

$$\mathbb{E}\left[e^{-\lambda k_\infty^{(-|\tilde{\mu}|)}}\right] = \frac{\sqrt{\lambda/2}}{\sinh\left(\sqrt{\lambda/2}\right)\left(\cosh(\sqrt{\lambda/2})\right)^{2|\tilde{\mu}|}}$$

Then

$$\begin{aligned}
-\log\mathbb{E}\left[e^{-\lambda k_\infty^{(-|\tilde{\mu}|)}}\right] &= -\log(\sqrt{\lambda/2}) + \log\left(\sinh\left(\sqrt{\lambda/2}\right)\right) + 2|\tilde{\mu}|\log\left(\cosh\left(\sqrt{\lambda/2}\right)\right) \\
&= -\log(\sqrt{\lambda/2}) + \log\left(\frac{e^{\sqrt{\lambda/2}}}{2}\left(1-e^{-2\sqrt{\lambda/2}}\right)\right) \\
&\quad + 2|\tilde{\mu}|\log\left(\frac{e^{\sqrt{\lambda/2}}}{2}\left(1+e^{-2\sqrt{\lambda/2}}\right)\right) \\
&= -\log(\sqrt{\lambda/2}) - (1+2|\tilde{\mu}|)\log 2 + (1+2|\tilde{\mu}|)\sqrt{\lambda/2} \\
&\quad + \log\left(1-e^{-2\sqrt{\lambda/2}}\right) + 2|\tilde{\mu}|\log\left(1+e^{-2\sqrt{\lambda/2}}\right) \\
&\underset{\lambda\to\infty}{\sim} \frac{(1+2|\tilde{\mu}|)}{\sqrt{2}}\sqrt{\lambda} \qquad \text{(because } \frac{\log(x)}{x} \to 0 \text{ as } x\to\infty\text{)} \\
&\underset{\lambda\to\infty}{\sim} \frac{2\sqrt{B}}{\overleftarrow{\psi}(\lambda)}
\end{aligned}$$

where $\overleftarrow{\psi}$ is the inverse function of ψ.

Then $-\log\mathbb{E}\left[e^{-\lambda k_\infty^{(-|\tilde{\mu}|)}}\right]$ satisfies the equivalence (4.4.2) in proposition 4.4.2 with $\alpha = -1$ (because $\phi \in R_{-1}(0+)$). Then from proposition 4.4.2 we have:

$$-\log\tilde{F}(x) \underset{x\to 0+}{\sim} \frac{B}{x}$$

□

4.4.3 The behavior of $1-\tilde{F}(x)$ for large x

By applying Karamata's Tauberian theorem 4.4.4 with the random variable $k_\infty^{(-|\tilde{\mu}|)}$, we find that the index α is equal to 1. So we are in the case b) in proposition 4.4.4 and then we can't deduce an equivalent for $1-\tilde{F}(x)$ for large x.

So we aim to use the analytic Tauberian theorem 4.4.7 in order to deduce the behavior of $1-\tilde{F}(x)$ for large x.

In the sequel, we denote respectively by $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ the Laplace transform of S_1 and $C_{2|\tilde{\mu}|}$.

Lemma 4.4.10. *The abscissa of convergence of Φ_1 and Φ_2 are given respectively by*

$$a_1 = -2\pi^2 \qquad \text{and} \qquad a_2 = -\frac{\pi^2}{2} \tag{4.4.5}$$

Proof :
From the definition of the variable S_1, we have

$$\begin{aligned}
\Phi_1(\lambda) := \mathbb{E}\left[e^{-\lambda S_1}\right] &= \mathbb{E}\left[\prod_{n=1}^{\infty} \exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right] \\
&= \prod_{n=1}^{\infty}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]\right) \qquad \left(\text{because } \Gamma_{1,n} \text{ are independent }\right) \qquad (4.4.6) \\
&= \prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2\pi^2n^2}\right)^{-1} \qquad \text{(if } \lambda > -2\pi^2\text{)} \qquad (4.4.7)
\end{aligned}$$

The condition $\lambda > -2\pi^2$ in (4.4.7) is deduced from (4.2.2), this condition ensures that each term in the infinite product in (4.4.6) is finite and is equal to $\left(1+\frac{\lambda}{2\pi^2n^2}\right)^{-1}$.

We now prove that $\lambda = -2\pi^2$ is the abscissa of convergence of Φ_1.

i) We first proof that for $\lambda > -2\pi^2$, $\Phi_1(\lambda)$ is finite. In fact, for $\lambda > -2\pi^2$ the Laplace transform $\Phi_1(\lambda)$ is explicitly given by the product (4.4.7) which is finite. Indeed,

$$\log\left(\prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2\pi^2n^2}\right)^{-1}\right) = \sum_{n=1}^{\infty}\log\left(\left(1+\frac{\lambda}{2\pi^2n^2}\right)^{-1}\right).$$

Notice that the general term of this series is equivalent to $\frac{C}{n^2}$, where C is a constant. Then the above series is convergent.

ii) It remains to proof that for $\lambda < -2\pi^2$, the Laplace transform $\Phi_1(\lambda)$ is infinite. From (4.4.6) we have:

$$\Phi_1(\lambda) = \prod_{n=1}^{\infty}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]\right)$$

For $\lambda < -2\pi^2$ there exists $n_0 \in \mathbb{N}$ such that $-2(n_0+1)^2\pi^2 \leqslant \lambda < -2n_0^2\pi^2$.

Then

$$\prod_{n=1}^{\infty}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]\right) = \underbrace{\prod_{n=1}^{n_0}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]\right)}_{=\infty}\underbrace{\prod_{n=n_0+1}^{\infty}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]\right)}_{<\infty}$$

is not finite.

Notice that the first product on the right hand-side is infinite. In fact, for each $1 \leqslant n \leqslant n_0$, the expectation $\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]$ is infinite, then the product is infinite.

The second product on the right hand-side is finite because for each $n \geqslant n_0+1$, we have $\frac{\lambda}{2\pi^2n^2} > -1$ and then:

$$\prod_{n=n_0+1}^{\infty}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{1,n}}{2n^2\pi^2})\right]\right) = \prod_{n=n_0+1}^{\infty}\left(1+\frac{\lambda}{2\pi^2n^2}\right)^{-1}$$

where we prove by the same argument as in case i) that the last product is finite.

Similarly for the random variable $C_{2|\tilde{\mu}|}$. We have:

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda C_{2|\tilde{\mu}|}}\right] &= \mathbb{E}\left[\prod_{n=1}^{\infty}\exp\left(-\frac{\lambda\Gamma_{2|\tilde{\mu}|,n}}{2\pi^2(n-\frac{1}{2})^2}\right)\right] \\
&= \prod_{n=1}^{\infty}\left(\mathbb{E}\left[\exp(-\frac{\lambda\Gamma_{2|\tilde{\mu}|,n}}{2\pi^2(n-\frac{1}{2})^2})\right]\right) \qquad \text{(because } \Gamma_{2|\tilde{\mu}|,n} \text{ are independent)} \qquad (4.4.8) \\
&= \prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2\pi^2(n-\frac{1}{2})^2}\right)^{-2|\tilde{\mu}|} \qquad \left(\text{if } \lambda > -\frac{\pi^2}{2}\right). \qquad (4.4.9)
\end{aligned}$$

Similarly by the same argument in the case of the random variable S_1, we can deduce that $a_2 = -\frac{\pi^2}{2}$ is the abscissa of convergence of $C_{2|\tilde{\mu}|}$.

□

Now we have the following theorem which states that the tail probability decays exponentially.

Proposition 4.4.11. *We have*

$$\lim_{x\to\infty}\frac{1}{x}\log[1-\tilde{F}(x)]=-\frac{\pi^2}{2} \tag{4.4.10}$$

Proof :

From the equality in law in lemma 4.2.2, we can deduce that:

$$\mathbb{E}\left[e^{\lambda k_\infty^{(-|\tilde{\mu}|)}}\right]=\mathbb{E}\left[e^{\lambda(S_1+C_{2|\tilde{\mu}|})}\right] \qquad (\text{for } \lambda\in\mathbb{R}),$$

So $\mathbb{E}\left[e^{\lambda k_\infty^{\tilde{\mu}}}\right]$ is finite iff $\mathbb{E}\left[e^{\lambda(S_1+C_{2|\tilde{\mu}|})}\right]$ is finite. Then from lemma 4.4.10, we deduce that $a_0=-\frac{\pi^2}{2}$ is the abscissa of convergence of $k_\infty^{(-|\tilde{\mu}|)}$. As the zeros of the hyperbolic sine and the hyperbolic cosine functions are imaginar complex numbers, then we have the sole singularity $-\frac{\pi^2}{2}$ at the axis of convergence $Re(\lambda)=-\frac{\pi^2}{2}$.
Therefore from theorem 4.4.7, we deduce that

$$\lim_{x\to\infty}\frac{1}{x}\log[1-\tilde{F}(x)]=-\frac{\pi^2}{2}$$

□

4.5 Application to the detection a change in the constant coefficient σ

We now show how the normalized indicator NI can detect a regime change in σ. For that we aim to compare the behavior of its tails for different σ. To emphasize the dependence of the normalized indicator of the constant coefficient σ, we will use the notation $NI_\delta(\sigma)$.

- Compare the behavior of $\mathbb{P}[NI_\delta\leqslant x]$ for small x

From corollary 4.1.2, the behavior of the cumulative distribution function of NI_δ can be deduced from that of $k_\infty^{(\tilde{\mu})}$. Then from proposition 4.4.9, the behavior of $\mathbb{P}[NI_\delta\leqslant x]$ for small x is given by:

$$-\log\mathbb{P}[NI_\delta\leqslant x]=-\log\mathbb{P}\left[k_\infty^{-|\tilde{\mu}|}\leqslant\frac{x}{\sigma^2}\right]\underset{x\to0+}{\sim}\frac{(1+2|\tilde{\mu}|)^2\,\sigma^2}{8x}. \tag{4.5.1}$$

We remark that for σ_0 less than σ_1, $\mathbb{P}[NI_\delta(\sigma_0)\leqslant x]$ decays exponentially to zero just a little faster than $\mathbb{P}[NI_\delta(\sigma_1)\leqslant x]$.

- Compare the behavior of $\mathbb{P}[NI_\delta > x]$ for large x

Similarly as above, we can deduced from corollary 4.1.2 and proposition 4.4.11 that:

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}[NI_\delta > x] = \lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}\left[k_\infty^{-|\tilde{\mu}|} > \frac{x}{\sigma^2}\right] = -\frac{\pi^2}{2\sigma^2}. \tag{4.5.2}$$

Also we remark that for σ_0 less than σ_1, $\mathbb{P}[NI_\delta(\sigma_0) > x]$ decays exponentially to zero faster than $\mathbb{P}[NI_\delta(\sigma_1) > x]$.

We have showed that the tails behavior depend on the value of σ. So we can deduce that we have different behavior of these tails for different value of σ. Therefore we can deduce that the normalized indicator NI has the capacity to detect a regime change in σ.

Numerical results:

We now show numerically the above result by estimating the density of the variable NI_δ in order to verify that the behavior of its tails for different value of σ behaves like what we have got in (4.5.1) and (4.5.2).

Notice that we will be careful about the choice of the parameters μ,σ_0, σ_1 and δ as in observation 4.1.3 in order to have the convergence of the cumulative distribution of NI_δ to that of $\sigma^2 k_\infty^{(-|\tilde{\mu}|)}$ and then to deduce from (4.4.11) and (4.4.4) the two equivalents (4.5.1) and (4.5.2).

First we illustrate in figures 4.3(a) and 4.3(b) the estimated densities of the random variable NI_δ in the case where $\tilde{\mu} = -3$. Then from observation 4.1.3, the approximation of $\mathbb{P}[NI_\delta \leqslant x]$ by $\mathbb{P}[\sigma^2 k_\infty^{(-|\tilde{\mu}|)} \leqslant x]$ is good when

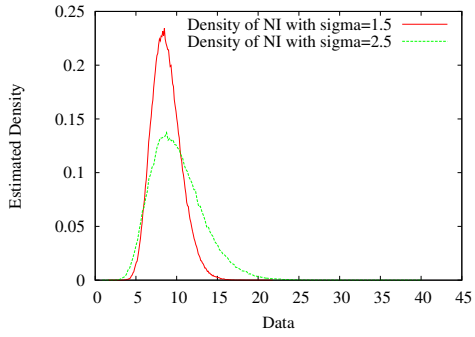
$$\sigma^2\delta \geqslant 2. \tag{4.5.3}$$

In figure 4.3(a), we estimate the density of $NI_\delta(\sigma_0)$ and $NI_\delta(\sigma_1)$ for small $\delta = 0.8$, where $\sigma_0 = 1.5$ and $\sigma_1 = 2.5$. In figure 4.3(b), we estimate the density of $NI_\delta(\sigma_0)$ and $NI_\delta(\sigma_1)$ for large $\delta = 2$, where $\sigma_0 = 1$ and $\sigma_1 = 1.5$. Notice that σ_0 and σ_1 are selected such that $\sigma_0^2\delta$ and $\sigma_1^2\delta$ satisfy (4.5.3).

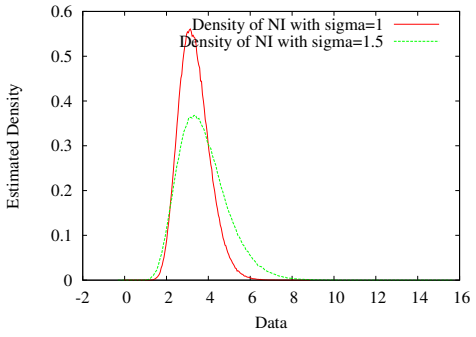
Second we illustrate in figures 4.3(d) and 4.3(c) the estimated densities of the random variable NI_δ in the case where $\tilde{\mu} = -5$. Similarly as above, we can deduce from observation 4.1.3 that the cumulative distribution function of NI is close to that of $\sigma^2 k_\infty^{(-|\tilde{\mu}|)}$ when

$$\sigma^2\delta \geqslant 0.8. \tag{4.5.4}$$

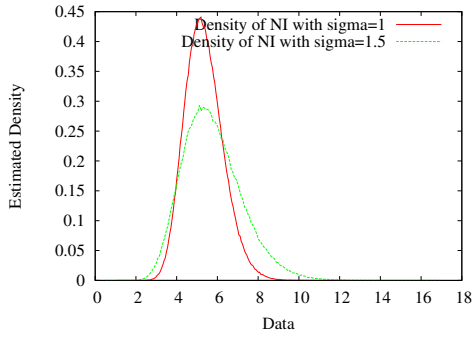
In figure 4.3(d), we estimate the density of $NI_\delta(\sigma_0)$ and $NI_\delta(\sigma_1)$ for small $\delta = 0.8$, where $\sigma_0 = 1$ and $\sigma_1 = 1.5$. In figure 4.3(c), we estimate the density of $NI_\delta(\sigma_0)$ and $NI_\delta(\sigma_1)$ for large $\delta = 2$, where $\sigma_0 = 0.6$ and $\sigma_1 = 1.2$. Here σ_0 and σ_1 are selected such that $\sigma_0^2\delta$ and $\sigma_1^2\delta$ satisfy (4.5.4).



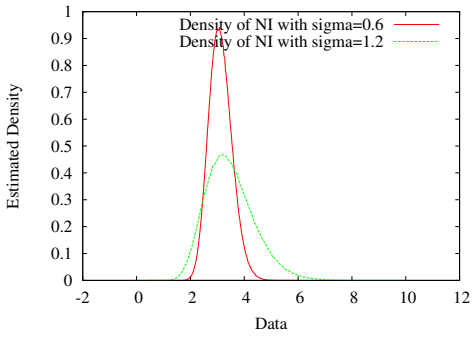
(a) $\delta = 0.8$,$\sigma_0 = 1.5$,$\sigma_1 = 2.5$ and $\mu = -15$



(b) $\delta = 2$,$\sigma_0 = 1$,$\sigma_1 = 1.5$ and $\mu = -6$



(c) $\delta = 0.8$,$\sigma_0 = 1$,$\sigma_1 = 1.5$ and $\mu = -6$



(d) $\delta = 2$,$\sigma_0 = 0.6$,$\sigma_1 = 1.2$ and $\mu = -6$

Figure 4.3: Esitmated density of NI indicator

From the above numerical example, we remark that the right tail probability of NI decays exponentially fast to zero with small σ_0 than the right tail probability of NI with σ_1 and this result is theoretically proved in (4.5.2). The left tail probability of NI with σ_0 decays to zero a little faster than the left tail probability of NI with σ_1, also this result is given theoretically in (4.5.1).

4.6 Appendix B

Tauberian Theory

Definition 4.6.1. *A measurable function $f : (0,\infty) \to (0,\infty)$ is regularly varying at the origin with index α, and we write $f \in R_\alpha(0+)$ if for every $a > 0$,*

$$\lim_{x\to 0} \frac{f(ax)}{f(x)} = a^\alpha$$

Definition 4.6.2. *A measurable function $f : (0,\infty) \to (0,\infty)$ is slowly varying at infinity if for every $a > 0$, $\lim_{x\to\infty} \frac{f(ax)}{f(x)} \to 1$*

Definition 4.6.3. *A measurable function $f : (0,\infty) \to (0,\infty)$ is regularly varying at ∞ if for every $a > 0$, the limit $\lim_{x\to\infty} \frac{f(ax)}{f(x)}$ exists.*

Lemma 4.6.4. *(Characterization of a regularly varying function [2, p.9]).*

If the function f is regularly varying at infinity, then there exists a real number ρ, called the index, such that

$$\lim_{x\to\infty} \frac{f(ax)}{f(x)} = a^\rho \tag{4.6.1}$$

for every $a > 0$. Moreover, $l(x) = f(x)x^{-\rho}$ is slowly varying at ∞.

Proposition 4.6.5. *If l is slowly varying, X is so large that $l(x)$ is locally bounded in $[X,\infty]$, and $\alpha > -1$, then*

$$\int_X^x t^\alpha l(t)dt \sim x^{\alpha+1} l(x)/\left(\alpha+1\right)$$

Proof :
We refer to [4, p.26]. □

Definition 4.6.6. *A function f on $[0,\infty)$ is said to be ultimately monotone if it is monotone on some $[x_0,\infty)$ for some $x_0 \geqslant 0$.*

If $U : \mathbb{R} \to \mathbb{R}$ has locally bounded variation, is right-continuous, and vanishes on $(-\infty, 0)$, we define its Laplace -Stieltjes transform $\hat{U}$ by:

$$\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} dU(x)$$

where the integral converges absolutely for $s > c$ or more generally for all complex $s = a + ib$ with $a > c$. The constant c is defined as:

$$c = \inf\{a \in \mathbb{R} : \int_0^\infty e^{-ax} dF(x) < \infty\}$$

and dU denote the associated Stieltjes measure for U.

The most important case is when U is non-decreasing on $\mathbb{R}$, with $U = 0$ on $(-\infty, 0)$. For such U, statements about $\hat{U}$ are to be considered to include the assertion that it is finite for the arguments in question.

We now present a theorem from [1, p.58].

Theorem 4.6.7. *Let X is a Banach space and $U : \mathbb{R} \mapsto X$ be a local bounded semi-variation. Assume $c < \infty$. Then*

$$\tilde{U}^{(n)} = \int_0^\infty e^{-\lambda t}(-t)^n dU(x).$$

where $\tilde{U}^{(n)}$ denotes the n^{th} derivative of $\tilde{U}$.

We now give Karamata Tauberian Theorem from [4, p.37]

Theorem 4.6.8. *(Karamata Tauberian Theorem). Let U be a non-decreasing right-continuous function on $\mathbb{R}$ with $U(x) = 0$ for all $x < 0$. If l varies slowly and $c \geqslant 0, \rho \geqslant 0$, the following are equivalent:*

$$U(x) \sim \frac{cx^\rho l(x)}{\Gamma(1+\rho)} \qquad (x \to \infty) \tag{4.6.2}$$

$$\hat{U}(\lambda) \sim c\lambda^{-\rho} l(1/\lambda) \qquad (\lambda \to 0+) \tag{4.6.3}$$

Suppose that U is absolutely continuous with density u, say:

$$U(x) = \int_0^x u(y)dy$$

When U has a density $U' = u$ it is desirable to obtain estimate of u. This problem cannot be treated in full generality, so we may obtain a "differentiated form" of the asymptotic relation under the "ultimately monotone" condition as we show in the next theorem.

Theorem 4.6.9. *(Monotone Density Theorem).*

Let $U(x) = \int_0^x u(y)dy$. If $U(x) \sim cx^\rho l(x)$ $(x \to \infty)$, where $c \in \mathbb{R}, \rho > 0$, l is slowly varying at infinity and if u is ultimately monotone, then

$$u(x) \sim \rho x^{\rho-1} l(x) \qquad (x \to \infty) \tag{4.6.4}$$

Proof :
We refer to [4, p.39]. □

Conversely, if we have (4.6.4), then Proposition 4.6.5 on "integrating asymptotic relations" yields $U(x) \sim cx^{\rho}l(x) \quad (x \to \infty)$ even if u is not ultimately monotone.

Application to probability theory

In this statement we are interested by the probability case i.e where U is the cumulative distribution function of a real random variables.

Let X be a non-negative random variable with distribution function $F(x) := \mathbb{P}[X \leqslant x]$. The induced law of X on $\mathbb{R}$ is the Lebesgue-Stieltjes measure $dF(x) = \mathbb{P}[X \in dx]$. We shall identify the law with F.

For F supported by $[0, \infty)$, the Laplace-Stieltjes transform

$$\hat{F}(\lambda) := \mathbb{E}\left[e^{-\lambda X}\right] = \int_0^{\infty} e^{-\lambda x} dF(x).$$

is finite for all $\lambda \geqslant 0$.

Preliminaries

Here we list some elementary properties which are useful in the proof of theorem 4.6.14.

Lemma 4.6.10. *Let X be a positive random variable with probability distribution function F. Let $\hat{F}$ its Laplace transform, then $\hat{F}$ possesses derivatives of all orders given by*

$$\hat{F}^{(n)}(\lambda) = (-1)^n \int_0^{\infty} e^{-\lambda x} x^n dF(x).$$

Proof :
Apply theorem 4.6.7. □

Remark 4.6.11. *The above lemma implies that F possesses a finite n^{th} moment if and only of a finite limit $\hat{F}^{(n)}(0)$ exists. For $n = 1$, $\mathbb{E}[X] = -\hat{F}^{(1)}(0)$ and for $n = 2$, $\mathbb{E}[X^2] = -\hat{F}^{(2)}(0)$.*

Lemma 4.6.12. *For any $\alpha > 0$*

$$\int_0^{\infty} x^{\alpha} dF(x) = \alpha \int_0^{\infty} x^{\alpha - 1}(1 - F(x))dx \tag{4.6.5}$$

Proof :
We have that:

$$\begin{aligned}\int_0^\infty x^{\alpha-1}\left(1-F(x)\right)dx &= \int_0^\infty x^{\alpha-1}\int_x^\infty dF(y)dx = \int_0^\infty x^{\alpha-1}\int_0^\infty 1_{(y>x)}dF(y)dx\\ &=^{\text{Fubini}} \int_0^\infty\int_0^\infty x^{\alpha-1}1_{(y>x)}dxdF(y)\\ &= \frac{1}{\alpha}\int_0^\infty y^\alpha dF(y)\end{aligned}$$

Then (4.6.5) follows. □

Lemma 4.6.13.

$$\int_0^\infty e^{-\lambda x}\left(1-F(x)\right)dx = \frac{1-\hat{F}(\lambda)}{\lambda} \tag{4.6.6}$$

Proof :
Integrating by parts $\hat{F}(\lambda) = \int_0^\infty e^{-\lambda x}dF(x)$, we get:

$$\frac{\hat{F}(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x}F(x)dx$$

Then

$$\frac{1-\hat{F}(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x}\left(1-F(x)\right)dx$$

□

Let X be a positive random variable with distribution function F and denote by $\hat{F}$ its Laplace transform. Then we have the following result:

Theorem 4.6.14. *Let l is a slowly varying at infinity, then*

a) For $0 \leqslant \alpha < 1$, the following are equivalent

$$\begin{aligned} &i)\ 1-\hat{F}(\lambda) \sim \lambda^\alpha l(1/\lambda) && (\lambda\to 0)\\ &ii)\ 1-F(x) \sim \frac{l(x)}{x^\alpha\Gamma(1-\alpha)} && (x\to\infty)\end{aligned}$$

b) For $\alpha = 1$, the following are equivalent

$$i)\ 1-\hat{F}(\lambda) \sim \lambda^\alpha l(1/\lambda) \qquad (\lambda\to 0)$$

$$ii) \begin{cases} \displaystyle\int_0^x t dF(t) \sim l(x) & (x \to \infty) \\ \displaystyle\int_0^x (1 - F(t)) dt \sim l(x) & (x \to \infty) \end{cases}$$

Proof :

From (4.6.6), we remark that $\frac{1-\hat{F}(\lambda)}{\lambda}$ is the Laplace transform of the function $\displaystyle\int_0^x (1 - F(t))\, dt$.

- $0 \leqslant \alpha < 1$:

  From $i)$ we have that $\frac{1-\hat{F}(\lambda)}{\lambda} \sim \lambda^{\alpha-1} l(1/\lambda)$, then by Karamata's Tauberian Theorem 4.6.8 we deduce that $i)$ is equivalent to:

  $$\int_0^x (1 - F(t))\, dt \underset{x\to\infty}{\sim} \frac{x^{1-\alpha} l(x)}{\Gamma(2-\alpha)}$$

  As $1 - F(t)$ is ultimately monotone and $\alpha < 1$, then by the monotone density theorem 4.6.9 we get:

  $$1 - F(x) \underset{x\to\infty}{\sim} \frac{(1-\alpha)}{\Gamma(2-\alpha)} x^{-\alpha} l(x) = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} l(x)$$

  Therefore $i)$ implies $ii)$. Conversely an integration shows that $ii)$ implies $i)$, hence the equivalence between $i)$ and $ii)$ in $a)$.

- $\alpha = 1$:

  From $i)$ we have that $\frac{1-\hat{F}(\lambda)}{\lambda} \sim l(1/\lambda)$, then by Karamata's Tauberian Theorem 4.6.8 we deduce that $i)$ is equivalent to:

  $$\int_0^x (1 - F(t))\, dt \underset{x\to\infty}{\sim} l(x)$$

  On the other hand, notice that from (4.6.5) the two statement in $ii)$ are equal as $x \to \infty$. Then $i)$ implies $ii)$ in $b)$.

□

Application: The case of the exponential law

We now aim to apply the above theorem 4.6.14 to the exponential law.

Let X be a exponential random variable with parameter $\mu = 1$ and $l(x) = \frac{x}{1+x}$.

We have:

$$1 - \hat{F}(\lambda) = \frac{\lambda}{1+\lambda} = \lambda l(1/\lambda) \tag{4.6.7}$$

As l is slowly varying function at infinity, then from 4.6.14 we have:

$$\int_0^x (1 - F(t))dt \underset{x\to\infty}{\sim} l(x)$$

So we have an information about the integrated tail probability for large x, but we don't have an equivalent for the tail probability $1 - F(x)$ for large x.

So we aim to define another random variable Y in function of X such that its first moment is infinite and we apply again the theorem 4.6.14 in order to get an information about the tail probability of Y.

Let $Y = e^{\beta X}$, where $\beta > 1$ and X be the exponential law with parameter $\mu = 1$. We remark that the first moment of Y is infinite.

As above, we denote by F_Y the distribution function of Y and by $\hat{F}_Y$ its Laplace transform.Then for $\lambda > 0$, we have:

$$\hat{F}_Y(\lambda) = \mathbb{E}[e^{-\lambda Y}] = \mathbb{E}[e^{-\lambda e^{\beta X}}] = \int_0^\infty e^{-\lambda e^{\beta x}} e^{-x} dx$$

Using the change of variable $u = e^{\beta x}$, we obtain that

$$\hat{F}_Y(\lambda) = \frac{1}{\beta} \int_1^\infty \frac{e^{-\lambda u}}{u^{\frac{1}{\beta}+1}} du$$

Firstly, we aim to find a slowly varying function l at infinity such that

$$1 - \hat{F}_Y(\lambda) \sim \lambda^\alpha l(1/\lambda) \qquad (\lambda \to 0)$$

where $0 \leqslant \alpha < 1$.

We have:

$$1 - \hat{F}_Y(\lambda) = 1 - \frac{1}{\beta} \int_1^\infty \frac{e^{-\lambda u}}{u^{\frac{1}{\beta}+1}} du = \frac{1}{\beta} \int_1^\infty \frac{1 - e^{-\lambda u}}{u^{\frac{1}{\beta}+1}} du \tag{4.6.8}$$

Let $f(x) = \frac{1}{\beta}\int_1^\infty \frac{1-e^{-\frac{u}{x}}}{u^{\frac{1}{\beta}+1}}du$. Then for $a>0$, we have:

$$\lim_{x\to\infty}\frac{f(ax)}{f(x)} = \lim_{x\to\infty}\frac{\int_1^\infty \frac{1-e^{-\frac{u}{ax}}}{u^{\frac{1}{\beta}+1}}du}{\int_1^\infty \frac{1-e^{-\frac{u}{x}}}{u^{\frac{1}{\beta}+1}}du} = \lim_{x\to\infty}\frac{1}{a^{1/\beta}}\frac{(\frac{1}{x})^{1/\beta}\int_{1/ax}^\infty \frac{1-e^{-\zeta}}{\zeta^{\frac{1}{\beta}+1}}d\zeta}{(\frac{1}{x})^{1/\beta}\int_{1/x}^\infty \frac{1-e^{-v}}{v^{\frac{1}{\beta}+1}}dv} \to \frac{1}{a^{1/\beta}}$$

The last equality comes from the change of variable $\zeta = \frac{u}{ax}$ in the numerator and $v = \frac{u}{x}$ in the denominator.

As $\lim_{x\to\infty}\frac{f(ax)}{f(x)} = \frac{1}{a^{1/\beta}}$, then from lemma 4.6.4, the function $l(x) = x^{1/\beta}f(x)$ is slowly varying at infinity.

From (4.6.8), we have that:

$$1-\hat{F}(\lambda) = \lambda^{1/\beta}l(1/\lambda)$$

where $1/\beta$ is less than 1.

Then from theorem 4.6.14, the tail behavior of F_Y near infinity is given by:

$$1-F_Y(x) \sim \frac{l(x)}{x^{1/\beta}\Gamma(1-1/\beta)} \sim \frac{1}{\Gamma(1-1/\beta)}f(x) \qquad (x\to\infty)$$

$$\sim \frac{1}{\Gamma(1-1/\beta)}\frac{1}{\beta}\int_1^\infty \frac{1-e^{-\frac{u}{x}}}{u^{\frac{1}{\beta}+1}}du \qquad (4.6.9)$$

On the other hand, we have:

$$1-F_Y(x) = \mathbb{P}[Y>x] = \mathbb{P}[X > \frac{1}{\beta}\log(x)] = \frac{1}{x^{1/\beta}} \qquad (4.6.10)$$

Finally we check that (4.6.9) is equivalent to (4.6.10) at infinity. Let $\delta = \frac{1}{x}$, then

$$\int_1^\infty \frac{1-e^{-\frac{u}{x}}}{u^{\frac{1}{\beta}+1}}du = \int_1^\infty \frac{1-e^{-\delta u}}{u^{\frac{1}{\beta}+1}}du = \delta^{\frac{1}{\beta}}\int_\delta^\infty \frac{1-e^{-\zeta}}{\zeta^{\frac{1}{\beta}+1}}d\zeta, \qquad (4.6.11)$$

where the last equality comes from the change of variable $\zeta = \delta u$.

As $\delta^{\frac{1}{\beta}}\int_\delta^\infty \frac{1-e^{-\zeta}}{\zeta^{\frac{1}{\beta}+1}}d\zeta = \delta^{\frac{1}{\beta}}\left[\int_0^\infty \frac{1-e^{-\zeta}}{\zeta^{\frac{1}{\beta}+1}}d\zeta - \int_0^\delta \frac{1-e^{-\zeta}}{\zeta^{\frac{1}{\beta}+1}}d\zeta\right]$,

so, when $x\to\infty$ (i.e δ near to 0), (4.6.11) is equivalent to

$$\delta^{\frac{1}{\beta}}\left[A+O(\delta^{1/\beta})\right] \sim A.\delta^{1/\beta} + O\left(\delta^{2/\beta}\right) \sim A.\frac{1}{x^{1/\beta}} + O\left(\frac{1}{x^{2/\beta}}\right)$$

where A is a constant. Therefore the equivalence between (4.6.9) and (4.6.10) is proven.

Joint density of Bessel process and the integral of its square

The following Lemma is given in [6](formulae 1.9.8 p.378 and 1.0.6 p.373)

Lemma 4.6.15. *Let $R_s^{(n)}$ be a n-dimensional Bessel process started at x, then*

$$\mathbb{P}_x\left[\int_0^t \left(R_s^{(n)}\right)^2 ds \in dz, R_t^{(n)} \in dy\right] = \frac{y^{\nu+1}}{x} is_z\left(\nu, t, 0, (x^2+y^2)/2, xy/2\right) dzdy$$

and

$$\mathbb{P}_x\left[R_t^{(n)} \in dy\right] = y^{\nu+1}x^{-\nu}t^{-1}e^{-(x^2+y^2)/2t} I_\nu(xy/t)dy$$

when $\nu = (n/2)-1$ is called the index of the Bessel process R.

Special functions

These functions are defined in [6].

Special Inverse Laplace Transforms:

$$is_z(\nu,t,r,y,x) := \mathcal{L}_a^{-1}\Big[\frac{\sqrt{2a}}{sh(t\sqrt{2a})}\exp\big(-r\sqrt{2a} - \frac{y\sqrt{2a}\ ch(t\sqrt{2a})}{sh(t\sqrt{2a})}\big) I_\nu\big(\frac{2x\sqrt{2a}}{sh(t\sqrt{2a})}\big)\Big]$$

where

$$I_\nu(x) := \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$$

and

$$I_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}(x/2)^\nu \qquad \text{as} \quad x \to 0 \tag{4.6.12}$$

Error function

$$Erfc(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}dv$$

and

$$Erfc(x) \approx \frac{1}{\sqrt{\pi}x}e^{-x^2} \qquad (\text{as } x \to \infty)$$

Gamma functions The incomplete gamma function is defined as:

$$\Gamma(s,x) := \int_x^\infty u^{s-1}e^{-u}ds$$

and the gamma function is defined as:

$$\gamma(x) := \int_0^\infty u^{x-1}e^{-u}du, \qquad (x>0)$$

Chapter 5

Quadratic variation estimator and Bollinger Bands indicator

We have showed in previous chapters that the Bollinger Bandwidth indicator can be used as an estimator to detect the change time at which the volatility changes its value. Also it is well known in the literature that the quadratic variation is an estimator of the volatility, that is, can be used to detect the change time of the volatility. But the quadratic variation estimator is optimal if we select the time increment as small as possible.

We consider the case of a trader who does not perfectly detect τ but, at least, uses an optimal detection procedures to decide when he reinvests his portfolio. So we assume that the trader uses one of the above estimators in order to detect the change time, and then we are interested in this chapter to compare the performance of the various detection strategies in the case of large value for the time increment.

We assume that the change time τ has an exponential law with parameter λ. We show that in the case of large time increment ($\Delta = 10^{-2}$), the optimal trading strategy based on Bollinger Bands indicator can overperfom that of quadratic variation indicator in the case of large value for λ. Also we show that in this case of large λ, Bollinger Bands can detect the change time faster than quadratic variation.

5.1 Detection methods for the change time

We show in this section how can detect the change time of the volatility using respectively the Bollinger Bandwidth and the quadratic variation estimators.

Let us start by a short abstract about the quadratic variation.

We assume that the price process (S_t) evolves according to,

$$dS_t = S_t \left(\mu dt + \sigma dB_t\right). \tag{5.1.1}$$

where μ and σ are constants.

Then the log-price process $Y_t = \log(S_t)$ satisfies:

$$d\log S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

The main object of interest is the quadratic variation QV:

$$QV_t := \langle Y, Y \rangle_t = \int_0^t \sigma^2 ds = \sigma^2 t.$$

over a fixed time period $[0, t]$. The usual estimator of the QV_t is the realized volatility (RV), which is simply the sum of observed squared log-returns

$$RV_t = \sum_{k=0}^{n-1} \left(Y_{t_{k+1}} - Y_{t_k}\right)^2$$

where we consider a regular partition of the interval $[0, t]$ with time increments Δ:

$$0 = t_0 < t_1 < \ldots\ldots\ldots\ldots\ldots < t_n = t, \qquad t_k = k\Delta.$$

$(t_k, k = 0, 1....n)$ are the times at which the price of the asset is available during period t, and n is the number of intra-period observations used in computing the estimator. In theory, sampling at increasingly higher frequency should deliver, at the limit, a consistent estimator of the quadratic variation. Further, the RV_t converges to the quadratic variation QV_t, and this convergence it is optimal by selecting Δ as small as possible. Unfortunately this theory cannot be applied directly to real financial data.

We now return to our underlying mathematical model which assume that the price process satisfies the following SDE:

$$dS_t = \mu S_t dt + S_t \left(\sigma_1 + (\sigma_0 - \sigma_1)\, \mathbb{1}_{t \leqslant \tau}\right) dB_t, \tag{5.1.2}$$

We present in this section two methods for computing the stopping rule (alarm time) $\tilde{\tau}$ which detects the instant τ at which the volatility changes its value. The first one is based on the quadratic variation estimator and the second one, is based on the Bollinger Bandwidth indicator.

We first suppose that the trader uses the quadratic variation estimator and at each time t_k he knows the last δ observations. Then the estimated $\hat{\sigma}$ at each time t_k can be obtained by:

$$\hat{\sigma}^2_{t_k} = \frac{1}{V\Delta}\sum_{i=0}^{N}\left(Y_{(k+i)\Delta} - Y_{(k-i-1)\Delta}\right)^2 := \frac{1}{N\Delta}RV_{t_k}, \qquad (t_k = k\Delta). \tag{5.1.3}$$

where $t_k \in \{t-\delta, ..., t\}$ and the parameter δ is the size of the time window used to compute the (5.1.3), and is also the size of the time window used to compute the moving average for the Bollinger Bands.

For simplifying the notations, we denote in the sequel by $\tilde{\tau}^Q$ the alarm time $\tilde{\tau}$ computed by using quadratic variation estimator. Then $\tilde{\tau}^Q$ can be expressed by:

$$\tilde{\tau}^Q = \inf\{t_k; \frac{1}{N\Delta}RV_{t_k} > \varrho\}, \tag{5.1.4}$$

where ϱ is a threshold to be controlled and it depends on σ_0 and σ_1. Indeed, the threshold ϱ can be defined as:

$$\varrho = c\ \sigma_0^2 + (1-c)\sigma_1^2 \qquad c \in]0,1[. \tag{5.1.5}$$

We now assume that the trader uses Bollinger bands estimator to estimate the time τ. Here we denote by $\tilde{\tau}^B$, the alarm time $\tilde{\tau}$ detected by using the Bollinger Bands indicator. Then $\tilde{\tau}^B$ can be expressed by:

$$\tilde{\tau}^B = \inf\{t_k / BWI_{t_k} > \beta\} \tag{5.1.6}$$

where β is a constant to be fixed. For first intuition, the value of β can be chosen as in section 3.3 in chapter 1.

Notice that the alarms time $\tilde{\tau}^Q$ and $\tilde{\tau}^B$ can occur before the instant τ, that is it corresponds to a false alarm, or after τ. So, the amount of time which $\tilde{\tau}^Q$ and $\tilde{\tau}^B$ miss the true time change τ is given by $|\tilde{\tau}^Q - \tau|$ and $|\tilde{\tau}^B - \tau|$. We restrict ourselves to the detection procedure introduced by Karatzas which consists in minimizing respectively the amount $|\tilde{\tau}^Q - \tau|$ and $|\tilde{\tau}^B - \tau|$.

The Karatzas's detection method consists in computing the optimal stopping rule (alarm time) $\tilde{\tau}^Q$ that minimizes the expected miss

$$EK(Q) = \mathbb{E}[\ |\tilde{\tau}^Q - \tau|\ ], \tag{5.1.7}$$

and

$$EK(B) = \mathbb{E}[\ |\tilde{\tau}^B - \tau|\ ]. \tag{5.1.8}$$

As the alarm times $\tilde{\tau}^Q$ and $\tilde{\tau}^B$ depend respectively on the thresholds ϱ and β, then the expected miss $EK(Q)$ and $EK(B)$ can be influence by the choice of these parameters. So it must choose the good parameters in order to minimize $EK(Q)$ and $EK(B)$. But as the trader's goal is to maximize his gain, so it will be better to choice the optimal parameters c, β and δ as the parameters with which the trader maximizes the expected utility of his terminal wealth $\mathbb{E}[\log(W_T)]$.

5.2 Compare the performance of the various detection strategies

We compare the performance of two mathematical detected strategies aimed at detecting the time at which the volatility changes. We aim to compare the performance of traders who use one of the two volatility estimators to detect τ in order to decide his portfolio investment strategy. Supposing that σ_0 is less than σ_1, then his strategy consists in putting all his money in the stock until the detected time, and in the bond after this time.

Consider a trader who takes decisions at discrete times of a regular partition of the interval $[0, T]$ with step $\Delta = \frac{T}{N}$.

$$0 = t_0 < t_1 < \ldots\ldots\ldots\ldots\ldots < t_N = T, \qquad t_k = k\Delta.$$

We denote by $\pi^Q_{t_k}$ (resp.$\pi^B_{t_k}$) $\in \{0, 1\}$ the proportion of the agent's wealth invested in the risky asset at time $t_k, k \in [1, N]$, using respectively the quadratic variation and Bollinger Bands indicators to detect the change time τ. Consequently,

$$\pi^Q_{t_k} = \mathbb{1}_{(t_k < \tilde{\tau}^Q)}, \tag{5.2.1}$$

and the corresponding wealth $W^Q_{t_k}$ at time t_k satisfies:

$$W^Q_{t_{k+1}} = W^Q_{t_k}\left[\frac{S_{t_{k+1}}}{S_{t_k}}\pi^Q_{t_k} + \frac{S^0_{t_{k+1}}}{S^0_{t_k}}(1 - \pi^Q_{t_k})\right] \tag{5.2.2}$$

Similarly we have:

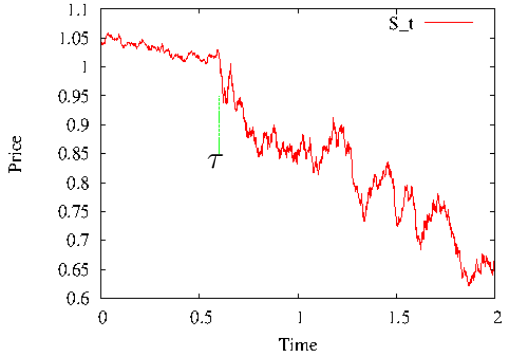
$$\pi^B_{t_k} = \mathbb{1}_{(t_k < \tilde{\tau}^B)}, \tag{5.2.3}$$

and the corresponding wealth $W^B_{t_k}$ at time t_k satisfies:

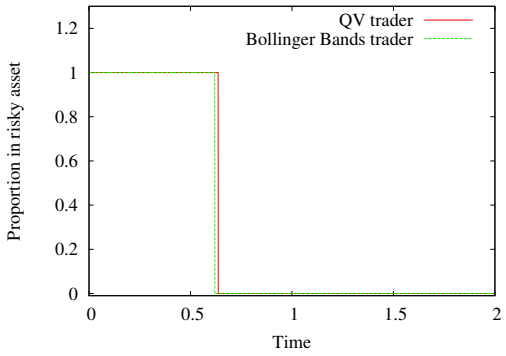
$$W^B_{t_{k+1}} = W^B_{t_k}\left[\frac{S_{t_{k+1}}}{S_{t_k}}\pi^B_{t_k} + \frac{S^0_{t_{k+1}}}{S^0_{t_k}}(1 - \pi^B_{t_k})\right]. \tag{5.2.4}$$

The parameters used in this numerical example is described below:

$\mu = 0.01$ $r = 0$	$\sigma_0 = 0.05$ $T = 2$	$\sigma_1 = 0.2$ $\tau = 0.6$

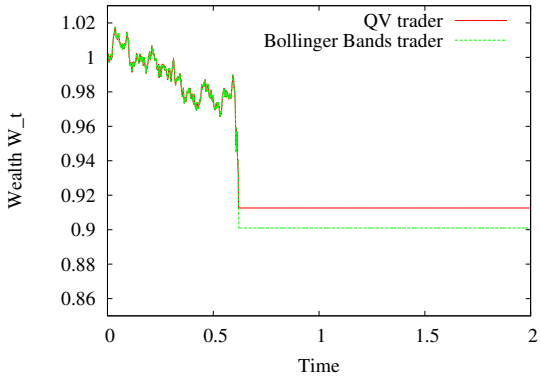


(a) A typical trajectory of the stock price



(b) Proportion of the wealth invested in the risky asset

Figure 5.1: A nominal trajectory



(a)

Figure 5.2: Time evolution of the wealth.

Fig.5.1(a) shows the underlying trajectory of the stock price, when the change occurs at $\tau = 0.6$. Figs 5.1(b) shows the allocation strategies of traders using quadratic variation and, respectively, Bollinger Bands estimators detection methods. In Fig 5.2(a), we show the time evolution of the wealth.

We aim to compute the optimal parameters for each method, this means we aim to

compute the optimal parameters c and δ of the quadratic variation estimator which maximize $\mathbb{E}[U(W_T^Q)]$. Similarly we compute the optimal parameters β and δ of the Bollinger Bands estimator which maximize $\mathbb{E}[U(W_T^B)]$. Finally we compare the optimal $\mathbb{E}[U(W_T^Q)]$ and $\mathbb{E}[U(W_T^B)]$ which are computed with the optimal parameters.

5.2.1 Comparison results for large data $\Delta = 10^{-2}$

We first assume that the trader displays a logarithmic utility function and we assume that τ is deterministic. The parameters used to obtain the following results are given by:

$\mu = 0.01$	$\sigma_0 = 0.05$	$\sigma_1 = 0.2$	$\Delta = 10^{-2}$
$r = 0$	$T = 2$	$\tau = 0.6$	

- **Results about quadratic variation estimator**

Empirical determination of a good weight c

One can optimize the choice of c by means of Monte Carlo simulations. For each δ, we have simulated 10^6 trajectories of the asset price and computed the time evolution of the expectation logarithm of wealth for different value of c. Then the optimal choice of c is that maximize the $\mathbb{E}[\log(W_T^Q)]$. In tabular 5.1, we show some optimal c for different value of δ.

δ	optimal weight c
0.03	0.76
0.05	0.70
0.06	0.70
0.08	0.77
0.1	0.82
0.12	0.87

Table 5.1

Empirical determination of a good windowing δ

In this paragraph we aim to determine the optimal choice of δ. From the previous paragraph we have computed for different δ, the optimal associated choice of the weight c. So now in this paragraph we compute by Monte Carlo simulations the expectation $\mathbb{E}[\log(W_t^Q)]$ for different value of δ by fixing for each δ the associated optimal choice of c. In all our simulations, the Monte Carlo error on $\mathbb{E}[\log(W_t^Q)]$ is of order 3×10^{-5}.

Then from figure 5.3 the optimal choice of δ is around 0.05 and the optimal associated choice of c is around 0.7.

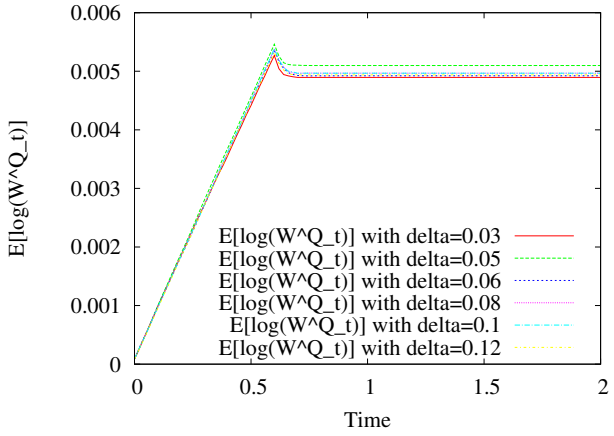


Figure 5.3: Comparison of the expected values of the logarithm of wealth for different value of δ and for deterministic τ.

- **Results about Bollinger Bands estimator**

Empirical determination of a good level β

Similarly as in the quadratic variation case, one can optimize the choice of β by means of Monte Carlo simulations. We present below for some δ the associated optimal threshold β.

δ	optimal β
0.03	0.043
0.05	0.061
0.06	0.058
0.08	0.061
0.1	0.059
0.12	0.064

Table 5.2

Empirical determination of a good windowing δ

We now aim to determine the optimal choice of δ. Similarly we compute by Monte Carlo simulations expected logarithm of wealth $\mathbb{E}[\log(W_t^B)]$ for different value of δ by

fixing for each δ its optimal choice of β as in tabular 5.2. In all our simulations, the Monte Carlo error on $\mathbb{E}[\log(W_t^B)]$ is also of order 3×10^{-5}.

Then from figure 5.4 the optimal choice of δ is around 0.03 and the optimal associated choice of β is around 0.043.

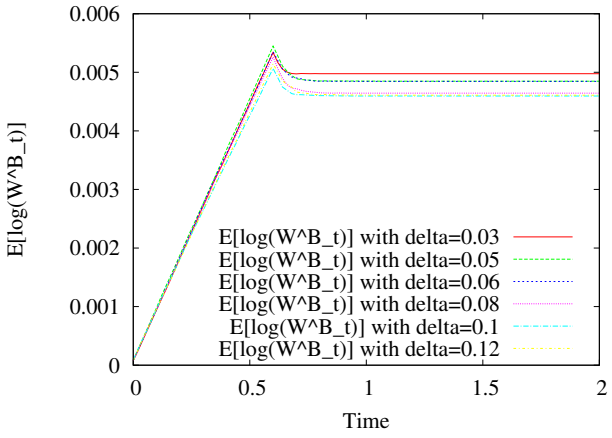


Figure 5.4: Comparison of the expected values of the logarithm of wealth for different value of δ and for deterministic τ.

- **Comparison of performance when the traders use optimal strategy with quadratic variation and Bollinger Bands.**

Finally we aim to compare the performances of optimal trading strategy based on the quadratic variation estimator with optimal trading strategy based on Bollinger Bands estimator. In figure 5.5, we illustrate the expected logarithm of wealth $\mathbb{E}[\log(W_t^Q)]$ and $\mathbb{E}[\log(W_t^B)]$ when we have taken for both the optimal value of δ. We remark that quadratic variation indicator can overperform Bollinger Bands indicator but the difference between their performances is not large (not too significant).

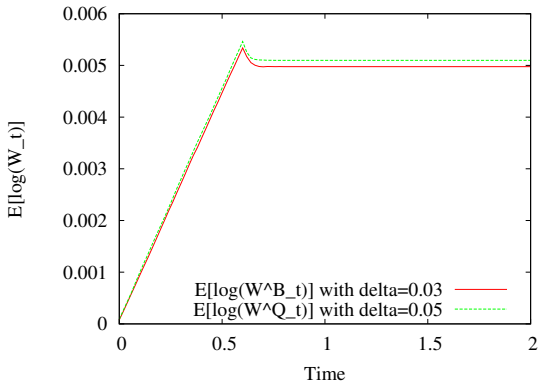


Figure 5.5: Comparison between quadratic variation and Bollinger Bands for logarithm utility and for deterministic τ.

To conclude this analysis, we examine the effects of the utility functions on the optimal choice of δ for both of estimators. We now assume that the trader displays a power utility function like $U(x) = x^{1/2}$. As in the case of logarithm utility function, we compute the optimal trading strategy based on quadratic variation estimator and the optimal trading strategy based on Bollinger Bands estimator, that is we compute for both estimators the optimal choice of δ. From Monte Carlo simulations, the optimal choice of delta for quadratic variation is $\delta = 0.06$ and the optimal delta for Bollinger Bands is $\delta = 0.05$.

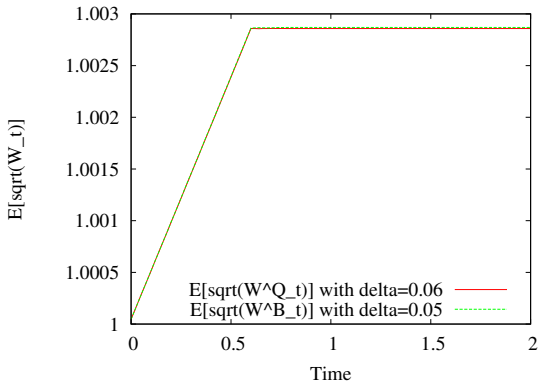
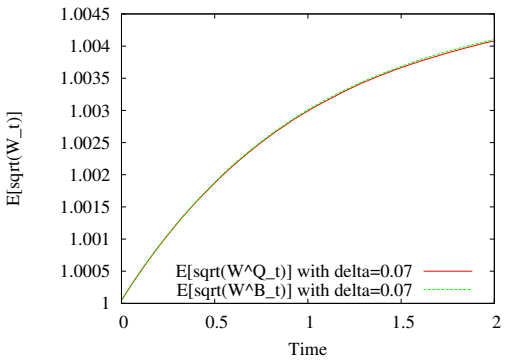


Figure 5.6: Comparison between quadratic variation and Bollinger Bands for power utility and for deterministic τ.

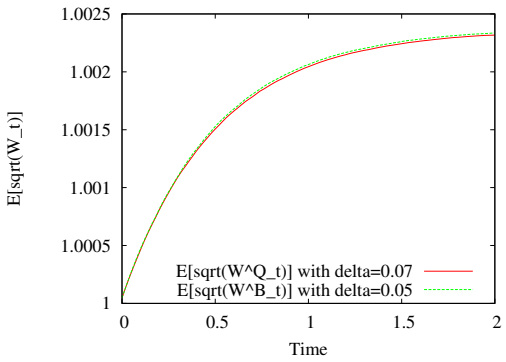
It is clear from this figure that the optimal trading strategies have the same performances.

- **Comparison of performances where τ has an exponential law with parameter λ**

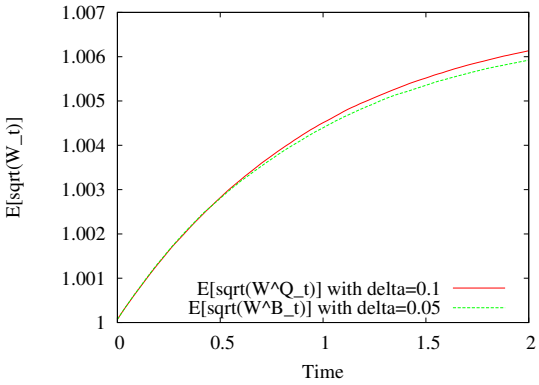
We assume that the trader displays a power utility and we aim to examine the effect of the parameter λ on the performances of the strategies.



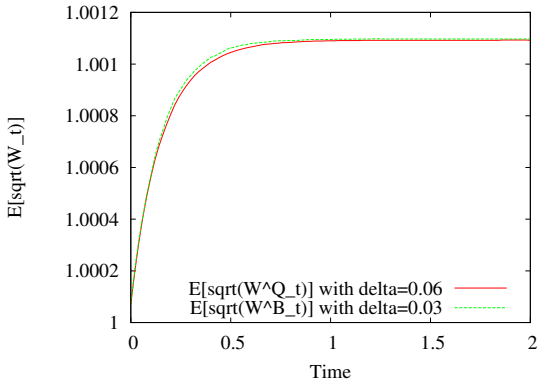
(a) $\mu = 0.01, \sigma_0 = 0.05, \sigma_1 = 0.2, \lambda = 1$



(b) $\mu = 0.01, \sigma_0 = 0.05, \sigma_1 = 0.2, \lambda = 2$



(c) $\mu = 0.02, \sigma_0 = 0.15, \sigma_1 = 0.3, \lambda = 1$



(d) $\mu = 0.02, \sigma_0 = 0.15, \sigma_1 = 0.3, \lambda = 6$

Figure 5.7: Comparison of performances for different λ.

In the above figures, we compare the performances of optimal trading strategies based respectively on quadratic variation and Bollinger Bands indicators, when we have taken the optimal choice of δ for both indicators.

In figures 5.7(a) and 5.7(b), we take a small value of σ_0 and σ_1, and we change the value of λ. We remark that for $\lambda = 1$, the performances are almost equal and for $\lambda = 2$, the performances are also almost equal, but the Bollinger Bands slightly overperfoms the quadratic variation indicator. From tabular 5.3, we have the expected miss of the detection time τ given respectively for quadratic variation as in (5.1.7), and for bollinger Bands as in (5.1.8). We remark that in the case of $\lambda = 2$, the expected

miss $EK(B)$ of Bollinger Bands is smaller than the expected miss $EK(Q)$ of quadratic variation, that means that Bollinger Bands can detect the change time τ faster than quadratic variation.

In figures 5.7(c) and 5.7(d), we take a large value of σ_0 and σ_1, and we change the value of λ. We remark that for $\lambda = 1$, the quadratic variation overperfoms the Bollinger Bands indicator, but for $\lambda = 6$, the performances are almost equal and Bollinger Bands slightly overperfoms the quadratic variation indicator. Also from tabular 5.4, we remark that for $\lambda = 6$, the expected miss $EK(B)$ is smaller than $EK(Q)$.

λ	$EK(Q)$	$EK(B)$
1	0.237	0.267
2	0.124	0.105

Table 5.3: The expected miss $EK(Q)$ and $EK(B)$ corresponding to figures 5.7(a) and 5.7(b)

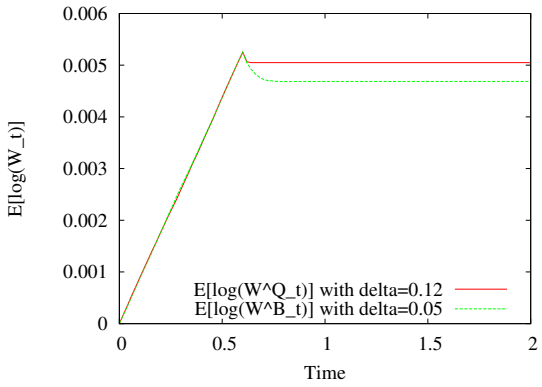
.

λ	$EK(Q)$	$EK(B)$
1	0.291	0.366
6	0.097	0.089

Table 5.4: The expected miss $EK(Q)$ and $EK(B)$ corresponding to figures 5.7(c) and 5.7(d).

5.2.2 Comparison result for small data $\Delta = 10^{-4}$

As we have said that it is well known in the literature that quadratic variation indicator is an optimal estimator of the volatility in the case of small value of time increment Δ. So in this section we just give an simple example which show clearly how the optimal trading strategy based on quadratic variation overperfom the trading strategy based on Bollinger Bands. We assume that the change time is deterministic, $\tau = 0.6$.



(a) $\mu = 0.01, \sigma_0 = 0.05, \sigma_1 = 0.2$

Figure 5.8: Comparison of performances for $\Delta = 10^{-4}$.

Part II

Portfolio allocation problem

Chapter 6

Optimal portfolio allocation problem with random time change

6.1 Introduction

The objective of this chapter is to study the optimal portfolio allocation strategy in the case where the model has a change of volatility at a random time τ. We aim to make explicit the optimal wealth and strategy in the case where the model is perfectly known by the trader. Of course this situation is unrealistic. But, it is worth computing the best performance that one can expect within our setting. This performance represents an optimal benchmark for mis-specified allocation strategies relying either on a mathematical model or on technical analysis.

We want to exhibit the mathematical optimal strategy issued from the stochastic control theory. But, it appears that we can't apply the classical stochastic control theory without care because:

- The diffusion coefficient of the dynamics of the risky asset changes at the random time τ. This change makes the filtration generated by the prices different from the filtration generated by the Brownian motion.

- The trader's strategy needs to be adapted to the filtration generated by the prices.

In order to circumvent the difficulties of our optimal allocation problem, we will use a techniques which with we can take in account the change time τ in order to decompose the initial allocation problem into an allocation problem before the change of volatility and an allocation problem after the change of volatility.

6.2 Description of the mathematical model

The financial market consists of one risky asset and a bank account. The bank account has dynamics

$$\begin{cases} dS_t^0 = S_t^0 r dt \\ S_0^0 = 1 \end{cases}$$

and the risky asset has dynamics

$$\begin{cases} \frac{dS_t}{S_t} = \mu(S_t)dt + \Big(\sigma_1(S_t) + (\sigma_0(S_t) - \sigma_1(S_t))\, \mathbb{1}_{t\leqslant\tau}\Big)dB_t \\ S_0 = S^0 \end{cases} \tag{6.2.1}$$

where $(B_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion on a given probability space $(\Omega, F, \mathbb{P})$(below we will precise the filtration). At the random time τ , which is neither known, nor directly observable, the diffusion term changes from the function σ_0 to the function σ_1. we also assume that the Brownian motion $(B_t)_{t\geqslant 0}$ and the random variable τ are independent.

We now take sufficient assumptions on the coefficients in order that (6.2.1) has a solution. These are standard assumptions that we should be able to show by Feller's test for explosions that the solution of (6.2.1) does not explode, that is, the solution does not touch 0 or ∞ in finite time.

Let the coefficients $\sigma_0, \sigma_1, \mu : (0,\infty) \mapsto \mathbb{R}$ satisfy the following conditions:

i) $\exists\ c, C > 0$ such that, $\forall x$, we have $0 < c \leqslant \sigma_0^2(x) < \sigma_1^2(x) \leqslant C$,

ii) μ, σ_0 and σ_1 are continuous and bounded,

iii) $v_0(0+) = v_0(\infty) = \infty$, where v_0 is a function defined on $(0,\infty)$ as follows:

$$v_0(x) = \int_1^x \exp\left(-2\int_1^y \frac{\mu(u)}{\sigma_0^2(u)u}du\right)\int_1^y \frac{2}{\exp\left(-2\int_1^z \frac{\mu(u)}{\sigma_0^2(u)u}du\right)\sigma_0^2(z)z^2}dzdy,$$

iv) $v_1(0+) = v_1(\infty) = \infty$, where v_1 is a function defined on $(0,\infty)$ as follows:

$$v_1(x) = \int_1^x \exp\left(-2\int_1^y \frac{\mu(u)}{\sigma_1^2(u)u}du\right)\int_1^y \frac{2}{\exp\left(-2\int_1^z \frac{\mu(u)}{\sigma_1^2(u)u}du\right)\sigma_1^2(z)z^2}dzdy.$$

Then from Feller's test for explosions given in [18, p.348] and under the above assumptions, we can deduce that the explosion time is infinite and therefore the solution of (6.2.1) does not explode.

6.3 Stochastic differential equation for (S_t)

We aim in this section to define a filtration $\mathbb{G}$, such that the stochastic integral in the price dynamic is well defined in $\mathbb{G}$.

Let us introduce the process $Z_t = \mathbb{1}_{t>\tau}, 0 \leqslant t \leqslant T$ and $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{t\in[0,T]}$ be the filtration generated by this process. We also denote by $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ the filtration generated by the Brownian motion B. We define the enlarged progressive filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ as:

$$\mathbb{G} = \mathbb{F}^B \vee \mathbb{F}^Z.$$

We now prove that $(B_t)_{t\geqslant 0}$ is a $\mathbb{G}$ Brownian motion. This result can be deduced from Jacod's countable expansion theorem (see Protter [25], chap.II,theorem 5). Here, we give an elementary proof with reflects our simple setting.

Proposition 6.3.1. *$(B_t)_{t\geqslant 0}$ is a $\mathbb{G}$ Brownian motion .*

Proof :
By Levy characterization of Brownian motion, it's enough to show that $(B_t)_{t\geqslant 0}$ is continuous $\mathbb{G}$ local-martingale with:

$$< B >_t = t \qquad a.s. \tag{6.3.1}$$

We obviously have only to prove that B_t is $\mathbb{G}$ martingale. $\forall 0 \leqslant s \leqslant t$,

$$\begin{aligned}\mathbb{E}[B_t|\mathcal{G}_s] &= \mathbb{E}[(B_t - B_s + B_s)|\mathcal{G}_s] \\ &= \mathbb{E}[(B_t - B_s)|\mathcal{G}_s] + \mathbb{E}[B_s|\mathcal{G}_s]\end{aligned}$$

As $\mathcal{G}_s = \mathcal{F}_s^B \vee \sigma(\mathbb{1}_{\theta>\tau}, \theta \leqslant s)$ and (B_s) and τ are independent, we have:

$$\mathbb{E}[(B_t - B_s)|\mathcal{G}_s] = \mathbb{E}[(B_t - B_s)|\mathcal{F}_s^B] = 0$$

Furthermore, (B) is $\mathbb{G}$-adapted, therefore $(B_t)_{t\geqslant 0}$ is a $\mathbb{G}$ Brownian motion.

□

From the definition of the process (Z_t), the dynamics of the price process (S_t) can be obviously expressed in terms of Z as follows:

$$\frac{dS_t}{S_t} = \mu(S_t)dt + \Big(\sigma_0(S_t)(1 - Z_t) + \sigma_1(S_t)Z_t\Big)dB_t. \tag{6.3.2}$$

As Z_t is $\mathbb{G}$-adapted and (B_t) is a $\mathbb{G}$-Brownian motion, then the stochastic integral in the above dynamic (6.3.2) is well defined in the filtration $\mathbb{G}$.

In the sequel, the filtration generated by the price is denoted by $\mathbb{F}^S = (\mathcal{F}_t^S)_{t\in[0,T]}$.

Lemma 6.3.2. *The process (Z_t) is $\mathbb{F}^S$-adapted.*

Proof :

Let $Y_t = \log(S_t)$ and denote by $g(t)$ the left-derivative of the quadratic variation for the process Y at time t:

$$g(t) = \begin{cases} \sigma_0^2(S_t) & \text{if } Z_t = 0 \\ \\ \sigma_1^2(S_t) & \text{if } Z_t = 1 \end{cases} \tag{6.3.3}$$

then $Z_t = \mathbb{1}_{g(t)=\sigma_1^2(S_t)}$. As $g(t)$ is the left-derivative of the quadratic variation for the process Y, thus it is $\mathbb{F}^S$-adapted, and therefore the result follows. □

Lemma 6.3.3. *The filtration generated by the observations $\mathbb{F}^S$ is equal to the augmented filtration $\mathbb{G} = \mathbb{F}^B \vee \mathbb{F}^Z$.*

Proof :
From lemma 6.3.2, (Z_t) is $\mathbb{F}^S$-adapted. Furthermore, we have

$$\log(S_t) = \int_0^t \Big(\mu(S_\theta) - \frac{1}{2}\Big(\sigma_0^2(S_\theta)(1-Z_\theta) + \sigma_1^2(S_\theta)Z_\theta\Big)\Big)d\theta + \int_0^t (\sigma_0(S_\theta)(1-Z_\theta) + \sigma_1(S_\theta)Z_\theta)dB_\theta.$$

Remark that under the assumption $i)$ on σ_0 and σ_1, that is, $\forall x, 0 < c \leqslant \sigma_0^2(x) < \sigma_1^2(x)$, we have that the integrand in the stochastic integral is bounded below by a strictly positive constant. Then it is enough to differentiate the above equation and divide by the integrand of the stochastic integral (which is strictly greater than zero), in order to deduce that (B_t) is written in terms of $\mathbb{F}^S$-adapted processes, therefore it is $\mathbb{F}^S$-adapted, and so $\mathbb{G} := \mathbb{F}^B \vee \mathbb{F}^Z \subset \mathbb{F}^S$. The other inclusion is obvious. Therefore the result follows.

□

6.4 Right-discontinuity of the filtration $(\mathcal{F}_t^S)_{t\geqslant 0}$

The objective of this section is to prove that $(\mathcal{F}_t^S)$ is not a right continuous filtration. This could follow from results in the general theory of stochastic processes. We propose here a simple proof adapted to our framework.

The following proposition stated, e.g.,in Karatzas and Shreve in [18, p.6] will be useful to prove the discontinuity of the filtration $(\mathcal{F}_t^S)$. It is known as Galmarino test. A

proof can be found in Dellacherie and Meyer [10, p.234], or in Revuz and Yor [26, chap.I, section 4] for the canonical space framework. Here we give a different proof.

Proposition 6.4.1. *Let X be a stochastic process and T an $(\mathcal{F}_t^X)$ stopping time. Suppose that for some pair $w_0, w_0' \in \Omega$, we have $X_t(w_0) = X_t(w_0')\ \forall t \in [0, T(w_0)] \cap [0, \infty[$. Then $T(w_0) = T(w_0')$.*

Proof :

We denote $\theta_0 = T(w_0)$. Let us define :

$$D_{w_0,w_0'} = \{A \in \mathcal{F}_{\theta_0}^X / w_0 \in A \Longleftrightarrow w_0' \in A\}$$

$D_{w_0,w_0'}$ is the set of elements $A \in \mathcal{F}_{\theta_0}^X$ such that : ($w_0 \in A$ and $w_0' \in A$) or ($w_0 \notin A$ and $w_0' \notin A$).

We split the proof into two steps:

- First step: $D_{w_0,w_0'}$ is a σ-algebra

  We have:

  - $\Omega \in D_{w_0,w_0'}$.
  - $D_{w_0,w_0'}$ is closed under complementation.
  - $D_{w_0,w_0'}$ is closed under countable unions. We show that if $\forall n \in \mathbb{N}, A_n \in D_{w_0,w_0'}$ then $\cup_{n\in\mathbb{N}} A_n \in D_{w_0,w_0'}$. In fact:

  i) $\forall n \in \mathbb{N}, A_n \in D_{w_0,w_0'} \Rightarrow A_n \in \mathcal{F}_{\theta_0}^X$ and therefore $\cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_{\theta_0}^X$.

  ii) It remains to prove that $w_0 \in \cup_{n\in\mathbb{N}} A_n \Longleftrightarrow w_0^{'} \in \cup_{n\in\mathbb{N}} A_n$.
  If $w_0 \in \cup_{n\in\mathbb{N}} A_n$, then there exists a n_0 such that $w_0 \in A_{n_0}$. As $A_{n_0} \in D_{w,w_0}$, then $w_0' \in A_{n_0}$. This implies that $w_0' \in \cup_{n\in\mathbb{N}} A_n$.

  Hence from *i)* and *ii)*, $\cup_{n\in\mathbb{N}} A_n \in D_{w_0,w_0'}$. Therefore $D_{w_0,w_0'}$ is a σ-algebra.

- Second step: $\mathcal{F}_{\theta_0}^X \subset D_{w_0,w_0'}$

  By hypotheses, $\forall t \leqslant \theta_0$ we have $X_t(w_0) = X_t(w_0')$. Then $\forall C$ Borel set in $\mathbb{R}$, we have $\{X_t \in C\} \in D_{w_0,w_0'}$. This implies that X_t is $D_{w_0,w_0'}$ measurable for all $0 \leqslant t \leqslant \theta_0$. As $\mathcal{F}_{\theta_0}^X$ is the smallest σ-algebra with respect to which X_t is measurable for every $0 \leqslant t \leqslant \theta_0$, therefore we conclude that $\mathcal{F}_{\theta_0}^X \subset D_{w_0,w_0'}$.

Let us define the set:

$$Q_{\theta_0} = \{w \in \Omega; T(w) = \theta_0\}$$

To end the proof, we must be able to show that $Q_{\theta_0} \in D_{w_0,w_0'}$.

As T is an stopping time, we have $I_{\theta_0} := \{w \in \Omega; T(w) \leqslant \theta_0\} \in \mathcal{F}_{\theta_0}^X$. Also we have $J_{\theta_0} := \{w \in \Omega; T(w) < \theta_0\} \in \mathcal{F}_{\theta_0}^X$, because $J_{\theta_0} = \cup_{n\in\mathbb{N}}\{w \in \Omega; T(w) \leqslant \theta_0 - \frac{1}{n}\} \in \mathcal{F}_{\theta_0}^X$.
As $Q_{\theta_0} = I_{\theta_0} \cap J_{\theta_0}^c$, then $Q_{\theta_0} \in \mathcal{F}_{\theta_0}^X \subset D_{w_0,w_0'}$. As $w_0 \in Q_{\theta_0}$ and $Q_{\theta_0} \in D_{w_0,w_0'}$, then w_0' $\in Q_{\theta_0}$. Hence $T(w_0) = T(w_0')$.

□

Let us denote by Ω_B and Ω_Z respectively the space for the Brownian motion (B_t) and the process (Z_t).

Proposition 6.4.2. *Suppose that the underlying probability space is the canonical product space $\Omega = \Omega_B \times \Omega_Z$. Let S_t be a solution of (6.2.1). Then τ is an $(\mathcal{F}_{t^+}^S)$ stopping time but not an $(\mathcal{F}_t^S)$ stopping time.*

Proof :

We split the proof into two steps:

- First step: we prove that τ is an $(\mathcal{F}_{t^+}^S)$ stopping time.

  For $t \geqslant 0$, we have $\forall\ \theta > t$, $\{\tau \leqslant t\} \subset \{\tau < \theta\} = \{g(\theta) = \sigma_1^2(S_\theta)\} \in \mathcal{F}_\theta^S$. Then $\{\tau \leqslant t\} \subset \bigcap_{\theta>t} \{g(\theta) = \sigma_1^2(S_\theta)\} \in \mathcal{F}_{t^+}^S$. Thus τ is an $(\mathcal{F}_+^S)$ stopping time.

- Second step: we prove that τ is not an $(\mathcal{F}_t^S)$ stopping time.

  We recall that our probability space is the canonical space. Let $w_0 = (w, \hat{w}), w_1 = (w, \tilde{w}) \in \Omega$ and assume that $\tau(\hat{w}) < \tau(\tilde{w})$, this is possible because τ is not deterministic. We also consider the process Y_t defined in Ω as below:

  For $w^* = (w, w') \in \Omega$,

$$Y_t(w^*) := \log(S_t)(w^*) = \int_0^{t\wedge\tau(w')} \sigma_0(S_\theta(w))dB_\theta(w) + \int_{t\wedge\tau(w')}^t \sigma_1(S_\theta(w, w'))dB_\theta(w)$$
$$+ \int_0^{t\wedge\tau(w')} \left(\mu(S_\theta(w)) - \frac{1}{2}\sigma_0^2(S_\theta(w))\right) d\theta + \int_{t\wedge\tau(w')}^t \left(\mu(S_\theta(w)) - \frac{1}{2}\sigma_1^2(S_\theta(w))\right) d\theta.$$

Notice that the trajectories $Y_t(w_0)$ and $Y_t(w_1)$ are equal until $\tau(\hat{w})$, so if we assume that τ is an $\mathbb{F}^S$ stopping time, then we can deduce from proposition 6.4.1 that $\tau(\hat{w}) = \tau(\tilde{w})$, which exhibits a contradiction with the assumption $\tau(\hat{w}) < \tau(\tilde{w})$. Thus τ is not an $\mathbb{F}^S$-stopping time.

□

Corollary 6.4.3. *Within the framework of proposition 6.4.2, the filtration* $(\mathcal{F}_t^S)_{t\geqslant 0}$ *is right-discontinuous.*

Proof :
For that we prove that there exists a set $A \in \mathcal{F}_{t^+}^S$ but $A \notin \mathcal{F}_t^S$. On the one side, from proposition 6.4.2, τ is not an $(\mathcal{F}_t^S)$ stopping time, then there exists t_1 such that $\{\tau \leqslant t_1\} \notin \mathcal{F}_{t_1}^S$. On the other hand, τ is an $(\mathcal{F}_{t^+}^S)$ stopping time, thus $\{\tau \leqslant t_1\} \in \mathcal{F}_{t_1^+}^S$. Therefore, we can choose $A = \{\tau \leqslant t_1\}$.

□

6.5 The optimal portfolio allocation strategy under a change of volatility

In this section our aim is to make explicit the optimal wealth and strategy for a trader who knows all the parameters of the mathematical model. As the main difference with the classical stochastic control theory comes from the discontinuity of the filtration generated by the prices or more precisely from the change at time τ, so we aim to resolve this control problem by using a specific feature of the change time τ.

Definition 6.5.1. *(Trading strategy) A trading strategy is a two-dimensional stochastic process* $h = \{(H_t^0, H_t), t \in [0,T]\}$ *satisfying*

- h *is progressively measurable*
- h *is adapted, i.e.* $\forall\ t\ h_t$ *is* $\mathcal{F}_t^S$*-measurable.*

The financial interpretation of the trading strategy is that H_t^0 is the number of bonds held by the trader at time t and H_t is the number of stocks that he holds at time t. So, the wealth process $(W_t, t \geqslant 0)$ of the portfolio is such that for all t,

$$W_t = H_t^0 S_t^0 + H_t S_t.$$

Definition 6.5.2. *A pair $(H_t^0; H_t)$ is called self-financing if the corresponding wealth process W_t is a continuous and adapted process such that*

$$W_t = W_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u. \tag{6.5.1}$$

This means that the composition of the portfolio changes from time t to time $t + \Delta t$ with no endowment.

Let π_t be the fraction of the current wealth W_t that the trader decides to invest in the risky asset at time t, so $1 - \pi_t$ is the fraction of wealth invested in the bond. we suppose that the trading strategy is self-financing, then the wealth process satisfies the following SDE :

$$\begin{aligned} dW_t^\pi &= H_t^0 dS_t^0 + H_t dS_t \\ &= rH_t^0 S_t^0 dt + H_t S_t \left(\mu(S_t)dt + (\sigma_0(S_t) + (\sigma_1(S_t) - \sigma_0(S_t))Z_t)dB_t\right) \\ &= W_t^\pi \left[(1 - \pi_t)\, r dt + \pi_t \left(\mu(S_t)dt + (\sigma_0(S_t) + (\sigma_1(S_t) - \sigma_0(S_t))Z_t)dB_t\right)\right]. \end{aligned} \tag{6.5.2}$$

Let us denote by $\mathbb{E}_{t,x}^{\zeta,z}$ the expectation operator conditional on $W_t = x, S_t = \zeta$ and $Z_t = z$. We consider a utility function U which is, either the logarithmic utility function, or an element of the set $\mathcal{U}$ of the increasing and concave functions of class $C^1((0, +\infty); \mathbb{R})$ which satisfy: $U(0) = 0$, and there exist real numbers $K > 0$ and $0 \leqslant \alpha \leqslant 1$ such that

$$0 < U'(x) \leqslant K\left(1 + x^{-\alpha}\right) \quad \text{for all } x > 0.$$

The performance of an admissible trading strategy π associated with a wealth process W, is measured over the finite horizon T by the cost function J defined as:

$$J(t, x, \zeta, z, \pi) = \mathbb{E}_{t,x}^{\zeta,z}\left[U(W_T^\pi)\right]$$

and the investor's objective is to maximize his expected utility of wealth at the terminal time T. He solves the following optimization problem :

$$V(t, x, \zeta, z) = \sup_{\pi \in \mathcal{A}} J(t, x, \zeta, z, \pi), \tag{6.5.3}$$

where $\mathcal{A}$ denotes the set of the admissible controls $(\pi_t)_{t\in[0,T]}$ which are $\mathbb{F}^S$-adapted, take their values in a compact $\mathbb{U} \subset \mathbb{R}$.

In the following, the function $V(t,x,\zeta,0)$ will be interpreted as the value function before the change time, that is, $z=0$ and the function $V(t,x,\zeta,1)$ will be interpreted as the value function after the change time, that is, $z=1$.

First, we present the density approach which have been introduced by Pham and Jiao in [17]. We shall work in this approach with a density hypothesis on the conditional law of τ given the filtration generated by the Brownian motion. We show that the pair $(V(t,x,\zeta,0),V(t,x,\zeta,1))$ is the unique viscosity solution of a Hamilton-Jacobi-Bellman system.

Second, we show by verification theorem that under smoothness assumptions on the value function, the pair $(V(t,x,\zeta,0),V(t,x,\zeta,1))$ is solution of a Hamilton-Jacobi-Bellman system.

6.5.1 Density approach

In this approach, Pham and Jiao [17] have taken benefit of the specific feature of the change time τ and how we can separate the initial optimisation problem into a problem after the change-time and a problem before the change time, by relying on the density hypothesis on the change time τ.

They introduce the enlarged progressive filtration $\mathbb{G}^{'}=\mathbb{F}^{B}\vee\mathbb{F}^{D}$, where $\mathbb{F}^{D}$ is the filtration generated by the process $D_t=\mathbb{1}_{t\geqslant\tau}\ \ 0\leqslant t\leqslant T$. Then the optimal investment problem is to maximize the expected utility of wealth at the terminal time T over all the admissible strategy $\pi\in\mathcal{A}^{'}$, where $\mathcal{A}^{'}$ is the set of a $\mathbb{G}^{'}$-predictable processes.

The main advantage of the set $\mathcal{A}^{'}$ is that we can use the following important result given in [14] and [16] about the decomposition of any $\mathbb{G}^{'}$-predictable process. Let L be a $\mathbb{G}^{'}$-adapted process. Then there exists an $\mathbb{F}^{B}$-adapted process L^{b}, and a family of processes $\{L^{a}_{t}(\theta),\ \theta\leqslant t\leqslant T,\theta\in[0,T]\}$, where $L^{a}_{t}(\theta)$ is measurable with respect to $\mathcal{F}^{B}_{t}\otimes\mathcal{B}(\mathbb{R}^{+})$, such that

$$L_t=L^{b}_{t}\mathbb{1}_{t\leqslant\tau}+L^{a}_{t}(\tau)\mathbb{1}_{t>\tau},\qquad\forall t\in[0,T]. \tag{6.5.4}$$

Density hypothesis:

For any $t\in[0,T]$, the conditional distribution of ζ given $\mathcal{F}^{B}_{t}$ admits a density with respect to the Lebesgue measure, i.e. there exists a family of $\mathcal{F}^{B}_{t}\otimes\mathcal{B}(\mathbb{R}^{+})$ measurable positive function $(w,\theta)\to\alpha_t(w,\theta)$ which satisfies

$$\mathbb{P}\left[\zeta\in d\theta|\mathcal{F}^{B}_{t}\right]=\alpha_t(\theta)d\theta,\qquad t\in[0,T]\qquad\qquad(\mathbf{DH})$$

The family $\alpha_t(.)$ is called the conditional density of ζ with respect to Lebesgue measure given $\mathcal{F}^{B}_{t}$.

By definition of the conditional expectation, we have for any (bounded) Borel function f,

$$\mathbb{E}\left[f(\zeta)|\mathcal{F}_t^B\right] = \int_0^\infty f(\theta)\alpha_t(\theta)d\theta \qquad \text{a.s.}$$

The conditional distribution of ζ is also characterized by the survival probability function

$$G(t) = \mathbb{P}(\zeta > t|\mathcal{F}_t^B) = \int_t^\infty \alpha_t(\theta)d\theta.$$

In the sequel, q denotes the density of the change time τ. As the change time τ is supposed independent of the Brownian motion B, then $\alpha_t(\theta)$ is simply a known deterministic function $q(\theta)$ of $\theta \in \mathbb{R}^+$ and the survival probability is a deterministic function given as:

$$G(t) = \mathbb{P}[\tau > t|\mathcal{F}_t^B] = \mathbb{P}[\tau > t] = \int_t^\infty q(\theta)d\theta. \tag{6.5.5}$$

In the sequel we will use the notation $G(t,s)$ and $q(t,s)$ in order to describe the survival probability and the density of τ at time s such that $\tau > t$.

Let us introduce the following processes:

$$dS_t^{(0)} = S_t^{(0)}\left[\mu(S_t^{(0)})dt + \sigma_0(S_t^{(0)})dB_t\right], \tag{6.5.6}$$

$$dS_t^{(1)} = S_t^{(1)}\left[\mu(S_t^{(1)})dt + \sigma_1(S_t^{(1)})dB_t\right], \tag{6.5.7}$$

and

$$dW_t^{\pi^{(0)}} = W_t^{\pi^{(0)}}\left[\left(\left(\mu(S_t^{(0)}) - r\right)\pi_t^{(0)} + r\right)dt + \pi_t^{(0)}\sigma_0(S_t^{(0)})dB_t\right], \tag{6.5.8}$$

$$dW_t^{\pi^{(1)}} = W_t^{\pi^{(1)}}\left[\left(\left(\mu(S_t^{(1)}) - r\right)\pi_t^{(1)} + r\right)dt + \pi_t^{(1)}\sigma_1(S_t^{(1)})dB_t\right], \tag{6.5.9}$$

where $\pi^{(0)}$ is $\mathbb{F}^B$-adapted, representing the proportion of wealth invested before the change, and $\pi_t^{(1)}$ is $\mathbb{F}^B$-adapted, representing the proportion of wealth invested after the change at time τ.

Notice that with this approach, the set of admissible strategies is the set $\mathcal{A}'$ of $\mathbb{G}'$-predictible processes. So we first solve this problem under the constraint that $\pi \in \mathcal{A}'$; then, from the definition of $\mathcal{A}'$, we can deduce that the optimal strategy π given by

$$\pi_t = \pi_t^{(0)} \mathbb{1}_{t \leqslant \tau} + \pi_t^{(1)} \mathbb{1}_{t > \tau},$$

is a $\mathbb{F}^S$-adapted process and thus belongs to the set $\mathcal{A}$.

We now present the main result obtained in [17] namely the decomposition of the maximisation problem into two problems as follows.

$$V(t,x,\zeta,0) = \sup_{\pi^{(0)} \in \mathcal{A}^{(0)}} \mathbb{E}_{t,x}^{\zeta,0} \left[U\left(W_T^{\pi^{(0)}}\right) G(t,T) + \int_t^T V(\theta, W_\theta^{\pi^{(0)}}, S_\theta^{(0)}, 1) q(t,\theta) d\theta, \right].$$

where $\mathcal{A}^{(0)}$ is the set of the proportion of wealth invested before the change time; $\mathcal{A}^{(0)}$ is a $\mathbb{F}^B$-adapted process and G the survival probability. Notice that the integrand of the integral is described in terms of the value function after the rupture.

In the sequel, we shall study these two optimisation problems in the particular case of the exponential law of τ with parameter λ.

In this case for the law of τ, the above decomposition can be rewritten as follows:

$$V(t,x,\zeta,0) = \sup_{\pi^{(0)} \in \mathcal{A}^{(0)}} \mathbb{E}_{t,x}^{\zeta,0} \left[U\left(W_T^{\pi^{(0)}}\right) e^{-\lambda(T-t)} + \int_t^T V(\theta, W_\theta^{\pi^{(0)}}, S_\theta^{(0)}, 1) \lambda e^{-\lambda(\theta-t)} d\theta \right].$$

We show in the above equation how the solution of the before-change time problem $V(t,x,\zeta,0)$ depends on the solution of the after-change time problem $V(t,x,\zeta,1)$. Each of these optimisation problems is performed in market model driven by the Brownian motion and with coefficients adapted with respect to the Brownian reference filtration.

Let us now consider the system

$$(S) \begin{cases} \frac{\partial v}{\partial t}(t,x,\zeta,0) + \sup_{\pi \in \mathbb{U}} \mathcal{L}_\pi^{(0)} v(t,x,\zeta,0) + \lambda \left(v(t,x,\zeta,1) - v(t,x,\zeta,0) \right) = 0 \\ \frac{\partial v}{\partial t}(t,x,\zeta,1) + \sup_{\pi \in \mathbb{U}} \mathcal{L}_\pi^{(1)} v(t,x,\zeta,1) = 0 \end{cases}$$

with the boundary condition

$$v(T,x,\zeta,z) = U(x), \qquad x, \zeta \in \mathbb{R}^+, \ z \in \{0,1\}. \tag{6.5.10}$$

where

$$\mathcal{L}_\pi^{(0)} v(t,x,\zeta,0) = x(\pi\mu(\zeta)+(1-\pi)r)\frac{\partial v}{\partial x} + \frac{1}{2}x^2\pi^2\sigma_0^2(\zeta)\frac{\partial^2 v}{\partial^2 x} + \zeta\mu(\zeta)\frac{\partial v}{\partial \zeta} + \frac{1}{2}\sigma_0^2(\zeta)\zeta^2\frac{\partial^2 v}{\partial \zeta^2}, \tag{6.5.11}$$

and

$$\mathcal{L}_\pi^{(1)} v(t,x,\zeta,1) = x(\pi\mu(\zeta)+(1-\pi)r)\frac{\partial v}{\partial x} + \frac{1}{2}x^2\pi^2\sigma_1^2(\zeta)\frac{\partial^2 v}{\partial^2 x} + \zeta\mu(\zeta)\frac{\partial v}{\partial \zeta} + \frac{1}{2}\sigma_1^2(\zeta)\zeta^2\frac{\partial^2 v}{\partial \zeta^2}. \tag{6.5.12}$$

6.5.2 Existence and Uniqueness of the viscosity solution

Under the assumptions on the utility function U, we can easily show that there exists a constant C such that, for all positive real numbers $x, \bar{x}$ we have:

$$\mid U(x) - U(\bar{x}) \mid \leqslant C \mid x - \bar{x} \mid \left(1 + x^{-\alpha} + \bar{x}^{-\alpha}\right).$$

As in the classical case, we can show from the above inequality and under the bounded and continuous assumptions on the functions μ, σ_0 and σ_1 that the value functions $V(t,x,\zeta,0)$ and $V(t,x,\zeta,1)$ are continuous on $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+\times\{0,1\}$.

Definition 6.5.3. *A pair of continuous functions $(v(t,x,\zeta,0), v(t,x,\zeta,1))$ on $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+$ is a viscosity supersolution (resp. subsolution) to HJB system (S) if*

$$\begin{aligned}
&\frac{\partial \Phi}{\partial t}(\bar{t},\bar{x},\bar{\zeta},0) + \mathcal{L}_\pi^{(0)}\Phi(\bar{t},\bar{x},\bar{\zeta},0) + \lambda\left(\Phi(\bar{t},\bar{x},\bar{\zeta},1) - \Phi(\bar{t},\bar{x},\bar{\zeta},0)\right) \geqslant \ (\textit{resp.} \leqslant)\ 0\\
&\frac{\partial \Phi}{\partial t}(\bar{t},\bar{x},\bar{\zeta},1) + \mathcal{L}_\pi^{(1)}\Phi(\bar{t},\bar{x},\bar{\zeta},1) \geqslant \ (\textit{resp.} \leqslant)\ 0,
\end{aligned}$$

for all C^2 pair functions $(\Phi(t,x,\zeta,0), \Phi(t,x,\zeta,1))$ on $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+$ and any $(\bar{t},\bar{x},\bar{\zeta})$ such that $(\Phi(\bar{t},\bar{x},\bar{\zeta},0), \Phi(\bar{t},\bar{x},\bar{\zeta},1)) = (v(\bar{t},\bar{x},\bar{\zeta},0), v(\bar{t},\bar{x},\bar{\zeta},1))$ and $v \geqslant$ (resp.$\leqslant$) Φ on $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+$.

Definition 6.5.4. *A pair of continuous functions $(v(t,x,\zeta,0), v(t,x,\zeta,1))$ on $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+$ is a viscosity solution to the above HJB system (S) if it is both a viscosity supersolution and subsolution to this system.*

Theorem 6.5.5. *Assume that σ_0 and σ_1 are in $C_b^1(\mathbb{R})$. Then the pair $(V(t,x,\zeta,0), V(t,x,\zeta,1))$ is the unique viscosity solution of the HJB system (S) and satisfying the boundary condition (6.5.10).*

Proof :
We first prove the existence of a viscosity solution of system (S). By truncation of the coefficients of the SDE satisfied by (S_t), we obtain that the process $(W_t^{\pi^{(1)}}, S_t^{(1)})$ satisfy the classical assumptions which ensure that the value function $V(t,x,\zeta,1)$ is a viscosity solution to the second HJB of system (S) (cf.theorem 5.2 of [32, chap.4]). We therefore conclude the existence result by the stability result given in proposition $I.3$ of Lions [20].

Similarly, we can conclude that the value function $V(t,x,\zeta,0)$ is a viscosity solution of the first HJB of system (S).

We now prove the uniqueness of the viscosity solution of HJB system (S). As the value functions $V(t,x,\zeta,0)$ and $V(t,x,\zeta,1)$ are continuous and lipschitz, then by applying theorem 6.2 in chapter IV of [32], the uniqueness follows.

□

6.5.3 A verification theorem

Under smoothness assumptions on the value function, we show by verification theorem that the pair $(V(t,x,\zeta,0), V(t,x,\zeta,1))$ is a solution of the above HJB system. We assume that τ follows an exponential law with parameter λ, that is the process (Z_t) is an absorbed Markov process at state 1.

We start by making explicit the infinitesimal operator of the $\mathbb{R}^+ \times \mathbb{R}^+ \times \{0,1\}$ valued process (W^π, S, Z). Then we show our verification theorem 6.5.6 and we prove that the optimal trading strategy is to follow the optimal strategy which characterizes $V(t,x,\zeta,0)$ and then to switch when the change occurs to the optimal strategy which characterizes $V(t,x,\zeta,1)$.

Let v be a smooth function defined on $[0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \{0,1\}$. Let $\pi_t = p \in \mathbb{U}, \forall t$. By applying Itô's formula to the process (W^π, S, Z), we obtain that its infinitesimal operator $\mathcal{L}_\pi$ is

$$\mathcal{L}_\pi v(t,x,\zeta,z) = \begin{cases} \mathcal{L}_\pi^{(0)} v(t,x,\zeta,0) + \lambda(v(t,x,\zeta,1) - v(t,x,\zeta,0)) & \text{if} \quad z=0 \\ \mathcal{L}_\pi^{(1)} v(t,x,\zeta,1) & \text{if} \quad z=1 \end{cases}$$

where $\mathcal{L}_\pi^{(0)}$ and $\mathcal{L}_\pi^{(1)}$ are defined respectively as in (6.5.11) and (6.5.12) and they are the infinitesimal operators associated to the process (W^π, S) respectively for fixed $z=0$ and $z=1$.

In the sequel, we aim to show that a smooth solution $(v(t,x,\zeta,0), v(t,x,\zeta,1))$ of the Hamilton-Jacobi-Bellman system (S), coincides with the pair of value functions $(V(t,x,\zeta,0), V(t,x,\zeta,1))$. The proof of this result is decomposed in two parts : for the part concerning the function $V(t,x,\zeta,1)$, the proof is simple and relies essentially on Itô's formula. For the part concerning $V(t,x,\zeta,0)$, the proof is more technical because of the volatility change at time τ.

As $\pi_t^{(0)}$ is $\mathbb{F}^B$-adapted, then there exists a measurable function ρ defined on the canonical space such that:

$$\pi_t^{(0)} = \rho(B_\theta, \theta \leqslant t).$$

Let us introduce $\theta_n^{(0)}$ which will be useful in the sequel:

$$\theta_n^{(0)} = \inf\{s \geqslant t : \int_t^s \left| \frac{\partial v}{\partial x}\left(u, W_u^{\pi^{(0)}}, S_u^{(0)}, 0\right) W_u^{\pi^{(0)}} \pi_u^{(0)} \sigma_0(S_u^{(0)}) \right|^2 du \geqslant n\}$$

Theorem 6.5.6. *(Verification theorem)*

a) Suppose that there exists a pair of functions $(v(t,x,\zeta,0), v(t,x,\zeta,1))$ in $C^{1,2}([0,T[\times\mathbb{R}^+\times\mathbb{R}^+) \cap C([0,T]\times\mathbb{R}^+\times\mathbb{R}^+)$, satisfying the HJB system (S):

$$\begin{cases} \frac{\partial v}{\partial t}(t,x,\zeta,0) + \sup_{\pi\in\mathbb{U}} \mathcal{L}_\pi^{(0)} v(t,x,\zeta,0) + \lambda(v(t,x,\zeta,1) - v(t,x,\zeta,0)) = 0 \\ \frac{\partial v}{\partial t}(t,x,\zeta,1) + \sup_{\pi\in\mathbb{U}} \mathcal{L}_\pi^{(1)} v(t,x,\zeta,1) = 0 \end{cases}$$

with boundary condition $v(T,x,\zeta,z) = U(x), \qquad x,\zeta \in \mathbb{R}^+,\ z \in \{0,1\}$

Then, for all $t \in [0,T], x, \zeta \in \mathbb{R}^+$

$$\begin{cases} V(t,x,\zeta,0) \leqslant v(t,x,\zeta,0) \\ V(t,x,\zeta,1) \leqslant v(t,x,\zeta,1) \end{cases}$$

b) Assume there exists a maximizer $(\pi_*^{(0)}(t,x,\zeta), \pi_*^{(1)}(t,x,\zeta))$ of

$$(p_0,p_1) \to \left(\mathcal{L}_{p_0}^{(0)} v(t,x,\zeta,0), \mathcal{L}_{p_1}^{(1)} v(t,x,\zeta,1)\right)$$

such that

$$\begin{cases} \frac{\partial v}{\partial t}(t,x,\zeta,0) + \mathcal{L}_{\pi_*^{(0)}}^{(0)} v(t,x,\zeta,0) + \lambda(v(t,x,\zeta,1) - v(t,x,\zeta,0)) = 0 \\ \\ \frac{\partial v}{\partial t}(t,x,\zeta,1) + \mathcal{L}_{\pi_*^{(1)}}^{(1)} v(t,x,\zeta,1) = 0 \end{cases}$$

and $V(t,x,\zeta,0) = V(t,x,\zeta,1) = U(x)$.

Then $(V(t,x,\zeta,0), V(t,x,\zeta,1)) = (v(t,x,\zeta,0), v(t,x,\zeta,1))$ *for all* $t \in [0,T], x, \zeta \in \mathbb{R}^+$ *and* π^* *defined as:*

$$\pi_t^* = \pi_*^{(0)}(t, W_t^{\pi^*}, S_t)\mathbb{1}_{Z_t=0} + \pi_*^{(1)}(t, W_t^{\pi^*}, S_t)\mathbb{1}_{Z_t=1}. \tag{6.5.13}$$

is the optimal strategy.

Proof :

I) Show that for all $t \in [0,T],\ x, \zeta \in \mathbb{R}^+$ we have : $V(t,x,\zeta,1) \leqslant v(t,x,\zeta,1)$

Let $\pi \in \mathcal{A}$ be an arbitrary control process and define the stopping time $\tau_n = T \wedge \alpha_n$ where :

$$\alpha_n = \inf\{s \geqslant t : \int_t^s \left| \frac{\partial v}{\partial x}(u, W_u^\pi, S_u, 1)\, W_u^\pi \pi_u \sigma_1(S_u) \right|^2 du \geqslant n\}$$

As $v(t,x,\zeta,1)$ is smooth, by applying Itô's formula to $s \mapsto v(s, W_s^\pi, S_s, 1)$ between t and τ_n, we obtain:

$$\begin{aligned}
\mathbb{E}_{t,x}^{\zeta,1}\left[v(\tau_n, W_{\tau_n}^\pi, S_{\tau_n}, 1)\right] &= v(t,x,\zeta,1) + \mathbb{E}_{t,x}^{\zeta,1}\left[\int_t^{\tau_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, 1) d\theta\right] \\
&\quad + \mathbb{E}_{t,x}^{\zeta,1}\left[\int_t^{\tau_n} \frac{\partial v}{\partial x}(\theta, W_\theta^\pi, S_\theta, 1)\sigma_1(S_\theta)\pi_u W_\theta^\pi dB_\theta\right] \\
&\leqslant v(t,x,\zeta,1)
\end{aligned} \tag{6.5.14}$$

From the definition of τ_n, the expectation of the stochastic integral cancels. The last inequality comes from the fact that $v(t,x,\zeta,1)$ verifies the second HJB equation. We now take $n \to \infty$, $\tau_n \to T$, then by Fatou lemma we have:

$$\mathbb{E}_{t,x}^{\zeta,1}\left[\liminf_n v\left(\tau_n, W_{\tau_n}^\pi, S_{\tau_n}, Z_{\tau_n}\right)\right] \leqslant \liminf_n \mathbb{E}_{t,x}^{\zeta,1}\left[v\left(\tau_n, W_{\tau_n}^\pi, S_{\tau_n}, Z_{\tau_n}\right)\right] \leqslant v(t,x,\zeta,1)$$

Therefore from the boundary condition, we obtain that:

$$V(t,x,\zeta,1) := \mathbb{E}_{t,x}^{\zeta,1}[U(W_T^\pi)] \leqslant v(t,x,\zeta,1).$$

II) Now we show that for all $t \in [0,T]$, $x, \zeta \in \mathbb{R}^+ \times \mathbb{R}^+$ we have: $V(t,x,\zeta,0) \leqslant v(t,x,\zeta,0)$

Let us introduce the stopping time $T_n = T \wedge \theta_n$ where :

$$\theta_n = \inf\left\{ s \geqslant t : \int_t^s \left| \frac{\partial v}{\partial x}(u, W_u^\pi, S_u, 0) W_u^\pi \pi_u (\sigma_0(S_u)(1-Z_u) + \sigma_1(S_u) Z_u) \right|^2 du \geqslant n \right\}$$

By Itô formula, we have:

$$\begin{aligned}
\mathbb{E}_{t,x}^{\zeta,0}\left[v\left(T_n, W_{T_n}^\pi, S_{T_n}, Z_{T_n}\right)\right] &= v(t,x,\zeta,0) + \mathbb{E}_{t,x}^{\zeta,0}\left[\int_t^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta) d\theta\right] \\
&\quad + \mathbb{E}_{t,x}^{\zeta,0}\left[\int_t^{T_n} \left(\frac{\partial v}{\partial x}(\theta, W_\theta^\pi, S_\theta, Z_\theta) W_\theta^\pi \pi_\theta (\sigma_0(1-Z_\theta) + \sigma_1 Z_\theta)\right) dB_\theta\right] \\
&\quad + \mathbb{E}_{t,x}^{\zeta,0}\Big[\left(v(\tau, W_\tau^\pi, S_\tau, 1) - v(\tau, W_\tau^\pi, S_\tau, 0)\right) \mathbb{1}_{\tau \leqslant T_n}\Big] \\
&=: v(t,x,\zeta,0) + A + B + C. \qquad (6.5.15)
\end{aligned}$$

From the definition of T_n, the expectation of the stochastic integral term cancels. It remains to make explicit the terms A and C.

We start by making explicit the term A:

$$\begin{aligned}
A &= \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_t^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta) d\theta\right) \mathbb{1}_{\tau \leqslant T_n}\right] \\
&\quad + \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_t^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta) d\theta\right) \mathbb{1}_{\tau > T_n}\right] \\
&= \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_t^{\tau} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta) d\theta\right) \mathbb{1}_{\tau \leqslant T_n}\right] \\
&\quad + \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_\tau^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta) d\theta\right) \mathbb{1}_{\tau \leqslant T_n}\right] \\
&\quad + \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_t^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta) d\theta\right) \mathbb{1}_{\tau > T_n}\right] \\
&= A_n^1 + A_n^2 + A_n^3.
\end{aligned}$$

i) Make explicit the term A_n^1:

Recall that $T_n = T \wedge \theta_n$, then we have

$$\begin{aligned} A_n^1 &:= \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_t^\tau \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta)d\theta\right)\mathbb{1}_{\tau \leqslant T_n}\right] \\ &= \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_t^\tau \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_{\pi^{(0)}}^{(0)} v\right)(\theta, W_\theta^{\pi^{(0)}}, S_\theta^{(0)}, 0)d\theta\right)\mathbb{1}_{\tau \leqslant T \wedge \theta_n^{(0)}}\right] \end{aligned}$$

As $\pi^{(0)}, W^{\pi^{(0)}}, S^{(0)}$ and $\theta_n^{(0)}$ are functionals of $(B_\theta, \theta \geqslant 0)$, then by independence of $(B_\theta, \theta \geqslant 0)$ and τ, we therefore have:

$$A_n^1 = \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \int_t^s \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_{\pi^{(0)}}^{(0)} v\right)(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)d\theta.\lambda e^{-\lambda(s-t)}ds\right]$$

where $W_\theta^{\pi^{(0)}}(x)$ is the solution of the SDE (6.5.8) with initial condition $W_t^{\pi^{(0)}}(x) = x$ and $\mathbb{E}^B$ is the expected operator w.r.t the law of $(B_\theta, \theta \geqslant 0)$.

As $v(t, x, \zeta, 0)$ is solution of the first HJB of system (S), then

$$A_n^1 \leqslant \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \int_t^s -\lambda\left(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\right)d\theta.\lambda e^{-\lambda(s-t)}ds\right].$$

ii) Make explicit the term A_n^2:

$$\begin{aligned} A_n^2 &:= \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_\tau^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi v\right)(\theta, W_\theta^\pi, S_\theta, Z_\theta)d\theta\right)\mathbb{1}_{\tau \leqslant T_n}\right] \\ &= \mathbb{E}_{t,x}^{\zeta,0}\left[\left(\int_\tau^{T_n} \left(\frac{\partial v}{\partial \theta} + \mathcal{L}_\pi^{(1)} v\right)(\theta, W_\theta^\pi, S_\theta, 1)d\theta\right)\mathbb{1}_{\tau \leqslant T_n}\right] \qquad \text{(from the definition of } \mathcal{L}_\pi) \\ &\leqslant 0. \end{aligned}$$

Where the last inequality comes from the fact that $v(t, x, \zeta, 1)$ is solution of the second HJB equation of system (S).

iii) Make explicit the term A_n^3:

As above we can deduce that:

$$A_n^3 = \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \left(\frac{\partial v}{\partial\theta} + \mathcal{L}^{(0)}_{\pi^{(0)}} v\right)(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0) d\theta . e^{-\lambda(T\wedge\theta_n^{(0)} - t)}\right]$$
$$\leqslant \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} -\lambda\left(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\right) d\theta . e^{-\lambda(T\wedge\theta_n^{(0)} - t)}\right].$$

Therefore

$$A := A_n^1 + A_n^2 + A_n^3$$
$$\leqslant \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \int_t^s -\lambda\left(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\right) d\theta . \lambda e^{-\lambda(s-t)} ds\right]$$
$$+ \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} -\lambda\left(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\right) d\theta . e^{-\lambda(T\wedge\theta_n^{(0)} - t)}\right].$$

It remains to make explicit the term C. Indeed,

$$C = \mathbb{E}^{\zeta,0}_{t,x}\Big[\left(v(\tau, W_\tau^\pi, S_\tau, 1) - v(\tau, W_\tau^\pi, S_\tau, 0)\right) \mathbb{1}_{\tau\leqslant T_n}\Big]$$
$$= \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \left(v(s, W_s^{\pi^{(0)}}(x), S_s^{(0)}, 1) - v(s, W_s^{\pi^{(0)}}(x), S_s^{(0)}, 0)\right) . \lambda e^{-\lambda(s-t)} ds\right]$$
$$= \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \left(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\right) d\theta . \lambda e^{-\lambda(T\wedge\theta_n^{(0)} - t)}\right]$$
$$+ \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \int_t^s \left(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\right) d\theta . \lambda^2 e^{-\lambda(s-t)} ds\right].$$

Where the last equality is given from an integration by part.

Notice that $A + C$ is equal to zero, then from (6.5.15) we obtain:

$$\mathbb{E}^{\zeta,0}_{t,x}\left[v\left(T_n, W^\pi_{T_n}, S_{T_n}, Z_{T_n}\right)\right] \leqslant v(t, x, \zeta, 0).$$

We now take the limit as n increases to infinity. Since $T_n \to T$ a.s and from the boundary condition, we can deduce from Fatou lemma that

$$V(t, x, \zeta, 0) \leqslant v(t, x, \zeta, 0).$$

We now prove the statement b). By repeating the above argument in I), we observe that the control $\pi_*^{(1)}$ achieves equality in (6.5.14) and therefore $V(t,x,\zeta,1) = v(t,x,\zeta,1)$. Similarly, by repeating the above steps in II) , we can conclude that $V(t,x,\zeta,0) = v(t,x,\zeta,0)$ and thus the optimal control is given by (6.5.13).

□

Remark 6.5.7. *Notice that under some assumptions on the density q of τ, the above verification theorem still applicable.*

Indeed, if we rewrite the terms A and C in terms of the density q, we obtain:

$$A = \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}}\int_t^s \left(\frac{\partial v}{\partial\theta} + \mathcal{L}^{(0)}_{\pi^{(0)}} v\right)(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)d\theta.q(s)ds\right]$$
$$+ \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \left(\frac{\partial v}{\partial\theta} + \mathcal{L}^{(0)}_{\pi^{(0)}} v\right)(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)d\theta.\mathbb{P}[\tau > T\wedge\theta_n^{(0)}]\right]$$

and

$$C = \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}} \Big(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\Big)\, d\theta.q(T\wedge\theta_n^{(0)})\right]$$
$$- \mathbb{E}^B\left[\int_t^{T\wedge\theta_n^{(0)}}\int_t^s \Big(v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 1) - v(\theta, W_\theta^{\pi^{(0)}}(x), S_\theta^{(0)}, 0)\Big)\, d\theta.q'(s)ds\right]$$

We can notice from the above case with exponential law that the first term in A was compensated with the second term of C and the second term in A was compensated with the first term of C. So this leads us to do the same with this case. Thus we can deduce the following assumptions on q:

- $q'(x) < 0 \quad \forall x,$
- $-\frac{q'(x)}{q(x)} \leqslant \alpha \quad \forall x,$
- $q(x) \geqslant \beta\displaystyle\int_x^\infty q(s)ds \quad \forall x.$

6.5.4 Application

In this section, we present a particular example for the dynamics of the price process (S_t), where the coefficients μ, σ_0 and σ_1 are constants.

We consider the following dynamics of (S_t):

$$dS_t = \mu S_t dt + S_t \left(\sigma_1 + (\sigma_0 - \sigma_1)\, \mathbb{1}_{t \leqslant \tau}\right) dB_t.$$

Therefore the wealth process satisfies the following SDE:

$$dW_t^\pi = W_t^\pi \left[(1-\pi_t)\, r dt + \pi_t \left(\mu dt + (\sigma_0 + (\sigma_1 - \sigma_0) Z_t) dB_t\right)\right].$$

and the optimization problem becomes:

$$V(t,x,z) = \sup_{\pi \in \mathcal{A}} J(t,x,z,\pi),$$

where

$$J(t,x,z,\pi) = \mathbb{E}_{t,x}^z \left[U(W_T^\pi)\right].$$

As the coefficients μ, σ_0 and σ_1 do not depend on the price S_t, then the parameter ζ does not appear in the above SDE of the wealth process and thus in the value function.

In this particular example, the HJB system becomes:

$$(S) \begin{cases} \frac{\partial v}{\partial t}(t,x,0) + \sup_{\pi \in \mathbb{U}} \mathcal{L}_\pi^{(0)} v(t,x,0) + \lambda \left(v(t,x,1) - v(t,x,0)\right) = 0 \\ \frac{\partial v}{\partial t}(t,x,1) + \sup_{\pi \in \mathbb{U}} \mathcal{L}_\pi^{(1)} v(t,x,1) = 0 \end{cases}$$

with the boundary condition

$$v(T,x,z) = U(x), \qquad x \in \mathbb{R}^+,\ z \in \{0,1\}. \tag{6.5.16}$$

where

$$\mathcal{L}_\pi^{(0)} v(t,x,0) = x(\pi\mu + (1-\pi)r)\frac{\partial v}{\partial x} + \frac{1}{2} x^2 \pi^2 \sigma_0^2 \frac{\partial^2 v}{\partial^2 x},$$

and

$$\mathcal{L}_\pi^{(1)} v(t,x,1) = x(\pi\mu + (1-\pi)r)\frac{\partial v}{\partial x} + \frac{1}{2} x^2 \pi^2 \sigma_1^2 \frac{\partial^2 v}{\partial^2 x}.$$

We now consider the power utility function which with we obtain an explicit smooth solution of the above system (S). Then by verification theorem, we show that the value function is solution of the HJB system and we make explicit the optimal strategy.

Let the power utility function $U(x) = \frac{x^\gamma}{\gamma}$, $0 < \gamma < 1$. We first solve the second equation of the system (S) and then we deduce the solution of the first equation of (S).

For the second equation, we can resolve it easily as in the case of the Merton example. Therefore we obtain that:

$$\begin{aligned} v(t,x,1) &= \frac{x^\gamma}{\gamma}\exp\left(\gamma\left(r+\frac{(\mu-r)^2}{2\left(1-\gamma\right)\sigma_1^2}\right)(T-t)\right) \\ &= \frac{x^\gamma}{\gamma}\exp\left(\gamma\rho(T-t)\right), \end{aligned}$$

where

$$\rho = \sup_{p_1\in\mathbb{U}}[p_1(\mu-r)+r+\frac{1}{2}p_1^2\sigma_1^2(\gamma-1)] = r+\frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma_1)^2\sigma^2}.$$

.

It remains to solve the first equation of (S). Similarly, we shall look for a solution of the form $v(t,x,0) = x^\gamma\psi(t)$. Plugging into in the first equation of (S), we get a differential equation for ψ:

$$\psi'(t)+(\gamma\rho_0-\lambda)\psi(t) = \frac{-\lambda}{\gamma}\exp(\gamma\rho(T-t))$$

where

$$\rho_0 = \sup_{p_0\in\mathbb{U}}[p_0(\mu-r)+r+\frac{1}{2}p_0^2\sigma_0^2(\gamma-1)] = r+\frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma)^2\sigma_0^2}.$$

As $V(T,x,0) = U(x)$, we obtain that $\psi(T) = \frac{1}{\gamma}$ and therefore the unique solution of the first equation of (S) is given by

$$\psi(t) = \frac{1}{\gamma}\left[\left(\frac{\lambda}{\zeta}+1\right)\exp\left(\left(\gamma\rho_0-\lambda\right)(T-t)\right)-\frac{\lambda}{\zeta}\exp\left(\gamma\rho\left(T-t\right)\right)\right].$$

where ζ is a constant: $\zeta = (\gamma\rho_0-\gamma\rho-\lambda)$ and ρ_0,ρ are given by:

$$\rho_0 = r+\frac{(\mu-r)^2}{2(1-\gamma)\sigma_0^2},$$

$$\rho = r+\frac{(\mu-r)^2}{2(1-\gamma)\sigma_1^2}.$$

Therefore,

$$v(t,x,0) = \frac{x^\gamma}{\gamma}\left[\left(\frac{\lambda}{\zeta}+1\right)\exp\left(\left(\gamma\rho_0-\lambda\right)(T-t)\right)-\frac{\lambda}{\zeta}\exp\left(\gamma\rho\left(T-t\right)\right)\right].$$

Consider now the strategy processes $(\pi_t^*)_{t\geqslant 0}$ defined by

$$\pi_t^* = \begin{cases} \frac{\mu-r}{\sigma_0^2(1-\gamma)} & \text{if} \quad Z_t = 0 \\ \frac{\mu-r}{\sigma_1^2(1-\gamma)} & \text{if} \quad Z_t = 1 \end{cases} \tag{6.5.17}$$

Notice that the functions $v(t,x,0)$ and $v(t,x,1)$ are smooth. Then from Theorem 6.5.6 we deduce that for the power utility function the pair of value functions $(V(t,x,0),V(t,x,1))$ is solution of the HJB system (S) and is given explicitly by $(V(t,x,0),V(t,x,1)) = (v(t,x,0),v(t,x,1))$ and the optimal strategy is given by π_t^*:

$$\pi_t^* = \frac{\mu - r}{\sigma_0^2(1-\gamma)}\mathbb{1}_{Z_t=0} + \frac{\mu - r}{\sigma_1^2(1-\gamma)}\mathbb{1}_{Z_t=1}.$$

As Z_t is $\mathbb{F}^S$-adapted, we can deduce that π_t^* is also $\mathbb{F}^S$ adapted, then $\pi^* \in \mathcal{A}$.

6.6 Perspectives

We plan to consider the more realistic case where there are a sequence (τ_n) of change times and a sequence (θ_n) of decision times at which the trader reinvests his portfolio, either in the risky asset (S_t) or in the bank account (S_t^0). We study this case with the presence of the transaction costs. We begin a new study for this problem, with a first difficulty in the identification of the Hamilton-Jacobi-Bellman system corresponding to this new framework and studying the impact of the parameter of the change times laws on the strategies and their performances.

Bibliography

[1] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[2] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[3] Philippe Biane, Jim Pitman, and Marc Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc. (N.S.)*, 38(4):435–465 (electronic), 2001.

[4] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[5] Christophette Blanchet-Scalliet, Awa Diop, Rajna Gibson, Denis Talay, and Etienne Tanré. Technical analysis techniques versus mathematical models: boundaries of their validity domains. In *Monte Carlo and quasi-Monte Carlo methods 2004*, pages 15–30. Springer, Berlin, 2006.

[6] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[7] T. Chan, David S. Dean, Kalvis M. Jansons, and L. C. G. Rogers. On polymer conformations in elongational flows. *Comm. Math. Phys.*, 160(2):239–257, 1994.

[8] K. L. Chung. A cluster of great formulas. *Acta Math. Acad. Sci. Hungar.*, 39(1-3):65–67, 1982.

[9] Alain Comtet, Cécile Monthus, and Marc Yor. Exponential functionals of Brownian motion and disordered systems. *J. Appl. Probab.*, 35(2):255–271, 1998.

[10] Claude Dellacherie and Paul-André Meyer. *Probabilités et potentiel*. Hermann, Paris, 1975. Chapitres I à IV, Édition entièrement refondue, Publications de

l'Institut de Mathématique de l'Université de Strasbourg, No. XV, Actualités Scientifiques et Industrielles, No. 1372.

[11] Catherine Donati-Martin, Raouf Ghomrasni, and Marc Yor. On certain Markov processes attached to exponential functionals of Brownian motion; application to Asian options. *Rev. Mat. Iberoamericana*, 17(1):179–193, 2001.

[12] Catherine. Donati-Martin and Marc. Yor. Fubini's theorem for double Wiener integrals and the variance of the Brownian path. *Ann. Inst. H. Poincaré Probab. Statist.*, 27(2):181–200, 1991.

[13] Daniel Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990.

[14] Nicole El Karoui, Monique Jeanblanc, and Ying Jiao. What happens after a default: the conditional density approach. *Stochastic Process. Appl.*, 120(7):1011–1032, 2010.

[15] Lancelot F. James, Bernard Roynette, and Marc Yor. Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surv.*, 5:346–415, 2008.

[16] Thierry Jeulin. *Semi-martingales et grossissement d'une filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[17] Ying Jiao and Huyên Pham. Optimal investment with counterparty risk: a default-density model approach. *Finance and Stochastics*, 15(4):725–753, 2011.

[18] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[19] Paul Levy. Sur certains processus stochastiques homogènes. *Compositio Math.*, 7:283–339, 1940.

[20] Pierre-Louis. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness. *Comm. Partial Differential Equations*, 8(11):1229–1276, 1983.

[21] Hiroyuki Matsumoto and Marc Yor. Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.*, 2:312–347, 2005.

[22] Kenji Nakagawa. Tail probability of random variable and Laplace transform. *Appl. Anal.*, 84(5):499–522, 2005.

[23] Kenji Nakagawa. Application of Tauberian theorem to the exponential decay of the tail probability of a random variable. *IEEE Trans. Inform. Theory*, 53(9):3239–3249, 2007.

[24] Mihael Perman. An excursion approach to Ray-Knight theorems for perturbed Brownian motion. *Stochastic Process. Appl.*, 63(1):67–74, 1996.

[25] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.

[26] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991.

[27] Ken-iti Sato. Modes and moments of unimodal distributions. *Ann. Inst. Statist. Math.*, 39(2):407–415, 1987.

[28] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[29] Aurel Wintner. Cauchy's stable distributions and an "explicit formula" of Mellin. *Amer. J. Math.*, 78:819–861, 1956.

[30] Stephen James Wolfe. On the continuity properties of L functions. *Ann. Math. Statist.*, 42:2064–2073, 1971.

[31] Makoto Yamazato. Unimodality of infinitely divisible distribution functions of class L. *Ann. Probab.*, 6(4):523–531, 1978.

[32] Jiongmin. Yong and Xun Yu. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer-Verlag, 1999.

[33] Marc Yor. *Some aspects of Brownian motion. Part I.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992. Some special functionals.

[34] Marc Yor. Interpretations in terms of Brownian and Bessel meanders of the distribution of a subordinated perpetuity. In *Lévy processes*, pages 361–375. Birkhäuser Boston, Boston, MA, 2001.