Uniform controllability of discrete partial differential equations

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LABORATOIRE : MAPMO

THÈSE présentée par :

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soutenue le : **26 octobre 2012**

pour obtenir le grade de : **Docteur de l'université d'Orléans**

Discipline : **MATHÉMATIQUES**

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Abstract

In this thesis, we study uniform controllability properties of semi-discrete approximations for parabolic systems.

In a first part, we address the minimization of the L^q-norm ($q > 2$) of semidiscrete controls for parabolic equation. As shown in [LT06], under the main approximation assumptions that the discretized semigroup is uniformly analytic and that the degree of unboundedness of control operator is lower than $1/2$, uniform observability is achieved in L^2 for semidiscrete approximations for the parabolic systems. Our goal is to overcome the limitation of [LT06] about the order $1/2$ of unboundedness of the control operator. Namely, we show that the uniform observability property also holds in L^q ($q > 2$) even in the case of a degree of unboundedness greater than $1/2$. Moreover, a minimization procedure to compute the approximation controls is provided. An example of application is implemented for the one-dimensional heat equation with Dirichlet boundary control.

The study of L^q optimality above is in a general context. However, the discrete observability inequalities that are obtained are not so precise than the ones derived then with Carleman estimates. In a second part, in the discrete setting of one-dimensional finite-differences we prove a Carleman estimate for a semi-discrete version of the parabolic operator $\partial_t - \partial_x(c\partial_x)$ which allows one to derive observability inequalities that are far more precise. Here we consider in case that the diffusion coefficient has a jump which yields a transmission problem formulation. Carleman estimate are L^2 weighted energy estimates. Here the weight is chosen so as to accommodate the jump of the diffusion coefficient. Moreover, there is a strong connexion between the Carleman large parameter and the (small) discretization parameter . Consequence of this Carleman estimate, we deduce consistent null-controllability results for classes of linear and semi-linear parabolic equations.

Acknowledgements

Foremost, I would like to express my sincerest gratitude to my two supervisors Professor Emmanuel Trélat and Professor Jérôme Le Rousseau for continuous support of my PhD study and research, for their patience, motivation, enthusiam, and immense knowledge. They offered the best advices as well as encouragement throughout the course of my thesis. I feel lucky for having them as supervisors.

I would like to acknowledge Professor Michel Zinmeister for giving me a chance to work in good research enviroment of University of Orleans from PUF program.

I thank to Region Center for the financial support, without which this work would have never been possible.

Many thanks to all members of laboratory MAPMO, who have accompanied with me during a long three years of working and studing here. A special acknowledge to the secrectaries of MAPMO for their kindness.

I am grateful to my close friends Dung Duong Hoang, Peipei Shang, Tan Cao Hoang who are always by my side to support and encourage me to overcome the difficult stages in the course of this research as well as my life . I also thank to Vietnamese students who have shared with me a lot of good memories during three years in France.

Finally, I would like to express the profound gratitude from my deep heart to my beloved parents and little brother for their love and spiritual support throughout my life.

–Thuy NGUYEN–

Contents

Chapter 1

Introduction

1.1 Controllability and observability

A control system takes the following form for an evolution equation

$$\dot{y}(t) = Ay(t) + Bv(t), \qquad y(0) = y_0.$$

The controllability problem is, roughly speaking, the following one. Given $T > 0$ and two states y_0 and y_T, is it possible to steer the solution of the evolution systems from the given initial state $y(0) = y_0$ to the final state $y(T) = y_T$ by means of a control $v(t)$? Here, B is the control operator.

We give examples of such control systems in case of systems of ordinary differential equations and partial differential equations.

Example 1.1.1. *The controlled harmonic oscillator*

$$x'' + x = u,$$

which may be written as a system in the following way

$$\begin{cases} x' = y \\ y' = u - x. \end{cases}$$

Then, A and B are matrices respectively

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example 1.1.2. *The heat equation*

$$\begin{cases} \partial_t y - \triangle y = \mathbf{1}_\omega v & in \quad \Omega \subset \mathbb{R}^n, \\ y = 0 & in \quad (0,T) \times \partial\Omega, \\ y|_{t=0} = y_0, \end{cases}$$

where ω is an open subset of Ω, an open set in $\mathbb{R}^n$.

Example 1.1.3. *The wave equation*

$$\begin{cases} c^{-2}\partial_t^2 y - \triangle y = v & in \quad \Omega \subset \mathbb{R}^n, \\ y = 0 & in \quad (0,T) \times \partial\Omega, \\ y|_{t=0} = y_0, \qquad \partial_t y|_{t=0} = y_1. \end{cases}$$

The main results of this thesis concern linear systems. In this section, we propose to review some basic concepts and properties for controllability of linear control systems. First, we consider with finite-dimensional control systems. Then, we consider with infinite-dimensional control systems modeled by partial differential equations.

1.1.1 Controllability of finite-dimensional linear control systems

We start the section by recalling some well known results in the finite-dimensional context.

Let $T > 0$ be fixed. Consider the linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \tag{1.1.1}$$

where $y(t) \in \mathbb{R}^n$, A is a $(n \times n)$- matrix, B is a $(n \times m)$- matrix, with real coefficients, and $u(.) \in L^2(0,T;\mathbb{R}^m)$.

Let $x_0 \in \mathbb{R}^n$. The systems (1.1.1) is said to be controllable from x_0 in time T if and only if, for every $x_1 \in \mathbb{R}^n$, there exists $u(.) \in L^2(0,T;\mathbb{R}^m)$ so that the solution $x(.)$ of (1.1.1), with $x(0) = x_0$, associated with the control $u(.)$, satisfies $x(T) = x_1$.

It is well known that the system (1.1.1) is controllable in time T if and only if the $n \times n$ matrix

$$\int_0^T e^{(T-t)A} BB^* e^{(T-t)A^*} dt,$$

called *Gramian* of the system, is nonsingular (here, M^* denotes the transpose of a matrix M). Since we are in finite dimension, this is equivalent to the existence of $\alpha > 0$ so that

$$\int_0^T \left\| B^* e^{(T-t)A} \psi \right\|^2 dt \geq \alpha \left\| \psi \right\|^2, \tag{1.1.2}$$

for every $\psi \in \mathbb{R}^n$ (observability inequality).

It is also well known that, if such a linear system is controllable from x_0 in time $T > 0$, then it is controllable in time T', for every $T' > 0$, and from every initial state $x_0' \in \mathbb{R}^n$. Indeed, another necessary and sufficient condition for the controllability to hold is the Kalman condition

$$rank(B, AB, \cdots, A^{n-1}B) = n,$$

which is independent on x_0 and T.

1.1.2 Controllability of infinite-dimensional linear control systems

In this section we give some known facts on controllability of infinite dimensional linear control systems in Banach spaces. We refer the reader to the excellent textbook [TW07] (see also [CT06, S05, EZ12] and references therein).

The notation $L(E, F)$ stands for the set of linear continuous mappings from E to F, where E and F are reflexive Banach spaces.

Let X be a reflexive Banach space. In what follows, we denote by $\|.\|_X$ the norm on X. Let $S(t)$ denote a strongly continuous semigroup on X, of generator $(A, D(A))$. Let X_{-1} denote the completion of X for norm $\|x\|_{-1} = \left\|(\beta I - A)^{-1} x\right\|$, where $\beta \in \rho(A)$ is fixed. Note that X_{-1} does not depend on the specific value of $\beta \in \rho(A)$. The space X_{-1} is isomorphic to $(D(A^*))'$, the dual space of $D(A^*)$. The semigroup $S(t)$ extends to a semigroup on X_{-1}, still denoted by $S(t)$, whose generator is an extension of the operator A, also denoted by A. With this notation, A is a linear operator from X to X_{-1}.

Let U be a reflexive Banach space. Denote by $\|.\|_U$ the norm on U. A linear continuous operator $B : U \to X_{-1}$ is called the control operator. Note that B is said to be bounded if $B \in L(U, X)$, and is called unbounded otherwise.

The control operator B is admissible for the semigroup $S(t)$ if every solution of

$$y' = Ay(t) + Bu(t), \tag{1.1.3}$$

with $y(0) = y_0 \in X$ and $u(.) \in L^2(0, +\infty; U)$, satisfies $y(t) \in X$, for every $t \geq 0$. The solution of equation (2.1.1) is understood in the mild sense, i.e,

$$y(t) = S(t)y(0) + \int_0^T S(t-s)Bu(s)ds, \tag{1.1.4}$$

for every $t \geq 0$.

For $T > 0$, define $L_T : L^2(0, T; U) \to X_{-1}$ by

$$L_T u = \int_0^T S(T-s)Bu(s)ds. \tag{1.1.5}$$

A control operator $B \in \mathcal{L}(U, X_{-1})$ is admissible, if and only if $ImL_T \subset X$, for some (and hence for every) $T > 0$.

In contrast to the case of linear finite-dimensional control systems there exist many types of controllability properties. We provide three different notions (they all coincide in the finite dimensional case).

Let $B \in \mathcal{L}(U, X_{-1})$ denote an admissible control operator.

Definition 1.1.4. *For $y_0 \in X$, and $T > 0$, the system (2.1.1) is said to be exactly controllable from y_0 in time T if, for every $y_1 \in X$, there exists $u(.) \in L^2(0, T; U)$ so that the solution of (2.1.1) , with $y(0) = y_0$, associated with the control $u(.)$, satisfies $y(T) = y_1$.*

Definition 1.1.5. *The system (2.1.1) is said to be approximately controllable from y_0 in time T if, for every $y_1 \in X$ and every $\epsilon > 0$, there exists $u(.) \in L^2(0, T; U)$ so that the solution of (2.1.1) , with $y(0) = y_0$, associated with the control $u(.)$, satisfies $\|y(T) - y_1\|_X \leq \epsilon$.*

Definition 1.1.6. *For $T > 0$, the system (2.1.1) is said to be null controllable in time T if, for every $y_0 \in X$, there exists $u(.) \in L^2(0, T; U)$ so that the solution of (2.1.1), with $y(0) = y_0$, associated with the control $u(.)$, satisfies $y(T) = 0$.*

Clearly, exact controllability implies null controllability and approximate controllability. However, the converse is false in general.

Let us now analyze the null controllability problem that we shall mostly study in this thesis. We have the following theorem.

Theorem 1.1.7. *System (2.1.1) is null-controllable in time T if and only if there exists $\alpha > 0$ so that*

$$\int_0^T \|B^* S(t)^* \psi\|_U^2 \, dt \geq \|S(T)^* \psi\|_X^2 \tag{1.1.6}$$

for every $\psi \in D(A^*)$. Inequality (1.1.6) is called an observation *or* observability *inequality.*

For linear equations, null-controllability is achieved in general by the proof of such an observability inequality. Several methods can be used to derive such an the observability inequality including Carleman estimates, the method of multipliers, microlocal analysis... At present, a powerful approach to prove the observability inequality for general parabolic system is through the derivation of a global Carleman estimate which will be considered in a part of this thesis.

1.2 Case of parabolic equations

In this section, we mention some tools to achieve controllability results for parabolic equations. Firstly, we present the HUM method through which one can construct a control for a general system. The second one that we use in this thesis is Carleman estimate which yields observability inequality of heat type equations.

1.2.1 The HUM control

In this section, we refer for the well-known Hilbert Uniqueness Method (in short HUM), introduced in [L88], consisting of minimizing a cost function, namely, the L^2-norm of the control.

Assume that B is admissible and the control system (2.1.1) is null controllable in time T. We define the space H as the completion of $\mathcal{D}(\Omega)$ (with Ω is a domain where System (2.1.1) acts on) with respect to the norm

$$\|\psi\|_H = \left(\int_0^T \|B^* S(t)^* \psi\|_U^2 \, dt \right)^{1/2}.$$

Let $y_0 \in X$. For every $\psi \in H$, set

$$J(\psi) = \frac{1}{2}\int_0^T \|B^*S(t)^*\psi\|_U^2\, dt + \langle S(T)^*\psi, y_0\rangle_X$$

The functional J is strictly convex, and, from the observability inequality (1.1.6), is coercive in the space H. Then, it follows that J has a unique minimizer $\varphi \in H$. Define the control u by

$$u(t) = B^*S(T-t)^*\varphi$$

for every $t \in [0,T]$, and let $y(.)$ be the solution of (2.1.1), such that $y(0) = y_0$, associated with the control u. Then, one has $y(T) = 0$, and moreover, u is the control of minimal of L^2-norm, among all controls whose associated trajectory satisfies $y(T) = 0$.

This proves that observability implies controllability, and gives a way to construct the control of minimal of L^2-norm. This is more or less the contents of the Hilbert Uniqueness Method. Hence, in what follows, we refer to the control (1.2.1) as to the HUM control.

Remark 1.2.1. *From a theoretical viewpoint, the existence of the null control is due to the obsevability inequality* (1.1.6) *and therefore is independent of how large the space H is. However, an efficient computing of the null control for a numerical approximation scheme is intimately related to this large space H that can hardly be approximated by standard techniques in numerical analysic. Recently, A. Munch and collaborators have developed some feasible numericals for heat type equation such that the transmutation method, variational approach, dual and primal algorithms allow to more efficiently compute the null control (see in series [FCM09], [FCM10], [MZ10], [PM10]).*

1.2.2 Carleman estimates and null controllability in continuous case

Here, we shall mainly survey controllability results for parabolic equations, for which Carleman estimates have now become an essential technique.

Let Ω, ω be connected non-empty open subsets of $\mathbb{R}^n$ with $\omega \Subset \Omega$. We consider the following parabolic problem in $(0,T)\times\Omega$, with $T > 0$,

$$\begin{aligned} &\partial_t y - \triangle y = \mathbf{1}_\omega v && \text{in } Q = (0,T)\times\Omega, \\ &y = 0 && \text{on } \Sigma = (0,T)\times\partial\Omega, \\ &y(0) = y_0 && \text{in } \Omega. \end{aligned} \tag{1.2.1}$$

For the above systems, the null controllability was proved independently by G. Lebeau-L. Robbiano [LR95] on the one hand and A. Fursikov- O. Yu. Imanuvilov [FI96] on the other hand. More precisely, the controllability result is stated through the following Theorem.

Theorem 1.2.2. *([LR95, FI96]) Let $\omega \neq \emptyset$ and $T > 0$. For all $y_0 \in L^2(\Omega)$, there exists a control $v \in L^2(Q)$ such that $y(T) = 0$ and $\|v\|_{L^2(Q)} \leq C \|y_0\|_{L^2(Q)}$ where $C > 0$ only depends on Ω, ω and T.*

The two proofs of this result are different: one uses a local Carleman approach and another uses a global Carleman approach. The approach of [LR95] is by means of Carleman estimates for the elliptic operator $(-\partial_t^2 - \triangle_x)$ in a domain $Z = (0, S_0) \times \Omega$ with $S_0 > 0$, these estimates are local as they apply to compactly supported functions in Ω. The second approach, introduced in [FI96], is based on Carleman estimates for the parabolic operator $\partial_t - \triangle$. These estimates are said to be global for they apply to functions that are defined in the whole domain $(0, T) \times \Omega$ with particular boundary conditions, here homogeneous Dirichlet boundary conditions.

What is a Carleman estimate?

A local Carleman estimate is an L^2-weighted estimate of the type

$$\|e^{\tau\varphi} P v\|_{L^2(\Omega)} \geq C \|e^{\tau\varphi} v\|_{L^2(\Omega)}, \tag{1.2.2}$$

where $P = P(x, D_x)$ is a differential operator, $\varphi = \varphi(x)$ is a weight function, $\tau > 0$ is a large parameter and $v \in C_c^\infty(\Omega)$. The weight of exponential from $e^{\tau\varphi}$ allows one to provide "strong" domination of the function v where φ is of smaller values. This can be done by choosing the parameter τ large.

For applications, the parameter τ plays an essential role. The choice of the weight function $\varphi(x)$ is important in order to adjust to given geometric configurations.

Estimates of the form (1.2.2) were first established by T. Carleman in 1939 [Car39] for proving the unique continuation property of some elliptic partial differential equations (PDE) in dimension two. Since then there have been many investigations for the use of Carleman estimate used to study of inverse problems and controllability issues for PDEs. Below, we shall mention two kinds of Carleman estimates which yield the null controllability for linear parabolic equations.

Local Carleman estimates

We present a local Carleman estimate for an elliptic operator of order two, in the simple case where $P = -\triangle = D.D$.

Let φ be a real-valued function. We define the following conjugated operator $P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi}$ to be considered as a semi-classical differential operator with large parameter τ. We have

$$\begin{aligned} P &= -\triangle - \left|\tau\varphi'\right|^2 + \left\langle \tau\varphi', \bigtriangledown \right\rangle + \left\langle \bigtriangledown, \tau\varphi' \right\rangle \\ &= D.D - \left|\tau\varphi'\right|^2 + 2i\left\langle \tau\varphi', D \right\rangle + \tau \triangle \varphi. \end{aligned}$$

Its full symbol is given by $|\xi|^2 - \tau^2 |\varphi'|^2 + 2i\tau \langle \varphi', \xi \rangle + \tau \triangle \varphi$. Its principal symbol is given by (the same strength is given to τ and $|\xi|$ in the semi-classical setting)

$$p_\varphi = \sigma(P_\varphi) = |\xi|^2 - \tau^2 \left|\varphi'\right|^2 + 2i\tau \left\langle \varphi', \xi \right\rangle.$$

We define the following symmetric operators

$$\begin{aligned} Q_2 &= (P_\varphi + P_\varphi^*)/2 = D.D - \left|\tau\varphi'\right|^2, \\ Q_1 &= (P_\varphi - P_\varphi^*)/2i = \left\langle \tau\varphi', D \right\rangle + \left\langle D, \tau\varphi' \right\rangle \end{aligned}$$

with respective principal symbols

$$q_2 = |\xi|^2 - \tau^2 \left|\varphi'\right|, \qquad q_1 = 2\tau \left\langle \xi, \varphi' \right\rangle.$$

We have $p_\varphi = q_2 + iq_1$ and $P_\varphi = Q_2 + iQ_1$.

We choose φ that satisfies the following assumption.

Assumption 1.2.3. *(L.Hörmander) Let V be a bounded open set in $\mathbb{R}^n$. We say that the weight function $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies the Hörmander sub-elliptic assumption in $\bar{V}$ if $|\varphi'| > 0$ in $\bar{V}$ and*

$$\forall (x,\xi) \in \bar{V} \times \mathbb{R}^n, \qquad p_\varphi(x,\xi) = 0 \Rightarrow \{q_2, q_1\}(x,\xi) \geq C > 0.$$

Theorem 1.2.4. *(L.Hörmander) Let V be a bounded open set in $\mathbb{R}^n$ and let φ satisfy*

the sub-ellipticity Assumption 1.2.3 in $\bar{V}$, then there exist $\tau_0 > 0$ and $C > 0$ such that

$$\tau^3 \left\| e^{\varphi/h} u \right\|^2_{L^2(\mathbb{R}^n)} + \tau \left\| e^{\varphi/h} \bigtriangledown_x u \right\|^2_{L^2(\mathbb{R}^n)} \leq C \left\| e^{\varphi/h} P u \right\|^2_{L^2(\mathbb{R}^n)},$$

for $u \in \boldsymbol{C}^\infty_c(\bar{V})$ and $\tau \geq \tau_0$.

For the purpose of proving the null controllability of the heat equation, local Carleman estimates given in Theorem 1.2.4 allow one to prove the following spectral inequality

$$\|u\|^2_{L^2(\Omega)} \leq C e^{C\sqrt{\mu}} \|u\|^2_{L^2(\omega)},$$

where ω is an open subset of Ω, $C > 0$ and u a linear combination of eigenfunctions of $-\triangle$ associated to eigenvalues less than $\mu > 0$. The spectral inequality allows one to obtain an iterative construction of the control function v working in increasingly larger finite-dimensional subspaces. This yields the controllability result as stated in Theorem 1.2.2. For more details we refer to the original approach of Lebeau and Robbiano [LR95]. See also [LL11a] and [M10].

Global Carleman estimates

The approach of [FI96] allows one to treat also the controllability of more general parabolic equations by means of a global Carleman estimate. We refer to the surveys of [FCG06] and [LL11a] for some background introduction to global Carleman estimates.

The null controllability of the linear parabolic system (1.2.1) is equivalent to the observability of the associated adjoint states. More precisely, for each $q^0 \in L^2(\Omega)$, let us consider the so-called adjoint system

$$\begin{cases} -\partial_t q - \triangle q = 0 & \text{in } Q = (0,T) \times \Omega, \\ q = 0 & \text{on } \Sigma = (0,T) \times \partial\Omega, \\ q(T) = q^0 & \text{in } \Omega. \end{cases} \tag{1.2.3}$$

Then (1.2.1) is null controllable with controls in $L^2((0,T) \times \omega)$ if and only if there exists $C > 0$ such that

$$\|q(0)\|^2_{L^2(\Omega)} \leq C \int_{(0,T)\times\omega} |q|^2 \, dxdt, \tag{1.2.4}$$

for all q solution to (1.2.3), which is the observability inequality.

The global Carleman estimate that we present below can be very useful for proving

such an observability inequality. Here, we state a global estimate as derived by A. Fursikov-O.Yu. Imanuvilov.

We choose a global weight function satisfying the following requirements

$$\begin{aligned}
&\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K}, \text{ with} \quad K > \|\psi\|_\infty, \quad \left|\psi'(x)\right| \neq 0, \quad x \in \Omega\backslash\omega_0,\\
&\psi|_{\partial\Omega} = 0, \qquad \partial_n\psi|_{\partial\Omega} < 0, \qquad \psi(x) > 0, \quad x \in \Omega,\\
&\eta(t) = \frac{1}{t(T-t)}
\end{aligned}$$

and we choose the positive parameter λ sufficiently large. For the construction of such a function ψ we refer to [FI96].

A global Carleman estimate is then of the following form.

Theorem 1.2.5. *[FI96] There exists $K > 0$, $\tau_0 > 0$, depending on ω, Ω such that*

$$\begin{aligned}
\tau^3 \left\|\eta(t)^{3/2} e^{\tau\eta\varphi} q\right\|^2_{L^2(Q)} + \tau \left\|\eta(t)^{1/2} e^{\tau\eta\varphi} \nabla q\right\|^2_{L^2(Q)}\\
\leq K\Big(\|e^{\tau\eta\varphi}(q_t + \triangle q)\|^2_{L^2(Q)} + \tau^3 \left\|\eta(t)^{3/2} e^{\tau\eta\varphi} q\right\|^2_{L^2\left((0,T)\times\omega\right)}\Big),
\end{aligned}$$

for $\tau \geq (T + T^2)\tau_0$.

Note that the global aspect of the estimate imposes an "observation" term over $(0, T) \times \omega$, with $\omega \Subset \Omega$ in the r.h.s of the estimate (compare with the form of a local estimate above).

From this global Carleman estimate, one deduces the observability inequality (1.2.4) (we refer to [FCG06] for a proof). The null controllability of System (1.2.1) then follows.

1.3 Uniform observability and discrete controls

This section is devoted to analyzing controllability of parabolic sytems after discretization. If one considers controllability properties for parabolic differential equations, a natural question is then that of uniform controllability after discretization. Above we reviewed that controllability and observability are dual aspects of the same problem. We shall therefore focus on uniform observability which is shown to hold when the observability constant of the discritized approximation system does not depend on step-size h.

In the first part of this thesis, the main goal is to establish conditions to obtain an uniform observability inequality in a general context. Furthermore, by making use of

this observability inequality, we provide some constructive way to compute a suitable semi-discrete control function.

Because of the generality of the discrete schemes studied in this first part one can hope to obtain sharper observability inequalities if considering particular schemes. In a second part, specializing the analysis to finite-difference schemes we obtain improved observability inequalities. This is done by means of discrete Carleman estimates.

Before stating the different results obtained here we wish to explain the reason why there is no hope to obtain observability inequalities of the same quality as what is obtained in the continuous case.

1.3.1 A counter-example to null-controllability

As mentioned in Theorem 1.2.2, the heat equation in a bounded domain is null controllable from any open, non-empty subset of the domain or its boundary [FI96, LR95]. One can expect that with common numerical approximation schemes the heat equation should be controllable, uniformly with respect to the discretization parameter under similar weak geometric conditions for the location of the action of the control. A positive uniform null controllability result is provided in [LZ98] concerning the case of a boundary control in 1D in the case of a finite-difference scheme. Yet, a counter-example due to O. Kavian shows that the null controllability property fails to hold in 2D again for a finite-difference semi-discretization scheme for the heat equation in a square where the control acts on a strict subset of the domain. Here, we shall present this counter-example for a distributed control.

Set $\Omega = [0,1] \times [0,1]$. We consider the following heat equation in $(0,T) \times \Omega$, with $T > 0$

$$\partial_t y - \triangle y = \mathbf{1}_\omega v \quad \text{in } (0,T) \times \Omega, \qquad y|_{\partial\Omega} = 0 \quad \text{and} \quad y|_{t=0} = y_0, \tag{1.3.1}$$

where $v = v(x_1, x_2, t)$ is the control and $\mathbf{1}_\omega$ is the characteristic function of the set ω where the control is applied.

The Laplace operator is approximated by a five-point finite difference scheme. The semi-discrete systems we consider is then

$$\partial_t y - \triangle_h y = \mathbf{1}_\omega v, \qquad y|_{\partial\Omega} = 0, \qquad y|_{t=0} = y_0, \tag{1.3.2}$$

where y is discrete in the space variable. Its values are known on the uniform mesh of $[0,1]^2$, h is the discrete step size in the both directions. In fact, there exists an eigenvector e_0 of the discrete Laplacian $\triangle_h$ with corresponding eigenvalue $\lambda_{0k} = \frac{4}{h^2}$,

taking alternating values ± 1 along the diagonal and vanishing uniformly outside of this diagonal. Assume that the control region ω does not meet the diagonal of the domain $\Omega = [0,1] \times [0,1]$ which yields $\langle \mathbf{1}_\omega v, e_0 \rangle = 0$. We pick an initial state $y_0 = e_0$. The solution of (1.3.1) can be written as

$$y(t) = e^{-t\triangle_h} e_0 + \int_0^T e^{-(T-s)\triangle_h} \mathbf{1}_\omega v. \tag{1.3.3}$$

Starting with the eigenvector e_0, we introduce a family of eigenvectors $(e_j)_{j=1,\dots,N-1}$ of the discrete Laplacian to constitute an orthonormal basic of the space of discrete function (for the discrete L^2 norm). Thus, we can express the solution (1.3.3) in following form

$$\begin{aligned} y(t) &= e^{-t\lambda_{0k}} e_0 + \int_0^T \sum_{0 \le j \le N-1} e^{-(t-s)\lambda_j} \langle \mathbf{1}_\omega v, e_j \rangle e_j ds \\ &= e^{-t\lambda_{0h}} e_0 + \int_0^T \sum_{1 \le j \le N-1} e^{-(t-s)\lambda_j} \langle \mathbf{1}_\omega v, e_j \rangle e_j ds. \end{aligned}$$

If system (1.3.1) is exactly null-controllable in time T, i. e, $y(T) = 0$ for some control v, it is equivalent to having $\langle y(T), e_j \rangle = 0, \forall j$. Yet, we have

$$\begin{aligned} \langle y(T), e_0 \rangle &= \left\langle e^{-T\lambda_{0k}} e_0, e_0 \right\rangle + \int_0^T \left\langle \sum_{j \neq 0} e^{-(T-s)\lambda_j} \langle \mathbf{1}_\omega v, e_j \rangle e_j, e_0 \right\rangle ds \\ &= e^{-T\frac{4}{h^2}} \|e_0\|^2 \neq 0. \end{aligned} \tag{1.3.4}$$

This constitute a contradiction for the expected null controllability.

Remark 1.3.1.

- *There is a positive counterpart of the example above provided in [Zua06]. Namely, the null controllability result is shown to hold in case that the control region ω meets the diagonal of the domain (distributed control) or the control is required to be supported everywhere on the boundary of the domain (boundary control). We emphasize here that, despite the fact that no geometric restrictions are needed for the continuous heat equation, in the sense that null controllability holds from an arbitrarily small open subset of the boundary, this is not the case for the semi-discrete heat equation.*

- *The result which we obtained in (1.3.4) is consistent with the result obtained in [BHL10a]-[BHL10b], [LT06] and [N12]. Namely, as shown in [BHL10a] in 1D as*

well as [BHL10b] in arbitrary dimension, one can prove that one can find a control function v such that

$$|y_h(T)|_{L^2(\Omega)} \leq Ce^{-C/h^2} \left|y_0^h\right|_{L^2(\Omega)}. \tag{1.3.5}$$

In comparison, in the result obtained in [LT06] and [N12], a weaker inequality of the form (1.3.5) is proved with e^{-C/h^2} replaced by h^β, for some explicit $\beta > 0$. Yet, it can be applied for general parabolic problems.

1.3.2 Optimal control in the L^p framework

Consider an infinite dimensional linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \tag{1.3.6}$$

where the state $y(t)$ belongs to a reflexive Banach space X, the control $u(t)$ belongs to a reflexive Banach space U, $A : D(A) \to X$ is an operator, with dense domain and B is a control operator (in general, unbounded) on U. Discretizing this partial differential equation by using, for instance, a finite-difference or a finite-element scheme, leads to a family of finite dimensional linear control systems

$$\dot{y_h}(t) = A_h y_h(t) + B_h u_h(t), \qquad y_h(0) = y_{0h}, \tag{1.3.7}$$

where $y_h(t) \in X_h$ and $u_h(t) \in U_h$, for $0 < h < h_0$.

Here, we investigate a method by which we can achieve the control of minimal L^q-norm ($q > 2$). Namely, we will establish some conditions ensuring the existence and convergence of the discrete control of the minimal L^q-norm

$$\min \frac{1}{q} \int_0^T \|u_h(t)\|^q dt \quad (q > 2). \tag{1.3.8}$$

Moreover, in the framework of parabolic equation, our goal is to study the uniform controllability property of the family of discretized control systems (1.3.7) in L^q ($q > 2$).

In L^2-setting, by using finite-difference schemes for space semi-discretization, some relevant references concerning this problem have been in vestigated by many authors in series of articles [IZ99], [LZ98], [LZ02], [NZ03], [Zua99], [Zua02], [Zua04], [Zua05], [Zua06], [BHL10a], [BHL10b] and [LT06]. Recently, finite-element schemes for space

semi-discretization where also studied on [M12]. As opposed to the results for space semi-discretization only we also mention several works devoted to observability property for time semidiscrete and fully discrete linear systems such as [EZZ08], [EV09], and [BHL11].

The continuous framework

Firstly, we consider the problem of controls of the minimal L^q-norm ($q > 2$) in the continuous case. Namely, our aim is to minimize the following functional

$$\begin{cases} \text{Minimize } J(u) = \frac{1}{q}\int_0^T \|u\|^q\, dt \quad (q \geq 2) \\ \text{Subject to } u \in E \end{cases} \tag{1.3.9}$$

where $E = \{u \in U :$ u steering System (1.3.6) from y_0 to $y(T) = 0\}$.

In case of $q = 2$, this corresponds the well-known Hilbert Uniqueness Method which we presented in Section 1.2.1. However, this method is not appropriate for the case $q > 2$. Fortunately, thanks to the Fenchel-Rockafellar duality theorem following the approach of [CGL94], [GLH08], the problem can be reduced to the minimization of the corresponding conjugate function. More precisely, we state here our result in the case of $q > 2$ as follows.

Theorem 1.3.2. *1. We have the identity:*

$$\inf_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\, dt = -\inf_{\psi_T\in X^*}(\frac{1}{p}\int_0^T \|B^*\psi\|^p\, dt + < \psi(0), y_0 >),$$

where ψ is the solution of

$$-\dot{\psi} = A^*\psi$$
$$\psi(T) = \psi_T.$$

Equivalently, we have

$$\min_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\, dt = -\min_{\psi\in H}(\frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\, dt + < S(T)^*\psi, y_0 >).$$

2. If u_{op} is optimal of problem (1.3.9) then

$$u_{op}(t) = \|B^*S(T-t)^*\varphi\|^{p-2}\, B^*S(T-t)^*\varphi,$$

where $\varphi \in H$ is the unique minimizer of the functional

$$J^*(\psi) = \frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\, dt + < S(T)^*\psi, y_0 > .$$

The space H is defined as the completion of $\mathcal{D}(\Omega)$ w.r.t the norm

$$\|\psi\|_H = \left(\int_0^T \|B^*S(T-t)^*\psi\|_U^p\, dt\right)^{1/p}.$$

Remark 1.3.3. *According to the above arguments, if the control $\bar{u}$ is defined by*

$$\bar{u}(t) = \|B^*S(T-t)^*\varphi\|^{p-2}\, B^*S(T-t)^*\varphi, \tag{1.3.10}$$

with φ minimizer of J^*, for every $t \in [0,T]$ and if $y(.)$ is the solution of (1.3.6), such that $y(0) = y_0$, associated with the control $\bar{u}$, then we have $y(T)$=0. Therefore, $\bar{u}$ is the control of minimal L^q-norm $(q > 2)$, among all controls whose associated trajectory satisfies $y(T) = 0$.

The discrete case

Here, we address the minimization of the L^q-norm $(q > 2)$ of semidiscrete controls for parabolic equations.

In this context, assume that the control operator $B \in L(U, D((-\hat{A}^*)^\gamma)')$. The degree γ is so-called the degree of unboundedness of the control operator B.

As shown in [LT06], under standard assumptions on the discretization process and for a null-controllable parabolic system (1.3.6), if **the degree of unboundedness of the control operator B is lower than 1/2** then the semidiscrete approximation models are uniformly controllable and moreover it is shown how approximated control can be obtained using a minimization proceduce. The result of [LT06] is only satisfied for Neumann control but not for Dirichlet control. The main goal of our work is to overcome the limitation of [LT06] about the order 1/2 of unboundedness of the control operator and further to extend the result for Dirichlet control . We study the existence

of the minimum of the cost function of discretized control with power $q > 2$ instead of power $q = 2$, in the case the operator A generates an analytic semigroup. Namely, we show that the uniform controllability property also holds in L^q $(q > 2)$ even in the case of a degree of unboundedness greater than $1/2$.

In recent works in the L^2 setting, as in [LT06], uniform controllability is derived using semigroup arguments and introducing a vanishing term of the form $h^\beta \|\psi_h(T)\|^2_{L^2(\Omega)}$ for some $\beta > 0$ in the observability inequality. Additionally, by means of discrete Carleman inequalities, the authors of [BHL10a], [BHL10b], [BL12] obtain a weak uniform observability inequality for parabolic equations in arbitrary dimension. Moreover, the work of [M12] yields an uniform observability inequality in the abstract setting of unitary groups by using a finite-element semi-discretization.

Now, we state the following result of this thesis (see detail in Chapter 2).

Theorem 1.3.4. *Under the main assumptions stated in Section 2.3, if the control system $\dot{y} = Ay + Bu$ is null-controllable in time $T > 0$, with a control in L^q $(q > 2)$ then there exist $\beta > 0, h_1 > 0$, and positive real numbers C, C' satisfying*

$$\left\|e^{TA_h^*}\psi_h\right\|^p_{X_h} \leq \int_0^T \left\|B_h^* e^{tA_h^*}\psi_h\right\|^p_U dt + h^\beta \|\psi_h\|^p_{X_h} \leq C' \|\psi_h\|^p_{X_h}, \tag{1.3.11}$$

for every $h \in (0, h_1)$ and every $\psi_h \in X_h$, (where p, q are conjugate, i.e, $\frac{1}{p} + \frac{1}{q} = 1$).

In these conditions, for every $y_0 \in X$, and every $h \in (0, h_1)$, there exists $\varphi_h \in X_h$ minimizing the functional

$$J_h(\psi_h) = \frac{1}{p}\int_0^T \left\|B_h^* e^{tA_h^*}\psi_h\right\|^p_U dt + \frac{1}{p}h^\beta \|\psi_h\|^p_{X_h} + < e^{TA_h^*}\psi_h, P_h y_0 >_{X_h}, (1 \leq p < 2)$$

and the sequence $(\tilde{Q}_h u_h)_{0<h<h_1}$, where the control u_h is defined by

$$u_h(t) = \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} B_h^* e^{(T-t)A_h^*}\varphi_h, \tag{1.3.12}$$

for every $t \in [0, T]$ converges weakly (up to a subsequence), in the space $L^q(0, T; U)$ to a control u such that the solution of :

$$\dot{y} = Ay + Bu, \ y(0) = y_0, \tag{1.3.13}$$

satisfies $y(T) = 0$. For every $h \in (0, h_1)$, let $y_h(.)$ denote the solution of

$$\dot{y_h} = A_h y_h + B_h u_h, \ y_h(0) = P_h y_0. \tag{1.3.14}$$

Then,

- *the final state is given by*

$$y_h(T) = -h^\beta \|\varphi_h\|^{p-2} \varphi_h \tag{1.3.15}$$

- *The sequence* $(\tilde{P}_h y_h)_{0<h<h_1}$ *converges strongly (up to subsequence) in the space* $L^q(0,T;X)$, *to* $y(.)$.

Furthermore, there exists $M > 0$ *such that*

$$\int_0^T \|u(t)\|_U^p \leq M^{p/(p-1)} \|y_0\|_X^{p/(p-1)},$$

and, for every $h \in (0, h_1)$,

$$\begin{aligned}
&\int_0^T \|u_h(t)\|_{U_h}^p \leq M^{p/(p-1)} \|y_0\|_X^{p/(p-1)}, \\
&h^\beta \|\varphi_h\|_{X_h}^p \leq M^{p/(p-1)} \|y_0\|_X^{p/(p-1)}, \\
&\|y_h(T)\|_{X_h} \leq M^{1/(p-1)} h^{\beta/p} \|y_0\|_X^{1/(p-1)}.
\end{aligned} \tag{1.3.16}$$

(The approximation operators such as A_h, A_h^*, B_h, B_h^* and P_h will be precisely given in Section 2.3 of Chapter 2).

Remark 1.3.5.

- *Note that the result of [LT06] is only satisfied for Neumann control, but not for Dirichlet control. Our improvement now covers the case of Dirichlet control.*

- *The left hand side of (1.3.11) is considered as a 'weak' uniform observability type inequality for (2.1.2). It is 'weak' because of the additional viscosity term* $h^\beta \|\psi_h\|_{X_h}^p$.

- *We observe from (1.3.15) that* $y_h(T)$ *is not equal to zero. Yet,* $y_h(T)$ *goes to zero as the mesh size* $h \to 0$ *and the error estimate is expressed by inequality (2.3.25). This is consistent with the counter-example presented above.*

Theorem 1.3.4 states that the controls u_h defined by (2.3.22) tend to a control $\underline{u}$ realizing null-controllability for System (1.3.13). A natural question arises: since there exists a control u of minimal of L^q-norm ($q > 2$) such that $y(T) = 0$ (see Section 2.2.2 for details), what assumptions are needed to have $\underline{u} = u$? An answer is provided through the following proposition. Apart from the condition on y_0 known in [LT06], we here give an extra condition on the control u defined by (2.2.22).

Proposition 1.3.6. *With the notations of Theorem 1.3.4, if the sequence of real numbers* $\|\varphi_h\|_{X_h}$, $0 < h < h_1$, *is moreover bounded, then the control* u *is the unique control of*

minimal L^q-norm, as defined by (2.2.22), such that y(T)=0. Moreover, the sequence $(\tilde{Q}_h u_h)_{0<h<h_1}$ converges strongly (up to a subsequence) in the space $L^q(0,T;U)$ to the control u.

A sufficient condition on $y_0 \in X$, ensuring the boundedness of the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$, is the following : there exists $\eta > 0$ such that the control system $\dot{y} = Ay + Bu$ is null controllable in time t, for every $t \in [T-\eta, T+\eta]$, and the trajectory $t \mapsto S(t)y_0$ in X, for $t \in [T-\eta, T+\eta]$, is not contained in a hyperplane of X.

Another sufficient condition ensuring the boundedness of the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$ concerns the control function and in the following: there exists $\eta > 0$ such that the control system $\dot{y} = Ay + Bu$ is null controllable in time t, for every $t \in [T-\eta, T+\eta]$, and with the control u as defined in (2.2.22), the trajectory $t \mapsto S(t-\xi)Bu(\xi)$ in X, for $t \in [T-\eta, T+\eta]$, every $\xi \in (0,t)$ is not contained in a hyperplane of X.

Perspectives

One interesting open question is the following: how the above results change if we remove the assumption of uniform analyticity of the discretized semigroup. Another open question is to investigate whether these results still hold for semi-linear systems or for nonlinear systems.

1.3.3 Carleman estimates in the discrete case

In the first part, we dealt within a very general framework, and derived a general uniform observability inequality with a viscosity h^β. The second part of the thesis focuses on heat type equations, for which Carleman estimates allow one to derive more precise observability inequalities with a better viscosity in $e^{-C/h}$ in the case of general discrete schemes for parabolic equations. In this second part we shall only consider finite-difference discretizations.

Let Ω, ω be connected non-empty open subsets of $\mathbb{R}^n$ with $\omega \Subset \Omega$. We consider the following parabolic problem in $(0,T) \times \Omega$, with $T > 0$

$$\partial_t y - \nabla_x(c\nabla_x y) = \mathbf{1}_\omega \quad \text{in } (0,T)\times\Omega, \qquad y|_{\partial\Omega} = 0 \quad \text{and} \quad y|_{t=0} = y_0, \qquad (1.3.17)$$

where the diffusion coefficient c satisfies $c = c(x) \geq c_{min} > 0$.

In the continuous case Carleman estimates have many applications in the study of inverse problems, control theory of PDEs [LR95, FI96], unique continuation results

[Rob91, FL96, Tat95], and stabilization results [LR97, Bel03] to cite a few. One of the recent applications of continuous Carleman estimates is to obtain controllability results for parabolic equations. One expects that uniform controllability results can also be obtained from discrete versions of Carleman estimates.

Let us consider the elliptic operator on Ω given by

$$A = -\nabla_x(c\nabla_x .),$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. We shall introduce a finite-difference approximation of the operator A. For a mesh $\mathfrak{M}$ that we shall describe in Chapter 3, associated with a discretization step h, a consistent finite-difference approximation of A is given by $\mathcal{A}^{\mathfrak{M}} = -\bar{D}(c_d D.)$. It acts on a finite dimensional space $\mathbb{R}^{\mathfrak{M}}$, of dimension $|\mathfrak{M}|$, and is selfadjoint for a suitable inner product in $\mathbb{R}^{\mathfrak{M}}$. In this section, we present Carleman estimates for the discrete operator $\mathcal{A}^{\mathfrak{M}}$ or related operators in which the diffusion coefficient $c(x)$ is either smooth or discontinuous.

Smooth diffusion coefficients

An earlier attempt to derive discrete Carleman estimates for continuous diffusion coefficients can be found in [KS91]. With the result presented in [KS91], there is no connexion between the large Carleman parameter and the discretization step size. In recent years, some authors derived discrete Carleman estimates firstly for elliptic operators in 1D [BHL10a] and secondly extended that result to arbitrary dimension [BHL10b]. Additionally, the authors of [BL12] prove discrete Carleman for parabolic operators. An application of discrete Carleman estimate we mention below is to obtain a weak uniform controllability for parabolic equations.

An important point in the proof of Carleman estimate is the construction of a suitable weight function ψ whose gradient does not vanish in the complement of the observation region. The weight function is chosen to be smooth in the case of a smooth diffusion coefficient $c(x)$. Namely, to treat Carleman estimate for semi-discrete elliptic operators $P^{\mathfrak{M}} = -\partial_t^2 + \mathcal{A}^{\mathfrak{M}}$ with the continuous diffusion coefficient $c(x)$, the authors of [BHL10a] choose a function ψ that satisfies the following property.

Assumption 1.3.7. *Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\bar{\Omega}$ in $\mathbb{R}^n$*

and set $\tilde{Q} = (0,T) \times \tilde{\Omega}$. *The function* ψ *is in* $C^2(\overline{\tilde{Q}}, \mathbb{R})$ *and satisfies, for some* $c > 0$

$$\begin{aligned} &\psi > 0 \text{ and } \quad |\nabla \psi| \geq c \text{ in } \quad \tilde{Q} \qquad \partial_{n_x} \psi(t,x) < 0 \text{ in } \quad (0,T) \times V_{\partial\Omega}, \\ &\partial_t \psi \geq c \text{ on } \quad \{0\} \times (\Omega \backslash \omega) \qquad \nabla_x \psi = 0 \text{ and } \quad \partial_t \psi \leq -c \text{ on } \quad \{T\} \times \Omega, \end{aligned}$$

where $V_{\partial\Omega}$ *is a sufficiently small neighborhood of* $\partial\Omega$ *in* $\tilde{\Omega}$, *in which the outward unit normal* n_x *to* Ω *is extended from* $\partial\Omega$.

Such a function can be obtained by following the technique of [FI96], i.e., making use of Morse functions. With such a function ψ, we define the weight function $\varphi := e^{\lambda\psi}$. Now, we state the following discrete Carleman estimate for semi-discrete elliptic operators $P^{\mathfrak{M}} = -\partial_t^2 + \mathcal{A}^{\mathfrak{M}}$ as

Theorem 1.3.8. *[BHL10a] For* $\lambda \geq 1$ *sufficient large, there exist* $C > 0$ *and* $s_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, *depending on* Ω, ω, T *and* c *such that we have*

$$\begin{aligned} &s^3 \left\| e^{s\varphi} u \right\|^2_{L^2(Q)} + s \left\| e^{s\varphi} \partial_t u \right\|^2_{L^2(Q)} + s \left\| e^{s\varphi} Du \right\|^2_{L^2(Q)} + s \left| e^{s\varphi(0,.)} \partial_t u(0,.) \right|^2_{L^2(\Omega)} \\ &\qquad + s e^{2s\varphi(T,.)} \left| \partial_t u(T,.) \right|^2_{L^2(\Omega)} + s^3 e^{2s\varphi(T,.)} \left| u(T,.) \right|^2_{L^2(\Omega)} \\ &\leq C \Big(\left\| e^{s\varphi} Pu \right\|^2_{L^2(Q)} + s e^{2s\varphi(T,.)} \left| Du(T,.) \right|^2_{L^2(\Omega)} + s \left| e^{s\varphi(0,.)} \partial_t u(0,.) \right|^2_{L^2(\omega)} \Big) \end{aligned} \tag{1.3.18}$$

for $s \geq s_0$, $0 < h \leq h_0$ *and* $sh \leq \epsilon_0$ *and for all* $u \in C^2([0,T] \times \mathbb{C}^{\mathfrak{M} \cup \partial\mathfrak{M}})$, *satisfying* $u|_{\{0\}\times\Omega}$, $u|_{(0,T)\times\partial\Omega} = 0$.

Remark 1.3.9. *In the course of the proof of this Carleman estimate, the Carleman large parameter* s *has to be connected to the mesh size* h: *they obtained a condition of the form* $sh \leq \epsilon_0$, *with* $\epsilon_0 = \epsilon(\Omega, \omega, \gamma)$. *This condition cannot be avoided as it would imply null controllability for the associated heat equation in contradiction with the counter-example presented in Section* (1.3.1).

With this Carleman estimate they proved a Lebeau-Robbiano type inequality from which the following controllability result can be deduced.

Theorem 1.3.10. *[BHL10a] There exit* $h_0 > 0$, $C_T > 0$ *and* $C_1, C_2 > 0$, *such that with* $0 < h < h_0$, *and all initial data* $y_0 \in \mathbb{C}^{\mathfrak{M}}$, *there exists a semi-discrete control function* v

such that the solution to

$$\partial_t y - \bar{D}(c_d D y) = \mathbf{1}_\omega v, \qquad y^{\partial\mathfrak{M}} = 0, \qquad y|_{t=0} = y_0, \tag{1.3.19}$$

satisfying $\|v\|^2_{L^2(Q)} \leq C_T \, |y_0|^2_{L^2(\Omega)}$ *and furthermore* $|y(T)|^2_{L^2(\Omega)} \leq C_1 e^{-C_2/h^2} \, |y_0|^2_{L^2(\Omega)}$.

The observability estimate they then obtain is of the form

$$|q(0)|^2_{L^2(\Omega)} \leq C_T \Big(\int_0^T \int_\omega |q(t)|^2 \, dt \Big)^{\frac{1}{2}} + C e^{-C/h^2} \, |q(T)|^2_{L^2(\Omega)} \, ,$$

for any q solution to the adjoint system (1.3.19). As compared to result (1.2.4) in continuous case the observability estimate we state above is weak because of an additional term. Yet this term is exponentialy small in the limit $h \to 0$.

Next, we present a Carleman estimate for the semi-discrete parabolic operators $P^{\mathfrak{M}}_{\pm} = -\partial_t \pm \mathcal{A}^{\mathfrak{M}}$. In comparison with the elliptic case, the choice of weight functions for parabolic operators is a little different. Namely, the function $\psi = \psi(x)$ fulfills the following assumption

$$\begin{aligned} \psi > 0 \text{ in } \quad \tilde{\Omega} \qquad & |\nabla \psi| \geq c \text{ in } \quad \tilde{\Omega}, \\ \partial_{n_x} \psi(x) \leq -c < 0, \qquad & \partial^2_x \psi(x) \geq 0 \text{ in } \quad V_{\partial\Omega}, \end{aligned}$$

where $V_{\partial\Omega}$ is a sufficiently small neighborhood of $\partial\Omega$ in $\tilde{\Omega}$, in which the outward unit normal n_x to Ω is extended from $\partial\Omega$. Let $K > \|\psi\|_\infty$ and set

$$\begin{aligned} &\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \quad \phi(x) = e^{\lambda\psi(x)}, \\ &r(t,x) = e^{s(t)\varphi(x)}, \quad \rho(t,x) = (r(t,x))^{-1}, \end{aligned} \tag{1.3.20}$$

with

$$s(t) = \tau\theta(t), \quad \tau > 0, \quad \theta(t) = ((t+\delta T)(T+\delta T - t))^{-1},$$

for $0 < \delta < \frac{1}{2}$. From that, we state a Carleman estimate for the semi-discrete parabolic operators as follows.

Theorem 1.3.11. *[BL12] For the parameter* $\lambda \geq 1$ *sufficiently large, there exist* $C, \tau_0 \geq$

1, $h_0 > 0, \epsilon_0 > 0$, *depending on* ω, ω_0, T, c *we have*

$$\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q)} + \tau\sum_{i\in[1,d]}\left(\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}D_i u\right\|^2_{L^2(Q)} + \left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}\overline{D_i u}^i\right\|^2_{L^2(Q)}\right)$$
$$+\tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2(Q)} \leq C_{\lambda,\mathfrak{K}}\left(\left\|e^{\tau\theta\varphi}P^{\mathfrak{M}}u\right\|^2_{L^2(Q)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2((0,T)\times\omega)}\right.$$
$$\left.+h^{-2}\left|e^{\tau\theta\varphi}u|_{t=0}\right|^2_{L^2(\Omega)} + h^{-2}\left|e^{\tau\theta\varphi}u|_{t=T}\right|^2_{L^2(\Omega)}\right),$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h < h_0$, $0 < \delta \leq 1/2$ *and* $u \in C^1([0,T], \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}})$, *satisfying* $u|_{(0,T)\times\partial\Omega} = 0$, *if* $\tau h(\delta T^2)^{-1} \leq \epsilon_0$.

Remark 1.3.12. *Here, the parameter* δ *is introduced to avoid the singularity of the weight function at times* $t = 0$ *and* $t = T$. *This parameter connect to the other parameters through the condition* $\tau h(\delta T^2)^{-1} \leq \epsilon_0$ *which implies that* $s(t)h \leq \epsilon_0$ *similarly to Theorem 1.3.8.*

Some techniques, which the authors of [BHL10a] and [BL12] established for semi-discrete elliptic and parabolic operators, shall be useful in obtaining a Carleman estimate for a semi-discrete parabolic operator with discontinuous diffusion coefficients below.

Discontinuous diffusion coefficients

The question of controllability of partial differential systems with discontinuous coefficients and its dual counterpart, observability, are not fully solved yet. Here, we shall consider a parabolic operator in which the higher-order terms have the form $\partial_t - \nabla(c(x)\nabla)$ and the discontinuous coefficient refers here to the coefficient c in the elliptic operator in space x.

To our knowledge, in continuous case, this question was first addressed in [DOP02] for a parabolic operator $P = \partial_t - \nabla(c(x)\nabla)$, with a monotonicity assumption: the observation takes place in the region where the diffusion coefficient c is the "lowest". In the one-dimensional case, the mononicity assumption was relaxed for general piecewise C^1 coefficients [BDL07] and for coefficients with bounded variations [L07]. Recently, the case of an arbitrary dimension without any monotonicity condition in the elliptic case was solved in [LR10], in the parabolic case in [LR11].

Yet, the discrete analogues of these results are still to be developed.

In this thesis, we consider Carleman estimates for parabolic equations in case of

- the heat equation in one space dimension;

- a piecewise C^1 coefficient c with jumps at a finite number of points in Ω;
- a finite-difference discretization in space.

When considering a discontinuous coefficient c the parabolic problem (1.3.17) can be understood as a transmission problem. For instance, assume that c exhibits a jump at $a \in \Omega$. Then we write

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) = \mathbf{1}_\omega v & \text{in } (0,T) \times \big((0,a) \cup (a,1)\big), \\ c\partial_x y|_{a^+} = c\partial_x y|_{a^-}, \qquad y|_{a^+} = y|_{a^-}, \\ y|_{\partial\Omega} = 0, \qquad \text{and} \qquad y|_{t=0} = y_0. \end{cases}$$

The second line is thus a transmission condition implying the continuity of the solution and of the flux at $x = a$.

When one gives a finite-difference version of this transmission problem, a similar condition can be given for the continuity of the solution. Yet, for the flux, it is only achieved up to a consistent term. In what follows, in the finite-difference approximation, we shall in fact write

$$\begin{cases} y(a^-) = y(a^+) = y_{n+1}, \\ (c_d Dy)_{n+\frac{3}{2}} - (c_d Dy)_{n+\frac{1}{2}} = h\big(\bar{D}(c_d Dy)\big)_{n+1}, \end{cases}$$

(the discrete notation will be given Section (3.1.1)). Note that the flux condition converges to the continuous one if $h \to 0$, h being the discretization parameter. This difference between the continuous and the discrete case will be the source of several technical points.

As mentioned above, the Carleman weight function is chosen to be smooth in the case of a smooth diffusion coefficient $c(x)$. However, in case of a non-smooth diffusion coefficient, we shall introduce a particular type of weight functions, which are constructed through the following lemma.

Set $\Omega_1 = (0,a), \Omega_2 = (a,1)$. We enlarge the open intervals Ω_1, Ω_2 to large open sets $\tilde{\Omega}_1, \tilde{\Omega}_2$.

Lemma 1.3.13. *Let $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ be a smooth open and connected neighborhoods of intervals $\overline{\Omega}_1$, $\overline{\Omega}_2$ of $\mathbb{R}$ and let $\omega \subset \Omega_2$ be a non-empty open set. Then, there exists a function $\psi \in C(\bar{\Omega})$ such that*

$$\psi(x) = \begin{cases} \psi_1 \text{ in } \overline{\Omega}_1, \\ \psi_2 \text{ in } \overline{\Omega}_2, \end{cases}$$

with $\psi_i \in C^\infty(\overline{\widetilde{\Omega}_i}), i = 1, 2$, $\psi > 0$ *in* $\Omega, \psi = 0$ *on* $\Gamma, \psi_2' \neq 0$ *in* $\overline{\Omega}_2 \setminus \omega$, $\psi_1' \neq 0$ *in* $\overline{\Omega}_1$ *and the function* ψ *satisfies the following trace properties, for some* $\alpha_0 > 0$,

$$(\boldsymbol{A}u, u) \geq \alpha_0 |u|^2 \quad u \in \mathbb{R}^2,$$

with the matrix $\boldsymbol{A}$ *defined by*

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

with

$$\begin{aligned} a_{11} &= [\psi' \star]_a, \\ a_{22} &= [c\psi' \star]_a^2 (\psi')(a^+) + [c^2 (\psi')^3 \star]_a, \\ a_{12} &= a_{21} = [c\psi' \star]_a (\psi')(a^+), \end{aligned}$$

(see the above notation introduced in Chapter 3).

Here, we choose a weight function that yields an observation in the region $\omega \Subset \Omega_2$ in the Carleman estimate below. This choice is of course arbitrary. Such a weight function for continuous version can found in [BDL07].

Choosing a function ψ, as in the lemma above, we set the weight functions $\varphi(x), \phi(x), r(t, x)$ and $\rho(t, x)$ as given in (1.3.20). Now, we can state the semi-discrete Carleman estimate for the operator $P^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}}$ with a non-smooth diffusion coefficient c as follows (see details in Chapter 3).

Theorem 1.3.14. *Let* $\omega_0 \Subset \Omega_{02}$ *be a non-empty open set and we set* $f := \bar{D}(c_d Du)$. *For the parameter* $\lambda > 1$ *sufficiently large, there exist* C, $\tau_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, *depending on* ω_0 *so that the following estimate holds*

$$\begin{aligned} &\tau^{-1} \left\| \theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u \right\|^2_{L^2(Q_0)} + \tau \left\| \theta^{\frac{1}{2}} e^{\tau\theta\varphi} Du \right\|^2_{L^2(Q_0)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2(Q_0)} \\ &\leq C_{\lambda,\mathfrak{K}} \left(\left\| e^{\tau\theta\varphi} P^{\mathfrak{M}} u \right\|^2_{L^2(Q_0)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2((0,T)\times\omega_0)} \right. \\ &\qquad \left. + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=0} \right|^2_{L^2(\Omega_0)} + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=T} \right|^2_{L^2(\Omega_0)} \right), \end{aligned} \tag{1.3.21}$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$ *and* $\tau h(\alpha T)^{-1} \leq \epsilon_0$ *and for all* $u \in C^\infty(0,T;\mathbb{C}^{\mathfrak{M}})$ *satisfying* $u|_{\partial\Omega_0} = 0$.

The proof is a combination of the derivation of a discrete Carleman estimate as in [BHL10a, BL12] and tecniques of [BDL07] for operators with discontinuous coefficients in the one-dimensional case.

From the semi-discrete Carleman estimate we obtain above we deduce following weak observability estimate

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} \|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C}{h}} |q(T)|^2_{L^2(\Omega)},$$

for any q solution to the adjoint system

$$\partial_t q + \mathcal{A}^{\mathfrak{M}} q + aq = 0, \qquad q|_{\partial\Omega} = 0.$$

Here a is L^∞ potential function. A precise statement is given in Chapter 3.

Remark 1.3.15. *Similarly to the result in continuous case [BDL07] the observability estimate we state here is weak because of an additional term that describes the obstruction to the null-controllability* [1]*. This term is exponentially small in agreement with the results obtained in [BHL10a, BHL10b] in the smooth coefficient case.*

From the weak observability estimate given above we obtain a controllability result for the linear operator $P^{\mathfrak{M}}$. This result can be extended to classes of semi-linear equations

$$\big(\partial_t + \mathcal{A}^{\mathfrak{M}}\big) y + G(y) = \mathbf{1}_\omega v, \quad y \in (0,T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0,$$

with $G(x) = xg(x)$, where $g \in L^\infty(\mathbb{R})$ and

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \qquad \text{with} \qquad 0 \leq r < \frac{3}{2}.$$

We shall state controllability results with a control that satisfies

$$\|v\|_{L^2(Q)} \leq C\,|y_0|\,.$$

Note that the size of the control function is uniform with respect to the discretization parameter h.

[1] The counter-example we gave in Section (1.3.1) concerns dimensions greater than one. Yet Carleman estimate are insensitive to dimension.

Perspectives

In this second part, we prove a semi-discrete Carleman estimate for the operator $P^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}}$ with a non-smooth diffusion coefficient c in one space dimension. A similar question, in higher dimension, remains open.

Recently, the author [BDL11] proved Carleman estmiates for stratified media in $\mathbb{R}^n$ characterized by discontinuities of the coefficients in one direction. We expect to obtain such a Carleman estimate for stratified media with a finite-difference semi-discretization from combination of techniques of [BDL11] and techniques in the present thesis.

Chapter 2

Uniform controllability of semidiscrete approximations for parabolic systems in Banach spaces

2.1 Introduction

Consider an infinite dimensional linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \qquad y(0) = y_0, \tag{2.1.1}$$

where the state $y(t)$ belongs to a reflexive Banach space X, the control $u(t)$ belongs to a reflexive Banach space U, $A : D(A) \to X$ is an operator, with dense domain and B is a control operator (in general, unbounded) on U. Such a framework for studying control problems in the infinite dimensional setting has been well formalized in the textbook [TW07]. Discretizing this partial differential equation by using, for instance, a finite-difference or a finite-element scheme, leads to a family of finite dimensional linear control systems

$$\dot{y_h}(t) = A_h y_h(t) + B_h u_h(t), \qquad y_h(0) = y_{0h}, \tag{2.1.2}$$

where $y_h(t) \in X_h$ and $u_h(t) \in U_h$, for $0 < h < h_0$.

The control system (2.1.1) is said to be null-controllable in time T if there exists a solution $y(.)$ of (2.1.1) associated with a control u such that $y(T) = 0$. We refer for the well-known Hilbert Uniqueness Method (in short, HUM), introduced in [L88], consisting of minimizing the L^2-norm of the control. In this work, however, we investigate a method by which we can achieve the control of minimal L^q-norm ($q > 2$). In fact our problem

comes from an open problem of S. Labbé and E. Trélat in [LT06] where discretization issues of the HUM method are investigated for parabolic systems in a general framework. There is a limitation in the hypothesis in [LT06] following that the result of minimizing the discrete control can be only obtained in L^2-norm. Herein they refered to the condition on the degree of unboundedness of control operator which is only satisfied for Newmann control but not satisfied for Dirichlet control. This motivates us to find a solution to overcome their limitation and to extend their results for Dirichlet control. Here, we investigate a method to obtain the control of minimal L^q-norm ($q > 2$). Namely, we will establish some conditions ensuring the existence and convergence of the discrete control of the minimal L^q-norm

$$\min \frac{1}{q} \int_0^T \|u_h(t)\|^q dt \quad (q > 2). \tag{2.1.3}$$

Necessary conditions for optimal control in a finite dimensional state spaces were derived by Pontryagin et al. [PS62] (see also [T05]). The Maximum Principle as a set of necessary conditions for optimal control in infinite dimensional space was studied afterwards by many authors. Yet the Maximum Principle does not hold in general in infinite dimensional spaces. Many contributions provide conditions to ensure that the Maximum Principle remains true. Li and Yao [LY85] used the Eidelheit separation theorem and the Uhl theorem to extend the Maximum Principle to a large class of problems in infinite dimensional spaces when the target set is convex and the final time T is fixed. Additionally, the authors of [F87], [FF91], [LY91], by making use of Ekeland's variational principle, gave conditions on the reachable set and on the target set so as to obtain an extension of Maximum Principle. Considering system (2.1.1) in the case the final state and final time are fixed, the finite-codimensional condition of [F87], [FF91] and [LY91] is not satisfied in general. As a consequence, here, we cannot adapt the Maximum Principle. Yet, using the Fenchel-Rockafellar duality Theorem, following the approach of [CGL94], [GLH08], the constrained minimization of the function can be replaced by the unconstrained minimization problem of the corresponding conjugate function. This is the direction we shall follow here.

If one consider controllability properties for parabolic differential equations, a natural question is then that of the uniform controllability after discretization. In the framework of parabolic equation, the main goal of this article is to establish conditions ensuring a uniform controllability property of the family of discretized control systems (2.1.2) in L^q ($q > 2$) and to compute numerically the control function.

It is well known that controllabilty and observability are dual aspects of the same

problem. We shall therefore focus on uniform observability which is shown to hold when the observability constant of the finite-dimensional approximation systems does not depend on h. Relevant references concerning observability of discretized equations are [IZ99], [LZ98], [LZ02], [NZ03], [Zua99], [Zua02], [Zua04], [Zua05], [Zua06], [BHL10a], [BHL10b] and [LT06]. For finite-difference schemes, a uniform observability property holds for one-dimensional heat equation [LZ02], beam equation [LZ98], Schrodinger equations [Zua05], but does not hold for 1-D wave equations [IZ99]. This is due to the fact that the discrete dynamics generates high frequency spurious solutions for which the group velocity vanishes. To overcome these difficulties, Zuazua [Zua05] showed some remedies such as Tychonoff's regularization, multigrid method, mixed finite element and filtering of high-frequency, etc.

To our knowledge, in the 1-D heat equation case, due to the fact that the dissipative effect of the 1-D heat equation acts as a filtering mechanism by itself it is strong enough to exclude high-frequency spurious oscillations [LZ98]. However, the situation is more complex in multi-dimensional cases. A counter-example due to O. Kavian is shown in [Zua06] for the simplest finite-difference semi-discretization scheme for the heat equation in the square.

In recent works in the L^2 setting, by means of discrete Carleman inequalities, the authors of [BHL10a], [BHL10b] obtain a weak uniform observability inequality for parabolic equations in arbitrary dimension, by a adding term of the form $e^{-Ch^{-2}} \|\psi_h(T)\|^2_{L^2(\Omega)}$ in the bound, a term that vanishes asymptotically as $h \to 0$. Moreover, as in [LT06], the uniform controllability is derived using semigroup arguments and introducing a vanishing term of the form $h^\beta \|\psi_h(T)\|^2_{L^2(\Omega)}$ for some $\beta > 0$.

Apart from the results for space semi-discretization, we also mention several works devoting to achieve observability property for time semidiscrete and fully discrete linear systems such as [EZZ08], [EV09] and [BHL11].

The discretization framework in this paper is the same as in [LT06], [LT00]. According to [LT06], under standard assumptions on the discretization process and for a null-controllable parabolic system (2.1.1), if **the degree of unboundedness of the control operator is lower than 1/2** then the semidiscrete approximation models are uniformly controllable and moreover it is shown how approximated control can be obtained using a minimization proceduce. The main goal of our work is to overcome the limitation of [LT06] about the order 1/2 of unboundedness of the control operator. We study the existence of the minimum of the cost function of discretized control with power $q > 2$ instead of power $q = 2$, in the case the operator A generates an analytic semigroup. Our main result, Theorem 2.3.5, states that for null-controllable parabolic systems (2.1.1) and under standard approximation assumptions, if the discretized semigroup is uniformly

analytic, and if **the degree of unboundedness of the control operator B with respect to A is greater than 1/2**, then a uniform observability inequality in L^p $(1 < p < 2)$ is proved (p, q are conjugate). Moreover, a minimization procedure to compute approximated controls is provided. A degree of unboundedness greater than 1/2 appears in standard situation for example if one considers boundary controls.

The outline of the paper is as follows. In Section 2, we briefly review some well-known facts on controllability of linear partial differential equations in reflexive Banach spaces. Furthermore, we consider the existence and uniqueness of solution of the minimization problem in continuous case. By making use of the Fenchel-Rockafellar duality theorem, we gives a constructive way to build the control of minimal L^q-norm. The main result is stated in Section 3 and proved in Section 4. An example of application and numerical simulations are provided in Section 5, for the one-dimensional heat equation with Dirichlet boundary control. An appendix is devoted to the proof of a technical lemma.

2.2 Preliminary

We shall first present some basic knowledges about controllability of infinite dimensional linear control systems in reflexive Banach spaces. Second, we shall introduce a method allowing to achieve the control of minimal L^q-norm $(q > 2)$ in the continuous framework. This result in the continuous case will be the starting point for the study of a similar approach in the discrete case in Section 3.

2.2.1 A short review on controllability of linear partial differential equations in reflexive Banach spaces

For issues related to controllability in reflexive Banach spaces we refer the reader to the excellent textbook [TW07](see also [CT06, S05] and references therein).

The notation $L(E, F)$ stands for the set of linear continuous mappings from E to F, where E and F are reflexive Banach spaces.

Let X be a reflexive Banach space. In what follows, we denote by $\|.\|_X$ the norm on X. Let $S(t)$ denote a strongly continuous semigroup on X, of generator $(A, D(A))$. Let X_{-1} denote the completion of X for norm $\|x\|_{-1} = \left\|(\beta I - A)^{-1}x\right\|$, where $\beta \in \rho(A)$ is fixed. Note that X_{-1} does not depend on the specific value of $\beta \in \rho(A)$. The space X_{-1} is isomorphic to $(D(A^*))'$, the dual space of $D(A^*)$. The semigroup $S(t)$ extends to a semigroup on X_{-1}, still denoted by $S(t)$, whose generator is an extension of the operator A, still denoted by A. With this notation, A is a linear operator from X to X_{-1}.

Let U be a reflexive Banach space. Denote by $\|.\|_U$ the norm on U.

A linear continuous operator $B : U \to X_{-1}$ is admissible for the semigroup $S(t)$ if every solution of

$$y' = Ay(t) + Bu(t), \tag{2.2.1}$$

with $y(0) = y_0 \in X$ and $u(.) \in L^q(0, +\infty; U)$ $(q > 2)$, satisfies $y(t) \in X$, for every $t \geq 0$. The solution of equation (2.2.1) is understood in the mild sense, i.e,

$$y(t) = S(t)y(0) + \int_0^T S(t-s)Bu(s)ds, \tag{2.2.2}$$

for every $t \geq 0$.

For $T > 0$, define $L_T : L^q(0, T; U) \to X_{-1}$ by

$$L_T u = \int_0^T S(T-s)Bu(s)ds. \tag{2.2.3}$$

A control operator $B \in L(U, X_{-1})$ is admissible, if and only if $ImL_T \subset X$, for some (and hence for every) $T > 0$.

The adjoint L_T^* of L_T satisfies

$$\begin{aligned} &L_T^* : X^* \to (L^q(0, T; U))^* = L^p(0, T; U^*) \\ &L_T^*\psi(t) = B^* S(T-t)^*\psi \end{aligned} \tag{2.2.4}$$

(p and q are conjugate) a.e on [0,T] for every $\psi \in D(A^*)$. Moreover, we have

$$\|L_T^*\psi\| = \sup_{\|u\|_q \leq 1} \int_0^T \langle B^* S(T-s)^*\psi, u(s)\rangle ds, \tag{2.2.5}$$

for every $\psi \in X^*$.

Let $B \in L(U, X_{-1})$ denote an admissible control operator.

We shall use the following lemma (for a proof we refer to [chapter 10, [TW07]])

Lemma 2.2.1. *Let Z_1, Z_2, Z_3 be reflexive Banach spaces and $f \in L(Z_1, Z_3)$, $g \in L(Z_2, Z_3)$. Then the following statements are equivalent:*

- *$Imf \subset Img$.*
- *There exists a constant $C > 0$ such that : $\|f^*z\|_{Z_1} \leq C \|g^*z\|_{Z_2}$ for every $z \in Z_3$.*

- *There exists an operator* $h \in L(Z_1, Z_2)$ *such that* $f = gh$.

We now state the concept of exact null controllablity as follows.

For $T > 0$, System (2.2.1) is said to be null controllable in time T if for every $y_0 \in X$, there exists $u(.) \in L^q(0,T;U)$ so that the corresponding solution of (2.2.1), with $y(0) = y_0$ satisfies $y(T) = 0$.

According to [TW07], System (2.2.1) is null controllable in time T if and only if $ImS(T) \subset ImL_T$. It follows from the above Lemma that there exists $C > 0$ such that

$$\begin{aligned} C \left\| S(T)^* \psi \right\|_X \leq \left\| L_T^* \psi \right\| &= \sup_{\|u\|_q \leq 1} \int_0^T < B^* S(T-s)^* \psi, u(s) > ds \\ &\leq \sup_{\|u\|_q \leq 1} \int_0^T \|B^* S(T-t)^* \psi\| \|u(t)\| dt \\ &\leq \Big(\int_0^T \|B^* S(t)^* \psi\|^p dt \Big)^{\frac{1}{p}}. \end{aligned}$$

Thus, system (2.2.1) is null controllable with a control in L^p for the time T if and only if

$$\int_0^T \|B^* S(t)^* \psi\|^p dt \geq C \left\| S(T)^* \psi \right\|_X^p . \tag{2.2.6}$$

2.2.2 Minimal L^q-norm $(q > 2)$ for the continous case

The Maximum Principle is a well-known method to tackle optimal control problems. Applying the Maximum Principle built in infinite dimensional spaces, such as [F87], [FF91] and [LY91] is not possible here. Unfortunately, the finite-codimensional condition of [F87], [FF91] and [LY91] is an obstacle in adapting the Maximum Principle to our problem.

The goal of this section is to show that, using duality arguments and the Fenchel-Rockafellar theorem, we can obtain controls of minimal L^q-norm $(q > 2)$ for the continuous framework.

Consider the system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \text{ on } Q_T = (0,T) \times \Omega, \\ y(0) = y_0, \end{cases} \tag{2.2.7}$$

where B is admissible and A generates an analytic semigroup $S(t)$ in the reflexive Banach space X. Assume that system (2.2.7) is null controllable in time $T > 0$.

Our aim is to mimimize the following function

$$\begin{cases} \text{Minimize } J(u) = \frac{1}{q}\int_0^T \|u\|^q \, dt \quad (q > 2), \\ \text{Subject to } u \in E, \end{cases} \tag{2.2.8}$$

where $E = \{u \in L^q(0,T;U) : \text{ u steering the system from } y_0 \text{ to } y(T) = 0\}$. Clearly, E is nonempty.

Theorem 2.2.2. *Problem (2.2.8) has a unique solution u.*

Proof. First of all, we show the existence of the solution of the optimal control problem.

Consider a minimizing sequence $(u_n)_{n\in N}$ of controls on $[0,T]$, i.e,

$$\int_0^T \|u_n\|^q \, dt \text{ converges to } \inf J(u) \text{ as } n \to +\infty.$$

It follows that the sequence $(u_n)_{n\in N}$ is bounded in $L^q(0,T;U)$. Since U is reflexive space and $q < +\infty$, then $L^q(0,T;U)$ is reflexive as well. Thus, up to a sequence, $(u_n)_{n\in N}$ converges weakly to some u in L^q. Note that the trajectory y_n (resp. y) associated with the control u_n (resp. u) on [0,T] through the system

$$\dot{y}_n = Ay_n + Bu_n, \; y_n(0) = y_0,$$

and the solution of the above system is expressed in form

$$y_n(t) = S(t)y_0 + \int_0^T S(T-s)Bu_n(s)ds \; .$$

Passing to the limit, we find

$$\dot{y} = Ay + Bu, \; y(0) = y_0,$$

and the solution y associated with control u in the form

$$y(t) = S(t)y_0 + \int_0^T S(T-s)Bu(s)ds.$$

As u_n converges weakly to u in L^q, we obtain the inequality

$$\int_0^T \|u(t)\|^q \, dt \leq \liminf_{n \to +\infty} \int_0^T \|u_n(t)\|^q \, dt.$$

But $(u_n)_{n \in N}$ is a minimizing sequence, it follows

$$\int_0^T \|u(t)\|^q \, dt = \inf_{v \in E} \int_0^T \|v(t)\|^q \, dt,$$

i.e, the control u is the minimizer of (2.2.8). This ensures the existence of an optimal control.

Moreover, the cost function is strictly convex which yields uniqueness of the solution.

□

By making use of convex duality, the problem is reduced to the minimization of the corresponding conjugate function as stated in the following theorem.

Theorem 2.2.3. *1. We have the identity:*

$$\inf_{u \in E} \frac{1}{q} \int_0^T \|u\|^q \, dt = - \inf_{\psi_T \in X^*} \Big(\frac{1}{p} \int_0^T \|B^* \psi\|^p \, dt + < \psi(0), y_0 > \Big), \tag{2.2.9}$$

where ψ is the solution of

$$-\dot{\psi} = A^* \psi \tag{2.2.10}$$

$$\psi(T) = \psi_T. \tag{2.2.11}$$

Equivalently, we have

$$\min_{u \in E} \frac{1}{q} \int_0^T \|u\|^q \, dt = - \min_{\psi \in H} \Big(\frac{1}{p} \int_0^T \|B^* S(T-t)^* \psi\|^p \, dt + < S(T)^* \psi, y_0 > \Big). \tag{2.2.12}$$

2. If u_{op} is solution of the problem (2.2.8) then u_{op} is of the following form

$$u_{op}(t) = \|B^* S(T-t)^* \varphi\|^{p-2} \, B^* S(T-t)^* \varphi,$$

where $\varphi \in H$ is the unique minimizer of the functional

$$J^*(\psi) = \frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\, dt + < S(T)^*\psi, y_0 > .$$

The space H is defined as the completion of $\mathcal{D}(\Omega)$ w.r.t the norm

$$\|\psi\|_H = \left(\int_0^T \|B^*S(T-t)^*\psi\|_U^p\, dt\right)^{1/p}.$$

Proof. 1. Let $\bar{y}$ be solution of (2.2.7) with $u = 0$ and we introduce the operator $N \in L(L^q(Q_T), X)$ with $Nu = z_u(., T)$ for all $u \in L^q(Q_T)$, where z_u is solution to

$$\dot{z} = Az + Bu \tag{2.2.13}$$

$$z(x, 0) = 0. \tag{2.2.14}$$

Accordingly, the solution y of (2.2.7) can be decomposed in the form

$$y = z_u + \bar{y}. \tag{2.2.15}$$

The adjoint N^* is given as follows. For each $\psi_T \in X^*$, $N^*\psi_T = B^*\psi$ where ψ is solution of (2.2.10) – (2.2.11).

Let us introduce the following functions F and G

$$F(z) = \begin{cases} 0 & \text{for } z = -\bar{y}(T) \\ +\infty & \text{otherwise} \end{cases},$$

$$G(u) = \frac{1}{q}\int_0^T \|u\|^q\, dt.$$

Then, problem (2.2.8), where the infimum is taken over all u satisfying E, is equivalent to the following minimization problem

$$\inf_{u \in L^q(Q_T)} \big(F(Nu) + G(u)\big). \tag{2.2.16}$$

System (2.2.7) is null controllable then there exists a control $u \in L^q$ such that $z_u = -\bar{y}(T)$. With a such control, it follows $F(Nu) + G(u) < \infty$ and $0 \in int(DomF - NDomG)$.

Functions F^* and G^* are the convex conjugate of F and G, respectively. Denote $\psi_T = \psi(T)$ and observe that

$$F^*(\psi_T) = - < \psi_T, \bar{y}_T >,$$

for all $\psi_T \in X^*$. Additionally,

$$G^*(\omega) = \frac{1}{p} \int_0^T \|\omega\|^p \, dt.$$

Therefore,

$$G^*(N^*\psi_T) + F^*(-\psi_T) = \frac{1}{p} \int_0^T \|B^*\psi\|^p \, dt + < \psi_T(x), \bar{y}_T(x) > . \tag{2.2.17}$$

We have B is admissible then

$$\begin{aligned} G^*(N^*\psi_T) + F^*(-\psi_T) &= \frac{1}{p} \int_0^T \|B^*\psi\|^p \, dt + < \psi_T(x), \bar{y}_T(x) > \\ &\leq C \|\psi\|^p + \|\psi_T\| \, \|\bar{y}_T\| \leq \infty \end{aligned}$$

Futher, we can choose $\psi_T \in DomF^*$, $-B\psi \in DomG^*$ which ψ is solution of (2.2.10) – (2.2.11) with $\psi(T) = \psi_T$ such that $0 \in int(N^*DomF^* + DomG^*)$.

Then we can apply the duality theorem of W.Fenchel and T.R.Rockafellar (see e.g Theorem 4.2 p.60 in [ET99]). It yields

$$\inf_{u \in L^q(Q_T)} (F(Nu) + G(u)) = - \inf_{\psi_T \in X^*} (G^*(N^*\psi_T) + F^*(-\psi_T)), \tag{2.2.18}$$

By means of multiplying the state equation (2.2.10) by $\bar{y}$ and then integrating by parts, we obtain

$$< \psi_T, \bar{y}_T > = < \psi(0), y_0 > .$$

Equation (2.2.17) can thus be written as follows

$$\begin{aligned} G^*(N^*\psi_T) + F^*(-\psi_T) &= \frac{1}{p}\int_0^T \|B^*\psi\|^p\, dt + < \psi(0), y_0 > \\ &= \frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi_T\|^p\, dt + < S(T)^*\psi_T, y_0 >, \end{aligned}$$

where ψ is the solution of (2.2.10) – (2.2.11). It follows from (2.2.16) – (2.2.18) and from the above equation that

$$\inf_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\, dt = -\inf_{\psi_T}\Big(\frac{1}{p}\int_0^T \|B^*\psi\|^p\, dt + < \psi(0), y_0 > \Big),$$

where ψ is the solution of (2.2.10) – (2.2.11). It can be as well written as

$$\min_{u\in E}\frac{1}{q}\int_0^T \|u\|^q\, dt = -\min_{\psi\in H}\Big(\frac{1}{p}\int_0^T \|B^*S(T-t)^*\psi\|^p\, dt + < S(T)^*\psi, y_0 > \Big).$$

2. If we denote by (u_{op}), (φ_T) the unique solutions to "LHS of (2.2.12)" and "RHS of (2.2.12)" respectively, then one finds

$$0 = \frac{1}{q}\int_0^T \|u_{op}\|^q\, dt + \frac{1}{p}\int_0^T \|B^*\varphi_T\|^p\, dt + < \varphi_T(0), y_0 > . \tag{2.2.19}$$

We apply the Young inequality to the first two terms in the RHS of (2.2.19):

$$\frac{1}{q}\int_{Q_T} \|u_{op}\|^q\, dt + \frac{1}{p}\int_{Q_T} \|B^*\varphi_T\|^p\, dt \geq \int_{Q_T} < u_{op}, B^*\varphi_T > dt. \tag{2.2.20}$$

Then, the "RHS of (2.2.19)" $\geq \int_{Q_T} < u_{op}, B^*\varphi_T > dt + < \varphi_T(0), y_0 >$.

Furthermore, by multiplying the two sides of (2.2.10) by y and applying the Green formula, we obtain

$$< B^*\varphi_T, u > + < \varphi_T(0), y_0 >= 0. \tag{2.2.21}$$

On the one hand, "RHS of (2.2.19)" ≥ 0 (due to (2.2.21)). On the other hand, "RHS of (2.2.19)" $= 0$ (due to (2.2.19)).

This implies an equality in (2.2.20), i.e

$$\|u_{op}\|^q = \|B^*\varphi_T\|^p .$$

It follows that

$$u_{op}(t) = \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi,$$

where $\varphi \in H$ is the minimizer of the functional J^*.

□

Remark 2.2.4. *The functional J^* is convex, and from inequality (2.2.6), it is coercive in space H. Then, it follows that J^* has a unique minimizer $\varphi \in H$. According to the above arguments, the control $\bar{u}$ is defined by*

$$\bar{u}(t) = \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi, \tag{2.2.22}$$

for every $t \in [0,T]$ and if $y(.)$ is the solution of (2.2.8), such that $y(0) = y_0$, associated with the control $\bar{u}$, then we have $y(T)=0$. Therefore, $\bar{u}$ is the control of minimal L^q-norm $(q > 2)$, among all controls whose associated trajectory satisfies $y(T) = 0$.

Here, we emphasize that observability in L^p-norm $(1 < p < 2)$ implies controllability and gives a way to build the control of minimal L^q-norm $(q > 2)$. A similar result was known in L^2-norm through HUM (see [L88]).

2.3 Main result

We are concerned in this work with the uniform controllability property for semidiscretizations of parabolic systems. As shown in [LT06], this property is known to hold if the degree of unboundedness of control operator $\gamma \in [0, 1/2)$. In this section, we also establish some appropriate assumptions and conditions ensuring that the uniform controllability still holds even in the case $\gamma \in \left[1/2, \frac{1}{p}\right)$. It is the condition $\gamma \in \left[1/2, \frac{1}{p}\right)$ that motivates the study of minimal L^q-norm $(q > 2)$ instead of L^2-norm, with p and q conjugate.

Let X and U be reflexive Banach spaces, and let $A : D(A) \to X$ be a linear operator and self-adjoint, generating a strongly continuous semigroup $S(t)$ on X. Let $B \in L(U, D(A^*)')$ be a control operator. We now express all assumptions that will be

used in what follows. We keep most of the usual approximation assumptions as well as the assumptions about analytic semigroup stated in [LT06]. However, assumptions on γ such as (21), (22) and (30) in [LT06], do not hold if $\gamma > 1/2$. To overcome this limitation, here we give a remedy by means of replacing the above assumptions by other appropriate conditions. This will be explained more clearly through Remark 3.1 below. We make the following assumptions, of which (H1)-(H2) concern the continuous case and (H3)-(H4) concern the discretization scheme.

(H1) The semigroup S(t) is analytic.

Therefore, (see [P83]) there exist positive real number C_1 and ω such that

$$\|S(t)\|_X \leqslant C_1 e^{\omega t} \|y\|_X , \quad \|AS(t)y\|_X \leqslant C_1 \frac{e^{\omega t}}{t} \|y\|_X , \tag{2.3.1}$$

for all $t > 0$ and $y \in D(A)$, and such that, if we set $\hat{A} = A - \omega I$, for $\theta \in [0, 1]$ and there holds

$$\left\|(-\hat{A}^{\theta})S(t)y\right\|_X \leq C_1 \frac{e^{\omega t}}{t^{\theta}} \|y\|_X , \tag{2.3.2}$$

for all $t > 0$ and $y \in D(A)$.

Of course, inequalities (2.3.1) hold as well if one replaces A by A^*, $S(t)$ by $S(t)^*$, for $y \in D(A^*)$.

Moreover, if $\rho(A)$ denotes the resolvent set of A, then there exists $\delta \in \left(0, \frac{\pi}{2}\right)$ such that $\rho(A) \supset \Delta_\delta = \left\{\omega + \rho e^{i\theta} | \theta > 0, |\theta| \leq \frac{\pi}{2} + \delta\right\}$.

For $\lambda \in \rho(A)$, denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent of A . It follows from the previous estimates that there exists $C_2 > 0$ such that

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{C_2}{|\lambda - \omega|}, \quad \|AR(\lambda, A)\|_{L(X))} \leq C_2, \tag{2.3.3}$$

for every $\lambda \in \Delta_\delta$, and

$$\left\|R(\lambda, \hat{A}))\right\|_{L(X)} \leq \frac{C_2}{|\lambda|}, \quad \left\|\hat{A}R(\lambda, \hat{A})\right\|_{L(X)} \leq C_2, \tag{2.3.4}$$

for every $\lambda \in \Delta_\delta + \omega$. Similarly, inequalities (2.3.3) and (2.3.4) hold as well with A^* and $\hat{A}^*$.

(H2) The degree of unboundedness of B is γ. Assume that $\gamma \in \left[1/2, \frac{1}{p}\right)$ (where p,q are conjugate, i.e $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < 2$). This means that

$$B \in L(U, D((-\hat{A}^*)^{\gamma})'). \tag{2.3.5}$$

In these conditions, the domain of B^* is $D(B^*) = D((-\hat{A}^*)^{\gamma})$, and there exists $C_3 > 0$

such that

$$\|B^*\psi\| \leq C_3 \left\| ((-\hat{A}^*)^\gamma)\psi \right\|_X, \tag{2.3.6}$$

for every $\psi \in D((-\hat{A}^*)^\gamma)$.

(H3) We consider two families $(X_h)_{0<h<h_0}$ and $(U_h)_{0<h<h_0}$ of finite dimensional spaces, where h is the discretization parameter.

For every $h \in (0, h_0)$, there exist linear mappings $P_h : D((-\hat{A}^*)^{\frac{1}{2}})' \to X_h$ and $\tilde{P}_h : X_h \to D((-\hat{A}^*)^{\frac{1}{2}})$ and $(\hat{A}^*)^{-\gamma+\frac{1}{2}} : D(-(\hat{A}^*)^{\frac{1}{2}}) \to D(-(\hat{A}^*)^\gamma)$ (resp., there exist linear mappings $Q_h : U \to U_h$ and $\tilde{Q_h} : U_h \to U$), satisfying the following requirements:

$(H_{3.1})$ For every $h \in (0, h_0)$. The following properties hold

$$P_h\tilde{P}_h = id_{X_h} \text{ and } Q_h\tilde{Q_h} = id_{U_h}. \tag{2.3.7}$$

$(H_{3.2})$ There exist $s > 0$ and $C_4 > 0$ such that there holds, for every $h \in (0, h_0)$,

$$\left\| \left(I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h \right) \psi \right\|_X \leq C_4 h^s \left\| A^*\psi \right\|_X , \tag{2.3.8}$$

$$\left\| ((-\hat{A}^*)^\gamma) \left(I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h \right) \psi \right\|_X \leq C_4 h^{s(1-\gamma)} \left\| A^*\psi \right\|_X , \tag{2.3.9}$$

for every $\psi \in D(A^*)$ and

$$\left\| (I - \tilde{Q_h}Q_h)u \right\|_U \to 0, \quad \text{as } h \to 0, \tag{2.3.10}$$

for every $u \in U$, and

$$\left\| (I - \tilde{Q_h}Q_h)B^*\psi \right\|_U \leq C_4 h^{s(1-\gamma)} \|A^*\psi\|_X, \tag{2.3.11}$$

for every $\psi \in D(A^*)$.

For every $h \in (0, h_0)$, the vector space X_h (resp. U_h) is endowed with the norm $\|.\|_{X_h}$ (resp. $\|.\|_{U_h}$) defined by:

$\|y_h\|_{X_h} = \left\| \tilde{P}_h y_h \right\|_X$ for $y_h \in X_h$ (resp.$\|u_h\|_{U_h} = \left\| \tilde{Q_h} u_h \right\|_U$).

Therefore, we have the properties

$$\left\| \tilde{P}_h \right\|_{L(X_h,X)} = \left\| \tilde{Q_h} \right\|_{L(U_h,U)} = 1 \text{ and } \left\| (\hat{A}^*)^{-\gamma+\frac{1}{2}} x \right\|_X \leq C \left\| x \right\|_X , \tag{2.3.12}$$

$$\|P_h\|_{L(X,X_h)} \leq C_5 \text{ and } \|Q_h\|_{L(U,U_h)} \leq C_5. \tag{2.3.13}$$

($H_{3.3}$) For every $h \in (0, h_0)$, there holds

$$P_h = \tilde{P_h}^* \text{ and } Q_h = \tilde{Q_h}^*, \tag{2.3.14}$$

where the adjoint operators are considered with respect to the pivot spaces X, U, X_h, U_h.

($H_{3.4}$) There exists C_6 such that

$$\left\| B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P_h} \psi_h \right\|_U \leq C_6 h^{-\gamma s} \|\psi_h\|_{X_h}, \tag{2.3.15}$$

for all $h \in (0, h_0)$ and $\psi_h \in X_h$.

For every $h \in (0, h_0)$, we define the approximation operators $A_h^* : X_h \to X_h$ of A^* and $B_h^* : X_h \to U_h$ of B^*, by

$$A_h^* = P_h A^* \tilde{P_h} \text{ and } B_h^* = Q_h B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P_h}. \tag{2.3.16}$$

(H_4) The following properties hold:

($H_{4.1}$) Let $A_h : X_h \to X_h$ be a finite dimensional approximation of A that satisfies the requirement: the family of operators e^{tA_h} is uniformly analytic, in the sense that there exists $C_7 > 0$ such that

$$\left\| e^{tA_h} \right\|_{L(X_h)} \leq C_7 e^{\omega t}, \tag{2.3.17}$$

$$\left\| A_h e^{tA_h} \right\|_{L(X_h)} \leq C_7 \frac{e^{\omega t}}{t}, \tag{2.3.18}$$

for all $t > 0$ and $h \in (0, h_0)$.

($H_{4.2}$) There exists $C_8 > 0$ such that, for every $f \in X$ and every $h \in (0, h_0)$, the respective solutions of $\hat{A}^* \psi = f$ and $\hat{A}_h^* \psi_h = P_h f$ satisfy

$$\|P_h \psi - \psi_h\|_{X_h} \leq C_8 h^s \|f\|_X . \tag{2.3.19}$$

In other words, there holds $\left\| P_h \hat{A}^{*-1} - \hat{A_h}^{*-1} P_h \right\|_{L(X, X_h)} \leq C_8 h^s$.

Remark 2.3.1. *Note that the result of [LT06] is satisfied for Newmann control but not for Dirichlet control. Our improvement now covers the case of Dirichlet control.*

Remark 2.3.2. *In comparison with [LT06], the important point to note here is the*

appearance of $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ *in* (2.3.8), (2.3.9) *and* (2.3.15).

Note that inequality (22) of [LT06] makes sense for $\gamma < \frac{1}{2}$ *and thus* $im\tilde{P}_h \subset D((-\hat{A}^*)^{1/2}) \subset D((-\hat{A}^*)^{\gamma})$.

In our context, on account of $\gamma \geq \frac{1}{2}$, *inequality* (2.3.9), *which is similar to inequality (22) of [LT06], only makes sense if we add* $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ *so that* $im(\hat{A}^{*-\gamma+\frac{1}{2}}\tilde{P}_h) \subset D((-\hat{A}^*)^{\gamma})$. *The choice of* $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$ *seems to be the best adapted to our theory. This modification of the assumptions of [LT06] allows us to overcome the limitation* $\gamma < 1/2$.

Remark 2.3.3. *By means of the condition of the degree of unboundedness of operator* B *and* (2.3.6), B *is admissible. Indeed, we have*

$$\begin{aligned}
\|L_T^*\psi\| &= \sup_{\|u\|_q \leq 1} \int_0^T < B^*S^*(T-s)x, u(s) > ds \\
&\leq \Big(\int_0^T \|B^*S(t)^*\psi\|^p dt \Big)^{\frac{1}{p}} \\
&\leq C_3 \Big(\int_0^T \Big\| (-\hat{A}^*)^{\gamma} S(t)^*\psi \Big\|^p dt \Big)^{\frac{1}{p}} \\
&\leq C_3 \Big(\int_0^T \frac{e^{\omega t}}{t^{p\gamma}} \|\psi\|^p dt \Big)^{\frac{1}{p}} dt \\
&\leq C_T \|\psi\| \quad (since \quad p\gamma < 1).
\end{aligned}$$

Remark 2.3.4. . *Assumptions* (H_3) *(except for the inequalities* (2.3.8), (2.3.9), (2.3.15)*) and* $(H_{4.2})$ *hold for most of the classical numerical approximation schemes, such as Galerkin methods, centered finite-difference schemes,...Inequalities* (2.3.8), (2.3.9) *and* (2.3.15) *have to be checked in each case. Moreover, as noted in [LT00], assumption* $(H_{4.1})$ *of uniform analyticity is not standard, and has to be checked in each specific case as well.*

The main result of the article is the following

Theorem 2.3.5. *Under the previous assumptions, if the control system* $\dot{y} = Ay + Bu$ *is null-controllable in time* $T > 0$, *then there exist* $\beta > 0, h_1 > 0$, *and positive real numbers* C, C' *satisfying*

$$C \Big\| e^{TA_h^*} \psi_h \Big\|_{X_h}^p \leq \int_0^T \Big\| B_h^* e^{tA_h^*} \psi_h \Big\|_U^p dt + h^{\beta} \|\psi_h\|_{X_h}^p \leq C' \|\psi_h\|_{X_h}^p, \tag{2.3.20}$$

for every $h \in (0, h_1)$ *and every* $\psi_h \in X_h$, $(1 \leq p < 2)$.

In these conditions, for every $y_0 \in X$, and every $h \in (0, h_1)$, there exists $\varphi_h \in X_h$ minimizing the functional

$$J_h(\psi_h) = \frac{1}{p}\int_0^T \left\| B_h^* e^{tA_h^*}\psi_h \right\|_U^p dt + \frac{1}{p}h^\beta \|\psi_h\|_{X_h}^p + < e^{TA_h^*}\psi_h, P_h y_0 >_{X_h}, (1 \le p < 2) \quad (2.3.21)$$

and the sequence $(\tilde{Q}_h u_h)_{0<h<h_1}$, where the control u_h is defined by

$$u_h(t) = \left\| B_h^* e^{(T-t)A_h^*}\varphi_h \right\|^{p-2} B_h^* e^{(T-t)A_h^*}\varphi_h, \quad (2.3.22)$$

for every $t \in [0, T]$ converges weakly (up to a subsequence), in the space $L^q(0, T; U)$ to a control u such that the solution of :

$$\dot{y} = Ay + Bu,\ y(0) = y_0, \quad (2.3.23)$$

satisfies $y(T) = 0$. For every $h \in (0, h_1)$, let $y_h(.)$ denote the solution of

$$\dot{y_h} = A_h y_h + B_h u_h,\ y_h(0) = P_h y_0. \quad (2.3.24)$$

Then,

- *the final state is given by*
$$y_h(T) = -h^\beta \|\varphi_h\|^{p-2}\varphi_h;$$
- *the sequence $(\tilde{P}_h y_h)_{0<h<h_1}$ converges strongly (up to subsequence) in the space $L^q(0, T; X)$, to $y(.)$.*

Furthermore, there exists $M > 0$ such that

$$\int_0^T \|u(t)\|_U^p \le M^{p/(p-1)} \|y_0\|_X^{p/(p-1)},$$

and, for every $h \in (0, h_1)$,

$$\begin{aligned} &\int_0^T \|u_h(t)\|_{U_h}^p \le M^{p/(p-1)} \|y_0\|_X^{p/(p-1)}, \\ &h^\beta \|\varphi_h\|_{X_h}^p \le M^{p/(p-1)} \|y_0\|_X^{p/(p-1)}, \\ &\|y_h(T)\|_{X_h} \le M^{1/(p-1)} h^{\beta/p} \|y_0\|_X^{1/(p-1)}. \end{aligned} \quad (2.3.25)$$

Remark 2.3.6. *In the course of the proof we shall find that β can be chosen according to $0 \le \beta \le s(1-\gamma)\theta$ where θ satisfies $0 < \theta + (1-\theta)\gamma < \frac{1}{p}$. The numerical example of*

Section 5 seems to indicate that this range of values for β is far from optimal. Further investigation are needed on this aspect. This is important as the choice of β has a large impact on the convergence of the method when $h \to 0$ (see (2.3.25)).

Remark 2.3.7. *The left hand side of (2.3.20) is an uniform observability type inequality for control System (2.3.24) with the additional viscosity term $h^\beta \|\psi_h\|^p_{X_h}$. Here, this is a viscosity in h^β which is quite bad for numerics but completely general. Futher, in more particular cases (1D heat equation), we derive a better viscosity term, in $e^{-C/h}$ (see detail in Chapter 3).*

Remark 2.3.8. *Our objective is to investigate the results in [LT06] in the case $\gamma > 1/2$. Moreover, we mention here an upper bound for $\gamma \leq \frac{1}{p}$. This constraint is derived from the proof of the theorem hereinafter.*

Remark 2.3.9. *A similar result holds if the control system (2.1.1) is exactly controllable in time T. However, due to assumption (H_1), the semigroup $S(t)$ enjoys in general regularity properties. Therefore, the solution $y(.)$ of the control system may belong to a subspace of X, whatever the control u is. For instance, in the case of the heat equation with a Dirichlet or Neumann boundary control, the solution is a smooth function of the state variable x, as soon as $t > 0$, for every control and initial condition $y_0 \in L^2$. Hence, exact controllability does not hold in this case L^2. That is why we only focus on exact null-controllability.*

Theorem 2.3.5 stated that the controls u_h defined by (2.3.22) tend to a control u realizing exact null-controllability for System (2.3.23). A question arises naturally that under which assumptions the control u is the control, is defined by (2.2.22), such that $y(T) = 0$? A answer will be expressed through the following proposition. Apart from the condition on y_0 known in [LT06], we here give an extra condition on the control u defined by (2.2.22).

Proposition 2.3.10. *With the notations of Theorem 2.3.5, if the sequence of real numbers $\|\varphi_h\|_{X_h}$, $0 < h < h_1$, is moreover bounded, then the control u is the unique control of minimal L^q-norm, as defined by (2.2.22), such that $y(T)=0$. Moreover, the sequence $(\tilde{Q}_h u_h)_{0<h<h_1}$ converges strongly (up to a subsequence) in the space $L^q(0,T;U)$ to the control u.*

A sufficient condition on $y_0 \in X$, ensuring the boundedness of the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$, is the following : there exists $\eta > 0$ such that the control system $\dot{y} = Ay + Bu$ is null controllable in time t, for every $t \in [T-\eta, T+\eta]$, and the trajectory $t \mapsto S(t)y_0$ in X, for $t \in [T-\eta, T+\eta]$, is not contained in a hyperplane of X.

Another sufficient condition also ensuring the boundedness of the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$ concerns the control function and in the following : there exists $\eta > 0$ such that the control system $\dot{y} = Ay + Bu$ is null controllable in time t, for every $t \in [T-\eta, T+\eta]$, and with the control u as defined in (2.2.22), the trajectory $t \mapsto S(t-\xi)Bu(\xi)$ in X, for $t \in [T-\eta, T+\eta]$, every $\xi \in (0,t)$ is not contained in a hyperplane of X.

2.4 Proof of the main results

2.4.1 Proof of Theorem 2.3.5

Proof. For convenience, we first state the following useful approximation Lemma, whose proof readily follows that of [LT06], [LT00]. The proof of this Lemma is provided in the Appendix.

Lemma 2.4.1. *There exists $C_9 > 0$ such that, for all $t \in (0,T]$ and $h \in (0,h_0)$, there holds*

$$\left\|(e^{tA_h^*}P_h - P_hS(t)^*)\psi\right\|_{X_h} \leq C_9\frac{h^s}{t}\|\psi\|_X, \tag{2.4.1}$$

$$\left\|\tilde{Q}_hB_h^*e^{tA_h^*}\psi_h\right\|_U \leq \frac{C_9}{t^\gamma}\|\psi_h\|_{X_h}, \tag{2.4.2}$$

for every $\theta \in [0,1]$

$$\left\|\tilde{Q}_hB_h^*e^{tA_h^*}\psi_h - B^*S(t)^*\tilde{P}_h\psi_h\right\|_U \leq C_9\frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma}}\|\psi_h\|_{X_h} \tag{2.4.3}$$

every $\psi_h \in X_h$.

We carry out proving the theorem as follows

As the degree of unboundedness γ of the control operator B is lower than $\frac{1}{p}$ then there exists $\theta \in (0,1)$ such that $0 < \theta + (1-\theta)\gamma < \frac{1}{p}$.

For all $h \in (0, h_0)$ and $\psi_h \in X_h$ we have

$$\begin{aligned}\int_0^T \left\|\tilde{Q}_h B_h^* e^{tA_h^*}\psi_h\right\|_U^p dt &= \int_0^T \left(\left\|\tilde{Q}_h B_h^* e^{tA_h^*}\psi_h\right\|_U^p - \left\|B^* S(t)^* \tilde{P}_h \psi_h\right\|_U^p \right) dt \\ &\quad + \int_0^T \left\|B^* S(t)^* \tilde{P}_h \psi_h\right\|_U^p dt. \end{aligned} \tag{2.4.4}$$

We estimate the two terms of right hand side of (2.4.4).

The control system is null controllable in time T, then there exists a positive real number $C > 0$ such that

$$\int_0^T \left\|B^* S(t)^* \tilde{P}_h \psi_h\right\|^p dt \geq C \left\|S(T)^* \tilde{P}_h \psi_h\right\|_X^p . \tag{2.4.5}$$

We introduce the following useful inequality which is employed along the proof

$$|y^p - z^p| < p(y^{p-1} + z^{p-1}) \, |y - z| \, , \tag{2.4.6}$$

where $y, z \in R^+, p > 1$.

This inequality follows from applying the mean-value theorem for $f(x) = x^p (p > 1, x \in R^+)$. Indeed, there exists $\xi \in (y, z)$ such that

$$\begin{aligned} |y^p - z^p| &= \left|f'(\xi)\right| |y - z| \\ &= p \left|\xi^{(p-1)}\right| . \, |y - z| \\ &< p(y^{p-1} + z^{p-1}) . \, |y - z| \, . \end{aligned}$$

By making use of the above inequality as well as inequalities (2.3.13), (2.3.1), (2.3.17)

and (2.4.1) yield

$$\begin{aligned}
&\left|\left\|P_hS(T)^*\tilde{P}_h\psi_h\right\|_{X_h}^p - \left\|e^{TA_h^*}\psi_h\right\|_{X_h}^p\right| \\
\leq\ & p(\left\|P_hS(T)^*\tilde{P}_h\psi_h\right\|_{X_h}^{p-1} + \left\|e^{TA_h^*}\psi_h\right\|_{X_h}^{p-1}) \\
&\times \left|\left\|P_hS(T)^*\tilde{P}_h\psi_h\right\|_{X_h} - \left\|e^{TA_h^*}\psi_h\right\|_{X_h}\right| \\
\leq\ & p(C_5C_1e^{\omega t}\left\|\psi_h\right\|_{X_h}^{p-1} + C_7\left\|\psi_h\right\|_{X_h}^{p-1}).\left\|P_hS(T)^*\tilde{P}_h\psi_h - e^{TA_h^*}\psi_h\right\|_{X_h} \\
\leq\ & C_p\left\|\psi_h\right\|_{X_h}^{p-1}C_9C_5h^s\left\|\psi_h\right\|_{X_h} \\
\leq\ & C_{14}h^s\left\|\psi_h\right\|_{X_h}^p.
\end{aligned}$$

It follows from the above estimate and (2.3.13) that

$$\left\|e^{TA_h^*}\psi_h\right\|_{X_h}^p - C_{14}h^s\|\psi_h\|_{X_h}^p \leq \left\|P_hS(T)^*\tilde{P}_h\psi_h\right\|_{X_h}^p \leq C_5^p\left\|S(T)^*\tilde{P}_h\psi_h\right\|_X^p. \tag{2.4.7}$$

Combining (2.4.5) with (2.4.7), we obtain

$$\int_0^T \left\|B^*S(t)^*\tilde{P}_h\psi_h\right\|_U^p dt \geq C_{15}\left\|e^{TA_h^*}\psi_h\right\|_{X_h}^p - C_{14}h^s\|\psi_h\|_{X_h}^p. \tag{2.4.8}$$

For the first term on the right hand side of (2.4.4), one has, using (2.3.6), (2.4.2), (2.4.3) and applying inequality (2.4.6)

$$\begin{aligned}
&\left|\left\|\tilde{Q}_hB_h^*e^{tA_h^*}\psi_h\right\|_U^p - \left\|B^*S(t)^*\tilde{P}_h\psi_h\right\|_U^p\right| \\
\leq\ & p\Big(\left\|\tilde{Q}_hB_h^*e^{tA_h^*}\psi_h\right\|_U^{p-1} + \left\|B^*S(t)^*\tilde{P}_h\psi_h\right\|_U^{p-1}\Big) \\
&\times \left|\left\|\tilde{Q}_hB_h^*e^{tA_h^*}\psi_h\right\|_U - \left\|B^*S(t)^*\tilde{P}_h\psi_h\right\|_U\right| \\
\leq\ & p\Big(\frac{C_9^{p-1}}{t^{\gamma(p-1)}}\left\|\psi_h\right|_{X_h}^{p-1} + C_3^{p-1}\frac{e^{\omega t(p-1)}}{t^{\gamma(p-1)}}\left\|\psi_h\right\|_{X_h}^{p-1}\Big) \\
&\times \left|\left\|\tilde{Q}_hB_h^*e^{tA_h^*}\psi_h - B^*S(t)^*\tilde{P}_h\psi_h\right\|_U\right| \\
\leq\ & \frac{C_{16}}{t^{\gamma(p-1)}}\left\|\psi_h\right\|_{X_h}^{p-1}.C_9\frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma}}\left\|\psi_h\right\|_{X_h} \\
\leq\ & C_{17}\frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma+\gamma(p-1)}}\left\|\psi_h\right\|_{X_h}^p.
\end{aligned}$$

Since $\gamma < \frac{1}{p}$ $(p \geq 1)$ we have $\theta+(1-\theta)\gamma+\gamma(p-1) < 1$ and we can get, by integration,

$$\left| \int_0^T \left(\left\| \tilde{Q}_h B_h^* e^{tA_h^*} \tilde{P}_h \psi_h \right\|_U^p - \|B^* S(t)^* \psi_h\|_U^p \right) dt \right| \leq C_{18} h^{s(1-\gamma)\theta} \|\psi_h\|_{X_h}^p.$$

It leads to

$$\int_0^T \left\| \tilde{Q}_h B_h^* e^{tA_h^*} \psi_h \right\|_U^p dt \geq \int_0^T \|B^* S(t)^* \psi_h\|_U^p dt - C_{18} h^{s(1-\gamma)\theta} \|\psi_h\|_{X_h}^p. \tag{2.4.9}$$

We choose a real number β such that $0 \leq \beta \leq s(1-\gamma)\theta$. It is seen clearly that inequality (2.3.20) follows by combining results (2.4.8), (2.4.9) with (2.4.4).

For $h \in (0, h_1)$, the functional J_h is convex, and inequality (2.3.20), is coercive. It therefore admits a minimum at $\varphi_h \in X_h$ so that

$$0 = \bigtriangledown J_h(\varphi_h) = G_h(T)\varphi_h + h^\beta \|\varphi_h\|^{p-2} \varphi_h + e^{TA_h} P_h y_0,$$

where $G_h(T) = \int_0^T \left\| B_h^* e^{tA_h^*} \varphi_h \right\|^{p-2} e^{tA_h} B_h B_h^* e^{tA_h^*} dt$ is the Gramian of the semidiscrete system.

The discrete control $u_h(t) = \left\| B_h^* e^{(T-t)A_h^*} \varphi_h \right\|^{p-2} B_h^* e^{(T-t)A_h^*} \varphi_h$ is chosen then, the solution $y_h(.)$ satisfies

$$\begin{aligned} y_h(T) &= e^{TA_h} y_h(0) + \int_0^T e^{(T-t)A_h} B_h u_h(t) dt \\ &= e^{TA_h} P_h y_0 + G_h(T)\varphi_h \\ &= -h^\beta \|\varphi_h\|^{p-2} \varphi_h. \end{aligned}$$

Note that, since $J_h(0) = 0$, there must hold, at the minimum, $J_h(\varphi_h) \leq 0$. Hence, using observability inequality (2.3.20) and the Cauchy-Schwarz inequality, one gets

$$\begin{aligned} c \left\| e^{TA_h^*} \varphi_h \right\|_{X_h}^p &\leq \int_0^T \left\| B_h^* e^{tA_h^*} \varphi_h \right\|_{U_h}^p + h^\beta \|\varphi_h\|_{X_h}^p \\ &\leq 2 \left\| e^{TA_h^*} \varphi_h \right\|_{X_h} \|P_h y_0\|_{X_h}, \end{aligned}$$

and thus, we obtain

$$\left\|e^{TA_h^*}\varphi_h\right\|_{X_h} \leq \left(\frac{2}{c}\right)^{1/(p-1)} \|P_h y_0\|_{X_h}^{1/(p-1)}. \tag{2.4.10}$$

As a consequence, we have

$$\int_0^T \left\|B_h^* e^{tA_h^*}\varphi_h\right\|_{U_h}^p \leq (\frac{2^p}{c})^{1/(p-1)}(\|P_h y_0\|_{X_h}^{p/(p-1)}), \tag{2.4.11}$$

and $h^\beta \|\varphi_h\|_{X_h}^p \leq \left(\frac{2^p}{c}\right)^{1/(p-1)} \|P_h y_0\|_{X_h}^{p/(p-1)}$, and estimates (2.3.25) follow. □

2.4.2 Proof of Proposition 2.3.10

Proof. If the sequence $\left(\left\|\tilde{P}_h\varphi_h\right\|_X\right)_{0<h<h_1}$ is bounded then up to a subsequence, it converges weakly to an element $\varphi \in X$. It follows from estimate (2.4.3) that $u(t) = \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi$ for every $t \in [0,T]$. Moreover, $\tilde{Q}_h u_h$ tends strongly to u in $L^q(0,T;U)$. Indeed, for $t \in [0,T]$,

$$\begin{aligned}
& \tilde{Q}_h u_h(t) - u(t) \\
= \; & \tilde{Q}_h \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} B_h^* e^{(T-t)A_h^*}\varphi_h \\
& - \|B^*S(T-t)^*\varphi\|^{p-2} B^*S(T-t)^*\varphi \\
= \; & \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} (\tilde{Q}_h B_h^* e^{(T-t)A_h^*} - B^*S(T-t)^*\tilde{P}_h)\varphi_h \\
& + \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} B^*S(T-t)^*(\tilde{P}_h\varphi_h - \varphi) \\
& + B^*S(T-t)^*\varphi(\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} - \|B^*S(T-t)^*\varphi\|^{p-2}).
\end{aligned} \tag{2.4.12}$$

Since the φ_h are bounded then the $\|u_h\|$ are bounded. It follows that $\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2}$ are bounded as well.

By making use of estimate (2.4.3), the first term of right hand side of (2.4.12) tends to zero clearly. For the second term, for every $t \in [0,T]$ the operator $B^*S(T-t)^*$ is compact, as a strong limit of finite rank operators and since $\tilde{P}_h\varphi_h - \varphi$ tends to weakly to zero, it follows that the second term of the right hand side of (2.4.12) tends to zero.

Furthermore, through inequality (2.4.6) we get

$$\begin{aligned} & \left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-2} - \|B^* S(T-t)^*\varphi\|^{p-2} \\ < \; & (2-p)\Big(\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-3} + \|B^* S(T-t)^*\varphi\|^{p-3}\Big) \\ & \times\Big(\left\|B_h^* e^{(T-t)A_h^*}\varphi_h - B^* S(T-t)^*\varphi\right\|\Big). \end{aligned}$$

It is seen easily that the third term tends to zero because of the unboundedness of $\left\|B_h^* e^{(T-t)A_h^*}\varphi_h\right\|^{p-3}$ and estimate (2.4.3).

The control u is such that y(T)=0, hence the vector φ must be solution of $\nabla J^*(\psi) = 0$, where J is defined as in Theorem 2.2.3. Since J is convex, φ is the minimum of J^*, that is, u is the control such that $y(T) = 0$.

We next prove, by contradiction, that the sufficient conditions provided in the statement of the proposition implies that the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$ is bounded. Since the proof of the first sufficient condition is found in [LT06], we give here the proof only for the second sufficient condition. If the sequence $(\|\varphi_h\|_{X_h})_{0<h<h_1}$ is not bounded, then, up to subsequence, $\tilde{P}_h(\varphi_h/\|\varphi_h\|_{X_h})$ converges weakly to Φ in X, as h tends to 0. For every $t \in [T-\eta, T+\eta]$, the control system is null controllable in time t; and thus, from estimate (2.4.11), the sequence $\int_0^t < B_h^* e^{(t-s)A_h^*}\varphi_h, Q_h u(s) >_{U_h} ds$ is bounded, uniformly for $h \in (0, h_1)$. Thus, passing to the limit, one gets

$$\int_0^t < \Phi, S(t-s)Bu(s) >_X ds = 0.$$

This equality is equivalent to the fact that : there exists $\xi \in (0,t)$ such that $< \Phi, S(t-\xi)Bu(\xi) >_X = 0$. This contradicts the fact that the trajectory $t \mapsto S(t-\xi)Bu(\xi)$, $t \in [T-\eta, T+\eta]$ and every $\xi \in (0,t)$, is not contained in a hyperplane of X. □

2.5 Numerical simulation for the heat equation with Dirichlet boundary control

In this section, we give an example of a situation where the assumptions in Theorem 2.3.5 are satisfied, and we provide comments numerical simulations to illustrate our results.

Let $\Omega \subset R^n$ be an open bounded domain with sufficiently smooth boundary Γ. We consider the Dirichlet mixed problem for the heat equation:

$$\dot{y} = \Delta y + c^2 y \text{ in } (0,T) \times \Omega$$

$$y(0,.) = y_0 \text{ in } \Omega$$

$$y = u \text{ in } (0,T) \times \Gamma = \Sigma,$$

with boundary control $u \in L^6(0,T;L^2(\Gamma))$ and $y_0 \in L^2(\Omega)$.

Set $X = L^2(\Omega)$ and $U = L^2(\Gamma)$. It can be written in form (2.1.1): the self-adjoint operator

$$A : D(A) = H^2 \cap H_0^1 \to L^2(\Omega).$$

is defined by $Ay = \Delta y + c^2 y$ and the adjoint $B^* \in L(D(A^*), U)$ of the control operator B is given by

$$B^*\psi = -\frac{\partial \psi}{\partial \nu}, \quad \psi \in D(A^*).$$

Moreover, according to [LT00] (Section 3.1), the degree of unboundedness of B is $\gamma = \frac{3}{4} + \epsilon > \frac{1}{2}$ $(\epsilon > 0)$.

The conditions $u \in L^6(0,T;L^2(\Gamma))$ and $\gamma > 1/2$ exceed the hypothesis stated in [LT06]. The results of [LT06] can not be applied to this example. The assumptions of the results presented here are however satisfied as we shall see below. Constructing the control as in Theorem 2.3.5 we shall observe the convergence to zero of $y_h(T)$.

2.5.1 A one-dimensional finite-difference semi-discretized model

We introduce a semi-discretized model for the above heat equation through 1D finite-difference.

For simplicity, we set $\Omega = (0,1)$, $\Gamma = \{0,1\}$, c=1 and T=1.

Given $n \in \mathbb{N}$ we define $h = \frac{1}{n+1} > 0$. We consider the following mesh

$$\Omega_h = \{x_0 = 0;\ x_i = ih,\ i = 1, ..., n;\ x_{n+1} = 1\},$$

which divides [0,1] into n+2 subintervals $\omega_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ $j = 0, ..., n+1$ where $x_{-\frac{1}{2}} = x_0$, $x_{n+\frac{3}{2}} = x_{n+1}$ and $x_{j+\frac{1}{2}} = \dfrac{x_j + x_{j+1}}{2}$ for $j = 0, \ldots, n$. Let χ_{ω_j} denote the characteristic function of the interval ω_j. X_h is finite-dimensional subspace of X.

Let $\tilde{X}_h$ denote the set of functions on $I = (0, 1)$ whose restriction to each subinterval I_i, $i = 0, \ldots, n+1$, is polynominal with degree less than or equal to two.

For every positive integer m, every strictly ordered vector $\tilde{X} \in \mathbb{R}^m$ and every $\tilde{Y} \in Z^m$, where Z is a reflexive Banach space, let $\zeta_{\tilde{X}}(\tilde{Y})$ denote the Lagrange interpolation polynom of $\tilde{Y}$ at points $\tilde{X}$, that is

$$\zeta_{\tilde{X}}(\tilde{Y})(x) = \sum_{i=1}^{m} y_i \Big(\prod_{\substack{j=1 \\ j\neq i}} (x - x_j) / \prod_{\substack{j=1 \\ j\neq i}} (x_i - x_j) \Big)$$

for every $x \in \mathbb{R}$. Note that

$$\zeta_{\tilde{X}}(\tilde{Y})(x_i) = y_i, \qquad i = 1, \ldots, m$$

For every $X \in \mathbb{R}^{n+2}$, we use notation $X.\chi_h = \sum_{i=0}^{n+1} x_i \chi_{\omega_i}$

Define $p_h : \mathbb{R}^{n+2} \to \tilde{X}_h$ by

$$p_h(U) = \sum_{i=0}^{n+1} \zeta_{(x_j)_{j\in V_i}} \big(U_j\big)_{j\in V_i} \chi_{\omega_i},$$

for every $U \in \mathbb{R}^{n+2}$.

Set $X_h = p_h(\mathbb{R}^{n+2})$.

Define $\tilde{P}_h$ (resp., $\tilde{Q}_h$) as the canonical injection from X_h into $D((-A)^{1/2})$ (resp., from U_h to U). For $x_h \in X_h$ and $u_h \in U_h$, set, $\tilde{P}_h(x_h) = x_h$ and $\tilde{Q}_h(u_h) = u_h$. For $y \in D((-A)^{1/2})' = H^1(\Omega)'$, set $P_h y = y_h$ and, for $u \in U$, set, $Q_h u = u_h$.

We now consider the finite difference approximation of the above heat equation

$$\begin{aligned}
&\dot{y}_j = \frac{1}{h^2}[y_{j+1} + y_{j-1} - 2y_j] + y_j, \quad 0 < t < T, \quad j = 1, ..., n, \\
&y_j(0) = y_{j0} \ , j = 1, ..., n, \\
&y_0(t) = u_h^1(t), \quad y_{n+1}(t) = u_h^2(t) \ , 0 < t < T,
\end{aligned}$$

where $y \in \mathbb{R}^{n+2}$, $y_0 \in \mathbb{R}^{n+2}$; $u_h^1, u_h^2 \in \mathbb{R}$.

We can write the above discrete system in following form

$$\begin{aligned} \dot{y}_h &= A_h y_h + B_h u_h, \\ y_h(0,.) &= P_h y_0(.), \end{aligned} \tag{2.5.1}$$

where

$$\mathbf{A_h} = \frac{1}{h^2} \begin{pmatrix} d & 1 & 0 & \dots & 0 & 0 \\ 1 & d & 1 & \dots & 0 & 0 \\ 0 & 1 & d & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \vdots & 1 & d & 1 & 0 \\ 0 & \dots & 0 & 1 & d & 1 \\ 0 & 0 & \dots & 0 & 1 & d \end{pmatrix}_{n\times n},$$

with d=$h^2 - 2$ and

$$\mathbf{B_h} = \frac{1}{h^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \\ 0 & 1 \end{pmatrix}_{n\times 2}, \mathbf{u_h} = \begin{pmatrix} u_h^1 \\ u_h^2 \end{pmatrix},$$

and $y_0 h = (y_{0h}(j))_{1\le j\le n} = (y_0(jh))_{1\le j\le n}$.

Next we check that assumptions $(H3)$-$(H4)$ are fulfilled for the above system. Assumptions $(H_{3.1})$ and $(H_{3.3})$ are satisfied. To check assumptions $(H_{3.2})$ and $(H_{3.4})$, we recall the following usual approximation properties (see [LT00], Section 5) that are satisfied by the usual discretization scheme

(i) $\|\Pi_h y - y\|_{H^l(\Omega)} \le c h^{s-l} \|y\|_{H^s(\Omega)}$, $s \le r+1$, $\quad s-l \ge 0, \quad 0 \le l \le 1,$
and the inverse approximation properties

(ii) $\|y_h\|_{H^\alpha(\Omega)} \le c h^{-\alpha} \|y_h\|_L^2 (\Omega)$, $0 \le \alpha \le 1$.

(iii) $h^{-1} \|y - \Pi_h y\|_{L^2_{(\Gamma)}} + \left\|(I - \Pi_h)\frac{\partial y}{\partial \nu}\right\|_{L^2(\Gamma)} \le c h^{s-\frac{3}{2}} \|y\|_{H^s(\Omega)}$, $\frac{3}{2} < s < r+1$, $y \in H^s(\Omega)$.

(iv) $\|y_h\|_{L^2(\Gamma)} + h \left\|\frac{\partial y_h}{\partial \nu}\right\|_{L^2(\Gamma)} \le C h^{-\frac{1}{2}} \|y_h\|_{L^2(\Omega)}$, $y_h \in V_h$.
where r is the order of approximation (degree of polynomials) and Π_h is the orthogonal projection of $L^2(\Omega)$ onto V_h .

By means of property (i), we easily get inequality (2.3.8)

$$\begin{aligned}\left\|(I-\hat{A}^{*-\gamma+\frac{1}{2}}\tilde{P}_hP_h)\psi\right\|_{L^2(\Omega)} &\leq Ch^2\left\|\psi\right\|_{H^2(\Omega)}\\ &\leq Ch^2\left\|\psi\right\|_{D(A^*)}\\ &\leq Ch^2\left\|A^*\psi\right\|_X,\end{aligned}$$

in the case s=2.

We next verify inequality (2.3.9) as follows

$$\begin{aligned}&\left\|((-\hat{A}^*)^\gamma)\left(I-(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_hP_h\right)\psi\right\|_X\\ \leq\ & C\left\|(I-(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_hP_h)\psi\right\|_{D((-\hat{A}^*)^\gamma)}\\ \leq\ & Ch^{s-l}\left\|\psi\right\|_{D(A^*)}\\ \leq\ & Ch^{s(1-\gamma)}\left\|A^*\psi\right\|_X,\end{aligned}$$

where we have employed property (i) with s=2 and where $D(A^*)=H^s(\Omega)$, $D((-\hat{A}^*)^\gamma)=H^l(\Omega)$.

For inequality (2.3.11), we employ property (iii) with s=2 as

$$\begin{aligned}\left\|(I-\tilde{Q}_hQ_h)B^*\psi\right\|_{L^2(\Gamma)} &= \left\|(I-\tilde{Q}_hQ_h)\frac{\partial\psi}{\partial\nu}\right\|_{L^2(\Gamma)}\\ &\leq Ch^{1/2}\left\|\psi\right\|_{H^2(\Omega)}\\ &\leq Ch^{s(1-\gamma)}\left\|\psi\right\|_{D(A^*)}\\ &\leq Ch^{s(1-\gamma)}\left\|A^*\psi\right\|_X.\end{aligned}$$

For inequality (2.3.15), by making use of property (iv) and inequality (2.3.12) we have

$$\begin{aligned}\left\|B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h\psi_h\right\|_U &= \left\|\frac{\partial((\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h\psi_h)}{\partial\nu}\right\|_{L^2(\Gamma)}\\ &\leq Ch^{-\frac{3}{2}}\left\|(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h\psi_h\right\|_{L^2(\Omega)}\\ &\leq Ch^{-\frac{3}{2}}\left\|\psi_h\right\|_{X_h}.\end{aligned}$$

Inequality (2.3.15) is thus satisfied for s=2, $\gamma=\frac{3}{4}+\epsilon$. Moreover, assumption $(H_{4.2})$

is satisfied with s=2 (see [LT00]).

The assumptions of Theorem 2.3.5 are thus adapted to this example, with $\beta = 0.16$, for instance. Note that the choice of β comes from the proof of Theorem 2.3.5.

For greater convenience in numerical computations, we shall exploit spectral decompositions in what follows.

The solution of discrete system (2.5.1) at time T reads

$$y_h(T) = e^{TA_h} P_h y_0 + \int_0^T e^{(T-t)A_h} B_h u_h(t) dt. \tag{2.5.2}$$

As shown in Theorem 2.3.5, a control u_h is built as

$$u_h(t) = \left\| B_h^* e^{(T-t)A_h^*} \varphi_h \right\|^{p-2} B_h^* e^{(T-t)A_h^*} \varphi_h,$$

where φ_h is the minimizer of the functional

$$J_h(\psi_h) = \frac{1}{p} \int_0^T \left\| B_h^* e^{tA_h^*} \psi_h \right\|_U^p dt + \frac{1}{p} h^\beta \|\psi_h\|_{X_h}^p + < e^{TA_h^*} \psi_h, P_h y_0 >_{X_h} .$$

Replacing the above discrete control into (2.5.2) we obtain

$$y_h(T) = e^{TA_h} P_h y_0 + \int_0^T \left\| B_h^* e^{tA_h^*} \varphi_h \right\|^{p-2} e^{tA_h} B_h B_h^* e^{tA_h^*} \varphi_h dt.$$

The eigenvectors and eigenvalues of the operator A_h are respectively given by

$$w_k(j) = sin(j\pi hk), \qquad \lambda_k = \frac{1}{h^2}\big(2cos(\pi hk) + (h^2 - 2)\big) \neq 0$$

for $1 \leq j, k \leq n$.

We normalize the eigenvectors $(w_k)_{1 \leq k \leq n}$ and then consider them as an orthonormal

basis of n-dimensional space X_h. Exploiting the spectral decomposition we write

$$\begin{aligned} y_h(T) &= \sum_{j=0}^{n} e^{T\lambda_j} < P_h y_0, w_j > w_j \\ &+ \int_0^T \Big\| \sum_{j=0}^{n} B^* e^{t\lambda_j} < \varphi_h, w_j > w_j \Big\|^{p-2} \sum_{j=0}^{n} e^{tA_h} B_h B_h^* e^{t\lambda_j} < \varphi_h, w_j > w_j dt \\ &= \sum_{j=0}^{n} e^{T\lambda_j} < P_h y_0, w_j > w_j \\ &+ \frac{1}{h^{2p}} \int_0^T \Big((\sum_{j=0}^{n} e^{t\lambda_j} < \varphi_h, w_j > w_j)^2 + (\sum_{j=0}^{n} e^{t\lambda_j} < \varphi_h, w_j > w_j(n))^2 \Big)^{\frac{p-2}{2}} \\ &\times \Big(\sum_{j=0}^{n} \sum_{k=0}^{n} e^{t(\lambda_k+\lambda_j)} (w_j(1)w_k(1) + w_j(n)w_k(n)) < \varphi_h, w_j > w_k \Big) dt, \end{aligned} \tag{2.5.3}$$

where φ_h is minimizer of

$$\begin{aligned} J_h(\psi_h) &= \frac{1}{ph^{2p}} \int_0^T \Big((\sum_{j=0}^{n} e^{t\lambda_j} < \psi_h, w_j > w_j(1))^2 + (\sum_{j=0}^{n} e^{t\lambda_j} < \psi_h, w_j > w_j(n))^2 \Big)^{\frac{p}{2}} dt \\ &+ \frac{1}{p} h^\beta \|\psi_h\|_{X_h}^p + \sum_{j=0}^{n} e^{t\lambda_j} < \psi_h, w_j >< P_h y_0, w_j > . \end{aligned} \tag{2.5.4}$$

Equivalently we have

$$\begin{aligned} 0 &= \bigtriangledown J_h(\psi_h) \\ &= \int_0^T \Big\| B_h^* e^{tA_h^*} \psi_h \Big\|^{p-2} e^{tA_h} B_h B_h^* e^{tA_h^*} \psi_h dt + h^\beta \|\psi_h\|^{p-2} \psi_h + e^{TA_h} P_h y_0, \\ &= \frac{1}{h^{2p}} \int_0^T \Big((\sum_{j=0}^{n} e^{t\lambda_j} < \varphi_h, w_j > w_j(1))^2 + (\sum_{j=0}^{n} e^{t\lambda_j} < \varphi_h, w_j > w_j(n))^2 \Big)^{\frac{p-2}{2}} \\ &\times \Big(\sum_{j=0}^{n} \sum_{k=0}^{n} e^{t(\lambda_k+\lambda_j)} (w_j(1)w_k(1) + w_j(n)w_k(n)) < \varphi_h, w_j > w_k \Big) dt \\ &+ h^\beta \|\psi_h\|^{p-2} \psi_h + \sum_{j=0}^{n} e^{T\lambda_j} < P_h y_0, w_j > w_j. \end{aligned} \tag{2.5.5}$$

Observe that thanks to the spectral decomposition the various expressions we just wrote are quite straightforward to evaluate. Namely, expressions (2.5.3), (2.5.4) and (2.5.5) are useful for the numerical computations below.

Note the using this spectral decomposition is classical (see eg. [MZ10] where it is also exploited).

2.5.2 Numerical simulation

The minimization procedure stated in Theorem 2.3.5 is implemented for d=1, using a simple gradient method that has the advantage to not require complex computations. Moreover, this method can be applied with any power p. We now describe the simple gradient method which we employ as follows

1. Set iteration counter $k = 0$, and make an initial guess, x_0 for the minimum.
2. Repeat:
3. Compute a 'descent direction' d_k.
4. Choose 'step length' α_k to 'loosely' minimize $h(\alpha) = f(x_k + \alpha d_k)$ over $\alpha \in \mathbb{R}_+$.
5. Update $x_{k+1} = x_k + \alpha_k d_k$ and $k = k + 1$.
6. Stop criterion: $\|\bigtriangledown f(x_k)\| < 10^{-2}$.

At step (4), in order to obtain an acceptable step length α_k we choose a line search algorithm where the Wolfe condition is satisfied, that is $f(x_k + \alpha d_k) \leq f(x_k) + c\alpha \bigtriangledown {f_k}^T d_k$ (note that $c = 10^{-4}$ is chosen here). This condition ensures that the step length α_k yields a sufficient decrease for f. However it is not sufficient on its own to ensure that a reasonable value is generated, since all α small enough will satisfy the Wolfe condition. To avoid the selection of steps that are too small, the additional curvature condition is usually imposed. For simplicity, we impose in practice $\alpha \geq 10^{-4}$.

At step (3), there are many methods to compute a 'descent direction' d_k such as

1. The Fletcher-Reeves method where

$$d_{k+1} = - \bigtriangledown f(x_{k+1}) + \frac{\|\bigtriangledown f(x_{k+1})\|^2}{\|\bigtriangledown f(x_k)\|^2} d_k$$

2. The BroydenFletcherGoldfarbShanno method (in short: BFGS method) where a direction d_{k+1} is obtained by solving $B_{k+1} d_{k+1} = - \bigtriangledown f(x_{k+1})$ where

$$B_{k+1} = B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k},$$

with

$$s_k = x_{k+1} - x_k,$$
$$y_k = \bigtriangledown f(x_{k+1}) - \bigtriangledown f(x_k).$$

3. The Polak-Ribière method where

$$d_{k+1} = - \bigtriangledown f(x_{k+1}) + \frac{\bigtriangledown f(x_{k+1})\big(\bigtriangledown f(x_{k+1}) - \bigtriangledown f(x_k)\big)}{\|\bigtriangledown f(x_k)\|^2} d_k.$$

All above methods lead to reasonable results. However, the best choice for our example turns out to be the Polak-Ribière method (in the sense that acceptable numerical results are obtained in the shortest time).

Below we present the numerical results where the line search method at step (4) and Polak-Ribiere method at step (3) have been used.

name	n	h	y_0
1D-10	10	0.1	$y_0(x) = e^{-x^2}$
1D-50	50	0.02	$y_0(x) = e^{-x^2}$
1D-100	100	0.01	$y_0(x) = e^{-x^2}$
1D-200	200	0.005	$y_0(x) = e^{-x^2}$

Table 1: Data for the one-dimensional heat equation.

name	$\|\varphi_h\|_X$	$h^{\frac{\beta}{p}} (\beta = 0.16)$	$\|\bigtriangledown J_h\|$	$\|y_h(T)\|$
1D-10	7.3100e-07	0.7359	0.0089	0.0410
1D-50	8.9224e-07	0.5936	0.0098	0.0324
1D-100	9.1999e-07	0.5412	0.0084	0.0297
1D-200	9.2306e-07	0.4936	0.0076	0.0266

Table 2: Numerical results for one dimensional equation for $\beta = 0.16$.

Numerical simulations are carried out with a space-discretization step equal to 0.005, with the data of Table 1. The numerical results are provided in Table 2 for $\beta = 0.16$ and Table 3 for $\beta = 2$. Note that the fifth column corresponds to the numerical results as computed by (2.5.3).

The convergence of the method can be slow. According to the result of Theorem 2.3.5, the final state $y_h(T)$ is equal to $-h^\beta \|\varphi_h\|^{p-2} \varphi_h$ in which φ_h is the minimizer of J_h. We note that the maximum value for β which the theorem asserts the convergence is very small. For such a small value of β (here, $\beta = 0.16$), it follows from the estimate (2.3.25) that $\|y_h(T)\|$ converges to 0 with order '$\beta/p = 0.1333$' which is quite small. Namely, in order to divide $\|y_h(T)\|$ by 10 then one has to divide h by 32.10^{+6}. To achieve sufficient precision one need to take h small which lead to time consuming computation, which is reasonable for 1D problem.

Moreover, we provide the numerical results for $\beta = 2$ in Table 3 for reference. Although the case $\beta = 2$ is not covered by our theory, the method seems to converge as well for this value of beta. Naturally the greater the value of β is, the quicker the convergence is. This observation

motivates further developments to improve the results we have presented here.

name	$\|\varphi_h\|_X$	$h^{\beta/p}(\beta=2)$	$\|\bigtriangledown J_h\|$	$\|y_h(T)\|$
1D-10	1.6037e-06	0.0215	0.0081	5.7311e-04
1D-50	3.3748e-06	0.0015	0.0093	3.0939e-05
1D-100	3.9861e-06	$4.6415e-04$	0.0064	6.2477e-06
1D-200	4.1893e-06	$1.4620e-04$	0.0078	1.8722e-06

Table 3: Numerical results for one dimensional equation for $\beta = 2$.

2.6 Proof of Lemma 2.4.1

In comparison with Lemma 4.1 in [LT06], a different point in proof of this Lemma is the appearance of $(\hat{A}^*)^{-\gamma+\frac{1}{2}}$.

Proof. • First of all, we will prove (2.4.2)

For every $\psi \in D(A^*)$, one has

$$\left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h\psi - B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h P_h S(t)^*\psi\right\|_U \leq \left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h\psi\right\|_U + \left\|B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h P_h S(t)^*\psi\right\|_U. \quad (2.6.1)$$

We estimate each term of the right hand side of (2.6.1). Since $B_h^* = Q_h B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h$ and thus, using (2.3.13) (2.3.15) (2.3.17) one gets

$$\begin{aligned}\left\|\tilde{Q}_h B_h^* e^{tA_h^*} P_h\psi\right\|_U &\leq C_5\left\|B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h e^{tA_h^*} P_h\psi\right\|_U \\ &\leq C_5 C_6 h^{-\gamma s}\left\|e^{tA_h^*} P_h\psi\right\|_{X_h} \\ &\leq C_5^2 C_6 C_7 h^{-\gamma s} e^{\omega t}\|\psi\|_X. \end{aligned} \quad (2.6.2)$$

On the other hand, it follows from (2.3.1), (2.3.13), (2.3.15) that

$$\begin{aligned}\left\|B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_h P_h S(t)^*\psi\right\|_U &\leq C_6 h^{-\gamma s}\|P_h S(t)^*\psi\|_{X_h} \\ &\leq C_5 C_6 h^{-\gamma s}\|S(t)^*\psi\|_X \\ &\leq C_1 C_5 C_6 h^{-\gamma s} e^{\omega t}\|\psi\|_X. \end{aligned} \quad (2.6.3)$$

Hence, combining (2.6.2), (2.6.3) with (2.6.1), there exists $C_{10} > 0$ such that

$$\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \leq C_{10} h^{-\gamma s} \|\psi\|_X. \tag{2.6.4}$$

for every $\psi \in D(A^*)$, every $t \in [0,T]$, and every $h \in (0, h_0)$.

Moreover, we get another estimate of this term. By using (2.3.6), (2.3.11), (2.3.13), (2.3.15), (2.4.1) one gets

$$\begin{aligned}
& \left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \\
= \; & \left\| \tilde{Q}_h Q_h B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \\
\leq \; & \left\| \tilde{Q}_h Q_h B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h (e^{tA_h^*} P_h \psi - P_h S(t)^* \psi) \right\|_U \\
& + \left\| \tilde{Q}_h Q_h B^* ((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h - I) S(t)^* \psi \right\|_U \\
& + \left\| (\tilde{Q}_h Q_h - I) B^* S(t)^* \psi \right\|_U \\
& + \left\| B^* (I - (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h) S(t)^* \psi \right\|_U \\
\leq \; & C_5 C_6 h^{\gamma s} \left\| e^{tA_h^*} P_h \psi - P_h S(t)^* \psi \right\|_{X_h} \\
& + C_5 C_3 \left\| (-\hat{A})^{\gamma} ((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h - I) S(t)^* \psi \right\|_X \\
& + C_4 h^{s(1-\gamma)} \|A^* S(t)^* \psi\|_X \\
& + C_3 \left\| (-\hat{A})^{\gamma} ((\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h - I) S(t)^* \psi \right\|_X \\
\leq \; & C_5 C_6 C_9 \frac{h^{s(1-\gamma)}}{t} \|\psi\|_X + (C_3(C_5+1)+1) C_4 h^{s(1-\gamma)} \|A^* S(t)^* \psi\|_X \\
\leq \; & C_{11} \frac{h^{s(1-\gamma)}}{t} \|\psi\|_X.
\end{aligned} \tag{2.6.5}$$

Then, raising (2.6.4) to the power $1-\gamma$, (2.6.5) to the power γ and then multiplying both result estimates, we obtain

$$\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi - B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U \leq \frac{C_{12}}{t^\gamma} \|\psi\|_X.$$

Hence,

$$\left\| \tilde{Q}_h B_h^* e^{tA_h^*} P_h \psi \right\|_U \leq \frac{C_{12}}{t^\gamma} \|\psi\|_X + \left\| B^* (\hat{A}^*)^{-\gamma+\frac{1}{2}} \tilde{P}_h P_h S(t)^* \psi \right\|_U. \tag{2.6.6}$$

It follows from (2.3.2), (2.3.5), (2.3.9) that

$$\begin{aligned}\left\|B^*(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_hP_hS(t)^*\psi\right\|_U &\leq \left\|B^*(I-(\hat{A}^*)^{-\gamma+\frac{1}{2}}\tilde{P}_hP_h)S(t)^*\psi\right\|_U + \|B^*S(t)^*\psi\|_U \\ &\leq C_{13}\frac{e^{\omega t}}{t^\gamma}\|\psi\|_X. \end{aligned} \tag{2.6.7}$$

Combining (2.6.6) with (2.6.7) and by setting $\psi = \tilde{P}_h\psi_h$ we get (2.4.2).

- Finally, we prove (2.4.3). On the one hand, reasoning as above for obtaining (2.6.5), we get

$$\left\|\tilde{Q}_hB_h^*e^{tA_h^*}P_h\psi - B^*S(t)^*\psi\right\|_U \leq C\frac{h^{s(1-\gamma)}}{t}\|\psi\|_X, \tag{2.6.8}$$

for every $\psi \in D(A^*)$, every $t \in [0,T]$ and every $h \in (0,h_0)$.

On the other hand, from (2.4.2) and setting $\psi = \tilde{P}_h\psi_h$ one obtains

$$\begin{aligned}\left\|\tilde{Q}_hB_h^*e^{tA_h^*}P_h\psi - B^*S(t)^*\psi\right\|_U &\leq \left\|\tilde{Q}_hB_h^*e^{tA_h^*}P_h\psi\right\|_U + \|B^*S(t)^*\psi\|_U \\ &\leq \frac{C_9}{t^\gamma}\|\psi_h\| + C_3\left\|(-\hat{A}^*)^\gamma S(t)^*\psi\right\|_X \\ &\leq \frac{C}{t^\gamma}\|\psi_h\|_X. \end{aligned} \tag{2.6.9}$$

Raising (2.6.8) to the power θ, (2.6.9) to the power $1-\theta$ and then multiplying both resulting estimates, we obtain (2.4.3).

The proof of the inequality (2.4.1) is found in [LT06], [LT00].

□

2.7 Conclusion

We have shown that the appropriate duality techniques can be applied to solve (2.1.3), namely the Fenchel-Rockafellar theorem.

Additionally, it is also stated that under appropriate standard assumptions on the discretization process, for null controllable linear control system, if the semigroup of approximating system is uniformly analytic, and if the degree of unboundedness of the control operator is greater than $\frac{1}{2}$ then a unform observability type inequality is proved. Our result overcomes the limitation of [LT06] on the 1/2 of degree unboundedness of the control operator. As a consequence, a minimization procedure was provided to build the aproximation controls in L^q norm ($q > 2$). This is implemented in the case of the one dimensional heat equation with Dirichlet boundary control.

Note that, we only stress our problem on the case $\gamma \geq 1/2$. Some relevant problems for which $\gamma < 1/2$ that can be found in [LT06] and [EV09]. More precisely, while authors of [LT06] achieved the uniform observability result on space semidiscretization, authors of [EV09] provided the result for suitable time-discretization schemes which can then be combined with the existing ones on the observability of space semi-discrete systems, yielding observability properties for full discretization schemes.

In this work, we dealt within a very general framework, and derived a general uniform observability inequality with a viscosity in h^β. Next chapter, we investigate more particular equations (1D heat equation), for which Carleman estimate allows one to untimately derive uniform observability inequality with a better viscosity in $e^{-C/h}$.

Chapter 3

Carleman estimates for semi-discrete parabolic operators with a discontinuous diffusion coefficient and application to controllability

3.1 Introduction and settings

Let Ω, ω be connected non-empty open interval of $\mathbb{R}$ with $\omega \Subset \Omega$. We consider the following parabolic problem in $(0,T)\times\Omega$, with $T>0$,

$$\partial_t y - \partial_x(c\partial_x y) = \mathbf{1}_\omega v \ \text{ in } (0,T)\times\Omega,\ y|_{\partial\Omega}=0,\ \text{and } y|_{t=0}=y_0, \tag{3.1.1}$$

where the diffusion coefficient $c=c(x)>0$.

System (3.1.1) is said to be null controllable from $y_0 \in L^2(\Omega)$ in time T if there exists $v\in L^2((0,T)\times\Omega)$, such that $y(T)=0$.

In the continuous framework, we refer to [FI96] and [LR95] who proved such a controllability result by means of a global/local Carleman observability estimates in the case the diffusion coefficient c is smooth. The authors of [BDL07] produced this controllability result in the case of a discontinuous coefficient in the one-dimensional case later extended to arbitrary dimension by [LR10]. Additionally, a result of controllability in the case of a coefficient with bounded variation (BV) was shown in [FCZ02, L07].

The authors of [LZ98] show that uniform controllability holds in the one-dimensional case

with constant diffusion coefficient c and for a constant step size finite-difference scheme. Here, "uniform" is meant with respect to the discretization parameter h. The situation becomes more complex in higher dimension. In fact, a counter-example to null-controllability due to O. Kavian is provided in [Zua06] for a finite-difference discretization scheme for the heat equation in a square.

In recent works, by means of discrete Carleman estimate, the authors of [BHL10a], [BHL10b] and [BL12] obtained weak observability inequalities in the case of a smooth diffusion coefficient $c(x)$. Such observability estimates are charaterized by an additional term that vanishes exponentially fast. Morever, also with a constant diffusion coffiencient c, under the assumption that the discretized semigroup is uniformly analytic and that the degree of unboundedness of control operator is lower than $1/2$, a uniform observability property of semi-discrete approximations for System (3.1.1) is achieved in L^2 [LT06]. Besides that, such a result continues to hold even with the condition that the degree of unboundedness of control operator is greater than $1/2$ [N12].

In the case of a non-smooth coefficient, our aim is to investigate the uniform controllability of System (3.1.1) after discretization. It is well known that controllability and observability are dual aspects of the same problem. We shall therefore focus on uniform observability which is shown to hold when the observability constant of the finite dimensional approximation systems does not depend on the step-size h.

In the present paper we prove a Carleman estimate for aystem (3.1.1) in the case of:

- the heat equation in one space dimension;
- a piecewise $\mathbf{C}^1$ coefficient c with jumps at a finite number of points in Ω;
- a finite-difference discretization in space.

The main idea of the proof is combination of the derivation of a discrete Carleman estimate as in [BHL10a, BL12] and tecniques of [BDL07] for operators with discontinuous coefficients in the one-dimensional case. A similar question in n-dimensional case, $n \geq 2$, remains open, to our knowledge.

When considering a discontinuous coefficient c the parabolic problem (3.1.1) can be understood as a transmission problem. For instance, assume that c exhibits a jump at $a \in \Omega$. Then we write

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) = \mathbf{1}_\omega v & \text{in } (0,T) \times \big((0,a) \cup (a,1)\big), \\ c\partial_x y|_{a^+} = c\partial_x y|_{a^-}, \qquad y|_{a^+} = y|_{a^-}, \\ y|_{\partial\Omega} = 0, \qquad \text{and} \qquad y|_{t=0} = y_0. \end{cases}$$

The second line is thus a transmission condition implying the continuity of the solution and of the flux at $x = a$.

When one gives a its finite-difference version of this transmission problem, a similar condition can be given for the continuity of the solution. Yet, for the flux, it is only achieved up to a consistent term. In what follows, in the finite-difference approximation, we shall in fact write

$$\begin{cases} y(a^-) = y(a^+) = y_{n+1}, \\ (c_d Dy)_{n+\frac{3}{2}} - (c_d Dy)_{n+\frac{1}{2}} = h\big(\bar{D}(c_d Dy)\big)_{n+1}, \end{cases}$$

(the discrete notation will be given below). Note that the flux condition converges to the continuous one if $h \to 0$, h being the discretization parameter. This difference between the continuous and the discrete case will be the source of several technical points.

An important point in the proof of Carleman estimate is the construction of a suitable weight function ψ whose gradient does not vanish in the complement of the observation region. The weight function is chosen to be smooth in the case of a smooth diffusion coefficient $c(x)$. In general, the technique to construct such a function is based on Morse functions (see some details in [FI96]). In one space dimension, this construction is in fact straightforward. In the case of a discontinuous diffusion coefficient, authors of [BDL07] introduced an *ad hoc* transmission condition on the weight function: its derivative exhibits jumps of its derivative at the singular points of the coefficient. In this paper, we construct a weight function based on these techniques in one-dimentional discrete case.

From the semi-discrete Carleman we obtain, we give an observability inequality for semi-discrete parabolic problems with potential. As compared to the result in continuous case [BDL07] the observability estimate we state here is weak because of an additional term that describes the obstruction to the null-controllability. This term is exponentially small in agreement with the results obtained in [BHL10a, BHL10b] in the smooth coefficient case. A precise statement is given in Section 3.6.

Finally, the observability inequality allows one to obtain controllability results for semi-discrete parabolic with semi-linear terms. In continuous case, this was achieved in [BDL07]. Taking advantage of one-dimensional situation, the results we state are uniform with respect to the discretization parameter h (see Section 3.6).

3.1.1 Discrete settings

We restrict our analysis to one dimension in space. Let us consider the operator formally defined by $\mathcal{A} = -\partial_x(c\partial_x)$ on the open interval $\Omega = (0, L) \subset \mathbb{R}$. We let $a' \in \Omega$ and set $\Omega_1 := (0, a')$

and $\Omega_2 := (a', L)$. The diffussion coefficient c is assumed to be piecewise regular such that

$$0 < c_{min} \leq c \leq c_{max} \tag{3.1.2}$$

$$c = \begin{cases} c_0 \text{ in } \Omega_1, \\ c_1 \text{ in } \Omega_2, \end{cases}$$

with $c_i \in C^1(\overline{\Omega_i}), i = 1, 2$.

The domain of $\mathcal{A}$ is $D(\mathcal{A}) = \left\{u \in H_0^1(\Omega); \quad c\partial_x u \in H^1(\Omega)\right\}$.

Let $T > 0$. We shall use the following notation $\Omega' = \Omega_1 \cup \Omega_2$, $Q = (0, T) \times \Omega$, $Q' = (0, T) \times \Omega'$, $Q_i = (0, T) \times \Omega_i$, $i = 1, 2$, $\Gamma = \{0, L\}$, and $\Sigma = (0, T) \times \Gamma$. We also set $S = \{a'\}$. We consider the following parabolic problem

$$\begin{cases} \partial_t y + \mathcal{A}y = f \text{ in } Q', \\ y(0, x) = y_0(x) \text{ in } \Omega \quad . \end{cases}$$

(real valued coefficient and solution), for $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$, with the following transmission conditions at a'

$$(TC) \begin{cases} y(a'^-) = y(a'^+), \\ c(a'^-)\partial_x y(a'^-) = c(a'^+)\partial_x y(a'^+). \end{cases}$$

Now, we introduce finite-difference approximations of the operator $\mathcal{A}$. Let $0 = x'_0 < x'_1 < \ldots < \underline{x'_{n+1} = a'} < \ldots < x'_{n+m+1} < x'_{n+m+2} = L$. We refer to this discretization as to the primal mesh $\mathfrak{M} := (x'_i)_{1 \leq i \leq n+m+1}$. We set $|\mathfrak{M}| := n + m + 1$. We set $h'_{i+\frac{1}{2}} = x'_{i+1} - x'_i$ and $x'_{i+\frac{1}{2}} = (x'_{i+1} + x'_i)/2$, $i = 0, \ldots, n+m+1$, and $h' = \max_{0 \leq i \leq n+m+1} h'_{i+\frac{1}{2}}$. We call $\overline{\mathfrak{M}} := (x'_{i+\frac{1}{2}})_{0 \leq i \leq n+m+1}$ the dual mesh and set $h'_i = x'_{i+\frac{1}{2}} - x'_{i-\frac{1}{2}} = (h'_{i+\frac{1}{2}} + h'_{i-\frac{1}{2}})/2$, $i = 0, \ldots, n+m+1$.

In this paper, we shall address to some families of non uniform meshes, that will be precisely defined in Section 3.1.2.

We introduce the following notation

$$[\rho_1 \star]_a = \rho_1(a^+) - \rho_1(a^-), \tag{3.1.3}$$

$$[\star \rho_2]_a = \rho_2(n + \frac{3}{2}) - \rho_2(n + \frac{1}{2}), \tag{3.1.4}$$

$$[\rho_1 \star \rho_2]_a = \rho_1(a^+)\rho_2(n + \frac{3}{2}) - \rho_1(a^-)\rho_2(n + \frac{1}{2}). \tag{3.1.5}$$

We follow some notation of [BHL10a] for discrete functions in the one-dimensional case. We denote by $\mathbb{C}^{\mathfrak{M}}$ and $\mathbb{C}^{\overline{\mathfrak{M}}}$ the sets of discrete functions defined on $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ respectively. If $u \in \mathbb{C}^{\mathfrak{M}}$

(resp. $\mathbb{C}^{\overline{\mathfrak{M}}}$), we denote by u_i (resp. $u_{i+\frac{1}{2}}$) its value corresponding to x'_i (resp. $x'_{i+\frac{1}{2}}$). For $u \in \mathbb{C}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{i=1}^{n+m+1} 1|_{[x'_{i-\frac{1}{2}}, x'_{i+\frac{1}{2}}]} u_i \in L^\infty(\Omega).$$

And for $u \in \mathbb{C}^{\mathfrak{M}}$ we define $\int_\Omega u := \int_\Omega u^{\mathfrak{M}}(x)dx = \sum_{i=1}^{n+m+1} h'_i u_i$.

For $u \in \mathbb{C}^{\overline{\mathfrak{M}}}$ we define

$$u^{\overline{\mathfrak{M}}} = \sum_{i=0}^{n+m+1} 1|_{[x'_i, x'_{i+1}]} u_{i+\frac{1}{2}}.$$

As above, for $u \in \mathbb{C}^{\overline{\mathfrak{M}}}$, we define $\int_\Omega u := \int_\Omega u^{\overline{\mathfrak{M}}}(x)dx = \sum_{i=0}^{n+m+1} h'_{i+\frac{1}{2}} u_{i+\frac{1}{2}}$. In particular we define the following L^2 inner product on $\mathbb{C}^{\mathfrak{M}}$ (resp. $\mathbb{C}^{\overline{\mathfrak{M}}}$)

$$(u,v)_{L^2} = \int_\Omega u^{\mathfrak{M}}(x) v^{\mathfrak{M}}(x)dx, \text{ resp. } (u,v)_{L^2} = \int_\Omega u^{\overline{\mathfrak{M}}}(x) v^{\overline{\mathfrak{M}}}(x)dx.$$

For some $u \in \mathbb{C}^{\mathfrak{M}}$, we shall need to associate boundary conditions $u^{\partial\mathfrak{M}} = \{u_0, u_{n+m+2}\}$. The set of such extended discrete functions is denoted by $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_0 = u_{n+m+2} = 0$, in short $u^{\partial\mathfrak{M}} = 0$. We can define translation operators $\tau^\pm$, a difference operator D and an averaging operator as the map $\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}} \to \mathbb{C}^{\overline{\mathfrak{M}}}$ given by

$$\begin{aligned}
&(\tau^+ u)_{i+\frac{1}{2}} := u_{i+1}, \qquad (\tau^- u)_{i+\frac{1}{2}} := u_i, \qquad i = 0, \dots n+m+1,\\
&(Du)_{i+\frac{1}{2}} := \frac{1}{h'_{i+\frac{1}{2}}}(\tau^+ u - \tau^- u)_{i+\frac{1}{2}}, \qquad \tilde{u} := \frac{1}{2}(\tau^+ + \tau^-)u.
\end{aligned}$$

We also define, on the dual mesh, translation operators $\tau^\pm$, a difference operator $\bar{D}$ and an averaging operator as the map $\mathbb{C}^{\overline{\mathfrak{M}}} \to \mathbb{C}^{\mathfrak{M}}$ given by

$$\begin{aligned}
&(\tau^+ u)_i := u_{i+\frac{1}{2}}, \qquad (\tau^- u)_i := u_{i-\frac{1}{2}}, \qquad i = 1, \dots n+m+1,\\
&(\bar{D}u)_i := \frac{1}{h'_i}(\tau^+ u - \tau^- u)_i, \qquad \bar{u} := \frac{1}{2}(\tau^+ + \tau^-)u.
\end{aligned}$$

3.1.2 Families of non-uniform meshes

In this paper, we address non-uniform meshes that are obtained as the smooth image of an uniform grid.

More precisely, let $\Omega_0 =]0,1[$ and let $\vartheta : \mathbb{R} \to \mathbb{R}$ be an increasing map such that

$$\vartheta(\Omega_0) = \Omega, \quad \vartheta \in \mathcal{C}^\infty, \quad \inf \vartheta' > 0 \text{ and } \quad \vartheta(a) = a' \tag{3.1.6}$$

with a to be kept fixed in what follows and chosen such that $a \in (0,1) \cap \mathbb{Q}$, i.e $a = \frac{p}{q}$ with $p, q \in \mathbb{N}^*$. Clearly, we have $q > p$. We impose the function ϑ to be affine on $[a-\delta, a+\delta]$ $\vartheta|_{[a-\delta,a+\delta]}$ (for some $\delta > 0$).

Given $r \in \mathbb{N}^*$ and set $m = (q-p)r$ and $n = pr$. The parameter r is used to refine the mesh when increased. Set $a = x_{n+1} = x_{pr+1}$. The interval $\overline{\Omega_{01}} = [0, a]$ is then discretized with $n = pr$ interior grid points (excluding 0 and a). The interval $\overline{\Omega_{02}} = [a, 1]$ is discretized with $m = (q-p)r$ exterior grid points (excluding a and 1). Let $\mathfrak{M}_0 = (ih)_{1 \leq i \leq n+m+1}$ with $h = \frac{1}{n+m+2}$ be uniform mesh of Ω_0 and $\overline{\mathfrak{M}_0}$ be the associated dual mesh. We define a non-uniform mesh $\mathfrak{M}$ of Ω as image of $\mathfrak{M}_0$ by the map ϑ, settings

$$\begin{aligned} &x_i' = \vartheta(ih), \quad \forall i \in \{0, ..., n\} \cup \{n+2, ..., n+m+2\} \\ &x_{n+1}' := a' = \vartheta(a). \end{aligned} \tag{3.1.7}$$

The dual mesh $\overline{\mathfrak{M}}$ and the general notation are then those of the previous section.

3.1.3 Main results

With the notation we have introduced, a consistent finite-difference approximation of $\mathcal{A}u$ with homogeneous boundary condition is

$$\mathcal{A}^{\mathfrak{M}} u = -\bar{D}(c_d D u)$$

for $u \in \mathbb{C}^{\mathfrak{M} \cup \partial\mathfrak{M}}$ satisfying $u|_{\partial\Omega} = u^{\partial\mathfrak{M}} = 0$. We have

$$(\mathcal{A}^{\mathfrak{M}} u)_i = -\frac{c_d(x_{i+\frac{1}{2}}) \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} - c_d(x_{i-\frac{1}{2}}) \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}}}{h_i}, \quad i = 1, .., n+m+1.$$

For a suitable weight function φ (to be defined below), the announced semi-discrete Carleman estimate for the operator $\mathcal{P}^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}}$ with a discontinuous diffusion coefficient c, for the non-uniform meshes we consider, is of the form

$$\begin{aligned} &\tau^{-1} \left\| \theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u \right\|^2_{L^2(Q)} + \tau \left\| \theta^{\frac{1}{2}} e^{\tau\theta\varphi} D u \right\|^2_{L^2(Q)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2(Q)} \\ &\quad \leq C_{\lambda,\mathfrak{K}} \left(\left\| e^{\tau\theta\varphi} P^{\mathfrak{M}} u \right\|^2_{L^2(Q)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2((0,T)\times\omega)} \right. \\ &\qquad \left. + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=0} \right|^2_{L^2(\Omega)} + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=T} \right|^2_{L^2(\Omega)} \right), \end{aligned}$$

for properly chosen functions $\theta = \theta(t)$ and $\varphi = \varphi(x)$, for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$ and $\tau h(\alpha T)^{-1} \leq \epsilon_0$, $0 < \alpha < T$ and for all $u \in C^\infty(0, T; \mathbb{C}^{\mathfrak{M}})$ where τ_0, h_0, ϵ_0 only depend on the data. We refer to Theorem 3.5.6 below for a precise result. The proof of this estimate will be first carried out for piecewise uniform meshes (see Theorem 3.4.1), and then adapted to the case of the non-uniform meshes we introduced in Section 3.1.2.

From the semi-discrete Carleman estimate we obtain allows we deduce following weak observability estimate

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} \|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C}{h}} |q(T)|^2_{L^2(\Omega)},$$

for any q solution to the adjoint system

$$\partial_t q + \mathcal{A}^{\mathfrak{M}} q + aq = 0, \qquad q|_{\partial\Omega} = 0.$$

A precise statement is given in Section 3.6.

Moreover, from the weak observability estimate given above we obtain a controllability result for the linear operator $P^{\mathfrak{M}}$. This result can be extended to classes of semi-linear equations

$$\big(\partial_t + \mathcal{A}^{\mathfrak{M}}\big) y + G(y) = \mathbf{1}_\omega v, \quad y \in (0, T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0,$$

with $G(x) = xg(x)$, where $g \in L^\infty(\mathbb{R})$ and

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \qquad \text{with} \qquad 0 \leq r < \frac{3}{2}.$$

We shall state controllability results with a control that satisfies

$$\|v\|_{L^2(Q)} \leq C\, |y_0|\,.$$

Thanks to one space dimension the size of the control function is uniform with respect to the discretization parameter h.

3.1.4 Sketch of proof of Theorem 3.4.1 (Carleman estimate for uniform mesh)

The main idea of the proof lays in the combination of the derivation of a discrete Carleman estimate as in [BHL10a, BL12] and techniques used in [BDL07] to achieve such estimates for operators with discontinuous coefficients in the one-dimensional case.

We set $v = e^{-s\varphi}u$ yielding $e^{s\varphi}Pe^{-s\varphi}v = e^{s\varphi}f_1$ in Q_0' if $Pu = f_1$

We obtain $g = Av + Bv$ in Q_0', with A and iB 'essentially' selfadjoint.

We write $\|g\|^2_{L^2} = \|Av\|^2_{L^2} + \|Bv\|^2_{L^2} + 2(Av, Bv)_{L^2}$ and the main part of the proof is dedicated to computing the inner product $(Av, Bv)_{L^2(Q_0')}$, involving (discrete) integration by parts.

We proceed with these computations separately in each domain Ω_{01}, Ω_{02}. As in [BL12] we obtain terms involving boundary points $x = 0$ and $x = 1$ such as $v(0)$,$v(1)$,$\partial_t v(0)$,$\partial_t v(1)$, $(Dv)_{n+m+\frac{1}{2}}, (Dv)_{n+m+\frac{3}{2}}$. In our case we obtain additional terms involving the jump point a such as $v(a), \partial_t v(a)$, $\tilde{v}_{n+\frac{1}{2}}, \tilde{v}_{n+\frac{3}{2}}$, $(Dv)_{n+\frac{1}{2}}, (Dv)_{n+\frac{3}{2}}$. Main difficulties of our work come from dealing with these new terms. To reduce the number of terms to control, we find relations among connecting these various values at jump point allowing to focus our computations on terms only involving $v(a), \partial_t v(a)$ and $(Dv)_{n+\frac{1}{2}}$. Those relations are stated in **Lemma 3.3.17**. In the limit $h \to 0$ they give back the transmission conditions for the function $v = e^{-s\varphi}u$ used crucial way in [BDL07]. The idea of this technique comes from a similar technique shown in continuos case by [BDL07].

The discrete setting could allow computation on the whole Ω. Yet such computation would yield constant that would depend on discrete derivatives of the diffusions coefficient, yielding non-uniformity with respect to the discretization parameter h. This explains why we resort to working on both Ω_0 and Ω_1 separately and deal with the interface terms that appear. As in [BDL07] the weight function is chosen to obtain positive contributions for these terms.

Sketch of proof Theorem

1. We compute the inner product (Av, Bv) in a series of terms and collect them together in an estimate (see **Lemma 3.4.4**–**Lemma 3.4.12**). In that estimate, we need to tackle two parts: volume integrals, integrals involving boundary points and the jump point. Volume integrals and boundary terms are dealt with similar to [BL12]. Terms at the jump point require special case.

2. Treatment of terms the jump point

   - Terms at jump point involving $\partial_t v$: when treating the term Y_{13} we obtain a positive integral of $(\partial_t v(a))^2$ in the LHS of the estimate as shown in **Lemma 3.4.15**. We keep this term in the LHS of the estimate.

   - Other terms: We collect together the terms at the jump point that already exist in the continuous case. As in [BDL07] we obtain a quadratic form because of the choice of the weight function (jump of its slope). This allows us to obtain positive two integrals involving $v^2(a)$, $(Dv)^2_{n+\frac{1}{2}}$ in the LHS of our estimate (see **Lemma 3.4.14**).

- The remaining terms at the jump point are placed in the RHS of estimate. After that, we apply Young's inequality to them (as shown in **Lemma 3.4.16**) and they then can be absorded by the positive integrals involving $v^2(a)$, $(Dv)^2_{n+\frac{1}{2}}$, $(\partial_t v(a))^2$ in the LHS of estimate as described above.

3.1.5 Outline

In section 3.2, we construct the weight functions to be used in the Carleman estimate. In section 3.3 we have gathered some preliminary discrete calculus results and we present how transmission conditions can be expressed in the discretization scheme. Section 3.4 is devoted to the proof the semi-discrete parabolic Carleman estimate in the case of a discontinuous diffusion cofficient for piecewise uniform meshes in the one-dimensional case. To ease the reading, a large number of proofs of intermediate estimates have been provided in Section 3.7. This result is then extended to non-uniform meshes in Section 3.5. Finally, in Section 3.6, as consequences of the Carleman estimate, we present the weak observability estimate and associated some controllability results.

3.2 Weight functions

We shall first introduce a particular type of weight functions, which are constructed through the following lemma.

We enlarge the open intervals Ω_1, Ω_2 to large open sets $\tilde{\Omega}_1, \tilde{\Omega}_2$.

Lemma 3.2.1. *Let $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ be a smooth open and connected neighborhoods of intervals $\overline{\Omega}_1$, $\overline{\Omega}_2$ of $\mathbb{R}$ and let $\omega \subset \Omega_2$ be a non-empty open set. Then, there exists a function $\psi \in C(\bar{\Omega})$ such that*

$$\psi(x) = \begin{cases} \psi_1 \text{ in } \overline{\Omega}_1, \\ \psi_2 \text{ in } \overline{\Omega}_2, \end{cases}$$

with $\psi_i \in C^\infty(\overline{\tilde{\Omega}_i}), i = 1, 2$, $\psi > 0$ in $\Omega, \psi = 0$ on $\Gamma, \psi_2' \neq 0$ in $\overline{\Omega}_2 \setminus \omega$, $\psi_1' \neq 0$ in $\overline{\Omega}_1$ and the function ψ satisfies the following trace properties, for some $\alpha_0 > 0$,

$$(\boldsymbol{A}u, u) \geq \alpha_0 |u|^2 \quad u \in \mathbb{R}^2,$$

with the matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

with

$$
\begin{aligned}
a_{11} &= [\psi'\star]_{a'},\\
a_{22} &= [c\psi'\star]^2_{a'}(\psi')(a'^{+}) + [c^2(\psi')^3\star]_{a'},\\
a_{12} &= a_{21} = [c\psi'\star]_{a'}(\psi')(a'^{+}),
\end{aligned}
$$

(see the notation (3.1.3) - (3.1.5) *introduced in Section 3.1.1).*

Remark 3.2.2. *Here we choose a weight function that yields an observation in the region $\omega \Subset \Omega_2$ in the Carleman estimate of Section 3.4. This choice is of course arbitrary.*

Proof. We refer to Lemma 1.1 in [BDL07] for a similar proof. □

Choosing a function ψ, as in the previous lemma, for $\lambda > 0$ and $K > \|\psi\|_\infty$, we define the following weight functions

$$
\begin{aligned}
&\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \quad \phi(x) = e^{\lambda\psi(x)},\\
&r(t,x) = e^{s(t)\varphi(x)}, \quad \rho(t,x) = (r(t,x))^{-1},
\end{aligned}
\tag{3.2.1}
$$

with

$$
s(t) = \tau\theta(t), \quad \tau > 0, \quad \theta(t) = ((t+\alpha)(T+\alpha-t))^{-1},
$$

for $0 < \alpha < T$.

We have

$$
\max_{[0,T]}\theta = \theta(0) = \theta(T) = \alpha^{-1}(T+\alpha)^{-1},
\tag{3.2.2}
$$

and $\min_{[0,T]}\theta \geq T^{-2}$. We note that

$$
\partial_t\theta = (2t-T)\theta^2.
\tag{3.2.3}
$$

For the Carleman estimate and the observation/control results we choose here to treat the case of an distributed-observation in $\omega \Subset \Omega$. The weight function is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi}$, with ψ fulfilling the following assumption. Construction of such a weight function is classical (see e.g [FI96]).

Assumption 3.2.3. *Let $\omega \Subset \Omega$ be an open set. Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\bar{\Omega}$ in $\mathbb{R}$. The function $\psi = \psi(x)$ is in $C^p\big(\overline{\tilde{\Omega}}, \mathbb{R}\big)$, p sufficiently large, and satisfies, for*

some $c > 0$,

$$\psi > 0 \text{ in } \tilde{\Omega}, \qquad |\nabla\psi| \geq c \text{ in } \tilde{\Omega}\backslash\omega_0,$$
$$\partial_n\psi(x) \leq -c < 0, \qquad \partial_x^2\psi(x) \leq 0 \text{ in } V_{\partial\Omega}.$$

where $V_{\partial\Omega}$ *is a sufficiently small neighborhood of* $\partial\Omega$ *in* $\tilde{\Omega}$*, in which the outward unit normal* n *to* Ω *is extended from* $\partial\Omega$*.*

3.3 Some preliminary discrete calculus results for uniform meshes

Here, to prepare for Section 3.4, we only consider constant-step discretizations, i.e., $h_{i+\frac{1}{2}} = h$, $i = 0, \ldots, n+m+1$.

We use here the following notation: $\Omega_0 = (0,1)$, $\Omega_{01} = (0,a)$, $\Omega_{02} = (a,1)$, $\Omega_0' = \Omega_{01} \cup \Omega_{02}$, $Q_0 = (0,T) \times \Omega_0$, $Q_0' = (0,T) \times \Omega_0'$, $Q_{0i} = (0,T) \times \Omega_{0i}$ with $i = 1,2$ and $\partial\Omega_0 = \{0,1\}$.

This section aims to provide calculus rules for discrete operators such as $D_i, \bar{D}_i$ and also to provide estimates for the successive applications of such operators on the weight functions. To avoid cumbersome notation we introduce the following continuous difference and averaging operators on continuous functions. For a function f defined on Ω_0 we set

$$\tau^+ f(x) := f(x + h/2), \qquad \tau^- f(x) := f(x - h/2),$$
$$\mathtt{D} f(x) := (\tau^+ - \tau^-) f(x)/h, \qquad \hat{f}(x) = (\tau^+ + \tau^-) f(x)/2.$$

Remark 3.3.1. *To iterate averaging symbols we shall sometimes write* $Af = \hat{f}$*, and thus* $A^2 f = \hat{\hat{f}}$*.*

3.3.1 Discrete calculus formulae

We present calculus results for finite-difference operators that were defined in the introductory section. Proofs can be found in Appendix of [BHL10a] in the one-dimension case.

Lemma 3.3.2. *Let the functions* f_1 *and* f_2 *be continuously defined in a neighborhood of* $\bar{\Omega}$*. We have:*

$$\mathtt{D}(f_1 f_2) = \mathtt{D}(f_1)\hat{f}_2 + \hat{f}_1 \mathtt{D}(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{C}^{\mathfrak{M}}$ *and*

$g_1, g_2 \in \mathbb{C}^{\overline{\mathfrak{M}}}$ *is*

$$D(f_1 f_2) = D(f_1)\tilde{f}_2 + \tilde{f}_1 D(f_2),\ \bar{D}(g_1 g_2) = \bar{D}(g_1)\bar{g_2} + \bar{g_1}\bar{D}(g_2).$$

Lemma 3.3.3. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\bar{\Omega}$. We have:*

$$\widehat{f_1 f_2} = \hat{f}_1 \hat{f}_2 + \frac{h^2}{4}\mathtt{D}(f_1)\mathtt{D}(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{C}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{C}^{\overline{\mathfrak{M}}}$ is

$$\widetilde{f_1 f_2} = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} D(f_1) D(f_2),\ \overline{g_1 g_2} = \bar{g_1}\bar{g_2} + \frac{h^2}{4}\bar{D}(g_1)\bar{D}(g_2).$$

Some of the following properties can be extended in such a manner to discrete functions. We shall not always write it explicitly.

Averaging a function twice gives the following formula.

Lemma 3.3.4. *Let the function f be continuously defined over $\mathbb{R}$. We then have*

$$A^2 f := \hat{\hat{f}} = f + \frac{h^2}{4}\mathtt{DD}f.$$

The following proposition covers discrete integrations by parts and related formula.

Proposition 3.3.5. *Let $f \in \mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ and $g \in \mathbb{C}^{\overline{\mathfrak{M}}}$. We have the following formulae:*

$$\int_{\Omega_0} f(\bar{D}g) = -\int_{\Omega_0} (Df)g + f_{n+m+2}g_{n+m+\frac{3}{2}} - f_0 g_{\frac{1}{2}},$$
$$\int_{\Omega_0} f\bar{g} = \int_{\Omega_0} \tilde{f}g - \frac{h}{2} f_{n+m+2}g_{n+m+\frac{3}{2}} - \frac{h}{2} f_0 g_{\frac{1}{2}}.$$

Lemma 3.3.6. *Let f be a smooth function defined in a neighborhood of $\bar{\Omega}$. We have*

$$\tau^{\pm} f = f \pm \frac{h}{2}\int_0^T \partial_x f(. \pm \sigma h/2)d\sigma,$$
$$A^j f = f + C_j h^2 \int_{-1}^1 (1 - |\sigma|)\partial_x^2 f(. + l_j \sigma h)d\sigma,$$
$$D^j f = \partial_x^j f + C'_j h^2 \int_{-1}^1 (1 - |\sigma|)^{j+1}\partial_x^{j+2} f(. + l_j\sigma h)d\sigma,\ j = 1, 2,\ l_1 = \frac{1}{2},\ l_2 = 1.$$

3.3.2 Calculus results related to the weight functions

We now present some technical lemmata related to discrete operators performed on the Carleman weight functions that is of the form $e^{s\varphi}$, $\varphi = e^{\lambda\psi} - e^{\lambda K}$, where ψ satisfies the properties listed in Section 3.2 in the domain Ω_0. For concision, we set $r(t,x) = e^{s(t)\varphi(x)}$ and $\rho = r^{-1}$, with $s(t) = \tau\theta(t)$. From Section 3.2, we have $\psi_{|\Omega_{01}} = \psi_{1|\Omega_{01}}$, $\psi_{|\Omega_{01}} = \psi_{2|\Omega_{01}}$ where $\psi_i \in \mathbb{C}^2(\bar{\tilde{\Omega}}_{0i})$. Then $\rho = e^{-s\varphi}$ can be replaced by

$$\begin{aligned} \rho_1 = e^{-s\varphi_1} \text{ with } \quad \varphi_1 = e^{\lambda\psi_1} - e^{\lambda K} \text{ in domain } \quad \Omega_{01} \\ \rho_1 = e^{-s\varphi_2} \text{ with } \quad \varphi_2 = e^{\lambda\psi_2} - e^{\lambda K} \text{ in domain } \quad \Omega_{02} \end{aligned}$$

And $r = \rho^{-1}$ is also replaced by

$$\begin{aligned} r_1 = \rho_1^{-1} \text{ in domain } \quad \Omega_{01} \\ r_2 = \rho_2^{-1} \text{ in domain } \quad \Omega_{02} \end{aligned}$$

The positive parameters τ and h will be large and small respectively and we are particularly interested in the dependence on τ, h and λ in the following basic estimates in each domain Ω_{01}, Ω_{02}.

We assume $\tau \geq 1$ and $\lambda \geq 1$.

Lemma 3.3.7. *Let $\alpha,\ \beta \in \mathbb{N}$, i=1,2. We have*

$$\begin{aligned} \partial_x^\beta(r_i\partial_x^\alpha\rho_i) &= \alpha^\beta(-s\phi_i)^\alpha\lambda^{\alpha+\beta}(\bigtriangledown\psi_i)^{\alpha+\beta} \\ &+ \alpha\beta(s\phi_i)^\alpha\lambda^{\alpha+\beta-1}\mathcal{O}_\lambda(1) + s^{\alpha-1}\alpha(\alpha-1)\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^\alpha). \end{aligned}$$

Let $\sigma \in [-1,1]$, we have

$$\partial_x^\beta\big(r_i(t,.)(\partial_x^\alpha\rho_i)(t,.+\sigma h)\big) = \mathcal{O}_\lambda(s^\alpha(1+(sh)^\beta))e^{\mathcal{O}_\lambda(sh)}.$$

Provided $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$ we have $\partial_x^\beta\big(r_i(t,.)(\partial^\alpha\rho_i)(t,.+\sigma h)\big) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. The same expressions hold with r and ρ interchanged and with s changed into -s.

A proof is given in [BHL10a, proof of Lemma 3.7] in the time independent case. Additionally, we provide a result below to the time-dependent case whose proof is refered to [BL12, proof of Lemma 2.8]. Note that the condition $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$ implies that $s(t)h \leq \mathfrak{K}$ for all $t \in [0,T]$.

Lemma 3.3.8. *Let $\alpha \in \mathbb{N}$, i=1,2. We have*

$$\partial_t(r_i\partial_x^\alpha \rho_i) = s^\alpha T\theta \mathcal{O}_\lambda(1).$$

With Leibniz formula we have the following estimates

Corollary 3.3.9. *Let $\alpha, \beta, \delta \in \mathbb{N}$, i=1,2. We have*

$$\begin{aligned}\partial_x^\delta(r_i^2(\partial_x^\alpha \rho_i)\partial_x^\beta \rho_i) &= (\alpha+\beta)^\delta(-s\phi_i)^{\alpha+\beta}\lambda^{\alpha+\beta+\delta}(\bigtriangledown\psi_i)^{\alpha+\beta+\delta} \\ &+ \delta(\alpha+\beta)(s\phi_i)^{\alpha+\beta}\lambda^{\alpha+\beta+\delta-1}\mathcal{O}(1) \\ &+ s^{\alpha+\beta-1}(\alpha(\alpha-1)+\beta(\beta-1))\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^{\alpha+\beta}).\end{aligned}$$

The proofs of the following properties can be found in Appendix A of [BHL10a].

Proposition 3.3.10. *Let $\alpha \in \mathbb{N}$, i=1,2. Provided $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$\begin{aligned}&r_i\tau^\pm\partial_x^\alpha\rho_i = r_i\partial_x^\alpha\rho_i + s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(sh) = s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\ &r_iA^k\partial_x^\alpha\rho_i = r_i\partial_x^\alpha\rho_i + s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(1),\ k=0,1,2,\\ &r_iA^kD\rho_i = r_i\partial_x\rho_i + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\ k=0,1,\\ &r_iD^2\rho_i = r_i\partial_x^2\rho_i + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1).\end{aligned}$$

The same estimates hold with ρ_i and r_i interchanged.

Lemma 3.3.11. *Let $\alpha, \beta \in \mathbb{N}$ and $k=1,2; j=1,2; i=1,2$. Provided $0 < \tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$\begin{aligned}&D^k(\partial_x^\beta(r_i\partial_x^\alpha\rho_i)) = \partial_x^{k+\beta}(r_i\partial_x^\alpha\rho_i) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^\alpha),\\ &A^j\partial_x^\beta(r_i\partial_x^\alpha\rho_i) = \partial^\beta(r_i\partial_x^\alpha\rho_i) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^\alpha).\end{aligned}$$

Let $\sigma \in [-1,1]$, we have $D^k\partial^\beta(r_i(t,.)\partial^\alpha\rho_i(t,.+\sigma h)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. The same estimates hold with r_i and ρ_i interchanged.

Lemma 3.3.12. *Let $\alpha, \beta, \delta \in \mathbb{N}$ and $k=1,2; j=1,2; i=1,2$. Provided $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$A^j\partial_x^\delta(r_i^2(\partial_x^\alpha\rho_i)\partial_x^\beta\rho_i) = \partial_x^\delta(r_i^2(\partial_x^\alpha\rho_i)\partial_x^\beta\rho_i) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}),$$

$$D^k\partial_x^\delta(r_i^2(\partial_x^\alpha\rho_i)\partial_x^\beta\rho_i) \quad = \quad \partial_x^{k+\delta}(r_i^2(\partial_x^{\alpha+\beta}\rho_i) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}).$$

Let $\sigma, \sigma' \in [-1,1]$. We have

$$A^j\partial^\delta(r_i(t,.)^2(\partial^\alpha\rho_i(t,.+\sigma h))\partial^\beta\rho_i(y,.+\sigma' h)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}),$$
$$D^k\partial^\delta(r_i(t,.)^2(\partial^\alpha\rho_i(t,.+\sigma h))\partial^\beta\rho_i(t,.+\sigma' h)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{\alpha+\beta}).$$

The same estimates hold with r_i and ρ_i interchanged.

Proposition 3.3.13. *Let $\alpha \in \mathbb{N}$ and $k = 0,1,2; j = 0,1,2; i = 1,2$. Provided $0 < sh \leq \mathfrak{K}$, we have*

$$D^kA^j\partial_x^\alpha(r_i\widehat{D\rho_i}) = \partial_x^{k+\alpha}(r_i\partial_x\rho_i) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$D^k(r_iD^2\rho_i) = \partial_x^k(r_i\partial^2\rho_i) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$r_iA^2\rho_i = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,\ D^k(r_iA^2\rho_i) = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.$$

The same estimates hold with r_i and ρ_i interchanged.

Proposition 3.3.14. *Provided $0 < \tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$ and σ is bounded, we have*

$$\partial_t(r_i(.,x)(\partial^\alpha\rho_i)(.,x+\sigma h)) = Ts^\alpha\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$\partial_t(r_iA^2\rho_i) = T(sh)^2\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$\partial_t(r_iD^2\rho_i) = Ts^2\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

The same estimates hold with r_i and ρ_i interchanged.

Proposition 3.3.15. *Let $\alpha,\ \beta \in \mathbb{N}$ and $k = 0,1,2; j = 0,1,2; i = 1,2$, provided $0 < sh \leq \mathfrak{K}$, we have*

$$A^jD^k\partial^\beta(r_i^2(\partial^\alpha)\widehat{D\rho_i}) \quad = \quad \partial_x^{k+\beta}(r_i^2(\partial^\alpha\rho)\partial\rho_i) + s^{\alpha+1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^{\alpha+1}\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

$$A^jD^k\partial^\beta(r_i^2(\partial^\alpha)A^2\rho_i) \quad = \quad \partial_x^{k+\beta}(r_i(\partial^\alpha\rho_i)) + s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s^\alpha\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

$$A^jD^k\partial^\beta(r_i^2(\partial^\alpha)D^2\rho_i) \quad = \quad \partial_x^{k+\beta}(r_i^2(\partial^\alpha\rho)\partial^2\rho_i) + s^{\alpha+2}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^{\alpha+2}\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

and we have

$$A^j D^k \partial^\alpha (r_i^2 \widehat{D\rho_i} D^2 \rho_{0i}) = \partial_x^{k+\alpha}(r_i^2(\partial\rho_i)\partial^2\rho_i) + s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$A^j D^k \partial^\alpha (r_i^2 \widehat{D\rho_i} A^2 \rho_i) = \partial_x^{k+\alpha}(r_i \partial\rho_i) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = s\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

3.3.3 Transmission conditions

We consider here discrete version of the transmission conditions (TC) at the point a. For $u \in \mathbb{C}^{\mathfrak{M}}$ we set $f := \bar{D}(c_d Du)$ we then have

$$\begin{cases} u(a^-) = u(a^+) = u_{n+1}, \\ (c_d Du)_{n+\frac{3}{2}} - (c_d Du)_{n+\frac{1}{2}} = hf_{n+1} \end{cases}.$$

Remark 3.3.16. *These transmission conditions provide the continuity for u and the discrete flux at the singular point of coefficient up to a consistent factor.*

From these conditions, we obtain the following lemma whose proof is given in Section 3.7.

Lemma 3.3.17. *For the parameter λ chosen sufficiently large and sh sufficiently small and with $u = \rho v$ we have*

$$[\star c_d Dv]_a = (c_d Dv)_{n+\frac{3}{2}} - (c_d Dv)_{n+\frac{1}{2}} = J_1 v_{n+1} + J_2 (c_d Dv)_{n+\frac{1}{2}} + J_3 h(rf)_{n+1} \tag{3.3.1}$$

where

$$J_1 = \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)\lambda s[\star c\phi\psi']_a + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$J_2 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \qquad J_3 = \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big).$$

Furthermore, we have

$$\partial_t J_1 = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\partial_t J_2 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \qquad \partial_t J_3 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh).$$

For simplicity, (3.3.1) can be written in form

$$[\star c_d Dv]_a = \lambda s[\star c\phi\psi']_a v_{n+1} + r_0, \tag{3.3.2}$$

where $r_0 = \lambda s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}} + h\Big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)(rf)_{n+1}$.

3.4 Carleman estimate for uniform meshes

In this section, we prove a Carleman estimate in case of picewise uniform meshes, i.e, constant-step discretizations in each subinterval $(0, a)$ and $(a, 1)$. The case of non-uniform meshes is treated in Section 3.5.

We let $\omega_0 \Subset \Omega_{02}$ be a nonempty open subset. We set the operator $P^{\mathfrak{M}}$ to be $\mathcal{P}^{\mathfrak{M}} = -\partial_t + \mathcal{A}^{\mathfrak{M}} = -\partial_t - \bar{D}(c_d D)$, continuous in the variable $t \in (0, T)$ with $T > 0$, and discrete in the variable $x \in \Omega_0$.

The Carleman weight function is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi} - e^{\lambda K}$ where ψ satisfies the properties listed in Section 3.2 in the domain Ω_0. Here, to treat the semi-discrete case, we use the enlarged neighborhoods $\tilde{\Omega}_{01}$, $\tilde{\Omega}_{02}$ of Ω_{01}, Ω_{02} as introduced in Lemma 3.2.1. This allows one to apply multiple discrete operators such as D and A on the weight functions. In particular, we take ψ such that $\partial_x\psi \geq 0$ in V_0 and $\partial_x\psi \leq 0$ in V_1 where V_0 and V_1 are neighborhoods of 0 and 1 respectively. This then yields on $\partial\Omega_0$

$$(r\overline{D\rho})_0 \leq 0, \quad (r\overline{D\rho})_{n+m+2} \geq 0 \tag{3.4.1}$$

Theorem 3.4.1. *Let $\omega_0 \Subset \Omega_{02}$ be a non-empty open set and we set $f := \bar{D}(c_d Du)$. For the parameter $\lambda > 1$ sufficiently large, there exists C, $\tau_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, depending on ω_0 so that the following estimate holds*

$$\begin{aligned}
&\tau^{-1}\left\|\theta^{-\frac{1}{2}} e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q_0)} + \tau\left\|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} Du\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\right\|^2_{L^2(Q_0)} \\
&\leq C_{\lambda,\mathfrak{K}}\left(\left\|e^{\tau\theta\varphi} P^{\mathfrak{M}} u\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\right\|^2_{L^2((0,T)\times\omega_0)}\right. \\
&\qquad \left. + h^{-2}\left|e^{\tau\theta\varphi} u|_{t=0}\right|^2_{L^2(\Omega_0)} + h^{-2}\left|e^{\tau\theta\varphi} u|_{t=T}\right|^2_{L^2(\Omega_0)}\right),
\end{aligned} \tag{3.4.2}$$

for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$ and $\tau h(\alpha T)^{-1} \leq \epsilon_0$ and for all $u \in C^\infty(0, T; \mathbb{C}^{\mathfrak{M}})$ satisfying $u|_{\partial\Omega_0} = 0$.

Remark 3.4.2. *Observation was chosen in Ω_{02} here. This is an arbitrary choice (see Remark 3.2.2).*

Proof. We set $f_1 := -P^{\mathfrak{M}} = \partial_t u + \bar{D}(c_d Du)$ and $f = \bar{D}(c_d Du)$. At first, we shall work with the function $v = ru$, i.e., $u = \rho v$, that satisfies

$$r\Big(\partial_t(\rho v) + \bar{D}\big(c_d D(\rho v)\big)\Big) = r f_1 \text{ in } Q_0'. \tag{3.4.3}$$

We have

$$r\partial_t(\rho v) = \partial_t v + r(\partial_t \rho)v = \partial_t v - \tau(\partial_t\theta)\varphi v.$$

We write: $g = Av + Bv$,

where $Av = A_1v + A_2v + A_3v$, $Bv = B_1v + B_2v + B_3v$ with

$$\begin{aligned}
&A_1v = r\bar{\overline{\rho}}\bar{D}(c_dDv),\ A_2v = cr(\bar{D}D\rho)\bar{\overline{v}},\ A_3v = -\tau(\partial_t\theta)\varphi v,\\
&B_1v = 2cr\overline{D\rho}\ \overline{Dv},\ B_2v = -2sc\phi''v,\ B_3v = \partial_t v,\\
&g = rf_1 - \frac{h}{4}r\overline{D\rho}(\bar{D}c_d)(\tau^+Dv - \tau^-Dv) - \frac{h^2}{4}(\bar{D}c_d)r(\bar{D}D\rho)\overline{Dv}\\
&\qquad -h\mathcal{O}(1)r\overline{D\rho}\ \overline{Dv} - \big(r(\bar{D}c_d)\ \overline{D\rho} + h\mathcal{O}(1)r(\bar{D}D\rho)\big)\bar{\overline{v}} - 2sc(\phi'')v,
\end{aligned}$$

as derived in [BL12].

Equation (3.4.3) now reads $Av + Bv = g$ and we write

$$\|Av\|^2_{L^2(Q'_0)} + \|Bv\|^2_{L^2(Q'_0)} + 2(Av, Bv)_{L^2(Q'_0)} = \|g\|^2_{L^2(Q'_0)}\,. \tag{3.4.4}$$

First we need an estimation of $\|g\|^2_{L^2(Q'_0)}$. The proof can be adapted from [BHL10a].

Lemma 3.4.3. *For $\tau h(\max_{[0,T]}\theta) \le \mathfrak{K}$ we have*

$$\|g\|^2_{L^2(Q'_0)} \le C_{\lambda,\mathfrak{K}}(\|rf_1\|^2_{L^2(Q'_0)}) + \|sv\|^2_{L^2(Q'_0)} + h^2\,\|sDv\|^2_{L^2(Q'_0)}\,. \tag{3.4.5}$$

Most of the remaining of the proof will be dedicated to computing the inner product $(Av, Bv)_{L^2(Q'_0)}$. Developing the inner-product $(Av, Bv)_{L^2(Q'_0)}$, we set $I_{ij} = (A_iv, B_jv)_{L^2(Q'_0)}$. The proofs of the following lemmata are provided in Section 3.7.

Lemma 3.4.4 (Estimate of I_{11})**.** *For $\tau h(max_{[0,T]}\theta) \le \mathfrak{K}$ we have*

$$I_{11} \ge -\int_{Q'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{11} + Y_{11},$$

where $X_{11} = \int_{Q_0'} \nu_{11}(Dv)^2$ *with* ν_{11} *of the form* $s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ *and*

$$\begin{aligned}
Y_{11} &= Y_{11}^{(1)} + Y_{11}^{(2,1)} + Y_{11}^{(2,2)},\\
Y_{11}^{(1)} &= \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(1)(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}}\\
&\quad - \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(0)(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}},\\
Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\bar{c}_d(a)\Big((c\psi')(a^+)(Dv)^2_{n+\frac{3}{2}} - (c\psi')(a^-)(Dv)^2_{n+\frac{1}{2}}\Big),\\
Y_{11}^{(2,2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{1}{2}} - \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{3}{2}}.
\end{aligned}$$

Lemma 3.4.5 (Estimate of I_{12})**.** *For* $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$*, the term* I_{12} *is of the following form*

$$I_{12} = 2\int_{Q_0'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{12} + Y_{12},$$

with

$$\begin{aligned}
Y_{12} &= \int_0^T s\lambda^2\phi(a)v(a)[c(\psi')^2 \star c_d Dv]_a\\
&+ \int_0^T \delta_{12}v(a)(cDv)_{n+\frac{3}{2}} + \int_0^T \bar{\delta}_{12}v(a)(cDv)_{n+\frac{1}{2}},
\end{aligned}$$

where $\delta_{12}, \bar{\delta}_{12}$ *are of the form* $s\big(\lambda\phi(a)\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big)$ *and*

$$X_{12} = \int_{Q_0'} \nu_{12}(Dv)^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv,$$

where

$$\nu_{12} = s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(h + (sh)^2).$$

Lemma 3.4.6 (Estimate of I_{13})**.** *There exists* $\epsilon_1(\lambda) > 0$ *such that, for* $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_1(\lambda)$*, the term* I_{13} *can be estimated from below in following way:*

$$I_{13} \geq -\int_{\Omega_0'} \mathcal{C}_{\lambda,\mathfrak{K}}(1)(Dv(T))^2 - X_{13} + Y_{13}.$$

with

$$X_{13} = \int_{Q_0'} \big(s(sh) + T(sh)^2\theta\big)\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2,$$

$$Y_{13} = -\int_0^T r\bar{\tilde{\rho}}(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} + \int_0^T r\bar{\tilde{\rho}}(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}}.$$

Lemma 3.4.7 (Estimate of I_{21})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{21} can be estimated as*

$$I_{21} \geq 3\int_{Q_0'} \lambda^4 s^3 \phi^3 c^2 (\psi')^4 v^2 - X_{21} + Y_{21},$$

with

$$X_{21} = \int_{Q_0'} \mu_{21} v^2 + \int_{Q_0'} \nu_{21}(Dv)^2,$$

where

$$\mu_{21} = (s\lambda\phi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \quad \nu_{21} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,$$

and

$$\begin{aligned}
Y_{21} &= Y_{21}^{(1,1)} + Y_{21}^{(1,21)} + Y_{21}^{(1,22)} + Y_{21}^{(2)},\\
Y_{21}^{(1,1)} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}},\\
Y_{21}^{(1,21)} &= \int_0^T s^3\lambda^3\phi^3(a)[c^2(\psi')^3 \star (\tilde{v})^2]_a,\\
Y_{21}^{(1,22)} &= \int_0^T \big(s^2\mathcal{O}_{\lambda}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big)\big((\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}}\big),\\
Y_{21}^{(2)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a).
\end{aligned}$$

Lemma 3.4.8 (Estimate of I_{22})**.** *For $sh \leq \mathfrak{K}$, we have*

$$I_{22} = -2\int_{Q_0'} c^2 s^3 \lambda^4 \phi^3 (\psi')^4 v^2 - X_{22} + Y_{22},$$

with

$$\begin{aligned}
Y_{22} &= Y_{22}^{(1)} + Y_{22}^{(2)},\\
Y_{22}^{(1)} &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{1}{2}} + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{3}{2}},\\
Y_{22}^{(2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2(a),
\end{aligned}$$

and

$$X_{22} = \int_{Q_0'} \mu_{22} v^2 + \int_{Q_0'} \nu_{22}(Dv)^2,$$

where

$$\mu_{22} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \quad \nu_{22} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.$$

Lemma 3.4.9 (Estimate of I_{23})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{23} can be estimated from below in the following way*

$$I_{23} \geq \int_{\Omega_0'} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2_{|_{t=0}} + \mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2_{|_{t=T}}\big) - X_{23} + Y_{23},$$

with

$$\begin{aligned} X_{23} \quad = \quad & \int_{Q_0} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v)^2 \\ & + \int_{Q_0'} (sh)^2 s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2, \end{aligned}$$

and

$$\begin{aligned} Y_{23} &= Y_{23}^{(1)} + Y_{23}^{(2)} + Y_{23}^{(3)}, \\ Y_{23}^{(1)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{1}{2}}) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{3}{2}}), \\ Y_{23}^{(2)} &= \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a), \\ Y_{23}^{(3)} &= \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a)|_{t=0}^{t=T}. \end{aligned}$$

Lemma 3.4.10 (Estimate of I_{31})**.** *For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$I_{31} = -X_{31} + Y_{31},$$

with

$$X_{31} = \int_{Q_0'} T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2,$$

and

$$\begin{aligned} Y_{31} &= Y_{31}^{(1)} + Y_{31}^{(2)}, \\ Y_{31}^{(1)} &= \int_0^T T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h}{2}(Dv)_{n+\frac{1}{2}} + \int_0^T T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h}{2}(Dv)_{n+\frac{3}{2}}, \\ Y_{31}^{(2)} &= \int_0^T T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a). \end{aligned}$$

Lemma 3.4.11 (Estimate of I_{32})**.** *[BL12] For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{32} can be estimated*

from below in the following way

$$I_{32} = -X_{32} = \int_{Q_0'} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2.$$

Lemma 3.4.12 (Estimate of I_{33}). *[BL12, proof of Lemma 3.9] For $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{33} can be estimated from below in the following way*

$$I_{33} \geq -X_{33} = \frac{1}{2}\tau \int_{Q_0'} c'\varphi(\partial_t^2\theta)v^2.$$

Continuation of the proof of Theorem 4.1. Collecting the terms we have obtained in the previous lemmata, from (3.4.4) and (3.4.5) for $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_1(\lambda)$ we find

$$\begin{aligned} &\|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + 2\int_{Q_0'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 + 2\int_{Q_0'} c^2s^3\lambda^4\phi^3(\psi')^4v^2 \\ &\quad +2\big(Y_{11}^{(1)} + Y_{21}^{(1,1)}\big) + 2\big(Y_{11}^{(2,1)} + Y_{21}^{(1,21)}\big) + 2Y_{13} \\ \leq\ & C_{\lambda,\mathfrak{K}}\Big(\|rf_1\|^2_{L^2(Q_0')} + \int_{\Omega_0'} s^2\big(v^2_{|t=0} + v^2_{|t=T}\big) + \int_{\Omega_0'} (Dv(T))^2\Big) + 2X + 2\underline{Y}, \end{aligned}$$

with

$$\begin{aligned} \underline{Y} &= -\Big(Y_{11}^{(2,2)} + Y_{12} + Y_{21}^{(1,22)} + Y_{21}^{(2)} + Y_{22} + Y_{23} + Y_{31}\Big), \\ X &= X_{11} + X_{12} + X_{13} + X_{21} + X_{22} + X_{23} + X_{31} + X_{32} + X_{33} \\ &\quad + C_{\lambda,\mathfrak{K}}\big(\|sv\|^2_{L^2(Q_0')} + h^2\|sDv\|^2_{L^2(Q_0')}\big). \end{aligned}$$

With the following lemma, we may in fact ignore the term $Y_{11}^{(1)} + Y_{21}^{(1,1)}$ in the previous inequality.

Lemma 3.4.13. *For all λ there exists $0 < \epsilon_2(\lambda) < \epsilon_1(\lambda)$ such that for $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_2(\lambda)$ we have $Y_{11}^{(1)} + Y_{21}^{(1,1)} \geq 0$.*

Recalling that $\nabla\psi \geq C > 0$ in $\Omega\backslash\omega_0$ we may thus write

$$\begin{aligned} &\|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + \int_{Q_0'} s(Dv)^2 + \int_{Q_0'} s^3v^2 \\ &\qquad +2\big(Y_{11}^{(2,1)} + Y_{21}^{(1,21)}\big) + 2Y_{13} \\ \leq\ & C_{\lambda,\mathfrak{K}}\left(\|rf_1\|^2_{L^2(Q_0')} + 2\int_0^T\int_{\omega_0} s(Dv)^2 + 2\int_0^T\int_{\omega_0} s^3v^2 \right. \\ &\left. + \int_{\Omega_0'} s^2(v^2_{|t=0} + v^2_{|t=T}) + \int_{\Omega_0'} (Dv(T))^2\right) + 2X + 2\underline{Y}. \end{aligned} \tag{3.4.6}$$

Lemma 3.4.14. *With the function ψ satisfing the properties of Lemma 3.2.1 and for $\tau h(max_{[0,T]}\theta) \leq \mathfrak{K}$, we have*

$$Y_{11}^{(2,1)} + Y_{21}^{(1,21)} \geq C\alpha_0 \int_0^T s\lambda\phi(a)(c_d Dv)_{n+\frac{1}{2}}^2 + C\alpha_0 \int_0^T s^3\lambda^3\phi^3(a)v_{n+1}^2 + \mu_1 + \mu_r,$$

with α_0 as given in Lemma 3.2.1 and where

$$\begin{aligned}\mu_r &= \int_0^T s\mathcal{O}_\lambda(1)r_0^2 + \int_0^T s^2\mathcal{O}_\lambda(1)r_0 v_{n+1} + \int_0^T s\mathcal{O}_\lambda(1)r_0(c_d Dv)_{n+\frac{1}{2}} \\ &+ \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_d Dv)_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0 v_{n+1},\end{aligned}$$

with r_0 as given in Lemma 3.3.17 and

$$\mu_1 = \mu_1^{(1)} + \mu_1^{(2)},$$

where

$$\begin{aligned}\mu_1^{(1)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{3}{2}}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}^2, \\ \mu_1^{(2)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{3}{2}}v_{n+1} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}v_{n+1}.\end{aligned}$$

For a proof see Section 3.7.

Lemma 3.4.15. *With $0 < \epsilon_3(\lambda) < \epsilon_2(\lambda)$ sufficiently small we obtain*

$$\begin{aligned}Y_{13} &\geq \int_0^T \mathcal{C}_{\lambda,\mathfrak{K}}h(\partial_t v(a))^2 + \int_0^T \big(sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)v^2(a) \\ &+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1}.\end{aligned}$$

where $C_{\lambda,\mathfrak{K}}$ is positive constant whose value depends on λ and sh.

For a proof see Section 3.7.

If we choose $\lambda_2 \geq \lambda_1$ sufficiently large, then for $\lambda = \lambda_2$ (fixed for the rest of the proof) and $0 < \tau h(\max_{[0,T]}\theta) \leq \epsilon_3$, from (3.4.6) and Lemma 3.4.14 and Lemma 3.4.15, we can thus achieve

the following inequality

$$
\begin{aligned}
& \|Av\|^2_{L^2(Q'_0)} + \|Bv\|^2_{L^2(Q'_0)} + \int_{Q'_0} s\,|Dv|^2\,dt + \int_{Q'_0} s^3v^2dt \\
+ \quad & C\alpha_0 \int_0^T s(c_dDv)^2_{n+\frac{1}{2}} + C\alpha_0 \int_0^T s^3v^2(a) + \int_0^T C_{\lambda,\mathfrak{K}} h(\partial_t v(a))^2 \\
\leq \quad & C_{\lambda,\mathfrak{K}} \left(\|rf_1\|^2_{L^2(Q'_0)} + 2\int_0^T \int_{\omega_0} s(Dv)^2 + 2\int_0^T \int_{\omega_0} s^3v^2 \right. \\
& \left. + \int_{\Omega'_0} s^2(v^2_{|t=0} + v^2_{|t=T}) + \int_{\Omega'_0} (Dv(T))^2 + sv^2(a)|_{t=0}^{t=T} \right) \\
+ \quad & \int_0^T \big(sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)v^2(a) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_dDv)_{n+\frac{1}{2}} \\
+ \quad & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} + 2\underline{X} + 2\underline{Y} + 2\underline{Z} \qquad , \qquad (3.4.7)
\end{aligned}
$$

where $\underline{Z} = \mu_r + \mu_1$ with μ_r and μ_1 are given as in Lemma 3.4.14 and where

$$
\begin{aligned}
\underline{X} \quad = \quad & \int_{Q'_0} \bar{\mu}v^2 \quad + \quad \int_{Q'_0} \bar{\nu}(Dv)^2 \\
+ \quad & X_{12} + X_{13} + X_{23} + X_{31} + X_{32} + X_{33},
\end{aligned}
$$

with $\bar{\mu} = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and $\bar{\nu}$ of the form $s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$.

By using the Young's inequality, we estimate in turn all the terms of $\underline{Y}$, $\underline{Z}$ and the two terms at the RHS of (3.4.7) through the following Lemma whose proof can be found in Section 3.7.

Lemma 3.4.16. *For $sh \leq \mathfrak{K}$, we have*

$$
\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} \leq \epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)h(\partial_t v(a))^2 + C_\epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)h(rf_1)^2_{n+1},
$$

$$
\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_dDv)_{n+\frac{1}{2}} \leq \epsilon \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)h(\partial_t v(a))^2 + C_\epsilon \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}}.
$$

$$
\left|Y_{11}^{(2,2)}\right| \leq \int_0^T \alpha_{11}v^2_{n+1} + \int_0^T \beta_{11}h(\partial_t v)^2_{n+1} + \int_0^T \gamma_{11}(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T \eta_{11}h(rf_1)^2_{n+1},
$$

$$\alpha_{11} = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\right) \qquad \beta_{11} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3,$$
$$\gamma_{11} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 \qquad \eta_{11} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3.$$

$$|Y_{12}| \leq \int_0^T \alpha_{12} v_{n+1}^2 + \int_0^T \beta_{12} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{12}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{12} h(rf_1)_{n+1}^2,$$

$$\alpha_{12} = \left(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\right) \qquad \beta_{12} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\gamma_{12} = \mathcal{O}_{\lambda,\mathfrak{K}}(1) \qquad \eta_{12} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh).$$

$$\left|Y_{21}^{(1,22)}\right| \leq \int_0^T \alpha_{21} v_{n+1}^2 + \int_0^T \beta_{21} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{21}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{21} h(rf_1)_{n+1}^2,$$

$$\alpha_{21} = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\right) \qquad \beta_{21} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3,$$
$$\gamma_{21} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 \qquad \eta_{21} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3.$$

$$\left|Y_{22}^{(1)}\right| \leq \int_0^T \alpha_{22} v_{n+1}^2 + \int_0^T \beta_{22} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{22}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{22} h(rf_1)_{n+1}^2,$$

$$\alpha_{22} = \left(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\right) \qquad \beta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3,$$
$$\gamma_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 \qquad \eta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3.$$

$$\left|Y_{23}^{(1)}\right| \leq \int_0^T \alpha_{23} v_{n+1}^2 + \int_0^T \beta_{23} h(\partial_t v(a))^2 + \int_0^T \gamma_{23}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{22} h(rf_1)_{n+1}^2,$$

$$\alpha_{23} = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\right) \qquad \beta_{23} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\gamma_{23} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 \qquad \eta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.$$

$$\left|Y_{31}^{(1)}\right| \leq \int_0^T \alpha_{31} v_{n+1}^2 + \int_0^T \beta_{23} h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_{23}(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_{22} h(rf_1)_{n+1}^2,$$

$$\alpha_{31} = \left(s^2T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\right) \qquad \beta_{23} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),$$
$$\gamma_{23} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) \qquad \eta_{22} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh).$$

$$\mu_1 \leq \int_0^T \alpha_1 v_{n+1}^2 + \int_0^T \beta_1 h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_1 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_1 h(rf_1)_{n+1}^2,$$

$$\alpha_1 = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\right) \qquad \beta_1 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,$$
$$\gamma_1 = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) \qquad \eta_1 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.$$

$$\mu_r \leq \int_0^T \alpha_r v_{n+1}^2 + \int_0^T \beta_r h(\partial_t v)_{n+1}^2 + \int_0^T \gamma_r (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \eta_r h(rf_1)_{n+1}^2,$$

$$\alpha_r = \left(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh)^2 + \epsilon s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\right) \qquad \beta_r = \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh),$$
$$\gamma_r = \left(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\right) \qquad \eta_r = \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh).$$

Futhermore, we can estimate the term in X_{12} as follows

$$\begin{aligned}
\int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv &\leq \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\tilde{v})^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 \\
&\leq \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{|v|}^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 \\
&= \int_{Q_0} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2,
\end{aligned}$$

by Lemma 3.3.3 and as $\int_{\Omega_0'} \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{|v|}^2 = \int_{\Omega_0} \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2$.

Observe that

$$1 \leq T^2\theta \text{ and } \left|\partial_t^2\theta\right| \leq CT^2\theta^3.$$

We can now choose ϵ_4 and h_0 sufficiently small, with $0 < \epsilon_4 \leq \epsilon_3(\lambda_2)$, $0 < h_0 \leq h_1(\lambda_2)$, and $\tau_2 \geq 1$ sufficiently large, such that for $\tau \geq \tau_2(T+T^2)$, $0 < h \leq h_0$, and $\tau h(max_{[0,T]}\theta) \leq \epsilon_4$, from (3.4.7) and Lemma 3.4.16 we get

$$\begin{aligned}
& \|Av\|^2_{L^2(Q_0')} + \|Bv\|^2_{L^2(Q_0')} + \int_{Q_0'} s\,|Dv|^2 + \int_{Q_0'} s^3 v^2 \\
+ \;& C\alpha_0 \int_0^T s(c_d Dv)^2_{n+\frac{1}{2}} + C\alpha_0 \int_0^T s^3 v^2_{n+1} + C_{\lambda,\mathfrak{K}} \int_0^T h(\partial_t v(a))^2 \\
\leq \;& C_{\lambda,\mathfrak{K}} \Bigg(\|rf_1\|^2_{L^2(Q_0')} + \int_0^T \int_{\omega_0} s(Dv)^2 + \int_0^T \int_{\omega_0} s^3 v^2 \\
& + h^{-2}\Big(\int_{\Omega_0'} v^2_{|_{t=0}} + \int_{\Omega_0'} v^2_{|_{t=T}} \Big) + sv^2(a)|_{t=0}^{t=T} \Bigg) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1) h (rf_1)^2_{n+1} \\
+ \;& \int_{Q_0} s\mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2 + \int_{Q_0} s^2 T\theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2.
\end{aligned} \tag{3.4.8}$$

where we used that $(Dv)^2 \leq Ch^{-2}((\tau^+ v)^2 + (\tau^- v)^2)$ and the last three terms whose integral taken on domain Q_0 come from the term in X_{12}, X_{13} and X_{23} respectively.

As $\tau \geq \tau_2(T+T^2)$ then $s \geq \tau_2 > 0$ and furthermore we observe that

$$\begin{aligned}
\left\| s^{-\frac{1}{2}} \partial_t v \right\|^2_{L^2(Q_0')} &\leq \mathcal{C}_{\lambda,\mathfrak{K}} \Big(\left\| s^{-\frac{1}{2}} Bv \right\|^2_{L^2(Q_0')} + \left\| s^{\frac{1}{2}} v \right\|^2_{L^2(Q_0')} + \left\| s^{\frac{1}{2}} Dv \right\|^2_{L^2(Q_0')} \Big) \\
&\leq \mathcal{C}_{\lambda,\tau,\mathfrak{K}} \Big(\|Bv\|^2_{L^2(Q_0')} + \left\| s^{\frac{3}{2}} v \right\|^2_{L^2(Q_0')} + \left\| s^{\frac{1}{2}} Dv \right\|^2_{L^2(Q_0')} \Big).
\end{aligned}$$

We then add the following terms $\int_0^T hs^3 v^2_{n+1}$ and $\int_0^T hs^{-1}(\partial_t v(a))^2$ on both the right hand side and the left hand side of (3.4.8). This allows us to change the domain of integration from Q_0' to Q_0 for the discrete integrals on the primal mesh. No additional term is required for discrete integrals on the dual mesh. For sh sufficiently small and $s \geq 1$ sufficiently large, these terms at the right hand side are then absorbed by the terms at the left hand side. More precisely, with $0 < \epsilon_0 \leq \epsilon_4$ sufficiently small and for $\tau \geq \tau_2(T+T^2)$, $0 < h \leq h_0$, and $0 < \tau h(max_{[0,T]}\theta) \leq \epsilon_0$ we thus obtain

$$
\begin{aligned}
&\left\|s^{-\frac{1}{2}}\partial_t v\right\|^2_{L^2(Q_0)} \quad + \quad \int_{Q_0} s(Dv)^2 \quad + \quad \int_{Q_0} s^3 v^2 \\
\leq\ & C_{\lambda,\mathfrak{K}}\left(\|rf_1\|^2_{L^2(Q_0)} + \int_0^T\int_{\omega_0} s(Dv)^2 + \int_0^T\int_{\omega_0} s^3 v^2\right. \\
&\left. +h^{-2}\Big(\int_{\Omega_0} v^2_{|t=0} + \int_{\Omega_0} v^2_{|t=T}\Big) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T}\right). \qquad (3.4.9)
\end{aligned}
$$

Now we shall estimate the term $s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=T}$. We have

$$
\left\|v_{|t=T}\right\|^2_{L^2(\Omega_0)} = \sum_{j=1}^{n+m+1} h v^2_{j|t=T} \geq h\left\|v_{|t=T}\right\|^2_{L^\infty(\Omega_0)}.
$$

It follows that, as sh is bounded

$$
\begin{aligned}
\left|s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=T}\right| \leq C_{\lambda,\mathfrak{K}}\ s\left\|v_{|t=T}\right\|^2_{L^\infty(\Omega_0)} &\leq C_{\lambda,\mathfrak{K}}\ sh^{-1}\left\|v_{|t=T}\right\|^2_{L^2(\Omega_0)} \\
&\leq C_{\lambda,\mathfrak{K}}\ h^{-2}\left\|v_{|t=T}\right\|^2_{L^2(\Omega_0)}.
\end{aligned}
$$

Similarly, we treat the term $s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=0}$ as

$$
\left|s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)_{|t=0}\right| \leq C_{\lambda,\mathfrak{K}}\ h^{-2}\left\|v_{|t=0}\right\|^2_{L^2(\Omega_0)}.
$$

Therefore, (3.4.9) can be written as

$$
\begin{aligned}
&\left\|s^{-\frac{1}{2}}\partial_t v\right\|^2_{L^2(Q_0)} \quad + \quad \left\|s^{\frac{1}{2}}Dv\right\|^2_{L^2(Q_0)} + \quad \left\|s^{\frac{3}{2}}v\right\|^2_{L^2(Q_0)} \\
\leq\ & C_{\lambda,\mathfrak{K}}\left(\|rf_1\|^2_{L^2(Q_0)} + \left\|s^{\frac{1}{2}}Dv\right\|^2_{L^2\left((0,T)\times\omega_0\right)} + \left\|s^{\frac{3}{2}}v\right\|^2_{L^2\left((0,T)\times\omega_0\right)}\right. \\
&\left. +h^{-2}\Big(\int_{\Omega_0} v^2_{|t=0} + \int_{\Omega_0} v^2_{|t=T}\Big)\right).
\end{aligned}
$$

We next remove the volume norm $\left\|s^{\frac{1}{2}}Dv\right\|^2_{L^2((0,T)\times\omega_0)}$ by proceeding as in the proof of The-

orem 4.1 in [BHL10a] we thus write

$$\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q_0)}+\tau\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2(Q_0)}+\tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2(Q_0)}$$
$$\leq C_{\lambda,\mathfrak{K}}\left(\left\|e^{\tau\theta\varphi}P^{\mathfrak{M}}u\right\|^2_{L^2(Q_0)}+\tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}u\right\|^2_{L^2((0,T)\times\omega_0)}\right.$$
$$\left.+h^{-2}\left|e^{\tau\theta\varphi}u|_{t=0}\right|^2_{L^2(\Omega_0)}+h^{-2}\left|e^{\tau\theta\varphi}u|_{t=T}\right|^2_{L^2(\Omega_0)}\right), \tag{3.4.10}$$

As we have $\max_{[0,T]}\theta\leq\frac{1}{T\alpha}$, we see that a sufficient condition for $\tau h\big(\max_{[0,T]}\theta\big)\leq\epsilon_0$ then becomes $\tau h(T\alpha)^{-1}\leq\epsilon_0$. To finish the proof, we need to express all the terms in the estimate above in terms of the original function u. We can proceed exactly as in the end of proof of Theorem 4.1 in [BHL10a].

3.5 Carleman estimates for regular non uniform meshes

In this section we focus on extending the above result to the class of non piecewise uniform meshes introduced in Section 3.1.2. We choose a function ϑ satisfying (3.1.6) and further $\vartheta|_{[a-\delta,a+\delta]}$ is chosen affine (for some $\delta>0$ to remain fixed in the sequel). The way we proceed here is similar to what is done in [BHL10a]. In this framework, we shall prove a non-uniform Carleman estimate for the parabolic operator $\mathcal{P}^{\mathfrak{M}}=-\partial_t+\mathcal{A}^{\mathfrak{M}}$ on the mesh $\mathfrak{M}$ by using the result on uniform meshes of Section 3.4.

By using first-order Taylor formulae we obtain the following result.

Lemma 3.5.1. *Let us define $\zeta\in\mathbb{R}^{\overline{\mathfrak{M}}}$ and $\overline{\zeta}\in\mathbb{R}^{\mathfrak{M}}$ as follows*

$$\zeta_{i+\frac{1}{2}}=\frac{h'_{i+\frac{1}{2}}}{h},i\in\{0,\dots,n+m+1\}\qquad\overline{\zeta_i}=\frac{h'_i}{h},i\in\{1,\dots,n+m+1\}$$

These two discrete functions are connected to the geometry of the primal and dual meshes $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ and we have

$$0<\inf_{\Omega_0}\vartheta'\leq\zeta_{i+\frac{1}{2}}\leq\sup_{\Omega_0}\vartheta',\qquad\forall i\in 0,\dots,n+m+1$$
$$0<\inf_{\Omega_0}\vartheta'\leq\overline{\zeta}_i\leq\sup_{\Omega_0}\vartheta',\qquad\forall i\in 1,\dots,n+m+1$$

$$\left|\bar{D}\zeta\right|_{L_\infty(\Omega)}\leq\frac{\|\vartheta''\|_{L_\infty}}{\inf_{\Omega_0}\vartheta'},\qquad\left|D\bar{\zeta}\right|_{L_\infty(\Omega)}\leq\frac{\|\vartheta''\|_{L_\infty}}{\inf_{\Omega_0}\vartheta'}.$$

We introduce some notation. To any $u\in\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$, we associate the discrete function denoted by $\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u\in\mathbb{C}^{\mathfrak{M}_0\cup\partial\mathfrak{M}_0}$ defined on the uniform mesh $\mathfrak{M}_0$ which takes the same values as u at the

corresponding nodes. More precisely, if $u = \sum_{i=1}^{n+m+1} \mathbf{1}_{[x'_{i-\frac{1}{2}}, x'_{i+\frac{1}{2}}]} u_i$, we let

$$\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u = \sum_{i=1}^{n+m+1} \mathbf{1}_{[(i-\frac{1}{2})h, (i+\frac{1}{2})h]} u_i$$

and $(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u)_0 = u_0$, $(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u)_{n+m+2} = u_{n+m+2}$. Similarly, for $u \in \mathbb{C}^{\overline{\mathfrak{M}}}$, $u = \sum_{i=1}^{n+m+1} \mathbf{1}_{[x'_i, x'_{i+1}]} u_{i+\frac{1}{2}}$, we set

$$\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}} u = \sum_{i=0}^{n+m+1} \mathbf{1}_{[ih, (i+1)h]} u_{i+\frac{1}{2}}.$$

The operators $\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}$ and $\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}}$ are invertible and we denote by $\mathcal{Q}_{\mathfrak{M}_0}^{\mathfrak{M}}$ and $\mathcal{Q}_{\overline{\mathfrak{M}_0}}^{\overline{\mathfrak{M}}}$ their respective inverses. We give commutation properties between these operators and discrete-difference operators through the following Lemmata whose proofs can be found in [BHL10a].

Lemma 3.5.2. *[BHL10a, see the proof of Lemma 5.2]*

1. *For any $u \in \mathbb{C}^{\mathfrak{M} \cup \partial\mathfrak{M}}$ and any $v \in \mathbb{C}^{\overline{\mathfrak{M}}}$, we have*

$$D(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u) = \mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}}(\zeta D u), \qquad \bar{D}\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}} v = \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}(\bar{\zeta}\ \bar{D} v)$$

2. *For any $u \in \mathbb{C}^{\mathfrak{M} \cup \partial\mathfrak{M}}$ we have*

$$\bar{D}(c_d D u) = (\bar{\zeta})^{-1} \mathcal{Q}_{\mathfrak{M}_0}^{\mathfrak{M}} \Big(\bar{D}\big((\mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}} \frac{c_d}{\zeta}) D(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u) \big) \Big).$$

Lemma 3.5.3. *[BHL10a, see proof of Lemma 5.3]*

For any $u \in \mathbb{C}^{\mathfrak{M}}$ and any $v \in \mathbb{C}^{\overline{\mathfrak{M}}}$, we have

$$(\sup_{\Omega_0} \vartheta')^{-1} |u|^2_{L^2(\Omega)} \leq \left| \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} u \right|^2_{L^2(\Omega_0)} \leq (\inf_{\Omega_0} \vartheta')^{-1} |u|^2_{L^2(\Omega)}$$
$$(\sup_{\Omega_0} \vartheta')^{-1} |v|^2_{L^2(\Omega)} \leq \left| \mathcal{Q}_{\overline{\mathfrak{M}}}^{\overline{\mathfrak{M}_0}} u \right|^2_{L^2(\Omega_0)} \leq (\inf_{\Omega_0} \vartheta')^{-1} |v|^2_{L^2(\Omega)}$$

Futhermore, the same inequalities hold by replacing Ω by ω and Ω_0 by ω_0, respectively.

For any continuous function f defined on $\overline{\Omega}$ (resp. on $\overline{\Omega_0}$) we denote by $\Pi_{\mathfrak{M}} f = (f(x'_i))_{0 \leq i \leq n+m+2} \in \mathbb{C}^{\mathfrak{M} \cup \partial\mathfrak{M}}$ the sampling of f on $\mathfrak{M}$ (resp. $\Pi_{\mathfrak{M}_0} f = (f(ih))_{0 \leq i \leq n+m+2} \in \mathbb{C}^{\mathfrak{M}_0 \cup \partial\mathfrak{M}_0}$ the sampling of f on $\mathfrak{M}_0$).

Lemma 3.5.4. *[BHL10a, see the proof of Lemma 5.4]*

Let f be a continuous function defined on Ω

$$\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\Pi_{\mathfrak{M}}f=\Pi_{\mathfrak{M}_0}(f\circ\vartheta).$$

In particular, for $u\in\mathbb{C}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ we have

$$\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\Big((\Pi_{\mathfrak{M}}f)u\Big)=\Pi_{\mathfrak{M}_0}(f\circ\vartheta)(\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}u).$$

Moreover, by making use of Taylor formulae we get the following result

Lemma 3.5.5. *With ζ defined as in Lemma 3.5.1 we have*

$$\left\|\bar{D}D\nu\right\|_{\infty}<\infty,\qquad \left\|\bar{D}\tilde{\nu}\right\|_{\infty}<\infty,\qquad 0<\|\nu\|_{\infty},\|\tilde{\nu}\|_{\infty}<\infty$$

where $\nu:=\frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\bar{\zeta}}$.

Proof. From the definition of ζ, $\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}$ and D acting on $\mathbb{C}^{\mathfrak{M}_0}$, $\bar{D}$ acting on $\mathbb{C}^{\overline{\mathfrak{M}_0}}$ we have

$$\begin{aligned}\Big(\bar{D}D(\frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\bar{\zeta}})\Big)_i &:= (\bar{D}D\nu)_i\\ &= \frac{\nu_{i+1}-2\nu_i+\nu_{i-1}}{h^2}\\ &= \frac{1}{h}\frac{(h'_i-h'_{i+1})h'_{i-1}-(h'_{i-1}-h'_i)h'_{i+1}}{h'_{i-1}h'_ih'_{i+1}}.\end{aligned}\tag{3.5.1}$$

We find

$$\begin{aligned}h'_i&=x'_{i+\frac{1}{2}}-x'_{i-\frac{1}{2}}=\frac{x'_{i+1}-x'_{i-1}}{2}=\frac{\vartheta\big((i+1)h\big)-\vartheta\big((i-1)h\big)}{2}=\frac{\vartheta_{i+1}-\vartheta_{i-1}}{2},\\ h'_{i+1}&=\frac{\vartheta\big((i+2)h\big)-\vartheta\big(ih\big)}{2}=\frac{\vartheta_{i+2}-\vartheta_i}{2},\\ h'_{i-1}&=\frac{\vartheta(ih)-\vartheta\big((i-2)h\big)}{2}=\frac{\vartheta_i-\vartheta_{i-2}}{2}.\end{aligned}$$

By using Taylor formulae we write

$$\begin{aligned}\vartheta_{i+2}&=\vartheta_i+(2h)\vartheta'_i+\frac{(2h)^2}{2}\vartheta''_i+\frac{(2h)^3}{6}\vartheta'''_i+\frac{(2h)^4}{24}\vartheta^{(4)}_i+\mathcal{O}(h^5),\\ \vartheta_{i-2}&=\vartheta_i-(2h)\vartheta'_i+\frac{(2h)^2}{2}\vartheta''_i-\frac{(2h)^3}{6}\vartheta'''_i+\frac{(2h)^4}{24}\vartheta^{(4)}_i+\mathcal{O}(h^5),\\ \vartheta_{i+1}&=\vartheta_i+h\vartheta'_i+\frac{h^2}{2}\vartheta''_i+\frac{h^3}{6}\vartheta'''_i+\frac{(h)^4}{24}\vartheta^{(4)}_i+\mathcal{O}(h^5),\\ \vartheta_{i-1}&=\vartheta_i-h\vartheta'_i+\frac{h^2}{2}\vartheta''_i-\frac{h^3}{6}\vartheta'''_i+\frac{(h)^4}{24}\vartheta^{(4)}_i+\mathcal{O}(h^5).\end{aligned}$$

Thus we have

$$\begin{aligned} h_i' &= 2h\vartheta_i' + \frac{2h^3}{6}\vartheta_i''' + \mathcal{O}(h^5), \\ h_{i+1}' &= 2h\vartheta_i' + \frac{(2h)^2}{2}\vartheta_i'' + \frac{(2h)^3}{6}\vartheta''' + \frac{(2h)^4}{24}\vartheta^{(4)} + \mathcal{O}(h^5), \\ h_{i-1}' &= 2h\vartheta_i' - \frac{(2h)^2}{2}\vartheta_i'' + \frac{(2h)^3}{6}\vartheta''' - \frac{(2h)^4}{24}\vartheta^{(4)} + \mathcal{O}(h^5). \end{aligned}$$

From (3.5.1) we obtain

$$\bar{D}D(\frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\bar{\zeta}})_i = \frac{N}{D},$$

where

$$\begin{aligned} N &= (h_i' - h_{i+1}')h_{i-1}' - (h_{i-1}' - h_i')h_{i+1}' \\ &= \Big(-\frac{(2h)^2}{2}\vartheta_i'' - h^3\vartheta_i''' - \frac{(2h)^4}{24}\vartheta_i^{(4)} + \mathcal{O}(h^5)\Big)\big((2h)\vartheta_i' - \frac{(2h)^2}{2}\vartheta_i'' + \mathcal{O}(h^3)\big) \\ &- \Big(-\frac{(2h)^2}{2}\vartheta_i'' - h^3\vartheta_i''' - \frac{(2h)^4}{24}\vartheta_i^{(4)} + \mathcal{O}(h^5)\Big)\big((2h)\vartheta_i' + \frac{(2h)^2}{2}\vartheta_i'' + \mathcal{O}(h^3)\big) \\ &= \frac{(2h)^4}{2}(\vartheta_i'')^2 + \mathcal{O}(h^5), \end{aligned}$$

and

$$D = h \times h_{i-1}' \times h_i' \times h_{i+1}' = (2h)^4(\vartheta_i')^3 + \mathcal{O}(h^5).$$

Thus, we have

$$\left|\bar{D}D(\frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0}\bar{\zeta}})_i\right| \lesssim (\inf\vartheta')^{-3} < \infty,$$

which proves the first result. Next, we proceed with the second result in the same manner as above. We have

$$(\bar{D}\tilde{\nu})_i = \frac{\tilde{\nu}_{i+\frac{1}{2}} - \tilde{\nu}_{i+\frac{1}{2}}}{h} = \frac{\nu_{i+1} - \nu_{i-1}}{2h} = \frac{1}{2h}\big(\frac{h}{h_{i+1}'} - \frac{h}{h_{i-1}'}\big) = \frac{h_{i-1}' - h_{i+1}'}{h_{i+1}'h_{i-1}'}.$$

By using the computations of h_{i-1}', h_{i+1}' above we find

$$\big|(\bar{D}\tilde{\nu})_i\big| = \left|\frac{-(2h)^2\vartheta'' + \mathcal{O}(h^4)}{(2h)^2\vartheta'^2 + \mathcal{O}(h^3)}\right| \lesssim \frac{\|\vartheta''\|_\infty}{(\inf\vartheta')^2} < \infty,$$

which yields the second result.

Moreover, with the properties of ζ shown as in Lemma 3.5.1 we can assert

$$0 < \|\nu\|_\infty, \|\tilde{\nu}\|_\infty < \infty.$$

□

From Lemmata 3.5.2 – 3.5.4 we thus obtain the following discrete Carleman estimate for the operator $P^{\mathfrak{M}} = -\partial_t - D(c_d D.)$ on the mesh $\mathfrak{M}$.

Theorem 3.5.6. *Let $\omega \Subset \Omega_2$ be a non-empty open set and we set $f := \bar{D}(c_d Du)$. For the parameter $\lambda > 1$ sufficiently large, there exists C, $\tau_0 \geq 1$, $h_0 > 0$, $\epsilon_0 > 0$, depending on ω such that for any mesh $\mathfrak{M}$ obtained from ϑ by (3.1.6) – (3.1.7), we have*

$$\begin{aligned}
&\tau^{-1} \left\| \theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u \right\|^2_{L^2(Q)} + \tau \left\| \theta^{\frac{1}{2}} e^{\tau\theta\varphi} Du \right\|^2_{L^2(Q)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2(Q)} \\
&\leq C_{\lambda,\mathfrak{K}} \Big(\left\| e^{\tau\theta\varphi} P^{\mathfrak{M}} u \right\|^2_{L^2(Q)} + \tau^3 \left\| \theta^{\frac{3}{2}} e^{\tau\theta\varphi} u \right\|^2_{L^2((0,T)\times\omega)} \\
&\qquad + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=0} \right|^2_{L^2(\Omega)} + h^{-2} \left| e^{\tau\theta\varphi} u|_{t=T} \right|^2_{L^2(\Omega)} \Big), \qquad (3.5.2)
\end{aligned}$$

for all $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$ and $\tau h(\alpha T)^{-1} \leq \epsilon_0$ and for all $u \in C^\infty(0, T; \mathbb{C}^{\mathfrak{M}})$ satisfying $u|_{\partial\Omega} = 0$.

Proof. We set $w = \mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} u$ defined on the uniform mesh $\mathfrak{M}_0$. By using Lemma 3.5.2 we have

$$\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\bar{\zeta} P^{\mathfrak{M}} u) = -(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} \bar{\zeta}) \partial_t w - \bar{D}\Big((\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}} \frac{c_d}{\zeta}) Dw \Big). \qquad (3.5.3)$$

We observe that the right-hand side of (3.5.3) is a semi-discrete parabolic operator of the form $P^{\mathfrak{M}_0} = \xi'(-\partial_t - \frac{1}{\xi'} D(\xi_d D.))$ applied to w, where

$$\xi' = \mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} \bar{\zeta}, \quad \xi_d = \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}} \frac{c_d}{\zeta}. \qquad (3.5.4)$$

We set $\nu := \frac{1}{\xi'} = \frac{1}{\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}} \bar{\zeta}}$ and we find

$$\bar{\tilde{\nu}} = \nu + h^2 \bar{D} D\nu = \nu + h^2 \mathcal{O}(1),$$

by using Lemma 3.3.3 and Lemma 3.5.5.

Thus, the operator $P^{\mathfrak{M}_0}$ can be written in form as

$$P^{\mathfrak{M}_0} w = \xi'\big(- \partial_t w - \bar{\tilde{\nu}} \bar{D}(\xi_d Dw) + h^2 \mathcal{O}(1) \bar{D}(\xi_d Dw) \big).$$

Moreover, using Lemma 3.3.2 we have

$$\bar{\tilde{\nu}} \bar{D}(\xi_d Dw) = \bar{D}(\tilde{\nu} \xi_d Dw) - \bar{D}(\tilde{\nu}) \overline{\xi_d Dw}.$$

We set $P_0^{\mathfrak{M}_0} w := -\partial_t w - \bar{D}(\tilde{\nu}\xi_d Dw) = -\partial_t w - \bar{D}(b_d Dw)$ with $b_d = \tilde{\nu}\xi_d$. From the properties of $\tilde{\nu}$ and ξ_d it follows that

$$0 < b_{min} \leq b \leq b_{max} \quad \text{and} \quad \left\|\bar{D}(b_d)\right\|_\infty < \infty.$$

First, we shall obtain a Carleman estimate for $P_0^{\mathfrak{M}_0}$. Then we shall deduce a Carleman estimate for the operator

$$P^{\mathfrak{M}_0} w = \xi' \big(P_0^{\mathfrak{M}_0} w + \bar{D}(\tilde{\nu})\overline{\xi_d Dw} + h^2 \mathcal{O}(1)\bar{D}(\xi_d Dw) \big) \tag{3.5.5}$$

Now, we consider the function $\psi \circ \vartheta : (t,x) \mapsto \psi\big(t, \vartheta(x)\big)$. By using the properties listed in Lemma 3.2.1 and (3.1.6), we shall see that $\psi \circ \vartheta$ is a suitable weight function associated to the control domain $\omega_0 = \vartheta^{-1}(w)$ in Ω_0, i.e., that $\psi \circ \vartheta$ satisfies Lemma 3.2.1 for the domaims Ω_0 and ω_0.

The important property to checking is the trace property. The remaining properties are left to the reader. We set

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

with

$$\begin{aligned}
b_{11} &= [(\psi \circ \vartheta)' \star]_a \\
b_{22} &= [b(\psi \circ \vartheta)' \star]_a^2 (\psi \circ \vartheta)'(a^+) + [b^2 (\psi \circ \vartheta)'^3 \star]_a \\
b_{12} = b_{21} &= [b(\psi \circ \vartheta)' \star]_a (\psi \circ \vartheta)'(a^+)
\end{aligned}$$

where $b = \frac{1}{\mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} \bar{\zeta}} \mathcal{Q}_{\mathfrak{M}}^{\mathfrak{M}_0} \frac{c}{\bar{\zeta}} = \frac{c \circ \vartheta}{\bar{\zeta}^2}$. Morever, we have $\vartheta'_+(a) = \vartheta'_-(a)$ and $\bar{\zeta}_{n+1} = \vartheta'_{n+1}$ (recall that $\vartheta|_{[a-\delta, a+\delta]}$ is an affine function). It follows that

$$\begin{aligned}
b_{11} &= [\psi'(\vartheta)\star]_a \vartheta'(a) = [\psi' \star]_{a'} \vartheta'(a), \\
b_{22} &= [\frac{c \circ \vartheta}{\bar{\zeta}^2} \psi'(\vartheta)\vartheta' \star]_a^2 \psi'(\vartheta)(a^+)\vartheta'(a^+) + [(\frac{c \circ \vartheta}{\bar{\zeta}^2})^2 \big(\psi'(\vartheta)\vartheta'\big)^3 \star]_a \\
&= [c\psi' \star]_{a'} \psi'(a^+) \frac{1}{\vartheta'(a)} + [c^2 (\psi')^3 \star]_{a'} \frac{1}{\vartheta'(a)}, \\
b_{12} = b_{21} &= [\frac{c \circ \vartheta}{\bar{\zeta}^2} \psi'(\vartheta)\vartheta' \star]_a \psi'(\vartheta(a^+))\vartheta'(a^+) \\
&= [c\psi' \star]_{a'} \psi'(a'^+).
\end{aligned}$$

We can see that $(Bw, w) = (Aw, w) \geq \alpha_0 \|w\|^2$. This means that $\psi \circ \vartheta$ satisfies the trace

property.

Through Theorem 3.4.1, we obtained a discrete uniform Carleman estimate for $P_0^{\mathfrak{M}_0}$ and the Carleman weight function is of the form $r_0 = e^{s\varphi_0}$, with $\varphi_0 = \varphi \circ \vartheta = e^{\lambda\psi_0} - e^{\lambda K}$ where $\psi_0 = \psi \circ \vartheta$ on the uniform mesh $\mathfrak{M}_0$. We can deduce the same result on the non-uniform mesh $\mathfrak{M}$. Namely, through (3.4.2) we see that the following estimate holds

$$\begin{aligned}
&\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi_0}\partial_t w\right\|^2_{L^2(Q_0)} + \tau\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}Dw\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2(Q_0)}\\
&\leq C\left(\left\|e^{\tau\theta\varphi_0}P_0^{\mathfrak{M}_0}w\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2((0,T)\times\omega_0)}\right.\\
&\qquad\left. + h^{-2}\left|e^{\tau\theta\varphi_0}w|_{t=0}\right|^2_{L^2(\Omega_0)} + h^{-2}\left|e^{\tau\theta\varphi_0}w|_{t=T}\right|^2_{L^2(\Omega_0)}\right),
\end{aligned} \tag{3.5.6}$$

and the constant C is uniform in h for τ sufficiently large and with $\tau h(\alpha T)^{-1} \leq \epsilon_0$, for ϵ_0 sufficiently smaill. Note that, setting $\tilde{\epsilon}_0 = (\inf_{\Omega_0}\vartheta')\epsilon_0$, we see that the condition $\tau h'(\alpha T)^{-1} \leq \tilde{\epsilon}_0$ on the size of the non-uniform mesh $\mathfrak{M}$ implies the condition $\tau h(\alpha T)^{-1} \leq \epsilon_0$ for the uniform mesh $\mathfrak{M}_0$.

From (3.5.5) – (3.5.6) we deduce the following Carleman estimate for $P^{\mathfrak{M}_0}$

$$\begin{aligned}
&\tau^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi_0}\partial_t w\right\|^2_{L^2(Q_0)} + \tau\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}Dw\right\|^2_{L^2(Q_0)} + \tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2(Q_0)}\\
&\leq C\left(\left\|e^{\tau\theta\varphi_0}P^{\mathfrak{M}_0}w\right\|^2_{L^2(Q_0)} + \left\|e^{\tau\theta\varphi_0}\bar{D}(\tilde{\nu})\overline{\xi_d Dw}\right\|^2_{L^2(Q_0)} + h^4\left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)}\right.\\
&\left.+\tau^3\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2((0,T)\times\omega_0)} + h^{-2}\left|e^{\tau\theta\varphi_0}w|_{t=0}\right|^2_{L^2(\Omega_0)} + h^{-2}\left|e^{\tau\theta\varphi_0}w|_{t=T}\right|^2_{L^2(\Omega_0)}\right).
\end{aligned} \tag{3.5.7}$$

Now, by using Lemma 3.5.5 we estimate $\left\|e^{\tau\theta\varphi_0}\bar{D}(\tilde{\nu})\overline{\xi_d Dw}\right\|^2_{L^2(Q_0)}$ in the RHS of (3.5.7) as

$$\left\|e^{\tau\theta\varphi_0}\bar{D}(\tilde{\nu})\overline{\xi_d Dw}\right\|^2_{L^2(Q_0)} \leq C\left\|e^{s\varphi_0}\overline{\xi_d Dw}\right\|^2_{L^2(Q_0)}.$$

We see that

$$\overline{\xi_d Dw} = \frac{1}{2}\Big(\bar{\tau}_+(\xi_d Dw) + \bar{\tau}_-(\xi_d Dw)\Big).$$

Hence we find

$$
\begin{aligned}
&\big\|e^{s\varphi_0}\overline{\xi_d Dw}\big\|^2_{L^2(Q_0)}\\
\leq\ & C\Big(\|e^{s\varphi_0}\bar{\tau}_+(\xi_d Dw)\|^2_{L^2(Q_0)} + \|e^{s\varphi_0}\bar{\tau}_-(\xi_d Dw)\|^2_{L^2(Q_0)}\Big)\\
\leq\ & C\Big(\Big\|e^{s\varphi_0}\bar{\tau}_+\big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\big)\Big\|^2_{L^2(Q_0)} + \Big\|e^{s\varphi_0}\bar{\tau}_-\big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\big)\Big\|^2_{L^2(Q_0)}\Big)\\
\leq\ & C\Big(\Big\|e^{s\varphi_0}\bar{\tau}_+\big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\big)\Big\|^2_{L^2(Q_0)} + \Big\|e^{s\varphi_0}\bar{\tau}_-\big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\big)\Big\|^2_{L^2(Q_0)}\Big)\\
\leq\ & C\Big(\Big\|e^{s\varphi_0}\bar{\tau}_+\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(c_d Du)\Big\|^2_{L^2(Q_0)} + \Big\|e^{s\varphi_0}\bar{\tau}_-\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(c_d Du)\Big\|^2_{L^2(Q_0)}\Big)\\
\leq\ & C\Big(\Big\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\Big(e^{s\varphi}\bar{\tau}_+(c_d Du)\Big)\Big\|^2_{L^2(Q_0)} + \Big\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\Big(e^{s\varphi}\bar{\tau}_-(c_d Du)\Big)\Big\|^2_{L^2(Q_0)}\Big)\\
\leq\ & C(\inf\vartheta')^{-1}\Big(\|e^{s\varphi}\bar{\tau}_+(c_d Du)\|^2_{L^2(Q)} + \|e^{s\varphi}\bar{\tau}_-(c_d Du)\|^2_{L^2(Q)}\Big),
\end{aligned}
$$

by using (3.5.4) and Lemmata 3.5.2 – 3.5.4.

We treat $\|e^{s\varphi}\bar{\tau}_+(c_d Du)\|_{L^2(Q)}$ (the term $\|e^{s\varphi}\bar{\tau}_-(c_d Du)\|_{L^2(Q)}$ can be treated similarly). We find

$$
\|e^{s\varphi}\bar{\tau}_+(c_d Du)\|_{L^2(Q)} = \|r\bar{\tau}_+(c_d Du)\|_{L^2(Q)} \leq \|(\tau_- r)(c_d Du)\|_{L^2(Q)} \leq C\,\|(\tau_- r)Du\|_{L^2(Q)}\,. \quad (3.5.8)
$$

We have $\tau_- r = r(\rho\tau_- r) = r\big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$ (due to Proposition 3.3.10). From that we can write

$$
\big\|e^{\tau\theta\varphi_0}\bar{D}(\tilde{\nu})\overline{\xi_d Dw}\big\|^2_{L^2(Q_0)} \leq C(\inf\vartheta')^{-1}\,\|e^{s\varphi}Du\|_{L^2(Q)}\,,
$$

which allows one to absorb by the term at the LHS of the Carleman estimate by choosing τ sufficiently large.

Next, we estimate $h^4 \left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)}$ in the RHS of (3.5.7) as

$$\begin{aligned}
& h^4 \left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)} \\
= \;& h^2 \left\|e^{s\varphi_0}\bar{\tau}_+(\xi_d Dw) - e^{s\varphi_0}\bar{\tau}_-(\xi_d Dw)\right\|^2_{L^2(Q_0)} \\
\leq \;& Ch^2\Big(\|e^{s\varphi_0}\bar{\tau}_+(\xi_d Dw)\|^2_{L^2(Q_0)} + \|e^{s\varphi_0}\bar{\tau}_-(\xi_d Dw)\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|e^{s\varphi_0}\bar{\tau}_+\big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\big)\right\|^2_{L^2(Q_0)} + \left\|e^{s\varphi_0}\bar{\tau}_-\big(\xi_d D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\big)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|e^{s\varphi_0}\bar{\tau}_+\big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\big)\right\|^2_{L^2(Q_0)} + \left\|e^{s\varphi_0}\bar{\tau}_-\big(\xi_d \mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\big)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|e^{s\varphi_0}\bar{\tau}_+\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(c_d Du)\right\|^2_{L^2(Q_0)} + \left\|e^{s\varphi_0}\bar{\tau}_-\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(c_d Du)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2\Big(\left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\left(e^{s\varphi}\bar{\tau}_+(c_d Du)\right)\right\|^2_{L^2(Q_0)} + \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}\left(e^{s\varphi}\bar{\tau}_-(c_d Du)\right)\right\|^2_{L^2(Q_0)}\Big) \\
\leq \;& Ch^2(\inf\vartheta')^{-1}\Big(\|e^{s\varphi}\bar{\tau}_+(c_d Du)\|^2_{L^2(Q)} + \|e^{s\varphi}\bar{\tau}_-(c_d Du)\|^2_{L^2(Q)}\Big),
\end{aligned}$$

by using (3.5.4) and Lemmata 3.5.2 – 3.5.4. We proceed with an estimate as in (3.5.8). We thus obtain

$$h^4 \left\|e^{\tau\theta\varphi_0}\bar{D}(\xi_d Dw)\right\|^2_{L^2(Q_0)} \leq Ch^2(\inf\vartheta')^{-1}\left\|e^{s\varphi}Du\right\|_{L^2(Q)},$$

which allows one to absorb by the term in the LHS of Carleman estimate by choosing τ sufficiently large.

Futhermore, by using the previous Lemmata 3.5.1 – 3.5.4 and considering each term in (3.5.7) separately, we see that we have the following estimates

- For the first term in LHS of (3.5.7)

$$\begin{aligned}
\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi_0}\partial_t w\right\|^2_{L^2(Q_0)} &= \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u)\right\|^2_{L^2(Q_0)} \\
&\geq (\sup_{\Omega_0}\vartheta')^{-1}\left\|\theta^{-\frac{1}{2}}e^{\tau\theta\varphi}\partial_t u\right\|^2_{L^2(Q)},
\end{aligned}$$

  and a similar inequality holds for $\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2(Q_0)}$.

- For the second term of LHS of (3.5.7) we use Lemma 3.5.2 and Lemma 3.5.3 as follows

$$\begin{aligned}
\left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}Dw\right\|^2_{L^2(Q_0)} &= \left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi_0}D(\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}u)\right\|^2_{L^2(Q_0)} = \left\|\theta^{\frac{1}{2}}\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(e^{\tau\theta\varphi})\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta Du)\right\|^2_{L^2(Q_0)} \\
&= \left\|\theta^{\frac{1}{2}}\mathcal{Q}^{\overline{\mathfrak{M}_0}}_{\overline{\mathfrak{M}}}(\zeta e^{\tau\theta\varphi}Du)\right\|^2_{L^2(Q_0)} \gtrsim \left\|\theta^{\frac{1}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2(Q)}.
\end{aligned}$$

- By using (3.5.3) and Lemma 3.5.3 we have

$$\begin{aligned}\left\|e^{\tau\theta\varphi_0}P^{\mathfrak{M}_0}w\right\|^2_{L^2(Q_0)} &= \left\|e^{\tau\theta\varphi_0}\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\bar{\zeta}P^{\mathfrak{M}}u)\right\|^2_{L^2(Q_0)} = \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(e^{\tau\theta\varphi}\bar{\zeta}P^{\mathfrak{M}}u)\right\|^2_{L^2(Q_0)}\\ &\lesssim \left\|e^{\tau\theta\varphi}\bar{\zeta}P^{\mathfrak{M}}u\right\|^2_{L^2(Q)} \lesssim \left\|e^{\tau\theta\varphi}P^{\mathfrak{M}}u\right\|^2_{L^2(Q)}.\end{aligned}$$

- For the third term of RHS of (3.5.7)

$$\begin{aligned}\left|e^{\tau\theta\varphi_0}w|_{t=0}\right|^2_{L^2(\Omega_0)} &= \left|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(e^{\tau\theta\varphi}w|_{t=0})\right|^2_{L^2(\Omega_0)}\\ &\lesssim \left|e^{\tau\theta\varphi}w|_{t=0}\right|^2_{L^2(\Omega)}\end{aligned}$$

  and a similar inequality holds for $\left|e^{\tau\theta\varphi_0}w|_{t=T}\right|^2_{L^2(\Omega_0)}$, $\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}w\right\|^2_{L^2((0,T)\times\omega_0)}$.

- Finally, since $\vartheta(\omega_0)=\omega$ we have

$$\begin{aligned}\left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi_0}Du\right\|^2_{L^2((0,T)\times\omega_0)} &= \left\|\mathcal{Q}^{\mathfrak{M}_0}_{\mathfrak{M}}(\theta^{\frac{3}{2}}e^{\tau\theta\varphi}Du)\right\|^2_{L^2((0,T)\times\omega_0)}\\ &\lesssim \left\|\theta^{\frac{3}{2}}e^{\tau\theta\varphi}Du\right\|^2_{L^2((0,T)\times\omega}\end{aligned}$$

The proof is complete. □

3.6 Controllability results

The Carleman estimate proved in the previous Section allows to give observability estimate that yields results of controllability to the trajectories for classes of semi-linear heat equations.

3.6.1 The linear case

We consider the following semi-discrete parabolic problem with potential

$$\partial_t y + \mathcal{A}^{\mathfrak{M}}y + ay = \mathbf{1}_\omega v, \quad t\in(0,T) \qquad y|_{\partial\Omega}=0 \tag{3.6.1}$$

The adjoint system associated with the controlled system with potential (3.6.1) is given by

$$-\partial_t q + \mathcal{A}^{\mathfrak{M}}y + ay = 0, \quad t\in(0,T) \qquad q|_{\partial\Omega}=0 \tag{3.6.2}$$

We assume that a piecewise C^1 diffusion coefficient c satisfies (3.1.2) and $\Omega=(0,1)$. From Carleman estimate (3.4.2) we obtain a following observability estimate.

Proposition 3.6.1. *There exists positive constants C_0, C_1 and C_2 such that for all $T > 0$ and all potential fucntion a, under the condition $h \leq \min(h_0, h_1)$ with*

$$h_1 = C_0\big(1 + \frac{1}{T} + \|a\|_\infty^{\frac{2}{3}}\big)^{-1}$$

any solution of (3.6.2) satisfies

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} \|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C_1}{h}+T\|a\|_\infty} |q(T)|^2_{L^2(\Omega)}, \tag{3.6.3}$$

with $C_{obs} = e^{C_2\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{\frac{2}{3}}\right)}$.

Remark 3.6.2. *In comparision the observability inequality in continuous case which performed in [BDL07], we find that the observability inequality obtained here is weak since there is an additional term depending upon h at right-hand-side of inequality (3.6.3).*

From the result of Proposition 6.1 we deduce the following controllability result for system (3.6.1).

Proposition 3.6.3. *There exists positive constants C_1, C_2, C_3 and for $T > 0$ a map $L_{T,a} : \mathbb{R}^{\mathfrak{M}} \to L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ such that if $h \leq \min(h_0, h_2)$ with*

$$h_1 = C_0\big(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{\frac{2}{3}}\big)^{-1}$$

for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function v given by $v = L_a(y_0)$ such that the solution to (3.6.1) satisfies

$$|y(T)|_{L^2(\Omega)} \leq C_0 e^{-C_2/h} |y_0|_{L^2(\Omega)}$$

and

$$\|v\|_{L^2(Q)} \leq C_0 |y_0|_{L^2(\Omega)}$$

, with $C_0 = e^{C_3\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{\frac{2}{3}}\right)}$.

Note that the final state is of size $e^{-C/h}|y_0|_{L^2(\Omega)}$. The proof of these proposition are given in [BL12].

3.6.2 The semilinear case

We consider the following semilinear semi-discrete control problem

$$\big(\partial_t + \mathcal{A}^{\mathfrak{M}}\big)y + G(y) = \mathbf{1}_\omega v, \quad y \in (0,T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0 \tag{3.6.4}$$

where $\omega \subset \Omega$. The function $G : \mathbb{R} \to \mathbb{R}$ is assumed of the form

$$G(x) = xg(x), \qquad x \in \mathbb{R}, \tag{3.6.5}$$

with g Lipschitz continuous. Here, we consider the function g in two cases: $g \in L^\infty(\mathbb{R})$ and the more general case as

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \qquad \text{with} \qquad 0 \leq r < \frac{3}{2} \tag{3.6.6}$$

The results of semi-discrete parabolic with potential above allows one to obtain controllability results for parabolic equation with semi-linear terms whose proofs are given in [BL12]

Theorem 3.6.4. *(sublinear case) We assume that G satisfies (3.6.5) with $g \in L^\infty(\mathbb{R})$ and c satisfies (3.1.2). There exists positive constants C_0, C_1 such that for all $T > 0$ and h chosen sufficiently small, for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function v with*

$$\|v\|_{L^2(Q)} \leq C\, |y_0|_{L^2(\Omega)}$$

such that the solution to the semi-linear parabolic equation (3.6.4) satisfies

$$|y(T)|_{L^2(\Omega)} \leq Ce^{-C_0/h}\, |y_0|_{L^2(\Omega)}$$

with $C_0 = e^{C_1\left(1+\frac{1}{T}+T\|g\|_\infty+\|g\|_\infty^{\frac{2}{3}}\right)}$.

Theorem 3.6.5. *(superlinear case) Let $\Omega = (0,1)$, c satisfy (3.1.2) and G satisfy (3.6.5) - (3.6.6). There exists C_0 such that, for $T > 0$ and $M > 0$, there exists positive constants C, h_0 such that for $0 < h \leq h_0$ and for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$ satisfying $|y_0|_{H_1(\Omega)} \leq M$ there exists a semi-discrete control function v such that the solution to the semi-linear parabolic equation*

$$\big(\partial_t - \bar{D}cD\big)y + G(y) = \mathbf{1}_\omega v, \quad y \in (0,T) \qquad y|_{\partial\Omega} = 0, \quad y(0) = y_0 \tag{3.6.7}$$

satisfies

$$|y(T)|_{L^2(\Omega)} \leq Ce^{-C_0/h}\,|y_0|_{L^2(\Omega)}$$

where $C = C(T, M)$.

Observe that the constants are uniform with respect to discretization parameter h.

3.7 Proofs of Lemma 3.3.17 and intermediate results in Section 3.4

3.7.1 Proof of Lemma 3.3.17

We have

$$(c_d Du)_{n+\frac{3}{2}} - (c_d Du)_{n+\frac{1}{2}} = hf_{n+1}.$$

As $Du = \tilde{\rho} Dv + D\rho \tilde{v}$ we obtain

$$\begin{aligned} &r_{n+1}\Big(\tilde{\rho}_{n+\frac{3}{2}}(cDv)_{n+\frac{3}{2}} - \tilde{\rho}_{n+\frac{1}{2}}(cDv)_{n+\frac{1}{2}} + (D\rho)_{n+\frac{3}{2}}(c\tilde{v})_{n+\frac{3}{2}} - (D\rho)_{n+\frac{1}{2}}(c\tilde{v})_{n+\frac{1}{2}}\Big) \\ &= h(rf)_{n+1}. \end{aligned} \tag{3.7.1}$$

We write

$$r_{n+1}\tilde{\rho}_{n+\frac{3}{2}} = \frac{r_{n+1}\rho_{n+1} + r_{n+1}\rho_{n+2}}{2} = \frac{1 + (((\tau^+)^2\rho)r)_{n+1}}{2} := K_{11},$$

$$\begin{aligned} r_{n+1}(c_d D\rho)_{n+\frac{3}{2}} &= (r\tau^+\rho)_{n+1}(c_d r D\rho)_{n+\frac{3}{2}} \\ &= (r\tau^+\rho)_{n+1}\Big((c_d r\partial\rho)_{n+\frac{3}{2}} + (c_d r D\rho)_{n+\frac{3}{2}} - (c_d r\partial\rho)_{n+\frac{3}{2}}\Big) \\ &= K_{21}\Big((c_d r\partial\rho)_{n+\frac{3}{2}} + K_{22}\Big), \end{aligned}$$

where $K_{21} = (r\tau^+\rho)_{n+1}$ and $K_{22} = (c_d r D\rho)_{n+\frac{3}{2}} - (c_d r\partial\rho)_{n+\frac{3}{2}}$.

Similarly,

$$r_{n+1}\tilde{\rho}_{n+\frac{1}{2}} = \frac{r_{n+1}\rho_{n+1} + r_{n+1}\rho_n}{2} = \frac{1 + (((\tau^-)^2\rho)r)_{n+1}}{2} := K_{31},$$

$$
\begin{aligned}
r_{n+1}(c_dD\rho)_{n+\frac{1}{2}} &= (r\tau^-\rho)_{n+1}(c_drD\rho)_{n+\frac{1}{2}} \\
&= (r\tau^-\rho)_{n+1}\Big((c_dr\partial\rho)_{n+\frac{1}{2}} + (c_drD\rho)_{n+\frac{1}{2}} - (c_dr\partial\rho)_{n+\frac{1}{2}}\Big) \\
&:= K_{41}\Big((c_dr\partial\rho)_{n+\frac{1}{2}} + K_{42}\Big),
\end{aligned}
$$

where $K_{41} = (r\tau^-\rho)_{n+1}$ and $(c_drD\rho)_{n+\frac{1}{2}} - (c_dr\partial\rho)_{n+\frac{1}{2}}$.

Additionally,

$$
\begin{aligned}
(\tilde{v})_{n+\frac{1}{2}} &= v_{n+1} + \frac{v_n - v_{n+1}}{2} = v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{1}{2}}, \\
(\tilde{v})_{n+\frac{3}{2}} &= v_{n+1} + \frac{v_{n+2} - v_{n+1}}{2} = v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{3}{2}}.
\end{aligned}
$$

From (3.7.1) we thus write

$$
\begin{aligned}
& K_{11}(c_dDv)_{n+\frac{3}{2}} - K_{31}(c_dDv)_{n+\frac{1}{2}} \\
+ \;& K_{21}\Big((cr\partial\rho)_{n+\frac{3}{2}} + K_{22}\Big)\Big(v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{3}{2}}\Big) \\
- \;& K_{41}\Big((cr\partial\rho)_{n+\frac{1}{2}} + K_{42}\Big)\Big(v_{n+1} + \mathcal{O}(h)(Dv)_{n+\frac{1}{2}}\Big) \\
= \;& h(rf)_{n+1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& K_{11}\Big((c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}}\Big) + (K_{11} - K_{31})(c_dDv)_{n+\frac{1}{2}} \\
+ \;& K_{21}[\star cr\partial\rho]_a v_{n+1} + (K_{21} - K_{41})(cr\partial\rho)_{n+\frac{1}{2}} v_{n+1} + (K_{21}K_{22} - K_{41}K_{42})v_{n+1} \\
+ \;& K_{21}\Big((cr\partial\rho)_{n+\frac{3}{2}} + K_{22}\Big)\mathcal{O}(h)(Dv)_{n+\frac{3}{2}} + K_{41}\Big((cr\partial\rho)_{n+\frac{1}{2}} + K_{42}\Big)\mathcal{O}(h)(Dv)_{n+\frac{1}{2}} \\
= \;& h(rf)_{n+1}.
\end{aligned}
$$

Moreover, as $r\partial\rho = -\lambda s\phi\partial\psi = s\mathcal{O}_\lambda(1)$ we have

$$
\begin{aligned}
& K_{11}\Big((c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}}\Big) + (K_{11} - K_{31})(c_dDv)_{n+\frac{1}{2}} \\
= \;& K_{21}\lambda s[\star c\phi\partial\psi]_a v_{n+1} - Kv_{n+1} + \Big(K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h)\Big)\Big((c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}}\Big) \\
+ \;& \Big(K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) + K_{41}\mathcal{O}_\lambda(sh) + K_{41}K_{42}\mathcal{O}(h)\Big)(c_dDv)_{n+\frac{1}{2}} \\
+ \;& h(rf)_{n+1},
\end{aligned}
$$

where

$$\begin{aligned} K &= (K_{21}-K_{41})(cr\partial\rho)_{n+\frac{1}{2}} + K_{21}K_{22} - K_{41}K_{42} \\ &= (K_{21}-K_{41})s\mathcal{O}_\lambda(1) + K_{21}K_{22} - K_{41}K_{42}. \end{aligned}$$

From that, we can write

$$\begin{aligned} & L\Big((c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}}\Big) \\ = \;& K_{21}\lambda s[\star c\phi\partial\psi]_a v_{n+1} - Kv_{n+1} + H(c_dDv)_{n+\frac{1}{2}} + h(rf)_{n+1}, \end{aligned}$$

where

$$\begin{aligned} L &= K_{11} - K_{21}\mathcal{O}_\lambda(sh) - K_{21}K_{22}\mathcal{O}(h), \\ K &= (K_{21}-K_{41})s\mathcal{O}_\lambda(1) + K_{21}K_{22} - K_{41}K_{42}, \\ H &= K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) + K_{41}\mathcal{O}_\lambda(sh) + K_{41}K_{42}\mathcal{O}(h) - K_{11} + K_{31}. \end{aligned}$$

As $L = 1 + \mathcal{O}_{\lambda,\Re}(sh) \neq 0$ (see below) then we read

$$\begin{aligned} & (c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}} \\ = \;& \Big(L^{-1}K_{21}\lambda s[\star c\phi\partial\psi]_a - L^{-1}K\Big)v_{n+1} + L^{-1}H(c_dDv)_{n+\frac{1}{2}} + L^{-1}h(rf)_{n+1}. \end{aligned}$$

We set

$$\begin{aligned} J_1 &= L^{-1}K_{21}\lambda s[\star c\phi\partial\psi]_a - L^{-1}K, \\ J_2 &= L^{-1}H, \qquad\qquad J_3 = L^{-1}. \end{aligned}$$

We thus have

$$(c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}} = J_1v_{n+1} + J_2(c_dDv)_{n+\frac{1}{2}} + J_3h(rf)_{n+1} \tag{3.7.2}$$

By using Proposition 3.3.10 we find

$$
\begin{aligned}
K_{11} &= \frac{1+(((\tau^+)^2\rho)r)_{n+1}}{2} = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
K_{31} &= \frac{1+(((\tau^-)^2\rho)r)_{n+1}}{2} = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
K_{21} &= (r\tau^+\rho)_{n+1} = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
K_{41} &= (r\tau^-\rho)_{n+1} = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
K_{22} &= (c_d r D\rho)_{n+\frac{3}{2}} - (c_d r\partial\rho)_{n+\frac{3}{2}} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,\\
K_{42} &= (c_d r D\rho)_{n+\frac{1}{2}} - (c_d r\partial\rho)_{n+\frac{1}{2}} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.
\end{aligned}
$$

From that we estimate

$$
\begin{aligned}
K &= (K_{21}-K_{41})s\mathcal{O}_\lambda(1) + K_{21}K_{22} - K_{41}K_{42} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
H &= K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) + K_{41}\mathcal{O}_\lambda(sh) + K_{41}K_{42}\mathcal{O}(h) - K_{11} + K_{31}\\
&= \mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
L &= K_{11} - K_{21}\mathcal{O}_\lambda(sh) + K_{21}K_{22}\mathcal{O}(h) = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh).
\end{aligned}
$$

For sh sufficiently small we have $L^{-1} = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and then we obtain

$$
\begin{aligned}
J_1 &= L^{-1}K_{21}\lambda s[\star c\phi\partial\psi]_a - L^{-1}K\\
&= \Big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)\lambda s[\star c\phi\partial\psi]_a + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
J_2 &= L^{-1}H = \mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
J_3 &= L^{-1} = 1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh).
\end{aligned}
$$

By using Proposition 3.3.14, Lemma 3.3.8 and Lemma 3.3.6 yield

$$
\begin{aligned}
\partial_t K_{11} &= \partial_t\Big(((\tau^+)^2\rho)r)_{n+1}\Big) = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
\partial_t K_{31} &= \partial_t\Big(((\tau^-)^2\rho)r)_{n+1}\Big) = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
\partial_t K_{21} &= \partial_t(r\tau^+\rho)_{n+1} = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
\partial_t K_{41} &= \partial_t(r\tau^-\rho)_{n+1} = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
\partial_t K_{22} &= \partial_t\Big((c_d r D\rho)_{n+\frac{3}{2}} - (c_d r\partial\rho)_{n+\frac{3}{2}}\Big) = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,\\
\partial_t K_{42} &= \partial_t\Big((c_d r D\rho)_{n+\frac{1}{2}} - (c_d r\partial\rho)_{n+\frac{1}{2}}\Big) = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,
\end{aligned}
$$

which give

$$\begin{aligned}\partial_t L^{-1} = -\frac{\partial_t L}{L^2} &= \Big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)\,(\partial_t K_{11}+\partial_t K_{21}\mathcal{O}_\lambda(sh)+K_{21}(\partial_t s)\mathcal{O}_\lambda(h)\\ &\qquad +\partial_t K_{21}K_{22}\mathcal{O}(h)+\partial_t K_{22}K_{21}\mathcal{O}(h))\\ &= T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\end{aligned}$$

where sh sufficiently small and

$$\begin{aligned}&\partial_t H = \partial_t K_{21}\mathcal{O}_\lambda(sh)+K_{21}(\partial_t s)\mathcal{O}_\lambda(h)+\partial_t K_{21}K_{22}\mathcal{O}(h)+K_{21}\partial_t K_{22}\mathcal{O}(h)\\ +\;&\partial_t K_{41}\mathcal{O}_\lambda(sh)+K_{41}(\partial_t s)\mathcal{O}_\lambda(h)+\partial_t K_{41}K_{42}\mathcal{O}(h)+K_{41}\partial_t K_{42}\mathcal{O}(h)-\partial_t K_{11}+\partial_t K_{31}\\ =\;&T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

It follows that we have

$$\begin{aligned}&\partial_t J_1 = sT\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\ &\partial_t J_2 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \qquad\qquad \partial_t J_3 = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(sh).\end{aligned}$$

Furthermore, we can write (3.7.2) in the simple form

$$\begin{aligned}&(c_d Dv)_{n+\frac{3}{2}}-(c_d Dv)_{n+\frac{1}{2}}\\ =\;&\lambda s[\star c\phi\partial\psi]_a v_{n+1}+\lambda s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}\\ +\;&\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}+\Big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)h(rf)_{n+1},\end{aligned}$$

which yields the conclusion.

3.7.2 Proof of Lemma 3.4.4

By using Lemma 3.3.2 in each domain Ω_{01}, Ω_{02} , we have

$$\begin{aligned}
I_{11} &= 2\int_{Q_0'} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}\bar{D}(c_dDv)\overline{Dv} \\
&= 2\int_{Q_{01}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}\bar{D}(c_dDv)\overline{Dv} + 2\int_{Q_{02}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}\bar{D}(c_dDv)\overline{Dv} \\
&= 2\int_{Q_{01}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}\bar{c}_d\bar{D}(Dv)\overline{Dv} + 2\int_{Q_{01}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2 \\
&+ 2\int_{Q_{02}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}\bar{c}_d\bar{D}(Dv)\overline{Dv} + 2\int_{Q_{02}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2 \\
&= \sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}\bar{c}_d\bar{D}(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2.
\end{aligned}$$

We then apply a discrete integration by parts (Proposition 3.3.5) in each domain Ω_{01}, Ω_{02} with $\partial\Omega_{01} = \{0, a\}$ and $\partial\Omega_{02} = \{a, 1\}$ for the first two terms and we obtain

$$\begin{aligned}
I_{11} &= -\sum_{i=1}^{2}\int_{Q_{0i}} D(c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2 \\
&+ \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} - \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^+)(Dv)^2_{n+\frac{3}{2}} \\
&+ \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^-)(Dv)^2_{n+\frac{1}{2}} - \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(0)(Dv)^2_{\frac{1}{2}} \\
&= -\sum_{i=1}^{2}\int_{Q_{0i}} D(c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} cr^2\bar{\tilde{\rho}}\ \overline{D\rho}(\bar{D}c_d)(\overline{Dv})^2 + Y_{11}.
\end{aligned}$$

where

$$\begin{aligned}
Y_{11} &= Y_{11}^{(1)} + Y_{11}^{(2)} \\
Y_{11}^{(1)} &= \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} - \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^+)(Dv)^2_{n+\frac{3}{2}} \\
Y_{11}^{(2)} &= \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(a^-)(Dv)^2_{n+\frac{1}{2}} - \int_0^T (c\bar{c}_dr^2\bar{\tilde{\rho}}\ \overline{D\rho})(0)(Dv)^2_{\frac{1}{2}}
\end{aligned}$$

Lemma 3.7.1. *(see Lemma B.3 in [BHL10a]) Provided $sh \le \mathfrak{K}$ we have*

$$\begin{aligned}
&D(c_i\bar{c}_{di}r_i^2\bar{\tilde{\rho}}_i\ \overline{D\rho_i}) = -s\lambda^2(c_i^2\phi_i(\psi_i')^2)_d + s\lambda\phi_{di}\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
&c_ir_i^2\bar{\tilde{\rho}}_i\overline{D\rho_i}(\bar{D}c_{di}) = s\lambda\phi_i\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \\
&r_i^2\bar{\tilde{\rho}}_i\overline{D\rho_i} = r_i\partial\rho_i + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = -s\lambda\phi_i\psi_i' + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \\
&r_i^2\bar{\tilde{\rho}}_i\overline{D\rho_i} = r_i\bar{\tilde{\rho}}_ir_i\overline{D\rho_i} = (1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh))r_i\overline{D\rho_i}.
\end{aligned}$$

Moreover, by Lemma 3.3.3 and Proposition 3.3.5 in each domain Ω_{01}, Ω_{02} we obtain

$$\int_{\Omega_0'} s\lambda\phi(\overline{Dv})^2 \leq \int_{\Omega_0'} s\lambda\phi\overline{(Dv)^2} = \int_{\Omega_0'} s\lambda\tilde{\phi}(Dv)^2 - \frac{h}{2}\sum_{i=1}^{2} BT_i \leq \int_{\Omega_0'} s\lambda\tilde{\phi}(Dv)^2$$

since

$$\begin{aligned} BT_1 &= s\lambda\phi(a)(Dv)^2_{n+\frac{1}{2}} + s\lambda\phi(a)(Dv)^2_{\frac{1}{2}} \geq 0 \\ BT_2 &= s\lambda\phi(1)(Dv)^2_{n+m+\frac{3}{2}} + s\lambda\phi(a)(Dv)^2_{n+\frac{3}{2}} \geq 0 \end{aligned}$$

and $\tilde{\phi} = \phi + h^2\mathcal{O}_\lambda(1)$ then we can write

$$\int_{\Omega_{0i}} s\lambda\phi(\overline{Dv})^2 \leq \int_{\Omega_{0i}} s\lambda\phi(Dv)^2 + \int_{\Omega_{0i}} s\lambda h^2\mathcal{O}_\lambda(1)(Dv)^2$$

Similarly, we have

$$\left|\int_{\Omega_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\overline{Dv})^2\right| \leq \int_{\Omega'} s\left|\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\right|\overline{(Dv)^2} \leq \int_{\Omega'} s\left|\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\right|(Dv)^2.$$

Thus

$$I_{11} \geq -\int_{Q'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{11} + Y_{11},$$

where $X_{11} = \int_{Q_0'} \nu_{11}(Dv)^2$ with ν_{11} of the form $s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and

$$\begin{aligned} Y_{11} &= Y_{11}^{(1)} + Y_{11}^{(2,1)} + Y_{11}^{(2,2)}, \\ Y_{11}^{(1)} &= \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(1)(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} \\ &\quad - \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(0)(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}}, \\ Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\bar{c}_d(a)\Big(-(c\psi')(a^-)(Dv)^2_{n+\frac{1}{2}} + (c\psi')(a^+)(Dv)^2_{n+\frac{3}{2}}\Big), \\ Y_{11}^{(2,2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{1}{2}} - \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2_{n+\frac{3}{2}}. \end{aligned}$$

3.7.3 Proof of Lemma 3.4.5

We set $q = r\bar{\bar{\rho}}c\phi''$. By using a discrete integrations by parts (Proposition 3.3.5) and Lemma 3.3.2 in each domain Ω_{01}, Ω_{02} we have

$$
\begin{aligned}
I_{12} &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv\bar{D}(c_dDv)\\
&= 2\sum_{i=1}^{2}\int_{Q_{0i}} s\tilde{q}c_d(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} sDqc_d\tilde{v}Dv\\
&- \int_0^T sq(a^-)v(a)(c_dDv)_{n+\frac{1}{2}} + \int_0^T sq(a^+)v(a)(c_dDv)_{n+\frac{3}{2}}\\
&= 2\sum_{i=1}^{2}\int_{Q_{0i}} s\tilde{q}c_d(Dv)^2 + 2\sum_{i=1}^{2}\int_{Q_{0i}} sDqc_d\tilde{v}Dvdt + Y_{12},
\end{aligned}
$$

since $v|_{\partial\Omega_0} = 0$ and with $\partial\Omega_{01} = \{0, a\}$, $\partial\Omega_{02} = \{a, 1\}$.

Lemma 3.7.2. *(see the proof as given in Lemma 4.4 of [BHL10a]) Let $i = 1, 2$. Provided $sh \leq \mathfrak{K}$ we have*

$$
\begin{aligned}
&\phi_i'' = \lambda^2\phi_i(\psi_i')^2 + \lambda\phi_i\mathcal{O}(1),\\
&q_i = r_i\bar{\tilde{\rho}}_i c\phi_i^{''} = \lambda^2 c\phi_i(\psi_i')^2 + \lambda\phi_i\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,\\
&\tilde{q}_i = \lambda^2(c\phi_i(\psi_i')^2)_d + \lambda\phi_i\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2 + h),\\
&Dq_i = D(r_i\bar{\tilde{\rho}}_i)\,\widetilde{c\phi_i^{''}} + \widetilde{(r_i\bar{\tilde{\rho}}_i)}\,D(c\phi_i^{''}) = \mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}
$$

Note that the proof and the use of Lemma 3.7.2 are carried out in each domain Ω_{01}, Ω_{02} independently.

It follows that

$$
I_{12} = 2\sum_{i=1}^{2}\int_{Q_{0i}} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \nu_{12}(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv + Y_{12},
$$

Then

$$
I_{12} = 2\int_{Q_0'} s\lambda^2(c^2\phi(\psi')^2)_d(Dv)^2 - X_{12} + Y_{12},
$$

with

$$
\begin{aligned}
Y_{12} &= \int_0^T s\lambda^2\phi(a)v(a)[c(\psi')^2 \star c_dDv]_a\\
&+ \int_0^T \delta_{12}v(a)(cDv)_{n+\frac{3}{2}} + \bar{\delta}_{12}v(a)(cDv)_{n+\frac{1}{2}},
\end{aligned}
$$

where $\delta_{12}, \bar{\delta}_{12}$ are of form $s\big(\lambda\phi(a)\mathcal{O}(1) + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big)$ and

$$X_{12} = \int_{Q_0'} \nu_{12}(Dv)^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\tilde{v}Dv,$$

where

$$\nu_{12} = s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(h + (sh)^2).$$

3.7.4 Proof of Lemma 3.4.6

We carry out a discrete integration by parts (Proposition 3.3.5) in each domain Ω_{01}, Ω_{02} with $\partial\Omega_{01} = \{0, a\}$ and $\partial\Omega_{02} = \{a, 1\}$ as follows

$$\begin{aligned}
I_{13} &= \int_{Q_{01}} r\bar{\tilde{\rho}}\bar{D}(c_d Dv)\partial_t v + \int_{Q_{02}} r\bar{\tilde{\rho}}\bar{D}(c_d Dv)\partial_t v \\
&= -\int_{Q_{01}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv - \int_{Q_{02}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv \\
&+ \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(0)\partial_t v(0)(c_d Dv)_{\frac{1}{2}} \\
&+ \int_0^T (r\bar{\tilde{\rho}})(1)\partial_t v(1)(c_d Dv)_{n+m+\frac{3}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} \\
&= -\int_{Q_{01}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv - \int_{Q_{02}} D(r\bar{\tilde{\rho}}\partial_t v)c_d Dv \\
&+ \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} \\
&= \underbrace{-\sum_{i=1}^2 \int_{Q_{0i}} D(r\bar{\tilde{\rho}})\partial_t\tilde{v}c_d Dv}_{\bar{Q}_1} \underbrace{-\sum_{i=1}^2 \int_{Q_{0i}} \widetilde{r\bar{\tilde{\rho}}}\,(\partial_t Dv)c_d Dv}_{\bar{Q}_2} + Y_{13},
\end{aligned}$$

by Lemma 3.3.2 and with

$$Y_{13} = \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}},$$

as $v|_{\partial\Omega_0} = 0$.

By applying Proposition 3.3.13 in each domain Ω_{01}, Ω_{02} we find

$$\begin{aligned}
D(r_i\bar{\tilde{\rho}}_i) &= \mathcal{O}_{\lambda,\mathfrak{K}}(sh), \\
r_i\bar{\tilde{\rho}}_i &= 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 = \mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

On the one hand, we have

$$\begin{aligned}
|\bar{Q}_1| &\leq \sum_{i=1}^{2}\int_{Q_{0i}} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t\tilde{v})^2 + \sum_{i=1}^{2}\int_{Q_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(Dv)^2 \\
&\leq \sum_{i=1}^{2}\int_{Q_{0i}} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\,\widetilde{(\partial_t v)^2} + \sum_{i=1}^{2}\int_{Q_{0i}} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(Dv)^2 \\
&= \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(Dv)^2,
\end{aligned}$$

by $(\partial_t\tilde{v})^2 \leq \widetilde{(\partial_t\tilde{v})^2}$ in each domain Ω_{01}, Ω_{02} and $\sum_{i=1}^{2}\int_{\Omega_{0i}}\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{(\partial_t\tilde{v})^2} = \int_{\Omega_0}\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t\tilde{v})^2$.

On the other hand, by an integrations by parts w.r.t t we write as

$$\begin{aligned}
\bar{Q}_2 &= -\frac{1}{2}\sum_{i=1}^{2}\int_{Q_{0i}} \widetilde{r\bar{\tilde{\rho}}}\, c_d.\partial_t(Dv)^2 \\
&= \frac{1}{2}\sum_{i=1}^{2}\int_{Q_{0i}} \partial_t(\widetilde{r\bar{\tilde{\rho}}})c_d(Dv)^2 - \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_{0i}} \widetilde{r\bar{\tilde{\rho}}}\, c_d.(Dv)^2|_{t=0}^{t=T}.
\end{aligned}$$

We observe that for $sh \leq \epsilon_1(\lambda)$ with $\epsilon_1(\lambda)$ sufficiently small we have $r\bar{\tilde{\rho}} > 0$ by Proposition 3.3.13. The sign of the term at $t = T$ and $t = 0$ are thus prescribed. Furthermore, Proposition 3.3.14 leads to $\partial_t(r_i\bar{\tilde{\rho}}_i) = T(sh)^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)$, so that, for $sh \leq \mathfrak{K}$ we obtain

$$\bar{Q}_2 \geq \sum_{i=1}^{2}\int_{Q_{0i}} T(sh)^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 - \mathcal{C}_{\lambda,\mathfrak{K}}(1)\sum_{i=1}^{2}\int_{\Omega_{0i}}(Dv(T))^2.$$

Thus,

$$I_{13} \geq -\int_{\Omega_0'}\mathcal{C}_{\lambda,\mathfrak{K}}(1)(Dv(T))^2 - X_{13} + Y_{13}.$$

with

$$\begin{aligned}
X_{13} &= \int_{Q_0'}\big(s(sh) + T(sh)^2\theta\big)\mathcal{O}_{\lambda,\mathfrak{K}}(1)(Dv)^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v)^2. \\
Y_{13} &= \int_0^T (r\bar{\tilde{\rho}})(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T (r\bar{\tilde{\rho}})(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}}
\end{aligned}$$

3.7.5 Proof of Lemma 3.4.7

We set $q = c^2 r^2 (\bar{D}D\rho)\overline{D\rho}$. Observing that $\overline{Dv} = \bar{D}\tilde{v}$ we get

$$\begin{aligned}
I_{21} &= 2\int_{Q_{01}} \underbrace{c^2 r^2 (\bar{D}D\rho)\overline{D\rho}}_{q}\,\tilde{\tilde{v}}\;\overline{Dv} + 2\int_{Q_{02}} \underbrace{c^2 r^2 (\bar{D}D\rho)\overline{D\rho}}_{q}\,\tilde{\tilde{v}}\;\overline{Dv} \\
&= \int_{Q_{01}} q\bar{D}(\tilde{v})^2 + \int_{Q_{02}} q\bar{D}(\tilde{v})^2 \\
&= -\int_{Q_{01}} Dq(\tilde{v})^2 - \int_{Q_{02}} Dq(\tilde{v})^2 \\
&+ \int_0^T q(a^-)(\tilde{v})^2_{n+\frac{1}{2}} - \int_0^T q(0)(\tilde{v})^2_{\frac{1}{2}} \\
&+ \int_0^T q(1)(\tilde{v})^2_{n+m+\frac{3}{2}} - \int_0^T q(a^+)(\tilde{v})^2_{n+\frac{3}{2}} \\
&= -\sum_{i=1}^{2}\int_{Q_{0i}} Dq\;\widetilde{v^2} + \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} (Dq)(Dv)^2 + Y_{21}^{(1)} \\
&= -\sum_{i=1}^{2}\int_{Q_{0i}} \overline{Dq}(v)^2 + \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} (Dq)(Dv)^2 + Y_{21}^{(1)} + Y_{21}^{(2)},
\end{aligned}$$

by means of Proposition 3.3.5, Lemma 3.3.2, Lemma 3.3.3 in each domain Ω_{01}, Ω_{02} independently and where

$$\begin{aligned}
&Y_{21} = Y_{21}^{(1)} + Y_{21}^{(2)} = Y_{21}^{(1,1)} + Y_{21}^{(1,2)} + Y_{21}^{(2)}, \\
&Y_{21}^{(1,1)} = \int_0^T q(1)(\tilde{v})^2_{n+m+\frac{3}{2}} - \int_0^T q(0)(\tilde{v})^2_{\frac{1}{2}}, \\
&Y_{21}^{(1,2)} = \int_0^T q(a^-)(\tilde{v})^2_{n+\frac{1}{2}} - \int_0^T q(a^+)(\tilde{v})^2_{n+\frac{3}{2}}, \\
&Y_{21}^{(2)} = -\frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T v^2(0)(Dq)_{\frac{1}{2}} \\
&-\frac{h}{2}\int_0^T v^2(1)(Dq)_{n+m+\frac{3}{2}} - \frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{3}{2}} \\
&= -\frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{3}{2}},
\end{aligned}$$

as $v|_{\partial\Omega_0} = 0$.

We note that $\tilde{v}_{\frac{1}{2}} = \frac{h}{2}(Dv)_{\frac{1}{2}}$, $\tilde{v}_{n+m+\frac{3}{2}} = -\frac{h}{2}(Dv)_{n+m+\frac{3}{2}}$. On the one hand, by Proposition 3.3.10 we have $q = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)r\overline{D\rho}$ in each domain Ω_{01}, Ω_{02}. It follows that

$$
\begin{aligned}
Y_{21}^{(1,1)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(r\overline{D\rho})(1)(\tilde{v})^2_{n+m+\frac{3}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(r\overline{D\rho})(0)(\tilde{v})^2_{\frac{1}{2}} \\
&= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})(1)(Dv)^2_{n+m+\frac{3}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(r\overline{D\rho})(0)(Dv)^2_{\frac{1}{2}}.
\end{aligned}
$$

On the other hand, by Proposition 3.3.15, Corollary 3.3.9 we have $q = -c^2(s\phi\lambda)^3(\psi')^3 + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2$ in each domain Ω_{01}, Ω_{02}. We thus obtain

$$
\begin{aligned}
Y_{21}^{(1,2)} &= \int_0^T \big(s\lambda\phi(a)\big)^3\Big(-(c^2\psi'^3)(a^-)(\tilde{v})^2_{n+\frac{1}{2}} + (c^2\psi'^3)(a^+)(\tilde{v})^2_{n+\frac{3}{2}}\Big) \\
&\quad + \int_0^T \Big(s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\Big)\Big((\tilde{v})^2_{n+\frac{1}{2}} - (\tilde{v})^2_{n+\frac{3}{2}}\Big) \\
&= Y_{21}^{(1,21)} + Y_{21}^{(1,22)},
\end{aligned}
$$

where

$$
Y_{21}^{(1,21)} = \int_0^T s^3\lambda^3\phi^3(a)[c^2(\psi')^3 \star \tilde{v}^2]_a.
$$

Lemma 3.7.3. *(see Lemma B.8 in [BHL10a]) Provided $sh \le \mathfrak{K}$ we have*

$$
\begin{aligned}
&Dq_i = s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&\overline{Dq_i} = -3s^3\lambda^4\phi_i^3c^2(\psi_i')^4 + (s\lambda\phi_i)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2.
\end{aligned}
$$

Note that the proof and the use of Lemma 3.7.3 are done in each domain Ω_{01}, Ω_{02} separately.

We then obtain

$$
\begin{aligned}
Y_{21}^{(2)} &= -\frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T v^2(a)(Dq)_{n+\frac{3}{2}} \\
&= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a).
\end{aligned}
$$

We thus write I_{21}

$$
I_{21} \ge 3\int_{Q_0'} \lambda^4 s^3\phi^3c^2(\psi')^4(v)^2 - \int_{Q_0'}\mu_{21}(v)^2 - \int_{Q_0'}\nu_{21}(Dv)^2 + Y_{21},
$$

where

$$
\begin{aligned}
&\mu_{21} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \quad \nu_{21} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \\
&Y_{21} = Y_{21}^{(1,1)} + Y_{21}^{(1,21)} + Y_{21}^{(1,22)} + Y_{21}^{(2)}.
\end{aligned}
$$

3.7.6 Proof of Lemma 3.4.8

We set $q = c^2 r(\bar{D}D\rho)\phi''$ and by Lemma 3.3.4 we have $\overline{\tilde{v}} = v + h^2\bar{D}Dv/4$ in each domain Ω_{01}, Ω_{02}. It follows that

$$\begin{aligned} I_{22} &= -2\int_{Q_{01}} sq\bar{\tilde{v}}v - 2\int_{Q_{02}} sq\bar{\tilde{v}}v \\ &= -2\int_{Q_{01}} sqv^2 - \int_{Q_{01}} \frac{sh^2}{2} q(\bar{D}Dv)v \\ &\quad -2\int_{Q_{02}} sqv^2 - \int_{Q_{02}} \frac{sh^2}{2} q(\bar{D}Dv)v. \end{aligned}$$

Applying a discrete integration by parts (Proposition 3.3.5) and Lemma 3.3.2 in each domain Ω_{01}, Ω_{02} yield

$$\begin{aligned} I_{22} &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2} D(qv)Dv + Y_{22}^{(1)} \\ &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2}\tilde{q}(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2} D(q)\tilde{v}Dv + Y_{22}^{(1)} \\ &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2}\tilde{q}(Dv)^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{4} D(q)D(v^2) + Y_{22}^{(1)} \\ &= -2\sum_{i=1}^{2}\int_{Q_{0i}} sqv^2 + \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{2}\tilde{q}(Dv)^2 - \sum_{i=1}^{2}\int_{Q_{0i}} \frac{sh^2}{4}\bar{D}Dqv^2 + Y_{22}^{(1)} + Y_{22}^{(2)}, \end{aligned}$$

where

$$\begin{aligned} Y_{22}^{(1)} &= -\int_0^T \frac{sh^2}{2} q(a^-)v(a)(Dv)_{n+\frac{1}{2}} + \int_0^T \frac{sh^2}{2} q(a^+)v(a)(Dv)_{n+\frac{3}{2}}, \\ Y_{22}^{(2)} &= \int_0^T \frac{sh^2}{4} v^2(a)(Dq)_{n+\frac{1}{2}} - \int_0^T \frac{sh^2}{4} v^2(a)(Dq)_{n+\frac{3}{2}}, \end{aligned}$$

as $v|_{\partial\Omega_0} = 0$.

In each domain Ω_{01}, Ω_{02}, we have $\phi'' = \mathcal{O}_\lambda(1)$ and from Proposition 3.3.13 we have $q = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)$ and $Dq = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)$. We thus obtain

$$\begin{aligned} Y_{22}^{(1)} &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{1}{2}} + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{3}{2}}, \\ Y_{22}^{(2)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2(a). \end{aligned}$$

Lemma 3.7.4. *(see Lemma B.9 and Lemma B.10 in [BHL10a]) Provided $sh \leq \mathfrak{K}$ we have*

$$c^2 r_i \bar{D} D\rho_i = c^2(r_i \partial^2 \rho_i + s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2) = c^2 (s\lambda\phi_i)^2 (\psi_i')^2 + s\mathcal{O}_\lambda(1) + s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,$$
$$h^2 \bar{D} D q_i = s(sh)\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

Note that the proof and use of above Lemma 3.7.4 are done in each domain Ω_{01}, Ω_{02} separately.

Futhermore, we have $\phi'' = \lambda^2(\psi')^2\phi + \lambda\phi\mathcal{O}(1)$ in each domain Ω_{01}, Ω_{02}. It follows that

$$\begin{aligned} sq_i &= s\Big(c^2(s\lambda\phi_i)^2(\psi_i')^2 + s\mathcal{O}_\lambda(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\Big)\Big(\lambda^2(\psi_i')^2\phi_i + \lambda\phi_i\mathcal{O}(1)\Big) \\ &= c^2 s^3\lambda^4(\psi_i')^4\phi_i^3 + s^3\lambda^3\phi_i^3\mathcal{O}(1) + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \end{aligned}$$

in each domain Ω_{01}, Ω_{02}.

We thus write I_{22} as

$$I_{22} = -2\int_{Q_0'} c^2 s^3\lambda^4\phi^3(\psi')^4 v^2 + \int_{Q_0'} \mu_{22} v^2 + \int_{Q_0'} \nu_{22}(Dv)^2 + Y_{22},$$

where

$$\mu_{22} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2, \quad \nu_{22} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2,$$
$$Y_{22} = Y_{22}^{(1)} + Y_{22}^{(2)}.$$

3.7.7 Proof of Lemma 3.4.9

By means of a discrete integration by parts (Proposition 3.3.5) in each domain Ω_{01}, Ω_{02}, we obtain

$$\begin{aligned} I_{23} &= \sum_{i=1}^{2}\int_{Q_{0i}} cr(\bar{D}D\rho)\bar{\tilde{v}}\partial_t v \\ &= \sum_{i=1}^{2}\int_{Q_{0i}} \widetilde{cr(\bar{D}D\rho)\partial_t v}\ \tilde{v} \\ &- \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(0)\partial_t v(0)\tilde{v}_{\frac{1}{2}} - \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^-)\partial_t v(a)\tilde{v}_{n+\frac{1}{2}} \\ &- \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^+)\partial_t v(a)\tilde{v}_{n+\frac{3}{2}} - \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(1)\partial_t v(1)\tilde{v}_{n+m+\frac{3}{2}} \\ &= \bar{Q}_1 + \bar{Q}_2 + Y_{23}^{(1)}, \end{aligned}$$

by Lemma 3.3.3 and where

$$\bar{Q}_1 = \sum_{i=1}^{2} \int_{Q_{0i}} \widetilde{(cr(\bar{D}D\rho))}\, \partial_t \tilde{v}\tilde{v},$$
$$\bar{Q}_2 = \sum_{i=1}^{2} \frac{h^2}{4} \int_{Q_{0i}} D(cr\bar{D}D\rho)(D\partial_t v)\tilde{v},$$
$$Y_{23}^{(1)} = -\frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^-)\partial_t v(a)\tilde{v}_{n+\frac{1}{2}} - \frac{h}{2}\int_0^T (cr(\bar{D}D\rho))(a^+)\partial_t v(a)\tilde{v}_{n+\frac{3}{2}}$$

as $\partial_t v|_{\partial\Omega_0} = 0$.

With an integrations by parts w.r.t t we have

$$\bar{Q}_1 = \frac{-1}{2}\sum_{i=1}^{2}\int_{Q_{0i}} \partial_t\widetilde{(cr\bar{D}D\rho)}(\tilde{v})^2 + \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_{0i}} \widetilde{(cr(\bar{D}D\rho))}\,(\tilde{v})^2|_{t=0}^{t=T}.$$

By means of Proposition 3.3.13 and Lemma 3.3.7 in each domain Ω_{01}, Ω_{02} we get

$$\widetilde{cr_i(\bar{D}D\rho_i)} = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$r_i\bar{D}D\rho_i = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

and we further have

Lemma 3.7.5. *(see Lemma A.1 in [BL12])*

$$\partial_t\widetilde{(cr_i\bar{D}D\rho_i)} = Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

Note that the proof and use of Lemma 3.7.5 are done in each domain Ω_{01}, Ω_{02} separately.

It follows that

$$\bar{Q}_1 = \sum_{i=1}^{2}\int_{Q_{0i}} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2} + \sum_{i=1}^{2}\int_{\Omega_{0i}} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|_{t=0}} + \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|_{t=T}}\big)$$

as $|\tilde{v}|^2 \leq \widetilde{|v|^2}$ in each domain Ω_{01}, Ω_{02}.

Moreover, we observe that $\sum_{i=1}^{2}\int_{\Omega_{0i}} \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2} = \int_{\Omega_0}\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2$. Then,

$$\bar{Q}_1 = \int_{Q_0} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{\Omega_0} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|_{t=0}} + \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{v^2}_{|_{t=T}}\big). \tag{3.7.3}$$

We have

$$Y_{23}^{(1)} = \int_0^T s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{1}{2}}) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)\frac{h}{2}(\tilde{v}_{n+\frac{3}{2}}).$$

By an integration by parts w.r.t t and Lemma 3.3.2 in each domain Ω_{01}, Ω_{02} we find

$$\bar{Q}_2 = \underbrace{-\sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}}\partial_t(D(cr\bar{D}D\rho)\tilde{v})Dv}_{\bar{Q}_2^1} + \underbrace{\sum_{i=1}^{2}\frac{h^2}{8}\int_{\Omega_{0i}} D(cr\bar{D}D\rho)D(v)^2|_{t=0}^{t=T}}_{\bar{Q}_2^2}.$$

By means of Lemma 3.3.2 and a discrete intergration by parts in space (Proposition 3.3.5) in each domain Ω_{01}, Ω_{02} we see that

$$\begin{aligned}
\bar{Q}_2^1 &= \sum_{i=1}^{2}\frac{h^2}{8}\int_{Q_{0i}}\partial_t(\bar{D}D(cr\bar{D}D\rho))v^2 - \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} D(cr\bar{D}D\rho)(\partial_t\tilde{v})Dv \\
&\quad -\frac{h^2}{8}\int_0^T v^2(a)\partial_t(D(cr\bar{D}D\rho))_{n+\frac{1}{2}} + \frac{h^2}{8}\int_0^T v^2(a)\partial_t(D(cr\bar{D}D\rho))_{n+\frac{3}{2}} \\
&= \sum_{i=1}^{2}\frac{h^2}{8}\int_{Q_{0i}}\partial_t(\bar{D}D(cr\bar{D}D\rho))v^2 - \sum_{i=1}^{2}\frac{h^2}{4}\int_{Q_{0i}} D(cr\bar{D}D\rho)(\partial_t\tilde{v})Dv + Y_{23}^{(2)}
\end{aligned}$$

as $v|_{\partial\Omega_0} = 0$.

Lemma 3.7.6. *(Lemma A.2 in [BL12]) Provided $sh \leq \mathfrak{K}$ we have*

$$\begin{aligned}
&h^2\bar{D}D(c_i r_i(\bar{D}D\rho_i)) = s(sh)\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&h^2\partial_t(\bar{D}D(c_i r_i\bar{D}D\rho_i)) = Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&h\partial_t(D(c_i r_i\bar{D}D\rho_i)) = Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1), \\
&D(c_i r_i\bar{D}D\rho_i) = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

Note that all above terms are done in each domain Ω_{01}, Ω_{02} separately.

We thus obtain

$$Y_{23}^{(2)} = \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a).$$

Applying the Young's inequality and using that $|\partial_t\tilde{v}|^2 \leq \widetilde{|\partial_t v|^2}$ in each domain Ω_{01}, Ω_{02}, we

have

$$\begin{aligned}\bar{Q}_2^1 &\geq \int_{Q_0'} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\,\widetilde{|\partial_t v|^2} + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2 + Y_{23}^{(2)}\\ &\geq \int_{Q_0'} Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0} s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2|\partial_t v|^2 + \int_{Q_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2 + Y_{23}^{(2)}\end{aligned} \tag{3.7.4}$$

as $\sum_{i=1}^2 \int_{\Omega_{0i}} \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,\widetilde{|\partial_t v|^2} = \int_\Omega \mathcal{O}_{\lambda,\mathfrak{K}}(1)\,|\partial_t v|^2$.

By using Proposition 3.3.5, Lemma 3.7.6 in each domain Ω_{01}, Ω_{02} separately yield

$$\begin{aligned}\bar{Q}_2^2 &= -\sum_{i=1}^2 \frac{h^2}{8}\int_{\Omega_{0i}} \bar{D}D(cr\bar{D}D\rho)(v)^2|_{t=0}^{t=T}\\ &\quad + \frac{h^2}{8}v^2(a)\big(D(cr\bar{D}D\rho)\big)_{n+\frac{1}{2}}|_{t=0}^{t=T} - \frac{h^2}{8}v^2(a)\big(D(cr\bar{D}D\rho)\big)_{n+\frac{3}{2}}|_{t=0}^{t=T}\\ &= \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=T} + \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=0} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a)|_{t=0}^{t=T}\\ &= \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=T} + \int_{\Omega_0'} s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(v)^2|_{t=0} + Y_{23}^{(3)},\end{aligned} \tag{3.7.5}$$

as $v|_{\partial\Omega_0} = 0$ where

$$Y_{23}^{(3)} = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a)|_{t=0}^{t=T}.$$

Collecting (3.7.3), (3.7.4) and (3.7.5) we obtain

$$I_{23} \geq \int_{\Omega_0} s^2\big(\mathcal{O}_{\lambda,\mathfrak{K}}(1)v_{|_{t=0}}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(1)v_{|_{t=T}}^2\big) - X_{23} + Y_{23},$$

where X_{23} and Y_{23} are as given in the statement of Lemma 3.4.9.

3.7.8 Proof of Lemma 3.4.10

By means of a discrete integration by parts (Proposition 3.3.5) in each domain Ω_{01}, Ω_{02} separately, we get

$$\begin{aligned}I_{31} &= -2\tau\int_{Q_{01}} (\partial_t\theta)\varphi cr\overline{D\rho}v\overline{Dv} - 2\tau\int_{Q_{02}} (\partial_t\theta)\varphi cr\overline{D\rho}v\overline{Dv}\\ &= -2\tau\int_{Q_{01}} (\partial_t\theta)\,\widetilde{\varphi cr\overline{D\rho}v}\;Dv - 2\tau\int_{Q_{02}} (\partial_t\theta)\,\widetilde{\varphi cr\overline{D\rho}v}\;Dv + Y_{31}^{(1)},\end{aligned}$$

with

$$Y_{31}^{(1)} = \tau\frac{h}{2}\int_0^T (\partial_t\theta)(cr\overline{D\rho}\varphi v)(a^-)(Dv)_{n+\frac{1}{2}} + \tau\frac{h}{2}\int_0^T (\partial_t\theta)(cr\overline{D\rho}\varphi v)(a^+)(Dv)_{n+\frac{3}{2}}$$

as $v|_{\partial\Omega_0} = 0$.

We have $\widetilde{\varphi cr\overline{D\rho}v}=\widetilde{\varphi cr\overline{D\rho}}\,\tilde{v} + \frac{h^2}{4}D(\varphi cr\overline{D\rho})Dv$ in each domain Ω_{01}, Ω_{02}. It follows that

$$\begin{aligned}
I_{31} &= -\tau\sum_{i=1}^{2}\int_{Q_{0i}} (\partial_t\theta)\,\widetilde{(cr\overline{D\rho}\varphi)}\,D(v)^2 - \sum_{i=1}^{2}\tau\frac{h^2}{2}\int_{Q_{0i}}(\partial_t\theta)D(cr\overline{D\rho}\varphi)(Dv)^2 + Y_{31}^{(1)}\\
&= \tau\sum_{i=1}^{2}\int_{Q_{0i}} (\partial_t\theta)\overline{(D(cr\overline{D\rho}\varphi))}v^2 - \sum_{i=1}^{2}\tau\frac{h^2}{2}\int_{Q_{0i}}(\partial_t\theta)D(cr\overline{D\rho}\varphi)(Dv)^2 + Y_{31}^{(1)}\\
&- \tau\int_0^T (\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac{1}{2}} + \tau\int_0^T (\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac{3}{2}}\\
&= \tau\sum_{i=1}^{2}\int_{Q_{0i}} (\partial_t\theta)\overline{(D(cr\overline{D\rho}\varphi))}v^2 - \sum_{i=1}^{2}\tau\frac{h^2}{2}\int_{Q_{0i}}(\partial_t\theta)D(cr\overline{D\rho}\varphi)(Dv)^2 + Y_{31}^{(1)} + Y_{31}^{(2)}
\end{aligned}$$

by using a discrete integration by parts in each domain Ω_{01}, Ω_{02} separately and

$$Y_{31}^{(2)} = -\tau\int_0^T (\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac{1}{2}} + \tau\int_0^T (\partial_t\theta)v^2(a)\,\widetilde{(cr\overline{D\rho}\varphi)}_{n+\frac{3}{2}}$$

as $v|_{\partial\Omega} = 0$.

By using the Lipschitz continuity and Proposition 3.3.13 we get

$$\begin{aligned}
&D(cr_i\overline{D\rho_i}\varphi_i) = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&\overline{D(cr_i\overline{D\rho_i}\varphi_i)} = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&\widetilde{cr_i\overline{D\rho_i}\varphi_i}= s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&cr_i\overline{D\rho_i} = c(r_i\partial\rho_i + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2) = c(-s\lambda\phi_i\psi_i' + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2) = s\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

The proof is done in each domain Ω_{01}, Ω_{02} separately. Note that $\max_t \partial_t\theta = T\theta^2$.

It thus follows that

$$I_{31} = \int_{Q_0'} T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2 + \int_{Q_0'} T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(Dv)^2 + Y_{31},$$

where

$$
\begin{aligned}
&Y_{31} = Y_{31}^{(1)} + Y_{31}^{(2)},\\
&Y_{31}^{(1)} = \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \frac{h}{2} (Dv)_{n+\frac{1}{2}} + \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v(a) \frac{h}{2} (Dv)_{n+\frac{3}{2}},\\
&Y_{31}^{(2)} = \int_0^T T\theta s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2(a).
\end{aligned}
$$

3.7.9 Proof of Lemma 3.4.13

We see that

$$
\begin{aligned}
& Y_{11}^{(1)} + Y_{21}^{(1,1)}\\
= & \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(1)(r\overline{D\rho})_1 (Dv)^2_{n+m+\frac{3}{2}}\\
- & \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)(c\bar{c}_d)(0)(r\overline{D\rho})_0 (Dv)^2_{\frac{1}{2}}\\
+ & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (r\overline{D\rho})_0 (Dv)^2_{\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (r\overline{D\rho})_1 (Dv)^2_{n+m+\frac{3}{2}}.
\end{aligned}
$$

Moreover, by (3.4.1) we have $Y_{11}^{(1)} + Y_{21}^{(1,1)} \geq 0$ for sh sufficiently small.

We next focus our attention on the trace term at $'a'$ on $Y_{11}^{(2,1)} + Y_{21}^{(1,21)}$ as follows

3.7.10 Proof of Lemma 3.4.14

$$
\begin{aligned}
(\tilde{v})^2_{n+\frac{3}{2}} &= \big(\frac{v_{n+1} + v_{n+2}}{2}\big)^2 = \big(v_{n+1} + \frac{h}{2}(Dv)_{n+\frac{3}{2}}\big)^2\\
&= v^2_{n+1} + \frac{h^2}{4}(Dv)^2_{n+\frac{3}{2}} + h v_{n+1}(Dv)_{n+\frac{3}{2}}\\
&= v^2_{n+1} + \frac{h^2}{4(c_d)^2_{n+\frac{3}{2}}}(c_d Dv)^2_{n+\frac{3}{2}} + v_{n+1}\frac{h}{(c_d)_{n+\frac{3}{2}}}(c_d Dv)_{n+\frac{3}{2}}. \qquad (3.7.6)
\end{aligned}
$$

Similarly, we have

$$
(\tilde{v})^2_{n+\frac{1}{2}} = v^2_{n+1} + \frac{h^2}{4(c_d)^2_{n+\frac{1}{2}}}(c_d Dv)^2_{n+\frac{1}{2}} - v_{n+1}\frac{h}{(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}}. \qquad (3.7.7)
$$

We thus write $Y_{21}^{(1,21)}$ as follows:

$$
\begin{aligned}
& Y_{21}^{(1,21)} \\
= & \int_0^T (s\lambda\phi(a))^3 [c^2(\psi')^3 \star |\tilde{v}|^2]_a \\
= & \int_0^T (s\lambda\phi(a))^3 (c^2\psi'^3)(a^+)\Big(v_{n+1}^2 + \frac{h^2}{4(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2 + v_{n+1}\frac{h}{2(c_d)_{n+\frac{3}{2}}}(c_d Dv)_{n+\frac{3}{2}}\Big) \\
- & \int_0^T (s\lambda\phi(a))^3 (c^2\psi'^3)(a^-)\Big(v_{n+1}^2 + \frac{h^2}{4(c_d)_{n+\frac{1}{2}}^2}(cDv)_{N+\frac{1}{2}}^2 - v_{n+1}\frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}}\Big) \\
= & \int_0^T (s\lambda\phi(a))^3 [c^2\psi'^3 \star]_a v_{n+1}^2 \\
+ & \int_0^T (s\lambda\phi(a))^3 \Big((c^2\psi'^3)(a^+)\frac{h^2}{4(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2 - (c^2\psi'^3)(a^-)\frac{h^2}{4(c_d)_{n+\frac{1}{2}}^2}(c_d Dv)_{n+\frac{1}{2}}^2\Big) \\
+ & \int_0^T (s\lambda\phi(a))^3 \Big((c^2\psi'^3)(a^+)\frac{h}{2(c_d)_{n+\frac{3}{2}}}(c_d Dv)_{n+\frac{3}{2}} - (c^2\psi'^3)(a^-)\frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}}\Big) v(a).
\end{aligned}
\tag{3.7.8}
$$

Moreover, the term $Y_{11}^{(2,1)}$ is given by

$$
\begin{aligned}
Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\Big(- \psi'(a^-)c(a^-)\bar{c}_d(a)(Dv)_{n+\frac{1}{2}}^2 + \psi'(a^+)c(a^+)\bar{c}_d(a)(Dv)_{n+\frac{3}{2}}^2\Big) \\
&= \int_0^T s\lambda\phi(a)\Big(- \psi'(a^-)\frac{c(a^-)\bar{c}_d(a)}{(c_d)_{n+\frac{1}{2}}^2}(c_d Dv)_{n+\frac{1}{2}}^2 + \psi'(a^+)\frac{c(a^+)\bar{c}_d(a)}{(c_d)_{n+\frac{3}{2}}^2}(c_d Dv)_{n+\frac{3}{2}}^2\Big).
\end{aligned}
$$

We estimate as

$$
\begin{aligned}
\frac{c(a^-)\bar{c}_d(a)}{(c_d)_{n+\frac{1}{2}}^2} &= \frac{\big((c_d)_{n+\frac{1}{2}} + \mathcal{O}(h)\big)\big((c_d)_{n+\frac{1}{2}} + (c_d)_{n+\frac{3}{2}}\big)}{2(c_d)_{n+\frac{1}{2}}^2} \\
&= \frac{\big((c_d)_{n+\frac{1}{2}} + \mathcal{O}(h)\big)\big(2(c_d)_{n+\frac{1}{2}} + \mathcal{O}(h)\big)}{2(c_d)_{n+\frac{1}{2}}^2} \\
&= 1 + h\mathcal{O}(1).
\end{aligned}
$$

Similarly,

$$
\frac{c(a^+)\bar{c}_d(a)}{(c_d)_{n+\frac{3}{2}}^2} = 1 + h\mathcal{O}(1).
$$

We thus obtain $Y_{11}^{(2,1)}$

$$\begin{aligned} Y_{11}^{(2,1)} &= \int_0^T s\lambda\phi(a)\Big(-\psi'(a^-)\big(1+h\mathcal{O}(1)\big)(c_dDv)^2_{n+\frac{1}{2}} + \psi'(a^+)\big(1+h\mathcal{O}(1)\big)(c_dDv)^2_{n+\frac{3}{2}}\Big) \\ &= \int_0^T s\lambda\phi(a)[\psi'\star(c_dDv)^2]_a \\ &+ \int_0^T s\lambda\phi(a)\psi'(a^+)\mathcal{O}(h)(c_dDv)^2_{n+\frac{3}{2}} + \int_0^T s\lambda\phi(a)\psi'(a^-)\mathcal{O}(h)(c_dDv)^2_{n+\frac{1}{2}}. \end{aligned} \tag{3.7.9}$$

Combining (3.7.8) with (3.7.9) we obtain

$$\begin{aligned} & Y_{11}^{(2,1)} + Y_{21}^{(1,21)} \\ = & \int_0^T s\lambda\phi(a)[\psi'\star(c_dDv)^2]_a + \int_0^T s^3\lambda^3\phi^3(a)[(\psi')^3c^2\star]_a v^2_{n+1} \\ + & \int_0^T s\lambda\phi(a)\psi'(a^+)\mathcal{O}(h)(c_dDv)^2_{n+\frac{3}{2}} + \int_0^T s\lambda\phi(a)\psi'(a^-)\mathcal{O}(h)(c_dDv)^2_{n+\frac{1}{2}} \\ + & \int_0^T (s\lambda\phi(a))^3\Big((c^2\psi'^3)(a^+)\frac{h^2}{4(c_d)^2_{n+\frac{3}{2}}}(c_dDv)^2_{n+\frac{3}{2}} - (c^2\psi'^3)(a^-)\frac{h^2}{4(c_d)^2_{n+\frac{1}{2}}}(c_dDv)^2_{n+\frac{1}{2}}\Big) \\ + & \int_0^T (s\lambda\phi(a))^3\Big((c^2\psi'^3)(a^+)\frac{h}{2(c_d)_{n+\frac{3}{2}}}(c_dDv)_{n+\frac{3}{2}} - (c^2\psi'^3)(a^-)\frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_dDv)_{n+\frac{1}{2}}\Big)v(a) \\ = & \mu+\mu_1, \end{aligned}$$

where

$$\mu = \int_0^T s\lambda\phi(a)[\psi'\star(c_dDv)^2]_a + \int_0^T s^3\lambda^3\phi^3(a)[c^2(\psi')^3\star]_a v^2_{n+1}.$$

and μ_1 can be written as

$$\begin{aligned} \mu_1 &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{3}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}} \\ &+ \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{3}{2}}v_{n+1} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}v_{n+1}. \end{aligned}$$

We can write

$$\begin{aligned} & [(\psi')\star(c_dDv)^2]_a \\ = & [(\psi')\star]_a(c_dDv)^2_{n+\frac{1}{2}} + [\star(c_dDv)]^2_a\psi'(a^+) + 2[\star(c_dDv)]_a\psi'(a^+)(c_dDv)_{n+\frac{1}{2}}. \end{aligned}$$

Indeed, we have

$$[(\psi')\star(c_dDv)^2]_a = (c_dDv)^2_{n+\frac{3}{2}}\psi'(a^+) - (c_dDv)^2_{n+\frac{1}{2}}\psi'(a^-),$$

and

$$\begin{aligned}
& [(\psi')\star]_a(c_dDv)^2_{n+\frac{1}{2}} + [\star(c_dDv)]^2_a\psi'(a^+) + 2[\star(c_dDv)]_a\psi'(a^+)(c_dDv)_{n+\frac{1}{2}} \\
= \;& (c_dDv)^2_{n+\frac{1}{2}}\psi'(a^+) - (c_dDv)^2_{n+\frac{1}{2}}\psi'(a^-) \\
+ \;& (c_dDv)^2_{n+\frac{3}{2}}\psi'(a^+) + (c_dDv)^2_{n+\frac{1}{2}}\psi'(a^+) - 2(c_dDv)_{n+\frac{3}{2}}(c_dDv)_{n+\frac{1}{2}}\psi'(a^+) \\
+ \;& 2(c_dDv)_{n+\frac{3}{2}}(c_dDv)_{n+\frac{1}{2}}\psi'(a^+) - 2(c_dDv)^2_{n+\frac{1}{2}}\psi'(a^+) \\
= \;& (c_dDv)^2_{n+\frac{3}{2}}\psi'(a^+) - (c_dDv)^2_{n+\frac{1}{2}}\psi'(a^-).
\end{aligned}$$

Moreover, by using Lemma 3.3.17, we obtain

$$\begin{aligned}
& [(\psi')\star(c_dDv)^2]_a \\
= \;& [(\psi')\star]_a(c_dDv)^2_{n+\frac{1}{2}} + [\star(c_dDv)]^2_a\psi'(a^+) + 2[\star(c_dDv)]_a\psi'(a^+)(c_dDv)_{n+\frac{1}{2}} \\
= \;& [(\psi')\star]_a(c_dDv)^2_{n+\frac{1}{2}} + \Big(\lambda^2 s^2[\star(c\phi\psi')]^2_a v^2_{n+1} + r_0^2 + 2\lambda s r_0[\star c\phi\psi']_a v_{n+1}\Big)\psi'(a^+) \\
& \qquad +2\Big(s\lambda[\star c\phi\psi']_a v_{n+1} + r_0\Big)\psi'(a^+)(c_dDv)_{n+\frac{1}{2}},
\end{aligned}$$

which gives

$$\begin{aligned}
\mu \;=\; & \int_0^T s\lambda\phi(a)[\psi'\star]_a(c_dDv)^2_{n+\frac{1}{2}} \\
+ \;& \int_0^T 2s^2\lambda^2\phi(a)[\star c\phi\psi']_a\psi'(a^+)v_{N+1}(c_dDv)_{n+\frac{1}{2}} \\
+ \;& \int_0^T s^3\lambda^3\phi(a)\Big([\star c\phi\psi']^2_a\psi'(a^+) + [c^2(\psi')^3\star]_a\phi^2(a)\Big)v^2_{n+1} \\
+ \;& \int_0^T s\lambda\phi(a)\psi'(a^+)r_0^2 + 2\int_0^T s^2\lambda^2\phi(a)[\star c\phi\psi']_a\psi'(a^+)r_0v_{n+1} \\
+ \;& 2\int_0^T s\lambda\phi(a)\psi'(a^+)r_0(c_dDv)_{n+\frac{1}{2}} \quad .
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
& [\star c\phi\psi']_a = c\phi\psi'|_{n+\frac{3}{2}} - c\phi\psi'|_{n+\frac{1}{2}} = c\phi\psi'|_{a^+} - c\phi\psi'|_{a^-} + h\mathcal{O}_\lambda(1) = \phi(a)[c\psi'\star]_a + h\mathcal{O}_\lambda(1), \\
& [\star c\phi\psi']^2_a = [c\phi\psi'\star]^2_a + 2[c\phi\psi'\star]_a h\mathcal{O}_\lambda(1) + h^2\mathcal{O}_\lambda(1) = \phi^2(a)[c\psi'\star]^2_a + h\mathcal{O}_\lambda(1).
\end{aligned}$$

We thus write μ as

$$\begin{aligned}
\mu &= \int_0^T s\lambda\phi(a)[\psi'\star]_a(c_dDv)^2_{n+\frac{1}{2}} \\
&+ \int_0^T 2s^2\lambda^2\phi^2(a)[c\psi'\star]_a\psi'(a^+)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^3\lambda^3\phi^3(a)\Big([c\psi'\star]^2_a\psi'(a^+) + [c^2(\psi')^3\star]_a\Big)v^2_{n+1} \\
&+ \int_0^T s\lambda\phi(a)\psi'(a^+)r_0^2 + 2\int_0^T s^2\lambda^2\phi^2(a)[c\psi'\star]_a\psi'(a^+)r_0v_{n+1} \\
&+ 2\int_0^T s\lambda\phi(a)\psi'(a^+)r_0(c_dDv)_{n+\frac{1}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\Re}(sh)v^2_{n+1} \\
&+ \int_0^T s\mathcal{O}_{\lambda,\Re}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\Re}(sh)r_0v_{n+1} \\
&= \int_0^T s\lambda\phi(a)[\psi'\star]_a(c_dDv)^2_{n+\frac{1}{2}} \\
&+ \int_0^T 2s^2\lambda^2\phi^2(a)[c\psi'\star]_a\psi'(a^+)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^3\lambda^3\phi^3(a)\Big([c\psi'\star]^2_a\psi'(a^+) + [c^2(\psi')^3\star]_a\Big)v^2_{n+1} \quad + \mu_r
\end{aligned}$$

where μ_r can be written as

$$\begin{aligned}
\mu_r &= \int_0^T s\mathcal{O}_\lambda(1)r_0^2 + \int_0^T s^2\mathcal{O}_\lambda(1)r_0v_{n+1} + \int_0^T s\mathcal{O}_\lambda(1)r_0(c_dDv)_{n+\frac{1}{2}} \\
&+ \int_0^T s^2\mathcal{O}_{\lambda,\Re}(sh)v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\Re}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\Re}(sh)r_0v_{n+1}.
\end{aligned}$$

We have thus achieved

$$\mu = \int_0^T s\lambda\phi(a)\big(\mathbf{A}u(t,a), u(t,a)\big) + \mu_r,$$

with $u(t,a) = \big((c_dDv)_{n+\frac{1}{2}}, s\lambda\phi(a)v_{n+1}\big)^t$ and the symmetric matrix $\mathbf{A}$ defined in Lemma 3.2.1.

From the choice made for the weight function β in Lemma 3.2.1 we find that:

$$\mu \geq C\alpha_0\int_0^T s\lambda\phi(a)(c_dDv)^2_{n+\frac{1}{2}} + C\alpha_0\int_0^T s^3\lambda^3\phi^3(a)v^2_{n+1} + \mu_r,$$

with $\alpha_0 > 0$.

3.7.11 Proof of Lemma 3.4.15

By using Lemma 3.3.17 we have

$$\begin{aligned} Y_{13} &= -\int_0^T r\bar{\bar{\rho}}(a^+)\partial_t v(a)(c_d Dv)_{n+\frac{3}{2}} + \int_0^T r\bar{\bar{\rho}}(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} \\ &= -\int_0^T r\bar{\bar{\rho}}(a^+)\partial_t v(a)\Big((c_d Dv)_{n+\frac{1}{2}} + J_1 v_{n+1} + J_2(c_d Dv)_{n+\frac{1}{2}} + J_3 h(rf)_{n+1}\Big) \\ &\quad + \int_0^T r\bar{\bar{\rho}}(a^-)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}}, \end{aligned}$$

where J_1, J_2 and J_3 are given as in Lemma 3.3.17.

Since $J_2 = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and $r\bar{\bar{\rho}} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ we can write

$$\begin{aligned} Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T r\bar{\bar{\rho}}(a^+)J_1 v(a)\partial_t v(a) \\ &\quad - \int_0^T r\bar{\bar{\rho}}(a^+)J_3\partial_t v(a)h(rf)_{n+1}. \end{aligned}$$

Futhermore, as $f = f_1 - \partial_t(\rho v)$ we thus find

$$\begin{aligned} Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} - \int_0^T r\bar{\bar{\rho}}(a^+)J_1 v(a)\partial_t v(a) \\ &\quad - \int_0^T r\bar{\bar{\rho}}(a^+)J_3\partial_t v(a)h\big(rf_1 - r\partial_t(\rho v)\big)_{n+1}. \end{aligned}$$

With an integration by parts w.r.t t for the second term above we obtain

$$\begin{aligned} Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \frac{1}{2}\int_0^T \partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big)v^2(a) \\ &\quad - \frac{1}{2}r\bar{\bar{\rho}}(a^+)J_1 v^2(a)|_{t=0}^{t=T} - \int_0^T r\bar{\bar{\rho}}(a^+)J_3\partial_t v(a)h(rf_1)_{n+1} \\ &\quad + \int_0^T r\bar{\bar{\rho}}(a^+)J_3\partial_t v(a)hr_{n+1}\big(\rho\partial_t v + \partial_t\rho v\big)_{n+1} \\ &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \frac{1}{2}\int_0^T \partial_t\big(r\bar{\bar{\rho}}(a^+)J_1\big)v^2(a) \\ &\quad + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} \\ &\quad + \int_0^T \big(1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)h\big(\partial_t v(a)\big)^2 + \frac{1}{2}\int_0^T r\bar{\bar{\rho}}(a^+)J_3 h(r\partial_t\rho)_{n+1}\partial_t\big(v^2(a)\big), \end{aligned}$$

where $r\bar{\bar{\rho}}, J_3$ are of the form $1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ and J_1 of the form $s\mathcal{O}_{\lambda,\mathfrak{K}}(1)$.

We apply an integration by parts in time for the last term

$$\begin{aligned} Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \frac{1}{2}\int_0^T \partial_t\big(r\bar{\tilde{\rho}}(a^+)J_1\big)v^2(a) \\ &+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1} \\ &+ \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)h\big(\partial_t v(a)\big)^2 - \frac{1}{2}\int_0^T \partial_t\big(r\bar{\tilde{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}\big)hv^2(a) \\ &+ \frac{1}{2}r\bar{\tilde{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}hv^2(a,.)|_{t=0}^{t=T}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\partial_t s = s(2t-T)\theta = sT\theta\mathcal{O}(1), \\ &\partial_t\rho = -\varphi(x)(\partial_t s)\rho = -\varphi(x)s(2t-T)\theta\rho, \\ &r\partial_t\rho = -\varphi(x)s(2t-T)\theta \\ &\partial_t(r\partial_t\rho) = sT^2\theta^2\mathcal{O}(1), \end{aligned} \tag{3.7.10}$$

by using (3.2.2)- (3.2.3).

Now we estimate the terms $\partial_t\big(r\bar{\tilde{\rho}}(a^+)J_1\big)$ and $\partial_t\big(r\bar{\tilde{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}\big)$. By recalling $\partial_t J_1 = sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, $\partial_t J_3 = T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ as well as using Proposition 3.3.14 and (3.7.10) we obtain

$$\begin{aligned} \partial_t\big(r\bar{\tilde{\rho}}(a^+)J_1\big) &= \partial_t\big(r\bar{\tilde{\rho}}(a^+)\big)J_1 + r\bar{\tilde{\rho}}(a^+)\partial_t J_1 \\ &= sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh), \end{aligned}$$

and

$$\begin{aligned} &\partial_t\big(r\bar{\tilde{\rho}}(a^+)J_3(r\partial_t\rho)_{n+1}\big) \\ =\ &\partial_t\big(r\bar{\tilde{\rho}}(a^+)\big)J_3(r\partial_t\rho)_{n+1} + r\bar{\tilde{\rho}}(a^+)\partial_t J_3(r\partial_t\rho)_{n+1} + r\bar{\tilde{\rho}}(a^+)J_3\partial_t\big((r\partial_t\rho)_{n+1}\big) \\ =\ &sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(1). \end{aligned}$$

Thus Y_{13} can be written

$$
\begin{aligned}
Y_{13} &= \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a)\\
&+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1}\\
&+ \int_0^T \big(1+\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)h\big(\partial_t v(a)\big)^2 + \int_0^T T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2(a)\\
&+ \frac{1}{2}(1+\mathcal{O}_{\lambda,\mathfrak{K}}(s(t)h))h\Big(-\varphi(a)s(t)(2t-T)\theta(t)v^2(a,.)\Big)|_{t=0}^{t=T}.
\end{aligned}
$$

We observe that for $0 < sh < \epsilon_3(\lambda)$ with $\epsilon_3(\lambda)$ sufficiently small we have

$$
r\bar{\tilde{\rho}} = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}(sh) > 0.
$$

Additionally, $\varphi(x) < 0$ then the last term of Y_{13} are non-negative. From that, we estimate Y_{13} as follows

$$
\begin{aligned}
Y_{13} &\geq \int_0^T \mathcal{C}_{\lambda,\mathfrak{K}}h(\partial_t v(a))^2 + \int_0^T \big(sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + T^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)v^2(a)\\
&+ s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a)|_{t=0}^{t=T} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\partial_t v(a)(c_d Dv)_{n+\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)\partial_t v(a)h(rf_1)_{n+1}.
\end{aligned}
$$

3.7.12 Proof of Lemma 3.4.16

On the one hands, as $f = f_1 - \partial_t(\rho v)$ we write

$$
\begin{aligned}
(rf)_{n+1} &= (rf_1)_{n+1} - (r\partial_t(\rho v))_{n+1}\\
&= (rf_1)_{n+1} - \Big((r\rho)\partial_t v + (r\partial_t\rho)v\Big)_{n+1}\\
&= (rf_1)_{n+1} - (\partial_t v)_{n+1} - sT\theta\mathcal{O}_\lambda(1)v_{n+1}.
\end{aligned}
$$

We thus obtain

$$
|(rf)_{n+1}|^2 \leq C\Big((rf_1)_{n+1}^2 + (\partial_t v)_{n+1}^2 + s^2T^2\theta^2\mathcal{O}_\lambda(1)v_{n+1}^2\Big). \tag{3.7.11}
$$

On the other hands,

$$[\rho_1 \star \rho_2] = [\rho_1 \star]_a(\rho_2)_{n+\frac{1}{2}} + \rho_1(a^+)[\star\rho_2]_a, \tag{3.7.12}$$

and we recall

$$(c_dDv)_{n+\frac{3}{2}} - (c_dDv)_{n+\frac{1}{2}} = [\star c_dDv]_a = \lambda s[\star c\phi\psi']_a v_{n+1} + r_0,$$

where r_0 is given in Lemma 3.3.17 as

$$r_0 = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}} + h\mathcal{O}_{\lambda,\mathfrak{K}}(1)(rf)_{n+1},$$

We then have

$$\begin{aligned}(c_dDv)^2_{n+\frac{3}{2}} &= (c_dDv)^2_{n+\frac{1}{2}} + [\star c_dDv]^2_a + 2[\star c_dDv]_a(c_dDv)_{n+\frac{1}{2}} \\ &= (c_dDv)^2_{n+\frac{1}{2}} + \lambda^2 s^2[\star c\phi\psi']^2_a v^2_{n+1} + r_0^2 + 2\lambda s[\star c\phi\psi']_a r_0 v_{n+1} \\ &\quad +2\lambda s[\star c\phi\psi']_a v_{n+1}(c_dDv)_{n+\frac{1}{2}} + 2r_0(c_dDv)_{n+\frac{1}{2}}. \end{aligned} \tag{3.7.13}$$

and we compute

$$\begin{aligned} r_0^2 &= s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(rf)^2_{n+1} \\ &\quad +s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}}v_{n+1} + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}v_{n+1} \\ &\quad +\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}h(rf)_{n+1}. \end{aligned}$$

By applying Cauchy-Schwartz inequality we have

$$(c_dDv)^2_{n+\frac{3}{2}} \leq \mathcal{O}(1)(c_dDv)^2_{n+\frac{1}{2}} + s^2\mathcal{O}_\lambda(1)v^2_{n+1} + \mathcal{O}(1)r_0^2 \tag{3.7.14}$$

$$r_0^2 \leq s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v^2_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(rf)^2_{n+1}, \tag{3.7.15}$$

$$sr_0v_{n+1} \leq \big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big)v^2_{n+1} + \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}} + h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)^2_{n+1}, \tag{3.7.16}$$

$$s^2 r_0 v_{n+1} \leq \big(s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(1)\big) v_{n+1}^2 + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}^2 + C_\epsilon h \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)_{n+1}^2, \tag{3.7.17}$$

$$\begin{aligned} sT\theta r_0 v_{n+1} &\leq \big(s^2 T\theta \mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big) v_{n+1}^2 \\ &+ s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}^2 + h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)_{n+1}^2, \end{aligned} \tag{3.7.18}$$

$$sr_0 (cDv)_{n+\frac{1}{2}} \leq s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1}^2 + \big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big)(c_d Dv)_{n+\frac{1}{2}}^2 + C_\epsilon \mathcal{O}_{\lambda,\mathfrak{K}}(sh) h (rf)_{n+1}^2, \tag{3.7.19}$$

$$(\partial_t v(a)) r_0 \leq \mathcal{O}_{\lambda,\mathfrak{K}}(1) h (\partial_t v(a))^2 + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}^2 + s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh) v_{n+1}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(1) h (rf)_{n+1}^2. \tag{3.7.20}$$

We estimate following terms

The first term, by using (3.7.14) we have

$$\begin{aligned} \left|Y_{11}^{(2,2)}\right| &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{3}{2}}^2 \\ &\leq \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 r_0^2. \end{aligned}$$

Moreover, by using (3.7.15) we obtain

$$\begin{aligned} &\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 r_0^2 \\ \leq\ &\int_0^T s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4 v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf)_{n+1}^2. \end{aligned}$$

Then, by using (3.7.11) we estimate $Y_{11}^{(2,2)}$

$$\begin{aligned}\left|Y_{11}^{(2,2)}\right| &\leq \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v_{n+1}^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf)_{n+1}^2 \\ &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\Big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 h(\partial_t v)_{n+1}^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf_1)_{n+1}^2.\end{aligned}$$

For the next term, using (3.7.12) and Lemma 3.3.17 we obtain

$$\begin{aligned}Y_{12} &= \int_0^T s\mathcal{O}_\lambda(1)v(a)[c\psi'^2 \star (c_d Dv)]_a \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)(cDv)_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)(cDv)_{n+\frac{3}{2}} \\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)(c_d Dv)_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\Big((c_d Dv)_{n+\frac{1}{2}} + s\mathcal{O}_\lambda(1)v(a) + r_0\Big) \\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)(c_d Dv)_{n+\frac{1}{2}} + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)r_0.\end{aligned}$$

Using (3.7.16) yields

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)r_0 \\ \leq &\int_0^T \Big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big)v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(rf)_{n+1}^2.\end{aligned}$$

By using (3.7.11) we obtain

$$\begin{aligned}|Y_{12}| &\leq \int_0^T \big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\big)v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v)_{n+1}^2 \\ &+ \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(1)(c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf_1)_{n+1}^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\tilde{v}_{n+\frac{1}{2}} &= v_{n+1} - \frac{h}{2(c_d)_{n+\frac{1}{2}}}(c_d Dv)_{n+\frac{1}{2}} = v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{1}{2}}, \\ \tilde{v}_{n+\frac{3}{2}} &= v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{3}{2}}.\end{aligned} \tag{3.7.21}$$

By using (3.7.21), (3.7.14) we obtain

$$\begin{aligned}(\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}} &\leq \mathcal{O}(1)v^2_{n+1} + \mathcal{O}(h^2)(cDv)^2_{n+\frac{3}{2}} + \mathcal{O}(h^2)(cDv)^2_{n+\frac{1}{2}} \\ &\leq \big(\mathcal{O}(1) + \mathcal{O}(sh)^2\big)v^2_{n+1} + \mathcal{O}(h^2)(cDv)^2_{n+\frac{1}{2}} + \mathcal{O}(h^2)r_0^2\end{aligned}$$

Thus, we have the following estimate

$$\begin{aligned}\left|Y_{21}^{(1,22)}\right| &= \int_0^T \Big(s^2\mathcal{O}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\Big)\Big((\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}}\Big) \\ &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big((\tilde{v})^2_{n+\frac{1}{2}} + (\tilde{v})^2_{n+\frac{3}{2}}\Big) \\ &\leq \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0^2.\end{aligned}$$

Futhermore, using (3.7.15) we have

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0^2 \\ \leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(rf)^2_{n+1}.\end{aligned}$$

By using (3.7.11) we get

$$\begin{aligned}\left|Y_{21}^{(1,22)}\right| &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\Big)v^2_{n+1} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3h(\partial_tv)^2_{n+1} \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(rf_1)^2_{n+1}.\end{aligned}$$

For the term $Y_{22}^{(1)}$ we have

$$\begin{aligned}Y_{22}^{(1)} &= \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{1}{2}} + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h^2}{2}(Dv)_{n+\frac{3}{2}} \\ &= \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}} + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{3}{2}}\Big)v(a) \\ &= \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}} + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2r_0\Big)v(a) \\ &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a)(c_dDv)_{n+\frac{1}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v^2(a) + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v(a)r_0.\end{aligned}$$

Using (3.7.16) we achieve

$$\begin{aligned}
&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v(a) r_0 \\
\leq\ & \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf)_{n+1}^2.
\end{aligned}$$

Using (3.7.11), we estimate $Y_{22}^{(1)}$ as:

$$\begin{aligned}
\left|Y_{22}^{(1)}\right| \ \leq\ & \int_0^T \Big(s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^4\Big) v_{n+1}^2 + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 h(\partial_t v)_{n+1}^2 \\
+\ & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (c_d Dv)_{n+\frac{1}{2}}^2 + \int_0^T h\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3 (rf_1)_{n+1}^2.
\end{aligned}$$

And, using (3.7.21) and Lemma 3.3.17 we obtain

$$\begin{aligned}
Y_{23}^{(1)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\tilde{v}_{n+\frac{1}{2}} + \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\tilde{v}_{n+\frac{3}{2}} \\
=\ & \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\Big(v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{1}{2}}\Big) \\
+\ & \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)(\partial_t v(a))\frac{h}{2}\Big(v_{n+1} + \mathcal{O}(h)(c_d Dv)_{n+\frac{3}{2}}\Big) \\
=\ & \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v(a))v(a) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{1}{2}} \\
+\ & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{3}{2}} \\
=\ & \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v(a))v(a) + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{1}{2}} \\
+\ & \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))\Big((c_d Dv)_{n+\frac{1}{2}} + \lambda s[\star c\phi\psi']_a v_{n+1} + r_0\Big).
\end{aligned}$$

In addition, with s, λ enough large, sh enough small and with applying Young's inequality and (3.7.20) yield

$$\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(\partial_t v(a))v(a) \leq \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v(a))^2 + \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)v^2(a).$$

$$\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))(c_d Dv)_{n+\frac{1}{2}} \leq \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v(a))^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_d Dv)_{n+\frac{1}{2}}^2.$$

$$\begin{aligned}\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(\partial_t v(a))r_0 &\leq \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(\partial_t v(a))^2 + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(c_d Dv)^2_{n+\frac{1}{2}} \\ &+ \int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(v)^2_{n+1} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(rf)^2_{n+1}.\end{aligned}$$

Using (3.7.11), we estimate $Y_{23}^{(1)}$

$$\begin{aligned}Y_{23}^{(1)} &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1) + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v^2_{n+1} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v(a))^2 \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_d Dv)^2_{n+\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(rf_1)^2_{n+1}.\end{aligned}$$

Applying Lemma 3.3.17 we have

$$\begin{aligned}Y_{31}^{(1)} &= \int_0^T T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h}{2}(Dv)_{n+\frac{1}{2}} + \int_0^T T\theta s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a)\frac{h}{2}(Dv)_{n+\frac{3}{2}} \\ &= \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v(a)(c_d Dv)_{n+\frac{1}{2}} \\ &+ \int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v(a)\Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(1)v(a) + r_0\Big)\end{aligned}$$

By using (3.7.18) we obtain

$$\begin{aligned}\int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}r_0 &\leq \int_0^T \Big(s^2T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 + sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2\Big)v^2_{n+1} \\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_d Dv)^2_{n+\frac{1}{2}} + \int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2 h(rf)^2_{n+1}.\end{aligned}$$

We have

$$\int_0^T sT\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)_{n+\frac{1}{2}}v_{n+1} \leq \int_0^T sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_d Dv)^2_{n+\frac{1}{2}}.$$

With (3.7.11) we thus estimate $Y_{31}^{(1)}$ as

$$\begin{aligned}\left|Y_{31}^{(1)}\right| &\leq \int_0^T \Big(s^2T\theta\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\Big)v_{n+1}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(\partial_t v)_{n+1}^2\\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf_1)_{n+1}^2.\end{aligned}$$

Next, by using (3.7.14) we estimate $\mu_1^{(1)}$ as

$$\begin{aligned}\mu^{(1)} &= \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{3}{2}}^2+\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2\\ &\leq \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2+\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0^2\end{aligned}$$

By making use of (3.7.15) we have

$$\begin{aligned}&\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0^2\\ \leq &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3v_{n+1}^2+\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf)_{n+1}^2.\end{aligned}$$

Using (3.7.11) we obtain

$$\begin{aligned}\mu_1^{(1)} &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v_{n+1}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(\partial_t v)_{n+1}^2\\ &+ \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf_1)_{n+1}^2.\end{aligned}$$

By making use Lemma 3.3.17 we have

$$\begin{aligned}\mu_1^{(2)} &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{3}{2}}+\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}}\\ &= \int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}}+\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2+\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1}.\end{aligned}$$

Applying Young's inequality and using (3.7.17) yield

$$\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}}\leq\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2+\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2.$$

$$
\begin{aligned}
&\int_0^T s^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)r_0v_{n+1}\\
\leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v_{n+1}^2+\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf)_{n+1}^2.
\end{aligned}
$$

Using (3.7.11) we have

$$
\begin{aligned}
\mu_1^{(2)} \leq\ &\int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v_{n+1}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(\partial_tv)_{n+1}^2\\
+\ &\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf_1)_{n+1}^2.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
\mu_1 \leq\ &\int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+sT^2\theta^2\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^3\Big)v_{n+1}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(\partial_tv)_{n+1}^2\\
+\ &\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2h(rf_1)_{n+1}^2.
\end{aligned}
$$

Now, we estimate some terms of μ_r. By using (3.7.15)- (3.7.19) we have

$$
\begin{aligned}
&\int_0^T s\mathcal{O}_{\lambda}(1)r_0^2\\
\leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2v_{n+1}^2+\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)^2(c_dDv)_{n+\frac{1}{2}}^2+\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}^2.
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T s^2\mathcal{O}_{\lambda}(1)r_0v_{n+1} \leq\ &\int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+\epsilon s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big)v_{n+1}^2\\
+\ &\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)_{n+\frac{1}{2}}^2+C_\epsilon\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}^2.
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T s\mathcal{O}_{\lambda}(1)r_0(c_dDv)_{n+\frac{1}{2}} \leq\ &\int_0^T s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}^2+C_\epsilon\int_0^T \mathcal{O}_{\lambda,\mathfrak{K}}(sh)h(rf)_{n+1}^2\\
+\ &\int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)+\epsilon s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big)(c_dDv)_{n+\frac{1}{2}}^2.
\end{aligned}
$$

$$\int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v_{n+1}(c_dDv)_{n+\frac{1}{2}} \leq \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)(c_dDv)^2_{n+\frac{1}{2}} + \int_0^T s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)v^2_{n+1}.$$

Using (3.7.11) we have:

$$\begin{aligned}
\mu_r &\leq \int_0^T \Big(s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + sT^2\theta^2\mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh)^2 + \epsilon s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big)v^2_{n+1} \\
&+ \int_0^T \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh)h(\partial_t v)^2_{n+1} + \int_0^T \mathcal{O}_{\epsilon,\lambda,\mathfrak{K}}(sh)h(rf_1)^2_{n+1} \\
&+ \int_0^T \Big(s\mathcal{O}_{\lambda,\mathfrak{K}}(sh) + \epsilon s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\Big)(c_dDv)^2_{n+\frac{1}{2}}.
\end{aligned}$$

Bibliography

[BDL07] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem*, J. Math. Anal. Appl. 336 (2007), 865887.

[BDL11] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau, *Carleman estimates for stratified media*, J. Functional Analysis 260(2011), 3645-3677.

[Bel03] M. Bellassoued. Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization. Asymptotic Anal., 35:257279, 2003.

[BHL10a] Boyer F, Hubert F and J. Le Rousseau , *Discrete Carleman estimates for the elliptic operators and uniform controllability of semi discretized parabolic equations*, J. Math. Pur. Appl., vol 93 (2010), pages 240-276.

[BHL10b] Boyer F, Hubert F and J. Le Rousseau , *Discrete Carleman estimates for the elliptic operators in arbitrary dimension and applications*, SIAM J. Control Optim, vol 48 (2010), pages 5357-5397.

[BHL11] Boyer F, Hubert F and J. Le Rousseau , *Uniform null-controllability properties for space/time-discretized parabolic equations*, Numer. Math., Vol. 118, no 4, pp. 601-661 (2011).

[BL12] Boyer F, J. Le Rousseau, *Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations*, preprint (2012).

[Car39] T. Carleman, *Sur une problème d'unicité pour les systèmes d'équation aux dérivées partielles à deux variables indépendantes*, Ark. Math. Astr. Fys. 26B(17) (1939)1-9.

[CGL94] Carthel C, Glowinski R and Lions J L, *On exact and approximate Boundary Controllabilities for the heat equation: a numerical approach*, J. Optimal. Theory Appl., vol 82 (1994), pages 429-484.

[CT06] Y. Chitour, E. Trélat, *Controllability of partial differential equations, Advanced Topics in Control Systems Theory*, Lecture Notes in Control and Inform. Sci, vol 328, Spinger, London (2006).

[DOP02] A. Doubova, A. Osses, J. P. Puel, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients*, ESIAM Control Optim. Calc. Var. 8 (2002) 621-661.

[ET99] I. Ekeland, R. Temam, *Convex analysic and variational problems*, Classics in Applied Mathematics 28, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1999.

[EZ12] Ervedoza S., Zuazua E., *The Wave Equation: Control and Numerics , " Control and stabilization of PDE's "*, P. M. Cannarsa , J. M. Coron, eds., Lecture Notes in Mathematics 2048, CIME Subseries, Springer Verlag, 2012, pp. 245-340., published in 2012.

[EV09] S. Ervedoza, J. Valein *On the observability of abstract time-discrete linear parabolic equations*, Revista Matematica Complutense, 23, no. 1 p. 163-190.

[EZZ08] S. Ervedoza, C. Zheng and E. Zuazua *On the observability of time-discrete conservative linear systems*, Journal of Functional Analysis, 2008, 254(12), pages 3037-3078.

[F87] H. O. Fattorini, *A unified theory of necessary conditions for nonlinear nonconvex control systems*, Appl. Math. Optim. , 15 (1987), pp. 141-185.

[FCG06] E. Fernández-Cara, S. Guerrero, *Global Carleman inequalities for parabolic systems an application to controllability*, SIAM J. Control Optim. 45(4) (2006)1395-1446.

[FCM09] E. Fernandez- Cara and A. Munch, *Numerical null controllability of the 1D heat equation: primal algorithms*. Prepint (2009).

[FCM10] E. Fernandez- Cara and A. Munch, *Numerical null controllability of the 1D heat equation: dual algorithms*. Prepint (2010).

[FCZ02] $E.Fernández-Cara$ and E. Zuazua, *On the null controllability of the one-dimensional heat equation with BV coefficients*, Comput. Appl. Math. 21 (2002), 167190.

[FF91] H. O. Fattorini and H. Frankowska, *Necessary conditions for infinite dimensional problems, Mathematics of Control*, Signals, and Systems, (1991), Spinger - Verlag New York.

[FI96] A. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, vol. 34, Seoul National University, Korea, 1996, Lecture notes.

[FL96] C. Fabre and G. Lebeau. Prolongement unique des solutions de lequation de Stokes. Comm. Partial Dierential Equations, 21:573596, 1996.

[GLH08] R. Glowinski, J.L. Lions and J. He, *Exact and approximate controllability for distributed parameter systems: a numerical approach*, Encyclopedia of Mathematical and its Applications, 117. Cambridge University Press, Cambridge, 2008.

[IZ99] J.A Infante, E. Zuazua, *Boundary observability for the space semi-discretizations of the 1-D wave equation*, M2AN Math. Model. Number. Anal. 33 (2) (1999) 407-438.

[KS91] M. V. Klibanov and F. Santosa, *A computational quasi-reversibilty method for Cauchy problems for Laplace's equation*, SIAM J. Appl. Math. 51 (1991), 1653-1675.

[L07] J. Le Rousseau, *Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients*, J. Differential Equations 233 (2007), 417447.

[L88] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev. 30 (1988) 1-68.

[LL11a] J. Le Rousseau and G. Lebeau, *On carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*. ESAIM Control Optim. Calc. Var. (2011), 26 pages.

[LR95] G. Lebeau and L. Robbiano, *Contrôle exact de léquation de la chaleur*, Comm. Partial Differential Equations 20 (1995), 335356.

[LR97] G. Lebeau and L. Robbiano. Stabilisation de lequation des ondes par le bord. Duke Math. J., 86:465491, 1997.

[LR10] J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficents with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, Arch. Rational Mech. Anal., Vol. 195, 953-990, 2010.

[LR11] J. Le Rousseau and, L. Robbiano. *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Invent. Math., 183(2011) 245-336.

[LT00] I.Lasiecka, R.Triggiani, *Control theory for partial differential equation: continuous and approximation theories*. I. Abstract parabolic systems, Encyclopedia of Mathematics and its Applications, vol. 74, Cambridge University Press, Cambridge, 2000.

[LT06] S. Labbé , E. Trélat, *Uniform controllability of semidiscrete approximations of parabolic control system*, Systems and Control Letters, vol 55 (2006), pages 597-609.

[LY85] X. Li and Y. L. Yao, *Maximum Principle of distributed parameter systems with time lags*, Lecture Notes in Control and Information Sciences, vol. 75, Spinger- Verlag, New York, 1985, pp. 410-427.

[LY91] X. Li and J. Yong, *Necessary conditions for optimal control of distributed parameter systems*, SIAM J. Control Optim., vol. 29, number 4, 1991, pp. 895-909.

[LZ98] A. Lopez, E. Zuazua, *Some new results related to the null controllability of the 1-D heat equation*, Sem. EDP, Ecole Polytechnique, VIII, 1998, 1-22.

[LZ02] L. Leon. E. Zuazua, *Boundary controllability of the finite-difference space semi-discretizations of the beam equation*, ESAIM Control Optim. Calc. Var. 8 (2002) 827-862.

[M10] Luc Miller, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Contin. Dyn. Syst. B 14 (2010), no. 4, 1465-1485.

[M12] Luc Miller, *Resolvent conditions for the control of unitary groups and their approximations*, Journal of Spectral Theory 2, 1 (2012) 1–55.

[MZ10] A. Munch and E. Zuazua, *Numerical approximation of the null controls for the heat equation through transmutation*, Inverse Problems (2010), 43 pp.

[N12] T. Nguyen, *The uniform controllability property of semidiscrete approximations for the parabolic distributed parameter systems in Banach*, preprint. (2012).

[NZ03] M. Negreanu, E. Zuazua, *Uniform boundary controllability of discre 1-D wave equation*, Systems Control Lett. 48 (3-4) (2003) 261-279.

[P83] A.Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. 44 (1983).

[PM10] P. Pedregal, A. Munch, *Numerical null controllability of the heat equation through a variational approach*, 2010

[PS62] Pontryagin, L.S. et al. *The Mathematical Theory of Optimal Processes*, vol. 4. Interscience, 1962

[Rob91] L. Robbiano. Théorème dunicité adapté au contrôle des solutions des problèmes hyperboliques. ‘ Comm. Partial Differential Equations, 16:789800, 1991.

[S05] O. Staffans, *Well-posed linear systems*, Encyclopedia of Mathematical and its Applications, 103. Cambridge University Press, Cambridge, 2005.

[T05] E. Trélat, (French version) *Optimal control : Theory and applications, Concrete Mathematics*, Vuibert, Paris, 2005, 246 pages.

[Tat95] D. Tataru. Carleman estimates and unique continuation for the Schroedinger equation. Dierential Integral Equations, 8:901905, 1995.

[TW07] M.Tucsnak, G.Weiss, *Observability and controllability of infinite dimensional systems*, 2007.

[Zua99] E. Zuazua, *Boundary observability for finite-difference space semi-discretizations of the 2-D wave equation in the square*, J. Math. Pures Appl. (9) 78 (1999) 523-563.

[Zua02] E. Zuazua, *Controllability of the partial differential equations and its semi-discrete approximations*, Discrete Contin. Dyn. Syst. 8 (2) (2002) 469-513.

[Zua04] E. Zuazua, *Optimal and approximate control of finite-difference approximation schemes for 1-D wave equation*, Rendiconti di Matematica VIII, 24, II (2004) 201-237.

[Zua05] E. Zuazua, *Propagation, Observation, Control and Numerical Approximation of Wave approximated by finite-difference method*, SIAM Review 47, 2 (2005) 197-243.

[Zua06] E. Zuazua, *Control and numerical approximation of the wave and heat equations*, International Congress of Mathematicians, Madrid, Spain III (2006), 1389-1417.

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Contrôlabilité uniforme des équations aux dérivées partielles disécrétisées

Résumé :

Dans cette thèse, nous étudions les propriétés de contrôlabilité uniforme des semi-discrets approximations de systèmes paraboliques. Dans une première partie, nous nous intéressons à la minimisation de L^q-norme $(q > 2)$ des contrôles semidiscrete pour l'équation parabolique. Notre objectif est de dépasser la limitation de [LT06] à propos de l'ordre 1/2 de l'absence de limites d'opérateur de contrôle. Plus précisément, nous montrons que la propriété d'observabilité uniforme est également titulaire dans L^q $(q > 2)$, même dans le cas d'un degré d'absence de limites supérieure à 1/2. En outre, une procédure de minimisation pour calculer les commandes d'approximation est fournie. L'étude de l'optimalité L^q dans la première partie est dans un contexte général. Cependant, les inégalités d'observabilité discrets qui sont obtenus ne sont pas aussi précises que celles dérivées puis avec des estimations de Carleman. Dans une seconde partie, dans le contexte particulier de unidimensionnels-finis différences nous démontrons une inégalité de Carleman pour une version semi-discret de l'opérateur parabole $\partial_t - \partial_x(c\partial_x)$ qui permet pour dériver les inégalités d'observabilité qui sont beaucoup plus précis. On considère ici que dans le cas où le coefficient de diffusion a un saut qui donne une formulation du problème de transmission. Conséquence de cette inégalité de Carleman, on en déduit cohérentes nul contrôlabilité des résultats pour les classes de linéaires et semi-linéaire des équations paraboliques.

Mots clés : Contrôlabilité, inégalité d'observabilité discrétisation, inégalité de Carleman.

Contrôlabilité uniforme des équations aux dérivées partielles disécrétisées

Abstract :

In this thesis, we study uniform controllability properties of semi-discrete approximations for parabolic systems. In a first part, we address the minimization of the L^q-norm $(q > 2)$ of semidiscrete controls for parabolic equation. Our goal is to overcome the limitation of [LT06] about the order 1/2 of unboundedness of the control operator. Namely, we show that the uniform observability property also holds in L^q $(q > 2)$ even in the case of a degree of unboundedness greater than 1/2. Moreover, a minimization procedure to compute the approximation controls is provided. The study of L^q optimality in the first part is in a general context. However, the discrete observability inequalities that are obtained are not so precise than the ones derived then with Carleman estimates. In a second part, in the discrete setting of one-dimensional finite-differences we prove a Carleman estimate for a semi-discrete version of the parabolic operator $\partial_t - \partial_x(c\partial_x)$ which allows one to derive observability inequalities that are far more precise. Here we consider in case that the diffusion coefficient has a jump which yields a transmission problem formulation. Consequence of this Carleman estimate, we deduce consistent null-controllability results for classes of linear and semi-linear parabolic equations.

Keywords : Controllability, discretization, observability inequality, Carleman estimate.

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