Non local evolution equations and phase transition problems

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Equations d'évolution non locales et problèmes de transition de phase

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À mes parents.

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Equations d'évolution non locales et problèmes de transition de phase

Résumé

L'objet de cette thèse est d'étudier le comportement en temps long de solutions d'équations d'évolution non locales ainsi que la limite singulière d'équations et de systèmes d'équations aux dérivées partielles, où intervient un petit paramètre ε.

Au Chapitre 1, nous considérons une équation de réaction-diffusion non locale avec conservation au cours du temps de l'intégrale en espace de la solution ; cette équation a été initialement proposée par Rubinstein et Sternberg pour modéliser la séparation de phase dans un mélange binaire. Le problème de Neumann associé possède une fonctionnelle de Lyapunov, c'est-à-dire une fonctionnelle qui décroit selon les orbites. Après avoir prouvé que la solution est confinée dans une région invariante, nous étudions son comportement en temps long. Nous nous appuyons sur une inégalité de Lojasiewicz pour montrer qu'elle converge vers une solution stationnaire quand t tend vers l'infini. Nous évaluons également le taux de la convergence et calculons précisément la solution stationnaire limite en dimension un d'espace.

Le Chapitre 2 est consacré à l'étude de l'équation différentielle non locale que l'on obtient en négligeant le terme de diffusion dans l'équation d'Allen-Cahn non locale étudiée au Chapitre 1. Sans le terme de diffusion, la solution ne peut pas être plus régulière que la fonction initiale. C'est la raison pour laquelle on ne peut pas appliquer la méthode du Chapitre 1 pour l'étude du comportement en temps long de la solution. Nous présentons une nouvelle méthode basée sur la théorie des réarrangements et sur l'étude du profil de la solution. Nous montrons que la solution est stable pour les temps grands et présentons une caractérisation détaillée de sa limite asymptotique quand t tend vers l'infini. Plus précisément, la fonction limite est une fonction en escalier, qui prend au plus deux valeurs, qui coïncident avec les points stables d'une équation différentielle associée. Nous montrons aussi par un contre-exemple non trivial que, quand une hypothèse sur la fonction initiale n'est pas satisfaite, la fonction limite peut prendre trois valeurs, qui correspondent aux points instable et stables de l'équation différentielle associée.

Nous étudions au Chapitre 3 une équation différentielle ordinaire non locale qui a été proposée par M. Nagayama. Une difficulté essentielle est que le dénominateur dans le terme de réaction non local peut s'annuler. Nous appliquons un théorème de point fixe lié à une application contractante pour démontrer que le problème à valeur initiale correspondant possède une solution unique qui reste confinée dans un ensemble invariant. Ce problème possède une fonctionnelle de Lyapunov, qui est un ingrédient essentiel pour démontrer que la solution converge vers une solution stationnaire constante par morceaux quand t tend vers l'infini.

Au Chapitre 4, nous considérons un modèle d'interface diffuse pour la croissance de tumeurs, où intervient une équation d'ordre quatre de type Cahn-Hilliard. Après avoir introduit un modèle de champ de phase associé, on étudie formellement la limite singulière de la solution quand le coefficient du terme de réaction tend vers l'infini. Plus précisément, nous montrons que la solution converge vers la solution d'un problème à frontière libre.

Mot clés. Flot de gradient, équations non locales, inégalité de Lojasiewicz, stabilisation des solutions, équations d'Allen-Cahn avec conservation de l'intégrale, comportement en temps long, équations de réaction-diffusion, perturbations singulières, mouvement de l'interface, modèles de croissance de tumeurs, développements asymptotiques.

AMS subject classifications. 35K57, 35K50, 35K20, 35R35, 35R37, 35B40, 35B25.

Non local evolution equations and phase transition problems

Abstract

The aim of this thesis is to study the large time behavior of solutions of nonlocal evolution equations and to also study the singular limit of equations and systems of parabolic partial differential equations involving a small parameter ε.

In Chapter 1, we consider a nonlocal reaction-diffusion equation with mass conservation, which was originally proposed by Rubinstein and Sternberg as a model for phase separation in a binary mixture. The corresponding Neumann problem possesses a Lyapunov functional, namely a functional which decreases in time along solution orbits. After having proved that the solution is confined in an invariant region, we study its large time behavior and apply a Lojasiewicz inequality to show that it converges to a stationary solution as t tends to infinity. We also evaluate the rate of convergence and precisely compute the limiting stationary solution in one space dimension.

Chapter 2 is devoted to the study of a nonlocal evolution equation which one obtains by neglecting the diffusion term in the nonlocal Allen-Cahn equation studied in Chapter 1. Without the diffusion term, the solution can not be expected to be more regular than the initial function. Moreover, because of the absence of the diffusion term, the method of Chapter 1 can not be applied to study the large time behavior of the solution. We present a new method based up on rearrangement theory and the study of the solution profile. We show that the solution stabilizes for large times and give a detailed characterization of its asymptotic limit as t tends to infinity. More precisely, it turns out that the limiting function is a step function, which takes at most two values, which are stable points of a corresponding ordinary differential equation. We also show by means of a nontrivial counterexample that, when a certain hypothesis on the initial function does not hold, the limiting function may take three values. One of them is the unstable point and the two others are the stable points of the ordinary differential equation.

We study in Chapter 3 a nonlocal ordinary differential equation which has been proposed by M. Nagayama. The nonlocal term involves a denominator which may vanish. We apply a contraction fixed point theorem to prove the existence of a unique solution which stays confined in an invariant region. We also show that the corresponding initial value problem possesses a Lyapunov functional and prove that the solution stabilizes for large times to a step function, which takes at most two values.

In Chapter 4, we consider a diffuse-interface tumor-growth model which involves a fourth order Cahn-Hilliard type equation. Introducing a related phase-field model, we formally study the singular limit of the solution as the reaction coefficient tends to infinity. More precisely, we show that the solution converges to the solution of a moving boundary problem.

Key words. Gradient flow, non local equations, Lojasiewicz inequality, stabilisation of solutions, mass-conserved Allen-Cahn equation, large time behavior, reaction-diffusion system, singular perturbation, interface motion, matched asymptotic expansions, tumor-growth model.

AMS subject classifications. 35K57, 35K50, 35K20, 35R35, 35R37, 35B40, 35B25.

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Introduction

Cette thèse porte sur l'étude d'équations et de systèmes d'équations paraboliques non linéaires, locales ou non locales, ainsi que sur celle d'équations différentielles ordinaires non linéaires et non locales ; il s'agit d'une part d'analyser le comportement en temps long des solutions, et d'autre part d'en décrire la limite singulière quand un paramètre lié au coefficient de termes de réaction tend vers l'infini.

Les problèmes d'évolution dont nous faisons l'étude possèdent le plus souvent des solutions stationnaires multiples, qui forment parfois un continuum ; si l'on souhaite étudier leur comportement asymptotique en temps long, une première étape consiste à démontrer une propriété de compacité des orbites, de façon à pouvoir conclure que des sous-suites convergent vers une limite quand t tend vers l'infini ; le problème qui se pose ensuite est de déterminer s'il s'agit seulement de la convergence de sous-suites ou si c'est toute l'orbite de solution qui converge vers une solution stationnaire bien définie. Dans le cas où le problème a la forme d'un système gradient, c'est-à-dire qu'il possède une fonctionnelle de Lyapunov, une possibilité est de s'appuyer sur l'inégalité de Łojasiewicz. C'est cette inégalité que Łojasiewicz [9], [10] a utilisé pour prouver la stabilisation des solutions bornées de systèmes gradients dans $\mathbb{R}^n$, qui sont en fait des systèmes d'équations différentielles ordinaires. Cette idée a été étendue plus tard aux systèmes gradients en dimension infinie par L. Simon [15], qui en a démontré une version appropriée pour établir la stabilisation de la solution de l'équation d'Allen-Cahn. De très nombreux travaux ont suivi, comme par exemple ceux de Haraux et Jendoubi [5] et Jendoubi [7] qui ont considéré plus particulièrement le cas de l'équation des ondes non linéaires tandis que Feireisl et Simondon [3] ont démontré la stabilisation d'équations paraboliques dégénérées. Par la suite Hofmann et Rybka [14] ont démontré une version de l'inégalité de Łojasiewicz pour l'équation de Cahn-Hilliard equation, ce qui leur a permis d'obtenir également des résultats de stabilisation. Dans cette thèse, nous étudions le comportement en temps long des solutions d'une équation de réaction-diffusion avec un terme non local ; dans ce but nous nous appuyons sur une inégalité de Łojasiewicz.

Nous abordons aussi l'équation différentielle non locale que l'on obtient en négligeant le terme de diffusion dans l'équation d'Allen-Cahn non locale étudiée au Chapitre 1 si bien qu'il ne semble pas possible d'appliquer l'inégalité de Łojasiewicz. On souhaite d'une part établir la stabilisation des orbites de solutions et d'autre part obtenir une caractérisation aussi précise que possible des fonctions de l'ensemble ω-limite pour une classe aussi large que possible de conditions initiales. C'est ce qui nous amène à appliquer la théorie des réarrangement [8] afin de transformer notre problème d'évolution en un problème unidimensionnel dont la solution est décroissance en espace. Pour ce problème, il est presque immédiat que les orbites sont relativement compactes dans tous les L^p ; après avoir étudié le

comportement asymptotique des solutions, il nous faut finalement transposer nos résultats au problème original. Nous étudions également une variante de l'équation différentielle non locale, dans laquelle le dénominateur peut s'annuler.

Nous nous intéressons finalement à la limite singulière de solutions d'équations ou de systèmes d'équations de réaction-diffusion ou de modèles de champ de phase. Quand on modélise l'évolution d'interfaces, on peut choisir ou bien de s'appuyer sur des modèles où les interfaces sont diffuses, ou bien sur des modèles où elles sont abruptes. En particulier, Lowengrub et son groupe ont développé à la fois des des modèles d'interfaces diffuses et des modèles d'interfaces abruptes pour décrire la croissance de tumeurs cancéreuses. Un but serait d'établir un lien entre ces modèles et en particulier de savoir dire quels modèles réguliers correspondent à quels problèmes à frontière libre limites. Ces problèmes s'appuient souvent sur des équations de type Cahn-Hilliard ; comme il nous semble plus aisé de rechercher la limite singulière de modèles de champ de phase, une idée essentielle est de plonger tout d'abord l'équation de Cahn-Hillard dans un modèle de champ de phase correspondant dont on déduit formellement la limite singulière. Nos résultats sont seulement formels ; des résultats rigoureux sont encore hors d'atteinte du fait de la complexité du système.

Chapitre 1 : Convergence vers des états stationnaires pour les solutions d'une équation de réaction-diffusion non locale

Ce chapitre fait l'objet d'un article écrit en collaboration avec S. Bousaïd (Université de Batna, Algérie) et D. Hilhorst (Université de Paris-Sud).

Nous nous intéressons à un modèle qui est initialement proposé par Rubinstein et Sternberg [13] pour modliser la séparation de phase dans un mélange binaire,

$$\begin{cases} u_t = \Delta u + f(u) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(u) & \text{dans } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{sur } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

où Ω est un ouvert connexe borné de $\mathbb{R}^N, N \geq 1$, de frontière régulière, et où $u_0 \in L^\infty(\Omega)$. Nous supposons que f est un polynôme de la forme

$$f(s) = \sum_{i=0}^{n} a_i s^i \qquad \text{où} \quad n \geq 1, \text{ impaire} \quad \text{et} \quad a_n < 0.$$

Ce problème possède la fonctionnelle de Lyapunov

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u).$$

Les résultats essentiels de cette partie sont les suivants. On démontre tout d'abord l'existence et l'unicité de la solution, et on prouve qu'elle devient instantanément très régulière.

Bien que ce problème, qui est non local, n'admette pas de principe de comparaison, on peut démontrer l'existence d'ensembles invariants.

Notre résultat principal est le suivant : les solutions se stabilisent quand t tend vers l'infini et la convergence des solutions vers les solutions stationnaires est exponentielle ou polynômial. De plus on peut calculer précisément la fonction limite de le cas de la dimension d'un espace. Ce résultat est loin d'être trivial dans la mesure où ce problème possède un grand nombre de solutions stationnaires.

L'outil principal pour les démonstrations est l'inégalité de Lojasiewicz. La première version de cette inégalité a été initialement proposée par Lojasiewicz [9] dans le cas d'un espace de dimension finie (voir également [1, Theorem 1.1]).

Théorème 0.0.1 (Lojasiewicz)**.** *Soient $U \subset \mathbb{R}^N$ un ouvert, $G : U \to \mathbb{R}$ une fonction réelle analytique, et $a \in U$. Alors, il existe des constantes $\theta \in (0, \frac{1}{2}], c, \sigma > 0$ telles que pour tout $z \in U, \|z - u\| \leq \sigma$,*

$$|G(z) - G(a)|^{1-\theta} \leq c\|\nabla G(z)\|_{\mathbb{R}^N}.$$

Lojasiewicz a appliqué cette inégalité pour démontrer la convergence vers un équilibre des solutions bornées du système gradient

$$\dot{u} + \nabla G(u) = 0.$$

L'idée de Lojasiewicz a été ensuite généralisée à des espaces de dimension infinie et utilisée pour prouver la convergence vers des états stationnaires de solutions bornées d'un bon nombre d'équations d'évolution comme par exemple un équation de réaction-diffusion (Simon [15]), une équation des ondes (Haraux et Jendoubi [5] et Jendoubi [7]), une équation parabolique dégénérée (Feireisl et Simondon [3]) et l'équation de Cahn-Hilliard (Hoffmann et Rybka [14]).

Le problème non local nous amène à considérer l'inégalité de Lojasiewicz sur l'espace de Hilbert V, défini par

$$V = \Big\{ u \in H^1(\Omega) : \int_\Omega u(x)dx = 0 \Big\}.$$

et nous vérifions que la version suivante de l'inégalité de Lojasiewicz est satisfaite

Théorème 0.0.2. *Soit $\varphi \in V$ un point critique de E (c'est-à-dire $E'(\varphi) = 0$) tel que $s_1 \leq \varphi \leq s_2$. Alors, il existe des constantes $\theta \in (0, \frac{1}{2}]$ et $C, \sigma > 0$ telles que*

$$|E(u) - E(\varphi)|^{1-\theta} \leq C\|E'(u)\|_{V^*}$$

pour tout $\|u - \varphi\|_V \leq \sigma$.

La preuve de ce theorème est basée sur des résultats de Chill [1].

Chapitre 2 : Comportement en temps long des solutions d'une équation différentielle non locale.

Ce chapitre fait l'objet d'un article écrit en collaboration avec D. Hilhorst (Université de Paris-Sud), H. Matano (Université de Tokyo) and H. Weber (Université de Warwick).

On considère le problème à valeurs intiales pour l'équation différentielle non locale

$$(P)\quad \begin{cases} u_t = f(u) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(u) & \text{dans } \Omega \times (0,\infty), \\ \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 1$, $f \in C^1(\mathbb{R})$ et $u_0 \in L^\infty(\Omega)$. On suppose qu'il existe $m < M$ tel que $f'(m) = f'(M) = 0$ et que

$$f' < 0 \ \text{ sur } \ (-\infty, m) \cup (M, +\infty), \quad f' > 0 \ \text{ sur } \ (m, M).$$

On suppose de plus qu'il existe $s_* < s^*$ satisfaisant

$$\begin{cases} s_* < m < M < s^* \\ f(s_*) = f(M), f(s^*) = f(m). \end{cases}$$

Ces hypothèses impliquent en particulier que la fonction f est de type bistable. Nous démontrons tout d'abord l'existence et l'unicité de la solution et nous prouvons que cette solution reste dans un ensemble invariant à tous les temps. Nous étudions ensuite son comportement asymptotique en temps long et démontrons que si u_0 satisfait l'une des deux hypothèses

(i) $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0(x)\,dx \notin [s_*, s^*]$

(ii) $s_* \leq u_0 \leq s^*$ et $|\{x \in \Omega : \quad |\{u_0(x) = s\}| = 0 \qquad \text{pour tout } \ s \in (m, M)$

alors, quand t tend vers l'infini, la solution converge vers une fonction en escalier qui prend au plus deux valeurs, qui coïncident avec les points stables d'une équation différentielle ordinaire associée. De plus, nous montrons par un contre-exemple que quand l'hypothèse (ii) sur la fonction initiale n'est pas satisfaite, la fonction limite peut prendre trois valeurs. L'un d'eux est le point instable et les deux autres sont les points stables de l'équation différentielle ordinaire associée.

Des difficultés essentielles dans l'étude de ce problème sont liées à l'absence de terme de diffusion ; en particulier, il est plus difficile de démontrer la compacité relative des orbites de solution. De plus, il est plus difficile sinon impossible d'appliquer la méthode du premier chapitre basée sur l'application d'une inégalité de Lojasiewicz.

Pour résoudre ces difficultés, on s'appuie sur la théorie des réarrangements : on réarrange les valeurs de la solution u de façon à obtenir une solution $u^\sharp$ qui est définie sur $(0, |\Omega|) \times [0, +\infty)$ et est telle que pour tout $t \geq 0$, $u^\sharp(\cdot, t)$ est décroissante. Nous démontrons que la norme BV de $u^\sharp$ est bornée uniformément en temps si bien que l'ensemble ω-limite de $u^\sharp$ est non vide. L'unicité des éléments de l'ensemble ω-limite est prouvée en considérant l'ensemble des points instables, c'est-à-dire l'ensemble

$$\{x \in (0, |\Omega|) : f'(u^\sharp(x,t)) \geq 0\},$$

et en montrant qu'il est décroissant au sens des inclusions. Finalement, on prouve que la stabilisation de $u^\sharp$ en grands temps implique celle de u.

Chapitre 3 : Sur un problème d'évolution non local proposé par M. Nagayama

On étudie un problème à valeur initiale proposé par M. Nagayama pour modéliser le mouvement d'une bulle

$$\begin{cases} u_t = u^2(1-u) - u(1-u)\dfrac{\displaystyle\int_\Omega u^2(1-u)}{\displaystyle\int_\Omega u(1-u)} & \text{dans} \quad \Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 1$. On suppose que la fonction initiale u_0 satisfait l'une des deux hypothèses

$(\mathbf{H_1})$ $u_0 \in L^\infty(\Omega)$, $1 \leq u_0(x)$ pour presque partout $x \in \Omega$; et $u_0 \not\equiv 1$.

$(\mathbf{H_2})$ $u_0 \in L^\infty(\Omega)$, $u_0(x) \leq 0$ pour presque partout $x \in \Omega$; et $u_0 \not\equiv 0$.

Une difficulté essentielle provient du fait que le dénominateur peut s'annuler dans l'un des termes de réaction. Nous considérons le problème obtenu par réarrangement décroissant unidimensionnel correspondant et appliquons un théorème de point fixe lié à une application contractante pour démontrer que le problème à valeur initiale possède une solution unique qui reste confinée dans un ensemble invariant. Ce problème possède une fonctionnelle de Lyapunov, ce qui est un ingrédient essentiel pour démontrer que la solution converge une solution stationnaire constante par morceaux quand t tend vers l'infini.

Chapitre 4 : Limite asymptotique formelle d'un modèle d'interface diffuse pour la croissance de tumeurs

Cette partie de la thèse correspond à des travaux en collaboration avec D. Hilhorst (Université de Paris-Sud), J. Kampmann (Technische Universität Dortmund) and K. Van der Zee (Technische Universiteit Eindhoven).

L'étude de la croissance de tumeurs est un sujet d'intérêt essentiel en biologie médicale. Dans une phase initiale de son développement, la croissance d'une tumeur est due à la consommation de nutriments qui diffuse à travers le tissu environnant,et cette consommation est modélisée par des termes de réaction. Pour la modélisation, on peut choisir ou bien s'appuyer sur des modèles où les interfaces sont diffuses, ou bien sur des modèles où elles sont abruptes. Un grand nombre de travaux est consacré à ce phénomène, comme par example [2, 16, 12, 11]. Le but de ce chapitre est de prendre comme point de départ un modèle d'interface diffuse et d'effectuer une dérivation formelle du problème à frontière libre limite.

Nous considérons ici un modèle d'interface diffuse pour la croissance de tumeurs qui améliore un modèle étudié par [6]. Il fait intervenir quatre constituants : une phase de cellules contaminées, $u \approx 1$, une phase de cellules saines $u \approx -1$, une phase d'eau extra-cellulaire riche en nutriments $\sigma \approx 1$ et une phase d'eau extra-cellulaire pauvre en nutriments

$\sigma \approx 0$. Les équations sont données par

$$\begin{aligned} u_t &= \Delta\mu + \varepsilon^{-1} p(u)(\sigma - \delta\mu) \\ \mu &= -\varepsilon^{-1} f(u) - \varepsilon\Delta u \\ \sigma_t &= \Delta\sigma - \varepsilon^{-1} p(u)(\sigma - \delta\mu) \end{aligned}$$

où $\delta > 0$ est un paramètre de régularisation ; la fonction de croissance $p(u)$ est définie par

$$p(u) := \begin{cases} 2p_0\sqrt{W(u)} & u \in [-1,1] \\ 0 & \text{ailleurs,} \end{cases}$$

où $W(u) := -\int_{-1}^{u} f(s)\,ds$. On suppose que la fonction bistable $f(u)$ possède deux racines stables ± 1, une racine unstable 0 et que $\int_{-1}^{1} f(s)ds = 0$.

Le but de ce chapitre est de déterminer la limite singulière de ces équations quand $\varepsilon \downarrow 0$; ceci nous amène à considérer le problème plus général (P_ε), qui a la forme d'un modèle de champ de phase :

$$\boxed{\begin{aligned} \alpha\mu_t^\varepsilon + u_t^\varepsilon &= \Delta\mu^\varepsilon + \varepsilon^{-1} p(u^\varepsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon) && \text{dans } \ \Omega \times (0,+\infty), \\ \alpha u_t^\varepsilon &= \Delta u^\varepsilon + \varepsilon^{-2} f(u^\varepsilon) + \varepsilon^{-1}\mu^\varepsilon && \text{dans } \ \Omega \times (0,+\infty), \\ \sigma_t^\varepsilon &= \Delta\sigma^\varepsilon - \varepsilon^{-1} p(u^\varepsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon) && \text{dans } \ \Omega \times (0,+\infty), \end{aligned}}$$

avec des conditions aux limites de Neumann homogènes et des conditions initiales. On retrouve le modèle d'origine en posant $\alpha = 0$ dans le Problème (P_ε).

La deuxième équation dans le modèle de champ de phase est une équation de type Allen-Cahn pour u^ε. Quand $\varepsilon \downarrow 0$, u^ε converge vers une fonction limite u qui prend deux valeurs -1 et 1 et une interface, que l'on note $\Gamma(t)$, sépare les régions $\Omega^+(t) = \{x \in \Omega : u(x,t) = 1\}$ et $\Omega^-(t) = \{x \in \Omega : u(x,t) = -1\}$. On suppose ici que $\Gamma(t)$ est une hypersurface sans bords régulière qui se déplace avec la vitesse V_n.

Nous nous proposons de rechercher formellement le problème à frontière libre limite que nous notons (P_0) ; les fonctions inconnues sont u, μ, σ et $\Gamma(t)$, et l'on suppose que u, μ, σ sont les limites de $u^\varepsilon, \mu^\varepsilon, \sigma^\varepsilon$ quand $\varepsilon \downarrow 0$; nous montrons que le problème (P_0) est donné par

$$\boxed{\begin{aligned} u(x,t) &= \begin{cases} 1 & \text{dans} \quad \Omega^+(t), t \in [0,T) \\ -1 & \text{dans} \quad \Omega^-(t), t \in [0,T) \end{cases} \\ \alpha\mu_t &= \Delta\mu && \text{dans } \ \Omega \setminus \Gamma(t) \times \{t\}, \\ \sigma_t &= \Delta\sigma && \text{sur } \ \Omega \setminus \Gamma(t) \times \{t\}, \\ \alpha V_n &= -(N-1)\kappa + \mathcal{C}\mu && \text{sur } \ \Gamma(t), \\ [\![\mu]\!] &= [\![\sigma]\!] = 0 && \text{sur } \ \Gamma(t), \\ [\![\frac{\partial\mu}{\partial n}]\!] &= -2V_n + 2\sqrt{2}p_0(\sigma - \delta\mu) && \text{sur } \ \Gamma(t), \\ [\![\frac{\partial\sigma}{\partial n}]\!] &= -2\sqrt{2}p_0(\sigma - \delta\mu) && \text{sur } \ \Gamma(t), \end{aligned}} \qquad (1)$$

avec des conditions aux limites de Neumann homogènes et des conditions initiales. Ici, κ est la courbure moyenne de l'interface et $[\![\cdot]\!]$ désigne le saut au travers de l'interface.

Nos résultats principaux sont les suivants.

Théorème 0.0.3. *Le Problème (P_ε) possède la fonctionnelle de Lyapunov E_ε, définie par*

$$E_\varepsilon(u,\mu,\sigma) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u) + \frac{\alpha\mu^2}{2} + \frac{\sigma^2}{2\delta}\Big).$$

Théorème 0.0.4. *Le Problème (P_0) possède la fonctionnelle de Lyapunov E, définie par*

$$E(\Gamma,\mu,\sigma) := \frac{2}{\mathcal{C}}\int_\Gamma 1\,\mathrm{d}\Gamma + \int_\Omega \Big(\frac{\alpha\mu^2}{2} + \frac{\sigma^2}{2\delta}\Big),$$

Théorème 0.0.5. *Soit $(u^\varepsilon,\mu^\varepsilon,\sigma^\varepsilon)$ une solution du Problème (P_ε). On suppose que le Problème (P_0) possède une solution classique unique (Γ,μ,σ) sur l'intervalle $[0,T]$. Si*

$$u^\varepsilon \longrightarrow u, \quad \mu^\varepsilon \longrightarrow \mu, \quad \sigma^\varepsilon \longrightarrow \sigma \quad \textit{dans un sens suffisamment fort,}$$

quand $\varepsilon \to 0$, alors (Γ,μ,σ) coïncide avec la solution classique du Problème (P_0) sur l'intervalle $[0,T]$.

Une difficulté essentielle de la démonstration est liée au fait que nous devons trouver la limite du terme réaction $\varepsilon^{-1}p(u^\varepsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon)$ quand $\varepsilon \to 0$.

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Chapter 1

Convergence to steady state for the solutions of a non local reaction-diffusion equation

Résumé. Nous considérons une équation de réaction-diffusion avec conservation de l'intégrale, qui a été proposée par Rubinstein et Sternberg pour modéliser la séparation de phase dans un mélange binaire. Nous étudions le comportement en temps long de la solution et démontrons qu'elle converge vers une solution stationnaire quand t tend vers l'infini. Nous évaluons également le taux de la convergence et calculons précisément la solution stationnaire limite en dimension un d'espace.

Abstract. We consider a nonlocal reaction-diffusion equation with mass conservation, which was originally proposed by Rubinstein and Sternberg as a model for phase separation in a binary mixture. We study the large time behavior of the solution and show that it converges to a stationary solution as t tends to infinity. We also evaluate the rate of convergence and precisely compute the limit stationary solution in one space dimension.

1.1 Introduction

We consider the non local initial value problem

$$(P)\begin{cases} u_t = \Delta u + f(u) - \fint_\Omega f(u) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a connected open set with smooth boundary $\partial\Omega$; ∂_ν is the outer normal derivative to $\partial\Omega$ and

$$\fint_\Omega f(u) := \frac{1}{|\Omega|}\int_\Omega f(u(x))\,dx.$$

This model is **mass conserved**, namely

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \text{for all } \ t > 0,$$

and it possesses a free energy functional which coincides with the usual Allen-Cahn functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx,$$

where $F(u) := \displaystyle\int_0^u f(s)ds$.

Problem (P) was introduced by Rubinstein and Sternberg [25] as a model for phase separation in a binary mixture. We first prove in this paper the existence, uniqueness and regularity of the solutions. Although this problem is a non local problem, we can prove l'existence of invariant set. The principle result is the large time behavior. We show that the solution converges to a stationary solution as t tends to infinity. We also evaluate the rate of this convergence and precisely compute the limit stationary solution in one space dimension.

The main tool to study the large time behavior is a Lojasiewicz inequality that was first proposed by Lojasiewicz himself [19], [21]. He showed that all bounded solutions of gradient systems in $\mathbb{R}^N$, (which are an ODE systems), converge to a stationary solution. This idea was subsequently developed in infinite-dimensional spaces for proving the convergence to steady state of bounded solution of several local equations such as reaction-diffusion equation (Simon [27]), wave equation (Haraux and Jendounbi [12]), degenerated parabolic equation (Feireisl et Simondon [11]) and Cahn-Hilliard equation (Hofmann and Rybka [26]); let us also mention the book by Huang [16]. In this paper, we present a version of Lojasiewicz inequality for a non local problem.

In [25], the authors consider the model in which f is bistable type, typical example $f(s) = s - s^3$. In this paper, we assume that the function f is of the following form

$$f(s) = \sum_{i=0}^{n} a_i s^i, \quad \text{where } \ n \geq 1 \ \text{ is an odd number, } a_n < 0. \tag{1.1}$$

Note that there exists a constant $c_1 > 0$ satisfying

$$f'(s) \leq c_1 \quad \text{for all} \quad s \in \mathbb{R}. \tag{1.2}$$

Constants $\mathbf{s_1}, \mathbf{s_2}$: Let $s_1 < s_2$ be two constants such that

$$f(s_2) < f(s) < f(s_1) \qquad \text{for all} \quad s \in (s_1, s_2), \tag{1.3}$$

Note that we can choose s_1, s_2 such that s_1 is arbitrarily small and s_2 is arbitrarily large.

Assumption on initial data: We will make the following hypotheses on the initial data: $(\mathbf{H_0})$: $u_0 \in L^2(\Omega)$ and $s_1 \leq u_0 \leq s_2$ a.e on Ω.

Theorem 1.1.1. *Assume that Hypotheses* $(\mathbf{H_0})$ *holds. Then, Problem* (P) *possesses a unique solution* $u \in C([0,\infty); L^2(\Omega))$ *which satisfies for every* $T > 0$,

$$u \in L^\infty(Q_T) \cap L^2(0,T; H^1(\Omega)) \quad \textit{and} \quad u_t \in L^2(0,T; (H^1(\Omega))^*),$$

where $Q_T := \Omega \times (0,T)$. *Moreover* $u \in C^\infty(\overline{\Omega} \times (0,+\infty))$,

$$s_1 \leq u(x,t) \leq s_2 \qquad \textit{for all} \quad x \in \overline{\Omega}, t > 0;$$

and

$$\{u(t), t \geq 1\} \quad \textit{is relatively compact in} \quad C^m(\overline{\Omega}) \quad \textit{for all} \quad m \in \mathbb{N}.$$

Theorem 1.1.2. *Let* $(\mathbf{H_0})$ *hold and let* u *be the unique solution of Problem* (P). *Then there exists a smooth function* φ *such that*

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{C^m(\overline{\Omega})} = 0 \quad \textit{as} \quad t \to \infty, \quad \textit{for all} \quad m \in \mathbb{N}.$$

Moreover,

$$\int_\Omega \varphi = \int_\Omega u_0,$$

and φ *is a smooth solution of the stationary problem*

$$(S) \begin{cases} \Delta\varphi = -f(\varphi) + \fint_\Omega f(\varphi) & \textit{in} \quad \Omega, \\ \partial_\nu \varphi = 0 & \textit{on } \partial\Omega. \end{cases}$$

The proofs of Theorem 1.1.1 and Theorem 1.1.2 are based on the auxiliary problem $(\overline{P})$, which will be introduced in Section 2. The advantage is that Problem $(\overline{P})$ has a bounded nonlinearity. We will first study the existence, the uniqueness and the large time behavior of solutions to Problem $(\overline{P})$, then show that the solution of Problem $(\overline{P})$ coincides with the unique solution of Problem (P). Finally, we deduce the results of Problem (P) from the results of Problem $(\overline{P})$.

The remainder of the paper is organized as follows: in Section 1.3, we introduce Problem $(\overline{P})$ and we prove the existence and uniqueness of the solution of Problem $(\overline{P})$; we also give a regularity result. Theorem 1.1.1 is proved in Section 1.2.3. Section 1.3 is devoted to prove a version of Lojasiewicz inequality for the energy functional E of Problem $(\overline{P})$ (cf. Theorem 1.3.8). In Section 1.4, we apply the Lojasiewicz inequality to prove Theorem 1.1.2. We also precisely compute the limit stationary solution in one space dimension in this section. The rate of this convergence is established in this Section 1.5.

1.2 Existence and uniqueness of solution

1.2.1 Problem $(\overline{P})$

In order to prove the existence of the solution of Problem (P), we consider an auxiliary problem which has a bounded nonlinearity. First, we introduce the function $\bar{f} \in C_c^\infty(\mathbb{R})$ as follows,

$$\bar{f}(s) = \begin{cases} 0 & \text{if } s \le s_1 - 2, \\ f(s) & \text{if } s \in [s_1 - 1, s_2 + 1], \\ 0 & \text{if } s \ge s_2 + 2. \end{cases} \tag{1.4}$$

We then define the auxiliary problem $(\overline{P})$ by

$$(\overline{P}) \begin{cases} \overline{u}_t = \Delta \overline{u} + \bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u}) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu \overline{u} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \overline{u}(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Note that there exists a constant $c_2 > 0$ such that

$$|\bar{F}(s)|, |\bar{f}(s)|, |\bar{f}(s)s|, |\bar{f}'(s)| < c_2 \quad \text{for all} \quad s \in \mathbb{R}, \tag{1.5}$$

where $\bar{F}(s) := \int_0^s \bar{f}(\tau) d\tau$.

Lemma 1.2.1. *Let $u_0 \in L^2(\Omega)$. Then, for any $T > 0$ arbitrary, Problem $(\overline{P})$ possesses a unique solution $\overline{u}$ which satisfies*

$$\overline{u} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)); \quad \overline{u}_t \in L^2(0,T; (H^1(\Omega))^*). \tag{1.6}$$

Moreover,

$$\overline{u} \in C([0,\infty); L^2(\Omega)).$$

Remark 1.2.2. Consider the operator $-\Delta$ with homogeneous Neumann boundary conditions. Denote by

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_i \le \cdots$$

the eigenvalues of the Laplacian, and $w_i, i = 1, \ldots$ the corresponding unit eigenfunctions. Remark that the w_i's are smooth functions up to boundary and that they form an orthonormal basis of $L^2(\Omega)$. Moreover

$$w_1 = \frac{1}{|\Omega|^{1/2}}, \quad \int_\Omega w_i = 0 \quad \text{for} \quad i \ge 2.$$

Proof of Lemma 1.2.1. We apply the Galerkin method and look for an approximate solution of the form

$$\overline{u}_m(t) = \sum_{i=1}^m g_{mi}(t)\, w_i \tag{1.7}$$

satisfying

$$\int_\Omega \overline{u}_{mt} w_j + \int_\Omega \nabla \overline{u}_m \nabla w_j = \int_\Omega \bar{f}(\overline{u}_m) w_j - \fint_\Omega \bar{f}(\overline{u}_m) \int_\Omega w_j \tag{1.8}$$

for $j = 1, \dots, m$ and

$$\overline{u}_m(0) = \overline{u}_{m0} := \sum_{i=1}^{m} g_{mi}^0 w_i \to u_0 \text{ in } L^2(\Omega) \text{ as } m \to \infty. \tag{1.9}$$

Since $\int_\Omega w_j = 0$ for $2 \le j \le m$, we deduce that the equations (1.8) form a nonlinear differential system for the functions $g_{m1}, \dots, g_{mm}$:

$$g'_{m1}(t) = 0, \tag{1.10}$$

$$g'_{mj} + \lambda_j g_{mj} = \int_\Omega \bar{f}\big(\sum_{i=1}^{m} g_{mi}(t)\, w_i\big) w_j \qquad \text{for} \quad 2 \le j \le m. \tag{1.11}$$

The condition (1.9) forms the m scalar initial conditions

$$g_{m1}(0) = g_{m1}^0, \tag{1.12}$$

$$g_{mj}(0) = g_{mj}^0 \quad \text{for} \quad 2 \le j \le m. \tag{1.13}$$

It follows from (1.10) and (1.12) that

$$g_{m1}(t) = g_{m1}^0 \quad \text{for all} \quad t \ge 0. \tag{1.14}$$

Substituting (1.14) in (1.11), we obtain a nonlinear differential system of $m-1$ variables. Then the nonlinear differential system (1.11) with the initial condition (1.13) has a maximal solution defined on some interval $(0, T_m)$. In fact $T_m = \infty$ because of the following *a priori* estimates.

***A priori* estimates for $\overline{u}_m$:** First, we note from (1.7) that

$$\int_\Omega \overline{u}_m(t) = g_{m1}(t)|\Omega|^{\frac{1}{2}};$$

and from (1.9) that

$$g_{m1}^0|\Omega|^{\frac{1}{2}} = \int_\Omega \overline{u}_{m0}(x)\, dx \to \int_\Omega u_0(x)\, dx \quad \text{as} \quad m \to \infty.$$

These together with (1.14) implies that

$$\int_\Omega \overline{u}_m(t)\, dx = \int_\Omega \overline{u}_{m0}(x)\, dx \to \int_\Omega u_0(x)\, dx \quad \text{as} \quad m \to \infty \tag{1.15}$$

for all $0 \le t < T_m$. Consequently, there exists a positive constant c_3 independent of m such that

$$\Big| \int_\Omega \overline{u}_m(t)\, dx \Big| \le c_3 \quad \text{for all} \quad 0 \le t < T_m. \tag{1.16}$$

Multiplying (1.8) by g_{mj} and summing on $j = 1, \dots, m$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\overline{u}_m|^2 + \int_\Omega |\nabla \overline{u}_m|^2 = \int_\Omega \bar{f}(\overline{u}_m)\overline{u}_m - \fint_\Omega \bar{f}(\overline{u}_m) \int_\Omega \overline{u}_m. \tag{1.17}$$

Note from (1.5) that

$$|\bar{f}(s)s|, |\bar{f}(s)| \le c_2 \qquad \text{for all} \quad s \in \mathbb{R}.$$

This together with (1.16) and (1.17) follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\overline{u}_m|^2 + \int_\Omega |\nabla \overline{u}_m|^2 \le c_2|\Omega| + c_2c_3.$$

Therefore, we integrate this inequality from 0 to t with t arbitrary, and then take $t = T$ to deduce that

$$\|\overline{u}_m\|_{L^\infty(0,T;L^2(\Omega))}, \|\overline{u}_m\|_{L^2(0,T;H^1(\Omega))} \le K_1(\|u_0\|_{L^2(\Omega))} + T), \tag{1.18}$$

where K_1 is a constant independent of m.

***A priori* estimates for $\overline{u}_{mt}$:** Next, we give an estimate for $\overline{u}_{mt}$. Fix any $\eta \in H^1(\Omega)$, with $\|\eta\|_{H^1(\Omega)} \le 1$. We write

$$\eta = \eta_1 + \eta_2,$$

where $\eta_1 \in \text{span}\{w_i\}_{i=1}^m$ and $\displaystyle\int_\Omega \eta_2 w_i = 0$ for all $i = 1, \dots, m$. Note that $\eta_2 \in H^1(\Omega)$ and that

$$\int_\Omega \nabla\eta_2 \nabla w_i = -\int_\Omega \eta_2 \Delta w_i = \lambda_i \int_\Omega \eta_2 w_i = 0 \ \text{ for all } \ 1 \le i \le m.$$

Thus, η_2 is also orthogonal to $\text{span}\{w_i\}_{i=1}^m$ with respect to the scalar product in $H^1(\Omega)$. In particular, η_2 is orthogonal to η_1 in $H^1(\Omega)$, hence

$$\|\eta_1\|_{H^1(\Omega)} \le \|\eta\|_{H^1(\Omega)} \le 1. \tag{1.19}$$

We deduce from (1.8) that

$$\int_\Omega \overline{u}_{mt}\eta = \int_\Omega \overline{u}_{mt}\eta_1 = -\int_\Omega \nabla\overline{u}_m \nabla\eta_1 + \int_\Omega \bar{f}(\overline{u}_m)\eta_1 - \fint_\Omega \bar{f}(\overline{u}_m)\int_\Omega \eta_1.$$

Therefore, in view of (1.5); we have

$$\begin{aligned}\left|\int_\Omega \overline{u}_{mt}\eta\right| &\le \|\nabla\overline{u}_m\|_{L^2(\Omega)}\|\nabla\eta_1\|_{L^2(\Omega)} + 2c_2\|\eta_1\|_{L^1(\Omega)} \\ &\le \|\nabla\overline{u}_m\|_{L^2(\Omega)}\|\nabla\eta_1\|_{L^2(\Omega)} + c_4\|\eta_1\|_{L^2(\Omega)} \\ &\le \|\nabla\overline{u}_m\|_{L^2(\Omega)} + c_4,\end{aligned}$$

where the last inequality follows from (1.19); c_4 is a constant independent of m. Since $\eta \in H^1(\Omega)$ is an arbitrary function such that $\|\eta\|_{H^1(\Omega)} \le 1$, it follows that

$$\|\overline{u}_{mt}\|_{(H^1(\Omega))^*} \le \|\nabla\overline{u}_m\|_{L^2(\Omega)} + c_4.$$

This together with (1.18) implies that there exists a positive constant K_2 independent of m such that

$$\|\overline{u}_{mt}\|_{L^2(0,T;(H^1(\Omega))^*)} \le K_2. \tag{1.20}$$

Passing to the limit: It follows from (1.18) and (1.20) that there exists a subsequence of $\overline{u}_m$, still denoted by $\overline{u}_m$, such that

$$\begin{aligned}&\overline{u}_m \rightharpoonup \overline{u} \ \text{ weakly in } \ L^2(0,T;H^1(\Omega)), \text{ weak-star in } \ L^\infty(0,T;L^2(\Omega)),\\ &\overline{u}_m \to \overline{u} \ \text{ strongly in } \ L^2(0,T;L^2(\Omega)),\\ &\overline{u}_{mt} \rightharpoonup \overline{u}_t \ \text{ weakly in } \ L^2(0,T;(H^1(\Omega))^*),\end{aligned}$$

as $m \to +\infty$. On the other hand, in view of (1.5), we have $|\bar{f}'(s)| \le c_2$. Therefore

$$\int_0^T \int_\Omega |\bar{f}(\overline{u}_m) - \bar{f}(\overline{u})|^2 \le (c_2)^2 \int_0^T \int_\Omega |\overline{u}_m - \overline{u}|^2,$$

so that

$$\bar{f}(\overline{u}_m) \to \bar{f}(\overline{u}) \ \text{ strongly in } \ L^2(0,T;L^2(\Omega)).$$

Passing to the limit as $m \to +\infty$ in (1.8), we deduce that

$$\overline{u}_t = \Delta \overline{u} + \bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u}) \text{ in } L^2(0,T;(H^1(\Omega))^*).$$

Since $\overline{u} \in L^2(0,T;H^1(\Omega))$ and since $\overline{u}_t \in L^2(0,T;(H^1(\Omega))^*)$, it follows that $\overline{u} \in C([0,T];L^2(\Omega))$. Moreover, $\overline{u}(x,0) = u_0(x)$ by classical arguments.

We have thus proven the existence part of Lemma 1.2.1. The uniqueness of the solution of Problem $(\overline{P})$ follows from Lemma 1.2.3 below, which will complete the proof of Lemma 1.2.1. □

Lemma 1.2.3. *Let $u_{01}, u_{02} \in L^2(\Omega)$ and let $\overline{u}_1, \overline{u}_2$ be two solutions of Problem $(\overline{P})$ corresponding to u_{01}, u_{02} respectively. Then there exists a constant $c > 0$ such that*

$$\int_\Omega |\overline{u}_1(t) - \overline{u}_2(t)|^2 \le \exp(ct) \int_\Omega |u_{01} - u_{02}|^2 \quad \textit{for all} \quad t \ge 0. \tag{1.21}$$

Proof. Set

$$w := \overline{u}_1 - \overline{u}_2.$$

We take the duality product of the difference of the equations for $\overline{u}_1$ and $\overline{u}_2$ by w to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) + \int_\Omega |\nabla w(t)|^2 = \int_\Omega \big[\bar{f}(\overline{u}_1) - \bar{f}(\overline{u}_2)\big] w - \int_\Omega w \fint_\Omega \big[\bar{f}(\overline{u}_1) - \bar{f}(\overline{u}_2)\big].$$

It follows from (1.5) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) + \int_\Omega |\nabla w(t)|^2 &\le c_2 \int_\Omega w^2 + \frac{c_2}{|\Omega|}\Big(\int_\Omega |w|\Big)^2 \\ &\le c_5 \int_\Omega w^2.\end{aligned}$$

Using Gronwall's lemma, we have

$$\int_\Omega |\overline{u}_1(t) - \overline{u}_2(t)|^2 \le \exp(2c_5 t) \int_\Omega |\, u_{01} - u_{02}|^2, \tag{1.22}$$

which completes the proof of Lemma 1.2.3.

□

Lemma 1.2.4 (Conservation of the integral)**.** *Let $u_0 \in L^2(\Omega)$, then*

$$\int_\Omega \overline{u}(x,t)dx = \int_\Omega u_0(x)dx \quad \textit{for all} \quad t > 0. \tag{1.23}$$

Proof. We take the duality product of the equation for $\overline{u}$ by 1 to obtain

$$\frac{d}{dt}\int_\Omega \overline{u} + \int_\Omega \nabla\overline{u}\,\nabla 1 = \int_\Omega \bar{f}(\overline{u})1 - \fint_\Omega \bar{f}(\overline{u})\int_\Omega 1.$$

Therefore,

$$\frac{d}{dt}\int_\Omega \overline{u}(x,t)\,dx = 0,$$

which implies that the identity (1.23) holds. □

1.2.2 More regularity properties of $\overline{u}$

We now prove more regularity properties of the solution of Problem $(\overline{P})$. For this purpose, we first recall some technical lemmas which will be used in what follows. We set $Q_a^b = \Omega \times (a,b)$ for $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$.

Lemma 1.2.5. *Let $u_0 \in L^2(\Omega), f \in L^p(Q_T)$ for some $p \in (1,\infty)$ and let u be the solution of the time evolution problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Then

(i) for each $0 < \delta < 1$, there exists a positive constant $C_1(\delta,\Omega,T)$ such that

$$\|u\|_{W_p^{2,1}(Q_\delta^T)} \le C_1(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^p(Q_T)}),$$

(ii) if we suppose that $u_0 \in W^{2,p}(\Omega)$ then we obtain the estimate

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_2(\|u_0\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(Q_T)}).$$

Remark 1.2.6. If $T = 1$, then the constant C_1 depends only on δ and Ω.

Lemma 1.2.7. *One has the following embedding*

$$W_p^{2,1}(Q_T) \subset C^{\lambda,\lambda/2}(\bar{Q}_T) \text{ with } \lambda = 2 - \frac{N+2}{p} \text{ if } p > \frac{N+2}{2} \text{ and } p \neq N+2.$$

Lemma 1.2.5 and Lemma 1.2.7 follow from [18, chapter 4, section 3 and chapter 2, section 3] which are stated in [4, p. 206]. Lemma 1.2.5(ii) is also stated in [6, Formula 5.4, page 899]

Next, we prove an estimate of $\|\overline{u}(t)\|_{L^2(\Omega)}$ which does not depend on T.

Lemma 1.2.8. *Let $u_0 \in L^2(\Omega)$ and let $\overline{u}$ be the solution of Problem $(\overline{P})$. Then there exists a constant $C > 0$ such that*

$$\|\overline{u}(t)\|_{L^2(\Omega)} \le C(\|u_0\|_{L^2(\Omega)} + 1) \quad \text{for all } \ t \ge 0.$$

Proof. Set

$$m_0 := \fint_\Omega u_0 = \fint_\Omega \overline{u}(t).$$

It follows from Poincaré-Wirtinger inequality that there exists a constant $c_6 = c_6(\Omega)$ such that

$$\int_\Omega |\overline{u} - m_0|^2 \leq c_6 \int_\Omega |\nabla \overline{u}|^2 dx,$$

or equivalently

$$\int_\Omega |\overline{u}|^2 \leq m_0^2 |\Omega| + c_6 \int_\Omega |\nabla \overline{u}|^2 dx.$$

Hence,

$$-\int_\Omega |\nabla \overline{u}|^2 dx \leq \frac{m_0^2 |\Omega|}{c_6} - \frac{1}{c_6} \int_\Omega |\overline{u}|^2. \tag{1.24}$$

We take the duality of the partial differential equation for $\overline{u}$ in Problem $(\overline{P})$ by $\overline{u}$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\overline{u}|^2 + \int_\Omega |\nabla \overline{u}|^2 &= \int_\Omega \bar{f}(\overline{u})\overline{u} - \fint_\Omega \bar{f}(\overline{u}) \int_\Omega \overline{u} \\
&\leq \int_\Omega \left(\frac{\varepsilon}{2} |\overline{u}|^2 + \frac{1}{2\varepsilon} \bar{f}(\overline{u})^2 \right) + c_2 m_0 \\
&\leq \frac{\varepsilon}{2} \int_\Omega |\overline{u}|^2 + \frac{(c_2)^2}{2\varepsilon} |\Omega| + c_2 m_0
\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\overline{u}|^2 \leq -\int_\Omega |\nabla \overline{u}|^2 + \frac{\varepsilon}{2} \int_\Omega |\overline{u}|^2 + \frac{(c_2)^2}{2\varepsilon} |\Omega| + c_2 m_0,$$

which together with (1.24) implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\overline{u}|^2 \leq (\frac{\varepsilon}{2} - \frac{1}{c_6}) \int_\Omega |\overline{u}|^2 + \frac{(c_2)^2}{2\varepsilon} |\Omega| + c_2 m_0 + \frac{m_0^2 |\Omega|}{c_6}.$$

We choose ε small enough to deduce that there exist $c_7, c_8 > 0$ such that

$$\frac{d}{dt}\int_\Omega |\overline{u}|^2 \leq -c_7 \int_\Omega |\overline{u}|^2 + c_8.$$

Applying Gronwall inequality we obtain

$$\int_\Omega |\overline{u}(t)|^2 \leq \exp(-c_7 t) \int_\Omega |u_0|^2 + \frac{c_8}{c_7},$$

which completes the proof of Lemma 1.2.8.

□

Lemma 1.2.9. *Let $u_0 \in L^2(\Omega)$, $m \in \mathbb{N}$ and let $0 < \delta < \frac{1}{2}$, $\alpha \in (0,1)$ be arbitrary. Then,*

$$\|\overline{u}\|_{C^{2m+1+\alpha, m+\frac{1+\alpha}{2}}(Q_\delta^\infty)} \leq C(m, \delta, \Omega)(\|u_0\|_{L^2(\Omega)} + 1).$$

Proof. First, we prove Lemma 1.2.9 for $m = 0$. Set $p := \dfrac{N+2}{1-\alpha}$. Since $|\bar{f}| \leq c_2$, we apply Lemma 1.2.5(i) and the embedding in Lemma 1.2.7 on domain Q_0^1 to obtain

$$\begin{aligned}\|\overline{u}\|_{C^{1+\alpha,(1+\alpha)/2}(Q_\delta^1)} &\leq C_1\left(\|u_0\|_{L^2(\Omega)} + \left\|\bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u})\right\|_{L^p(Q_0^1)}\right)\\ &\leq C_1\left(\|u_0\|_{L^2(\Omega)} + |\Omega|^{\frac{1}{p}}\left\|\bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u})\right\|_{L^\infty(Q_0^1)}\right)\\ &\leq C_1(\|u_0\|_{L^2(\Omega)} + 2c_2|\Omega|^{\frac{1}{p}}).\end{aligned}$$

Similarly, we apply Lemma 1.2.5(i) and the embedding in Lemma 1.2.7 on the domains Q_k^{k+1} and $Q_{k+1/2}^{k+3/2}$ to obtain (cf. Lemma 1.2.8)

$$\|\overline{u}\|_{C^{1+\alpha,(1+\alpha)/2}(Q_{k+\delta}^{k+1})} \leq C_1(\|\overline{u}(k)\|_{L^2(\Omega)} + 2c_2|\Omega|^{\frac{1}{p}}) \leq \bar{C}(\|u_0\|_{L^2(\Omega)} + 1)$$

and a similar one on the domain $Q_{k+1/2}^{k+3/2}$. Finally, we deduce from the fact that k can be chosen arbitrary that

$$\|\overline{u}\|_{C^{1+\alpha,(1+\alpha)/2}(Q_\delta^\infty)} \leq \bar{C}(\|u_0\|_{L^2(\Omega)} + 1).$$

By standard bootstrap arguments [18, Theorem 10.1, p. 351], we deduce the result of Lemma 1.2.9 for any $m \in \mathbb{N}$.

□

The following corollary is an immediate consequence of Lemma 1.2.9.

Corollary 1.2.10. *Let $u_0 \in L^2(\Omega)$, then $\{\overline{u}(t) : t \geq 1\}$ is relatively compact in $H^m(\Omega)$ and in $C^m(\overline{\Omega})$ for all $m \geq 0$.*

1.2.3 Derivation of Theorem 1.1.1 from the results about Problem $(\overline{P})$

The key lemma is the following:

Lemma 1.2.11. *Assume that u_0 satisfies Hypotheses $(\mathbf{H_0})$, namely*

$$s_1 \leq u_0(x) \leq s_2 \qquad \text{for a.e.} \quad x \in \Omega.$$

Then

$$s_1 \leq \overline{u}(x,t) \leq s_2 \qquad \text{for all} \;\; x \in \overline{\Omega}, t > 0.$$

The main idea of the proof of Lemma 1.2.11 is to approximate the initial function u_0 by a sequence of smooth functions and to first obtain the result for approximate problems. Then, we deduce the result for the original problem. To begin with, we need the following lemma.

Lemma 1.2.12. *Let u_0 satisfy Hypothesis $(\mathbf{H_0})$. Then, there exists a sequence u_{n0} in $C^\infty(\overline{\Omega})$ such that $s_1 < u_{n0} < s_2$ on $\overline{\Omega}$, and that*

$$u_{n0} \to u_0 \;\; \text{in} \;\; L^2(\Omega) \;\; \text{as} \;\; n \to \infty.$$

The proof of this lemma will be given later. Next, we use it to prove Lemma 1.2.11.

Proof of Lemma 1.2.11. We denote by $\overline{u}_n$ the corresponding solutions of Problem $(\overline{P})$ with initial function u_{n0}. We will prove this lemma through several steps.

Claim 1: We prove that

$$\overline{u}_n \in C^{2,1}(\overline{\Omega} \times (0,T]) \cap C(\overline{\Omega} \times [0,T]). \tag{1.25}$$

Indeed, since $|\bar{f}| \leq c_2$, we deduce from Lemma 1.2.5(ii) that $\overline{u}_n \in W_p^{2,1}(Q_T)$ for all $p > 1$, which by the Sobolev embedding in Lemma 1.2.7 implies that $\overline{u}_n \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q}_T)$ for all $\alpha \in (0,1)$. Applying a standard bootstrap argument (cf. [18, Theorem 10.1, p, 351]), we deduce that $\overline{u}_n \in C^{3+\alpha,1+\frac{1+\alpha}{2}}(\overline{\Omega} \times (0,T])$. Hence

$$\overline{u}_n \in C^{2,1}(\overline{\Omega} \times (0,T]) \cap C(\overline{\Omega} \times [0,T]).$$

Claim 2: We prove that

$$s_1 < \overline{u}_n(x,t) < s_2 \qquad \text{for all } \ x \in \overline{\Omega}, t > 0. \tag{1.26}$$

For the purpose of contradiction, we suppose that there exists a first time $t_0 > 0$ such that $\overline{u}_n(x_0,t_0) = s_1$ or $\overline{u}_n(x_0,t_0) = s_2$ for some $x_0 \in \overline{\Omega}$. Without loss of generality, assume that $\overline{u}_n(x_0,t_0) = s_2$. By the continuity of $\overline{u}_n$ and the definition of t_0, we have

$$s_1 \leq \overline{u}_n(x,t_0) \leq s_2 \quad \text{for all } x \in \overline{\Omega}, \text{ and } \overline{u}_n(x,t) < s_2 \text{ for all } x \in \overline{\Omega} \text{ and } 0 \leq t < t_0. \tag{1.27}$$

Since $\partial_\nu \overline{u}_n = 0$, we deduce from Hopf's maximum principle that $x_0 \in \Omega$. Therefore the function $\overline{u}_n(\cdot,t_0)$ attains its maximum at $x_0 \in \Omega$, which implies that $\Delta \overline{u}_n(x_0,t_0) \leq 0$. By (2.10), we have

$$\overline{u}_{nt}(x_0,t_0) = \lim_{\Delta t \to 0^+} \frac{\overline{u}_n(x_0,t_0-\Delta t) - \overline{u}_n(x_0,t_0)}{-\Delta t} \geq 0,$$

which we substitute in Problem $(\overline{P})$ to obtain $\fint_\Omega (\bar{f}(s_2) - \bar{f}(\overline{u}_n(x,t_0)))\ dx \geq 0$. Since $s_1 \leq \overline{u}_n(x,t_0) \leq s_2$ for all $x \in \Omega$, it follows from (1.3) that $\bar{f}(s_2) \leq \bar{f}(\overline{u}_n(x,t_0))$ for all $x \in \Omega$ so that $\bar{f}(s_2) = \bar{f}(\overline{u}_n(x,t_0))$. Using (1.3) again, we obtain $\overline{u}_n(x,t_0) = s_2$ for all x in Ω. Consequently, we have

$$\int_\Omega \overline{u}_n(x,t_0)\ dx = s_2 > \int_\Omega u_0(x)\ dx,$$

which contradicts the integral preserving property in Lemma 1.2.4. We obtain (1.26).

Claim 3: Next, we prove that

$$s_1 \leq \overline{u}(x,t) \leq s_2 \qquad \text{for all } \ x \in \overline{\Omega}, t > 0.$$

It follows from Lemma 1.2.3 that

$$\int_\Omega |\overline{u}(t) - \overline{u}_n(t)|^2 \leq \exp(ct) \int_\Omega |u_0 - \overline{u}_{n0}|^2$$

for some constant $c > 0$. Therefore,

$$\overline{u}_n \to \overline{u} \ \text{ in } \ L^2(Q_T) \ \text{ as } \ n \to +\infty.$$

Consequently, there exists a subsequence $\{\overline{u}_{nj}\}_{j\geq 0}$ of $\{\overline{u}_n\}_{n\geq 0}$ such that

$$\overline{u}_{nj} \to \overline{u} \ \text{ almost everywhere on } \quad Q_T,$$

which together with (1.26) implies that

$$s_1 \leq \overline{u}(x,t) \leq s_2 \ \text{ almost everywhere on } \quad Q_T.$$

Moreover, in view of Lemma 1.2.9, $\overline{u}$ is smooth on Q_T and $T > 0$ is arbitrary so that

$$s_1 \leq \overline{u}(x,t) \leq s_2 \qquad \text{for all } \ x \in \overline{\Omega}, t > 0.$$

□

Proof of Lemma 1.2.12. First, we extend the function u_0 to w_0 defined on $\mathbb{R}^N$ by

$$w_0(x) = \begin{cases} u_0(x) & \text{if } \ x \in \Omega \\ \dfrac{s_1 + s_2}{2} & \text{elsewhere.} \end{cases}$$

Obviously,

$$s_1 \leq w_0(x) \leq s_2 \quad \text{a.e.} \quad x \in \mathbb{R}^N.$$

For all n large enough such that

$$s_1 + \frac{1}{n} < s_2 - \frac{1}{n},$$

we define

$$w_{n0} := \min\left\{ s_2 - \frac{1}{n}, \max\{w_0, s_1 + \frac{1}{n}\} \right\}.$$

Then, we have

$$s_1 + \frac{1}{n} \leq w_{n0}(x) \leq s_2 - \frac{1}{n} \quad \text{a.e.} \quad x \in \mathbb{R}^N, \tag{1.28}$$

and moreover,

$$\begin{aligned} \|w_{n0} - u_0\|_{L^2(\Omega)} &= \|w_{n0} - w_0\|_{L^2(\Omega)} \\ &\leq |\Omega|^{\frac{1}{2}} \|w_{n0} - w_0\|_{L^\infty(\Omega)} \leq \frac{|\Omega|^{\frac{1}{2}}}{n}. \end{aligned}$$

As in [1, Theorem 2.29, page 36], we can choose a smooth function u_{n0} such that

$$\|u_{n0} - w_{n0}\|_{L^2(\Omega)} \leq \frac{|\Omega|^{\frac{1}{2}}}{n}.$$

More precisely,

$$u_{n0}(x) := \int_{\mathbb{R}^N} J_\varepsilon(x-y) w_{n0}(y) dy, \quad \text{for } \ \varepsilon \ \text{ small enough,}$$

where

$$J_\varepsilon(x) = \varepsilon^{-N} J(x/\varepsilon),$$

with $J \in C_c^\infty(\mathbb{R}^N)$ is a nonnegative function satisfying

$$J(x) = 0 \qquad \text{if} \quad |x| \geq 1, \quad \text{and} \quad \int_{\mathbb{R}^N} J(x)dx = 1.$$

Note that J_ε is nonnegative and that

$$\int_{\mathbb{R}^N} J_\varepsilon(x)dx = 1. \tag{1.29}$$

Therefore, in view of (1.28) and (1.29)

$$s_1 + \frac{1}{n} \leq u_{n0} \leq s_2 - \frac{1}{n} \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Moreover,

$$\|u_{n0} - u_0\|_{L^2(\Omega)} \leq \|u_{n0} - w_{n0}\|_{L^2(\Omega)} + \|w_{n0} - u_0\|_{L^2(\Omega)} \leq \frac{2|\Omega|^{\frac{1}{2}}}{n}.$$

It follows that

$$u_{n0} \to u_0 \qquad \text{in} \quad L^2(\Omega) \quad \text{as} \quad n \to +\infty.$$

The proof of Lemma 1.2.12 is complete. □

Lemma 1.2.13. *Let u be a solution of Problem (P) such that*

$$u \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \quad \textit{and} \quad u_t \in L^2(0,T;(H^1(\Omega))^*),$$

Then,

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)\,dx \quad \textit{for all} \quad t > 0. \tag{1.30}$$

Proof. The proof is similar to the one of Lemma 1.2.4. We omit it. □

Lemma 1.2.14. *Let $u_{01}, u_{02} \in L^\infty(\Omega)$ such that*

$$\int_\Omega u_{01}(x)\,dx = \int_\Omega u_{02}(x)\,dx,$$

and let u_1, u_2 be two corresponding solutions of Problem (P), which satisfy

$$u_1, u_2 \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \qquad u_{1t}, u_{2t}, \in L^2(0,T;(H^1(\Omega))^*).$$

Then there exists a constant $c > 0$ such that

$$\int_\Omega |u_1(t) - u_2(t)|^2 \leq \exp(ct) \int_\Omega |\ u_{01} - u_{02}|^2 \quad \textit{for all} \quad t \geq 0. \tag{1.31}$$

Proof. We denote by $w := u_1 - u_2$, and use the mass conservation property (cf. Lemma 1.2.13) to deduce that

$$\int_\Omega w(x,t)\,dx = \int_\Omega (u_{01}(x) - u_{02}(x))\,dx = 0 \quad \text{for all} \quad t > 0. \tag{1.32}$$

We multiply the difference of the equations for u and u_2 by w, then integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) + \int_\Omega |\nabla w(t)|^2 = \int_\Omega \big[f(u_1) - f(u_2)\big]w - \int_\Omega w \fint_\Omega \big[f(u_1) - f(u_2)\big].$$

It follows from (2.5) and that (1.32) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) + \int_\Omega |\nabla w(t)|^2 = \int_\Omega \big[f(u_1) - f(u_2)\big]w \le c_1 \int_\Omega w^2.$$

Using Gronwall's lemma, we have

$$\int_\Omega |u_1(t) - u_2(t)|^2 \le \exp(2c_1 t) \int_\Omega |\, u_{01} - u_{02}|^2.$$

□

Proof of Theorem 1.1.1. The uniqueness of solution of Problem (P) is a consequence of Lemma 1.2.14. Since by Lemma 1.2.11,

$$s_1 \le \overline{u}(x,t) \le s_2 \quad \text{for all} \quad x \in \Omega, t > 0.$$

It follows from the definition of $\bar{f}$ in (2.4) that for all $t > 0$

$$\bar{f}(\overline{u}(x,t)) = f(\overline{u}(x,t)) \quad \text{for all} \quad x \in \Omega,$$

so that $\overline{u}$ satisfies the equations

$$\begin{cases} \overline{u}_t = \Delta\overline{u} + \bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u}) = \Delta\overline{u} + f(\overline{u}) - \fint_\Omega f(\overline{u}) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu \overline{u} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \overline{u}(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Thus, $\overline{u}$ coincides with the unique solution of Problem (P). Therefore, all the properties which we have proved for $\overline{u}$ hold for the unique solution u of Problem (P). In particular, in view of Lemma 1.2.9 and of Corollary 1.2.10, we have that $u \in C^\infty(\overline{\Omega} \times (0, +\infty))$ and

$$\{u(t), t \ge 1\} \quad \text{is relatively compact in} \quad C^m(\overline{\Omega}) \quad \text{for all} \quad m \in \mathbb{N}.$$

□

1.3 A version of Lojasiewicz inequality

The main result of this section is the Lojasiewicz inequality stated in Theorem 1.3.8 below. More precisely, we prove a version of Lojasiewicz for the functional

$$E(\overline{u}) := \frac{1}{2}\int_\Omega |\nabla \overline{u}|^2\, dx - \int_\Omega \bar{F}(\overline{u}) dx,$$

where we recall that $\bar{F}(s) := \int_0^s \bar{f}(\tau)\, d\tau$. We also note that E is a Lyapounov functional of Problem $(\overline{P})$ (cf. Lemma 1.4.1 below).

This section is organized as follows: In Section 1.3.1, as a preparation for the proof of Theorem 1.3.8, we prove the differentiability of E and compute its derivative. The definition and some equivalent conditions of a critical point are given. The Lojasiewicz inequality is proved in Section 1.3.2.

1.3.1 Some preparations

We define the spaces

$$H = \{u \in L^2(\Omega), \int_\Omega u(x)dx = 0\}, \quad \text{equipped with the norm} \quad \|\cdot\|_H := \|\cdot\|_{L^2(\Omega)},$$

$$V = \Big\{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\Big\}, \quad \text{equipped with the norm} \quad \|\cdot\|_V := \|\cdot\|_{H^1(\Omega)}.$$

Let V^* be the dual space of V. We identify H with its dual to obtain:

$$V \hookrightarrow H \hookrightarrow V^*,$$

where the embeddings $V \hookrightarrow H$, $H \hookrightarrow V^*$ are continuous, dense and compact (see e.g. [17, p. 677]). We use $\langle\cdot,\cdot\rangle$ to denote the duality product between V^* and V. We note $\mathcal{L}(X,Y)$ the space of bounded linear operators from a Banach space X to a second Banach space Y, and we write $\mathcal{L}(X) := \mathcal{L}(X,X)$.

We also define the spaces

$$\mathcal{L}^p(\Omega) := \{u \in L^p(\Omega) : \quad \int_\Omega u(x)\,dx = 0\}, \tag{1.33}$$

equipped with the norm $\|\cdot\|_{\mathcal{L}^p(\Omega)} := \|\cdot\|_{L^p(\Omega)}$ and

$$X_p := \{u \in W^{2,p}(\Omega) : \partial_\nu u = 0, \int_\Omega u(x)dx = 0\}, \tag{1.34}$$

equipped with the norm $\|\cdot\|_{X_p} := \|\cdot\|_{W^{2,p}(\Omega)}$. Throughout the sequel, we denote by $C \geq 0$ a generic constant which may vary from line to line. We start with the following result.

Lemma 1.3.1. *Let $u, h \in L^1(\Omega)$, $p \in [1,\infty)$ and let g be a continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$|g(s)|, |g'(s)| \leq C \quad \textit{for all} \quad s \in \mathbb{R}.$$

Then

$$\int_0^1 g(u+\tau h)d\tau \to g(u) \qquad \textit{in} \quad L^p(\Omega) \quad \textit{as} \quad \|h\|_{L^1(\Omega)} \to 0.$$

Proof. It is sufficient to prove that

$$B := \Big\|\int_0^1 g(u+\tau h)d\tau - g(u)\Big\|^p_{L^p(\Omega)} \to 0 \quad \text{as} \quad \|h\|_{L^1(\Omega)} \to 0.$$

Since g is bounded,

$$\begin{aligned} B &\leq \Big\|\int_0^1 |g(u+\tau h) - g(u)|d\tau\Big\|^p_{L^p(\Omega)} \leq C\Big\|\int_0^1 |g(u+\tau h) - g(u)|d\tau\Big\|_{L^1(\Omega)} \\ &\leq C\Big\|\int_0^1 C\tau|h|d\tau\Big\|_{L^1(\Omega)} \leq C\|h\|_{L^1(\Omega)}. \end{aligned}$$

This completes the proof of Lemma 1.3.1. □

Lemma 1.3.2. *The functional E is twice continuously Fréchet differentiable on V. We denote by E', L the first and second derivative of E, respectively. Then*

(i) The first derivative

$$E' : V \longrightarrow V^* \text{ is given by}$$

$$\langle E'(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h - \int_\Omega \bar{f}(u)h \qquad \text{for all } \ u, h \in V. \tag{1.35}$$

(ii) The second derivative

$$L : V \longrightarrow \mathcal{L}(V, V^*) \ \text{ is given by}$$

$$\langle L(u)h, k\rangle_{V^*,V} = \int_\Omega \nabla h \nabla k - \int_\Omega \bar{f}'(u)hk \qquad \text{for all } u, h, k \in V. \tag{1.36}$$

Consequently,

$$\langle L(u)h, k\rangle_{V^*,V} = \langle h, L(u)k\rangle_{V,V^*}. \tag{1.37}$$

Proof. We write E as the difference of E_1 and E_2, where

$$E_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx \ \text{ and } \ E_2(u) = \int_\Omega \bar{F}(u)\, dx. \tag{1.38}$$

Obviously, E_1 is twice continuously Fréchet differentiable. Its derivatives are easily identified in the formulas (1.35) and (1.36). We now compute the first and second derivative of E_2.

(i) By Taylor's formula,

$$\bar{F}(u+h) - \bar{F}(u) = h\int_0^1 \bar{f}(u+\tau h)d\tau =: h\zeta \ \text{ for all } \ u, h \in V,$$

where

$$\zeta(x) := \int_0^1 \bar{f}(u(x) + \tau h(x))d\tau$$

It follows that

$$\begin{aligned}\left|E_2(u+h) - E_2(u) - \int_\Omega \bar{f}(u)h\, dx\right| &\le \int_\Omega |\zeta - \bar{f}(u)|\,|h|\, dx \\ &\le C\|\zeta - \bar{f}(u)\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)} \\ &\le C\|\zeta - \bar{f}(u)\|_{L^2(\Omega)}\|h\|_V.\end{aligned}$$

We deduce from Lemma 1.3.1 that

$$\zeta = \int_0^1 \bar{f}(u+\tau h)d\tau \to \bar{f}(u) \ \text{ in } \ L^2(\Omega) \ \text{ as } \ \|h\|_V \to 0.$$

Therefore

$$\left|E_2(u+h) - E_2(u) - \int_\Omega \bar{f}(u)h\, dx\right| = o(\|h\|_V) \ \text{ as } \ \|h\|_V \to 0.$$

This implies that the first derivative E_2' exists and

$$\langle E_2'(u), h\rangle_{V^*,V} = \int_\Omega \bar{f}(u)h\, dx.$$

(ii) The Fréchet differentiability of E_2' is shown in a similar way. Choose $p \in (2, +\infty)$ such that V is continuously embedded in $L^p(\Omega)$. Let T be the linear mapping from V to V^* given by

$$\langle T\,h, k\rangle_{V^*,V} = \int_\Omega \bar{f}'(u)h\,k\,dx.$$

We will use below a generalized Hölder inequality based on the identity

$$\frac{1}{p} + \frac{1}{p} + \frac{p-2}{p} = 1.$$

For every $u, h, k \in V$ and for

$$\eta(x) := \int_0^1 \bar{f}'(u(x) + \tau h(x))d\tau$$

we have

$$\begin{aligned}\Big|\langle E_2'(u+h) - E_2'(u) - Th, k\rangle_{V^*,V}\Big| &\\ &\le \int_\Omega |\eta - \bar{f}'(u)|\,|h|\,|k|\,dx\\ &\le \|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_{L^p(\Omega)}\|k\|_{L^p(\Omega)}\\ &\le C\|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V\|k\|_V. \qquad (1.39)\end{aligned}$$

Consequently, we have

$$\|E_2'(u+h) - E_2'(u) - T\,h\|_{V^*} \le C\|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V. \qquad (1.40)$$

Since $1 < p/(p-2) < +\infty$, we deduce from Lemma 1.3.1 that

$$\|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)} \to 0 \qquad \text{as} \quad \|h\|_V \to 0,$$

which together with (1.40) follows that

$$\|E_2'(u+h) - E_2'(u) - Th\|_{V^*} = o(\|h\|_V).$$

Therefore,

$$\langle E_2''(u)h, k\rangle_{V^*,V} = \int_\Omega \bar{f}'(u)h\,k \quad \text{for all} \quad u, h, k \in V.$$

We also note that

$$\begin{aligned}|\langle (E_2''(u) - E_2''(v))h, k\rangle_{V^*,V}| &\le \int_\Omega |\bar{f}'(u) - \bar{f}'(v)|\,|h|\,|k|\,dx\\ &\le C\|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V\|k\|_V.\end{aligned}$$

Hence

$$\|E_2''(u) - E_2''(v)\|_{\mathcal{L}(V,V^*)} \le C\|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/p-2}(\Omega)},$$

which implies the continuity of E_2''. Finally, (1.37) is an immediate consequence of (1.36). □

We define a continuous bilinear form from $V \times V \to \mathbb{R}$ by

$$a(u,v) = \int_\Omega \nabla u \nabla v \, dx.$$

The following lemma is an immediate consequence of Lax-Milgram theorem [3, Corollary 5.8, page 140]. We omit its proof.

Lemma 1.3.3. *There exists an isomorphism A from V onto V^* such that*

$$a(u,v) = \langle Au, v\rangle_{V^*,V} \quad \text{for all} \quad u, v \in V. \tag{1.41}$$

Corollary 1.3.4. *The first and second derivatives of E can be represented in V^* as:*

$$E'(u) = Au - \bar{f}(u) + \fint_\Omega \bar{f}(u), \tag{1.42}$$

$$L(u)h = Ah - \bar{f}'(u)h + \fint_\Omega \bar{f}'(u)h, \tag{1.43}$$

for all $u, h \in V$.

Proof. Since $\bar{f}$ is bounded, $\bar{f}(u) - \fint_\Omega \bar{f}(u) \in H \hookrightarrow V^*$. Therefore,

$$Au - \bar{f}(u) + \fint_\Omega \bar{f}(u) \in V^*.$$

We also note that

$$\int_\Omega \left(\fint_\Omega \bar{f}(u) \right) h = \fint_\Omega \bar{f}(u) \int_\Omega h = 0 \quad \text{for all} \quad h \in V,$$

thus,

$$\langle Au - \bar{f}(u) + \fint_\Omega \bar{f}(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h - \int_\Omega \bar{f}(u)h.$$

This together with (1.35) implies that

$$E'(u) = Au - \bar{f}(u) + \fint_\Omega \bar{f}(u).$$

Identity (1.43) may be proved in a similar way. □

Lemma 1.3.5. *Let $\mathcal{L}^p(\Omega), X_p$ be the Banach spaces as in* (1.33) *and* (1.34)*. Assume that $p \geq 2$. Then, for any $g \in \mathcal{L}^p(\Omega)$, there exists a unique solution $u \in X_p$ of the equation*

$$Au = g \quad \text{in} \quad V^*.$$

Moreover, we have

$$A = -\Delta \qquad \text{on} \qquad X_p. \tag{1.44}$$

Proof. It follows from Lemma 1.3.3 that the equation

$$Au = g \ \text{ in } \ V^* \tag{1.45}$$

has a unique solution $u \in V$ so that it is enough to prove that $u \in X_p$. For this purpose, we consider the elliptic problem

$$\begin{cases} -\Delta \tilde{u} = g & \text{in } \Omega, \\ \partial_\nu \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $g \in H$, we apply the Fredholm alternative to deduce that this problem possesses a unique solution $\tilde{u} \in V$. Note that $g \in \mathcal{L}^p(\Omega)$, so that we deduce from [2] that $\tilde{u} \in W^{2,p}(\Omega)$ so that also $\tilde{u} \in X_p$. On the other hand, for all $v \in V$, we have

$$\langle A\tilde{u}, v\rangle_{V^*,V} = a(\tilde{u}, v) = \int_\Omega \nabla\tilde{u}\nabla v \, dx = \langle -\Delta\tilde{u}, v\rangle_{V^*,V} = \langle g, v\rangle_{V^*,V}.$$

Therefore, $\tilde{u}$ coincides with the unique solution of equation (1.45). In other words,

$$u = \tilde{u} \in X_p.$$

We also conclude that $A = -\Delta \ \text{ on } \ X_p$, since for all $w \in X_p, v \in V$,

$$\langle -\Delta w, v\rangle_{V^*,V} = \int_\Omega \nabla w \nabla v \, dx = \langle Aw, v\rangle_{V^*,V}.$$

□

Definition 1.3.6. *We say that $\varphi \in V$ is a critical point of E if*

$$E'(\varphi) = 0 \qquad in \qquad V^*.$$

Lemma 1.3.7. *For every $\varphi \in V$, the following assertions are equivalent:*

(i) φ is a critical point of E,

(ii) $\varphi \in X_2$ and φ satisfies the equations

$$\begin{cases} -\Delta\varphi = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) & in \ \ \Omega, \\ \partial_\nu \varphi = 0 & on \ \ \partial\Omega. \end{cases}$$

Moreover, φ is $C^\infty(\overline{\Omega})$.

Proof. $(ii) \Rightarrow (i)$. By Formula (1.42), we have

$$E'(\varphi) = A\varphi - \bar{f}(\varphi) + \fint_\Omega \bar{f}(\varphi),$$

which together with (1.44) implies that

$$E'(\varphi) = -\Delta\varphi - \bar{f}(\varphi) + \fint_\Omega \bar{f}(\varphi) = 0.$$

Namely, φ is a critical point of E.

$(i) \Rightarrow (ii)$. Assume that $\varphi \in V$ is a critical point of E. We deduce from (1.42) that

$$A(\varphi) = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) \quad \text{in} \quad V^*.$$

Since $\bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) \in H$, it follows from Lemma 1.3.5 that $\varphi \in X_2$ satisfies the equations

$$\begin{cases} -\Delta\varphi = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, we deduce that $\varphi \in C^\infty(\overline{\Omega})$ from the boundedness of $\bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi)$, Sobolev embedding theorem and a standard bootstrap argument. □

1.3.2 Lojasiewicz inequality

We recall the definition of a direct topological sum. Let X be a Banach space and let X_1, X_2 be two subspaces of X. We say that X is the direct topological sum of X_1 and X_2, which we write

$$X = X_1 \oplus X_2,$$

if X_1, X_2 are closed subspaces of X, such that

$$\begin{cases} X_1 \cap X_2 = \{0\} \\ X = X_1 + X_2. \end{cases}$$

An equivalent condition is that there exists a continuous projection P in X (i.e $P \in \mathcal{L}(X)$ and $P^2 = P$) such that

$$X_1 = \ker P \quad \text{and} \quad X_2 = \operatorname{Rg} P.$$

We refer to [24, page 133] for more details. Next, we check that the following Lojasiewicz inequality is satisfied.

Theorem 1.3.8 (Lojasiewicz inequality)**.** *Let $\varphi \in V$ be a critical point of the functional E such that $s_1 \leq \varphi \leq s_2$. Then there exist constants $\theta \in (0, \frac{1}{2}]$ and $C, \sigma > 0$ such that*

$$|E(u) - E(\varphi)|^{1-\theta} \leq C\|E'(u)\|_{V^*}, \tag{1.46}$$

for all $\|u - \varphi\|_V \leq \sigma$. In this case, we say that E satisfies the Lojasiewicz inequality in φ. The number θ will be called the Lojasiewicz exponent.

Chill proved in [8, Theorem 3.10] a general version of the Lojasiewicz inequality. Moreover, in [8, Corollary 3.11], he gave a set of hypotheses for which E satisfies the Lojasiewicz inequality near a critical point. Our task is to verify these hypotheses and then to deduce the result of Theorem 1.3.8. We list here the hypotheses of Chill in [8, Corollary 3.11].

$(\mathbf{H_1})$: (see [8, Hypothesis 3.2, page 580]). The kernel $\ker L(\varphi)$ is a complemented subspace of V, i.e., there exists a projection $P \in \mathcal{L}(V)$ such that $\operatorname{Rg} P = \ker L(\varphi)$. We denote by $P^* \in \mathcal{L}(V^*)$ the adjoint projection of P.

$(\mathbf{H_2})$: (see [8, Hypothesis 3.4, page 580]). There exists a Banach space W with the following properties

(i) W is continuously embedded in V^* and invariant under P^* (i.e. $P^*(W) \subset W$),

(ii) $E' \in C^1(V, W)$,

(iii) $\operatorname{Rg} L(\varphi) = \ker P^* \cap W$.

($\mathbf{H_3}$) : There exist Banach spaces $X \subset V$ and $Y \subset W$ such that

(i) the spaces X and Y are invariant under projection P and P^*, respectively,

(ii) the restriction of the derivative E' to X is analytic in a neighborhood of φ with values in Y,

(iii) $\ker L(\varphi)$ is contained in X and finite-dimensional,

(iv) $\operatorname{Rg} L(\varphi)\big|_X = \ker P^* \cap Y$.

It follows from [8, Corollary 3.11] that if ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$) hold, then E satisfies for some $\theta \in (0, \frac{1}{2}]$,

$$|E(u) - E(\varphi)|^{1-\theta} \leq C\|E'(u)\|_W \quad \text{for all} \quad \|u - \varphi\|_V \leq \sigma. \tag{1.47}$$

We note here that the norm of the space W appears on the right-hand side of (1.47). Therefore, in order to prove Inequality (1.46) in Theorem 1.3.8, we choose $W = V^*$, so that instead of considering ($\mathbf{H_2}$), ($\mathbf{H_3}$), we will consider ($\mathbf{H_2'}$), ($\mathbf{H_3'}$) below:

($\mathbf{H_2'}$) : $\operatorname{Rg} L(\varphi) = \ker P^*$.

($\mathbf{H_3'}$) : There exist Banach spaces $X \subset V$ and $Y \subset V^*$ such that

(i) X and Y are invariant under projection P and P^*, respectively,

(ii) the restriction of the derivative E' to X is analytic in a neighborhood of φ with value in Y,

(iii) $\ker L(\varphi)$ is contained in X and finite dimensional,

(iv) $\operatorname{Rg} L(\varphi)\big|_X = \ker P^* \cap Y$.

Our task now is to verify ($\mathbf{H_1}$), ($\mathbf{H_2'}$), ($\mathbf{H_3'}$). We start with the following lemma.

Lemma 1.3.9. *Let φ be a critical point of E. Then,*

(i) $L(\varphi)$ is a Fredholm operator from V to V^* of index 0 i.e, $\operatorname{Rg} L(\varphi)$ is closed in V^* and

$$\dim \ker L(\varphi) = \operatorname{codim}(\operatorname{Rg} L(\varphi)) < +\infty,$$

where $\operatorname{codim} \operatorname{Rg} L(\varphi) := \dim(V^*/\operatorname{Rg} L(\varphi))$.

(ii) $\ker L(\varphi) \subset \{u \in C^\infty(\overline{\Omega}) : \partial_\nu u = 0 \quad on \quad \partial\Omega; \int_\Omega u = 0\}$.

(iii) For all $u \in \ker L(\varphi)$ and $v \in \operatorname{Rg} L(\varphi)$, we have

$$\langle u, v\rangle_{V,V^*} = 0.$$

Proof. (i) We first prove that the linear operator

$$\begin{aligned} T : V &\longrightarrow V^* \\ h &\longmapsto -\bar{f}'(\varphi)h + \fint_\Omega \bar{f}'(\varphi)h \end{aligned}$$

is compact. Indeed, note that we have for all $h \in V$

$$\begin{aligned}\|Th\|_H &\le \|\bar{f}'(\varphi)h\|_{L^2(\Omega)} + \left\| \fint_\Omega \bar{f}'(\varphi)h \right\|_{L^2(\Omega)} \\ &\le C\|h\|_{L^2(\Omega)} \le C\|h\|_V.\end{aligned}$$

Therefore, T is continuous from V to H, which together with the compactness of the embedding $H \hookrightarrow V^*$ implies that T is compact from V to V^*.

Next, since A is an isomorphism from V onto V^*, it is also a Fredholm operator of index

$$\operatorname{ind} A := \dim \ker A - \operatorname{codim} \operatorname{Rg} A = 0.$$

It follows that $L(\varphi) = A + T$, as a sum of a Fredholm operator and a compact operator, is also a Fredholm operator with the same index (cf. [3, p. 168]). Therefore, the conclusion of (i) is proved.

(ii) Let $h \in \ker L(\varphi)$, then $L(\varphi)h = 0$ in V^*, or equivalently

$$Ah = \bar{f}'(\varphi)h - \fint_\Omega \bar{f}'(\varphi)h \quad \text{in} \quad V^*.$$

We first claim that $h \in C^\infty(\overline{\Omega})$. Since $\bar{f}'(\varphi)h - \fint_\Omega \bar{f}'(\varphi)h \in H$, we deduce from Lemma 1.3.7 that $h \in X_2$ satisfies

$$\begin{cases} -\Delta h = \bar{f}'(\varphi)h - \fint_\Omega \bar{f}'(\varphi)h & \text{in } \Omega, \\ \partial_\nu h = 0 & \text{on } \partial\Omega. \end{cases}$$

We also note from Lemma 1.3.7 that $\varphi \in C^\infty(\overline{\Omega})$ so that $\bar{f}'(\varphi) \in C^\infty(\overline{\Omega})$. We then apply a Sobolev embedding theorem and a bootstrap argument to deduce that $h \in C^\infty(\overline{\Omega})$. We also note that $\int_\Omega h = 0$ since $h \in V$. Therefore, we obtain the conclusion of (ii).

(iii) Since $v \in \operatorname{Rg} L(\varphi)$, there exists $w \in V$ such that $v = L(\varphi)w$. In view of (1.37), we have

$$\langle u, v\rangle_{V,V^*} = \langle u, L(\varphi)w\rangle_{V,V^*} = \langle L(\varphi)u, w\rangle_{V^*,V} = 0,$$

which implies (iii).

□

Lemma 1.3.10. *V^* is the topological direct sum of $\ker L(\varphi) \subset V \hookrightarrow V^*$ and $\operatorname{Rg} L(\varphi)$. Namely,*

$$V^* = \ker L(\varphi) \oplus \operatorname{Rg} L(\varphi).$$

Proof. Let $u \in \ker L(\varphi) \cap \operatorname{Rg} L(\varphi)$ be arbitrary; then in view of Lemma 1.3.9(iii), we have

$$\|u\|_H = \langle u, u\rangle_{V^*,V} = 0.$$

Consequently,

$$\ker L(\varphi) \cap \operatorname{Rg} L(\varphi) = \{0\}.$$

It follows from Lemma 1.3.9(i) that $\dim \ker L(\varphi) = \operatorname{codim} \operatorname{Rg} L(\varphi)$, so that

$$\begin{cases} \ker L(\varphi) \cap \operatorname{Rg} L(\varphi) = \{0\} \\ V^* = \ker L(\varphi) + \operatorname{Rg} L(\varphi). \end{cases}$$

On the other hand, $\operatorname{Rg} L(\varphi)$ is closed in V^* because of Lemma 1.3.9(i) and $\ker L(\varphi)$ is closed in V^* since it is finite-dimensional. Therefore, we deduce that V^* is the topological direct sum of $\ker L(\varphi)$ and $\operatorname{Rg} L(\varphi)$. □

Lemma 1.3.11 (Projection P)**.** *There exists a projection $P \in \mathcal{L}(V)$ such that the adjoint projection $P^* \in \mathcal{L}(V^*)$ satisfies*

$$\operatorname{Rg} P = \operatorname{Rg} P^* = \ker L(\varphi) \quad \textit{and} \quad \ker P^* = \operatorname{Rg} L(\varphi).$$

Consequently, $(\mathbf{H_1})$ *and* $(\mathbf{H_2'})$ *hold.*

Proof. By Lemma 1.3.10, we can define a projection $Q \in \mathcal{L}(V^*)$ onto $\ker L(\varphi)$ along $\operatorname{Rg} L(\varphi)$ i.e., $Q^2 = Q$;

$$\ker Q = \operatorname{Rg} L(\varphi) \quad \text{and} \quad \operatorname{Rg} Q = \ker L(\varphi).$$

Let P and P_H be the restriction of Q to V and H, respectively.

Claim 0: We first prove that that $P \in \mathcal{L}(V), P_H \in \mathcal{L}(H)$. Indeed, since

$$P_H(H) \subset Q(V^*) = \ker L(\varphi) \subset \{u \in C^\infty(\overline{\Omega}), \int_\Omega u = 0\} \subset H,$$

P_H is linear mapping from H to H and $P_H^2 = P_H$. Moreover,

$$\operatorname{Rg} P_H = \operatorname{Rg} Q \cap H = \ker L(\varphi), \quad \ker P_H = \ker Q \cap H = \operatorname{Rg} L(\varphi) \cap H. \tag{1.48}$$

It follows from [24, Theorem 5.15, page 133] that

$$\begin{cases} \ker L(\varphi) \cap (\operatorname{Rg} L(\varphi) \cap H) = \{0\} \\ \ker L(\varphi) + (\operatorname{Rg} L(\varphi) \cap H) = H \end{cases} \tag{1.49}$$

In order to prove P_H is continuous from H to H, in view of [24, Theorem 5.16, page 133], it is enough to prove that $\ker L(\varphi)$ and $(\operatorname{Rg} L(\varphi) \cap H)$ are closed in H. Clearly, $\ker L(\varphi)$ is closed in H since it is finite-dimensional. In order to prove $(\operatorname{Rg} L(\varphi) \cap H)$ is closed in H, we denote by j the continuous, compact embedding from H to V^*. Then $\operatorname{Rg} L(\varphi) \cap H = j^{-1}(\operatorname{Rg} L(\varphi))$ is the inverse image of a closed set under a continuous mapping, so that $\operatorname{Rg} L(\varphi) \cap H$ is closed in H. We conclude that $P_H \in \mathcal{L}(H)$. Similarly, $P \in \mathcal{L}(V)$ and

$$\operatorname{Rg} P = \ker L(\varphi), \quad \ker P = \operatorname{Rg} L(\varphi) \cap V.$$

Claim 1: We now prove that P_H is symmetric, i.e

$$\langle P_H u_1, u_2 \rangle_{H,H} = \langle u_1, P_H u_2 \rangle_{H,H} \quad \text{for all} \quad u_1, u_2 \in H.$$

We note from Lemma 1.3.9(iii) that $\ker L(\varphi)$ and $\operatorname{Rg}(\varphi) \cap H$ are orthogonal with respect to scalar product in H, which together with (1.48) implies that $\operatorname{Rg} P_H$ is orthogonal to $\ker P_H$ with respect to scalar product in H. On the other hand, $P_H u - u \in \ker P_H$ for all $u \in H$ so that

$$\langle P_H u_1, P_H u_2 - u_2 \rangle = 0.$$

Therefore,

$$\begin{aligned} \langle P_H u_1, u_2 \rangle_{H,H} &= \langle P_H u_1, (u_2 - P_H u_2) + P_H u_2 \rangle = \langle P_H u_1, P_H u_2 \rangle_{H,H} \\ &= \langle (P_H u_1 - u_1) + u_1, P_H u_2 \rangle_{H,H} = \langle u_1, P_H u_2 \rangle_{H,H}. \end{aligned}$$

Claim 2: Let $P^* \in \mathcal{L}(V^*)$ be the adjoint projection of P. We now claim that $P^* = Q$, i.e,

$$P^*(u) = Q(u) \text{ in } V^* \text{ for all } u \in V^*.$$

Let $u \in H \hookrightarrow V^*$; and $v \in V$; by the definition of the dual operator we have

$$\langle P^*u, v\rangle_{V^*,V} = \langle u, Pv\rangle_{V^*,V}.$$

Since $u, Pv \in H$, it follows that

$$\langle P^*u, v\rangle_{V^*,V} = \langle u, Pv\rangle_{V^*,V} = \langle u, P_H v\rangle_{H,H} = \langle P_H u, v\rangle_{H,H} = \langle Qu, v\rangle_{V^*,V}$$

It follows that $P^*(u) = Q(u)$ in V^* for all $u \in H$. On the other hand, since H is dense in V^* we deduce that $P^* = Q$. Consequently,

$$\operatorname{Rg} P^* = \operatorname{Rg} Q = \ker L(\varphi) \quad \text{and} \quad \ker P^* = \ker Q = \operatorname{Rg} L(\varphi).$$

□

Before proving Theorem 1.3.8, we recall the definition of an analytic map on a neighborhood of a point (cf. [28, Definition 8.8, p. 362]). A map T from a Banach space X into a Banach space Y is called analytic on a neighborhood of $z \in X$ if there exists $\varepsilon > 0$ such that for all $h \in X$, $\|h\|_X \leq \varepsilon$,

$$T(z+h) - T(z) = \sum_{i \geq 1} T_i(z)[h, \dots, h] \text{ in } Y,$$

where $T_i(z)$ is a symmetric i-linear form on X with values in Y and

$$\sum_{i \geq 1} \|T_i(z)\|_{\mathcal{L}_i(X,Y)} \|h\|_X^i < \infty.$$

Here, $\mathcal{L}_i(X, Y)$ is the space of bounded i-linear operators from X^i to Y.

Proof of Theorem 1.3.8. It follows from Lemma 1.3.11 that $(\mathbf{H_1})$ and $(\mathbf{H_2'})$ hold. We now verify $(\mathbf{H_3'})$. For this purpose, let $p > \max(2, N)$ and set

$$X := X_p, \qquad Y := \mathcal{L}^p(\Omega).$$

Since $P(X_p) \subset P(V) = \ker L(\varphi) \subset X_p$ by Lemma 1.3.9(ii) and since $P^*(\mathcal{L}^p(\Omega)) \subset P^*(V^*) = \ker L(\varphi) \subset \mathcal{L}^p(\Omega)$, we deduce that X_p and $\mathcal{L}^p(\Omega)$ are invariant by P and P^* respectively. In other words, $(\mathbf{H_3'})(\mathbf{i})$ holds. Moreover, $(\mathbf{H_3'})(\mathbf{iii})$ hold because of Lemma 1.3.9(i) and (ii). Now, it remains to verify $(\mathbf{H_3'})(\mathbf{ii})$ and $(\mathbf{H_3'})(\mathbf{iv})$.

Verifying $(\mathbf{H_3'})(\mathbf{ii})$: First, note that

$$E'(u) = -\Delta u - \bar{f}(u) + \fint_\Omega \bar{f}(u) \in \mathcal{L}^p(\Omega),$$

for all $u \in X_p$. We now claim that E' is analytic in a neighborhood of φ. Indeed, let ε be small enough such that for all $h \in X_p$ with $\|h\|_{X_p} \leq \varepsilon$, we have

$$\|h\|_{C(\overline{\Omega})} \leq C\|h\|_{X_p} < 1.$$

Since

$$\bar{f}(s) = f(s) = \sum_{i=0}^{n} a_i s^i \qquad \text{for all} \quad s \in (s_1 - 1, s_2 + 1),$$

we perform a Taylor's expansion to deduce for all $h \in X_p$ such that $\|h\|_{X_p} \leq \varepsilon$

$$\bar{f}(\varphi(x) + h(x)) - \bar{f}(\varphi(x)) = \sum_{i=1}^{n} \frac{\bar{f}^{(i)}(\varphi(x))}{i!} h^i(x).$$

It follows that

$$\begin{aligned} E'(\varphi + h) - E'(\varphi) &= -\Delta h + \sum_{i=1}^{n} \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i - \sum_{i=1}^{n} \fint_{\Omega} \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i \, dx \\ &= \sum_{i=1}^{n} T_i[h, \ldots, h], \end{aligned}$$

where

$$T_1[h] := -\Delta h + \bar{f}'(\varphi)h - \fint_{\Omega} \bar{f}'(\varphi)h$$

and

$$T_i[h, \ldots, h] := \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i - \fint_{\Omega} \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i \qquad \text{for all} \quad 1 < i \leq n.$$

We now prove that $T_i \in \mathcal{L}_i(X_p, \mathcal{L}^p(\Omega))$. For all $h_1, \ldots, h_i \in X_p$, and $1 < i \leq n$, we have

$$\begin{aligned} \Big\| T_i[h_1, .., h_i] \Big\|_{\mathcal{L}^p(\Omega)} &\leq C \Big\| T_i[h_1, .., h_i] \Big\|_{\mathcal{L}^\infty(\Omega)} \\ &\leq C \Big\| \frac{\bar{f}^{(i)}(\varphi)}{i!} h_1 \ldots h_i \Big\|_{\mathcal{L}^\infty(\Omega)} + C \Big\| \fint_{\Omega} \frac{\bar{f}^{(i)}(\varphi)}{i!} h_1 \ldots h_i \Big\|_{\mathcal{L}^\infty(\Omega)} \\ &\leq C \prod_{i=1}^{i} \|h_i\|_{L^\infty(\Omega)} \leq C \prod_{i=1}^{i} \|h_i\|_{X_p}, \end{aligned}$$

which implies that $T_i \in \mathcal{L}_i(X_p, \mathcal{L}^p(\Omega))$ for all $1 < i \leq n$. In the case $i = 1$, since $-\Delta$ is linear, continuous from X_p to $\mathcal{L}^p(\Omega)$, we easily deduce that $T_1 \in \mathcal{L}(X_p, \mathcal{L}^p(\Omega))$. Therefore, E' is analytic on a neighborhood of φ.

Verifying $(\mathbf{H}'_\mathbf{3})(\mathbf{iv})$: It is clear that

$$\operatorname{Rg}\left(L(\varphi)|_{X_p}\right) \subset \operatorname{Rg} L(\varphi) \cap \mathcal{L}^p(\Omega).$$

Now, we claim that

$$\operatorname{Rg}\left(L(\varphi)|_{X_p}\right) \supset \operatorname{Rg} L(\varphi) \cap \mathcal{L}^p(\Omega).$$

Let $g \in \operatorname{Rg} L(\varphi) \cap \mathcal{L}^p(\Omega)$, then there exists $u \in V$ such that $L(\varphi)u = g$. It is enough to prove that $u \in X_p$. Indeed, note that $L(\varphi)u = g$ is equivalent to

$$Au = \bar{f}'(\varphi)u - \fint_{\Omega} \bar{f}'(\varphi)u + g \quad \text{in} \quad V^*.$$

Since $\bar{f}'(\varphi)u - \fint_{\Omega} \bar{f}'(\varphi)u + g \in H$, we deduce from Lemma 1.3.5 that $u \in X_2$ and

$$\begin{cases} -\Delta u = \bar{f}'(\varphi)u - \fint_\Omega \bar{f}'(\varphi)u + g & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

We then apply a Sobolev embedding theorem and a bootstrap argument to deduce that $u \in X_p$, so that $(\mathbf{H}_\mathbf{3}')(\mathbf{iv})$ holds. The proof of Theorem 1.3.8 is complete. □

1.4 Large time behavior

This section is devoted to the proof of Theorem 1.1.2. We also compute the limit stationary solution in the one-dimensional case in Theorem 1.4.5 below.

Lemma 1.4.1. *Let $\overline{u}$ be the solution of Problem $(\overline{P})$ corresponding to the initial condition $u_0 \in L^2(\Omega)$. Then*

(i) For all $0 < s \le t < \infty$,

$$E(\overline{u}(s)) = E(\overline{u}(t)) + \int_s^t \int_\Omega |\overline{u}_t|^2 \, dx. \tag{1.50}$$

(ii) Further, $E(\overline{u}(\cdot))$ is continuous, nonincreasing on $(0, +\infty)$, and there exists e such that

$$\lim_{t\to\infty} E(\overline{u}(t)) = e.$$

Proof. (i) Because of Lemma 1.2.9, $\overline{u}$ is a smooth function on $\overline{\Omega} \times (0, \infty)$ so that

$$\begin{aligned} \frac{d}{dt}E(\overline{u}(t)) &= \int_\Omega \big(-\Delta\overline{u} - \bar{f}(\overline{u})\big)\overline{u}_t \\ &= \int_\Omega \big(-\Delta\overline{u} - \bar{f}(\overline{u}) + \fint_\Omega \bar{f}(\overline{u})\big)\overline{u}_t \\ &= -\int_\Omega \overline{u}_t^2(x,t)\, dx \le 0. \end{aligned}$$

As a consequence, for all $0 < s \le t < \infty$

$$E(\overline{u}(s)) = E(\overline{u}(t)) + \int_s^t \int_\Omega |\overline{u}_t|^2 \, dx.$$

(ii) We recall that the function $\bar{F}$ is bounded on $\mathbb{R}$. Therefore the function $t \mapsto E(\overline{u}(t))$, which is nonincreasing and bounded from below, converges to a limit as $t \to \infty$. □

Definition 1.4.2. *We define the ω-limit set of u_0 by*

$$\omega(u_0) := \{\varphi \in H^1(\Omega) : \exists t_n \to \infty, \overline{u}(t_n) \to \varphi \ \text{ in } \ H^1(\Omega) \text{ as } n \to \infty\}.$$

Corollary 1.4.3. *Let $u_0 \in L^2(\Omega)$, then $\omega(u_0)$ is non-empty.*

Proof. This corollary is an immediate consequence of Corollary 1.2.10. □

Lemma 1.4.4. *Let $u_0 \in L^2(\Omega)$. Then*

(i) $\omega(u_0)$ is a compact set of $H^1(\Omega)$.

(ii) For all $\varphi \in \omega(u_0)$

$$E(\varphi) = e,$$

where e *is defined as in Lemma 1.4.1(ii).*

(iii) Let $\varphi \in \omega(u_0)$ *then* φ *is a stationary solution of Problem* $(\overline{P})$*, which implies that*

$$\begin{cases} -\Delta\varphi = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\varphi \in C^\infty(\overline{\Omega})$ *and*

$$\int_\Omega \varphi = \int_\Omega u_0.$$

If we assume further that $s_1 \le u_0(x) \le s_2$ *a.e., in* Ω*, then*

$$s_1 \le \varphi(x) \le s_2 \quad \text{for all} \quad x \in \overline{\Omega}.$$

(iv) $d(\overline{u}(t), \omega(u_0)) \to 0$ *as* $t \to \infty$*, where*

$$d(\overline{u}(t), \omega(u_0)) := \inf_{\varphi \in \omega(u_0)} \|\overline{u}(t) - \varphi\|_{H^1(\Omega)}.$$

Proof. (i) This is an immediate consequence of Corollary 1.2.10.

(ii) Let $\varphi \in \omega(u_0)$ and let $\{\overline{u}(t_n)\}$ be such that

$$\overline{u}(t_n) \to \varphi \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad n \to +\infty.$$

We deduce from the continuity of E on $H^1(\Omega)$ that

$$E(\varphi) = \lim_{n\to\infty} E(\overline{u}(t_n) = e,$$

where e is as in Lemma 1.4.1.

(iii) We need to prove that φ is a stationary solution. We denote here by $\overline{u}(t; w)$ the solution of Problem $(\overline{P})$ corresponding to initial function w. Let t_n be such that

$$\overline{u}(t_n; u_0) \to \varphi \ \text{ in } \ H^1(\Omega) \ \text{ as } \ n \to \infty.$$

This implies in particular that

$$\overline{u}(t_n; u_0) \to \varphi \ \text{ in } \ L^2(\Omega) \ \text{ as } \ n \to \infty.$$

We use (1.31) to deduce that for all $t \ge 0$,

$$\overline{u}(t; \overline{u}(t_n; u_0)) \to \overline{u}(t; \varphi) \ \text{ in } \ L^2(\Omega) \ \text{ as } \ n \to \infty.$$

In other words,

$$\overline{u}(t + t_n; u_0) \to \overline{u}(t; \varphi) \ \text{ in } \ L^2(\Omega) \ \text{ as } \ n \to \infty.$$

In view of Corollary 1.2.10, $\{\overline{u}(\tau; u_0) : \tau \ge 1\}$ is relatively compact in $H^1(\Omega)$, so that

$$\overline{u}(t_n + t; u_0) \to \overline{u}(t; \varphi) \ \text{ in } \ H^1(\Omega) \ \text{ as } \ n \to \infty.$$

It follows that $\overline{u}(t;\varphi) \in \omega(\overline{u}_0)$. This together with (ii) implies that for all $t \geq s \geq 0$.

$$E(\overline{u}(t;\varphi)) = E(\overline{u}(s;\varphi)) = e.$$

In view of Lemma 1.4.1 , we have $t \geq s > 0$.

$$0 = E(\overline{u}(t;\varphi)) - E(\overline{u}(s;\varphi)) = -\int_s^t \int_\Omega |\overline{u}_t(\varphi)|^2 \, dxdt.$$

Consequently, for all $t > 0$, $\overline{u}_t(t;\varphi) = 0$. In other words, $\overline{u}(t;\varphi)$ the solution of Problem $(\overline{P})$ with the initial function φ is independent of time. Therefore, φ is a stationary solution of Problem $(\overline{P})$, which implies that

$$\begin{cases} -\Delta\varphi = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

We deduce from the Sobolev embedding theorem and a bootstrap argument that $\varphi \in C^\infty(\overline{\Omega})$. The identity

$$\int_\Omega \varphi = \int_\Omega u_0,$$

follows from the mass conservation property (cf. Lemma 1.2.4).

Now, if $s_1 \leq u_0 \leq s_2$ almost everywhere on Ω, then it follows from Lemma 1.2.11 that

$$s_1 \leq \overline{u}(x,t) \leq s_2 \qquad \text{for all } \ x \in \overline{\Omega}, t > 0,$$

so that also,

$$s_1 \leq \varphi(x) \leq s_2 \ \text{ for all } \ x \in \overline{\Omega}.$$

(iv) For the purpose of contradiction, we assume that there exists a sequence $t_n \to \infty$ and $\varepsilon_0 > 0$ such that

$$d(\overline{u}(t_n), \omega(u_0)) \geq \varepsilon_0 \ \text{ for all } \quad n > 0. \tag{1.51}$$

By Corollary 1.2.10, there exists a subsequence $t_{n_k} \to \infty$ and $w \in H^1(\Omega)$ such that

$$\overline{u}(t_{n_k}) \to w \in \omega(u_0) \ \text{ in } \quad H^1(\Omega) \ \text{ as } \quad k \to \infty,$$

Therefore, $d(\overline{u}(t_{n_k}), \omega(u_0)) = 0$ as $k \to \infty$, which is in contradiction with (1.51). □

Proof of Theorem 1.1.2. We will first prove Theorem 1.1.2 in the case

$$\int_\Omega u_0(x) = 0.$$

By the mass conservation property, we have that

$$\int_\Omega \overline{u}(x,t) = 0.$$

Consequently, $\overline{u}(t) \in V$ for all $t > 0$. Recall from Lemma 1.4.4(ii) that

$$E\big|_{\omega(u_0)} = e. \tag{1.52}$$

It follows from Lemma 1.4.4(iii) and Lemma 1.3.7 that for all $\varphi \in \omega(u_0)$,

$$s_1 \leq \varphi \leq s_2$$

and that φ is a critical point of E. We apply Theorem 1.3.8 to deduce that E satisfies the Lojasiewicz inequality in the neighborhood of every $\varphi \in \omega(u_0)$. In other words, we have that for every $\varphi \in \omega(u_0)$ there exist constants $\theta \in (0, \frac{1}{2}], C \geq 0$ and $\delta > 0$ such that

$$|E(v) - E(\varphi)|^{1-\theta} \leq C\|E'(v)\|_{V^*} \text{ whenever } \|v - \varphi\|_V \leq \delta. \tag{1.53}$$

Since E is continuous on V (cf. the proof of Lemma 1.4.4(ii)), we may choose δ small enough so that

$$|E(v) - E(\varphi)| < 1 \text{ whenever } \|v - \varphi\|_V \leq \delta. \tag{1.54}$$

It follows from the compactness of $\omega(u_0)$ in V that there exists a neighborhood $\mathcal{U}$ of $\omega(u_0)$ composed of finitely many balls $B_j, j = 1, ..., J$, with center φ_j and radius δ_j. In each of the ball B_j, inequality (1.54) and the Lojasiewicz inequality (1.53) hold for some constants θ_j and C_j. We define $\bar{\theta} = \min\{\theta_j, j = 1, ..., J\}$ and $\bar{C} = \max\{C_j, j = 1, ..., J\}$ to deduce from (1.52), (1.53) and (1.54) that

$$|E(v) - e|^{1-\bar{\theta}} \leq \bar{C}\|E'(v)\|_{V^*} \text{ for } v \in \mathcal{U}.$$

It follows from Lemma 1.4.4(iv) that there exists $t_0 \geq 0$ such that $\overline{u}(t) \in \mathcal{U}$ for all $t \geq t_0$. Hence, for every $t \geq t_0$, there holds

$$\begin{aligned} -\frac{d}{dt}|E(\overline{u}(t)) - e|^{\bar{\theta}} &= \bar{\theta}|E(\overline{u}(t)) - e|^{\bar{\theta}-1}\Big(-\frac{dE}{dt}(\overline{u}(t))\Big) \\ &\geq \frac{\bar{\theta}}{\bar{C}}\frac{\|\overline{u}_t\|^2_{L^2(\Omega)}}{\|E'(\overline{u}(t))\|_{V^*}}, \end{aligned} \tag{1.55}$$

where we have also used (1.50). Note that for all $t \geq t_0, \overline{u}(t) \in C^\infty(\overline{\Omega})$, so that $E'(\overline{u}(t)) \in H$ and it can be written of the form

$$E'(\overline{u}(t)) = -\Delta\overline{u} - \bar{f}(\overline{u}) + \fint_\Omega \bar{f}(\overline{u}) = -\overline{u}_t.$$

Applying continuous embedding $H \hookrightarrow V^*$, we have

$$\|E'(\overline{u}(t))\|_{V^*} \leq \bar{\bar{C}}\|E'(\overline{u}(t))\|_{L^2(\Omega)} = \bar{\bar{C}}\|\overline{u}_t\|_{L^2(\Omega)} \text{ for all } t \geq t_0, \tag{1.56}$$

where $\bar{\bar{C}}$ is a positive constant. Combining (1.55) and (1.56) we obtain

$$-\frac{d}{dt}|E(\overline{u}(t)) - e|^{\bar{\theta}} \geq C_0\|\overline{u}_t\|_{L^2(\Omega)}.$$

Here, $C_0 = \dfrac{\bar{\theta}}{\bar{C}\bar{\bar{C}}}$. Thus

$$\|\overline{u}(t_1) - \overline{u}(t_2)\|_{L^2(\Omega)} \leq \int_{t_1}^{t_2} \|\overline{u}_t\|_{L^2(\Omega)} \leq \frac{1}{C_0}(|E(\overline{u}(t_1)) - e|^{\bar{\theta}} - |E(\overline{u}(t_2)) - e|^{\bar{\theta}})$$

for all $t_0 \le t_1 \le t_2$. Therefore, $\|\overline{u}(t_1) - \overline{u}(t_2)\|_{L^2(\Omega)}$ tends to zero as $t_1 \to \infty$ so that $\{\overline{u}(t)\}$ is a Cauchy sequence in H. Consequently, there exists $\varphi \in H$ such that $\lim_{t\to\infty} \overline{u}(t) = \varphi$ exists in H, which together with Corollary 1.2.10 allows us to conclude that

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{C^m(\overline{\Omega})} = 0.$$

In the general case, when

$$\int_\Omega u_0(x)\,dx \neq 0,$$

instead of considering Problem $(\overline{P})$, we consider the Problem $(\widehat{P})$:

$$(\widehat{P}) \begin{cases} \widehat{u}_t = \Delta\widehat{u} + \hat{f}(\widehat{u}) - \fint_\Omega \hat{f}(\widehat{u}) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu \widehat{u} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \widehat{u}(x,0) = u_0(x) - m_0, & x \in \Omega. \end{cases}$$

where $m_0 := \fint_\Omega u_0$, and $\hat{f}(s) := \bar{f}(s + m_0)$. We note that

$$s_1 - m_0 \le \hat{u}(x,0) \le s_2 - m_0, \qquad \int_\Omega \hat{u}(x,0) = 0.$$

Moreover,

$$u = \overline{u} = \widehat{u} + m_0$$

and $\hat{f}$ is analytic on $(s_1 - 1 - m_0, s_2 + 1 - m_0)$. Repeating the above arguments for Problem $(\widehat{P})$, we deduce that there exists a smooth stationary solution ψ of Problem $(\widehat{P})$ such that

$$\lim_{t\to\infty} \|\widehat{u}(t) - \psi\|_{C^m(\overline{\Omega})} = 0.$$

It follows that for $\varphi := \psi + m_0$, we have

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{C^m(\overline{\Omega})} = 0.$$

The proof of Theorem 1.1.2 is complete. □

Theorem 1.4.5. *Let the same assumption as in Theorem 1.1.2 hold and let φ be the limit stationary solution as in Theorem 1.1.2. We assume further that $N = 1$ (i.e. we consider Problem (P) in the one-dimensional case) and that*

$$f'(s) \le 0 \qquad \textit{for all} \quad s \in [s_1, s_2].$$

Then

$$\varphi = \fint_\Omega u_0.$$

Proof. We recall that φ is a smooth solution of stationary problem

$$\begin{cases} \varphi_{xx} = -f(\varphi) + \fint_\Omega f(\varphi) & \text{in } \Omega, \\ \varphi_x = 0 & \text{on } \partial\Omega. \end{cases}$$

Set $w := \varphi_x$, then w satisfies the problem

$$\begin{cases} w_{xx} + f'(\varphi)w = 0 & \text{in } \Omega, \\ \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $c(x) := f'(\varphi) \leq 0$, we apply the maximum principle [10, Theorem 2, page 346] to w and to $-w$ to deduce that $w \equiv 0$. Consequently, φ is constant. This together mass conservation property implies that

$$\varphi = \fint_{\Omega} u_0.$$

□

1.5 Rate of the convergence

In this section, we evaluate the rate of the convergence of the solution to the stationary solution The proof is based once more on the Lojasiewicz inequality. We consider two cases, first the case that the Lojasiewicz exponent $\theta = \frac{1}{2}$ and then the case that $\theta \in (0, \frac{1}{2})$, as it is done by Haraux and Jendoubi [12]; Haraux, Jendoubi and Kavian [13].

1.5.1 The case that $\theta = \frac{1}{2}$

Lemma 1.5.1 (see [12], Lemma 2.2)**.** *Let $t_0 \geq 0$ be arbitrary. Assume that there exist positive constants γ and a such that*

$$\int_t^{+\infty} \|u_t\|^2_{L^2(\Omega)}\, ds \leq a \exp(-\gamma t) \qquad \text{for all } \ t \geq t_0.$$

Then for all $\tau \geq t \geq t_0$, we have that

$$\|u(t) - u(\tau)\|_{L^2(\Omega)} \leq \sqrt{a}b \exp(-\frac{\gamma t}{2}),$$

where $b := \dfrac{\exp(\frac{\gamma}{2})}{\exp(\frac{\gamma}{2}) - 1}$.

Theorem 1.5.2. *Let* $(\mathbf{H_0})$ *hold. Assume further that Theorem 1.3.8 holds for $\theta = \frac{1}{2}$; then there exist positive constants K, δ such that*

$$\|u(t) - \varphi\|_{L^2(\Omega)} \leq K \exp(-\delta t) \quad \text{for all } \ t > 0.$$

Proof. As in the proof of Theorem 1.1.2, it is sufficient to prove this result for the function $\overline{u}$ with the assumption that

$$\int_\Omega u_0 = 0.$$

Since $\overline{u}(x,t)$ is smooth for all $t > 0$, we have

$$\frac{d}{dt}(E(\overline{u}) - E(\varphi)) = \langle E'(\overline{u}), \overline{u}_t \rangle = -\langle E'(\overline{u}), E'(\overline{u}) \rangle = -\|E'(\overline{u})\|^2_H. \tag{1.57}$$

Note that

$$\overline{u}(t) \to \varphi \ \text{ in } \ V \ \text{ as } \ t \to \infty,$$

we deduce that for σ as in Theorem 1.3.8 there exists $T_0 > 0$ such that for all $t \geq T_0$

$$\|\overline{u}(t) - \varphi\|_V \leq \sigma.$$

Therefore, by Theorem 1.3.8, we have for all $t \geq T_0$

$$(E(\overline{u}(t)) - E(\varphi))^{\frac{1}{2}} = |E(\overline{u}(t)) - E(\varphi)|^{\frac{1}{2}} \leq C\|E'(\overline{u}(t))\|_{V^*}.$$

By using the continuous embedding $H \hookrightarrow V^*$, we obtain

$$(E(\overline{u}(t)) - E(\varphi))^{\frac{1}{2}} \leq C_1\|E'(\overline{u}(t))\|_H,$$

which implies that

$$(E(\overline{u}(t)) - E(\varphi)) \leq C_1^2\|E'(\overline{u}(t))\|_H^2,$$

or equivalently,

$$-\|E'(\overline{u}(t))\|_H^2 \leq -\frac{1}{C_1^2}(E(\overline{u}(t)) - E(\varphi)).$$

This together with (1.57) implies that

$$\frac{d}{dt}(E(\overline{u}(t)) - E(\varphi)) \leq -C_2(E(\overline{u}(t)) - E(\varphi)) \ \text{ for all } \ t \geq T_0, \tag{1.58}$$

where $C_2 := 1/C_1^2$. We also note that

$$y(t) := \left(E(\overline{u}(T_0)) - E(\varphi)\right)\exp(-C_2(t - T_0))$$

is the unique solution of the differential equation

$$\begin{cases} \dfrac{d}{dt}y(t) = -C_2 y \ \text{ for } \ t \geq T_0, \\ \\ y(T_0) = E(\overline{u}(T_0)) - E(\varphi). \end{cases}$$

Therefore, by [14, Theorem 6.1, p.31] and the differential inequality (1.58), we deduce that for all $t \geq T_0$

$$E(\overline{u}(t)) - E(\varphi) \leq \left(E(\overline{u}(T_0)) - E(\varphi)\right)\exp(-C_2(t - T_0)).$$

In view of (1.50), this implies that for all $t \geq T_0$

$$\int_t^\infty \|\overline{u}_t(s)\|_H^2\, ds \leq \left(E(\overline{u}(T_0) - E(\varphi)\right)\exp(-C_2(t - T_0)).$$

Setting $a := \left(E(\overline{u}(T_0) - E(\varphi)\right)\exp(C_2T_0) > 0$, we obtain the inequality

$$\int_t^\infty \|\overline{u}_t(s)\|_{L^2(\Omega)}^2 = \int_t^\infty \|\overline{u}_t(s)\|_H^2\, ds \leq a\exp(-C_2t) \ \text{ for all } \ t \geq T_0.$$

The conclusion of Theorem 1.5.2 then follows from Lemma 1.5.1. □

1.5.2 The case that $\theta \in (0, \frac{1}{2})$

We will apply the following lemma.

Lemma 1.5.3 (see [13], Lemma 3.3)**.** *Let $t_0 > 0$ be arbitrary. Assume that there exist two positive constant $\alpha >$ and K such that*

$$\int_t^\infty \|u_t\|^2_{L^2(\Omega)} \le K t^{-2\alpha-1} \qquad \text{for all } \ t \ge t_0.$$

Then, we have

$$\|u(t) - u(\tau)\|_{L^2(\Omega)} \le \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } \ \tau \ge t \ge t_0.$$

Theorem 1.5.4. *Let $(\mathbf{H_0})$ hold. Assume further that Theorem 1.3.8 holds for $\theta \in (0, \frac{1}{2})$ and set $\alpha := \dfrac{\theta}{1-2\theta} > 0$. Then there exists a positive constant K such that*

$$\|u(t) - \varphi\|_{L^2(\Omega)} \le \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } \ t > 0.$$

Proof. As in the proof of Theorem 1.1.2, it is sufficient to prove this result for the function $\overline{u}$ in the case that

$$\int_\Omega u_0 = 0.$$

Since $\overline{u}(x,t)$ is smooth for all $t > 0$, we have

$$\frac{d}{dt}(E(\overline{u}) - E(\varphi)) = \langle E'(\overline{u}), \overline{u}_t \rangle = -\langle E'(\overline{u}), E'(\overline{u}) \rangle = -\|E'(\overline{u})\|^2_H. \tag{1.59}$$

Note that

$$\overline{u}(t) \to \varphi \ \text{ in } \ V \ \text{ as } \ t \to \infty,$$

we deduce that for σ as in Theorem 1.3.8 there exists $T_0 > 0$ such that for all $t \ge T_0$

$$\|\overline{u}(t) - \varphi\|_V \le \sigma.$$

Therefore, by Theorem 1.3.8, we have for all $t \ge T_0$

$$|E(\overline{u}(t)) - E(\varphi)|^{1-\theta} \le C \|E'(\overline{u}(t))\|_{V^*}.$$

By applying the continuous embedding $H \hookrightarrow V^*$, we obtain

$$(E(\overline{u}(t)) - E(\varphi))^{1-\theta} = |E(\overline{u}(t)) - E(\varphi)|^{1-\theta} \le C_1 \|E'(\overline{u}(t))\|_H,$$

which implies that

$$(E(\overline{u}(t)) - E(\varphi))^{2(1-\theta)} \le C_1^2 \|E'(\overline{u}(t))\|^2_H,$$

or equivalently,

$$-\|E'(\overline{u})\|^2_H \le -\frac{1}{C_1^2}(E(\overline{u}) - E(\varphi))^{2(1-\theta)}.$$

This together with (1.59) implies that

$$\frac{d}{dt}(E(\overline{u}) - E(\varphi)) \le -C_2 (E(\overline{u}) - E(\varphi))^{2(1-\theta)} \ \text{ for all } \ t \ge T_0, \tag{1.60}$$

where $C_2 := 1/C_1^2$. We also note that

$$y(t) := \left((E(\overline{u}(T_0) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t-T_0) \right)^{-1/(1-2\theta)}$$

is the unique solution of the differential equation

$$\begin{cases} \dfrac{d}{dt}y(t) = -C_2 y^{2(1-\theta)} \quad \text{for} \quad t \geq T_0, \\ \\ y(T_0) = E(\overline{u}(T_0) - E(\varphi)). \end{cases}$$

Therefore, by [14, Theorem 6.1, p.31] and the differential inequality (1.60), we deduce that

$$\begin{aligned} E(\overline{u}(t)) - E(\varphi) &\leq \left((E(\overline{u}(T_0) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t-T_0) \right)^{-1/(1-2\theta)} \\ &= \left((E(\overline{u}(T_0) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)t \right)^{-1/(1-2\theta)} \\ &= \left((E(\overline{u}(T_0) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)\frac{t}{2} + C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \\ &\leq \left(C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \quad \text{for all } t \geq 2T_0. \end{aligned}$$

It follows that for all $t \geq 2T_0$

$$\int_t^\infty \|\overline{u}_t(s)\|^2 \leq \left(C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)}.$$

We set $K := \left(\dfrac{C_2(1-2\theta)}{2} \right)^{-1/(1-2\theta)}$ and $\alpha := \dfrac{\theta}{1-2\theta} > 0$, then

$$\int_t^\infty \|\overline{u}_t(s)\|^2 \, ds \leq K t^{-2\alpha-1},$$

which by Lemma 1.5.3 implies that

$$\|\overline{u}(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all} \quad t \geq 2T_0.$$

□

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Chapter 2

On the large time behavior of the solutions of a nonlocal ordinary differential equation

Résumé. Nous analysons le comportement en temps long des solutions d'une équation différentielle ordinaire non locale. Un résultat de convergence est prouvé pour une grande classe de fonctions initiales. Nous caractérisons également la fonction limite, et montrons qu'il s'agit d'une fonction en escalier.

Abstract. We analyse the large time behavior of solutions of a nonlocal ordinary differential equation. A convergence result is proved for a large class of initial functions. We also characterize the limit function and we prove that the limit function is given by a step function.

2.1 Introduction

We consider the initial value problem for the equation

$$(P)\quad \begin{cases} u_t = f(u) - \fint_\Omega f(u) & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where Ω is a bounded, open set of $\mathbb{R}^N$ with $N \geq 1$ and

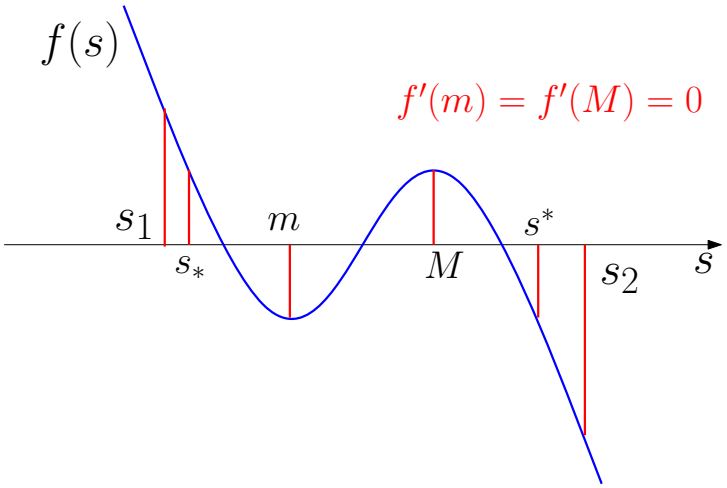
$$\fint_\Omega f(u) := \frac{1}{|\Omega|}\int_\Omega f(u(x))\,dx.$$

We assume that $f \in C^1(\mathbb{R})$ is such that there exists $m < M$ sastifying $f'(m) = f'(M) = 0$ and

$$f' < 0 \ \text{ on } \ (-\infty, m) \cup (M, +\infty), \quad f' > 0 \ \text{ on } \ (m, M);$$

We suppose further that there exist $s_* < s^*$ satisfying

$$\begin{cases} s_* < m < M < s^* \\ f(s_*) = f(M), \quad f(s^*) = f(m). \end{cases} \tag{2.1}$$



Note that there exists a constant $c_0 > 0$ such that

$$f'(s) \leq c_0 \quad \text{for all} \quad s \in \mathbb{R}. \tag{2.2}$$

A typical example is $f(s) = s - s^3$.

Constants s_1, s_2: Let s_1, s_2 be two constants such that $s_1 < s_* < s^* < s_2$. The following property holds

$$f(s_2) < f(s) < f(s_1) \qquad \text{for all} \quad s \in (s_1, s_2). \tag{2.3}$$

Assumption on the initial function: For the existence and the uniqueness of the solution, we suppose that the following hypothesis is satisfied

$(\mathbf{H_1})$ $u_0 \in L^\infty(\Omega)$ and $s_1 \le u_0(x) \le s_2$ for a.e. $x \in \Omega$.
For the large time behavior, we assume furthermore one of the two hypotheses
$(\mathbf{H_2})$ $\fint_\Omega u_0 \notin [s_*, s^*]$;
$(\mathbf{H_3})$ $s_* \le u_0 \le s^*$ and $|\{x \in \Omega : u_0(x) = s\}| = 0$ for all $s \in (m, M)$.

Theorem 2.1.1. *Assume that* $(\mathbf{H_1})$ *holds. Then Problem* (P) *possesses a unique solution* $u \in C^1([0,+\infty); L^\infty(\Omega))$. *Moreover, for all* $t \ge 0$, *we have*

$$s_1 \le u(x,t) \le s_2 \qquad \text{for a.e. } x \in \Omega.$$

Theorem 2.1.2. *Let* $(\mathbf{H_1})$ *hold and let* $p \in [1,+\infty)$ *be arbitrary.*

(i) If we assume further that $(\mathbf{H_2})$ *holds, then*

$$u(t) \to \psi \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad t \to \infty,$$

where

$$\psi \equiv \fint_\Omega u_0(x)\,dx.$$

(ii) If we assume further that $(\mathbf{H_3})$ *holds then*

$$u(t) \to \psi \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad t \to \infty,$$

where ψ *is a step function. More precisely,* ψ *takes at most two values* a_-, a_+ *such that* $f(a_-) = f(a_+)$ *and that*

$$f'(a_-) \le 0, \quad f'(a_+) \le 0.$$

The essential difficulty of this problem is due to the lack of diffusion term, so that the solution is not very smooth. In particular, it is more difficult to prove the relative compactness of the solution orbits. Applying the rearrangement theory, we introduce the problem $(P^\sharp)$ in one space dimension which possesses a unique solution $u^\sharp$ which is such that $\{u^\sharp(t)\}$ is relatively compact in $L^1((0,|\Omega|))$. We study the large time behavior of the solution of Problem $(P^\sharp)$ and then deduce the result for the original problem.

The paper is organized as follows: in Section 2.2 we prove Theorem 2.1.1. In Section 2.3, we prove some properties of the solution u. We also prove in this section that Problem (P) possesses a Lyapunov functional. In Section 2.4, applying the rearrangement theory, we introduce Problem $(P^\sharp)$ and present some properties of its solution. In section 2.5, the uniqueness of the elements of the ω-limit set $\omega(u_0^\sharp)$ for Problem $(P^\sharp)$ is proved. Theorem 2.1.2 is proved in Section 2.6. We also show by means of a nontrivial counterexample that, when Hypothesis $(\mathbf{H_3})$ on the initial function does not hold, the limiting function may take three values. In Section 2.7, we give some auxiliary propositions.

2.2 Existence and uniqueness of the solution of Problem (P)

In order to prove the existence of the solution of Problem (P), we first consider an auxiliary problem which has a bounded nonlinearity. We introduce the function $\bar f \in C^1(\mathbb{R})$ as follows,

$$\bar f(s) = \begin{cases} 0 & \text{if } s \le s_1 - 2, \\ f(s) & \text{if } s \in [s_1 - 1, s_2 + 1], \\ 0 & \text{if } s \ge s_2 + 2. \end{cases} \tag{2.4}$$

We then define the auxiliary problem $(\overline{P})$ by

$$(\overline{P}) \quad \begin{cases} \overline{u}_t = \bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u}) & \text{in } \Omega \times (0,\infty), \\ \\ \overline{u}(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Notation: In this section, we denote by X the Banach space $C(\overline{\Omega})$ or $L^q(\Omega)$ for $1 \le q \le +\infty$. We write $u(t;u_0)$ for the solution of Problem (P) with initial function u_0 and $\overline{u}(t;u_0)$ for the solution of Problem $(\overline{P})$ with initial function u_0.

Lemma 2.2.1. *Let $u_0 \in X$, then Problem $(\overline{P})$ possesses a unique solution in $C^1([0,+\infty);X)$.*

Remark 2.2.2. Before proving Lemma 2.2.1, we observe that there exists a constant $c_1 > 0$ such that

$$|\bar{f}'(s)| \le c_1 \quad \text{for all} \quad s \in \mathbb{R}. \tag{2.5}$$

Therefore,

$$|\bar{f}(s) - \bar{f}(\tilde{s})| \le c_1 |s - \tilde{s}| \quad \text{for all} \quad s, \tilde{s} \in \mathbb{R}. \tag{2.6}$$

Proof of Lemma 2.2.1. Set

$$G(v) := \bar{f}(v) - \fint_\Omega \bar{f}(v).$$

We will prove that $G : X \to X$ is a Lipschitz map. Indeed, let $v_1, v_2 \in X$, we have

$$G(v_1) - G(v_2) = (\bar{f}(v_1) - \bar{f}(v_2)) - \left(\fint_\Omega (\bar{f}(v_1) - \bar{f}(v_2)) \right).$$

Therefore,

$$\begin{aligned} \|(G(v_1) - G(v_2)\|_X &\le \|\bar{f}(v_1) - \bar{f}(v_2)\|_X + \left\| \fint_\Omega (\bar{f}(v_1) - \bar{f}(v_2)) \right\|_X \\ &\le \|\bar{f}(v_1) - \bar{f}(v_2)\|_X + \frac{\|1\|_X}{|\Omega|} \int_\Omega |\bar{f}(v_1) - \bar{f}(v_2)| \\ &= \|\bar{f}(v_1) - \bar{f}(v_2)\|_X + \frac{\|1\|_X}{|\Omega|} \|\bar{f}(v_1) - \bar{f}(v_2)\|_{L^1(\Omega)}. \end{aligned}$$

We deduce from (2.6) that

$$\begin{aligned} \|G(v_1) - G(v_2)\|_X &\le c_1 \|v_1 - v_2\|_X + \frac{\|1\|_X}{|\Omega|} c_1 \|v_1 - v_2\|_{L^1(\Omega)} \\ &\le c_1 \|v_1 - v_2\|_X + \frac{\|1\|_X}{|\Omega|} c_1 c_2 \|v_1 - v_2\|_X \\ &= \left(c_1 + \frac{\|1\|_X}{|\Omega|} c_1 c_2 \right) \|v_1 - v_2\|_X. \end{aligned}$$

We define $L := (c_1 + \dfrac{\|1\|_X}{|\Omega|} c_1 c_2) > 0$, then

$$\|G(v_1) - G(v_2)\|_X \le L \|v_1 - v_2\|_X \quad \text{for all} \quad v_1, v_2 \in X. \tag{2.7}$$

In other words, G is a Lipschitz map from X to X. Now, the conclusion of Lemma 2.2.1 follows from [5, Theorem 7.3, page 184]. □

Corollary 2.2.3. *The solution is continuous with respect to the initial data. More precisely, let $u_{10}, u_{20} \in X$ and let $\overline{u}_1, \overline{u}_2$ be solutions of Problem $(\overline{P})$ corresponding to u_{10}, u_{20}, respectively. Then*

$$\|\overline{u}_1(t) - \overline{u}_2(t)\|_X \leq \|u_{10} - u_{20}\|_X \exp(Lt) \quad \text{for all} \quad t \geq 0, \tag{2.8}$$

where L is the Lipchitz constant given in the proof of Lemma 2.2.1.

Lemma 2.2.4 (Mass conservation)**.** *Let the same assumption as in Lemma 2.2.1 hold. Then,*

$$\int_\Omega \overline{u}(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \text{for all} \quad t \geq 0.$$

Proof. Integrating the differential equation in Problem $(\overline{P})$ from 0 to t, we get

$$\overline{u}(t) - \overline{u}(0) = \int_0^t \overline{u}_t = \int_0^t \left[\bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u})\right],$$

so that

$$\int_\Omega \overline{u}(x,t)\,dx - \int_\Omega u_0(x)\,dx = \int_0^t \int_\Omega \left[\bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u})\right] = 0.$$

This completes the proof of Lemma 2.2.4. □

Lemma 2.2.5. *Suppose that u_0 satisfies Hypothesis $(\mathbf{H_1})$, namely,*

$$s_1 \leq u_0(x) \leq s_2 \quad \text{for a.e.} \quad x \in \Omega.$$

Then for all $t \geq 0$,

$$s_1 \leq \overline{u}(x,t) \leq s_2 \quad \text{for a.e.} \quad x \in \Omega.$$

The main idea of the proof of Lemma 2.2.5 is to approximate the initial function u_0 by a sequence of continuous functions and to first obtain the result for solution of approximate problems. Then, we deduce the result for the original problem. To begin with, we state the following result.

Lemma 2.2.6. *Given $u_0 \in L^\infty(\Omega)$ satisfying $s_1 \leq u_0(x) \leq s_2$ for a.e. $x \in \Omega$. Then there exists a sequence $\{u_{n0}\}$ in $C(\overline{\Omega})$ such that $s_1 < u_{n0} < s_2$ on $\overline{\Omega}$, and that*

$$u_{n0} \to u_0 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty.$$

The proof of this lemma will be given later. Next, we will use Lemma 2.2.6 to prove Lemma 2.2.5.

Proof of Lemma 2.2.5. Let $\overline{u}_n$ be the solution of Problem $(\overline{P})$ with the initial function u_{n0}. Since $u_{n0} \in C(\overline{\Omega})$, $\overline{u}_n \in C^1([0,+\infty); C(\overline{\Omega}))$ and hence $\overline{u}_n$ is continuous on $\overline{\Omega} \times [0,+\infty)$. We will first prove that

$$s_1 < \overline{u}_n(x,t) < s_2 \text{ for all } x \in \overline{\Omega}, t \geq 0. \tag{2.9}$$

For the purpose of contradiction, we suppose that there exists a first time $t_0 > 0$ such that

$$\overline{u}_n(x_0,t_0) = s_1 \quad \text{or} \quad \overline{u}_n(x_0,t_0) = s_2$$

for some $x_0 \in \overline{\Omega}$. Without loss of generality, assume that $\overline{u}_n(x_0, t_0) = s_2$. By the continuity of $\overline{u}_n$ and the definition of t_0, we have

$$s_1 \leq \overline{u}_n(x, t_0) \leq s_2 \ \text{ for all } x \in \overline{\Omega}, \text{ and } \overline{u}_n(x, t) < s_2 \text{ for all } x \in \overline{\Omega} \text{ and } 0 \leq t < t_0. \quad (2.10)$$

By (2.10), we have

$$\overline{u}_{nt}(x_0, t_0) = \lim_{\Delta t \to 0^+} \frac{\overline{u}_n(x_0, t_0 - \Delta t) - \overline{u}_n(x_0, t_0)}{-\Delta t} \geq 0,$$

which we substitute in Problem $(\overline{P})$ to obtain

$$⨍_{\Omega} (\bar{f}(s_2) - \bar{f}(\overline{u}_n(x, t_0))) \, dx \geq 0. \quad (2.11)$$

Since $s_1 \leq \overline{u}_n(x, t_0) \leq s_2$ for all $x \in \overline{\Omega}$, it follows that

$$\bar{f}(s_2) \leq \bar{f}(\overline{u}_n(x, t_0)) \ \text{ for all } \ x \in \overline{\Omega}.$$

which together with (2.11) implies that

$$\bar{f}(s_2) = \bar{f}(\overline{u}_n(x, t_0)) \ \text{ for all } \ x \in \overline{\Omega},$$

and hence

$$\overline{u}_n(x, t_0) = s_2 \ \text{ for all } \ x \in \overline{\Omega}.$$

Therefore,

$$⨍_{\Omega} \overline{u}_{n0}(x) \, dx < s_2 = ⨍_{\Omega} \overline{u}_n(x, t_0) \, dx,$$

which contradicts the mass conservation property stated in Lemma 2.2.4. This completes the proof of (2.9).

We now show that for all $t \geq 0$,

$$s_1 \leq \overline{u}(x, t) \leq s_2 \quad \text{almost everywhere on } \ \Omega.$$

In view of (2.8), we have

$$\|\overline{u}(t) - \overline{u}_n(t)\|_{L^1(\Omega)} \leq \|u_0 - u_{n0}\|_{L^1(\Omega)} \exp(Lt) \quad \text{for all } \ t \geq 0.$$

Therefore for all $t \geq 0$,

$$\overline{u}_n(t) \to \overline{u}(t) \ \text{ in } \ L^1(\Omega) \ \text{ as } \ n \to \infty.$$

Consequently, for each $t \geq 0$, there exists a subsequence $\{\overline{u}_{nj}(t)\}_{j \geq 0}$ of $\{\overline{u}_n(t)\}_{n \geq 0}$ such that

$$\overline{u}_{nj}(x, t) \to \overline{u}(x, t) \ \text{ for a.e. } \ x \in \Omega \ \text{ as } \ j \to \infty,$$

which together with (2.9) follows that for each $t \geq 0$,

$$s_1 \leq \overline{u}(x, t) \leq s_2 \ \text{ for a.e. } \ x \in \Omega.$$

We completed the proof of Lemma 2.2.5. □

Proof of Lemma 2.2.6. First, we extend the function u_0 to w_0 defined on $\mathbb{R}^N$ by

$$w_0(x) = \begin{cases} u_0(x) & \text{if } \ x \in \Omega \\ \dfrac{s_1 + s_2}{2} & \text{elsewhere.} \end{cases}$$

Obviously,

$$s_1 \leq w_0(x) \leq s_2 \quad \text{for a.e.} \quad x \in \mathbb{R}^N.$$

For all n large enough such that

$$s_1 + \frac{1}{n} < s_2 - \frac{1}{n},$$

we define

$$w_{n0} := \min\left\{ s_2 - \frac{1}{n}, \max\{w_0, s_1 + \frac{1}{n}\} \right\}.$$

Then, we have

$$s_1 + \frac{1}{n} \leq w_{n0}(x) \leq s_2 - \frac{1}{n} \quad \text{a.e.} \quad x \in \mathbb{R}^N, \tag{2.12}$$

and moreover,

$$\begin{aligned} \|w_{n0} - u_0\|_{L^1(\Omega)} &= \|w_{n0} - w_0\|_{L^1(\Omega)} \\ &\leq |\Omega| \|w_{n0} - w_0\|_{L^\infty(\Omega)} \leq \frac{|\Omega|}{n}. \end{aligned}$$

As in [2, Theorem 2.29, page 36], we can choose a smooth function u_{n0} such that

$$\|u_{n0} - w_{n0}\|_{L^1(\Omega)} \leq \frac{|\Omega|}{n}.$$

More precisely,

$$u_{n0}(x) := \int_{\mathbb{R}^N} J_\varepsilon(x - y) w_{n0}(y) dy, \quad \text{for } \ \varepsilon \ \text{ small enough,} \tag{2.13}$$

where

$$J_\varepsilon(x) = \varepsilon^{-N} J(\frac{x}{\varepsilon}),$$

with $J \in C_c^\infty(\mathbb{R}^N)$ is a nonnegative function satisfying

$$J(x) = 0 \qquad \text{if} \quad |x| \geq 1, \quad \text{and} \quad \int_{\mathbb{R}^N} J(x) dx = 1.$$

Note that J_ε is nonnegative and that

$$\int_{\mathbb{R}^N} J_\varepsilon(x)\, dx = 1. \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$s_1 + \frac{1}{n} \leq u_{n0} \leq s_2 - \frac{1}{n} \quad \text{for all } \ x \in \mathbb{R}^N.$$

Moreover,

$$\|u_{n0} - u_0\|_{L^1(\Omega)} \le \|u_{n0} - w_{n0}\|_{L^1(\Omega)} + \|w_{n0} - u_0\|_{L^1(\Omega)} \le \frac{2|\Omega|}{n}.$$

It follows that

$$u_{n0} \to u_0 \quad \text{in } L^1(\Omega) \text{ as } n \to +\infty.$$

The proof of Lemma 2.2.6 is complete. □

Lemma 2.2.7 (Mass conservation)**.** *Let $u \in C^1([0,+\infty); L^\infty(\Omega))$ be a solution of Problem (P), then*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \textit{for all } t \ge 0.$$

Proof. The proof is similar to the one of Lemma 2.2.4. We omit it. □

Proof of Theorem 2.1.1. According to Lemma 2.2.1, $\overline{u} \in C^1([0,+\infty); L^\infty(\Omega))$. Moreover, because of Lemma 2.2.5, for all $t \ge 0$ we have

$$s_1 \le \overline{u}(x,t) \le s_2 \text{ for a.e. } x \in \Omega.$$

This together with the definition of $\bar{f}$ in (2.4) implies that for all $t \ge 0$

$$\bar{f}(\overline{u}(x,t)) = f(\overline{u}(x,t)) \text{ for a.e. } x \in \Omega.$$

It follows that $\overline{u}$ satisfies

$$\begin{cases} \overline{u}_t = \bar{f}(\overline{u}) - \fint_\Omega \bar{f}(\overline{u}) = f(\overline{u}) - \fint_\Omega f(\overline{u}) & \text{in } L^\infty(\Omega) \\ \\ \overline{u}(x,0) = u_0(x). \end{cases}$$

Therefore, $\overline{u}$ is a solution of Problem (P) in $C^1([0,+\infty); L^\infty(\Omega))$.

Next, we prove the uniqueness of the solution of Problem (P). For this purpose, let $\widetilde{u}_0 \in L^\infty(\Omega)$ such that

$$\int_\Omega \widetilde{u}_0(x)\,dx = \int_\Omega u_0(x)\,dx$$

and let $\widetilde{u} \in C^1([0,+\infty); L^\infty(\Omega))$ be a solutions of Problem (P) corresponding to the initial function $\widetilde{u}_0$. We denote by $w := u - \widetilde{u}$, and use mass conservation property (cf. Lemma 2.2.7) to deduce that

$$\int_\Omega w(x,t)\,dx = \int_\Omega (u_0(x) - \widetilde{u}_0(x))\,dx = 0 \text{ for all } t > 0. \tag{2.15}$$

We multiply the difference of the equations for u and $\widetilde{u}$ by w, then integrate over Ω to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) &= \int_\Omega \big[f(u) - f(\widetilde{u})\big]w - \int_\Omega w \fint_\Omega \big[f(u) - f(\widetilde{u})\big] \\ &= \int_\Omega \big[f(u) - f(\widetilde{u})\big]w, \end{aligned}$$

where the last equality follows from (2.15). It follows from (2.2) and that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) \le c_0 \int_\Omega w^2,$$

which together with Gronwall's lemma implies the uniqueness of the solution. □

Lemma 2.2.8. *We define*

$$Z := \{w \in X \quad \text{such that} \quad s_1 \leq w(x) \leq s_2 \quad \text{for a.e.} \quad x \in \Omega\}$$

and equip with Z the distance d given by $d(w_1, w_2) := \|w_1 - w_2\|_X$. Then the mapping

$$w \longrightarrow u(t; w)$$

from Z to Z is continuous, where we recall that $u(t; w)$ is the solution of Problem (P) with the initial function w.

Proof. First we remark from the proof of Theorem 2.1.1 that the solution of Problem (P) with initial function $w \in Z$ coincides with the solution of Problem $(\overline{P})$ with the same initial function w. Namely, $u(t; w) = \overline{u}(t; w)$ for all $w \in Z$. This together with (2.8) follows that

$$\|u(t; w_1) - u(t; w_2)\|_X = \|\overline{u}(t; w_1) - \overline{u}(t; w_2)\|_X \leq \exp(Lt)\|w_1 - w_2\|_X,$$

for all $w_1, w_2 \in Z$. This completes the proof of Lemma 2.2.8. □

2.3 Some properties of the solution

In this section, as a preparation for next section, we consider an ordinary differential equation which is related to Problem (P). This study puts forth some properties of the solution of Problem (P), which are stated in Lemma 2.3.2 below. To begin with, we define the function

$$\lambda(t) := \fint_\Omega f(u(t)), \tag{2.16}$$

where u is the unique solution of Problem (P) given in Theorem 2.1.1. Note that

$$\lambda(t) = \fint_\Omega \bar{f}(u(t)), \tag{2.17}$$

since

$$s_1 \leq u(t) \leq s_2 \quad \text{almost everywhere on} \quad \Omega.$$

We consider the problem

$$(ODE) \begin{cases} Y' = \bar{f}(Y) - \lambda(t) \\ \\ Y(0) = s. \end{cases}$$

Lemma 2.3.1. *Let $s \in \mathbb{R}$. Problem (ODE) possesses a global unique solution in $C^1([0, +\infty))$, which is denoted by $Y(t; s)$. Moreover,*

(i) if $s < \tilde{s}$ then $Y(t; s) < Y(t; \tilde{s})$ for all $t \geq 0$;

(ii) if $s_1 \leq s \leq s_2$ then $s_1 \leq Y(t; s) \leq s_2$ for all $t \geq 0$. In this case, $Y(t, s)$ satisfies

$$Y' = f(Y) - \lambda(t);$$

(iii) we define the function $h^t : \mathbb{R} \to \mathbb{R}$ by $h^t(s) := Y(t; s)$. Then h^t is bijective, increasing from $\mathbb{R}$ to $\mathbb{R}$.

Proof. First, we prove that λ is continuous on $[0,+\infty)$. Indeed, it follows from (2.17) and (2.6) that

$$\begin{aligned}|\lambda(t)-\lambda(\tau)| &\le \fint_\Omega |\bar f(u(t))-\bar f(u(\tau))| \\ &\le \frac{c_1}{|\Omega|}\|u(t)-u(\tau)\|_{L^1(\Omega)} \le c_1\|u(t)-u(\tau)\|_{L^\infty(\Omega)}.\end{aligned}$$

Since $u\in C^1([0,+\infty);L^\infty(\Omega))$, it follows that λ is continuous on $[0,+\infty)$. Therefore, the function H defined by

$$H(Y,t) := \bar f(Y)-\lambda(t)$$

is continuous on $(-\infty,+\infty)\times[0,+\infty)$. Moreover H satisfies a global Lipschitz condition:

$$|H(Y_1,t)-H(Y_2,t)| = |\bar f(Y_1)-\bar f(Y_2)| \le c_1|Y_1-Y_2|$$

for all $Y_1,Y_2\in\mathbb{R}$, $t\ge 0$. By the standard theory of ordinary differential equations (for instance, see [3, Corollary 8.3, page 57]), Problem (ODE) possesses a unique solution in $C^1([0,+\infty))$.

(i) We define $W(t) := Y(t;\tilde s)-Y(t;s)$. Then W satisfies the equations

$$\begin{cases} W' = \bar f(Y(t;\tilde s))-\bar f(Y(t;s)) \\ W(0)=\tilde s - s>0. \end{cases}$$

Note that

$$\begin{aligned}\bar f(Y(t;\tilde s))-\bar f(Y(t;s)) &= \int_0^1 \frac{d}{d\tau}\Big[\bar f(\tau Y(t;\tilde s)+(1-\tau)Y(t;s))\Big]d\tau \\ &= [Y(t;\tilde s)-Y(t;s)]\int_0^1 f'(\tau Y(t;\tilde s)+(1-\tau)Y(t;s))d\tau \quad = W(t)\theta(t),\end{aligned}$$

where

$$\theta(t) := \int_0^1 f'(\tau Y(t;\tilde s)+(1-\tau)Y(t;s))d\tau.$$

Thus, W satisfies the initial value problem

$$\begin{cases} W' = \theta(t)W(t) \\ W(0)=\tilde s - s>0, \end{cases}$$

so that

$$W(t) = W(0)\exp\left(\int_0^t \theta(\tau)d\tau\right) > 0.$$

It follows that

$$Y(t,\tilde s) > Y(t,s) \quad \text{for all} \quad t\ge 0.$$

(ii) First, we prove that

$$\bar f(s_2)\le \lambda(t)\le \bar f(s_1) \quad \text{for all} \quad t\ge 0, \tag{2.18}$$

or equivalently,

$$f(s_2)\le \lambda(t)\le f(s_1) \quad \text{for all} \quad t\ge 0.$$

This follows from (2.3) and the fact that

$$s_1 \leq u(x,t) \leq s_2 \ \text{ for a.e. } \ x \in \Omega, t \geq 0$$

We now define the differential operator $\mathcal{L}$ by

$$\mathcal{L}(Y) := Y' - H(Y,t) = Y' - \bar{f}(Y) + \lambda(t).$$

We deduce from (2.18) that

$$\mathcal{L}(s_1) \leq 0 = \mathcal{L}(Y) \leq \mathcal{L}(s_2).$$

These differential inequalities and [9, Theorem 6.1, page 31] imply that

$$s_1 \leq Y(t;s) \leq s_2 \ \text{ for all } \ t > 0,$$

provided that $s_1 \leq s \leq s_2$.

(iii) Let $t_0 > 0$ be arbitrary. It follows from (i) that if $s < \tilde{s}$, then

$$Y(t_0;s) < Y(t_0;\tilde{s});$$

hence, we have

$$h^{t_0}(s) < h^{t_0}(\tilde{s}).$$

Consequently, h^{t_0} is injective, increasing on $\mathbb{R}$.

Now we claim that h^{t_0} is surjective. Let $S \in \mathbb{R}$ be arbitrary. We consider the problem

$$\begin{cases} Y' = \bar{f}(Y) - \lambda(t) \\ Y(t_0) = S. \end{cases}$$

This problem possesses a unique solution in $C^1([0,+\infty))$ (cf. [3, Corollary 8.3, page 57]). We denote by $Y(t;t_0;S)$ that solution. We also remark that the solution of Problem (ODE) can be written as $Y(t;0;s)$. Moreover, by the uniqueness of the solution, we have that

$$Y(t_0;0;Y(0;t_0;S)) = S.$$

Therefore, $h^{t_0}(Y(0;t_0;S)) = S$, so that h^{t_0} is surjective. □

Lemma 2.3.2. *Let* $(\mathbf{H_1})$ *hold. Then for all* $t \geq 0$*, we have*

$$u(t) = h^t(u_0) \ \ \textit{almost everywhere on} \ \ \Omega.$$

Proof. We define

$$z(x,t) := h^t(u_0(x)) = Y(t;u_0(x)) \ \text{ on } \ \Omega \times [0,+\infty).$$

We need to prove that for all $t \geq 0$

$$z(t) = u(t) \ \text{ almost everywhere on } \ \Omega.$$

First, by the definition of $Y(t;\cdot)$, we see that for all $x \in \Omega$; $z(x,\cdot) \in C^1([0,+\infty))$ and

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \bar{f}(z(x,t)) - \lambda(t) & \text{in } \Omega \times [0,\infty), \\ \\ z(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{2.19}$$

The right-hand-side of the differential equation in (2.19) is bounded in $L^\infty(0,T;L^\infty(\Omega))$ which together with the continuity of λ implies that $z \in C^1([0,+\infty);L^\infty(\Omega))$.

We integrate from 0 to t the difference of the equations for u and for z to obtain

$$u(t) - z(t) = \int_0^t (f(u(\tau)) - \bar{f}(z(\tau)) = \int_0^t (\bar{f}(u(\tau)) - \bar{f}(z(\tau)).$$

It follows from (2.6) that

$$\|u(t) - z(t)\|_{L^\infty(\Omega)} \le \int_0^t \|\bar{f}(u(\tau)) - \bar{f}(z(\tau))\|_{L^\infty(\Omega)} \le c_1 \int_0^t \|u(\tau) - z(\tau)\|_{L^\infty(\Omega)},$$

which together with Gronwall's inequality completes the proof of Lemma 2.3.2. □

2.4 Problem $(P^\sharp)$ obtained by the rearrangement theory

This section is devoted to introduce Problem $(P^\sharp)$ by using the rearrangement theory. Problem $(P^\sharp)$ is a version of Problem (P) in one space dimension. We first recall some known results about rearrangement in Section 2.4.1. Then in Section 2.4.2, we introduce Problem $(P^\sharp)$ and present some properties of its solution, which we state in Theorem 2.4.10.

2.4.1 Rearrangement theory

Let w be a function from Ω to $\mathbb{R}$. Then the distribution function of w is given by

$$\mu_w(s) := |\{x \in \Omega : w(x) > s\}|.$$

Definition 2.4.1. *The (one-dimensional) decreasing rearrangement of w, denoted by $w^\sharp$, is defined on $[0,|\Omega|]$ by*

$$\begin{cases} w^\sharp(0) := \operatorname{ess\,sup}(w) \\ w^\sharp(y) = \inf\{s : \mu_w(s) < y\}, \quad y > 0, \end{cases} \tag{2.20}$$

where

$$\operatorname{ess\,sup}(w) := \inf\{a : |\{x \in \Omega : w(x) > a\}| = 0\}.$$

Remark 2.4.2. $w^\sharp$ is nonincreasing. (cf. [11, Proposition 1.1.1, page 3].)

Remark 2.4.3. $w^\sharp$ is a function defined for all $y \in [0,|\Omega|]$ and it is uniquely defined by the distribution function μ_w. Consequently, if $w_1, w_2 : \Omega \to \mathbb{R}$ are such that

$$w_1(x) = w_2(x) \quad \text{for a.e.} \quad x \in \Omega,$$

then

$$w_1^\sharp(y) = w_2^\sharp(y) \quad \text{for all} \quad y \in [0,|\Omega|].$$

We recall some properties of $w^\sharp$ which are stated in [11, Chapter 1].

Proposition 2.4.4. *(i) $w^\sharp, w$ are equi-measurable, i.e. $\mu_{w^\sharp} = \mu_w$,*
(ii) if G is a Borel measurable function such that either $G \geq 0$ or $G(w) \in L^1(\Omega)$ then

$$\int_0^{|\Omega|} G(w^\sharp) = \int_\Omega G(w).$$

(iii) Let $w_1, w_2 \in L^p(\Omega)$ for $1 \leq p \leq +\infty$ then

$$\|w_1^\sharp - w_2^\sharp\|_{L^p((0,|\Omega|))} \leq \|w_1 - w_2\|_{L^p(\Omega)}.$$

Proof. (i) see [11, Corollary 1.1.3, page 4].
(ii) see [11, Theorem 1.1.1, page 5; Corollary 1.1.3 page 6].
(iii) see [11, Proposition 1.2.1, page 8 and Theorem 1.2.3, page 12].

□

Remark 2.4.5. Since w and $w^\sharp$ are equi-measurable, we have the following properties:

$$|\{w^\sharp \geq s\}| = |\{w \geq s\}|;$$

$$|\{w^\sharp = s\}| = |\{w = s\}|.$$

Remark 2.4.6. If $s_1 \leq w \leq s_2$ almost everywhere on Ω, then

$$s_1 \leq w^\sharp(y) \leq s_2 \qquad \text{for all } \ y \in [0, |\Omega|].$$

Proof of Remark 2.4.6. Since $w^\sharp$ is nonincreasing, it is sufficient to prove that

$$s_1 \leq w^\sharp(|\Omega|) \qquad \text{and} \qquad w^\sharp(0) \leq s_2.$$

First, in view of the definition of $w^\sharp$ in (2.20), we have

$$\begin{cases} w^\sharp(0) = \operatorname{ess\,sup} w \leq s_2 \\ w^\sharp(|\Omega|) = \inf\{s : \mu_w(s) < |\Omega|\}. \end{cases}$$

We only need to show that

$$\inf\{s : \mu_w(s) < |\Omega|\} \geq s_1.$$

Note that for all $\tau \in (-\infty, s_1]$, we have $\mu_w(\tau) = |\Omega|$. Therefore $\tau \notin \{s : \mu_w(s) < |\Omega|\}$. Consequently, we have

$$\{s : \mu_w(s) < |\Omega|\} \subset (s_1, +\infty)$$

so that

$$\inf\{s : \mu_w(s) < |\Omega|\} \geq s_1.$$

□

The following proposition will be used later.

Proposition 2.4.7. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ is increasing, bijective and let $w : \Omega \to \mathbb{R}$. Then*

$$(\Phi(w))^\sharp(y) = \Phi(w^\sharp(y)) \qquad \text{for all } \ y \in [0, |\Omega|].$$

The proof of Proposition 2.4.7 will be given in Section 2.7.

2.4.2 Problem $(P^\sharp)$

We consider the problem:

$$(P^\sharp)\quad \begin{cases} v_t = f(v) - \fint_0^{|\Omega|} f(v) & \text{in } (0,|\Omega|)\times(0,\infty), \\ \\ v(y,0) = u_0^\sharp(y) & y \in (0,|\Omega|). \end{cases}$$

Problem $(P^\sharp)$ is a one-dimensional space variant of Problem (P), so that all results stated in Section 2.2 for Problem (P) also apply to Problem $(P^\sharp)$. We note that all hypotheses $(\mathbf{H_1}), (\mathbf{H_2}), (\mathbf{H_3})$ for the function u_0 are satisfied by $u_0^\sharp$. More precisely, we have the following result.

Lemma 2.4.8. *If u_0 satisfies $(\mathbf{H_1})$ (respectively $(\mathbf{H_2}), (\mathbf{H_3})$) then, $u_0^\sharp$ satisfies the same statement, respectively.*

Proof. First, it follows from Remark 2.4.6 that if $s_1 \le u_0 \le s_1$ then $s_1 \le u_0^\sharp \le s_2$. We also note that

$$\fint_\Omega u_0 = \frac{1}{|\Omega|}\int_\Omega u_0 = \frac{1}{|\Omega|}\int_0^{|\Omega|} u_0^\sharp = \fint_0^{|\Omega|} u_0^\sharp.$$

Therefore, the hypotheses $(\mathbf{H_1})$, $(\mathbf{H_2})$ are invariant by operation $\sharp$.

Now, we consider $(\mathbf{H_3})$. Assume that $|\{u_0 = s\}| = 0$ for all $s \in (m,M)$. According to Remark 2.4.5, we deduce that

$$|\{u^\sharp = s\}| = |\{u = s\}| = 0 \qquad \text{for all } \ s \in (m,M).$$

The proof of Lemma 2.4.8 is complete. □

Remark 2.4.9. *If u_0 satisfies (H_3), then $s_* \le u_0^\sharp(y) \le s^*$ for all $y \in (0,|\Omega|)$ and $u_0^\sharp$ is strictly decreasing on $\{x : u_0^\sharp(x) \in (m,M)\}$.*

Theorem 2.4.10. *Let $(\mathbf{H_1})$ hold. We define*

$$u^\sharp(y,t) := (u(t))^\sharp(y) \quad \text{on } \ (0,|\Omega|)\times[0,+\infty);$$

then $u^\sharp$ is the unique solution of Problem $(P^\sharp)$ in $C^1([0,\infty); L^\infty((0,|\Omega|)))$. Moreover,

$$s_1 \le u^\sharp(y,t) \le s_2 \qquad \text{for all } \ y \in (0,|\Omega|), t \ge 0; \tag{2.21}$$

and

$$u^\sharp(y,t) = h^t(u_0^\sharp(y)) \qquad \text{for all } \ y \in (0,|\Omega|), t \ge 0. \tag{2.22}$$

Proof. First we prove (2.21) and (2.22). Since

$$s_1 \le u(x,t) \le s_2 \ \text{ for a.e } \ x \in \Omega, t \ge 0$$

we deduce from Remark 2.4.6 that

$$s_1 \le u^\sharp(y,t) \le s_2 \text{ for all } \ y \in (0,|\Omega|), t \ge 0.$$

In order to prove (2.22), we recall from Lemma 2.3.2 that $u(t) = h^t(u_0)$ almost everywhere on Ω. We then apply Proposition 2.4.7 to obtain for all $t \geq 0$,

$$u^\sharp(t) = (h^t(u_0))^\sharp = h^t(u_0^\sharp) \quad \text{on} \quad (0, |\Omega|). \tag{2.23}$$

Now, it remains to prove that $u^\sharp$ is the unique solution of Problem $(P^\sharp)$. As a consequence of (2.23) and Lemma 2.3.1, we have for all $y \in (0, |\Omega|), t \geq 0$

$$\begin{aligned}\frac{\partial u^\sharp}{\partial t}(y,t) &= \bar{f}(u^\sharp(y,t)) - \lambda(t) \\ &= f(u^\sharp(y,t)) - \fint_\Omega f(u(t)) = f(u^\sharp(y,t)) - \fint_0^{|\Omega|} f(u^\sharp(t)),\end{aligned} \tag{2.24}$$

where the last identity follows from Proposition 2.4.4(ii). Note that the right-hand-side of (2.24) is bounded in $L^\infty(0,T;L^\infty((0,|\Omega|)))$ which together with the continuity of f implies that $u^\sharp \in C^1([0,+\infty);L^\infty((0,|\Omega|)))$. Therefore, $u^\sharp$ coincides with the unique solution of Problem $(P^\sharp)$ in $C^1([0,\infty);L^\infty((0,|\Omega|)))$. The proof of Theorem 2.4.10 is complete. □

Corollary 2.4.11. *Let* $(\mathbf{H_1})$ *hold. Then for all* $y \in (0,|\Omega|)$ *the function* $u^\sharp(y,\cdot)$ *is the unique solution of Problem* (ODE) *in* $C^1([0,+\infty))$ *with initial condition* $u_0^\sharp(y)$*. Moreover, for all* $y_1, y_2 \in (0,|\Omega|)$*,*

(i) if $u_0^\sharp(y_1) = u_0^\sharp(y_2)$ *then* $u^\sharp(y_1,t) = u^\sharp(y_2,t)$ *for all* $t \geq 0$*,*

(ii) if $u_0^\sharp(y_1) < u_0^\sharp(y_2)$ *then* $u^\sharp(y_1,t) < u^\sharp(y_2,t)$ *for all* $t \geq 0$*.*

Proof. This corollary is an immediate consequence of (2.22) and Lemma 2.3.1. □

Theorem 2.4.12 (Lyapunov functional)**.** *Let* $v_0 \in L^\infty((0,|\Omega|))$ *be such that* $s_1 \leq v_0(y) \leq s_2$ *for a.e.* $x \in (0,|\Omega|)$*. Let* $v \in C^1([0,+\infty);L^\infty((0,|\Omega|)))$ *the unique solution of Problem* $(P^\sharp)$ *corresponding to the initial function* v_0*. We define*

$$\mathcal{E}(v(t)) = -\int_0^{|\Omega|} F(v(y,t))\,dy,$$

where $F(s) = \int_0^s f(\tau)d\tau$*. Then,*

(i) we have for all $\tau_2 > \tau_1 \geq 0$*,*

$$\mathcal{E}(v(\tau_2)) - \mathcal{E}(v(\tau_1)) = -\int_{\tau_1}^{\tau_2}\int_0^{|\Omega|} |v_t|^2\,dydt;$$

(ii) $\mathcal{E}(v(\cdot))$ *is continuous and nonincreasing on* $[0,+\infty)$*, and there exists the limit of* $\mathcal{E}(v(t))$ *as* $t \to \infty$*.*

Proof. (i) We have

$$\frac{d}{dt}\mathcal{E}(v(t)) = -\frac{d}{dt}\int_0^{|\Omega|} F(v) = -\int_0^{|\Omega|} f(v)v_t.$$

We also remark that $\int_\Omega v_t = 0$, so that

$$\frac{d}{dt}\mathcal{E}(v(t)) = -\int_0^{|\Omega|}\left(f(v) - \fint_0^{|\Omega|} f(v)\right)v_t = -\int_0^{|\Omega|}|v_t|^2.$$

Intergrating this identity form τ_1 to τ_2, we obtain

$$\mathcal{E}(v(\tau_2)) - \mathcal{E}(v(\tau_1)) = -\int_{\tau_1}^{\tau_2}\int_0^{|\Omega|} |v_t|^2\, dydt.$$

(ii) As a consequence of (i), $\mathcal{E}(v(\cdot))$ is continuous and nonincreasing. Moreover, $\mathcal{E}$ is bounded from below since $-F$ is bounded from below. Therefore there exists a limit of $\mathcal{E}(v(t))$ as $t \to \infty$, which completes the proof of (ii) □

2.5 ω-limit set of Problem $(P^\sharp)$

This section is devoted to study the set $\omega(u_0^\sharp)$, which is defined as follows:

Definition 2.5.1. *We define the ω-limit set of $u_0^\sharp$ by*

$$\omega(u_0^\sharp) := \{\varphi \in L^1((0,|\Omega|)) : \exists t_n \to \infty, u^\sharp(t_n) \to \varphi \quad in \quad L^1((0,|\Omega|)) \ as\ n \to \infty\}.$$

In Section 5.1, we prove that $\omega(u_0^\sharp)$ is non-empty and study some properties of $\omega(u_0^\sharp)$. We then prove in section 5.2 that $\omega(u_0^\sharp)$ is composed of a single element.

2.5.1 Properties of ω-limit set

In the following, we first prove that ω-limit set is nonempty. To begin with, we recall the definition of the BV-norm of functions of one variable (cf. [7, Remark (ii), page 170; Section 5.10.1 page 216]).

Definition 2.5.2. *Let $-\infty \le a < b \le \infty$ and let w be a L^1-measure from (a,b) to $\mathbb{R}$. The BV-norm of w is defined by*

$$\|w\|_{BV(a,b)} := \|w\|_{L^1((a,b))} + \operatorname{ess} V_a^b w,$$

where

$$\operatorname{ess} V_a^b w := \sup\left\{\sum_{j=1}^{j=m} |w(t_{j+1}) - w(t_j)|\right\}.$$

Here the supremum is taken over all finite partitions $a < t_1 < \cdots < t_{m+1} < b$ such that each t_i is a point of approximate continuity of w.

Lemma 2.5.3. *The set*

$$\{u^\sharp(t) : \quad t \ge 0\}$$

is relatively compact in $L^1((0,|\Omega|))$. Consequently, $\omega(u_0^\sharp)$ is nonempty.

Proof. It follows from (2.21) that

$$s_1 \le u^\sharp(y,t) \le s_2 \qquad \text{for all} \quad y \in [0,|\Omega|], t \ge 0.$$

Moreover, for all $t \ge 0$, $u^\sharp(\cdot,t)$ is nonincreasing on $[0,|\Omega|]$, so that we can deduce from Definition 2.5.2 that there exists a constant $c > 0$ such that

$$\|u^\sharp(t)\|_{BV((0,|\Omega|))} \le c \qquad \text{for all} \quad t \ge 0.$$

By the compact embedding $BV((0,|\Omega|)) \hookrightarrow L^1((0,|\Omega|))$ (cf. [7, Theorem 4, p. 176]), we deduce that the set $\{u^\sharp(t) : t \ge 0\}$ is relatively compact in $L^1((0,|\Omega|))$. This completes the proof of Lemma 2.5.3. □

Lemma 2.5.4. *Assume that* $(\mathbf{H_1})$ *hold. Let* $\varphi \in \omega(u_0^\sharp)$*, then the following assertions hold:*

(i) $s_1 \le \varphi(y) \le s_2$ *for a.e.* $y \in (0, |\Omega|)$*; and* $\int_0^{|\Omega|} \varphi = \int_0^{|\Omega|} u_0^\sharp$.

(ii) $\mathcal{E}(\cdot)$ *is constant on* $\omega(u_0^\sharp)$.

(iii) φ *is a stationary solution, which implies that there exists a constant* k_φ *such that*

$$f(\varphi(y)) = k_\varphi \quad \text{for a.e.} \quad y \in (0, |\Omega|).$$

Proof. Before proving (i), (ii) (iii), we deduce from the definition of $\omega(u_0^\sharp)$ that there exists a sequence $t_n \to +\infty$ such that

$$u^\sharp(t_n) \to \varphi \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty. \tag{2.25}$$

(i) It follows from (2.25) that there exists a subsequence of $\{t_n\}$ (still denoted by $\{t_n\}$) such that as $n \to +\infty$

$$u^\sharp(y, t_n) \to \varphi(y) \ \text{ for a.e. } \ y \in (0, |\Omega|).$$

Moreover, we have that

$$s_1 \le u^\sharp(y, t_n) \le s_2 \ \text{ for all } \ y \in (0, |\Omega|), n \ge 0,$$

so that

$$s_1 \le \varphi(y) \le s_2 \ \text{ for a.e. } \ y \in (0, |\Omega|).$$

The identity $\int_0^{|\Omega|} \varphi = \int_0^{|\Omega|} u_0^\sharp$ follows from the mass conservation property (cf. Lemma 2.2.7) and the convergence of $u^\sharp(t_n)$ to φ in $L^1((0,|\Omega|))$ as $n \to \infty$.

(ii) First, it follows from Theorem 2.4.12(ii) that

$$\lim_{t\to\infty} \mathcal{E}(u^\sharp(t)) = \mathcal{E}_\infty.$$

for some constant $\mathcal{E}_\infty$. We define $c_3 := \sup_{s\in[s_1,s_2]} |f(s)|$. We have

$$|\mathcal{E}(u^\sharp(t_n)) - \mathcal{E}(\varphi)| \le \int_0^{|\Omega|} |F(u^\sharp(t_n)) - F(\varphi)| \le c_3 \|u^\sharp(t_n) - \varphi\|_{L^1((0,|\Omega|))}.$$

Letting $n \to \infty$, we deduce from (2.25) that

$$\mathcal{E}(\varphi) = \lim_{n\to\infty} \mathcal{E}(u^\sharp(t_n)) = \mathcal{E}_\infty, \tag{2.26}$$

In other words,

$$\mathcal{E}\big|_{\omega(u_0^\sharp)} = \mathcal{E}_\infty. \tag{2.27}$$

(iii) We have to prove that φ is a stationary solution. We denote here by $v(t; w)$ the solution of Problem $(P^\sharp)$ corresponding to initial function w. By (2.25), we have

$$v(t_n; u_0^\sharp) \to \varphi \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty.$$

It follows from Lemma 2.2.8 which expresses the continuity of the semi-group with respect to the initial data that for all $t \ge 0$,

$$v(t; v(t_n; u_0^\sharp)) \to v(t; \varphi) \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty.$$

In other words,

$$v(t+t_n;u_0^\sharp)\to v(t;\varphi)\ \text{ in }\ L^1((0,|\Omega|))\ \text{ as }\ n\to\infty.$$

It follows that $v(t;\varphi)\in\omega(u_0^\sharp)$. This together with (2.27) implies that for all $t>s\geq 0$.

$$\mathcal{E}(u(t;\varphi))=\mathcal{E}(u(s;\varphi)).$$

In view of Theorem 2.4.12(i), we have that

$$0=\mathcal{E}(v(t;\varphi))-\mathcal{E}(v(s;\varphi))=-\int_s^t\int_0^{|\Omega|}|v_t(t;\varphi)|^2\,dydt\qquad\text{for all }\ t>s\geq 0,$$

which implies that

$$v_t(\varphi)=0\qquad\text{a.e in }\ (0,|\Omega|)\times\mathbb{R}^+.$$

Thus the solution $v_t(t;\varphi)$ of Problem $(P^\sharp)$ with the initial function φ is independent of time. Consequently, we have that

$$f(\varphi(y))=\fint_0^{|\Omega|}f(\varphi)=:k_\varphi\ \text{ for a.e. }\ y\in(0,|\Omega|).$$

□

Corollary 2.5.5. *Let $\varphi\in\omega(u_0^\sharp)$. Then up to a modification on a set of zero measure, φ is a step function. More precisely, we have one of the following cases:*

(i) If $k_\varphi\notin[f(m),f(M)]$, then $\varphi=\fint_0^{|\Omega|}u_0^\sharp(y)\,dy$.

(ii) If $k_\varphi\in(f(m),f(M))$, then

$$\varphi=a_-\chi_{A_-}+a_0\chi_{A_0}+a_+\chi_{A_+},$$

where

$$s_*<a_-<m,a_0\in(m,M),M<a_+<s^*\ \text{ satisfy }\ f(a_-)=f(a_0)=f(a_+)=k_\varphi \tag{2.28}$$

and A_-,A_0,A_+ are pairwise disjoint subsets of $(0,|\Omega|)$ such that $A_-\cup A_0\cup A_+=(0,|\Omega|)$.

(iii) If $k_\varphi=f(m)$, then

$$\varphi(y)=m\chi_{A_-}+s^*\chi_{A_+},$$

where A_- and A_+ are disjoint and $A_-\cup A_+=(0,|\Omega|)$; (note that here $a_-=a_0:=m$ and $a_+:=s^*$).

(iv) If $k_\varphi=f(M)$, then

$$\varphi(y)=s_*\chi_{A_-}+M\chi_{A_+},$$

where A_- and A_+ are disjoint and $A_-\cup A_+=(0,|\Omega|)$; (note that here $a_-:=s_*$ and $a_0=a_+:=M$).

Proof. (i) If $k_\varphi \notin [f(m), f(M)]$, then the equation $f(s) = k_\varphi$ has a unique solution a. This together with Lemma 2.5.4(iii) follows that

$$\varphi(y) = a \quad \text{for a.e.} \quad y \in (0, |\Omega|).$$

Because of the mass conservation property, we have

$$a = \fint_0^{|\Omega|} \varphi = \fint_0^{|\Omega|} u_0^\sharp.$$

Therefore,

$$\varphi(y) = \fint_0^{|\Omega|} u_0^\sharp = \fint_\Omega u_0.$$

(ii) If $k_\varphi \in (f(m), f(M))$, then the equation $f(s) = k_\varphi$ has exactly three solutions which are denoted by a_-, a_0, a_+ with $a_- < a_0 < a_+$. This together with Lemma 2.5.4(iii) implies that φ is a step function and the sets A_-, A_0, A_+ are defined by

$$A_- := \varphi^{-1}(a_-), \quad A_0 := \varphi^{-1}(a_0) \quad \text{and} \quad A_+ := \varphi^{-1}(a_+).$$

(iii), (iv) The proof (iii) and (iv) are similar to the proof of (i) and (ii). We omit them. □

2.5.2 $\omega(u_0^\sharp)$ is composed of a single element

The main result of this section is the following.

Theorem 2.5.6. *Let* $(\mathbf{H_1})$ *hold.*

(i) If we assume that $(\mathbf{H_2})$ *holds, then* $\omega(u_0^\sharp)$ *only contains one element* φ *given by*

$$\varphi(y) \equiv \fint_0^{|\Omega|} u_0^\sharp = \fint_\Omega u_0.$$

(ii) If we assume that $(\mathbf{H_3})$ *holds, then* $\omega(u_0^\sharp)$ *only contains one element* φ*, where* φ *is a step function. More precisely,* φ *takes at most two values* a_-, a_+ *such that* $f(a_-) = f(a_+)$ *and that*

$$f'(a_-) \le 0, \quad f'(a_+) \le 0.$$

Proof of Theorem 2.5.6(i). Let $\varphi \in \omega(u_0^\sharp)$. We first prove that

$$k_\varphi \notin (f(m), f(M)).$$

Indeed, if $k_\varphi \in (f(m), f(M))$, then it follows from Corollary 2.5.5(ii) that

$$\varphi = a_- \chi_{A_-} + a_0 \chi_{A_0} + a_+ \chi_{A_+},$$

so that

$$\fint_0^{|\Omega|} u_0^\sharp = \fint_0^{|\Omega|} \varphi \in [a_-, a_+] \subset [s_*, s^*],$$

which contradicts $\displaystyle\fint_0^{|\Omega|} u_0^\sharp \notin [s_*, s^*]$. Similarly, $k_\varphi \neq f(m)$, $k_\varphi \neq f(M)$. Therefore, $k_\varphi \notin [f(m), f(M)]$. It follows from Corollary 2.5.5(i) that

$$\varphi = \fint_0^{|\Omega|} u_0^\sharp.$$

Therefore, $\omega(u_0^\sharp)$ only contains the element $\varphi = \fint_0^{|\Omega|} u_0^\sharp$. The proof of Theorem 2.5.6(i) is complete. □

The remaining part of this section is devoted to the proof of Theorem 2.5.6(ii). We first define for each $t \geq 0$,

$$\begin{aligned}
\Omega_-(t) &:= \{y \in (0, |\Omega|), u^\sharp(y,t) \leq m\},\\
\Omega_0(t) &:= \{y \in (0, |\Omega|), m < u^\sharp(y,t) < M\},\\
\Omega_+(t) &:= \{y \in (0, |\Omega|), u^\sharp(y,t) \geq M\}.
\end{aligned}$$

Lemma 2.5.7. *Suppose that* $(\mathbf{H_1}), (\mathbf{H_3})$ *hold. For every* $t' > t \geq 0$ *the following assertions hold*

(i) if $u^\sharp(y,t) \leq m$, *then* $u^\sharp(y,t') \leq m$,

(ii) if $u^\sharp(y,t) \geq M$, *then* $u^\sharp(y,t') \geq M$.

Proof. First, we prove that

$$\bar{f}(m) \leq \lambda(t) \leq \bar{f}(M), \tag{2.29}$$

or equivalently

$$f(m) \leq \lambda(t) \leq f(M).$$

Indeed, we apply Theorem 2.4.10 with $s_1 := s_* - \dfrac{1}{n}$ and $s_2 := s^* + \dfrac{1}{n}$ to obtain that

$$s_* - \frac{1}{n} \leq u^\sharp(y,t) \leq s^* + \frac{1}{n} \quad \text{for all} \quad y \in (0, |\Omega|), t \geq 0 \quad \text{and for all} \quad n > 0,$$

which in turn implies that

$$s_* \leq u^\sharp(y,t) \leq s^* \quad \text{for all} \quad y \in (0, |\Omega|), t \geq 0,$$

so that (2.29) holds.

(i) We recall that

$$\mathcal{L}(Y) := Y' - \bar{f}(Y) + \lambda(t).$$

By using (2.29), we have

$$\mathcal{L}(m) \geq 0.$$

Since by Corollary 2.4.11, $u^\sharp(\cdot, t)$ is the unique solution of Problem (ODE) with the initial condition $u_0^\sharp(y)$, we have that

$$\mathcal{L}(u^\sharp(y,\cdot)) = 0 \text{ for all } \quad t \geq 0.$$

The assertion (i) follows from [9, Theorem 6.1, Page 31] and the differential inequality

$$\mathcal{L}(u^\sharp(y,\cdot)) = 0 \leq \mathcal{L}(m) \qquad \text{for all} \quad t \geq 0.$$

(ii) The proof of (ii) is similar to that of (i). We omit it here. □

The following corollary is an immediate consequence of Lemma 2.5.7.

Corollary 2.5.8. *Suppose that* $(\mathbf{H_1}), (\mathbf{H_3})$ *hold. For every* $t' > t \geq 0$,

$$\Omega_-(t) \subset \Omega_-(t'), \quad \Omega_+(t) \subset \Omega_+(t') \quad \text{and} \quad \Omega_0(t) \supset \Omega_0(t').$$

In other words, $\Omega_-(t), \Omega_+(t)$ *are monotonically expanding in* t *and* $\Omega_0(t)$ *is monotonically shrinking in* t.

We define

$$\Omega_-(\infty) := \cup_{t\geq 0}\Omega_-(t), \Omega_0(\infty) := \cap_{t\geq 0}\Omega_0(t) \text{ and } \Omega_+(\infty) := \cup_{t\geq 0}\Omega_+(t). \tag{2.30}$$

Corollary 2.5.9. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold. Then* $\Omega_-(\infty), \Omega_0(\infty), \Omega_+(\infty)$ *are pairwise disjoint and*

$$(0, |\Omega|) = \Omega_-(\infty) \cup \Omega_0(\infty) \cup \Omega_+(\infty).$$

Moreover,

$$\begin{aligned}
\Omega_-(\infty) &= \{y \in (0, |\Omega|) : \quad \textit{there exists} \ \ t_0 \geq 0 \ \ \textit{such that} \ \ u^\sharp(y,t) \leq m \ \ \textit{for all} \ \ t \geq t_0\} \\
\Omega_0(\infty) &= \{y \in (0, |\Omega|) : m < u^\sharp(y,t) < M \ \ \textit{for all} \ \ t \geq 0\}, \\
\Omega_+(\infty) &= \{y \in (0, |\Omega|) : \quad \textit{there exists} \ \ t_0 \geq 0 \ \ \textit{such that} \ \ u^\sharp(y,t) \geq M \ \ \textit{for all} \ \ t \geq t_0\}.
\end{aligned}$$

Proof. For $y \in (0, |\Omega|)$,

(i) either $m < u^\sharp(y,t) < M$ for all $t \geq 0$ i.e $x \in \Omega_0(\infty)$;

(ii) or there exists $t_0 \geq 0$ such that $u^\sharp(y,t_0) \leq m$, i.e. $x \in \Omega_-(\infty)$. In view of Lemma 2.5.7(i), this is equivalent to the fact that $u^\sharp(y,t) \leq m$ for all $t \geq t_0$.

(iii) or there exists $t_0 \geq 0$ such that $u^\sharp(y,t_0) \geq M$, i.e. $x \in \Omega_+(\infty)$. In view of Lemma 2.5.7(ii), this is equivalent to the fact that $u^\sharp(y,t) \geq M$ for all $t \geq t_0$.

These arguments complete the proof of Corollary 2.5.9.

□

Lemma 2.5.10. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold. Let* $\varphi \in \omega(u_0^\sharp)$. *Then the following assertions hold*

(i)
$$\varphi(y) \leq m \quad \textit{for a.e.} \ \ x \in \Omega_-(\infty);$$

(ii)
$$m \leq \varphi(y) \leq M \quad \textit{for a.e.} \ \ x \in \Omega_0(\infty);$$

(iii)
$$\varphi(y) \geq M \quad \textit{for a.e.} \ \ x \in \Omega_+(\infty).$$

Proof. First, we deduce from the definition of $\omega(u_0^\sharp)$ that there exists a sequence $t_n \to +\infty$ such that

$$u^\sharp(y, t_n) \to \varphi(y) \quad \text{for a.e } y \in (0, |\Omega|) \ \text{ as } \ n \to +\infty.$$

This together with Corollary 2.5.9 completes the proof of Lemma 2.5.10. □

Lemma 2.5.11. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold. Then*

$$|\Omega_0(\infty)| = 0.$$

Proof. If $\Omega_0(\infty)$ is not empty, we will prove that there exists $\alpha \in (m, M)$ such that

$$u_0^\sharp(y) = \alpha \quad \text{for all} \quad y \in \Omega_0(\infty).$$

Indeed, for the purpose of contradiction, we assume that there exits $y_1, y_2 \in \Omega_0(\infty)$ such that

$$u_0^\sharp(y_1) \neq u_0^\sharp(y_2).$$

Without loss of generality, we suppose that $y_1 < y_2$; then $u_0^\sharp(y_2) < u_0^\sharp(y_1)$ since $u_0^\sharp$ is nonincreasing. Moreover, by the definition of $\Omega_0(\infty)$ and Corollary 2.4.11(ii), we have

$$m < u^\sharp(y_2, t) < u_0^\sharp(y_1, t) < M \qquad \text{for all} \quad t \geq 0. \tag{2.31}$$

Therefore,

$$u_0^\sharp(y_2) \leq u_0^\sharp(y) \leq u_0^\sharp(y_1) \qquad \text{for all} \quad y \in [y_1, y_2].$$

It follows from Corollary 2.4.11(ii) and (2.31) that

$$m < u^\sharp(y, t) < M \qquad \text{for all} \quad y \in [y_1, y_2], t \geq 0.$$

Consequently,

$$[y_1, y_2] \subset \Omega_0(\infty). \tag{2.32}$$

Let $\varphi \in \omega(u_0^\sharp)$. It follows from Lemma 2.5.10(ii) and (2.32) that

$$m \leq \varphi(y) \leq M \qquad \text{for a.e.} \quad y \in [y_1, y_2].$$

On the other hand, we deduce from Corollary 2.5.5 that φ takes at most one value in $[m, M]$. Therefore, there exists $a \in [m, M]$ such that

$$\varphi(y) = a \qquad \text{for a.e.} \quad y \in [y_1, y_2].$$

Let $\{t_n\}$ be such that

$$u^\sharp(y, t_n) \to \varphi(y) \quad \text{for a.e.} \quad y \in (0, |\Omega|) \quad \text{as} \quad n \to +\infty;$$

then

$$u^\sharp(y, t_n) \to a \quad \text{for a.e.} \quad y \in [y_1, y_2] \quad \text{as} \quad n \to +\infty.$$

Since $u^\sharp(t_n)$ is nonincreasing, it follows that as $n \to +\infty$,

$$u^\sharp(y, t_n) \to a \quad \text{for all } y \in (y_1, y_2) \subset \Omega_0(\infty). \tag{2.33}$$

We now prove that $u_0^\sharp$ is constant on (y_1, y_2). Let $y_1 < \bar{y}_1 < \bar{y}_2 < y_2$. Then we have

$$u_0^\sharp(\bar{y}_1) \geq u_0^\sharp(\bar{y}_2).$$

Next, we define

$$r(t) := u^\sharp(\bar{y}_1, t) - u^\sharp(\bar{y}_2, t);$$

then in view of Corollary 2.4.11,

$$r(t) \geq 0 \quad \text{for all} \quad t \geq 0,$$

and

$$r'(t) = u_t^\sharp(\bar{y}_1, t) - u_t^\sharp(\bar{y}_2, t) = f(u^\sharp(\bar{y}_1, t)) - f(u^\sharp(\bar{y}_2, t)) \geq 0,$$

where the last inequality holds since $\bar{y}_1, \bar{y}_2 \in \Omega_0(\infty)$ so that

$$M > u^\sharp(\bar{y}_1, t) \geq u^\sharp(\bar{y}_2, t) > m \qquad \text{for all } \ t \geq 0,$$

and since f is increasing on $[m, M]$. Moreover, in view of (2.33), we have

$$\lim_{n\to\infty} r(t_n) = 0.$$

Thus, the function r which is nonnegative and nondecreasing is identically equal to zero. In particular, $u_0^\sharp(\bar{y}_1) = u_0^\sharp(\bar{y}_2)$. Therefore, $u_0^\sharp$ is constant on (y_1, y_2), which in view of Remark 2.4.9 contradicts Hypotheses $(\mathbf{H_3})$. Thus, we conclude that there exists $\alpha \in (m, M)$ such that

$$u_0^\sharp(y) = \alpha \qquad \text{for all } \ y \in \Omega_0(\infty),$$

which again together with Hypothesis $(\mathbf{H_3})$ implies that

$$|\Omega_0(\infty)| = 0.$$

□

Lemma 2.5.12. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold and let* $\varphi \in \omega(u_0^\sharp)$. *Then,* $k_\varphi \in [f(m), f(M)]$ *and*

$$\varphi = a_-\chi_{\Omega_-(\infty)} + a_+\chi_{\Omega_+(\infty)},$$

where $a_- \in [s_*, m], a_+ \in [M, s^*]$ *and* $f(a_-) = f(a_+) = k_\varphi$. *More precisely,*

(i) If $f(m) < k_\varphi < f(M)$

$$\varphi = a_-\chi_{\Omega_-(\infty)} + a_+\chi_{\Omega_+(\infty)},$$

where $a_- \in (s_*, m), a_+ \in (M, s^*)$ *and* $f(a_-) = f(a_+) = k_\varphi$.

(ii) If $k_\varphi = f(m)$ *then*

$$\varphi = m\chi_{\Omega_-(\infty)} + s^*\chi_{\Omega_+(\infty)}.$$

(Note that here $a_- = m$ *and* $a_+ = s^*$*).*

(iii) If $k_\varphi = f(M)$ *then*

$$\varphi = s_*\chi_{\Omega_-(\infty)} + M\chi_{\Omega_+(\infty)}.$$

(Note that here $a_- = s_*$ *and* $a_+ = M$*).*

Proof. It follows from the proof of Lemma 2.5.7 that for all $t \geq 0$

$$s_* \leq u^\sharp(y, t) \leq s^* \ \text{ for a.e. } \ y \in (0, |\Omega|)$$

so that

$$s_* \leq \varphi(y) \leq s^* \ \text{ for a.e. } \ y \in (0, |\Omega|).$$

Consequently, we deduce from the definition of s_*, s^* in (2.1) that

$$k_\varphi = \fint_0^{|\Omega|} f(\varphi) \in [f(m), f(M))].$$

(i) If $f(m) < k_\varphi < f(M)$, it follows from Corollary 2.5.5(ii) and Lemma 2.5.11 that φ is a step function which takes two values a_-, a_+ where $a_- \in (s_*, m), a_+ \in (M, s^*)$ and $f(a_-) = f(a_+) = k_\varphi$. Thus by Lemma 2.5.10,

$$\varphi(y) = a_- \qquad \text{for a.e. } x \in \Omega_-(\infty).$$

and

$$\varphi(y) = a_+ \qquad \text{for a.e. } x \in \Omega_+(\infty).$$

This together with Lemma 2.5.11 implies that

$$\varphi = a_- \chi_{\Omega_-(\infty)} + a_+ \chi_{\Omega_+(\infty)}.$$

The proofs of (ii) and (iii) are similar to that of (i). We omit them here.

□

Proof of Theorem 2.5.6(ii). We prove that $\omega(u_0^\sharp)$ is exactly composed of one element. By contradiction, we assume that $\omega(u_0^\sharp)$ contains more than one element. We deduce from Lemma 2.5.12 that for all $k \in [f(m), f(M)]$, $\omega(u_0^\sharp)$ contains at most one element φ such that $k_\varphi = k$, so that we can choose two functions $\varphi, \overline{\varphi} \in \omega(u_0^\sharp)$ such that

$$f(m) \le k_\varphi < k_{\overline{\varphi}} \le f(M).$$

It follows from Lemma 2.5.12 that

$$\varphi = a_- \chi_{\Omega_-(\infty)} + a_+ \chi_{\Omega_+(\infty)},$$

and that

$$\overline{\varphi} = \overline{a}_- \chi_{\Omega_-(\infty)} + \overline{a}_+ \chi_{\Omega_+(\infty)},$$

where

$$a_-, \overline{a}_- \in [s_*, m], \quad a_+, \overline{a}_+ \in [M, s^*];$$

and

$$f(a_-) = f(a_+) = k_\varphi, \quad f(\overline{a}_-) = f(\overline{a}_+) = k_{\overline{\varphi}}.$$

Since f is strictly decreasing on $(-\infty, m]$ and $[M, +\infty)$, we deduce that $a_- > \overline{a}_-, a_+ > \overline{a}_+$. Then using mass conservation property, we obtain

$$\fint_0^{|\Omega|} u_0^\sharp = a_- |\Omega_-(\infty)| + a_+ |\Omega_+(\infty)| > \overline{a}_- |\Omega_-(\infty)| + \overline{a}_+ |\Omega_+(\infty)| = \fint_0^{|\Omega|} u_0^\sharp$$

which gives a contradiction. Therefore, $\omega(u_0^\sharp)$ only contains one element. The proof of Theorem 2.5.6(ii) is complete. □

Corollary 2.5.13. *Let the same assumptions as in Theorem 2.5.6 hold. Then for all* $p \in [1, +\infty)$,

$$u^\sharp(t) \to \varphi \quad \text{in} \quad L^p((0, |\Omega|)) \quad \text{as} \quad t \to \infty,$$

where φ *is defined as in Theorem 2.5.6.*

Proof. It immediately follows from Theorem 2.5.6 that

$$u^\sharp(t) \to \varphi \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ t \to \infty.$$

The convergence of $u^\sharp(t)$ in $L^p((0,|\Omega|))$ for all $p \in (1,+\infty)$ follows from the estimates

$$\begin{aligned}\|u^\sharp(t)-\varphi\|^p_{L^p((0,|\Omega|))} &\le \|u^\sharp(t)-\varphi\|_{L^1((0,|\Omega|))}\||u^\sharp(t)-\varphi|^{p-1}\|_{L^\infty((0,|\Omega|))}\\ &\le |s_2-s_1|^{p-1}\|u^\sharp(t)-\varphi\|_{L^1((0,|\Omega|))}.\end{aligned}$$

□

2.6 Proof of Theorem 2.1.2 and a nontrivial counterexample

We have proved the stabilization of the solution of Problem $(P^\sharp)$. We now have to deduce the stabilization of the solution of Problem (P). We need the following result.

Lemma 2.6.1. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold. Then for all* $t,\tau \ge 0$ *we have*

$$\|u^\sharp(t)-u^\sharp(\tau)\|_{L^2((0,|\Omega|))} = \|u(t)-u(\tau)\|_{L^2(\Omega)}.$$

Proof. Define $G(s) := h^t(s)h^\tau(s)$. We apply Proposition 2.4.4(ii) for G to deduce that

$$\int_0^{|\Omega|} h^t(u_0^\sharp)h^\tau(u_0^\sharp) = \int_\Omega h^t(u_0)h^\tau(u_0).$$

or equivalently,

$$\int_0^{|\Omega|} u^\sharp(t)u^\sharp(\tau) = \int_\Omega u(t)u(\tau). \tag{2.34}$$

We apply again Proposition 2.4.4(ii) to the function $G(s) := s^2$ and use equality (2.34) to deduce that

$$\begin{aligned}\|u^\sharp(t)-u^\sharp(\tau)\|^2_{L^2((0,|\Omega|))} &= \int_0^{|\Omega|}(u^\sharp(t))^2 + \int_0^{|\Omega|}(u^\sharp(\tau))^2 - 2\int_0^{|\Omega|}u^\sharp(t)u^\sharp(\tau)\\ &= \int_\Omega (u(t))^2 + \int_\Omega (u(\tau))^2 - 2\int_\Omega u(t)u(\tau)\\ &= \|u(t)-u(\tau)\|^2_{L^2(\Omega)}.\end{aligned}$$

This completes the proof of Lemma 2.6.1. □

Proof of Theorem 2.1.2. It follows from Corollary 2.5.13 that

$$u^\sharp(t) \to \varphi \ \text{ in } \ L^2((0,|\Omega|)) \qquad \text{as } \ t \to \infty, \tag{2.35}$$

so that $u^\sharp(t)$ is a Cauchy sequence in $L^2((0,|\Omega|))$. This together with Lemma 2.6.1 implies that $u(t)$ is also a Cauchy sequence in $L^2(\Omega)$. Consequently, there exists $\psi \in L^2(\Omega)$ such that

$$u(t) \to \psi \ \text{ in } \ L^2(\Omega) \ \text{ as } \ t \to \infty.$$

Applying Proposition 2.4.4(iii), we have

$$u^\sharp(t) \to \psi^\sharp \ \text{ in } \ L^2((0,|\Omega|)),$$

which together with (2.35) implies that

$$\psi^\sharp = \varphi \qquad \text{a.e. in } \ (0,|\Omega|).$$

Therefore,

$$|\{\psi = s\}| = |\{\psi^\sharp = s\}| = |\{\varphi = s\}| \ \text{ for all } \ s \in \mathbb{R},$$

which in view of Theorem 2.5.6 allows us to conclude that

(i) if $(\mathbf{H_2})$ holds, then

$$\psi(x) \equiv \fint_\Omega u_0(y)\,dy;$$

(ii) if $(\mathbf{H_3})$ holds, then ψ is a step function, which takes at most two values a_-, a_+ satisfying

$$f(a_-) = f(a_+); \quad f'(a_-) \le 0, \quad f'(a_+) \le 0.$$

As in the proof of Corollary 2.5.13, we deduce that for all $p \in [1, +\infty)$

$$u(t) \to \psi \ \text{ in } \ L^p(\Omega) \ \text{ as } \ t \to \infty.$$

This completes the proof of Theorem 2.1.2. □

Counterexample: We present below a counterexample which shows that if the hypothesis $(\mathbf{H_3})$ does not hold, the limiting function may take three values.

Let $\Omega := (-1,1)$ and consider $f(s) = s - s^3$. Let u_0 be an odd function such that

$$|\{x : u_0(x) < 0\}| > 0, \quad |\{x : u_0(x) = 0\}| > 0, \quad |\{x : u_0(x) > 0\}| > 0.$$

We denote by $y(t; y_0)$ the unique solution of the autonomous differential equation

$$y' = y - y^3 \ \text{ and } \ y(0) = y_0.$$

Since

$$-y'(t,y_0) = -y(t,y_0) + (y(t,y_0))^3 = -y(t,y_0) - (-y(t,y_0))^3,$$

it follows from the uniqueness of the solution of differential equations that

$$y(t; -y_0) = -y(t; y_0).$$

We also recall that

(i) if $y_0 < 0$ then $\lim\limits_{t\to\infty} y(t; y_0) = -1$,

(ii) if $y_0 > 0$ then $\lim\limits_{t\to\infty} y(t; y_0) = 1$,

(iii) if $y_0 = 0$ then $y(t; y_0) \equiv 0$. Hence, $\lim\limits_{t\to\infty} y(t; y_0) = 0$.

Define $u(x,t) = y(t; u_0(x))$, then $f(u(\cdot,t))$ is an odd function, so that

$$\int_\Omega f(u(x,t))\,dx = 0 \quad \text{for all} \quad t \geq 0.$$

We deduce that

$$u_t = f(u) = f(u) - \fint_\Omega f(u),$$

and therefore that u is the unique solution of Problem (P). Thus, for all $x \in \Omega$,

(i) if $u_0(x) < 0$ then $\lim_{t\to\infty} u(x,t) = -1$,

(ii) if $u_0(x) > 0$ then $\lim_{t\to\infty} u(x,t) = 1$,

(iii) if $u_0(x) = 0$ then $\lim_{t\to\infty} u(x,t) = 0$.

Define

$$\psi(x) = \begin{cases} -1 & \text{if } u_0(x) < 0 \\ 0 & \text{if } u_0(x) = 0 \\ 1 & \text{if } u_0(x) > 0. \end{cases} \tag{2.36}$$

Then

$$u(t) \to \psi \quad \text{in} \quad L^p(\Omega) \quad \text{for all} \quad p \in [1, +\infty) \quad \text{as} \quad t \to \infty.$$

2.7 Appendix

For the sake of completeness, we present a complete proof of Proposition 2.4.7. The equality

$$(\Phi(w))^\sharp = \Phi(w^\sharp) \qquad \text{a.e. in} \quad (0, |\Omega|). \tag{2.37}$$

was proved by Kesavan (see [11, Proposition 1.1.4, page 7]). We prove here that (2.37) holds for all $y \in [0, |\Omega|]$. We need the following lemma.

Lemma 2.7.1. *Let $\mathcal{A}$ be a bounded set of $\mathbb{R}$ and let Φ be a increasing, bijective function from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\Phi(\inf \mathcal{A}) = \inf\{\Phi(s) : \quad s \in \mathcal{A}\}.$$

Proof. First, we prove that

$$\Phi(\inf \mathcal{A}) \leq \inf\{\Phi(s) : \quad s \in \mathcal{A}\}.$$

Indeed, let $a = \inf \mathcal{A}$, then $a \leq s$ for all $s \in \mathcal{A}$. We also remark that Φ is increasing on $\mathbb{R}$, so that

$$\Phi(a) \leq \Phi(s) \quad \text{for all} \quad s \in \mathcal{A}.$$

Consequently, we have

$$\Phi(\inf \mathcal{A}) = \Phi(a) \leq \inf\{\Phi(s) : \quad s \in \mathcal{A}\}. \tag{2.38}$$

Next, we prove that

$$\Phi(\inf \mathcal{A}) \geq \inf\{\Phi(s) : \quad s \in \mathcal{A}\}.$$

For this purpose, set $b := \inf\{\Phi(s) : \quad s \in \mathcal{A}\}$. Therefore,

$$b \leq \Phi(s) \quad \text{for all} \quad s \in \mathcal{A}.$$

Consequently, we have

$$\Phi^{-1}(b) \leq s \quad \text{for all} \quad s \in \mathcal{A}.$$

We deduce that

$$\Phi^{-1}(b) \leq \inf \mathcal{A},$$

and hence $b \leq \Phi(\inf \mathcal{A})$. Thus,

$$\inf\{\Phi(s) : \quad s \in \mathcal{A}\} = b \leq \Phi(\inf \mathcal{A}). \tag{2.39}$$

The conclusion of Lemma 2.7.1 follows from (2.38) and (2.39). □

Proof of Proposition 2.4.7. First, we prove that

$$(\Phi(w))^\sharp(0) = \Phi(w^\sharp)(0).$$

Note that

$$\begin{aligned}(\Phi(w))^\sharp(0) = \operatorname{ess\,sup} \Phi(w) &= \inf\{a : |\{\Phi(w) > a\}| = 0\} \\ &= \inf\{a : |\{w > \Phi^{-1}(a)\}| = 0\} \\ &= \inf\{\Phi(\Phi^{-1}(a)) : |\{w > \Phi^{-1}(a)\}| = 0\} \\ &= \inf\{\Phi(\widetilde{a}) : |\{w > \widetilde{a}\}| = 0\}\end{aligned}$$

which together with Lemma 2.7.1 implies that

$$(\Phi(w))^\sharp(0) = \Phi(\inf\{\widetilde{a} : |\{w > \widetilde{a}\}| = 0\}) = \Phi(\operatorname{ess\,sup} w) = \Phi(w^\sharp(0)).$$

Let $0 < y \leq |\Omega|$, we now prove that

$$(\Phi(w))^\sharp(y) = \Phi(w^\sharp)(y).$$

We have

$$\begin{aligned}(\Phi(w))^\sharp(y) &= \inf\{s : |\{\Phi(w) > s\}| < y\} \\ &= \inf\{s : |\{w > \Phi^{-1}(s)\}| < y\} \\ &= \inf\{\Phi(\Phi^{-1}(s)) : |\{w > \Phi^{-1}(s)\}| < y\} \\ &= \inf\{\Phi(\widetilde{s}) : |\{w > \widetilde{s}\}| < y\} \\ &= \Phi(\inf\{\widetilde{s} : |\{w > \widetilde{s}\}| < y\}) = \Phi(w^\sharp(y)).\end{aligned}$$

This completes the proof of Proposition 2.4.7. □

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Chapter 3

Large time behavior for a nonlocal differential equation proposed by M. Nagayama

Résumé. Nous étudions une équation différentielle ordinaire nonlocale, où l'intégrale de la solution est conservée en temps et démontrons la stabilisation des orbites de solutions.

Abstract. We study a nonlocal ordinary differential equation with mass conservation and prove that the solution stabilizes as t tends to $+\infty$.

3.1 Introduction

We study the initial value problem

$$(P)\begin{cases} u_t = u^2(1-u) - u(1-u)\dfrac{\displaystyle\int_\Omega u^2(1-u)}{\displaystyle\int_\Omega u(1-u)} & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded open set. This problem has been proposed by M. Nagayama [10]

After having proved that Problem (P) possesses some invariant regions, we study the large time behavior of its solution. An essential difficulty is to analyze the denominator in the nonlocal term in Problem (P) since it may vanish. Another difficulty of this problem is due to the lack of a diffusion term, so that the solution does not regularize in time. Applying the rearrangement theory, we prove the relative compactness of the solution orbits of the corresponding one-dimensional problem $(P^\sharp)$. We study the large time behavior of the solution of Problem $(P^\sharp)$ and then deduce the result for the original problem.

Constants $\mathbf{s_1}, \mathbf{s_2}$: Let s_1, s_2 be two constants such that

$$s_1 < 0 < 1 < s_2.$$

Assumptions on initial condition: We will consider one of the following assumptions on initial function.

$(\mathbf{H_1})$ $u_0 \in L^\infty(\Omega)$, $1 \leq u_0(x) \leq s_2$ for a.e. $x \in \Omega$; and $u_0 \not\equiv 1$.

$(\mathbf{H_2})$ $u_0 \in L^\infty(\Omega)$, $s_1 \leq u_0(x) \leq 0$ for a.e. $x \in \Omega$; and $u_0 \not\equiv 0$.

We define

$$m_0 := \fint_\Omega u_0 := \frac{1}{|\Omega|}\int_\Omega u_0.$$

Note that if u_0 satisfies $(\mathbf{H_1})$ or $(\mathbf{H_2})$ then

$$m_0 \notin [0,1].$$

Two functions $\mathbf{f}, \mathbf{g}$: We introduce two functions f, g given by

$$f(s) := s^2(1-s) \qquad \text{and} \qquad g(s) := s(1-s).$$

Then, the differential equation can be written as

$$u_t = f(u) - g(u)\frac{\displaystyle\int_\Omega f(u)}{\displaystyle\int_\Omega g(u)}.$$

For two real numbes $a < b$, we define

$$K(a,b) := \max(\sup_{[a,b]}|f(s)|, \sup_{[a,b]}|g(s)|), \sup_{[a,b]}|f'(s)|, \sup_{[a,b]}|g'(s)|) \tag{3.1}$$

Our main results can be summarized as follows:

Theorem 3.1.1. *Let* $(\mathbf{H_1})$ *hold. Then Problem* (P) *possesses a unique solution in* $C^1([0,+\infty);L^\infty(\Omega))$. *Moreover, for each* $t \geq 0$,

$$1 \leq u(t) \leq s_2 \qquad \text{almost everywhere in } \Omega.$$

If $u_0 \in C(\overline{\Omega})$, *then* $u \in C^1([0,+\infty);C(\overline{\Omega}))$.

Theorem 3.1.2. *Let* $(\mathbf{H_1})$ *hold. Then there exists* $\psi \in L^\infty(\Omega)$, *such that for all* $1 \leq p < +\infty$.

$$u(t) \to \psi \qquad \text{in } L^p(\Omega) \text{ as } t \to +\infty.$$

Moreover, ψ *is a step function, which takes two values* 1 *and* $\gamma > 1$.

In the case that Assumption $(\mathbf{H_2})$ holds, we have the similar results.

Theorem 3.1.3. *Let* $(\mathbf{H_2})$ *hold. Then Problem* (P) *possesses a unique solution in* $C^1([0,+\infty);L^\infty(\Omega))$. *Moreover, for each* $t \geq 0$,

$$s_1 \leq u(t) \leq 0 \qquad \text{almost everywhere in } \Omega.$$

If $u_0 \in C(\overline{\Omega})$, *then* $u \in C^1([0,+\infty);C(\overline{\Omega}))$.

Theorem 3.1.4. *Let* $(\mathbf{H_2})$ *hold. Then there exists* $\psi \in L^\infty(\Omega)$, *such that for all* $1 \leq p < +\infty$.

$$u(t) \to \psi \qquad \text{in } L^p(\Omega) \text{ as } t \to +\infty.$$

Moreover, ψ *is a step function, which takes two values* $\delta < 0$ *and zero.*

In this paper we only give the proofs of Theorem 3.1.1 and Theorem 3.1.2. The proofs of Theorem 3.1.3 and Theorem 3.1.4 are similar.

The paper is organized as follows: in Section 3.2 we prove Theorem 3.1.1. In Section 3.3, we prove some properties of the solution u and show that Problem (P) possesses a Lyapunov functional. In Section 3.4, applying the rearrangement theory, we introduce the corresponding problem $(P^\sharp)$ and present some properties of its solution. Then we prove that the ω-limit set $\omega(u_0^\sharp)$ is composed of a single element. In Section 3.5, we proved Theorem 3.1.2.

3.2 Existence and uniqueness of solution of Problem (P)

This section is devoted to the proof of Theorem 3.1.1. We first present some *a priori estimates* and show that Problem (P) possesses a local in time solution. We then extend this local solution to a global solution. We start with the following result.

Lemma 3.2.1. *Let* $\phi \in L^\infty(\Omega)$ *be such that*

$$\fint_\Omega \phi = m_0 \notin [0,1].$$

Then

$$\int_\Omega g(\phi) \leq m_0|\Omega|(1-m_0) < 0.$$

Proof. Since g is concave, we apply Jensen's inequality to deduce that

$$g(\fint_\Omega u) \geq \fint_\Omega g(u).$$

It follows that

$$\int_\Omega g(u) \leq |\Omega| g(m_0) = |\Omega| m_0(1-m_0),$$

which completes the proof of Lemma 3.2.1. □

Notation: We set

$$G(\phi) := f(\phi) - g(\phi)\frac{\displaystyle\int_\Omega f(\phi)}{\displaystyle\int_\Omega g(\phi)}; \tag{3.2}$$

and use the notation X for either of the spaces $L^1(\Omega)$, $L^\infty(\Omega)$ or $C(\overline{\Omega})$.

Lemma 3.2.2. *Let $a < b$ be two real numbers and let $v_1, v_2 \in X$ be such that*

$$a \leq v_1(x), v_2(x) \leq b \qquad \text{for a.e } \ x \in \Omega.$$

We assume further that

$$\alpha = \alpha(v_1, v_2) := \min\left(|\int_\Omega g(v_1)|, |\int_\Omega g(v_2)| \right) > 0. \tag{3.3}$$

Then

$$\|G(v_1) - G(v_2)\|_X \leq L_1 \|v_1 - v_2\|_X,$$

where

$$L_1 := K(a,b) + \frac{(K(a,b))^3 |\Omega|(3|\Omega|+2)}{\alpha^2}.$$

We recall that $K(a,b)$ is defined by (3.1).

Remark 3.2.3. Before proving Lemma 3.2.2, we remark that for all $s, \bar{s} \in [a,b]$,

$$|f(s) - f(\bar{s})| \leq K(a,b)\,|s - \bar{s}| \tag{3.4}$$

and

$$|g(s) - g(\bar{s})| \leq K(a,b)\,|s - \bar{s}|. \tag{3.5}$$

Proof of Lemma 3.2.2. We have

$$\begin{aligned} G(v_1) - G(v_2) &= [f(v_1) - f(v_2)] - \left[g(v_1)\frac{\displaystyle\int_\Omega f(v_1)}{\displaystyle\int_\Omega g(v_1)} - g(v_2)\frac{\displaystyle\int_\Omega f(v_2)}{\displaystyle\int_\Omega g(v_2)} \right] \\ &= [f(v_1) - f(v_2)] - \frac{g(v_1)\displaystyle\int_\Omega f(v_1)\int_\Omega g(v_2) - g(v_2)\int_\Omega f(v_2)\int_\Omega g(v_1)}{\displaystyle\int_\Omega g(v_1)\int_\Omega g(v_2)} \\ &=: A_1 - \frac{A_2}{A_3}, \end{aligned}$$

where

$$A_1 := f(v_1) - f(v_2);$$

$$A_2 := g(v_1)\int_\Omega f(v_1)\int_\Omega g(v_2) - g(v_2)\int_\Omega f(v_2)\int_\Omega g(v_1);$$

and

$$A_3 := \int_\Omega g(v_1)\int_\Omega g(v_2).$$

In the following, we will give estimates for A_1, A_2 and A_3. For simplicity, we will write K instead of $K(a,b)$ in the remaining of the proof of this lemma. For the term A_1, we use (3.4) to obtain

$$\|A_1\|_X = \|f(v_1) - f(v_2)\|_X \le K\|v_1 - v_2\|_X. \tag{3.6}$$

Next, we write A_2 as

$$\begin{aligned} A_2 := g(v_1)\int_\Omega f(v_1)\int_\Omega g(v_2) - g(v_2)\int_\Omega f(v_1)\int_\Omega g(v_2) \\ + g(v_2)\int_\Omega f(v_1)\int_\Omega g(v_2) - g(v_2)\int_\Omega f(v_2)\int_\Omega g(v_2) \\ + g(v_2)\int_\Omega f(v_2)\int_\Omega g(v_2) - g(v_2)\int_\Omega f(v_2)\int_\Omega g(v_1), \end{aligned}$$

or equivalently,

$$\begin{aligned} A_2 := [g(v_1) - g(v_2)]\int_\Omega f(v_1)\int_\Omega g(v_2) \\ + g(v_2)\int_\Omega [f(v_1) - f(v_2)]\int_\Omega g(v_2) \\ + g(v_2)\int_\Omega f(v_2)\int_\Omega [g(v_2) - g(v_1)]. \end{aligned}$$

This together with (3.1) implies that

$$\|A_2\|_X \le (K)^3|\Omega|^2\|v_1 - v_2\|_X + 2(K)^3|\Omega|\|v_1 - v_2\|_{L^1(\Omega)}.$$

We also note that

$$\|v_1 - v_2\|_{L^1(\Omega)} \le (1 + |\Omega|)\|v_1 - v_2\|_X$$

so that

$$\|A_2\|_X \le (K)^3|\Omega|(3|\Omega| + 2)\|v_1 - v_2\|_{L^\infty(\Omega)}. \tag{3.7}$$

For the term A_3, by hypotheses (3.3), we have

$$|A_3| \ge \alpha^2 > 0. \tag{3.8}$$

We combine (3.6), (3.7) and (3.8) to obtain

$$\|G(v_1) - G(v_2)\|_X \le \Big(K + \frac{(K)^3|\Omega|(3|\Omega| + 2)}{\alpha^2}\Big)\|v_1 - v_2\|_X,$$

which completes the proof of Lemma 3.2.2. □

Corollary 3.2.4. *Let the same assumptions as in Lemma 3.2.2 hold. We assume furthermore that*

$$⨍_\Omega v_1 = ⨍_\Omega v_2 = m_0 \notin [0,1]$$

and define

$$\alpha_0 := -m_0|\Omega|(1-m_0) > 0.$$

Then

$$\|G(v_1) - G(v_2)\|_X \le L_2 \|v_1 - v_2\|_X,$$

where

$$L_2 := K(a,b) + \frac{(K(a,b))^3|\Omega|(3|\Omega|+2)}{\alpha_0^2}.$$

Proof. It follows from Lemma 3.2.1 that

$$\int_\Omega g(v_1) \le -\alpha_0 < 0 \quad \text{and} \quad \int_\Omega g(v_2) \le -\alpha_0 < 0.$$

Therefore,

$$\alpha(v_1, v_2) = \min(|\int_\Omega g(v_1)|, |\int_\Omega g(v_2)|) \ge \alpha_0 > 0,$$

so that $L_1 \le L_2$. This together with Lemma 3.2.2 completes the proof of Corollary 3.2.4. □

Next, we show the mass conservation property of the solution.

Lemma 3.2.5 (Mass conservation)**.** *Assume that $u \in C^1([0,T];X)$ is a solution of Problem (P). Then the mass conservation property holds, namely*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx. \tag{3.9}$$

Proof. Integrating the differential equation in Problem (P) from 0 to t, we get

$$u(t) - u_0 = \int_0^t u_t = \int_0^t G(u).$$

It follows that

$$\int_\Omega u(x,t)\,dx - \int_\Omega u_0(x)\,dx = \int_0^t \int_\Omega G(u) = 0,$$

where the last identity holds since

$$\int_\Omega G(u) = 0.$$

This completes the proof of Lemma 3.2.5.

□

Lemma 3.2.6 (Boundedness of the solution)**.** *Let $(\mathbf{H_1})$ hold. We assume further that $u_0 \in C(\overline{\Omega})$ and that Problem (P) possesses a solution $u \in C^1([0,T];C(\overline{\Omega}))$ for some $T > 0$. Then,*

(i) the function

$$\lambda(t) := \frac{\displaystyle\int_\Omega f(u)}{\displaystyle\int_\Omega g(u)}$$

is continuous on $[0,T]$*;*

(ii) we have that

$$1 \le u(x,t) \le s_2 \qquad \textit{for all } \ x \in \overline{\Omega}, t \ge 0.$$

Proof. (i) Since $u \in C^1([0,T]; C(\overline{\Omega}))$, it follows that $\displaystyle\int_\Omega f(u)$ and $\displaystyle\int_\Omega g(u)$ are continuous on $[0,T]$. Therefore, λ is continuous on $[0,T]$.

(ii) We first prove that

$$u(x,t) \ge 1 \qquad \text{for all } \ x \in \overline{\Omega}.$$

We define the differential operator

$$\mathcal{L}(y) := y' - f(y) + g(y)\lambda(t),$$

then for every $x \in \overline{\Omega}$, we have

$$\mathcal{L}(u(x,\cdot)) = 0 = \mathcal{L}(1) \qquad \text{and} \qquad u_0(x) \ge 1.$$

We deduce from [8, Theorem 6.1, page 31] that $u(x,t) \ge 1$ for all $t \ge 0$ and $x \in \overline{\Omega}$ arbitrary. In other words, we have

$$u(x,t) \ge 1 \qquad \text{for all } \ x \in \overline{\Omega}, t \ge 0. \tag{3.10}$$

Next, we claim that

$$u(x,t) \le s_2 \qquad \text{for all } \ x \in \overline{\Omega}, t \ge 0.$$

Let $\varepsilon > 0$ be arbitrary. It is sufficient to show that

$$u(x,t) < s_2 + \varepsilon \qquad \text{for all } \ x \in \Omega, t \ge 0.$$

We suppose for the purpose of contradiction that there exists a first time $t_0 > 0$ such that

$$u(x_0,t_0) = s_2 + \varepsilon$$

for some $x_0 \in \overline{\Omega}$. By the definition of t_0, and the continuity of u on $\Omega \times [0,T]$, we have

$$u(x,t_0) \le s_2 + \varepsilon \qquad \text{for all } \ x \in \overline{\Omega}. \tag{3.11}$$

and

$$u(x,t) < s_2 + \varepsilon \qquad \text{for all } \ x \in \Omega, 0 \le t < t_0.$$

Since

$$u_t(x_0,t_0) \ge 0,$$

it follows that

$$f(u(x_0,t_0)) - g(u(x_0,t_0))\frac{\displaystyle\int_\Omega f(u(t_0))}{\displaystyle\int_\Omega g(u(t_0))} \ge 0,$$

or equivalently,

$$u(x_0,t_0))(1-u(x_0,t_0))\left[u(x_0,t_0)-\frac{\displaystyle\int_\Omega u^2(t_0)(1-u(t_0))}{\displaystyle\int_\Omega u(t_0)(1-u(t_0))}\right]\geq 0.$$

Note that

$$u(x_0,t_0)(1-u(x_0,t_0))=(s_2+\varepsilon)(1-(s_2+\varepsilon))<0,$$

so that

$$u(x_0,t_0)-\frac{\displaystyle\int_\Omega u^2(t_0)(1-u(t_0))}{\displaystyle\int_\Omega u(t_0)(1-u(t_0))}\leq 0.$$

Hence,

$$u(x_0,t_0)\int_\Omega u(t_0)(1-u(t_0))\geq\int_\Omega u^2(t_0)(1-u(t_0)),$$

or equivalently,

$$\int_\Omega [u(x_0,t_0)-u(x,t_0)]u(x,t_0)[1-u(x,t_0)]\geq 0. \tag{3.12}$$

On the other hand, it follows from (3.10) and (3.11) that

$$[u(x_0,t_0)-u(x,t_0)]u(x,t_0)[1-u(x,t_0)]\leq 0,$$

which together with (3.12) and the continuity of $u(\cdot,t_0)$ on $\overline{\Omega}$ implies that

$$u(x,t_0)=u(x_0,t_0)=s_2+\varepsilon \qquad \text{for all } \ x\in\overline{\Omega}.$$

Consequently, we have that

$$\int_\Omega u(x,t_0)dx=(s_2+\varepsilon)|\Omega|>s_2|\Omega|\geq\int_\Omega u_0(x)dx,$$

which contradicts the mass conservation property. Therefore,

$$u(x,t)\leq s_2 \qquad \text{for all } \ x\in\overline{\Omega},t\geq 0.$$

□

Lemma 3.2.7 (Local existence of solution)**.** *(i) Let* $(\mathbf{H_1})$ *hold. We assume further that* $u_0\in C(\overline{\Omega})$*. Then there exists* $T_0=T_0(s_2,m_0)$ *such that Problem* (P) *possesses a solution in* $C^1([0,T_0];C(\overline{\Omega}))$*.*

(ii) Let $(\mathbf{H_1})$ *hold. Then there exists* $T_0=T_0(s_2,m_0)$ *such that Problem* (P) *possesses a solution in* $C^1([0,T_0];L^\infty(\Omega))$*.*

Moreover, in (i) and (ii), we can choose

$$T_0:=\min\left\{\left(K+\frac{(K)^2|\Omega|}{\alpha_0}\right)^{-1},\frac{1}{2}\Big(K+\frac{(K)^3|\Omega|(3|\Omega|+2)}{\alpha_0^2}\Big)^{-1}\right\}$$

where $K:=K(-s_2-1;s_2+1)$ *and* $\alpha_0:=-m_0|\Omega|(1-m_0)$*.*

Proof. (i) We will use the contraction mapping theorem to prove the local existence of solution. For this purpose, we set

$$\mathcal{V}(u_0) := \{v \in C(\overline{\Omega}) : \fint_\Omega v = m_0 \quad \text{and} \quad \|v - u_0\|_{C(\overline{\Omega})} \leq 1\}.$$

and

$$\mathcal{A} := \{w \in C([0,T_0]; C(\overline{\Omega})) : w(t) \in \mathcal{V}(u_0) \ \text{ for all } \ t \in [0,T_0]\},$$

where $T_0 > 0$ will be choosen later. We define the operator $\mathcal{T}$ by

$$\mathcal{T}w(t) := u_0 + \int_0^t G(w) \qquad \text{for all } \ w \in \mathcal{A},$$

where G is given by (3.2). In the following, we will prove that $\mathcal{T}$ is well-defined and has a fixed point. We begin with some properties of G.

Claim 1: We prove that G is Lipschitz continuous on $\mathcal{V}(u_0)$. For all $v \in \mathcal{V}(u_0)$, we have

$$\|v\|_{C(\overline{\Omega})} \leq \|u_0\|_{C(\overline{\Omega})} + 1 \leq s_2 + 1. \tag{3.13}$$

Therefore, we apply Corollary 3.2.4 to deduce that for all $v_1, v_2 \in \mathcal{V}(u_0)$,

$$\|G(v_1) - G(v_2)\|_{C(\overline{\Omega})} \leq L_3 \|v_1 - v_2\|_{C(\overline{\Omega})},$$

where

$$L_3 = L_3(s_2, m_0) := \Big(K + \frac{(K)^3|\Omega|(3|\Omega| + 2)}{\alpha_0^2}\Big) > 0,$$

with $K := K(-s_2 - 1, s_2 + 1)$ and $\alpha_0 := -m_0|\Omega|(1 - m_0)$.

Claim 2: We prove that G is bounded on $\mathcal{V}(u_0)$. It follows from (3.13) and from the definition of $K(-s_2 - 1, s_2 + 1)$ that for all $v \in \mathcal{V}(u_0)$,

$$\|f(v)\|_{C(\overline{\Omega})} \leq K(-s_2 - 1, s_2 + 1), \|g(v)\|_{C(\overline{\Omega})} \leq K(-s_2 - 1, s_2 + 1).$$

We also deduce from Lemma 3.2.1 that

$$\int_\Omega g(v) < -\alpha_0 < 0 \ \text{ for all } \ v \in \mathcal{V}(u_0).$$

Therefore, we have

$$\|G(v)\|_{C(\overline{\Omega})} \leq K + \frac{(K)^2|\Omega|}{\alpha_0} =: M \ \text{ for all } \ v \in \mathcal{V}(u_0),$$

so that G is bounded on $\mathcal{V}(u_0)$.

We define

$$T_0 = T(s_2; m_0) := \min(\frac{1}{M}, \frac{1}{2L_3})$$

We claim that $\mathcal{T}$ is a contraction mapping on $\mathcal{A}$.

Claim 3: We prove that $\mathcal{T}$ is well-defined. Indeed, let $w \in \mathcal{A}$, then by the definition of $\mathcal{A}$, we have for all $0 \leq t \leq T_0$, $w(t) \in \mathcal{V}(u_0)$. It follows from Lemma 3.2.1 that

$$\int_\Omega g(w(t)) \leq \alpha_0 < 0 \qquad \text{for all } \ w \in \mathcal{A}, 0 \leq t \leq T_0.$$

Therefore,

$$G(w(t)) = f(w(t)) - g(w(t))\frac{\displaystyle\int_\Omega f(w(t))}{\displaystyle\int_\Omega g(w(t)))}$$

is well-defined. Consequently, $\mathcal{T}$ is well-defined.

Claim 4: We claim that

$$\mathcal{T}(\mathcal{A}) \subset \mathcal{A}.$$

Let $w \in \mathcal{A}$, for all $0 \leq t \leq \bar{t} \leq T_0$, we have

$$\|\mathcal{T}w(t) - \mathcal{T}w(\bar{t})\|_{C(\overline{\Omega})} \leq \int_t^{\bar{t}} \|G(w(s))\|_{C(\overline{\Omega})} \leq M|t - \bar{t}|,$$

so that $\mathcal{T}w \in C([0,T]; C(\overline{\Omega}))$. It remains now to prove that $\mathcal{T}w(t) \in \mathcal{V}(u_0)$. Since

$$\int_\Omega G(w(s)) = 0 \qquad \text{for all } \ s \in [0, T_0],$$

it follows that

$$⨍_\Omega \mathcal{T}w(t) = \int_\Omega u_0 + \int_0^t ⨍_\Omega G(w(s))ds = ⨍_\Omega u_0 = m_0.$$

Moreover,

$$\|\mathcal{T}w(t) - u_0\|_{C(\overline{\Omega})} \leq \int_0^t \|G(w(s))\|_{C(\overline{\Omega})}\, ds \leq MT_0 \leq 1.$$

Therefore, $\mathcal{T}w(t) \in \mathcal{V}(u_0)$. Thus, we conclude that

$$\mathcal{T}(\mathcal{A}) \subset \mathcal{A}.$$

Claim 5: Now, we prove that $\mathcal{T}$ is a contraction mapping on $\mathcal{A}$. Indeed, let $w_1, w_2 \in \mathcal{A}$, $t \in [0, T_0]$, we have

$$\begin{aligned}
\|\mathcal{T}(w_1)(t) - \mathcal{T}(w_2)(t)\|_{C(\overline{\Omega})} &\leq \int_0^t \|G(w_1(s)) - G(w_2(s))\|_{C(\overline{\Omega})} ds \\
&\leq L_3 \int_0^t \|w_1(s) - w_2(s)\|_{C(\overline{\Omega})} ds \\
&\leq T_0 L_3 \|w_1 - w_2\|_{C([0,T_0];C(\overline{\Omega}))} \\
&\leq \frac{1}{2}\|w_1 - w_2\|_{C([0,T_0];C(\overline{\Omega}))}.
\end{aligned}$$

Hence, we have

$$\|\mathcal{T}w_1 - \mathcal{T}w_2\|_{C([0,T_0];C(\overline{\Omega}))} \leq \frac{1}{2}\|w_1 - w_2\|_{C([0,T_0];C(\overline{\Omega}))},$$

which implies that $\mathcal{T}$ is contraction on $\mathcal{A}$.

It is easy to see that $\mathcal{A}$ is closed in $C([0,T_0];C(\overline{\Omega}))$. Then, we apply the contraction mapping theorem [12, Theorem 2.2, page 44] to conclude that $\mathcal{T}$ has a unique fixed point. In other words, there exists $u \in C([0,T_0];C(\overline{\Omega}))$ such that

$$u(t) := u_0 + \int_0^t G(u).$$

Therefore, Problem (P) possesses a unique solution in $C^1([0,T_0];C(\overline{\Omega}))$.

(ii) The proof of (ii) is similar to that of (i). We omit it here.

□

We are ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. The local existence of solution is given in Lemma 3.2.7. Let $[0,T_{max})$ be the maximal interval of existence of solution. We now prove that

$$T_{\max} = +\infty.$$

The case that $u_0 \in C(\overline{\Omega})$**:** By contradiction, assume that $T_{\max} < +\infty$. Let $t_0 \in [T_{max} - \frac{T_0}{2}, T_{max})$. It follows from Lemma 3.2.5 and Lemma 3.2.6(ii) that

$$⨍_\Omega u(t_0) = m_0 \quad \text{and} \quad 1 \leq u(t_0) \leq s_2.$$

Then we apply Lemma 3.2.7(i) to deduce that we can extend solution to interval $[0, T_{max} + \frac{T_0}{2}]$, which give a contradiction.

The case that $u_0 \in L^\infty(\Omega)$**:** We first prove that

$$1 \leq u(x,t) \leq s_2 \quad \text{for a.e } x \in \Omega, 0 \leq t < T_{max} \tag{3.14}$$

The main idea of the proof is to approximate the initial function u_0 by a sequence of continuous functions u_{n0} and to first obtain the result for approximate problems. Then, we deduce the result for the original problem. To begin with, we need the following lemma.

Lemma 3.2.8. *Suppose that u_0 satisfies Hypothesis $(\mathbf{H_1})$. Then there exists a sequence $\{u_{n0}\}$ in $C(\overline{\Omega})$ such that*

$$1 \leq u_{n0} \leq s_2 \quad \text{on } \overline{\Omega},$$

and that

$$u_{n0} \to u_0 \quad \text{in } L^1(\Omega) \quad \text{as } n \to \infty.$$

The proof of this lemma will be given later. Now, we will use this lemma to prove (3.14). It follows from (i) that Problem (P) with initial condition u_{n0} possesses a unique solution u_n in $C([0,+\infty);C(\overline{\Omega}))$. Moreover,

$$1 \leq u_n(x,t) \leq s_2 \quad \text{for all } x \in \overline{\Omega}, t \geq 0. \tag{3.15}$$

We define

$$b := \max(\|u\|_{C([0,T];L^\infty(\Omega))}, s_2) \quad \text{and} \quad a := -b.$$

Since

$$u_{n0} \to u_0 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to +\infty,$$

then

$$\fint_\Omega u_{n0} \to \fint_\Omega u_0 = m_0 \quad \text{as} \quad n \to +\infty.$$

Therefore, for ε is small enough such that $[m_0-\varepsilon, m_0+\varepsilon]\cap[0,1] = \emptyset$, we can assume without loss of generality that for all $n \geq 0$,

$$\fint_\Omega u_{n0} \in [m_0 - \varepsilon, m_0 + \varepsilon].$$

We define

$$\beta := \inf_{s\in[m_0-\varepsilon, m_0+\varepsilon]} -|\Omega| s(1-s),$$

then $\beta > 0$. By Lemma 3.2.1, we have

$$|\int_\Omega g(u_n(t))| > \beta \quad \text{for all} \quad n > 0$$

and

$$|\int_\Omega g(u(t))| > \beta.$$

We apply Lemma 3.2.2 to deduce that

$$\|G(u_n(t)) - G(u(t))\|_{L^\infty(\Omega)} \leq L_4 \|u_n(t) - u(t)\|_{L^1(\Omega)} \quad \text{for all} \quad 0 \leq t \leq T_{max}, \tag{3.16}$$

where

$$L := \Big(K(a,b) + \frac{(K(a,b))^3|\Omega|(3|\Omega|+2)}{\beta^2}\Big) > 0$$

We integrate the difference of the equations for u and u_n from 0 to t, to obtain

$$u_n(t) - u(t) = [u_{n0} - u_0] + \int_0^t \big[G(u_n(s)) - G(u(s))\big]\, ds.$$

Applying (3.16), we deduce that

$$\|u_n(t) - u(t)\|_{L^1(\Omega)} \leq \|u_{n0} - u_0\|_{L^1(\Omega)} + L_4 \int_0^t \|u_n(s) - u(s)\|_{L^1(\Omega)}\, ds.$$

By Gronwall's inequality, we deduce that

$$\|u_n(t) - u(t)\|_{L^1(\Omega)} \leq \|u_{n0} - u_0\|_{L^1(\Omega)} \exp(L_4 t).$$

Thus for all $0 \leq t < T_{max}$,

$$u_n(t) \to u(t) \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty.$$

Consequently, for each $t \geq 0$, there exists a subsequence $\{u_{nj}(t)\}_{j\geq 0}$ of $\{u_n(t)\}_{n\geq 0}$ such that

$$u_{nj}(x,t) \to u(x,t) \quad \text{for a.e.} \quad x \in \Omega \quad \text{as} \quad j \to \infty,$$

which together with (3.15) implies that for each $t \geq 0$,

$$1 \leq u(x,t) \leq s_2 \quad \text{for a.e.} \quad x \in \Omega.$$

Therefore, we obtain (3.14). Now we are ready to prove $T_{max} = +\infty$. By contradiction, assume that $T_{\max} < +\infty$. Let $t_0 \in [T_{max} - \frac{T_0}{2}, T_{max})$, then

$$⨍_\Omega u(t_0) = m_0, \qquad \text{and} \qquad 1 \leq u(t_0) \leq s_2.$$

We apply Lemma 3.2.6(ii) to deduce that we can extend solution to interval $[0, T_{max} + \frac{T_0}{2}]$, which give a contradiction.

We have proven the existence part of Theorem 3.1.1. The uniqueness of the solution follows from Lemma 3.2.9 below, which will complete the proof of Theorem 3.1.1. □

Lemma 3.2.9. *Let $u_{01}, u_{02} \in X$ such that*

$$\int_\Omega u_{01}(x)\,dx = \int_\Omega u_{02}(x)\,dx = m_0 \notin [0,1],$$

and that

$$1 \leq u_{01}, u_{02} \leq s_2.$$

Let u_1, u_2 be two corresponding solutions of Problem (P) in $C^1([0,+\infty); X)$. Then

$$\|u_1(t) - u_2(t)\|_X \leq \exp(L_5 t)\|u_{01} - u_{02}\|_X, \tag{3.17}$$

where

$$L_5 := K(1, s_2) + \frac{(K(1,s_2))^3|\Omega|(3|\Omega| + 2)}{\alpha_0^2}.$$

Proof. By the mass conservation property, we have

$$⨍_\Omega u_1(t) = ⨍_\Omega u_2(t) = m_0 \quad \text{for all} \quad t > 0. \tag{3.18}$$

Moreover, we have

$$1 \leq u_1(t), u_2(t) \leq s_2.$$

We apply Corollary 3.2.4 to deduce that

$$\|G(u_1(t)) - G(u_2(t))\|_X \leq L_5\|u_1(t) - u_2(t)\|_X \quad \text{for all} \quad 0 \leq t \leq T, \tag{3.19}$$

where

$$L_5 := K(1, s_2) + \frac{(K(1,s_2))^3|\Omega|(3|\Omega| + 2)}{\alpha_0^2} > 0.$$

We integrate the difference of the equations for u_1 and u_2 from 0 to t, to obtain

$$u_1(t) - u_2(t) = u_{01} - u_{02} + \int_0^t \left[G(u_1(s)) - G(u_2(s))\right] ds.$$

It follows from (3.19) that

$$\|u_1(t) - u_2(t)\|_X \leq \|u_{01} - u_{02}\|_X + L_5 \int_0^t \|u_1(s) - u_2(s)\|_X\, ds.$$

This together with Gronwall's inequality completes the proof of Lemma 3.2.9. □

Proof of Lemma 3.2.8. First, we extend the function u_0 to w_0 defined on $\mathbb{R}^N$ by

$$w_0(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega \\ \dfrac{1+s_2}{2} & \text{elsewhere.} \end{cases}$$

Obviously,

$$1 \le w_0(x) \le s_2 \quad \text{for a.e.} \quad x \in \mathbb{R}^N.$$

For all n large enough such that

$$1 + \frac{1}{n} < s_2 - \frac{1}{n},$$

we define

$$w_{n0} := \min\left\{ s_2 - \frac{1}{n}, \max\{w_0, 1 + \frac{1}{n}\} \right\}.$$

Then, we have

$$1 + \frac{1}{n} \le w_{n0}(x) \le s_2 - \frac{1}{n} \quad \text{a.e.} \quad x \in \mathbb{R}^N, \tag{3.20}$$

and moreover,

$$\begin{aligned} \|w_{n0} - u_0\|_{L^1(\Omega)} &= \|w_{n0} - w_0\|_{L^1(\Omega)} \\ &\le |\Omega| \|w_{n0} - w_0\|_{L^\infty(\Omega)} \le \frac{|\Omega|}{n}. \end{aligned}$$

As in [2, Theorem 2.29, page 36], we can choose a smooth function u_{n0} such that

$$\|u_{n0} - w_{n0}\|_{L^1(\Omega)} \le \frac{|\Omega|}{n}.$$

More precisely,

$$u_{n0}(x) := \int_{\mathbb{R}^N} J_\varepsilon(x-y) w_{n0}(y) dy, \quad \text{for } \varepsilon \text{ small enough,} \tag{3.21}$$

where

$$J_\varepsilon(x) = \varepsilon^{-N} J(\frac{x}{\varepsilon}),$$

with $J \in C_c^\infty(\mathbb{R}^N)$ is a nonnegative function satisfying

$$J(x) = 0 \qquad \text{if} \quad |x| \ge 1, \quad \text{and} \quad \int_{\mathbb{R}^N} J(x) dx = 1.$$

Note that J_ε is nonnegative and that

$$\int_{\mathbb{R}^N} J_\varepsilon(x)\, dx = 1. \tag{3.22}$$

It follows from (3.20), (3.21) and (3.22) that

$$s_1 + \frac{1}{n} \le u_{n0} \le s_2 - \frac{1}{n} \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover,

$$\|u_{n0} - u_0\|_{L^1(\Omega)} \leq \|u_{n0} - w_{n0}\|_{L^1(\Omega)} + \|w_{n0} - u_0\|_{L^1(\Omega)} \leq \frac{2|\Omega|}{n}.$$

It follows that

$$u_{n0} \to u_0 \quad \text{in } L^1(\Omega) \text{ as } n \to +\infty.$$

The proof of Lemma 3.15 is complete. □

The following lemma expresses the continuity of the semi-group with respect to the initial data.

3.3 Some preparations

In this section, as a preparation for the next section, we consider an ordinary differential equation which is related to Problem (P). This study puts forth some properties of the solution of Problem (P), which are stated in Lemma 3.3.2 below. To begin with, we define the function

$$\lambda(t) := \frac{\displaystyle\int_\Omega f(u(x,t))dx}{\displaystyle\int_\Omega g(u(x,t))dx}, \tag{3.23}$$

where u is the unique solution of Problem (P) given in Theorem 3.1.1. Note that λ is well-defined thanks to Lemma 3.2.1. Since $u \in C^1([0,+\infty); L^\infty(\Omega))$ and since

$$1 \leq u(x,t) \leq s_2 \quad \text{for a.e } x \in \Omega, t \geq 0,$$

it follows that $\lambda(t)$ is continuous on $[0,+\infty)$ and that

$$1 \leq \lambda(t) \leq s_2 \quad \text{for all } t \geq 0.$$

We define the function H on $\mathbb{R} \times [0,+\infty)$ by

$$H(Y,t) := \bar{f}(Y) - \bar{g}(Y)\lambda(t),$$

where $\bar{f}, \bar{g} \in C_c^\infty(\mathbb{R})$ are such that for $s \in [0, s_2+1]$

$$\bar{f}(s) = f(s); \quad \bar{g}(s) = g(s).$$

Note that there exists a constant c_2 such that

$$|H|, |\frac{\partial H}{\partial Y}| \leq c_2 \quad \text{on } \mathbb{R} \times [0,+\infty). \tag{3.24}$$

We consider the problem

$$(ODE) \begin{cases} Y' = H(Y,t) \\ \\ Y(0) = s. \end{cases}$$

Lemma 3.3.1. *Let $s \in \mathbb{R}$. Problem (ODE) possesses a global unique solution $Y(t;s) \in C^1([0,+\infty))$. Moreover,*

(i) if $s < \tilde{s}$ then $Y(t;s) < Y(t;\tilde{s})$ for all $t \geq 0$;

(ii) we define the function $h^t : \mathbb{R} \to \mathbb{R}$ by $h^t(s) := Y(t;s)$. Then h^t is bijective, increasing from $\mathbb{R}$ to $\mathbb{R}$.

Proof. In view of (3.24), we have that

$$|H(Y_1,t) - H(Y_2,t)| \leq c_2|Y_1 - Y_2|$$

for all $Y_1, Y_2 \in \mathbb{R}, t \geq 0$. It follows from standard theory of ordinary differential equations (for instance, see [3, Corollary 8.3, page 57]) that Problem (ODE) possesses a unique solution in $C^1([0,+\infty))$.

(i) We define $W(t) := Y(t;\tilde{s}) - Y(t;s)$. Then W satisfies the equations

$$\begin{cases} W' = H(Y(t;\tilde{s}),t) - H(Y(t;s),t) \\ W(0) = \tilde{s} - s > 0, \end{cases}$$

Note that

$$\begin{aligned} H(Y(t;\tilde{s}),t) - H(Y(t;s),t) &= [Y(t;\tilde{s}) - Y(t;s)] \int_0^1 \frac{\partial H}{\partial Y}(\tau Y(t;\tilde{s}) + (1-\tau)Y(t;s),t)d\tau \\ &:= W(t)\theta(t), \end{aligned}$$

where

$$\theta(t) := \int_0^1 \frac{\partial H'}{\partial Y}(\tau Y(t;\tilde{s}) + (1-\tau)Y(t;s),t)d\tau.$$

Thus, W satisfies the initial value problem

$$\begin{cases} W' = \theta(t)W(t) \\ W(0) = \tilde{s} - s > 0, \end{cases}$$

so that

$$W(t) = W(0)\exp\left(\int_0^t \theta(\tau)d\tau\right) > 0.$$

It follows that

$$Y(t,\tilde{s}) > Y(t,s) \quad \text{for all} \quad t \geq 0.$$

(ii) Let $t_0 > 0$ be arbitrary. It follows from (i) that if $s < \tilde{s}$, then

$$Y(t_0;s) < Y(t_0;\tilde{s});$$

hence, we have

$$h^{t_0}(s) < h^{t_0}(\tilde{s}).$$

Consequently, h^{t_0} is injective, increasing on $\mathbb{R}$.

Now we claim that h^{t_0} is surjective. Let $S \in \mathbb{R}$ be arbitrary. We consider the problem

$$\begin{cases} Y' = H(Y,t) \\ Y(t_0) = S. \end{cases}$$

This problem possesses a unique solution in $C^1([0,+\infty))$ (cf. [3, Corollary 8.3, page 57]). We denote by $Y(t;t_0;S)$ this solution. We also remark that the solution of Problem (ODE) can be written as $Y(t;0;s)$. Moreover, by the uniqueness of the solution, we have

$$Y(t_0;0;Y(0;t_0;S)) = S.$$

Therefore, $h^{t_0}(Y(0;t_0;S)) = S$, so that h^{t_0} is surjective. □

Lemma 3.3.2. *Let* $(\mathbf{H_1})$ *hold. Then for all* $t \geq 0$*, we have*

$$u(t) = h^t(u_0) \quad a.e. \ in \ \ \Omega.$$

Proof. We define

$$z(x,t) := h^t(u_0(x)) = Y(t;u_0(x)) \ \text{ on } \ \Omega \times [0,+\infty).$$

We need to prove that for all $t \geq 0$

$$z(t) = u(t) \ \text{ almost everywhere on } \ \Omega.$$

It follows from the definition of $Y(t;\cdot)$ that for all $x \in \Omega$; $z(x,\cdot) \in C^1([0,+\infty))$ and

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \bar{f}(z(x,t)) - \bar{g}(z(x,t))\lambda(t) & \text{in } \Omega \times [0,\infty), \\ \\ z(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{3.25}$$

The right hand side of the differential equation in (3.25) is bounded in $L^\infty(0,T;L^\infty(\Omega))$ which together with the continuity of λ implies that $z \in C^1([0,+\infty);L^\infty(\Omega))$.

Since $1 \leq u(t) \leq s_2$,

$$f(u(t)) = \bar{f}(u(t)) \quad \text{and} \quad g(u(t)) = \bar{g}(u(t)).$$

Therefore,

$$u_t = H(u,t).$$

We integrate from 0 to t the difference of the equations for u and for z to obtain

$$u(t) - z(t) = \int_0^t (H(u(\tau),\tau) - H(z(\tau),\tau),$$

In view of (3.24), we have

$$\|u(t) - z(t)\|_{L^\infty(\Omega)} \leq \int_0^t \|H(u(\tau),\tau) - H(z(\tau),\tau)\|_{L^\infty(\Omega)} \leq c_2 \int_0^t \|u(\tau) - z(\tau)\|_{L^\infty(\Omega)},$$

which together with Gronwall's inequality completes the proof of Lemma 3.3.2. □

3.4 Problem $(P^\sharp)$ and the ω-limit set for Problem $(P^\sharp)$

As it was done in Chapter 2, we consider the problem

$$(P^\sharp)\quad \begin{cases} v_t = f(v) - g(v)\dfrac{\displaystyle\int_\Omega f(v)}{\displaystyle\int_\Omega g(v)} & \text{in } \ (0,|\Omega|)\times(0,\infty), \\ \\ v(x,0) = u_0^\sharp(x) & y\in(0,|\Omega|), \end{cases}$$

where $w^\sharp$ denotes the one-dimensional decreasing rearrangement of the function $w:\Omega\to\mathbb{R}$. We refer to Section 2.4.1 in chapter 2 for more details.

Problem $(P^\sharp)$ is a one-dimensional variant of Problem (P), so that all results stated in Section 3.2 for Problem (P) also apply to Problem $(P^\sharp)$. We note that Hypothesis $(\mathbf{H_1})$ for the function u_0 is also satisfied by $u_0^\sharp$.

Theorem 3.4.1. *Let* $(\mathbf{H_1})$ *hold. We define*

$$u^\sharp(y,t) := (u(t))^\sharp(y) \quad on \ \ (0,|\Omega|)\times[0,+\infty),$$

then $u^\sharp$ *is the unique solution of Problem* $(P^\sharp)$ *in* $C^1([0,\infty);L^\infty((0,|\Omega|)))$. *Moreover,*

$$1\le u^\sharp(y,t)\le s_2 \quad \text{for all } \ y\in(0,|\Omega|), t\ge 0, \tag{3.26}$$

and

$$u^\sharp(y,t) = h^t(u_0^\sharp(y)) \quad \text{for all } \ y\in(0,|\Omega|), t\ge 0. \tag{3.27}$$

Proof of Theorem 3.4.1. First we prove (3.26) and (3.27). Since

$$1\le u(x,t)\le s_2 \ \text{ for a.e } \ x\in\Omega, t\ge 0$$

we deduce from Remark 2.4.6 in Chapter 2 that

$$1\le u^\sharp(y,t)\le s_2 \text{ for all } \ y\in(0,|\Omega|), t\ge 0.$$

In order to prove (3.27), we recall from Lemma 3.3.2 that

$$u(t) = h^t(u_0) \qquad \text{for a.e. in } \ \Omega.$$

We then apply Proposition 2.4.7 in Chapter 2 to obtain for all $t\ge 0$,

$$u^\sharp(t) = (h^t(u_0))^\sharp = h^t(u_0^\sharp) \ \text{ on } \ (0,|\Omega|). \tag{3.28}$$

Now, it remains to prove that $u^\sharp$ is the unique solution of Problem $(P^\sharp)$. As a consequence of (3.28) and Lemma 3.3.1, we have for all $y\in(0,|\Omega|), t\ge 0$

$$\begin{aligned}\frac{\partial u^\sharp}{\partial t}(y,t) &= \bar f(u^\sharp(y,t)) - \bar g(u^\sharp(y,t))\lambda(t)\\ &= f(u^\sharp(y,t)) - g(u^\sharp(y,t))\frac{\displaystyle\int_\Omega f(u(t))}{\displaystyle\int_\Omega g(u(t))} = f(u^\sharp(y,t)) - g(u^\sharp(y,t))\frac{\displaystyle\int_0^{|\Omega|} f(u^\sharp(t))}{\displaystyle\int_0^{|\Omega|} g(u^\sharp(t))}\end{aligned}$$

where the last identity follows from Proposition 2.4.4(ii) in chapter 2. Since the right-hand-side of the above equation is bounded in $L^\infty(0,+\infty;L^\infty((0,|\Omega|)))$ and since by the continuity of f,g,λ, we have $u^\sharp \in C^1([0,\infty);L^\infty((0,|\Omega|)))$. Therefore, $u^\sharp$ coincide with the unique solution of Problem $(P^\sharp)$ in $C^1([0,+\infty);L^\infty((0,|\Omega|)))$. □

Corollary 3.4.2. *Let* $(\mathbf{H_1})$ *hold. Then for all* $y \in (0,|\Omega|)$ *the function* $u^\sharp(y,\cdot)$ *is the unique solution of Problem* (ODE) *in* $C^1([0,+\infty))$ *with the initial condition* $u_0^\sharp(y)$*. Moreover, for all* $y_1, y_2 \in (0,|\Omega|)$*,*

(i) if $u_0^\sharp(y_1) = u_0^\sharp(y_2)$ *then* $u^\sharp(y_1,t) = u^\sharp(y_2,t)$ *for all* $t \geq 0$*;*

(ii) if $u_0^\sharp(y_1) < u_0^\sharp(y_2)$ *then* $u^\sharp(y_1,t) < u^\sharp(y_2,t)$ *for all* $t \geq 0$*.*

Proof. This corollary is an immediate consequence of Theorem 3.4.1 and Lemma 3.3.1. □

Theorem 3.4.3 (Lyapunov functional)**.** *Let* $v_0 \in L^\infty((0,|\Omega|))$ *be such that* $1 \leq v_0 \leq s_2$*. Let* v *be the unique solution of Problem* $(P^\sharp)$ *corresponding to the initial function* v_0*. We define*

$$\mathcal{E}(v(t)) = \int_0^{|\Omega|} v^2.$$

(i) We have for all $\tau_2 > \tau_1 \geq 0$*,*

$$\mathcal{E}(v(\tau_2)) - \mathcal{E}(v(\tau_1)) = -\int_{\tau_1}^{\tau_2}\int_0^{|\Omega|} v(1-v)\left(v - \frac{\int_0^{|\Omega|} f(v)}{\int_0^{|\Omega|} g(v)}\right)^2.$$

(ii) $\mathcal{E}(v(\cdot))$ *is continuous and non increasing on* $[0,+\infty)$*, and there exists the limit of* $\mathcal{E}(v(t))$ *as* $t \to \infty$*.*

Proof. (i) We have

$$\frac{d}{dt}\mathcal{E}(v(t)) = -\frac{d}{dt}\int_0^{|\Omega|} v^2 = \int_0^{|\Omega|} v v_t.$$

We also remark that $\int_0^{|\Omega|} v_t = 0$, so that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(v(t)) &= \int_0^{|\Omega|}\left(v - \frac{\int_0^{|\Omega|} f(v)}{\int_0^{|\Omega|} g(v)}\right)v_t \\ &= \int_0^{|\Omega|} v(1-v)\left(v - \frac{\int_0^{|\Omega|} f(v)}{\int_0^{|\Omega|} g(v)}\right)^2 \leq 0,\end{aligned}$$

where the last inequality holds since

$$1 \leq v(y,t) \leq s_2 \qquad \text{for a.e. } y \in (0,|\Omega|), t \geq 0.$$

Integrating this identity form τ_1 to τ_2, we obtain

$$\mathcal{E}(v(\tau_2)) - \mathcal{E}(v(\tau_1)) = \int_{\tau_1}^{\tau_2}\int_0^{|\Omega|} v(1-v)\left(v - \frac{\int_0^{|\Omega|} f(v)}{\int_0^{|\Omega|} g(u)}\right)^2.$$

(ii) As a consequence of (i), $\mathcal{E}(v(\cdot))$ is continuous and nonincreasing. Moreover, $\mathcal{E}$ is bounded from below by 0. Therefore there exists the limit of $\mathcal{E}(v(t))$ as $t \to \infty$, which completes the proof of (ii). □

The remaining part of this section is devoted to study the set $\omega(u_0^\sharp)$, which is defined as follows:

Definition 3.4.4. *We define the ω-limit set of $u_0^\sharp$ by*

$$\omega(u_0^\sharp) := \{\varphi \in L^1((0,|\Omega|)) : \exists t_n \to \infty, u^\sharp(t_n) \to \varphi \ \ in \ \ L^1((0,|\Omega|)) \ as \ n \to \infty\}.$$

Lemma 3.4.5. *The set*

$$\{u^\sharp(t) : \quad t \geq 0\}$$

is relatively compact in $L^1((0,|\Omega|))$. Consequently, $\omega(u_0^\sharp)$ is nonempty.

Proof. The proof of this lemma which is based on the fact that $u^\sharp(t)$ is nonincreasing in space for all t is similar to that of Lemma 2.5.3 in Chapter 2. We omit it here. □

Lemma 3.4.6. *Assume that $(\mathbf{H_1})$ hold. Then, $\omega(u_0^\sharp)$ is a connected set of $L^1(\Omega)$. Moreover, for a function $\varphi \in \omega(u_0^\sharp)$, the following assertions hold:*

(i) $1 \leq \varphi(y) \leq s_2$ *for almost* $y \in (0,|\Omega|)$ *and* $\int_0^{|\Omega|} \varphi = \int_0^{|\Omega|} u_0^\sharp$.

(ii) There exists a constant $\mathcal{E}_\infty$ such that

$$\mathcal{E}_{|\omega(u_0^\sharp)} = \mathcal{E}_\infty.$$

(iii) φ is a stationary solution of Problem $(P^\sharp)$, which implies that

$$\varphi = \gamma \chi_{(0,A)} + \chi_{(A,|\Omega|)}, \tag{3.29}$$

where A, γ are positive constants such that $\gamma > 1$.

Proof. Before proving (i), (ii) (iii), we deduce from the definition of $\omega(u_0^\sharp)$ that there exists a sequence $t_n \to +\infty$ such that

$$u^\sharp(t_n) \to \varphi \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty. \tag{3.30}$$

(i) It follows from (3.30) that there exists a subsequence of $\{t_n\}$ (still denoted by $\{t_n\}$) such that as $n \to +\infty$

$$u^\sharp(y,t_n) \to \varphi(y) \ \text{ for a.e. } \ y \in (0,|\Omega|).$$

Moreover, we have that

$$1 \leq u^\sharp(y,t_n) \leq s_2 \ \text{ for all } \ y \in (0,|\Omega|),$$

so that also

$$1 \leq \varphi(y) \leq s_2 \ \text{ for a.e. } \ y \in (0,|\Omega|).$$

The identity $\int_0^{|\Omega|} \varphi = \int_0^{|\Omega|} u_0^\sharp$ follows from the mass conservation property.

(ii) It follows from Theorem 3.4.3(ii) that there exists the limit of $\mathcal{E}(u^\sharp(t))$ as $t \to \infty$. Set

$$\mathcal{E}_\infty := \lim_{t\to\infty} \mathcal{E}(u^\sharp(t)).$$

Note that

$$|\mathcal{E}(u^\sharp(t_n)) - \mathcal{E}(\varphi)| \le \int_0^{|\Omega|} |u^\sharp(t_n) + \varphi|\,|u^\sharp(t_n) - \varphi| \le c\|u^\sharp(t_n) - \varphi\|_{L^1((0,|\Omega|))}.$$

Therefore, letting $n \to \infty$, we deduce that

$$\mathcal{E}(\varphi) = \lim_{n\to\infty} \mathcal{E}(u^\sharp(t_n)) = \mathcal{E}_\infty, \tag{3.31}$$

In other words,

$$\mathcal{E}|_{\omega(u_o^*)} = \mathcal{E}_\infty, \tag{3.32}$$

(iii) We have to prove that φ is a stationary solution. We denote here by $v(t;w)$ the unique solution of Problem $(P^\sharp)$ together with the initial condition w. By (3.30), we have

$$v(t_n; u_0^\sharp) \to \varphi \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty.$$

It follows from Lemma 3.2.9 which expresses the continuity of the semi-group with respect to the initial data that for all $t \ge 0$,

$$v(t; v(t_n; u_0^\sharp)) \to v(t;\varphi) \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty.$$

In other words,

$$v(t + t_n; u_0^\sharp) \to v(t;\varphi) \ \text{ in } \ L^1((0,|\Omega|)) \ \text{ as } \ n \to \infty.$$

It follows that $v(t;\varphi) \in \omega(u_0^\sharp)$. This together with (3.32) implies that for all $t > s \ge 0$.

$$\mathcal{E}(v(t;\varphi)) = \mathcal{E}(v(s;\varphi)).$$

Therefore, in view of Theorem 3.4.3(i), we have

$$0 = \mathcal{E}(v(t;\varphi)) - \mathcal{E}(v(s;\varphi)) = -\int_s^t \int_0^{|\Omega|} v(t;\varphi)(1 - v(t;\varphi)) \left[v(t;\varphi) - \frac{\int_0^{|\Omega|} f(v(t;\varphi))}{\int_0^{|\Omega|} f(v(t;\varphi))}\right]^2$$

for al $t > s \ge 0$, which implies that

$$v(t;\varphi)(1 - v(t;\varphi)) \left[v(t;\varphi) - \frac{\int_0^{|\Omega|} f(v(t;\varphi))}{\int_0^{|\Omega|} f(v(t;\varphi))}\right]^2 = 0 \qquad \text{a.e in } \ (0,|\Omega|) \times \mathbb{R}^+.$$

Therefore,

$$v(t;\varphi)(1 - v(t;\varphi)) \left[v(t;\varphi) - \frac{\int_0^{|\Omega|} f(v(t;\varphi))}{\int_0^{|\Omega|} f(v(t;\varphi))}\right] = 0 \qquad \text{a.e in } \ (0,|\Omega|) \times \mathbb{R}^+,$$

and hence

$$v_t(\varphi) = 0 \qquad \text{a.e in } \ (0,|\Omega|) \times \mathbb{R}^+.$$

Thus the solution $v(t;\varphi)$ of Problem $(P^\sharp)$ with the initial function φ is independent of time. Consequently, we have that

$$\varphi(1 - \varphi) \left[\varphi - \frac{\int_0^{|\Omega|} f(\varphi)}{\int_0^{|\Omega|} h(\varphi)}\right] = 0 \qquad \text{for a.e. } \ y \in (0,|\Omega|).$$

We also note that

$$1 \leq \varphi(y) \leq s_2 \qquad \text{for a.e. } y \in (0, |\Omega|),$$

so that

$$(1-\varphi)\left[\varphi - \frac{\displaystyle\int_0^{|\Omega|} f(\varphi)}{\displaystyle\int_0^{|\Omega|} h(\varphi)}\right] = 0 \qquad \text{for a.e. } y \in (0, |\Omega|).$$

It follows that φ is a step function which takes the two values 1 and $\gamma = \gamma(\varphi)$. Moreover, since $u^\sharp(t_n)$ is nonincreasing, we deduce that the function φ which is the limit almost everywhere of $u^\sharp(t_n)$ is also nonincreasing. Therefore, φ can be represented in the form

$$\varphi = \gamma \chi_{(0,A)} + \chi_{(A,|\Omega|)}, \tag{3.33}$$

where A is a constant. Since $\varphi \geq 1$ and since $⨍_\Omega \varphi = ⨍_\Omega u_0 > 1$, we deduce that $A > 0$ and $\gamma > 1$. □

Lemma 3.4.7. *Let* $(\mathbf{H_1})$ *hold and let* $\varphi \in \omega(u_0^\sharp)$. *Then*

$$\gamma = \frac{\mathcal{E}_\infty - m_0|\Omega|}{(m_0-1)|\Omega|} > 1, \tag{3.34}$$

and

$$A = \frac{|m_0-1|^2|\Omega|^2}{\mathcal{E}_\infty - 2m_0|\Omega| + |\Omega|} > 0, \tag{3.35}$$

where $\mathcal{E}_\infty$ *is as in Lemma 3.4.6(ii). Consequently,* $\omega(u_0^\sharp)$ *only contains one element.*

Proof. First, we prove the identities (3.34) and (3.35). Note that

$$\int_0^{|\Omega|} \varphi = m_0|\Omega| \qquad \text{and} \qquad \mathcal{E}(\varphi) = \mathcal{E}_\infty.$$

Thus, we deduce from (3.33) that

$$\begin{cases} |\Omega| - A + \gamma A = m_0|\Omega| \\ |\Omega| - A + \gamma^2 A = \mathcal{E}_\infty, \end{cases}$$

or equivalently,

$$\begin{cases} (\gamma-1)A = (m_0-1)|\Omega| \\ (\gamma+1)(\gamma-1)A = \mathcal{E}_\infty - |\Omega|. \end{cases}$$

It follows that

$$\gamma = \frac{\mathcal{E}_\infty - |\Omega|}{(m_0-1)|\Omega|} - 1 = \frac{\mathcal{E}_\infty - m_0|\Omega|}{(m_0-1)|\Omega|}$$

and that

$$A = \frac{(m_0-1)|\Omega|}{(\gamma-1)} = \frac{(m_0-1)^2|\Omega|^2}{\mathcal{E}_\infty - 2m_0|\Omega| + |\Omega|}.$$

The knowledge of the constants γ and A completely determines the stationary solution φ given by (3.29) Consequently, $\omega(u_0^\sharp)$ only contains the element φ. □

Corollary 3.4.8. *Let the same assumptions as in Theorem 3.4.7 hold. Then for all* $p \in [1, +\infty)$,

$$u^\sharp(t) \to \varphi \quad in \quad L^p((0, |\Omega|)) \quad as \quad t \to \infty,$$

where φ *is determined by* (3.29), (3.34) *and* (3.35).

Proof. It immediately follows from Theorem 3.4.7 that

$$u^\sharp(t) \to \varphi \ \text{ in } \ L^1((0, |\Omega|)).$$

Since $\{u^\sharp(t)\}$ and φ are bounded in $L^\infty((0, |\Omega|))$, the convergence in every $L^p((0, |\Omega|))$ for $p \in [1, \infty)$ follows. □

3.5 Proof of Theorem 3.1.2

We have proved the stabilization of the solution of the Problem $(P^\sharp)$. We now have to deduce the stabilization of the solution of Problem (P). We need the following result.

Lemma 3.5.1. *Suppose* $(\mathbf{H_1})$ *hold. Then for all* $t, \tau \geq 0$ *we have*

$$\|u^\sharp(t) - u^\sharp(\tau)\|_{L^2((0,|\Omega|))} = \|u(t) - u(\tau)\|_{L^2(\Omega)}.$$

Proof. Define $G(s) := h^t(s)h^\tau(s)$. We apply Proposition 2.4.3(ii) in Chapter 2 for G to deduce that

$$\int_0^{|\Omega|} h^t(u_0^\sharp)h^\tau(u_0^\sharp) = \int_\Omega h^t(u_0)h^\tau(u_0).$$

or equivalently,

$$\int_0^{|\Omega|} u^\sharp(t)u^\sharp(\tau) = \int_\Omega u(t)u(\tau). \tag{3.36}$$

We apply again Proposition 2.4.3(ii) in Chapter 2 for $G(s) := s^2$ and use equality (3.36) to deduce that

$$\begin{aligned}
\|u^\sharp(t) - u^\sharp(\tau)\|^2_{L^2((0,|\Omega|))} &= \int_0^{|\Omega|} (u^\sharp(t))^2 + \int_0^{|\Omega|} (u^\sharp(\tau))^2 - 2\int_0^{|\Omega|} u^\sharp(t)u^\sharp(\tau) \\
&= \int_\Omega (u(t))^2 + \int_\Omega (u(\tau))^2 - 2\int_\Omega u(t)u(\tau) \\
&= \|u(t) - u(\tau)\|^2_{L^2(\Omega)}.
\end{aligned}$$

The proof of Lemma 3.5.1 is complete.

□

Proof of Theorem 3.1.2. It follows from Corollary 3.4.8 that

$$u^\sharp(t) \to \varphi \ \text{ in } \ L^2((0, |\Omega|)) \quad \text{ as } \ t \to \infty, \tag{3.37}$$

so that $u^\sharp(t)$ is a Cauchy sequence in $L^2((0, |\Omega|))$. This together with Lemma 3.5.1 implies that $u(t)$ is also a Cauchy sequence in $L^2(\Omega)$. Consequently, there exists $\psi \in L^2(\Omega)$ such that

$$u(t) \to \psi \ \text{ in } \ L^2(\Omega) \ \text{ as } \ t \to \infty.$$

Applying Proposition 2.4.3(iii) in Chapter 2, we have

$$u^\sharp(t) \to \psi^\sharp \ \text{ in } \ L^2((0,|\Omega|)),$$

This together with (3.37) implies that

$$\psi^\sharp = \varphi \qquad \text{a.e. in } \ (0,|\Omega|).$$

Therefore,

$$|\{\psi = s\}| = |\{\psi^\sharp = s\}| = |\{\varphi = s\}| \ \text{ for all } \ s \in \mathbb{R},$$

which in view of Theorem 3.4.7 allows us to conclude that ψ is step function which takes the two values 1 and γ. Moreover, it follows that for all $p \in [1, +\infty)$

$$u(t) \to \psi \ \text{ in } \ L^p(\Omega) \ \text{ as } \ t \to \infty.$$

□

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Chapter 4

Formal asymptotic limit of a diffuse-interface tumor-growth model

Résumé. Nous considérons un modèle d'interface diffuse pour la croissance de tumeurs, qui a la forme d'un système de champ de phase. Nous caractérisons la limite singulière de ce modèle. Plus précisément, nous prouvons formellement que quand le coefficient du terme de réaction tend vers l'infini , la solution converge vers la solution d'un problème à frontière libre limite.

Abstract. We consider a diffuse-interface tumor-growth model, which has the form of a phase-field system. We characterize the singular limit of this problem. More precisely, we formally prove that as the coefficient of the reaction term tends to infinity, the solution converges to the solution of a free boundary problem.

4.1 Introduction

4.1.1 Diffuse-interface tumor-growth models

Diffuse-interface tumor-growth models have been modeled and studied in several articles [33, 49, 20, 8, 37, 36]. We also refer to the overviews in [19, 38, 21, 42]. The basic model is composed of a fourth order parabolic equation for the tumor cell concentration $u : \Omega \to \mathbb{R}$ coupled to an elliptic equation for the nutrient concentration $\sigma : \Omega \to \mathbb{R}$:

$$u_t = \Delta(-\varepsilon^{-1} f(u) - \varepsilon \Delta u) + \varepsilon^{-1} p_0 \sigma u \tag{4.1a}$$

$$-\Delta\sigma + \varepsilon^{-1} p_0 \sigma u = 0, \tag{4.1b}$$

where ε^2 is the diffusivity corresponding to the surface energy, the positive constant p_0 is a proliferation growth parameter, and f is a bistable function.

Introducing the chemical potential $\mu : \Omega \to \mathbb{R}$, given by

$$\mu := -\varepsilon^{-1} f(u) - \varepsilon \Delta u,$$

(4.1a)-(4.1b) becomes

$$u_t = \Delta\mu + \varepsilon^{-1} p_0 \sigma u, \tag{4.2a}$$

$$\mu = -\varepsilon^{-1} f(u) - \varepsilon \Delta u, \tag{4.2b}$$

$$0 = \Delta\sigma - \varepsilon^{-1} p_0 \sigma u. \tag{4.2c}$$

The above system models the evolution of the first stage of a growing tumor [44]. In this stage a tumor grows because of the consumption of nutrients that diffuse through the surrounding tissue. This stage is referred to as avascular growth, as the tumor has not yet acquired its own blood supply to nurture itself. Consumption of nutrients is modeled in (4.2a) and (4.2c) via the reactive terms. To describe the evolution of the tumor boundary a diffuse-interface description is employed. This is classically modeled in (4.2a) with a diffusion via the chemical potential μ which depends in a nonlinear manner on u and contains the higher-order regularization $\varepsilon\Delta u$, see (4.2b).

Diffuse-interface tumor-growth models fall within the broader class of multiconstituent tumor-growth models based on continuum mixture theory, such as presented in [5, 4, 12, 34, 7]. The derivation of diffuse-interface models within continuum mixture theory has been reviewed in [42], and requires the set up of balance laws for each constituent as well as the specification of constraints on the constitutive choices imposed by the second law of thermodynamics. Typically, only the cellular and fluidic constituents of a tumor are modeled as parts of a mixture, while nutrients are considered separately. Recently however, a diffuse-interface tumor growth model has been proposed that incorporates all constituents within the mixture and is proven to be thermodynamically consistent, see [37]. In fact, the model is of gradient-flow type.

The model from [37] is a modification of (4.2) and has a natural four-constituent interpretation: a tumorous phase $u \approx 1$, a healthy cell phase $u \approx -1$, a nutrient-rich extracellular water phase $\sigma \approx 1$ and a nutrient-poor extracellular water phase $\sigma \approx 0$. It is given by

$$u_t = \Delta\mu + \varepsilon^{-1} p(u)(\sigma - \delta\mu) \tag{4.3a}$$

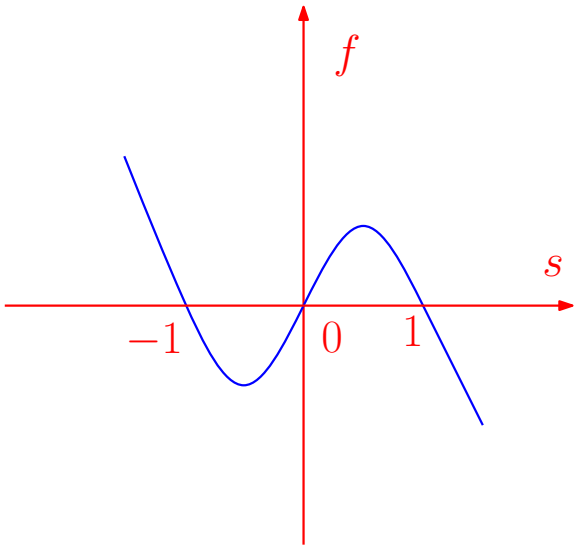
$$\mu = -\varepsilon^{-1} f(u) - \varepsilon \Delta u \tag{4.3b}$$

$$\sigma_t = \Delta\sigma - \varepsilon^{-1} p(u)(\sigma - \delta\mu) \tag{4.3c}$$

where $\delta > 0$ is a small regularization parameter, and the growth function $p(u)$ is defined by

$$p(u) := \begin{cases} 2p_0\sqrt{W(u)} & u \in [-1,1] \\ 0 & \text{elsewhere.} \end{cases}$$

Here $W(u) := -\int_{-1}^{u} f(s)\,ds$ is the classical Cahn–Hilliard double well free-energy density. We assume that the bistable function $f(u)$ has two stable roots ± 1, an unstable root 0 and mean zero: $\int_{-1}^{1} f(s)ds = 0$.



Note that, compared to (4.2a)-(4.2c), the reactive terms have been modified to be thermodynamically consistent. They include a regularization part $\delta\mu$ and they have been localized to the interface (since $p(u)$ is nonzero if $u \in (-1,1)$); see [37] for more details.

In this work, we shall be interested in the singular limit $\varepsilon \downarrow 0$ of (4.3a)-(4.3c) together with homogeneous Neumann boundary conditions. For articles involving singular limits we refer to [3, 13, 17, 14, 46, 43, 9, 39, 26, 2, 23, 48, 15, 1]; we should also mention the overviews [18, 32, 41, 40], and the numerical studies [28, 27, 29, 30, 31, 16].

The unknown pair (u, σ) is a dissipative gradient flow for the energy functional

$$\mathcal{E}(u,\sigma) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \varepsilon W(u) + \frac{\sigma^2}{2\delta}\Big).$$

We refer to Theorem 4.1.1 for the proof of this property in a slightly more general context.

4.1.2 The main results

In order to study the singular limit of Problem (4.3a)-(4.3c) as $\varepsilon \downarrow 0$, we introduce the following phase-field model

$$\begin{aligned} \alpha\mu_t^\varepsilon + u_t^\varepsilon &= \Delta\mu^\varepsilon + \varepsilon^{-1}p(u^\varepsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon) && \text{in } \Omega\times(0,+\infty), && \text{(4.4a)}\\ \varepsilon^{-1}\mu^\varepsilon - \alpha u_t^\varepsilon &= -\varepsilon^{-2}f(u^\varepsilon) - \Delta u^\varepsilon && \text{in } \Omega\times(0,+\infty), && \text{(4.4b)}\\ \sigma_t^\varepsilon &= \Delta\sigma^\varepsilon - \varepsilon^{-1}p(u^\varepsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon) && \text{in } \Omega\times(0,+\infty), && \text{(4.4c)} \end{aligned}$$

together with the boundary and initial conditions

$$\boxed{\begin{aligned} &\frac{\partial \mu^\varepsilon}{\partial \nu} = \frac{\partial u^\varepsilon}{\partial \nu} = \frac{\partial \sigma^\varepsilon}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,+\infty), && \text{(4.4d)}\\ &\alpha\mu^\varepsilon(\cdot,0) = \alpha\mu_0^\varepsilon, \quad u^\varepsilon(\cdot,0) = u_0^\varepsilon, \quad \sigma^\varepsilon(\cdot,0) = \sigma_0^\varepsilon, && \text{on } \Omega. && \text{(4.4e)} \end{aligned}}$$

Here, Ω is a smooth bounded domain of $\mathbb{R}^N (N \geq 2)$, ν is the outer unit normal vector to $\partial\Omega$ and α is a positive constant. We denote by (P_ε) the problem (4.4a)-(4.4e). Setting $\alpha = 0$ in the singular limit of Problem (P_ε), we will obtain the singular limit of Problem (4.3a)-(4.3c). Problem (P_ε) possesses the Lyaponov functional

$$E_\varepsilon(u,\mu,\sigma) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u) + \frac{\alpha\mu^2}{2} + \frac{\sigma^2}{2\delta}\Big).$$

We will prove in section 4.2 that Problem (P_ε) is a gradient flow associated to the functional $E_\varepsilon(u,\mu,\sigma)$.

Theorem 4.1.1. *Let $(u^\varepsilon,\mu^\varepsilon,\sigma^\varepsilon)$ be a smooth solution of Problem (P_ε). Then $E_\varepsilon(u^\varepsilon,\mu^\varepsilon,\sigma^\varepsilon)$ is decreasing along solution orbits.*

We will show in the following that, if in some sense

$$\mu^\varepsilon \longrightarrow \mu, \quad u^\varepsilon \longrightarrow u, \quad \sigma^\varepsilon \longrightarrow \sigma,$$

then the triple (μ,u,σ) is characterized by a limit free boundary problem, where the interface motion equation appears as the limit of the equation (4.4b). A rigorous proof of the convergence of the solution of the equation (4.4b) may for instance be found in [1]. According to [1], the function u only takes the two values -1 or 1 and the interface which separates the regions where $\{u=-1\}$ and $\{u=1\}$ partially moves according to its mean curvature.

Assumption on initial conditions: We assume that as $\varepsilon \downarrow 0$,

$$\mu_0^\varepsilon \longrightarrow \mu_0, \quad u_0^\varepsilon \longrightarrow u_0, \quad \sigma_0^\varepsilon \longrightarrow \sigma_0,$$

in some sense and that there exists a closed smooth hypersurface without boundary $\Gamma_0 \subset\subset \Omega$ which divides Ω into two subdomains $\Omega^+(0)$ and $\Omega^-(0)$ such that

$$u_0 = \begin{cases} -1 & \text{in } \Omega^-(0), \\ \\ 1 & \text{in } \Omega^+(0). \end{cases} \tag{4.5}$$

We also assume that $\Omega^+(0)$ is the region enclosed by Γ_0 and that $\Omega^-(0)$ is the region enclosed between $\partial\Omega$ and Γ_0.

Now, we are ready to introduce a free boundary problem namely the singular limit of Problem (P_ε) as $\varepsilon \downarrow 0$:

$$\boxed{\begin{aligned} u(x,t) &= \begin{cases} 1 & \text{in } \Omega^+(t), t \in (0,T] \\ -1 & \text{in } \Omega^-(t), t \in (0,T] \end{cases} && && \text{(4.6a)}\\ \alpha V_n &= -(N-1)\kappa + \mathcal{C}\mu && \text{on } \Gamma(t), t \in (0,T] && \text{(4.6b)}\\ \alpha\mu_t + u_t &= \Delta\mu + 2\sqrt{2}p_0(\sigma - \delta\mu)\delta_0(x-\Gamma(t)) && \text{in } \Omega \times (0,T], && \text{(4.6c)}\\ \sigma_t &= \Delta\sigma - 2\sqrt{2}p_0(\sigma - \delta\mu)\delta_0(x-\Gamma(t)) && \text{in } \Omega \times (0,T], && \text{(4.6d)} \end{aligned}}$$

together with the boundary and initial conditions

$$\boxed{\begin{aligned} &\frac{\partial \mu}{\partial \nu} = \frac{\partial \sigma}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T], && (4.6e)\\ &\alpha\mu(\cdot,0) = \alpha\mu_0, \quad \sigma(\cdot,0) = \sigma_0, && \text{on } \Omega, && (4.6f)\\ &\Gamma(0) = \Gamma_0, && && (4.6g) \end{aligned}}$$

Here, $\Gamma(t) \subset\subset \Omega$ is a closed hypersurface; $\Omega^+(t)$ is the region enclosed by $\Gamma(t)$; $\Omega^-(t) = \Omega \setminus (\Omega^+(t) \cup \Gamma(t))$; δ_0 is the Dirac distribution; $V_n : \Gamma(t) \to \mathbb{R}^N$ is the normal velocity of the evolving interface $\Gamma(t)$, κ is the mean curvature at each point of $\Gamma(t)$ and

$$\mathcal{C} = \left[\int_{-1}^{1} \sqrt{W(s)/2}\ \mathrm{d}s\right]^{-1}.$$

We denote by (P_0) the problem (4.6a)-(4.6g) and define

$$\Gamma_T := \bigcup_{t\in(0,T]} \Gamma(t) \times \{t\}.$$

Definition 4.1.2. *We say that the triple (Γ_T, μ, σ) is a solution of Problem (P_0) if*

(i) the set $\bigcup_{0\le t\le T} \Gamma(t) \times \{t\}$ is smooth, namely $\Gamma(t)$ is a smooth hypersurface which lies entirely within Ω for all $t \in [0,T]$ and $\Gamma(t)$ varies smoothly in time;

(ii) for all test functions

$$\psi \in \mathcal{F}_T := \{\psi \in C^{2,1}(\overline{\Omega}\times[0,T]) \ \text{ such that } \ \frac{\partial \psi}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega\times[0,T] \ \text{ and } \ \psi(T) = 0\},$$

we have

$$\int_0^T \int_\Omega (-\alpha\mu - u)\psi_t - \int_\Omega (\alpha\mu_0 + u_0)\psi(0) = \int_0^T \int_\Omega \mu\Delta\psi + 2\sqrt{2}p_0 \int_0^T \int_{\Gamma(t)} (\sigma - \delta\mu)\psi,$$

and

$$\int_0^T \int_\Omega -\sigma\psi_t - \int_\Omega \sigma_0\psi(0) = \int_0^T \int_\Omega \sigma\Delta\psi - 2\sqrt{2}p_0 \int_0^T \int_{\Gamma(t)} (\sigma - \delta\mu)\psi.$$

Now, in order to state the next result, we need some notations. Let $n^+(t), n^-(t)$ be the outer unit normal vectors to $\partial\Omega^+(t)$ and $\partial\Omega^-(t)$, respectively. Note that $n^+ = -n^-$ on Γ_T, so we may define $n := n^+ = -n^-$ on Γ_T. We define $[\![\cdot]\!]$ the jump across $\Gamma(t)$, by $[\![\phi]\!] := \phi^+ - \phi^-$, where $\phi^\pm$ should be understood as the following limit

$$\phi^\pm(\cdot) := \lim_{\rho\to 0^-} \phi(\cdot + \rho n^\pm(t)) \ \text{ on } \ \Gamma(t).$$

We also define

$$Q_T^+ := \bigcup_{t\in(0,T]} \Omega^+(t) \times \{t\}, \qquad \text{and} \qquad Q_T^- := \bigcup_{t\in(0,T]} \Omega^-(t) \times \{t\}.$$

Theorem 4.1.3. *Assume that Problem* (P_0) *possesses a solution* (Γ_T, μ, σ) *such that* Γ_T *is smooth on the time interval* $(0,T)$ *and that* μ *and* σ *are smooth up to* Γ_T *on both sides of* Γ_T*. Then the triple* (Γ_T, μ, σ) *satisfies:*

$$\begin{aligned}
\alpha\mu_t &= \Delta\mu && \text{in } Q_T^+ \cup Q_T^-, && (4.7a)\\
\sigma_t &= \Delta\sigma && \text{on } Q_T^+ \cup Q_T^-, && (4.7b)\\
[\![\mu]\!] &= [\![\sigma]\!] = 0 && \text{on } \Gamma_T, && (4.7c)\\
[\![\frac{\partial\mu}{\partial n}]\!] &= -2V_n + 2\sqrt{2}p_0(\sigma - \delta\mu) && \text{on } \Gamma_T, && (4.7d)\\
[\![\frac{\partial\sigma}{\partial n}]\!] &= -2\sqrt{2}p_0(\sigma - \delta\mu) && \text{on } \Gamma_T, && (4.7e)\\
\alpha V_n &= -(N-1)\kappa + \mathcal{C}\mu && \text{on } \Gamma_T, && (4.7f)
\end{aligned}$$

together with the boundary and initial conditions

$$\begin{aligned}
&\frac{\partial\mu}{\partial\nu} = \frac{\partial\sigma}{\partial\nu} = 0 && \text{on } \partial\Omega \times (0,T], && (4.7g)\\
&\mu(\cdot,0) = \mu_0, \quad \sigma(\cdot,0) = \sigma_0, && \text{on } \Omega, && (4.7h)\\
&\Gamma(0) = \Gamma_0. && && (4.7i)
\end{aligned}$$

In this case, we say that (Γ_T, μ, σ) *is a classical solution of Problem* (P_0) *on the time interval* $[0,T]$*.*

Problem (P_0) possesses the Lyapunov functional

$$E(\Gamma, \mu, \sigma) := \frac{2}{\mathcal{C}}\int_\Gamma 1 \,\mathrm{d}\Gamma + \int_\Omega \Big(\frac{\alpha\mu^2}{2} + \frac{\sigma^2}{2\delta}\Big),$$

which is analogous to the Lyapunov functional satisfied by Problem (P_ε).

Theorem 4.1.4. *Let* (Γ_T, μ, σ) *be a classical solution of Problem* (P_0)*. Then* $E(\Gamma, \mu, \sigma)$ *is decreasing along solution orbits.*

Finally, we will formally prove the following result.

Theorem 4.1.5. *Let* $(\mu^\varepsilon, u^\varepsilon, \sigma^\varepsilon)$ *be solution of Problem* (P_ε)*. We suppose that Problem* (P_0) *possesses a unique classical solution on the interval* $[0,T]$*. If* $\varepsilon \to 0$*,*

$$\mu^\varepsilon \longrightarrow \mu, \quad u^\varepsilon \longrightarrow u, \quad \sigma^\varepsilon \longrightarrow \sigma \quad \textit{in a strong enough sense,}$$

then (Γ_T, μ, σ) *coincide with the classical solution of Problem* (P_0) *and* u *is given by* (4.6a)*.*

We note that the singular limit corresponds to a moving boundary problem which is similar to other sharp-interface tumor-growth models [35, 22, 10, 11].

The remainder of the paper is organized as follows: in section 2 we prove Theorem 4.1.1 and Theorem 4.1.3; in section 3, we formally justify Theorem 4.1.5.

4.2 Proof of the main results

4.2.1 Proof of Theorem 4.1.1

It is sufficient to prove that

$$\frac{d}{dt}E_\varepsilon(u^\varepsilon,\mu^\varepsilon,\sigma^\varepsilon)\le 0. \tag{4.8}$$

For simplicity, we write u,μ,σ instead of $u^\varepsilon,\mu^\varepsilon,\sigma^\varepsilon$. Now, the inequality (4.8) follows from the following computations:

$$\begin{aligned}
\frac{d}{dt}E_\varepsilon(u,\mu,\sigma) &= \int_\Omega \Big(\varepsilon\nabla u\nabla u_t+\varepsilon^{-1}W'(u)u_t+\alpha\mu\mu_t+\frac{\sigma\sigma_t}{\delta}\Big)\\
&= \int_\Omega \Big([-\varepsilon^{-1}f(u)-\varepsilon\Delta u]u_t+\alpha\mu\mu_t+\frac{\sigma\sigma_t}{\delta}\Big)\\
&= \int_\Omega \Big((\mu-\alpha\varepsilon u_t)u_t+\alpha\mu\mu_t\Big)+\int_\Omega\frac{\sigma}{\delta}\Big(\Delta\sigma-\varepsilon^{-1}p(u)(\sigma-\delta\mu)\Big)\\
&= -\int_\Omega \alpha\varepsilon u_t^2+\int_\Omega\mu(u_t+\alpha\mu_t)+\int_\Omega\frac{\sigma}{\delta}\Big(\Delta\sigma-\varepsilon^{-1}p(u)(\sigma-\delta\mu)\Big)\\
&= -\int_\Omega \alpha\varepsilon u_t^2+\int_\Omega\mu\Big(\Delta\mu+\varepsilon^{-1}p(u)(\sigma-\delta\mu)\Big)+\int_\Omega\frac{\sigma}{\delta}\Big(\Delta\sigma-\varepsilon^{-1}p(u)(\sigma-\delta\mu)\Big)\\
&= -\int_\Omega \alpha\varepsilon u_t^2-\int_\Omega|\nabla\mu|^2-\int_\Omega\frac{|\nabla\sigma|^2}{\delta}+\varepsilon^{-1}\int_\Omega p(u)\Big(\mu(\sigma-\delta\mu)-\frac{\sigma}{\delta}(\sigma-\delta\mu)\Big)\\
&= -\int_\Omega \alpha\varepsilon u_t^2-\int_\Omega|\nabla\mu|^2-\int_\Omega\frac{|\nabla\sigma|^2}{\delta}-\varepsilon^{-1}\int_\Omega p(u)\Big(\sqrt{\delta}\mu-\frac{\sigma}{\sqrt{\delta}}\Big)^2\le 0.
\end{aligned}$$

4.2.2 Proof of Theorem 4.1.3

First, we recall that $n^+(t), n^-(t)$ are the outer unit normal vectors to $\partial\Omega^+(t)$ and $\partial\Omega^-(t)$, respectively and $n := n^+ = -n^-$ on Γ_T. We define $V_n = V.n_+$, where V is the velocity of displacement of the interface Γ_T.

Equations for μ

We recall that u,μ satisfy

$$\int_0^T\int_\Omega(-\alpha\mu-u)\psi_t-\int_\Omega(\alpha\mu_0+u_0)\psi(0)=\int_0^T\int_\Omega\mu\Delta\psi+2\sqrt{2}p_0\int_0^T\int_{\Gamma(t)}(\sigma-\delta\mu)\psi, \tag{4.9}$$

for all $\psi\in\mathcal{F}_T$. We define the terms A_1, A_2 and the diffusion term B by

$$A_1:=\int_0^T\int_\Omega-\alpha\mu\psi_t,\quad A_2:=\int_0^T\int_\Omega-u\psi_t,\quad\text{and}\quad B:=\int_0^T\int_\Omega\mu\Delta\psi.$$

Analysis of the terms $\mathbf{A_1}$ and $\mathbf{A_2}$: Our analysis of the terms A_1 and A_2 relies on the Reynolds transport theorem, by which we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm(t)}\phi\psi=\int_{\Omega^\pm(t)}\Big(\phi_t\psi+\phi\psi_t\Big)\pm\int_{\Gamma(t)}V_n\phi^\pm\psi,$$

for all smooth function ψ and for function ϕ. These equations for the integrals over $\Omega^{\pm}(t)$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^+(t)\cup\Omega^-(t)}\phi\psi = \int_{\Omega^+(t)\cup\Omega^-(t)}\Big(\phi_t\psi + \phi\psi_t\Big) + \int_{\Gamma(t)} V_n[\![\phi]\!]\psi.$$

Hence we have

$$\int_{\Omega^+(t)\cup\Omega^-(t)} -\phi\psi_t = \int_{\Omega^+(t)\cup\Omega^-(t)}\phi_t\psi + \int_{\Gamma(t)} V_n[\![\phi]\!]\psi - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^+(t)\cup\Omega^-(t)}\phi\psi. \tag{4.10}$$

In our case, we choose $\phi := \alpha\mu$ in (4.10) and integrate from 0 to T. This yields

$$\begin{aligned} A_1 &= \int_0^T\int_{\Omega^+(t)\cup\Omega^-(t)}\alpha\mu_t\psi + \int_0^T\int_{\Gamma(t)} V_n[\![\alpha\mu]\!]\psi - \int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^+(t)\cup\Omega^-(t)}\alpha\mu\psi \\ &= \int_0^T\int_{\Omega^+(t)\cup\Omega^-(t)}\alpha\mu_t\psi + \int_0^T\int_{\Gamma(t)}\alpha V_n[\![\mu]\!]\psi + \int_{\Omega^+(0)\cup\Omega^-(0)}\alpha\mu(0)\psi(0). \end{aligned} \tag{4.11}$$

Similarly, we apply the formula (4.10) for $\phi := u$ to obtain

$$\begin{aligned} A_2 &= \int_0^T\int_{\Omega^+(t)\cup\Omega^-(t)} u_t\psi + \int_0^T\int_{\Gamma(t)} V_n[\![u]\!]\psi + \int_{\Omega^+(0)\cup\Omega^-(0)} u(0)\psi(0) \\ &= 2\int_0^T\int_{\Gamma(t)} V_n\psi + \int_{\Omega^+(0)\cup\Omega^-(0)} u(0)\psi(0). \end{aligned} \tag{4.12}$$

Analysis of the term B: We write B as the sum

$$B = \int_0^T\Big(\int_{\Omega^+(t)}\mu\Delta\psi + \int_{\Omega^-(t)}\mu\Delta\psi\Big).$$

Integration by parts yields

$$\begin{aligned} \int_{\Omega^-(t)}\mu\Delta\psi &= -\int_{\Omega^-(t)}\nabla\mu\nabla\psi + \int_{\Gamma(t)}\frac{\partial\psi}{\partial n^-}\mu^- \\ &= \int_{\Omega^-(t)}\Delta\mu\psi - \int_{\Gamma(t)}\frac{\partial\mu^-}{\partial n^-}\psi - \int_{\partial\Omega}\frac{\partial\mu}{\partial\nu}\psi + \int_{\Gamma(t)}\frac{\partial\psi}{\partial n^-}\mu^- \\ &= \int_{\Omega^-(t)}\Delta\mu\psi + \int_{\Gamma(t)}\frac{\partial\mu^-}{\partial n}\psi - \int_{\partial\Omega}\frac{\partial\mu}{\partial\nu}\psi - \int_{\Gamma(t)}\frac{\partial\psi}{\partial n}\mu^- \end{aligned}$$

and

$$\int_{\Omega^+(t)}\mu\Delta\psi = \int_{\Omega^+(t)}\Delta\mu\psi - \int_{\Gamma(t)}\frac{\partial\mu^+}{\partial n}\psi + \int_{\Gamma(t)}\frac{\partial\psi}{\partial n}\mu^+$$

which implies that

$$\int_{\Omega^+(t)\cup\Omega^-(t)}\mu\Delta\psi = \int_{\Omega^+(t)\cup\Omega^-(t)}\Delta\mu\psi - \int_{\Gamma(t)}[\![\frac{\partial\mu}{\partial n}]\!]\psi + \int_{\Gamma(t)}\frac{\partial\psi}{\partial n}[\![\mu]\!] - \int_{\partial\Omega}\frac{\partial\mu}{\partial\nu}\psi.$$

Integrating this identity from 0 to T, we obtain

$$B = \int_0^T\int_{\Omega^+(t)\cup\Omega^-(t)}\Delta\mu\psi - \int_0^T\int_{\Gamma(t)}[\![\frac{\partial\mu}{\partial n}]\!]\psi + \int_0^T\int_{\Gamma(t)}\frac{\partial\psi}{\partial n}[\![\mu]\!] - \int_0^T\int_{\partial\Omega}\frac{\partial\mu}{\partial\nu}\psi. \tag{4.13}$$

Conclusion: Combining (4.9), (4.11) (4.12) and (4.13), we then have for all $\psi \in \mathcal{F}_T$,

$$\begin{aligned}\int_0^T \int_{\Omega^+(t)\cup\Omega^-(t)} \alpha\mu_t\psi + \int_0^T \int_{\Gamma(t)} V_n(\alpha[\![\mu]\!]+2)\psi \\ + \int_{\Omega^+(t)\cup\Omega^-(t)} \alpha(\mu(0)-\mu_0)\psi(0) + \int_{\Omega^+(t)\cup\Omega^-(t)} (u(0)-u_0)\psi(0) \\ = \int_0^T \int_{\Omega^+(t)\cup\Omega^-(t)} \Delta\mu\,\psi - \int_0^T \int_{\Gamma(t)} [\![\frac{\partial\mu}{\partial n}]\!]\psi + \int_0^T \int_{\Gamma(t)} \frac{\partial\psi}{\partial n}[\![\mu]\!] - \int_0^T \int_{\partial\Omega} \frac{\partial\mu}{\partial\nu}\psi \\ + \int_0^T \int_{\Gamma(t)} 2\sqrt{2}p_0(\sigma-\delta\mu)\psi.\end{aligned} \tag{4.14}$$

By using test functions with suitable supports, namely $\psi \in C_0^\infty(Q_T^+)$ and $\psi \in C_0^\infty(Q_T^-)$, we deduce that

$$\alpha\mu_t = \Delta\mu \ \text{ in } Q_T^+ \cup Q_T^-. \tag{5b}$$

Similarly, by taking $\psi \in C_0^\infty(Q_T)$ such that $\dfrac{\partial\psi}{\partial n} = 0$ on Γ_T (we refer to Remark 4.2.1 below for the construction of such functions), we obtain

$$V_n(2+\alpha[\![\mu]\!]) = -[\![\frac{\partial\mu}{\partial n}]\!] + 2\sqrt{2}p_0(\sigma-\delta\mu) \ \text{ on } \ \Gamma_T. \tag{4.15}$$

Now, we take $\psi \in C_0^\infty(Q_T)$ to deduce that

$$\int_0^T \int_{\Gamma(t)} \frac{\partial\psi}{\partial n}[\![\mu]\!] \quad = 0 \ \text{ for all } \ \psi \in C_0^\infty(Q_T). \tag{4.16}$$

Therefore,

$$[\![\mu]\!] = 0 \ \text{ on } \ \Gamma_T. \tag{4.17}$$

It follows from (4.15) and (4.17) that

$$2V_n = -[\![\frac{\partial\mu}{\partial n}]\!] + 2\sqrt{2}p_0(\sigma-\delta\mu) \qquad \text{on } \ \Gamma_T.$$

Now, for the initial conditions, we use the test function $\psi \in \mathcal{F}_T$ such that $\psi = 0$ on $\partial\Omega\times(0,T)$ to obtain

$$u(0) + \alpha\mu(0) = u_0 + \alpha\mu_0,$$

which in view of (4.5) implies that

$$\mu(0) = \mu_0, \quad u(0) = u_0$$

Finally, the remaining term in (4.14) allows us to conclude that

$$\frac{\partial\mu}{\partial\nu} = 0.$$

Therefore, μ satisfies the equations:

$$\begin{aligned}&\alpha\mu_t = \Delta\mu && \text{in } Q_T^+ \cup Q_T^-,\\ &2V_n = -[\![\frac{\partial\mu}{\partial n}]\!] + 2\sqrt{2}p_0(\sigma-\delta\mu) && \text{on } \Gamma_T,\\ &[\![\mu]\!] = 0 && \text{on } \ \Gamma_T,\end{aligned}$$

together with the boundary condition and the initial condition:

$$\frac{\partial \mu}{\partial \nu} = 0, \quad \mu(0) = \mu_0.$$

Remark 4.2.1. *Let $\widetilde{\psi} \in C_0^\infty(Q_T)$. In the following, we construct a function $\psi \in C_0^\infty(Q_T)$ such that*

$$\frac{\partial \psi}{\partial n}|_{\Gamma_T} = 0 \qquad \text{and} \quad \psi = \widetilde{\psi} \qquad \text{on} \qquad \Gamma_T. \tag{4.18}$$

Let $\widetilde{d}(x,t)$ be the signed distance function to $\Gamma(t)$ (see (4.23) below). Since $\bigcup_{0\le t\le T} \Gamma(t)\times\{t\}$ is smooth, there exists δ small enough such that $\widetilde{d}(x,t)$ is smooth in

$$\mathcal{V} := \{(x,t) \in \Omega \times [0,T], |\widetilde{d}(x,t)| < \delta\};$$

moreover for all $(x,t) \in \mathcal{V}$ there exists a unique (y,t) in $\Gamma(t)\times\{t\}$ such that $|\widetilde{d}(x,t)| = |x-y|$ and $(\nabla \widetilde{d}(x,t),t) = n(y,t)$. More precisely,

$$(y,t) = J(x,t) := (x - \nabla \widetilde{d}(x,t)\widetilde{d}(x,t), t),$$

where the projection operator J is a smooth map from $\mathcal{V}$ into $\mathbb{R}^{N+1}$. We define ψ on $\mathcal{V}$ by

$$\psi(\cdot) := \widetilde{\psi}(\mathcal{J}(\cdot)).$$

Then ψ is smooth on $\mathcal{V}$ and (4.18) holds. Moreover, we can extend ψ to a smooth function on Q_T.

Equations for σ

Since the computations in this section are similar to the previous ones, we will only give a sketch of the necessary steps. For $\psi \in C_0^\infty(Q_T)$, we have

$$\int_0^T \int_\Omega -\sigma\psi_t = \int_0^T \int_\Omega \sigma\Delta\psi - 2\sqrt{2}p_0 \int_0^T \int_{\Gamma(t)} (\sigma - \delta\mu)\psi. \tag{4.19}$$

We define two terms

$$C := \int_0^T \int_\Omega -\sigma\psi_t \quad \text{and} \quad D := \int_0^T \int_\Omega \sigma\Delta\psi.$$

One can easily deduce that

$$C = \int_0^T \int_{\Omega^+(t)\cup\Omega^-(t)} \sigma_t\psi + \int_0^T \int_{\Gamma(t)} V_n[\![\sigma]\!]\psi,$$

and

$$D = \int_0^T \int_{\Omega^+(t)\cup\Omega^-(t)} \Delta\sigma\psi - \int_0^T \int_{\Gamma(t)} [\![\frac{\partial\sigma}{\partial n}]\!]\psi + \int_0^T \int_{\Gamma(t)} \frac{\partial\psi}{\partial n}[\![\sigma]\!].$$

It follows that

$$\begin{aligned}\int_0^T \int_{\Omega^+(t)\cup\Omega^-(t)} \sigma_t\psi + \int_0^T \int_{\Gamma(t)} V_n[\![\sigma]\!]\psi = \int_0^T \int_{\Omega^+(t)\cup\Omega^-(t)} \Delta\sigma\psi \\ - \int_0^T \int_{\Gamma(t)} [\![\frac{\partial\sigma}{\partial n}]\!]\psi + \int_0^T \int_{\Gamma(t)} \frac{\partial\psi}{\partial n}[\![\sigma]\!] - 2\sqrt{2}p_0 \int_0^T \int_{\Gamma(t)} (\sigma - \delta\mu)\psi.\end{aligned}$$

and hence we have

$$\begin{aligned} \sigma_t &= \Delta\sigma && \text{in } Q_T^+ \cup Q_T^-, \\ [\![\frac{\partial\sigma}{\partial n}]\!] &= -2\sqrt{2}p_0(\sigma - \delta\mu) && \text{on } \Gamma_T, \\ [\![\sigma]\!] &= 0 && \text{on } \Gamma_T, \end{aligned}$$

This concludes the proof of Theorem 4.1.3

4.2.3 Proof of Theorem 4.1.4

We prove below that

$$\frac{d}{dt}E(\Gamma,\mu,\sigma) \le 0. \tag{4.20}$$

The inequality (4.20) follows from the following computations and [24, Theorem 4.3 p.355 and formula 4.12 p.356]):

$$\begin{aligned} &\frac{d}{dt}E(\Gamma,\mu,\sigma) \\ &= \frac{2}{\mathcal{C}}\int_\Gamma (N-1)\kappa\, V_n + \int_{\Omega^+\cup\Omega^-}\Big(\alpha\mu\mu_t + \delta^{-1}\sigma\sigma_t\Big) \\ &= \frac{2}{\mathcal{C}}\int_\Gamma [\mathcal{C}\mu - \alpha V_n]\, V_n + \int_{\Omega^+\cup\Omega^-}\Big(\alpha\mu\mu_t + \delta^{-1}\sigma\sigma_t\Big) \\ &= \int_\Gamma 2\mu V_n - \int_\Gamma \frac{2\alpha}{\mathcal{C}}V_n^2 + \int_{\Omega^+\cup\Omega^-}\Big(\mu\Delta\mu + \delta^{-1}\sigma\Delta\sigma\Big) \\ &= \int_\Gamma \mu\Big(-[\![\frac{\partial\mu}{\partial n}]\!] + 2\sqrt{2}p_0(\sigma-\delta\mu)\Big) - \int_\Gamma \frac{2\alpha}{\mathcal{C}}V_n^2 + \int_{\Omega^+\cup\Omega^-}\Big(\mu\Delta\mu + \delta^{-1}\sigma\Delta\sigma\Big) \\ &= \int_\Gamma 2\sqrt{2}p_0\mu(\sigma-\delta\mu) - \int_\Gamma \frac{2\alpha}{\mathcal{C}}V_n^2 - \int_{\Omega^+\cup\Omega^-}|\nabla\mu|^2 + \int_{\Omega^+\cup\Omega^-}\delta^{-1}\sigma\Delta\sigma \\ &= \int_\Gamma 2\sqrt{2}p_0\mu(\sigma-\delta\mu) - \int_\Gamma \frac{2\alpha}{\mathcal{C}}V_n^2 - \int_{\Omega^+\cup\Omega^-}|\nabla\mu|^2 - \int_{\Omega^+\cup\Omega^-}\frac{|\nabla\sigma|^2}{\delta} + \int_\Gamma \delta^{-1}\sigma[\![\frac{\partial\sigma}{\partial n}]\!] \end{aligned}$$

which in turn implies that

$$\begin{aligned} &\frac{d}{dt}E(\Gamma,\mu,\sigma) \\ &= \int_\Gamma 2\sqrt{2}p_0\Big(\mu(\sigma-\delta\mu) - \delta^{-1}\sigma(\sigma-\delta\mu)\Big) - \int_\Gamma \frac{2\alpha}{\mathcal{C}}V_n^2 - \int_{\Omega^+\cup\Omega^-}|\nabla\mu|^2 - \int_{\Omega^+\cup\Omega^-}\frac{|\nabla\sigma|^2}{\delta} \\ &= -\int_\Gamma \frac{2\alpha}{\mathcal{C}}V_n^2 - \int_{\Omega^+\cup\Omega^-}|\nabla\mu|^2 - \int_{\Omega^+\cup\Omega^-}\frac{|\nabla\sigma|^2}{\delta} - \int_\Gamma 2\sqrt{2}p_0\Big(\sqrt{\delta}\mu - \frac{\sigma}{\sqrt{\delta}}\Big)^2 \le 0. \end{aligned}$$

4.3 Formal derivation of Theorem 4.1.5

This section is devoted to prove formally theorem 4.1.5. We shall derive in turn equations for $u, \Gamma(t), \mu, \sigma$.

4.3.1 Equation for u

First, we formally show that u only takes two values ± 1. To that purpose, we rewrite Equation (4.4b) in the form

$$\alpha u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} f(u^\varepsilon) + \varepsilon^{-1}\mu^\varepsilon.$$

By setting $\tau := t/\varepsilon^2$, we obtain

$$\alpha u_\tau^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) + \varepsilon\mu^\varepsilon.$$

When ε is small, we neglect the effect of diffusion term $\varepsilon^2\Delta u^\varepsilon$ and of the term $\varepsilon\mu^\varepsilon$ with respect to the term $f(u^\varepsilon)$, which yields the ordinary differential equation

$$\alpha \frac{du^\varepsilon}{d\tau} \cong f(u^\varepsilon). \tag{4.21}$$

Note that $\tau \to \infty$ as $\varepsilon \to 0$. Remembering that ± 1 are two stable zeros of this equation. We formally deduce that as $\varepsilon \downarrow 0$

$$\begin{cases} u^\varepsilon(x,t) \ \text{ approaches } \ -1 & \text{if } \ u^\varepsilon(x,0) < 0 \\ \\ u^\varepsilon(x,t) \ \text{ approaches } \ \ 1 & \text{if } \ u^\varepsilon(x,0) > 0. \end{cases} \tag{4.22}$$

This implies that the function u which is the limit of u^ε only takes two values ± 1.

4.3.2 Formal derivation of the interface equation

We define

$$\Omega^-(t) = \{x \in \Omega : \ u(x,t) = -1\}, \qquad \Omega^+(t) = \{x \in \Omega : \ u(x,t) = 1\},$$

and

$$\Gamma(t) := \Omega \backslash (\Omega^-(t) \cup \Omega^+(t)).$$

Since roughly speaking, the regions $\{u = -1\}$ and $\{u = 1\}$ are the "limit" of the regions $\{u^\varepsilon \approx -1\}$ and $\{u^\varepsilon \approx 1\}$ as $\varepsilon \to 0$, $\Gamma(t)$ can be considered as the limit as $\varepsilon \to 0$ of $\Gamma^\varepsilon(t)$ which is the interface between the two regions

$$\{x \in \Omega : \quad u^\varepsilon(x,t) \approx -1\} \quad \text{and} \quad \{x \in \Omega : \quad u^\varepsilon(x,t) \approx 1\}.$$

We recall that 0 is an unstable equilibria of Equation (4.21), and define

$$\Gamma^\varepsilon(t) = \{x \in \Omega : \quad u^\varepsilon(x,t) = 0\} \quad \text{for each } \ t \geq 0.$$

In what follows, we will use an formal asymptotic expansion to derive the equation describing $\Gamma(t)$. We need some preparations.

1. *Signed distance function:* We assume that the interface $\Gamma(t)$ is a smooth, closed hypersurface without boundary of $\mathbb{R}^N$. Further, we suppose that $\Omega^+(t)$ is the region enclosed by $\Gamma(t)$ and that $\Omega^-(t)$ is the region enclosed between $\partial\Omega$ and $\Gamma(t)$. Let $\tilde{d}(x,t)$ be the signed distance function to $\Gamma(t)$ defined by

$$\tilde{d}(x,t) = \begin{cases} \operatorname{dist}(x, \Gamma(t)) & \text{for } \ x \in \Omega^-(t), \\ \\ -\operatorname{dist}(x, \Gamma(t)) & \text{elsewhere.} \end{cases} \tag{4.23}$$

Note that $\tilde{d} = 0$ on Γ_T and $|\nabla \tilde{d}| = 1$ in a neighborhood of Γ_T.

2. *Outer expansion:* It is reasonable to assume that outside a neighbourhood of Γ_T, u^ε has the expansion

$$u^\varepsilon(x,t) = \pm 1 + \varepsilon u_1^\pm(x,t) + \varepsilon^2 u_2^\pm(x,t) + \dots \tag{4.24}$$

3. *Inner expansion:* Near Γ_T, we assume that u^ε has form

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \dots \tag{4.25}$$

Here $U_j(x,t,z), j \geq 0$ are defined for $x \in \Omega$, $t \geq 0, z \in \mathbb{R}$ and $\xi := \tilde{d}(x,t)/\varepsilon$.

4. *Normalization conditions:* The stretched space variable ξ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{u^\varepsilon \approx -1\}$ and $\{u^\varepsilon \approx 1\}$. We normalize U_0 in such a way that

$$U_0(x,t,0) = 0.$$

5. *Matching conditions:* For $\xi \to \pm\infty$, we require two expansions (4.24) and (4.25) to be consistent, i.e.

$$U_0(x,t,-\infty) = 1, \qquad U_0(x,t,+\infty) = -1;$$

and

$$U_k(x,t,-\infty) = u_k^+(x,t), \qquad U_k(x,t,+\infty) = u_k^-(x,t)$$

for all $k \geq 1$.

Formal interface motion equation We will substitute the inner expansion (4.25) into (4.4b). We will then compare the terms of the same order to determine equations of U_0 and U_1. To that purpose, we start by some computations.

$$\begin{aligned}
u_t^\varepsilon &= U_{0t} + U_{0z}\frac{\widetilde{d}_t}{\varepsilon} + \varepsilon U_{1t} + U_{1z}\widetilde{d}_t + \dots,\\
\nabla u^\varepsilon &= \nabla U_0 + U_{0z}\frac{\nabla\widetilde{d}}{\varepsilon} + \varepsilon\nabla U_1 + U_{1z}\nabla\widetilde{d} + \dots,\\
\Delta u^\varepsilon &= \Delta U_0 + 2\frac{\nabla\widetilde{d}}{\varepsilon}\cdot\nabla U_{0z} + U_{0z}\frac{\Delta\widetilde{d}}{\varepsilon} + U_{0zz}\frac{|\nabla\widetilde{d}|}{\varepsilon^2} + \varepsilon\Delta U_1\\
&\qquad + 2\nabla\widetilde{d}\cdot\nabla U_{1z} + U_{1z}\Delta\widetilde{d} + U_{1zz}\frac{|\nabla\widetilde{d}|}{\varepsilon} + \dots,\\
f(u^\varepsilon) &= f(U_0) + \varepsilon f'(U_0)U_1 + O(\varepsilon^2),\\
\mu^\varepsilon &= \mu + O(\varepsilon).
\end{aligned}$$

Substituting $u_t^\varepsilon, \Delta u^\varepsilon, f(u^\varepsilon), \mu^\varepsilon$ in (4.4b), collecting all terms of order ε^{-2} then yields

$$\begin{cases} U_{0zz} + f(U_0) = 0 \\ \\ U_0(-\infty) = 1, \quad U_0(0) = 0, \quad U_0(+\infty) = -1. \end{cases} \tag{4.26}$$

This problem has a unique solution U_0. Furthermore, U_0 is independent of (x,t), i.e. $U_0(x,t,z) = U_0(z)$ and thus, we write U_0', U_0'' instead of U_{0z}, U_{0zz}. We have the following lemma.

Lemma 4.3.1. *The solution U_0 of equation (4.26) also fulfills the differential equation*

$$U_0' = -\sqrt{2W(U_0)}.$$

As a consequence, $\int_{\mathbb{R}}(U_0'(z))^2\,dz$ can be written in the form:

$$\int_{\mathbb{R}}(U_0'(z))^2\,dz = \sqrt{2}\int_{-1}^{1}\sqrt{W(s)}\,ds.$$

Proof. Multiplying the above mentioned differential equation (4.26) for U_0 by U_0', we get

$$U_0''U_0' + f(U_0)U_0' = 0. \tag{4.27}$$

Keeping in mind that $W'(u) = -f(u)$, (4.27) can be read as

$$\left(\frac{(U_0')^2}{2}\right)' - (W(U_0))' = 0. \tag{4.28}$$

Integrating this equation from $-\infty$ to z, we obtain

$$\frac{(U_0'(z))^2}{2} = W(U_0(z)). \tag{4.29}$$

Moreover, U_0 is non increasing, therefore, we deduce that

$$U_0'(z) = -\sqrt{2W(U_0(z))}.$$

Consequently, we have

$$\int_{\mathbb{R}}(U_0'(z))^2\,dz = -\int_{\mathbb{R}}U_0'(z)\sqrt{2W(U_0(z))}\,dz = \sqrt{2}\int_{-1}^{1}\sqrt{W(s)}\,ds.$$

This completes the proof of Lemma 4.3.1. □

We now collect the terms of order ε^{-1} in the substituted equation (4.4b). Because we have $|\nabla\widetilde{d}| = 1$ in a neighbourhood of $\Gamma(t)$, we obtain

$$U_{1zz} + f'(U_0)U_1 = U_0'(\alpha\widetilde{d}_t - \Delta\widetilde{d}) - \mu. \tag{4.30}$$

A solvability condition for this equation is given by the following lemma.

Lemma 4.3.2 (see [1, Lemma 2.2])**.** *Let $A(z)$ be a bounded function for $z \in \mathbb{R}$. Then the existence of a solution ϕ for the problem*

$$\begin{cases}\phi_{zz} + f'(U_0(z))\phi = A(z)\ ,\ z \in \mathbb{R}\\ \phi(0) = 0, \phi \in L^\infty(\mathbb{R})\end{cases} \tag{4.31}$$

is equivalent to

$$\int_{\mathbb{R}}A(z)U_0'(z)\,dz = 0. \tag{4.32}$$

Therefore, the existence of a solution U_1 of (4.30) is equivalent to

$$\int_{\mathbb{R}} \Big[(U_0')^2(z)(\alpha\widetilde{d}_t - \Delta\widetilde{d})(x,t) - \mu(x,t)U_0'(z) \Big]\, dz = 0 \tag{4.33}$$

for all (x,t) in a neighbourhood of the interface Γ_T. Thus,

$$(\alpha\widetilde{d}_t - \Delta\widetilde{d})(x,t) = \frac{\mu(x,t)\int_{\mathbb{R}} U_0'(z)\,dz}{\int_{\mathbb{R}} (U_0'(z))^2\, dz} = -\frac{2\mu(x,t)}{\int_{\mathbb{R}} (U_0'(z))^2\, dz}. \tag{4.34}$$

It follows from Lemma 4.3.1 that

$$(\alpha\widetilde{d}_t - \Delta\widetilde{d})(x,t) = -\frac{\sqrt{2}\mu(x,t)}{\int_{-1}^{1}\sqrt{W(s)}\,ds}. \tag{4.35}$$

Note that, on $\Gamma(t)$ we have $n = n^+|_\Gamma = \nabla\widetilde{d}$, $\kappa = \dfrac{\operatorname{div}(n)}{N-1} = \dfrac{\Delta\widetilde{d}}{N-1}$, and $\tilde{d}_t = -V_n$. Therefore, we deduce that $\Gamma(t)$ satisfies indeed the interface motion equation (4.6b):

$$\boxed{\alpha V_n = -(N-1)\kappa + \frac{\sqrt{2}\mu}{\int_{-1}^{1}\sqrt{W(s)}\,ds} = -(N-1)\kappa + \mathcal{C}\mu \quad \text{on } \Gamma_T,}$$

where $\mathcal{C} := \Big[\int_{-1}^{1}\sqrt{W(s)/2}\,ds\Big]^{-1}$.

4.3.3 Equations for μ, σ

We will suppose that the following convergence holds in a strong enough sense:

$$\mu^\varepsilon \longrightarrow \mu, \qquad\qquad \sigma^\varepsilon \longrightarrow \sigma$$

as $\varepsilon \downarrow 0$ and derive the limit of the reaction term in (4.4a) and (4.4c). To that purpose, we first prove a stronger version of Lemma 2.1 by Du et al. [25] (see also [6, 39]).

Lemma 4.3.3. *Let $\gamma \subset\subset \Omega$ be a smooth hypersurface without boundary, d be the signed distance to γ, and let $g \in L^1(\mathbb{R})$. Furthermore, let $\phi^\varepsilon \in L^\infty(\Omega)$ and $V \subset \Omega$ be a neighborhood γ such that*

$$\begin{aligned} &\|\phi^\varepsilon\|_{L^\infty(\Omega)} \le C, \\ &\phi^\varepsilon \quad \textit{is continuous on } V, \\ &\phi^\varepsilon \longrightarrow \phi \qquad \textit{uniformly in } V. \end{aligned}$$

We then have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_U g\big(d(x)/\varepsilon\big)\phi^\varepsilon(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(\tau)\,\mathrm{d}\tau \int_\gamma \phi$$

for a small enough neighborhood $U \subset V$ of γ.

Proof. For simplicity, we prove this lemma in the three-dimensional case and assume that the hypersurface γ has a parametrization α. More precisely, we assume that there exists an open set W of $\mathbb{R}^2$ such that the mapping α from $\overline{W}$ onto γ is smooth and that α^{-1} is continuous from γ onto $\overline{W}$. We write the function α as

$$\alpha(z_1, z_2) = (\alpha_1(z_1, z_2), \alpha_2(z_1, z_2), \alpha_3(z_1, z_2)) \quad \text{for all} \quad (z_1, z_2) \in \overline{W}.$$

For $\delta > 0$ small enough, we consider η from $\overline{W} \times [-\delta, \delta]$ to $\mathbb{R}^3$, which satisfies

$$\begin{cases} \eta_\tau(z_1, z_2, \tau) = \nabla d(\eta(z_1, z_2, \tau)), \\ \eta(z_1, z_2, 0) = \alpha(z_1, z_2). \end{cases}$$

We write

$$\eta(z_1, z_2, \tau) = (\eta_1(z_1, z_2, \tau), \eta_2(z_1, z_2, \tau), \eta_3(z_1, z_2, \tau))$$

with $\eta_i : \overline{W} \times [-\delta, \delta] \to \mathbb{R}$. We define $U := \eta(\{\overline{W} \times [-\delta, \delta]\})$ and choose δ small enough so that $U \subset V$. Note that

$$\frac{d}{d\tau} d(\eta(z_1, z_2, \tau)) = \nabla d(\eta(z_1, z_2, \tau)) \eta_\tau(z_1, z_2, \tau) = |\nabla d(\eta(z_1, z_2, \tau))|^2 = 1,$$

and that $d(\eta(z_1, z_2, 0)) = d(\alpha(z_1, z_2)) = 0$. Thus we conclude that $d(\eta(z_1, z_2, \tau)) = \tau$. We define $J(z_1, z_2, \tau)$ as the determinant of the Jacobian matrix of η at (z_1, z_2, τ) and perform the change of coordinates $\eta(z_1, z_2, \tau) = x$ to obtain

$$\begin{aligned} &\int_U g\left(\frac{d(x)}{\varepsilon}\right) \phi^\varepsilon(x)\, dx \\ &\qquad = \int_{-\delta}^{\delta} d\tau \int_W g\left(\frac{d(\eta(z_1, z_2, \tau))}{\varepsilon}\right) \phi^\varepsilon(\eta(z_1, z_2, \tau)) |J(z_1, z_2, \tau)| dz_1 dz_2 \\ &\qquad = \int_{-\delta}^{\delta} d\tau \int_W g\left(\frac{\tau}{\varepsilon}\right) \phi^\varepsilon(\eta(z_1, z_2, \tau)) |J(z_1, z_2, \tau)| dz_1 dz_2. \end{aligned}$$

By applying the change of coordinates $\tau = \varepsilon\widetilde{\tau}$, we have

$$\begin{aligned} &\int_U g\left(\frac{d(x)}{\varepsilon}\right) \phi^\varepsilon(x)\, dx \\ &\qquad = \varepsilon \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} d\widetilde{\tau} \int_W g\left(\widetilde{\tau}\right) \phi^\varepsilon(\eta(z_1, z_2, \varepsilon\widetilde{\tau})) |J(z_1, z_2, \varepsilon\widetilde{\tau})|\, dz_1 dz_2. \end{aligned}$$

Therefore,

$$\begin{aligned} A_\varepsilon := \frac{1}{\varepsilon} &\int_U g\left(\frac{d(x)}{\varepsilon}\right) \phi^\varepsilon(x)\, dx \\ &= \int_{-\infty}^{\infty} \int_W \mathbf{1}_{(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon})}(\widetilde{\tau})\, g\left(\widetilde{\tau}\right) \phi^\varepsilon(\eta(z_1, z_2, \varepsilon\widetilde{\tau})) |J(z_1, z_2, \varepsilon\widetilde{\tau})| d\widetilde{\tau} dz_1 dz_2. \end{aligned}$$

In the following, we will apply the dominated convergence theorem to deduce the limit of A_ε as $\varepsilon \downarrow 0$. Set

$$H_\varepsilon(z_1, z_2, \widetilde{\tau}) := \mathbf{1}_{(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon})}(\widetilde{\tau})\, g\left(\widetilde{\tau}\right) \phi^\varepsilon(\eta(z_1, z_2, \varepsilon\widetilde{\tau})) |J(z_1, z_2, \varepsilon\widetilde{\tau})|.$$

For $-\frac{\delta}{\varepsilon} \le \widetilde{\tau} \le \frac{\delta}{\varepsilon}$, we have $-\delta \le \varepsilon\widetilde{\tau} \le \delta$, so that for all $\varepsilon > 0$

$$\left|\mathbf{1}_{(-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon})}(\widetilde{\tau})\,|J(z_1,z_2,\varepsilon\widetilde{\tau})|\right| \le \sup_{z_1,z_2\in\overline{W},\ -\delta\le\tau\le\delta} |J(z_1,z_2,\tau)| =: C_1.$$

Moreover, $\|\phi^\varepsilon\|_{L^\infty(\Omega)} \le C$ for all $\varepsilon > 0$, therefore,

$$|H_\varepsilon(z_1,z_2,\widetilde{\tau})| \le C|g(\widetilde{\tau})| \quad \text{ on } \ \overline{W}\times\mathbb{R}. \tag{4.36}$$

Next, since ϕ^ε converges uniformly to ϕ on U and since J is continuous, we have for all $\widetilde{\tau}\in\mathbb{R}, (z_1,z_2)\in W$,

$$\mathbf{1}_{(-\frac{\delta}{\varepsilon},\frac{\delta}{\varepsilon})}(\widetilde{\tau})\,\phi^\varepsilon(\eta(z_1,z_2,\varepsilon\widetilde{\tau})) \to \phi(\eta(z_1,z_2,0)) = \phi(\alpha(z_1,z_2)),$$

$$J(z_1,z_2,\varepsilon\widetilde{\tau}) \to J(z_1,z_2,0),$$

as $\varepsilon\downarrow 0$. It follows that as $\varepsilon\downarrow 0$,

$$H_\varepsilon(z_1,z_2,\widetilde{\tau}) \to g(\widetilde{\tau})\phi(\eta(z_1,z_2,0))|J(z_1,z_2,0)| \ \text{ for all } \ \widetilde{\tau}\in\mathbb{R}, (z_1,z_2)\in W. \tag{4.37}$$

Combining (4.36) and (4.37), we have

$$\lim_{\varepsilon\downarrow 0} A_\varepsilon = \int_{-\infty}^{\infty} g(\widetilde{\tau})d\widetilde{\tau}\int_W \phi(\alpha(z_1,z_2))|J(z_1,z_2,0)|\,dz_1dz_2. \tag{4.38}$$

Next, we computes $|J(z_1,z_2,0)|$. For this purpose, we write

$$\frac{\partial\eta}{\partial z_1} = (\frac{\partial\eta_1}{\partial z_1},\frac{\partial\eta_2}{\partial z_1},\frac{\partial\eta_3}{\partial z_1}),$$

$$\frac{\partial\eta}{\partial z_2} = (\frac{\partial\eta_1}{\partial z_2},\frac{\partial\eta_2}{\partial z_2},\frac{\partial\eta_3}{\partial z_2}),$$

$$\frac{\partial\eta}{\partial\tau} = (\frac{\partial\eta_1}{\partial\tau},\frac{\partial\eta_2}{\partial\tau},\frac{\partial\eta_3}{\partial\tau}).$$

Note that $\dfrac{\partial\eta}{\partial\tau}(z_1,z_2,0)$ is the outer normal vector to γ at the point $\eta(z_1,z_2,0) = \alpha(z_1,z_2)$ and that $\{\dfrac{\partial\eta}{\partial u}(z_1,z_2,0), \dfrac{\partial\eta}{\partial v}(z_1,z_2,0)\}$ is a basis of the tangent space of γ at point $\eta(z_1,z_2,0) = \alpha(z_1,z_2)$. Therefore,

$$\begin{aligned}
|J(z_1,z_2,0)| &= \left|(\frac{\partial\eta}{\partial z_1}\wedge\frac{\partial\eta}{\partial z_2}).\frac{\partial\eta}{\partial\tau}\right| = \left|\frac{\partial\eta}{\partial z_1}\wedge\frac{\partial\eta}{\partial z_2}\right|\left|\frac{\partial\eta}{\partial\tau}\right| \\
&= \left|\frac{\partial\eta}{\partial z_1}\wedge\frac{\partial\eta}{\partial z_2}\right||\nabla d(\eta(z_1,z_2,0))| = \left|\frac{\partial\eta}{\partial z_1}\wedge\frac{\partial\eta}{\partial z_2}\right|(z_1,z_2,0) \\
&= \left|\frac{\partial\alpha}{\partial z_1}\wedge\frac{\partial\alpha}{\partial z_2}\right|(z_1,z_2)
\end{aligned}$$

where $\wedge$ is the vector product. This together with (4.38) implies that

$$\lim_{\varepsilon\downarrow 0} A_\varepsilon = \int_{-\infty}^{\infty} g(\widetilde{\tau})d\widetilde{\tau}\int_W \phi(\alpha(z_1,z_2))\left|\frac{\partial\alpha}{\partial z_1}\wedge\frac{\partial\alpha}{\partial z_2}\right|dz_1dz_2.$$

On the other hand, in view of the definition of the integral of surface (see [47, Formula (131), p. 283]), we have

$$\int_\gamma \phi \, d\gamma = \int_W \phi(\alpha(z_1, z_2)) \left| \frac{\partial \alpha}{\partial z_1} \wedge \frac{\partial \alpha}{\partial z_2} \right| dz_1 dz_2.$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} A_\varepsilon = \int_{-\infty}^{\infty} g(\widetilde{\tau}) d\widetilde{\tau} \int_\gamma \phi$$

which completes the proof of the lemma. □

Application to reaction term: Now we apply Lemma 4.3.3 to formally compute the limit as $\varepsilon \downarrow 0$ of

$$\frac{1}{\varepsilon} \int_0^T \int_\Omega p(u^\varepsilon)(\sigma^\varepsilon - \delta \mu^\varepsilon)\psi, \quad \text{for } \psi \in \mathcal{F}_T.$$

Because of the outer and inner expression of u^ε in (4.24) and (4.25), we deduce that for ε small enough

$$u^\varepsilon(x,t) \approx \begin{cases} \pm 1 & \text{if } (x,t) \text{ is far from } \Gamma_T \\ U_0(\dfrac{\tilde{d}(x,t)}{\varepsilon}) & \text{if } (x,t) \text{ is closed to } \Gamma_T. \end{cases}$$

Therefore

$$p(u^\varepsilon(x,t)) \approx \begin{cases} 0 & \text{if } (x,t) \text{ is far from } \Gamma_T \\ p(U_0(\dfrac{\tilde{d}(x,t)}{\varepsilon})) & \text{if } (x,t) \text{ is closed to } \Gamma_T. \end{cases}$$

Thus we can apply Lemma 4.3.3 by setting

$$g(\xi) := p(U_0(\xi)) \quad \text{and} \quad \phi^\varepsilon := (\sigma^\varepsilon - \delta \mu^\varepsilon)\psi,$$

where $\psi \in \mathcal{F}_T$. This yields

$$\begin{aligned} P^0(\psi(t)) &:= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_\Omega p(u^\varepsilon)(\sigma^\varepsilon - \delta \mu^\varepsilon)\psi = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{U(t)} p(u^\varepsilon)(\sigma^\varepsilon - \delta \mu^\varepsilon)\psi \\ &= \int_{-\infty}^{\infty} p(U_0(\xi)) \, \mathrm{d}\xi \int_{\Gamma(t)} (\sigma(t) - \delta \mu(t))\psi(t), \end{aligned}$$

where $U(t)$ is a small enough neighborhood of $\Gamma(t)$. Recalling that in view of the definition of p and of Lemma 4.3.1

$$p(U_0) = 2p_0 \sqrt{W(U_0)} = -\sqrt{2} p_0 \, U_0',$$

we get

$$\begin{aligned} P^0(\psi(t)) &= -\sqrt{2} p_0 \int_{-\infty}^{\infty} U_0'(\xi) \, \mathrm{d}\xi \int_{\Gamma(t)} (\sigma(t) - \delta \mu(t))\psi(t) \\ &= 2\sqrt{2} p_0 \int_{\Gamma(t)} (\sigma(t) - \delta \mu(t))\psi(t). \end{aligned}$$

Hence, we formally conclude that, for all $\psi \in \mathcal{F}_T$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^T \int_\Omega p(u^\varepsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon)\psi = 2\sqrt{2}p_0 \int_0^T \mathrm{d}t \int_{\Gamma(t)} (\sigma(t) - \delta\mu(t))\psi(t). \tag{4.39}$$

Conclusion: Now, we recall the definition of a weak solution of the equation for μ^ε:

$$\int_0^T \int_\Omega (-\alpha\mu^\varepsilon - u^\varepsilon)\psi_t - \int_\Omega (\alpha\mu_0^\varepsilon + u_0^\varepsilon)\psi(0) = \int_0^T \int_\Omega \left(\mu^\varepsilon \Delta\psi + \varepsilon^{-1} p(u^\epsilon)(\sigma^\varepsilon - \delta\mu^\varepsilon)\psi\right)$$

for ψ in $\mathcal{F}_T$ and take the limit $\varepsilon \to 0$ on both sides, to obtain in view of (4.39)

$$\int_0^T \int_\Omega (-\alpha\mu - u)\psi_t - \int_\Omega (\alpha\mu_0 + u_0)\psi(0) = \int_0^T \int_\Omega \mu\Delta\psi + 2\sqrt{2}p_0 \int_0^T \mathrm{d}t \int_{\Gamma(t)} (\sigma - \delta\mu)(t)\psi(t).$$

This together a similar argument for the equation for σ completes the proof of Theorem 4.1.5.

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