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Explosion des solutions de Schrödinger de masse critique sur une variété riemannienne.

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Résumé

Ce travail cherche à comprendre comment l'ajout d'une géométrie non euclidienne dans un problème de Schrödinger non linéaire influe sur l'existence et l'unicité des solutions explosives de masse critique. On s'inspire pour beaucoup des travaux de Merle et Raphaël sur la méthode de modulation des paramètres d'invariance géométrique pour une EDP qui possède de bonnes lois de conservations. On s'appuie ici plus particulièrement sur l'article de Raphaël et Szeftel [RS11] qui prouve l'existence et l'unicité d'une solution de masse critique en dimension 2 pour l'équation de Schrödinger non linéaire avec potentiel d'inhomogénéité devant la non-linéarité, et qui explose par ailleurs au maximum de l'inhomogénéité.

Dans un premier temps, il s'agit de reprendre la méthode dans son ensemble afin de l'adapter à des cas où la Laplacien n'est plus plat, mais remplacé par un opérateur du type $\mathcal{L} = \frac{1}{g(x)} div (G(x) \nabla) + V(x)$ afin de comprendre le rôle qu'une déformation de type métrique peut jouer dans le processus d'explosion des solutions de Schrödinger.

Ayant mis en avant le rôle de la courbure au point d'explosion, comme des conditions sur les dérivées de G et g, on reprend dans un deuxième temps l'étude du début dans le cas plus général d'une variété riemannienne. Le Laplacien est alors remplacé par un opérateur de Laplace-Beltrami Δ_g pour une métrique g, et grâce à un ansatz sur la solution qui intègre maintenant la transformation induite par la métrique, on est capable d'énoncer un résultat d'existence et d'unicité similaire à celui de [RS11], en termes de conditions géométriques sur la variété elle même. Par soucis de simplicité, on se limite néanmoins au rôle local de la métrique, en supposant la métrique définie globalement dans une certaine carte, et asymptotiquement équivalente à la métrique euclidienne.

Mots-clefs : équation de Schrödinger non linéaire en dimension 2, lois de conservations et d'invariances géométriques du flot de Schrödinger, inégalités optimales de Gagliardo-Nirenberg et définition variationnelle du Ground-State, Théorème de Merle, méthode de modulation des paramètres d'invariance géométrique, géométrie riemannienne en dimension 2.

Blowing-up solutions of the Shrödinger equation on a riemannian manifold.

Abstract

The present work aims at investigating the effects of a non-euclidean geometry on existence and uniqueness results for critical and blowing-up NLS solutions. We will use many ideas from the works of Merle and Raphaël, particularly ideas for modulation of the geometric invariants parameters. We will rely more specifically on [RS11] for existence and unicity of a critical mass solution in dimension two of the nonlinear Schrödinger equation with inhomogeneous potential acting on the nonlinearity, and which blows up where the inhomogeneity reaches its maximum.

At first, we consider a generalized Laplacian operator $\mathcal{L} = \frac{1}{g(x)} div(G(x)\nabla) + V(x)$ and deploy the classical ansatz method to point out difficulties inherited from the non-flat metric terms, and in particular the key role played by the curvature at the blow-up point.

In a second part, we reproduce the method when modifying the geometrical ansatz on which the parametrix is constructed, and investigating more precisely what is needed for existence and then unicity when dealing with an operator $\Delta_g + V$, where Δ_g is the Laplace-Beltrami operator associated with the metric g. For simplicity, we shall only consider the role of g locally around the blow up point we are constructed, by assuming g is globally defined in some map, and asymptotically equals the usual euclidean metric.

Keywords : nonlinear Schrödinger equation in dimension 2, conservations and geometrical invariance laws, optimal Gagliardo-Nirenberg inequalities and variational definition of the Ground-State, Merle's Theorem, modulation metod of geometrical invariance parameters, basic of riemannian geometry in dimension 2.

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Introduction

What is so important about studying the Schrödinger equation and its solutions properties? Since the equation itself was first formulated in the mid 20's by Austrian Physicist Erwin Schrödinger, it has been a first step into the quantum world for many students as a quantum mechanics equivalent of the Newton's second law of motion : how does a quantum system evolve through time ?

As a key topic for quantum mechanics, it has of course been discussed for decades, and many interpretations, refinements have been proposed so the equation would describe even more complex physical systems. For instance, while the simple equation is linear, has good invariance properties that passes on the solutions, adding nonlinear or semilinear terms may yields to modelling nonlinear optics, lasers or plasmas, which proves much more challenging to handle. In many cases, the seemingly simplicity of the resulting equation goes with an ordeal when trying to find solutions analytically, and one is quickly compelled to use modelling through computer science to solve an issue.

In the last thirty years or so however, the nonlinear Schrödinger equation has been the center of many mathematical studies which aim at giving tools for a better understanding of PDE's. The theory of global existence or blow-up solutions for NLS has been developed in relation with the theory of Ground States, that is solitons which are special periodic solutions to the Hamiltonian system. It is therefore of great interest to investigate the stability of these special solutions, and to describe the flow around these solutions.

Among special cases, the self-focusing nonlinear Schrödinger equation $i\partial_t u + \Delta u + u|u|^{p-1} = 0$ has been one of the most studied, as it intervenes in plasma theory or nonlinear optics. It has also been described in terms of stability depending on which side it is from a L^2 -subcritical region $1 , or a <math>L^2$ supercritical one $p > \frac{4}{n}$. For any initial condition $u_0 \in H^1$, the associated NLS problem has a unique global in time solution. The critical cas $p = \frac{4}{n}$ has been dealt with thanks to the Merle Theorem which states all the solutions are given by the Ground State up to geometrical invariances modifications, and blow up in finite time. The so-called *log-log* result of Merle and Raphaël then deal with some supercritical cases, which are close enough to the critical mass, and provided some energy bound holds, then there is still a unique blowing up solution.

The critical case appears to hold a particular role in the classification of NLS solutions, as one may see that for any datum $u_0 \in H^1$, if $||u_0||_{L^2} < ||Q||_{L^2}$, where Q stands for the corresponding NLS Ground State, then the associated solution u is global in time. The result was proven on \mathbb{R}^n , and as a Master Thesis, I have proven the result still holds on a compact and complete riemannian manifold. The stability of the solutions around the critical case, even in some non euclidean special context is the key to what we are doing here using the fact that even with some deformations on the nonlinear terms or on the metric, the solution should look like a deformation of some arranged Ground State.

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1 Introduction and modulation tools

1.1 The Inhomogeneous NLS with a smooth metric

We study the 2 dimension NLS equation with nonlinearity potential k, a metric defined by nonnegative function g, symmetric matrix G, and a potential V :

$$i\partial_t u + \frac{1}{g(x)} div \big(G(x)\nabla u \big) + k(x)|u|^2 u + V(x) u = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^2$$
(1.1)

$$u(0,x) = u_0(x), \quad u_0 : \mathbb{R}^2 \longrightarrow \mathbb{C} \in H^1$$
(1.2)

We first recall some results established in the case $g \equiv 1$, $G \equiv I$ and $V \equiv 0$, when $\mathcal{L} = \Delta$ is the usual Laplace operator.

1.1.1 The flat homogeneous case.

For any dimension N, when $k(x) \equiv k_0$, the L²-critical NLS problem may be written as

$$\int i\partial_t u + \Delta u + |u|^{4/N} u = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^N$$
(1.3)

$$\begin{cases} u(0,x) = u_0(x), & u_0 : \mathbb{R}^N \longrightarrow \mathbb{C} \in H^1 \end{cases}$$
(1.4)

and given an initial datum $u_0 \in H^1(\mathbb{R}^N)$, it is a well known fact (see [GV79] or [Kat87]) that equation (1.1)-(1.2) is well-posed, with solutions $u(t) \in C([0,T), H^1(\mathbb{R}^N))$ that satisfy the following conservation laws

Conservation of Mass :
$$\int |u(t)|^2 dx = \int |u_0|^2 dx$$

Conservation of Energy :
$$E(u(t)) = \frac{1}{2} \int |\nabla u(t)|^2 dx - \frac{1}{2 + \frac{4}{N}} \int |u(t)|^{2 + \frac{4}{N}} dx = E(u_0)$$

Conservation of Momentum :
$$M(u(t)) = \mathcal{I}m \int \overline{u} \nabla u \, dx = M(u_0)$$

and a large group of H^1 symmetries which leaves the flow invariant, that is, if u solves (1.3), then so does

$$v(t,x) = \lambda_0^{N/2} u(t+t_0, \lambda_0 x + x_0 - \beta_0 t) e^{i\frac{\beta_0}{2} \cdot \left(x - \frac{\beta_0}{2} t\right)} e^{i\gamma_0}$$

for any $(\lambda_0, t_0, x_0, \beta_0, \gamma_0) \in \mathbb{R}^+_* \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$

Moreover, a last symmetry which does not lies in the H^1 energy space, but in the virial space $\Sigma = \{x \ u \in L^2\} \cap H^1$ is the pseudo-conformal invariance. Whenever u solves (1.3) then so does

$$v(t,x) = \frac{1}{|t|^{N/2}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}$$

Let Q be the unique nonnegative radially symetric solution to the elliptic equation (see [BL83] for existence and [Kwo89] for uniqueness)

$$\Delta Q + Q^{1 + \frac{4}{N}} - Q = 0$$

Q is the so-called *ground state* solution, and all solutions to

$$\Delta \phi + \phi |\phi|^{\frac{4}{N}} - \phi = 0, \quad \phi \in H^1(\mathbb{R}^N), \quad \phi(x) > 0 \quad \text{are a translate of } Q$$

In [Wei83], thanks to the the variational characterization of Q, Weinstein gives a simple but powerful criterion to find out whether a NLS solution with H^1 initial datum blows up or not :

either $||u_0||_{L^2} < ||Q_{k_0}||_{L^2}$ then $T = +\infty$

or $||u_0||_{L^2} \ge ||Q_{k_0}||_{L^2}$ then both $T < +\infty$ and $T = +\infty$ may happen

Then, at the critical mass threshold, the pseudo conformal symmetry applied to the periodic solitary wave solution $u(t, x) = Q(x) e^{it}$ yields a minimal mass blow up solution

$$S(t,x) = \frac{1}{|t|^{N/2}} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \quad \left\|S(t)\right\|_{L^2(\mathbb{R}^N)} = \|Q\|_{L^2(\mathbb{R}^N)}$$

that blows up at time t = 0. In[Mer93], Merle proves that solution is the only critical mass blow up solution : if $u \in H^1$ is solution to (1.3)(1.4), with $||u_0||_{L^2} = ||Q||_{L^2}$ and blowing up at t = T, then u(t) is equal to S(t), up to the symmetries of the flow, so that u is given by

$$u(t,x) = \left(\frac{\omega}{T-t}\right)^{N/2} Q\left(\frac{\omega(x-x_1)}{T-t} - \omega x_0\right) e^{i\left[\theta - \frac{|x-x_1|^2 - 4\omega^2}{4(T-t)}\right]}$$
$$\omega > 0, \quad \theta \in S^1, \quad x_0 \in \mathbb{R}^N, \quad x_1 \in \mathbb{R}^N,$$

1.1.2 The flat inhomogeneous case.

In the usual euclidean case $g \equiv 1$, the introduction of a smooth bounded inhomogenity $k : \mathbb{R}^2 \to \mathbb{R}^+$ raises the problem of how to treat NLS solutions when symmetries are lost, among which the pseudo conformal symmetry and thus the explicit description of blowing up solutions that had been deduced. Notice both conservation of Mass and conservation of Energy still hold, while the Momentum now varies with time t:

Conservation of Mass :
$$\int |u(t)|^2 dx = \int |u_0|^2 dx$$

Conservation of Energy :
$$E(u(t)) = \frac{1}{2} \int |\nabla u(t)|^2 dx - \frac{1}{2 + \frac{4}{N}} \int k(x) |u(t)|^{2 + \frac{4}{N}} dx = E(u_0)$$

Variation of Momentum :
$$\frac{d}{dt} \mathcal{I}m \int \overline{u} \nabla u \, dx = \frac{1}{2} \int \nabla k(x) |u(t)|^{2 + \frac{4}{N}}$$

Merle also initiated the study of the inhomogeneous case in [Mer96]. In the case N = 2, given $\kappa > 0$, one now defines Q_{κ} as

$$Q_{\kappa}(x) = \frac{1}{\kappa^{1/2}} Q(x), \quad x \in \mathbb{R}^2$$

Let

$$k_2 = \max_{x \in \mathbb{R}^2} k(x) < +\infty$$

then for *small* initial data that satisfy

$$||u_0||_{L^2} < M_k = ||Q_{k_2}||_{L^2}$$

the associated (1.3)-(1.4) solutions are global and bounded in H^1 , while blow up may occur for solutions with *big* initial data $||u_0||_{L^2} \ge M_k$. Moreover, Merle obtained a more precise result about the localization of the concentration point, and a non existence criterion for critical blow up elements :

Theorem. (|Mer96|)

Assume that $k \in C^1(\mathbb{R}^2)$.

- Localization of the concentration point : Let $u \in H^1$ with $||u||_{L^2} = M_k$ be a solution to (1.3) blowing up at T = 0. Assume that $\{x \in \mathbb{R}^2 \text{ such that } k(x) = k_2\}$ is finite. Assume that there is $\delta > 0$ and R > 0 such that $k(x) \le k_2 - \delta$ for $|x| \ge R$. Then there is $x_0 \in \mathbb{R}^2$ such that $k(x_0) = k_2$ and

$$\left|u(t)\right|^{2} \rightharpoonup \left\|Q_{k(x_{0})}\right\|_{L^{2}}^{2} \delta_{x=x_{0}} \quad as \quad t \to 0$$

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- Criterion of non existence : Assume that $k(x_0) = k_2$ and

$$\nabla k(x) \cdot (x - x_0) \le -|x - x_0|^{1 + \alpha_0} \quad \text{near } x_0 \text{ and for some } \alpha_0 > 0 \tag{1.5}$$

then there is no critical mass blow up solution at x_0 .

In particular, it follows from this theorem that blow up must occur at some point for which k reaches its maximum, so it satisfies $\nabla k(x_0) = 0$, and the repulsivity condition (1.5) implies a blow up solution may not exist at some point for which $\nabla k(x_0) \neq 0$.

Now, Banica, Carles and Duyckaerts [BCD11] prove the existence of critical elements for any point x_0 where k is smooth enough, that is $\nabla k(x_0) = \nabla^2 k(x_0) = 0$: after linearizing the problem close to the explicit S(t) approximate solution, they use modulation theory and energy estimates to treat pertubatively the unstable modes and integrate the system backwards from the singularity.

In that approach, the problem is treated pertubatively from the homogeneous case. Then, in the case of smooth k with non degenerate Hessian at x_0 , $\nabla^2 k(x_0) < 0$, Raphaël and Szeftel prove in [RS11] both a necessary condition for existence of a critical blow up solution, and a existence and uniqueness theorem, focusing their proof on the case N = 2.

They assume k satisfies

$$k \in C^5 \cap W^{1,\infty}, \quad 0 < k_1 \le k(x) \le 1, \quad \max_{x \in \mathbb{R}^2} k(x) = 1 \quad \text{is attained}$$
(1.6)

so that the critical mass is $M_k = ||Q||_{L^2}$.

Proposition. Let u with $||u||_{L^2} = ||Q||_{L^2}$ be a solution to (1.3)-(1.4) for N = 2, which blows up at time T = 0, then there exists $x_0 \in \mathbb{R}^2$ such that

 $k(x_0) = 1$

and u blows up at x_0 in the sense

$$|u(t)|^{2} \rightharpoonup ||Q||_{L^{2}}^{2} \delta_{x=x_{0}} \quad as \quad t \to 0$$
(1.7)

Moreover, the energy E_0 of u satisfies

$$E_0 + \frac{1}{8} \int \nabla^2 k(x_0) .(y, y) Q^4 > 0$$
(1.8)

and they claim the energy bound (1.8) is sharp. Then, they focus on the case $\nabla^2 k(x_0) < 0$ which is expected to be the most delicate one. The main result is stated

Theorem. (Existence and Uniqueness of a critical element at a nondegenerate critical point). Let $x_0 \in \mathbb{R}^2$ with

$$k(x_0) = 1$$
 and $\nabla^2 k(x_0) < 0$

Then for all E_0 satisfying (1.8), there exists a unique up to phase shift H^1 critical mass blow up solution to (1.1) which blows up at time T = 0 and the point x_0 in the sense of (1.7), with energy E_0 . Moreover

$$\lim_{t \to 0} \mathcal{I}m\left(\int \nabla u \,\overline{u}\right) = 0 \tag{1.9}$$

1.1.3 The almost euclidean and inhomogeneous case.

We assume that the inhomogeneity k is bounded and reaches its maximum at some point $x_0 \in \mathbb{R}^2$. Whithout loss of generality, we may suppose $x_0 = 0$ and $k(x_0) = 1$. We will now aim at constructing a solution u of (1.1)-(1.2) that blows up at $x_0 = 0$.

We will also assume the metric is almost euclidean at point $x_0 = 0$, which should also be a critical point for both the scalr function g and the matrix function G.

Since the degenerate cases are more easily dealt with, we wish to assume all tensors $\nabla^2 k(0)$, $\nabla^2 g(0)$ and $\nabla^2 G(0)$ to be non-degenerate. However the non-degeneracy cases $\nabla^2 g(0) \neq 0$ and $\nabla^2 G(0) \neq 0$ compel to assume technical hypotheses about the scalar function g and the matrix function G which may not seem natural or meaningful. So first we give a set of reasonable hypotheses (\mathcal{H}_0) under which the following main theorem will hold. Then we give a refined set of minimal hypotheses (\mathcal{H}) that we will later look into so we may link it to geometric elements.

From now on, and in all the sequel, notice we will use a more compact Einstein notation for matrices.

- $(H1)_0: \quad k \in C^5 \cap W^{1,\infty}, \quad 0 < k_1 \le k \le 1, \quad k(0) = 1, \quad \nabla k(0) = 0, \quad \nabla^2 k(0) < 0$
- $(H2)_0: \quad g, V \in C^5 \cap W^{5,\infty}(\mathbb{R}^2, \mathbb{R}), \quad g > 0, \quad g(0) = 1, \quad \nabla g(0) = 0, \quad V \ge V(0)$
- $(H3)_0: \quad G \in C^5 \cap W^{5,\infty}(\mathbb{R}^2, \mathbb{M}(2, 2)), \quad G_{ij} = G_{ji}, \quad G_{ij}(0) = \delta_{ij}, \quad \nabla G_{ij}(0) = 0, \quad i, j = 1, 2$

$$(H4)_0: \quad \nabla^2 g(0) = \nabla^2 G_{ij}(0) = 0, \quad i, j = 1, 2$$

 $(H5)_0$: Whatever is requested to ensure the Smoothing Effect and The Strichartz estimates in 8.3:

(1)
$$\forall p \in \mathbb{N}^2$$
, $\left|\partial^p (G_{ij} - I_{ij})\right| \lesssim \langle x \rangle^{-\tau(|p|)}$, $1 \le i, j \le 2, \tau : \mathbb{N} \to \mathbb{N}^*, \tau(m) > m+1, \forall m \ge (2) \exists u > 0, \forall (x, \xi) \in \mathbb{R}^4$. $\sum_{i=1}^{n} C_{i} \le \xi_i \le 2, \tau : \mathbb{N} \to \mathbb{N}^*, \tau(m) > m+1, \forall m \ge (2) \exists u > 0, \forall (x, \xi) \in \mathbb{R}^4$.

$$(2) \exists \nu > 0, \forall (x, \zeta) \in \mathbb{R}, \quad \sum_{i,j} G_{ij} \zeta_i \zeta_j \ge \nu |\zeta|, \quad \exists C > 0, \quad \overline{C} \ge g \ge C |ax|$$

(3) $\forall (x,\xi) \in T^*\mathbb{R}^2 \setminus \{0\}$, the flow is neither trapped backwards, nor forwards : $\lim_{t \to \pm \infty} |x(t,x,\xi)| = +\infty$

where $(x(t, x, \xi), \xi(t, x, \xi))$ would denote here the bicharacteristic originating at (x, ξ) , that is the integral curve of the Hamilton vector field associated to the principal symbol of \mathcal{L} , with initial condition (x, ξ) . Also notice assumption (1) here implies assumption (8.8) since $\tau(|p| = 1) > 1$. Assumptions $(H1)_0 - (H4)_0$ are describing the local behavior of metric terms around the blow up point, while assumption $(H5)_0$ describe their asymptotic behavior.

As one will realise later, a little less restrictive set of hypotheses (\mathcal{H}) may replace $(H1)_0$ - $(H4)_0$ to complete the proof :

$$\begin{array}{ll} (H1): & k \in C^5 \cap W^{1,\infty}, \quad 0 < k_1 \leq k \leq 1, \quad k(0) = 1, \quad \nabla k(0) = 0, \quad \nabla^2 k(0) < \frac{1}{2} \, \nabla^2 Tr(G)(0) \\ (H2): & g, V \in C^5 \cap W^{5,\infty}(\mathbb{R}^2, \mathbb{R}), \quad g > 0, \quad g(0) = 1, \quad \nabla g(0) = 0, \quad \nabla^2 k(0) < \nabla^2 g(0), \quad V \geq V(0) \\ (H3): & G \in C^5 \cap W^{5,\infty}(\mathbb{R}^2, \mathbb{M}(2, 2)), \quad G_{ij} = G_{ji}, \quad G_{ij}(0) = \delta_{ij}, \quad \nabla G_{ij}(0) = 0, \quad i, j = 1, 2 \\ (H4): & \Delta g(0) = 2 \, K_{21} \, \nabla^2 G_{ij}(0).(\partial_i, \partial_j) + \Delta \big(Tr(G) \big)(0) + 2 \, K_{22} \, \nabla^2 G_{\tilde{p} \, \tilde{p}}(0).(\partial_p, \partial_p) \\ (H5): & \nabla^2 \bigg(k - \frac{1}{2} \, Tr(G) \bigg)(0) + M_{12}^{[2]}(0) < 0, \quad M_{12}^{[2]}(0) = \frac{1}{2} \, \nabla^2 G_{12}(0) . \left(\begin{pmatrix} (\partial_1, \partial_1 - \partial_2) & (\partial_2, \partial_1 - \partial_2) \\ (\partial_1, \partial_2 - \partial_1) & (\partial_2, \partial_2 - \partial_1) \end{pmatrix} \\ (H6): & \Delta g(0) > \frac{1}{6} \, \Delta \Big(k - \frac{1}{2} \, Tr(G) \Big)(0) \, \|Q\|_{L^2}^2 \, \left\| \frac{|y|}{2} \, |\nabla Q| \right\|_{L^2}^2 \\ \end{array}$$

with constants

$$K_{21} = \frac{1}{8} \int \left[\left(3 + \frac{|y|^2}{2} \right) Q^2 - 3|y|^2 |\nabla Q|^2 \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2}$$

$$K_{22} = \frac{1}{8} \int \left[\left(1 - \frac{|y|^2}{2} \right) Q^2 - |y|^2 |\nabla Q|^2 \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2}$$

0

Notice, those hypotheses will be needed in some differents part of the proof, in the following way :

(H1)-(H3) are the natural requirements to get a second order expansion of the equation and all the terms computed in rescaled variables.

(H4) is the main issue of this paper : as we shall see, introduction of metric terms G and g has brought a $\mathcal{O}(\lambda^2)$ term in the expansion of the b law which will prove hard to deal with. That is why, for now we choose to nullify it in (1.42).

(H5) is a condition needed in the Uniqueness proof to get (5.74), ensuring the matrix c_0 defined in (1.50) will be negative definite.

(H6) is an additional identity designed to ensure (5.38) in the Uniqueness proof as one shall see in (5.93).

As one may see, it means (H1)-(H4) are existence conditions, while (H5) and (H6) are uniqueness conditions. However, all these seem rather technical assumptions, which are not easily linked to anything relative to what was changed in the geometry when introducing the metric terms. In section 6 we will introduce geometric aspects that will clarify and refine the problem.

Indeed, considering the pseudo-Laplacian operator $\mathcal{L} = \frac{1}{g} div(G\nabla) + V$ in (\mathbb{R}^2, dx) , is pretty much the same to considering the Laplace-Beltrami operator in the riemannian manifold (\mathbb{R}^2, h) with $h^{ij} = \frac{1}{g} G_{ij} dx_i \otimes dx_j$, the previous hypothesis may now be rewritten in terms of geometric properties at the origin, that is where the blow-up is designed to occur.

Thus, we denote by $R_{ijkl} = h_{lp} \left[\left(\nabla_{\partial_j} (\nabla_{\partial_i}) - \nabla_{\partial_i} (\nabla_{\partial_j}) \right) \partial_k \right]^p \partial_l$ the coordinates of the riemannian curvature tensor at x = 0 in a geodesic normal chart around the origin x = 0. All previous assumptions regarding the local behavior around blow up point may then be assembled as the more geometric set of hypotheses (\mathcal{H}_R)

$$\begin{array}{ll} (H1)_R: & k \in C^5 \cap W^{1,\infty}, \quad 0 < k_1 \le k \le 1, \quad k(0) = 1, \quad \nabla k(0) = 0 \\ (H2)_R: & g, V \in C^5 \cap W^{5,\infty}(\mathbb{R}^2, \mathbb{R}), \quad g > 0, \quad g(0) = 1, \quad V \ge V(0) \\ (H3)_R: & G \in C^5 \cap W^{5,\infty}(\mathbb{R}^2, \mathbb{M}(2,2)), \quad G_{ij} = G_{ji}, \quad G_{ij}(0) = \delta_{ij}, \\ & \partial_k \partial_l G_{ij}(0) = \partial_k \partial_l g(0) \, \delta_{ij} \quad i, j, k, l = 1, 2 \\ (H4)_R: & \Delta g(0) = 0, \quad R_{1212} = 0, \quad \nabla^2 (k - g)(0) < 0 \end{array}$$

Remark 1. $\nabla g(0) = \nabla G_{ij} = 0$, i, j = 1, 2 are a natural consequence of computations at the origin in a geodesic normal set of coordinates. See Appendix A γ for more details.

Eventually, since our problem lives in \mathbb{R}^2 , in which every metric is conformal, one may also consider a metric defined by $h^{ij}(x) = e^{\varphi(x)} \delta_{ij}$, for some smooth function φ . For such a metric which represents a pertubation of the euclidean structure around the origin, the previous hypotheses may be rewritten as the set $(\mathcal{H})_{\varphi}$

$$\begin{aligned} &(H1)_{\varphi}: \quad k \in C^{5} \cap W^{1,\infty}, \quad 0 < k_{1} \leq k \leq 1, \quad k(0) = 1, \quad \nabla k(0) = 0, \\ &(H2)_{\varphi}: \quad g, V \in C^{5} \cap W^{5,\infty}(\mathbb{R}^{2},\mathbb{R}), \quad g > 0, \quad g(0) = 1, \quad V \geq V(0) \\ &(H3)_{\varphi}: \quad G \in C^{5} \cap W^{5,\infty}(\mathbb{R}^{2},\mathbb{M}(2,2)), \quad G_{ij} = G_{ji}, \quad G_{ij}(0) = \delta_{ij}, \quad i, j = 1, 2 \\ &\partial_{k}\partial_{l}G_{ij}(0) = \partial_{k}\partial_{l}\left(g - \varphi\right)(0)\,\delta_{ij} \quad i, j, k, l = 1, 2 \\ &(H4)_{\varphi}: \quad \Delta\varphi(0) = \frac{1 + 2\,K_{23}}{2 + 2\,K_{23}}\,\Delta g(0), \quad \nabla^{2}(k - g)(0) + \nabla^{2}\varphi(0) < 0, \\ &(H5)_{\varphi}: \quad \Delta k(0) < \left(1 + \kappa_{22} - \frac{1 + 2\,K_{23}}{2 + 2\,K_{23}}\right)\Delta g(0) \end{aligned}$$

with constants

$$K_{23} = \frac{1}{2} \int \left[Q^2 - |y|^2 |\nabla Q|^2 \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2}, \qquad \kappa_{22} = 6 \left\| Q \right\|_{L^2}^{-2} \left\| \frac{|y|}{2} |\nabla Q| \right\|_{L^2}^{-2}$$

and notice that $\left[(H4)_{\varphi} + \Delta g(0) \ge 0\right] \Rightarrow (H5)_{\varphi}.$

Back to our analysis, one should note the laplacian operator $\mathcal{L}_0 = \frac{1}{g(x)} div (G(x) \nabla)$ is symmetric with respect to the scalar product of space $L^2(g(x) dx)$, since one may obtain after two integration by parts

$$\left(\mathcal{L}u\,,\,v\right)_{L^2(gdx)} = \int \mathcal{L}u\,v\,g(x)\,dx = -\int G(x)\nabla u\,\cdot\nabla v\,dx = \int u\,\mathcal{L}v\,g(x)\,dx = \left(u\,,\,\mathcal{L}v\right)_{L^2(gdx)}$$

Notice in the sequel we will sometimes denote the measures g(x) dx by g dx and $g(\lambda y + \alpha) dy$ by g dy to shorten computations.

We then recall the classical result that a solution u of (1.1)-(1.2) satisfies the following conservation laws

• Conservation of Mass :

$$\|u(t)\|_{L^{2}(g(x)dx)} = \|u_{0}\|_{L^{2}(g(x)dx)}, \quad \forall \ t \in [0,T)$$
(1.10)

• Conservation of energy :

$$E(u(t)) = \frac{1}{2} \int G(x) \nabla u(t) \cdot \nabla \overline{u}(t) \, dx - \frac{1}{4} \int k(x) \, |u(t)|^4 \, g(x) \, dx - \frac{1}{2} \int V(x) \, |u(t)|^2 \, g(x) \, dx$$

$$= E(u_0) = E_0, \quad \forall \ t \in [0, T)$$
(1.11)

In the sequel we will also use the homogeneous and flat metric energy as follows

$$E^{0}(f) = \frac{1}{2} \int |\nabla f|^{2} - \frac{1}{4} \int |f|^{4}$$
(1.12)

for which the usual Ground State function Q satisfies $E^0(Q) = 0$.

Theorem 1.1. (Existence and Uniqueness of a critical element at a nondegenerate critical point).

Let $x_0 \in \mathbb{R}^2$ with functions k, g, V and G satisfying one of the previous set of hypotheses $(\mathcal{H})_0$ or (\mathcal{H}) or $(\mathcal{H})_R$ or $(\mathcal{H})_{\varphi}$ at x_0 .

Then for all E_0 such as $E_0 + \frac{1}{8}C_E > 0$, with

$$C_E = \Delta(k+g)(0) \left\| \frac{|y|}{2} Q^2 \right\|_{L^2}^2 - \frac{5}{2} \Delta g(0) \left\| \frac{|y|}{2} Q \right\|_{L^2}^2 + 4V(0) \left\| Q \right\|_{L^2}^2$$

there exists a unique up to phase shift H^1 critical mass blow up solution to (1.1) which blows up at time T = 0 and the point x_0 in the sense of (1.7), with energy E_0 . Moreover

$$\lim_{t \to 0} \mathcal{I}m\left(\int \nabla u \,\overline{u}\right) = 0 \tag{1.13}$$

Notice the energy constant C_E has been expressed using the (\mathcal{H}_R) set of hypotheses, although we will find more convenient to do otherwise when it appears in (2.9).

1.2 The slow modulated ansatz

Let $\mathcal{P}_0 = (b, \lambda, \alpha, \beta, \gamma)$ be the set of geometric parameters that embody the invariants of the 2-dimensional inhomogeneous nonlinear problem. On the one hand λ is a scalling parameter that will measure the blow up speed, α is a translation parameter that approximates the blow up point position, and γ is a phase angle. On the other hand b is a pseudoconformal speed and β a galilean invariance parameter. Now our goal is to form an approximate solution of the 2-dimensional inhomogeneous non-linear schrödinger equation with respect to these parameters. By introducing the inhomogeneity k, we lost the pseudo conformal symmetry of (NLS) equation, so that we need to find a dynamical way of controlling the solution through the set of small invariants parameters $\mathcal{P} = (b, \lambda, \alpha, \beta)$.

Thus, we aim to forming an approximate solution with repect of all the \mathcal{P} parameters. To do so, we consider the following ansatz for u, a critical mass blow up solution :

$$u(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} v\left(t, \frac{x - \alpha(t)}{\lambda}\right) e^{i\gamma(t)} \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda^2}$$
(1.14)

Note we will use the following notation :

$$\Lambda f = f + y \, . \nabla f$$

v(s, y) is now solution of :

$$i\partial_{s}v + \mathcal{L}_{\lambda\alpha}v - v + \frac{k(\lambda y + \alpha)}{k(\alpha)}v|v|^{2} = i\frac{\lambda_{s}}{\lambda}\Lambda v + i\frac{\alpha_{s}}{\lambda}\cdot\left(\nabla v + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}v\right) + \widetilde{\gamma}_{s}v$$

$$\mathcal{L}_{\lambda\alpha} = \frac{1}{g(\lambda y + \alpha)}div\left(G(\lambda y + \alpha)\nabla\right) + \lambda^{2}V(\lambda y + \alpha)$$
(1.15)

with $\tilde{\gamma}_s = \gamma_s - 1$. Let us now introduce the two last parameters :

$$w(s,y) = v(s,y) \ e^{i \ b \ \frac{|y|^2}{4} - i\beta \cdot y} \tag{1.16}$$

In the case $g \equiv 1, G \equiv I, V \equiv 0, w(s, y)$ must now satisfy :

$$i\partial_{s}w + \Delta w - w + \frac{k(\lambda y + \alpha)}{k(\alpha)}w|w|^{2} + \left(b_{s} - b\frac{\lambda_{s}}{\lambda} - b\left(\frac{\lambda_{s}}{\lambda} + b\right)\right)\frac{|y|^{2}}{4}w$$
$$-\left\{\left(\beta_{s} - \frac{\lambda_{s}}{\lambda}\beta + \frac{b}{2}\left(\frac{\alpha_{s}}{\lambda} - 2\beta\right)\right).y + i\frac{\lambda}{2}\frac{\alpha_{s}}{\lambda}.\frac{\nabla k(\alpha)}{k(\alpha)}\right\}w - i\left(\frac{\lambda_{s}}{\lambda} + b\right)\Lambda w$$
$$+ i\left(2\beta - \frac{\alpha_{s}}{\lambda}\right).\nabla w - \left(\widetilde{\gamma}_{s} + |\beta|^{2} - \frac{\alpha_{s}}{\lambda}.\beta\right)w = 0$$
$$(1.17)$$

This leads us to fix the laws of the \mathcal{P} parameters such as :

$$\frac{\lambda_s}{\lambda} = -b, \quad b_s + b^2 = 0, \quad \frac{\alpha_s}{\lambda} = 2\beta, \quad \beta_s + b\beta = \mathcal{B}(\lambda, \alpha), \quad \widetilde{\gamma}_s = |\beta|^2 \tag{1.18}$$

One should note, it is an unexpected algoratic cancellation that allows a rather simple modification of these laws to work out when introducing the inhomogeneity k.

However, when introducing the metric through g and G, one should modify a bit the b law along with the β law. Still, we keep the following ansatz as a geometrical decomposition for u solution of (1.1):

$$u(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} v\left(t, \frac{x - \alpha(t)}{\lambda}\right) e^{i\gamma(t)} \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda^2}$$

$$w(s,y) = v(s,y) \ e^{i \ b \frac{|y|^2}{4} - i\beta \cdot y}.$$
(1.19)

Remark 2. We will see later this ansatz is not as well-adapted as it was in the flat case, for the metric part bring in our computations some drastic changes in the b law.

Note the pertubation induced by g and G could have been dealt with otherwise. For instance one may choose to introduce a pertubated set of orthogonality conditions (3.7)-(3.11) which drive the modulations laws, as it is done later on in (5.80)-(5.81) for both the b and the β laws.

For the sake of simplicity, and compactness of our further computations, we now introduce some notations we will use a lot in the sequel. In particular, we introduce the pseudo Einstein notation we will use to leave our matrices terms out of the summation symbol \sum .

F being a well defined function of the \mathcal{P} parameters, we will denote by $F^{(p)}$ the expansion of F at order p with respect to \mathcal{P} , and $F^{[p]}$ the term of order p in that expansion.

$$\Delta_{G} v = div \left(G(\lambda y + \alpha) \nabla v \right) = \partial_{i} \left(G(\lambda y + \alpha) \partial_{j} v \right)$$

$$= \Delta v + \sum_{p=2}^{4} \left(G_{ij}^{[p]} \partial_{ij}^{2} v + \lambda \partial_{i} G_{ij}^{[p]} \partial_{j} v \right) + \mathcal{O} \left(\mathcal{P}^{5} ||v||_{H^{2}} \right)$$

$$= \Delta v + \sum_{p=2}^{4} \Delta_{G}^{[p]} v + \mathcal{O} \left(\mathcal{P}^{5} ||v||_{H^{2}} \right)$$

$$\Delta_{G}^{[p]} = G_{ij}^{[p]} \partial_{ij}^{2} + \lambda \partial_{i} G_{ij}^{[p]} \partial_{j}$$

$$G_{ij}^{[p]} = \frac{1}{p!} \nabla^{p} G_{ij}(0) . (\lambda y + \alpha)^{p}$$

$$\partial_{i} G_{ij}^{[p]} = \frac{1}{(p-1)!} \nabla^{p} G_{ij}(0) . (\partial_{i}, (\lambda y + \alpha)^{p-1})$$
(1.20)

and

$$\begin{split} \Delta_{G,g} v &= \frac{1}{g(\lambda y + \alpha)} div \big(G(\lambda y + \alpha) \, \nabla v \big) = \frac{1}{g(\lambda y + \alpha)} \, \partial_i \big(G(\lambda y + \alpha) \, \partial_j v \big) \\ &= \Delta v + \big(\Delta_G^{[2]} - g^{[2]} \Delta \big) \, v + \big(\Delta_G^{[3]} - g^{[3]} \Delta \big) \, v \\ &+ \left(\Delta_G^{[4]} - g^{[2]} \, \Delta_G^{[2]} + \left(\frac{(g^{[2]})^2}{2} - g^{[4]} \right) \Delta \right) \, v + \mathcal{O} \Big(\mathcal{P}^5 \, \|v\|_{H^2} \Big) \end{split}$$
(1.21)
$$g(\lambda y + \alpha) = 1 + g^{[2]} + g^{[3]} + g^{[4]} + \mathcal{O} \big(\mathcal{P}^5 \big) \\ g^{[p]} &= \frac{1}{p!} \, \nabla^p g(0) . (\lambda y + \alpha)^p \end{split}$$

Eventually, the expansion of our operator $\mathcal{L}_{\lambda\alpha}$ will be given by

$$\mathcal{L}_{\lambda\alpha}v = \frac{1}{g(\lambda y + \alpha)} div (G(\lambda y + \alpha) \nabla v) + \lambda^2 V(\lambda y + \alpha) v$$

$$= \Delta v + \sum_{p=2}^{4} \mathcal{L}_{\lambda\alpha}^{[p]} v + \mathcal{O} (\mathcal{P}^5 ||v||_{H^2})$$

$$\mathcal{L}_{\lambda\alpha}^{[2]} = \Delta_G^{[2]} - g^{[2]} \Delta + \lambda^2 V(0)$$

$$\mathcal{L}_{\lambda\alpha}^{[3]} = \Delta_G^{[3]} - g^{[3]} \Delta + \lambda^2 \nabla V(0).(\lambda y + \alpha)$$

$$\mathcal{L}_{\lambda\alpha}^{[4]} = \Delta_G^{[4]} - g^{[2]} \Delta_G^{[2]} + \left(\frac{(g^{[2]})^2}{2} - g^{[4]}\right) \Delta + \frac{\lambda^2}{2} \nabla^2 V(0).(\lambda y + \alpha)^2$$

(1.22)

Notice in the sequel we will often write g or G_{ij} instead of $g(\lambda y + \alpha)$ or $G_{ij}(\lambda y + \alpha)$, so one should be

cautious when derivating with respect to y that a λ factor appears.

Assuming g, G and V satisfy respectively (H2) and (H3), v must solve

$$i\partial_s v + \mathcal{L}_{\lambda\alpha} v - v + \frac{k(\lambda y + \alpha)}{k(\alpha)} v|v|^2 = i\frac{\lambda_s}{\lambda} \Lambda v + i\frac{\alpha_s}{\lambda} \cdot \left(\nabla v + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}v\right) + \widetilde{\gamma}_s v \tag{1.23}$$

For further computations, we need some more notations that will shorten our expressions, so let

$$\psi = b \left(1 + \lambda^2 \phi\right) \frac{|y|^2}{4} - \beta . y,$$

$$\tilde{G}_{ij} = (1 - g) I_{ij} + (G_{ij} - I_{ij})$$

Notice one has then, for any function v

$$\begin{aligned} \mathcal{L}_{\lambda\alpha} v &= \Delta v + \frac{1}{g} \left[\Delta_{\tilde{G}} \, v + \nabla g \, . \, \nabla v \right] + \lambda^2 \, V \, v \\ \text{with} \quad \Delta_{\tilde{G}} \, v &= \partial_i \big(\tilde{G}_{ij} \partial_j v \big) \end{aligned}$$

Now, we claim we have

$$i \partial_{s} v e^{i \psi} = i \partial_{s} w + \left[b_{s} \frac{|y|^{2}}{4} - \beta_{s} \cdot y \right] w$$

$$\nabla v e^{i \psi} = \nabla w + i \left[-b \frac{y}{2} + \beta \right] w$$

$$\Lambda v e^{i \psi} = \Lambda w + i \left[-b \frac{|y|^{2}}{2} + \beta \cdot y \right] w$$

$$\Delta_{G,g} v e^{i \psi} = \Delta_{G,g} w + \frac{2i}{g} G_{ij} \left(-b \frac{y_{i}}{2} + \beta_{i} \right) \partial_{j} w - i \frac{b}{2} \frac{1}{g} \left[G_{ii} + \lambda \partial_{i} G_{ij} y_{j} \right] w$$

$$- \frac{1}{g} G_{ij} \left(-b \frac{y_{i}}{2} + \beta_{i} \right) \left(-b \frac{y_{j}}{2} + \beta_{j} \right) w$$
(1.24)

Putting all previous terms together, using equation (1.23), we now see w must satisfy

$$i\partial_{s}w + \mathcal{L}_{\lambda\alpha}w - w + \frac{k(\lambda y + \alpha)}{k(\alpha)}w|w|^{2} + (b_{s} + b^{2})\frac{|y|^{2}}{4}w$$

$$-\left\{\left(\beta_{s} + b\beta\right) \cdot y + i\left[\lambda\beta \cdot \frac{\nabla k(\alpha)}{k(\alpha)} + \frac{b}{2g}\left(\lambda\partial_{i}G_{ij}y_{j} + (1 - g)G_{ii} + (G_{ii} - 2)\right)\right]\right\}w$$

$$-\frac{1}{g}\left\{\tilde{G}_{ij}\left[p_{ij}^{[2]}w + i\left(by_{i}\partial_{j}w - 2\beta_{i}\partial_{j}w\right)\right] + ib\left(1 - g\right)\left(G_{ij} - I_{ij}\right)y_{i}\partial_{j}w\right\}$$

$$-\left(\frac{\lambda_{s}}{\lambda} + b\right)\left[i\Lambda w + 2b\frac{|y|^{2}}{4}w - \beta \cdot yw\right] - \left(\widetilde{\gamma}_{s} - |\beta|^{2}\right)w$$

$$-\left(\frac{\alpha_{s}}{\lambda} - 2\beta\right)\cdot\left[i\left(\nabla w + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\right) + b\frac{y}{2} - \beta w\right] = 0$$

$$(1.25)$$

where

$$p_{ij}^{[2]} = p_{ij}^{[2]}(y) = b^2 \frac{y_i y_j}{4} - b \beta_i y_j + \beta_i \beta_j$$
(1.26)

For now, we claim we may choose the ${\mathcal P}$ parameters laws such as :

$$\frac{\lambda_s}{\lambda} + b = 0, \quad b_s + b^2 = B_1(\lambda, \alpha, b, \beta), \quad \frac{\alpha_s}{\lambda} = 2\beta,$$

$$\beta_s + b\beta = B_2(\lambda, \alpha, b, \beta), \quad \widetilde{\gamma}_s = |\beta|^2$$
(1.27)

so that w(s, y) is solution of equation

$$i\partial_{s}w + \mathcal{L}_{\lambda\alpha}w - w + \frac{k(\lambda y + \alpha)}{k(\alpha)}w|w|^{2} + B_{1}\frac{|y|^{2}}{4}w$$

$$-\left\{B_{2} \cdot y + i\left[\lambda\beta \cdot \frac{\nabla k(\alpha)}{k(\alpha)} + \frac{b}{2g}\left(\lambda\partial_{i}G_{ij}y_{j} + (1 - g)G_{ii} + (G_{ii} - 2)\right)\right]\right\}w$$

$$-\frac{1}{g}\left\{\tilde{G}_{ij}\left[p_{ij}^{[2]}w + i\left(by_{i}\partial_{j}w - 2\beta_{i}\partial_{j}w\right)\right] + ib\left(1 - g\right)\left(G_{ij} - I_{ij}\right)y_{i}\partial_{j}w\right)\right\} = 0$$

$$(1.28)$$

by introducing pertubation terms, we rewrite the equation as

$$i\partial_{s}w + \mathcal{L}_{\lambda\alpha}w - w + \frac{k(\lambda y + \alpha)}{k(\alpha)}w|w|^{2} + B_{1}\frac{|y|^{2}}{4}w$$

$$-\left\{B_{2} \cdot y + i\left[\lambda\beta \cdot \frac{\nabla k(\alpha)}{k(\alpha)} + \frac{b}{2}\Gamma_{12}\right]\right\}w - \left(\Gamma_{21}w + i\left(\Gamma_{22}\right)_{j}\partial_{j}w\right) = 0$$
(1.29)
$$\Gamma_{12} = \frac{1}{g}\left(\lambda\partial_{i}G_{ij}\tilde{y}_{j} + (1 - g)G_{ii} + (G_{ii} - 2)\right)$$

$$\Gamma_{21} = \frac{1}{g}\tilde{G}_{ij}p_{ij}^{[2]}$$

$$\left(\Gamma_{22}\right)_{j} = \frac{1}{g}\left[\tilde{G}_{ij}\left(by_{i} - 2\beta_{i}\right) + b\left(1 - g\right)\left(G_{ij} - I_{ij}\right)y_{i}\right]$$

Until the end of the first section, we look forward to building a fourth order approximation $P_{\mathcal{P}}$ for function w. Then, introducing $Q_{\mathcal{P}} = P_{\mathcal{P}} e^{-i\psi}$, the exact solution u of (1.1) will be approximated by

$$u(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} Q_{\mathcal{P}}\left(t, \frac{x - \alpha(t)}{\lambda}\right) e^{i\gamma(t)} + \mathcal{O}\left(\mathcal{P}^5 e^{-c_5 \left|\frac{x - \alpha}{\lambda}\right|}\right)$$
(1.30)

Note that when introducing the inhomogeneity k, a β -law change was enough to ensure the existence of an approximate solution. Now we also introduce metric terms g and G in the equation, so now both changes on the b and β laws seem to be needed to easily ensure the existence of such an approximate solution. However, the Backward Propagation of smallness method which consists in integrating backward from blow-up time our approximate profile to ensure the whole geometrical decomposition (1.30) describes our solution u near blow-up point, one needs the b-law to be at most $\mathcal{O}(|\alpha|^2 + \lambda^4)$ for the sake of its integrability.

Note also the metric brings some extra mass to the approximate profile as a sign that our solution u now naturally lies in a $L^2(dg)$ space where dg is slightly modified from euclidean measure.

All small terms we have obtained in the previous equation will need to be expanded near the origin, which we will do in the next section.

Remark 3. Note in [RS11], it is assumed from the beginnig that

$$\mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^3\left(\frac{b}{\lambda} - \frac{1}{C_0}\right) \lesssim \mathcal{P}^5$$
(1.31)

with $C_0 > 0$ a constant to be set later. All computations are therefore much more simple. However, this may only be proved after quite some work, that is at the end of the fourth section in (4.61) of Lemma 4.7. So we decide here to do as much computations as possible to make clear where this is needed.

In the [RS11] proof, that assumption may be used only from the third section, since it is previously used only to make computations more simple. Yet, we will use it more dramatically, since it comes in handy from the construction of our approximate profile at order 3.

1.3 Expansion near the origin

We will now expand equation (1.28) near the origin up to fourth order. Let us recall that λ should be thought as the scalling parameter that drives the blow up of the solution u of (1.1) for $\|\nabla u\|_{L^2} = \frac{1}{\lambda} \|\nabla v\|_{L^2} \sim \frac{1}{\lambda}$. Assuming u is blowing up at $x_0 = 0$, when $t \to T = 0$, provided λ and α are chosen as continuous function of time, they must satisfy :

$$\lambda \longrightarrow 0, \quad t \to T = 0 \quad (i.e. \quad s \to +\infty), \qquad \lambda(0) = T = 0$$

$$\alpha \longrightarrow 0, \quad t \to T = 0 \quad (i.e. \quad s \to +\infty), \qquad \alpha(0) = x_0 = 0 \tag{1.32}$$

We list here expansions up to order 4 with respect to the \mathcal{P} parameters of all terms of equation (1.28). Recall such a description has already been given for the laplacian term $\mathcal{L}_{\lambda\alpha}w$ in (1.22). First, to deal with the $\frac{k(\lambda y+\alpha)}{k(\alpha)}$ term :

$$\frac{k(\lambda y + \alpha)}{k(\alpha)} = \frac{1 + \frac{1}{2}\nabla^2 k(0).(\lambda y + \alpha)^2 + \frac{1}{6}\nabla^3 k(0).(\lambda y + \alpha)^3 + \frac{1}{24}\nabla^4 k(0).(\lambda y + \alpha)^4 + \mathcal{O}(\mathcal{P}^5)}{1 + \frac{1}{2}\nabla^2 k(0).(\alpha)^2 + \frac{1}{6}\nabla^3 k(0).(\alpha)^3 + \frac{1}{24}\nabla^4 k(0).(\alpha)^4 + \mathcal{O}(\mathcal{P}^5)} = 1 + k^{[2]} + k^{[3]} + k^{[4]} + \mathcal{O}(\mathcal{P}^5)$$

$$k^{[2]} = \frac{\lambda^2}{2}\nabla^2 k(0).(y, y) + \lambda\nabla^2 k(0).(y, \alpha)$$

$$k^{[3]} = \frac{\lambda^3}{6}\nabla^3 k(0).(y, y, y) + \frac{\lambda^2}{2}\nabla^3 k(0).(y, y, \alpha) + \frac{\lambda}{2}\nabla^3 k(0).(y, \alpha, \alpha)$$

$$k^{[4]} = \frac{1}{24}\nabla^4 k(0).\left[(\lambda y + \alpha)^4 - (\alpha)^4\right] - \left[\frac{\lambda^2}{4}\nabla^2 k(0).(y, y) + \frac{\lambda}{2}\nabla^2 k(0).(y, \alpha)\right]\nabla^2 k(0).(\alpha, \alpha)$$
(1.33)

then there only remains two terms we wish to expand, one term appearing in the imaginary part of the multiplier of w

$$\lambda \beta \cdot \frac{\nabla k(\alpha)}{k(\alpha)} + \frac{b}{2g} \left(\lambda \partial_i G_{ij} y_j + (1-g) G_{ii} + (G_{ii} - 2) \right) = \Gamma_1^{[3]} + \Gamma_1^{[4]} + \mathcal{O}(\mathcal{P}^5) \Gamma_1^{[3]} = \lambda \nabla^2 k(0).(\alpha, \beta) + \frac{b \lambda}{2} \partial_i G_{ij}^{[2]} y_j - b g^{[2]} + \frac{b}{2} G_{ii}^{[2]} \Gamma_1^{[4]} = \frac{\lambda}{2} \nabla^3 k(0).(\alpha, \alpha, \beta) + \frac{b \lambda}{2} \partial_i G_{ij}^{[3]} y_j - b g^{[3]} + \frac{b}{2} G_{ii}^{[3]}$$
(1.34)

and another one which gathers all non-diagonal terms

$$\frac{1}{g} \left\{ \tilde{G}_{ij} \left[p_{ij}^{[2]} w + i \left(b \, y_i \, \partial_j w - 2 \, \beta_i \, \partial_j w \right) \right] + i \, b \left(1 - g \right) \left(G_{ij} - I_{ij} \right) y_i \, \partial_j w \right\}
= \Gamma_{21}^{[4]} w + i \left[\left(\Gamma_{22} \right)_j^{[3]} + \left(\Gamma_{22} \right)_j^{[4]} \right] \partial_j w + \mathcal{O} (\mathcal{P}^5)$$
(1.35)
$$\Gamma_{21}^{[4]} = \left(G_{ij}^{[2]} - g^{[2]} \, I_{ij} \right) p_{ij}^{[2]}
\left(\Gamma_{22} \right)_j^{[3]} = \left(G_{ij}^{[2]} - g^{[2]} \, I_{ij} \right) \left(b \, y_i - 2 \, \beta_i \right)
\left(\Gamma_{22} \right)_j^{[4]} = \left(G_{ij}^{[3]} - g^{[3]} \, I_{ij} \right) \left(b \, y_i - 2 \, \beta_i \right)$$

Since we are only working here in dimension 2, one will find convenient that given $p \in \{1, 2\}$, we denote by \tilde{p} the associated index such as $\{p; \tilde{p}\} = \{1, 2\}$.

Moreover, when needed, it will sometimes be convenient to use the notation (∂_1, ∂_2) to denote the canonical basis for \mathbb{R}^2 .

We will also denote by Tr the trace linear operator as $Tr(A) = A_{ii} = \sum A_{ii}$.

In the sequel we will make use of all these notations, referring to this section.

1.4 Modulation and construction of the approximation

Let us recall, that the Ground State of the 2 dimension NLS is the unique H^1 , positive and radial solution of the elliptic equation :

$$\Delta Q - Q + Q^3 = 0 \tag{1.36}$$

First one should note that taking scalar product of $\Delta Q = Q - Q^3$ with Q on the one hand, and its scalar product with ΛQ on the other hand, then computing it both directly and after integrating by parts, one get

$$\int Q^2 = \int |\nabla Q|^2 = \int \frac{Q^4}{2}$$
(1.37)

Now, Q being radial, it may be expressed as a function $Q(y) = h(|y|^2) = h(y_1^2 + y_1^2)$ with h a well defined real function of class C^1 . In the sequel the radiality property will help to simplify computations.

We will now form an approximate solution of the equation (1.28) or (1.29) up to the fourth order with respect of the \mathcal{P} parameters as an asymptotic expansion near the Ground State. Let :

$$P_{\mathcal{P}} = Q + \sum_{j=1}^{4} \left(T_j + iS_j \right)$$
 (1.38)

where T_j and S_j are regular functions of order j.

We now introduce the linear expansion of the 2 dimension NLS operator near Q, namely $L = (L_+, L_-)$ with :

$$L_{+}f = -\Delta f + f - 3Q^{2}f, \qquad L_{-}f = -\Delta f + f - Q^{2}f$$
(1.39)

We now state the result :

Proposition 1.1. Let $C_0 > 0$. One can find a universal constant c > 0 and a small constant $\eta^*(C_0) > 0$ such that whenever $|\mathcal{P}| = |(b, \lambda, \alpha, \beta)| \leq \eta^*$, what follows is satisfied :

We have regulars and well-localized functions $(T_j, S_j)_{1 \le j \le 4}$ of order j with respect to \mathcal{P} such that :

$$P_{\mathcal{P}} = Q + \sum_{j=1}^{4} \left(T_j + iS_j \right)$$

is an approximate solution of (1.28) or (1.29), that is :

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$$i\left(-b^{2}+B_{1}\right)\partial_{b}P_{\mathcal{P}}-ib\lambda\partial_{\lambda}P_{\mathcal{P}}+2i\lambda\beta\partial_{\alpha}P_{\mathcal{P}}+i\left(-b\beta+B_{2}\right)\partial_{\beta}P_{\mathcal{P}}$$

$$+\mathcal{L}_{\lambda\alpha}w-w+\frac{k(\lambda y+\alpha)}{k(\alpha)}w|w|^{2}+B_{1}\frac{|y|^{2}}{4}w$$

$$-\left\{B_{2}\cdot y+i\left[\lambda\beta\cdot\frac{\nabla k(\alpha)}{k(\alpha)}+\frac{b}{2}\Gamma_{12}\right]\right\}w-\left(\Gamma_{21}w+i\left(\Gamma_{22}\right)_{j}\partial_{j}w\right)=-\tilde{\Psi}_{\mathcal{P}}$$

$$(1.40)$$

$$\Gamma_{12}=\frac{1}{g}\left(\lambda\partial_{i}G_{ij}\tilde{y}_{j}+(1-g)G_{ii}+(G_{ii}-2)\right)$$

$$\Gamma_{21}=\frac{1}{g}\tilde{G}_{ij}p_{ij}^{[2]}$$

$$\left(\Gamma_{22}\right)_{j}=\frac{1}{g}\left[\tilde{G}_{ij}\left(by_{i}-2\beta_{i}\right)+b\left(1-g\right)\left(G_{ij}-I_{ij}\right)y_{i}\right]$$

with $\tilde{\Psi}_{\mathcal{P}}$ polynomial in \mathcal{P} , and well-localized in y :

$$\forall p \in \mathbb{N}^2, \exists C_p > 0, \quad \left| \partial^p \tilde{\Psi}_{\mathcal{P}}(y) \right| \lesssim \mathcal{P}^5 e^{-C_p |y|}$$
(1.41)

In order to satisfy this construction, the b and β laws are set in such a way that $B_1 \in \mathbb{R}$ and $B_2 \in \mathbb{R}^2$ are :

$$B_{1} = \frac{\lambda^{2}}{2} K_{G,g} + b_{0}(\alpha, \alpha) + \lambda^{2} l_{0}(\alpha) + \mathcal{O}(\mathcal{P}^{4})$$

$$K_{G,g} = \Delta g(0) - (2 K_{21} + 1) \partial_{ii}^{2} G_{ii}(0) - (2 K_{22} + 1) \partial_{\tilde{p} \tilde{p}}^{2} G_{pp}(0) - 4 K_{21} \partial_{12}^{2} G_{12}(0)$$

$$K_{21} = \frac{1}{8} \int \left[\left(3 + \frac{|y|^{2}}{2} \right) Q^{2} - 3 |y|^{2} |\nabla Q|^{2} \right] \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2}$$

$$K_{22} = \frac{1}{8} \int \left[\left(1 - \frac{|y|^{2}}{2} \right) Q^{2} - |y|^{2} |\nabla Q|^{2} \right] \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2}$$

$$b_{0}(\alpha, \alpha) = \frac{1}{2} \nabla^{2} \left(k - \frac{1}{2} Tr(G) \right) (0) . (\alpha, \alpha) \left\| Q \right\|_{L^{2}}^{2} \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2}$$

$$l_{0}(\alpha) = \alpha . \nabla \left[\Delta \left(g - \frac{1}{2} Tr(G) \right) - \left(K_{21} \partial_{ij}^{2} G_{ij} + K_{22} \partial_{\tilde{p} \tilde{p}}^{2} G_{pp} \right) \right] (0)$$
(1.42)

 $B_2 = \lambda c_0(\alpha) + \lambda^3 C_3 + \mathcal{O}(\mathcal{P}^4)$

with $C_3 \in \mathbb{R}^2$ a constant, and c_0 a linear form on \mathbb{R}^2 .

Proof of Proposition 1.1:

<u>Order 0</u> (1.40) now becomes $\Delta Q - Q + Q^3 = 0$ which is the Ground State equation. This proves (1.38) is an order 1 pertubation of the Ground State equation.

<u>Order 1</u> Expanding the non-linear term up to first order and using (1.33) leads to :

$$\frac{k(\lambda y + \alpha)}{k(\alpha)} P_{\mathcal{P}} |P_{\mathcal{P}}|^2 = \left(1 + \mathcal{O}(\mathcal{P}^2)\right) \left(Q^3 + 3Q^2 T_1 + iQ^2 S_1 + \mathcal{O}(\mathcal{P}^2)\right)$$

= $Q^3 + 3Q^2 T_1 + iQ^2 S_1 + \mathcal{O}(\mathcal{P}^2)$ (1.43)

Then, first order terms in (1.40) must verify :

$$-L_{+}T_{1} - iL_{-}S_{1} = 0 \tag{1.44}$$

Hence we may choose $T_1 = 0$, $S_1 = 0$.

Construction of T_j and S_j , $j \ge 2$: Before going any further, we need a bit more of explanations regarding the constructions of the T_j and S_j functions. When calculating the *j*-th order of equation (1.40) we exhibit equations on L_+T_j and L_-S_j . So we need to recall some facts about operators L_+ and L_- . Let us begin with :

$$Ker(L_{-}) = span(Q), \quad Ker(L_{+}) = span(\partial_{j}Q, j = 1, 2) = span(\nabla Q)$$
(1.45)

Now using (1.22), at order j equation (1.40) is equivalent to :

$$\begin{cases} L_{+}T_{j} = F_{j}, & F_{j} = \mathcal{O}(\mathcal{P}^{j}) \text{ in } L^{2} \\ L_{-}S_{j} = G_{j}, & G_{j} = \mathcal{O}(\mathcal{P}^{j}) \text{ in } L^{2} \end{cases}$$
(1.46)

Then, L_+ and L_- are self-adjoint operators. This implies :

$$\exists T_j \in H^2, \quad L_+T_j = F_j \Leftrightarrow F_j \in Im(L_+) = Ker(L_+)^{\perp} \Leftrightarrow \left(F_j, \nabla Q\right) = 0$$

$$\exists S_j \in H^2, \quad L_-S_j = G_j \Leftrightarrow G_j \in Im(L_-) = Ker(L_-)^{\perp} \Leftrightarrow \left(G_j, Q\right) = 0$$
(1.47)

We now have found a process to decide wether or not T_j and S_j exist.

<u>Order 2</u> Expanding the non-linear term up to second order and using (1.33) leads to :

$$\frac{k(\lambda y + \alpha)}{k(\alpha)} P_{\mathcal{P}} |P_{\mathcal{P}}|^2 = \left(1 + k^{[2]} + \mathcal{O}(\mathcal{P}^3)\right) \left(Q^3 + 3Q^2 T_2 + iQ^2 S_2 + \mathcal{O}(\mathcal{P}^3)\right)$$

= $Q^3 + 3Q^2 T_2 + iQ^2 S_2 + k^{[2]} Q^3 + \mathcal{O}(\mathcal{P}^3)$ (1.48)

Thus equation (1.40) at order 2 implies :

$$\begin{cases} L_{+}T_{2} = \mathcal{L}_{\lambda\alpha}^{[2]}Q + k^{[2]}Q^{3} + B_{1}^{[2]}\frac{|y|^{2}}{4}Q - B_{2}^{[2]} \cdot y \ Q =: F_{2} \\ L_{-}S_{2} = 0 \end{cases}$$
(1.49)

We can choose at once $S_2 = 0$. Still, we need to ensure $(F_2, \nabla Q) = 0$.

 $\frac{\text{Computation of } \left(F_2, \ \partial_p Q\right), \quad 1 \le p \le 2}{B_1^{[2]} \frac{|y|^2}{4} Q} \text{ being radial, we have } \left(B_1^{[2]} \frac{|y|^2}{4} Q, \ \partial_p Q\right) = 0. \text{ Now from (1.22) and (1.37), one may compute} \\ \left(\mathcal{L}_{\lambda\alpha}^{[2]} Q, \ \partial_p Q\right) = \frac{\lambda}{4} \left[\nabla^2 Tr(G)(0).(\alpha, \partial_p) + \nabla^2 G_{12}(0).(\alpha, \partial_{\tilde{p}} - \partial_p)\right] \int Q^2$

and from (1.33)

$$\left(k^{[2]}Q^3, \ \partial_j Q\right) = -\frac{\lambda}{2}\nabla^2 k(0).(\alpha, \partial_p) \int Q^2$$

Now by integration by parts, we have :

$$\left(-B_2^{[2]} \cdot y \ Q, \ \partial_p Q\right) = \left(B_2^{[2]}\right)_p \ \int \frac{Q^2}{2}$$

Then, one easily check that choosing

$$\left(B_2^{[2]}\right)_p = \lambda \,\nabla^2 \left(k - \frac{1}{2} \,Tr(G)\right)(0).(\alpha, \partial_p) + \frac{\lambda}{2} \,\nabla^2 G_{12}(0).(\alpha, \partial_p - \partial_{\tilde{p}}) \tag{1.50}$$

ensures the existence of T_2 .

Order 3 Expanding the non-linear term up to third order yields :

$$\frac{k(\lambda y + \alpha)}{k(\alpha)} P_{\mathcal{P}} |P_{\mathcal{P}}|^{2} = \left(1 + k^{[2]} + k^{[3]} + \mathcal{O}(\mathcal{P}^{4})\right) \left(Q^{3} + 3Q^{2}T_{2} + iQ^{2}S_{2} + 3Q^{2}T_{3} + iQ^{2}S_{3} + \mathcal{O}(\mathcal{P}^{4})\right)$$

$$= Q^{3} + \left[3Q^{2}T_{2} + iQ^{2}S_{2} + k^{[2]}Q^{3}\right] + \left[3Q^{2}T_{3} + iQ^{2}S_{3} + k^{[3]}Q^{3}\right] + \mathcal{O}(\mathcal{P}^{4})$$

$$(1.51)$$

Thus, using (1.34) equation (1.40) at order 3 yields :

$$\begin{cases} L_{+}T_{3} = \mathcal{L}_{\lambda\alpha}^{[3]}Q + k^{[3]}Q^{3} + B_{1}^{[3]}\frac{|y|^{2}}{4}Q - B_{2}^{[3]}.y \ Q =: F_{3} \\ L_{-}S_{3} = -\Gamma_{1}^{[3]}Q - \left(\Gamma_{22}^{[3]}\right)_{j}\partial_{j}Q + \partial_{pa}^{[2]}T_{2} =: G_{3} = G_{31} + \partial_{pa}^{[2]}T_{2} \end{cases}$$
(1.52)

where :

$$\partial_{pa}^{[j]} f = \left(\partial_s f\right)^{[j+1]} = \left[\left(\left(-b^2 + B_1\right)\partial_b - b\,\lambda\,\partial_\lambda + 2\,\lambda\,\beta\,\partial_\alpha + \left(-b\beta + B_2\right)\partial_\beta\right)f\right]^{[j+1]}$$
(1.53)

Here $\partial_{pa}^{[2]}T_2$ is a source term coming from approximation of $\partial_s T_2$ at order 3.

$$\partial_{pa}^{[j]}T_j = \left(\partial_s T_j\right)^{[j+1]}$$

 $\frac{\text{Computation of } \left(F_3, \ \partial_p Q\right), \quad 1 \le p \le 2}{\text{First}, \ B_1^{[3]} \frac{|y|^2}{4} Q \text{ being radial, we have :}}$

$$\left(B_1^{[3]}\frac{|y|^2}{4}Q, \ \partial_p Q\right) = 0$$

Then from (1.22), (1.37) integration by parts and the radially symmetry of Q, one may compute

$$\begin{pmatrix} \mathcal{L}_{\lambda\alpha}^{[3]}Q, \ \partial_p Q \end{pmatrix} = \frac{\lambda^3}{4} \left[\partial_p \Delta Tr(G)(0) + 2 \,\partial_{12}^2 \partial_i G_{i\tilde{p}}(0) \right] \int \frac{|y|^2}{4} |\nabla Q|^2 - \frac{\lambda^3}{2} \,\partial_p \Delta G_{\tilde{p}\tilde{p}}(0) \int \frac{|y|^2}{4} \left(Q^2 - \frac{Q^4}{4} \right) + \frac{\lambda}{4} \left[\nabla^3 Tr(G)(0).(\alpha, \alpha, \partial_p) + \nabla^3 G_{12}(0).(\alpha, \alpha, \partial_{\tilde{p}}) \right] \int \frac{Q^2}{2}$$

and since $\left(Q^3, \partial_p Q\right) = 0$

Now we can decide that $B_2^{[3]} = \lambda^3 C_3 + \lambda c_1(\alpha, \alpha)$ at order 3, with $C_3 \in \mathbb{R}^2$ a constant and $c_1 : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a bilinear symmetric map. As before, the last term is calculated by integration by parts :

$$\left(-B_2^{[3]} \cdot y \ Q, \ \partial_p Q\right) = \lambda^3 (C_3)_p \int \frac{Q^2}{2} \, dy + \lambda (c_1(\alpha, \alpha))_p \int \frac{Q^2}{2} \, dy$$

Then we can ensure T_3 existence by choosing :

$$(C_3)_p = \frac{\lambda^3}{2} \int \frac{|y|^2}{4} \left[\partial_p \Delta k(0) Q^4 - \partial_p \Delta Tr(G)(0) |\nabla Q|^2 \right] ||Q||_{L^2}^{-2} + \lambda^3 \int \frac{|y|^2}{4} \left[\partial_p \Delta G_{\tilde{p}\tilde{p}}(0) \left(Q^2 - \frac{Q^4}{4} \right) - \partial_{12}^2 \partial_i G_{i\tilde{p}}(0) |\nabla Q|^2 \right] ||Q||_{L^2}^{-2} (1.54) (c_1(\alpha, \alpha))_p = \frac{\lambda}{2} \left[\nabla^3 \left(k - \frac{1}{2} Tr(G) \right) (0) \cdot (\alpha, \alpha, \partial_p) - \frac{1}{2} \nabla^3 G_{12}(0) \cdot (\alpha, \alpha, \partial_{\tilde{p}}) \right]$$

 $\frac{\text{Computation of } \left(G_3, Q\right)}{\text{Let us begin with recalling some identities :}}$

$$L_{+}(\nabla Q) = 0, \quad L_{+}(\Lambda Q) = -2Q$$

$$L_{-}Q = 0, \quad L_{-}(yQ) = -2\nabla Q, \quad L_{-}(|y|^{2}Q) = -4\Lambda Q$$
(1.55)

 L_+ being self-adjoint, this implies

$$\begin{pmatrix} G_3, Q \end{pmatrix} = \begin{pmatrix} G_{31} + \partial_{pa}^{[2]} T_2, Q \end{pmatrix}$$
$$= \begin{pmatrix} G_{31}, Q \end{pmatrix} - \frac{1}{2} \begin{pmatrix} L_+ \partial_{pa}^{[2]} T_2, \Lambda Q \end{pmatrix}$$

Now to ensure the existence of S_3 , we just need to make $(G_3, Q) = 0$. First, L_+ being a linear operator not depending on \mathcal{P} , one gets

$$L_+ \partial_{pa}^{[2]} T_2 = \partial_{pa}^{[2]} \left(L_+ T_2 \right)$$

and

$$\left(L_{+} \partial_{pa}^{[2]} T_{2}, \Lambda Q\right) = \partial_{pa}^{[2]} \left(L_{+} T_{2}, \Lambda Q\right)$$

Now, using (1.49) and (1.22) and the radial property of the Ground State as we did before

$$\left(\mathcal{L}_{\lambda\alpha}^{[2]}Q, \Lambda Q \right) = \lambda^2 \left[\Delta \left(g - \frac{1}{2} Tr(G) \right) (0) - \left(\partial_{ij}^2 G_{ij}(0) K_{21} + \nabla^2 G_{pp}(0) . (\partial_{\tilde{p}}, \partial_{\tilde{p}}) K_{22} \right) \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^2$$

$$+ \frac{1}{2} \nabla^2 \left(g - \frac{1}{2} Tr(G) \right) (0) . (\alpha, \alpha) \int Q^2$$

$$K_{21} = \frac{1}{8} \int \left[\left(3 + \frac{|y|^2}{2} \right) Q^2 - 3 |y|^2 |\nabla Q|^2 \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2}$$

$$K_{22} = \frac{1}{8} \int \left[\left(1 - \frac{|y|^2}{2} \right) Q^2 - |y|^2 |\nabla Q|^2 \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2} .$$

$$(1.56)$$

Moreover, thanks to the following algebraic cancellation

$$(y_j y_l Q^3, \Lambda Q) = 0, \quad \text{for } j, l = 1, 2$$
 (1.57)

we get from (1.33)

$$\left(k^{[2]} Q^3, \Lambda Q\right) = 0$$

Eventually, we have

$$\left(B_1^{[2]} \frac{|y|^2}{4} Q - B_2^{[2]} \cdot y Q, \ \Lambda Q\right) = -B_1^{[2]} \int \frac{|y|^2}{4} Q^2$$

so that

$$\left(L_{+}T_{2}, \Lambda Q \right) = \lambda^{2} \left[\Delta \left(g - \frac{1}{2} Tr(G) \right)(0) - \left(\partial_{ij}^{2} G_{ij}(0) K_{21} + \nabla^{2} G_{pp}(0) . (\partial_{\tilde{p}}, \partial_{\tilde{p}}) K_{22} \right) \right] \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{2}$$

$$+ \frac{1}{2} \nabla^{2} \left(g - \frac{1}{2} Tr(G) \right)(0) . (\alpha, \alpha) \int Q^{2} - B_{1}^{[2]} \int \frac{|y|^{2}}{4} Q^{2}$$

$$(1.58)$$

Then, using (1.34), (1.35), and again the radial symmetry of Q, one may finally compute

$$\left(-\Gamma_{1}^{[3]}Q - \left(\Gamma_{22}^{[3]}\right)_{j}\partial_{j}Q, Q \right) = \frac{b}{4}\nabla^{2}g(0).(\alpha, \alpha) \int Q^{2} - \frac{b\lambda^{2}}{2}\Delta g(0) \int \frac{|y|^{2}}{4}Q^{2} - \lambda \left[\nabla^{2}(k-g)(0).(\alpha, \beta) + \nabla^{2}G_{ij}(0).(\alpha, \partial_{i})\beta_{j} \right] \int Q^{2}$$

$$(1.59)$$

The first and the last terms of that expression raise the first issue, as we see it does not fit in our construction of the solution. Indeed, the first one seems far from looking like some $\partial_s f$, while the last one is a priori not symmetric in α and β .

Fortunately, it seems possible to get rid of those, claiming they were not third order terms in the first place. We thus assume here, α and β are very small regarding the others parameters, namely we use Remark 3

$$\mathcal{P} |\alpha|^2 \lesssim \mathcal{P}^5, \quad \mathcal{P} |\alpha| |\beta| \lesssim \mathcal{P}^5$$

Now, for any symmetric bilinear form ϕ

$$-\frac{1}{2}\partial_{pa}^{[2]}(\lambda^2) = -\frac{1}{2}\left[-b\,\lambda\,\partial_\lambda(\lambda^2)\right] = b\,\lambda^2$$

so that altogether the previous computations yield

$$\left(G_3 \,,\, Q \right) = -\frac{1}{2} \partial_{pa}^{[2]} \left\{ \lambda^2 \left[\frac{1}{2} \Delta \left[g - Tr(G) \right](0) - \left(\partial_{ij}^2 G_{ij}(0) \,K_{21} + \nabla^2 G_{pp}(0) . (\partial_{\tilde{p}}, \partial_{\tilde{p}}) \,K_{22} \right) \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^2 + \frac{1}{2} \nabla^2 \left(g - \frac{1}{2} \,Tr(G) \right)(0) . (\alpha, \alpha) \,\int Q^2 - B_1^{[2]} \,\int \frac{|y|^2}{4} \,Q^2 \right\}$$

$$(1.60)$$

so that to get $(G_3, Q) = 0$ as targeted, it is enough to choose

$$B_{1}^{[2]} = \frac{\lambda^{2}}{2} \left[\Delta g(0) - \left(2K_{21} + 1 \right) \partial_{ii}^{2} G_{ii}(0) - \left(2K_{22} + 1 \right) \partial_{\tilde{p}\,\tilde{p}} G_{pp}(0) - 4K_{21} \partial_{12}^{2} G_{12}(0) \right] \\ + \frac{1}{2} \nabla^{2} \left(g - \frac{1}{2} Tr(G) \right) (0) \cdot (\alpha, \alpha) \|Q\|_{L^{2}}^{2} \|\frac{|y|}{2} Q\|_{L^{2}}^{-2}$$

$$(1.61)$$

Notice that (1.58) and (1.61) now yield

$$(L_+T_2, \Lambda Q) = \frac{\lambda^2}{2} \Delta g(0) \int \frac{|y|^2}{4} Q^2$$
 (1.62)

Notice the λ^2 term that now appeared in $B_1^{[2]}$ - in other words, in the *b* law. That is the term which we will have to nullify later to make *b* integrable, so the proof may be going on.

Order 4 Expanding the non-linear term up to fourth order yields :

$$\frac{k(\lambda y + \alpha)}{k(\alpha)} P_{\mathcal{P}} |P_{\mathcal{P}}|^{2}$$

$$= Q^{3} + \left[3Q^{2}T_{2} + iQ^{2}S_{2} + k^{[2]}Q^{3} \right] + \left[3Q^{2}T_{3} + iQ^{2}S_{3} + k^{[3]}Q^{3} \right]$$

$$+ \left[3Q^{2}T_{4} + iQ^{2}S_{4} + 3Q\left(T_{2}^{2} + k^{[2]}QT_{2}\right) + k^{[4]}Q^{3} \right] + \mathcal{O}(\mathcal{P}^{5})$$
(1.63)

Thus, using (1.22) equation (1.40) at order 4 yields :

$$\begin{pmatrix}
L_{+}T_{4} = \mathcal{L}_{\lambda\alpha}^{[4]}Q + k^{[4]}Q^{3} + 3Q(T_{2}^{2} + k^{[2]}QT_{2}) \\
+ B_{1}^{[4]}\frac{|y|^{2}}{4}Q - B_{2}^{[4]} \cdot y Q - \Gamma_{21}^{[4]}Q =: F_{4} \\
L_{-}S_{4} = -\Gamma_{1}^{[4]}Q - (\Gamma_{22}^{[4]})_{j}\partial_{j}Q + \partial_{pa}^{[3]}T_{3} =: G_{4}
\end{cases}$$
(1.64)

 $\begin{array}{l} \begin{array}{l} \mbox{Computation of } \left(F_4 \,,\, \partial_p Q\right), \quad 1 \leq p \leq 2 \\ \hline \mbox{First, let } f_4 := 3\,Q\left(T_2^2 + k^{[2]}\,Q\,T_2\right) \mbox{ a polynomial function with respect to } T_2, \mbox{ that is a well-localized function in } y, \mbox{ of order } 4 \mbox{ with respect to the } \mathcal{P} \mbox{ parameters.} \\ \hline \mbox{Then, } B_1^{[4]} \, \frac{|y|^2}{4} Q \mbox{ being radial, we have :} \end{array}$

$$\left(B_1^{[4]} \,\frac{|y|^2}{4}Q\,,\,\partial_p Q\right) = 0$$

Here we do not compute the terms in detail, since we will not need it. Notice as for the term f_4 , we get here some source terms coming from the second order computations.

As before, one gets

$$-\int B_2^{[4]} \cdot y \, Q \, \partial_p Q = \left(B_2^{[4]}\right)_p \, \int \frac{Q^2}{2}$$

hence it is clear one can ensure the existence of T_4 provided

$$\left(B_{2}^{[4]}\right)_{p} = -2 \left\|Q\right\|_{L^{2}}^{-2} \left(\mathcal{L}_{\lambda\alpha}^{[4]}Q + k^{[4]}Q^{3} + f_{4} - \Gamma_{21}^{[4]}Q, \ \partial_{p}Q\right)$$
(1.65)

Computation of (G_4, Q) First, let $G_4 = G_{41} + \partial_{pa}^{[3]} T_3$. As before, we need to nullify $(G_4, Q) = (G_{41}, Q) - \frac{1}{2} \partial_{pa}^{[3]} (L_+ T_3, \Lambda Q)$. First, like our previous computations at order 3, we have

$$\left(\mathcal{L}_{\lambda\alpha}^{[3]}Q, \Lambda Q \right) = \lambda^2 \alpha \cdot \nabla \left[\Delta \left(g - \frac{1}{2} Tr(G) \right) - \left(K_{21} \partial_{ij}^2 G_{ij} + K_{22} \partial_{\tilde{p}\,\tilde{p}}^2 G_{pp} \right) \right] (0) \left\| \frac{|y|}{2} Q \right\|_{L^2}^2$$

$$+ \frac{1}{2} \nabla^2 \left(g - \frac{1}{2} Tr(G) \right) (0) \cdot (\alpha, \alpha, \alpha) \int Q^2$$

$$(1.66)$$

$$d \text{ according to } (1.22) \text{ and } (1.57)$$

and according to (1.33) and (1.57)

$$\left(k^{[3]} Q^3, \Lambda Q\right) = 0 \tag{1.67}$$

so that

$$\begin{pmatrix} L_{+}T_{3}, \Lambda Q \end{pmatrix} = \lambda^{2} \alpha \cdot \nabla \left[\Delta \left(g - \frac{1}{2} Tr(G) \right) - \left(K_{21} \partial_{ij}^{2} G_{ij} + K_{22} \partial_{\tilde{p}\,\tilde{p}}^{2} G_{pp} \right) \right] (0) \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{2}$$

$$+ \frac{1}{2} \nabla^{2} \left(g - \frac{1}{2} Tr(G) \right) (0) \cdot (\alpha, \alpha, \alpha) \int Q^{2} - B_{1}^{[3]} \int \frac{|y|^{2}}{4} Q^{2}$$

$$(1.68)$$

Then, using (1.34) and (1.35), one may compute

$$\left(-\Gamma_{1}^{[4]}Q - \left(\Gamma_{22}^{[4]}\right)_{j}\partial_{j}Q, Q \right) = -\frac{b\lambda^{2}}{2}\nabla(\Delta g)(0) \cdot \alpha \int \frac{|y|^{2}}{4}Q^{2} - \frac{\lambda}{2}\left[\nabla^{3}(k-g)(0) \cdot (\alpha, \alpha, \beta) + \nabla^{3}G_{ij}(0) \cdot (\alpha, \alpha, \partial_{j})\beta_{i}\right] \int Q^{2} + \lambda^{3}\left[\nabla(\Delta g)(0) \cdot \beta - \partial_{j}\Delta G_{ij}(0)\beta_{i}\right] \int \frac{|y|^{2}}{4}Q^{2} + \frac{b}{12}\nabla^{3}g(0) \cdot (\alpha, \alpha, \alpha) \int Q^{2} = \mathcal{O}\left(\mathcal{P}^{3}\left(|\alpha| + |\beta|\right) + \mathcal{P}\left|\alpha\right|^{3} + \mathcal{P}\left|\alpha\right|^{2}\left|\beta\right|\right) = \mathcal{O}\left(\mathcal{P}^{5}\right)$$

$$(1.69)$$

where we have used again Remark 3. Hence one may discard those previous remaining terms so one has

$$\begin{pmatrix} G_4 , Q \end{pmatrix} = -\frac{1}{2} \partial_{pa}^{[3]} \left\{ \lambda^2 \alpha . \nabla \left[\Delta \left(g - \frac{1}{2} Tr(G) \right) - \left(K_{21} \partial_{ij}^2 G_{ij} + K_{22} \partial_{\tilde{p}\,\tilde{p}}^2 G_{pp} \right) \right] (0) \left\| \frac{|y|}{2} Q \right\|_{L^2}^2 + \frac{1}{2} \nabla^2 \left(g - \frac{1}{2} Tr(G) \right) (0) . (\alpha, \alpha, \alpha) \int Q^2 - B_1^{[3]} \int \frac{|y|^2}{4} Q^2 \right\}$$

$$(1.70)$$

Consequently, to ensure $(G_4, Q) = 0$ it is enough to choose

$$B_{1}^{[3]} = \lambda^{2} \alpha \cdot \nabla \left[\Delta \left(g - \frac{1}{2} Tr(G) \right) - \left(K_{21} \partial_{ij}^{2} G_{ij} + K_{22} \partial_{\tilde{p}\tilde{p}}^{2} G_{pp} \right) \right] (0) + \frac{1}{2} \nabla^{2} \left(g - \frac{1}{2} Tr(G) \right) (0) \cdot (\alpha, \alpha, \alpha) \|Q\|_{L^{2}}^{2} \|\frac{|y|}{2} Q\|_{L^{2}}^{-2}$$

$$(1.71)$$

Notice that (1.68) and (1.71) now yield

$$\left(L_{+}T_{3},\Lambda Q\right) = 0 \tag{1.72}$$

This concludes the construction of our approximate solution. \Box

Exponential estimate of the remaining terms We now look forward to proving (1.41).

We recall that the description (8.1) of the kernel of the linearized operator (L_+, L_-) ensures the standard uniform elliptic estimates :

$$\forall f \in (\nabla Q)^{\perp}, \quad \| e^{\delta |y|} L_{+}^{-1} f \|_{H^{2}} \lesssim \| e^{2\delta |y|} f \|_{L^{2}} \forall f \in (Q)^{\perp}, \quad \| e^{\delta |y|} L_{-}^{-1} f \|_{H^{2}} \lesssim \| e^{2\delta |y|} f \|_{L^{2}}$$

$$(1.73)$$

for a universal constant $\delta > 0$.

So far, we have constructed a fourth order approximate solution of (1.40), which means there is a welllocalized function Ψ , and regulars functions F_5 , G_5 of order at least 5 with respect to the \mathcal{P} parameters such that :

$$\begin{cases} L_{+}\mathcal{R}e\Psi = F_{5} & : \quad F_{5} \in (\nabla Q)^{\perp} \\ L_{-}\mathcal{I}m\Psi = G_{5} & : \quad G_{5} \in (Q)^{\perp} \end{cases}$$
(1.74)

Considering that $\|e^{\delta |y|} f\|_{H^2} \sim \sum_{|p| \leq 2} \|e^{\delta |y|} \partial^p f\|_{L^2}$, we have for every $y \in \mathbb{R}^2$ $\|e^{\delta_p |y|} \partial^p \mathcal{R}e\Psi\|_{L^2} \lesssim \|e^{\delta |y|} F_5\|_{L^2} \lesssim \mathcal{P}^5$, $\|e^{\delta_p |y|} \partial^p \mathcal{I}m\Psi\|_{L^2} \lesssim \|e^{\delta |y|} G_5\|_{L^2} \lesssim \mathcal{P}^5$

2 Estimation of mass and energy for the approximate profile

We denote by $P_{\mathcal{P}}$ the fourth order approximate solution we have formed in the previous section. Let :

$$Q_{\mathcal{P}} = P_{\mathcal{P}} e^{-ib \frac{|y|^2}{4} + i\beta \cdot y} \tag{2.1}$$

Then according to (1.40) and (1.23) $Q_{\mathcal{P}}$ must solve

$$i(-b^{2}+B_{1})\partial_{b}Q_{\mathcal{P}}-ib\lambda\partial_{\lambda}Q_{\mathcal{P}}+i(-b\beta+B_{2})\partial_{\beta}Q_{\mathcal{P}}+2i\lambda\beta\partial_{\alpha}Q_{\mathcal{P}}$$
$$+\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}}-Q_{\mathcal{P}}+\frac{k(\lambda y+\alpha)}{k(\alpha)}Q_{\mathcal{P}}|Q_{\mathcal{P}}|^{2}-i\lambda\beta\cdot\frac{\nabla k(\alpha)}{k(\alpha)}Q_{\mathcal{P}}$$
$$+ib\Lambda Q_{\mathcal{P}}-2i\beta\cdot\nabla Q_{\mathcal{P}}-|\beta|^{2}Q_{\mathcal{P}}=-\Psi_{\mathcal{P}}$$
$$(2.2)$$

with :

$$\Psi_{\mathcal{P}} = \tilde{\Psi}_{\mathcal{P}} \ e^{-ib \frac{|y|^2}{4} + i\beta \cdot y} \tag{2.3}$$

Of course, like (1.41) one gets

$$\forall p \in \mathbb{N}^2, \exists C_p > 0, \quad \left| \partial^p \Psi_{\mathcal{P}}(y) \right| \lesssim \mathcal{P}^5 e^{-C_p |y|}$$

$$\tag{2.4}$$

Before going any further, recalling $P_{\mathcal{P}} = Q + T_2 + T_3 + T_4 + i(S_3 + S_4) + \mathcal{O}(\mathcal{P}^5 e^{-|y|})$ and $\psi = b \frac{|y|^2}{4} - \beta \cdot y$, one should note

$$Q_{\mathcal{P}} = Q + U_2 + U_3 + U_4 + i\left(-\psi Q + V_3 + V_4\right) + \mathcal{O}(\mathcal{P}^5)$$

$$U_2 = T_2 - \frac{\psi^2}{2} Q$$

$$U_3 = T_3$$

$$U_4 = T_4 + \frac{\psi^4}{24} Q + \psi S_3 - \frac{\psi^2}{2} T_2$$

$$V_3 = S_3 + \frac{\psi^3}{6} Q - \psi T_2$$

$$V_4 = S_4 - \psi T_3$$
(2.5)

and in particular

$$Q_{\mathcal{P}} = Q e^{-i\psi} + \mathcal{O}(\mathcal{P}^2) = \Sigma + i\Theta$$

$$\Sigma = Q \cos(\psi) + \mathcal{O}(\mathcal{P}^2 e^{-C_2|y|}) = Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2|y|})$$

$$\Theta = -Q \sin(\psi) + \mathcal{O}(\mathcal{P}^2 e^{-C_2|y|}) = -\psi Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2|y|})$$

$$|Q_{\mathcal{P}}|^2 = |P_{\mathcal{P}}|^2$$
(2.6)

We move to the next step, with the computation of the $Q_{\mathcal{P}}$ invariants, that is its mass and energy. We state :

Lemma 2.1. Mass of $Q_{\mathcal{P}}$:

$$\int |Q_{\mathcal{P}}|^2 g(\lambda y + \alpha) dy = (1 + \lambda^2 \kappa) \int Q^2 + \frac{1}{2} \nabla^2 g(0).(\alpha, \alpha) \int Q^2 + \frac{\lambda^2}{2} \nabla (\Delta g)(0).\alpha \int \frac{|y|^2}{4} Q^2 + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3)$$

$$\kappa = \frac{\Delta g(0)}{2} \int \frac{|y|^2}{4} Q^2 ||Q||_{L^2}^{-2}$$
(2.7)

Energy of $Q_{\mathcal{P}}$:

$$E(Q_{\mathcal{P}}) = \frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{\lambda^2}{8} \left(C_E + \phi_E(\alpha) \right) - J_E(\alpha, \alpha) + \mathcal{O}\left(\mathcal{P}^4 + |\alpha|^3\right)$$
(2.8)

with

$$C_{E} = \Delta (k+g)(0) \int \frac{|y|^{2}}{2} Q^{4} - \left[\Delta \left[g + \frac{1}{2} Tr(G) \right](0) + \partial_{ii}^{2} G_{ii}(0) + \partial_{12}^{2} G_{12}(0) \right] \int \frac{|y|^{2}}{2} Q^{2} + 4 V(0) \int Q^{2}$$

$$\phi_{E}(\alpha) = \nabla \left[\Delta (k+g) \right](0) \cdot \alpha \int \frac{|y|^{2}}{2} Q^{4} - \nabla \left[\frac{1}{2} \Delta Tr(G) + \partial_{ii}^{2} G_{ii}(0) + \partial_{12}^{2} G_{12} \right](0) \cdot \alpha \int \frac{|y|^{2}}{2} Q^{2}$$
(2.9)

$$+ 4 \nabla V(0) \cdot \alpha \int Q^{2}$$

$$J_{E}(\alpha, \alpha) = \frac{1}{4} \nabla^{2} \left(g - \frac{1}{2} Tr(G) \right)(0) \cdot (\alpha, \alpha) \int Q^{2}$$

Proof of Lemma 2.1

Computation of the mass

First, note that $|P_{\mathcal{P}}|^2 = (Q + T_2 + T_3)^2 + \mathcal{O}(\mathcal{P}^4) = Q^2 + 2QT_2 + 2QT_3 + \mathcal{O}(\mathcal{P}^4)$. Then

$$|Q_{\mathcal{P}}|^2 = |P_{\mathcal{P}}|^2 = Q^2 + 2QT_2 + 2QT_3 + \mathcal{O}(\mathcal{P}^4)$$

so that using (1.55):

$$\int |Q_{\mathcal{P}}|^2 dy = \int Q^2 dy - (L_+ T_2, \Lambda Q) - (L_+ T_3, \Lambda Q) + \mathcal{O}(\mathcal{P}^4)$$

= $\int Q^2 - \frac{\lambda^2}{2} \Delta g(0) \int \frac{|y|^2}{4} Q^2 + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3)$ (2.10)

where we have used (1.62) and (1.72). Then,

$$\begin{split} &\int |Q_{\mathcal{P}}|^2 g(\lambda y + \alpha) dy = \int |Q_{\mathcal{P}}|^2 dy + \int \left(g^{[2]} + g^{[3]}\right) Q^2 + \mathcal{O}\left(\mathcal{P}^4 + |\alpha|^3\right) \\ &= \int |Q_{\mathcal{P}}|^2 dy + \lambda^2 \left[\Delta g(0) + \frac{1}{2} \nabla \left(\Delta g\right)(0) \cdot \alpha\right] \int \frac{|y|^2}{4} Q^2 + \frac{1}{2} \nabla^2 g(0) \cdot (\alpha, \alpha) \int Q^2 + \mathcal{O}\left(\mathcal{P}^4 + |\alpha|^3\right) \\ & \text{Eventually,} \end{split}$$

$$\int |Q_{\mathcal{P}}|^2 g(\lambda y + \alpha) dy = \int Q^2 + \frac{\lambda^2}{2} \left[\Delta g(0) + \nabla (\Delta g)(0) \cdot \alpha \right] \int \frac{|y|^2}{4} Q^2 + \frac{1}{2} \nabla^2 g(0) \cdot (\alpha, \alpha) \int Q^2 + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3) = (1 + \lambda^2 \kappa) \int Q^2 + \frac{1}{2} \nabla^2 g(0) \cdot (\alpha, \alpha) \int Q^2 + \frac{\lambda^2}{2} \nabla (\Delta g)(0) \cdot \alpha \int \frac{|y|^2}{4} Q^2 + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3) \kappa = \frac{\Delta g(0)}{2} \left\| \frac{|y|}{2} Q \right\|_{L^2}^2 \|Q\|_{L^2}^{-2}$$
(2.11)

Computation of the energy.

First note that after changing variables in (1.11), one gets

$$\begin{split} E(Q_{\mathcal{P}}) &= \frac{1}{2} \int G(\lambda y + \alpha) \nabla Q_{\mathcal{P}} \cdot \overline{\nabla Q_{\mathcal{P}}} \, dy - \frac{1}{4} \int \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^4 \, g(\lambda y + \alpha) dy - \frac{\lambda^2}{2} \int V(\lambda y + \alpha) \, |Q_{\mathcal{P}}|^2 \, g(\lambda y + \alpha) dy \\ &= \frac{1}{2} \int |\nabla Q_{\mathcal{P}}|^2 \, dy + \frac{1}{2} \int \left(G_{ij}^{[2]} + G_{ij}^{[3]} \right) \partial_i Q_{\mathcal{P}} \overline{\partial_j Q_{\mathcal{P}}} \, dy - \frac{1}{4} \int |Q_{\mathcal{P}}|^4 \, g(\lambda y + \alpha) dy \\ &- \frac{1}{4} \int \left(\frac{k(\lambda y + \alpha)}{k(\alpha)} - 1 \right) |Q_{\mathcal{P}}|^4 \, g(\lambda y + \alpha) dy - \frac{\lambda^2}{2} \int V(\lambda y + \alpha) \, |Q_{\mathcal{P}}|^2 \, g(\lambda y + \alpha) dy + \mathcal{O}(\mathcal{P}^4) \\ &= \frac{1}{2} \int |\nabla Q_{\mathcal{P}}|^2 \, dy + \frac{1}{2} \int \left(G_{ij}^{[2]} + G_{ij}^{[3]} \right) \partial_i Q \partial_j Q \, dy - \frac{1}{4} \int |Q_{\mathcal{P}}|^4 \, g(\lambda y + \alpha) dy \\ &- \frac{1}{4} \int \left(\frac{k(\lambda y + \alpha)}{k(\alpha)} - 1 \right) Q^4 \, dy - \frac{\lambda^2}{2} \left(1 + \alpha \cdot \nabla \right) V(0) \int Q^2 \, dy + \mathcal{O}(\mathcal{P}^4) \end{split}$$

Then, one needs to compute

$$\nabla Q_{\mathcal{P}} = \left(\nabla P_{\mathcal{P}} - i P_{\mathcal{P}} \nabla \psi\right) e^{-i\psi}$$

= $\left(\nabla Q + \nabla T_2 + \nabla T_3 + i \left[-\left(Q + T_2\right) \nabla \psi + \nabla S_3\right]\right) e^{-i\psi} + \mathcal{O}(\mathcal{P}^4)$
with $\nabla \psi = b \frac{y}{2} - \beta$

so that

$$\begin{aligned} |\nabla Q_{\mathcal{P}}|^{2} &= |\nabla Q|^{2} + 2 \nabla Q \cdot \nabla T_{2} + 2 \nabla Q \cdot \nabla T_{3} + Q^{2} |\nabla \psi|^{2} + \mathcal{O}(\mathcal{P}^{4}) \\ &= |\nabla Q|^{2} + 2 \nabla Q \cdot \nabla T_{2} + 2 \nabla Q \cdot \nabla T_{3} + \left|\beta - b \frac{y}{2}\right|^{2} Q^{2} + \mathcal{O}(\mathcal{P}^{4}) \\ &= |\nabla Q|^{2} + 2 \nabla Q \cdot \nabla T_{2} + 2 \nabla Q \cdot \nabla T_{3} + |\beta|^{2} Q^{2} + b^{2} \frac{|y|^{2}}{4} Q^{2} - b \beta \cdot \frac{y}{2} Q^{2} + \mathcal{O}(\mathcal{P}^{4}) \end{aligned}$$

and hence, using the radial symmetry of ${\cal Q}$

$$\begin{split} &\frac{1}{2} \int |\nabla Q_{\mathcal{P}}|^2 \, dy + \frac{1}{2} \int \left(G_{ij}^{[2]} + G_{ij}^{[3]} \right) \partial_i Q \partial_j Q \, dy \\ &= \frac{1}{2} \int |\nabla Q|^2 \, dy - \int \left(T_2 + T_3 \right) \Delta Q \, dy + \frac{|\beta|^2}{2} \int Q^2 + \frac{b^2}{8} \int |y|^2 \, Q^2 \, dy - \int b \, \beta \, . \, \frac{y}{2} \, Q^2 \, dy \\ &+ \frac{\lambda^2}{8} \left(1 + \alpha \, . \, \nabla \right) \Big[\Delta Tr(G) + 2 \left(\partial_{ii}^2 G_{ii} + \partial_{12}^2 G_{12} \right) \Big] (0) \, \int \frac{|y|^2}{4} \, Q^2 \, dy \\ &+ \frac{1}{8} \, \nabla^2 Tr(G)(0).(\alpha, \alpha) \, \int Q^2 \, dy + \mathcal{O} \big(\mathcal{P}^4 + |\alpha|^3 \big) \end{split}$$

Moreover, using the following approximation

$$|Q_{\mathcal{P}}|^4 = \left(Q + T_2 + T_3\right)^4 + \mathcal{O}\left(\mathcal{P}^4 e^{-|y|}\right)$$
$$= Q^4 + 4Q^3T_2 + 4Q^3T_3 + \mathcal{O}\left(\mathcal{P}^4 e^{-|y|}\right)$$

one has

$$-\frac{1}{4}\int |Q_{\mathcal{P}}|^{4}g(\lambda y + \alpha)dy = -\frac{1}{4}\int Q^{4}g(\lambda y + \alpha)dy - \int (T_{2} + T_{3})Q^{3}dy + \mathcal{O}(\mathcal{P}^{4})$$

$$= -\frac{1}{4}\int Q^{4}dy - \frac{\lambda^{2}}{8}(1 + \alpha \cdot \nabla)(\Delta g)(0)\int \frac{|y|^{2}}{2}Q^{4}dy - \frac{1}{4}\nabla^{2}g(0).(\alpha, \alpha)\int Q^{2}$$

$$-\int (T_{2} + T_{3})Q^{3}dy + \mathcal{O}(\mathcal{P}^{4})$$

and with (1.33)

$$-\frac{1}{4} \int \left(\frac{k(\lambda y + \alpha)}{k(\alpha)} - 1\right) |Q_{\mathcal{P}}|^4 \, dy = -\frac{1}{4} \int \left(k^{[2]} + k^{[3]}\right) Q^4 \, dy + \mathcal{O}(\mathcal{P}^4)$$
$$= -\frac{\lambda^2}{8} \left(1 + \alpha \cdot \nabla\right) \left(\Delta k\right)(0) \int \frac{|y|^2}{2} Q^4 + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3)$$

Then using the Ground State equation along with (1.55)

$$-\int (T_2 + T_3) \left[\Delta Q + Q^3\right] dy = -\int (T_2 + T_3) Q \, dy = \frac{1}{2} \left(L_+ T_2 + L_+ T_3, \Lambda Q\right)$$

Summing everything up, using (1.62), (1.72), we have :

$$E(Q_{\mathcal{P}}) = \frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{\lambda^2}{8} \left(C_E + \phi_E(\alpha) \right) - J_E(\alpha, \alpha) + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3)$$

where we have used $E^0(Q) = 0$, $\int |\nabla Q|^2 = \int Q^2$ from (1.37), and where C_E , ϕ_E , J_E are defined by (2.9).

3 Estimation of Energy. Existence of critical elements.

Introduction of the nonlinear decomposition and Modulation of the param-3.1eters

We assume $u(t) \in H^1$ is solution of (1.1)-(1.2) on $[t_0, t_1], t_1 < 0$. We look for u such as :

$$u(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} (Q_{\mathcal{P}(t)} + \epsilon) \left(t, \frac{x - \alpha(t)}{\lambda}\right) e^{i\gamma(t)}$$
(3.1)

Moreover, we assume the uniform bound (bootstrap's initialization):

$$|\mathcal{P}(t)| + \|\epsilon(t)\|_{H^1} \lesssim \lambda(t), \quad \forall \ t \in [t_0, t_1]$$
(3.2)

and that u(t) has almost critical mass

$$\left| \int |u|^2 g(x) dx - \left(1 + \lambda^2 \kappa\right) \int Q^2 \right| \lesssim \lambda(t)^4, \quad \forall \ t \in [t_0, t_1]$$
(3.3)

this may also be written when its mass is estimated in some $L^2(dg)$:

$$\left| \int |u|^2 \, dg - \int Q^2 \right| \lesssim \lambda(t)^4, \quad \forall \ t \in [t_0, t_1]$$
(3.4)

where

$$dg(t) = C(\lambda(t)) g(x) dx, \quad C(\lambda(t)) = (1 + \kappa \lambda^2(t))^{-1}$$
(3.5)

is a time dependant metric that embodies the ammount of mass our approximate solution $Q_{\mathcal{P}}$ gained through introduction of g as we found out in (2.7).

Now let :

$$Q_{\mathcal{P}} = \Sigma + i\Theta, \qquad \epsilon = \epsilon_1 + i\epsilon_2 \tag{3.6}$$

Recall we already stated in (2.5) that

$$\Sigma = Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2 |y|}), \quad \Theta = Q\left(-b \frac{|y|^2}{4} + \beta \cdot y\right) + \mathcal{O}(\mathcal{P}^2 e^{-C_2 |y|})$$

A standard modulation idea based on the implicit function theorem, see [MR05], prove one may ensure unicity of decomposition (3.1) provided we set the following orthogonality conditions :

$$T\left(\epsilon_{2}, \nabla\Sigma\right) - \left(\epsilon_{1}, \nabla\Theta\right) = 0$$
(3.7)

$$\begin{pmatrix} \epsilon_1, \ y \Sigma \end{pmatrix} + \begin{pmatrix} \epsilon_2, \ y \Theta \end{pmatrix} = 0$$

$$- \begin{pmatrix} \epsilon_1, \ \Lambda \Theta \end{pmatrix} + \begin{pmatrix} \epsilon_2, \ \Lambda \Sigma \end{pmatrix} = 0$$

$$(3.8)$$

$$(3.9)$$

$$-\left(\epsilon_{1}, \Lambda\Theta\right) + \left(\epsilon_{2}, \Lambda\Sigma\right) = 0 \tag{3.9}$$

$$\begin{pmatrix} \left(\epsilon_{1}, |y|^{2} \Sigma\right) + \left(\epsilon_{2}, |y|^{2} \Theta\right) = 0 \\ -\left(\epsilon_{1}, \varrho_{2}\right) + \left(\epsilon_{2}, \varrho_{1}\right) = 0 \end{cases}$$
(3.10) (3.11)

$$-(\epsilon_1, \varrho_2) + (\epsilon_2, \varrho_1) = 0$$
(3.11)

with ϱ the unique H^1 , radial solution of $L_+ \varrho = |y|^2 Q$ and :

$$\varrho_1 + i\varrho_2 = \varrho(y)e^{-ib\frac{|y|^2}{4} + i\beta \cdot y} \tag{3.12}$$

In the case $\mathcal{P} = (0, 0, 0, 0)$, these orthogonality conditions give a first order approximation of the kernel of linearized operator $L = (L_+, L_-)$ around the Ground-State Q. Indeed, we remind :

$$L_{-}Q = 0, \quad L_{-}(yQ) = -2\nabla Q, \quad L_{-}(|y|^{2}Q) = -4\Lambda Q$$

$$L_{+}(\nabla Q) = 0, \quad L_{+}(\Lambda Q) = -2Q, \quad L_{+}\varrho = |y|^{2}Q$$
(3.13)

and :

$$\left(\varrho, Q\right) = -\frac{1}{2} \left(L_{+}\varrho, \Lambda Q\right) = -\frac{1}{2} \left\|y Q\right\|_{L^{2}}^{2}$$

$$(3.14)$$

Note it is a well-known argument that \mathcal{P} may be chosen C^1 with respect to time, see [MR05]. Let

$$s(t) = \int_{t_0}^t \frac{d\tau}{\lambda(\tau)^2}$$
(3.15)

be the rescaled time.

<u>Computation of equation of ϵ </u>. We remind $Q_{\mathcal{P}} + \epsilon$ is solution of (1.15). Let us begin with the non-linear term development :

$$\begin{aligned} Q_{\mathcal{P}} + \epsilon \Big|^2 (Q_{\mathcal{P}} + \epsilon) &= |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} + \tilde{M}_1(\epsilon) + i \tilde{M}_2(\epsilon) + R_1(\epsilon) + i R_2(\epsilon) \\ \tilde{M}_1(\epsilon) &= |Q_{\mathcal{P}}|^2 \epsilon_1 + 2\Sigma^2 \epsilon_1 + 2\Sigma \Theta \epsilon_2 \\ \tilde{M}_2(\epsilon) &= |Q_{\mathcal{P}}|^2 \epsilon_2 + 2\Theta^2 \epsilon_2 + 2\Sigma \Theta \epsilon_1 \\ R_1(\epsilon) &= 3\Sigma \epsilon_1^2 + \Sigma \epsilon_2^2 + 2\Theta \epsilon_1 \epsilon_2 + |\epsilon|^2 \epsilon_1 \\ R_2(\epsilon) &= 3\Theta \epsilon_2^2 + \Theta \epsilon_1^2 + 2\Sigma \epsilon_1 \epsilon_2 + |\epsilon|^2 \epsilon_2 \end{aligned}$$

$$(3.16)$$

Let :

$$M_j(\epsilon) = -\mathcal{L}_{\lambda\alpha}\epsilon_j + \epsilon_j - \frac{k(\lambda y + \alpha)}{k(\alpha)}\tilde{M}_j(\epsilon), \quad j = 1, 2$$
(3.17)

Using (1.15), ϵ must satisfy :

$$i \partial_{s} Q_{\mathcal{P}} + \left[\mathcal{L}_{\lambda \alpha} Q_{\mathcal{P}} - Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^{2} Q_{\mathcal{P}} \right] \\ + i \partial_{s} \epsilon - \left(M_{1}(\epsilon) + i M_{2}(\epsilon) \right) + \frac{k(\lambda y + \alpha)}{k(\alpha)} \left(R_{1}(\epsilon) + i R_{2}(\epsilon) \right) \\ = i \frac{\lambda_{s}}{\lambda} \Lambda (Q_{\mathcal{P}} + \epsilon) + i \frac{\alpha_{s}}{\lambda} \cdot \left[\nabla (Q_{\mathcal{P}} + \epsilon) + \frac{\lambda}{2} \frac{\nabla k(\alpha)}{k(\alpha)} (Q_{\mathcal{P}} + \epsilon) \right] \\ + \tilde{\gamma}_{s} \left(Q_{\mathcal{P}} + \epsilon \right)$$

$$(3.18)$$

Moreover, with (1.33) and (3.2):

$$\frac{k(\lambda y + \alpha)}{k(\alpha)} R_j(\epsilon) = R_j(\epsilon) + \mathcal{O}(\mathcal{P}^5)$$
(3.19)

Then we have thanks to (2.2)

$$i \left(b_{s}+b^{2}-B_{1}\right)\partial_{b}Q_{\mathcal{P}}+i\left(\lambda_{s}+b\,\lambda\right)\partial_{\lambda}Q_{\mathcal{P}}+i\left(\beta_{s}+b\,\beta-B_{2}\right)\partial_{\beta}Q_{\mathcal{P}}+i\left(\alpha_{s}-2\lambda\,\beta\right)\partial_{\alpha}Q_{\mathcal{P}}$$

$$+i\,\partial_{s}\epsilon-\left(M_{1}(\epsilon)+iM_{2}(\epsilon)\right)+\left(R_{1}(\epsilon)+iR_{2}(\epsilon)\right)+i\,b\,\Lambda\epsilon-2\,i\,\beta.\left(\nabla\epsilon+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\epsilon\right)-|\beta|^{2}\epsilon$$

$$=i\left(\frac{\lambda_{s}}{\lambda}+b\right)\left(\Lambda Q_{\mathcal{P}}+\Lambda\epsilon\right)+\left(\tilde{\gamma}_{s}-|\beta|^{2}\right)\left(Q_{\mathcal{P}}+\epsilon\right)$$

$$+i\left(\frac{\alpha_{s}}{\lambda}-2\,\beta\right).\left(\nabla Q_{\mathcal{P}}+\nabla\epsilon+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}Q_{\mathcal{P}}+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\epsilon\right)-\Psi_{\mathcal{P}}$$

$$(3.20)$$

Then, one needs only to take on the one hand the real and on the other hand the imaginary part of (3.20) to get the equations of ϵ_1 and ϵ_2 respectively.

3.2 A first estimation of the \mathcal{P} parameters.

We now make use of the signs hypothesis we assumed in (\mathcal{H}) . We show how this forces the regime of the \mathcal{P} parameters in such a way that λ is the greatest parameter and thus controls the \mathcal{P} set.

We then prove a consistency result, which shows the approximate laws we built in the first section for those parameters are indeed like we constructed them to be, that is at least fourth order approximations. To get those results, we will write the conservation laws for the approximation.

Lemma 3.1. For $s \in [s_0, s_1]$, we may estimate :

• Refined variational control :

Provided $E_0 + \frac{1}{8}C_E > 0$ and $\nabla^2 (k - \frac{1}{2}Tr(G))(0) < 0$

$$b^{2} + |\beta|^{2} + |\alpha|^{2} + \|\epsilon\|_{H^{1}}^{2} \lesssim \lambda^{2} \left(E_{0} + \frac{1}{8} C_{E} \right) + \mathcal{O}(\mathcal{P}^{4} + |\alpha|^{3})$$
(3.21)

• Control of the geometrical parameters : Let

$$Mod(t) := \left(L_b - K_b(\lambda, \alpha), \ L_\lambda, \ L_\alpha, \ L_\beta, \ L_{\tilde{\gamma}} - K_{\tilde{\gamma}}(\lambda, \alpha) \right)$$

$$L_b = b_s + b^2 - B_1, \quad L_\lambda = \frac{\lambda_s}{\lambda} + b, \quad L_\alpha = \frac{\alpha_s}{\lambda} - 2\beta,$$

$$L_\beta = \beta_s + b\beta - B_2, \quad L_{\tilde{\gamma}} = \tilde{\gamma}_s - |\beta|^2$$
(3.22)

be the vector of modulations equations with

$$K_{b}(\lambda,\alpha) = -\nabla^{2}k(0).(\alpha,\alpha) \frac{\|Q\|_{L^{2}}^{2}}{2\|\frac{|y|}{2}Q\|_{L^{2}}^{2}} + \lambda^{2}k_{1}[\alpha], \qquad k_{1}[\alpha] = -\frac{1}{2}\nabla(\Delta g)(0).\alpha$$

$$K_{\tilde{\gamma}}(\lambda,\alpha) = K_{b}(\lambda,\alpha) \frac{\left(\frac{|y|^{2}}{4}Q,\varrho\right)}{\left(Q,\varrho\right)}$$
(3.23)

Then:

$$|Mod(t)| \lesssim \mathcal{O}\Big(\mathcal{P}^4 + \mathcal{P} |\alpha|^2 + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\Big)$$
(3.24)

Remark 4. Note the vector of modulation laws $Law(t) = (L_b, L_\lambda, L_\alpha, L_\beta, L_{\widetilde{\gamma}})$ only differs from a $|\alpha|^2$ term, so that

$$\begin{aligned} |Law(t)| &\lesssim \mathcal{P}^4 + |\alpha|^2 + \mathcal{P}^2 \|\epsilon\|_{H^1} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3 \\ |Law(t)| &= |Mod(t)| + |\alpha|^2 \end{aligned}$$

Proof of Lemma 3.1 : Step 1 The L^2 norm being conserved (conservation of Mass) :

$$\int |Q_{\mathcal{P}} + \epsilon|^2 g(\lambda y + \alpha) dy = k(\alpha) \int |u|^2 g(x) dx$$
(3.25)

so that using (3.3), (2.7) and (3.5) show :

$$2\mathcal{R}e\int \epsilon \overline{Q_{\mathcal{P}}} g(\lambda y + \alpha)dy + \int |\epsilon|^{2} g(\lambda y + \alpha)dy$$

$$= k(\alpha)\int |u|^{2} g(x)dx - \int |Q_{\mathcal{P}}|^{2} g(\lambda y + \alpha)dy$$

$$= k(\alpha)\left[\int |u|^{2} g(x)dx - (1 + \lambda^{2} \kappa) \int Q^{2}\right] + (k(\alpha) - 1)(1 + \lambda^{2} \kappa) \int Q^{2}$$

$$- \frac{1}{2} \nabla^{2} g(0).(\alpha, \alpha) \int Q^{2} - \frac{\lambda^{2}}{2} \nabla(\Delta g)(0).\alpha \int \frac{|y|^{2}}{4} Q^{2} + \mathcal{O}(\mathcal{P}^{4} + |\alpha|^{3})$$

$$= \frac{1}{2} \nabla^{2} (k - g)(0).(\alpha, \alpha) \int Q^{2} - \frac{\lambda^{2}}{2} \nabla(\Delta g)(0).\alpha \int \frac{|y|^{2}}{4} Q^{2} + \mathcal{O}(\mathcal{P}^{4} + |\alpha|^{3})$$
(3.26)

Now with conservation of energy, we have :

$$\begin{split} &\frac{1}{2}\int G(\lambda y+\alpha)\nabla\left(Q_{\mathcal{P}}+\epsilon\right).\overline{\nabla\left(Q_{\mathcal{P}}+\epsilon\right)}\,dy - \frac{1}{4}\int\frac{k(\lambda y+\alpha)}{k(\alpha)}|Q_{\mathcal{P}}+\epsilon|^4\,g(\lambda y+\alpha)dy\\ &-\frac{\lambda^2}{2}\int V(\lambda y+\alpha)\,|Q_{\mathcal{P}}+\epsilon|^2\,g(\lambda y+\alpha)dy\\ &=k(\alpha)\,\lambda^2\left(\frac{1}{2}\int G(x)\,\nabla u\,.\overline{\nabla u}\,dx - \frac{1}{4}\int k(x)\,|u|^4\,dx - \frac{1}{2}\int V(x)\,|u|^2\,dx\right)\\ &=k(\alpha)\,\lambda^2\,E_0 \end{split}$$

Then, notice expanding the non linear terms yield with the 2-dimensional Gagliardo-Nirenberg estimate

$$\begin{split} &\frac{1}{4} \int \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}} + \epsilon|^4 \, g(\lambda y + \alpha) dy \\ &= \frac{1}{4} \int \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^4 \, g(\lambda y + \alpha) dy + \frac{1}{2} \int |Q_{\mathcal{P}}|^2 \Big[\Big(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_1^2 + 4 \frac{\Sigma\Theta}{|Q_{\mathcal{P}}|^2} \epsilon_1 \epsilon_2 + \Big(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_2^2 \Big] \\ &+ \mathcal{R}e \int \frac{k(\lambda y + \alpha)}{k(\alpha)} \, \epsilon \, \overline{|Q_{\mathcal{P}}|^2 Q_{\mathcal{P}}} \, g(\lambda y + \alpha) dy + \mathcal{O} \big(\|\epsilon\|_{H^1}^3 + \mathcal{P}^2 \, \|\epsilon\|_{L^2}^2 \big) \end{split}$$

and

$$\frac{\lambda^2}{2} \int V(\lambda y + \alpha) |Q_{\mathcal{P}} + \epsilon|^2 g(\lambda y + \alpha) dy$$

= $\frac{\lambda^2}{2} \int V(\lambda y + \alpha) |Q_{\mathcal{P}}|^2 g(\lambda y + \alpha) dy + \mathcal{R}e \int \epsilon \overline{\lambda^2 V(\lambda y + \alpha) Q_{\mathcal{P}}} g(\lambda y + \alpha) dy + \mathcal{O}(\mathcal{P}^2 ||\epsilon||_{L^2}^2)$

so that using the symmetry of G, integration by parts, and eventually (2.8) we get

$$\begin{split} &\frac{1}{2} \int G(\lambda y + \alpha) \nabla \left(Q_{\mathcal{P}} + \epsilon\right) \cdot \overline{\nabla (Q_{\mathcal{P}} + \epsilon)} \, dy - \frac{1}{4} \int \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}} + \epsilon|^4 g(\lambda y + \alpha) dy \\ &- \frac{\lambda^2}{2} \int V(\lambda y + \alpha) |Q_{\mathcal{P}} + \epsilon|^2 g(\lambda y + \alpha) dy \\ &= \frac{1}{2} \int G(\lambda y + \alpha) \nabla Q_{\mathcal{P}} \cdot \overline{\nabla Q_{\mathcal{P}}} \, dy - \frac{1}{4} \int \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^4 g(\lambda y + \alpha) dy - \frac{\lambda^2}{2} \int V(\lambda y + \alpha) |Q_{\mathcal{P}}|^2 g(\lambda y + \alpha) dy \\ &+ \frac{1}{2} \int G(\lambda y + \alpha) \nabla \epsilon \cdot \overline{\nabla \epsilon} \, dy - \frac{1}{2} \int |Q_{\mathcal{P}}|^2 \Big[\Big(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_1^2 + 4 \frac{\Sigma \Theta}{|Q_{\mathcal{P}}|^2} \epsilon_1 \epsilon_2 + \Big(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_2^2 \Big] \\ &- \mathcal{R}e \int \epsilon \, \overline{div} (G(\lambda y + \alpha) \nabla Q_{\mathcal{P}}) \, dy - \mathcal{R}e \int \epsilon \, \left[\frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} + \lambda^2 V(\lambda y + \alpha) Q_{\mathcal{P}} \right] g(\lambda y + \alpha) dy \\ &+ \mathcal{O}(||\epsilon||_{H^1}^3 + \mathcal{P}^2 ||\epsilon||_{L^2}^2) \\ &= E(Q_{\mathcal{P}}) + \frac{1}{2} \int G(\lambda y + \alpha) \nabla \epsilon \cdot \overline{\nabla \epsilon} \, dy - \frac{1}{2} \int |Q_{\mathcal{P}}|^2 \Big[\Big(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_1^2 + 4 \frac{\Sigma \Theta}{|Q_{\mathcal{P}}|^2} \epsilon_1 \epsilon_2 + \Big(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_2^2 \Big] \\ &- \mathcal{R}e \int \epsilon \, \overline{\left(\mathcal{L}_{\lambda \alpha} Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} \right)} g(\lambda y + \alpha) dy + \mathcal{O}(||\epsilon||_{H^1}^3 + \mathcal{P}^2 ||\epsilon||_{L^2}^2) \\ &= \frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{\lambda^2}{8} \left(C_E + \phi_E(\alpha) \right) - J_E(\alpha, \alpha) + \frac{1}{2} \int G(\lambda y + \alpha) \nabla \epsilon \cdot \overline{\nabla \epsilon} \, dy \\ &- \frac{1}{2} \int |Q_{\mathcal{P}}|^2 \Big[\Big(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} e_1 \epsilon_2 + \Big(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_2^2 \Big] \\ &- \mathcal{R}e \int \epsilon \, \overline{\left(\mathcal{L}_{\lambda \alpha} Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} \right)} g(\lambda y + \alpha) dy + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3 + ||\epsilon||_{H^1}^3 + \mathcal{P}^2 ||\epsilon||_{L^2}^2) \\ &= \frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{\lambda^2}{8} \left(C_E + \phi_E(\alpha) \right) - J_E(\alpha, \alpha) \\ &+ \frac{1}{2} \int |\nabla \epsilon|^2 \, dy - \frac{1}{2} \int |Q_{\mathcal{P}}|^2 \Big[\Big(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_1^2 + 4 \frac{\Sigma\Theta}{|Q_{\mathcal{P}}|^2} \epsilon_1 \epsilon_2 + \Big(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \Big) \epsilon_2^2 \Big] \\ &- \mathcal{R}e \int \epsilon \, \overline{\left(\mathcal{L}_{\lambda \alpha} Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} \right)} g(\lambda y + \alpha) dy + \mathcal{O}(\mathcal{P}^4 + |\alpha|^3 + ||\epsilon||_{H^1}^3 + \mathcal{P}^2 ||\epsilon||_{H^1}^2) \\ &= \frac{B^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{\lambda^2}{8} \left(C_E + \phi_E(\alpha) \right) - J_E(\alpha, \alpha) \\ &+ \frac{1}{2} \int |\nabla \epsilon|^2$$

With (3.3), since $k(\alpha) \lambda^2 E_0 = \lambda^2 E_0 + \mathcal{O}(\mathcal{P}^4)$, we get :

$$\lambda^{2} \left(E_{0} + \frac{1}{8} \left(C_{E} + \phi_{E}(\alpha) \right) \right) + J_{E}(\alpha, \alpha)$$

$$= \frac{b^{2}}{8} \int |y|^{2} Q^{2} + \frac{|\beta|^{2}}{2} \int Q^{2} - \mathcal{R}e \int \epsilon \left[\mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^{2} Q_{\mathcal{P}} \right] g(\lambda y + \alpha) dy$$

$$+ \frac{1}{2} \int |\nabla \epsilon|^{2} - \frac{1}{2} \int |Q_{\mathcal{P}}|^{2} \left[\left(1 + \frac{2\Sigma^{2}}{|Q_{\mathcal{P}}|^{2}} \right) \epsilon_{1}^{2} + 4 \frac{\Sigma\Theta}{|Q_{\mathcal{P}}|^{2}} \epsilon_{1} \epsilon_{2} + \left(1 + \frac{2\Theta^{2}}{|Q_{\mathcal{P}}|^{2}} \right) \epsilon_{2}^{2} \right]$$

$$+ \mathcal{O} \left(\mathcal{P}^{4} + |\alpha|^{3} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{H^{1}}^{2} \right)$$

$$(3.27)$$

Step 2

We sum conservation of Mass (3.26) and conservation of Energy (3.27):

$$\lambda^{2} \left(E_{0} + \frac{1}{8} \left[C_{E} + \phi_{E}(\alpha) - \nabla(\Delta g)(0) \cdot \alpha \int \frac{|y|^{2}}{2} Q^{2} \right] \right) + \tilde{J}_{E}(\alpha, \alpha)$$

$$= \frac{b^{2}}{8} \int |y|^{2} Q^{2} + \frac{|\beta|^{2}}{2} \int Q^{2} + \mathcal{R}e \int \epsilon \left[\left(-\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}} + Q_{\mathcal{P}} - \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^{2} Q_{\mathcal{P}} \right) g(\lambda y + \alpha) dy \right]$$

$$+ \frac{1}{2} \int |\epsilon|^{2} + \frac{1}{2} \int |\nabla\epsilon|^{2} - \frac{1}{2} \int |Q_{\mathcal{P}}|^{2} \left[\left(1 + \frac{2\Sigma^{2}}{|Q_{\mathcal{P}}|^{2}} \right) \epsilon_{1}^{2} + 4 \frac{\Sigma\Theta}{|Q_{\mathcal{P}}|^{2}} \epsilon_{1} \epsilon_{2} + \left(1 + \frac{2\Theta^{2}}{|Q_{\mathcal{P}}|^{2}} \right) \epsilon_{2}^{2} \right]$$

$$+ \mathcal{O} \left(\mathcal{P}^{4} + |\alpha|^{3} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{H^{1}}^{2} \right)$$

$$(3.28)$$

where

$$\tilde{J}_{E}(\alpha,\alpha) = J_{E}(\alpha,\alpha) + \frac{1}{4}\nabla^{2}(k-g)(0).(\alpha,\alpha)\int Q^{2} = \frac{1}{4}\nabla^{2}\left(k-\frac{1}{2}Tr(G)\right)(0).(\alpha,\alpha)\int Q^{2} dx$$

With (2.2), we have the following estimation in L^2 :

$$-\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}} + Q_{\mathcal{P}} - \frac{k(\lambda y + \alpha)}{k(\alpha)}|Q_{\mathcal{P}}|^2 Q_{\mathcal{P}}$$
$$= i b \Lambda Q_{\mathcal{P}} - 2 i \beta \cdot \nabla Q_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^2)$$

so that, thanks to orthogonality conditions (3.9) and (3.7):

$$\mathcal{R}e\int \epsilon \overline{\left(-\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}}+Q_{\mathcal{P}}-\frac{k(\lambda y+\alpha)}{k(\alpha)}|Q_{\mathcal{P}}|^{2}Q_{\mathcal{P}}\right)} g(\lambda y+\alpha)dy$$

= $b \mathcal{I}m\left(\epsilon, \overline{\Lambda Q_{\mathcal{P}}}\right) - 2\beta \mathcal{I}m\left(\epsilon, \overline{\nabla Q_{\mathcal{P}}}\right) + \mathcal{O}(\mathcal{P}^{2}\|\epsilon\|_{L^{2}})$
= $\mathcal{O}(\mathcal{P}^{2}\|\epsilon\|_{L^{2}})$ (3.29)

Thanks to (3.2), $Q_{\mathcal{P}}$ being close to Q, we see the quadratic form in ϵ that appears in (3.28) is to be thought as a small pertubation of the energy in a neighbourhood of Q. First, let

$$L_{+}^{[g]} = -\mathcal{L}_{\lambda\alpha} + 1 - 3Q^{2}$$

$$L^{[g]} = -\mathcal{L}_{\lambda\alpha} + 1 - Q^{2}$$
(3.30)

be the operators that linearize the NLS operator we are studying here. Integration by parts yields :

$$\frac{1}{2} \Big[\Big(L_{+}^{[g]} \epsilon_1, \epsilon_1 \Big) + \Big(L_{-}^{[g]} \epsilon_2, \epsilon_2 \Big) \Big] = \frac{1}{2} \int |\epsilon|^2 g(\lambda y + \alpha) dy + \frac{1}{2} \int G(\lambda y + \alpha) \nabla \epsilon \cdot \overline{\nabla \epsilon} g(\lambda y + \alpha) dy \\ - \frac{1}{2} \int Q^2 \Big(3\epsilon_1^2 + \epsilon_2^2 \Big) g(\lambda y + \alpha) dy - \frac{\lambda^2}{2} \int V(\lambda y + \alpha) |\epsilon|^2 g(\lambda y + \alpha) dy$$

Thus, using expansions $\Sigma = Q + U_2 + U_3 + \mathcal{O}(\mathcal{P}^4)$ and $\Theta = -\psi Q + V_3 + \mathcal{O}(\mathcal{P}^4)$, we have

$$\begin{split} &\frac{1}{2} \int |\epsilon|^2 g(\lambda y + \alpha) dy + \frac{1}{2} \int G(\lambda y + \alpha) \, \nabla \epsilon \, \cdot \overline{\nabla \epsilon} \, g(\lambda y + \alpha) dy - \frac{\lambda^2}{2} \, \int V(\lambda y + \alpha) \, |\epsilon|^2 \, g(\lambda y + \alpha) dy \\ &- \frac{1}{2} \int |Q_{\mathcal{P}}|^2 \left[\left(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} \right) \epsilon_1^2 + 4 \frac{\Sigma \Theta}{|Q_{\mathcal{P}}|^2} \epsilon_1 \epsilon_2 + \left(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \right) \epsilon_2^2 \right] \\ &= \frac{1}{2} \left[\left(L_+^{[g]} \epsilon_1, \epsilon_1 \right) + \left(L_-^{[g]} \epsilon_2, \epsilon_2 \right) \right] + \frac{1}{2} \int \left(3Q^2 - \left(1 + \frac{2\Sigma^2}{|Q_{\mathcal{P}}|^2} \right) |Q_{\mathcal{P}}|^2 \right) \epsilon_1^2 - 2 \int \Sigma \Theta \, \epsilon_1 \, \epsilon_2 \\ &+ \frac{1}{2} \left(Q^2 - \left(1 + \frac{2\Theta^2}{|Q_{\mathcal{P}}|^2} \right) |Q_{\mathcal{P}}|^2 \right) \epsilon_2^2 \\ &= \frac{1}{2} \left[\left(L_+^{[g]} \epsilon_1, \epsilon_1 \right) + \left(L_-^{[g]} \epsilon_2, \epsilon_2 \right) \right] - 2 \int \Sigma \Theta \, \epsilon_1 \, \epsilon_2 + \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2}^2) \end{split}$$

Then, using hypothesis (3.2), one easily shows $\int \Sigma \Theta \epsilon_1 \epsilon_2 = \mathcal{O}(\mathcal{P} \|\epsilon\|_{L^2}^2) = \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2})$ So that (3.28) becomes :

$$\lambda^{2} \left(E_{0} + \frac{1}{8} C_{E} + \mathcal{O}(|\alpha|) \right)$$

$$= \frac{b^{2}}{8} \int |y|^{2} Q^{2} + \frac{|\beta|^{2}}{2} \int Q^{2} - \frac{1}{4} \nabla^{2} \left(k - \frac{1}{2} Tr(G) \right) (0) \cdot (\alpha, \alpha) \int Q^{2}$$

$$+ \frac{1}{2} \left[\left(L_{+}^{[g]} \epsilon_{1}, \epsilon_{1} \right) + \left(L_{-}^{[g]} \epsilon_{2}, \epsilon_{2} \right) \right] + \mathcal{O} \left(\mathcal{P}^{4} + |\alpha|^{3} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}} \right)$$
(3.31)

To conclude, we need to deal with the bilinear form we introduced in (3.31). A coercivity result will do it :

Lemma 3.2. We can find a universal constant $c_0 > 0$ such as $\forall \epsilon \in H^1$:

$$\frac{1}{2} \left[\left(L_{+} \epsilon_{1}, \epsilon_{1} \right) + \left(L_{-} \epsilon_{2}, \epsilon_{2} \right) \right] \\
\geq c_{0} \|\epsilon\|_{H^{1}}^{2} - \frac{1}{c_{0}} \left\{ \left(\epsilon_{1}, Q \right)^{2} + \left(\epsilon_{1}, |y|^{2} Q \right)^{2} + \left(\epsilon_{1}, y Q \right)^{2} + \left(\epsilon_{2}, \varrho \right)^{2} \right\}$$
(3.32)

Thanks to orthogonality conditions (3.10), (3.8) we can prove :

$$\left(\epsilon_1, |y|^2 Q\right) = 0, \qquad \left(\epsilon_1, y Q\right) = 0$$

$$(3.33)$$

In [MR05], it is proved that the orthogonality condition (3.11) yields $(\epsilon_2, \rho) = 0$. Moreover (3.26) shows

$$\left(\epsilon_{1}, Q\right)\right|^{2} \lesssim \left[2\mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right)\right]^{2} = \mathcal{O}\left(\mathcal{P}^{4} + \|\epsilon\|_{H^{1}}^{4}\right)$$
(3.34)

Hence by (3.32):

$$\frac{1}{2}\left[\left(L_{+}\epsilon_{1},\epsilon_{1}\right)+\left(L_{-}\epsilon_{2},\epsilon_{2}\right)\right] \geq \frac{c_{0}}{2}\|\epsilon\|_{H^{1}}^{2}+\mathcal{O}\left(\mathcal{P}^{4}+\|\epsilon\|_{H^{1}}^{4}\right)$$
(3.35)

then,

:

$$\frac{1}{2} \left[\left(L_{+}^{[g]} \epsilon_{1}, \epsilon_{1} \right) + \left(L_{-}^{[g]} \epsilon_{2}, \epsilon_{2} \right) \right] = \frac{1}{2} \left[\left(L_{+} \epsilon_{1}, \epsilon_{1} \right) + \left(L_{-} \epsilon_{2}, \epsilon_{2} \right) \right] + \mathcal{O} \left(\mathcal{P}^{2} \| \epsilon \|_{H^{1}}^{2} \right) \\
\geq \frac{c_{0}}{2} \| \epsilon \|_{H^{1}}^{2} + \mathcal{O} \left(\mathcal{P}^{2} \| \epsilon \|_{H^{1}}^{2} \right)$$
(3.36)

Injecting this into (3.31):

$$\lambda^{2} \left(E_{0} + \frac{1}{8} C_{E} + \mathcal{O}(|\alpha|) \right)$$

$$\geq \frac{b^{2}}{8} \int |y|^{2} Q^{2} + \frac{|\beta|^{2}}{2} \int Q^{2} + \frac{c_{0}}{2} \|\epsilon\|_{H^{1}}^{2} - \frac{1}{4} \nabla^{2} \left(k - \frac{1}{2} Tr(G)\right)(0).(\alpha, \alpha) \int Q^{2}$$

$$+ \mathcal{O}\left(\mathcal{P}^{4} + |\alpha|^{3} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}}\right)$$
(3.37)

Eventually, provided $E_0 + \frac{1}{8}C_E > 0$ and $\nabla^2 \left(k - \frac{1}{2}Tr(G)\right)(0) < 0$, since $\lambda^2 \alpha = o(\lambda^2)$, for α small enough (that is for a time t close enough to 0) :
$$b^{2} + |\beta|^{2} + |\alpha|^{2} + \|\epsilon\|_{H^{1}}^{2} \lesssim \lambda^{2} \left(E_{0} + \frac{1}{8}C_{E}\right) + \mathcal{O}\left(\mathcal{P}^{4} + |\alpha|^{3}\right)$$
(3.38)

And (3.21) is proved thanks to (3.2).

Computation of the \mathcal{P} laws.

Quantities α , β , λ , b, $\tilde{\gamma}$ have been introduced in the first section as geometrical parameters, adjusted to our situation through a system of ODE satisfied by \mathcal{P} . Each law was chosen so it was simplifying our computations to build approximate solution $P_{\mathcal{P}}$ by using symmetric invariances. Now, assuming we have an exact solution u of (1.1) on $[t_0, t_1]$, we want to check the correspondant parameters defined by geometrical decomposition (3.1) follow the laws we build our approximate profile with, up to fourth order, and as long as u is defined in H^1 - which is assumed here to be $t \in [t_0, t_1]$.

In the previous section, we checked as long as u is defined, those parameters were controlled in some sense. We now work out a consistency type of result, checking the parameters laws are not blowing up, or strongly modified on the way to blow-up time.

Here we will reproduce analysis led in Appendix A of [RS11] with more details about computations.

First, we simplify the ϵ real and imaginary parts equations obtained thanks to (3.20):

$$\begin{aligned} &(b_s + b^2 - B_1)\partial_b \Sigma + (\beta_s + b\beta - B_2)\partial_\beta \Sigma \\ &+ \partial_s \epsilon_1 - M_2(\epsilon) + b\Lambda\epsilon_1 - 2\beta \cdot \nabla\epsilon_1 \\ &= \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\Sigma + \left(\tilde{\gamma}_s - |\beta|^2\right)\left(\Theta + \epsilon_2\right) + \left(\frac{\alpha_s}{\lambda} - 2\beta\right) \cdot \nabla\Sigma \\ &+ \mathcal{I}m(\tilde{\psi}) - R_2(\epsilon) \end{aligned}$$

$$(3.39)$$

and

with :

$$\tilde{\psi}(y) = \mathcal{R}_0(\mathcal{P})F_1(y) + \mathcal{O}\left(\mathcal{P}^2 + |Mod(t)|\right) \left\{ |\epsilon(y)| + |(1+|y|)\nabla\epsilon(y)| \right\}$$
(3.41)

where \mathcal{R}_0 is polynomial in \mathcal{P} such as $|\mathcal{R}_0| \leq |\mathcal{P}|^4$, and F_1 is a regular function such as $|F_1(y)| \leq e^{-c|y|}$.

Remark 5. Terms in ∂_{λ} and ∂_{α} are hidden in $\tilde{\psi}$. Indeed :

$$(\lambda_s + b\lambda)\partial_\lambda \Sigma = \lambda \Big(\frac{\lambda_s}{\lambda} + b\Big)\partial_\lambda \Sigma = \mathcal{O}\Big(\mathcal{P}^2 \big| Mod(t) \big| \Big)$$

$$(\alpha_s - 2\lambda\beta)\partial_\alpha \Sigma = \lambda \Big(\frac{\alpha_s}{\lambda} - 2\beta\Big)\partial_\alpha \Sigma = \mathcal{O}\Big(\mathcal{P}^2 \big| Mod(t) \big| \Big)$$

$$(3.42)$$

Computation of the \mathcal{P} laws now relies on combinations of scalar products of equations (3.39) and (3.40), making use of orthogonality conditions (3.7)-(3.11). Thus, as proved in [RS11]

Lemma 3.3. We claim the following computations hold

$$-\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ \nabla\Theta\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ \nabla\Sigma\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ y\Sigma\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ y\Theta\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$-\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ \Lambda\Theta\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ \Lambda\Sigma\right)$$

$$=-2\mathcal{R}e\left(\epsilon,\ \overline{Q_{\mathcal{P}}}\right)+\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ |y|^{2}\Sigma\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ |y|^{2}\Theta\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$-\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ \varrho_{2}\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ \varrho_{1}\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

We focus only on the third scalar product, for it brings the remaining term $-2\mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right)$ which, according to (3.26), contains some second and third order terms that will perturb the \mathcal{P} laws :

$$-2\mathcal{R}e\Big(\epsilon, \overline{Q_{\mathcal{P}}}\Big) = \int |\epsilon|^2 - \frac{1}{2}\nabla^2\Big(k-g\Big)(0).(\alpha,\alpha)\int Q^2 + \frac{\lambda^2}{2}\nabla\Big(\Delta g\Big)(0).\alpha\int \frac{|y|^2}{4}Q^2 + \mathcal{O}\Big(\mathcal{P}^4 + |\alpha|^3 + \mathcal{P}^2 \|\epsilon\|_{L^2}^2\Big)$$

First things first we compute the equation satisfied by $\Lambda Q_{\mathcal{P}}$ based on (2.2). We state

Lemma 3.4. For $Q_{\mathcal{P}} = \Sigma + i\Theta$ solution of (2.2), we have

$$-\mathcal{L}_{\lambda\alpha}\Lambda\Sigma + \Lambda\Sigma - Q^{2}\Lambda\Sigma - 2\Sigma\left(\Sigma\Lambda\Sigma + \Theta\Lambda\Theta\right)$$

= $\mathcal{R}e\Lambda\Psi_{\mathcal{P}}^{2} - 2\left(\Sigma - 2\beta\cdot\nabla\Theta + b\Lambda\Theta - \mathcal{R}e\Psi_{\mathcal{P}}^{2}\right) - b\Lambda^{2}\Theta$
+ $2\beta\cdot\nabla(\Lambda\Theta) + \mathcal{O}(\mathcal{P}^{2})$ (3.44)

and

$$-\mathcal{L}_{\lambda\alpha}\Lambda\Theta + \Lambda\Theta - Q^2\Lambda\Theta - 2\Theta\left(\Sigma\Lambda\Sigma + \Theta\Lambda\Theta\right)$$

= $\mathcal{I}m\Lambda\Psi_{\mathcal{P}}^2 - 2\left(\Theta + 2\beta \cdot \nabla\Sigma - b\Lambda\Sigma - \mathcal{I}m\Psi_{\mathcal{P}}^2\right) + b\Lambda^2\Sigma$
- $2\beta \cdot \nabla(\Lambda\Sigma) + \mathcal{O}(\mathcal{P}^2)$ (3.45)

where
$$\mathcal{L}_{\lambda\alpha}f = \frac{1}{g}\operatorname{div}(G\nabla f) + \lambda^2 V f = \frac{1}{g(\lambda y + \alpha)}\operatorname{div}(G(\lambda y + \alpha)\nabla f) + \mathcal{O}(\mathcal{P}^2|f|)$$

Proof of Lemma 3.4 :

Rewriting (2.2) up to second order, $Q_{\mathcal{P}}$ satisfy :

$$\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}} - Q_{\mathcal{P}} + Q_{\mathcal{P}}|Q_{\mathcal{P}}|^2 + i b \Lambda Q_{\mathcal{P}} - 2 i \beta \cdot \nabla Q_{\mathcal{P}} = -\Psi_{\mathcal{P}}^2$$
$$|\Psi_{\mathcal{P}}^2| \lesssim \mathcal{P}^2 e^{-C_2 |y|}$$

Then we check

$$\begin{split} \Lambda Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 &= \Lambda \left[Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 \right] - 2 \,\mathcal{R}e \left(y \, \cdot \, \nabla Q_{\mathcal{P}} \, \overline{Q_{\mathcal{P}}} \right) Q_{\mathcal{P}} \\ &= \Lambda \left[Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 \right] + 2 \, Q_{\mathcal{P}} \, |Q_{\mathcal{P}}|^2 - 2 \, Q_{\mathcal{P}} \left(\Sigma \, \Lambda \Sigma + \Theta \, \Lambda \Theta \right) \end{split}$$

and

$$div[G\nabla(y.\nabla Q_{\mathcal{P}})] = \Lambda[div(G\nabla Q_{\mathcal{P}})] + (G_{ij} - y.\nabla G_{ij})\partial_{ij}^2 Q_{\mathcal{P}} - \partial_{im}^2 G_{ij}y_m \partial_j Q_{\mathcal{P}} = \Lambda[div(G\nabla Q_{\mathcal{P}})] + div(G\nabla Q_{\mathcal{P}}) + \mathcal{O}(\mathcal{P}^2)$$

so that

$$\mathcal{L}_{\lambda\alpha}(\Lambda Q_{\mathcal{P}}) = \Lambda \left[\mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} \right] + \left(2 + \frac{y \cdot \nabla g}{g} \right) \mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^2)$$
$$= \Lambda \left[\mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} \right] + 2 \mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^2)$$

A straightforward computation now shows

 $\begin{aligned} \mathcal{L}_{\lambda\alpha}\Lambda Q_{\mathcal{P}} &-\Lambda Q_{\mathcal{P}} + \Lambda Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 + 2 Q_{\mathcal{P}} \left(\Sigma \Lambda \Sigma + \Theta \Lambda \Theta \right) + i b \Lambda^2 Q_{\mathcal{P}} - 2 i \beta \cdot \nabla \left(\Lambda Q_{\mathcal{P}} \right) \\ &= \Lambda \Big[\mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} - Q_{\mathcal{P}} + Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 + i b \Lambda Q_{\mathcal{P}} - 2 i \beta \cdot \nabla Q_{\mathcal{P}} \Big] + 2 \left(\mathcal{L}_{\lambda\alpha} Q_{\mathcal{P}} - Q_{\mathcal{P}} + Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 \right) + 2 Q_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^2) \\ &= -\Lambda \Psi_{\mathcal{P}}^2 + 2 \left(Q_{\mathcal{P}} + 2 i \beta \cdot \nabla Q_{\mathcal{P}} - i b \Lambda Q_{\mathcal{P}} - \Psi_{\mathcal{P}}^2 \right) + \mathcal{O}(\mathcal{P}^2) \end{aligned}$

which leads to (3.44) and (3.45) by taking real and imaginary parts. \Box

Proof of Lemma 3.3 :

First we recall from (3.16) and (3.17) we may write up to second order

$$M_{1}(\epsilon) = -\mathcal{L}_{\lambda\alpha}\epsilon_{1} + \epsilon_{1} - \left[(Q^{2} + 2\Sigma^{2})\epsilon_{1} + 2\Sigma\Theta\epsilon_{2} \right] + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}})$$

$$M_{2}(\epsilon) = -\mathcal{L}_{\lambda\alpha}\epsilon_{2} + \epsilon_{2} - \left[(Q^{2} + 2\Theta^{2})\epsilon_{2} + 2\Sigma\Theta\epsilon_{1} \right] + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}})$$
(3.46)

We continue to recall small expands up to second order that will prove useful. Remind from (2.1) we have

$$\Sigma = Q \cos\left(-b \frac{|y|^2}{4} + \beta \cdot y\right) + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$$

= $Q + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$
 $\Theta = Q \sin\left(-b \frac{|y|^2}{4} + \beta \cdot y\right) + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$
= $Q\left(-b \frac{|y|^2}{4} + \beta \cdot y\right) + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$ (3.47)

We also have

$$\nabla\Sigma = \nabla Q + \mathcal{O}(\mathcal{P}^{2} e^{-C_{2} |y|})$$

$$\nabla\Theta = \left(-b\frac{y}{2} + \beta\right)Q + \left(-b\frac{|y|^{2}}{4} + \beta \cdot y\right)\nabla Q + \mathcal{O}(\mathcal{P}^{2} e^{-C_{2} |y|})$$

$$\Lambda\Sigma = \Lambda Q + \mathcal{O}(\mathcal{P}^{2} e^{-C_{2} |y|})$$

$$\Lambda\Theta = \left(-b\frac{|y|^{2}}{2} + \beta \cdot y\right)Q + \left(-b\frac{|y|^{2}}{4} + \beta \cdot y\right)\Lambda Q + \mathcal{O}(\mathcal{P}^{2} e^{-C_{2} |y|})$$
(3.48)

We now have everything we need to proceed to the computation of the third scalar product in (3.43). Using (3.46), then integration by parts we see

$$\begin{pmatrix} M_1(\epsilon), \Lambda \Sigma \end{pmatrix} = \begin{pmatrix} \epsilon_1, -\mathcal{L}_{\lambda\alpha}\Lambda\Sigma + \Lambda\Sigma - (Q^2 + 2\Sigma^2)\Lambda\Sigma \end{pmatrix} - \begin{pmatrix} \epsilon_2, 2\Sigma\Theta\Lambda\Sigma \end{pmatrix} \\ + \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2}) \\ \begin{pmatrix} M_2(\epsilon), \Lambda\Theta \end{pmatrix} = \begin{pmatrix} \epsilon_2, -\mathcal{L}_{\lambda\alpha}\Lambda\Theta + \Lambda\Theta - (Q^2 + 2\Theta^2)\Lambda\Theta \end{pmatrix} - \begin{pmatrix} \epsilon_1, 2\Sigma\Theta\Lambda\Theta \end{pmatrix} \\ + \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2})$$

Then injecting (3.44) and (3.45) yields

$$\begin{split} & \left(M_{1}(\epsilon),\Lambda\Sigma\right) + \left(M_{2}(\epsilon),\Lambda\Theta\right) \\ &= \left(\epsilon_{1},-\mathcal{L}_{\lambda\alpha}\Lambda\Sigma + \Lambda\Sigma - Q^{2}\Lambda\Sigma - 2\Sigma\left(\Sigma\Lambda\Sigma + \Theta\Lambda\Theta\right)\right) \\ &+ \left(\epsilon_{2},-\mathcal{L}_{\lambda\alpha}\Lambda\Theta + \Lambda\Theta - Q^{2}\Lambda\Theta - 2\Theta\left(\Sigma\Lambda\Sigma + \Theta\Lambda\Theta\right)\right) + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}}) \\ &= -2\left(\epsilon_{1},\Sigma - 2\beta,\nabla\Theta + b\Lambda\Theta - \mathcal{R}e\Psi_{\mathcal{P}}^{2}\right) - 2\left(\epsilon_{2},\Theta + 2\beta,\nabla\Sigma - b\Lambda\Sigma - \mathcal{I}m\Psi_{\mathcal{P}}^{2}\right) \\ &+ b\left[-\left(\epsilon_{1},\Lambda^{2}\Theta\right) + \left(\epsilon_{2},\Lambda^{2}\Sigma\right)\right] + \left[\left(\epsilon_{1},2\beta,\nabla(\Lambda\Theta)\right) - \left(\epsilon_{2},2\beta,\nabla(\Lambda\Sigma)\right)\right] \\ &+ \left(\epsilon_{1},\mathcal{R}e\Lambda\Psi_{\mathcal{P}}^{2}\right) + \left(\epsilon_{2},\mathcal{I}m\Lambda\Psi_{\mathcal{P}}^{2}\right) + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}}) \\ &= -2\mathcal{R}e\left(\epsilon,\overline{Q_{\mathcal{P}}}\right) + 2b\left[-\left(\epsilon_{1},\Lambda\Theta\right) + \left(\epsilon_{2},\Lambda\Sigma\right)\right] + 4\beta\cdot\left[\left(\epsilon_{1},\nabla\Theta\right) - \left(\epsilon_{2},\nabla\Sigma\right)\right] \\ &- \left(-b\Lambda\epsilon_{1} + 2\beta,\nabla\epsilon_{1},\Lambda\Theta\right) + \left(-b\Lambda\epsilon_{2} + 2\beta,\nabla\epsilon_{2},\Lambda\Sigma\right) + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}}) \end{split}$$

Eventually, using orthogonality condition (3.7) and (3.9) we get our result. \Box

It remains to find some linear equations that will help us to estimate the vector Mod(t) of the laws of the \mathcal{P} parameters. Again, to do so we need some second order expansions which we list here

$$\partial_{b}\Sigma = \left(-b\frac{|y|^{2}}{4} + \beta \cdot y\right)\frac{|y|^{2}}{4}Q + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right)$$

$$\partial_{b}\Theta = -\frac{|y|^{2}}{4}Q + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right)$$

$$\partial_{\beta}\Sigma = -\left(-b\frac{|y|^{2}}{4} + \beta \cdot y\right)yQ + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right)$$

$$\partial_{\beta}\Theta = yQ + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right)$$
(3.49)

Now we will get our equations on the vector of modulations laws $Mod(t) = (L_b - K_b(\lambda, \alpha), L_\lambda, L_\alpha, L_\beta, L_{\tilde{\gamma}} - K_{\tilde{\gamma}}(\lambda, \alpha)).$

$$\underline{\mathbf{Law of b}} \qquad \text{We compute } \left(\widehat{\mathbb{Q}}, -\Lambda\Theta \right) + \left(\widehat{\mathbb{Q}}, \Lambda\Sigma \right), \text{ with } \widehat{\mathbb{Q}} = (3.39), \widehat{\mathbb{Q}} = (3.40): \\
 \left(b_s + b^2 - B_1 \right) \left\{ \int \frac{|y|^2}{4} Q^2 + \mathcal{O}(\mathcal{P}^2) \right\} - 2 \mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) \\
 = -\left(\widetilde{\gamma}_s - |\beta|^2 \right) \left[\left(\epsilon_1, \Lambda\Sigma \right) + \left(\epsilon_2, \Lambda\Theta \right) \right] - \left(\frac{\alpha_s}{\lambda} - 2\beta \right) \cdot \beta \int \frac{Q^2}{2} \\
 + \left(R_1(\epsilon), \Lambda\Sigma \right) + \left(R_2(\epsilon), \Lambda\Theta \right) + \mathcal{O}\left(\mathcal{P}^2\left(||\epsilon||_{L^2} + |Mod(t)| \right) + \mathcal{P}^4 \right)$$
(3.50)

Note from Hölder estimates

$$\left(R_1(\epsilon), \Lambda\Sigma\right) + \left(R_2(\epsilon), \Lambda\Theta\right) = \mathcal{O}(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3)$$

Let

$$A_1 = \int \frac{|y|^2}{4} Q^2, \quad a_1 = \beta \int \frac{Q^2}{2} = \mathcal{O}(\mathcal{P}), \quad a_2 = \left(\epsilon_1, \Lambda \Sigma\right) + \left(\epsilon_2, \Lambda \Theta\right) = \mathcal{O}(\|\epsilon\|_{L^2})$$

we compute

$$\begin{aligned} (b_s + b^2 - B_1) \left\{ \int \frac{|y|^2}{4} Q^2 + \mathcal{O}(\mathcal{P}^2) \right\} &- 2 \,\mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) \\ &= (b_s + b^2 - B_1) \int \frac{|y|^2}{4} Q^2 + \int |\epsilon|^2 - \frac{1}{2} \,\nabla^2 \big(k - g\big)(0).(\alpha, \alpha) \int Q^2 + \frac{\lambda^2}{2} \,\nabla \big(\Delta g\big)(0).\alpha \,\int \frac{|y|^2}{4} \,Q^2 \\ &+ \mathcal{O}\Big(\mathcal{P}^2 \left|b_s + b^2 - B_1\right| + \mathcal{P}^4\Big) \\ &= (b_s + b^2 - B_1 - K_b) \int \frac{|y|^2}{4} \,Q^2 + \int |\epsilon|^2 + \mathcal{O}\Big(\mathcal{P}^2 \left|Mod(t)\right| + \mathcal{P}^4 + |\alpha|^3\Big) \\ &K_b(\lambda, \alpha) = \frac{\|Q\|_{L^2}^2}{2\|\frac{|y|}{2}Q\|_{L^2}^2} \,\nabla^2 \big(k - g\big)(0).(\alpha, \alpha) - \frac{\lambda^2}{2} \,\nabla \big(\Delta g\big)(0).\alpha \end{aligned}$$

Our first equation, which is linear with respect to the Mod(t) coordinates, is then

$$A_1 L_b + a_1 L_\alpha + a_2 L_{\tilde{\gamma}} = \mathcal{O}\Big(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3 + \mathcal{P}^2 \left(\|\epsilon\|_{L^2} + |Mod(t)| \right) + \mathcal{P}^4 + |\alpha|^3 \Big)$$
(3.51)

$$\underline{\text{Law of }\lambda} \qquad \text{We compute } \left(\widehat{\mathbb{Q}}, |y|^{2} \Sigma\right) + \left(\widehat{\mathbb{Q}}, |y|^{2} \Theta\right) \text{ with } \widehat{\mathbb{Q}} = (3.39), \widehat{\mathbb{Q}} = (3.40): \\
0 = \left(\frac{\lambda_{s}}{\lambda} + b\right) \left\{ -\int |y|^{2} Q^{2} + \mathcal{O}(\mathcal{P}^{2}) \right\} + \left(\widetilde{\gamma}_{s} - |\beta|^{2}\right) \left[-\left(\epsilon_{1}, |y|^{2} \Theta\right) + \left(\epsilon_{2}, |y|^{2} \Sigma\right) \right] \\
+ \left(R_{1}(\epsilon), |y|^{2} \Theta\right) - \left(R_{2}(\epsilon), |y|^{2} \Sigma\right) + \mathcal{O}\left(\mathcal{P}^{2}\left(||\epsilon||_{L^{2}} + |Mod(t)|\right) + \mathcal{P}^{4}\right)$$
(3.52)

Let

$$A_{2} = -\int |y|^{2} Q^{2}, \quad a_{3} = -(\epsilon_{1}, |y|^{2} \Theta) + (\epsilon_{2}, |y|^{2} \Sigma) = \mathcal{O}(||\epsilon||_{L^{2}})$$

also note

$$(\widetilde{\gamma}_s - |\beta|^2) \left[-\left(\epsilon_1, |y|^2 \Theta\right) + \left(\epsilon_2, |y|^2 \Sigma\right) \right] = \left(\widetilde{\gamma}_s - |\beta|^2 - K_{\widetilde{\gamma}}\right) \left[-\left(\epsilon_1, |y|^2 \Theta\right) + \left(\epsilon_2, |y|^2 \Sigma\right) \right] + \mathcal{O}(\mathcal{P} |\alpha|^2)$$

Again, with Hölder estimates we have

$$A_2 L_{\lambda} + a_3 L_{\tilde{\gamma}} = \mathcal{O}\left(\|\epsilon\|_{H^1}^3 + \mathcal{P}^2\left(\|\epsilon\|_{L^2} + |Mod(t)|\right) + \mathcal{P}^4 + \mathcal{P}|\alpha|^2\right)$$

$$\underline{\text{Law of } \alpha} \qquad \text{We compute } \left(\widehat{\mathbb{Q}}, y \Sigma \right) + \left(\widehat{\mathbb{Q}}, y \Theta \right) \text{ with } \widehat{\mathbb{Q}} = (3.39), \widehat{\mathbb{Q}} = (3.40): \\
0 = \left(\left(\frac{\alpha_s}{\lambda} - 2\beta \right) \cdot \nabla Q, y Q \right) + \mathcal{O} \left(\mathcal{P}^2 \left| \frac{\alpha_s}{\lambda} - 2\beta \right| \right) \\
+ \left(\widetilde{\gamma}_s - |\beta|^2 \right) \left[-\left(\epsilon_1, y \Theta \right) + \left(\epsilon_2, y \Sigma \right) \right] + \left(R_1(\epsilon), y \Theta \right) - \left(R_2(\epsilon), y \Sigma \right) \\
+ \mathcal{O} \left(\mathcal{P}^2 \left(||\epsilon||_{L^2} + |Mod(t)| \right) + \mathcal{P}^4 \right) \\
= \left(\left(\frac{\alpha_s}{\lambda} - 2\beta \right) \cdot \nabla Q, y Q \right) + \mathcal{O} \left(\mathcal{P}^2 \left| \frac{\alpha_s}{\lambda} - 2\beta \right| \right) \\
+ \left(\widetilde{\gamma}_s - |\beta|^2 - K_{\widetilde{\gamma}} \right) \left[-\left(\epsilon_1, y \Theta \right) + \left(\epsilon_2, y \Sigma \right) \right] + \left(R_1(\epsilon), y \Theta \right) - \left(R_2(\epsilon), y \Sigma \right) \\
+ \mathcal{O} \left(\mathcal{P}^2 \left(||\epsilon||_{L^2} + |Mod(t)| \right) + \mathcal{P}^4 + \mathcal{P} |\alpha|^2 \right)$$
(3.53)

Let

$$A_4 = -\int \frac{Q^2}{2}, \quad a_4 = -\left(\epsilon_1, \ y \Theta\right) + \left(\epsilon_2, \ y \Sigma\right) = \mathcal{O}(\|\epsilon\|_{L^2})$$

from integration by parts and Hölder estimates we have

$$A_4 L_{\alpha} + a_4 L_{\tilde{\gamma}} = \mathcal{O}\Big(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3 + \mathcal{P}^2 \left(\|\epsilon\|_{L^2} + |Mod(t)|\right) + \mathcal{P}^4 + \mathcal{P} |\alpha|^2\Big)$$

$$\underline{\text{Law of }\beta} \qquad \text{We compute } \left(\widehat{\mathbb{1}}, -\nabla\Theta \right) + \left(\widehat{\mathbb{2}}, \nabla\Sigma \right) \text{ with } \widehat{\mathbb{1}} = (3.39), \widehat{\mathbb{2}} = (3.40): \\
\left(\left(\beta_s + b\beta - B_2 \right) \cdot y \, Q, \, \nabla Q \right) + \mathcal{O} \left(\mathcal{P}^2 \left| \beta_s + b\beta - B_2 \right| \right) \\
= \left(\frac{\lambda_s}{\lambda} + b \right) \left\{ -\beta \int \frac{Q^2}{2} + \mathcal{O} (\mathcal{P}^2) \right\} - \left(\widetilde{\gamma}_s - |\beta|^2 \right) \left[\left(\epsilon_1, \, \nabla\Sigma \right) + \left(\epsilon_2, \, \nabla\Theta \right) \right] \\
- \left(R_1(\epsilon), \, \nabla\Sigma \right) - \left(R_2(\epsilon), \, \nabla\Theta \right) + \mathcal{O} \left(\mathcal{P}^2 \left(||\epsilon||_{L^2} + |Mod(t)| \right) + \mathcal{P}^4 \right) \\
= \left(\frac{\lambda_s}{\lambda} + b \right) \left\{ -\beta \int \frac{Q^2}{2} + \mathcal{O} (\mathcal{P}^2) \right\} - \left(\widetilde{\gamma}_s - |\beta|^2 - K_{\widetilde{\gamma}} \right) \left[\left(\epsilon_1, \, \nabla\Sigma \right) + \left(\epsilon_2, \, \nabla\Theta \right) \right] \\
- \left(R_1(\epsilon), \, \nabla\Sigma \right) - \left(R_2(\epsilon), \, \nabla\Theta \right) + \mathcal{O} \left(\mathcal{P}^2 \left(||\epsilon||_{L^2} + |Mod(t)| \right) + \mathcal{P}^4 + \mathcal{P} |\alpha|^2 \right)$$
(3.54)

Let

$$A_5 = -\int \frac{Q^2}{2}, \quad a_5 = -\beta \int \frac{Q^2}{2} = \mathcal{O}(\mathcal{P}), \quad a_6 = -\left(\epsilon_1, \ \nabla \Sigma\right) - \left(\epsilon_2, \ \nabla \Theta\right) = \mathcal{O}(\|\epsilon\|_{L^2})$$

so we have

$$A_5 L_{\beta} + a_5 L_{\lambda} + a_6 L_{\tilde{\gamma}} = \mathcal{O}\Big(\|\epsilon\|_{H^1}^3 + \mathcal{P}^2\left(\|\epsilon\|_{L^2} + |Mod(t)|\right) + \mathcal{P}^4 + \mathcal{P}|\alpha|^2\Big)$$

 $\underbrace{ \mathbf{Law of } \widetilde{\gamma} }_{\text{We compute } \left(\textcircled{0}, -\varrho_2 \right) + \left(\textcircled{0}, \varrho_1 \right) \text{ with } \textcircled{0} = (3.39), \textcircled{0} = (3.40) : }_{\text{We recall that } \varrho \text{ was introduced as the unique radial } H^1 \text{ solution to } L_+ \varrho = |y|^2 Q, \text{ and } \varrho_1, \varrho_2 \text{ are therefore the real, radial and } H^1 \text{ functions introduced as }$

$$\varrho_1 + i \, \varrho_2 = \varrho(y) \, e^{-i \, b \, \frac{|y|^2}{4} + i \, \beta \, \cdot \, y}$$

so that their second order extensions are given as before by

$$\varrho_1 = \varrho + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right), \quad \varrho_2 = \left(-b \frac{|y|^2}{4} + \beta \cdot y\right) \varrho + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$$

Thus, the computation we announced leads to

$$-(b_{s}+b^{2}-B_{1})\left\{\left(\frac{|y|^{2}}{4}Q, \varrho\right)+\mathcal{O}(\mathcal{P}^{2})\right\}=-\left(\widetilde{\gamma}_{s}-|\beta|^{2}\right)\left[\left(Q, \varrho\right)+\left(\epsilon_{1}, \varrho_{1}\right)+\left(\epsilon_{2}, \varrho_{2}\right)\right]$$

$$-\left(R_{1}(\epsilon), \varrho_{1}\right)-\left(R_{2}(\epsilon), \varrho_{2}\right)+\mathcal{O}\left(\mathcal{P}^{2}\left(\|\epsilon\|_{L^{2}}+|Mod(t)|\right)+\mathcal{P}^{4}\right)$$
so that

so that

$$-(b_{s}+b^{2}-B_{1}-K_{b})\left(\frac{|y|^{2}}{4}Q, \varrho\right)+\left(\widetilde{\gamma}_{s}-|\beta|^{2}-K_{\widetilde{\gamma}}\right)\left[\left(Q, \varrho\right)+\left(\epsilon_{1}, \varrho_{1}\right)+\left(\epsilon_{2}, \varrho_{2}\right)\right]$$

$$=-\left(R_{1}(\epsilon), \varrho_{1}\right)-\left(R_{2}(\epsilon), \varrho_{2}\right)+\mathcal{O}\left(\mathcal{P}^{2}\left(\|\epsilon\|_{L^{2}}+|Mod(t)|\right)+\mathcal{P}^{4}+\mathcal{P}|\alpha|^{2}\right)$$
(3.56)

Let

$$A_{6} = \int \frac{|y|^{2}}{4} Q \varrho, \quad A_{7} = \int Q \varrho$$
$$a_{7} = \left(\epsilon_{1}, \varrho_{1}\right) + \left(\epsilon_{2}, \varrho_{2}\right) = \mathcal{O}(||\epsilon||_{L^{2}})$$

so we have

$$A_{6} L_{b} + (A_{7} + a_{7}) L_{\tilde{\gamma}} = \mathcal{O}\left(\|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2}\left(\|\epsilon\|_{L^{2}} + |Mod(t)|\right) + \mathcal{P}^{4} + \mathcal{P}|\alpha|^{2}\right)$$

Conclusion All these almost linear identities we got may be written as a matrix equation. Let $X = {}^{t}Mod(t) = {}^{t}(L_{b}, L_{\lambda}, L_{\alpha}, L_{\beta}, L_{\tilde{\gamma}}) \in \mathbb{R}^{7}$ be the vector of the laws modulations, and let

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_5 & 0 \\ A_6 & 0 & 0 & 0 & 0 & 0 & A_7 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 & (a_1)_1 & (a_1)_2 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & (a_4)_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & (a_4)_2 \\ 0 & (a_5)_1 & (a_5)_2 & 0 & 0 & 0 & (a_6)_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (a_6)_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix}$$
(3.57)

Eventually, let $Z \in \mathbb{R}^8$ which satisfies

$$Z_{j} = \mathcal{O}\Big(\|\epsilon\|_{L^{2}}^{2} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2}\left(\|\epsilon\|_{L^{2}} + |Mod(t)|\right) + \mathcal{P}^{4} + \mathcal{P}|\alpha|^{2}\Big), \quad 1 \le j \le 8$$

Now our previous computations simply become

$$(A+a) X = Z, \quad a = \mathcal{O}(\mathcal{P} + \|\epsilon\|_{L^2})$$

so that

$$(A + a + \mathcal{O}(\mathcal{P}^2)) X = Y, \text{ with} Y_j = \mathcal{O}\Big(\|\epsilon\|_{L^2}^2 + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{H^1}^3 + \mathcal{P}^4 + \mathcal{P} |\alpha|^2 \Big), \quad 1 \le j \le 8$$
Since A is invertible, with $A^{-1} = \begin{pmatrix} 1/A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/A_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/A_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/A_7 \end{pmatrix}, \text{ and since } a + \mathcal{O}(\mathcal{P}^2) \to 0$

as $t \to 0$

$$X = \sum_{j \ge 0} \left(-A[a + \mathcal{O}(\mathcal{P}^2)] \right)^j A^{-1} Y = \left(A^{-1} + \mathcal{O}(\mathcal{P} + \|\epsilon\|_{L^2}) \right) Y$$

and hence

$$X_{j} = \mathcal{O}\Big(\|\epsilon\|_{L^{2}}^{2} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{4} + \mathcal{P} |\alpha|^{2}\Big), \quad 1 \le j \le 8$$

This completes the proof of (3.24) for we have

$$\left| \left(L_b - K_b, L_\lambda, L_\alpha, L_\beta, L_{\widetilde{\gamma}} - K_{\widetilde{\gamma}} \right) \right| \lesssim \|\epsilon\|_{L^2}^2 + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{H^1}^3 + \mathcal{P}^4 + \mathcal{P} |\alpha|^2$$

3.3 Refined Energy identity

We proceed now, following the method of [RS11]. Note the following energy estimate will prove useful in the uniqueness part of the proof, when all a priori bounds assumed here will have been shown. In particular, to take all benefit from this estimate, it is crucial to first obtain $b \sim \lambda$, which will then be investigated in the next section.

Let u be a solution to (1.1)-(1.2) on $[t_0, 0)$, and w a be an approximate solution to that same equation :

$$i \partial_t w + \mathcal{L}w + k(x) |w|^2 w = \psi$$
(3.58)

with a priori bounds :

$$\|w\|_{L^2(g\,dx)} \lesssim 1, \quad \|\nabla w\|_{L^2(g\,dx)} \lesssim \frac{1}{\lambda}, \quad \|w\|_{H^{3/2}(g\,dx)} \lesssim \frac{1}{\lambda^{3/2}}$$
 (3.59)

Then, we decompose $u = w + \tilde{u}$ so that :

$$i\partial_t \tilde{u} + \mathcal{L}\tilde{u} + k(x)\left(|u|^2 u - |w|^2 w\right) = -\psi$$
(3.60)

We assume the a priori bounds on \tilde{u} :

$$\|\nabla \tilde{u}\|_{L^2(g\,dx)} \lesssim \lambda, \quad \|\tilde{u}\|_{L^2(g\,dx)} \lesssim \lambda^2 \tag{3.61}$$

and on the geometrical parameters :

$$|\lambda \lambda_t + b| \lesssim \lambda^4, \quad b \sim \lambda, \quad |\lambda \alpha_t| \lesssim \lambda, \quad |b_t| \lesssim 1$$
(3.62)

where $\lambda > 0$, $\alpha \in \mathbb{R}^2$, $b \ll 1$. We let A > 0 a large enough constant, to be chosen later, and ϕ a cut off function such that :

$$\phi'(r) = \begin{cases} r & \text{for } r \le 1, \\ 3 - e^{-r} & \text{for } r \ge 2 \end{cases}$$
(3.63)

Let

$$F_{4}(u) = \frac{1}{4} |u|^{4}, \quad f_{4}(u) = |u|^{2} u, \text{ so that } F'(u) \cdot h = \mathcal{R}e(f_{4}(u)\overline{h})$$

$$F_{2}(u) = \frac{1}{2} |u|^{2}, \text{ so that } F'_{2}(u) \cdot h = \mathcal{R}e(u\overline{h})$$
(3.64)

We now want to look at the variation of the conservation laws on an exact solution u with respect to the approximate profile w we have just introduced. So let

$$m(u) = \frac{1}{2} \int |u|^2 g(x) dx,$$

$$M_{\phi,A}(u) = \mathcal{I}m \int A \, G \nabla \phi \left(\frac{x-\alpha}{A\lambda}\right) \cdot \nabla u \, \overline{u} \, g(x) dx$$

and recall E(u) was defined in (1.11). A simple computation shows

$$\begin{split} E(u) &- E(w) + \frac{1}{\lambda^2} \left(m(u) - m(w) \right) + \frac{b}{2\lambda} \left(M_{\phi,A}(u) - M_{\phi,A}(w) \right) \\ &= -\mathcal{R}e \int \left[\mathcal{L}w + \frac{1}{\lambda^2} w + k(x) \, |w|^2 \, w \right] \overline{\tilde{u}} \, g \, dx + \frac{b}{2\lambda} \, \mathcal{I}m \int A \, G \nabla \phi \left(\frac{x - \alpha}{A\lambda} \right) \cdot \left[\overline{\tilde{u}} \, \nabla w + \tilde{u} \, \overline{\nabla w} \right] \, g \, dx \\ &+ \frac{1}{2} \int G(x) \, \nabla \tilde{u} \, . \, \overline{\nabla \tilde{u}} \, dx + \frac{1}{2} \, \int \frac{|\tilde{u}|^2}{\lambda^2} \, g \, dx - \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u} \right] \, g \, dx \\ &- \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u} \right] \, g \, dx + \frac{b}{2\lambda} \int A \, G \nabla \phi \left(\frac{x - \alpha}{A\lambda} \right) \cdot \nabla \tilde{u} \, \overline{\tilde{u}} \, g \, dx \end{split}$$

The first line term can easily be estimated. On the one hand, using (2.5), the Ground State equation (1.36) and both orthogonality conditions (3.7) and (3.9)

$$\mathcal{R}e\int \left[\mathcal{L}w + \frac{1}{\lambda^2}w + k(x)|w|^2w\right]\overline{\tilde{u}} g \,dx = \frac{1}{k(\alpha)}\frac{1}{\lambda^2}\int \left(b\Lambda Q - 2\beta \cdot \nabla Q\right)\epsilon_2 g dy + \mathcal{O}\left(\|\epsilon\|_{L^2}\right)$$
$$= \frac{1}{k(\alpha)}\frac{1}{\lambda^2}\int \left(b\Lambda\Theta - 2\beta \cdot \nabla\Theta\right)\epsilon_1 \,dy + \mathcal{O}\left(\|\epsilon\|_{L^2}\right) = \mathcal{O}\left(\|\epsilon\|_{L^2}\right) = \mathcal{O}\left(\|\tilde{u}\|_{L^2}\right)$$

On the other hand, definition (3.63) of function ϕ imply

$$\begin{aligned} \nabla \phi(y) &= y, \quad |y| \leq 1, \qquad |\nabla \phi(y)| \lesssim 1, \quad |y| > 1\\ \Delta \phi(y) &= 2, \quad |y| \leq 1, \qquad |\Delta \phi(y)| \lesssim e^{-|y|}, \quad |y| > 1 \end{aligned}$$

so that, since $G(\lambda y + \alpha) = I + \mathcal{O}(\mathcal{P}^2)$

$$\begin{split} \mathcal{I}m \int A \, G \nabla \phi \left(\frac{x-\alpha}{A\lambda}\right) \cdot \left[\tilde{u} \, \nabla w + \tilde{u} \, \overline{\nabla w}\right] \, g \, dx \\ &= \frac{1}{k(\alpha)} \frac{1}{\lambda} \left\{ \int_{|y| \le A} \left(-\epsilon_2 \, \Lambda \Sigma + \epsilon_1 \, \Lambda \Theta \right) \, g dy + \int_{|y| > A} 2 \, A \, \nabla \phi \left(\frac{y}{A}\right) \cdot \left(-\epsilon_2 \, \nabla \Sigma + \epsilon_1 \, \nabla \Theta \right) \, g dy \\ &+ \int_{|y| > A} \Delta \phi \left(\frac{y}{A}\right) \left(-\epsilon_2 \, \Sigma + \epsilon_1 \, \Theta \right) \, g dy \right\} + \mathcal{O} \big(\mathcal{P} \, \|\epsilon\|_{L^2} \big) \\ &= \frac{1}{k(\alpha)} \frac{1}{\lambda} \left\{ \int_{|y| \le A} \left(-\epsilon_2 \, \Lambda \Sigma + \epsilon_1 \, \Lambda \Theta \right) dy + \int_{|y| > A} R_{\phi, A} \, dy \right\} + \mathcal{O} \big(\mathcal{P} \, \|\epsilon\|_{L^2} \big) \end{split}$$

with

$$R_{\phi,A} = 2A\nabla\phi\left(\frac{y}{A}\right) \cdot \left(-\epsilon_2\nabla\Sigma + \epsilon_1\nabla\Theta\right) + \Delta\phi\left(\frac{y}{A}\right)\left(-\epsilon_2\Sigma + \epsilon_1\Theta\right)$$

and obviously, thanks to orthogonality condition (3.9)

$$\int_{|y| \le A} \left(-\epsilon_2 \Lambda \Sigma + \epsilon_1 \Lambda \Theta \right) dy \to \int \left(-\epsilon_2 \Lambda \Sigma + \epsilon_1 \Lambda \Theta \right) dy = 0, \quad \text{as} \quad A \to +\infty$$
$$\int_{|y| > A} R_{\phi,A} \, dy \to 0, \quad \text{as} \quad A \to +\infty$$

Then, we claim the following

Lemma 3.5. (Generalized energy estimate) Let

$$\mathcal{I} = \frac{1}{2} \int G(x) \nabla \tilde{u} \cdot \overline{\nabla \tilde{u}} \, dx + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} g \, dx + \frac{b}{2\lambda} \int A \, G \nabla \phi \left(\frac{x - \alpha}{A\lambda}\right) \cdot \nabla \tilde{u} \, \overline{\tilde{u}} \, g \, dx$$
$$- \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u} \right] g(x) dx$$
$$- \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u} \right] g(x) dx$$
(3.65)

then, we have

$$\begin{aligned} \frac{d\mathcal{I}}{dt} &= -\frac{1}{\lambda^2} \mathcal{I}m\left(\int k(x) \, w^2 \,\overline{\tilde{u}^2} \, g \, dx\right) - \mathcal{R}e\left(\int k(x) \, \partial_t w \,\overline{(2 \, |\tilde{u}|^2 \, w + \tilde{u}^2 \, \overline{w})} \, g \, dx\right) \\ &+ \frac{b}{\lambda^2} \left[\int \frac{|\tilde{u}|^2}{\lambda^2} \, g \, dx + \mathcal{R}e \int \frac{1}{g^2} \left(g \, G \nabla\right)^2 \phi\left(\frac{x - \alpha}{A\lambda}\right) . \left(\nabla \tilde{u} \, , \, \overline{\nabla \tilde{u}}\right) \, dx - \frac{1}{4A^2} \int \Delta_G \Delta_{G,g,2} \, \phi\left(\frac{x - \alpha}{A\lambda}\right) \, \frac{|\tilde{u}|^2}{\lambda^2} \, dx\right] \\ &+ \frac{b}{\lambda} \mathcal{R}e \int A \, G \, \nabla \phi\left(\frac{x - \alpha}{A\lambda}\right) . k(x) \left(2 \, w \, |\tilde{u}|^2 + \overline{w} \, \tilde{u}^2\right) \nabla \overline{w} \, g \, dx \end{aligned} \tag{3.66} \\ &+ \mathcal{I}m \, \int \left[\mathcal{L}\psi - \frac{\psi}{\lambda^2} + k(x) \left(2 \, |w|^2 \, \psi - w^2 \, \overline{\psi}\right) + i \, \frac{b}{\lambda} \, A \, G \, \nabla \phi\left(\frac{x - \alpha}{A\lambda}\right) . \nabla \psi + i \, \frac{b}{2\lambda^2} \, \Delta_{G,g,2} \, \phi\left(\frac{x - \alpha}{A\lambda}\right) \, \psi\right] \overline{\tilde{u}} \, g \, dx \\ &+ \mathcal{O}\left(\lambda^2 \, \|\psi\|_{L^2}^2 + \frac{1}{\lambda^2} \, \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2\right) \end{aligned}$$

Remark 6. (i) Note that (3.66) brings a better control, for it keeps track of the quadratic terms in \tilde{u} . It is all about getting a control of the form :

$$\frac{d}{dt}\left\{ \left(G\nabla\tilde{u}\,,\,\nabla\tilde{u}\right)_{L^2_{dx}} + \frac{1}{\lambda^2}\,\|\tilde{u}\|^2_{L^2_{g\,dx}} \right\} \ge \frac{b}{\lambda^2} \left(\int_{|x-\alpha| \lesssim \lambda} G\nabla\tilde{u}\,.\,\overline{\nabla\tilde{u}}\,dx + \frac{1}{\lambda^2}\,\|\tilde{u}\|^2_{L^2(gdx)} + l.o.t.\right) \tag{3.67}$$

(ii) From hypothseis (H2) on g, notice that for a function f

$$\|f\|_{\mathcal{H}(g\,dx)} = \mathcal{O}(\|f\|_{\mathcal{H}(dx)}) \|f\|_{\mathcal{H}(g\,dy)} = \|f\|_{\mathcal{H}(dy)} + \mathcal{O}(\lambda^2 \|f\|_{\mathcal{H}(dy)})$$
(3.68)

where \mathcal{H} is any L^p or H^k space.

(iii) Before proving the lemma, let us take some time to talk about hypotheses (3.59) to (3.62). Indeed, based on the previous sections, one should check those assumptions are satisfied by the approximate profile we have built, that is when

$$w = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} Q_{\mathcal{P}(t)}\left(t, \frac{x - \alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
$$\tilde{u} = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \epsilon\left(t, \frac{x - \alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$

So here it comes

1. From (2.7)

$$\|w\|_{L^{2}(g\,dx)}^{2} = \frac{1}{k(\alpha)} \int |Q_{\mathcal{P}}|^{2} g\,dy = \int Q^{2} + \mathcal{O}(\lambda^{2} + |\alpha|^{2}) \lesssim 1$$

2. From computation of energy

$$\|\nabla w\|_{L^2(g\,dx)}^2 = \frac{1}{k(\alpha)\lambda^2} \int |\nabla Q_{\mathcal{P}}|^2 g\,dy = \frac{1}{\lambda^2} \left[\int |\nabla Q|^2 + \mathcal{O}\left(\lambda^2 + |\alpha|^2\right) \right]$$

hence $\|\nabla w\|_{L^2(g\,dx)} \lesssim \frac{1}{\lambda}$

3. Fourier transform computation leads to

$$\begin{split} \widehat{w}(\xi) &= \frac{\lambda \, e^{-i\alpha \, \cdot \, \xi}}{k(\alpha)^{1/2}} \, \widehat{Q_{\mathcal{P}}}(t, \lambda \, \xi) = \frac{\lambda \, e^{-i\alpha \, \cdot \, \xi}}{k(\alpha)^{1/2}} \left[\widehat{Q}(\lambda \, \xi) + \mathcal{O}(\mathcal{P}) \right] \\ so \ that \ \ \|w\|_{H^{3/2}}^2 &= \int (1 + |\xi|^2)^{3/2} \, \lambda^2 \, |\widehat{Q}(\lambda \, \xi)|^2 \, d\xi + \mathcal{O}(\mathcal{P}^3) \lesssim \frac{1}{\lambda^3} \, \|Q\|_{H^{3/2}}^2 + \mathcal{O}(1) \\ hence \ \ \|w\|_{H^{3/2}(g \, dx)} \lesssim \|w\|_{H^{3/2}} \lesssim \frac{1}{\lambda^{3/2}} \end{split}$$

4. From (3.51), since $|Mod(t)| \leq \mathcal{P}^4 + \mathcal{P} |\alpha|^2$ thanks to (3.24), and using (3.21)

$$\left(1 + \mathcal{O}\left(\|\epsilon\|_{L^2} + \mathcal{P}^2\right)\right) \int |\epsilon|^2 = \mathcal{O}\left(\mathcal{P}^4 + \mathcal{P} |\alpha|^2\right)$$

hence $\|\tilde{u}\|_{L^2(g\,dx)} = \frac{1}{k(\alpha)^{1/2}} \|\epsilon\|_{L^2(g\,dy)} \lesssim \lambda^2 + \lambda^{1/2} |\alpha|$

5. From (3.21), the best we can say so far is $\|\nabla \tilde{u}\|_{L^2(g\,dx)} \lesssim \frac{1}{\lambda} \|\nabla \epsilon\|_{L^2} (1+|\alpha|^2) \lesssim 1$

6. Eventually, from (3.24) and (3.21) one successively gets

$$\begin{aligned} |\lambda \lambda_t + b| &= \left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \lambda^4 \\ |\lambda \alpha_t| &= \left| \frac{\alpha_s}{\lambda} \right| \lesssim \left| \frac{\alpha_s}{\lambda} - 2\beta \right| + |\beta| \lesssim \lambda \\ |b_t| &= \frac{1}{\lambda^2} \left| b_s \right| \lesssim \frac{1}{\lambda^2} \left[\left| b_s + b^2 - B_1 \right| + \left| b^2 - B_1 \right| \right] \lesssim \lambda^2 + 1 \lesssim 1 \end{aligned}$$

Provided we have proven $|\alpha| \leq \lambda^2$, all needed assumptions are satisfied but two which are left to prove. Indeed, both $b \sim \lambda$ and $\|\tilde{u}\|_{H^1(g\,dx)} \leq \lambda^2$ have to be shown. We will do so in the fourth section.

Proof of Lemma 3.5 :

Step 1 The computation is essentially the same as in [RS11], except for the term Δ which is replaced by \mathcal{L} . Thus, derivating the first part of \mathcal{I} , using (3.60) and (3.64) yield

$$\begin{split} &\frac{d}{dt} \left\{ \frac{1}{2} \int G(x) \,\nabla \tilde{u} \,. \overline{\nabla \tilde{u}} \,dx + \frac{1}{2} \,\int \frac{|\tilde{u}|^2}{\lambda^2} g(x) dx - \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \,. \tilde{u} \right] g \,dx \right\} \\ &- \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \,. \tilde{u} \right] g \,dx \right\} \\ &= \mathcal{R}e \Big(\partial_t \tilde{u} \,, \overline{-\frac{1}{g} \,div(G \nabla \tilde{u}) + \frac{1}{\lambda^2} \,\tilde{u} - k(x) \left[f_4(w + \tilde{u}) - f_4(w) \right] - V(x) \left[w + \tilde{u} - w \right]} \Big)_{L^2(g \,dx)} \\ &+ \mathcal{R}e \Big(\partial_t w \,, \overline{-k(x) \left[f_4(w + \tilde{u}) - f_4(w) - f_4'(w) \,. \tilde{u} \right] - V(x) \left[w + \tilde{u} - w - \tilde{u} \right]} \Big)_{L^2(g \,dx)} \\ &- \frac{\lambda_t}{\lambda^3} \,\int |\tilde{u}|^2 \,g \,dx \\ &= \mathcal{I}m \Big(\psi \,, \overline{\mathcal{L}\tilde{u} - \frac{1}{\lambda^2} \,\tilde{u} + k(x) \left[f_4(u) - f_4(w) \right]} \Big)_{L^2(g \,dx)} - \frac{1}{\lambda^2} \,\mathcal{I}m \Big(k(x) \left[f_4(u) - f_4(w) \right] \,, \overline{\tilde{u}} \Big)_{L^2(g \,dx)} \\ &- \frac{\lambda_t}{\lambda^3} \,\int |\tilde{u}|^2 \,g \,dx - \mathcal{R}e \Big(\partial_t w \,, \overline{k(x) \left[f_4(w + \tilde{u}) - f_4(w) - f_4'(w) \,. \tilde{u} \right]} \Big)_{L^2(g \,dx)} \end{split}$$

Then, notice that $f_4'(w) \cdot \tilde{u} = 2 |w|^2 \tilde{u} + w^2 \overline{\tilde{u}}$, and

$$\mathcal{I}m\left(\tilde{u}, \overline{k(x)\left[f_4(w+\tilde{u}) - f_4(w) - f_4'(w).\tilde{u}\right]}\right)_{L^2(g\,dx)}$$

= $\mathcal{I}m\int k(x)\left[2\,\overline{w}\,\tilde{u}\,|\tilde{u}|^2 + |\tilde{u}|^4 + w\,\overline{\tilde{u}}\,|\tilde{u}|^2\right]\,g\,dx$
= $\mathcal{I}m\int k(x)\,\overline{w}\,\tilde{u}\,|\tilde{u}|^2\,g\,dx$

so that the last line can be rewritten

$$\begin{split} \mathcal{I}m\Big(\psi, \overline{\mathcal{L}\tilde{u} - \frac{1}{\lambda^2}}\,\tilde{u} + k(x)\left[2\,|w|^2\,\tilde{u} + \overline{\tilde{u}}\,w^2\right]\Big)_{L^2(g\,dx)} &- \frac{1}{\lambda^2}\,\mathcal{I}m\int k(x)\,\overline{\tilde{u}}^2\,w^2\,g\,dx\\ &- \frac{\lambda_t}{\lambda^3}\,\int |\tilde{u}|^2\,g\,dx - \mathcal{R}e\Big(\partial_t w\,,\overline{k(x)}\left[\overline{w}\,\tilde{u}^2 + 2\,w\,|\tilde{u}|^2\right]\Big)_{L^2(g\,dx)} - \mathcal{R}e\Big(\partial_t w\,,\overline{k(x)}\,\tilde{u}\,|\tilde{u}|^2\Big)_{L^2(g\,dx)}\\ &+ \mathcal{I}m\Big(\psi - \frac{1}{\lambda^2}\,\tilde{u}\,,\overline{k(x)}\left[f_4(u) - f_4(w) - f_4\,'(w)\,.\,\tilde{u}\right]\Big)_{L^2(g\,dx)} \end{split}$$

We now estimate some of those remaining terms. First, from (3.62) one gets

$$-\frac{\lambda_t}{\lambda^3} \int |\tilde{u}|^2 g \, dx = \frac{b}{\lambda^4} \int |\tilde{u}|^2 g \, dx - \left(\lambda \, \lambda_t + b\right) \int |\tilde{u}|^2 g \, dx = \frac{b}{\lambda^4} \int |\tilde{u}|^2 g \, dx + \mathcal{O}\left(\|\tilde{u}\|_{H^1(g \, dx)}^2\right)$$

Then, the last two terms are to be treated with a priori bounds (3.59) and (3.61)

$$\begin{aligned} \left| \mathcal{I}m\left(\psi - \frac{1}{\lambda^{2}}\tilde{u}, \overline{k(x)}\left[f_{4}(u) - f_{4}(w) - f_{4}'(w).\tilde{u}\right]\right)_{L^{2}(g\,dx)} \right| \\ &= \left| \mathcal{I}m\left(\psi - \frac{1}{\lambda^{2}}\tilde{u}, \overline{k(x)}\left[2w\,|\tilde{u}|^{2} + \tilde{u}\,|\tilde{u}|^{2} + \overline{w}\,\tilde{u}^{2}\right]\right)_{L^{2}(g\,dx)} \right| \\ &\lesssim \|\psi\|_{L^{2}_{g\,dx}}\left(\|\tilde{u}\|^{2}_{L^{6}_{g\,dx}}\|w\|_{L^{6}_{g\,dx}} + \|\tilde{u}\|^{3}_{L^{6}_{g\,dx}}\right) + \frac{1}{\lambda^{2}}\|\tilde{u}\|^{3}_{L^{6}_{g\,dx}}\|w\|_{L^{2}_{g\,dx}} \\ &\lesssim \|\psi\|_{L^{2}_{g\,dx}}\left(\|\tilde{u}\|^{2/3}_{L^{2}_{g\,dx}}\|\nabla\tilde{u}\|^{4/3}_{L^{2}_{g\,dx}}\|w\|^{1/3}_{L^{2}_{g\,dx}}\|\nabla w\|^{1/3}_{L^{2}_{g\,dx}} + \|\tilde{u}\|^{3}_{L^{6}_{g\,dx}}\right) + \frac{1}{\lambda^{2}}\|\tilde{u}\|_{L^{2}_{g\,dx}}\|\nabla\tilde{u}\|^{2}_{L^{2}_{g\,dx}} \\ &\lesssim \lambda^{2}\|\psi\|_{L^{2}_{g\,dx}} + \|\tilde{u}\|^{2}_{H^{1}_{g\,dx}} \end{aligned}$$

where we have used the Gagliardo-Nirenberg estimate

$$||u||_{L^6} \le C ||u||_{L^2}^{1/3} ||\nabla u||_{L^2}^{2/3}.$$

To deal with the last term, we use (3.58) to replace $\partial_t w$, then by integration by parts, using (3.59) we have

$$\begin{aligned} \left| \mathcal{R}e\Big(\partial_{t}w\,,\overline{k(x)\,\tilde{u}\,|\tilde{u}|^{2}}\Big)_{L^{2}(g\,dx)} \right| \\ &= \left| \mathcal{I}m\int div\big(G\nabla w\big)\,k\,|\tilde{u}|^{2}\,\tilde{u}\,dx + \mathcal{I}m\int k\,\big[k\,|w|^{2}\,w + V\,w + \psi\big]\,g\,dx \right| \\ &\lesssim \left\| div\big(G\nabla w\big) \right\|_{H^{-1/2}_{dx}} \|\,|\tilde{u}|^{2}\,\tilde{u}\,\|_{H^{1/2}_{dx}} + \left(\|\,|w|^{2}\,w\,\|_{L^{2}_{g\,dx}} + \|\,\psi\,\|_{L^{2}_{g\,dx}} + \|\,V\,w\,\|_{L^{2}_{g\,dx}}\Big)\,\|\,|\tilde{u}|^{2}\,\tilde{u}\,\|_{L^{2}_{g\,dx}} \\ &\lesssim \|w\|_{H^{3/2}_{dx}} \|\,|\tilde{u}|^{2}\,\tilde{u}\,\|_{H^{1/2}_{dx}} + \|w\|^{3}_{L^{6}_{g\,dx}} \|\,\tilde{u}\|^{3}_{L^{6}_{g\,dx}} + \|\psi\|_{L^{2}_{g\,dx}} \|\,\tilde{u}\|^{3}_{L^{6}_{g\,dx}} \\ &\lesssim \frac{1}{\lambda^{3/2}} \|\,\tilde{u}\,\|_{L^{2}_{dx}}^{1/2} \|\,\tilde{u}\,\|_{H^{1}_{dx}}^{5/2} + \frac{1}{\lambda^{2}} \|\,\tilde{u}\,\|_{L^{2}_{g\,dx}}^{2} \|\,\tilde{u}\,\|_{L^{2}_{g\,dx}}^{2} + \|\psi\|_{L^{2}_{g\,dx}} \|\,\tilde{u}\,\|_{L^{2}_{g\,dx}}^{2} \|\,\tilde{u}\,\|_{L^{2}_{g\,dx}}^{2} \\ &\lesssim \lambda^{2} \,\|\psi\|_{L^{2}_{g\,dx}} + \|\,\tilde{u}\,\|_{H^{1}_{g\,dx}}^{2} \end{aligned}$$

where we have used twice the pseudo derivative estimate

$$\begin{split} \left\| D^{1/2} (|\tilde{u}|^2 \, \tilde{u}) \right\|_{L^2} &\lesssim \left\| D^{1/2} (|\tilde{u}|^2) \right\|_{L^{8/3}} \|\tilde{u}\|_{L^8} + \| \, |\tilde{u}|^2 \, \|_{L^4} \, \left\| D^{1/2} \tilde{u} \right\|_{L^4} \\ &\lesssim \left\| D^{1/2} \tilde{u} \right\|_{L^4} \|\tilde{u}\|_{L^8}^2 \end{split}$$

then the Sobolev embedding result Theorem 8.2

$$H^{3/4}(\mathbb{R}^2) \subseteq L^8(\mathbb{R}^2)$$

to get

$$\left\| D^{1/2} \tilde{u} \right\|_{L^4} \| \tilde{u} \|_{L^8}^2 \lesssim \| \tilde{u} \|_{H^1} \left(\| \tilde{u} \|_{H^1}^{3/4} \| \tilde{u} \|_{L^2}^{1/4} \right)^2 \lesssim \| \tilde{u} \|_{L^2}^{1/2} \| \tilde{u} \|_{H^1}^{5/2}$$

Eventually

$$\frac{d}{dt} \left\{ \frac{1}{2} \int G(x) \nabla \tilde{u} \cdot \overline{\nabla \tilde{u}} \, dx + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} g \, dx - \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u} \right] g \, dx \right. \\
\left. - \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u} \right] g \, dx \right\}$$

$$= -\frac{1}{\lambda^2} \mathcal{I}m \int k(x) w^2 \, \overline{\tilde{u}^2} g \, dx - \mathcal{R}e \int k(x) \, \partial_t w \, \overline{(2 |\tilde{u}|^2 w + \tilde{u}^2 \overline{w})} g \, dx + \frac{b}{\lambda^2} \int \frac{|\tilde{u}|^2}{\lambda^2} g \, dx \\
+ \mathcal{I}m \int \left[\mathcal{L}\psi - \frac{\psi}{\lambda^2} + k(x) \, (2 |w|^2 \psi - w^2 \, \overline{\psi}) \right] \overline{\tilde{u}} g \, dx + \mathcal{O}\left(\lambda^2 \|\psi\|_{L^2}^2 + \frac{1}{\lambda^2} \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 \right)$$
(3.69)

$$\nabla \tilde{\phi}(t,x) = \frac{b}{\lambda} A \nabla \phi \left(\frac{x-\alpha}{A\lambda}\right)$$
(3.70)

then

$$\frac{1}{2} \frac{d}{dt} \left(\frac{b}{\lambda} \mathcal{I}m \int A \, G \nabla \phi \left(\frac{x - \alpha}{A \, \lambda} \right) \cdot \nabla \tilde{u} \, \overline{\tilde{u}} \, g \, dx \right) \\
= \frac{1}{2} \mathcal{I}m \int G \partial_t \nabla \tilde{\phi} \cdot \nabla \tilde{u} \, \overline{\tilde{u}} \, g \, dx + \mathcal{R}e \int i \partial_t \tilde{u} \left[\frac{1}{2} \, div \left(G \, \nabla \tilde{\phi} \right) \tilde{u} + G \, \nabla \tilde{\phi} \cdot \nabla \tilde{u} \right] \, g \, dx \tag{3.71} \\
+ \mathcal{R}e \int i \partial_t \tilde{u} \, \overline{\tilde{u}} \, G \nabla \tilde{\phi} \cdot \nabla g \, dx$$

First, thanks to (3.62) and the computation :

$$\partial_t \nabla \tilde{\phi} = A \left\{ \frac{1}{\lambda^3} \left[\lambda^2 b_t + b^2 \right] \nabla \phi \left(\frac{x - \alpha}{A \lambda} \right) - \frac{1}{\lambda^3} \lambda \lambda_t \left[\frac{b}{\lambda} \nabla^2 \phi \left(\frac{x - \alpha}{A \lambda} \right) \cdot \left(\frac{x - \alpha}{A} \right) + b \nabla \phi \left(\frac{x - \alpha}{A \lambda} \right) \right] - \frac{b}{\lambda^3} \left[\lambda \nabla^2 \phi \left(\frac{x - \alpha}{A \lambda} \right) \cdot \left(\frac{\alpha_t}{A} \right) + b \nabla \phi \left(\frac{x - \alpha}{A \lambda} \right) \right] \right\}$$

we have :

$$|\partial_t \nabla \tilde{\phi}| \lesssim \frac{1}{\lambda} \tag{3.72}$$

so that, since g is smooth

$$\left|\frac{1}{2}\mathcal{I}m\int\partial_{t}\nabla\tilde{\phi}.\nabla\tilde{u}\,\overline{\tilde{u}}\,g\,dx\right|\lesssim\frac{1}{\lambda}\|\tilde{u}\|_{L^{2}_{g\,dx}}\|\nabla\tilde{u}\|_{L^{2}_{g\,dx}}=\mathcal{O}\left(\frac{1}{\lambda^{2}}\|\tilde{u}\|^{2}_{L^{2}_{g\,dx}}+\|\tilde{u}\|^{2}_{H^{1}_{g\,dx}}\right)$$
(3.73)

Now, with the second term

$$\mathcal{R}e\int i\partial_{t}\tilde{u}\left[\frac{1}{2}\operatorname{div}(G\,\nabla\tilde{\phi})\,\tilde{u} + G\,\nabla\tilde{\phi}.\,\nabla\tilde{u}\right]\,g\,dx + \mathcal{R}e\int i\partial_{t}\tilde{u}\,\overline{\tilde{u}}\,G\nabla\tilde{\phi}.\,\nabla g\,dx$$

$$= \mathcal{R}e\int \frac{1}{g^{2}}\left(g\,G\nabla\right)^{2}\tilde{\phi}.\left(\nabla\tilde{u}\,,\,\overline{\nabla\tilde{u}}\right)\,dx - \frac{1}{4}\int\Delta_{G}\left[\frac{1}{g^{2}}\operatorname{div}\left(g^{2}\,G\nabla\tilde{\phi}\right)\right]\,|\tilde{u}|^{2}\,dx$$

$$- \mathcal{R}e\int \left[k(x)\left(f_{4}(u) - f_{4}(w)\right) + \psi\right]\left(\frac{1}{2g^{2}}\operatorname{div}\left(g^{2}\,G\nabla\tilde{\phi}\right)\,\overline{\tilde{u}} + G\nabla\tilde{\phi}.\,\overline{\nabla\tilde{u}}\right)\,g\,dx$$

$$- \frac{1}{2}\int G\nabla\tilde{\phi}.\,\nabla(g\,V)\,|\tilde{u}|^{2}\,dx$$

$$(3.74)$$

where assuming $g, V \in W^{1,\infty}$ yields

$$\left|\frac{1}{2}\int G\nabla\tilde{\phi} \cdot \nabla(g\,V)\,|\tilde{u}|^2\,dx\right| \lesssim \|g\,V\|_{W^{1,\infty}}\,\|\tilde{u}\|_{L^2}^2 = \mathcal{O}\Big(\frac{1}{\lambda^2}\,\|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2\Big)$$

thus

$$\mathcal{R}e \int i\partial_{t}\tilde{u} \left[\frac{1}{2} \operatorname{div}(G\,\nabla\tilde{\phi})\,\tilde{u} + G\,\nabla\tilde{\phi}\,\cdot\nabla\tilde{u} \right] g\,dx + \mathcal{R}e \int i\partial_{t}\tilde{u}\,\tilde{\bar{u}}\,G\nabla\tilde{\phi}\,\cdot\nabla g\,dx \\ = \frac{b}{\lambda^{2}}\,\mathcal{R}e \int \frac{1}{g^{2}}\left(g\,G\nabla\right)^{2}\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\cdot\left(\nabla\tilde{u}\,,\,\overline{\nabla\tilde{u}}\right)\,dx - \frac{1}{4A^{2}}\,\frac{b}{\lambda^{4}}\,\int\Delta_{G}\Delta_{G,g,2}\,\phi\left(\frac{x-\alpha}{A\lambda}\right)\,|\tilde{u}|^{2}\,dx \\ - \frac{b}{2\lambda^{2}}\,\mathcal{R}e\int\Delta_{G,g,2}\,\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\bar{\bar{u}}\,k(x)\left(f_{4}(u) - f_{4}(w)\right)\,g\,dx \\ - \frac{b}{\lambda}\,\mathcal{R}e\int\Lambda\,G\nabla\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\cdot\overline{\nabla\tilde{u}}\,k(x)\left(f_{4}(u) - f_{4}(w)\right)\,g\,dx \\ - \frac{b}{2\lambda^{2}}\,\mathcal{R}e\int\Delta_{G,g,2}\,\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\bar{\bar{u}}\,\psi\,g\,dx - \frac{b}{\lambda}\,\mathcal{R}e\int\Lambda\,G\nabla\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\cdot\overline{\nabla\tilde{u}}\,\psi\,g\,dx \\ + \mathcal{O}\left(\frac{1}{\lambda^{2}}\,\|\tilde{u}\|_{L^{2}}^{2} + \|\tilde{u}\|_{H^{1}}^{2}\right)$$

$$(3.75)$$

where $\Delta_{_{G,g,\nu}}\,\phi=\frac{1}{g^{\nu}}\,div\bigl(g^{\nu}\,G\nabla\phi\bigr)$

We aim at using the same estimation as in [RS11], using (3.59) to (3.62) along with integration by parts, Holder estimates and Gagliardo-Nirenberg inequalities. First we decompose

$$f_4(u) - f_4(w) = \left(|w|^2 + |\tilde{u}|^2 + \overline{w}\,\tilde{u} + w\,\overline{\tilde{u}}\right)(w + \tilde{u}) - |w|^2\,w$$
$$= \left(2\,|w|^2\,\tilde{u} + w^2\,\overline{\tilde{u}}\right) + \left(2\,w\,|\tilde{u}|^2 + \tilde{u}^2\,\overline{w} + |\tilde{u}|^2\,\tilde{u}\right)$$

Then using (3.21)

$$\begin{vmatrix} -\frac{b}{2\lambda^{2}} \int \Delta_{G,g,2} \phi\left(\frac{x-\alpha}{A\lambda}\right) \bar{\tilde{u}} k(x) \left(2w |\tilde{u}|^{2} + \tilde{u}^{2} \overline{w} + |\tilde{u}|^{2} \tilde{u}\right) g dx \\ -\frac{b}{\lambda} \int A G \nabla \phi\left(\frac{x-\alpha}{A\lambda}\right) \cdot \overline{\nabla \tilde{u}} k(x) \left(2w |\tilde{u}|^{2} + \tilde{u}^{2} \overline{w} + |\tilde{u}|^{2} \tilde{u}\right) g dx \end{vmatrix} \\ \lesssim \frac{1}{\lambda} \left(\int |\tilde{u}|^{3} |w| g dx + \int |\tilde{u}|^{4} g dx\right) + \int \left(|\tilde{u}|^{2} |w| + |\tilde{u}|^{3}\right) |\nabla \tilde{u}| g dx \\ \lesssim \frac{1}{\lambda} \left(\|\tilde{u}\|_{L^{6}}^{3} \|w\|_{L^{2}} + \|\tilde{u}\|_{L^{4}}^{4}\right) + \left\||\tilde{u}|^{2} |w| + |\tilde{u}|^{3} \|_{L^{2}} \|\nabla \tilde{u}\|_{L^{2}} \\ \lesssim \frac{1}{\lambda} \left(\|\tilde{u}\|_{L^{6}}^{3} \|w\|_{L^{2}} + \|\tilde{u}\|_{L^{2}}^{4}\right) + \left(\|\tilde{u}\|_{L^{6}}^{2} \|w\|_{L^{6}} + \|\tilde{u}\|_{L^{6}}^{3}\right) \|\nabla \tilde{u}\|_{L^{2}} \\ \lesssim \frac{1}{\lambda} \left(\|\tilde{u}\|_{L^{2}}^{2} \|\nabla \tilde{u}\|_{L^{2}}^{2} + \|\tilde{u}\|_{L^{2}}^{2} \|\nabla \tilde{u}\|_{L^{2}}^{2}\right) + \left(\|\tilde{u}\|_{L^{2}}^{2/3} \|\nabla \tilde{u}\|_{L^{2}}^{4/3} \|w\|_{L^{2}}^{1/3} \|\nabla w\|_{L^{2}}^{4/3} + \|\tilde{u}\|_{L^{2}} \|\nabla \tilde{u}\|_{L^{2}}^{2}\right) \\ \lesssim \|\tilde{u}\|_{H^{1}}^{2} = \mathcal{O}\left(\frac{1}{\lambda^{2}} \|\tilde{u}\|_{L^{2}}^{2} + \|\tilde{u}\|_{H^{1}}^{2}\right)$$

Moreover

$$\begin{split} &-\frac{b}{2\lambda^2} \int \Delta_{G,g,2} \phi \Big(\frac{x-\alpha}{A\lambda} \Big) \ \overline{\tilde{u}} \ k(x) \left(2 \left| w \right|^2 \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \ g \, dx - \frac{b}{\lambda} \int A \, G \nabla \phi \Big(\frac{x-\alpha}{A\lambda} \Big) \, \cdot \overline{\nabla \tilde{u}} \ k(x) \left(2 \left| w \right|^2 \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \ g \, dx \\ &= \frac{b}{2\lambda} \, \mathcal{R}e \int A \, G \, \nabla \phi \Big(\frac{x-\alpha}{A\lambda} \Big) \, \cdot \left[g \, \nabla \Big(\frac{k}{g} \Big) \left(2 \left| w \, \tilde{u} \right|^2 + \left(w \, \overline{\tilde{u}} \right)^2 \right) + 2 \, k \left(2 \, w \, \left| \tilde{u} \right|^2 + \overline{w} \, \tilde{u}^2 \right) \, \nabla \overline{w} + 2 \, k \left(2 \, \left| w \right|^2 \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \, \nabla \overline{\tilde{u}} \right] \ g \, dx \\ &- \frac{b}{\lambda} \int A \, G \nabla \phi \Big(\frac{x-\alpha}{A\lambda} \Big) \, \cdot \overline{\nabla \tilde{u}} \ k(x) \left(2 \, \left| w \right|^2 \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \ g \, dx \\ &= \frac{b}{2\lambda} \, \mathcal{R}e \int A \, G \, \nabla \phi \Big(\frac{x-\alpha}{A\lambda} \Big) \, \cdot \left[g \, \nabla \Big(\frac{k}{g} \Big) \left(2 \, \left| w \, \tilde{u} \right|^2 + \left(w \, \overline{\tilde{u}} \right)^2 \Big) + 2 \, k \left(2 \, w \, \left| \tilde{u} \right|^2 + \overline{w} \, \tilde{u}^2 \right) \, \nabla \overline{w} \right] \ g \, dx \end{split}$$

and one gets the estimate

$$\left|\frac{b}{2\lambda}\mathcal{R}e\int A\,G\,\nabla\phi\left(\frac{x-\alpha}{A\,\lambda}\right).g\,\nabla\left(\frac{k}{g}\right)\left(2\,|w\,\tilde{u}|^{2}+\left(w\,\overline{\tilde{u}}\right)^{2}\right)\,g\,dx\right|$$
$$\lesssim \|\tilde{u}\|_{L^{4}}^{2}\,\|w\|_{L^{4}}^{2}\lesssim\frac{1}{\lambda}\,\|\tilde{u}\|_{L^{2}}\,\|\nabla\tilde{u}\|_{L^{2}}=\mathcal{O}\left(\frac{1}{\lambda^{2}}\,\|\tilde{u}\|_{L^{2}}^{2}+\|\tilde{u}\|_{H^{1}}^{2}\right)$$

Finally, integrating the last term of (3.75) by parts, we have

$$\begin{aligned} &-\frac{b}{\lambda} \mathcal{R}e \int A \, G \nabla \phi \left(\frac{x-\alpha}{A \, \lambda}\right) \cdot \overline{\nabla \tilde{u}} \, \psi \, g \, dx \\ &= \frac{b}{\lambda} \mathcal{R}e \int \overline{\tilde{u}} \, A \, G \nabla \phi \left(\frac{x-\alpha}{A \, \lambda}\right) \cdot \nabla \psi \, g \, dx + \frac{b}{\lambda^2} \, \mathcal{R}e \int \overline{\tilde{u}} \, \psi \, \Delta_{G,g,2} \, \phi \left(\frac{x-\alpha}{A \, \lambda}\right) \, g \, dx \end{aligned}$$

so that

$$\mathcal{R}e \int i\partial_{t}\tilde{u} \left[\frac{1}{2} \operatorname{div}(G\,\nabla\tilde{\phi})\,\tilde{u} + G\,\nabla\tilde{\phi}\,\cdot\nabla\tilde{u} \right] g\,dx + \mathcal{R}e \int i\partial_{t}\tilde{u}\,\bar{\tilde{u}}\,G\nabla\tilde{\phi}\,\cdot\nabla g\,dx \\ = \frac{b}{\lambda^{2}}\,\mathcal{R}e \int \frac{1}{g^{2}}\left(g\,G\nabla\right)^{2}\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\cdot\left(\nabla\tilde{u}\,,\,\overline{\nabla\tilde{u}}\right)\,dx - \frac{1}{4A^{2}}\,\frac{b}{\lambda^{4}}\,\int\Delta_{G}\Delta_{G,g,2}\,\phi\left(\frac{x-\alpha}{A\lambda}\right)\,|\tilde{u}|^{2}\,dx \\ + \frac{b}{\lambda}\,\mathcal{R}e \int A\,G\,\nabla\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\cdot k(x)\left(2\,w\,|\tilde{u}|^{2} + \overline{w}\,\tilde{u}^{2}\right)\nabla\overline{w}\,g\,dx \\ + \frac{b}{\lambda}\,\mathcal{R}e\int\bar{u}\,A\,G\nabla\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\cdot\nabla\psi\,g\,dx + \frac{b}{2\lambda^{2}}\,\mathcal{R}e\int\Delta_{G,g,2}\,\phi\left(\frac{x-\alpha}{A\lambda}\right)\,\bar{u}\,\psi\,g\,dx \\ + \mathcal{O}\left(\frac{1}{\lambda^{2}}\,\|\tilde{u}\|_{L^{2}}^{2} + \|\tilde{u}\|_{H^{1}}^{2}\right)$$

3.4 Working the bootstrap under further assumptions

Say we are now able to put a more precise estimation on the α , β parameters in out bootstrap

$$|\alpha| \lesssim \lambda^2, \quad |\beta| \lesssim \lambda^2, \quad b \sim \lambda$$
 (3.77)

All previous computations are much more simple. We have then from (1.42)

$$B_1 = \frac{\lambda^2}{2} K_{G,g} + \mathcal{O}(\lambda^4)$$

$$B_2 = \lambda c_0(\alpha) + \lambda^3 C_3 + \mathcal{O}(\lambda^4)$$
(3.78)

the λ^2 pertubation in the *b* law is something too big to be dealt with through the method we are following here. So we need to assume a somewhat strong condition on the metrics terms *g* and *G*:

$$K_{G,g} = 0$$
 (3.79)

3.4.1 Backward propagation of smallness

Once again, we stick with the proof in [RS11]. We use the last section to get a bootstrap result. Let u be a solution of (1.1)-(1.2) on $[\tilde{t_0}, 0)$, $\tilde{t_0} < t_1 < 0$, and assume, there is a geometrical decomposition of u on $[\tilde{t_0}, t_1]$ such that

$$u(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \left(Q_{\mathcal{P}(t)} + \epsilon \right) \left(t, \frac{x - \alpha(t)}{\lambda(t)} \right) e^{i \gamma(t)}$$

where ϵ satisfies orthogonality conditions (3.7)-(3.11) and $\|\epsilon(t)\|_{H^1} + |\mathcal{P}(t)| \ll 1$. Let

$$\tilde{u}(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \, \frac{1}{\lambda(t)} \, \epsilon \Big(t, \frac{x - \alpha(t)}{\lambda(t)} \Big) \, e^{i \, \gamma(t)}$$

Assume that the energy E_0 satisfy :

$$E_0 + \frac{1}{8} C_E > 0$$

we define C_0 as

$$C_0 = \sqrt{\frac{\|y\,Q\|_{L^2}^2}{8\,E_0 + C_E}} \tag{3.80}$$

We claim, as in [RS11] the Backwards propagation estimates, with corrected mass (3.4) : Lemma 3.6. Assuming there holds for some $t_1 < 0$ close enough to 0 :

$$\left| \|u\|_{L^{2}_{g\,dx}} - \left(1 + \lambda^{2}(t_{1})\,\kappa \right) \|Q\|_{L^{2}} \right| \lesssim \lambda^{4}(t_{1}), \tag{3.81}$$

$$\|\nabla \tilde{u}(t_1)\|_{L^2}^2 + \frac{\|\tilde{u}(t_1)\|_{L^2}^2}{\lambda^2(t)} \lesssim \lambda^2(t_1),$$
(3.82)

$$\left|\frac{\beta}{\lambda}(t_1)\right| + \left|\frac{\alpha}{\lambda}(t_1)\right| \lesssim \lambda(t_1), \quad \left|\lambda(t_1) + \frac{t_1}{C_0}\right| \lesssim \lambda^3(t_1), \quad \left|\frac{b(t_1)}{\lambda(t_1)} - \frac{1}{C_0}\right| \lesssim \lambda^2(t_1)$$
(3.83)

Then, there exists a backward time t_0 depending only on C_0 such that $\forall t \in [t_0, t_1]$,

$$\|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{\|\tilde{u}(t)\|_{L^2}^2}{\lambda^2(t)} \lesssim \|\nabla \tilde{u}(t_1)\|_{L^2}^2 + \frac{\|\tilde{u}(t_1)\|_{L^2}^2}{\lambda^2(t_1)} + \lambda^6(t)$$
(3.84)

$$\left|\frac{b}{\lambda}(t) - \frac{1}{C_0}\right| \lesssim \lambda^2(t) \tag{3.85}$$

$$\left|\lambda(t) + \frac{t}{C_0}\right| \lesssim \lambda^3(t) \tag{3.86}$$

$$\left|\frac{\alpha}{\lambda}(t)\right| \lesssim \lambda(t), \quad \left|\frac{\beta}{\lambda}(t)\right| \lesssim \lambda(t)$$
(3.87)

Proof of Lemma 3.6

Since $u \in C([t_0, t_1], H^1)$ is continuous, we can find a backwards time t_0 such that $\forall t \in [t_0, t_1]$:

$$\|\tilde{u}(t)\|_{L^2} \le K \lambda^2(t), \quad \|\tilde{u}(t)\|_{H^1} \le K \lambda(t)$$
(3.88)

$$\left|\frac{\beta(t)}{\lambda(t)}\right| + \left|\frac{\alpha(t)}{\lambda(t)}\right| \le K\,\lambda(t), \quad \left|\lambda(t) + \frac{t}{C_0}\right| \le K\,\lambda^3(t), \quad \left|\frac{b(t)}{\lambda(t)} - \frac{1}{C_0}\right| \le K\,\lambda^2(t) \tag{3.89}$$

for some large enough universal constant K > 0. Then we claim that (3.84)-(3.87) hold on $[t_0, t_1]$ which improve (3.88), (3.89) on $[t_0, t_1]$ for some $t_0 = t_0(C_0)$ small enough independent of t_1 .

Step 1 Monotonicity of the norm

We then apply previous lemma to function

$$w(t,x) = \tilde{Q}(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} Q_{\mathcal{P}(t)}\left(t, \frac{x - \alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
(3.90)

note we may apply previous lemma 3.5, since (3.81) to (3.83) ensure all needed hypotheses are satisfied : on the one hand (3.82) imply $\|\nabla \tilde{u}\|_{L^2} \leq \lambda^2$ on $[t_0, t_1]$, while on the other hand (3.83) imply $b \sim \lambda$ on $[t_0, t_1]$, which were the two missing assumptions for our approximate profile as we already pointed out.

Let \mathcal{I} be given by (3.65). We will check later (3.66) implies

$$\frac{d\mathcal{I}}{dt} \ge \frac{b}{\lambda^4} \int |\tilde{u}|^2 g \, dx + \mathcal{O}\left(K^4 \, \lambda^5 + \|\tilde{u}\|_{H^1}^2\right) \tag{3.91}$$

A first rough estimation of ${\mathcal I}$ using Hölder estimates show

$$|\mathcal{I}| \lesssim \|\nabla \tilde{u}\|_{L^2}^2 + \frac{\|\tilde{u}\|_{L^2}^2}{\lambda^2}$$
(3.92)

Then, looking a bit closer we may check using both result and proof of Lemma 3.2 that

$$\begin{split} &\frac{1}{2} \int G(x) \nabla \tilde{u} . \overline{\nabla \tilde{u}} \, dx + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} g \, dx \\ &= \frac{1}{2 \, k(\alpha) \, \lambda^2} \left[\int G(\lambda y + \alpha) \, \nabla \epsilon . \overline{\nabla \epsilon} \, g \, dy + \int |\epsilon|^2 \, g \, dy \right] \\ &= \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \left(L_+^{[g]} \epsilon_1 \, , \, \epsilon_1 \right) + \left(L_-^{[g]} \epsilon_2 \, , \, \epsilon_2 \right) + \int \left(Q^2 + \lambda^2 \, V \right) |\epsilon|^2 \, g \, dy + 2 \int Q^2 \, \epsilon_1^2 \, g \, dy \right\} \\ &= \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \left(L_+ \epsilon_1 \, , \, \epsilon_1 \right) + \left(L_- \epsilon_2 \, , \, \epsilon_2 \right) + \int Q^2 \left(3 \, |\epsilon_1|^2 + |\epsilon_2|^2 \right) \, g \, dy + \mathcal{O} \left(\mathcal{P}^2 \, \|\epsilon\|_{H^1}^2 \right) \right\} \\ &\gtrsim \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \left\| \epsilon \right\|_{H^1}^2 - \left(\epsilon_1 \, , \, Q \right)^2 \right\} \end{split}$$

Furthermore with (3.26), (3.77) and (3.81)

$$\begin{aligned} \left| \mathcal{R}e\Big(\epsilon, \ \overline{Q_{\mathcal{P}}}\Big) \right| \lesssim \|\epsilon\|_{L^2}^2 + \lambda^4 + |\alpha|^2 + \left| \int |u|^2 \ g \, dx - \left(1 + \lambda^2 \, \kappa\right) \, \int Q^2 \Big| \\ \lesssim \|\epsilon\|_{L^2}^2 + K^2 \, \lambda^4(t) \end{aligned}$$

therefore

$$\left(\epsilon_1, Q\right)^2 \lesssim o\left(\|\epsilon\|_{L^2}^2\right) + K^4 \lambda^8(t)$$

from which

$$\mathcal{I} \gtrsim \frac{1}{2k(\alpha)\lambda^2} \left\{ \|\epsilon\|_{H^1}^2 - K^4 \lambda^8(t) \right\} \gtrsim \|\nabla \tilde{u}\|_{L^2}^2 + \frac{\|\tilde{u}\|_{L^2}^2}{\lambda^2} - K^4 \lambda^6(t)$$
(3.93)

Integrating (3.91) between t and t_1 gives

$$\mathcal{I}(t_1) - \mathcal{I}(t) = \int_t^{t_1} \frac{d\mathcal{I}}{dt} \ge \int_t^{t_1} \left(K^4 \,\lambda^5(\tau) + \|\tilde{u}(\tau)\|_{H^1}^2 \right) d\tau$$

thus from (3.92) and (3.93) one gets

$$\begin{split} \|\nabla \tilde{u}(t)\|_{L^{2}}^{2} &+ \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} - K^{4} \lambda^{6}(t) \\ \lesssim \mathcal{I}(t) \\ \lesssim \mathcal{I}(t_{1}) &+ \int_{t}^{t_{1}} \left(K^{4} \lambda^{5}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}}^{2}\right) d\tau \\ \lesssim \|\nabla \tilde{u}(t_{1})\|_{L^{2}}^{2} &+ \frac{\|\tilde{u}(t_{1})\|_{L^{2}}^{2}}{\lambda^{2}(t_{1})} + \int_{t}^{t_{1}} \left(K^{4} \lambda^{5}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}}^{2}\right) d\tau \end{split}$$

Altogether, we conclude for $t_0 = t_0(C_0)$ small enough

$$\|\nabla \tilde{u}(t)\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} \lesssim \|\nabla \tilde{u}(t_{1})\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t_{1})\|_{L^{2}}^{2}}{\lambda^{2}(t_{1})} + K^{4} \lambda^{6}(t) + \int_{t}^{t_{1}} \left(K^{4} \lambda^{5}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}}^{2}\right) d\tau$$

and (3.84) follows from Gronwall lemma. In particular, from (3.82)

$$\|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{\|\tilde{u}(t)\|_{L^2}^2}{\lambda^2(t)} \lesssim \lambda^2(t)$$
(3.94)

which closes the bootstrap of (3.85).

Step 2 Integration of the laws for the parameters

From both (3.24) and (3.77) we have the following estimations

$$\left|b_{s}+b^{2}-B_{1}\right|+\left|\frac{\lambda_{s}}{\lambda}+b\right|+\left|\beta_{s}+b\beta-B_{2}\right|+\left|\frac{\alpha_{s}}{\lambda}-2\beta\right|\lesssim\lambda^{4}$$
(3.95)

We now aim at integrating these laws to get (3.85)-(3.87). To do so, we first see from (3.78)

$$\left(\frac{b}{\lambda}\right)_s = \frac{b_s + b^2 - B_1}{\lambda} - \frac{b}{\lambda}\left(\frac{\lambda_s}{\lambda} + b\right) + \frac{B_1}{\lambda} \lesssim \lambda^3$$

hence integration leads to

$$\frac{b}{\lambda}(s_1) - \frac{b}{\lambda}(s) = \int_s^{s_1} \left(\frac{b}{\lambda}\right)_s d\sigma \lesssim \int_s^{s_1} \lambda^3 \, d\sigma \lesssim \lambda^2(s_1) - \lambda^2(s) \lesssim \lambda^2(s)$$

Thanks to (3.86) it proves

$$\frac{1}{C_0} - \frac{b}{\lambda}(s) \lesssim \frac{1}{C_0} - \frac{b}{\lambda}(s_1) + \lambda^2(s) \lesssim \lambda^2(s)$$
(3.96)

By the same token that led to (3.31) then to (3.38), we sum conservation of energy and of mass at time t

$$\frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{1}{4} \nabla^2 \left(k - \frac{1}{2} Tr(G)\right)(0) \cdot (\alpha, \alpha) \int Q^2 = \lambda^2 \left(E_0 + \frac{1}{8} C_E\right) + \mathcal{O}(\lambda^4)$$

where we used our assumption $|\alpha| \leq \lambda^2$, and the fact that from (3.94)

$$\|\nabla \epsilon\|_{L^2}^2 = k(\alpha) \,\lambda^2 \,\|\nabla \tilde{u}\|_{L^2}^2 = \mathcal{O}(\lambda^4), \quad \text{so that} \quad \|\epsilon\|_{H^1}^2 = \mathcal{O}(\lambda^4)$$

Thus from the choice of C_0 we see it implies thanks to (3.96)

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} \lesssim \frac{1}{C_0^2} - \frac{b^2}{\lambda^2} \lesssim \frac{1}{C_0} - \frac{b}{\lambda} \lesssim \lambda^2$$

In particular using (3.96) again, that last estimation evenually yields

$$\left|\frac{b}{\lambda} - \frac{1}{C_0}\right| \lesssim \lambda^2$$

which concludes (3.85). Then by (3.95)

$$\left|\lambda_t + \frac{b}{\lambda}\right| = \left|\frac{1}{\lambda}\left(\frac{\lambda_s}{\lambda} + b\right)\right| \lesssim \lambda^3$$

And finally,

$$\left|\lambda(t) + \frac{t}{C_0}\right| \lesssim \left|\lambda(t_1) + \frac{t_1}{C_0}\right| + \int_t^{t_1} \left|\frac{b}{\lambda} - \frac{1}{C_0}\right|(\tau) \, d\tau + \lambda^3(t) \lesssim \lambda^3(t)$$

Step 3 Coercivity of the quadratic form in the $\frac{d\,\mathcal{I}}{dt}$ expression

We now come back to the proof of (3.91). To begin with, we compute explicitly the quadratic terms in (3.66) for $w = \tilde{Q}$ given by

$$\begin{aligned} \mathcal{K}(\tilde{u}) &= -\frac{1}{\lambda^2} \mathcal{I}m\left(\int k(x) \, w^2 \, \overline{\tilde{u}^2} \, g \, dx\right) - \mathcal{R}e\left(\int k(x) \, \partial_t w \, \overline{(2 \, |\tilde{u}|^2 \, w + \tilde{u}^2 \, \overline{w})} \, g \, dx\right) \\ &+ \frac{b}{\lambda^2} \left[\int \frac{|\tilde{u}|^2}{\lambda^2} \, g \, dx + \mathcal{R}e \int \frac{1}{g^2} \left(g \, G\nabla\right)^2 \phi\left(\frac{x-\alpha}{A\lambda}\right) . \left(\nabla \tilde{u} \, , \, \overline{\nabla \tilde{u}}\right) \, dx - \frac{1}{4A^2} \, \int \Delta_G \Delta_{G,g,2} \, \phi\left(\frac{x-\alpha}{A\lambda}\right) \, \frac{|\tilde{u}|^2}{\lambda^2} \, dx\right] \\ &+ \frac{b}{\lambda} \mathcal{R}e \int A \, G \, \nabla \phi\left(\frac{x-\alpha}{A\lambda}\right) . k(x) \left(2 \, w \, |\tilde{u}|^2 + \overline{w} \, \tilde{u}^2\right) \nabla \overline{w} \, g \, dx \end{aligned}$$

$$(3.97)$$

we claim

$$\mathcal{K}(\tilde{u}) \ge \frac{c_1}{\lambda^3} \left(\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} + \int |\epsilon|^2 \right) + \mathcal{O}\left(K^4 \lambda^5\right)$$
(3.98)

for some universal constant $c_1 > 0$.

First we compute

$$\begin{aligned} \partial_{t}\tilde{Q} &= -\frac{\alpha_{t} \cdot \nabla k(\alpha)}{2\,k(\alpha)}\,\tilde{Q} - \frac{\lambda_{t}}{\lambda}\,\tilde{Q} - \frac{\lambda\alpha_{t} + \lambda_{t}\,(x-\alpha)}{k(\alpha)^{1/2}\,\lambda^{3}} \cdot \nabla Q_{\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} \\ &+ i\,\gamma_{t}\,\tilde{Q} + \frac{1}{k(\alpha)^{1/2}\,\lambda}\,\mathcal{P}_{t}\,\frac{\partial Q_{\mathcal{P}}}{\partial \mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} \\ &= \left(-\frac{(\lambda\,\alpha_{t} - 2\,\beta) \cdot \nabla k(\alpha)}{2\,k(\alpha)\,\lambda} - \frac{2\,\beta \cdot \nabla k(\alpha)}{2\,k(\alpha)\,\lambda} - \frac{\lambda\,\lambda_{t} + b}{\lambda^{2}} + \frac{b}{\lambda^{2}}\right)\tilde{Q} \\ &+ i\left(\frac{1}{\lambda^{2}} + \frac{\lambda^{2}\,\tilde{\gamma}_{t} - |\beta|^{2}}{\lambda^{2}} + \frac{|\beta|^{2}}{\lambda^{2}}\right)\tilde{Q} \\ &+ \left[-\frac{\lambda\,\lambda_{t} + b}{\lambda^{2}}\left(\frac{x-\alpha}{\lambda}\right) + \frac{b}{\lambda^{2}}\left(\frac{x-\alpha}{\lambda}\right) - \frac{\lambda\,\alpha_{t} - 2\,\beta}{\lambda^{2}} - \frac{2\,\beta}{\lambda^{2}}\right] \cdot \frac{1}{k(\alpha)^{1/2}\,\lambda}\,\nabla Q_{\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} \\ &+ \frac{1}{k(\alpha)^{1/2}\,\lambda}\,\mathcal{P}_{t}\,\frac{\partial Q_{\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} \end{aligned}$$
(3.99)

then, by the same procedure that led to (1.41), we see

$$|Q_{\mathcal{P}} - Q| \lesssim |\mathcal{P}| e^{-|y|}, \text{ therefore } \left|\frac{\partial Q_{\mathcal{P}}}{\partial \mathcal{P}}\right| \lesssim e^{-|y|}$$

moreover we check that $|\mathcal{P}_t| \lesssim 1$, so that using (3.88) and (3.95)

$$\partial_t \tilde{Q} = \left(\frac{b}{\lambda^2} + \frac{i}{\lambda^2}\right) \tilde{Q} + \frac{b}{\lambda} \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla \tilde{Q} + \mathcal{O}\left(\frac{K}{\lambda} e^{-\frac{|x-\alpha(t)|}{\lambda(t)}}\right)$$

now putting this together and using the exponential decay of $Q_{\mathcal{P}}$

$$-\mathcal{R}e\left(\int k(x)\tilde{Q}_{t}\left(\overline{2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}}\right)g\,dx\right)$$

$$=\frac{1}{\lambda^{2}}\mathcal{I}m\left(\int k(x)\tilde{Q}\left(\overline{2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}}\right)g\,dx\right)-\frac{b}{\lambda^{2}}\mathcal{R}e\left(\int k(x)\left(2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}\right)\overline{\tilde{Q}}g\,dx\right)$$

$$-\frac{b}{\lambda}\mathcal{R}e\left(\int\left(\frac{x-\alpha}{\lambda}\right)k(x)\left(2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}\right).\overline{\nabla\tilde{Q}}g\,dx\right)+\frac{1}{\lambda^{3}}\mathcal{O}\left(K\,\lambda\,\|\epsilon\|_{L^{2}}^{2}\right)$$
(3.100)

and

$$\begin{split} &-\mathcal{R}e\left(\int k(x)\,\tilde{Q}_t\,\overline{(2\,|\tilde{u}|^2\,\tilde{Q}+\tilde{u}^2\,\overline{\tilde{Q}})}\,g\,dx\right) - \frac{1}{\lambda^2}\,\mathcal{I}m\left(\int k(x)\,w^2\,\overline{\tilde{u}^2}\,g\,dx\right) \\ &= -\frac{1}{k(\alpha)}\,\frac{b}{\lambda^4}\,\int \frac{k(\lambda y+\alpha)}{k(\alpha)}\left[\left(|Q_{\mathcal{P}}|^2+2\,\Sigma^2\right)\epsilon_1^2 + 4\,\Sigma\,\Theta\,\epsilon_1\,\epsilon_2 + \left(|Q_{\mathcal{P}}|^2+2\,\Theta^2\right)\epsilon_2^2\right]\,g\,dy \\ &- \frac{1}{k(\alpha)}\,\frac{b}{\lambda^4}\,\mathcal{R}e\,\int \frac{k(\lambda y+\alpha)}{k(\alpha)}\,(2\,|\epsilon|^2\,Q_{\mathcal{P}}+\epsilon^2\,\overline{Q_{\mathcal{P}}})\,y\,.\,\overline{\nabla Q_{\mathcal{P}}}\,g\,dy \end{split}$$

Injecting this into (3.97) yields

$$\begin{split} \mathcal{K}(\tilde{u}) &= \frac{1}{k(\alpha)} \frac{b}{\lambda^4} \left[\int |\epsilon|^2 \ g \ dy + \mathcal{R}e \int \frac{1}{g^2} \left(g \ G \nabla \right)^2 \phi \left(\frac{y}{A} \right) . \left(\nabla \epsilon \,, \, \overline{\nabla \epsilon} \right) \ dy - \frac{1}{4A^2} \int \Delta_G \Delta_{G,g,2} \phi \left(\frac{y}{A} \right) \ |\epsilon|^2 \ dy \\ &- \int \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[\left(|Q_{\mathcal{P}}|^2 + 2\Sigma^2 \right) \epsilon_1^2 + 4\Sigma \Theta \epsilon_1 \epsilon_2 + \left(|Q_{\mathcal{P}}|^2 + 2\Theta^2 \right) \epsilon_2^2 \right] g \ dy \\ &+ \mathcal{R}e \ \int \left(A \ G(\lambda y + \alpha) \ \nabla \phi \left(\frac{y}{A} \right) - y \right) \frac{k(\lambda y + \alpha)}{k(\alpha)} \left(2 \ |\epsilon|^2 \ Q_{\mathcal{P}} + \epsilon^2 \ \overline{Q_{\mathcal{P}}} \right) . \, \overline{\nabla Q_{\mathcal{P}}} \ g \ dy \right] \\ &+ \frac{1}{\lambda^3} \ \mathcal{O} \left(K \lambda \ \|\epsilon\|_{L^2}^2 \right) \end{split}$$

Notice that expanding the metric terms g and G yields

$$\begin{split} \mathcal{K}(\tilde{u}) &= \frac{1}{k(\alpha)} \frac{b}{\lambda^4} \left[\int \left[1 - \frac{1}{4A^2} \Delta^2 \phi \left(\frac{y}{A} \right) \right] |\epsilon|^2 \, dy + \mathcal{R}e \int \nabla^2 \phi \left(\frac{y}{A} \right) . \left(\nabla \epsilon \,, \, \overline{\nabla \epsilon} \right) \, dy \\ &- \int \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[\left(|Q_{\mathcal{P}}|^2 + 2\,\Sigma^2 \right) \epsilon_1^2 + 4\,\Sigma\,\Theta\,\epsilon_1\,\epsilon_2 + \left(|Q_{\mathcal{P}}|^2 + 2\,\Theta^2 \right) \epsilon_2^2 \right] \, dy \\ &+ \mathcal{R}e \, \int \left(A\,\nabla \phi \left(\frac{y}{A} \right) - y \right) \frac{k(\lambda y + \alpha)}{k(\alpha)} \left(2\,|\epsilon|^2\,Q_{\mathcal{P}} + \epsilon^2\,\overline{Q_{\mathcal{P}}} \right) . \, \overline{\nabla Q_{\mathcal{P}}} \, dy \right] \\ &+ \frac{1}{\lambda^3} \, \mathcal{O}\Big(K\,\lambda \, \|\epsilon\|_{L^2}^2 + \lambda^2 \, \|\epsilon\|_{H^1}^2 \Big) \end{split}$$

and

$$\begin{split} \mathcal{K}(\tilde{u}) &= \frac{1}{k(\alpha)} \frac{b}{\lambda^4} \left[\int \left[1 - \frac{1}{4A^2} \Delta^2 \phi\left(\frac{y}{A}\right) \right] |\epsilon|^2 \, dy + \mathcal{R}e \int \nabla^2 \phi\left(\frac{y}{A}\right) . \left(\nabla \epsilon \,, \, \overline{\nabla \epsilon}\right) \, dy \\ &- \int \left[\left(|Q_{\mathcal{P}}|^2 + 2\,\Sigma^2 \right) \epsilon_1^2 + 4\,\Sigma\,\Theta\,\epsilon_1\,\epsilon_2 + \left(|Q_{\mathcal{P}}|^2 + 2\,\Theta^2 \right) \epsilon_2^2 \right] \, dy \right] \\ &+ \frac{1}{\lambda^3} \, \mathcal{O}\left(K\,\lambda \, \|\epsilon\|_{L^2}^2 + \int_{|y|>A} |\epsilon|^2 \, dy + \lambda^2 \, \|\epsilon\|_{H^1}^2 \right) \end{split}$$

From proximity of $Q_{\mathcal{P}}$ to Q, from our choice of orthogonality conditions, from (3.32) and from (3.88), the above quadratic form is for A large enough a small deformation in A of energy which satisfy

$$\mathcal{K}(\tilde{u}) \gtrsim \frac{1}{\lambda^3} \left[\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} + \int |\epsilon|^2 - \left(\epsilon_1, Q\right)^2 \right] \gtrsim \frac{1}{\lambda^3} \left[\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} + \int |\epsilon|^2 \right] + \mathcal{O}\left(K^4 \lambda^5\right) \quad (3.101)$$

where we have used (3.34) to get the last line. (3.98) is proved.

Step 4 Control of the remainder terms in the $\frac{d\mathcal{I}}{dt}$ expression

We have left to deal with the ψ terms in the $\frac{d\mathcal{I}}{dt}$ expression. According to the definition of ψ we see with (1.15), (2.2) and (3.20)

$$\begin{split} \psi &= -i\partial_{t}\tilde{u} - \mathcal{L}\tilde{u} - k(x)\left(|\tilde{u}|^{2}\tilde{u} - |w|^{2}w\right) \\ &= \frac{1}{k(\alpha)^{1/2}\lambda^{3}}\left[i\partial_{s}\epsilon + \mathcal{L}_{\lambda\alpha}\epsilon - \epsilon + \frac{k(\lambda y + \alpha)}{k(\alpha)}\left(\widetilde{M}(\epsilon) + R(\epsilon)\right) \right. \\ &- i\frac{\lambda_{s}}{\lambda}\Lambda\epsilon - i\frac{\alpha_{s}}{\lambda}\cdot\nabla\epsilon - \widetilde{\gamma}_{s}\epsilon\right]\left(\frac{x - \alpha}{\lambda}\right)e^{i\gamma(t)} \\ &= \frac{-1}{k(\alpha)^{1/2}\lambda^{3}}\left[i(b_{s} + b^{2} - B_{1})\partial_{b}Q_{\mathcal{P}} + i\left(\frac{\lambda_{s}}{\lambda} + b\right)\lambda\partial_{\lambda}Q_{\mathcal{P}} + i\left(\frac{\alpha_{s}}{\lambda} - 2\beta\right)\lambda\partial_{\alpha}Q_{\mathcal{P}} \right. \\ &+ i(\beta_{s} + b\beta - B_{2})\partial_{\beta}Q_{\mathcal{P}} - i\left(\frac{\lambda_{s}}{\lambda} + b\right)\Lambda Q_{\mathcal{P}} - i\frac{\lambda}{2}\left(\frac{\alpha_{s}}{\lambda} - 2\beta\right)\cdot\left(\nabla Q_{\mathcal{P}} + \frac{\nabla k(\alpha)}{k(\alpha)}Q_{\mathcal{P}}\right) \\ &- \left(\widetilde{\gamma}_{s} - |\beta|^{2}\right)Q_{\mathcal{P}} + \psi_{\mathcal{P}}\right]\left(\frac{x - \alpha}{\lambda}\right)e^{i\gamma(t)} \end{split}$$
(3.102)

where $\psi_{\mathcal{P}}$ is the remainder term in the construction of $Q_{\mathcal{P}}$:

$$\psi_{\mathcal{P}} = \mathcal{O}\left(\lambda^5 \, e^{-\delta \, |y|}\right)$$

First see that (3.95) along with (3.88) and (3.89) gives a rough bound on ψ , that is for i = 0, 1, 2:

$$|\nabla^{i}\psi| \lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(|Mod(t)| + |K_{b}| + |K_{\tilde{\gamma}}| + \lambda^{5} \right) \lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(K \lambda^{2} \|\epsilon\|_{L^{2}} + K^{2} \lambda^{4} \right)$$
(3.103) and thus

$$\|\nabla^i \psi\|_{L^2} \lesssim \frac{1}{\lambda^{2+i}} \left(K \lambda^2 \|\epsilon\|_{L^2} + K^2 \lambda^4 \right)$$

In particular we find the bound

$$\lambda^2 \, \|\psi\|_{L^2}^2 \lesssim \lambda^2 \, \|\epsilon\|_{L^2}^2 + K^4 \, \lambda^6$$

then for any $\nu > 0$

$$\begin{split} & \left| \mathcal{I}m \int \left[i \frac{b}{\lambda} A \, G \nabla \phi \left(\frac{x - \alpha}{A \, \lambda} \right) . \nabla \psi + i \frac{b}{2\lambda^2} \, \Delta_{G,g,2} \phi \left(\frac{x - \alpha}{A \, \lambda} \right) \psi \right] \overline{\tilde{u}} \, g \, dx \\ & \lesssim \| \nabla \psi \|_{L^2} \| \tilde{u} \|_{L^2} + \frac{\| \psi \|_{L^2}}{\lambda} \| \tilde{u} \|_{L^2} \\ & \lesssim \frac{1}{\lambda^3} \left[\lambda^2 \, \| \epsilon \|_{L^2} + K^2 \, \lambda^4 \right] \| \epsilon \|_{L^2} = \frac{\| \epsilon \|_{L^2}^2}{\lambda} + \left(K^4 \, \lambda^{5 - \nu} \right)^{1/2} \left(\frac{\| \epsilon \|_{L^2}^2}{\lambda^{3 - \nu}} \right)^{1/2} \\ & \lesssim \lambda^{\nu} \, \frac{\| \epsilon \|_{L^2}^2}{\lambda^3} + K^4 \, \lambda^{5 - \nu} = o \left(\frac{\| \epsilon \|_{L^2}^2}{\lambda^3} \right) + K^4 \, \lambda^{5 - \nu} \end{split}$$

Yet bound (3.103) is not precise enough to take care of remainder terms in (3.65). Let us remind that the construction of $Q_{\mathcal{P}}$ induces

$$Q_{\mathcal{P}} = P_{\mathcal{P}} e^{-i b \frac{|y|^2}{4} + i \beta \cdot y} + \mathcal{O}(|\mathcal{P}|^2), \quad P_{\mathcal{P}} = Q + \mathcal{O}(|\mathcal{P}|^2)$$

so that using (3.95)

$$i(b_s + b^2 - B_1) \partial_b Q_{\mathcal{P}} = (b_s + b^2 - B_1) \frac{|y|^2}{4} Q + \mathcal{O}(|\mathcal{P}| Mod(t))$$
$$i(\beta_s + b\beta - B_2) \partial_\beta Q_{\mathcal{P}} = -(\beta_s + b\beta - B_2) \cdot y Q + \mathcal{O}(|\mathcal{P}| Mod(t))$$

now we have $\psi = \psi_1 + \psi_2$ where

$$\psi_{1} = \frac{1}{k(\alpha)^{1/2} \lambda^{3}} \left[\left(b_{s} + b^{2} - B_{1} \right) \frac{|y|^{2}}{4} Q - \left(\beta_{s} + b \beta - B_{2} \right) \cdot y Q - i \left(\frac{\lambda_{s}}{\lambda} + b \right) \Lambda Q - i \frac{\lambda}{2} \left(\frac{\alpha_{s}}{\lambda} - 2 \beta \right) \cdot \nabla Q - \left(\widetilde{\gamma}_{s} - |\beta|^{2} \right) Q \right] \left(\frac{x - \alpha}{\lambda} \right) e^{i \gamma(t)}$$
(3.104)

and for i=0,1,2 :

$$\begin{aligned} |\nabla^{i}\psi_{2}| &\lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(|\mathcal{P}| |Mod(t)| + K^{2} \lambda^{5} \right) \\ &\lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(\lambda^{3} \|\epsilon\|_{L^{2}} + K^{2} \lambda^{5} \right) \end{aligned}$$
(3.105)

This implies the remainder term ψ_2 in (3.65) may be estimated as the previous one

$$\begin{aligned} \left| \mathcal{I}m \int \left[\mathcal{L}\psi_2 - \frac{\psi_2}{\lambda^2} + k(x) \left(2 \, |w|^2 \, \psi_2 - w^2 \, \overline{\psi_2} \right) \right] \overline{\widetilde{u}} \right| \\ &\lesssim \left[\| \nabla^2 \psi_2 \|_{L^2} + \frac{\| \psi_2 \|_{L^2}}{\lambda^2} + \| \psi_2 \|_{L^{\infty}} \, \|w\|_{L^4}^4 \right] \| \widetilde{u} \|_{L^2} \\ &\lesssim o \left(\frac{\| \epsilon \|_{L^2}^2}{\lambda^3} \right) + K^4 \, \lambda^{5-\nu} \end{aligned}$$
(3.106)

while the one in ψ_1 is estimated using (3.13) and the orthogonality conditions (3.7) and (3.9) on ϵ which allow to gain a factor $\mathcal{O}(\mathcal{P})$

$$\begin{aligned} \left| \mathcal{I}m \int \left[\mathcal{L}\psi_1 - \frac{\psi_1}{\lambda^2} + k(x) \left(2 \left| w \right|^2 \psi_1 - w^2 \overline{\psi_1} \right) \right] \overline{\hat{u}} \right| \\ &\lesssim \frac{|Mod(t)| + |\alpha|^2}{\lambda^4} \left[\left(\epsilon_2 \,, \, L_- \left(\left| y \right|^2 Q \right) \right) + \left(\epsilon_2 \,, \, L_- \left(y \, Q \right) \right) + \left(\epsilon_1 \,, \, L_+ \left(\nabla Q \right) \right) \right. \\ &+ \left(\epsilon_2 \,, \, L_- \left(Q \right) \right) + \mathcal{O} \left(\mathcal{P} \left\| \epsilon \right\|_{L^2} \right) \right] + \frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} + b \right| \left| \left(\epsilon_1 \,, \, L_+ \left(\Lambda Q \right) \right) \right| \\ &\lesssim \frac{\lambda \left\| \epsilon \right\|_{L^2} + K^2 \lambda^4}{\lambda^4} \left| \mathcal{P} \right| \left\| \epsilon \right\|_{L^2} + \frac{\lambda^2 \left\| \epsilon \right\|_{L^2} + K^2 \lambda^4}{\lambda^4} \left(\lambda \left\| \epsilon \right\|_{L^2} + K^2 \lambda^4 \right) \\ &\lesssim o \left(\frac{\left\| \epsilon \right\|_{L^2}^2}{\lambda^3} \right) + K^4 \lambda^{5-\nu} \end{aligned}$$

$$(3.107)$$

where we have used the fact

$$\mathcal{L}_{\lambda\alpha}f = \Delta f + \mathcal{O}(\lambda^2)\left(|\nabla f| + |\nabla^2 f|\right)$$

and (3.13) together with the conservation of mass through estimate (3.24) to get the bound

$$\left| \begin{pmatrix} \epsilon_1, \Sigma \end{pmatrix} + \begin{pmatrix} \epsilon_2, \Theta \end{pmatrix} + \frac{1}{2} \|\epsilon\|_{L^2}^2 \right| \lesssim \lambda^4$$

hence $\left| \begin{pmatrix} \epsilon_1, Q \end{pmatrix} \right| \lesssim \lambda \|\epsilon\|_{L^2} + K^2 \lambda^4$, so that
 $\frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} + b \right| \left| \begin{pmatrix} \epsilon_1, L_+(\Lambda Q) \end{pmatrix} \right| \lesssim \frac{|Mod(t)| + |\alpha|^2}{\lambda^4} \left| \begin{pmatrix} \epsilon_1, Q \end{pmatrix} \right|$

Now injecting (3.105), (3.106), (3.107) and (3.98) into (3.66) we see

$$\begin{split} \frac{d\mathcal{I}}{dt} \gtrsim &\frac{1}{\lambda^3} \left(\int |\nabla \epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, g \, dy + \int |\epsilon|^2 \, g dy \right) + K^4 \, \lambda^{5-\nu} \\ \gtrsim &\frac{b}{\lambda^4} \, \int |\tilde{u}|^2 \, g dx + \int |\nabla \tilde{u}|^2 \, \frac{1}{\lambda} \, e^{-\frac{|x-\alpha|}{\sqrt{A} \, \lambda}} \, g \, dx + K^4 \, \lambda^{5-\nu} \\ \gtrsim &\frac{b}{\lambda^4} \, \int |\tilde{u}|^2 \, g dx + \mathcal{O}\bigg(\|\tilde{u}\|_{H^1}^2 + K^4 \, \lambda^{5-\nu} \bigg) \end{split}$$

and since it is true for any $\nu > 0$, it remains true when $\nu = 0$, proving (3.91) holds. This concludes the proof of the Lemma.

3.4.2 Existence of critical mass blow up solutions

As a first consequence of previous lemma, we may integrate the flow backward from the singularity to prove there are critical mass blow up solutions. The following proposition ends the existence part.

Proposition 3.7. (Existence of critical mass blow up solutions)

Let

$$\gamma_0 \in \mathbb{R}, \quad E_0 + \frac{1}{8} C_E > 0,$$
(3.108)

and C_0 given by (3.80), then there exists $t_0 < 0$ and a solution $u_c \in C([t_0, 0), H^{3/2})$ to (1.1)-(1.2) which blows up at T = 0 with

$$E(u_c) = E_0 \quad and \quad \|u_c\|_{L^2_{a,d_x}} = \|Q\|_{L^2}.$$
 (3.109)

Moreover, the solution admits on $[t_0, 0)$ a geometrical decomposition :

$$u_c(t,x) = \frac{1}{k(\alpha_c(t))^{1/2}} \frac{1}{\lambda_c(t)} \left(Q_{\mathcal{P}_c(t)} + \epsilon_c(t) \right) \left(t, \frac{x - \alpha_c(t)}{\lambda_c(t)} \right) e^{i \gamma_c(t)}$$

$$= \widetilde{Q}_c + \widetilde{u}_c$$
(3.110)

where ϵ_c satisfies the orthogonality conditions (3.7)-(3.11), and there holds the bounds :

$$\begin{aligned} \|\widetilde{u}_{c}\|_{L^{2}} &\lesssim \lambda_{c}^{4}, \quad \|\widetilde{u}_{c}\|_{H^{1}} \lesssim \lambda_{c}^{3}, \quad \|\widetilde{u}_{c}\|_{H^{3/2}} \lesssim \lambda_{c}^{3/2}, \\ \lambda_{c} + \frac{t}{C_{0}} &= \mathcal{O}(\lambda_{c}^{3}), \quad \frac{b_{c}}{\lambda_{c}} - \frac{1}{C_{0}} = \mathcal{O}(\lambda_{c}^{2}), \\ |\alpha_{c}| + |\beta_{c}| \lesssim \lambda_{c}^{2}, \quad \gamma_{c} = -\frac{C_{0}^{2}}{t} + \gamma_{0} + \mathcal{O}(\lambda_{c}) \end{aligned}$$
(3.111)

Proof of Proposition 3.7

Step 1 Backwards uniform bounds.

Let a sequence $t_n \to 0$ and u_n be the solution to (1.1) with initial data at $t = t_n$ given by :

$$u_n(t_n, x) = \frac{1}{\lambda_n} Q_{\mathcal{P}_n}\left(\frac{x}{\lambda_n}\right) e^{i\gamma_n(t_n)}$$
(3.112)

with $\mathcal{P}_n = (b_n(t_n), \lambda_n(t_n), \beta_n(t_n), \alpha_n(t_n))$ and :

$$b_n(t_n) = -\frac{t_n}{C_0^2}, \quad \lambda_n(t_n) = -\frac{t_n}{C_0}, \quad \alpha_n(t_n) = \beta_n(t_n) = 0, \quad \gamma_n(t_n) = \gamma_0 - \frac{C_0^2}{t_n}.$$
(3.113)

Recall the conservation of mass identity we have computed in (2.7) which here becomes

$$\|u_n(t_n)\|_{L^2_{g\,dx}}^2 = \left(1 + \kappa \,\lambda_n^2(t_n)\right) \,\int Q^2 + \mathcal{O}(t_n^4) \tag{3.114}$$

Furthermore, we have $\tilde{u}_n(t_n) = 0$ by construction. Hence u_n satisfies at $t_1 = t_n$ the assumptions of the backward propagation of smallness lemma, and thus we can find a time t_0 independent of n such that $\forall t \in [t_0, t_n), u_n$ admits a geometrical decomposition

$$u_n(t,x) = \frac{1}{k(\alpha_n(t))^{1/2}} \frac{1}{\lambda_n(t)} Q_{\mathcal{P}_n(t)} \Big(t, \frac{x - \alpha_n(t)}{\lambda_n(t)} \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big(t, \frac{x - \alpha_n(t)}{\lambda_n(t)} \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big(t, \frac{x - \alpha_n(t)}{\lambda_n(t)} \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big(t, \frac{x - \alpha_n(t)}{\lambda_n(t)} \Big) e^{i \gamma_n(t)} + \tilde{u}_n(t) \Big) e^{i \gamma_n(t)} + \tilde{u}_n$$

with uniform bounds in n :

$$\|\nabla \tilde{u}_n(t)\|_{L^2}^2 + \frac{\|\tilde{u}_n(t)\|_{L^2}^2}{\lambda_n^2(t)} \lesssim \lambda_n^6(t)$$
(3.115)

$$\left|\frac{b_n}{\lambda_n}(t) - \frac{1}{C_0}\right| \lesssim \lambda_n^2(t), \quad \left|\lambda_n(t) + \frac{t}{C_0}\right| \lesssim \lambda_n^3(t),$$

$$\left|\alpha_n(t)\right| + \left|\beta_n(t)\right| \lesssim \lambda_n^2(t)$$
(3.116)

From Strichartz bounds, this implies the uniform $H^{3/2}$ bound :

$$\|\tilde{u}_n(t)\|_{L^{\infty}\left([t,t_n],H^{3/2}\right)} \lesssim \lambda_n^{3/2}(t)$$
 (3.117)

that we will prove in step 2.

Now the H^1 compactness of $u_n(t_0)$ is a consequence of a standard localization procedure. Indeed let a cut off function $\chi(x) = 0$ for $|x| \le 1$ and $\chi(x) = 1$ for $|x| \ge 2$, and $\chi_R(x) = \chi(\frac{x}{R})$, then since

$$\partial_t u_n = i \left[\mathcal{L} u_n + k(x) \, |u_n|^2 \, u_n \right]$$

we have

$$\left| \frac{d}{dt} \int \chi_{R} |u_{n}|^{2} g \, dx \right| = 2 \left| \mathcal{I}m \int G(x) \, \nabla u_{n} \cdot \nabla \chi_{R} \, \overline{u_{n}} \, dx \right| \lesssim \frac{1}{R}$$

and

$$\left| \frac{d}{dt} \int \chi_R \left(\frac{1}{2} G(x) \nabla u_n \cdot \nabla \overline{u_n} - \frac{1}{4} k(x) |u_n|^4 g(x) - \frac{1}{2} V(x) |u_n|^2 g(x) \right) dx \right|$$
$$= \left| \mathcal{I}m \int \nabla \chi_R \cdot G(x) \nabla \overline{u_n} \overline{\left(\mathcal{L}u_n + k(x) |u_n|^2 u_n \right)} dx \right| \lesssim \frac{1}{R}$$

where we have used (3.115), (3.116) and (3.117). Integrating this backwards from t_1 to t_0 we have :

$$\int \chi_R |u_n(t_0)|^2 \lesssim \frac{1}{R}, \quad \int \chi_R |\nabla u_n(t_0)|^2 \lesssim \frac{1}{R}$$
(3.118)

Let's say a bit more on how we got the last estimation. We have used the Gagliardo-Nirenberg inequality as follows

$$\begin{split} \int \chi_{\scriptscriptstyle R} \, |\nabla u_n|^2 \, dx &\lesssim \left| \int \chi_{\scriptscriptstyle R} \left[\frac{1}{2} \, G(x) \, \nabla u_n \, . \, \nabla \overline{u}_n - g(x) \left(\frac{1}{4} \, k(x) \, |u_n|^4 - \frac{1}{2} \, V(x) \, |u_n|^2 \right) \right] \, dx \right| \\ &+ \int \left| \chi_{\scriptscriptstyle R}^{1/4} \, u_n \right|^4 \, dx + \int \chi_{\scriptscriptstyle R} \, |u_n|^2 \, dx \\ &\lesssim \frac{1}{R} + \int \chi_{\scriptscriptstyle R}^{1/2} \, |u_n|^2 \, dx \, \int \chi_{\scriptscriptstyle R}^{1/2} \, |\nabla u_n|^2 \, dx + \frac{1}{R} \\ &\lesssim \frac{1}{R} + \frac{1}{R} \, \int \chi_{\scriptscriptstyle R}^{1/2} \, |\nabla u_n|^2 \, dx \\ &\lesssim \frac{1}{R} + \frac{1}{R} \, \int \chi_{\scriptscriptstyle R} \, |\nabla u_n|^2 \, dx + \frac{1}{R} \, \int \left(\chi_{\scriptscriptstyle R}^{1/2} - \chi_{\scriptscriptstyle R} \right) \, |\nabla u_n|^2 \, dx \end{split}$$

then

$$\int \left(\chi_{\scriptscriptstyle R}^{1/2}-\chi_{\scriptscriptstyle R}\right) |\nabla u_n|^2 \, dx \lesssim \int_{R \le |x| \le 2\,R} |\nabla u_n|^2 \, dx \lesssim 1$$

so that

$$\int \chi_{\scriptscriptstyle R} \, |\nabla u_n|^2 \, dx \lesssim \frac{1}{R}$$

This last result (3.118) gives us a control at infinity of functions $u_n(t_0)$ that will help us to get H^1 compactness of the $(u_n(t_0))$ sequence. So we use Lemma 8.4 of Appendix B to get for any $\delta > 0$

$$H^{1}_{\delta}(gdx) = \left\{ u \in H^{1}(\mathbb{R}^{2}, gdx), \quad \left(\langle x \rangle^{\delta} + \langle D_{x} \rangle^{\delta} \right) \ \left(|u| + |\nabla u| \right) \in L^{2}(gdx) \right\} \underset{\text{compact}}{\hookrightarrow} H^{1}(\mathbb{R}^{2}, gdx)$$

Our next move is to claim (3.118) ensures $(u_n(t_0)) \subset H^1_{\delta}$. Once this is done, we know there is a subsequence of $u_n(t_0)$ which converges to some $u_c(t_0) \in H^1(\mathbb{R}^2)$.

First, since we have $u_n(t_0) \in H^{3/2}(\mathbb{R}^2)$. Now we prove $u_n(t_0) \in H^1_{\delta}(\mathbb{R}^2)$ whenever $0 \le \delta \le \frac{1}{2}$. Using (3.118) we say for every $j \in \mathbb{N}$

$$\int_{2^{j} \le |x| \le 2^{j+1}} \frac{\langle x \rangle^{2\delta}}{\langle 2^{j+1} \rangle^{2\delta}} |u_{n}(t_{0})|^{2} g \, dx \quad \lesssim \int \chi_{2^{j-1}} |u_{n}(t_{0})|^{2} g \, dx \lesssim \frac{1}{2^{j-1}}$$

and since $\delta < \frac{1}{2}$, summing this over j finally yields

$$\int \langle x \rangle^{2\delta} |u_n(t_0)|^2 g \, dx = \int_{0 \le |x| \le 1} \langle x \rangle^{2\delta} |u_n(t_0)|^2 g \, dx + \sum_{j=0}^{+\infty} \int_{2^j \le |x| \le 2^{j+1}} \langle x \rangle^{2\delta} |u_n(t_0)|^2 g \, dx$$
$$\lesssim \sum_{j=0}^{+\infty} \frac{1}{2^{(1-2\delta)j}} < +\infty$$

The same preocess applied to $\nabla u_n(t_0)$ instead of $u_n(t_0)$ yields $\int \langle x \rangle^{\delta} |\nabla u_n(t_0)|^2 < +\infty$ \Box

Eventually, we have proved

$$u_n(t_0) \longrightarrow u_c(t_0)$$
 in $H^1(\mathbb{R}^2)$, as $n \to +\infty$

Let u_c be the solution to (1.1)-(1.2) with initial data $u_c(t_0)$. From the H^1 continuity of the flow of our Schrödinger operator, we have

$$\forall t \in [t_0, 0), \quad u_n(t) \longrightarrow u_c(t) \text{ in } H^1(\mathbb{R}^2), \quad \text{as } n \to +\infty$$

Of course u_c admits a geometrical decomposition like in (3.110) and we claim as in [RS11] it implies

$$\forall t \in [t_0, 0), \quad \mathcal{P}_n(t) \longrightarrow \mathcal{P}_c(t), \quad \text{as } n \to +\infty$$

By passing to the limit in (3.115), (3.116) we obtain the H^1 bound along with the estimates on the parameters in (3.111). This implies in particular that u_c blows up at t = 0. Similarly the conservation of the L^2 norm ensures

$$||u_c||_{L^2_{g\,dx}} = \lim_{n \to +\infty} ||u_n(t_n)||_{L^2_{g\,dx}} = ||Q||_{L^2}$$

Recall that from the very method we used to approximate our solution through the geometric decomposition

$$Q_{\mathcal{P}} = P_{\mathcal{P}} e^{-i b \frac{|y|^2}{4} + i \beta \cdot y}, \quad P_{\mathcal{P}} = Q + \mathcal{O}(\lambda^2 e^{-C_2 |y|})$$

which along with (3.115) and the computation we already led in (2.8) yields

$$\begin{split} E(u_c(t)) &= \frac{1}{2} \int G(x) \, \nabla u_c \, . \, \nabla \overline{u}_c \, dx - \frac{1}{4} \int k(x) \, |u_c|^4 \, g \, dx - \frac{1}{2} \int V(x) \, |u_c|^2 \, g \, dx \\ &= \frac{1}{k(\alpha_c) \, \lambda_c^2} \left[\frac{1}{2} \int G(\lambda_c y + \alpha_c) \, \nabla Q_{\mathcal{P}_c} \, . \, \nabla \overline{Q_{\mathcal{P}_c}} \, g \, dy - \frac{1}{4} \int \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \, |Q_{\mathcal{P}_c}|^4 \, g \, dy \\ &- \frac{\lambda_c^2}{2} \int V(\lambda_c y + \alpha_c) \, |Q_{\mathcal{P}_c}|^2 \, g \, dy \right] + \mathcal{O}(\lambda_c) \\ &= \frac{1}{k(\alpha_c) \, \lambda_c^2} \left[\frac{b_c^2}{8} \int |y|^2 Q^2 - \frac{\lambda_c^2}{8} \, C_E \right] + \mathcal{O}(\lambda_c) \\ &= \frac{1}{8} \left(\left(\frac{b_c}{\lambda_c} \right)^2 - \frac{1}{C_0^2} \right) \|y \, Q\|_{L^2}^2 + E_0 + \mathcal{O}(\lambda_c) \xrightarrow{t \to 0} E_0 \end{split}$$

Now from the conservation of energy, we may conclude $E(u_c) = E_0$.

There is only left to prove the estimate about the phase parameter. This may be done using equation (1.15) along with (3.115) and (3.116) that lead to the rough bound

$$\left| (\widetilde{\gamma}_n)_s \right| \lesssim \lambda_n^2$$

so that

$$\left|\frac{d}{dt}\left(\gamma_n + \frac{C_0^2}{t}\right)\right| = \left|\frac{(\gamma_n)_s}{\lambda_n^2} - \frac{C_0^2}{t^2}\right| = \left|\frac{(\tilde{\gamma_n})_s}{\lambda_n^2} + \left(\frac{1}{\lambda_n^2} - \frac{C_0^2}{t^2}\right)\right| \lesssim 1$$

from which we may conclude after integrating

$$\gamma_n(t) + \frac{C_0^2}{t} = \gamma_0 + \mathcal{O}(t)$$

and taking the limit $n \to +\infty$

$$\gamma_c(t) + \frac{C_0^2}{t} = \gamma_0 + \mathcal{O}(t)$$

Step 2 $H^{3/2}$ bound

It remains to prove the bound (3.117). In particular, when done this implies the $H^{3/2}$ bound in (3.111) by taking the weak limit of $H^{3/2}$.

 \tilde{u}_n satisfies

$$i \partial_t \tilde{u}_n + \mathcal{L} \tilde{u}_n = -\psi_n - k(x) \,\tilde{u}_n \, |\tilde{u}_n|^2 - F_n$$

with

$$i \partial_t \tilde{Q}_n + \mathcal{L} \tilde{Q}_n + k(x) \, \tilde{Q}_n \, |\tilde{Q}_n|^2 = \psi_n$$

$$F_n = k(x) \, (\tilde{Q}_n + \tilde{u}_n) \, |\tilde{Q}_n + \tilde{u}_n|^2 - k(x) \, \tilde{Q}_n \, |\tilde{Q}_n|^2 - k(x) \tilde{u}_n \, |\tilde{u}_n|^2$$
(3.119)

Given $\tilde{u}_n(t_n) = 0$, using the Duhamel formula

$$\tilde{u}_n(t) = i \int_{t_n}^t e^{i(t-s)\mathcal{L}} \left(\psi_n(s) + k(x) \,\tilde{u}_n(s) \,| \tilde{u}_n(s) |^2 + F_n(s) \right) ds$$

then the Strichartz estimates coupled with the smoothing effect of the linear Schrödinger flow leaves us with three terms to control

$$\|\nabla^{3/2}\tilde{u}_n\|_{L^{\infty}_{[t,t_n]}L^2} \lesssim \|\nabla^{3/2}\psi_n\|_{L^{4/3}_{[t,t_n]}L^{4/3}} + \|(1+|x|^2)F_n\|_{L^2_{[t,t_n]}H^1} + \|\nabla^{3/2}(\tilde{u}_n|\tilde{u}_n|^2)\|_{L^{4/3}_{[t,t_n]}L^{4/3}}$$
(3.120)

See Appendix B 8.3.2 for more details about that last statement.

First, we deal with the error ψ_n which is to be estimated thanks to (3.102)

$$\|\nabla^{3/2}\psi_n\|_{L^{4/3}} \lesssim \frac{1}{\lambda_n^3} \left(|Mod_n(t)| + |\alpha_n|^2 + \lambda_n^5\right) \lesssim \lambda_n$$

where of course $Mod_n(t)$ corresponds to the modulation equation of $\mathcal{P}_n(t)$ which is estimated as in (3.24), and where we use (3.115) and (3.116) to get $|Mod_n(t)| \leq \lambda_n^4$. Hence

$$\|\nabla^{3/2}\psi_n\|_{L^{4/3}_{[t,t_n]}L^{4/3}} \lesssim \lambda_n^{7/4}.$$
(3.121)

The F_n term is local in y with linear and quadratic terms in \tilde{u}_n . Expanding (3.119) we see

$$F_n = \mathcal{O}\left[\left(\frac{|\tilde{u}_n|^2}{\lambda_n} + \frac{|\tilde{u}_n|}{\lambda_n^2}\right)e^{-c\frac{|x-\alpha_n|}{\lambda_n}}\right], \quad \nabla F_n = \mathcal{O}\left[\left(\frac{|\tilde{u}_n|^2}{\lambda_n^2} + \frac{|\tilde{u}_n\nabla\tilde{u}_n|}{\lambda_n} + \frac{|\tilde{u}_n|}{\lambda_n^3} + \frac{|\nabla\tilde{u}_n|}{\lambda_n^2}\right)e^{-c\frac{|x-\alpha_n|}{\lambda_n}}\right]$$

so that

$$\|(1+|x|^2)F_n\|_{L^2} + \|(1+|x|^2)\nabla F_n\|_{L^2} \lesssim \frac{1}{\lambda_n^3} \|\tilde{u}_n\|_{L^2} + \frac{1}{\lambda_n^2} \|\nabla \tilde{u}_n\|_{L^2} + \frac{1}{\lambda_n} \|\tilde{u}_n\|_{L^4} \|\nabla \tilde{u}_n\|_{L^4}$$

Again by Sobolev embedding 8.2 along with a Gagliardo-Nirenberg estimate

Sobolev
$$H^{1/2}(\mathbb{R}^2) \subseteq L^4(\mathbb{R}^2)$$

GN $\|\tilde{u}_n\|_{L^4} \lesssim \|\tilde{u}_n\|_{L^2}^{1/2} \|\nabla \tilde{u}_n\|_{L^2}^{1/2}$

and (3.115) yields

$$\begin{aligned} \|(1+|x|^2)F_n\|_{H^1} &\lesssim \frac{1}{\lambda_n^3} \lambda_n^4 + \frac{1}{\lambda_n^2} \lambda_n^3 + \frac{1}{\lambda_n} \left(\lambda_n^4\right)^{1/2} \left(\lambda_n^3\right)^{1/2} \|\nabla \tilde{u}_n\|_{H^{1/2}} \\ &\lesssim \lambda_n + \lambda_n^{5/2} \|\nabla^{3/2} \tilde{u}_n\|_{L^2} \end{aligned}$$

Thus

$$\|(1+|x|^2)F_n\|_{L^2_{[t,t_n]}H^1} \lesssim \lambda_n^{3/2} + \lambda_n^{5/2} \|\nabla^{3/2}\tilde{u}_n\|_{L^\infty_{[t,t_n]}L^2}$$
(3.122)

There only remains to deal with the nonlinear term. A little caution here, remind the symbol $\nabla^{3/2}$ we are working with is a generalized non-integer derivative that stands for

$$\nabla^{3/2} u(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \left(i \, \xi \right)^{3/2} \widehat{u}(\xi) \, d\xi$$

where $\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} \, u(x) \, dx$ is the Fourier transform, (3.123)
and $\overline{\mathcal{F}}u(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \, \widehat{u}(\xi) \, d\xi$ is its inverse

Then, standard non linear estimates show

$$\|\nabla^{3/2} \left(\tilde{u}_n \,| \tilde{u}_n |^2 \right) \|_{L^{4/3}} \lesssim \|\nabla^{3/2} \tilde{u}_n \|_{L^2} \,\| \tilde{u}_n \|_{H^1}^2 \lesssim \|\nabla^{3/2} \tilde{u}_n \|_{L^2} \,\lambda_n^6 \tag{3.124}$$

where we have used (3.115), see Appendix B 8.2.2 for details. Eventually (3.117) follows from (3.121), (3.122) and (3.124). This ends the proof of our Proposition, existence of solutions for equation (1.1) is now done provided we justify the boot strap is actually valid, that all terms involved are in control as we assumed they should be when moving toward blow up time. It is what we do in the next section.

4 Critical mass blow up solutions have conformal speed

All that is proved in the previous sections strongly depends on hypothesis (3.77), which was

$$|\alpha| \lesssim \lambda^2, \quad |\beta| \lesssim \lambda^2, \quad b \sim \lambda, \quad \|\epsilon\|_{H^1} \lesssim \lambda^2$$

$$(4.1)$$

Recall this particularly simplifies our parameters laws since they should satisfy (3.78). We now want to justify the use we have made of these assumptions in the first sections and to prove they are are in fact a rigidity in the regime that governs the (\mathcal{P}, ϵ) parameters of the decomposition (3.1). As a result this is also a first step to get uniqueness of the critical mass blow up solution u_c we have just constructed since it will determine the size of dispersion between two given solutions. To do so, we keep following the procedure of [RS11] with a variational study of the decomposition.

We let k satisfy assumption (H1).

In this section we start proceeding in a perturbation kind of way. In the next section we consider a solution for equation (1.1) which would be given by profile Q and a remaining term ϵ . We then prove the Lemma 4.1 that proves such a solution tends, as $t \to 0$ - that is as we move forward to blow up time - to satisfy all assumptions we made to build approximate solution $Q_{\mathcal{P}}$ in previous sections. Therefore, for some time close enough to blow up time, our solution will enter in a regime where its approximation may be sharpened by considering $Q_{\mathcal{P}}$ the approximate profile instead of Ground State Q. So this will prove, the geometric decomposition (4.10) still provides a good description, with all due estimations on parameters and remaining term ϵ , even when moving to blow up time.

Then, a localized virial type of estimation, along with a refined dispersion result on the tail of the (4.10) decompisation - the one induced by remaining term ϵ - will get us to a final argument in which we manage to ensure the control of the geometric parameters for (4.10), and so closing the loop on our boot strap argument.

4.1 Variational estimates and convergence of the concentration point.

The existence of a geometrical decomposition for a mass critical blowing up solution, and its consequences, among which is the convergence of the concentration point, are a well known result based on a variational analysis. We begin with adapting this one to our metric situation.

Lemma 4.1. (Variational control of minimal mass blow up solutions).

Let u(t) be a critical mass solution to (1.1)-(1.2) which blows up at T = 0. Then for t < 0 close enough to 0, u(t) admits a geometrical decomposition

$$u(t,x) = \frac{1}{\lambda(t)} \left(Q + \epsilon\right) \left(t, \frac{x - \alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)},\tag{4.2}$$

for some C^1 parameters $(\lambda(t), \alpha(t), \gamma(t)) \in \mathbb{R}^*_+ \times \mathbb{R}^2 \times \mathbb{R}$ with

1. Uniform bound on the decomposition :

$$|G(\alpha) - Id|^{1/2} + |g(\alpha) - 1|^{1/2} + |1 - k(\alpha)|^{1/2} + \|\epsilon(t)\|_{H^1(\mathbb{R}^2)} \lesssim \lambda(t) \longrightarrow 0 \quad as \quad t \to 0$$
(4.3)

2. Convergence of the concentration point :

$$\alpha(t) \longrightarrow \alpha^* \quad with \quad k(\alpha^*) = 1, \quad g(\alpha^*) = 1$$

$$(4.4)$$

3. Lower bound on the blow up rate :

$$\lambda(t) \le C(u_0) \left| t \right| \tag{4.5}$$

Proof of Lemma 4.1

Step 1 Let

$$v_0(t,x) = \lambda_0(t) u(t, \lambda_0(t) x) \quad \text{with} \quad \lambda_0(t) = \frac{\|\nabla Q\|_{L^2}}{\left(\int G(x) \nabla u(t) \cdot \nabla \overline{u(t)} \, dx\right)^{1/2}}$$

then

$$\int |v_0(t)|^2 g(\lambda_0 x) dx = \int |u(t)|^2 g(z) dz = ||Q||_{L^2}^2, \quad \text{and}$$
$$\int G(\lambda_0 x) \nabla v_0(t, x) \cdot \nabla \overline{v_0}(t, x) dx = \lambda_0^2 \int G(z) \nabla u(t, z) \cdot \nabla \overline{u}(t, z) dz = \int |\nabla Q|^2 dz$$
using assumption (H1), $h < 1$

so that using assumption $(H1), k \leq 1$

$$\begin{aligned} &\frac{1}{2} \int G(\lambda_0 x) \,\nabla v_0(t, x) \,\cdot \nabla \overline{v_0}(t, x) \,\, dx - \frac{1}{4} \int k(\lambda_0 x) \,|v_0(t, x)|^4 \,\,g(\lambda_0 x) \,dx - \frac{1}{2} \,\int V(\lambda_0 x) \,|v_0(t, x)|^2 \,\,g(\lambda_0 x) \,dx \\ &\geq \frac{1}{2} \,\int |\nabla v_0(t, x)|^2 \,\, dx - \frac{1}{4} \,\int |v_0(t, x)|^4 \,\,g(\lambda_0 x) \,dx - \frac{1}{2} \,\int V(\lambda_0 x) \,|v_0(t, x)|^2 \,\,g(\lambda_0 x) \,dx \\ &= \frac{1}{2} \,\int |\nabla v_0|^2 \,\, dx - \frac{1}{4} \,\int |v_0|^4 \,\, dx - \frac{V(0)}{2} \,\int |u|^2 \,\,g(z) \,\, dz + \mathcal{O}(\lambda_0) \end{aligned}$$

and the conservation of energy

$$\begin{split} &\frac{1}{2} \int G(\lambda_0 \, x) \, \nabla v_0(t, x) \, . \, \nabla \overline{v_0}(t, x) \, \, dx - \frac{1}{4} \int k(\lambda_0 \, x) \, |v_0(t, x)|^4 \, g(\lambda_0 \, x) \, dx - \frac{1}{2} \int V(\lambda_0 \, x) \, |v_0(t, x)|^2 \, g(\lambda_0 \, x) \, dx \\ &= \frac{\lambda_0^2}{2} \int G(z) \, \nabla u(t, z) \, . \, \nabla \overline{u}(t, z) \, \, dz - \frac{\lambda_0^2}{4} \int k(z) \, |u(t, x)|^4 \, g(z) \, dz - \frac{1}{2} \int V(z) \, |u(t, z)|^2 \, g(z) \, dz \\ &= \lambda_0^2 \, E_0 - \frac{1}{2} \left(1 - \lambda_0^2\right) \, \int V(z) \, |u|^2 \, g(z) \, dz \end{split}$$

Eventually using $(H2), V \ge V(0)$, one gets

$$\frac{1}{2} \int |\nabla v_0(t,x)|^2 \, dx - \frac{1}{4} \int |v_0(t,y)|^4 \, dx \le \frac{1}{2} \int \left(V(0) - V(z) \right) |u|^2 \, g(z) \, dz + \mathcal{O}\left(\lambda_0\right) \\ \le 0, \quad \text{as} \quad t \to 0$$

From a standard concentration compactness argument along with the variational characterisation of the Ground State Q, this implies we can find $(x_0(t), \gamma_0(t)) \in \mathbb{R}^2 \times \mathbb{R}$ such that

 $v_0(t, .+ x_0(t)) e^{i \gamma_0(t)} \longrightarrow Q$ in H^1 as $t \to 0$

In another words, u(t) admits near blow up time a decomposition

$$u(t,x) = \frac{1}{\lambda(t)} \left(Q + \epsilon \right) \Bigl(t, \frac{x - \alpha(t)}{\lambda(t)} \Bigr) \, e^{i \, \gamma(t)},$$

with $\lambda(t) = \lambda_0(t) \longrightarrow 0$, as $t \to 0$ from blow-up assumption, and since the decomposition is set so that

$$\left\| u(t,x) - \frac{1}{\lambda(t)} Q\left(\frac{x - \alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)} \right\|_{H^1(g\,dx)} \longrightarrow 0, \quad \text{as } t \to 0$$

one easily deduce from approximate values of mass and energy the smallness conditions

$$\|\epsilon\|_{H^1} + \left|g(\alpha) - 1\right| + \left|\frac{1}{2}Tr(G(\alpha)) - k(\alpha)\right| \longrightarrow 0, \quad \text{as } t \to 0$$

$$(4.6)$$

Using the implicit function theorem, the uniqueness of the decomposition (4.2) can be ensured through a suitable choice of orthogonality conditions. We then set the orthogonality conditions on ϵ to be

$$\left(\epsilon_1, |y|^2 Q\right) = 0, \quad \left(\epsilon_1, y Q\right) = 0, \quad \left(\epsilon_2, \varrho\right) = 0$$

where we wrote $\epsilon = \epsilon_1 + i \epsilon_2$.

Let $v = Q + \epsilon$, it almost satisfies (1.15)

$$i\partial_s v + \mathcal{L}_{\lambda\alpha} v - v + k(\lambda y + \alpha) v |v|^2 = i\frac{\lambda_s}{\lambda} \Lambda v + i\frac{\alpha_s}{\lambda} \cdot \nabla v + \widetilde{\gamma}_s v$$

where $\tilde{\gamma}_s = \gamma_s - 1$. Note that we are currently considering Q an approximate solution of (1.28) or (1.29). It is a very rough approximation, but since we do not know yet that $\alpha(t)$ converges to α^* , there is no point using our approximation Q_P for it takes sense only for $\alpha(t)$ close enough to α^* .

Now the implicit function theorem leads to the C^1 regularity of our parameters λ and α . Then expanding g, G, k around α and using the Ground State equation

$$i\left(\frac{\lambda_s}{\lambda}\Lambda Q + \frac{\alpha_s}{\lambda} \cdot \nabla Q\right) + \widetilde{\gamma}_s Q = \frac{1}{g(\alpha)} \left(G_{ij}(\alpha) - Id\right) \partial_{ij}^2 Q + \left(\frac{1}{g(\alpha)} - 1\right) \Delta Q + \left(k(\alpha) - 1\right) Q^3 + \frac{1}{g(\alpha)} G_{ij}(\alpha) \partial_{ij}^2 \epsilon + \mathcal{O}\left(\lambda + |\epsilon| + |\nabla\epsilon|\right)$$

Taking the scalar product of that last equation, first with ΛQ , then with ∇Q yields

$$-i\frac{\lambda_s}{\lambda}\frac{\|\Lambda Q\|_{L^2}^2}{\|Q\|_{L^2}^2} = \frac{1}{g(\alpha)}\left(\frac{1}{2}Tr(G(\alpha)) - k(\alpha)\right) + \left(\frac{1}{g(\alpha)} - 1\right) + \mathcal{O}(\lambda + \|\epsilon\|_{H^1}), \quad i\frac{\alpha_s}{\lambda} = \mathcal{O}(\lambda + \|\epsilon\|_{H^1})$$

so that

$$\left|\frac{\lambda_s}{\lambda}\right| + \left|\frac{\alpha_s}{\lambda}\right| \lesssim \left|\frac{1}{2}Tr(G(\alpha)) - k(\alpha)\right| + \left|g(\alpha) - 1\right| + \lambda(t) + \|\epsilon\|_{H^1}$$

$$\tag{4.7}$$

Step 2 Expansion of the conservation laws

To get a more precise estimation of our approximation remainings, we expand the conservation laws in the ϵ variables. From the critical mass assumption, we know we have

$$\int Q^2 \, dy = \int |u(t,x)|^2 \, g(x) \, dx = \int |v(s,y)|^2 \, g(\lambda y + \alpha) \, dy$$

and thus

$$2\int \epsilon_1 Q \,dy + \int |\epsilon|^2 \,dy = \left(1 - g(\alpha)\right)\int Q^2 \,dy + \mathcal{O}\left(\left(\lambda + |g(\alpha) - 1|\right)\|\epsilon\|_{L^2}\right) \tag{4.8}$$

Using the boundedness of the k's, g's and G's derivatives around α we have

$$g(\lambda y + \alpha) = 1 + (g(\alpha) - 1) + (g(\lambda y + \alpha) - g(\alpha)) = 1 + (g(\alpha) - 1) + \mathcal{O}(\lambda)$$

$$k(\lambda y + \alpha) = 1 - (1 - k(\alpha)) + \mathcal{O}(\lambda), \quad G(\lambda y + \alpha) = I + (G(\alpha) - I) + \mathcal{O}(\lambda)$$

As already said, and very similarly to the calculation which led to (6.66), the conservation of energy simply becomes by rescalling

$$\begin{split} \lambda^{2} E_{0} &= \frac{1}{2} \int G(\lambda \, y + \alpha) \, \nabla v \, \cdot \nabla \overline{v} \, dy - \frac{1}{4} \int k(\lambda \, y + \alpha) \, |v|^{4} \, g(\lambda y + \alpha) + \mathcal{O}(\lambda^{2}) \\ &= E^{0}(Q) + \frac{1}{2} \left(G_{ij}(\alpha) - I_{ij} \right) \int \partial_{i} Q \partial_{j} Q \, dy + \frac{1}{4} \left(1 - k(\alpha) \, g(\alpha) \right) \int Q^{4} \, dy + \frac{1}{2} \int |\nabla \epsilon|^{2} \, dy + \int \nabla \epsilon_{1} \, \cdot \nabla Q \, dy \\ &- \int \epsilon_{1} \, Q^{3} \, dy - \int Q^{2} \left(\epsilon_{1}^{2} + \frac{1}{2} \, |\epsilon|^{2} \right) \, dy + \mathcal{O}\left(\lambda^{2} + \|\epsilon\|_{H^{1}}^{3} + \left(\lambda + |G_{ij}(\alpha) - I_{ij}| + |1 - k(\alpha) \, g(\alpha)| \right) \|\epsilon\|_{H^{1}} \right) \\ &= \left(\frac{1}{2} \, Tr(G(\alpha)) - 1 \right) \int \frac{Q^{2}}{2} \, dy + \frac{1}{2} \left(1 - k(\alpha) \right) \int Q^{2} \, dy + \frac{1}{2} \int |\nabla \epsilon|^{2} \, dy - \int \epsilon_{1} \, Q \, dy \\ &- \frac{1}{2} \int Q^{2} \left(3 \, \epsilon_{1}^{2} + \epsilon_{2}^{2} \right) \, dy + \mathcal{O}\left(\lambda^{2} + \|\epsilon\|_{H^{1}}^{3} + \left(\lambda + |G_{ij}(\alpha) - I_{ij}| + |1 - k(\alpha) \, g(\alpha)| \right) \|\epsilon\|_{H^{1}} \right) \end{split}$$

which together with (4.8) leads to

$$\lambda^{2} E_{0} = \left(\frac{1}{2} Tr(G(\alpha)) + g(\alpha) (1 - k(\alpha)) - 1\right) \int \frac{Q^{2}}{2} dy + \frac{1}{2} \left[\left(L_{+}\epsilon_{1}, \epsilon_{1}\right) + \left(L_{-}\epsilon_{2}, \epsilon_{2}\right) \right] \\ + \mathcal{O}\left(\lambda^{2} + \|\epsilon\|_{H^{1}}^{3} + \left(\lambda + |G_{ij}(\alpha) - I_{ij}| + |g(\alpha) - 1| + |1 - k(\alpha)|\right) \|\epsilon\|_{H^{1}} \right)$$

Now using (4.6) and (3.32) of Lemma 3.2, one easily get the following estimate

$$\left|\frac{1}{2}Tr(G(\alpha)) - k(\alpha)\right| + |g(\alpha) - 1| + \|\epsilon\|_{H^1}^2 \lesssim \lambda^2$$

Step 3 Convergence of the concentration point and upper bound on the blow up rate. There is only (4.4) left to prove. It essentially follows from (4.7) since

$$\begin{split} |\alpha_t| &= \frac{1}{\lambda} \left| \frac{\alpha_s}{\lambda} \right| \lesssim \frac{\|\epsilon\|_{H^1} + \left| \frac{1}{2} Tr(G(\alpha)) - k(\alpha) \right| + |g(\alpha) - 1| + \lambda}{\lambda} \\ \text{so that} \quad \int_{-1}^0 |\alpha_t| < +\infty \end{split}$$

This implies $\alpha(t) \to \alpha^*$ as $t \to 0$ and $k(\alpha^*) = 1$, $g(\alpha^*) = 1$ from (4.3). Similarly since the blow up assumption at t = 0 is $\lambda(0) = 0$

$$\begin{aligned} |\lambda_t| &= \frac{1}{\lambda} \left| \frac{\lambda_s}{\lambda} \right| \lesssim \frac{\|\epsilon\|_{H^1} + \left| \frac{1}{2} Tr(G(\alpha)) - k(\alpha) \right| + |g(\alpha) - 1| + \lambda}{\lambda} \\ \text{so that} \quad \lambda(t) \lesssim \int_t^0 |\lambda_t| \lesssim |t| \end{aligned}$$

This ends the proof of our lemma.

4.2 Strict lower bound on the energy.

From previous section we have underlined fact that center of mass $\alpha(t)$ must stabilize as $t \to 0$ around some α^* where both k and g reach their maxima. Without loss of gnerality we may naturally assume

$$\begin{array}{ll}
\alpha^* = 0, & k(0) = 1, & \nabla k(0) = 0, & \nabla^2 k(0) < \frac{1}{2} \nabla^2 Tr(G)(0) \\
& g(0) = 1, & \nabla g(0) = 0, \\
& G(0) = I, & \nabla G(0) = 0, & K_{G,g} = 0
\end{array}$$
(4.9)

Then, for $T_0 > 0$ close enough to blow up time, $|\mathcal{P}(t)|$, $t \in [T_0, 0)$, is small enough for u to be in the regime described in section 3.1. This induces we can sharpen our previous approximation profile by replacing Q by $Q_{\mathcal{P}}$ which is a small deformation of Q. Hence for $t \in [T_0, 0)$ let

$$u(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \left(Q_{\mathcal{P}(t)} + \epsilon \right) \left(t, \frac{x - \alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)}$$

$$(4.10)$$

where ϵ satisfies orthogonalized conditions (3.7)-(3.11). We also introduced the global rescaled time

$$s(t) = \int_{T_0}^t \frac{d\tau}{\lambda^2(\tau)} \to +\infty, \quad \text{as} \quad t \to 0$$

Thus, for every $s \in [s_0, +\infty)$ applying Lemma 3.1 we get

$$b^{2} + |\beta|^{2} + |\alpha|^{2} + \|\epsilon\|_{H^{1}}^{2} \lesssim \lambda^{2} \left(E_{0} + \frac{1}{8} C_{E} \right) + \mathcal{O} \left(\mathcal{P}^{4} + \mathcal{P}^{2} |\alpha| + |\alpha|^{3} \right)$$
(4.11)

where C_E is given by (2.9), and

$$\left| Mod(t) \right| \lesssim \mathcal{P}^4 + \mathcal{P} \left| \alpha \right|^2 + \mathcal{P}^2 \left\| \epsilon \right\|_{H^1} + \left\| \epsilon \right\|_{L^2}^2$$

$$\tag{4.12}$$

We claim these estimations bring the following strict lower bound on the energy

Lemma 4.2. There holds the strict lower bound on the energy :

$$E_0 > -\frac{1}{8} C_E \tag{4.13}$$

Proof of Lemma 4.2

By contradiction, if $E_1 = E_0 + \frac{1}{8}C_E \leq 0$, from (4.1) and (4.11) we have

$$\begin{split} b^2 + |\beta|^2 + |\alpha|^2 + \|\epsilon\|_{H^1}^2 \lesssim \lambda^2 E_1 + \lambda^4, \\ \text{hence} \quad b^2 + |\beta|^2 + |\alpha|^2 + \|\epsilon\|_{H^1}^2 \lesssim \lambda^4 \end{split}$$

Then, thanks to (4.12) this yields

$$\frac{\lambda_s}{\lambda} = \mathcal{O}(\lambda^2)$$

so that

$$\frac{\lambda_t}{\lambda} = \mathcal{O}(1), \text{ and hence } \ln(\lambda(t)) = \mathcal{O}(1), \text{ as } t \to 0$$

which contradicts the fact $\lambda(t) \to 0$ as $t \to 0$, that is u blows up at t = 0. It ends proof of Lemma 4.2.

4.3 The localized virial identity

In order to improve our estimate (3.21) as announced in (4.1) we need to get a more precise estimate on ϵ locally on the singularity that will help us treat the remaining terms of Lemma 3.5.

Keep in mind, since no a priori upper bound holds on blow up speed, we may not rule out the regime for which $b \ll \lambda$ that would make the $\frac{b}{\lambda}$ terms a negligible part. This makes the (3.66) estimate useless for the gain might as well degenerate.

It is a Morawetz type computation in the spirit of the local virial estimate that will here help us.

Lemma 4.3. (Local Virial Control)

Let ϕ be given by (3.63). There exists universal constants $c, \underline{c} > 0$, and a large enough constant A > 0 such that for t close enough to 0, we have :

$$\left\{ -\left(\frac{b}{\lambda}\right) \frac{\|y\,Q\|_{L^{2}}^{2}}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A\,G(\lambda y + \alpha)\,\nabla\phi\left(\frac{y}{A}\right) \cdot \nabla\epsilon\,\overline{\epsilon}\,g\,dy \right\}_{s} \\
\geq \frac{c}{\lambda} \left\{ |\alpha|^{2} + \int |\nabla\epsilon|^{2}\,e^{-\frac{|y|}{\sqrt{A}}} + \int |\epsilon|^{2} \right\} + \mathcal{O}\left(A\,\|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3} + |\beta|^{2}\right) \tag{4.14}$$

Note from (3.21), the terms involved in the right hand side are uniformly bounded in time

$$\left| - \left(\frac{b}{\lambda}\right) \frac{\|y \, Q\|_{L^2}^2}{4} + \frac{1}{2\lambda} \, \mathcal{I}m \int A \, G(\lambda y + \alpha) \nabla \phi \left(\frac{y}{A}\right) . \, \nabla \epsilon \, \overline{\epsilon} \, g \, dy \right| \lesssim \frac{|b|}{\lambda} + \frac{\|\epsilon\|_{H^1}^2}{\lambda} \lesssim 1$$

From (3.21) again, and the finite time blow up assumption

$$\int_{s_0}^{+\infty} \left(\|\epsilon\|_{H^1}^2 + \mathcal{P}^3 + |\beta|^2 \right) ds \lesssim \int_{s_0}^{+\infty} \lambda^2(s) \, ds = \int_{t_0}^{0} dt \lesssim 1$$

Hence, integrating (4.14) between s_0 and $+\infty$

$$\int_{s_0}^{+\infty} \frac{1}{\lambda} \left\{ |\alpha|^2 + \int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} + \int |\epsilon|^2 \right\} ds \lesssim 1$$
(4.15)

On the other hand, thanks to (4.5)

$$\int_{s_0}^{+\infty} \lambda(s) \, ds = \int_{t_0}^{0} \frac{d\tau}{\lambda(\tau)} = +\infty$$

Proof of Lemma 4.3

The proof relies on an algebraic computation and the specific structure of the quadratic terms in ϵ appearing in (4.14).

Step 1 Computation of $\left(\frac{b}{\lambda}\right)_s$

We have

$$\left(\frac{b}{\lambda}\right)_{s} = \frac{b_{s} + b^{2} - B_{1}}{\lambda} - \frac{b}{\lambda}\left(\frac{\lambda_{s}}{\lambda} + b\right) + \frac{B_{1}}{\lambda} = \frac{1}{\lambda}\left(b_{s} + b^{2} - B_{1}\right) + \mathcal{O}(\mathcal{P}^{3})$$

where we have made use of both our hypotheses $b \sim \lambda$ and $B_1 = \mathcal{O}(\lambda^4)$ in (3.77).

We recall that (3.50) along with (3.26) now yields

$$(b_s + b^2 - B_1) \left\{ \int \frac{|y|^2}{4} Q^2 + \mathcal{O}(\mathcal{P}^2) \right\} - 2 \mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right)$$
$$= \left(R_1(\epsilon), \ \Lambda \Sigma\right) + \left(R_2(\epsilon), \ \Lambda \Theta\right) + \mathcal{O}\left(\mathcal{P}^4 + \mathcal{P}^2 \|\epsilon\|_{L^2}\right)$$

and

$$2\mathcal{R}e\Big(\epsilon,\overline{Q_{\mathcal{P}}}\Big) + \int |\epsilon|^2 - \frac{1}{2}\nabla^2\Big(k-g\Big)(0).(\alpha,\alpha)\int Q^2 = \mathcal{O}\Big(\mathcal{P}^4 + |\alpha|^3 + \mathcal{P}^2 |\alpha| + \mathcal{P}^2 \|\epsilon\|_{L^2}\Big)$$

so that

$$\begin{pmatrix} \frac{b}{\lambda} \\ \frac{b}{\lambda} \\ \frac{|y|}{2} Q \|_{L^{2}}^{2} = -\frac{1}{\lambda} \left[\int |\epsilon|^{2} - \frac{\|Q\|_{L^{2}}^{2}}{2} \nabla^{2} (k - g)(0) \cdot (\alpha, \alpha) - \left(R_{1}(\epsilon) , \Lambda \Sigma \right) - \left(R_{2}(\epsilon) , \Lambda \Theta \right) \right]$$

$$+ \mathcal{O} \left(\mathcal{P}^{3} + |\alpha|^{2} + |\beta|^{2} + \mathcal{P}^{2} \left(b - \frac{\lambda}{C_{0}} \right) + \mathcal{P} \|\epsilon\|_{L^{2}} + \|\epsilon\|_{H^{1}}^{2} \right)$$

$$= -\frac{1}{\lambda} \left[\int |\epsilon|^{2} - \left(3\Sigma \epsilon_{1}^{2} + 2\Theta \epsilon_{1} \epsilon_{2} + \Sigma \epsilon_{2}^{2} , \Lambda \Sigma \right) - \left(3\theta \epsilon_{2}^{2} + 2\Sigma \epsilon_{1} \epsilon_{2} + \Theta \epsilon_{1}^{2} , \Lambda \Theta \right) \right]$$

$$+ \frac{\|Q\|_{L^{2}}^{2}}{2\lambda} \nabla^{2} (k - g)(0) \cdot (\alpha, \alpha) + \mathcal{O} \left(\mathcal{P}^{3} + |\alpha|^{2} + |\beta|^{2} + \mathcal{P}^{2} \left(b - \frac{\lambda}{C_{0}} \right) + \mathcal{P} \|\epsilon\|_{L^{2}} + \|\epsilon\|_{H^{1}}^{2} \right)$$

$$(4.16)$$

Step 2 Computation of the localized virial identity.

Our goal is now to compute, for A > 0 large enough, the following identity

$$\begin{split} &\left(\frac{1}{\lambda}\mathcal{I}m\int A\,G(\lambda y+\alpha)\nabla\phi\left(\frac{y}{A}\right).\nabla\epsilon\,\bar{\epsilon}\,g\,dy\right)_{s} \\ &= -\frac{\lambda_{s}}{\lambda^{2}}\mathcal{I}m\int A\,G(\lambda y+\alpha)\nabla\phi\left(\frac{y}{A}\right).\nabla\epsilon\,\bar{\epsilon}\,g\,dy \\ &+ \frac{1}{\lambda}\mathcal{I}m\int \left[\Delta_{G,g,1}\phi\left(\frac{y}{A}\right)\epsilon+2\,A\,G(\lambda y+\alpha)\nabla\phi\left(\frac{y}{A}\right).\nabla\epsilon\right]\overline{\partial_{s}\epsilon}\,g\,dy \\ &+ \mathcal{I}m\int A\left\{\left(\frac{\lambda_{s}}{\lambda}+b\right)y+\left(\frac{\alpha_{s}}{\lambda}-2\,\beta\right)+\left(2\,\beta-b\,y\right)\right\}.\nabla g\,G\,\nabla\phi\left(\frac{y}{A}\right).\nabla\epsilon\,\bar{\epsilon}\,dy \\ &+ \mathcal{I}m\int A\,\nabla^{2}G.\left(\left(\frac{\lambda_{s}}{\lambda}+b\right)y+\left(\frac{\alpha_{s}}{\lambda}-2\,\beta\right)+\left(2\,\beta-b\,y\right),\nabla\phi\left(\frac{y}{A}\right)\right).\nabla\epsilon\,\bar{\epsilon}\,g\,dy \end{split}$$
(4.17)

From (3.21) and (3.24) we first see

$$\frac{\lambda_s}{\lambda^2} = \frac{1}{\lambda} \left(\frac{\lambda_s}{\lambda} + b \right) - \frac{b}{\lambda} = \mathcal{O}(1)$$

which implies

$$-\frac{\lambda_s}{\lambda^2} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi\left(\frac{y}{A}\right) \cdot \nabla \epsilon \,\overline{\epsilon} \, g \, dy = \mathcal{O}\left(A \, \|\epsilon\|_{H^1}^2\right)$$

so that (4.17) may be rewritten as

$$\begin{split} &\left(\frac{1}{\lambda}\mathcal{I}m\int A\,G(\lambda y+\alpha)\nabla\phi\left(\frac{y}{A}\right).\nabla\epsilon\,\overline{\epsilon}\,\,g\,dy\right)_{s} \\ &=\frac{1}{\lambda}\mathcal{I}m\int \left[\Delta_{{}^{G,g,1}}\phi\left(\frac{y}{A}\right)\left(\partial_{s}\epsilon_{1}\,\epsilon_{2}-\epsilon_{1}\,\partial_{s}\epsilon_{2}\right)+2\,A\,G(\lambda y+\alpha)\nabla\phi\left(\frac{y}{A}\right).\left(\partial_{s}\epsilon_{1}\,\nabla\epsilon_{2}-\partial_{s}\epsilon_{2}\,\nabla\epsilon_{1}\right)\right]\,g\,dy \\ &+\mathcal{O}\Big(A\,\left(1+\mathcal{P}\right)\|\epsilon\|_{H^{1}}^{2}\Big) \end{split}$$

and since

$$\Delta_{G,g,1} \phi\left(\frac{y}{A}\right) = \Delta\phi\left(\frac{y}{A}\right) + \mathcal{O}\left(\mathcal{P}^2\left(|\Delta\phi| + A |\nabla\phi|\right)\right)$$
$$A G(\lambda y + \alpha) \nabla\phi\left(\frac{y}{A}\right) = A \nabla\phi\left(\frac{y}{A}\right) + \mathcal{O}\left(A \mathcal{P}^2 |\nabla\phi|\right)$$

we may simply write (4.17) as

$$\begin{pmatrix} \frac{1}{\lambda} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi(\frac{y}{A}) \cdot \nabla \epsilon \,\overline{\epsilon} \, g \, dy \end{pmatrix}_{s}$$

= $\frac{1}{\lambda} \mathcal{I}m \int \Delta \phi(\frac{y}{A}) \left(\partial_{s} \epsilon_{1} \epsilon_{2} - \epsilon_{1} \, \partial_{s} \epsilon_{2} \right) + \frac{2}{\lambda} \mathcal{I}m \int A \nabla \phi(\frac{y}{A}) \cdot \left(\partial_{s} \epsilon_{1} \, \nabla \epsilon_{2} - \partial_{s} \epsilon_{2} \, \nabla \epsilon_{1} \right) + \mathcal{O}(A \, \|\epsilon\|_{H^{1}}^{2})$

Reintroducing (3.39)-(3.40) along with the estimates of Lemma 3.1 we get

$$\begin{split} &\left(\frac{1}{\lambda}\mathcal{I}m\int A\,G(\lambda y+\alpha)\nabla\phi\left(\frac{y}{A}\right).\nabla\epsilon\,\overline{\epsilon}\,g\,dy\right)_{s} \\ &=\frac{1}{\lambda}\mathcal{I}m\int \Delta\phi\left(\frac{y}{A}\right)\left(M_{2}(\epsilon)\,\epsilon_{2}+M_{1}(\epsilon)\,\epsilon_{1}\right)+\frac{2}{\lambda}\mathcal{I}m\int A\,\nabla\phi\left(\frac{y}{A}\right).\left(M_{2}(\epsilon)\,\nabla\epsilon_{2}+M_{1}(\epsilon)\,\nabla\epsilon_{1}\right) \\ &+\mathcal{O}\big(A\,\|\epsilon\|_{H^{1}}^{2}+\mathcal{P}^{3}+\mathcal{P}\,\|\epsilon\|_{L^{2}}\big) \\ &=\frac{2}{\lambda}\mathcal{R}e\int\nabla^{2}\phi\left(\frac{y}{A}\right).\left(\nabla\epsilon,\,\nabla\overline{\epsilon}\right)-\frac{1}{2A^{2}\lambda}\int\Delta^{2}\phi\left(\frac{y}{A}\right)|\epsilon|^{2} \\ &+\frac{1}{\lambda}\int A\,\nabla\phi\left(\frac{y}{A}\right).\left[\nabla\left(Q^{2}+2\Sigma^{2}\right)\epsilon_{1}^{2}+4\,\nabla\left(\Sigma\,\Theta\right)\epsilon_{1}\,\epsilon_{2}+\nabla\left(Q^{2}+2\,\Theta^{2}\right)\epsilon_{2}^{2}\right] \\ &+\mathcal{O}\big(A\,\|\epsilon\|_{H^{1}}^{2}+\mathcal{P}^{3}+\mathcal{P}\,\|\epsilon\|_{L^{2}}\big) \end{split}$$

where we have made use of integration by parts, along with the identity $\int u \Delta \epsilon_j \epsilon_j = \frac{1}{2} \int \Delta u \epsilon_j^2 - \int u |\nabla \epsilon_j|^2$ when j = 1, 2. Eventually, the second term of the local virial identity is estimated as

$$\begin{pmatrix} \frac{1}{2\lambda} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi(\frac{y}{A}) \cdot \nabla \epsilon \overline{\epsilon} g \, dy \end{pmatrix}_{s}$$

$$= \frac{1}{\lambda} \left[\mathcal{R}e \int \nabla^{2} \phi\left(\frac{y}{A}\right) \cdot \left(\nabla \epsilon, \nabla \overline{\epsilon}\right) - \frac{1}{4A^{2}} \int \Delta^{2} \phi\left(\frac{y}{A}\right) |\epsilon|^{2} \right]$$

$$+ \frac{1}{2\lambda} \int A \nabla \phi\left(\frac{y}{A}\right) \cdot \left[\nabla (Q^{2} + 2\Sigma^{2}) \epsilon_{1}^{2} + 4 \nabla (\Sigma \Theta) \epsilon_{1} \epsilon_{2} + \nabla (Q^{2} + 2\Theta^{2}) \epsilon_{2}^{2} \right]$$

$$+ \mathcal{O}(A \|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3} + \mathcal{P} \|\epsilon\|_{L^{2}})$$

$$(4.18)$$

Step 3 Conclusion

Summing up (4.16) and (4.18), using (3.48) to expand non-linear terms of (4.16) finally leads to

$$\left\{ -\frac{b}{\lambda} \frac{\|yQ\|_{L^{2}}^{2}}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi\left(\frac{y}{A}\right) \cdot \nabla \epsilon \bar{\epsilon} g \, dy \right\}_{s} \\
\frac{1}{\lambda} \left[\int \left[1 - \frac{1}{4A^{2}} \Delta^{2} \phi\left(\frac{y}{A}\right) \right] |\epsilon|^{2} + \mathcal{R}e \int \nabla^{2} \phi\left(\frac{y}{A}\right) \cdot \left(\nabla \epsilon, \nabla \bar{\epsilon}\right) - \int \left(3 Q^{2} \epsilon_{1}^{2} + Q^{2} \epsilon_{2}^{2}\right) \right] \\
- \frac{\|Q\|_{L^{2}}^{2}}{2\lambda} \nabla^{2} (k - g)(0) \cdot (\alpha, \alpha) + \frac{1}{\lambda} \int \left(A \nabla \phi\left(\frac{y}{A}\right) - y\right) \cdot \left[3 Q \nabla Q \epsilon_{1}^{2} + Q \nabla Q \epsilon_{2}^{2}\right] \\
+ \mathcal{O} \left(A \|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3} + \mathcal{P} \|\epsilon\|_{L^{2}}\right) \tag{4.19}$$

Now, the choice for the cut-off function ϕ in (3.63) implies

$$\mathcal{R}e\int \nabla^2 \phi\left(\frac{y}{A}\right) \cdot \left(\nabla\epsilon, \, \nabla\overline{\epsilon}\right) + \int |\epsilon|^2 - \frac{1}{4A^2} \int \Delta^2 \phi\left(\frac{y}{A}\right) |\epsilon|^2 \ge \int e^{-\frac{|y|}{2A}} |\nabla\epsilon|^2 + \int |\epsilon|^2 + \mathcal{O}\left(\frac{1}{A^2} \|\epsilon\|_{L^2}^2\right)$$

The previous quadratic term is hence to be seen as a small deformation (as always, around the Ground-State) of the linearized energy $(L_{+}\epsilon_{1}, \epsilon_{1}) + (L_{-}\epsilon_{2}, \epsilon_{2})$. It can be estimated thanks to the coercivity property (3.32), which will ensure the existence of some universal constant $\underline{c} > 0$ such that whenever A is chosen large enough

$$\underline{c}\left[\int |\epsilon|^2 + \int e^{-\frac{|y|}{\sqrt{A}}} |\nabla\epsilon|^2\right] \leq \int e^{-\frac{|y|}{2A}} |\nabla\epsilon|^2 + \int |\epsilon|^2 - \int \left(3 Q^2 \epsilon_1^2 + Q^2 \epsilon_2^2\right) + \mathcal{O}\left(\|\epsilon\|_{H^1}^4 + \mathcal{P}^4\right)$$

Altogether, previous statements, the bound (3.21) and assumption $\nabla^2 (k-g)(0) < 0$ shows that for A > 0 large enough, and t(A) < t < 0 close enough to 0, we have

$$\left\{ -\frac{b}{\lambda} \frac{\|yQ\|_{L^{2}}^{2}}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi\left(\frac{y}{A}\right) \cdot \nabla \epsilon \,\overline{\epsilon} \, g \, dy \right\}_{s} \\
\geq \frac{c}{\lambda} \left[|\alpha|^{2} + \int |\epsilon|^{2} + \int e^{-\frac{|y|}{\sqrt{A}}} |\nabla \epsilon|^{2} \right] + \mathcal{O}\left(A \|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3}\right) \tag{4.20}$$

This concludes the proof of Lemma 4.3. Note A > 0 may now be fixed once and for all.

4.4 Convergence to 0 of \tilde{u} in H^1 away from the concentration point.

We wish to prove that profile $Q_{\mathcal{P}}$ contains all of the solution mass (in a H^1 sense) when staying away from the concentration point (which, for the record, has been chosen as $x_0 = 0 \in \mathbb{R}^2$). In other words, the difference between the exact solution u and the approximate solution $Q_{\mathcal{P}}$ we have consctructed so far completely vanish in H^1 as $t \to 0$. More precisely, working with the original variables, let

$$\tilde{Q}(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} Q_{\mathcal{P}}\left(\frac{x-\alpha(t)}{\lambda}\right) e^{i\gamma(t)}$$

$$\tilde{u}(t,x) = \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \epsilon\left(t, \frac{x-\alpha(t)}{\lambda}\right) e^{i\gamma(t)}$$
(4.21)

Using its very definition, and estimate (3.21) we have

$$\|\tilde{u}(t)\|_{L^2} \lesssim \lambda(t), \quad \|\tilde{u}(t)\|_{H^1} \lesssim 1, \quad \forall \ t_0 \le t < 0$$
(4.22)

The goal is now to improve the energy bound (4.22) for the dispersive bound

 $\tilde{u}(t) \to 0$ in H^1 as $t \to 0$

The first step is dispersion away from the blow up point.

Lemma 4.4. (H^1 dispersion away from the concentration point). There holds :

$$\lim_{t \to 0} \|\tilde{u}(t)\|_{H^1(|x| \ge 3\eta)} = 0 \tag{4.23}$$

for some small constant $\eta > 0$ small enough that only depends on G and g.

Remark 7. Because of assumptions (H2), (H3) and in particular $\nabla g(0) = 0$, $\nabla G(0) = 0$ we choose the region $|x| \ge 3\eta$ as the one away from the blow up point, for we will need an estimate of the type :

$$\left\| x \cdot \nabla g \, G - (1 \, 0 \, 0 \, 1) \, g \, \nabla G x \right\| < 2 \, g \tag{4.24}$$

so we choose and fix some constant $\eta > 0$ that satisfies these, and keep it in the sequel.

Proof of Lemma 4.4

Let \tilde{Q} , \tilde{u} be given by (4.21), then \tilde{u} satisfies

$$i \partial_t \tilde{u} + \mathcal{L}\tilde{u} = -R - k(x) |\tilde{u}|^2 \tilde{u}$$

$$(4.25)$$

with

$$R = k(x) \left[\left(\tilde{Q} + \tilde{u} \right) \left| \tilde{Q} + \tilde{u} \right|^2 - \tilde{Q} \left| \tilde{Q} \right|^2 - \tilde{u} \left| \tilde{u} \right|^2 \right] + i \partial_t \tilde{Q} + \mathcal{L} \tilde{Q} + k(x) \left| \tilde{Q} \right|^2 \tilde{Q}$$
(4.26)

Step 1 $L_t^2 H_{loc}^{3/2}$ bound away from the concentration point.

Here we use some technical results a bit like we already did in the proof for Proposition 3.7. We will also use the smoothing effect of the linear Schrödinger flow to claim the space time bound

$$\int_{t_0}^0 \|\tilde{u}(\tau)\|_{H^{3/2}(2\eta \le r \le 4\eta)}^2 \, d\tau < +\infty \tag{4.27}$$

We introduce the Fourier multiplier $D = (1 - \Delta)^{1/2}$. Note that using our definition (3.123) for the Fourier transform, for every $s \in \mathbb{R}$ one has

$$D^{s}u(x) = \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^{2}} e^{ix \cdot \xi} \left(1 + |\xi|^{2}\right)^{s/2} \widehat{u}(\xi) d\xi$$

so that, the H^{s} norm may be written
 $\|u\|_{H^{s}} = \|D^{s}u\|_{L^{2}}$ (4.28)

Starting with the nonlinear term in (4.25), let $\tilde{\zeta}$ be the solution to

$$i \partial_t \tilde{\zeta} + \mathcal{L} \tilde{\zeta} = -k(x) \, |\tilde{u}|^2 \, \tilde{u}, \quad \tilde{\zeta}(0) = 0 \tag{4.29}$$

Now following the procedure explained in Appendix B 8.3.2, using successively Strichartz bounds and the smoothing effect of the linear Schrödinger flow, we have

$$\begin{split} \|\tilde{\zeta}\|_{L^{2}(t_{0},0)\,H_{loc}^{3/2}} &\lesssim \left\|k(x)\,|\tilde{u}|^{2}\,\tilde{u}\right\|_{L^{2}_{t}\,H^{1}_{loc}} \quad (\text{smoothing effect}) \\ \|\tilde{\zeta}\|_{L^{\infty}(t_{0},0)\,H^{1}} &= \|D\tilde{\zeta}\|_{L^{\infty}(t_{0},0)\,L^{2}} \lesssim \left\|D\big(k(x)\,|\tilde{u}|^{2}\,\tilde{u}\big)\right\|_{L^{4/3}(t_{0},0)\,L^{4/3}} \quad (\text{strichartz bound}) \end{split}$$

so that using paraproduct and Hölder estimates along with the Sobolev embedding Theorem result 8.2, and in particular (8.5)

$$\left\| D(k(x) \, |\tilde{u}|^2 \, \tilde{u}) \right\|_{L^{4/3}} \lesssim \left\| |\tilde{u}|^2 \, D\tilde{u} \right\|_{L^{4/3}} \lesssim \|\tilde{u}\|_{L^4}^2 \, \|D\tilde{u}\|_{L^4} \lesssim \|\tilde{u}\|_{H^1}^3$$

one gets

$$\|\tilde{\zeta}\|_{L^{2}(t_{0},0)H^{3/2}_{loc}} + \|\tilde{\zeta}\|_{L^{\infty}(t_{0},0)H^{1}} \lesssim \|D(k(x)|\tilde{u}|^{2}\tilde{u})\|_{L^{4/3}(t_{0},0)L^{4/3}} \lesssim \|\tilde{u}\|_{L^{\infty}(t_{0},0)H^{1}}^{3} \lesssim 1$$

$$(4.30)$$
where we have used the $L^{\infty}((t_0, 0), H^1)$ bound (4.22) in the last line.

Then let $\tilde{w} = D^{1/2}(\tilde{u} - \tilde{\zeta})$, and χ be a radial smooth cut-off function with

$$\chi'(r) = \int_0^r \chi''(\rho) \, d\rho, \quad \text{with} \quad \chi''(r) = \begin{cases} 0 & \text{for} \quad 0 \le r \le \eta, \\ 1 & \text{for} \quad 2\eta \le r \le 4\eta, \\ \frac{1}{r^2} & \text{for} \quad r \ge 5\eta \end{cases}$$
(4.31)

and such that

$$\forall r \ge 0, \quad \frac{|\chi'|^2}{r^2} \lesssim \chi''(r) \le 1.$$
 (4.32)

We claim (4.27) follows from (4.30) and

$$\int_{t_0}^0 \left\|\nabla \tilde{w}\right\|_{L^2(2\eta \le r \le 4\eta)}^2 \lesssim 1 \tag{4.33}$$

Since $D^{1/2}$ commutes with $\mathcal{L} = \frac{1}{g} \operatorname{div}(G \nabla) + V$, from (4.25) \tilde{w} satisfies

$$i\partial_t \tilde{w} + \mathcal{L}\tilde{w} = D^{1/2}R\tag{4.34}$$

where R is defined by (4.26).

Let $\mathbf{e}_r = (\cos(\theta), \sin(\theta))$ be the unit vector in the radial direction and $\mathbf{e}_{\theta} = (-\sin(\theta), \cos(\theta))$ the unit orthoradial vector. Then, let

$$\begin{pmatrix} G_{rr} & G_{r\theta} \\ G_{\theta r} & G_{\theta \theta} \end{pmatrix} = \begin{pmatrix} G \mathbf{e}_r \cdot \mathbf{e}_r & G \mathbf{e}_r \cdot \mathbf{e}_\theta \\ G \mathbf{e}_\theta \cdot \mathbf{e}_r & G \mathbf{e}_\theta \cdot \mathbf{e}_\theta \end{pmatrix}$$

Notice that since G is symmetric, so is the previous matrix, so that $G_{r\theta} = G_{\theta r}$.

We compute the associated localized virial identity :

$$\begin{split} &\frac{1}{2} \frac{d}{dt} \mathcal{I}m \int \nabla \chi . \nabla \tilde{w} \,\overline{\tilde{w}} \, g \, dx \\ &= -\mathcal{R}e \int \left[\mathcal{L}\tilde{w} - D^{1/2}R \right] \overline{\left(\frac{1}{2} \left[\Delta \chi + \frac{\nabla g}{g} . \nabla \chi \right] \tilde{w} + \nabla \chi . \nabla \tilde{w} \right)} \, g \, dx \\ &= \int \left[\chi'' \, G_{rr} + \frac{\chi'}{2} \left(G_{rr} \, \frac{\partial_r g}{g} - \partial_r G_{rr} \right) \right] |\partial_r \tilde{w}|^2 + \int \left[\frac{\chi'}{r} \, G_{\theta\theta} + \frac{\chi'}{2} \left(G_{\theta\theta} \, \frac{\partial_r g}{g} - \partial_r G_{\theta\theta} \right) \right] |\partial_\tau \tilde{w}|^2 \\ &+ \mathcal{R}e \int \left[\chi'' \, G_{r\theta} + \frac{\chi'}{r} \left(G_{r\theta} \left(1 + r \, \frac{\partial_r g}{g} \right) - r \, \partial_r G_{rr} \right) \right] \partial_r \overline{\tilde{w}} \, \partial_\tau \tilde{w} \\ &- \frac{1}{4} \int \mathcal{L} \left[\Delta \chi + \frac{\nabla g}{g} . \nabla \chi \right] |\tilde{w}|^2 \, g \, dx + \frac{1}{4} \int \left[\frac{\nabla g}{g} - \nabla (g \, V) \right] . \nabla \chi \, |\tilde{w}|^2 \, dx \\ &+ \mathcal{R}e \Big(D^{1/2} \mathcal{R} \,, \frac{1}{2} \left[\Delta \chi + \frac{\nabla g}{g} . \nabla \chi \right] \tilde{w} + \nabla \chi . \nabla \tilde{w} \Big)_{L^2_{g \, dx}} \end{split}$$
(4.35)

$$\frac{1}{2} \frac{d}{dt} \mathcal{I}m \int \nabla \chi \cdot \nabla \tilde{w} \,\overline{\tilde{w}} \, g \, dx \\
= -\mathcal{R}e \int \left[\mathcal{L}\tilde{w} - D^{1/2}R \right] \overline{\left(\frac{1}{2} \left[\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right] \tilde{w} + \nabla \chi \cdot \nabla \tilde{w} \right)} \, g \, dx \\
= \int \left[\chi'' \, G_{rr} + \frac{\chi'}{r} \, \frac{r \left(G_{rr} \, \partial_r g - g \, \partial_r G_{rr} \right)}{2g} \right] |\partial_r \tilde{w}|^2 + \int \frac{\chi'}{r} \left[G_{\theta\theta} + \frac{r \left(G_{\theta\theta} \, \partial_r g - g \, \partial_r G_{\theta\theta} \right)}{2g} \right] |\partial_\tau \tilde{w}|^2 \\
+ \mathcal{R}e \int \left[\chi'' \, G_{r\theta} + \frac{\chi'}{r} \, \frac{G_{r\theta} \, \partial_r \left(gr \right) - g \, r \, \partial_r G_{rr}}{g} \right] \, \partial_r \overline{\tilde{w}} \, \partial_\tau \tilde{w} \\
- \frac{1}{4} \int \mathcal{L} \left[\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right] |\tilde{w}|^2 \, g \, dx + \frac{1}{4} \int \left[\frac{\nabla g}{g} - \nabla \left(g \, V \right) \right] \cdot \nabla \chi \, |\tilde{w}|^2 \, dx \\
+ \mathcal{R}e \left(D^{1/2} R , \, \frac{1}{2} \left[\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right] \tilde{w} + \nabla \chi \cdot \nabla \tilde{w} \right]_{L^2_{g \, dx}}$$
(4.36)

where $\partial_{\tau} \tilde{w} = \frac{1}{r} \partial_{\theta} \tilde{w}$.

Note the metric terms G and g here pertubated our computation which aims to estimate \tilde{w} in space $L^2_{(t_0,0)}L^2_{(2\eta\leq r\leq 4\eta)}$ as in (4.33). Thanks to (4.32) χ'' is the dominant term, so that one may check provided $\eta > 0$ is chosen small enough, from hypothesis (H3) :

$$G_{\mu\mu} = 1 + \mathcal{O}(r^2), \quad \mu = r, \theta$$

 $G_{r\theta} = \mathcal{O}(r^2)$

and $|r\partial_r g| = |x \cdot \nabla g| = \mathcal{O}(r^2)$ hold whenever $2\eta \leq r \leq 4\eta$, which is enough to ensure

$$\chi'' G_{rr} + \frac{\chi'}{r} \frac{r \left(G_{rr} \partial_r g - g \partial_r G_{rr}\right)}{2g} = \chi'' + \mathcal{O}(r^2) = \chi'' + \mathcal{O}(\eta^2)$$
$$G_{\theta\theta} + \frac{r \left(G_{\theta\theta} \partial_r g - g \partial_r G_{\theta\theta}\right)}{2g} = 1 + \mathcal{O}(r^2) = 1 + \mathcal{O}(\eta^2)$$
$$\chi'' G_{r\theta} + \frac{\chi'}{r} \frac{G_{r\theta} \partial_r (gr) - gr \partial_r G_{rr}}{g} = \mathcal{O}(r^2) = \mathcal{O}(\eta^2)$$

hence

$$\begin{split} &\frac{1}{2} \frac{d}{dt} \mathcal{I}m \int \nabla \chi \,. \, \nabla \tilde{w} \,\overline{\tilde{w}} \,g \,dx \\ &\gtrsim \int_{2\eta \leq r \leq 4\eta} |\partial_r \tilde{w}|^2 + \int_{2\eta \leq r \leq 4\eta} |\partial_\tau \tilde{w}|^2 \\ &- \frac{1}{4} \int \mathcal{L} \Big[\Delta \chi + \frac{\nabla g}{g} \,. \, \nabla \chi \Big] \,|\tilde{w}|^2 \,g \,dx + \frac{1}{4} \,\int \Big[\frac{\nabla g}{g} - \nabla \big(g \,V\big) \Big] \,. \, \nabla \chi \,|\tilde{w}|^2 \,dx \\ &+ \mathcal{R}e \Big(D^{1/2} R \,, \, \frac{1}{2} \,\Big[\Delta \chi + \frac{\nabla g}{g} \,. \, \nabla \chi \Big] \,\tilde{w} + \nabla \chi \,. \, \nabla \tilde{w} \Big)_{L^2_{g \,dx}} \end{split}$$

then, one can easily deduce the following estimate

$$\begin{split} &\int_{t_0}^0 \left\|\nabla \tilde{w}\right\|_{L^2(2\eta \le r \le 4\eta)}^2 = \int_{t_0}^0 \int_{2\eta \le r \le 4\eta} \left(|\partial_r \tilde{w}|^2 + |\partial_\tau \tilde{w}|^2\right) \, dr d\tau \\ &\lesssim \frac{1}{4} \, \int_{t_0}^0 \int \left|\mathcal{L}\left[\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi\right] - \left[\frac{\nabla g}{g^2} - \frac{\nabla (g \, V)}{g}\right] \cdot \nabla \chi \right| |\tilde{w}|^2 \, g \, dx \\ &+ \int_{t_0}^0 \left|\mathcal{R}e\left(D^{1/2}R, \, \frac{1}{2}\left[\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi\right] \tilde{w} + \nabla \chi \cdot \nabla \tilde{w}\right)_{L^2_{g \, dx}}\right| \\ &+ \int_{t_0}^0 \left|\frac{1}{2} \, \frac{d}{dt} \, \mathcal{I}m \int \nabla \chi \cdot \nabla \tilde{w} \, \overline{\tilde{w}} \, g \, dx\right| \end{split}$$

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We now have to estimate the terms in (4.35). To begin with, using (4.22) and (4.30), from definition of \tilde{w} the boundary term in time is bounded

$$\left|\mathcal{I}m\int\nabla\chi\,\cdot\nabla\tilde{w}\,\overline{\tilde{w}}\,g\,dx\right|\lesssim\|\tilde{w}\|_{H^{1/2}}^2\lesssim\|\tilde{u}\|_{H^1}^2+\|\tilde{\zeta}\|_{H^1}^2\lesssim1\tag{4.37}$$

and similarly

$$\left| \frac{1}{4} \int_{t_0}^0 \int \left| \mathcal{L} \left[\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right] - \left[\frac{\nabla g}{g^2} - \frac{\nabla (g V)}{g} \right] \cdot \nabla \chi \right| |\tilde{w}|^2 g \, dx \right|$$

$$\lesssim \|\tilde{u}\|_{H^{1/2}}^2 + \|\tilde{\zeta}\|_{H^{1/2}}^2 \lesssim 1$$

$$(4.38)$$

Now with the R term in (4.35), first by Cauchy-Scwharz

$$\begin{split} & \left| \mathcal{R}e\Big(D^{1/2}R, \overline{\frac{1}{2}\left[\Delta\chi + \frac{\nabla g}{g} \cdot \nabla\chi\right]\tilde{w} + \nabla\chi \cdot \nabla\tilde{w}}\Big)_{L^2_{g\,dx}} \right| \\ & \lesssim \|x\,D^{1/2}R\|_{L^2(r\geq\eta)} \left[\left\|\frac{\Delta\chi}{r}\,\tilde{w}\right\|_{L^2(r\geq\eta)} + \left\|\frac{\chi'}{r}\,\partial_r\tilde{w}\right\|_{L^2} \right] \\ & \leq \frac{1}{\delta} \|x\,D^{1/2}R\|^2_{L^2(r\geq\eta)} + 2\,\delta\left(\left\|\frac{\Delta\chi}{r}\,\tilde{w}\right\|^2_{L^2(r\geq\eta)} + \left\|\frac{\chi'}{r}\,\partial_r\tilde{w}\right\|^2_{L^2} \right) \\ & \leq \frac{1}{\delta} \int_{|x|\geq\eta} |x|^2 \left|D^{1/2}R\right|^2 + 2\left(\|\tilde{w}\|^2_{L^2} + \int\chi'' \left|\partial_r\tilde{w}\right|^2 \right) \\ & \lesssim \delta \int\chi'' \left|\partial_r\tilde{w}\right|^2 + \frac{1}{\delta} \int_{|x|\geq\eta} |x|^2 \left|D^{1/2}R\right|^2 + 1 \end{split}$$

for some small $\delta > 0$, where we have used (4.32) and the H^1 bound (4.22) in the last line.

We now claim

Lemma 4.5.

$$\int_{t_0}^0 \left\| x \, D^{1/2} R \right\|_{L^2}^2 < +\infty \tag{4.39}$$

Proof of Lemma 4.5 : Recall $D = (1 - \Delta)^{1/2}$, with use of Pseudo Differential Calculus, one may prove

$$D^{1/2} |x|^2 D^{1/2} = x D x + a_{-1}(x, D), \quad a_{-1} \in S^{-1} \text{ is a symbol of order } -1$$
(4.40)

In particular, what will be useful for us now, is that $a_{-1}(x, D)$ is a bounded operator $L^{4/3} \to W^{1,4/3} \subseteq L^4$ thanks to Sobolev embedding result Theorem 8.1. For details about these statements, see Appendix B 8.4. Indeed, from (4.40) we have

$$\left\| x \, D^{1/2} \, R \right\|_{L^2}^2 = \left(D^{1/2} \, |x|^2 \, D^{1/2} \, R \, , \, R \right) \lesssim \left\| D^{1/2} \, x \, R \right\|_{L^2}^2 + \left\| R \right\|_{L^{4/3}}^2$$

We then change to rescaled variables

$$R(t,x) = \frac{1}{\lambda^3(t)k(\alpha)^{1/2}} S\left(\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
(4.42)

so that (4.26) becomes

$$S(s,y) = i \partial_s Q_{\mathcal{P}} + \mathcal{L}_{\lambda \alpha} Q_{\mathcal{P}} - Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}}$$

$$- i \frac{\lambda_s}{\lambda} \Lambda Q_{\mathcal{P}} - i \frac{\alpha_s}{\lambda} \left(\nabla Q_{\mathcal{P}} + \frac{\lambda}{2} \frac{\nabla k(\alpha)}{k(\alpha)} Q_{\mathcal{P}} \right) - \tilde{\gamma}_s Q_{\mathcal{P}}$$

$$+ \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[(Q_{\mathcal{P}} + \epsilon) |Q_{\mathcal{P}} + \epsilon|^2 - Q_{\mathcal{P}} |Q_{\mathcal{P}}|^2 - \epsilon |\epsilon|^2 \right]$$
(4.43)

(4.41)

which is well localized in y. Changing variables $(Z, \eta) = \left(\frac{z-\alpha}{\lambda}, \lambda \xi\right)$, with $dZ \, d\eta = dz \, d\xi$ and from Sobolev embedding

$$D^{1/2}x R = (2\pi)^{-2} \int e^{i(x-z) \cdot \xi} \langle \xi \rangle^{1/2} z R(z) dz d\xi$$

$$= \frac{(2\pi)^{-2}}{k(\alpha)^{1/2}} \int e^{i(y-Z) \cdot \eta} \frac{1}{\lambda^{1/2}} (\lambda^2 + |\eta|^2)^{1/4} \frac{e^{i\gamma}}{\lambda^3} (\lambda Z + \alpha) S(Z) dZ d\eta$$

$$= \frac{1}{\lambda^{5/2} k(\alpha)^{1/2}} (\lambda^2 - \Delta)^{1/4} \left(y + \frac{\alpha}{\lambda}\right) S(y)$$

$$= \frac{1}{\lambda^{5/2} k(\alpha)^{1/2}} D^{1/2} \left(y + \frac{\alpha}{\lambda}\right) S(y) + a_{-3/2} (D) \left(y + \frac{\alpha}{\lambda}\right) S(y)$$

$$a_{-3/2}(\eta) = \frac{\lambda^2 - 1}{\left[(\lambda^2 + |\eta|^2)^{1/4} + (\eta)^{1/2}\right] \left[(\lambda^2 + |\eta|^2)^{1/2} + (\eta)\right]} = \mathcal{O} \left(\langle \eta \rangle^{-3/2}\right)$$

$$\begin{split} & u_{-3/2}(\eta) = \left[(\lambda^2 + |\eta|^2)^{1/4} + \langle \eta \rangle^{1/2} \right] \left[(\lambda^2 + |\eta|^2)^{1/2} + \langle \eta \rangle \right] = \mathcal{O}\left(\langle \eta \rangle \right) \\ & a_{-3/2} : L^{4/3} \to W^{3/2, 4/3} \hookrightarrow H^1 \subset L^2 \end{split}$$

so we have

$$\int \left| D^{1/2} x R \right|^2 dx = \frac{1}{\lambda^3 k(\alpha)} \int \left| \left(\lambda^2 - \Delta \right)^{1/4} \left(y + \frac{\alpha}{\lambda} \right) S(y) \right|^2 dy$$

and thus from (3.21)

$$\left\| D^{1/2} x R \right\|_{L^2} \lesssim \frac{1}{\lambda^{3/2}} \left(\left\| D^{1/2} y S \right\|_{L^2} + \left\| D^{1/2} S \right\|_{L^2} + \left\| S \right\|_{L^{4/3}} \right)$$

Now thanks to (4.41), this yields

$$\left\| x D^{1/2} R \right\|_{L^2} \lesssim \frac{1}{\lambda^{3/2}} \left(\left\| D^{1/2} y S \right\|_{L^2} + \left\| D^{1/2} S \right\|_{L^2} + \left\| S \right\|_{L^{4/3}} \right)$$
(4.44)

We now explicitly expand the nonlinear terms in ϵ , in S, from definition (4.26) of R and (4.42), (4.43) of S, together with (2.2) and (2.3) we have

$$\begin{split} S &= -\psi_{\mathcal{P}} + i\left(b_s + b^2 - B_1\right)\partial_b Q_{\mathcal{P}} + i\lambda\left(\frac{\lambda_s}{\lambda} + b\right)\partial_\lambda Q_{\mathcal{P}} \\ &+ i\left(\beta_s + b\beta - B_2\right)\partial_\beta Q_{\mathcal{P}} + i\lambda\left(\frac{\alpha_s}{\lambda} - 2\beta\right)\partial_\alpha Q_{\mathcal{P}} - i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda Q_{\mathcal{P}} \\ &- i\left(\frac{\alpha_s}{\lambda} - 2\beta\right)\left[\nabla Q_{\mathcal{P}} + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}Q_{\mathcal{P}}\right] - \left(\widetilde{\gamma}_s - |\beta|^2\right)Q_{\mathcal{P}} \\ &+ \frac{k(\lambda y + \alpha)}{k(\alpha)}\left[Q_{\mathcal{P}} |\epsilon|^2 + 2\epsilon \operatorname{\mathcal{R}e}\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) + |Q_{\mathcal{P}}|^2\left(\epsilon + 2\epsilon_1\right)\right] \end{split}$$

Thus, (3.24) implies

$$S = \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[Q_{\mathcal{P}} |\epsilon|^2 + 2 \epsilon \mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) + |Q_{\mathcal{P}}|^2 (\epsilon + 2\epsilon_1) \right] + \mathcal{O}\left((|\alpha|^2 + \lambda^4) e^{-c_4 |y|} \right)$$

$$= Q_{\mathcal{P}} |\epsilon|^2 + 2 \epsilon \mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) + |Q_{\mathcal{P}}|^2 (\epsilon + 2\epsilon_1) + \mathcal{O}\left((\lambda^4 + |\alpha|^2) e^{-c_4 |y|} \right)$$
(4.45)

Then using standard commutator estimates together with the good localization in space of S(s, y), the bound on the geometrical parameters (3.21), (3.24) and the $\mathcal{O}(\lambda^3)$ control in the construction of $Q_{\mathcal{P}}$ (1.40), (1.41) to conclude :

$$\|x D^{1/2} R\|_{L^{2}} \lesssim \frac{1}{\lambda^{3}} \left(\int |\nabla \epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} + \|\epsilon\|_{L^{2}}^{2} + \lambda^{6} + |\alpha|^{2} \right)$$

$$\lesssim 1 + \frac{1}{\lambda^{3}} \left(\int |\nabla \epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} + \|\epsilon\|_{L^{2}}^{2} \right)$$

$$(4.46)$$

and (4.39) follows from (4.15) (remind $\frac{ds}{dt} = \frac{1}{\lambda^2}$).

Step 2 Strong H^1 convergence outside the blow up point.

The strong convergence (4.23) is now straightforward. Let a smooth cut off function $\tilde{\psi}$ with $\tilde{\psi} = 1$ on $|x| \ge 3\eta$ and $\tilde{\psi} = 0$ on $|x| \le 2\eta$, then $w = \tilde{\psi} \tilde{u}$ satisfies the following equation

$$i\partial_t w + \mathcal{L}w + k(x) |w|^2 w = F, \quad w(t) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \to 0$$

$$(4.47)$$

with
$$F = \frac{1}{g(x)} \left[\Delta_G \tilde{\psi} \, \tilde{u} + 2 \, G(x) \, \nabla \tilde{\psi} \, \cdot \nabla \tilde{u} \right] - k(x) \, \tilde{\psi} \left(1 - \tilde{\psi}^2 \right) |\tilde{u}|^2 \, \tilde{u} - \tilde{\psi} \, R$$

where R is given by (4.26). To get (4.23) we now only have to show

$$||w||_{L^{\infty}((t_0,0),H^1)} \to 0 \quad \text{as} \quad t_0 \to 0$$
(4.48)

First we write down Duhamel formula for (4.47)

$$w(t) = e^{it\mathcal{L}}w(t_0) - i\int_{t_0}^t e^{i(t-\tau)\mathcal{L}} \left[F(\tau)\right] d\tau$$

hence $\nabla w(t) = e^{it\mathcal{L}}\nabla w(t_0) - i\int_{t_0}^t e^{i(t-\tau)\mathcal{L}} \left[\nabla F(\tau)\right] d\tau$

then, we use the Strichartz estimates for the 2-dimensional Strichartz pairs (4,4), $(\infty, 2)$ along with the smoothing effect, see Appendix B 8.3.2, to get

$$\|\nabla w\|_{L^{4}((t_{0},0),L^{4})} + \|\nabla w\|_{L^{\infty}((t_{0},0),L^{2})}$$

$$\lesssim \|\nabla(k(x) |w|^{2} w)\|_{L^{4/3}((t_{0},0),L^{4/3})} + \||x| F\|_{L^{2}((t_{0},0),H^{1/2})}$$

$$(4.49)$$

We estimate the nonlinear term as follows, using the Gagliardo-Nirenberg estimate

$$\begin{aligned} \left\| \nabla(k(x) |w|^{2} w) \right\|_{L^{4/3}\left((t_{0},0),L^{4/3}\right)} \\ \lesssim \left\| |w|^{2} w \right\|_{L^{4/3}\left((t_{0},0),L^{4/3}\right)} + \left\| |w|^{2} \nabla w \right\|_{L^{4/3}\left((t_{0},0),L^{4/3}\right)} \\ \lesssim \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \left\| \tilde{u} \right\|_{L^{4}\left((t_{0},0),L^{4}\right)}^{2} \\ \lesssim \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \left\| \tilde{u} \right\|_{L^{\infty}\left((t_{0},0),H^{1}\right)} \left\| \tilde{u} \right\|_{L^{\infty}\left((t_{0},0),L^{2}\right)} \\ \lesssim \left\| \lambda \right\|_{L^{\infty}(t_{0},0)} \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \lesssim \delta \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \end{aligned}$$

$$(4.50)$$

for a constant $\delta > 0$ small enough, where we have used (4.22) in the last inequality. For the second term, we make use of the compact support property of $\nabla \tilde{\psi}$, $\Delta_G \tilde{\psi}$, $(1 - \tilde{\psi}^2)$ and (4.27) along with (4.39)

$$\begin{aligned} \left\| |x| F \right\|_{L^{2}\left((t_{0},0),H^{1/2}\right)} &\lesssim \left\| |x| R \right\|_{L^{2}\left((t_{0},0),H^{1/2}(|x|\geq\eta)\right)} + \left\| \tilde{u} \right\|_{L^{2}\left((t_{0},0),H^{3/2}(2\eta\leq r\leq 4\eta)\right)} \\ &\lesssim o(1) \quad \text{as} \quad t_{0} \to 0 \end{aligned}$$

$$(4.51)$$

Altogether, these last estimations prove

$$\left\|\nabla w\right\|_{L^{4}\left((t_{0},0),L^{4}\right)}+\left\|\nabla w\right\|_{L^{\infty}\left((t_{0},0),L^{2}\right)}\lesssim o(1) \quad \text{as} \quad t_{0}\to 0$$
(4.52)

which ends the proof of Lemma 4.4.

Convergence to 0 in average of \tilde{u} in H^1 . 4.5

The goal now is to propagate the H^1 convergence of \tilde{u} away from the concentration point (4.23) to the blow up region as well. We will now make use of the refined estimate (4.15) provided by localized virial estimate (4.14) which implies

$$\int_{t_0}^0 \frac{1}{\lambda^3} \|\tilde{u}(t)\|_{L^2}^2 dt = \int_{s_0}^{+\infty} \frac{1}{\lambda} \|\tilde{u}(s)\|_{L^2}^2 ds < +\infty$$

so that

$$\liminf_{t \to 0} \frac{\|\tilde{u}(t)\|_{L^2}}{\lambda(t)} = 0 \tag{4.53}$$

The first step is to obtain a convergence in average in time.

Lemma 4.6. $(H^1 \text{ dispersion in average in time.})$ There holds

$$\lim_{t \to 0} \frac{1}{|t|} \int_{t}^{0} \left(\frac{1}{|\tau|} \int_{\tau}^{0} \|\tilde{u}\|_{H^{1}(\mathbb{R}^{2})}^{2} d\sigma \right) d\tau = 0$$
(4.54)

Proof of Proposition 4.6: Thanks to (4.23), we may here restrict the H^1 norm to the region $|x| \leq 3\eta$. Step 1 Morawetz identity.

First, we claim the virial type bound

$$\int_{t}^{0} \int_{|x| \le 3\eta} \left| \nabla \tilde{u} \right|^{2} \lesssim o(|t|) + \|\tilde{u}(t)\|_{L^{2}} + \int_{t}^{0} \frac{|\alpha|^{2} + \|\epsilon(\tau)\|_{L^{2}}^{2} + \int |\nabla\epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}}}{\lambda^{2}(\tau)} d\tau$$
(4.55)

Let χ a smooth radial cut off function on \mathbb{R}^2 such that

$$supp(\chi) \subset \{|x| \le 4\eta\}$$

$$f_{\chi} := 2 G \nabla^2 \chi + \left(\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi\right) G - \Delta_G \chi I d \text{ is positive semidefinite}$$

$$\nabla \chi(x) = x, \text{ on } |x| \le 3\eta$$

$$(4.56)$$

thus $\partial_{jl}^2 \chi = 0$ for $j \neq l$, and $\partial_j^2 \chi = 1$ on $|x| \leq 3\eta$, so that the form f_{χ} is given by the matrix

$$F_{\chi} = 4 G - Tr(G) I_2 + x \cdot \frac{\nabla g}{g} G - (1 \ 0 \ 0 \ 1) \nabla Gx, \quad |x| \le 3\eta$$

From hypothesis (H2), g(0) = 1, $G(0) = I_2$, $\nabla g(0) = 0$, $\nabla G(0) = 0$, so that by continuity

$$F_{\chi} = 2 I_2 + \mathcal{O}(|x|^2), \quad |x| \le 3\eta$$

Hence we know it is reasonable to assume for x close enough to 0, here on region $|x| \le 3\eta, f_{\chi} \gtrsim 1$ holds, which considering our choice for χ is satisfied as long as $\eta > 0$ is chosen small enough to ensure $\left\| x \, \cdot \nabla g \, G - (1 \, 0 \, 0 \, 1) \, g \, \nabla G x \right\| < 2 \, g.$

$$\begin{split} \frac{d}{dt}\mathcal{I}m \int \nabla\chi \cdot \nabla\tilde{u}\,\overline{\tilde{u}}\,g\,dx \\ &= -2\,\mathcal{R}e\int \left(\mathcal{L}\tilde{u} + k(x)\,|\tilde{u}|^2\,\tilde{u} + R\right) \left[\frac{1}{2}\left(\Delta\chi + \frac{\nabla g}{g}\cdot\nabla\chi\right)\overline{\tilde{u}} + \nabla\chi\cdot\nabla\overline{\tilde{u}}\right]\,g\,dx \\ &= 2\,\mathcal{R}e\int G\,\nabla^2\chi \cdot \left(\nabla\tilde{u}\cdot\nabla\overline{\tilde{u}}\right)\,dx + \int \left(\Delta\chi + \frac{\nabla g}{g}\cdot\nabla\chi\right)G\,\nabla\tilde{u}\cdot\nabla\overline{\tilde{u}}\,dx - \int \Delta_G\,\chi\,|\nabla\tilde{u}|^2\,dx \\ &- \frac{1}{2}\int \left[\Delta_{G,g}\left(\Delta\chi + \frac{\nabla g}{g}\cdot\nabla\chi\right) - 2\,\nabla V\cdot\nabla\chi\right]|\tilde{u}|^2\,g\,dx \\ &- \frac{1}{2}\int \left[k(x)\left(\Delta\chi + \frac{\nabla g}{g}\cdot\nabla\chi\right) - \nabla k(x)\cdot\nabla\chi\right]|\tilde{u}|^4\,g\,dx \\ &- 2\,\mathcal{R}e\int R\left[\frac{1}{2}\left(\Delta\chi + \frac{\nabla g}{g}\cdot\nabla\chi\right)\overline{\tilde{u}} + \nabla\chi\cdot\nabla\overline{\tilde{u}}\right]g\,dx \end{split}$$
(4.57)

where \tilde{u} , \tilde{Q} and R are defined as in (4.21) and (4.26). We estimate the various terms of the right-hand side of (4.57). Using (4.22), and a Gagliardo-Nirenberg estimate, we have

$$\left| \int \left[\Delta_{G,g} \left(\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right) - 2 \nabla V \cdot \nabla \chi \right] |\tilde{u}|^2 g \, dx + \int \left[k(x) \left(\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right) - \nabla k(x) \cdot \nabla \chi \right] |\tilde{u}|^4 g \, dx \right| \\
\lesssim \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^4}^4 \\
\lesssim \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 \|\tilde{u}\|_{H^1}^2 \lesssim \lambda^2(t)$$
(4.58)

Then, recall from (4.45)

$$S = Q_{\mathcal{P}} |\epsilon|^2 + 2 \epsilon \mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) + |Q_{\mathcal{P}}|^2 (\epsilon + 2\epsilon_1) + \mathcal{O}\left(\left(\lambda^4 + |\alpha|^2\right) e^{-c_4 |y|}\right)$$

and hence, making use of assumption $\nabla \chi(x) \sim x$ near the origin, it is now clear that

$$\begin{aligned} \left| \mathcal{R}e \int R \left[\frac{1}{2} \left(\Delta \chi + \frac{\nabla g}{g} \cdot \nabla \chi \right) \overline{\tilde{u}} + \nabla \chi \cdot \nabla \overline{\tilde{u}} \right] g \, dx \right| \\ &= \frac{1}{\lambda^2} \left| \int S(y) \left[\frac{1}{2} \left(\Delta \chi (\lambda y + \alpha) + \frac{\nabla g}{g} \cdot \nabla \chi (\lambda y + \alpha) \right) \overline{\epsilon} + \frac{1}{\lambda} \nabla \chi (\lambda y + \alpha) \cdot \nabla \overline{\epsilon} \right] g \, dy \right| \\ &\lesssim \frac{1}{\lambda^2} \left[|\alpha|^2 + \lambda^3 + \|\epsilon\|_{L^2}^2 + \int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} \right] \end{aligned}$$
(4.59)

Eventually, we inject (4.58) and (4.59) in (4.57) and integrate in time, then we use assumption $f_{\chi} > 0$ and (4.5) to get

$$\int_{t}^{0} \int_{|x| \leq 3\eta} |\nabla \tilde{u}(\tau)|^{2} d\tau \lesssim \int_{t}^{0} \int f_{\chi} \left(\nabla \tilde{u} , \overline{\nabla \tilde{u}} \right) \\
\lesssim \left[\mathcal{I}m \int \nabla \chi . \nabla \tilde{u} \,\overline{\tilde{u}} \, g \, dx \right]_{t}^{0} + \int_{t}^{0} \frac{1}{\lambda^{2}(\tau)} \left[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} + \int |\nabla \epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} \right] d\tau + \int_{t}^{0} \lambda(\tau) \, d\tau \qquad (4.60)$$

$$\lesssim o(|t|) + \|\tilde{u}(t)\|_{L^{2}} + \int_{t}^{0} \frac{1}{\lambda^{2}(\tau)} \left[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} + \int |\nabla \epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} \right] d\tau$$

where we have used (4.22) which ensures

$$\left|\mathcal{I}m\int\nabla\chi\,\cdot\nabla\tilde{u}\,\overline{\tilde{u}}\,\,g\,dx\right|\lesssim\|\nabla\tilde{u}\|_{L^2}\,\|\tilde{u}\|_{L^2}\lesssim\|\tilde{u}\|_{L^2}$$

And (4.55) is proved.

Step 2 Averaged in time dispersion.

We now divide (4.55) by |t| and integrate in time. From the pointwise lower bound (4.5) and the dispersive bound (4.15), once recalling $\frac{ds}{dt} = \frac{1}{\lambda^2}$, we have

$$\begin{aligned} \frac{1}{|t|} \int_{t}^{0} \frac{1}{\lambda^{2}(\tau)} \Big[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} + \int |\nabla\epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} \Big] d\tau &\lesssim \int_{t}^{0} \frac{1}{\lambda^{3}(\tau)} \Big[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} + \int |\nabla\epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} \Big] d\tau \\ &\to 0 \quad \text{as } t \to 0 \end{aligned}$$

Similarly, from Cauchy-Schwarz, and then (4.5), (4.15) again

$$\begin{split} \frac{1}{|t|} \int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}}{|\tau|} \, d\tau &\lesssim \Big(\int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}}{|\tau|^{3}} \, d\tau\Big)^{1/2} \lesssim \Big(\int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}}{\lambda^{3}(\tau)} \, d\tau\Big)^{1/2} \\ &\lesssim \Big(\int_{s}^{+\infty} \frac{1}{\lambda} \, \|\epsilon\|_{L^{2}}^{2} \, d\sigma\Big)^{1/2} \to 0 \text{ as } t \to 0 \end{split}$$

Thus, (4.54) is proved, which concludes the proof of Lemma 4.6.

4.6 Control of the modulation parameters.

We now claim the H^1 dispersion (4.54) together with the conservation laws imply the sharpen control of the modulation parameters we already announced.

Proposition 4.7. (Pointwise dispersive bounds)

 $There \ holds \ the \ pointwise \ bounds:$

$$|\beta| + |\alpha| + \|\epsilon\|_{H^1} \lesssim \lambda^2 \tag{4.61}$$

$$\left|\frac{b(s)}{\lambda(s)} - \frac{1}{C_0}\right| \lesssim \lambda^2 \tag{4.62}$$

$$\lambda(t) = -\frac{t}{C_0} + \mathcal{O}(|t|^3), \quad \text{for } t_0 \le t \le 0.$$
(4.63)

Moreover, there exists $\gamma_0 \in \mathbb{R}$ such that :

$$\gamma(t) = -\frac{C_0^2}{t} + \gamma_0 + \mathcal{O}(|t|)$$
(4.64)

Remark 8. Note that (4.61) imply the zero momentum limit $\lim_{t\to 0} \mathcal{I}m \int \nabla u \,\overline{u} = 0$ so (1.13) is proved, and also sharpens the bound (4.22) by a factor λ .

Proof of Proposition 4.7:

Step 1 Control in average of α and β .

We claim

$$\lim_{t \to 0} \frac{1}{|t|} \int_{t}^{0} \left(\frac{1}{|\tau|} \int_{\tau}^{0} \left(\frac{|\alpha|^{2}}{\lambda^{2}} + \frac{|\beta|^{2}}{\lambda^{2}} \right) d\sigma \right) d\tau = 0$$
(4.65)

First, we deal with α . From the pointwise bound (4.5) and the dispersive bound (4.15)

$$\frac{1}{|t|} \int_t^0 \frac{|\alpha|^2}{\lambda^2} \, d\tau \lesssim \int_t^0 \frac{|\alpha|^2}{\lambda^3} \, d\tau = \int_s^{+\infty} \frac{|\alpha|^2}{\lambda} \, d\sigma \ \to 0 \quad \text{as } t \to 0$$

which proves (4.65) as far as α is concerned. Now dealing with β , we have

$$\left(\frac{\alpha \cdot \beta}{\lambda}\right)_{s} = \alpha_{s} \cdot \frac{\beta}{\lambda} + \beta_{s} \cdot \frac{\alpha}{\lambda} - \frac{\alpha \cdot \beta}{\lambda} \frac{\lambda_{s}}{\lambda}$$

$$= 2 \left|\beta\right|^{2} + \left(-b\beta + B_{2}\right) \cdot \frac{\alpha}{\lambda} + b \frac{\alpha \cdot \beta}{\lambda} + \left(\frac{\alpha_{s}}{\lambda} - 2\beta\right) \cdot \beta$$

$$+ \left(\beta_{s} + b\beta - B_{2}\right) \cdot \frac{\alpha}{\lambda} - \left(\frac{\lambda_{s}}{\lambda} + b\right) \frac{\alpha \cdot \beta}{\lambda}$$

$$= 2 \left|\beta\right|^{2} + B_{2} \cdot \frac{\alpha}{\lambda} + \mathcal{O}\left(|Mod(t)|\right)$$

$$= 2 \left|\beta\right|^{2} + c_{0}(\alpha) \cdot \alpha + \mathcal{O}\left(\lambda^{3} + \|\epsilon\|_{L^{2}}^{2}\right)$$

$$(4.66)$$

where we have used (1.42) for definition of B_2 , and (3.24) in the last line. Integrating (4.66) between s and $+\infty$, using (3.21) which imply that $\frac{\alpha \cdot \beta}{\lambda} \to 0$ as $s \to +\infty$, we obtain :

$$2\int_{s}^{+\infty} |\beta|^{2} d\sigma = -\frac{\alpha(s) \cdot \beta(s)}{\lambda(s)} - \int_{s}^{+\infty} c_{0}(\alpha) \cdot \alpha \, d\sigma + \int_{s}^{+\infty} \mathcal{O}\left(\lambda^{3}(\sigma) + \|\epsilon\|_{L^{2}}^{2}\right) d\sigma$$
$$\lesssim |\alpha(s)| + \int_{s}^{+\infty} \left(|\alpha(\sigma)|^{2} + \|\epsilon\|_{L^{2}}^{2}\right) d\sigma + \int_{s}^{+\infty} \lambda^{3}(\sigma) \, d\sigma$$

Now dividing that last estimate by $|\tau|$, changing variables - $\frac{ds}{dt} = \frac{1}{\lambda^2}$ - and using (4.15) yields

$$\begin{aligned} \frac{1}{|\tau|} \int_{\tau}^{0} \frac{|\beta|^2}{\lambda^2} \, d\sigma &\lesssim \frac{|\alpha(\tau)|}{|\tau|} + \frac{1}{|\tau|} \int_{\tau}^{0} \left(|\alpha(\sigma)|^2 + \|\epsilon\|_{L^2}^2 \right) \frac{d\sigma}{\lambda^2(\sigma)} + \frac{1}{|\tau|} \int_{\tau}^{0} \lambda(\sigma) \, d\sigma \\ &\lesssim \frac{|\alpha(\tau)|}{|\tau|} + \int_{\tau}^{0} \left(|\alpha(\sigma)|^2 + \|\epsilon\|_{L^2}^2 \right) \frac{d\sigma}{\lambda^3(\sigma)} + o(1) \\ &\lesssim \frac{|\alpha(\tau)|}{|\tau|} + o(1) \end{aligned}$$

Integrating once more, using Cauchy-Scwharz and (4.15) again, we have

$$\begin{aligned} \frac{1}{|t|} \int_t^0 \frac{1}{|\tau|} \int_\tau^0 \frac{|\beta|^2}{\lambda^2} \, d\sigma &\lesssim \frac{1}{|t|} \int_t^0 \frac{|\alpha(\tau)|}{|\tau|} \, d\tau + o(1) \lesssim \left(\int_t^0 \frac{|\alpha(\tau)|^2}{\lambda^3(\tau)} \, d\tau \right)^{1/2} + o(1) \\ &\lesssim \left(\int_s^{+\infty} \frac{1}{\lambda} \, |\alpha|^2 \, d\sigma \right)^{1/2} + o(1) \to 0 \quad \text{as } t \to 0 \end{aligned}$$

hence (4.65) follows for β as well.

Step 2 Limit of $\frac{b}{\lambda}$ on a subsequence as $t \to 0$.

From (4.65) and (4.54) we have now

$$\lim_{t \to 0} \frac{1}{|t|} \int_t^0 \left(\frac{1}{|\tau|} \int_\tau^0 \left(\frac{|\alpha|^2}{\lambda^2} + \frac{|\beta|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} \right) d\sigma \right) d\tau = 0$$

This obviously imply the existence of a sequence $t_n \to 0$ such that

$$\lim_{n \to +\infty} \frac{|\alpha(t_n)|}{\lambda(t_n)} + \frac{|\beta(t_n)|}{\lambda(t_n)} + \frac{\|\epsilon(t_n)\|_{H^1}}{\lambda(t_n)} = 0$$
(4.67)

Injecting this into (3.31) yields

$$\left|E_1 - \left(\frac{b}{\lambda}\right)^2 (t_n) \frac{1}{8} \int |y|^2 Q^2\right| \lesssim \left(\frac{|\alpha(t_n)|}{\lambda(t_n)}\right)^2 + \left(\frac{|\beta(t_n)|}{\lambda(t_n)}\right)^2 + \left(\frac{\|\epsilon(t_n)\|_{H^1}}{\lambda(t_n)}\right)^2 + \lambda(t_n)$$

so that

$$\lim_{n \to +\infty} \left(\frac{b(t_n)}{\lambda(t_n)}\right)^2 = \frac{E_0 + \frac{1}{8}C_E}{\frac{1}{8}\int |y|^2 Q^2} = \frac{1}{C_0^2}$$
(4.68)

Then, observe from (4.16) the bound

$$\left| \left(\frac{b}{\lambda}\right)_{s} \right| \lesssim \frac{1}{\lambda} \left[\int |\epsilon|^{2} + |\alpha|^{2} \right] + \mathcal{O}(\lambda^{2})$$

$$(4.69)$$

and thus, (4.15) ensures

$$\int_{s}^{+\infty} \left| \left(\frac{b}{\lambda} \right)_{s} \right| d\sigma < +\infty$$

Hence, $\frac{b}{\lambda}$ has a limit as $s \to +\infty$, so that from (4.68)

$$\frac{b}{\lambda} \to \pm \frac{1}{C_0}$$
 as $t \to 0$

Yet that limit cannot be $-\frac{1}{C_0}$, since it would contradict the finite blow up assumption. Indeed, since $\lambda(t) > 0$, and from (3.21), (3.24), one gets

$$\left|\lambda_t + \frac{b}{\lambda}\right| = \frac{1}{\lambda} \left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \lambda \tag{4.70}$$

so that $\frac{b}{\lambda} \to -\frac{1}{C_0}$ would mean at the same time $\lambda \to 0$ and $\lambda_t \ge \frac{1}{2C_0}$ which imply λ cannot reach 0. Eventually

$$\lim_{t \to 0} \frac{b(t)}{\lambda(t)} = \frac{1}{C_0} > 0 \tag{4.71}$$

At last, injecting this into (4.70) and then integrating in time yields in particular the pseudo conformal speed :

$$\lambda(t) = \frac{|t|}{C_0} (1 + o(1)), \quad \text{as } t \to 0$$
(4.72)

Step 3 Improved bounds.

We now claim the knowledge of the limit of $\frac{b}{\lambda}$, together with monotonicity result (4.14) and the conservation of energy will bring a spectacular improvement on the bounds of $\|\epsilon\|_{H^1}$.

Indeed, we integrate the local virial identity (4.14) between t and t_n and then let $t_n \to 0$.

First, the boundary term in t_n is estimated using (3.21), (4.71)

$$-\left(\frac{b(t_n)}{\lambda(t_n)}\right) \int \frac{|y|^2}{4} Q^2 + \frac{1}{2\lambda(t_n)} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi\left(\frac{y}{A}\right) \cdot \nabla \epsilon(t_n) \overline{\epsilon(t_n)} g \, dy$$

$$\rightarrow -\frac{1}{C_0} \int \frac{|y|^2}{4} Q^2 \quad \text{as } t_n \to 0$$
(4.73)

and thus $\forall \; s > 0$

$$\left(\frac{b(s)}{\lambda(s)} - \frac{1}{C_0}\right) \int \frac{|y|^2}{4} Q^2 - \frac{1}{2\lambda(s)} \mathcal{I}m \int A G(\lambda y + \alpha) \nabla \phi\left(\frac{y}{A}\right) \cdot \nabla \epsilon(s) \overline{\epsilon(s)} g \, dy$$

$$\geq \int_s^{+\infty} \left(\frac{c}{\lambda} \left[|\alpha|^2 + \|\epsilon\|_{L^2}^2 \right] + \mathcal{O}\left(\mathcal{P}^3 + |\beta|^2 + \|\epsilon\|_{H^1}^2 \right) \right) d\sigma$$

$$(2.21)$$

so that from (3.21)

$$\int_{s}^{+\infty} \frac{1}{\lambda} \left[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} \right] d\sigma$$

$$\lesssim \left(\frac{b(s)}{\lambda(s)} - \frac{1}{C_{0}} \right) + \|\epsilon\|_{H^{1}} + \int_{s}^{+\infty} \left(\lambda^{3} + |\beta|^{2} + \|\epsilon\|_{H^{1}}^{2} \right) d\sigma$$
(4.74)

We now recall (3.31) which implies

$$|\beta|^2 + |\alpha|^2 + \|\epsilon\|_{H^1}^2 \lesssim \frac{\lambda^2}{C_0^2} - b^2 + \lambda^4$$
(4.75)

and dividing by λ , we obtain

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} \lesssim \frac{1}{C_0^2} - \frac{b^2}{\lambda^2} + \lambda^2$$
(4.76)

Now, multiplying (4.74) by $\frac{b(s)}{\lambda(s)} + \frac{1}{C_0}$ and adding to (4.76), noticing the terms $\frac{b^2}{\lambda^2} - \frac{1}{C_0}$ cancel each other yields

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} + \int_s^{+\infty} \frac{1}{\lambda} \left[|\alpha|^2 + \|\epsilon\|_{L^2}^2 \right] d\sigma$$

$$\lesssim \lambda^2 + \int_s^{+\infty} \left(\lambda^3 + |\beta|^2 + \|\epsilon\|_{H^1}^2 \right) d\sigma$$

$$\lesssim \lambda^2 + \int_s^{+\infty} \lambda^2 \, d\sigma \lesssim |t| \lesssim \lambda$$
(4.77)

where we have used both (3.21) and (4.72). This yields the improved bound

$$|\beta|^2 + \|\epsilon\|_{H^1}^2 \lesssim \lambda^3$$

which reinjected into (4.77), using (4.72) once more, now shows

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} + \int_s^{+\infty} \frac{1}{\lambda} \left[|\alpha|^2 + \|\epsilon\|_{L^2}^2 \right] d\sigma$$

$$\lesssim \lambda^2 + \int_s^{+\infty} \left(\lambda^3 + |\beta|^2 + \|\epsilon\|_{H^1}^2 \right) d\sigma$$

$$\lesssim \lambda^2 + \int_s^{+\infty} \lambda^3 d\sigma$$

$$\lesssim \lambda^2 + \int_t^0 \lambda \, d\tau \lesssim \lambda^2$$
(4.78)

This proves the pointwise bound (4.61).

Now let's turn to the proof of (4.62). Integrating the localized virial identity (4.14) between s and $+\infty$, using (4.71) and (4.78) we may estimate

$$\left|\frac{b(s)}{\lambda(s)} - \frac{1}{C_0}\right| \lesssim \int_s^{+\infty} \frac{1}{\lambda} \left[|\alpha|^2 + \|\epsilon\|_{L^2}^2 \right] d\sigma + \|\epsilon\|_{H^1} + \int_s^{+\infty} \lambda^3 \, d\sigma$$
$$\lesssim \lambda^2$$

which proves (4.62). Then, (4.63) simply follows from (3.24), (4.61) and (4.62):

$$\mathcal{O}(\lambda^3) = \frac{\lambda_s}{\lambda} + b = \lambda \left(\lambda_t + \frac{b}{\lambda}\right) = \lambda \left(\lambda_t + \frac{1}{C_0} + \mathcal{O}(\lambda^2)\right)$$

and thus $\lambda_t + \frac{1}{C_0} = \mathcal{O}(|t|^2)$ (4.79)

which proves (4.63) by integration in time. Eventually, using also (3.24), (4.61) this implies

$$\begin{aligned} \left| \frac{d}{dt} \left(\gamma + \frac{C_0^2}{t} \right) \right| &= \frac{1}{\lambda^2} \left| \gamma_s - \frac{C_0^2 \lambda^2}{t^2} \right| = \frac{1}{\lambda^2} \left| \tilde{\gamma}_s - \left(\frac{C_0^2 \lambda^2}{t^2} - 1 \right) \right| \\ &= \frac{1}{\lambda^2} \left(|\beta|^2 + \mathcal{O} \left(|t|^2 + |\alpha|^2 + |Mod(t)| \right) \right) \lesssim 1 \end{aligned}$$

which finally proves (4.64).

This concludes the proof of Proposition 4.7.

Remark 9. We have finally proved the \mathcal{P} parameters are not of same orders, and that compared to λ , α and β are of order 2, b is of order 1 - that is $b \sim \lambda$ - and $\tilde{\gamma}$ is of order -1.

This ends the proof of (1.31) and (3.77).

5 Uniqueness

Once again, we follow linearly the paper [RS11]. So we now look forward proving the solution we built in Proposition 3.7 is actually the only one blowing up at T = 0, $\alpha^* = 0$, with energy E_0 and phase parameter γ_0 given by (4.64). In the sequel, we will call u_c that solution given by Proposition 3.7.

So far, we have proved $b \sim \lambda$ which provides the exact blow up speed, and we have obtained the dispersive behavior for the remains \tilde{u} of the approximation

$$\tilde{u} \to 0$$
, in H^1 as $t \to 0$

We now need to prove that given u a solution of (1.1)-(1.2), we have $u = u_c$. Again, as in [RS11], we will proceed in two steps.

First we will show the refined estimates of Proposition 4.7 together with the Backward Propagation of Smallness achieved in Lemma 3.6 imply the strong H^1 convergence

$$u_c - u \to 0$$
, in H^1 as $t \to 0$

Then, we will have to show the estimates obtained for $u - u_c$ will be strong enough to treat pertubatively the growth induced by the null space of $L^{[g]} = (L^{[g]}_+, L^{[g]}_-)$ when linearizing the equation close to u_c and running the estimates of Lemma 3.5.

Both these steps will make clear why the construction of an approximate solutions with an error term of at most $\mathcal{O}(\lambda^5)$ order was needed.

5.1 H^1 convergence to the critical element.

We claim the following dispersive property which somehow uses all previous estimates on the solution and is a key result for the proof of uniqueness.

Lemma 5.1. (H^1 convergence to the critical element)

There holds the strong convergence at blow up time :

$$u_c - u \to 0, \quad in \quad H^1 \quad as \quad t \to 0$$

$$(5.1)$$

More precisely

$$\left\|\nabla(u-u_c)\right\|_{L^2} + \frac{\|u-u_c\|_{L^2}}{|t|} \lesssim |t|^3 \quad as \quad t \to 0$$
(5.2)

Proof of Lemma 5.1

Step 1 Backward propagation of smallness and improved bounds on the solution.

We claim that the bounds of Proposition 4.7 coupled with Lemma 3.6 imply a refined bound

$$\|\tilde{u}\|_{L^2} \lesssim \lambda^4(t), \quad \|\nabla \tilde{u}\|_{L^2} \lesssim \lambda^3(t) \tag{5.3}$$

To prove it, first decompose u according to the geometrical decomposition (3.1). Then, let $t_n \to 0$ be an increasing sequence of times. From conservation of mass (3.26) we have

$$\|\epsilon\|_{L^2} \lesssim \left|2 \operatorname{\mathcal{R}e}\left(\epsilon, \overline{Q_{\mathcal{P}}}\right) + \int |\epsilon|^2\right| \lesssim |\alpha|^2 + \lambda^4$$

so that from Proposition 4.7

$$\|\tilde{u}\|_{L^2} \lesssim (1+|\alpha|^2) \|\epsilon\|_{L^2} \lesssim \lambda^4$$

Still thanks to Proposition 4.7, it is now clear that assumptions (3.81)-(3.83) of Lemma 3.6 are satisfied at any t_n . In particular, there is a time $t_0 < 0$ such that we have (3.84) for any $t_0 \le t \le t_n$:

$$\|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{\|\tilde{u}(t)\|_{L^2}^2}{\lambda^2(t)} \lesssim \|\nabla \tilde{u}(t_n)\|_{L^2}^2 + \frac{\|\tilde{u}(t_n)\|_{L^2}^2}{\lambda^2(t_n)} + \lambda^6(t)$$
(5.4)

Eventually, let $n \to +\infty$ and (4.61) yields (5.3).

Step 2 Comparison between the modulation parameters of u and u_c .

Let $b, \lambda, \alpha, \beta, \gamma$ denote the modulation parameters of u and $b, \lambda_c, \alpha_c, \beta_c, \gamma_c$ denote the modulation parameters of u_c . We claim

$$\frac{|\lambda - \lambda_c|}{|t|} + |b - b_c| + \frac{|\alpha - \alpha_c|}{|t|} + |\beta - \beta_c| + |\gamma - \gamma_c| \lesssim |t|^4$$

$$(5.5)$$

The proof of (5.5) is rather technical, and will be led later according to what is done in Appendix C of [RS11].

Step 3 Comparison between u and u_c .

(5.2) is now a simple consequence of (5.3) and (5.5). Indeed

$$\begin{aligned} &\frac{\|u - u_c\|_{L^2}}{|t|} + \|\nabla(u - u_c)\|_{L^2} \\ &\lesssim \frac{\|\tilde{u}\|_{L^2}}{|t|} + \|\nabla\tilde{u}\|_{L^2} + \frac{\|\tilde{u}_c\|_{L^2}}{|t|} + \|\nabla\tilde{u}_c\|_{L^2} + \frac{\|\tilde{Q} - \tilde{Q}_c\|_{L^2}}{|t|} + \|\nabla(\tilde{Q} - \tilde{Q}_c)\|_{L^2} \end{aligned}$$

Thus, thanks to (5.5) a simple computation yields

$$\|\tilde{Q} - \tilde{Q}_c\|_{L^2} \lesssim |t|^4, \quad \|\nabla(\tilde{Q} - \tilde{Q}_c)\|_{L^2} \lesssim |t|^3$$

and (3.111), (5.3) then imply (5.2) and concludes the proof of Lemma 5.1.

5.2 Energy estimates for the flow near u_c .

Let us now decompose :

$$u = u_c + \tilde{\tilde{u}}, \quad \tilde{\tilde{u}}(t, x) = \frac{1}{k(\alpha_c(t))^{1/2} \lambda_c(t)} \epsilon \left(t, \frac{x - \alpha_c(t)}{\lambda_c(t)}\right) e^{i \gamma_c(t)}$$
(5.6)

Here there is no further orthogonality conditions on ϵ , neither is there modulations equations on the parameters. Indeed, there is no uniform well localized bounds on the \tilde{u}_c part of u_c , and the only control there is on the $\partial_t \tilde{u}_c$ part is given by the $H^{3/2}$ bound (3.111). However, we claim from (3.111) and (5.2) that

$$\left\|\tilde{\tilde{u}}\right\|_{L^2} \lesssim \lambda^4, \quad \left\|\tilde{\tilde{u}}\right\|_{H^1} \lesssim \lambda^3 \tag{5.7}$$

is a sharp enough bound to treat pertubatively the instability generated by the null space of L. Let

$$N(t) := \sup_{t < \tau < 0} \left(\left\| \tilde{\tilde{u}}(\tau) \right\|_{H^1}^2 + \frac{\left\| \tilde{\tilde{u}}(\tau) \right\|_{L^2}^2}{\lambda_c^2(\tau)} \right)$$
(5.8)

and

$$Scal(t) := \left(\epsilon_1, Q\right)^2 + \left(\epsilon_2, \Lambda Q\right)^2 + \left(\epsilon_1, |y|^2 Q\right)^2 + \left(\epsilon_2, \varrho\right)^2 + \left(\epsilon_1, y Q\right)^2 + \left(\epsilon_2, \nabla Q\right)^2$$

$$(5.9)$$

We first claim the following energy bound :

Lemma 5.2. There holds for t close enough to 0:

$$N(t) \lesssim \sup_{t < \tau < 0} \frac{Scal(\tau)}{\lambda_c^2(\tau)} + \int_t^0 \frac{Scal(\tau)}{\lambda_c^3(\tau)} d\tau$$
(5.10)

Proof of Lemma 5.2

It is a consequence of the energy estimate (3.66) together with the a priori bound (3.111).

Step 1 Application of Lemma 3.5.

Let $\mathcal{I}(\tilde{\tilde{u}})$ be given by (3.65), we claim that

$$\frac{\left\|\tilde{\tilde{u}}\right\|_{L^{2}}^{2}}{\lambda_{c}^{3}} + \mathcal{O}\left(N(t) + \frac{Scal(t)}{\lambda_{c}^{3}}\right) \lesssim \frac{d\mathcal{I}}{dt}$$
(5.11)

We are willing to use Lemma 3.5 with $w = u_c = (u_c)_1 + i (u_c)_2$, then the bound (3.59) holds from (3.111) and ψ given by (3.58) is now identically zero, since u_c is an exact solution. Hence (3.66) becomes

$$\begin{aligned} \frac{d\mathcal{I}}{dt} &= -\frac{1}{\lambda_c^2} \mathcal{I}m\left(\int k(x) \, u_c^2 \, \bar{\tilde{u}}^2 \, g \, dx\right) - \mathcal{R}e\left(\int k(x) \, \partial_t u_c \, \overline{(2 \, |\tilde{\tilde{u}}|^2 \, u_c + \tilde{\tilde{u}}^2 \, \overline{u_c})} \, g \, dx\right) \\ &+ \frac{b_c}{\lambda_c^2} \left[\int \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} \, g \, dx + \mathcal{R}e \int \frac{1}{g^2} \left(g \, G\nabla\right)^2 \phi\left(\frac{x - \alpha_c}{A \, \lambda_c}\right) . \left(\nabla \tilde{\tilde{u}} \, , \, \overline{\nabla \tilde{\tilde{u}}}\right) \, dx \\ &- \frac{1}{4A^2} \int \Delta_G \Delta_{G,g,2} \, \phi\left(\frac{x - \alpha_c}{A \, \lambda_c}\right) \, \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} \, dx \\ &+ \lambda_c \, \mathcal{R}e \int A \, G \, \nabla \phi\left(\frac{x - \alpha_c}{A \, \lambda_c}\right) . \, k(x) \left(2 \, u_c \, |\tilde{\tilde{u}}|^2 + \overline{u_c} \, \tilde{\tilde{u}}^2\right) \, \nabla \overline{u_c} \, g \, dx \right] \\ &+ \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2\right) \end{aligned} \tag{5.12}$$

Remind we do not have good enough well localized bounds on \tilde{u}_c , so we need to rely on (3.111). Now we consider the first two terms in the right-hand side of (5.12) and expand $u_c = \tilde{Q}_c + \tilde{u}_c$, with $\tilde{Q}_c = \tilde{\Sigma}_c + i \tilde{\Theta}_c$

$$-\frac{1}{\lambda_{c}^{2}}\mathcal{I}m\left(\int k(x) u_{c}^{2}\overline{\tilde{u}}^{2} g \, dx\right) - \mathcal{R}e\left(\int k(x) \partial_{t}u_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} u_{c} + \tilde{\tilde{u}}^{2} \overline{u_{c}})} g \, dx\right)$$

$$= -\frac{1}{\lambda_{c}^{2}}\mathcal{I}m\left(\int k(x) \tilde{Q}_{c}^{2}\overline{\tilde{u}}^{2} g \, dx\right) - \mathcal{R}e\left(\int k(x) \partial_{t}\tilde{Q}_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} \tilde{Q}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{Q}_{c}})} g \, dx\right)$$

$$-\frac{1}{\lambda_{c}^{2}}\mathcal{I}m\left(\int k(x) \left(2 \tilde{u}_{c} \tilde{Q}_{c} + \tilde{u}_{c}^{2}\right)\overline{\tilde{u}}^{2} g \, dx\right) - \mathcal{R}e\left(\int k(x) \partial_{t}\tilde{Q}_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} \tilde{u}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{u}_{c}})} g \, dx\right)$$

$$-\mathcal{R}e\left(\int k(x) \partial_{t}\tilde{u}_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} \tilde{Q}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{Q}_{c}})} g \, dx\right) - \mathcal{R}e\left(\int k(x) \partial_{t}\tilde{u}_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} \tilde{u}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{u}_{c}})} g \, dx\right)$$

$$(5.13)$$

Now, we may rewrite the first two terms in the right-hand side of (5.13) the same way that led to (3.100)

$$-\frac{1}{\lambda_c^2} \mathcal{I}m\left(\int k(x) \tilde{Q}_c^2 \overline{\tilde{u}}^2 g \, dx\right) - \mathcal{R}e\left(\int k(x) \partial_t \tilde{Q}_c \overline{(2 |\tilde{u}|^2 \tilde{Q}_c + \tilde{u}^2 \overline{\tilde{Q}_c})} g \, dx\right)$$

$$= -\frac{b_c}{\lambda_c^2} \int k(x) \left[\left(|\tilde{Q}_c|^2 + 2 \tilde{\Sigma}_c^2 \right) \tilde{u}_1^2 + 4 \tilde{\Sigma}_c \tilde{\Theta}_c \tilde{\tilde{u}}_1 \tilde{\tilde{u}}_2 + \left(|\tilde{Q}_c|^2 + 2 \tilde{\Theta}_c^2 \right) \tilde{u}_2^2 \right] g \, dx$$

$$- \frac{b_c}{\lambda_c} \mathcal{R}e \int \left(\frac{x - \alpha_c}{\lambda_c} \right) k(x) \left(2 |\tilde{\tilde{u}}|^2 \tilde{Q}_c + \tilde{\tilde{u}}^2 \overline{\tilde{Q}_c} \right) \cdot \overline{\nabla \tilde{Q}_c} g \, dx$$

$$+ \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2 \right)$$
(5.14)

We then treat the next two terms in the right-hand side of (5.13) by Hölder estimates and (3.111) to get

$$\left| -\frac{1}{\lambda_c^2} \mathcal{I}m\left(\int k(x) \left(2\,\tilde{u}_c\,\tilde{Q}_c + \tilde{u}_c^2\right)\overline{\tilde{u}}^2 \,g\,dx\right) - \mathcal{R}e\left(\int k(x)\,\partial_t\tilde{Q}_c\,\overline{(2\,|\tilde{\tilde{u}}|^2\,\tilde{u}_c + \tilde{\tilde{u}}^2\,\overline{\tilde{u}_c})} \,g\,dx\right) \right|$$

$$\lesssim \frac{1}{\lambda_c^2} \,\|\tilde{Q}_c\|_{L^\infty} \,\|\tilde{u}_c\|_{L^2} \,\|\tilde{\tilde{u}}\|_{L^4}^2 + \frac{1}{\lambda_c^2} \,\|\tilde{u}_c\|_{L^4}^2 \,\|\tilde{\tilde{u}}\|_{L^4}^2 + \|\partial_t\tilde{Q}_c\|_{L^\infty} \,\|\tilde{u}_c\|_{L^2} \,\|\tilde{\tilde{u}}\|_{L^4}^2$$

$$(5.15)$$

furthermore, recall as in computation (3.99)

$$\partial_t \tilde{Q}_c = -\frac{(\alpha_c)_t \cdot \nabla k(\alpha_c)}{2 k(\alpha_c)} \tilde{Q}_c - \frac{(\lambda_c)_t}{\lambda_c} \tilde{Q}_c - \frac{\lambda_c (\alpha_c)_t + (\lambda_c)_t (x - \alpha_c)}{k(\alpha_c)^{1/2} \lambda_c^3} \cdot \nabla Q_{\mathcal{P}_c} \left(\frac{x - \alpha_c}{\lambda_c}\right) e^{i \gamma_c} + i (\gamma_c)_t \tilde{Q}_c + \frac{1}{k(\alpha_c)^{1/2} \lambda_c} (\mathcal{P}_c)_t \frac{\partial Q_{\mathcal{P}_c}}{\partial \mathcal{P}_c} \left(\frac{x - \alpha_c}{\lambda_c}\right) e^{i \gamma_c}$$
(5.16)

which is to be estimated only with bounds (3.111), and one may check the worst term is generated by $\frac{(\gamma_c)_t}{\lambda_c}$ in the last term so that

$$\|\partial_t \tilde{Q}_c\|_{L^{\infty}} \lesssim \frac{1}{\lambda_c^3}$$

thus, injecting into (5.15) we obtain

$$\left| -\frac{1}{\lambda_c^2} \mathcal{I}m\left(\int k(x) \left(2\,\tilde{u}_c\,\tilde{Q}_c + \tilde{u}_c^2\right)\overline{\tilde{u}}^2 \,g\,dx\right) - \mathcal{R}e\left(\int k(x)\,\partial_t\tilde{Q}_c\,\overline{(2\,|\tilde{\tilde{u}}|^2\,\tilde{u}_c + \tilde{\tilde{u}}^2\,\overline{\tilde{u}_c})} \,g\,dx\right) \right|$$

$$\lesssim \frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2$$

$$(5.17)$$

To deal with the last two terms in the right-hand side of (5.13), we need to focus a bit more on the equation satisfied by \tilde{u}_c and its remain

$$i \partial_t \tilde{u}_c = -\mathcal{L} \tilde{u}_c - k(x) \left(|u_c|^2 u_c - |\tilde{Q}_c|^2 \tilde{Q}_c \right) - \tilde{\psi}_c$$

$$\tilde{\psi}_c = i \partial_t \tilde{Q}_c + \mathcal{L} \tilde{Q}_c + k(x) |\tilde{Q}_c|^2 \tilde{Q}_c$$
(5.18)

Expanding this term like we did with (3.102), then using (3.24) and (3.111)

$$\|\tilde{\psi}_c\|_{L^2} \lesssim \frac{Mod_c(t) + |\alpha_c|^2 + \lambda_c^5}{\lambda_c^2} \lesssim \lambda_c^2$$

Now, using this together with (5.18), integration by parts, Hölder estimates and the $H^{3/2}$ bound in (3.111) yields

$$\left| -\mathcal{R}e\left(\int k(x) \,\partial_t \tilde{u}_c \,\overline{(2\,|\tilde{\tilde{u}}|^2\,\tilde{Q}_c + \tilde{\tilde{u}}^2\,\overline{\tilde{Q}_c})} \,g\,dx\right) - \mathcal{R}e\left(\int k(x) \,\partial_t \tilde{u}_c \,\overline{(2\,|\tilde{\tilde{u}}|^2\,\tilde{u}_c + \tilde{\tilde{u}}^2\,\overline{\tilde{u}_c})} \,g\,dx\right) \right|$$

$$\lesssim \|\tilde{u}_c\|_{H^{3/2}} \left[\|2\,|\tilde{\tilde{u}}|^2\,\tilde{Q}_c + \tilde{\tilde{u}}^2\,\overline{\tilde{Q}_c}\|_{H^{1/2}} + \|2\,|\tilde{\tilde{u}}|^2\,\tilde{u}_c + \tilde{\tilde{u}}^2\,\overline{\tilde{u}_c}\|_{H^{1/2}} \right]$$

$$+ \|k(x)\left(|u_c|^2\,u_c - |\tilde{Q}_c|^2\,\tilde{Q}_c\right) + \tilde{\psi}_c\|_{L^2} \left(1 + \|V\|_{L^{\infty}}\right) \left[\|\tilde{Q}_c\|_{L^{\infty}} \|\tilde{\tilde{u}}\|_{L^4}^2 + \|\tilde{u}_c\|_{L^6} \|\tilde{\tilde{u}}\|_{L^6}^2 \right]$$

$$\lesssim \frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2$$

$$(5.19)$$

There is but one last term in the right-hand side of (5.12) we need to compute

$$\mathcal{R}e\left(\int A G(x) \,\nabla\phi\left(\frac{x-\alpha_c}{A\,\lambda_c}\right) k(x) \left(2\,|\tilde{\tilde{u}}|^2 \,u_c + \tilde{\tilde{u}}^2 \,\overline{u_c}\right) . \,\overline{\nabla u_c} \,g \,dx\right)$$

$$= \mathcal{R}e\left(\int A G(x) \,\nabla\phi\left(\frac{x-\alpha_c}{A\,\lambda_c}\right) k(x) \left(2\,|\tilde{\tilde{u}}|^2 \,\tilde{Q}_c + \tilde{\tilde{u}}^2 \,\overline{\tilde{Q}_c}\right) . \,\overline{\nabla \tilde{Q}_c} \,g \,dx\right) + Error$$
(5.20)

where the remaining Error term may be handled with Hölder estimates and (3.111)

$$\begin{aligned} |Error| &\lesssim \left| \mathcal{R}e\left(\int AG(x) \nabla \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) k(x) \left(2 \, |\tilde{\tilde{u}}|^2 \, \tilde{u}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{u}_c} \right) . \, \overline{\nabla \tilde{Q}_c} \, g \, dx \right) \right| \\ &+ \left| \mathcal{R}e\left(\int AG(x) \, \nabla \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) k(x) \left(2 \, |\tilde{\tilde{u}}|^2 \, \widetilde{Q}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{Q}_c} \right) . \, \overline{\nabla \tilde{u}_c} \, g \, dx \right) \right| \\ &+ \left| \mathcal{R}e\left(\int AG(x) \, \nabla \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) k(x) \left(2 \, |\tilde{\tilde{u}}|^2 \, \widetilde{u}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{u}_c} \right) . \, \overline{\nabla \tilde{u}_c} \, g \, dx \right) \right| \\ &\lesssim \left(\| \nabla \widetilde{Q}_c \|_{L^\infty} \| \widetilde{u}_c \|_{L^2} + \| \widetilde{Q}_c \|_{L^2} \| \nabla \widetilde{u}_c \|_{L^2} \right) \| \tilde{\tilde{u}} \|_{L^4}^2 + \| \widetilde{u}_c \|_{L^6} \| \nabla \widetilde{u}_c \|_{L^2} \| \tilde{\tilde{u}} \|_{L^6}^2 \\ &\lesssim \frac{\| \tilde{\tilde{u}} \|_{L^2}^2}{\lambda_c^2} + \| \tilde{\tilde{u}} \|_{H^1}^2 \end{aligned}$$

$$(5.21)$$

Eventually, summing everything up, we obtain the following estimate

$$\begin{aligned} \frac{d\mathcal{I}}{dt} &= \frac{b_c}{\lambda_c^2} \left[\int \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} g \, dx + \mathcal{R}e \int \frac{1}{g^2} \left(g \, G \nabla \right)^2 \phi \left(\frac{x - \alpha}{A \lambda} \right) . \left(\nabla \tilde{\tilde{u}} \,, \overline{\nabla \tilde{\tilde{u}}} \right) \, dx \\ &- \frac{1}{4A^2} \int \Delta_G \Delta_{G,g,2} \, \phi \left(\frac{x - \alpha}{A \lambda} \right) \, \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} \, dx \\ &- \int k(x) \left[\left(|\tilde{Q}_c|^2 + 2 \, \tilde{\Sigma}_c^2 \right) \tilde{\tilde{u}}_1^2 + 4 \, \tilde{\Sigma}_c \, \tilde{\Theta}_c \, \tilde{\tilde{u}}_1 \, \tilde{\tilde{u}}_2 + \left(|\tilde{Q}_c|^2 + 2 \, \tilde{\Theta}_c^2 \right) \tilde{\tilde{u}}_2^2 \right] g \, dx \end{aligned}$$
(5.22)
$$&+ \lambda_c \, \mathcal{R}e \left(\int \left\{ A \, G(x) \, \nabla \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) - \left(\frac{x - \alpha_c}{\lambda_c} \right) \right\} k(x) \left(2 \, |\tilde{\tilde{u}}|^2 \, \tilde{Q}_c + \tilde{\tilde{u}}^2 \, \tilde{Q}_c \right) . \, \overline{\nabla \tilde{Q}_c} \, g \, dx \right) \right] \\&+ \mathcal{O} \left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2 \right) \end{aligned}$$

changing variables with (5.6), this may be rewritten

$$\begin{split} \frac{d\mathcal{I}}{dt} &= \frac{b_c}{k(\alpha_c)\,\lambda_c^4} \left[\int |\epsilon|^2 \; g \, dy + \int |\nabla \epsilon|^2 \; dy - \frac{1}{4A^2} \int \Delta_G \Delta_{G,g,2} \, \phi\left(\frac{y}{A}\right) |\epsilon|^2 \; dy \\ &+ \mathcal{R}e \int \left[\frac{1}{g^2} \left(g \, G \nabla \right)^2 \phi\left(\frac{y}{A}\right) - Id \right] \cdot \left(\nabla \epsilon \,, \, \overline{\nabla \epsilon} \right) \; dy \\ &- \int k(\lambda_c y + \alpha_c) \left[\left(3 Q^2 + \mathcal{O}(\lambda_c^2 e^{-|y|}) \right) \epsilon_1^2 + \mathcal{O}(\lambda_c^2 e^{-|y|}) \epsilon_1 \epsilon_2 + \left(Q^2 + \mathcal{O}(\lambda_c^2 e^{-|y|}) \right) \epsilon_2^2 \right] g \, dy \\ &+ \mathcal{R}e \; \int k(\lambda_c y + \alpha_c) \left[2 |\epsilon|^2 \, Q_{\mathcal{P}_c} + \epsilon^2 \, \overline{Q}_{\mathcal{P}_c} \right] \left\{ A \, G(\lambda_c y + \alpha_c) \, \nabla \phi\left(\frac{y}{A}\right) - y \right\} \cdot \overline{\nabla Q_{\mathcal{P}_c}} \; g \, dy \right] \\ &+ \mathcal{O}\left(\frac{\|\tilde{u}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{u}\|_{H^1}^2 \right) \end{split}$$

so that

$$\begin{split} \frac{d\mathcal{I}}{dt} &= \frac{b_c}{k(\alpha_c)\,\lambda_c^4} \left[\left(L_+\epsilon_1 \,,\,\epsilon_1 \right) + \left(L_-\epsilon_2 \,,\,\epsilon_2 \right) - \frac{1}{4A^2} \int \Delta_G \Delta_{G,g,2} \,\phi\!\left(\frac{y}{A}\right) |\epsilon|^2 \,\,dy \\ &+ \mathcal{R}e \int \left[\frac{1}{g^2} \left(g \, G \nabla \right)^2 \!\phi\!\left(\frac{y}{A}\right) - Id \right] \cdot \left(\nabla \epsilon \,,\,\overline{\nabla \epsilon} \right) \,\,dy \\ &+ \mathcal{R}e \,\int k(\lambda_c y + \alpha_c) \left[2 \, |\epsilon|^2 \, Q_{\mathcal{P}_c} + \epsilon^2 \,\overline{Q}_{\mathcal{P}_c} \right] \left\{ A \, G \, \nabla \phi\!\left(\frac{y}{A}\right) - y \right\} \cdot \overline{\nabla Q_{\mathcal{P}_c}} \,\,g \,\,dy \right] \\ &+ \mathcal{O}\!\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2 \right) \end{split}$$

And it now looks clear from the uniform proximity of $Q_{\mathcal{P}_c}$ to Q and the coercivity property (3.32) of Lemma 3.2, that we may conclude as we did for the proof of (3.101)

$$\frac{d\mathcal{I}}{dt} \gtrsim \frac{b_c}{\lambda_c^4} \left[\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} + \int |\epsilon|^2 + \mathcal{O}\left(Scal(t)\right) \right] + \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2\right)$$
(5.23)

This concludes the proof of (5.11).

Step 2 Coercivity of \mathcal{I} .

Recall from (3.65) the formula

$$\begin{split} \mathcal{I} &= \frac{1}{2} \, \int G(x) \, \nabla \tilde{\tilde{u}} \, . \, \overline{\nabla \tilde{\tilde{u}}} \, dx + \frac{1}{2} \, \int \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} \, g \, dx + \frac{b_c}{2\lambda_c} \int A \, G \nabla \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) \, . \, \nabla \tilde{\tilde{u}} \, \overline{\tilde{\tilde{u}}} \, g \, dx \\ &- \frac{1}{4} \, \int k(x) \, |u_c + \tilde{\tilde{u}}|^4 \, g \, dx + \frac{1}{4} \, \int k(x) \, |u_c|^4 \, g \, dx + \int k(x) \, |u_c|^2 \left[(u_c)_1 \, \tilde{\tilde{u}}_1 + (u_c)_2 \, \tilde{\tilde{u}}_2 \right] \, g \, dx \\ &- \frac{1}{2} \, \int V(x) \, |u_c + \tilde{\tilde{u}}|^2 \, g \, dx + \frac{1}{2} \, \int V(x) \, |u_c|^4 \, g \, dx + \int V(x) \left[(u_c)_1 \, \tilde{\tilde{u}}_1 + (u_c)_2 \, \tilde{\tilde{u}}_2 \right] \, g \, dx \end{split}$$

As we did proving (5.11), by expanding $u_c = \tilde{Q}_c + \tilde{u}_c$ one may get thanks to (5.7) the rough upper bound

$$|\mathcal{I}| \lesssim \frac{\|\tilde{\tilde{u}}(t)\|_{L^2}^2}{\lambda_c^2(t)} + \|\tilde{\tilde{u}}(t)\|_{H^1}^2 \to 0 \quad \text{as} \quad t \to 0$$
(5.24)

Then, very similarly to the proof of (3.93), using also control of interaction terms such as those we handled in (5.15), (5.19) and (5.21), we may obtain the lower bound

$$\mathcal{I}(t) \geq \frac{1}{2\lambda_c^2} \left[\left(L_+ \epsilon_1 \,, \, \epsilon_1 \right) + \left(L_- \epsilon_2 \,, \, \epsilon_2 \right) + o\left(\|\epsilon\|_{H^1}^2 \right) \right] \\
\geq \frac{c}{\lambda_c^2} \left[\|\epsilon\|_{L^2}^2 + \|\nabla\epsilon\|_{L^2}^2 - Scal(t) \right] \geq C \left[\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2 - Scal(t) \right]$$
(5.25)

To finally get (5.10), we now integrate (5.11) between t and 0. Using (5.25) we have

$$\int_t^0 \frac{d\mathcal{I}}{d\tau} \, d\tau = \mathcal{I}(0) - \mathcal{I}(t) \lesssim \frac{Scal(t)}{\lambda_c^2} - \frac{\|\tilde{u}\|_{L^2}^2}{\lambda_c^2} - \|\tilde{\tilde{u}}\|_{H^1}^2$$

Hence (5.23) and (5.24) yield

$$\frac{Scal(t)}{\lambda_c^2(t)} - \frac{\|\tilde{\tilde{u}}(t)\|_{L^2}^2}{\lambda_c^2(t)} - \|\tilde{\tilde{u}}(t)\|_{H^1}^2 \gtrsim \int_t^0 \frac{1}{\lambda_c} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^2}^2}{\lambda_c^2(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^1}^2\right) d\tau + \mathcal{O}\bigg[\int_t^0 \left(\frac{Scal(\tau)}{\lambda_c^3(\tau)} + N(\tau)\right) d\tau\bigg]$$

which involves using (5.7)

$$\begin{split} N(t) &\lesssim \sup_{t < \tau < 0} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^{1}}^{2} \right) + \int_{t}^{0} \frac{1}{\lambda_{c}} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^{1}}^{2} \right) d\tau \\ &\lesssim \sup_{t < \tau < 0} \frac{Scal(\tau)}{\lambda_{c}^{2}(\tau)} + \int_{t}^{0} \frac{Scal(\tau)}{\lambda_{c}^{3}(\tau)} \end{split}$$

This concludes the proof of Lemma 5.2.

5.3 Control of the scalar products and proof of Theorem 1.1

There now only remains to control the possible growth of the scalar product terms in (5.10). We claim:

Lemma 5.3. (A priori control of the null space). There holds for t close enough to 0

$$Scal(t) \lesssim |t|^{1/2} |t|^2 N(t)$$
 (5.26)

Let us assume Lemma 5.26 and conclude the proof of Theorem 1.1.

Proof of Theorem 1.1

From (5.10), (5.26) and the law $\lambda_c \sim |t|$ as shown in (3.111), we have for t close enough to 0

$$N(t) \lesssim |t|^{1/2} N(t) + \int_{t}^{0} \frac{N(\tau)}{\sqrt{|\tau|}} d\tau \lesssim |t|^{1/2} N(t)$$

and hence N(t) = 0 for t small enough. From definition (5.8) of N, this yields $u = u_c$ and achieves the proof of the Theorem (1.1).

Proof of Lemma 5.3

The proof follows by deriving the null space close to $Q_{\mathcal{P}_c}$ to sufficiently high order and reintegrating the corresponding modulation equations from blow up time. The worst behavior is on the even terms where the modulation equations are a deformation of the ones for L_+ , L_- , and roughly correspond to the system of ODE 's:

$$\left(\epsilon_{2}, \Lambda Q\right)_{s} = 2\left(\epsilon_{1}, Q\right), \quad \left(\epsilon_{1}, |y|^{2} Q\right)_{s} = -4\left(\epsilon_{2}, \Lambda Q\right), \quad \left(\epsilon_{2}, \varrho\right)_{s} = -\left(\epsilon_{1}, |y|^{2} Q\right)$$

with initial degeneracy provided by the L^2 norm conservation law and the a priori bound (5.7):

$$\left| \left(\epsilon_1 \,, \, Q \right) \right| \lesssim \int |\epsilon|^2 \lesssim \|\epsilon\|_{L^2} \,\lambda_c(t) \,\sqrt{N(t)} \lesssim \lambda_c^4(t) \,\lambda_c(t) \,\sqrt{N(t)}$$

The control of the worst parameter (related to the phase) requires

$$\begin{split} \left| \left(\epsilon_2(s) \,, \, \varrho \right) \right| &\lesssim \int_s^{+\infty} \int_{s_1}^{+\infty} \int_{s_2}^{+\infty} \left| \left(\epsilon_1(s_3) \,, \, Q \right) \right| \, ds_3 \, ds_2 \, ds_1 \\ &\lesssim \lambda_c(s) \, \sqrt{N(s)} \, \int_s^{+\infty} \int_{s_1}^{+\infty} \int_{s_2}^{+\infty} \lambda_c^4(s_3) \, ds_3 \, ds_2 \, ds_1 \\ &\lesssim \frac{\lambda_c(s) \, \sqrt{N(s)}}{s} \end{split}$$

This implies (5.26) and explains why we needed a small enough estimate $\|\epsilon\|_{L^2} \lesssim \lambda_c^4$ in (5.2), (5.7).

Let us now implement the above strategy which requires being careful with respect to polynomial losses, and in particular demands a high order approximation of the null space close to $Q_{\mathcal{P}_c}$.

Step 1 Approximate equation in conformal variables to the order $\mathcal{O}(\lambda_c^4)$.

Let v, w be defined as in the ansatz (1.19)

$$u(t,x) = \frac{1}{k(\alpha_{c}(t))^{1/2}} \frac{1}{\lambda_{c}(t)} v\left(t, \frac{x - \alpha_{c}(t)}{\lambda_{c}}\right) e^{i\gamma_{c}(t)} \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda_{c}^{2}}$$

$$w(s,y) = v(s,y) \ e^{i \ b_{c} \ \frac{|y|^{2}}{4} - i\beta_{c}.y}$$
(5.27)

Then we may compute as we did to get equation (1.25) and show that here again w is solution to

$$\begin{split} i\partial_s w + \mathcal{L}_{\lambda\alpha} w - w + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} w |w|^2 + ((b_c)_s + (b_c)^2) \frac{|y|^2}{4} w \\ &- \left\{ \left((\beta_c)_s + b_c \beta_c \right) \cdot y + i \left[\lambda_c \beta_c \cdot \frac{\nabla k(\alpha_c)}{k(\alpha_c)} + \frac{b_c}{2g} \left(\lambda_c \partial_i G_{ij} y_j + (1 - g) G_{ii} + (G_{ii} - 2) \right) \right] \right\} w \\ &- \frac{1}{g} \left\{ \tilde{G}_{ij} \left[(p_c)_{ij}^{[2]} w + i (b_c y_i \partial_j w - 2 (\beta_c)_i \partial_j w) \right] + i b_c (1 - g) (G_{ij} - I_{ij}) y_i \partial_j w \right\} \\ &- \left(\frac{(\lambda_c)_s}{\lambda_c} + b_c \right) \left[i \Lambda w + 2 b_c \frac{|y|^2}{4} w - \beta_c \cdot y w \right] - \left((\tilde{\gamma}_c)_s - |\beta_c|^2 \right) w \\ &- \left(\frac{(\alpha_c)_s}{\lambda_c} - 2 \beta_c \right) \cdot \left[i \left(\nabla w + \frac{\lambda_c}{2} \frac{\nabla k(\alpha_c)}{k(\alpha_c)} \right) + b_c \frac{y}{2} - \beta_c w \right] = 0 \end{split}$$

where

$$(p_c)_{ij}^{[2]} = (p_c)_{ij}^{[2]}(y) = (b_c)^2 \frac{y_i y_j}{4} - b_c (\beta_c)_i y_j + (\beta_c)_i (\beta_c)_j$$

Doing the same with u_c , we also define v_c and $w_c(s, y) = v_c(s, y) e^{i b_c \frac{|y|^2}{4} - i\beta_c \cdot y}$. We let $u = u_c + \tilde{\tilde{u}}$ and define

$$v = v_c + \epsilon, \quad w = w_c + \tilde{\epsilon}, \quad \text{i.e.} \quad \tilde{\epsilon} = \epsilon \ e^{i b_c \frac{|y|^2}{4} - i\beta_c \cdot y}$$
(5.28)

Since u_c satisfies (1.1), $\tilde{\epsilon}$ is solution to

$$\begin{split} i\partial_{s}\tilde{\epsilon} + \mathcal{L}_{\lambda_{c}\alpha_{c}}\tilde{\epsilon} - \tilde{\epsilon} + \frac{k(\lambda_{c}y + \alpha_{c})}{k(\alpha_{c})} \left(w|w|^{2} - |w_{c}|^{2} w_{c} \right) + \left((b_{c})_{s} + (b_{c})^{2} \right) \frac{|y|^{2}}{4} \tilde{\epsilon} \\ - \left\{ \left((\beta_{c})_{s} + b_{c} \beta_{c} \right) \cdot y + i \left[\lambda_{c} \beta_{c} \cdot \frac{\nabla k(\alpha_{c})}{k(\alpha_{c})} + \frac{b_{c}}{2g} \left(\lambda_{c} \partial_{i}G_{ij}y_{j} + (1 - g) G_{ii} + (G_{ii} - 2) \right) \right] \right\} \tilde{\epsilon} \\ - \frac{1}{g} \left\{ \tilde{G}_{ij} \left[(p_{c})_{ij}^{[2]} \tilde{\epsilon} + i \left(b_{c} y_{i} \partial_{j} \tilde{\epsilon} - 2 \left(\beta_{c} \right)_{i} \partial_{j} \tilde{\epsilon} \right) \right] + i b_{c} \left(1 - g \right) \left(G_{ij} - I_{ij} \right) y_{i} \partial_{j} \tilde{\epsilon} \right\} \\ - \left(\frac{(\lambda_{c})_{s}}{\lambda_{c}} + b_{c} \right) \left[i \Lambda \tilde{\epsilon} + 2 b_{c} \frac{|y|^{2}}{4} \tilde{\epsilon} - \beta_{c} \cdot y \tilde{\epsilon} \right] - \left((\tilde{\gamma}_{c})_{s} - |\beta_{c}|^{2} \right) \tilde{\epsilon} \\ - \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2 \beta_{c} \right) \cdot \left[i \left(\nabla \tilde{\epsilon} + \frac{\lambda_{c}}{2} \frac{\nabla k(\alpha_{c})}{k(\alpha_{c})} \right) + b_{c} \frac{y}{2} - \beta_{c} \tilde{\epsilon} \right] = 0 \end{split}$$
 (5.29)

Then, from (3.24), and (4.61)-(4.63), one has

$$|\alpha_c|^2 + |\beta_c|^2 + |Mod_c(t)| \lesssim \lambda_c^4$$

so that using (3.78), (5.29) may be rewritten as

$$i\partial_{s}\tilde{\epsilon} + \mathcal{L}_{\lambda_{c}\alpha_{c}}\tilde{\epsilon} - \tilde{\epsilon} + \frac{k(\lambda_{c}y + \alpha_{c})}{k(\alpha_{c})} (w|w|^{2} - |w_{c}|^{2} w_{c}) - i b_{c} \tilde{G}_{ij} y_{i} \partial_{j}\tilde{\epsilon} - \left\{ \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y + i \frac{b_{c}}{2} \left[\lambda_{c} \partial_{i} G_{ij} y_{j} + (1 - g) Tr(G) + \left(Tr(G) - 2\right) \right] \right\} \tilde{\epsilon}$$
(5.30)
$$= \mathcal{O} \left(\lambda_{c}^{4} \left(1 + |y|^{2}\right) \tilde{\epsilon} + \lambda_{c}^{4} \left(1 + |y|^{2}\right) \nabla \tilde{\epsilon} \right)$$

or

$$\begin{split} &i\partial_s \tilde{\epsilon} + \mathcal{L}_{\lambda_c \alpha_c} \tilde{\epsilon} - \tilde{\epsilon} + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \left(w|w|^2 - |w_c|^2 w_c \right) \\ &- \left\{ \left(\lambda_c c_0(\alpha_c) + \lambda_c^3 C_3 \right) \cdot y \ \tilde{\epsilon} + i \frac{b_c}{2} \left[\nabla^2 G_{ij}(0) \cdot \left(y, \partial_i(\tilde{\epsilon} y) \right) y_j - \nabla^2 g(0) \cdot \left(y, y \right) \Lambda \tilde{\epsilon} + \frac{1}{2} \nabla^2 \left(Tr(G) \right)(0) \cdot \left(y, y \right) \tilde{\epsilon} \right] \right\} \\ &= \mathcal{O} \left(\lambda_c^4 \left(1 + |y|^2 \right) \tilde{\epsilon} + \lambda_c^4 \left(1 + |y|^2 \right) \nabla \tilde{\epsilon} \right) \end{split}$$

Let $\tilde{\epsilon} = \tilde{\epsilon}_1 + i \tilde{\epsilon}_2$ and $w_c = (w_c)_1 + i (w_c)_2$. We now expand the nonlinear term in (5.30) as well as $w_c = P_{\mathcal{P}_c} + \tilde{\epsilon}_c$ and the expansion of the approximate solution (1.40) to obtain up to order $\mathcal{O}(\lambda_c^4)$

$$-i\partial_{s}\tilde{\epsilon} + M^{(4)}(\tilde{\epsilon}) + i b_{c}\tilde{G}_{ij}y_{i}\partial_{j}\tilde{\epsilon} + \left\{ \left(\lambda_{c}c_{0}(\alpha_{c}) + \lambda_{c}^{3}C_{3}\right) \cdot y + i \frac{b_{c}}{2} \left[\lambda_{c}\partial_{i}G_{ij}y_{j} + (1-g)Tr(G) + \left(Tr(G) - 2\right)\right] \right\}\tilde{\epsilon}$$
(5.31)
$$= -\psi_{c}$$

where $M^{(4)}$ is the fourth order expansion of M defined in (3.17) and

$$M^{(4)}(\tilde{\epsilon}) = M_1^{(4)}(\tilde{\epsilon}) + i M_2^{(4)}(\tilde{\epsilon})$$

which, according to (1.33) and its now simplified version induced by (4.61), are given by

$$M_{1}^{(4)}(\tilde{\epsilon}) = -\mathcal{L}_{\lambda_{c}\alpha_{c}}\tilde{\epsilon}_{1} + \tilde{\epsilon}_{1} - \left[3Q^{2} + 6QT_{2} + \frac{3}{2}\nabla^{2}k(0).(y,y)\lambda_{c}^{2}Q^{2} + 6QT_{3} + 3\nabla^{2}k(0).(y,\alpha_{c})\lambda_{c}Q^{2} + \frac{1}{2}\nabla^{3}k(0).(y,y,y)\lambda_{c}^{3}Q^{2}\right]\tilde{\epsilon}_{1} - 2QS_{3}\tilde{\epsilon}_{2}$$
(5.32)

and

$$M_{2}^{(4)}(\tilde{\epsilon}) = -\mathcal{L}_{\lambda_{c}\alpha_{c}}\tilde{\epsilon}_{2} + \tilde{\epsilon}_{2} - \left[Q^{2} + 2QT_{2} + \frac{1}{2}\nabla^{2}k(0).(y,y)\lambda_{c}^{2}Q^{2} + 2QT_{3} + \nabla^{2}k(0).(y,\alpha_{c})\lambda_{c}Q^{2} + \frac{1}{6}\nabla^{3}k(0).(y,y,y)\lambda_{c}^{3}Q^{2}\right]\tilde{\epsilon}_{2} - 2QS_{3}\tilde{\epsilon}_{1}$$
(5.33)

and where the remainder ψ_c satisfies

$$\psi_c = \mathcal{O}\left(\lambda_c^4 \left(1 + |y|^2\right)\tilde{\epsilon} + \lambda_c^4 \left(1 + |y|^2\right)\nabla\tilde{\epsilon} + \tilde{\epsilon}_c\,\tilde{\epsilon} + \tilde{\epsilon}_c^2\,\tilde{\epsilon} + w_c\,\tilde{\epsilon}^2 + \tilde{\epsilon}^3\right)$$

Step 2 Approximate null space.

Let $f(s, y) = O(e^{-|y|})$ be a smooth well localized slowly time dependent function. Then from equation (5.31)

$$\frac{d}{ds}\mathcal{I}m\left(\tilde{\epsilon}\,,\,\overline{f}\right) = -\mathcal{R}e\left(\tilde{\epsilon}\,,\,\overline{M^{(4)}(f) - i\,\partial_s f + m_3\,f}\right) + \mathcal{O}\left[\left(\psi\,,\,f\right)\right] \tag{5.34}$$

with

$$\left| \left(\psi, f \right) \right| \lesssim \lambda_{c}^{4} \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}_{c}\|_{L^{2}} \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}_{c}\|_{L^{4}}^{2} \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}_{c}\|_{L^{4}} \|\tilde{\epsilon}\|_{L^{4}} \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}\|_{L^{4}}^{2} \|\tilde{\epsilon}\|_{L^{2}}$$

$$\lesssim \lambda_{c}^{4} \|\tilde{\epsilon}\|_{L^{2}}$$

$$(5.35)$$

and where m_3 is the operator

$$m_{3} = \left\{ \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3} \right) \cdot y + i \frac{b_{c}}{2} \left[\lambda_{c} \partial_{i} G_{ij} y_{j} + (1 - g) Tr(G) + \left(Tr(G) - 2 \right) \right] \right\} + i \tilde{G}_{ij} y_{i} \partial_{j}$$

$$= \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3} \right) \cdot y + i \frac{\lambda_{c}^{3}}{2C_{0}} \left[\nabla^{2} G_{ij}(0) \cdot (\partial_{i}, y) y_{j} + \nabla^{2} \left(\frac{1}{2} Tr(G) - g \right) (0) \cdot (y, y) \right]$$

$$+ i \frac{\lambda_{c}^{3}}{2C_{0}} \nabla^{2} \left(G_{ij} - g I_{ij} \right) (0) \cdot (y, y) y_{i} \partial_{j} + \mathcal{O} \left(\lambda_{c}^{4} \right)$$
(5.36)

where we have used (4.62) to get $\frac{b_c \lambda_c^2}{2} = \frac{\lambda_c^3}{2C_0} + \mathcal{O}(\lambda_c^5)$.

We now claim that we can find some real valued smooth well localized functions $A_2(y)$, $B_2(y)$, $D_3(s, y)$, $E_2(s, y)$, $E_3(s, y)$, $F_2(s, y)$ with

$$A_2, B_2 = \mathcal{O}(\lambda_c^2 e^{-|y|}), \quad E_2, F_2 = \mathcal{O}(\lambda_c^2 e^{-|y|}), \quad E_3, D_3 = \mathcal{O}(\lambda_c^3 e^{-|y|})$$

such that we have the following approximate null space relations

$$M^{(4)}\left(\nabla Q + \lambda_c^2 A_2\right) - i \,\partial_s \left(\nabla Q + \lambda_c^2 A_2\right) = -a_1 \,\lambda_c^2 \, y \, Q + \mathcal{O}\left(\lambda_c^3 \, e^{-|y|}\right) \tag{5.37}$$

for some universal constant

$$a_1 > 0$$
 (5.38)

$$M^{(4)}\left[i\left(y\,Q+\lambda_c^2\,B_2\right)\right] - i\,\partial_s\left[i\left(y\,Q+\lambda_c^2\,B_2\right)\right] = -2\,i\left(\nabla Q+\lambda_c^2\,A_2\right) + \mathcal{O}\left(\lambda_c^3\,e^{-|y|}\right)$$
(5.39)

and for the even part

$$M^{(4)} \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) - i \partial_{s} \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) + \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) = -2 P_{\mathcal{P}_{c}} + \left(\lambda_{c} a_{21}(\alpha_{c}) + \lambda_{c}^{3} a_{31}\right) \cdot y Q + i \lambda_{c}^{3} a_{32} \varrho - i \frac{\lambda_{c}^{3}}{2C_{0}} h_{3} + \mathcal{O}\left(\lambda_{c}^{4} e^{-|y|}\right)$$
(5.40)

for a linear map a_{21} on \mathbb{R}^2 , a vector $a_{31} \in \mathbb{R}^2$ and a constant $a_{32} \in \mathbb{R}$, h_3 an even function to be fixed later

$$M^{(4)} [i (|y|^2 P_{\mathcal{P}_c} + F_2)] - i \partial_s [i (|y|^2 P_{\mathcal{P}_c} + F_2)] = -4 i (\Lambda P_{\mathcal{P}_c} + E_2) + i \lambda_c^2 a_{22} \rho + \mathcal{O}(\lambda_c^3 e^{-|y|})$$
(5.41)

for a constant $a_{22} \in \mathbb{R}$, and

$$M^{(4)}(\varrho) = |y|^2 Q + \mathcal{O}(\lambda_c^2 e^{-|y|})$$
(5.42)

The Algebraic derivation of (5.37), (5.39), (5.40), (5.41) and (5.42) is done in Appendix 5.B 5.5.

Step 3 Control of (ϵ_1, yQ) and (ϵ_2, yQ) . Let $A = \nabla Q + \lambda_c^2 A_2$ and $B = yQ + \lambda_c^2 B_2$. We now use (5.34) and (5.35) when f = A on one hand, and when f = iB on the orther hand to get

$$\begin{cases} \left(\tilde{\epsilon}_{2}, A\right)_{s} = a_{1} \lambda_{c}^{2} \left(\tilde{\epsilon}_{1}, y Q\right) + \mathcal{O}\left(\lambda_{c}^{3} e^{-|y|}\right) \\ \left(\tilde{\epsilon}_{1}, B\right)_{s} = -2 \left(\tilde{\epsilon}_{2}, A\right) + \mathcal{O}\left(\lambda_{c}^{3} e^{-|y|}\right) \end{cases}$$

$$(5.43)$$

Then, using the fact $\frac{ds}{dt} = \frac{1}{\lambda_c^2}$ together with (4.62) yields

$$\lambda_c(s) = \frac{C_0}{s} + \mathcal{O}\left(\frac{1}{s^2}\right) \tag{5.44}$$

Rewritting (5.43) in view of (5.44) and $B = yQ + \mathcal{O}(\lambda_c^2)$ thus leads to the following 2-dimensional ODE system

$$\begin{cases} \left(\tilde{\epsilon}_{1}, B\right)_{s} = -2\left(\tilde{\epsilon}_{2}, A\right) + F_{1} \\ \left(\tilde{\epsilon}_{2}, A\right)_{s} = a_{1} \lambda_{c}^{2}\left(\tilde{\epsilon}_{1}, y Q\right) + F_{2} \end{cases} \quad \text{with} \quad F_{1}, F_{2} = \mathcal{O}\left(\frac{\|\tilde{\epsilon}\|_{L^{2}}}{s^{3}}\right) \tag{5.45}$$

As we will do later in Appendix 5.A 5.4, we may now apply Lemma 8.10 with $\zeta = a_1 C_0^2 > 0$ thanks to (5.38), hence

$$\left| \left(\tilde{\epsilon}_1 \,, \, B \right) \right| + \left| \left(\tilde{\epsilon}_2 \,, \, A \right) \right| \lesssim \int_s^{+\infty} \frac{\ln(\sigma)}{\sigma^2} \, \| \tilde{\epsilon} \|_{L^2} \, d\sigma$$

Using the fact that $A = \nabla Q + \mathcal{O}(s^{-2})$ and $B = y Q + \mathcal{O}(s^{-2})$ yields

$$\left| \left(\tilde{\epsilon}_1 \,, \, y \, Q \right) \right| + \left| \left(\tilde{\epsilon}_2 \,, \, \nabla Q \right) \right| \lesssim \frac{\|\tilde{\epsilon}\|_{L^2}}{s^2} + \int_s^{+\infty} \frac{\ln(\sigma)}{\sigma^2} \, \|\tilde{\epsilon}\|_{L^2} \, d\sigma$$

then, the definition (5.8) together with (5.28) and (3.111) imply

$$\|\tilde{\epsilon}(s)\|_{L^2} = k(\alpha_c) \left\|\tilde{\tilde{u}}(s)\right\|_{L^2} \lesssim \frac{\sqrt{N(s)}}{s} \lesssim |t| \sqrt{N(t)}$$
(5.46)

so that we have

$$\begin{split} \left| \left(\epsilon_1(t) \,, \, y \, Q \right) \right| + \left| \left(\epsilon_2(t) \,, \, \nabla Q \right) \right| &\lesssim \frac{\left\| \tilde{\tilde{u}} \right\|_{L^2}}{s^2} + \frac{\sqrt{N(s)}}{s} \, \int_s^{+\infty} \frac{\ln(\sigma)}{\sigma^2} \, d\sigma \\ &\lesssim \frac{\sqrt{N(s)}}{s} \left(\frac{1}{s^2} + \frac{\ln(s)}{s} \right) \\ &\lesssim |t|^{1/2} \, |t| \, \sqrt{N(t)} \end{split}$$

This ends the proof of (5.26) for the odd directions.

Step 4 Control of
$$(\epsilon_1, Q)$$
, $(\epsilon_2, \Lambda Q)$, $(\epsilon_1, |y|^2 Q)$, (ϵ_2, ϱ) .

Notice here the $L^2(dy)$ scalar product will not be accuate enough to properly estimate the remaining terms. From the use of Conservation laws, it is the $L^2(g \, dy)$ scalar products that will be relevant.

We first have from the L^2 conservation law, the critical mass assumption and (5.28)

$$\int |v|^2 g \, dy = \int |w|^2 g \, dy = k(\alpha_c) \int Q^2 \, dy = \int |v_c|^2 g \, dy = \int |w_c|^2 g \, dy$$
$$\int |w|^2 g \, dy = \int |w_c + \tilde{\epsilon}|^2 g \, dy = \int |w_c|^2 g \, dy + \int |\tilde{\epsilon}|^2 g \, dy + 2 \operatorname{\mathcal{R}e} \int \tilde{\epsilon} \, \overline{w_c} g \, dy$$
$$\int |v|^2 g \, dy = \int |v_c + \epsilon|^2 g \, dy = \int |v_c|^2 g \, dy + \int |\epsilon|^2 g \, dy + 2 \operatorname{\mathcal{R}e} \int \epsilon \, \overline{v_c} g \, dy$$
$$(5.7) \text{ and the definition (5.8)}$$

thus from (5.7) and the definition (5.8)

$$\mathcal{R}e\int\tilde{\epsilon}\ \overline{w_c}\ g\ dy = \mathcal{R}e\int\epsilon\ \overline{v_c}\ g\ dy = -\frac{1}{2}\int|\epsilon|^2\ g\ dy = \mathcal{O}\big(\lambda_c^4\|\epsilon\|_{L^2}\big) = \mathcal{O}\big(\lambda_c^4|t|\sqrt{N(t)}\big)$$

Moreover, from (5.28) again and the fact that $\int |P_{\mathcal{P}_c}|^2 g \, dy = \int |Q_{\mathcal{P}_c}|^2 g \, dy$ we have

$$\int |P_{\mathcal{P}_c} + \tilde{\epsilon}|^2 g \, dy = \int |P_{\mathcal{P}_c}|^2 g \, dy + \int |\tilde{\epsilon}|^2 g \, dy + 2 \, \mathcal{R}e \int \tilde{\epsilon} \, \overline{P_{\mathcal{P}_c}} g \, dy$$
$$= \int |Q_{\mathcal{P}_c} + \epsilon|^2 g \, dy = \int |Q_{\mathcal{P}_c}|^2 g \, dy + \int |\epsilon|^2 g \, dy + 2 \, \mathcal{R}e \int \epsilon \, \overline{Q_{\mathcal{P}_c}} g \, dy$$

so that

$$2 \operatorname{\mathcal{R}e} \int \tilde{\epsilon} \ \overline{P_{\mathcal{P}_c}} \ g \, dy = 2 \operatorname{\mathcal{R}e} \int \epsilon \ \overline{Q_{\mathcal{P}_c}} \ g \, dy = -\int |\epsilon|^2 \ g \, dy = \mathcal{O}\left(\lambda_c^4 \left|t\right| \sqrt{N(t)}\right).$$
(5.47)

Going back to (5.34), doing the same computation with the $L^2(g dy)$ scalar product, one needs to be careful, for a remaining derivative part shows up

$$\begin{split} \frac{d}{ds}\mathcal{I}m\Big(\tilde{\epsilon}\,,\,\overline{f}\Big)_{L^{2}(g\,dy)} &= -\mathcal{R}e\int \tilde{\epsilon}\,\,\overline{\left(M^{(4)}(f) - i\,\partial_{s}f + m_{3}\,f\right)}\,\,g\,dy - \mathcal{I}m\int \tilde{\epsilon}\,\,\overline{f}\,\,\left((\lambda_{c})_{s}y + (\alpha_{c})_{s}\right).\,\nabla g\,dy \\ &+ \mathcal{O}\Big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\Big) \\ &= -\mathcal{R}e\int \tilde{\epsilon}\,\,\overline{\left(M^{(4)}(f) - i\,\partial_{s}f + m_{3}\,f\right)}\,\,g\,dy \\ &- \mathcal{I}m\int \tilde{\epsilon}\,\,\overline{f}\,\,\left(\left(\frac{(\lambda_{c})_{s}}{\lambda_{c}} + b_{c}\right)y + \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2\,\beta_{c}\right) - (b_{c}y - 2\,\beta_{c})\right).\,\lambda_{c}\,\nabla g\,dy + \mathcal{O}\Big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\Big) \\ &= -\mathcal{R}e\int \tilde{\epsilon}\,\,\overline{\left(M^{(4)}(f) - i\,\partial_{s}f + m_{3}\,f + i\,\frac{\lambda_{c}^{3}}{C_{0}}\,\nabla^{2}g(0).(y,y)\,f\right)}\,g\,dy + \mathcal{O}\Big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\Big) \end{split}$$

Using (5.34), (5.35) and (5.40) where we take $f = \Lambda P_{\mathcal{P}_c} + E_2 + E_3 + i D_3$, we have

$$\begin{split} &\frac{d}{ds}\mathcal{I}m\Big(\tilde{\epsilon}\,,\overline{\Lambda P_{\mathcal{P}_{c}}+E_{2}+E_{3}+i\,D_{3}}\Big)_{L^{2}(g\,dy)} \\ &=-\mathcal{R}e\int\tilde{\epsilon}\,\overline{\Big(-2\,P_{\mathcal{P}_{c}}+\left(\lambda_{c}\,a_{21}(\alpha_{c})+\lambda_{c}^{3}\,a_{31}\right).\,y\,Q+i\,\lambda_{c}^{3}\,a_{32}\,\varrho-i\,\frac{\lambda_{c}^{3}}{2C_{0}}\,h_{3}+i\,\frac{\lambda_{c}^{3}}{2C_{0}}\left[f_{3}+\nabla^{2}g(0).(y,y)\,\Lambda Q\right]\Big)}\,g\,dy \\ &=2\,\mathcal{R}e\int\tilde{\epsilon}\,\overline{P_{\mathcal{P}_{c}}}\,g\,dy-\left(\lambda_{c}\,a_{21}(\alpha_{c})+\lambda_{c}^{3}\,a_{31}\right).\,\int\tilde{\epsilon}_{1}\,y\,Q\,g\,dy-a_{32}\,\lambda_{c}^{3}\,\int\tilde{\epsilon}_{2}\,\varrho\,g\,dy \\ &+\frac{\lambda_{c}^{3}}{2C_{0}}\,\int\tilde{\epsilon}_{2}\left[f_{3}+\nabla^{2}g(0).(y,y)\,\Lambda Q-h_{3}\right]\,g\,dy+\mathcal{O}\big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\big) \\ &=2\,\mathcal{R}e\int\tilde{\epsilon}\,\overline{P_{\mathcal{P}_{c}}}\,g\,dy-\left(\lambda_{c}\,a_{21}(\alpha_{c})+\lambda_{c}^{3}\,a_{31}\right).\,\left(\tilde{\epsilon}_{1}\,,\,y\,Q\right)-a_{32}\,\lambda_{c}^{3}\,\left(\tilde{\epsilon}_{2}\,,\,\varrho\right)+\mathcal{O}\big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\big) \end{split}$$

with

$$f_{3} = \left[\nabla^{2} G_{ij}(0).(\partial_{i}, y) y_{j} + \nabla^{2} \left(\frac{1}{2} Tr(G) - g\right)(0).(y, y)\right] \Lambda Q + i \frac{\lambda_{c}^{3}}{2C_{0}} \nabla^{2} \left(G_{ij} - g I_{ij}\right)(0).(y, y) y_{i} \partial_{j} \Lambda Q$$

where we have now fixed once and for all

$$h_3 = f_3 + \nabla^2 g(0).(y, y) \Lambda Q$$
(5.48)

(5.44), (5.47), the definition (5.8) of N(t) and the result of the previous step now imply after integration

$$\begin{split} \left| \mathcal{I}m\Big(\tilde{\epsilon}\,,\overline{\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i\,D_{3}}\Big)_{L^{2}(g\,dy)} \right| \\ \lesssim \int_{s}^{+\infty} \left(\left| \mathcal{R}e\int \tilde{\epsilon} \,\overline{P_{\mathcal{P}_{c}}} \,g\,dy \right| + \frac{1}{\sigma^{3}} \left| \left(\tilde{\epsilon}_{1}\,,y\,Q\right) \right| + \frac{1}{\sigma^{3}} \left| \left(\tilde{\epsilon}_{2}\,,\varrho\right) \right| + \frac{\|\tilde{\epsilon}\|_{L^{2}}}{\sigma^{4}} \right) d\sigma \\ \lesssim \int_{s}^{+\infty} \left[\frac{\sqrt{N(\sigma)}}{\sigma} \left(\frac{1}{\sigma^{4}} + \frac{1}{\sigma^{7/2}} \right) + \frac{\|\tilde{\epsilon}(\sigma)\|_{L^{2}}}{\sigma^{3}} + \frac{\|\tilde{\epsilon}(\sigma)\|_{L^{2}}}{\sigma^{4}} \right] d\sigma \\ \lesssim \frac{\sqrt{N(s)}}{s} \int_{s}^{+\infty} \left(\frac{1}{\sigma^{7/2}} + \frac{1}{\sigma^{4}} \right) d\sigma \\ \lesssim |t|^{2} |t| \sqrt{N(t)} \end{split}$$

thus we have proved

$$\left| \mathcal{I}m\left(\tilde{\epsilon}, \overline{\Lambda P_{\mathcal{P}_c} + E_2 + E_3 + i D_3}\right)_{L^2(g \, dy)} \right| \lesssim |t|^2 \, |t| \, \sqrt{N(t)} \lesssim |t|^{1/2} \, |t| \, \sqrt{N(t)} \tag{5.49}$$

Using (5.34), (5.35) and (5.41) where we take $f = i (|y|^2 P_{\mathcal{P}_c} + F_2)$, we have

$$\frac{d}{ds} \mathcal{I}m\left(\tilde{\epsilon}, \overline{i\left(|y|^2 P_{\mathcal{P}c} + F_2\right)}\right)_{L^2(g \, dy)} = -\frac{d}{ds} \mathcal{R}e\left(\tilde{\epsilon}, \overline{|y|^2 P_{\mathcal{P}c} + F_2}\right)_{L^2(g \, dy)}$$
$$= 4 \mathcal{I}m\left(\tilde{\epsilon}, \overline{\Lambda P_{\mathcal{P}c} + E_2}\right)_{L^2(g \, dy)} - \lambda_c^2 a_{22}\left(\tilde{\epsilon}_2, \varrho\right) + \mathcal{O}\left(\lambda_c^3 \|\tilde{\epsilon}\|_{L^2}\right)$$

 $({\bf 5.44})$ and $({\bf 5.49})$ yield

$$\begin{aligned} \left| \mathcal{R}e\left(\tilde{\epsilon}\,,\,\overline{|y|^2 P_{\mathcal{P}_c} + F_2}\right)_{L^2(g\,dy)} \right| &\lesssim \int_s^{+\infty} \left(\left| \mathcal{I}m\left(\tilde{\epsilon}\,,\,\overline{\Lambda P_{\mathcal{P}_c} + E_2}\right)_{L^2(g\,dy)} \right| + \frac{1}{\sigma^2} \left| \left(\tilde{\epsilon}_2\,,\,\varrho\right) \right| + \frac{\|\tilde{\epsilon}\|_{L^2}}{\sigma^3} \right) d\sigma \\ &\lesssim \int_s^{+\infty} \left(\frac{\sqrt{N(\sigma)}}{\sigma} \frac{1}{\sigma^2} + \frac{\|\tilde{\epsilon}(\sigma)\|_{L^2}}{\sigma^2} + \frac{\|\tilde{\epsilon}(\sigma)\|_{L^2}}{\sigma^3} \right) d\sigma \\ &\lesssim |t| \, |t| \, \sqrt{N(t)} \end{aligned} \tag{5.50}$$

Using (5.34), (5.35) and (5.42) where we take $f = \rho$, we have

$$\left(\tilde{\epsilon}_{2}, \varrho\right)_{s} = -\left(\tilde{\epsilon}_{1}, |y|^{2} Q\right) + \mathcal{O}\left(\lambda_{c}^{2} \|\tilde{\epsilon}\|_{L^{2}}\right)$$

$$(5.51)$$

which eventually yields after integration using (5.50), the fact that $P_{\mathcal{P}_c} = Q + \mathcal{O}(|t|^2 e^{-|y|})$, and once agin the definition (5.8)

$$\left| \left(\tilde{\epsilon}_{2}, \varrho \right) \right| \lesssim \int_{s}^{+\infty} \left(\left| \left(\tilde{\epsilon}_{1}(\sigma), |y|^{2} Q \right) \right| + \frac{\|\tilde{\epsilon}\|_{L^{2}}}{\sigma^{2}} \right) d\sigma$$

$$\lesssim \frac{\sqrt{N(s)}}{s} \int_{s}^{+\infty} \left(\frac{1}{\sigma} + \frac{1}{\sigma^{2}} \right) d\sigma$$

$$\lesssim |t|^{1/2} |t| \sqrt{N(t)}$$
(5.52)

Notice that (5.50) and (5.49) can now be rewritten in terms of the $L^2(dy)$ scalar product since $\|\tilde{\epsilon}\|_{L^2} = |t|\sqrt{N(t)}$ and $P_{\mathcal{P}_c} = Q + \mathcal{O}(|t|^2 e^{-|y|})$, so we have

$$\left(\epsilon_2, \Lambda Q\right) = \mathcal{I}m\left(\tilde{\epsilon}, \overline{\Lambda P_{\mathcal{P}_c} + E_2 + E_3 + iD_3}\right)_{L^2(g\,dy)} + \mathcal{O}\left(|t|^2 \,\|\tilde{\epsilon}\|_{L^2}\right) \lesssim |t|^2 \,|t| \,\sqrt{N(t)}$$

$$\left(\epsilon_1, \,|y|^2 \,Q\right) = \mathcal{R}e\left(\tilde{\epsilon}, \,\overline{|y|^2 \, P_{\mathcal{P}_c} + F_2}\right)_{L^2(g\,dy)} + \mathcal{O}\left(|t|^2 \,\|\tilde{\epsilon}\|_{L^2}\right) \lesssim |t| \,|t| \,\sqrt{N(t)}$$

Finally, putting altogether (5.47), (5.49), (5.50) and (5.52) yields

$$\left| \left(\epsilon_1 \,, \, Q \right) \right| + \left| \left(\epsilon_2 \,, \, \Lambda Q \right) \right| + \left| \left(\epsilon_1 \,, \, |y|^2 \, Q \right) \right| + \left| \left(\epsilon_2 \,, \, \varrho \right) \right| \lesssim |t|^{1/2} \, |t| \, \sqrt{N(t)}$$

This ends the proof of (5.26) for the even directions.

5.4 Appendix 5.A: Proof of (5.5)

Here we now complete the proof of the step 2 of Lemma 5.1. Once again all we do here is to adapt the proof of Appendix C of [RS11] which we follow linearly. We will here prove a sligtly more precise estimate than (5.5), for we will need a refinement to control the phase parameter. We claim

$$|\lambda - \lambda_c| + |b - b_c| \lesssim |t|^6, \ |\alpha - \alpha_c| + |\beta - \beta_c| \lesssim |t|^5, \ |\gamma - \gamma_c| \lesssim |t|^4$$

$$(5.53)$$

Step 1 Improved bound on the modulation equations.

Let Mod(t) be the vector of modulation equations given by (3.22), for which we have proved estimation (3.24). The very same process we used in the proof of (3.24) leads in Appendix A of [RS11] to a slightly more general result :

$$Mod(t) = \mathcal{R}(\mathcal{P}) + \mathcal{O}\Big(\mathcal{P}^{7} + \left(\mathcal{P}^{2} + \left|Mod(t)\right|\Big) \|\epsilon\|_{L^{2}} + \|\epsilon\|_{L^{2}}^{2} + \|\epsilon\|_{H^{1}}^{3} + \left|(1 - c\,\lambda^{2})\,\|u\|_{L^{2}}^{2} - \|Q\|_{L^{2}}^{2}\Big|\Big)$$
(5.54)

where the parameter c is introduced through the computation in (3.26), and $\mathcal{R}(\mathcal{P})$ is polynomial in \mathcal{P} that satisfies

$$\left|\mathcal{R}(\mathcal{P})\right| \lesssim \mathcal{P}^4 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^3\left(b - \frac{\lambda}{C_0}\right)$$

Now using bounds (5.3) and (4.61), there is a polynomial $\mathcal{R}(\mathcal{P})$ in \mathcal{P} such that

$$Mod(t) = \mathcal{R}(\mathcal{P}) + \mathcal{O}\left(\mathcal{P}^7 + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\right)$$

$$= \mathcal{R}(\mathcal{P}) + \mathcal{O}\left(|t|^6\right)$$

$$\left|\mathcal{R}(\mathcal{P})\right| \lesssim \mathcal{P}^4 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^3\left(b - \frac{\lambda}{C_0}\right) \lesssim |t|^4$$
(5.55)

Unifortunately, this estimate is not strong enough, and we have to first refine both scalar products with $\nabla Q_{\mathcal{P}}$ and $\Lambda Q_{\mathcal{P}}$ in (3.43) to gain a cancellation on the null space for the (b, β) laws which should be set to satisfy

$$(b_s + b^2 - B_1 - K_b, \beta_s + b\beta - B_2) = \tilde{\mathcal{R}}(\mathcal{P}) + \mathcal{O}(\mathcal{P}^3 ||\epsilon||_{L^2} + ||\epsilon||_{L^2}^2 + ||\epsilon||_{H^1}^3)$$

= $\tilde{\mathcal{R}}(\mathcal{P}) + \mathcal{O}(|t|^7)$ (5.56)

For now, we assume (5.56) and finish the proof of (5.53).

We wish to compare Mod(t) and $Mod_c(t)$, so writing (5.55) for both, and using the fact \mathcal{R} vanishes at least at order 2 at the origin

$$\left| Mod(t) - Mod_c(t) \right| \lesssim |t|^2 \left| \mathcal{P} - \mathcal{P}_c \right| + \mathcal{O}\left(|t|^6 \right)$$
(5.57)

Similarly, using (5.56), the definition of K_b (3.23), and the degeneracy $\alpha, \alpha_c = \mathcal{O}(|t|^2)$, we have

$$\begin{aligned} \left| \left(b_s + b^2 - B_1 \right) - \left((b_c)_s + b_c^2 - (B_1)_c \right) \right| + \left| \left(\beta_s + b\beta - B_2 \right) - \left((\beta_c)_s + b_c\beta_c - (B_2)_c \right) \right| \\ \lesssim \left| \left(L_b - K_b, L_\beta \right) - \left(L_{b_c} - K_{b_c}, L_{\beta_c} \right) \right| + \left| \left(\lambda^2 - \lambda_c^2 \right) k_1 [\alpha] + \lambda_c^2 k_1 [\alpha - \alpha_c] \right| \\ \lesssim \left| \tilde{\mathcal{R}}(\mathcal{P}) + \mathcal{O}(|t|^7) \right| + |\alpha| \left| \alpha - \alpha_c \right| + |\alpha| \left| \lambda + \lambda_c \right| \left| \lambda - \lambda_c \right| + \lambda_c^2 \left| \alpha - \alpha_c \right| \\ \lesssim |t|^2 \left| \mathcal{P} - \mathcal{P}_c \right| + \mathcal{O}(|t|^7) \end{aligned}$$
(5.58)

Step 2 Estimates for $\lambda - \lambda_c$ and $b - b_c$.

Let us define

$$\underline{\mathcal{P}} = \mathcal{P} - \mathcal{P}_c = \mathcal{O}(|t|^2) \tag{5.59}$$

from (4.61), (4.62), (4.63).

We will now prove

$$|b - b_c| + |\lambda - \lambda_c| \lesssim |t|^6 + \int_t^0 |\underline{\mathcal{P}}| \, d\tau + |t| \left(\int_t^0 |\underline{\mathcal{P}}| \, \frac{d\tau}{\tau}\right)$$
(5.60)

First, we have

$$\lambda_t + \frac{b}{\lambda} = \frac{1}{\lambda} \left(\frac{\lambda_s}{\lambda} + b \right), \quad b_t + \frac{b^2}{\lambda^2} = \frac{1}{\lambda^2} \left(b_s + b^2 \right)$$
(5.61)

Then observe from (4.62), (4.63), (3.111)

$$\frac{b}{\lambda} - \frac{b_c}{\lambda_c} = \frac{1}{\lambda} \left(b - b_c \right) + b_c \left(\frac{1}{\lambda} - \frac{1}{\lambda_c} \right) = -\frac{C_0}{t} \underline{b} + \mathcal{O}\left(|t| \underline{\mathcal{P}} \right) - \frac{b_c}{\lambda_c} \frac{1}{\lambda} \underline{\lambda} = -\frac{C_0}{t} \underline{b} + \frac{1}{t} \underline{\lambda} + \mathcal{O}\left(|t| \underline{\mathcal{P}} \right)$$
(5.62)

Injecting this into (5.61) together with (5.57) for λ and the improved bound (5.58) for b yields

$$\underline{\lambda}_{t} - \frac{C_{0}}{t} \underline{b} + \frac{1}{t} \underline{\lambda} = \left(\lambda_{t} + \frac{b}{\lambda}\right) - \left((\lambda_{c})_{t} + \frac{b_{c}}{\lambda_{c}}\right) + \mathcal{O}\left(|t|\underline{\mathcal{P}}\right)$$

$$= \frac{1}{\lambda} \left(Mod(t) - Mod_{c}(t)\right) + \mathcal{O}\left(|t|\underline{\mathcal{P}}\right)$$

$$= F_{1}, \quad F_{1} = \mathcal{O}\left(|t|^{5} + |t|\underline{\mathcal{P}}\right)$$
(5.63)

$$\underline{b}_{t} - \frac{2}{t} \underline{b} + \frac{2}{C_{0} t} \underline{\lambda} = \left(b_{t} + \frac{b^{2}}{\lambda^{2}} \right) - \left((b_{c})_{t} + \frac{b_{c}^{2}}{\lambda^{2}_{c}} \right) + \mathcal{O}\left(|t| \underline{\mathcal{P}} \right)$$

$$= \frac{1}{\lambda^{2}} \left(L_{b} - L_{b_{c}} - |K_{b} - K_{b_{c}}| \right) + \frac{|\alpha|}{\lambda^{2}} |\alpha - \alpha_{c}| + \mathcal{O}\left(|t| \underline{\mathcal{P}} \right)$$

$$= F_{2}, \quad F_{1} = \mathcal{O}\left(|t|^{5} + |t| \underline{\mathcal{P}} \right)$$
(5.64)

We may now rewrite (5.63), (5.64) as

$$Z_t = \frac{1}{t} M Z + F$$

$$Z = \begin{pmatrix} \underline{\lambda} \\ \underline{b} \end{pmatrix}, \quad M = \begin{pmatrix} -1 & C_0 \\ -\frac{2}{C_0} & 2 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$
(5.65)

The eigenvalues of the matrix M are 0 and 1, hence the system may be rewritten in an eigenbasis

$$\tilde{Z}_t = \frac{1}{t} D \tilde{Z} + \tilde{F} \quad \text{with} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
(5.66)

and from (4.62), (4.63), (5.63), (5.64)

$$\left|\tilde{F}\right| \lesssim |t|^5 + \left|\underline{\mathcal{P}}\right|, \quad \frac{\tilde{Z}}{t} \to 0 \quad \text{as} \quad t \to 0$$

the explicit integration of (5.66) implies

$$\tilde{Z}_1 = \int_t^0 \tilde{F}_1(\tau) \, d\tau, \quad \tilde{Z}_2 = t \int_t^0 \frac{\tilde{F}_2(\tau)}{\tau} \, d\tau$$

Thus

$$\begin{aligned} \left| Z(t) \right| \lesssim \left| \tilde{Z}(t) \right| \lesssim \int_{t}^{0} \left| \tilde{F}(\tau) \right| d\tau + \left| t \right| \int_{t}^{0} \frac{\left| \tilde{F}(\tau) \right|}{\left| \tau \right|} d\tau \\ \lesssim \left| t \right|^{6} + \int_{t}^{0} \left| \underline{\mathcal{P}}(\tau) \right| d\tau + \left| t \right| \int_{t}^{0} \frac{\left| \underline{\mathcal{P}}(\tau) \right|}{\left| \tau \right|} d\tau \end{aligned}$$

which proves (5.60).

Step 3 Estimates for $\alpha - \alpha_c$, $\beta - \beta_c$.

We now claim the bound

$$\left|\alpha - \alpha_{c}\right| + \left|\beta - \beta_{c}\right| \lesssim \left|t^{6} \ln(|t|)\right| + \left|t\right| \left(\int_{t}^{0} \frac{\left|log(\tau)\right|}{|\tau|} \left|\underline{\mathcal{P}}(\tau)\right| d\tau\right)$$
(5.67)

First, we have

$$\alpha_t - 2\frac{\beta}{\lambda} = \frac{1}{\lambda} \left(\frac{\alpha_s}{\lambda} - 2\beta\right), \quad \beta_t + \frac{b\beta}{\lambda^2} - \frac{B_2}{\lambda^2} = \frac{1}{\lambda^2} \left(\beta_s + b\beta - B_2\right)$$
(5.68)

then from (3.111)

$$\frac{\beta}{\lambda} - \frac{\beta_c}{\lambda_c} = \frac{1}{\lambda} \left(\beta - \beta_c \right) + \beta_c \left(\frac{1}{\lambda} - \frac{1}{\lambda_c} \right) = -\frac{C_0}{t} \frac{\beta}{D} - \frac{\beta_c}{\lambda \lambda_c} \frac{\lambda}{\Delta} = -\frac{C_0}{t} \frac{\beta}{D} + \mathcal{O}(\underline{\mathcal{P}})$$
(5.69)

hence from both (3.111) and (4.62), using definition (1.42) of B_2 and (4.63)

$$\frac{b\beta}{\lambda^2} - \frac{B_2}{\lambda^2} - \frac{b_c\beta_c}{\lambda_c^2} + \frac{(B_2)_c}{\lambda_c^2} = \frac{b}{\lambda}\frac{\beta}{\lambda} - \frac{b_c}{\lambda_c}\frac{\beta_c}{\lambda_c} - \left(\frac{B_2}{\lambda^2} - \frac{(B_2)_c}{\lambda_c^2}\right) \\
= \left(\frac{1}{C_0} + \mathcal{O}(|t|^2)\right) \left(\frac{\beta}{\lambda} - \frac{\beta_c}{\lambda_c}\right) - \left(\frac{c_0(\alpha) + \lambda^2 C_3}{\lambda} - \frac{c_0(\alpha_c) + \lambda_c^2 C_3}{\lambda_c} + \mathcal{O}(\underline{\mathcal{P}})\right) \\
= -\frac{1}{t}\frac{\beta}{\underline{\mathcal{P}}} - \frac{1}{\lambda}\left(c_0(\alpha) - c_0(\alpha_c)\right) + c_0(\alpha_c)\left(\frac{1}{\lambda} - \frac{1}{\lambda_c}\right) + \mathcal{O}(\underline{\mathcal{P}}) \\
= -\frac{1}{t}\frac{\beta}{\underline{\mathcal{P}}} + \frac{C_0}{t}c_0(\underline{\alpha}) + \mathcal{O}(\underline{\mathcal{P}})$$
(5.70)

Thus, injecting this into (5.68), using bound (5.57) we have

$$\underline{\alpha}_{t} + \frac{2C_{0}}{t} \underline{\beta} = \frac{1}{\lambda} \left[\left(\frac{\alpha_{s}}{\lambda} - 2\beta \right) - \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2\beta_{c} \right) \right] - \frac{\lambda}{\lambda\lambda_{c}} \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2\beta_{c} \right) + \mathcal{O}(\underline{\mathcal{P}})
= \frac{1}{\lambda} \left(L_{\alpha} - L_{\alpha_{c}} \right) + \mathcal{O}(\underline{\mathcal{P}})
= G_{1}, \quad G_{1} = \mathcal{O}(|t|^{5} + \underline{\mathcal{P}})$$
(5.71)

and thanks to the improved bound (5.58)

$$\underline{\beta}_{t} - \frac{1}{t} \underline{\beta} + \frac{C_{0}}{t} c_{0}(\underline{\alpha}) = \frac{1}{\lambda^{2}} \left(L_{\beta} - L_{\beta_{c}} \right) - \frac{\underline{\lambda}^{2}}{(\lambda\lambda_{c})^{2}} L_{\beta_{c}} + \mathcal{O}(\underline{\mathcal{P}})$$

$$= G_{2}, \quad G_{2} = \mathcal{O}(|t|^{5} + \underline{\mathcal{P}})$$
(5.72)

Now we set

$$\underline{\eta} = \frac{\underline{\beta}}{\underline{t}} \tag{5.73}$$

so that we may rewrite the system of equations (5.71)-(5.72) as

$$\underline{\alpha}_t + 2 C_0 \, \underline{\eta} = G_1, \quad \underline{\eta}_t + \frac{C_0}{t^2} \, c_0(\underline{\alpha}) = \frac{1}{t} \, G_2$$

Recall from (1.50) that

$$\left(c_0(\underline{\alpha})\right)_p = \nabla^2 \left(k - \frac{1}{2} Tr(G)\right)(0) \cdot \left(\underline{\alpha}, \partial_p\right) + \frac{1}{2} \nabla^2 G_{12}(0) \cdot \left(\underline{\alpha}, \partial_p - \partial_{\bar{p}}\right)$$

so that matrix c_0 is given by

$$c_0 = \nabla^2 \left(k - \frac{1}{2} Tr(G) \right) (0) + \frac{1}{2} \begin{pmatrix} \nabla^2 G_{12}(0) \cdot (\partial_1, \partial_1 - \partial_2) & \nabla^2 G_{12}(0) \cdot (\partial_2, \partial_1 - \partial_2) \\ -\nabla^2 G_{12}(0) \cdot (\partial_1, \partial_1 - \partial_2) & -\nabla^2 G_{12}(0) \cdot (\partial_2, \partial_1 - \partial_2) \end{pmatrix}$$

Let $F = k - \frac{1}{2}Tr(G)$, one may compute

$$det(c_0) = det(\nabla^2 F(0)) - \frac{1}{4} \nabla^2 G_{12}(0) \cdot (\partial_1 - \partial_2, \partial_1 + \partial_2) \nabla^2 F(0) \cdot (\partial_1 - \partial_2, \partial_1 + \partial_2) + \frac{1}{4} \nabla^2 G_{12}(0) \cdot (\partial_1 - \partial_2)^2 \nabla^2 F(0) \cdot (\partial_1 - \partial_2)^2$$

hence from hypothesis (H5), matrix c_0 is negative definite, with eigenvalues $r_1, r_2 < 0$. Thus, we can find an eigenbasis, so that for j = 1, 2

$$\begin{cases} \left(\underline{\alpha}_{j}\right)_{t} = -2C_{0}\,\underline{\eta}_{j} + \mathcal{O}\left(t^{5} + \underline{\mathcal{P}}\right) \\ \left(\underline{\eta}_{j}\right)_{t} = -\frac{C_{0}\,r_{j}}{t^{2}}\,\underline{\alpha}_{j} + \mathcal{O}\left(t^{4} + \frac{\underline{\mathcal{P}}}{t}\right) \end{cases}$$
(5.74)

Then, we perform the change of variables $s = \frac{1}{|t|}$, and rewrite (5.74), for j = 1, 2

$$Z_j = \begin{pmatrix} \underline{\eta}_j \\ \frac{r_j C_0}{2} \underline{\alpha}_j \end{pmatrix}, \quad (Z_j)_s = \begin{pmatrix} 0 & -2 \\ \frac{\zeta_j}{s^2} & 0 \end{pmatrix} Z_j + F_j, \quad \zeta_j = -r_j C_0^2$$
(5.75)

with

$$F_j = ((F_j)_1, (F_j)_2), \quad (F_j)_1 = \mathcal{O}\left(\frac{1}{s^6} + \frac{\mathcal{P}}{s}\right), \quad (F_j)_2 = \mathcal{O}\left(\frac{1}{s^7} + \frac{\mathcal{P}}{s^2}\right)$$

Moreover, from (4.62), (4.63), (5.59)

$$\lim_{s \to +\infty} Z_j = 0 \quad \text{and} \quad F_j(s) = \mathcal{O}\left(\frac{1}{s^3}\right).$$

Now recall Lemma 8.10, see Appendix B, that now yields

$$|\underline{\eta}_{j}| + s |\underline{\alpha}_{j}| \lesssim \int_{s}^{+\infty} \left(\frac{1}{\sigma^{6}} + \frac{|\underline{\mathcal{P}}|}{\sigma}\right) \log(\sigma) \, d\sigma \lesssim \left|t^{5} \log|t|\right| + \int_{t}^{0} \frac{|\log(\tau)|}{|\tau|} |\underline{\mathcal{P}}(\tau)| \, d\tau$$

and hence (5.67) follows from (5.73).

Step 4 Bound on $\underline{\mathcal{P}}$.

We conclude from (5.60), (5.67) that

$$|\underline{\mathcal{P}}| \lesssim \left| t^6 \log|t| \right| + \int_t^0 |\underline{\mathcal{P}}(\tau)| \, d\tau + |t| \, \int_t^0 \frac{|\log(\tau)|}{|\tau|} \, |\underline{\mathcal{P}}(\tau)| \, d\tau$$

Injecting the a priori bound (5.59) in the previous estimate

$$\begin{split} |\underline{\mathcal{P}}| &\lesssim t^{6} \, |\log|t||^{4} + \int_{t}^{0} |\tau|^{2} \, d\tau + |t| \int_{t}^{0} |\tau| \, |\log(\tau)| \, d\tau \\ &\lesssim t^{6} \, |\log|t||^{4} + |t|^{3} + |t|^{3} \, |\log|t|| \\ &\lesssim |t|^{3} \, |\log|t|| \end{split}$$

and iterating 3 more times now imply

$$|\underline{\mathcal{P}}| \lesssim t^6 |\log|t||^4$$

Eventually, injecting this into (5.60), (5.67) clearly yields

$$|\lambda - \lambda_c| + |b - b_c| \lesssim |t|^6, \quad |\alpha - \alpha_c| + |\beta - \beta_c| \lesssim |t|^5$$
(5.76)

Step 5 Bound on the phase parameter.

There is only left to get control on the phase parameter, which is why we had to sharpen control on the scalling parameter λ in previous steps. We are willing to prove

$$|\gamma - \gamma_c| \lesssim |t|^4 \tag{5.77}$$

Indeed we have

$$\gamma_t - \frac{1 + |\beta|^2 + K_{\tilde{\gamma}}}{\lambda^2} = \frac{1}{\lambda^2} \left(\tilde{\gamma}_s - |\beta|^2 - K_{\tilde{\gamma}} \right)$$
(5.78)

and from (5.76)

$$\frac{1+|\beta|^2+K_{\tilde{\gamma}}}{\lambda^2} - \frac{1+|\beta_c|^2+K_{\tilde{\gamma}_c}}{\lambda_c^2} = \mathcal{O}\Big(\frac{|\lambda-\lambda_c|}{|t|^3} + |\beta-\beta_c| + |\alpha-\alpha_c|\Big) \\ = \mathcal{O}\big(|t|^3\big)$$
(5.79)

so that (5.57), (5.78) and (5.79) imply

$$\underline{\gamma}_t = \mathcal{O}\left(|t|^3 + |\underline{\mathcal{P}}|\right)$$

which after integration yields (5.77) and concludes the proof of (5.53) assuming (5.56).

Step 6 Proof of the improved bound (5.56)

To get the refinment (5.56) we now need to go back to the very definition of the remaining ϵ through the choice of the orthogonality conditions (3.7)-(3.11). Indeed, one way to sharpen the modulation laws estimate, is to pertubatively modify the remain ϵ through the orthogonality conditions (3.7) and (3.9) which respectively govern the law of β and b, that one should now replace by

$$\int \left(\epsilon_2, \nabla \Sigma + A \lambda^2\right) - \left(\epsilon_1, \nabla \Theta\right) = 0$$
(5.80)

$$\left(\left(\epsilon_{2},\,\Lambda\Sigma+B\,\lambda^{2}\right)-\left(\epsilon_{1},\,\Lambda\Theta\right)=0\tag{5.81}$$

where A and B are well-localized real functions to be chosen. We claim for a suitable choice for A and B, the computation of the modulations equations, like we did in (3.54) and (3.50) will lead us to get (5.56).

Let M_1 , M_2 and \tilde{M}_1 , \tilde{M}_2 given by (3.16) and (3.17), and then let the complex operator for $\epsilon = \epsilon_1 + i \epsilon_2$

$$\mathcal{M}(\epsilon) = \mathcal{M}_1(\epsilon_1, \epsilon_2) + i \,\mathcal{M}_2(\epsilon_1, \epsilon_2)$$

= $M_1(\epsilon_1, \epsilon_2) + i \,M_2(\epsilon_1, \epsilon_2) - i \,b \,\Lambda \epsilon + 2 \,i \,\beta \,. \nabla \epsilon$

These refined orthogonality conditions allow one to sharpen the scalar products computations (3.43) such that

$$\mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(\nabla Q_{\mathcal{P}} + \lambda^2 A) - i \partial_s (\nabla Q_{\mathcal{P}} + \lambda^2 A)}\right) = \mathcal{O}\left(|t|^3 \|\epsilon\|_{L^2}\right)$$
(5.82)

$$\mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(\Lambda Q_{\mathcal{P}} + \lambda^2 B) - i \partial_s (\Lambda Q_{\mathcal{P}} + \lambda^2 B)}\right) + 2 \mathcal{R}e\left(\epsilon, \overline{Q_{\mathcal{P}}}\right)$$

= $\mathcal{O}\left(|t|^3 \|\epsilon\|_{L^2}\right)$ (5.83)

First, note one has the following adjunction formula

$$\mathcal{R}e\left(\mathcal{M}(\epsilon), \overline{f}\right) = \mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(f)}\right)$$

then, recall the equations of real and imaginary parts of ϵ (3.39)-(3.40), and given an orthogonality condition $\mathcal{I}m(\epsilon, \overline{f}) = 0$ up to $\mathcal{O}(|t|^3 ||\epsilon||_{L^2})$, the linear term in the computation of these equations is

$$-\left(\mathcal{M}_{1}(\epsilon_{1},\epsilon_{2}),f_{1}\right)+\left(\epsilon_{2},\partial_{s}f_{1}\right)-\left(\mathcal{M}_{2}(\epsilon_{1},\epsilon_{2}),f_{2}\right)-\left(\epsilon_{1},\partial_{s}f_{2}\right)=-\mathcal{R}e\left(\epsilon,\overline{\mathcal{M}(f)-i\partial_{s}f}\right)$$

Proof of (5.82)

From (2.2), $Q_{\mathcal{P}}$ satisfies

$$i(-b^{2}+B_{1})\partial_{b}Q_{\mathcal{P}}+\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}}-Q_{\mathcal{P}}+\frac{k(\lambda y+\alpha)}{k(\alpha)}Q_{\mathcal{P}}|Q_{\mathcal{P}}|^{2}$$

+ $ib\Lambda Q_{\mathcal{P}}-2i\beta.\nabla Q_{\mathcal{P}}=\mathcal{O}(|t|^{3}e^{-|y|})$
(5.84)

By differentiating (5.84), since

$$\nabla \left(|Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} \right) = \tilde{M}_1(\nabla \Sigma, \nabla \Theta) + i \, \tilde{M}_2(\nabla \Sigma, \nabla \Theta)$$

and

$$\begin{split} \nabla \mathcal{L}_{\lambda \alpha} Q_{\mathcal{P}} &= \mathcal{L}_{\lambda \alpha} \big(\nabla Q_{\mathcal{P}} \big) + \frac{1}{g} \, \partial_i \big(\nabla G_{ij} \, \partial_j Q_{\mathcal{P}} \big) - \frac{\nabla g}{g} \, \Delta_{_{G,g}} Q_{\mathcal{P}} + \lambda^2 \, Q_{\mathcal{P}} \, \nabla V \\ &= \mathcal{L}_{\lambda \alpha} \big(\nabla Q_{\mathcal{P}} \big) + \nabla G_{ij} \, \partial_{ij}^2 Q_{\mathcal{P}} - \nabla g \, \Delta Q_{\mathcal{P}} + \mathcal{O} \big(|t|^3 \, e^{-|y|} \big) \end{split}$$

we obtain

$$i(-b^{2}+B_{1})\partial_{b}\nabla Q_{\mathcal{P}} + \mathcal{L}_{\lambda\alpha}(\nabla Q_{\mathcal{P}}) - \nabla Q_{\mathcal{P}} + \frac{k(\lambda y + \alpha)}{k(\alpha)} \left(\tilde{M}_{1}(\nabla Q_{\mathcal{P}}) + i\,\tilde{M}_{2}(\nabla Q_{\mathcal{P}}) + i\,b\,\Lambda(\nabla Q_{\mathcal{P}}) - 2\,i\,\beta\,\cdot\nabla^{2}Q_{\mathcal{P}} + \nabla G_{ij}\,\partial_{ij}^{2}Q_{\mathcal{P}} - \nabla g\,\Delta Q_{\mathcal{P}} + \frac{\lambda\,\nabla k(\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^{2}\,Q_{\mathcal{P}} + i\,b\,\nabla Q_{\mathcal{P}} = \mathcal{O}(|t|^{3}\,e^{-|y|})$$

$$(5.85)$$

Thus

$$\mathcal{M}(\nabla Q_{\mathcal{P}}) - i \partial_s (\nabla Q_{\mathcal{P}}) = \lambda \nabla G_{ij} (\lambda y + \alpha) \partial_{ij}^2 Q_{\mathcal{P}} - \lambda \nabla g (\lambda y + \alpha) \Delta Q_{\mathcal{P}} + \frac{\lambda \nabla k (\lambda y + \alpha)}{k(\alpha)} |Q_{\mathcal{P}}|^2 Q_{\mathcal{P}} + i b \nabla Q_{\mathcal{P}} + \mathcal{O}(|t|^3 e^{-|y|})$$
(5.86)

moreover thanks to (2.5) and Proposition 4.7 one easily check

$$\tilde{M}_{1}(\lambda^{2} A) = \lambda^{2} \left(3 Q^{2} A + \mathcal{O}(|t| e^{-|y|}) \right) = 3 \lambda^{2} Q^{2} A + \mathcal{O}(|t|^{3} e^{-|y|})$$
$$\tilde{M}_{2}(\lambda^{2} A) = \mathcal{O}(|t|^{3} e^{-|y|})$$

then, expanding the remaining terms

$$\mathcal{M}(\nabla Q_{\mathcal{P}} + \lambda^2 A) - i \partial_s (\nabla Q_{\mathcal{P}} + \lambda^2 A)$$

= $\lambda^2 \left[\nabla^2 (k+g)(0).(y, .) Q^3 + \nabla^2 G_{ij}(0).(y, .) \partial_{ij}^2 Q - \nabla^2 g(0).(y, .) Q + L_+(A) \right]$
+ $i b \nabla Q_{\mathcal{P}} + \mathcal{O}(|t|^3 e^{-|y|})$ (5.87)

From (8.3), (8.1), (8.2) we then choose A solution to

$$\begin{aligned} L_{+}(A) &= -\left[\nabla^{2}(k+g)(0).(y, .) Q^{3} + \nabla^{2}G_{ij}(0).(y, .) \partial_{ij}^{2}Q - \nabla^{2}g(0).(y, .) Q\right] + a yQ \\ a &= \frac{\left(\nabla^{2}(k+g)(0).(y, .) Q^{3} + \nabla^{2}G_{ij}(0).(y, .) \partial_{ij}^{2}Q - \nabla^{2}g(0).(y, .) Q, \nabla Q\right)}{\left(yQ, \nabla Q\right)} \\ a &= \Delta k(0) + \nabla^{2}\left(G_{11} - G_{22}\right)(0).(\partial_{1}, \partial_{1}) + \nabla^{2}\left(G_{22} - G_{11}\right)(0).(\partial_{2}, \partial_{2}) + 4\nabla^{2}G_{12}(0).(\partial_{1}, \partial_{2})\right) \end{aligned}$$

and thus from (5.87) for $f = \nabla Q_{\mathcal{P}} + \lambda^2 A$, using orthogonality conditions (3.8) and (5.80)

$$\begin{aligned} \mathcal{R}e\Big(\epsilon\,,\,\overline{\mathcal{M}(f)-i\,\partial_s f}\Big) &= b\,\mathcal{I}m\Big(\epsilon\,,\,\nabla\overline{Q_{\mathcal{P}}}\Big) + a\,\lambda^2\,\mathcal{R}e\Big(\epsilon\,,\,\overline{yQ}\Big) + \mathcal{O}\big(|t|^3\,e^{-|y|}\big) \\ &= \mathcal{O}\big(|t|^3\,\|\epsilon\|_{L^2}\big) \end{aligned}$$

This concludes the proof of (5.82)

Proof of (5.83)

First, we gather some computational results

$$\Lambda^{2}Q_{\mathcal{P}} = \Lambda Q_{\mathcal{P}} + y \cdot \nabla Q_{\mathcal{P}} + y \cdot \Lambda (\nabla Q_{\mathcal{P}})
\beta \cdot \nabla (\Lambda Q_{\mathcal{P}}) = 2\beta \cdot \nabla Q_{\mathcal{P}} + y \cdot \beta \cdot \nabla (\nabla Q_{\mathcal{P}})
\tilde{M}_{1}(\Lambda Q_{\mathcal{P}}) + i \tilde{M}_{2}(\Lambda Q_{\mathcal{P}}) = 3 |Q_{\mathcal{P}}|^{2} Q_{\mathcal{P}} + y \cdot \tilde{M}_{1}(\nabla Q_{\mathcal{P}}) + i y \cdot \tilde{M}_{2}(\nabla Q_{\mathcal{P}})
\mathcal{L}_{\lambda\alpha}(\Lambda Q_{\mathcal{P}}) = 3\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}} + y \cdot \mathcal{L}_{\lambda\alpha}(\nabla Q_{\mathcal{P}}) - 2\lambda^{2} V Q_{\mathcal{P}} - \frac{1}{g} \partial_{i}G_{ij}\partial_{j}Q_{\mathcal{P}}
= 3\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}} + y \cdot \mathcal{L}_{\lambda\alpha}(\nabla Q_{\mathcal{P}}) - 2\lambda^{2} V(0) Q - \lambda^{2} \nabla^{2}G_{ij}(0) \cdot (y, \partial_{i}) \partial_{j}Q
+ \mathcal{O}(|t|^{3} e^{-|y|})$$
(5.88)

thanks to which and (5.84), (5.85) we deduce

$$\begin{split} &i(-b^{2}+B_{1})\partial_{b}(\Lambda Q_{\mathcal{P}})+\mathcal{L}_{\lambda\alpha}(\Lambda Q_{\mathcal{P}})-\Lambda Q_{\mathcal{P}}+\frac{k(\lambda y+\alpha)}{k(\alpha)}\left[\tilde{M}_{1}(\Lambda Q_{\mathcal{P}})+i\,\tilde{M}_{2}(\Lambda Q_{\mathcal{P}})\right]\\ &+i\,b\,\Lambda^{2}Q_{\mathcal{P}}-2\,i\,\beta\,.\,\nabla(\Lambda Q_{\mathcal{P}})\\ &=3\left[i(-b^{2}+B_{1})\partial_{b}Q_{\mathcal{P}}+\mathcal{L}_{\lambda\alpha}Q_{\mathcal{P}}-Q_{\mathcal{P}}+\frac{k(\lambda y+\alpha)}{k(\alpha)}\,Q_{\mathcal{P}}|Q_{\mathcal{P}}|^{2}+i\,b\,\Lambda Q_{\mathcal{P}}-2\,i\,\beta\,.\,\nabla Q_{\mathcal{P}}\right]\\ &+y\,.\left[i(-b^{2}+B_{1})\partial_{b}\nabla Q_{\mathcal{P}}+\mathcal{L}_{\lambda\alpha}(\nabla Q_{\mathcal{P}})-\nabla Q_{\mathcal{P}}+\frac{k(\lambda y+\alpha)}{k(\alpha)}\left(\tilde{M}_{1}(\nabla Q_{\mathcal{P}})+i\,\tilde{M}_{2}(\nabla Q_{\mathcal{P}})\right)\right)\\ &+i\,b\,\Lambda(\nabla Q_{\mathcal{P}})-2\,i\,\beta\,.\,\nabla^{2}Q_{\mathcal{P}}\right]+2\,i\,(b^{2}-B_{1})\,\partial_{b}Q_{\mathcal{P}}+2\,Q_{\mathcal{P}}+i\,b\,y\,.\,\nabla Q_{\mathcal{P}}\\ &-2\,i\,b\,\Lambda Q_{\mathcal{P}}+2\,i\,\beta\,.\,\nabla Q_{\mathcal{P}}-2\,\lambda^{2}\,V\,Q_{\mathcal{P}}-\frac{1}{g}\partial_{i}G_{ij}\partial_{j}Q_{\mathcal{P}}\\ &=2\,i\,(b^{2}-B_{1})\,\partial_{b}Q_{\mathcal{P}}+2\,Q_{\mathcal{P}}-2\,i\,b\,\Lambda Q_{\mathcal{P}}+2\,i\,\beta\,.\,\nabla Q_{\mathcal{P}}\\ &-\left[\frac{\lambda\,y\,.\,\nabla k(\lambda y+\alpha)}{k(\alpha)}\,|Q_{\mathcal{P}}|^{2}\,Q_{\mathcal{P}}+y\,.\,\nabla G_{ij}\,\partial_{ij}^{2}Q_{\mathcal{P}}-y\,.\,\nabla g\,\Delta Q_{\mathcal{P}}+2\,\lambda^{2}\,V\,Q_{\mathcal{P}}+\partial_{i}G_{ij}\partial_{j}Q_{\mathcal{P}}\right]\\ &+\mathcal{O}(|t|^{3}\,e^{-|y|})\end{split}$$

Eventually, with (1.42) we obtain

$$\mathcal{M}(\Lambda Q_{\mathcal{P}} + \lambda^2 B) - i \partial_s (\Lambda Q_{\mathcal{P}} + \lambda^2 B)$$

$$= \lambda^2 \left(2 V(0) Q + \nabla^2 (k+g)(0).(y,y) Q^3 - \nabla^2 g(0).(y,y) Q + \nabla^2 G_{ij}(0) \cdot \left[(y,y) \partial_{ij}^2 Q + (y,\partial_i) \partial_j Q \right] + L_+(B) \right)$$

$$- 2 i b^2 \partial_b Q_{\mathcal{P}} - 2 Q_{\mathcal{P}} + 2 i b \Lambda Q_{\mathcal{P}} - 2 i \beta \cdot \nabla Q_{\mathcal{P}} + \mathcal{O}(|t|^3 e^{-|y|})$$
(5.89)

We hence choose B solution to

$$L_{+}(B) = -\left(2V(0)Q + \nabla^{2}(k+g)(0).(y,y)Q^{3} - \nabla^{2}g(0).(y,y)Q + \nabla^{2}G_{ij}(0).\left[(y,y)\partial_{ij}^{2}Q + (y,\partial_{i})\partial_{j}Q\right]\right) =: f_{2}$$

which is possible thanks to (8.2), since f_2 is even while ∇Q is odd, and thus $(f_2, \nabla Q) = 0$.

Then, observe as in (3.49) that

$$\partial_b Q_{\mathcal{P}} = -i \, \frac{|y|^2}{4} \, Q + \mathcal{O}\left(|t| \, e^{-|y|}\right)$$

so that (5.89) yields

$$\begin{split} &\mathcal{R}e\Big(\epsilon\,,\,\overline{\mathcal{M}\big(\Lambda Q_{\mathcal{P}}+\lambda^2\,B\big)-i\,\partial_s\big(\Lambda Q_{\mathcal{P}}+\lambda^2\,B\big)}\Big)+2\,\mathcal{R}e\Big(\epsilon\,,\,\overline{Q_{\mathcal{P}}}\Big)\\ &=-\frac{b^2}{2}\,\mathcal{R}e\Big(\epsilon\,,\,|y|^2\,\overline{Q_{\mathcal{P}}}\Big)-2\,\mathcal{R}e\Big(\epsilon\,,\,\overline{Q_{\mathcal{P}}}\Big)+2\,b\,\mathcal{I}m\Big(\epsilon\,,\,\Lambda\overline{Q_{\mathcal{P}}}\Big)-2\,\beta\,.\,\mathcal{I}m\Big(\epsilon\,,\,\nabla\overline{Q_{\mathcal{P}}}\Big)\\ &+2\,\mathcal{R}e\Big(\epsilon\,,\,\overline{Q_{\mathcal{P}}}\Big)+\mathcal{O}\big(|t|^3\,e^{-|y|}\big)\\ &=\mathcal{O}\big(|t|^3\,e^{-|y|}\big) \end{split}$$

where we have used (3.10), (5.80), (5.81). This concludes the proof of (5.83).

5.5 Appendix 5.B : Computation of the approximate null space of the linearized NLS operator around the Ground-State.

Proof of (5.37) and (5.38) We compute thanks to (5.32), (5.33) along with the control of the modulation parameters of (4.61)

$$M^{(4)}\left(\nabla Q + \lambda_c^2 A_2\right) - i \partial_s \left(\nabla Q + \lambda_c^2 A_2\right)$$

= $L_+\left(\nabla Q + \lambda_c^2 A_2\right) - \frac{1}{g} \left[\Delta_{\tilde{G}} \left(\nabla Q\right) + \nabla^2 Q \cdot \nabla g\right] - \lambda_c^2 V \nabla Q$
- $6 Q T_2 \nabla Q - \frac{3}{2} \nabla^2 k(0) \cdot (y, y) \lambda_c^2 Q^2 \nabla Q + \mathcal{O}\left(\lambda_c^3 e^{-|y|}\right)$ (5.90)

where we have used

$$\begin{split} \mathcal{L}_{\lambda_c \alpha_c} v &= \Delta v + \frac{1}{g} \left[\Delta_{\tilde{G}} \, v + \nabla g \, \cdot \nabla v \right] + \lambda^2 \, V \, v, \\ \text{with} \quad \tilde{G}_{ij} &= (1 - g) \, I_{ij} + (G_{ij} - I_{ij}) \end{split}$$

Recall $L_{+}(\nabla Q) = 0$, and then from (1.49) using again (4.61) the following identity which defined T_{2}

$$L_{+}T_{2} = \lambda_{c}^{2} \left[\frac{1}{2} \nabla^{2} (k+g)(0).(y,y) Q^{3} + \nabla^{2} G_{ij}(0). \left[\frac{1}{2} (y,y) \partial_{ij}^{2} Q + (y,\partial_{i}) \partial_{j} Q \right] - \frac{1}{2} \nabla^{2} g(0).(y,y) Q + V(0) Q \right].$$

Using (8.2) we may now introduce T_2^0 , such as $T_2 = \lambda_c^2 T_2^0$, a function which thus satisfy

$$L_{+}T_{2}^{0} = \frac{1}{2} \nabla^{2}(k+g)(0).(y,y) Q^{3} + \nabla^{2}G_{ij}(0).\left[\frac{1}{2}(y,y)\partial_{ij}^{2}Q + (y,\partial_{i})\partial_{j}Q\right] - \frac{1}{2} \nabla^{2}g(0).(y,y) Q + V(0) Q$$
 so that we may rewrite (5.90) as

rewrite (5.90)

$$\begin{split} &M^{(4)} \left(\nabla Q + \lambda_c^2 A_2 \right) - i \,\partial_s \left(\nabla Q + \lambda_c^2 A_2 \right) \\ &= -\lambda_c^2 \left[6 \,Q \,T_2^0 \,\nabla Q + \nabla^2 G_{ij}(0) . \left[\frac{1}{2} \,(y, y) \partial_{ij}^2 \nabla Q + (y, \partial_i) \partial_j \nabla Q \right] - \frac{1}{2} \,\nabla^2 g(0) . (y, y) \,\nabla Q + V(0) \,\nabla Q \right. \\ &+ \frac{3}{2} \,\nabla^2 k(0) . (y, y) \,Q^2 \,\nabla Q - L_+(A_2) \right] + \mathcal{O} \left(\lambda_c^3 \,e^{-|y|} \right) \end{split}$$

To fulfill (5.37) we may now simply choose A_2 solution to

$$\begin{aligned} L_{+}(A_{2}) &= 6 Q T_{2}^{0} \nabla Q + \tilde{A}_{2} - a_{1} y Q \\ \text{with} \quad \tilde{A}_{2} &= \nabla^{2} G_{ij}(0) \cdot \left[\frac{1}{2} (y, y) \partial_{ij}^{2} \nabla Q + (y, \partial_{i}) \partial_{j} \nabla Q\right] - \frac{1}{2} \nabla^{2} g(0) \cdot (y, y) \nabla Q + V(0) \nabla Q \\ &+ \frac{3}{2} \nabla^{2} k(0) \cdot (y, y) Q^{2} \nabla Q \end{aligned}$$
(5.91)

Q being is radial, T_2^0 is hence a second order polynomial function in y with radially symmetric coefficients (also in variable y), and therefore from (8.2) one should also set constant a_1 as

$$a_1 = \frac{\left(6 Q T_2^0 \nabla Q + \tilde{A}_2, \nabla Q\right)}{\left(yQ, \nabla Q\right)}$$
(5.92)

Yet there is left to ensure that $a_1 > 0$. To do so, one needs to compute a_1 more explicitly, which may be done provided one observe by differentiating twice the Ground-State equation (1.36) $L_+(\Delta Q) = 6 Q |\nabla Q|^2$. But first, note that Q being radial, some computations show

$$\begin{split} \left(\tilde{A}_{2}\,,\,\nabla Q\right) &= \int \nabla^{2} G_{ij}(0). \left[\frac{1}{2}\,(y,y)\partial_{ij}^{2}\nabla Q + (y,\partial_{i})\partial_{j}\nabla Q\right].\,\nabla Q + V(0)\,\int Q^{2} \\ &+ \int \left(3\,\Delta k(0)\,Q^{2} - \Delta g(0)\right)\frac{|y|^{2}}{4}\,|\nabla Q|^{2} \\ &= \frac{1}{2}\,\nabla^{2} G_{ii}(0). - \partial_{i},\partial_{i})\left[K_{23} - \int Q^{2}\right] + \frac{1}{2}\,\nabla^{2} G_{\tilde{p}\,\tilde{p}}(0).(\partial_{p},\partial_{p})\,K_{24} \\ &+ \frac{1}{2}\,\nabla^{2} G_{12}(0).(\partial_{1},\partial_{2})\left[-\int Q^{2} + \int \frac{|y|^{2}}{4}\,\left(1 - 3\,Q^{2}\right)\,|\nabla Q|^{2}\right] + V(0)\,\int Q^{2} \\ &+ \int \left(3\,\Delta k(0)\,Q^{2} - \Delta g(0)\right)\,\frac{|y|^{2}}{4}\,|\nabla Q|^{2} \end{split}$$

with

$$K_{23} := \int y_1^2 \nabla \partial_1^2 Q \cdot \nabla Q = \int y_2^2 \nabla \partial_2^2 Q \cdot \nabla Q = \frac{3}{2} \int \frac{|y|^2}{4} (2 - 3Q^2) |\nabla Q|^2$$
$$K_{24} := \int y_1^2 \nabla \partial_2^2 Q \cdot \nabla Q = \int y_2^2 \nabla \partial_1^2 Q \cdot \nabla Q = -\frac{1}{2} \int \frac{|y|^2}{4} (2 + 3Q^2) |\nabla Q|^2$$

Thus using again the fact that Q is radial and T_2^0 is somehow even, for j = 1, 2

$$\begin{split} & \left(6\,Q\,T_{2}^{0}\,\nabla Q\,,\,\nabla Q\right) = 6\,\left(T_{2}^{0}\,,\,Q\,|\nabla Q|^{2}\right) = \left(T_{2}^{0}\,,\,L_{+}(\Delta Q)\right) = \left(L_{+}(T_{2}^{0}\,,\,\Delta Q\right) \\ &= \left(\frac{1}{2}\,\nabla^{2}(k+g)(0).(y,y)\,Q^{3} + \nabla^{2}G_{ij}(0).\left[\frac{1}{2}\,(y,y)\partial_{ij}^{2}Q + (y,\partial_{i})\partial_{j}Q\right] - \frac{1}{2}\,\nabla^{2}g(0).(y,y)\,Q + V(0)\,Q\,,\,\Delta Q\right) \\ &= -\left(\frac{3}{2}\,\nabla^{2}(k+g)(0).(y,y)\,Q^{2}\,\nabla Q\,,\,\nabla Q\right) - \left(\frac{1}{2}\,\nabla[\nabla^{2}(k+g)(0).(y,y)]\,Q^{3}\,,\,\nabla Q\right) \\ &- \left(\nabla\left\{\nabla^{2}G_{ij}(0).\left[\frac{1}{2}\,(y,y)\partial_{ij}^{2}Q + (y,\partial_{i})\partial_{j}Q\right]\right\},\,\nabla Q\right) - V(0)\,\left(\nabla Q\,,\,\nabla Q\right) \\ &+ \frac{1}{2}\,\left(\nabla[\nabla^{2}g(0).(y,y)\,Q]\,,\,\nabla Q\right) \\ &= -\frac{3}{2}\,\int\nabla^{2}(k+g)(0).(y,y)\,Q^{2}\,|\nabla Q|^{2} + \frac{1}{8}\,\int\Delta\left[\nabla^{2}(k+g)(0).(y,y)\right]\,Q^{4} \\ &- \int\nabla^{2}G_{ij}(0).\left[\frac{1}{2}\,(y,y)\partial_{ij}^{2}\nabla Q + (y,\partial_{i})\partial_{j}\nabla Q\right]\,.\,\nabla Q - \frac{1}{4}\,\Delta(Tr(G))(0)\,\int Q^{2} - V(0)\,\int Q^{2} \\ &+ \Delta g(0)\,\left[\,-\int\frac{Q^{2}}{2}\,+\,\int\frac{|y|^{2}}{4}\,|\nabla Q|^{2}\right] \\ &= \Delta k(0)\,\left[\,\int\frac{Q^{2}}{2}\,-\,3\,\int\frac{|y|^{2}}{4}\,Q^{2}\,|\nabla Q|^{2}\,\right] + \Delta g(0)\,\int\frac{|y|^{2}}{4}\,\left(1-3\,Q^{2}\right)\,|\nabla Q|^{2} \\ &-\,\int\nabla^{2}G_{ij}(0).\left[\frac{1}{2}\,(y,y)\partial_{ij}^{2}\nabla Q + (y,\partial_{i})\partial_{j}\nabla Q\right]\,.\,\nabla Q - \frac{1}{4}\,\Delta(Tr(G))(0)\,\int Q^{2} - V(0)\,\int Q^{2} \end{split}$$

Altogether, this yields

$$a_1 = -\frac{1}{2} \Delta \left(k - \frac{1}{2} Tr(G) \right)(0) + 3 \Delta g(0) \|Q\|_{L^2}^{-2} \int \frac{|y|^2}{4} |\nabla Q|^2$$
(5.93)

Consequently, from hypothesis (H1), $\nabla^2 \left(k - \frac{1}{2}Tr(G)\right)(0) < 0$ is negative definite, and from hypothesis (H6), the second term in the RHS is positive. This now imply $a_1 > 0$, and ends the proof of (5.38).

Proof of (5.39) We compute thanks to (5.32), (5.33) along with the control of the modulation parameters of (4.61)

$$M^{(4)}(i(yQ + \lambda_c^2 B_2)) - i\partial_s(i(yQ + \lambda_c^2 B_2))$$

= $iL_-(yQ + \lambda_c^2 B_2) - i\lambda_c^2 \left[2QT_2^0 + \frac{1}{2}\nabla^2 k(0).(y,y)Q^2\right]yQ + \mathcal{O}(\lambda_c^3 e^{-|y|})$

so that to ensure (5.39), using $L_{-}(yQ) = -2\nabla Q$, we may simply choose B_2 solution to

$$L_{-}(B_{2}) = -2A_{2} + \left[2QT_{2}^{0} + \frac{1}{2}\nabla^{2}k(0).(y,y)Q^{2} + V(0)\right]yQ$$

which is solvable from (8.2) since its right-hand side is orthogonal to Q by definition of A_2 (5.91).

Proof of (5.40) We compute thanks to (5.32), (5.33) and (1.38) along with the control of the modulation

parameters of (4.61)

$$M^{(4)} \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) - i \partial_{s} \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) + \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y \left[\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right] = M^{(4)} \left(\Lambda P_{\mathcal{P}_{c}}\right) + L_{+}(E_{2} + E_{3}) + i L_{-}(D_{3}) - i \partial_{s} E_{2} - i \partial_{s} \Lambda T_{2} + \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y \Lambda Q + \mathcal{O}\left(\lambda_{c}^{4} e^{-|y|}\right)$$
(5.94)

Then, one easily checks from (1.49), (1.52) that $\partial_{\lambda}P_{\mathcal{P}} = \mathcal{O}(\mathcal{P})$, $\partial_{\alpha}P_{\mathcal{P}} = \mathcal{O}(\mathcal{P})$, $\partial_{b}P_{\mathcal{P}} = \mathcal{O}(\mathcal{P}^{2})$ and $\partial_{\beta}P_{\mathcal{P}} = \mathcal{O}(\mathcal{P}^{2})$. Thus, recall from (1.40), which is now simplified thanks to (4.61), (1.35) and (1.42), that $P_{\mathcal{P}_{c}}$ is solution to

$$-i b_c \lambda_c \partial_{\lambda_c} P_{\mathcal{P}_c} + \mathcal{L}_{\lambda_c \alpha_c} P_{\mathcal{P}_c} - P_{\mathcal{P}_c} + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} P_{\mathcal{P}_c} |P_{\mathcal{P}_c}|^2 - m_{31} P_{\mathcal{P}_c}$$
$$-i \frac{\lambda_c^3}{2C_0} \nabla^2 (G_{ij} - g I_{ij})(0).(y, y) y_i \partial_j P_{\mathcal{P}_c} = \mathcal{O}(\lambda_c^4 e^{-|y|})$$
with (5.95)

$$m_{31} = \left(\lambda_c \, c_0(\alpha_c) + \lambda_c^3 \, C_3\right) \cdot y + i \, \frac{\lambda_c^3}{2C_0} \left[\nabla^2 G_{ij}(0) \cdot (\partial_i, y) \, y_j + \nabla^2 \left(\frac{1}{2} \, Tr(G) - g\right)(0) \cdot (y, y)\right]$$

Therefore, as we did in (5.89), we derive (5.95) and use (5.88) to get

$$\begin{split} &-i b_c \lambda_c \,\partial_{\lambda_c} \left(\Lambda P_{\mathcal{P}_c}\right) + \mathcal{L}_{\lambda_c \alpha_c} \left(\Lambda P_{\mathcal{P}_c}\right) - \Lambda P_{\mathcal{P}_c} + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \tilde{M}(\Lambda P_{\mathcal{P}_c}) - m_3 \,\Lambda P_{\mathcal{P}_c} \\ &= 3 \left[-i b_c \lambda_c \,\partial_{\lambda_c} P_{\mathcal{P}_c} + \mathcal{L}_{\lambda_c \alpha_c} P_{\mathcal{P}_c} - P_{\mathcal{P}_c} + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} |P_{\mathcal{P}_c}|^2 P_{\mathcal{P}_c} - m_3 P_{\mathcal{P}_c} \right] \\ &+ 2i b_c \lambda_c \,\partial_{\lambda_c} P_{\mathcal{P}_c} + 2 \,P_{\mathcal{P}_c} + 2 \,m_{31} \,P_{\mathcal{P}_c} + y \,\cdot \nabla m_{31} \,P_{\mathcal{P}_c} + i \,\frac{\lambda_c^3}{2C_0} \,\nabla^2 \big(G_{ij} - g \,I_{ij}\big)(0).(y, y) \,y_i \,\partial_j P_{\mathcal{P}_c} \\ &+ y \,\cdot \left[-i b_c \lambda_c \,\partial_{\lambda_c} \big(\nabla P_{\mathcal{P}_c}\big) + \mathcal{L}_{\lambda_c \alpha_c} \big(\nabla P_{\mathcal{P}_c}\big) - \nabla P_{\mathcal{P}_c} + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \tilde{M}(\nabla P_{\mathcal{P}_c}) - m_3 \,\nabla P_{\mathcal{P}_c} \right] \\ &= 2 \,P_{\mathcal{P}_c} + 2 \,i \,b_c \lambda_c \,\partial_{\lambda_c} P_{\mathcal{P}_c} + 2 \,m_{31} \,P_{\mathcal{P}_c} + y \,\cdot \nabla m_{31} \,P_{\mathcal{P}_c} + i \,\frac{\lambda_c^3}{2C_0} \,\Lambda \Big[\nabla^2 \big(G_{ij} - g \,I_{ij}\big)(0).(y, y) \,y_i \,\partial_j P_{\mathcal{P}_c} \Big] \\ &- \Big[\frac{\lambda_c y \,\cdot \nabla k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \,|P_{\mathcal{P}_c}|^2 \,P_{\mathcal{P}_c} + y \,\cdot \nabla G_{ij} \partial_{ij}^2 P_{\mathcal{P}_c} - y \,\cdot \nabla g \,\Delta P_{\mathcal{P}_c} \Big] + \mathcal{O} \big(\lambda_c^4 \,e^{-|y|} \big) \\ &= 2 \,P_{\mathcal{P}_c} + 2 \,i \,b_c \lambda_c \,\partial_{\lambda_c} P_{\mathcal{P}_c} + 3 \,m_{31} \,P_{\mathcal{P}_c} + i \,\frac{\lambda_c^3}{2C_0} \,\Lambda \Big[\nabla^2 \big(G_{ij} - g \,I_{ij}\big)(0).(y, y) \,y_i \,\partial_j P_{\mathcal{P}_c} \Big] \\ &+ i \,\frac{\lambda_c^3}{2C_0} \,\Big[\nabla^2 G_{ij}(0).(\partial_i, y) \,y_j + \nabla^2 \Big(\frac{1}{2} \,Tr(G) - g \Big)(0).(y, y) \Big] \,P_{\mathcal{P}_c} \\ &- \Big[\frac{\lambda_c y \,\cdot \nabla k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \,|P_{\mathcal{P}_c}|^2 \,P_{\mathcal{P}_c} + y \,\cdot \nabla G_{ij} \partial_{ij}^2 P_{\mathcal{P}_c} - y \,\cdot \nabla g \,\Delta P_{\mathcal{P}_c} \Big] + \mathcal{O} \big(\lambda_c^4 \,e^{-|y|} \big) \end{split}$$

thus

$$- i b_c \lambda_c \partial_{\lambda_c} (\Lambda P_{\mathcal{P}_c}) + \mathcal{L}_{\lambda_c \alpha_c} (\Lambda P_{\mathcal{P}_c}) - \Lambda P_{\mathcal{P}_c} + \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \tilde{M} (\Lambda P_{\mathcal{P}_c}) - m_3 \Lambda P_{\mathcal{P}_c}$$

$$= 2 P_{\mathcal{P}_c} + 2 i b_c \lambda_c \partial_{\lambda_c} P_{\mathcal{P}_c} + 3 \left(\lambda_c c_0(\alpha_c) + \lambda_c^3 C_3\right) \cdot y P_{\mathcal{P}_c}$$

$$+ i \frac{\lambda_c^3}{2C_0} \left\{ 4 \left[\nabla^2 G_{ij}(0) \cdot (\partial_i, y) y_j + \nabla^2 \left(\frac{1}{2} Tr(G) - g \right)(0) \cdot (y, y) \right] P_{\mathcal{P}_c} + \Lambda \left[\nabla^2 \left(G_{ij} - g I_{ij} \right)(0) \cdot (y, y) y_i \partial_j P_{\mathcal{P}_c} \right] \right\}$$

$$- \left[\frac{\lambda_c y \cdot \nabla k(\lambda_c y + \alpha_c)}{k(\alpha_c)} |P_{\mathcal{P}_c}|^2 P_{\mathcal{P}_c} + y \cdot \nabla G_{ij} \partial_{ij}^2 P_{\mathcal{P}_c} - y \cdot \nabla g \Delta P_{\mathcal{P}_c} \right] + \mathcal{O} \left(\lambda_c^4 e^{-|y|} \right)$$

so that

$$i b_c \lambda_c \partial_{\lambda_c} (\Lambda P_{\mathcal{P}_c}) + M^{(4)} (\Lambda P_{\mathcal{P}_c}) + (\lambda_c c_0(\alpha_c) + \lambda_c^3 C_3) \cdot y \Lambda P_{\mathcal{P}_c}$$

$$= -2 P_{\mathcal{P}_c} - 2 i b_c \lambda_c \partial_{\lambda_c} T_2 - 3 (\lambda_c c_0(\alpha_c) + \lambda_c^3 C_3) \cdot y Q + \lambda_c^2 q_{21} + \lambda_c q_{22}(\alpha_c) + \lambda_c^3 q_{31}$$

$$- i \frac{\lambda_c^3}{2C_0} q_{32} + \mathcal{O}(\lambda_c^4 e^{-|y|})$$
(5.96)

where

$$q_{21} = \nabla^{2}(k+g)(0).(y,y) Q^{3} + \nabla^{2}G_{ij}(0).(y,y) \partial_{ij}^{2}Q - \nabla^{2}g(0).(y,y) Q$$

$$q_{22}(\alpha_{c}) = \nabla^{2}(k+g)(0).(y,\alpha_{c}) Q^{3} + \nabla^{2}G_{ij}(0).(y,\alpha_{c}) \partial_{ij}^{2}Q - \nabla^{2}g(0).(y,\alpha_{c}) Q$$

$$q_{31} = \frac{1}{2} \nabla^{3}(k+g)(0).(y,y,y) Q^{3} + \frac{1}{2} \nabla^{3}G_{ij}(0).(y,y,y) \partial_{ij}^{2}Q - \frac{1}{2} \nabla^{3}g(0).(y,y,y) Q$$

$$q_{32} = \left[\nabla^{2}G_{ij}(0).(\partial_{i},y) y_{j} + \nabla^{2}\left(\frac{1}{2}Tr(G) - g\right)(0).(y,y)\right] (4Q + \Lambda Q)$$

$$+ \nabla^{2}\left(G_{ij} - g I_{ij}\right)(0).(y,y) y_{i}\partial_{j} \Lambda Q + \Lambda \left[\nabla^{2}\left(G_{ij} - g I_{ij}\right)(0).(y,y) y_{i}\partial_{j} Q\right]$$
(5.97)

Eventually, with (4.62), (5.94) becomes

$$M^{(4)} \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) - i \partial_{s} \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) + \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y \left[\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right] = L_{+}(E_{2} + E_{3}) + i L_{-}(D_{3}) - i \partial_{s} E_{2} - 2 i b_{c} \lambda_{c} \partial_{\lambda_{c}} T_{2} - 2 P_{\mathcal{P}_{c}}$$

$$+ \lambda_{c}^{2} q_{21} + \lambda_{c} q_{22}(\alpha_{c}) + \lambda_{c}^{3} q_{31} - i \frac{\lambda_{c}^{3}}{2 C_{0}} q_{32} + \mathcal{O}\left(\lambda_{c}^{4} e^{-|y|}\right)$$
(5.98)

To ensure (5.40) we choose E_2 , E_3 and D_3 solutions to

$$L_{+}E_{2} = -\lambda_{c}^{2} q_{21} - \lambda_{c} q_{22}(\alpha_{c}) + 3\lambda_{c} c_{0}(\alpha_{c}) \cdot y Q + \lambda_{c} a_{21}(\alpha_{c}) \cdot y Q$$
(5.99)

with

$$a_{21}(\alpha_c) = \frac{-2\left(q_{22}(\alpha_c), \nabla Q\right)}{\int Q^2} - 3c_0(\alpha_c),$$
(5.100)

and

$$L_{+}E_{3} = -\lambda_{c}^{3} q_{31} + 3\lambda_{c}^{3} C_{3} \cdot y Q + \lambda_{c}^{3} a_{31} \cdot y Q, \qquad (5.101)$$

with

$$a_{31} = \frac{-2\left(q_{31}, \nabla Q\right)}{\int Q^2} - 3C_3 \tag{5.102}$$

and finally we wish to choose D_3 such as

$$L_{-}D_{3} = \frac{2}{C_{0}}\lambda_{c}^{2}\partial_{\lambda_{c}}T_{2} - \frac{1}{C_{0}}\lambda_{c}^{2}\partial_{\lambda_{c}}E_{2} + \frac{\lambda_{c}^{3}}{2C_{0}}q_{32} + \lambda_{c}^{3}a_{32}\varrho - \frac{\lambda_{c}^{3}}{2C_{0}}h_{3}$$
(5.103)

From (8.2), (5.99) and (5.101) are solvable since their right-hand side are both orthogonal to ∇Q . So there is only left to prove that the right-hand side of (5.103) may be chosen orthogonal to Q provided a_{32} is chosen wisely.

First remind from (1.62) and (3.79), with use of Lemma 4.7 that

$$(T_2, Q) = -\frac{1}{2} (L_+ T_2, \Lambda Q) = -\lambda_c^2 \Delta g(0) \frac{1}{2} \int \frac{|y|^2}{4} Q^2$$
(5.104)

Let $E_2 = E_{21} + E_{22}$ and $E_{22} = \lambda_c^2 E_{22}^0$ defined as
$$L_{+}E_{22} = -\lambda_c^2 \nabla^2 G_{ij}(0).(y,y) \partial_{ij}^2 Q$$

then from (5.99) along with a straightforward computation which yields

$$(E_{21}, Q) = -\frac{1}{2} (L_{+}E_{21}, \Lambda Q) = -\frac{\lambda_{c}^{2}}{2} (q_{21}, \Lambda Q)$$

$$= -\lambda_{c}^{2} \Delta g(0) \int \frac{|y|^{2}}{4} Q^{2}$$
(5.105)

hence

$$\frac{1}{C_0} \lambda_c^2 \partial_{\lambda_c} \left[2 \left(T_2, Q \right) - \left(E_{21}, Q \right) \right] = 0$$

so that (5.103) becomes

$$L_{-}D_{3} = \frac{\lambda_{c}^{2}}{C_{0}} \partial_{\lambda_{c}} \left(2T_{2} - E_{21} \right) + \frac{\lambda_{c}^{3}}{2C_{0}} \left(q_{32} - 4E_{22}^{0} - h_{3} \right) + \lambda_{c}^{3} a_{32} \rho$$

and thus to ensure the existence of D_3 it is enough to choose

$$a_{32} = \frac{\left(h_3 + 4E_{22}^0 - q_{32}, Q\right)}{2C_0\left(Q, \varrho\right)}$$
(5.106)

Proof of (5.41) We compute thanks to (5.32), (5.33) along with the control of the modulation parameters of (4.61) and the equation (5.95)

$$M^{(4)}(i(|y|^{2} P_{\mathcal{P}_{c}} + F_{2})) - i \partial_{s}(i(|y|^{2} P_{\mathcal{P}_{c}} + F_{2}))$$

$$= -i \left[-i b_{c} \lambda_{c} \partial_{\lambda_{c}} |y|^{2} P_{\mathcal{P}_{c}} + \mathcal{L}_{\lambda_{c} \alpha_{c}} (|y|^{2} P_{\mathcal{P}_{c}}) - |y|^{2} P_{\mathcal{P}_{c}} + \frac{k(\lambda_{c} y + \alpha_{c})}{k(\alpha_{c})} |y|^{2} P_{\mathcal{P}_{c}} |P_{\mathcal{P}_{c}}|^{2} \right]$$

$$+ i L_{-}F_{2} + \mathcal{O}(\lambda_{c}^{3} e^{-|y|})$$

$$= -i |y|^{2} \left[-i b_{c} \lambda_{c} \partial_{\lambda_{c}} P_{\mathcal{P}_{c}} + \mathcal{L}_{\lambda_{c} \alpha_{c}} P_{\mathcal{P}_{c}} - P_{\mathcal{P}_{c}} + \frac{k(\lambda_{c} y + \alpha_{c})}{k(\alpha_{c})} P_{\mathcal{P}_{c}} |P_{\mathcal{P}_{c}}|^{2} \right] - 4i \Lambda P_{\mathcal{P}_{c}}$$

$$- i \lambda_{c}^{2} q_{23} + i L_{-}F_{2} + \mathcal{O}(\lambda_{c}^{3} e^{-|y|})$$

$$= i \left[L_{-}F_{2} - 4 \Lambda P_{\mathcal{P}_{c}} - \lambda_{c}^{2} q_{23} \right] + \mathcal{O}(\lambda_{c}^{3} e^{-|y|})$$
th

with

$$q_{23} = \nabla^2 \big(Tr(G) \big)(0).(y,y) \, Q + 2 \, \nabla^2 G_{ij}(0). \Big[(y,y) \, \partial_j Q + (y,\partial_i) \, y_j \, Q \Big] - 2 \, \nabla^2 g(0).(y,y) \, \Lambda Q \tag{5.108}$$

To fulfill (5.41), we choose F_2 solution to

$$L_{-}F_{2} = -4E_{2} + \lambda_{c}^{2}q_{23} + \lambda_{c}^{2}a_{22}\varrho$$
(5.109)

using (5.105) and (3.14) it is now enough to take

$$a_{22} = \frac{\left(4 E_2^0 - q_{23}, Q\right)}{\left(Q, \varrho\right)} \tag{5.110}$$

which is obviously possible from (8.2).

Proof of (5.42) Ultimately, (5.42) simply relies on (3.13) and

$$M^{(4)}(\varrho) = L_{+}(\varrho) + \mathcal{O}(\lambda_{c}^{2} e^{-|y|})$$
$$= |y|^{2} Q + \mathcal{O}(\lambda_{c}^{2} e^{-|y|})$$

Recall Theorem 1.1 has been proven with a set of assumptions which could actually be written in term of curvature at the origin - our blow up point.

Indeed, in Appendix A 7 we have shown how introducing the metric factors g and G may be reinterpreted, at least near the origin, as considering the Laplace-Beltrami operator for a riemannian metric $h = g G^{ij} dx_i \otimes dx_j$, so that our solution has now to be thought as living on a riemannian manifold, for which we may focus on its geometrical aspects such as its curvature at the origin point.

In two dimensions as we are working with here, there is only one possibly non-zero component of the riemannian tensor curvature (which also corresponds to the Gaussian curvature up to some constant factor). We have already shown how our euclidean computations suggested we needed to assume the curvature is equal to zero at the origin $(R_{1212} = 0)$, meaning the metric terms g and G should be flat enough at the origin.

However, there are special cases in which such a flatness seems not to be required. It is already noticed in the work of Banica-Carles-Duyckaerts [BCD11] : given a metric $g_M = dr^2 + \phi(r)^2 d\theta^2$ expressed in polar coordinates, and a two-dimensional riemannian manifold (M, g_M) , they get the result :

Proposition 6.1. (part 1.3 [BCD11])

6

If \tilde{u} is a radially symmetric solution of NLS on M: $i \partial_t \tilde{u} + \Delta_M \tilde{u} + |\tilde{u}|^2 \tilde{u} = 0$

Then, u defined by $\tilde{u}(t,r) = \left(\frac{r}{\phi(r)}\right)^{1/2} u(t,r)$ satisfies the equation $i \partial_t u + \Delta u + k(x) |u|^2 u - V(x) u = 0$ with

$$k(r) = \frac{r}{\phi(r)}, \quad V(r) = \frac{1}{2} \frac{\phi''(r)}{\phi(r)} - \frac{1}{4} \left[\left(\frac{\phi'(r)}{\phi(r)} \right)^2 - \frac{1}{r^2} \right]$$

In the previous theorem, Δ_M denotes the Laplace-Beltrami operator for the metric g_M , and can thus be expressed as

$$\Delta_M = \partial_r^2 + \frac{\phi'(r)}{\phi(r)} \,\partial_r + \frac{1}{\phi(r)^2} \,\Delta_{S^1}$$

This proves for that types of metrics, a refined ansatz allows one to get rid of the annoying terms which has prevented us so far from solving the equation in a non-flat case. Here, we are willing to use the [BCD11] idea to somehow include the geometric deformation induced by the metric terms by modifying the geometric decomposition we are using to build the approximate solution with.

We consider a two-dimensional riemannian manifold (M, g), $M \simeq \mathbb{R}^2$, and we focus on the inhomogeneous and critical non-linear schrödinger problem

$$\begin{cases} i \partial_t u + \Delta_M u + k(x) |u|^2 u + V(x) u = 0, \quad (t, x) \in [0, T) \times M \\ u(0, x) = u_0(x), \quad u_0 : M \longrightarrow \mathbb{C} \in H^1(M) \end{cases}$$

$$\tag{6.1}$$

For details about elementary riemannian differential geometry, and in particular the exponential map and its expression in polar coordinates, one may refer to [Spi90].

Say we wish to build a solution of (6.1) that blows up at some point $p_0 \in M$, and at some time T. At first, we will be working on a small neighborhood $U_0 \subset M$ of p_0 , on which the exponential map at p_0 is a

 \mathcal{C}^{∞} -diffeomorphism and thus provides a set of coordinates, namely the geodesic normal coordinates for g. Notice one can take for U_0 any geodesic ball of radius strictly smaller than the radius of injectivity.

Then, for $x = \exp_{p_0} \left[x_0 \right] \in U_0$ we introduce $y = \exp_{p_0} \left[\frac{x_0 - \alpha}{\lambda} \right] \in V_0$ where λ and α are respectively scalling and translation parameters.

Consider now the euclidean space $T_{p_0}M \simeq \mathbb{R}^2$ the associated tangent plane at p_0 , and introduce the spherical coordinate system (r_0, θ_0) . Composition of (r_0, θ_0) with \exp_{p_0} is then the polar coordinate system $(r, \theta) = \exp_{p_0} \circ (r_0, \theta_0)$ where r is now the geodesic length of ray (r_0, θ_0) which has been projected on our surface M thanks to \exp_{p_0} . Let us also denote by (ρ, ω) the polar coordinates associated to the rescaled coordinates y. Using the geodesic polar coordinates, the metric g may now be expressed as

$$g = dr^{2} + f(r,\theta)^{2} d\theta^{2} = |dx|^{2} + \frac{1}{r^{2}} \left[1 - \left(\frac{f}{r}\right)^{2} \right] \left(x_{2}^{2} dx_{1}^{2} + x_{1}^{2} dx_{2}^{2} + 2 x_{1} x_{2} dx_{1} dx_{2} \right)$$
(6.2)

so that the Laplace-Beltrami operator may be written in geodesic polar coordinates

$$\Delta_g = \partial_r^2 + \frac{1}{f^2} \,\partial_\theta^2 + \frac{\partial_r f}{f} \,\partial_r - \frac{\partial_\theta f}{f^2} \,\partial_\theta$$

One may read in Part 3 of [Spi90], it is proved that for such a metric g, when written in geodesic polar coordinates as we just did, κ being the riemannian curvature of M at point p_0 , one has

$$f(r,\theta) = r - \frac{\kappa}{6} r^3 + o(r^3,\theta)$$

Now g being smooth, we assume a bit stronger regularity for the function f

$$f(r,\theta) = r - \frac{\kappa}{6}r^3 + \mathcal{O}(r^4,\theta)$$
(6.3)

The metric g may then be expanded as

$$g = Id - \frac{\kappa}{3} \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix} + r^3 g_3(r,\theta)$$

Remark 10. Locally, that expression is always true and relies only on a good choice of coordinates. Thus it seems no assumption other than that the metric g has no trapped geodesics is required to prove the result.

So the proper way to work this out would be to consider from now on some cut-off function $\chi \in C_0^{\infty}$, with compact support that would be $\chi = 1$ on $B(p_0, 1)$ and express the Laplace-Beltrami operator as $\Delta_g u = \Delta_g(\chi u) + \Delta_g((1-\chi)u)$. The first term may be expressed using the previous expressions. Then, dealing with the part that lives away from the bow up point would require to include some a priori concentration bound of the second term in the bootstrap.

Nevertheless, rather than doing that and facing the difficulty to glue all pieces together, we shall consider there is a global geodesic map in M for which the metric has the previous form with some compactly supported functions instead of the constants.

From now on, the situation is thus as follows : we are working with a set of coordinates defined on \mathbb{R}^2 which we assume to map \mathbb{R}^2 to M (so that $M \simeq \mathbb{R}^2$). Then, 0 will stand for the blow up point, around which these coordinates are defined as the geodesic normal coordinates. Basically, $\{|x| \leq 1\}$ will be the geodesic ball zone around blow up point, while $\{|x| \geq 1\}$ will be the area of properties at infinity.

For the sake of simplicity, we shall assume there is a global geodesic normal map given by the polar coordinates (r, θ) associated to variable x, for which at any point $x \in \mathbb{R}^2$, the metric and its volume form may be expressed

$$g = g_x = dr^2 + f(r,\theta)^2 d\theta^2 = Id + \frac{1}{r^2} \left[\left(\frac{f}{r}\right)^2 - 1 \right] \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix}, \quad g \in C^5(\mathbb{R}^2, \mathcal{M}_2(\mathbb{R})),$$
$$dg_x = \sqrt{|g_x|} \, dx = |f_x| \, dx, \quad \text{gaussian curvature of M at } \mathbf{x} = -\frac{\partial_r^2 f}{f^2}$$

and therefore the metric may be expressed as

$$g = Id + \left[-\frac{\kappa(r,\theta)}{3} + r c_3(r,\theta) \right] \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix} + r^4 g_4(r,\theta)$$
(6.4)

with $\kappa \in C_0^5(\mathbb{R}, \mathbb{R})$ satisfying $\kappa(0) = \kappa_0$ that stands for the Gaussian curvature at $x = 0, c_3 \in C_0^5(\mathbb{R}^2, \mathbb{R})$, and $g_4 \in C_0^5(\mathbb{R}^2, \mathcal{M}_2(\mathbb{R}))$. The idea is we may assume either that κ, c_3, g_4 have compact supports or that they decay fast enough at infinity to ensure the metric g is close enough to the euclidean metric so the usual analytical tools such as the smoothing effect and the Strichartz inequalities are availables.

In all the sequel, we consider potentials k and V, along with the metric g so the following assumptions hold :

(H1)
$$g \in C^5 \cap W^{1,\infty}$$
, $g_{ij} = g_{ji}$ $g = Id - \frac{\kappa(x)}{3} \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix} + |x|^3 g_3(x)$, κ , $g_3 \in C^5(\mathbb{R}^2)$,
(H2) $k \in C^5 \cap W^{1,\infty}$, $0 < k_1 < k(x) < 1$, $k(0) = 1$, $\nabla k(0) = 0$.

 $\begin{array}{ll} (H2) \quad k \in C^5 \cap W^{1,\infty}, \quad 0 < k_1 \le k(x) \le 1, \quad k(0) = 1, \quad \nabla k(0) = 0, \\ \nabla^2 k(0) + \max\left(0, \frac{\kappa_0}{3}\right) Id < 0, \quad \kappa_0 = \kappa(0) = \text{gaussian curvature of M at } 0, \end{array}$

$$(H3) \quad V \in C^3 \cap W^{1,\infty}, \quad V \ge V(0)$$

- (H4) $c_0 = Tr(g_3(0)) = 0, \quad \nabla^3 k(0) = 0, \quad \nabla V(0) = 0$
- (H5): Whatever is requested to ensure the Smoothing Effect and Strichartz in 8.3:

$$(1) \ \forall p \in \mathbb{N}^2, \ \left| \partial^p \kappa \right| \lesssim \langle x \rangle^{-\tau(|p|)-2}, \ \left| \partial^p g_3 \right| \lesssim \langle x \rangle^{-\tau(|p|)-3}, \ \tau : \mathbb{N} \to \mathbb{N}^*, \ \tau(m) > m+1, \ \forall m \ge 0$$
$$\left| \frac{\partial_r \kappa}{\kappa} \right| + \left| \frac{\partial_r g_3}{g_3} \right| = \mathcal{O}(1), \quad |x| \to +\infty$$
$$(2) \ \exists \ \nu > 0, \ \forall (x,\xi) \in \mathbb{R}^4, \ \sum_{i,j} g_{ij}(x) \xi_i \xi_j \ge \nu \ |\xi|^2, \ \exists \ C > 0, \ \frac{|dx|^2}{C} \le g \le C \ |dx|^2$$

(3) $\forall (x,\xi) \in T^* \mathbb{R}^2 \setminus \{0\}$, the flow is neither trapped backwards, nor forwards : $\lim_{t \to \pm \infty} |x(t,x,\xi)| = +\infty$

where $(x(t, x, \xi), \xi(t, x, \xi))$ would denote here the bicharacteristic originating at (x, ξ) , that is the integral curve of the Hamilton vector field associated to the principal symbol of Δ_g , with initial condition (x, ξ) . Assumptions (H1)-(H4) are describing the local behavior of g around the blow up point, while assumption (H5) describes its asymptotic behavior.

Remark 11. Notice the hypothesis (H4) is not required when dealing with a riemannian metric of constant curvature. Indeed when M resembles at 0 the sphere or the hyperbolic space of constant curvatures respectively 1 and -1, the function f defined in (6.3) which locally represents the metric may then be respectively given by $f(r,\theta) = \sin(r)$ or $\sinh(r)$, so it is an odd function hence there is no third order term for g and $c_0 = 0$. Also notice assumption (H1) together with (H5) – (1) here implies assumption (8.8) since $\tau(|p| = 1) > 1$.

We now state the result

Theorem 6.2. (Existence and Uniqueness of a critical element at a nondegenerate critical point).

Let $x_0 \in \mathbb{R}^2$ with functions k, V and g satisfying the previous set of hypotheses (H1) - (H3) and (H5). Then for all E_0 such as $E_0 + C_E > 0$, with

$$C_E = \frac{1}{8} \int \left[\nabla^2 k(0) + \frac{\kappa_0}{3} Id \right] (y, y) Q^4 dy + \frac{1}{2} \int \left[V(0) + \frac{\kappa_0}{3} \right] Q^2 dy$$

there exists a H^1 critical mass, $||u(t)||_{L^2(dg_x)} = ||Q||_{L^2}$, blow up solution to (6.1) with energy E_0 , which blows up at time T = 0, and at point x_0 in the sense

$$|u(t)|^2 \rightharpoonup ||Q||_{L^2}^2 \,\delta_{x=x_0}, \quad \text{as } t \to 0, \quad \text{in } L^2(dg_x)$$

Moreover one also has the momentum that goes to 0 at blow up time

$$\lim_{x \to 0} \mathcal{I}m\left(\int \nabla u \,\overline{u} \, dg_x\right) = 0 \tag{6.5}$$

Furthermore, whenever assumption (H4) also holds, the above solution is the unique up to phase shift H^1 critical mass, $\|u(t)\|_{L^2(dg_x)} = \|Q\|_{L^2}$, blow up solution to (6.1) with energy E_0 , which blows up at time T = 0, and at point x_0 .

We are now willing to take benefit from the form itself of the profile $Q_{\mathcal{P}}$ of the approximate solution we have constructed in the first sections. Indeed the function $v = Q_{\mathcal{P}} + \epsilon$ introduced in the ansatz was such that its profile $Q_{\mathcal{P}}$ is radially symmetric with respect to the rescaled variable y, at least up to the second order with respect to its modulation parameters. Thus the derivatives ∂_{ω} , $\partial^2_{\rho\omega}$, ∂^2_{ω} are at least $\mathcal{O}(\lambda^2)$ when applied to $Q_{\mathcal{P}}$.

Like we have already done in the first sections, we now introduce the ansatz

 $\frac{1}{t}$

$$u(t,x) = \frac{1}{\lambda} \frac{1}{k(\alpha)^{1/2}} v\left(t, \frac{x-\alpha}{\lambda}\right) e^{i\gamma}$$
(6.6)

where the new time variable s is such as $\frac{ds}{dt} = \frac{1}{\lambda^2}$. Then we will also introduce the conformal speed parameter b and the galilean invariance parameter β through

$$w(s,y) = v(s,y) e^{i b \frac{|y|^2}{4} - i\beta \cdot y}$$
(6.7)

Once again, we use the modulation method. We will choose an appropriate set of laws, that is ODE's, for these parameters to make the associated approximate solution being a precise enough profile for our NLS solution. Along with the local existence (of both the profile and its parameters) which we intend to extend up to the blowing time, we will consider as before, the set of estimates to be checked later (in section 6.7.5) that we include in the bootstrap process

$$|\alpha| \lesssim \lambda^{3/2}, \quad |\beta| \lesssim \lambda^{3/2}, \quad b \sim \lambda, \quad 0 < \mu \lesssim \lambda^3$$
(6.8)

where

$$\mu_s = b \,|\alpha|^2, \quad \mu(t) = \int_0^t \frac{b(\tau) \,|\alpha(\tau)|^2}{\lambda^2(\tau)} \,d\tau \tag{6.9}$$

is a parameter we will need to consider when building the approximate profile of the solution. In the sequel, to make short we shall often write remainings terms as fourth order terms such as

$$\lambda^{4} R = \mathcal{O}\left(\lambda^{4} + |\alpha|^{3} + \lambda^{3} |\alpha| + \lambda |\alpha|^{2}\right) \quad \text{or} \quad \lambda^{4} R = \mathcal{O}\left(\lambda^{4} + b^{4} + |\alpha|^{3} + \lambda^{3} \left(|\alpha| + |\beta|\right) + \lambda \left(|\alpha|^{2} + |\beta|^{2}\right)\right)$$

where R should be some regular and smooth function.

Next, the Laplace-Beltrami operator expressed in the polar coordinates associated to the rescaled variable y is

$$\begin{split} \left(\Delta_{g}\right)_{x} &= \frac{1}{\lambda^{2}} \left(\Delta_{g}\right)_{y} \\ &= \frac{1}{\left(\lambda\rho\right)^{2}} \left\{ \left(y_{1}^{2} g^{11} + y_{2}^{2} g^{22} + 2 y_{1} y_{2} g^{12}\right) \partial_{\rho}^{2} + \left(y_{2}^{2} g^{11} + y_{1}^{2} g^{22} - 2 y_{1} y_{2} g^{12}\right) \left(\frac{1}{\rho} \partial_{\rho} + \frac{1}{\rho^{2}} \partial_{\omega}^{2}\right) \right. \\ &\quad \left. + \left[y_{1} y_{2} \left(g^{11} - g^{22}\right) + \left(y_{2}^{2} - y_{1}^{2}\right) g^{12}\right] \left(\frac{1}{\rho^{2}} \partial_{\omega} - \frac{1}{\rho} \left[\partial_{\rho} \partial_{\omega} + \partial_{\omega} \partial_{\rho}\right]\right) \right\} \\ &\quad \left. + \frac{1}{\lambda^{2}} \left\{ \left(\frac{y_{1}}{\rho} \partial g^{1} + \frac{y_{2}}{\rho} \partial g^{2}\right) \partial_{\rho} + \left(-\frac{y_{2}}{\rho^{2}} \partial g^{1} + \frac{y_{1}}{\rho^{2}} \partial g^{2}\right) \partial_{\omega} \right\} \\ & \text{with} \quad \partial g^{1} = \frac{\partial_{y_{1}} \left(\sqrt{|g|} g^{11}\right) + \partial_{y_{2}} \left(\sqrt{|g|} g^{21}\right)}{\sqrt{|g|}} \quad \text{and} \quad \partial g^{2} = \frac{\partial_{y_{2}} \left(\sqrt{|g|} g^{22}\right) + \partial_{y_{1}} \left(\sqrt{|g|} g^{12}\right)}{\sqrt{|g|}} \end{split}$$

then one needs to expand (6.4) with respect to the modulation parameters

$$\begin{aligned} \kappa(\lambda y + \alpha) &= \kappa_0 + \nabla \kappa(0) \cdot y + \mathcal{O}(\lambda^2 + \lambda |\alpha|), \quad c_3(\lambda y + \alpha) = c_0 + \mathcal{O}(\lambda + |\alpha|) \\ g &= Id - \frac{\kappa_0}{3} M_g(\lambda, \alpha, y) + \lambda^3 \left(c_0 \rho - \frac{1}{3} \nabla \kappa(0) \cdot y\right) \begin{pmatrix} y_2^2 & -y_1 y_2 \\ -y_1 y_2 & y_1^2 \end{pmatrix} + \lambda^4 R_g(\lambda, \alpha, y), \\ g^{-1} &= Id + \frac{\kappa_0}{3} M_g(\lambda, \alpha, y) - \lambda^3 \left(c_0 \rho - \frac{1}{3} \nabla \kappa(0) \cdot y\right) \begin{pmatrix} y_2^2 & -y_1 y_2 \\ -y_1 y_2 & y_1^2 \end{pmatrix} + \lambda^4 R_{g^{-1}}(\lambda, \alpha, y), \end{aligned}$$
(6.10)
$$\sqrt{|g|} = 1 - \frac{\kappa_0}{6} \left[(\lambda \rho)^2 + 2\lambda \alpha \cdot y + |\alpha|^2 \right] + \frac{1}{6} \left(3 c_0 \rho - \nabla \kappa(0) \cdot y \right) \lambda^3 \rho^2 + \lambda^4 R_{|g|}(\lambda, \alpha, y) \\ M_g(\lambda, \alpha, y) &= \lambda^2 \begin{pmatrix} y_2^2 & -y_1 y_2 \\ -y_1 y_2 & y_1^2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \alpha_2 y_2 & -(\alpha_1 y_2 + \alpha_2 y_1) \\ -(\alpha_1 y_2 + \alpha_2 y_1) & 2\alpha_1 y_1 \end{pmatrix} + \begin{pmatrix} \alpha_2^2 & -\alpha_1 \alpha_2 \\ -\alpha_1 \alpha_1 & \alpha_1^2 \end{pmatrix} \end{aligned}$$

where R_g , $R_{g^{-1}}$, $R_{|g|}$ are regulars functions in y such that $\lambda^4 R_{g/g^{-1}/|g|} = \mathcal{O}(\lambda^4 + |\alpha|^3 + \lambda^3 |\alpha| + \lambda |\alpha|^2)$, and so the Laplace-Beltrami operator becomes

$$\begin{split} &\left(\Delta_{g}\right)_{x} = \frac{1}{\lambda^{2}} \left(\Delta_{g}\right)_{y} \\ &= \frac{1}{\lambda^{2}} \left\{ \left[1 + \frac{\kappa_{0}}{3} \left(\frac{\alpha^{\perp} \cdot y}{\rho} \right)^{2} + \lambda^{4} R_{11}^{\Delta} \right] \partial_{\rho}^{2} + \left[1 - \frac{\kappa_{0}}{3} \left(\lambda \rho \right)^{2} + \lambda^{3} \delta_{3} + \frac{\kappa_{0}}{3} \left(\frac{\alpha \cdot y}{\rho} \right)^{2} + \lambda^{4} R_{1}^{\Delta} \right] \frac{1}{\rho} \partial_{\rho} \\ &+ \left(1 + \frac{\kappa_{0}}{3} \left[\left(\lambda \rho \right)^{2} + 2 \lambda \alpha \cdot y + \frac{\left(\alpha \cdot y \right)^{2}}{\rho^{2}} \right] - \lambda^{3} \left[c_{0} \rho^{3} - \frac{\rho^{2}}{3} \nabla \kappa(0) \cdot y \right] + \lambda^{4} R_{22}^{\Delta} \right) \frac{1}{\rho^{2}} \partial_{\omega}^{2} \\ &- \left(\frac{\kappa_{0}}{3} \left[\lambda \alpha^{\perp} \cdot y + \frac{\left(\alpha \cdot y \right) \left(\alpha^{\perp} \cdot y \right)}{\rho^{2}} \right] + \lambda^{4} R_{12}^{\Delta} \right) \frac{1}{\rho} \left[\partial_{\rho} \partial_{\omega} + \partial_{\omega} \partial_{\rho} \right] \\ &- \left(\frac{\kappa_{0}}{3} \left[\lambda \alpha^{\perp} \cdot y + \frac{\left(\alpha \cdot y \right) \left(\alpha^{\perp} \cdot y \right)}{\rho^{2}} \right] - \frac{\lambda^{3}}{6} \nabla \kappa(0)^{\perp} \cdot y + \lambda^{4} R_{2}^{\Delta} \right) \frac{1}{\rho^{2}} \partial_{\omega} \right\} \end{split}$$

where $\delta_3 = \frac{1}{2} \left(3 c_0 \rho^3 - \rho^2 \nabla \kappa(0) \cdot y \right)$ and R_i^{Δ} , R_{ij}^{Δ} , i, j = 1, 2, are regulars and $L^2_{\text{loc}}(\mathbb{R}^2)$ functions in rescaled variable y such that $\lambda^4 R_{i/ij}^{\Delta} = \mathcal{O}(\lambda^4 + |\alpha|^3 + \lambda^3 |\alpha| + \lambda |\alpha|^2)$, and in addition we used the notation $\alpha^{\perp} = (-\alpha_2, \alpha_1)$ that stands for the orthogonal vector of α .

Next, between the two parts of the ansatz, following the method explained in part 1.3 of [BCD11], we first introduce

$$v(s,y) = \left(\frac{\rho}{\varphi}\right)^{1/2} \tilde{v}(s,y) \tag{6.11}$$

where φ is a function to be chosen such that the previous expression of the Laplace-Beltrami operator would be of the form

$$\begin{aligned} \left(\Delta_{g}\right)_{x} &= \frac{1}{\lambda^{2}} \left(\Delta_{g}\right)_{y} \\ &= \frac{1}{\lambda^{2}} \left\{ \left(1 + \mathcal{O}\left(\lambda^{4} + |\alpha|^{2}\right)\right) \partial_{\rho}^{2} + \left(\frac{\partial_{\rho}\varphi}{\varphi} + \mathcal{O}\left(\lambda^{3} + |\alpha|^{2}\right)\right) \partial_{\rho} + \left(1 + \mathcal{O}\left(\lambda^{2}\right)\right) \frac{1}{\rho^{2}} \partial_{\omega}^{2} + \mathcal{O}\left(\lambda^{3}\right) \left(\frac{1}{\rho} \partial_{\rho\omega}^{2} + \frac{1}{\rho^{2}} \partial_{\omega}\right) \right\} \end{aligned}$$

Such a transform is designed to flatten the Laplace Beltrami operator around the blow up point, so the geometric contributions, such as the riemannian curvature, appear more clearly in the equation. Some computations and later simplifications show that φ should be chosen as

$$\varphi(\lambda,\rho,\omega) = \rho \,\exp\left(-\frac{\kappa_0}{6}\,(\lambda\rho)^2 + \frac{c_0}{2}\,(\lambda\rho)^3\right) = \rho\left(1 - \frac{\kappa_0}{6}\,(\lambda\rho)^2 + \frac{c_0}{2}\,(\lambda\rho)^3 + \lambda^4\,\Phi\right) \tag{6.12}$$

where Φ is a regular function such that $\lambda^4 \Phi = \mathcal{O}(\lambda^4 + |\alpha|^3 + \lambda^3 |\alpha| + \lambda |\alpha|^2)$. Thanks to that change, one gets

$$(\Delta_g)_x v = \frac{1}{\lambda^2} \left(\frac{\rho}{\varphi}\right)^{\frac{1}{2}} \left[\mathcal{L}_g \tilde{v} + V_\varphi \tilde{v}\right], \quad \text{with}:$$

$$\mathcal{L}_g \tilde{v} = \Delta \tilde{v} + \left(\frac{\kappa_0}{3} \left(\frac{\alpha^{\perp} \cdot y}{\rho}\right)^2 + \lambda^4 \tilde{R}_{11}^{\Delta}\right) \partial_\rho^2 \tilde{v} + \left(-\frac{\lambda^3}{2} \rho^2 \nabla \kappa(0) \cdot y + \frac{\kappa_0}{3} \left(\frac{\alpha \cdot y}{\rho}\right)^2 + \lambda^4 \tilde{R}_{1}^{\Delta}\right) \frac{1}{\rho} \partial_\rho \tilde{v}$$

$$+ \left(\frac{\kappa_0}{3} \left[(\lambda \rho)^2 + 2\lambda \alpha \cdot y + \left(\frac{\alpha \cdot y}{\rho}\right)^2\right] + \frac{(\lambda \rho)^3}{3} \left[\nabla \kappa(0) \cdot \frac{y}{\rho} - 3c_0\right] + \lambda^4 \tilde{R}_{22}^{\Delta}\right) \frac{1}{\rho^2} \partial_\omega^2 \tilde{v}$$

$$- \left(\frac{\kappa_0}{3} \left[\lambda \alpha^{\perp} \cdot y + \frac{(\alpha \cdot y) (\alpha^{\perp} \cdot y)}{\rho^2}\right] - \frac{\lambda^3}{6} \nabla \kappa(0)^{\perp} \cdot y + \lambda^4 \tilde{R}_2^{\Delta}\right) \frac{1}{\rho^2} \partial_\omega \tilde{v}$$

$$- \left(\frac{\kappa_0}{3} \left[\lambda \alpha^{\perp} \cdot y + \frac{(\alpha \cdot y) (\alpha^{\perp} \cdot y)}{\rho^2}\right] + \lambda^4 \tilde{R}_{12}^{\Delta}\right) \frac{1}{\rho} \left[\partial_\rho \partial_\omega + \partial_\omega \partial_\rho\right] \tilde{v}$$

$$(6.13)$$

and

$$V_{\varphi} = \frac{1}{4} \left[\left(\frac{\partial_{\rho} \varphi}{\varphi} \right)^2 - \frac{1}{\rho^2} \right] - \frac{\partial_{\rho}^2 \varphi}{2\varphi} + \frac{1}{\rho^2} \left[\frac{3}{4} \left(\frac{\partial_{\omega} \varphi}{\varphi} \right)^2 - \frac{\partial_{\omega}^2 \varphi}{2\varphi} \right] + \lambda^4 \Phi_V = \frac{\kappa_0}{3} \lambda^2 - \frac{9c_0}{4} \lambda^3 \rho + \lambda^4 \tilde{\Phi}_V$$
(6.14)

It is then equivalent to say the function u is solution of NLS on M and that the function \tilde{v} is solution to

$$i \partial_s \tilde{v} + \mathcal{L}_g \tilde{v} - \tilde{v} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{v}|^2 \tilde{v} - i \frac{\lambda_s}{\lambda} \Lambda \tilde{v} - i \frac{\alpha_s}{\lambda} \cdot \left(\nabla \tilde{v} + \frac{\lambda}{2} \frac{\nabla k(\alpha)}{k(\alpha)} \tilde{v}\right) - \tilde{\gamma}_s \tilde{v} + \left(\lambda^2 V(\lambda y + \alpha) + V_\varphi + \frac{i}{2} \left[\frac{\lambda_s}{\lambda} \left(\frac{y \cdot \nabla \varphi}{\varphi} - 1\right) + \frac{\alpha_s}{\lambda} \cdot \left(\frac{\nabla \varphi}{\varphi} - \frac{y}{\rho^2}\right) - \frac{\partial_s \varphi}{\varphi}\right]\right) \tilde{v} = 0$$

$$(6.15)$$

Eventually, introduce the last part of the ansatz which we have used in the previous sections

$$\tilde{w}(s,y) = \tilde{v}(s,y) \, e^{i \, b \, \frac{|y|^2}{4} - i \, \beta \, . \, y}$$

we are now able to rewrite approximate equation 1.40 and see what have changed. Thus, up to the fourth order with respect to the parameters, function \tilde{w} is now solution to

$$i\partial_{s}\tilde{w} + \mathcal{L}_{g}\tilde{w} - \tilde{w} + \frac{\rho}{\varphi}\frac{k(\lambda y + \alpha)}{k(\alpha)}|\tilde{w}|^{2}\tilde{w} + (b_{s} + b^{2})\frac{\rho^{2}}{4}\tilde{w} + \left[\lambda^{2}V(\lambda y + \alpha) + V_{\varphi}\right]\tilde{w}$$

$$-\left\{\left(\beta_{s} + b\beta\right).y + i\left[\lambda\beta.\frac{\nabla k(\alpha)}{k(\alpha)} - \frac{b\lambda^{3}}{2}\rho^{2}\nabla\kappa(0).y + \frac{\kappa_{0}}{3}\left(\frac{b}{2} - \frac{\beta.y}{\rho^{2}}\right)\left(\frac{\alpha.y}{\rho}\right)^{2} + \frac{\kappa_{0}}{3}\lambda^{2}\beta.y\right]\right\}\tilde{w}$$

$$-i\left\{\left[-\frac{b\lambda}{2}\frac{\partial_{\lambda}\varphi}{\varphi} + \frac{b}{2}\left(\frac{y.\nabla\varphi}{\varphi} - 1\right) - \beta.\left(\frac{\nabla\varphi}{\varphi} - \frac{y}{\rho^{2}}\right)\right]\tilde{w} + \frac{\kappa_{0}}{3}\left[b\rho\left(\frac{\alpha^{\perp}.y}{\rho}\right)^{2}\partial_{\rho}\tilde{w} + f_{3}\partial_{\omega}\tilde{w}\right]\right\}$$

$$-\left(\frac{\lambda_{s}}{\lambda} + b\right)\left[\left(b\frac{\rho^{2}}{2} - \beta.y\right)\tilde{w} + i\left(\Lambda\tilde{w} + \frac{1}{2}\left[\left(1 - \frac{y.\nabla\varphi}{\varphi}\right) + \lambda\frac{\partial_{\lambda}\varphi}{\varphi}\right]\tilde{w}\right]\right] - \left(\tilde{\gamma}_{s} - |\beta|^{2}\right)\tilde{w}$$

$$-\left(\frac{\alpha_{s}}{\lambda} - 2\beta\right).\left[\left(b\frac{y}{2} - \beta\right)\tilde{w} + \frac{i}{2}\left[\frac{\lambda\nabla k(\alpha)}{k(\alpha)} - \left(\frac{\nabla\varphi}{\varphi} - \frac{y}{\rho^{2}}\right)\right]\tilde{w}\right] + \left(\mathcal{P}^{4}\mathcal{R}e + i\mathcal{P}^{\frac{9}{2}}\mathcal{I}m\right)\mathcal{R}_{E} = 0$$

where $f_3 = \mathcal{O}\left(\mathcal{P}^2\left(|\alpha| + |\beta|\right) + \mathcal{P}\left|\alpha\right||\beta|\right)$ and R_E is a regular function such that $\lambda^4 R_E = \mathcal{O}\left(\lambda^4 + (\lambda + b)\left(|\alpha|^2 + |\beta|^2\right) + (\lambda + b)^3\left(|\alpha| + |\beta|\right)\right)$. Some more calculations from (6.12) lead to

$$-\frac{b\lambda}{2}\frac{\partial_{\lambda}\varphi}{\varphi} + \frac{b}{2}\left(\frac{y\cdot\nabla\varphi}{\varphi} - 1\right) - \beta\cdot\left(\frac{\nabla\varphi}{\varphi} - \frac{y}{\rho^2}\right) = \frac{\kappa_0}{3}\lambda^2\beta\cdot y - \frac{3c_0}{2}\lambda^3\rho\beta\cdot y \tag{6.17}$$

It now requires very similar a proof to the one of Theorem 1.1 to build an approximate solution of equation 6.16 provided the parameters satisfy the set of ODE equations

$$b_s + b^2 = B_1(\lambda, \alpha), \quad \beta_s + b\,\beta = B_2(\lambda, \alpha, b, \beta), \quad \frac{\lambda_s}{\lambda} + b = 0, \quad \frac{\alpha_s}{\lambda} - 2\,\beta = 0, \quad \tilde{\gamma}_s - |\beta|^2 = 0 \tag{6.18}$$

Now we include into the initial bootstrap assumption that the $\mathcal{P} = (\lambda, \alpha, b, \beta)$ parameters satisfy to these equations up to some fourth order corrections. In section 6.3, we show the bootstrap runs well : the previous system of ODEs keeps consistency while time is going towards the blow-up time. Thus \tilde{w} is solution to

$$i\partial_{s}\tilde{w} + \mathcal{L}_{g}\tilde{w} - \tilde{w} + \frac{\rho}{\varphi}\frac{k(\lambda y + \alpha)}{k(\alpha)}|\tilde{w}|^{2}\tilde{w} + B_{1}\frac{\rho^{2}}{4}\tilde{w} - i\frac{\kappa_{0}}{3}\left[b\rho\left(\frac{\alpha^{\perp}\cdot y}{\rho}\right)^{2}\partial_{\rho}\tilde{w} + f_{3}\partial_{\omega}\tilde{w}\right] \\ - \left\{B_{2}\cdot y + i\left[\lambda\beta\cdot\frac{\nabla k(\alpha)}{k(\alpha)} - \frac{b\lambda^{3}}{2}\rho^{2}\nabla\kappa(0)\cdot y + \frac{\kappa_{0}}{3}\left(\frac{b}{2} - \frac{\beta\cdot y}{\rho^{2}}\right)\left(\frac{\alpha\cdot y}{\rho}\right)^{2} - \frac{2\kappa_{0}}{3}\lambda^{2}\beta\cdot y\right]\right\}\tilde{w}$$

$$+ \left[\lambda^{2}\left(V(0) + \frac{\kappa_{0}}{3}\right) + \lambda^{3}\left(\nabla V(0)\cdot y - \frac{9c_{0}}{4}\rho\right) + \lambda^{2}\nabla V(0)\cdot\alpha\right]\tilde{w} + \mathcal{P}^{4}\left(\mathcal{R}e + i\mathcal{P}^{\frac{1}{2}}\mathcal{I}m\right)\tilde{R}_{E} = 0$$

$$(6.19)$$

where \hat{R}_E is a regular function in y such that

$$\mathcal{P}^{4} \mathcal{R} e \tilde{R}_{E} = \mathcal{O} \left(\mathcal{P}^{4} + |\alpha|^{3} + \mathcal{P} \left(|\alpha|^{2} + |\beta|^{2} \right) + \mathcal{P}^{3} \left(|\alpha| + |\beta| \right) \right),$$

$$\mathcal{P}^{\frac{9}{2}} \mathcal{I} m \tilde{R}_{E} = \mathcal{O} \left(\mathcal{P}^{5} + \mathcal{P} |\alpha|^{3} + \mathcal{P}^{2} \left(|\alpha|^{2} + |\beta|^{2} \right) + \mathcal{P}^{3} \left(|\alpha| + |\beta| \right) \right)$$

Remark 12. Notice it has been made clear before that in order to get the estimate $b \sim \lambda$, it was necessary that the b law satisfies at least $B_1 = \mathcal{O}(\lambda^3)$. Unlike the previous sections in which we had to assume the riemannian curvature at 0, that is the blow up point, was null, it will no longer be necessary as the refined ansatz includes that geometric pertubation. However, that perturbation will later modify the conservation laws, on which we will therefore focus on.

Like in Proposition 1.1 we want to prove there is a solution

$$\tilde{w} = \tilde{P}_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^5), \quad \tilde{P}_{\mathcal{P}} = (T_2 + T_3 + T_4) + i(S_3 + S_4)$$

with T_j and S_j regular functions, of order j with respect to \mathcal{P} . One may then prove very similarly to Proposition 1.1 the following

Proposition 6.3. Let $C_0 > 0$. One can find a universal constant c > 0 and a small constant $\eta^*(C_0) > 0$ such that whenever $|\mathcal{P}| = |(b, \lambda, \alpha, \beta)| \leq \eta^*$, what follows is satisfied :

We have regulars and well-localized functions $(T_j, S_j)_{1 \le j \le 4}$ of order j with respect to \mathcal{P} such that :

$$\tilde{P}_{\mathcal{P}} = Q + \sum_{j=1}^{4} \left(T_j + iS_j \right)$$

is an approximate solution of (6.19), that is

$$i\left(-b^{2}+B_{1}\right)\partial_{b}\tilde{P}_{\mathcal{P}}-ib\,\lambda\,\partial_{\lambda}\tilde{P}_{\mathcal{P}}+2\,i\,\lambda\,\beta\,.\,\partial_{\alpha}\tilde{P}_{\mathcal{P}}+i\left(-b\,\beta+B_{2}\right).\,\partial_{\beta}\tilde{P}_{\mathcal{P}}+\mathcal{L}_{g}\tilde{P}_{\mathcal{P}}-\tilde{P}_{\mathcal{P}}\\ +\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}\,|\tilde{P}_{\mathcal{P}}|^{2}\,\tilde{P}_{\mathcal{P}}+B_{1}\,\frac{\rho^{2}}{4}\,\tilde{P}_{\mathcal{P}}+\left[\lambda^{2}\left(V(0)+\frac{\kappa_{0}}{3}\right)+\lambda^{3}\left(\nabla V(0)\,.\,y-\frac{9c_{0}}{4}\,\rho\right)+\lambda^{2}\,\nabla V(0)\,.\,\alpha\right]\tilde{P}_{\mathcal{P}}\\ -\left\{B_{2}\,.\,y+i\left[\lambda\,\beta\,.\,\frac{\nabla k(\alpha)}{k(\alpha)}-\frac{b\lambda^{3}}{2}\,\rho^{2}\,\nabla\kappa(0)\,.\,y+\frac{\kappa_{0}}{3}\left(\frac{b}{2}-\frac{\beta\,.\,y}{\rho^{2}}\right)\left(\frac{\alpha\,.\,y}{\rho}\right)^{2}-\frac{2\kappa_{0}}{3}\,\lambda^{2}\,\beta\,.\,y\right]\right\}\tilde{P}_{\mathcal{P}}\\ -i\,\frac{\kappa_{0}}{3}\,b\,\rho\left(\frac{\alpha^{\perp}\,.\,y}{\rho}\right)^{2}\partial_{\rho}\tilde{P}_{\mathcal{P}}+\mathcal{P}^{4}\left(\mathcal{R}e\tilde{R}_{E}+i\,\mathcal{P}^{\frac{1}{2}}\mathcal{I}m\tilde{R}_{E}\right)=-\tilde{\Psi}_{\mathcal{P}}$$

$$(6.20)$$

with $\Psi_{\mathcal{P}}$ polynomial in \mathcal{P} , and well-localized in y:

$$\forall p \in \mathbb{N}^2, \exists C_p > 0, \quad \left| \partial^p \tilde{\Psi}_{\mathcal{P}}(y) \right| \lesssim \mathcal{P}^5 e^{-C_p|y|}$$
(6.21)

In order to satisfy this construction, the b and β laws are set in such a way that $B_1 \in \mathbb{R}$ and $B_2 \in \mathbb{R}^2$ are :

$$B_{1} = \left[\frac{1}{2} \left(\nabla^{2} k(0) - \frac{\kappa_{0}}{3} Id\right) \cdot (\alpha, \alpha) - \frac{\kappa_{0}}{6} \mu\right] K_{1} + K_{0} \lambda^{3} + \mathcal{O}(\lambda^{4})$$

$$B_{2} = \lambda^{3} \left[\nabla V(0) + \|Q\|_{L^{2}}^{-2} \left(\nabla \kappa(0) \int \rho^{2} |\nabla Q|^{2} dy + \frac{1}{8} \int \nabla^{3} k(0) \cdot (y, y, .) Q^{4} dy\right)\right]$$

$$+ \lambda \nabla^{2} k(0) \cdot (\alpha, .) + \mathcal{O}(\lambda^{4})$$
with :
$$K_{0} = \frac{c_{0}}{8} \int \left[9 + \rho^{2} Q^{2}\right] \rho Q^{2} dy \left\|\frac{|y|}{2} Q\right\|_{L^{2}}^{-2}, \quad K_{1} = \|Q\|_{L^{2}}^{2} \left\|\frac{|y|}{2} Q\right\|_{L^{2}}^{-2}$$
(6.22)

Proof of Proposition 6.3 :

Clearly the equation is satisfied at order 0 since it is the Ground State equation for Q, and then it is enough to take $T_1 = S_1 = 0$ to ensure it is satisfied at order 1. Proceeding with the second order, the equation is satisfied provided

$$\begin{cases} L_{+}T_{2} = \frac{\kappa_{0}}{3} \left[\left(\frac{\alpha^{\perp} \cdot y}{\rho} \right)^{2} \partial_{\rho}^{2} Q + \left(\frac{\alpha \cdot y}{\rho} \right)^{2} \frac{1}{\rho} \partial_{\rho} Q \right] + \frac{\lambda^{2}}{2} \left[\nabla^{2} k(0) \cdot (y, y) + \frac{\kappa_{0}}{3} \rho^{2} \right] Q^{3} \\ + \lambda \nabla^{2} k(0) \cdot (y, \alpha) Q^{3} + B_{1}^{[2]} \frac{|y|^{2}}{4} Q + \lambda^{2} \left(V(0) + \frac{\kappa_{0}}{3} \right) Q - B_{2}^{[2]} \cdot y Q =: \tilde{F}_{2} \end{cases}$$

$$(6.23)$$

$$L_{-}S_{2} = 0$$

Hence one may choose at once $S_2 = 0$. Then, to ensure the existence of T_2 one has to check that $\left(\tilde{F}_2, \nabla Q\right) = 0$, which is obviously satisfied provided $B_2^{[2]} = 0$, since for any j = 1, 2, $\tilde{F}_2 \partial_j Q = \frac{y_j}{\rho} \tilde{F}_2 \partial_\rho Q$ is then polynomial in y with only monomials of odd degrees and radially symmetric coefficients, so that its integral over \mathbb{R}^2 is zero.

Focusing then on the third order, one may not neglect $\lambda \nabla^2 k(0).(\alpha, \beta) Q$ which according to (6.8) is $\mathcal{O}(\lambda^4)$, so one has to get

$$\begin{cases} L_{+}T_{3} = -\frac{\lambda^{3}}{2} \rho \,\nabla\kappa(0) \cdot y \,\partial_{\rho}Q + \frac{\lambda^{3}}{2} \left(\frac{1}{3} \nabla^{3}k(0) \cdot (y, y, y) - c_{0} \,\rho^{3}\right) Q^{3} + B_{1}^{[3]} \frac{\rho^{2}}{4} Q \\ + \frac{\lambda^{2}}{2} \nabla^{3}k(0) \cdot (y, y, \alpha) \,Q^{3} + \left[\lambda^{3} \left(\nabla V(0) \cdot y - \frac{9c_{0}}{4} \,\rho\right) + \lambda^{2} \,\nabla V(0) \cdot \alpha\right] Q \\ - B_{2}^{[3]} \cdot y \,Q =: \tilde{F}_{3} \\ L_{-}S_{3} = \partial_{pa} T_{2} - \lambda \,\nabla^{2}k(0) \cdot (\alpha, \beta) \,Q + \frac{2\kappa_{0}}{3} \,\lambda^{2} \,\beta \cdot y \,Q =: \tilde{G}_{3} \end{cases}$$
where $\partial_{pa} = -b\lambda\partial_{\lambda} + 2\lambda\beta \cdot \partial_{\alpha} + (-b^{2} + B_{1})\partial_{b} + (-b\beta + B_{2}) \cdot \partial_{\beta}$. Once again, the existence of T_{3} is ensured provided $\left(\tilde{F}_{3}, \nabla Q\right) = 0$ which may be done by taking

$$B_2^{[3]} = \lambda^3 \left[\nabla V(0) + \|Q\|_{L^2}^{-2} \left(\nabla \kappa(0) \int \rho^2 |\nabla Q|^2 \, dy + \frac{1}{8} \int \nabla^3 k(0) \cdot (y, y, \cdot) \, Q^4 \, dy \right) \right] + \lambda \, \nabla^2 k(0) \cdot (\alpha, \cdot) = 0$$

The existence of S_3 is then ensured provided

$$0 = \left(\tilde{G}_3, Q\right) = -\frac{1}{2} \partial_{pa} \left[\left(L_+ T_2, \Lambda Q \right) + \frac{1}{2} \nabla^2 k(0) . (\alpha, \alpha) \int Q^2 \, dy \right]$$

Using then (6.23), identities $\int y_j y_k Q^3 \Lambda Q = 0$, for any $1 \leq j, k \leq 2$, $L_+(\Lambda Q) = -2Q$, $(Q, \Lambda Q) = 0$ and ange of variable $(z_1, z_2) = (-y_2, y_1)$, one may compute change of variable $(z_1, z_2) = (-y_2, y_1)$, one may compute

$$\left(L_{+}T_{2}, \Lambda Q\right) = -\frac{\kappa_{0}}{6} |\alpha|^{2} \int Q^{2} dy - B_{1}^{[2]} \int \frac{|y|^{2}}{4} Q^{2}$$

Hence it suffices to set

$$B_1^{[2]} = \frac{1}{2} \left(\nabla^2 k(0) - \frac{\kappa_0}{3} Id \right) .(\alpha, \alpha) \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2} \|Q\|_{L^2}^2$$
(6.25)

By the way, notice that

$$\left(L_{+}T_{2}, \Lambda Q\right) = -\frac{1}{2} \nabla^{2} k(0).(\alpha, \alpha) \|Q\|_{L^{2}}^{2}$$
(6.26)

Eventually, focusing then on the fourth order, one has to get

$$\begin{cases} L_{+}T_{4} = F(T_{2}, S_{3}, \lambda, \alpha) + B_{1}^{[4]} \frac{\rho^{2}}{4} Q - B_{2}^{[3]} \cdot y \ Q =: \tilde{F}_{4} \\ L_{-}S_{4} = \partial_{pa} T_{3} + \frac{b\lambda^{3}}{2} \rho^{2} \nabla \kappa(0) \cdot y \ Q - \frac{\kappa_{0}}{3} \left(\frac{b}{2} - \frac{\beta \cdot y}{\rho^{2}}\right) \left(\frac{\alpha \cdot y}{\rho}\right)^{2} Q \\ - \frac{\kappa_{0}}{3} b \rho \left(\frac{\alpha^{\perp} \cdot y}{\rho}\right)^{2} \partial_{\rho} Q =: \tilde{G}_{4} \end{cases}$$
(6.27)

where $F(T_2, S_3, \lambda, \alpha)$ is a fourth order function of the parameters, and is sum of terms which are either radially symmetric in y, or polynomials in y with radially symmetric coefficients. All the same as before, one may ensure the existence of T_4 by adding some fourth orders terms to $B_2^{[4]}$, that we will not need to compute. As for the existence of S_4 , one needs now a bit of caution as it must be ensured that $(\tilde{G}_4, Q) = 0$. First, one may compute

$$\left(\tilde{G}_4, Q\right) = -\frac{1}{2} \partial_{pa} \left(L_+ T_3, \Lambda Q \right) + \frac{\kappa_0}{12} b |\alpha|^2 \int Q^2 dy$$

with no obvious antiderivative in time for the $b |\alpha|^2$ term, which is precisely why we have introduced the parameter μ in (6.9). To ensure the existence of S_4 now requires to solve

$$0 = \left(\tilde{G}_4, Q\right) = -\frac{1}{2} \partial_{pa} \left[\frac{c_0 \lambda^3}{8} \int \left[\rho^3 Q^4 + 9 \rho Q^2\right] dy - B_1^{[3]} \int \frac{\rho^2}{4} Q^2 dy - \frac{\kappa_0}{6} \mu \int Q^2 dy\right]$$

which is done provided

$$B_1^{[3]} = \left(\frac{c_0\lambda^3}{8}\int \left[9 + \rho^2 Q^2\right]\rho Q^2 dy - \frac{\kappa_0}{6}\mu \int Q^2 dy\right) \left\|\frac{|y|}{2}Q\right\|_{L^2}^{-2}$$
(6.28)

By the way, notice then that

$$\left(L_{+}T_{3}, \Lambda Q\right) = \frac{\kappa_{0}}{6} \,\mu \,\|Q\|_{L^{2}}^{2} \tag{6.29}$$

Eventually, to prove the estimates (6.21), one uses the same arguments as in (8.3). That ends the proof of Proposition 6.3.

6.1 The pseudo conservation laws of the approximate profile.

In the previous section, we have built the approximate profile $\tilde{Q}_{\mathcal{P}}$ of solution \tilde{v} , after the changes (6.6), (6.11) we have made from solution u of (1.1), so that $\tilde{Q}_{\mathcal{P}}$ has the form

$$\tilde{Q}_{\mathcal{P}} = \tilde{P}_{\mathcal{P}} e^{-ib\frac{\rho^{2}}{4} + i\beta \cdot y} = \left(Q + T_{2} + T_{3} + iS_{3} + \mathcal{O}(\mathcal{P}^{4})\right) e^{-ib\frac{\rho^{2}}{4} + i\beta \cdot y}$$

$$\tilde{Q}_{\mathcal{P}} = \left(1 - ib\frac{\rho^{2}}{4}\right) Q + \mathcal{O}(\mathcal{P}^{2} e^{-c|y|})$$
(6.30)

Using (6.12), (6.13) (6.15), equation (6.20) and estimate (6.21), $\tilde{Q}_{\mathcal{P}}$ is now solution of

$$i\left(-b^{2}+B_{1}\right)\partial_{b}\tilde{Q}_{\mathcal{P}}-ib\lambda\partial_{\lambda}\tilde{Q}_{\mathcal{P}}+2i\lambda\beta\cdot\partial_{\alpha}\tilde{Q}_{\mathcal{P}}+i\left(-b\beta+B_{2}\right)\cdot\partial_{\beta}\tilde{Q}_{\mathcal{P}}$$

$$+\mathcal{L}_{g}\tilde{Q}_{\mathcal{P}}-\tilde{Q}_{\mathcal{P}}+\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}|\tilde{Q}_{\mathcal{P}}|^{2}\tilde{Q}_{\mathcal{P}}+\left[\lambda^{2}V(\lambda y+\alpha)+V_{\varphi}\right]\tilde{Q}_{\mathcal{P}}-i\frac{\kappa_{0}}{3}\lambda^{2}\beta\cdot y\,\tilde{Q}_{\mathcal{P}}$$

$$+ib\Lambda\tilde{Q}_{\mathcal{P}}-2i\beta\cdot\left(\nabla\tilde{Q}_{\mathcal{P}}+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\,\tilde{Q}_{\mathcal{P}}\right)-\tilde{\gamma}_{s}\,\tilde{Q}_{\mathcal{P}}=-\Psi_{\mathcal{P}}$$

$$\text{where}\quad\Psi_{\mathcal{P}}=\left[\tilde{\Psi}_{\mathcal{P}}+i\mathcal{P}^{9/2}\mathcal{I}m\tilde{R}_{E}\right]e^{-ib\frac{\rho^{2}}{4}+i\beta\cdot y}$$

$$(6.31)$$

Since the Laplacian operator in rescaled variable has been flattened up to the third order in the equation of the approximate solution $\tilde{P}_{\mathcal{P}}$, the mass of profile $\tilde{Q}_{\mathcal{P}}$ is now to be computed with the measure $\frac{\rho}{\varphi}\sqrt{|g|}(\lambda y + \alpha)dy$

Lemma 6.4.

Mass of $\tilde{Q}_{\mathcal{P}}$:

$$\int |\tilde{Q}_{\mathcal{P}}|^2 \frac{\rho}{\varphi} \sqrt{|g|} dy = \left[1 + \frac{1}{2} \left(\nabla^2 k(0) - \frac{\kappa_0}{3} Id\right) \cdot (\alpha, \alpha) - \frac{\kappa_0}{6} \mu\right] \int Q^2 dy + \lambda^4 R_M$$
(6.32)

Energy of $\tilde{Q}_{\mathcal{P}}$:

$$E(\tilde{Q}_{\mathcal{P}}) = \frac{b^2}{8} \int \rho^2 Q^2 \, dy + \frac{|\beta|^2}{2} \int Q^2 - \frac{1}{4} \left(\nabla^2 k(0) . (\alpha, \alpha) - \frac{\kappa_0}{3} \, \mu \right) \int Q^2 \, dy - \lambda^2 \left(K_E + \phi_E(\alpha) - \lambda \, K_1 \right) + \lambda^4 \, R_E$$
(6.33)

where $R_{M/E}$ is a regular function such as $\lambda^4 R_{M/E} = \mathcal{O}(\lambda^4 + |\alpha|^3 + \lambda^3 |\alpha| + \lambda |\alpha|^2)$ and

$$K_E = \frac{1}{8} \left(\int \left[\nabla^2 k(0) + \frac{\kappa_0}{3} Id \right] (y, y) Q^4 dy + 4V(0) \int Q^2 dy \right), \quad K_1 = \frac{c_0}{8} \int \rho^3 Q^4 dy$$

$$\phi_E(\alpha) = \frac{1}{8} \left(\int \nabla^3 k(0) (y, y, \alpha) Q^4 dy + 4\nabla V(0) \cdot \alpha \int Q^2 dy \right)$$
(6.34)

Proof of Lemma 6.4 :

First, notice that

$$\frac{\rho}{\varphi}\sqrt{|g|}(\lambda y + \alpha) = 1 - \frac{\lambda^3}{6}\rho^2 \nabla \kappa(0) \cdot y - \frac{\kappa_0}{6} \left[2\lambda\alpha \cdot y + |\alpha|^2\right] + \lambda^4 R_m$$
(6.35)

Using (6.30), (6.26) and (6.29) one may calculate

$$\begin{split} &\int |\tilde{Q}_{\mathcal{P}}|^2 \frac{\rho}{\varphi} \sqrt{|g|} dy = \int |\tilde{P}_{\mathcal{P}}|^2 \frac{\rho}{\varphi} \sqrt{|g|} dy = \int \left(Q^2 + 2QT_2 + 2QT_3 + \mathcal{O}(\mathcal{P}^4)\right) \frac{\rho}{\varphi} \sqrt{|g|} dy \\ &= \int Q^2 \frac{\rho}{\varphi} \sqrt{|g|} dy - \left(L_+T_2, \Lambda Q\right) - \left(L_+T_3, \Lambda Q\right) + \lambda^4 \tilde{R}_M \\ &= \int Q^2 dy + \frac{1}{2} \left(\nabla^2 k(0) - \frac{\kappa_0}{3} Id\right) \cdot (\alpha, \alpha) \int Q^2 dy - \frac{\kappa_0}{6} \mu \int Q^2 dy + \lambda^4 R_M \end{split}$$

Similarly, the Energy should be

$$\begin{split} E(\tilde{Q}_{\mathcal{P}}) &= \frac{1}{2} \int g^{ij} (\lambda y + \alpha) \partial_i \tilde{Q}_{\mathcal{P}} \, \partial_j \overline{\tilde{Q}_{\mathcal{P}}} \, \frac{\rho}{\varphi} \sqrt{|g|} (\lambda y + \alpha) dy - \frac{1}{4} \int \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \, |\tilde{Q}_{\mathcal{P}}|^4 \, \frac{\rho}{\varphi} \sqrt{|g|} (\lambda y + \alpha) dy \\ &- \frac{\lambda^2}{2} \int V(\lambda y + \alpha) \, |\tilde{Q}_{\mathcal{P}}|^2 \, \frac{\rho}{\varphi} \, \sqrt{|g|} (\lambda y + \alpha) dy \end{split}$$
with

with

$$\begin{split} \nabla \tilde{Q}_{\mathcal{P}} &= \left\{ \nabla \tilde{P}_{\mathcal{P}} + \left(-b\frac{y}{2} + \beta \right) \tilde{P}_{\mathcal{P}} \right\} e^{-ib\frac{\rho^2}{4} + i\beta \cdot y} \\ &= \left\{ \left(\nabla Q + \nabla T_2 + \nabla T_3 \right) + i \left[\left(-b\frac{y}{2} + \beta \right) \left(Q + T_2 \right) + \nabla S_3 \right] + \mathcal{O}(\mathcal{P}^4) \right\} e^{-ib\frac{\rho^2}{4} + i\beta \cdot y} \end{split}$$
from interaction by parts and radial asymptotic of \mathcal{O}

hence from integration by parts and radial symmetry of Q

$$\int |\nabla \tilde{Q}_{\mathcal{P}}|^2 \, dy = \int |\nabla Q|^2 \, dy - 2 \int \Delta Q \left(T_2 + T_3 \right) dy + \frac{b^2}{4} \int \rho^2 \, Q^2 \, dy + |\beta|^2 \int Q^2 \, dy + \mathcal{O}(\mathcal{P}^4)$$

A simple computation based on the previous decomposition, expansion (6.35) and definition (6.10) shows that

$$\frac{1}{2} \int g^{ij} (\lambda y + \alpha) \partial_i \tilde{Q}_{\mathcal{P}} \, \partial_j \overline{\tilde{Q}_{\mathcal{P}}} \frac{\rho}{\varphi} \sqrt{|g|} (\lambda y + \alpha) dy$$

$$= \frac{1}{2} \int \left(1 - \frac{\kappa_0}{6} |\alpha|^2\right) |\nabla \tilde{Q}_{\mathcal{P}}|^2 \, dy + \frac{\kappa_0}{6} \int \left[\lambda^2 \left| \left(y_2 \partial_1 - y_1 \partial_2\right) \tilde{Q}_{\mathcal{P}} \right|^2 + \left| \left(\alpha_2 \partial_1 - \alpha_1 \partial_2\right) \tilde{Q}_{\mathcal{P}} \right|^2 \right] dy$$

$$+ \frac{\kappa_0}{3} \lambda \int \left[\alpha_2 y_2 |\partial_1 \tilde{Q}_{\mathcal{P}}|^2 + \alpha_1 y_1 |\partial_2 \tilde{Q}_{\mathcal{P}}|^2 - (\alpha_1 y_2 + \alpha_2 y_1) \mathcal{R}e(\partial_1 \tilde{Q}_{\mathcal{P}} \partial_2 \overline{\tilde{Q}_{\mathcal{P}}}) \right] dy$$

$$- \frac{\lambda^3}{2} \int \left[c_0 \rho - \frac{1}{3} \nabla \kappa(0) \cdot y\right] \left| \left(y_2 \partial_1 - y_1 \partial_2\right) \tilde{Q}_{\mathcal{P}} \right|^2 dy - \frac{1}{6} \int \left[2 \kappa_0 \lambda \alpha + \lambda^3 \rho^2 \nabla \kappa(0) \right] \cdot y \left| \nabla \tilde{Q}_{\mathcal{P}} \right|^2 dy + \lambda^4 R_E$$

from (6.30) and the radial symmetry property of Q, $(y_1\partial_2 - y_2\partial_1)\tilde{Q}_{\mathcal{P}} = \partial_{\omega}\tilde{Q}_{\mathcal{P}} = \mathcal{O}(\mathcal{P}^2 e^{-c|y|})$, so the energy becomes after some having calculated the potentials

$$\begin{split} E(\tilde{Q}_{\mathcal{P}}) &= E^{0}(Q) - \int \left(\Delta Q + Q^{3}\right) \left(T_{2} + T_{3}\right) dy + \frac{b^{2}}{8} \int \rho^{2} Q^{2} dy + \frac{|\beta|^{2}}{2} \int Q^{2} + \frac{\kappa_{0}}{12} |\alpha|^{2} \int Q^{2} dy \\ &- \frac{\lambda^{2}}{8} \left(\int \left[\left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} Id \right) \cdot (y, y) - c_{0} \lambda \rho^{3} \right] Q^{4} dy + 4 V(0) \int Q^{2} dy \right) \\ &- \frac{\lambda^{2}}{8} \left(\int \nabla^{3} k(0) \cdot (y, y, \alpha) Q^{4} dy + 4 \nabla V(0) \cdot \alpha \int Q^{2} dy \right) + \frac{\kappa_{0}}{12} \mu \int Q^{2} dy + \lambda^{4} R_{E} \end{split}$$

then to conclude the proof, one needs only to use $E^0(Q) = \frac{1}{2} \int |\nabla Q|^2 dy - \frac{1}{4} \int Q^4 dy = 0$, the Ground State equation along with (6.26) and (6.29).

Introduction of the nonlinear decomposition of the solution and initializa-6.2 tion of the bootstrap argument.

We begin with introducing the modulation method thanks to which we aim at building a NLS solution on time interval $[t_0, 0]$ blowing-up at time t = 0. In this section, we assume we have a solution u(t) of

$$\int i\partial_t u + \Delta_g u + k(x)|u|^2 u + V(x) u = 0, \quad (t,x) \in [t_0,t_1] \times M$$
(6.36)

$$u(0,x) = u_0(x), \quad u_0: M \longrightarrow \mathbb{C} \in H^1$$

$$(6.37)$$

with $t_0 < t_1 < 0$. In addition we assume we have the following nonlinear decomposition for u

$$u(t,x) = \left(\frac{\rho}{\varphi}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} (\tilde{Q}_{\mathcal{P}(t)} + \epsilon) \left(t, \frac{x - \alpha(t)}{\lambda}\right) e^{i\gamma(t)}$$
(6.38)

with the uniform bound (bootstrap's initialization) :

$$|\mathcal{P}(t)| + \|\epsilon(t)\|_{H^1} \lesssim \lambda(t), \quad \forall \ t \in [t_0, t_1]$$
(6.39)

We also assume u(t) has almost critical mass

$$\left| \int |u|^2 \, dg - \int Q^2 \right| \lesssim \lambda^4(t), \quad \forall \ t \in [t_0, t_1]$$
(6.40)

Remark 13. Notice the proper measure to evaluate the mass of u is indeed the volume form $dg = \sqrt{|g|}(x) dx$, since it is the one which makes of Δ_g a self-adjoint operator. In the sequel we will often denote that measure by $dg_x = \sqrt{|g|}(x) dx$ or $dh_y = \sqrt{|g|}(\lambda y + \alpha) dy$.

Now let :

$$\tilde{Q}_{\mathcal{P}} = \Sigma + i\Theta, \qquad \epsilon = \epsilon_1 + i\epsilon_2$$

From (6.30) it is straightforward that

$$\Sigma = Q + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right), \quad \Theta = Q\left(-b \frac{|y|^2}{4} + \beta \cdot y\right) + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$$

hence $\tilde{Q}_{\mathcal{P}} = \left[1 + i\left(-b \frac{\rho^2}{4} + \beta \cdot y\right)\right] Q + \mathcal{O}\left(\mathcal{P}^2 e^{-C_2 |y|}\right)$ (6.41)

A standard modulation idea based on the implicit function theorem, see [MR05], proves one may ensure unicity of decomposition (6.38) provided we set the following orthogonality conditions :

$$\left(\epsilon_2, \nabla\Sigma\right) - \left(\epsilon_1, \nabla\Theta\right) = 0$$
(6.42)

$$(\epsilon_1, y\Sigma) + (\epsilon_2, y\Theta) = 0$$
 (6.43)

$$\begin{cases} (1) & (1) & (1) & (1) \\ (\epsilon_1, y \Sigma) + (\epsilon_2, y \Theta) = 0 \\ -(\epsilon_1, \Lambda \Theta) + (\epsilon_2, \Lambda \Sigma) = 0 \\ (\epsilon_1, |y|^2 \Sigma) + (\epsilon_2, |y|^2 \Theta) = 0 \end{cases}$$
(6.45)

$$\left(\epsilon_1, |y|^2 \Sigma\right) + \left(\epsilon_2, |y|^2 \Theta\right) = 0 \tag{6.45}$$

$$-\left(\epsilon_{1}, \varrho_{2}\right) + \left(\epsilon_{2}, \varrho_{1}\right) = 0 \tag{6.46}$$

with ρ the unique H^1 , radial solution of $L_+\rho = |y|^2 Q$ and :

$$\varrho_1 + i\varrho_2 = \varrho(y)e^{-ib\frac{|y|^2}{4} + i\beta \cdot y}$$

In the case $\mathcal{P} = (0, 0, 0, 0)$, these orthogonality conditions give a first order approximation of the kernel of linearized operator $L = (L_+, L_-)$ around the Ground-State Q. Indeed, we remind :

$$L_{-}Q = 0, \quad L_{-}(yQ) = -2\nabla Q, \quad L_{-}(|y|^{2}Q) = -4\Lambda Q$$

$$L_{+}(\nabla Q) = 0, \quad L_{+}(\Lambda Q) = -2Q, \quad L_{+}\varrho = |y|^{2}Q$$
(6.47)

and :

$$(\varrho, Q) = -\frac{1}{2} (L_{+}\varrho, \Lambda Q) = -\frac{1}{2} ||yQ||_{L^{2}}^{2}$$
 (6.48)

Note it is a well-known argument that \mathcal{P} may be chosen C^1 with respect to time, see [MR05]. Let

$$s(t) = \int_{t_0}^t \frac{d\tau}{\lambda(\tau)^2} \tag{6.49}$$

be the rescaled time.

Computation of equation of ϵ . Let us begin with the non-linear term development :

$$\begin{split} \left| \tilde{Q}_{\mathcal{P}} + \epsilon \right|^2 \left(\tilde{Q}_{\mathcal{P}} + \epsilon \right) &= \left| \tilde{Q}_{\mathcal{P}} \right|^2 \tilde{Q}_{\mathcal{P}} + \tilde{M}_1(\epsilon) + i \tilde{M}_2(\epsilon) + R_1(\epsilon) + i R_2(\epsilon) \\ \tilde{M}_1(\epsilon) &= \left| \tilde{Q}_{\mathcal{P}} \right|^2 \epsilon_1 + 2\Sigma^2 \epsilon_1 + 2\Sigma \Theta \epsilon_2 \\ \tilde{M}_2(\epsilon) &= \left| \tilde{Q}_{\mathcal{P}} \right|^2 \epsilon_2 + 2\Theta^2 \epsilon_2 + 2\Sigma \Theta \epsilon_1 \\ R_1(\epsilon) &= 3\Sigma \epsilon_1^2 + \Sigma \epsilon_2^2 + 2\Theta \epsilon_1 \epsilon_2 + |\epsilon|^2 \epsilon_1 \\ R_2(\epsilon) &= 3\Theta \epsilon_2^2 + \Theta \epsilon_1^2 + 2\Sigma \epsilon_1 \epsilon_2 + |\epsilon|^2 \epsilon_2 \end{split}$$

$$(6.50)$$

Recall from (6.13) that $\mathcal{L}_g = \Delta + \mathcal{O}(\mathcal{P}^2 \nabla, \mathcal{P}^2 \nabla^2)$ and let :

$$M_j(\epsilon) = -\left(\mathcal{L}_g + V_g\right)\epsilon_j + \epsilon_j - \frac{\rho}{\varphi}\frac{k(\lambda y + \alpha)}{k(\alpha)}\tilde{M}_j(\epsilon), \quad j = 1, 2$$
(6.51)

with $V_g = \lambda^2 V(\lambda y + \alpha) + V_{\varphi}$ the corresponding potential. Remind using (6.12), (6.13), (6.15) and (6.17) the \tilde{v} equation may be rewritten as

$$i \partial_s \tilde{v} + \mathcal{L}_g \tilde{v} - \tilde{v} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{v}|^2 \tilde{v} - i \left(\frac{\kappa_0}{3} \lambda^2 \beta \cdot y - \frac{3c_0}{2} \lambda^3 \beta \cdot y\right) \tilde{v} + \lambda^2 \left[V(\lambda y + \alpha) + V_{\varphi} \right] \tilde{v} - i \frac{\lambda_s}{\lambda} \Lambda \tilde{v} - i \frac{\alpha_s}{\lambda} \cdot \left(\nabla \tilde{v} + \frac{\lambda}{2} \frac{\nabla k(\alpha)}{k(\alpha)} \right) - \tilde{\gamma}_s \tilde{v} = 0$$
(6.52)

Now since $\tilde{v} = \tilde{Q}_{\mathcal{P}} + \epsilon$ is solution of (6.52) and $\tilde{Q}_{\mathcal{P}}$ is solution of (6.31), ϵ must satisfy

$$i\partial_{s}\tilde{Q}_{\mathcal{P}} + \left[\mathcal{L}_{g}\tilde{Q}_{\mathcal{P}} + \lambda^{2}\left[V(\lambda y + \alpha) + V_{\varphi}\right]\tilde{Q}_{\mathcal{P}} - \tilde{Q}_{\mathcal{P}} + \frac{\rho}{\varphi}\frac{k(\lambda y + \alpha)}{k(\alpha)}|\tilde{Q}_{\mathcal{P}}|^{2}\tilde{Q}_{\mathcal{P}}\right] \\ + i\partial_{s}\epsilon - \left(M_{1}(\epsilon) + iM_{2}(\epsilon)\right) + \frac{\rho}{\varphi}\frac{k(\lambda y + \alpha)}{k(\alpha)}\left(R_{1}(\epsilon) + iR_{2}(\epsilon)\right) \\ = i\frac{\lambda_{s}}{\lambda}\Lambda\left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) + i\frac{\alpha_{s}}{\lambda}\cdot\left[\nabla\left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\left(\tilde{Q}_{\mathcal{P}} + \epsilon\right)\right] + \tilde{\gamma}_{s}\left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) \\ - i\left(\frac{\kappa_{0}}{3}\lambda^{2}\beta \cdot y - \frac{3c_{0}}{2}\lambda^{3}\beta \cdot y\right)\left(\tilde{Q}_{\mathcal{P}} + \epsilon\right)$$
(6.53)

Moreover, with (6.39):

$$\frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} R_j(\epsilon) = R_j(\epsilon) + \mathcal{O}(\mathcal{P}^2 |\epsilon|^2)$$
(6.54)

and so we have thanks to (6.31), and denoting $A = A_1 + i A_2$, for A = M, R

$$i\left(b_{s}+b^{2}-B_{1}\right)\partial_{b}\tilde{Q}_{\mathcal{P}}+i\left(\lambda_{s}+b\,\lambda\right)\partial_{\lambda}\tilde{Q}_{\mathcal{P}}+i\left(\beta_{s}+b\,\beta-B_{2}\right)\partial_{\beta}\tilde{Q}_{\mathcal{P}}+i\left(\alpha_{s}-2\lambda\,\beta\right)\partial_{\alpha}\tilde{Q}_{\mathcal{P}}$$

$$+i\,\partial_{s}\epsilon-M(\epsilon)+\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}R(\epsilon)+i\,b\,\Lambda\epsilon-2\,i\,\beta.\left(\nabla\epsilon+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\epsilon\right)$$

$$-|\beta|^{2}\epsilon-i\left(\frac{\kappa_{0}}{3}\,\lambda^{2}\,\beta.\,y-\frac{3c_{0}}{2}\,\lambda^{3}\,\beta.\,y\right)\epsilon$$

$$=i\left(\frac{\lambda_{s}}{\lambda}+b\right)\left(\Lambda\tilde{Q}_{\mathcal{P}}+\Lambda\epsilon\right)+i\left(\frac{\alpha_{s}}{\lambda}-2\,\beta\right).\left(\nabla\tilde{Q}_{\mathcal{P}}+\nabla\epsilon+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\tilde{Q}_{\mathcal{P}}+\frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\epsilon\right)$$

$$+\left(\tilde{\gamma}_{s}-|\beta|^{2}\right)\left(\tilde{Q}_{\mathcal{P}}+\epsilon\right)-\Psi_{\mathcal{P}}+\mathcal{O}\left(\mathcal{P}^{2}\left|Mod(t)\right|\right)$$

$$(6.55)$$

where Mod(t) is the vector of the \mathcal{P} -laws we introduce in (6.60). Then, one needs only to take on the one hand the real and on the other hand the imaginary part of (6.55) to get the equations of ϵ_1 and ϵ_2 respectively. From (6.41), (6.23) and a little calculations one easily gets

$$\partial_{b}\Sigma = \left(-b\frac{\rho^{2}}{4} + \beta \cdot y\right)\frac{\rho^{2}}{4}Q + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right), \quad \partial_{b}\Theta = -\frac{\rho^{2}}{4}Q + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right)$$

$$\partial_{\beta}\Sigma = -\left(-b\frac{\rho^{2}}{4} + \beta \cdot y\right)yQ + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right), \quad \partial_{\beta}\Theta = yQ + \mathcal{O}\left(\mathcal{P}^{2}e^{-C_{2}|y|}\right)$$

$$\partial_{\lambda}\tilde{Q}_{\mathcal{P}} = \mathcal{O}\left(\mathcal{P}e^{-c|y|}\right), \quad \text{so that} \quad (\lambda_{s} + b\lambda)\partial_{\lambda}\tilde{Q}_{\mathcal{P}} = \mathcal{O}\left(\mathcal{P}^{2}|Mod(t)|\right)$$

$$\partial_{\alpha}\tilde{Q}_{\mathcal{P}} = \mathcal{O}\left(\mathcal{P}e^{-c|y|}\right), \quad \text{so that} \quad (\alpha_{s} - 2\lambda\beta)\partial_{\alpha}\tilde{Q}_{\mathcal{P}} = \mathcal{O}\left(\mathcal{P}^{2}|Mod(t)|\right)$$
(6.56)

so the equation of ϵ may be simplified as

$$(b_s + b^2 - B_1) \frac{\rho^2}{4} Q - (\beta_s + b\beta - B_2) \cdot y Q - i \left(\frac{\lambda_s}{\lambda} + b\right) \Lambda Q - i \left(\frac{\alpha_s}{\lambda} - 2\beta\right) \cdot \nabla Q - (\tilde{\gamma}_s - |\beta|^2) Q + i \partial_s \epsilon - M(\epsilon) + R(\epsilon) + i b \Lambda \epsilon - 2 i \beta \cdot \nabla \epsilon$$

$$= -\Psi_{\mathcal{P}} + S_{\mathcal{P}} Mod(t) + \mathcal{O}\left(\left(\mathcal{P}^2 + |Mod(t)|\right) \left[|\epsilon| + (1 + |y|) |\nabla \epsilon|\right]\right)$$

$$(6.57)$$

where $\Psi_{\mathcal{P}}$ is defined in (6.31), and $S_{\mathcal{P}} = \mathcal{O}(\mathcal{P})$ is a matrix of order 1 with respect to \mathcal{P} .

6.3 A first estimation of the modulation parameters.

We now make use of the signs hypotheses we assumed in (H2). We show how this forces the regime of the \mathcal{P} parameters in such a way that λ is the greatest parameter and thus controls the \mathcal{P} set.

We then prove a consistency result, which shows the approximate laws we built in the first section for those parameters are indeed like we constructed them to be, that is at least fourth order approximations.

Proposition 6.5. For $s \in [s_0, s_1]$, we may estimate :

• Refined variational control :

Provided $E_0 + C_E + \phi_E(\alpha) - \lambda C_1 > 0$ and $\nabla^2 k(0) + \frac{\kappa_0}{3} Id < 0$

$$b^{2} + |\beta|^{2} + |\alpha|^{2} + \|\epsilon\|_{H^{1}}^{2} \lesssim \lambda^{2} \left(E_{0} + C_{E} + \phi_{E}(\alpha) - \lambda C_{1}\right) + \mathcal{O}\left(\mathcal{P}^{4} + |\alpha|^{3}\right)$$

with $C_{E} = \frac{1}{8} \int \left[\nabla^{2}k(0) + \frac{\kappa_{0}}{3}\right] \cdot (y, y) Q^{4} dy + \frac{1}{2} \int \left[V(0) + \frac{\kappa_{0}}{3}\right] Q^{2} dy$
 $C_{1} = \frac{c_{0}}{8} \int \left[\rho^{3} Q^{4} + 18 \rho Q^{2}\right] dy$ (6.58)

There is also the refinement

$$\|\epsilon\|_{L^2} \lesssim \lambda^4 + |\alpha|^2 + \mu \tag{6.59}$$

• Control of the geometrical parameters : Let

$$Mod(t) := \left(L_b - K_b(\alpha, \mu), L_\lambda, L_\alpha, L_\beta, L_{\tilde{\gamma}} - K_{\tilde{\gamma}}(\alpha, \mu) \right)$$
$$L_b = b_s + b^2 - B_1, \quad L_\lambda = \frac{\lambda_s}{\lambda} + b, \quad L_\alpha = \frac{\alpha_s}{\lambda} - 2\beta,$$
$$L_\beta = \beta_s + b\beta - B_2, \quad L_{\tilde{\gamma}} = \tilde{\gamma}_s - |\beta|^2$$
(6.60)

be the vector of modulations equations with

$$K_b(\alpha,\mu) = \frac{\kappa_0}{6} \frac{\|Q\|_{L^2}^2}{\left\|\frac{|y|}{2}Q\right\|_{L^2}^2} \left(|\alpha|^2 + \mu\right), \quad K_{\tilde{\gamma}}(\alpha,\mu) = \frac{\left(\frac{|y|^2}{4}Q,\varrho\right)}{\left(Q,\varrho\right)} K_b(\alpha,\mu)$$
(6.61)

Then:

$$|Mod(t)| \lesssim \mathcal{O}\Big(\mathcal{P}^4 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\Big)$$
(6.62)

moreover there is the improvement

$$\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \mathcal{O}\left(\mathcal{P}^5 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\right)$$
(6.63)

Remark 14. (i) Since $\lambda(t), \alpha(t) \to 0$, as $t \to 0$, as long as time t_0 is chosen close enough to blow up time - that is s_0 big enough - both asymptons $E_0 + C_E > 0$ and $\nabla^2 k(0) + \frac{\kappa_0}{3} < 0$ allow to prove the same result.

(ii) Though we have assumed from (6.8) that $|\alpha| \leq \lambda^{3/2}$, in this section, we will get all estimates up to $\mathcal{O}(\lambda^4 + |\alpha|^3)$ remaining terms, so we can obtain a first estimate $|\alpha| \leq \lambda$ that will help us get the announced refinement.

(iii) Note the vector of modulation laws $Law(t) = (L_b, L_\lambda, L_\alpha, L_\beta, L_{\tilde{\gamma}})$ only differs from a $|\alpha|^2$ term, so that from (6.8)

$$\left|Law(t)\right| \lesssim \mathcal{P}^{4} + |\alpha|^{2} + \mathcal{P}|\alpha|^{2} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}} + \|\epsilon\|_{L^{2}}^{2} + \|\epsilon\|_{H^{1}}^{3}$$
(6.64)

Proof of Proposition 6.5 :

Step 1

Recall the cancellation effect on the volume form when introducing the multiplier $\left(\frac{\rho}{\omega}\right)^{\frac{1}{2}}$

$$\frac{\rho}{\varphi}\sqrt{|g|} = 1 - \frac{\lambda^3}{6}\,\rho^2\,\nabla\kappa(0)\,.\,y - \frac{\kappa_0}{6}\left[2\,\lambda\,\alpha\,.\,y + |\alpha|^2\right] + \lambda^4\,R_m$$

Now, we compute the mass of solution u with nonlinear decomposition as in (6.38), with use of (6.39), (6.40), (6.35) and the radially symmetric form of $\tilde{Q}_{\mathcal{P}}$ up to the second order as shown in (6.41). The critical mass assumption for solution u and almost critical-mass assumption (6.40) for profile $\tilde{Q}_{\mathcal{P}}$ brings

$$\int |\tilde{Q}_{\mathcal{P}} + \epsilon|^2 \frac{\rho}{\varphi} \sqrt{|g|} \, dy = k(\alpha) \int |u|^2 \sqrt{|g|} \, dx = k(\alpha) \left[\int |u|^2 \sqrt{|g|} \, dx - \int Q^2 \right] + k(\alpha) \int Q^2 \, dy$$

so that using (8.2), see Appendix B 8.1, and assumption (H2) that k(0) = 1, $\nabla k(0) = 0$

$$\mathcal{R}e\int\epsilon\ \overline{\tilde{Q}_{\mathcal{P}}}\ \frac{\rho}{\varphi}\sqrt{|g|}\,dy + \frac{1}{2}\int|\epsilon|^2\,dy - \frac{\kappa_0}{12}\left(|\alpha|^2 + \mu\right)\int Q^2\,dy = \mathcal{O}\left(\mathcal{P}^4 + |\alpha|^3 + \mathcal{P}^3\,\|\epsilon\|_{L^2}\right) \tag{6.65}$$

This proves the (6.59) bound, as the previous expression shows

$$\|\epsilon\|_{L^2} + o\big(\|\epsilon\|_{L^2}\big) = \mathcal{O}\Big(\mathcal{P}^4 + |\alpha|^2 + \mu\Big)$$

Notice here that it is the almost critical mass assumption (6.40) that has put in the bootstrap process the previous estimate (6.65) which says what should be the size of the remaining term ϵ through its L^2 norm.

Next, we compute the energy of solution u. But first, notice that from (6.38) and (6.12)

$$\begin{aligned} \nabla u(t,x) &= \left(\frac{\rho}{\varphi}\right)^{\frac{1}{2}} \frac{1}{k(\alpha)^{\frac{1}{2}}} \frac{1}{\lambda^2} \left\{ \nabla \left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) + \frac{1}{2} \left(\frac{y}{\rho^2} - \frac{\nabla\varphi}{\varphi}\right) \left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) \right\} e^{i\gamma} \\ &= \left(\frac{\rho}{\varphi}\right)^{\frac{1}{2}} \frac{1}{k(\alpha)^{\frac{1}{2}}} \frac{1}{\lambda^2} \left\{ \nabla \left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) + \left(\frac{\kappa_0}{6} \lambda^2 - \frac{3c_0}{2} \lambda^3 \rho\right) y \left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) \right\} e^{i\gamma} \end{aligned}$$

so that

$$\begin{split} E(u) &= \frac{1}{2} \int g^{ij} \partial_i u \,\overline{\partial_j u} \,\sqrt{|g|} \,dx - \frac{1}{4} \int k(x) \,|u|^4 \,\sqrt{|g|} \,dx - \frac{1}{2} \int V(x) \,|u|^2 \,\sqrt{|g|} \,dx \\ &= \frac{1}{\lambda^2 \,k(\alpha)} \left\{ \frac{1}{2} \int g^{ij} (\lambda y + \alpha) \left[\partial_i (\tilde{Q}_{\mathcal{P}} + \epsilon) \,\partial_j (\overline{\tilde{Q}_{\mathcal{P}} + \epsilon}) + \left(\frac{\kappa_0}{3} \,\lambda^2 - 3c_0 \,\lambda^3 \,\rho \right) y_i \,\mathcal{R}e \left(\tilde{Q}_{\mathcal{P}} \,\partial_j \overline{\tilde{Q}_{\mathcal{P}}} \right) \right] \frac{\rho}{\varphi} \,\sqrt{|g|} \,dy \\ &- \frac{1}{4} \int \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \,|\tilde{Q}_{\mathcal{P}} + \epsilon|^4 \,\frac{\rho}{\varphi} \,\sqrt{|g|} \,dy - \frac{\lambda^2}{2} \int V(\lambda y + \alpha) \,|\tilde{Q}_{\mathcal{P}} + \epsilon|^2 \,\frac{\rho}{\varphi} \,\sqrt{|g|} \,dy + \mathcal{O}\left(\lambda^4 \,R + \|\epsilon\|_{H^1}^3 + \mathcal{P}^2 \,\|\epsilon\|_{L^2}\right) \right\} \end{split}$$

Notice expanding the non linear terms yields with the 2-dimensional Gagliardo-Nirenberg estimate, and using (6.35) and (6.39)

$$\begin{split} &\frac{1}{4} \int \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}} + \epsilon|^4 \frac{\rho}{\varphi} \sqrt{|g|} \, dy \\ &= \frac{1}{4} \int \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}}|^4 \frac{\rho}{\varphi} \sqrt{|g|} \, dy + \frac{1}{2} \int |\tilde{Q}_{\mathcal{P}}|^2 \Big[\Big(1 + \frac{2\Sigma^2}{|\tilde{Q}_{\mathcal{P}}|^2} \Big) \epsilon_1^2 + 4 \frac{\Sigma\Theta}{|\tilde{Q}_{\mathcal{P}}|^2} \epsilon_1 \epsilon_2 + \Big(1 + \frac{2\Theta^2}{|\tilde{Q}_{\mathcal{P}}|^2} \Big) \epsilon_2^2 \Big] \, dy \\ &+ \mathcal{R}e \int \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \, \epsilon \, |\tilde{Q}_{\mathcal{P}}|^2 \, \overline{\tilde{Q}_{\mathcal{P}}} \frac{\rho}{\varphi} \sqrt{|g|} \, dy + \mathcal{O} \Big(\mathcal{P}^4 + \|\epsilon\|_{H^1}^3 + \mathcal{P}^2 \, \|\epsilon\|_{L^2}^2 \Big) \end{split}$$

Eventually

$$\begin{aligned} \frac{\lambda^2}{2} \int V(\lambda y + \alpha) \, |\tilde{Q}_{\mathcal{P}} + \epsilon|^2 \frac{\rho}{\varphi} \sqrt{|g|} \, dy &= \frac{\lambda^2}{2} \int V(\lambda y + \alpha) \, |\tilde{Q}_{\mathcal{P}}|^2 \frac{\rho}{\varphi} \sqrt{|g|} \, dy \\ &+ \mathcal{O}\left(\mathcal{P}^2 \, \|\epsilon\|_{L^2}^2\right) \end{aligned}$$

so that using integration by parts this all together implies

$$\begin{split} \lambda^{2} k(\alpha) E(u) &= E(\tilde{Q}_{\mathcal{P}}) + \int \left(\frac{\kappa_{0}}{6} \lambda^{2} - \frac{3c_{0}}{2} \lambda^{3} \rho\right) Q \, y \, \cdot \nabla Q \, dy \\ &+ \frac{1}{2} \int |\nabla \epsilon|^{2} \, dy - \frac{1}{2} \, \int |\tilde{Q}_{\mathcal{P}}|^{2} \Big[\Big(1 + \frac{2\Sigma^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}}\Big) \epsilon_{1}^{2} + 4 \frac{\Sigma \Theta}{|\tilde{Q}_{\mathcal{P}}|^{2}} \epsilon_{1} \epsilon_{2} + \Big(1 + \frac{2\Theta^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}}\Big) \epsilon_{2}^{2} \Big] \, dy \\ &- \mathcal{R}e \int \epsilon \, \overline{\Big(\Big[\Delta_{g} + \lambda^{2} \, V(\lambda y + \alpha)\Big] \tilde{Q}_{\mathcal{P}} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}}|^{2} \, \tilde{Q}_{\mathcal{P}}\Big)} \frac{\rho}{\varphi} \sqrt{|g|} \, dy + \mathcal{O}\Big(\mathcal{P}^{4} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \, \|\epsilon\|_{L^{2}}\Big)} \end{split}$$

From conservation of energy for solution u, since $\lambda^2 k(\alpha) E(u) = \lambda^2 E_0 + \mathcal{O}(\mathcal{P}^4)$, using (6.33) this may be rewritten as

$$\begin{split} \lambda^{2} \left(E_{0} + \frac{1}{8} \int \left[\nabla^{2} k(0) + \frac{\kappa_{0}}{3} \right] \cdot (y, y) Q^{4} \, dy + \frac{1}{2} \int \left[V(0) + \frac{\kappa_{0}}{3} \right] Q^{2} \, dy - \frac{c_{0}\lambda}{8} \int \left[\rho^{3} Q^{4} + 18 \rho Q^{2} \right] dy \right) \\ + \lambda^{2} \left(\frac{1}{8} \int \nabla^{3} k(0) \cdot (y, y, \alpha) Q^{4} \, dy + \frac{1}{2} \nabla V(0) \cdot \alpha \int Q^{2} \, dy \right) \\ = \frac{b^{2}}{8} \int |y|^{2} Q^{2} \, dy + \frac{|\beta|^{2}}{2} \int Q^{2} \, dy - \frac{1}{4} \left(\nabla^{2} k(0) \cdot (\alpha, \alpha) - \frac{\kappa_{0}}{3} \mu \right) \int Q^{2} \, dy \\ + \frac{1}{2} \int |\nabla \epsilon|^{2} \, dy - \frac{1}{2} \int |\tilde{Q}_{\mathcal{P}}|^{2} \left[\left(1 + \frac{2\Sigma^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}} \right) \epsilon_{1}^{2} + 4 \frac{\Sigma \Theta}{|\tilde{Q}_{\mathcal{P}}|^{2}} \epsilon_{1} \epsilon_{2} + \left(1 + \frac{2\Theta^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}} \right) \epsilon_{2}^{2} \right] dy \\ - \mathcal{R}e \int \epsilon \left[\left[\Delta_{g} + \lambda^{2} V(\lambda y + \alpha) \right] \tilde{Q}_{\mathcal{P}} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}}|^{2} \tilde{Q}_{\mathcal{P}} \right] \frac{\rho}{\varphi} \sqrt{|g|} \, dy \\ + \mathcal{O} \left(\mathcal{P}^{4} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}} \right) \end{split}$$

$$(6.66)$$

Step 2

We sum conservation of Mass (6.65) and conservation of Energy (6.66):

$$\lambda^{2} \left(E_{0} + \frac{1}{8} \int \left[\nabla^{2} k(0) + \frac{\kappa_{0}}{3} \right] (y, y) Q^{4} dy + \frac{1}{2} \int \left[V(0) + \frac{\kappa_{0}}{3} \right] Q^{2} dy - \frac{c_{0}\lambda}{8} \int \left[\rho^{3} Q^{4} + 18 \rho Q^{2} \right] dy \right) + \lambda^{2} \left(\frac{1}{8} \int \nabla^{3} k(0) (y, y, \alpha) Q^{4} dy + \frac{1}{2} \nabla V(0) \alpha \int Q^{2} dy \right) = \frac{b^{2}}{8} \int |y|^{2} Q^{2} dy + \frac{|\beta|^{2}}{2} \int Q^{2} dy - \frac{1}{4} \left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} Id \right) (\alpha, \alpha) \int Q^{2} dy$$

$$+ \frac{1}{2} \int |\epsilon|^{2} dy + \frac{1}{2} \int |\nabla\epsilon|^{2} dy - \frac{1}{2} \int |\tilde{Q}_{\mathcal{P}}|^{2} \left[\left(1 + \frac{2\Sigma^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}} \right) \epsilon_{1}^{2} + 4 \frac{\Sigma\Theta}{|\tilde{Q}_{\mathcal{P}}|^{2}} \epsilon_{1} \epsilon_{2} + \left(1 + \frac{2\Theta^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}} \right) \epsilon_{2}^{2} \right] dy$$

$$- \mathcal{R}e \int \epsilon \left[\overline{\Delta \tilde{Q}_{\mathcal{P}} - \tilde{Q}_{\mathcal{P}}} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}}|^{2} \tilde{Q}_{\mathcal{P}} \right] dy + \mathcal{O} \left(\mathcal{P}^{4} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}} \right)$$

With (6.31), we have the following estimation in L^2 :

$$-\Delta \tilde{Q}_{\mathcal{P}} + \tilde{Q}_{\mathcal{P}} - \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}}|^2 \tilde{Q}_{\mathcal{P}} = i b \Lambda \tilde{Q}_{\mathcal{P}} - 2 i \beta \cdot \nabla \tilde{Q}_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^2)$$
(6.68)

so that, thanks to orthogonality conditions (6.44) and (6.42):

$$\mathcal{R}e\int \epsilon \overline{\left(-\Delta\tilde{Q}_{\mathcal{P}}+\tilde{Q}_{\mathcal{P}}-\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}|\tilde{Q}_{\mathcal{P}}|^{2}\tilde{Q}_{\mathcal{P}}\right)} dy$$

$$=b\,\mathcal{I}m\left(\epsilon,\overline{\Lambda\tilde{Q}_{\mathcal{P}}}\right)-2\beta\,\mathcal{I}m\left(\epsilon,\overline{\nabla\tilde{Q}_{\mathcal{P}}}\right)+\mathcal{O}(\mathcal{P}^{2}\|\epsilon\|_{L^{2}})=\mathcal{O}(\mathcal{P}^{2}\|\epsilon\|_{L^{2}})$$

$$(6.69)$$

Thanks to (6.39), $\tilde{Q}_{\mathcal{P}}$ being close to Q, we see the quadratic form in ϵ that appears in (6.67) is to be thought as a small perturbation of the energy in a neighbourhood of Q. From (6.41), (6.39) and definition of the linearized operator $L = (L_+, L_-)$, one may check

$$\frac{1}{2} \left[\left(L_{+}\epsilon_{1}, \epsilon_{1} \right) + \left(L_{-}\epsilon_{2}, \epsilon_{2} \right) \right] = \frac{1}{2} \int |\epsilon|^{2} dy + \frac{1}{2} \int |\nabla\epsilon|^{2} dy$$
$$- \frac{1}{2} \int |\tilde{Q}_{\mathcal{P}}|^{2} \left[\left(1 + \frac{2\Sigma^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}} \right) \epsilon_{1}^{2} + 4 \frac{\Sigma\Theta}{|\tilde{Q}_{\mathcal{P}}|^{2}} \epsilon_{1}\epsilon_{2} + \left(1 + \frac{2\Theta^{2}}{|\tilde{Q}_{\mathcal{P}}|^{2}} \right) \epsilon_{2}^{2} \right] dy + \mathcal{O}\left(\mathcal{P}^{2} \|\epsilon\|_{L^{2}}\right)$$

Thus (6.67) becomes :

$$\lambda^{2} \left(E_{0} + C_{E} + \phi_{E}(\alpha) - \lambda C_{1} \right) = \frac{b^{2}}{8} \int |y|^{2} Q^{2} + \frac{|\beta|^{2}}{2} \int Q^{2} + \frac{1}{2} \left[\left(L_{+}\epsilon_{1}, \epsilon_{1} \right) + \left(L_{-}\epsilon_{2}, \epsilon_{2} \right) \right] - \frac{1}{4} \left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} Id \right) \cdot (\alpha, \alpha) \int Q^{2} dy + \mathcal{O} \left(\mathcal{P}^{4} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \|\epsilon\|_{L^{2}} \right)$$
(6.70)

where

$$\begin{split} C_E &= \frac{1}{8} \int \left[\nabla^2 k(0) + \frac{\kappa_0}{3} \right] \cdot (y, y) \, Q^4 \, dy + \frac{1}{2} \int \left[V(0) + \frac{\kappa_0}{3} \right] Q^2 \, dy \\ C_1 &= \frac{c_0}{8} \int \left[\rho^3 \, Q^4 + 18 \, \rho \, Q^2 \right] dy \end{split}$$

To conclude, we need to deal with the bilinear form we introduced in (6.70). A coercivity result will do it :

Lemma 6.6. We can find a universal constant c > 0 such as $\forall \epsilon \in H^1$:

$$\frac{1}{2} \left[\left(L_{+} \epsilon_{1}, \epsilon_{1} \right) + \left(L_{-} \epsilon_{2}, \epsilon_{2} \right) \right]
\geq c \|\epsilon\|_{H^{1}}^{2} - \frac{1}{c} \left\{ \left(\epsilon_{1}, Q \right)^{2} + \left(\epsilon_{1}, |y|^{2} Q \right)^{2} + \left(\epsilon_{1}, y Q \right)^{2} + \left(\epsilon_{2}, \varrho \right)^{2} \right\}$$
(6.71)

Thanks to orthogonality conditions (6.45), (6.43) we can prove :

$$\left(\epsilon_1, |y|^2 Q\right) = 0, \qquad \left(\epsilon_1, y Q\right) = 0$$

In the same way than what is done in [MR05], we may also prove that the orthogonality condition (6.46) yields $(\epsilon_2, \rho) = 0$. Moreover (6.65) shows :

$$\left| \left(\epsilon_1 \,, \, Q \right) \right|^2 \lesssim \left[2\mathcal{R}e\left(\epsilon \,, \, \overline{\tilde{Q}_{\mathcal{P}}} \right) \right]^2 = \mathcal{O}\left(\mathcal{P}^4 + \|\epsilon\|_{H^1}^4 \right) \tag{6.72}$$

Hence by (6.71):

$$\frac{1}{2}\left[\left(L_{+}\epsilon_{1},\epsilon_{1}\right)+\left(L_{-}\epsilon_{2},\epsilon_{2}\right)\right] \geq \frac{c}{2}\|\epsilon\|_{H^{1}}^{2}+\mathcal{O}\left(\mathcal{P}^{4}+\|\epsilon\|_{H^{1}}^{4}\right)$$

Injecting this into (6.70):

$$\lambda^{2} \left(E_{0} + C_{E} + \phi_{E}(\alpha) - \lambda C_{1} \right) \geq \frac{b^{2}}{8} \int |y|^{2} Q^{2} \, dy + \frac{|\beta|^{2}}{2} \int Q^{2} \, dy + \frac{c}{2} \|\epsilon\|_{H^{1}}^{2} \\ - \frac{1}{4} \left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} \, Id \right) . (\alpha, \alpha) \int Q^{2} \, dy + \mathcal{O} \left(\mathcal{P}^{4} + |\alpha|^{3} + \|\epsilon\|_{H^{1}}^{3} + \mathcal{P}^{2} \, \|\epsilon\|_{L^{2}} \right)$$

Eventually, provided $E_0 + C_E + \phi_E(\alpha) - \lambda C_1 > 0$ and $\nabla^2 k(0) + \frac{\kappa_0}{3} Id < 0$ one gets

$$b^{2} + |\beta|^{2} + |\alpha|^{2} + \|\epsilon\|_{H^{1}}^{2} \lesssim \lambda^{2} \left(E_{0} + C_{E} + \phi_{E}(\alpha) - \lambda C_{1} \right) + \mathcal{O}(\mathcal{P}^{4})$$
(6.73)

And (6.58) is proved thanks to (6.39).

Computation of the \mathcal{P} laws.

Quantities α , β , λ , b, $\tilde{\gamma}$ have been introduced in the first section as geometrical parameters, adjusted to our situation through a system of ODE satisfied by \mathcal{P} . Each law was chosen so it was simplifying our computations to build approximate solution $\tilde{P}_{\mathcal{P}}$ by using symmetric invariances. Now, assuming we have an exact solution u of (6.1) on $[t_0, t_1]$, we want to check the correspondant parameters defined by geometrical decomposition (6.38) follow the laws we have built our approximate profile with, up to fourth order, and as long as u is defined in H^1 - which is assumed here to be ensured whenever $t \in [t_0, t_1]$.

In the previous section, we checked as long as u is defined, those parameters were controlled as in (6.73). We now work out a consistency type of result, checking the parameters laws are not blowing up, or strongly modified on the way to blow-up time.

Here we are reproducing analysis led in Appendix A of [RS11] with some more details about calculations. We also prove a slightly more general result than what is needed to get (6.62), and that will be useful when turning to the unicity proof.

We claim that we can find a polynomial vector $\mathcal{R}(\mathcal{P})$ with

$$\left|\mathcal{R}(\mathcal{P})\right| \lesssim \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^4 + \mathcal{P}^3\left(b - \frac{\lambda}{C_0}\right)$$
(6.74)

using (6.58), and such that

$$Mod(t) = \mathcal{R}(\mathcal{P}) + \mathcal{O}\left(\mathcal{P}^{7} + \left(\mathcal{P}^{2} + |Mod(t)|\right) \|\epsilon\|_{L^{2}} + \|\epsilon\|_{L^{2}}^{2} + \|\epsilon\|_{H^{1}}^{3} + |\|u\|_{L^{2}(dg)} - \|Q\|_{L^{2}}|\right)$$
(6.75)

First, we simplify the ϵ real and imaginary parts equations obtained thanks to (6.57):

$$-\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q - \left(\frac{\alpha_s}{\lambda}-2\beta\right) \cdot \nabla Q + \partial_s \epsilon_1 - M_2(\epsilon) + R_2(\epsilon) + b\Lambda\epsilon_1 - 2\beta \cdot \nabla\epsilon_1 = \mathcal{I}m(\tilde{\psi})$$
(6.76)

and

$$-\left(b_s+b^2-B_1\right)\frac{\rho^2}{4}Q+\left(\beta_s+b\beta-B_2\right)\cdot y\,Q+\left(\tilde{\gamma}_s-|\beta|^2\right)Q+\partial_s\epsilon_2+M_1(\epsilon)-R_1(\epsilon) +b\,\Lambda\epsilon_2-2\beta\,\cdot\nabla\epsilon_2=-\mathcal{R}e(\tilde{\psi})$$
(6.77)

where, using (6.31) and (6.21), there exists smooth, regulars in y and polynomials in \mathcal{P} , a function $F_{\mathcal{P}}$ and a matrix $S_{\mathcal{P}}$, such that the remainder $\tilde{\psi}$ has the following form

$$\tilde{\psi}(y) = F_{\mathcal{P}}(y) + S_{\mathcal{P}}(y) \operatorname{Mod}(t) + \mathcal{O}\left(\mathcal{P}^{2} + |\operatorname{Mod}(t)|\right) \left\{ |\epsilon(y)| + |(1+|y|)\nabla\epsilon(y)| \right\} + \mathcal{O}\left(\mathcal{P}^{7}\right) e^{-c|y|}$$
(6.78)

with

$$\left|\partial_{y}^{k}S_{\mathcal{P}}(y)\right| \lesssim \mathcal{P}e^{-c|y|}, \quad \left|\partial_{y}^{k}F_{\mathcal{P}}(y)\right| \lesssim \left[\mathcal{P}\left(|\alpha|^{2}+|\beta|^{2}\right) + \mathcal{P}^{5} + \mathcal{P}^{3}\left(b-\frac{\lambda}{C_{0}}\right)\right]e^{-c|y|}$$
(6.79)

Remark 15. Terms in ∂_{λ} and ∂_{α} are hidden in $\tilde{\psi}$. Indeed, from (6.41) since $\partial_{\lambda}\tilde{Q}_{\mathcal{P}}$, $\partial_{\alpha}\tilde{Q}_{\mathcal{P}} = \mathcal{O}(\mathcal{P}e^{-c_2|y|})$

$$(\lambda_s + b\lambda)\partial_\lambda \tilde{Q}_{\mathcal{P}} = \lambda \Big(\frac{\lambda_s}{\lambda} + b\Big)\partial_\lambda \tilde{Q}_{\mathcal{P}} = \mathcal{O}\Big(\mathcal{P}^3 \big| Mod(t) \big|\Big) (\alpha_s - 2\lambda\beta)\partial_\alpha \tilde{Q}_{\mathcal{P}} = \lambda \Big(\frac{\alpha_s}{\lambda} - 2\beta\Big)\partial_\alpha \tilde{Q}_{\mathcal{P}} = \mathcal{O}\Big(\mathcal{P}^3 \big| Mod(t) \big|\Big)$$

Computation of the \mathcal{P} laws now relies on combinations of scalar products of equations (6.76) and (6.77), making use of orthogonality conditions (6.42)-(6.46). Thus, as proved in [RS11]

Lemma 6.7. We claim the following computations hold

$$-\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ \nabla\Theta\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ \nabla\Sigma\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ y\Sigma\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ y\Theta\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$-\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ \Lambda\Theta\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ \Lambda\Sigma\right)$$

$$=-2\mathcal{R}e\left(\epsilon,\ \overline{\tilde{Q}_{\mathcal{P}}}\right)+\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ |y|^{2}\Sigma\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ |y|^{2}\Theta\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

$$-\left(-M_{2}(\epsilon)+b\Lambda\epsilon_{1}-2\beta\cdot\nabla\epsilon_{1},\ \varrho_{2}\right)+\left(M_{1}(\epsilon)+b\Lambda\epsilon_{2}-2\beta\cdot\nabla\epsilon_{2},\ \varrho_{1}\right)$$

$$=\mathcal{O}\left(\mathcal{P}^{2}\|\epsilon\|_{L^{2}}\right)$$

We will focus only on the third scalar product, the only one that brings a seemingly low-order term, which has actually already been proven otherwise, since according to (6.65)

$$-2\mathcal{R}e\left(\epsilon, \ \overline{\tilde{Q}_{\mathcal{P}}}\right) = \int |\epsilon|^2 + \mathcal{O}\left(\mathcal{P}^4 + |\alpha|^2 + \mu\right)$$

Remark 16. Notice the $\left(\frac{\rho}{\varphi}\right)^{\frac{1}{2}}$ term introduced in the goemetric decomposition (6.38) embodies the metric induced perturbation, which otherwise would have radically perturbated the \mathcal{P} laws through the previous scalar products.

First things first we compute the equation satisfied by $\Lambda \tilde{Q}_{\mathcal{P}}$ based on (6.31). We state

Lemma 6.8. For $\tilde{Q}_{\mathcal{P}} = \Sigma + i\Theta$ solution of (6.31), we have

$$-\Delta(\Lambda\Sigma) + \Lambda\Sigma - Q^2 \Lambda\Sigma - 2\Sigma (\Sigma \Lambda\Sigma + \Theta \Lambda\Theta)$$

= $\mathcal{R}e\Lambda\Psi^2_{\mathcal{P}} - 2 (\Sigma - 2\beta . \nabla\Theta + b\Lambda\Theta - \mathcal{R}e\Psi^2_{\mathcal{P}}) - b\Lambda^2\Theta$
+ $2\beta . \nabla(\Lambda\Theta) + \mathcal{O}(\mathcal{P}^2)$ (6.81)

and

$$-\Delta(\Lambda\Theta) + \Lambda\Theta - Q^2 \Lambda\Theta - 2\Theta \left(\Sigma \Lambda\Sigma + \Theta \Lambda\Theta\right)$$

= $\mathcal{I}m\Lambda\Psi_{\mathcal{P}}^2 - 2\left(\Theta + 2\beta \cdot \nabla\Sigma - b\Lambda\Sigma - \mathcal{I}m\Psi_{\mathcal{P}}^2\right) + b\Lambda^2\Sigma$ (6.82)
 $-2\beta \cdot \nabla(\Lambda\Sigma) + \mathcal{O}(\mathcal{P}^2)$

Proof of Lemma 6.8 :

Rewriting (6.31) up to second order, $\tilde{Q}_{\mathcal{P}}$ satisfy :

$$\begin{split} \Delta \tilde{Q}_{\mathcal{P}} &- \tilde{Q}_{\mathcal{P}} + \tilde{Q}_{\mathcal{P}} |\tilde{Q}_{\mathcal{P}}|^2 + i \, b \, \Lambda \tilde{Q}_{\mathcal{P}} - 2 \, i \, \beta \, . \, \nabla \tilde{Q}_{\mathcal{P}} = -\Psi_{\mathcal{P}}^2 \\ |\Psi_{\mathcal{P}}^2| \lesssim \mathcal{P}^2 \, e^{-C_2 \, |y|} \end{split}$$

Then we check

$$\begin{split} \Lambda \tilde{Q}_{\mathcal{P}} \, |\tilde{Q}_{\mathcal{P}}|^2 &= \Lambda \big[\tilde{Q}_{\mathcal{P}} \, |\tilde{Q}_{\mathcal{P}}|^2 \big] - 2 \, \mathcal{R}e \big(y \, \cdot \, \nabla \tilde{Q}_{\mathcal{P}} \, \tilde{Q}_{\mathcal{P}} \big) \, \tilde{Q}_{\mathcal{P}} \\ &= \Lambda \big[\tilde{Q}_{\mathcal{P}} \, |\tilde{Q}_{\mathcal{P}}|^2 \big] + 2 \, \tilde{Q}_{\mathcal{P}} \, |\tilde{Q}_{\mathcal{P}}|^2 - 2 \, \tilde{Q}_{\mathcal{P}} \, \big(\Sigma \, \Lambda \Sigma + \Theta \, \Lambda \Theta \big) \end{split}$$

and

$$\Delta(\Lambda \tilde{Q}_{\mathcal{P}}) = \Lambda[\Delta \tilde{Q}_{\mathcal{P}}] + 2\,\Delta \tilde{Q}_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^2)$$

A straightforward computation now shows

$$\begin{split} &\Delta\left(\Lambda\tilde{Q}_{\mathcal{P}}\right) - \Lambda\tilde{Q}_{\mathcal{P}} + \Lambda\tilde{Q}_{\mathcal{P}} \,|\tilde{Q}_{\mathcal{P}}|^{2} + 2\,\tilde{Q}_{\mathcal{P}}\left(\Sigma\,\Lambda\Sigma + \Theta\,\Lambda\Theta\right) + i\,b\,\Lambda^{2}\tilde{Q}_{\mathcal{P}} - 2\,i\,\beta\,.\,\nabla\left(\Lambda\tilde{Q}_{\mathcal{P}}\right) \\ &= \Lambda\left[\Delta\tilde{Q}_{\mathcal{P}} - \tilde{Q}_{\mathcal{P}} + \tilde{Q}_{\mathcal{P}}|\tilde{Q}_{\mathcal{P}}|^{2} + i\,b\,\Lambda\tilde{Q}_{\mathcal{P}} - 2\,i\,\beta\,.\,\nabla\tilde{Q}_{\mathcal{P}}\right] + 2\left(\Delta\tilde{Q}_{\mathcal{P}} - \tilde{Q}_{\mathcal{P}} + \tilde{Q}_{\mathcal{P}}|\tilde{Q}_{\mathcal{P}}|^{2}\right) + 2\,\tilde{Q}_{\mathcal{P}} + \mathcal{O}(\mathcal{P}^{2}) \\ &= -\Lambda\Psi_{\mathcal{P}}^{2} + 2\left(\tilde{Q}_{\mathcal{P}} + 2\,i\,\beta\,.\,\nabla\tilde{Q}_{\mathcal{P}} - i\,b\,\Lambda\tilde{Q}_{\mathcal{P}} - \Psi_{\mathcal{P}}^{2}\right) + \mathcal{O}(\mathcal{P}^{2}) \end{split}$$

which leads to (6.81) and (6.82) by taking real and imaginary parts. \Box

Proof of Lemma 6.7:

First we recall from (6.50) and (6.51) we may write up to second order

$$M_{1}(\epsilon) = -\Delta\epsilon_{1} + \epsilon_{1} - \left[(Q^{2} + 2\Sigma^{2})\epsilon_{1} + 2\Sigma\Theta\epsilon_{2} \right] + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}})$$

$$M_{2}(\epsilon) = -\Delta\epsilon_{2} + \epsilon_{2} - \left[(Q^{2} + 2\Theta^{2})\epsilon_{2} + 2\Sigma\Theta\epsilon_{1} \right] + \mathcal{O}(\mathcal{P}^{2} \|\epsilon\|_{L^{2}})$$
(6.83)

From (6.41) we have

$$\nabla\Sigma = \nabla Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2 |y|}), \quad \Lambda\Sigma = \Lambda Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2 |y|})$$

$$\nabla\Theta = \left(-\frac{b}{2}y + \beta\right)Q + \left(-\frac{b}{4}\rho^2 + \beta \cdot y\right)\nabla Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2 |y|})$$

$$\Lambda\Theta = \left(-\frac{b}{4}\rho^2 + \beta \cdot y\right)\Lambda Q + \left(-\frac{b}{2}\rho^2 + \beta \cdot y\right)Q + \mathcal{O}(\mathcal{P}^2 e^{-C_2 |y|})$$
(6.84)

We now have everything we need to proceed to the computation of the third scalar product in (6.80). Using (6.83), then integration by parts we see

$$\begin{pmatrix} M_1(\epsilon), \Lambda \Sigma \end{pmatrix} = \left(\epsilon_1, -\Delta(\Lambda \Sigma) + \Lambda \Sigma - \left(Q^2 + 2\Sigma^2\right)\Lambda \Sigma\right) - \left(\epsilon_2, 2\Sigma\Theta\Lambda\Sigma\right) + \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2}) \\ \left(M_2(\epsilon), \Lambda\Theta\right) = \left(\epsilon_2, -\Delta(\Lambda\Theta) + \Lambda\Theta - \left(Q^2 + 2\Theta^2\right)\Lambda\Theta\right) - \left(\epsilon_1, 2\Sigma\Theta\Lambda\Theta\right) + \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2})$$

Then injecting (6.81) and (6.82) yields

$$\begin{split} & \left(M_{1}(\epsilon),\Lambda\Sigma\right) + \left(M_{2}(\epsilon),\Lambda\Theta\right) \\ &= \left(\epsilon_{1},-\Delta(\Lambda\Sigma) + \Lambda\Sigma - Q^{2}\Lambda\Sigma - 2\Sigma\left(\Sigma\Lambda\Sigma + \Theta\Lambda\Theta\right)\right) \\ &+ \left(\epsilon_{2},-\Delta(\Lambda\Theta) + \Lambda\Theta - Q^{2}\Lambda\Theta - 2\Theta\left(\Sigma\Lambda\Sigma + \Theta\Lambda\Theta\right)\right) + \mathcal{O}(\mathcal{P}^{2} \, \|\epsilon\|_{L^{2}}) \\ &= -2\left(\epsilon_{1},\,\Sigma - 2\beta\,.\nabla\Theta + b\Lambda\Theta - \mathcal{R}e\Psi_{\mathcal{P}}^{2}\right) - 2\left(\epsilon_{2},\,\Theta + 2\beta\,.\nabla\Sigma - b\Lambda\Sigma - \mathcal{I}m\Psi_{\mathcal{P}}^{2}\right) \\ &+ b\left[-\left(\epsilon_{1},\,\Lambda^{2}\Theta\right) + \left(\epsilon_{2},\,\Lambda^{2}\Sigma\right)\right] + \left[\left(\epsilon_{1},\,2\beta\,.\nabla(\Lambda\Theta)\right) - \left(\epsilon_{2},\,2\beta\,.\nabla(\Lambda\Sigma)\right)\right] \\ &+ \left(\epsilon_{1},\mathcal{R}e\Lambda\Psi_{\mathcal{P}}^{2}\right) + \left(\epsilon_{2},\mathcal{I}m\Lambda\Psi_{\mathcal{P}}^{2}\right) + \mathcal{O}(\mathcal{P}^{2} \, \|\epsilon\|_{L^{2}}) \\ &= -2\mathcal{R}e\left(\epsilon,\overline{\tilde{Q}_{\mathcal{P}}}\right) + 2b\left[-\left(\epsilon_{1},\Lambda\Theta\right) + \left(\epsilon_{2},\Lambda\Sigma\right)\right] + 4\beta\,.\left[\left(\epsilon_{1},\,\nabla\Theta\right) - \left(\epsilon_{2},\,\nabla\Sigma\right)\right] \\ &- \left(-b\Lambda\epsilon_{1} + 2\beta\,.\nabla\epsilon_{1},\,\Lambda\Theta\right) + \left(-b\Lambda\epsilon_{2} + 2\beta\,.\nabla\epsilon_{2},\,\Lambda\Sigma\right) + \mathcal{O}(\mathcal{P}^{2} \, \|\epsilon\|_{L^{2}}) \end{split}$$

Eventually, using orthogonality condition (6.42) and (6.44) we get our result. \Box

It remains to find some linear equations that will help us to estimate the vector Mod(t) of the laws of the \mathcal{P} parameters. In the sequel we will use some derivatives estimates that have already been pointed out in (6.56) along with the rough estimates

$$\mathcal{J}\Sigma = \mathcal{J}Q + \mathcal{O}(\mathcal{P}e^{-c|y|}), \quad \mathcal{J}\Theta = \mathcal{O}(\mathcal{P}e^{-c|y|}), \quad \text{whenever } \mathcal{J} = Id, \nabla, \Lambda$$

 $\underline{\text{Law of b}} \quad \text{We compute } \left(\textcircled{0}, -\Lambda\Theta \right) + \left(\textcircled{0}, \Lambda\Sigma \right), \text{ with } \textcircled{0} = (6.76), \textcircled{0} = (6.77): \\
\left(b_s + b^2 - B_1 \right) \int \frac{|y|^2}{4} Q^2 \, dy - 2 \, \mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right) - \left[\left(R_1(\epsilon), \Lambda\Sigma \right) + \left(R_2(\epsilon), \Lambda\Theta \right) \right] \\
= \mathcal{R}_{11}(\mathcal{P}) + S_1(\mathcal{P}) \, Mod(t) + \mathcal{O}\left(\left(\mathcal{P}^2 + |Mod(t)| \right) \|\epsilon\|_{L^2} + \mathcal{P}^7 \right)$ (6.85)

with \mathcal{R}_{11} satisfying (6.74), where we have used the scalar products estimates (6.80) and orthogonality condition (6.44) to get

$$\left(\partial_s \epsilon_1, -\Lambda \Theta\right) + \left(\partial_s \epsilon_2, \Lambda \Sigma\right) = \left(\epsilon_1, \partial_s \Lambda \Theta\right) - \left(\epsilon_2, \partial_s \Lambda \Sigma\right) = \mathcal{O}\left(\mathcal{P}^2 \|\epsilon\|_{L^2}\right)$$

Note also that from Hölder estimates, with definition (6.50) we have

$$\left(R_1(\epsilon), \Lambda \Sigma\right) + \left(R_2(\epsilon), \Lambda \Theta\right) = \mathcal{O}(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3)$$

then one has from (6.65)

$$2 \mathcal{R}e \int \epsilon \ \overline{\tilde{Q}_{\mathcal{P}}} \, dy = -\int |\epsilon|^2 \, dy + \frac{\kappa_0}{6} \left(|\alpha|^2 + \mu \right) \int Q^2 \, dy + \mathcal{O}\left(\mathcal{P}^3 \, \|\epsilon\|_{L^2} + \left| \|u\|_{L^2(dg)} - \|Q\|_{L^2} \right| \right)$$

so we can compute

$$\begin{aligned} (b_s + b^2 - B_1) & \frac{\|yQ\|_{L^2}^2}{4} - 2 \,\mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right) \\ &= (b_s + b^2 - B_1 - K_b(\alpha, \mu)) \,\int \frac{|y|^2}{4} \,Q^2 + \int |\epsilon|^2 + \mathcal{R}_{12}(\mathcal{P}) + \mathcal{O}\left(\mathcal{P}^3 \,\|\epsilon\|_{L^2} + \left\|\|u\|_{L^2(dg)}^2 - \|Q\|_{L^2}^2\right) \right) \\ &\text{with} \quad K_b(\alpha, \mu) = \frac{2 \,\kappa_0}{3} \,\frac{\|Q\|_{L^2}^2}{\|\rho Q\|_{L^2}^2} \left(|\alpha|^2 + \mu\right) \end{aligned}$$

where \mathcal{R}_{12} polynomial satisfying (6.74), hence

$$(b_{s} + b^{2} - B_{1} - K_{b}) \int \frac{\rho^{2}}{4} Q^{2} dy = \mathcal{R}_{1}(\mathcal{P}) + S_{1}(\mathcal{P}) Mod(t) + \mathcal{O}\Big(\left(\mathcal{P}^{2} + |Mod(t)|\right) \|\epsilon\|_{L^{2}} + \|\epsilon\|_{H^{1}}^{3} + \|\epsilon\|_{L^{2}}^{2} + \left\|\|u\|_{L^{2}(dg)} - \|Q\|_{L^{2}}\right| + \mathcal{P}^{7} \Big)$$
(6.86)

where \mathcal{R}_1 is a polynomial satisfying (6.74).

$$\underline{\text{Law of }\lambda} \qquad \text{We compute } \left(\widehat{\mathbb{O}}, |y|^2 \Sigma \right) + \left(\widehat{\mathbb{O}}, |y|^2 \Theta \right) \text{ with } \widehat{\mathbb{O}} = (6.76), \ \widehat{\mathbb{O}} = (6.77): \\
\left(\frac{\lambda_s}{\lambda} + b \right) \int |y|^2 Q^2 \, dy - \left(R_1(\epsilon), |y|^2 \Theta \right) + \left(R_2(\epsilon), |y|^2 \Sigma \right) \\
= \mathcal{R}_2(\mathcal{P}) + S_2(\mathcal{P}) \, Mod(t) + \mathcal{O} \left(\left(\mathcal{P}^2 + |Mod(t)| \right) \|\epsilon\|_{L^2} + \mathcal{P}^7 + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3 \right)$$
(6.87)

where we have used the scalar products estimates (6.80), and orthogonality condition (6.45) to get

$$\left(\partial_s \epsilon_1, |y|^2 \Sigma\right) + \left(\partial_s \epsilon_2, |y|^2 \Theta\right) = -\left(\epsilon_1, \partial_s |y|^2 \Sigma\right) - \left(\epsilon_2, \partial_s |y|^2 \Theta\right) = \mathcal{O}\left(\mathcal{P}^2 \|\epsilon\|_{L^2}\right)$$

and $S_2(\mathcal{P})$ is polynomial of order 1, while \mathcal{R}_2 is a polynomial term in \mathcal{P} that only contains remaining terms of the form $F_{\mathcal{P}}$ through $\Psi_{\mathcal{P}}$, so that according to (6.79) it satisfies

$$\mathcal{R}_{2}(\mathcal{P}) = \mathcal{O}\left(\mathcal{P}^{5} + \mathcal{P}\left(|\alpha|^{2} + |\beta|^{2}\right) + \mathcal{P}^{3}\left(b - \frac{\lambda}{C_{0}}\right)\right)$$
(6.88)

Again, with Hölder estimates we have

$$-\left(R_1(\epsilon), |y|^2 \Theta\right) + \left(R_2(\epsilon), |y|^2 \Sigma\right) = \mathcal{O}\left(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\right)$$

$$\underline{\text{Law of } \alpha} \qquad \text{We compute } \left(\widehat{\mathbb{0}}, y \Sigma \right) + \left(\widehat{\mathbb{0}}, y \Theta \right) \text{ with } \widehat{\mathbb{0}} = (6.76), \ \widehat{\mathbb{0}} = (6.77): \\
\left(\frac{\alpha_s}{\lambda} - 2\beta \right) \int \frac{Q^2}{2} \, dy + \left(R_1(\epsilon), \ y \Theta \right) - \left(R_2(\epsilon), \ y \Sigma \right) \\
= \mathcal{R}_3(\mathcal{P}) + S_3(\mathcal{P}) \, Mod(t) + \mathcal{O} \left(\left(\mathcal{P}^2 + |Mod(t)| \right) \|\epsilon\|_{L^2} + \mathcal{P}^7 + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3 \right)$$
(6.89)

where we have used the scalar products estimates (6.80), and orthogonality condition (6.43) to get

$$\left(\partial_s \epsilon_1, y \Sigma\right) + \left(\partial_s \epsilon_2, y \Theta\right) = -\left(\epsilon_1, \partial_s y \Sigma\right) - \left(\epsilon_2, \partial_s y \Theta\right) = \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2})$$

also \mathcal{R}_3 is a polynomial satisfying (6.74), while $S_3(\mathcal{P})$ is polynomial of order 1. From Hölder estimates we also have

$$(R_1(\epsilon), y\Theta) - (R_2(\epsilon), y\Sigma) = \mathcal{O}(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3)$$

$$\underline{\text{Law of }\beta} \qquad \text{We compute } \left(\widehat{\mathbb{Q}}, -\nabla\Theta \right) + \left(\widehat{\mathbb{Q}}, \nabla\Sigma \right) \text{ with } \widehat{\mathbb{Q}} = (6.76), \ \widehat{\mathbb{Q}} = (6.77): \\
- \left(\beta_s + b\beta - B_2\right) \int \frac{Q^2}{2} \, dy - \left(R_1(\epsilon), \ \nabla\Sigma \right) - \left(R_2(\epsilon), \ \nabla\Theta \right) \\
= \mathcal{R}_4(\mathcal{P}) + S_4(\mathcal{P}) \, Mod(t) + \mathcal{O}\left(\left(\mathcal{P}^2 + |Mod(t)|\right) \|\epsilon\|_{L^2} + \mathcal{P}^7 + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\right)$$
(6.90)

where we have used the scalar products estimates (6.80), and orthogonality condition (6.42) to get

$$\left(\partial_s \epsilon_1, -\nabla \Theta\right) + \left(\partial_s \epsilon_2, \nabla \Sigma\right) = \left(\epsilon_1, \partial_s \nabla \Theta\right) - \left(\epsilon_2, \partial_s \nabla \Sigma\right) = \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2})$$

and \mathcal{R}_4 is a polynomial satisfying (6.74), while $S_4(\mathcal{P})$ is polynomial of order 1. From Hölder estimates we also have

$$(R_1(\epsilon), \nabla \Sigma) + (R_2(\epsilon), \nabla \Theta) = \mathcal{O}(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3)$$

Law of $\tilde{\gamma}$ We recall that ρ was introduced as the unique radial H^1 solution to $L_+\rho = |y|^2 Q$, and ρ_1, ρ_2 are therefore the real, radial and H^1 functions introduced as

$$\varrho_1 + i\,\varrho_2 = \varrho(y)\,e^{-i\,b\,\frac{|y|^2}{4} + i\,\beta\,\cdot\,y}$$

so that their second order extensions are given by

$$\varrho_1 = \varrho + \mathcal{O}(\mathcal{P}^2 e^{-c|y|}), \quad \varrho_2 = \left(-b\frac{\rho^2}{4} + \beta \cdot y\right)\varrho + \mathcal{O}(\mathcal{P}^2 e^{-c|y|})$$

We then compute $(@, -\varrho_2) + (@, \varrho_1)$ with @ = (6.76), @ = (6.77):

$$-(b_s+b^2-B_1)\int \frac{|y|^2}{4}Q\,\varrho\,dy + \left(\tilde{\gamma}_s-|\beta|^2\right)\int Q\,\varrho\,dy - \left(R_1(\epsilon)\,,\,\varrho_1\right) - \left(R_2(\epsilon)\,,\,\varrho_2\right)$$
$$= \mathcal{R}_5(\mathcal{P}) + S_5(\mathcal{P})\,Mod(t) + \mathcal{O}\left(\left(\mathcal{P}^2+|Mod(t)|\right)\|\epsilon\|_{L^2} + \mathcal{P}^7\right)$$

where we have used the scalar products estimates (6.80), and orthogonality condition (6.46) to get

$$\left(\partial_s \epsilon_1, -\varrho_2\right) + \left(\partial_s \epsilon_2, \varrho_1\right) = \left(\epsilon_1, \partial_s \varrho_2\right) - \left(\epsilon_2, \partial_s \varrho_1\right) = \mathcal{O}(\mathcal{P}^2 \|\epsilon\|_{L^2})$$

and \mathcal{R}_5 is a polynomial satisfying (6.74), while S_5 is of order 1 with respect to \mathcal{P} , and using again Hölder estimates

$$\left(R_1(\epsilon), \ \varrho_1\right) + \left(R_2(\epsilon), \ \varrho_2\right) = \mathcal{O}\left(\|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\right)$$

so that

$$-(b_s+b^2-B_1-K_b)\int \frac{|y|^2}{4}Q\,\varrho\,dy + \left(\tilde{\gamma}_s-|\beta|^2-K_{\tilde{\gamma}}\right)\int Q\,\varrho\,dy$$
$$=\mathcal{R}_5(\mathcal{P})+S_5(\mathcal{P})\,Mod(t)+\mathcal{O}\Big(\left(\mathcal{P}^2+|Mod(t)|\right)\|\epsilon\|_{L^2}+\mathcal{P}^7+\|\epsilon\|_{L^2}^2+\|\epsilon\|_{H^1}^3\Big)$$
with $K_{\tilde{\gamma}}=\frac{\left(\frac{\rho^2}{4}Q\,,\,\varrho\right)}{\left(Q\,,\,\varrho\right)}K_b$

Conclusion

We therefore have proven that there are an invertible matrix A, a polynomial vector $\tilde{\mathcal{R}}(\mathcal{P})$ satisfaying (6.74), and a one order with respect to \mathcal{P} function $S(\mathcal{P})$ so that

$$\begin{aligned} \left(A + \mathcal{S}(\mathcal{P})\right) Mod(t) &= \tilde{\mathcal{R}}(\mathcal{P}) + \mathcal{O}\Big(\left(\mathcal{P}^2 + |Mod(t)|\right) \|\epsilon\|_{L^2} + \mathcal{P}^7 + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3 + \left|\|u\|_{L^2(dh)} - \|Q\|_{L^2}\right|\Big) \\ \text{with} \quad A &= \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 0 & 0 \\ A_4 & 0 & 0 & 0 & 0 & 0 & A_5 \end{pmatrix}, \quad \begin{cases} A_1 &= \int \frac{|y|^2}{4} Q^2 \, dy & A_2 = \int |y|^2 Q^2 \, dy \\ A_3 &= \int \frac{Q^2}{2} \, dy & A_4 = -\int \frac{|y|^2}{4} Q \, \varrho \, dy \\ A_5 &= \int Q \, \varrho \, dy \end{aligned}$$

Since A is an invertible matrix, for t close enough to 0, $A + S(\mathcal{P})$ is also invertible. Let AS_2^{-1} be the second order Taylor expansion of $(A + S(\mathcal{P}))^{-1}$. One may now complete the proof of (6.62) by choosing $\mathcal{R}(\mathcal{P}) = AS_2^{-1}\tilde{\mathcal{R}}(\mathcal{P})$.

Now injecting this into the computation of the law of λ , using (6.88) yields improvement (6.63)

6.4 Refined Energy identity

Now we have proven the approximate construction using profile $\hat{Q}_{\mathcal{P}}$ transformed through the geometric parameters of \mathcal{P} is a valid process, which is made stable when going to the blow-up time, we may want to consider the conservation laws variation between that approximate solution and our exact one u. The following energy estimate will prove particularly useful in the next section that consists in integrating backwards the profile, and later in the uniqueness part of the proof, when all a priori bounds assumed here will have been shown. In particular, to take all benefit from this estimate, it is crucial to first obtain $b \sim \lambda$.

Let u be a solution to (6.1) on $[t_0, 0)$, and w a be an approximate solution to that same equation :

$$i \partial_t w + \Delta_g w + V(x) w + k(x) |w|^2 w = \psi$$
(6.91)

with a priori bounds :

$$\|w\|_{L^2(dg)} \lesssim 1, \quad \|\nabla w\|_{L^2(dg)} \lesssim \frac{1}{\lambda}, \quad \|w\|_{H^{3/2}(dg)} \lesssim \frac{1}{\lambda^{3/2}}$$
(6.92)

Then, we decompose $u = w + \tilde{u}$ so that :

$$i \partial_t \tilde{u} + \Delta_g \tilde{u} + V(x) \,\tilde{u} + k(x) \left(|u|^2 \, u - |w|^2 \, w \right) = -\psi \tag{6.93}$$

We assume the a priori bounds on \tilde{u} :

$$\|\nabla \tilde{u}\|_{L^2(dg)} \lesssim \lambda, \quad \|\tilde{u}\|_{L^2(dg)} \lesssim \lambda^2 \tag{6.94}$$

and on the geometrical parameters :

$$|\lambda \lambda_t + b| \lesssim \lambda^4, \quad b \sim \lambda, \quad |\lambda \alpha_t| \lesssim \lambda, \quad |b_t| \lesssim 1$$
(6.95)

where $\lambda > 0$, $\alpha \in \mathbb{R}^2$, $b \ll 1$. We let A > 0 be a large enough constant, to be chosen later, and ϕ a cut off function such that :

$$\phi'(r) = \begin{cases} r & \text{for } r \le 1, \\ 3 - e^{-r} & \text{for } r \ge 2 \end{cases}$$
(6.96)

Let also

$$F_{4}(u) = \frac{1}{4} |u|^{4}, \quad f_{4}(u) = |u|^{2} u, \text{ so that } F_{4}'(u) \cdot z = \mathcal{R}e(f_{4}(u) \overline{z})$$

$$F_{2}(u) = \frac{1}{2} |u|^{2}, \text{ so that } F_{2}'(u) \cdot z = \mathcal{R}e(u \overline{z})$$
(6.97)

We now want to look at the variation of the conservation laws on an exact solution u with respect to the approximate profile w we have just introduced. So let

$$\begin{split} m(u) &= \frac{1}{2} \int |u|^2 \sqrt{|g|} dx, \\ M_{\phi,A}(u) &= \mathcal{I}m \int A \, g^{ij} \, \partial_i \phi \left(\frac{x-\alpha}{A\lambda}\right) \partial_j u \, \overline{u} \, \sqrt{|g|} dx \end{split}$$

be respectively the mass of u, and a cut-off of momentum for u. Now using the energy expression E(u) and (1.11), a simple computation shows

$$\begin{split} E(u) &- E(w) + \frac{1}{\lambda^2} \left(m(u) - m(w) \right) + \frac{b}{2\lambda} \left(M_{\phi,A}(u) - M_{\phi,A}(w) \right) \\ &= \frac{1}{2} \int g^{ij} \partial_i \tilde{u} \ \overline{\partial_j \tilde{u}} \ dg_x + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} \ dg_x - \mathcal{R}e \int \left[\Delta_g w - \frac{w}{\lambda^2} + k(x) \left| w \right|^2 w + V(x) w \right] \overline{\tilde{u}} \ dg_x \\ &+ \frac{b}{2\lambda} \mathcal{I}m \int A g^{ij} \partial_i \phi \left(\frac{x - \alpha}{A\lambda} \right) \partial_j \tilde{u} \ \overline{\tilde{u}} \ dg_x - \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u} \right] \ dg_x \\ &+ \frac{b}{2\lambda} \mathcal{I}m \int A g^{ij} \partial_i \phi \left(\frac{x - \alpha}{A\lambda} \right) \left[\overline{\tilde{u}} \ \partial_j w + \partial_j \tilde{u} \ \overline{w} \right] \ dg_x - \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u} \right] \ dg_x \end{split}$$

The first line term can easily be estimated. On the one hand, considering the approximate solution is as we have described before

$$w = \left(\frac{\rho}{\varphi}\right)^{1/2} \frac{1}{k(\alpha)} \frac{1}{\lambda} \tilde{Q}_{\mathcal{P}}\left(t, \frac{x-\alpha}{\lambda}\right) e^{i\gamma}, \quad \tilde{u} = \left(\frac{\rho}{\varphi}\right)^{1/2} \frac{1}{k(\alpha)} \frac{1}{\lambda} \epsilon\left(t, \frac{x-\alpha}{\lambda}\right) e^{i\gamma}$$

At first, notice estimates (6.92), (6.94), (6.95) are easily obtain from the previous sections. Then using (6.15), both orthogonality conditions (6.42) and (6.44), and (6.41)

$$\mathcal{R}e\int \left[\Delta_g w - \frac{w}{\lambda^2} + k(x) |w|^2 w + V(x) w\right] \overline{\tilde{u}} \, dg_x = \frac{1}{k(\alpha)} \frac{1}{\lambda^2} \int \left(b \Lambda Q - 2\beta \cdot \nabla Q\right) \epsilon_2 \, dg_y + \mathcal{O}\left(\|\epsilon\|_{L^2(dy)}\right)$$
$$= \frac{1}{k(\alpha)} \frac{1}{\lambda^2} \int \left(b \Lambda \Theta - 2\beta \cdot \nabla \Theta\right) \epsilon_1 \, dy + \mathcal{O}\left(\|\epsilon\|_{L^2(dy)}\right) = \mathcal{O}\left(\|\epsilon\|_{L^2(dy)}\right) = \mathcal{O}\left(\|\tilde{u}\|_{L^2(dx)}\right)$$

On the other hand, definition (6.96) of function ϕ imply

$$\begin{aligned} \nabla \phi(y) &= y, \quad |y| \leq 1, \qquad |\nabla \phi(y)| \lesssim 1, \quad |y| > 1\\ \Delta \phi(y) &= 2, \quad |y| \leq 1, \qquad |\Delta \phi(y)| \lesssim e^{-|y|}, \quad |y| > 1 \end{aligned}$$

so that, since $g^{ij} = I_{ij} + \mathcal{O}(\mathcal{P}^2)$, from integrations by parts

$$\frac{1}{2}\mathcal{I}m\int A g^{ij}\partial_i\phi\left(\frac{x-\alpha}{A\lambda}\right)\left[\bar{\tilde{u}}\,\partial_jw+\partial_j\tilde{u}\,\overline{w}\right]\,dg_x = \mathcal{I}m\int \bar{\tilde{u}}\left[A g^{ij}\,\partial_i\phi\left(\frac{x-\alpha}{A\lambda}\right)\partial_jw+\frac{1}{2\lambda}\,\Delta_g\phi\left(\frac{x-\alpha}{A\lambda}\right)w\right]\,dg_x$$

$$=\frac{1}{\lambda\,k(\alpha)}\mathcal{I}m\int \bar{\epsilon}\left[A g^{ij}\partial_i\phi\left(\frac{y}{A}\right)\partial_j\tilde{Q}_{\mathcal{P}}+\frac{1}{2}\,\Delta_g\phi\left(\frac{y}{A}\right)\tilde{Q}_{\mathcal{P}}+\mathcal{O}(\mathcal{P}^3)\right]\,\frac{\rho}{\varphi}\,\sqrt{|g|}dy$$

$$=\frac{1}{\lambda\,k(\alpha)}\mathcal{I}m\left\{\int_{|y|\leq A}\bar{\epsilon}\,\Lambda\tilde{Q}_{\mathcal{P}}\,dy+\mathcal{O}\left(\|\epsilon\|_{L^2(|y|>A)}+\mathcal{P}\,\|\epsilon\|_{L^2}\right)\right\}=\mathcal{O}\left(\|\epsilon\|_{L^2}\right),\quad\text{as}\quad A\to+\infty$$

since thanks to orthogonality condition (6.44)

$$\lim_{A \to +\infty} \left[\mathcal{I}m \int_{|y| \le A} \bar{\epsilon} \Lambda \tilde{Q}_{\mathcal{P}} \, dy + \mathcal{O}\big(\|\epsilon\|_{L^2(|y| > A)} \big) \right] = 0$$

Now this should be enough motivation, thus we claim the following result

Proposition 6.9. (Generalized energy estimate) Let

$$\mathcal{I} = \frac{1}{2} \int g^{ij} \partial_i \tilde{u} \ \overline{\partial_j \tilde{u}} \ dg_x + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} \ dg_x + \frac{b}{2\lambda} \mathcal{I}m \int A g^{ij} \partial_i \phi\left(\frac{x-\alpha}{A\lambda}\right) \partial_j \tilde{u} \ \overline{\tilde{u}} \ dg_x \\
- \int k(x) \left[F_4(w+\tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u}\right] \ dg_x - \int V(x) \left[F_2(w+\tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u}\right] \ dg_x$$
(6.98)

then, we have

$$\frac{d\mathcal{I}}{dt} = -\frac{1}{\lambda^2} \mathcal{I}m \int k(x) w^2 \,\overline{\tilde{u}^2} \, dg_x - \mathcal{R}e \int k(x) \,\partial_t w \,\overline{(2\,|\tilde{u}|^2 \,w + \tilde{u}^2 \,\overline{w})} \, dg_x \\
+ \frac{b}{\lambda^2} \left[\int \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x + \mathcal{R}e \int \nabla_g^2 \phi \left(\frac{x-\alpha}{A\lambda}\right) \cdot \left(\nabla \tilde{u} \,, \, \nabla \overline{\tilde{u}}\right) \, dg_x - \frac{1}{4A^2} \int \Delta_g^2 \phi \left(\frac{x-\alpha}{A\lambda}\right) \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x \right] \\
+ \mathcal{I}m \int \left[\Delta_g \psi - \frac{\psi}{\lambda^2} + k(x) \left(2\,|w|^2 \,\psi - w^2 \,\overline{\psi}\right) + i \, \frac{b}{\lambda} \, A \, g^{ij} \, \partial_i \phi \left(\frac{x-\alpha}{A\lambda}\right) \partial_j \psi + i \, \frac{b}{2\lambda^2} \,\psi \, \Delta_g \phi \left(\frac{x-\alpha}{A\lambda}\right) \right] \overline{\tilde{u}} \, dg_x \\
+ \frac{b}{\lambda} \, \mathcal{R}e \int A \, g^{ij} \, \partial_i \phi \left(\frac{x-\alpha}{A\lambda}\right) k(x) \left(2\,w \,|\tilde{u}|^2 + \overline{w} \,\tilde{u}^2\right) \partial_j \overline{w} \, dg_x + \mathcal{O}\left(\lambda^2 \,\|\psi\|_{L^2}^2 + \frac{1}{\lambda^2} \,\|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2\right)$$
(6.99)

Remark 17. (i) Note that (6.99) brings a better control, for it keeps track of the quadratic terms in \tilde{u} . It is all about getting a control of the form :

$$\frac{d}{dt}\left\{\|\nabla_g \tilde{u}\|_{L^2(dg_x)}^2 + \frac{1}{\lambda^2} \|\tilde{u}\|_{L^2(dg_x)}^2\right\} \ge \frac{b}{\lambda^2} \left(\int_{|x-\alpha| \lesssim \lambda} g^{ij} \partial_i \tilde{u} \, \partial_j \overline{\tilde{u}} \, dg_x + \frac{1}{\lambda^2} \|\tilde{u}\|_{L^2(dg_x)}^2 + l.o.t.\right)$$

(ii) In the sequel, we will mainly use $\mathcal{O}\left(\frac{1}{\lambda^2} \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2\right)$ as estimation in H^1 for the tail \tilde{u} of solution u. Notice that in the context of a riemannian manifold, we should more strictly use $\mathcal{O}\left(\int \frac{|\tilde{u}|^2}{\lambda^2} dg_x + \int (\nabla \tilde{u}, \nabla \overline{\tilde{u}})_g dg_x\right)$. But since g is assumed smooth and $W^{1,\infty}$, there is no harm in using the previous estimate.

Proof of Proposition 6.9 :

Step 1 The computation is essentially the same as in [RS11], except for the term Δ which is replaced by Δ_g . Thus, derivating the first part of \mathcal{I} , using (6.93) and (6.97) yields

$$\begin{split} &\frac{d}{dt} \left\{ \frac{1}{2} \int g^{ij}(x) \,\partial_i \tilde{u} \,\partial_j \overline{\tilde{u}} \,dg_x + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} \,dg_x - \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u} \right] \,dg_x \right\} \\ &- \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u} \right] \,dg_x \right\} \\ &= \mathcal{R}e \Big(\partial_t \tilde{u} \,, \overline{-\Delta_g \tilde{u} + \frac{1}{\lambda^2} \tilde{u} - k(x) \left[f_4(w + \tilde{u}) - f_4(w) \right] - V(x) \left[w + \tilde{u} - w \right]} \Big)_{L^2(dg_x)} - \frac{\lambda_t}{\lambda^3} \int |\tilde{u}|^2 \,dg_x \\ &+ \mathcal{R}e \Big(\partial_t w \,, \overline{-k(x) \left[f_4(w + \tilde{u}) - f_4(w) - f_4'(w) \cdot \tilde{u} \right] - V(x) \left[w + \tilde{u} - w - \tilde{u} \right]} \Big)_{L^2(dg_x)} \\ &= \mathcal{I}m \Big(\psi \,, \overline{\Delta_g \tilde{u} - \frac{1}{\lambda^2} \tilde{u} + k(x) \left[f_4(u) - f_4(w) \right]} \Big)_{L^2(dg_x)} - \frac{1}{\lambda^2} \mathcal{I}m \Big(k(x) \left[f_4(u) - f_4(w) \right] \,, \overline{\tilde{u}} \Big)_{L^2(dg_x)} \\ &- \frac{\lambda_t}{\lambda^3} \int |\tilde{u}|^2 \,dg_x - \mathcal{R}e \Big(\partial_t w \,, \overline{k(x) \left[f_4(w + \tilde{u}) - f_4(w) - f_4'(w) \cdot \tilde{u} \right]} \Big)_{L^2(dg_x)} \end{split}$$

Then, notice that $f_4'(w) \cdot \tilde{u} = 2 |w|^2 \tilde{u} + w^2 \overline{\tilde{u}}$, and

$$\mathcal{I}m\Big(\tilde{u}\,,\overline{k(x)\left[f_4(w+\tilde{u})-f_4(w)-f_4'(w)\,.\,\tilde{u}\right]}\Big)_{L^2(dg_x)} = \mathcal{I}m\int k(x)\left[2\,\overline{w}\,\tilde{u}\,|\tilde{u}|^2+|\tilde{u}|^4+w\,\overline{\tilde{u}}\,|\tilde{u}|^2\right]\,dg_x$$
$$=\mathcal{I}m\int k(x)\,\overline{w}\,\tilde{u}\,|\tilde{u}|^2\,dg_x$$

so that the last line can be rewritten

$$\begin{split} &\mathcal{I}m\Big(\psi,\overline{\Delta_{g}\tilde{u}-\frac{1}{\lambda^{2}}\tilde{u}+k(x)\left[2\,|w|^{2}\,\tilde{u}+\overline{\tilde{u}}\,w^{2}\right]}\Big)_{L^{2}(dg_{x})}-\frac{1}{\lambda^{2}}\,\mathcal{I}m\int k(x)\,\overline{\tilde{u}}^{2}\,w^{2}\,dg_{x}\\ &-\frac{\lambda_{t}}{\lambda^{3}}\,\int |\tilde{u}|^{2}\,dg_{x}-\mathcal{R}e\Big(\partial_{t}w\,,\overline{k(x)\left[\overline{w}\,\tilde{u}^{2}+2\,w\,|\tilde{u}|^{2}\right]}\Big)_{L^{2}(dg_{x})}-\mathcal{R}e\Big(\partial_{t}w\,,\overline{k(x)\,\tilde{u}\,|\tilde{u}|^{2}}\Big)_{L^{2}(dg_{x})}\\ &+\mathcal{I}m\Big(\psi-\frac{1}{\lambda^{2}}\,\tilde{u}\,,\overline{k(x)\left[f_{4}(u)-f_{4}(w)-f_{4}\,'(w)\,.\,\tilde{u}\right]}\Big)_{L^{2}(dg_{x})} \end{split}$$

We now estimate some of those remaining terms. First, from (6.95) one gets

$$-\frac{\lambda_t}{\lambda^3} \int |\tilde{u}|^2 \, dg_x = \frac{b}{\lambda^4} \int |\tilde{u}|^2 \, dg_x - \left(\lambda \,\lambda_t + b\right) \int |\tilde{u}|^2 \, dg_x = \frac{b}{\lambda^4} \int |\tilde{u}|^2 \, dg_x + \mathcal{O}\left(\|\tilde{u}\|_{L^2(dg_x)}^2\right)$$
the last two terms are to be treated with a priori bounds (6.02) and (6.04)

Then, the last two terms are to be treated with a priori bounds (6.92) and (6.94)

$$\begin{aligned} \left| \mathcal{I}m\left(\psi - \frac{1}{\lambda^{2}}\tilde{u}, \overline{k(x)}\left[f_{4}(u) - f_{4}(w) - f_{4}'(w).\tilde{u}\right]\right)_{L^{2}(dg_{x})} \right| \\ &= \left| \mathcal{I}m\left(\psi - \frac{1}{\lambda^{2}}\tilde{u}, \overline{k(x)}\left[2w\left|\tilde{u}\right|^{2} + \tilde{u}\left|\tilde{u}\right|^{2} + \overline{w}\tilde{u}^{2}\right]\right)_{L^{2}(dg_{x})} \right| \\ &\lesssim \|\psi\|_{L^{2}(dg_{x})}\left(\|\tilde{u}\|_{L^{6}(dg_{x})}^{2}\|w\|_{L^{6}(dg_{x})} + \|\tilde{u}\|_{L^{6}(dg_{x})}^{3}\right) + \frac{1}{\lambda^{2}}\left\|\tilde{u}\|_{L^{6}(dg_{x})}^{3}\|w\|_{L^{2}(dg_{x})} \\ &\lesssim \|\psi\|_{L^{2}(dg_{x})}\left(\|\tilde{u}\|_{L^{2}(dg_{x})}^{2/3}\|\nabla\tilde{u}\|_{L^{2}(dg_{x})}^{4/3}\|w\|_{L^{2}(dg_{x})}^{1/3}\|\nabla w\|_{L^{2}(dg_{x})}^{1/3} + \|\tilde{u}\|_{L^{6}(dg_{x})}^{3}\right) + \frac{1}{\lambda^{2}}\left\|\tilde{u}\|_{L^{2}(dg_{x})}\left\|\nabla\tilde{u}\|_{L^{2}(dg_{x})}^{2}\right\| \\ &\lesssim \lambda^{2}\left\|\psi\|_{L^{2}(dg_{x})} + \|\tilde{u}\|_{H^{1}(dg_{x})}^{2} \right. \end{aligned}$$

where we have used the Gagliardo-Nirenberg estimate

$$\|u\|_{L^6} \le C \|u\|_{L^2}^{1/3} \|\nabla u\|_{L^2}^{2/3}$$

To deal with the last term, we use (6.91) to replace $\partial_t w$, and then by integration by parts, using (6.92) we have

$$\begin{split} \mathcal{R}e\Big(\partial_{t}w\,,\overline{k(x)\,\tilde{u}\,|\tilde{u}|^{2}}\Big)_{L^{2}(dg_{x})}\bigg| &= \left|\mathcal{I}m\int\left[\Delta_{g}w\,k(x)\,|\tilde{u}|^{2}\,\tilde{u}\,+k(x)\,|\tilde{u}|^{2}\,\tilde{u}\,\left(k(x)\,|w|^{2}\,w+V(x)\,w+\psi\right)\right]\,dg_{x}\right|\\ &\lesssim \left\|\partial_{i}\left(\sqrt{|g|}g^{ij}\,\partial_{j}w\right)\right\|_{H^{-1/2}_{dx}}\,\|\,|\tilde{u}|^{2}\,\tilde{u}\,\|_{H^{1/2}_{dx}} + \left(\|\,|w|^{2}\,w\,\|_{L^{2}(dg_{x})}\,+\,\|\psi\,\|_{L^{2}(dg_{x})}\,+\,\|V\,w\,\|_{L^{2}(dg_{x})}\right)\,\|\,|\tilde{u}|^{2}\,\tilde{u}\,\|_{L^{2}(dg_{x})}\\ &\lesssim \|w\|_{H^{3/2}_{dx}}\,\|\,|\tilde{u}|^{2}\,\tilde{u}\,\|_{H^{1/2}_{dx}}^{1/2}\,+\,\|w\|^{3}_{L^{6}(dg_{x})}\,\|\,\tilde{u}\|^{3}_{L^{6}(dg_{x})}\,+\,\|\psi\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\|^{3}_{L^{6}(dg_{x})}\\ &\lesssim \frac{1}{\lambda^{3/2}}\,\|\,\tilde{u}\,\|^{1/2}_{L^{2}_{dx}}\,\|\,\tilde{u}\,\|^{5/2}_{H^{1}_{dx}}\,+\,\frac{1}{\lambda^{2}}\,\|\,\tilde{u}\,\|^{2}_{H^{1}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,+\,\|\psi\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|^{2}_{H^{1}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|^{2}_{H^{1}(dg_{x})}\,\|\,\tilde{u}\,\|^{2}_{H^{1}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{H^{1}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|^{2}_{H^{1}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}\,\|_{L^{2}(dg_{x})}^{2}\,\|\,\tilde{u}$$

where we have used twice the pseudo derivative estimate

$$\begin{split} \left\| D^{1/2}(|\tilde{u}|^{2}\,\tilde{u}) \right\|_{L^{2}} &\lesssim \left\| D^{1/2}(|\tilde{u}|^{2}) \right\|_{L^{8/3}} \|\tilde{u}\|_{L^{8}} + \|\,|\tilde{u}|^{2}\,\|_{L^{4}} \,\left\| D^{1/2}\tilde{u} \right\|_{L^{4}} \\ &\lesssim \left\| D^{1/2}\tilde{u} \right\|_{L^{4}} \|\tilde{u}\|_{L^{8}}^{2} \end{split}$$

then the Sobolev embeddings Theorem results 8.2 and 8.3 give

$$H^{3/4}(\mathbb{R}^2) \subseteq L^8(\mathbb{R}^2), \text{ and } H^1(\mathbb{R}^2) \subseteq W^{1/2,4}(\mathbb{R}^2)$$

to get

$$\left\| D^{1/2} \tilde{u} \right\|_{L^4} \| \tilde{u} \|_{L^8}^2 \lesssim \| \tilde{u} \|_{H^1} \left(\| \tilde{u} \|_{H^1}^{3/4} \| \tilde{u} \|_{L^2}^{1/4} \right)^2 \lesssim \| \tilde{u} \|_{L^2}^{1/2} \| \tilde{u} \|_{H^1}^{5/2}$$

Eventually

$$\begin{split} &\frac{d}{dt} \left\{ \frac{1}{2} \int g^{ij} \partial_i \tilde{u} \, \partial_j \overline{\tilde{u}} \, dg_x + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x - \int k(x) \left[F_4(w + \tilde{u}) - F_4(w) - F_4'(w) \cdot \tilde{u} \right] \, dg_x \right. \\ &- \int V(x) \left[F_2(w + \tilde{u}) - F_2(w) - F_2'(w) \cdot \tilde{u} \right] \, dg_x \right\} \\ &= -\frac{1}{\lambda^2} \mathcal{I}m \, \int k(x) \, w^2 \, \overline{\tilde{u}^2} \, dg_x - \mathcal{R}e \, \int k(x) \, \partial_t w \, \overline{(2 \, |\tilde{u}|^2 \, w + \tilde{u}^2 \, \overline{w})} \, dg_x + \frac{b}{\lambda^2} \, \int \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x \\ &+ \mathcal{I}m \, \int \left[\Delta_g \psi - \frac{\psi}{\lambda^2} + k(x) \, (2 \, |w|^2 \, \psi - w^2 \, \overline{\psi}) \right] \overline{\tilde{u}} \, dg_x + \mathcal{O}\left(\lambda^2 \, \|\psi\|_{L^2}^2 + \frac{1}{\lambda^2} \, \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 \right) \end{split}$$

Step 2 There remains to deal with the localized momentum part. First, let

$$\nabla \tilde{\phi}(t,x) = \frac{b}{\lambda} A \nabla \phi \left(\frac{x-\alpha}{A \lambda} \right)$$

 then

$$\frac{1}{2} \frac{d}{dt} \left(\frac{b}{\lambda} \mathcal{I}m \int A g^{ij} \partial_i \phi \left(\frac{x - \alpha}{A \lambda} \right) \partial_j \tilde{u} \,\overline{\tilde{u}} \, dg_x \right) = \frac{1}{2} \frac{d}{dt} \left(\mathcal{I}m \int g^{ij} \partial_i \tilde{\phi} \, \partial_j \tilde{u} \,\overline{\tilde{u}} \, dg_x \right)$$
$$= \frac{1}{2} \mathcal{I}m \int g^{ij} \partial_t \partial_i \tilde{\phi} \, \partial_j \tilde{u} \,\overline{\tilde{u}} \, dg_x + \mathcal{R}e \int i \partial_t \tilde{u} \left[\frac{1}{2} \Delta_g \tilde{\phi} \, \tilde{u} + g^{ij} \, \partial_i \tilde{\phi} \, \partial_j \tilde{u} \right] \, dg_x$$

First, thanks to $\left(6.95\right)$ and the computation :

$$\begin{aligned} \partial_t \nabla \tilde{\phi} &= A \Biggl\{ \frac{1}{\lambda^3} \left[\lambda^2 \, b_t + b^2 \right] \nabla \phi \left(\frac{x - \alpha}{A \, \lambda} \right) - \frac{1}{\lambda^3} \, \lambda \, \lambda_t \, \left[\frac{b}{\lambda} \, \nabla^2 \phi \left(\frac{x - \alpha}{A \, \lambda} \right) \cdot \left(\frac{x - \alpha}{A} \, , \ \cdot \right) + b \, \nabla \phi \left(\frac{x - \alpha}{A \, \lambda} \right) \right] \\ &- \frac{b}{\lambda^3} \Biggl[\lambda \, \nabla^2 \phi \left(\frac{x - \alpha}{A \, \lambda} \right) \cdot \left(\frac{\alpha_t}{A} \, , \ \cdot \right) + b \, \nabla \phi \left(\frac{x - \alpha}{A \, \lambda} \right) \Biggr] \Biggr\} \end{aligned}$$

we have :

$$|\partial_t \nabla \tilde{\phi}| \lesssim \frac{1}{\lambda}$$

so that, since g is smooth

$$\left|\frac{1}{2}\mathcal{I}m\int g^{ij}\partial_t\partial_i\tilde{\phi}\,\partial_j\tilde{u}\,\overline{\tilde{u}}\,\,dg_x\right|\lesssim \frac{1}{\lambda}\,\|\tilde{u}\|_{L^2(dg_x)}\,\|\nabla\tilde{u}\|_{L^2(dg_x)}=\mathcal{O}\Big(\frac{1}{\lambda^2}\,\|\tilde{u}\|_{L^2(dg_x)}^2+\|\tilde{u}\|_{H^1(dg_x)}^2\Big)$$

Now, with the second term

$$\begin{aligned} &\mathcal{R}e\int i\partial_{t}\tilde{u}\left[\frac{1}{2}\Delta_{g}\tilde{\phi}\,\tilde{u}+g^{ij}\,\partial_{i}\tilde{\phi}\,\partial_{j}\tilde{u}\right]\,dg_{x} \\ &=\mathcal{R}e\int g^{ij}\,\partial_{j}\left(g^{lm}\,\partial_{l}\tilde{\phi}\right)\partial_{i}\tilde{u}\,\partial_{m}\overline{\tilde{u}}\,dg_{x}-\frac{1}{2}\,\mathcal{R}e\int g^{lm}\,\partial_{l}\tilde{\phi}\,\partial_{m}g^{ij}\,\partial_{i}\tilde{u}\,\partial_{j}\overline{\tilde{u}}\,dg_{x}-\frac{1}{4}\int\Delta_{g}^{2}\tilde{\phi}\,|\tilde{u}|^{2}\,dg_{x} \\ &+\frac{1}{2}\int g^{ij}\,\partial_{i}\tilde{\phi}\,\partial_{j}V\,|\tilde{u}|^{2}\,dg_{x}-\mathcal{R}e\int\left[k(x)\left(f_{4}(u)-f_{4}(w)\right)+\psi\right]\left(\frac{1}{2}\Delta_{g}\tilde{\phi}\,\overline{\tilde{u}}+g^{ij}\,\partial_{i}\tilde{\phi}\,\partial_{j}\overline{\tilde{u}}\right)\,dg_{x} \\ &=\mathcal{R}e\int\nabla_{g}^{2}\tilde{\phi}\,.\left(\nabla\tilde{u}\,,\,\nabla\overline{\tilde{u}}\right)\,dg_{x}-\frac{1}{2}\,\mathcal{R}e\int\left(\nabla\tilde{\phi}\,,\,\nabla g^{ij}\right)_{g}\,\partial_{i}\tilde{u}\,\partial_{j}\overline{\tilde{u}}\,dg_{x}-\frac{1}{4}\int\Delta_{g}^{2}\tilde{\phi}\,|\tilde{u}|^{2}\,dg_{x} \\ &+\frac{1}{2}\int\left(\nabla\tilde{\phi}\,,\,\nabla V\right)_{g}\,|\tilde{u}|^{2}\,dg_{x}-\mathcal{R}e\int\left[k(x)\left(f_{4}(u)-f_{4}(w)\right)+\psi\right]\left(\frac{1}{2}\Delta_{g}\tilde{\phi}\,\overline{\tilde{u}}+g^{ij}\,\partial_{i}\tilde{\phi}\,\partial_{j}\overline{\tilde{u}}\right)\,dg_{x} \end{aligned}$$

where we have denoted

$$abla_g = g^{ij} \partial_i, \quad \text{and} \quad (X, Y)_g = g^{ij} X_i Y_j$$

From here, first notice that from (6.95) $b \sim \lambda$, and since $\phi, V, g \in W^{1,\infty}(\mathbb{R}^2)$, we have

$$\begin{split} -\frac{1}{2} \mathcal{R}e \int \left(\nabla \tilde{\phi}, \nabla g^{ij}\right)_g \partial_i \tilde{u} \, \partial_j \overline{\tilde{u}} \, dg_x + \frac{1}{2} \, \int \left(\nabla \tilde{\phi}, \nabla V\right)_g |\tilde{u}|^2 \, dg_x &= \mathcal{O}\Big(\|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2\Big) \\ &= \mathcal{O}\Big(\frac{\|\tilde{u}\|_{L^2}^2}{\lambda^2} + \|\tilde{u}\|_{H^1}^2\Big) \end{split}$$

 ${\rm thus}$

$$\mathcal{R}e \int i\partial_{t}\tilde{u} \overline{\left[\frac{1}{2}\Delta_{g}\tilde{\phi}\,\tilde{u}+g^{ij}\,\partial_{i}\tilde{\phi}\,\partial_{j}\tilde{u}\right]} \, dg_{x}$$

$$= \frac{b}{\lambda^{2}}\mathcal{R}e \int \nabla_{g}^{2}\phi\left(\frac{x-\alpha}{A\lambda}\right).\left(\nabla\tilde{u},\,\nabla\bar{\tilde{u}}\right) \, dg_{x} - \frac{b}{4A^{2}\lambda^{4}} \int \Delta_{g}^{2}\phi\left(\frac{x-\alpha}{A\lambda}\right)|\tilde{u}|^{2} \, dg_{x}$$

$$- \mathcal{R}e \int \left[k(x)\left(f_{4}(u)-f_{4}(w)\right)+\psi\right]\left(\frac{b}{2\lambda^{2}}\Delta_{g}\phi\left(\frac{x-\alpha}{A\lambda}\right)\bar{\tilde{u}}+\frac{b}{\lambda}Ag^{ij}\,\partial_{i}\phi\left(\frac{x-\alpha}{A\lambda}\right)\partial_{j}\bar{\tilde{u}}\right) \, dg_{x}$$

$$+ \mathcal{O}\left(\frac{1}{\lambda^{2}}\|\tilde{u}\|_{L^{2}}^{2}+\|\tilde{u}\|_{H^{1}}^{2}\right)$$

$$(6.100)$$

There is left to deal with the non-linear terms. We will use (6.92) to (6.95) along with integration by parts, Holder estimates and Gagliardo-Nirenberg inequalities. First we decompose

$$f_4(u) - f_4(w) = \left(|w|^2 + |\tilde{u}|^2 + \overline{w}\,\tilde{u} + w\,\overline{\tilde{u}}\right)(w + \tilde{u}) - |w|^2\,w$$
$$= \left(2\,|w|^2\,\tilde{u} + w^2\,\overline{\tilde{u}}\right) + \left(2\,w\,|\tilde{u}|^2 + \tilde{u}^2\,\overline{w} + |\tilde{u}|^2\,\tilde{u}\right)$$

Then using (6.58)

$$\begin{split} \left| \int k(x) \left(2w \, |\tilde{u}|^2 + \tilde{u}^2 \,\overline{w} + |\tilde{u}|^2 \,\tilde{u} \right) \left[\frac{b}{2\lambda^2} \,\Delta_g \phi \left(\frac{x-\alpha}{A\lambda} \right) \,\overline{\tilde{u}} + \frac{b}{\lambda} \,A \, g^{ij} \partial_i \phi \left(\frac{x-\alpha}{A\lambda} \right) \partial_j \overline{\tilde{u}} \right] \,dg_x \right| \\ \lesssim \frac{1}{\lambda} \left(\int |\tilde{u}|^3 \, |w| \, dg_x + \int |\tilde{u}|^4 \, dg_x \right) + \int \left(|\tilde{u}|^2 \, |w| + |\tilde{u}|^3 \right) |\nabla \tilde{u}| \, dg_x \\ \lesssim \frac{1}{\lambda} \left(\|\tilde{u}\|_{L^6}^3 \, \|w\|_{L^2} + \|\tilde{u}\|_{L^4}^4 \right) + \left\| |\tilde{u}|^2 \, |w| + |\tilde{u}|^3 \, \right\|_{L^2} \|\nabla \tilde{u}\|_{L^2} \\ \lesssim \frac{1}{\lambda} \left(\|\tilde{u}\|_{L^6}^3 \, \|w\|_{L^2} + \|\tilde{u}\|_{L^4}^4 \right) + \left(\|\tilde{u}\|_{L^6}^2 \, \|w\|_{L^6} + \|\tilde{u}\|_{L^6}^3 \right) \|\nabla \tilde{u}\|_{L^2} \\ \lesssim \frac{1}{\lambda} \left(\|\tilde{u}\|_{L^2} \, \|\nabla \tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 \|\nabla \tilde{u}\|_{L^2}^2 \right) + \left(\|\tilde{u}\|_{L^2}^{2/3} \, \|\nabla \tilde{u}\|_{L^2}^{4/3} \, \|w\|_{L^2}^{1/3} \, \|\nabla w\|_{L^2}^{4/3} + \|\tilde{u}\|_{L^2} \, \|\nabla \tilde{u}\|_{L^2}^2 \right) \\ \lesssim \|\tilde{u}\|_{H^1}^2 = \mathcal{O}\left(\frac{1}{\lambda^2} \, \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2\right) \end{split}$$

Moreover

$$\begin{split} &-\int k(x) \left(2 \, |w|^2 \, \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \left[\frac{b}{2\lambda^2} \, \Delta_g \phi \left(\frac{x-\alpha}{A\lambda} \right) \, \overline{\tilde{u}} + \frac{b}{\lambda} \, A \, g^{ij} \, \partial_i \phi \left(\frac{x-\alpha}{A\lambda} \right) \partial_j \overline{\tilde{u}} \right] \, dg_x \\ &= \frac{b}{2\lambda} \, \mathcal{R}e \int A \, g^{ij} \, \partial_j \phi \left(\frac{x-\alpha}{A\lambda} \right) \left[\partial_i k \left(2 \, |w \, \tilde{u}|^2 + \left(w \, \overline{\tilde{u}} \right)^2 \right) + 2 \, k \left(2 \, w \, |\tilde{u}|^2 + \overline{w} \, \tilde{u}^2 \right) \partial_i \overline{w} + 2 \, k \left(2 \, |w|^2 \, \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \partial_i \overline{\tilde{u}} \right] \, dg_x \\ &- \frac{b}{\lambda} \int A \, g^{ij} \, \partial_i \phi \left(\frac{x-\alpha}{A\lambda} \right) \partial_j \overline{\tilde{u}} \, k(x) \left(2 \, |w|^2 \, \tilde{u} + w^2 \, \overline{\tilde{u}} \right) \, dg_x \\ &= \frac{b}{2\lambda} \, \mathcal{R}e \int A \, g^{ij} \, \partial_j \phi \left(\frac{x-\alpha}{A\lambda} \right) \left[\partial_i k \left(2 \, |w \, \tilde{u}|^2 + \left(w \, \overline{\tilde{u}} \right)^2 \right) + 2 \, k \left(2 \, w \, |\tilde{u}|^2 + \overline{w} \, \tilde{u}^2 \right) \partial_i \overline{w} \right] \, dg_x \end{split}$$

and since k is smooth, one gets the estimate

$$\begin{aligned} \left| \frac{b}{2\lambda} \mathcal{R}e \int A g^{ij} \partial_i \phi \left(\frac{x-\alpha}{A\lambda} \right) \partial_j k \left(2 |w \, \tilde{u}|^2 + \left(w \, \overline{\tilde{u}} \right)^2 \right) \, dg_x \right| &\lesssim \|\tilde{u}\|_{L^4}^2 \, \|w\|_{L^4}^2 \\ &\lesssim \frac{1}{\lambda} \, \|\tilde{u}\|_{L^2} \, \|\nabla \tilde{u}\|_{L^2} = \mathcal{O}\Big(\frac{1}{\lambda^2} \, \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 \Big) \end{aligned}$$

Finally, integrating the last term of (6.100) by parts, we have

$$\begin{aligned} -\frac{b}{\lambda} \mathcal{R}e \int A g^{ij} \partial_i \phi\Big(\frac{x-\alpha}{A\lambda}\Big) \partial_j \overline{\tilde{u}} \ \psi \ dg_x &= \frac{b}{\lambda} \mathcal{R}e \int \overline{\tilde{u}} \ A g^{ij} \partial_i \phi\Big(\frac{x-\alpha}{A\lambda}\Big) \partial_j \psi \ dg_x \\ &+ \frac{b}{\lambda^2} \mathcal{R}e \int \overline{\tilde{u}} \ \psi \ \Delta_g \phi\Big(\frac{x-\alpha}{A\lambda}\Big) \ dg_x \end{aligned}$$

so that

$$\mathcal{R}e \int i\partial_t \tilde{u} \left[\frac{1}{2} \Delta_g \tilde{\phi} \tilde{u} + g^{ij} \partial_i \tilde{\phi} \partial_j \tilde{u} \right] dg_x$$

$$= \frac{b}{\lambda^2} \mathcal{R}e \int \nabla_g^2 \phi \left(\frac{x - \alpha}{A\lambda} \right) . \left(\nabla \tilde{u} , \nabla \overline{\tilde{u}} \right) dg_x - \frac{b}{4A^2\lambda^4} \int \Delta_g^2 \phi \left(\frac{x - \alpha}{A\lambda} \right) |\tilde{u}|^2 dg_x$$

$$+ \frac{b}{\lambda} \mathcal{R}e \int \overline{\tilde{u}} A g^{ij} \partial_i \phi \left(\frac{x - \alpha}{A\lambda} \right) \partial_j \psi dg_x + \frac{b}{2\lambda^2} \mathcal{R}e \int \overline{\tilde{u}} \psi \Delta_g \phi \left(\frac{x - \alpha}{A\lambda} \right) dg_x$$

$$+ \frac{b}{\lambda} \mathcal{R}e \int A g^{ij} \partial_i \phi \left(\frac{x - \alpha}{A\lambda} \right) k(x) \left(2w |\tilde{u}|^2 + \overline{w} \, \tilde{u}^2 \right) \partial_j \overline{w} dg_x + \mathcal{O} \left(\frac{1}{\lambda^2} \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 \right)$$

$$(6.101)$$

6.5 Backward propagation of smallness

We use the last section to get a bootstrap result : the geometrical decomposition we have already used to approximate the NLS solution can be integrated backward from the blowing-up singularity time while keeping its parameters properties as defined in section 6.3.

That will allow us in the next section 6.6 to build the formal exact NLS solution with all wanted properties. Then the existence would be done, except the proof of (6.8).

Let u be a solution of (6.1) on $[\tilde{t}_0, 0)$, $\tilde{t}_0 < t_1 < 0$, and assume, there is a geometrical decomposition of u on $[\tilde{t}_0, t_1]$ such that

$$u(t,x) = \left(\frac{\rho}{\varphi(y,\lambda(t),\alpha(t))}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \left(\tilde{Q}_{\mathcal{P}(t)} + \epsilon\right) \left(t, \frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
(6.102)

where ϵ satisfies orthogonality conditions (6.42)-(6.46) and $\|\epsilon(t)\|_{H^1} + |\mathcal{P}(t)| \ll 1$. Let

$$\tilde{u}(t,x) = \left(\frac{\rho}{\varphi(y,\lambda(t),\alpha(t))}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \epsilon\left(t,\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
(6.103)

Assume that the energy E_0 satisfy :

$$E_0 + C_E > 0$$

We now may define constant C_0 as

$$C_0 = \sqrt{\frac{\|\rho Q\|_{L^2}^2}{8(E_0 + C_E)}}, \quad \text{so will hold} \quad \frac{b}{\lambda} - \frac{1}{C_0} = o(1), \text{ as } t \to 0$$
(6.104)

We claim, as in [RS11] the Backwards propagation estimates, with corrected mass (6.40)

Lemma 6.10. Assuming there holds for some $t_1 < 0$ close enough to 0 :

$$\left| \|u\|_{L^2(dg_x)} - \|Q\|_{L^2} \right| \lesssim \lambda^4(t_1), \tag{6.105}$$

$$\|\nabla \tilde{u}(t_1)\|_{L^2}^2 + \frac{\|\tilde{u}(t_1)\|_{L^2}^2}{\lambda^2(t)} \lesssim \lambda^2(t_1), \tag{6.106}$$

$$\left|\frac{\beta}{\lambda}(t_1)\right| + \left|\frac{\alpha}{\lambda}(t_1)\right| \lesssim \lambda^{1/2}(t_1), \quad \left|\lambda(t_1) + \frac{t_1}{C_0} - K_0 C_0 \iota_\lambda(t_1) + K_1 \iota_{k(\alpha)}^{[2]}(t_1)\right| \lesssim \lambda^3(t_1)$$
(6.107)

$$\frac{b(t_1)}{\lambda(t_1)} - \frac{1}{C_0} + K_0 C_0 \lambda(t_1) - K_1 \iota_{k(\alpha)}(t_1) \Big| \lesssim \lambda^2(t_1)$$
(6.108)

Then, there exists a backward time t_0 depending only on C_0 such that $\forall t \in [t_0, t_1]$,

$$\|\nabla \tilde{u}(t)\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} \lesssim \|\nabla \tilde{u}(t_{1})\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t_{1})\|_{L^{2}}^{2}}{\lambda^{2}(t_{1})} + \lambda^{4}(t)$$
(6.109)

$$\left|\frac{b}{\lambda}(t) - \frac{1}{C_0} + K_0 C_0 \lambda(t) - K_1 \iota_{k(\alpha)}(t)\right| \lesssim \lambda^2(t)$$
(6.110)

$$\left|\lambda(t) + \frac{t}{C_0} - K_0 C_0 \iota_\lambda(t) + K_1 \iota_{k(\alpha)}^{[2]}(t)\right| \lesssim \lambda^3(t)$$
(6.111)

$$\left|\frac{\alpha}{\lambda}(t)\right|^2 + \left|\frac{\beta}{\lambda}(t)\right|^2 \lesssim l_E(\alpha(t)) + K_2\,\lambda(t) - \frac{2K_1}{C_0}\,\iota_{k(\alpha)}(t) + \lambda^2(t) \tag{6.112}$$

with

$$\begin{split} \iota_{k(\alpha)}(t) &= \int_{0}^{t} \nabla^{2} k(0) . \left(\alpha(\tau), \alpha(\tau)\right) \frac{d\tau}{2\lambda^{3}(\tau)} = \mathcal{O}(|t|), \\ \iota_{\lambda}(t) &= \int_{0}^{t} \lambda(\tau) \, d\tau = \mathcal{O}(|t|^{2}), \quad \iota_{k(\alpha)}^{[2]}(t) = \int_{0}^{t} \iota_{k(\alpha)}(\tau) \, d\tau = \mathcal{O}(|t|^{2}), \quad l_{E}(\alpha) = \phi_{E}(\alpha) \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2}, \quad see \ (6.34) \\ K_{0} &= \frac{c_{0}}{8} \int \left[9 + \rho^{2} Q^{2}\right] \rho Q^{2} \, dy \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2}, \quad K_{1} = \|Q\|_{L^{2}}^{2} \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2}, \quad K_{2} = \frac{c_{0}}{8} \int \rho^{3} Q^{4} \, dy \left\| \frac{|y|}{2} Q \right\|_{L^{2}}^{-2} \end{split}$$

Proof of Lemma 6.10

Since $u \in C([t_0, t_1], H^1)$ is continuous, we can find a backwards time t_0 such that $\forall t \in [t_0, t_1]$:

$$\|\tilde{u}(t)\|_{L^2} \le K \lambda^2(t), \quad \|\tilde{u}(t)\|_{H^1} \le K \lambda(t)$$
(6.113)

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$$\left|\frac{\beta(t)}{\lambda(t)}\right| + \left|\frac{\alpha(t)}{\lambda(t)}\right| \le K \lambda^{1/2}(t), \quad \left|\lambda(t) + \frac{t}{C_0}\right| \le K \lambda^2(t), \quad \left|\frac{b(t)}{\lambda(t)} - \frac{1}{C_0}\right| \le K \lambda(t)$$
(6.114)

by the way, this also implies $|\iota_{\lambda}(t)| + |\iota_{k(\alpha)}(t)| + |\iota_{k(\alpha)}^{2}(t)| \le K \lambda(t)$ (6.115)

for some large enough universal constant K > 0. Then we claim that (6.109)-(6.112) hold on $[t_0, t_1]$, and thus improves (6.113), (6.114) on $[t_0, t_1]$ for some $t_0 = t_0(C_0)$ small enough and independent of t_1 .

Step 1 Monotonicity of the norm

We then apply previous lemma to function

$$w(t,x) = \tilde{Q}(t,x) = \left(\frac{\rho}{\varphi(y,\lambda(t),\alpha(t))}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \tilde{Q}_{\mathcal{P}(t)}\left(t,\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)} \frac{1}{\lambda(t)} \tilde{Q}_{\mathcal{P}(t)}\left(t,\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)} \frac{1}{\lambda(t)} \frac{1}{\lambda(t)} \tilde{Q}_{\mathcal{P}(t)}\left(t,\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)} \frac{1}{\lambda(t)} \frac{1}{\lambda(t)} \tilde{Q}_{\mathcal{P}(t)}\left(t,\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)} \frac{1}{\lambda(t)} \frac{1}{\lambda$$

note we may apply previous lemma 6.9, since (6.105) to (6.108) ensure all needed hypotheses are satisfied : on the one hand (6.106) imply $\|\nabla \tilde{u}\|_{L^2} \leq \lambda^2$ on $[t_0, t_1]$, while on the other hand (6.108) imply $b \sim \lambda$ on $[t_0, t_1]$, which were the two missing assumptions for our approximate profile as we have already pointed out.

Let \mathcal{I} be given by (6.98). We will check later (6.99) implies

$$\frac{d\mathcal{I}}{dt} \ge \frac{b}{\lambda^4} \int |\tilde{u}|^2 \, dg_x + \mathcal{O}\left(K^4 \,\lambda^3 + \|\tilde{u}\|_{H^1}^2\right) \tag{6.116}$$

A first rough estimation of \mathcal{I} using Hölder estimates show

$$|\mathcal{I}| \lesssim \|\nabla \tilde{u}\|_{L^2}^2 + \frac{\|\tilde{u}\|_{L^2}^2}{\lambda^2}$$
(6.117)

Then, looking a bit closer we may check using both result and proof of Lemma 6.6 that

$$\begin{split} &\frac{1}{2} \int \left(\nabla \tilde{u}, \nabla \overline{\tilde{u}}\right)_g \, dg_x + \frac{1}{2} \int \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x \\ &= \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \int \left[\left(\nabla \epsilon, \nabla \overline{\epsilon}\right)_g + \mathcal{O}\left(\mathcal{P}^2 \, \epsilon \, \nabla \overline{\epsilon}\right) \right] \frac{\rho}{\varphi} \sqrt{|g|} \, dy + \int |\epsilon|^2 \, \frac{\rho}{\varphi} \sqrt{|g|} \, dy \right\} \\ &= \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \int |\nabla \epsilon|^2 \, dy + \int |\epsilon|^2 \, dy + \mathcal{O}\left(\mathcal{P}^2 \, \|\epsilon\|_{H^1}^2\right) \right\} \\ &= \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \left(L_+\epsilon_1, \, \epsilon_1\right) + \left(L_-\epsilon_2, \, \epsilon_2\right) + \int Q^2 \left(3 \, |\epsilon_1|^2 + |\epsilon_2|^2\right) \, dy + \mathcal{O}\left(\mathcal{P}^2 \, \|\epsilon\|_{H^1}^2\right) \right\} \\ &\gtrsim \frac{1}{2 \, k(\alpha) \, \lambda^2} \left\{ \left\|\epsilon\|_{H^1}^2 - \left(\epsilon_1, \, Q\right)^2 \right\} \end{split}$$

Furthermore with (6.65), (6.8) and (6.105)

$$\left|\mathcal{R}e\left(\epsilon, \ \overline{\tilde{Q}_{\mathcal{P}}}\right)\right| \lesssim \|\epsilon\|_{L^{2}}^{2} + \lambda^{4} + |\alpha|^{2} + \mu + \left|\int |u|^{2} \ dg_{x} - \int Q^{2} \right| \lesssim \|\epsilon\|_{L^{2}}^{2} + K^{2} \lambda^{3}(t)$$

therefore

$$\left(\epsilon_1, Q\right)^2 \lesssim o\left(\|\epsilon\|_{L^2}^2\right) + K^4 \lambda^6(t)$$

from which

$$\mathcal{I} \gtrsim \frac{1}{2k(\alpha)\lambda^2} \left\{ \|\epsilon\|_{H^1}^2 - K^4 \lambda^6(t) \right\} \gtrsim \|\nabla \tilde{u}\|_{L^2}^2 + \frac{\|\tilde{u}\|_{L^2}^2}{\lambda^2} - K^4 \lambda^4(t)$$
(6.118)

Integrating (6.116) between t and t_1 gives

$$\mathcal{I}(t_1) - \mathcal{I}(t) = \int_t^{t_1} \frac{d\mathcal{I}}{dt} \ge \int_t^{t_1} \left(\frac{b}{\lambda^4} \|\tilde{u}\|_{L^2}^2 + K^4 \lambda^3(\tau) + \|\tilde{u}(\tau)\|_{H^1}^2 \right) d\tau \gtrsim \int_t^{t_1} \left(\frac{\|\tilde{u}\|_{L^2}^2}{\lambda^2} + \|\nabla\tilde{u}\|_{L^2}^2 + K^4 \lambda^3(\tau) \right) d\tau$$

thus from (6.117) and (6.118) one gets

$$\begin{aligned} \|\nabla \tilde{u}(t)\|_{L^{2}}^{2} &+ \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} - K^{4} \lambda^{4}(t) \lesssim \mathcal{I}(t) \lesssim \mathcal{I}(t_{1}) + \int_{t}^{t_{1}} \left(K^{4} \lambda^{3}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}}^{2}\right) d\tau \\ &\lesssim \|\nabla \tilde{u}(t_{1})\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t_{1})\|_{L^{2}}^{2}}{\lambda^{2}(t_{1})} + \int_{t}^{t_{1}} \left(K^{4} \lambda^{3}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}}^{2}\right) d\tau \end{aligned}$$

Altogether, we conclude for $t_0 = t_0(C_0)$ small enough

$$\|\nabla \tilde{u}(t)\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} \lesssim \|\nabla \tilde{u}(t_{1})\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t_{1})\|_{L^{2}}^{2}}{\lambda^{2}(t_{1})} + K^{4} \lambda^{4}(t) + \int_{t}^{t_{1}} \left(K^{4} \lambda^{3}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}}^{2}\right) d\tau$$

and (6.109) follows from Gronwall lemma. In particular, from (6.106)

$$\|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{\|\tilde{u}(t)\|_{L^2}^2}{\lambda^2(t)} \lesssim \lambda^2(t)$$
(6.119)

which closes the bootstrap of (6.113).

Step 2 Integration of the laws of the geometrical parameters

From both (6.62) and (6.8) we have the following estimates

$$\left|b_{s}+b^{2}-B_{1}-K_{b}\right|+\left|\frac{\lambda_{s}}{\lambda}+b\right|+\left|\beta_{s}+b\beta-B_{2}\right|+\left|\frac{\alpha_{s}}{\lambda}-2\beta\right|\lesssim\lambda^{4}$$
(6.120)

We now aim at integrating these laws to get (6.110)-(6.112). To do so, we first see from (6.22)

$$\left(\frac{b}{\lambda}\right)_s = \frac{b_s + b^2 - B_1 - K_b}{\lambda} - \frac{b}{\lambda} \left(\frac{\lambda_s}{\lambda} + b\right) + \frac{B_1 + K_b}{\lambda} = \lambda^2 K_0 + \frac{K_1}{2\lambda} \nabla^2 k(0).(\alpha, \alpha) + \mathcal{O}(\lambda^3)$$
 with

$$K_0 = \frac{c_0}{8} \int \left[9 + \rho^2 Q^2\right] \rho Q^2 dy \left\|\frac{|y|}{2} Q\right\|_{L^2}^{-2}, \quad K_1 = \|Q\|_{L^2}^2 \left\|\frac{|y|}{2} Q\right\|_{L^2}^{-2}$$

At first, from $\frac{ds}{dt} = \frac{1}{\lambda^2}$, using $\left(\frac{b}{\lambda}\right)_s = \mathcal{O}(\lambda^2)$, integration with (6.114) and (6.115) leads to

$$\frac{b}{\lambda}(s_1) - \frac{b}{\lambda}(s) = \int_s^{s_1} \left(\frac{b}{\lambda}\right)_s d\sigma = \int_s^{s_1} \mathcal{O}\left(K\,\lambda^2(\sigma)\right) d\sigma \lesssim K \int_t^{t_1} d\tau \lesssim K |t|$$

next, thanks to (6.107) and (6.108) it proves

$$\frac{1}{C_0} - \frac{b}{\lambda}(s) \lesssim \frac{1}{C_0} - \frac{b}{\lambda}(s_1) + K \left| t \right| \lesssim \lambda(t_1) + K \left| t \right| \lesssim K \lambda(s)$$
(6.121)

By the same token that led to (6.70) then to (6.73), we sum conservation of energy and of mass at time t

$$\frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{1}{4} \left(\nabla^2 k(0) + \frac{\kappa_0}{3} Id \right) .(\alpha, \alpha) \int Q^2 \, dy$$
$$= \lambda^2 \left(E_0 + C_E + \phi_E(\alpha) - \lambda C_1 \right) + \mathcal{O}(\lambda^4)$$

where we used the fact that from (6.119)

$$\|\nabla\epsilon(t)\|_{L^2}^2 = k(\alpha(t))\,\lambda^2\,\|\nabla\tilde{u}(t)\|_{L^2}^2 = \mathcal{O}\big(\lambda^4(t)\big), \quad \text{so that} \quad \|\epsilon(t)\|_{H^1}^2 = \mathcal{O}\big(\lambda^4(t)\big)$$

Thus from the choice of C_0 we see it implies thanks to (6.121) that

$$\frac{|\beta|^2}{\lambda^2}(t) + \frac{|\alpha|^2}{\lambda^2}(t) \lesssim \frac{1}{C_0^2} - \frac{b^2}{\lambda^2}(t) + \frac{8}{\|\rho Q\|_{L^2}^2} \left(\phi_E(\alpha(t)) - \lambda(t) C_1\right) + \mathcal{O}\left(\lambda^2(t)\right) \lesssim \frac{1}{C_0} - \frac{b}{\lambda}(t) + \mathcal{O}\left(\lambda(t)\right) \lesssim \lambda(t)$$

which proves $|\alpha(t)|^2 + |\beta(t)|^2 \leq \lambda^3(t)$. In particular, this also proves $\frac{1}{C_0} - \frac{b}{\lambda} > 0$, so that using (6.121) again, that last estimation evenually yields

$$\left|\frac{b}{\lambda}(t) - \frac{1}{C_0}\right| \lesssim |t|$$

which concludes (6.110). Then by (6.63)

$$\left|\lambda_t + \frac{b}{\lambda}\right| = \frac{1}{\lambda} \left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \lambda^3$$

And finally, from integration

$$\left|\lambda(t) + \frac{t}{C_0}\right| \lesssim \left|\lambda(t_1) + \frac{t_1}{C_0}\right| + \int_t^{t_1} \left|\frac{b}{\lambda} - \frac{1}{C_0}\right|(\tau) \, d\tau + \lambda^3(t) \lesssim \lambda^2(t)$$

Now, to refine (6.110), let

$$\iota = \int_{+\infty}^{s} \nabla^2 k(0).(\alpha, \alpha) \frac{d\sigma}{2\lambda} = \int_0^t \nabla^2 k(0).(\alpha, \alpha) \frac{d\tau}{2\lambda^3}$$
(6.122)

be an antiderivative for the remaining term in the $\left(\frac{b}{\lambda}\right)_s$ expression. Then using the previously obtained estimate $\left|\frac{1}{C_0} - \frac{b}{\lambda}(t)\right| \lesssim \lambda(t)$, using also (6.63) one gets $\lambda^2 = C_0 b \lambda + \mathcal{O}(\lambda^3) = -C_0 \lambda_s + \mathcal{O}(\lambda^3)$, so that

$$\left(\frac{1}{C_0} - \frac{b}{\lambda}\right)_s = \left(K_0 C_0 \lambda - K_1 \iota\right)_s + \mathcal{O}(\lambda^3)$$

which, using (6.108), proves (6.110) after integration.

Also notice reinjecting these estimates into μ and ι definitions yields

$$0 < \mu(t) \lesssim \lambda^3(t), \quad \iota(t) = \mathcal{O}(|t|) = \mathcal{O}(\lambda(t)), \quad T_0 < t < t_1 < 0$$

Eventually, let us give some refinements of the above bounds that will be useful in the end of the proof, so first rewrite (6.70) with use of Lemma 6.6

$$\begin{pmatrix} \frac{|\beta|^2}{\lambda^2}(t) - \frac{1}{2} \left(\nabla^2 k(0) + \frac{\kappa_0}{3} Id \right) \cdot \frac{(\alpha, \alpha)}{\lambda^2}(t) + c \|Q\|_{L^2}^{-2} \|\epsilon(t)\|_{H^1}^2 \right) \|Q\|_{L^2}^2 \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2} \\
\leq \frac{1}{C_0^2} - \frac{b^2}{\lambda^2}(t) + 2 \left\| \frac{\rho}{2} Q \right\|_{L^2}^{-2} \left(\phi_E(\alpha(t)) - \lambda(t) C_1 \right) + \mathcal{O}(\lambda^2(t)) \\
= \frac{1}{C_0^2} - \frac{b^2}{\lambda^2}(t) - 2 K_0 \lambda(t) + l_E(\alpha(t)) + K_2 \lambda(t) + \mathcal{O}(\lambda^2(t)) \\
= K_2 \lambda(t) - \frac{2K_1}{C_0} \iota_{k(\alpha)}(t) + l_E(\alpha(t)) + \mathcal{O}(\lambda^2(t)) \\
\text{with} \quad K_2 = \frac{9c_0}{8} \int \rho Q^2 dy \left\| \frac{\rho}{2} Q \right\|_{L^2}^{-2}, \quad l_E(\alpha) = 2 \left\| \frac{\rho}{2} Q \right\|_{L^2}^{-2} \phi_E(\alpha)$$
(6.123)

where we have used (6.121) so one has $\left(\frac{1}{C_0} - \frac{b}{\lambda} - K_0 C_0 \lambda + K_1 \iota\right) \left(\frac{1}{C_0} + \frac{b}{\lambda}\right) = \frac{1}{C_0^2} - \frac{b^2}{\lambda^2} - 2K_0 \lambda + \frac{2K_1}{C_0} \iota + \mathcal{O}(\lambda^2)$, and reinjecting the refined $\frac{1}{C_0} - \frac{b}{\lambda}$ bound into the $\lambda(t) + \frac{t}{C_0}$ bound yields

$$\frac{|\beta|^2}{\lambda^2}(t) + \frac{|\alpha|^2}{\lambda^2}(t) \lesssim K_2 \lambda(t) - \frac{2K_1}{C_0} \iota(t) + l_E(\alpha(t)) + \lambda^2(t)$$

$$\left|\lambda(t) + \frac{t}{C_0} - K_0 C_0 \int_{T_0}^t \lambda(\tau) d\tau + K_1 \int_{T_0}^t \iota(\tau) d\tau\right| \lesssim \lambda^3(t)$$
(6.124)

Step 3 Coercivity of the quadratic form in the $\frac{d\,\mathcal{I}}{dt}$ expression

We now come back to the proof of (6.116). To begin with, we compute explicitly the quadratic terms in (6.99) for $w = \tilde{Q}$ given by

$$\begin{aligned} \mathcal{K}(\tilde{u}) &= -\frac{1}{\lambda^2} \mathcal{I}m \int k(x) \, w^2 \, \overline{\tilde{u}^2} \, dg_x - \mathcal{R}e \, \int k(x) \, \partial_t w \, \overline{(2 \, |\tilde{u}|^2 \, w + \tilde{u}^2 \, \overline{w})} \, dg_x \\ &+ \frac{b}{\lambda^2} \left[\int \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x + \mathcal{R}e \int \nabla_g^2 \phi \left(\frac{x - \alpha}{A\lambda}\right) \cdot \left(\nabla \tilde{u} \,, \, \nabla \overline{\tilde{u}}\right) \, dg_x - \frac{1}{4A^2} \, \int \Delta_g^2 \phi \left(\frac{x - \alpha}{A\lambda}\right) \frac{|\tilde{u}|^2}{\lambda^2} \, dg_x \right] \\ &+ \frac{b}{\lambda} \mathcal{R}e \int A \, g^{ij} \, \partial_i \phi \left(\frac{x - \alpha}{A\lambda}\right) k(x) \left(2 \, w \, |\tilde{u}|^2 + \overline{w} \, \tilde{u}^2\right) \partial_j \overline{w} \, dg_x \end{aligned} \tag{6.125}$$

we claim

$$\mathcal{K}(\tilde{u}) \ge \frac{c_1}{\lambda^3} \left(\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} \, dy + \int |\epsilon|^2 \, dy \right) + \mathcal{O}\left(K^4 \, \lambda^3\right) \tag{6.126}$$

for some universal constant $c_1 > 0$.

First we compute

$$\begin{aligned} \partial_{t}\tilde{Q} &= -\frac{\alpha_{t} \cdot \nabla k(\alpha)}{2\,k(\alpha)}\,\tilde{Q} - \frac{\lambda_{t}}{\lambda}\,\tilde{Q} - \left(\frac{\rho}{\varphi}\right)^{1/2}\frac{\lambda\,\alpha_{t} + \lambda_{t}\,(x-\alpha)}{k(\alpha)^{1/2}\,\lambda^{3}} \cdot \nabla \tilde{Q}_{\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} + i\,\gamma_{t}\,\tilde{Q} \\ &+ \left(\frac{\rho}{\varphi}\right)^{1/2}\frac{1}{k(\alpha)^{1/2}\,\lambda}\,\mathcal{P}_{t}\,\frac{\partial\tilde{Q}_{\mathcal{P}}}{\partial\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} - \frac{\partial_{t}\varphi}{2\varphi}\,\tilde{Q} \\ &= \left(-\frac{(\lambda\,\alpha_{t} - 2\,\beta)\cdot\nabla k(\alpha)}{2\,k(\alpha)\,\lambda} - \frac{2\,\beta\cdot\nabla k(\alpha)}{2\,k(\alpha)\,\lambda} - \frac{\lambda\,\lambda_{t} + b}{\lambda^{2}} + \frac{b}{\lambda^{2}} - \frac{\partial_{t}\varphi}{2\varphi}\right)\tilde{Q} \\ &+ i\left(\frac{1}{\lambda^{2}} + \frac{\lambda^{2}\,\tilde{\gamma}_{t} - |\beta|^{2}}{\lambda^{2}} + \frac{|\beta|^{2}}{\lambda^{2}}\right)\tilde{Q} + \left(\frac{\rho}{\varphi}\right)^{1/2}\frac{1}{k(\alpha)^{1/2}\,\lambda}\,\mathcal{P}_{t}\,\frac{\partial\tilde{Q}_{\mathcal{P}}}{\partial\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} \\ &+ \left(\frac{\rho}{\varphi}\right)^{1/2}\left[-\frac{\lambda\,\lambda_{t} + b}{\lambda^{2}}\left(\frac{x-\alpha}{\lambda}\right) + \frac{b}{\lambda^{2}}\left(\frac{x-\alpha}{\lambda}\right) - \frac{\lambda\,\alpha_{t} - 2\,\beta}{\lambda^{2}} - \frac{2\,\beta}{\lambda^{2}}\right]\cdot\frac{1}{k(\alpha)^{1/2}\,\lambda}\,\nabla\tilde{Q}_{\mathcal{P}}\left(\frac{x-\alpha}{\lambda}\right)e^{i\,\gamma} \end{aligned}$$

then, by the same procedure that led to (6.21), we see
$$\left|\tilde{Q}_{\mathcal{P}} - Q\right| \lesssim \left|\mathcal{P}\right| e^{-|y|}, \quad \text{therefore} \quad \left|\frac{\partial \tilde{Q}_{\mathcal{P}}}{\partial \mathcal{P}}\right| \lesssim e^{-|y|}$$
(6.128)

moreover we check that $|\mathcal{P}_t| \lesssim 1$, so that using (6.113) and (6.120)

$$\partial_t \tilde{Q} = \left(\frac{b}{\lambda^2} + \frac{i}{\lambda^2}\right) \tilde{Q} + \frac{b}{\lambda} \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla \tilde{Q} + \mathcal{O}\left(\frac{K}{\lambda} e^{-\frac{|x-\alpha(t)|}{\lambda(t)}}\right)$$

now putting this together and using the exponential decay of $\tilde{Q}_{\mathcal{P}}$

$$-\mathcal{R}e\int k(x)\tilde{Q}_{t}\left(\overline{2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}}\right)dg_{x}$$

$$=\frac{1}{\lambda^{2}}\mathcal{I}m\int k(x)\tilde{Q}\left(\overline{2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}}\right)dg_{x}-\frac{b}{\lambda^{2}}\mathcal{R}e\int k(x)\overline{\tilde{Q}}\left(2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}\right)dg_{x}$$

$$-\frac{b}{\lambda}\mathcal{R}e\int k(x)\left(\frac{x-\alpha}{\lambda}\right).\overline{\nabla\tilde{Q}}\left(2|\tilde{u}|^{2}\tilde{Q}+\tilde{u}^{2}\overline{\tilde{Q}}\right)dg_{x}+\mathcal{O}\left(\frac{K}{\lambda^{2}}\|\epsilon\|_{L^{2}}^{2}\right)$$
(6.129)

so that

$$\begin{split} &-\mathcal{R}e\int k(x)\,\tilde{Q}_t\,\overline{(2\,|\tilde{u}|^2\,\tilde{Q}+\tilde{u}^2\,\overline{\tilde{Q}})}\,\,dg_x-\frac{1}{\lambda^2}\,\mathcal{I}m\int k(x)\,w^2\,\overline{\tilde{u}^2}\,\,dg_x\\ &=-\frac{1}{k(\alpha)}\,\frac{b}{\lambda^4}\int\frac{\rho}{\varphi}\,\frac{k(\lambda y+\alpha)}{k(\alpha)}\left[\left(|\tilde{Q}_{\mathcal{P}}|^2+2\,\Sigma^2\right)\epsilon_1^2+4\,\Sigma\,\Theta\,\epsilon_1\,\epsilon_2+\left(|\tilde{Q}_{\mathcal{P}}|^2+2\,\Theta^2\right)\epsilon_2^2\right]\,\frac{\rho}{\varphi}\,dg_y\\ &-\frac{1}{k(\alpha)}\,\frac{b}{\lambda^4}\mathcal{R}e\,\int\frac{\rho}{\varphi}\,\frac{k(\lambda y+\alpha)}{k(\alpha)}\,(2\,|\epsilon|^2\,\tilde{Q}_{\mathcal{P}}+\epsilon^2\,\overline{\tilde{Q}_{\mathcal{P}}})\,y\,.\,\overline{\nabla\tilde{Q}_{\mathcal{P}}}\,\frac{\rho}{\varphi}\,dg_y+\mathcal{O}\Big(\frac{K}{\lambda^2}\,\|\epsilon\|_{L^2}^2\Big) \end{split}$$

Remind the geometrical decompositions (6.102) and (6.103) imply the derivatives are decomposed as

$$\nabla u = \left(\frac{\rho}{\varphi}\right)^{\frac{1}{2}} \frac{1}{k(\alpha)^{\frac{1}{2}}} \frac{1}{\lambda^2} \left[\nabla + \Phi\right] \left(\tilde{Q}_{\mathcal{P}} + \epsilon\right) \left(t, \frac{x - \alpha}{\lambda}\right) e^{i\gamma}, \quad \Phi = \frac{1}{2} \left(\frac{y}{\rho^2} - \frac{\nabla\varphi}{\varphi}\right) = \left(\frac{\kappa_0}{6} - \frac{3c_0}{2}\lambda\rho\right) \lambda^2 y$$

Injecting this into (6.125) yields

$$\begin{split} \mathcal{K}(\tilde{u}) &= \frac{1}{k(\alpha)} \frac{b}{\lambda^4} \left[\int \left(1 - \frac{\Delta_g^2 \phi\left(\frac{y}{A}\right)}{4A^2} \right) |\epsilon|^2 \frac{\rho}{\varphi} \, dg_y + \mathcal{R}e \int \nabla_g^2 \phi\left(\frac{y}{A}\right) \cdot \left(\nabla \epsilon + \Phi \, \epsilon \,, \, \overline{\nabla \epsilon + \Phi \, \epsilon}\right) \frac{\rho}{\varphi} \, dg_y \\ &- \int \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[\left(|\tilde{Q}_{\mathcal{P}}|^2 + 2 \, \Sigma^2 \right) \epsilon_1^2 + 4 \, \Sigma \, \Theta \, \epsilon_1 \, \epsilon_2 + \left(|\tilde{Q}_{\mathcal{P}}|^2 + 2 \, \Theta^2 \right) \epsilon_2^2 \right] \frac{\rho}{\varphi} \, dg_y \\ &+ \mathcal{R}e \int \left(A \, g^{ij} \, \partial_i \phi\left(\frac{y}{A}\right) - y_j \right) \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \left(2 \, |\epsilon|^2 \, \tilde{Q}_{\mathcal{P}} + \epsilon^2 \, \overline{\tilde{Q}_{\mathcal{P}}} \right) \partial_j \overline{\tilde{Q}_{\mathcal{P}}} \frac{\rho}{\varphi} \, dg_y \right] + \frac{1}{\lambda^3} \, \mathcal{O}\left(K \, \lambda \, \|\epsilon\|_{L^2}^2 \right) \end{split}$$

Notice that expanding around the origin the metric terms according to (6.4), one has $\nabla_g = \nabla + \mathcal{O}(\lambda^2)$, $\Delta_g = \Delta + \mathcal{O}(\lambda^2)$, $\frac{\rho}{\varphi} dh_y = (1 + \mathcal{O}(\lambda^3)) dy$ and from (6.17) $\Phi = \mathcal{O}(\lambda^2)$ hence

$$\begin{split} \mathcal{K}(\tilde{u}) &= \frac{1}{k(\alpha)} \frac{b}{\lambda^4} \left[\int \left[1 - \frac{1}{4A^2} \,\Delta^2 \,\phi \Big(\frac{y}{A} \Big) \right] |\epsilon|^2 \,\, dy + \mathcal{R}e \int \nabla^2 \phi \Big(\frac{y}{A} \Big) . \left(\nabla \epsilon \,, \, \overline{\nabla \epsilon} \right) \,\, dy \\ &- \int \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[\left(|Q_{\mathcal{P}}|^2 + 2 \,\Sigma^2 \right) \epsilon_1^2 + 4 \,\Sigma \,\Theta \,\epsilon_1 \,\epsilon_2 + \left(|Q_{\mathcal{P}}|^2 + 2 \,\Theta^2 \right) \epsilon_2^2 \right] \,\, dy \\ &+ \mathcal{R}e \,\int \left(A \,\nabla \phi \Big(\frac{y}{A} \Big) - y \right) \frac{k(\lambda y + \alpha)}{k(\alpha)} \left(2 \,|\epsilon|^2 \,Q_{\mathcal{P}} + \epsilon^2 \,\overline{Q_{\mathcal{P}}} \right) . \,\overline{\nabla Q_{\mathcal{P}}} \,\, dy \right] \\ &+ \frac{1}{\lambda^3} \,\mathcal{O}\Big(K \,\lambda \, \|\epsilon\|_{L^2}^2 + \lambda^2 \, \|\epsilon\|_{H^1}^2 \Big) \end{split}$$

and using (6.96)

$$\begin{split} \mathcal{K}(\tilde{u}) &= \frac{1}{k(\alpha)} \frac{b}{\lambda^4} \left[\int \left[1 - \frac{1}{4A^2} \Delta^2 \phi\left(\frac{y}{A}\right) \right] |\epsilon|^2 \, dy + \mathcal{R}e \int \nabla^2 \phi\left(\frac{y}{A}\right) . \left(\nabla \epsilon \,, \, \overline{\nabla \epsilon}\right) \, dy \\ &- \int \left[\left(|Q_{\mathcal{P}}|^2 + 2\,\Sigma^2 \right) \epsilon_1^2 + 4\,\Sigma\,\Theta\,\epsilon_1\,\epsilon_2 + \left(|Q_{\mathcal{P}}|^2 + 2\,\Theta^2 \right) \epsilon_2^2 \right] \, dy \right] \\ &+ \frac{1}{\lambda^3} \, \mathcal{O}\!\left(K\,\lambda \, \|\epsilon\|_{L^2}^2 + \int_{|y|>A} |\epsilon|^2 \, dy + \lambda^2 \, \|\epsilon\|_{H^1}^2 \right) \end{split}$$

From proximity of $\tilde{Q}_{\mathcal{P}}$ to Q, from our choice of orthogonality conditions, from (6.71) and from (6.113), the above quadratic form is for A large enough a small deformation in A of energy which satisfy

$$\mathcal{K}(\tilde{u}) \gtrsim \frac{1}{\lambda^3} \left[\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} dy + \int |\epsilon|^2 dy - \left(\epsilon_1, Q\right)^2 \right]$$

$$\gtrsim \frac{1}{\lambda^3} \left[\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} dy + \int |\epsilon|^2 dy \right] + \mathcal{O}\left(K^4 \lambda^3\right)$$
(6.130)

where we have used $|(\epsilon_1, Q)|^2 \lesssim |\mathcal{R}e(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}})|^2 \lesssim (\lambda^4 + |\alpha|^2 + \mu)^2 \lesssim \lambda^6$ from (6.59), (6.65) and (6.8) along with Lemma 6.6 to get the last line. (6.126) is thus proved.

Step 4 Control of the remainder terms in the $\frac{d\mathcal{I}}{dt}$ expression

We have left to deal with the ψ terms in the $\frac{d\mathcal{I}}{dt}$ expression. According to the definition of ψ we see with (6.15), (6.31), (6.55) and (6.93)

$$\begin{split} \psi &= -i\partial_{t}\tilde{u} - \Delta_{g}\tilde{u} - V(x)\,\tilde{u} - k(x)\left(|\tilde{u}|^{2}\tilde{u} - |w|^{2}w\right) \\ &= \frac{-\left(\frac{\rho}{\varphi}\right)^{1/2}e^{i\gamma}}{k(\alpha)^{1/2}\,\lambda^{3}}\left[i\partial_{s}\epsilon + \left(\mathcal{L}_{g} + V_{g}\right)\epsilon - \epsilon + \frac{\rho}{\varphi}\,\frac{k(\lambda y + \alpha)}{k(\alpha)}\left(\tilde{M}(\epsilon) + R(\epsilon)\right) - i\frac{\lambda_{s}}{\lambda}\,\Lambda\epsilon - i\frac{\alpha_{s}}{\lambda}\,.\,\nabla\epsilon - \tilde{\gamma}_{s}\,\epsilon\right] \\ &= \left(\frac{\rho}{\varphi}\right)^{1/2}\frac{e^{i\gamma}}{k(\alpha)^{1/2}}\left[i(b_{s} + b^{2} - B_{1})\,\partial_{b}\tilde{Q}_{\mathcal{P}} + i\left(\frac{\lambda_{s}}{\lambda} + b\right)\lambda\,\partial_{\lambda}\tilde{Q}_{\mathcal{P}} + i\left(\frac{\alpha_{s}}{\lambda} - 2\,\beta\right)\lambda\,\partial_{\alpha}\tilde{Q}_{\mathcal{P}} \right. \end{split}$$
(6.131)
$$&+ i(\beta_{s} + b\,\beta - B_{2})\,\partial_{\beta}\tilde{Q}_{\mathcal{P}} - i\left(\frac{\lambda_{s}}{\lambda} + b\right)\Lambda\tilde{Q}_{\mathcal{P}} - i\frac{\lambda}{2}\left(\frac{\alpha_{s}}{\lambda} - 2\,\beta\right).\left(\nabla\tilde{Q}_{\mathcal{P}} + \frac{\nabla k(\alpha)}{k(\alpha)}\,\tilde{Q}_{\mathcal{P}}\right) \\ &- (\tilde{\gamma}_{s} - |\beta|^{2})\,\tilde{Q}_{\mathcal{P}} + \psi_{\mathcal{P}}\right]\left(\frac{x - \alpha}{\lambda}\right)e^{i\,\gamma(t)} \end{split}$$

where $\psi_{\mathcal{P}}$ is the remainder term in the construction of $\tilde{Q}_{\mathcal{P}}$:

$$\psi_{\mathcal{P}} = \mathcal{O}\left(\lambda^5 \, e^{-\delta \, |y|}\right)$$

First see that (6.8), (6.64) along with (6.113) and (6.114) gives a rough bound on ψ , that is for i = 0, 1, 2

$$|\nabla^{i}\psi| \lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(|Mod(t)| + |K_{b}| + |K_{\tilde{\gamma}}| + \lambda^{5} \right) \lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(K \lambda^{2} \|\epsilon\|_{L^{2}} + K^{2} \lambda^{3} \right)$$
(6.132)

and thus

$$\|\nabla^{i}\psi\|_{L^{2}} \lesssim \frac{1}{\lambda^{2+i}} \left(K \lambda^{2} \|\epsilon\|_{L^{2}} + K^{2} \lambda^{3}\right)$$

In particular we find the bound

$$\lambda^2 \|\psi\|_{L^2}^2 \lesssim \lambda^2 \|\epsilon\|_{L^2}^2 + K^4 \lambda^4$$

then for any $\nu > 0$

$$\begin{aligned} \left| \mathcal{I}m \int \left[i \frac{b}{\lambda} A g^{ij} \partial_i \phi \left(\frac{x - \alpha}{A \lambda} \right) \partial_j \psi + i \frac{b}{2\lambda^2} \psi \Delta_g \phi \left(\frac{x - \alpha}{A \lambda} \right) \right] \overline{u} \, dg_x \right| \\ \lesssim \|\nabla \psi\|_{L^2} \|\tilde{u}\|_{L^2} + \frac{\|\psi\|_{L^2}}{\lambda} \|\tilde{u}\|_{L^2} \lesssim \frac{1}{\lambda^3} \left[\lambda^2 \|\epsilon\|_{L^2} + K^2 \lambda^3 \right] \|\epsilon\|_{L^2} \\ = \frac{\|\epsilon\|_{L^2}^2}{\lambda} + \left(K^4 \, \lambda^{3-\nu} \right)^{1/2} \left(\frac{\|\epsilon\|_{L^2}^2}{\lambda^{3-\nu}} \right)^{1/2} \lesssim o \left(\frac{\|\epsilon\|_{L^2}^2}{\lambda^3} \right) + K^4 \, \lambda^{3-\nu} \end{aligned}$$

Yet bound (6.132) is not precise enough to deal with remainder terms in (6.98). Let us remind that the construction of $\tilde{Q}_{\mathcal{P}}$ induces

$$\tilde{Q}_{\mathcal{P}} = \tilde{P}_{\mathcal{P}} e^{-i b \frac{|y|^2}{4} + i \beta \cdot y} + \mathcal{O}(|\mathcal{P}|^2), \quad \tilde{P}_{\mathcal{P}} = Q + \mathcal{O}(|\mathcal{P}|^2)$$

so that using (6.120)

$$\begin{split} &i\left(b_s+b^2-B_1\right)\partial_b\tilde{Q}_{\mathcal{P}} = \left(b_s+b^2-B_1\right)\frac{|y|^2}{4}\,Q + \mathcal{O}\big(|\mathcal{P}|\,Mod(t)\big)\\ &i\left(\beta_s+b\,\beta-B_2\right)\partial_\beta\tilde{Q}_{\mathcal{P}} = -(\beta_s+b\,\beta-B_2)\,.\,y\,Q + \mathcal{O}\big(|\mathcal{P}|\,Mod(t)\big) \end{split}$$

now we have $\psi = \psi_1 + \psi_2$ where

$$\psi_1 = \frac{\left(\frac{\rho}{\varphi}\right)^{1/2} e^{i\gamma}}{k(\alpha)^{1/2} \lambda^3} \left[\left(b_s + b^2 - B_1\right) \frac{|y|^2}{4} Q - \left(\beta_s + b\beta - B_2\right) \cdot y Q - i\left(\frac{\lambda_s}{\lambda} + b\right) \Lambda Q - i\frac{\lambda}{2} \left(\frac{\alpha_s}{\lambda} - 2\beta\right) \cdot \nabla Q - \left(\tilde{\gamma}_s - |\beta|^2\right) Q \right]$$

and from (6.21), (6.62), (6.8) for i = 0, 1, 2

$$\left|\nabla^{i}\psi_{2}\right| \lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(\left|\mathcal{P}\right| \left| Mod(t) \right| + K^{2} \lambda^{5} \right) \lesssim \frac{1}{\lambda^{3+i}} e^{-\frac{|x-\alpha|}{\lambda}} \left(\lambda^{3} \left\|\epsilon\right\|_{L^{2}} + K^{2} \lambda^{5} \right)$$
(6.133)

This implies the remainder term ψ_2 in (6.98) may be directly estimated as before for any $\nu > 0$ as

$$\begin{aligned} \left| \mathcal{I}m \int \left[\Delta_{h}\psi_{2} - \frac{\psi_{2}}{\lambda^{2}} + k(x) \left(2 |w|^{2} \psi_{2} - w^{2} \overline{\psi_{2}} \right) \right] \overline{\tilde{u}} \, dg_{x} \right| \\ & \lesssim \left[\|\nabla^{2}\psi_{2}\|_{L^{2}} + \frac{\|\psi_{2}\|_{L^{2}}}{\lambda^{2}} + \|\psi_{2}\|_{L^{\infty}} \|w\|_{L^{4}}^{4} \right] \|\tilde{u}\|_{L^{2}} \\ & \lesssim \frac{\|\epsilon\|_{L^{2}}^{2}}{\lambda} + K^{2} \lambda^{5/2 - \nu/2} \frac{\|\epsilon\|_{L^{2}}^{2}}{\lambda^{3/2 - \nu/2}} \lesssim o\left(\frac{\|\epsilon\|_{L^{2}}^{2}}{\lambda^{3}}\right) + K^{4} \lambda^{5 - \nu} \end{aligned}$$
(6.134)

while the remaining term in ψ_1 is estimated using identities (6.47) and the orthogonality conditions (6.42) and (6.44) on ϵ which allow to gain a factor $\mathcal{O}(\mathcal{P})$

$$\begin{aligned} \left| \mathcal{I}m \int \left[\Delta_g \psi_1 - \frac{\psi_1}{\lambda^2} + k(x) \left(2 \left| w \right|^2 \psi_1 - w^2 \overline{\psi_1} \right) \right] \overline{\tilde{u}} \, dg_x \right| \\ \lesssim \frac{|Mod(t)| + |\alpha|^2}{\lambda^4} \left[\left(\epsilon_2 \,, \, L_-(|y|^2 \, Q) \right) + \left(\epsilon_2 \,, \, L_-(y \, Q) \right) + \left(\epsilon_1 \,, \, L_+(\nabla Q) \right) \right. \\ \left. + \left(\epsilon_2 \,, \, L_-(Q) \right) + \mathcal{O}(\mathcal{P} \, \|\epsilon\|_{L^2}) \right] \end{aligned}$$

$$\left. + \frac{1}{2} \left| \frac{\lambda_s}{\lambda_s} + b \right| \left| \left(\epsilon_1 \,, \, L_+(\Lambda Q) \right) \right| \end{aligned}$$

$$(6.135)$$

$$+ \frac{1}{\lambda^4} \left| \frac{-\delta}{\lambda} + b \right| \left| \left(\epsilon_1, L_+(\Lambda Q) \right) \right|$$

$$\lesssim \frac{\lambda^2 \|\epsilon\|_{L^2} + K^2 \lambda^3}{\lambda^4} \left| \mathcal{P} \right| \|\epsilon\|_{L^2} + \frac{\lambda^2 \|\epsilon\|_{L^2} + K^2 \lambda^5}{\lambda^4} \left(\lambda^2 \|\epsilon\|_{L^2} + K^2 \lambda^3 \right)$$

$$\lesssim o\left(\frac{\|\epsilon\|_{L^2}^2}{\lambda^3} \right) + K^4 \lambda^{3-\nu}$$

where we have used again the fact

$$\Delta_{h}f = \Delta f + \mathcal{O}(\lambda^{2})\left(|\nabla f| + |\nabla^{2}f|\right)$$

and (6.47) together with the conservation of mass through estimate (6.62) and the improved bounds (6.59), (6.63) to get the estimates

$$\left| \left(\epsilon_1, \Sigma \right) + \left(\epsilon_2, \Theta \right) + \frac{1}{2} \| \epsilon \|_{L^2}^2 \right| \lesssim \lambda^4 + |\alpha|^2 + \mu, \quad \text{hence} \quad \left| \left(\epsilon_1, Q \right) \right| \lesssim \lambda^2 \| \epsilon \|_{L^2} + K^2 \lambda^3, \quad \text{so that} \\ \frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} + b \right| \left| \left(\epsilon_1, L_+(\Lambda Q) \right) \right| \lesssim \frac{\lambda^2 \| \epsilon \|_{L^2} + K^2 \lambda^5}{\lambda^4} \left| \left(\epsilon_1, Q \right) \right|$$

Now injecting (6.133), (6.134), (6.135) and (6.126) into (6.99) we see

$$\begin{aligned} \frac{d\mathcal{I}}{dt} \gtrsim \frac{1}{\lambda^3} \left(\int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} dy + \int |\epsilon|^2 dy \right) + K^4 \lambda^{3-\nu} \\ \gtrsim \frac{b}{\lambda^4} \int |\tilde{u}|^2 dg_x + \int |\nabla \tilde{u}|^2 \frac{1}{\lambda} e^{-\frac{|x-\alpha|}{\sqrt{A}\lambda}} dg_x + K^4 \lambda^{3-\nu} \\ \gtrsim \frac{b}{\lambda^4} \int |\tilde{u}|^2 dg_x + \mathcal{O}\left(\|\tilde{u}\|_{H^1}^2 + K^4 \lambda^{3-\nu} \right) \end{aligned}$$

and since it is true for any $\nu > 0$, it remains true when $\nu = 0$, proving (6.116) holds. This concludes the proof of the Lemma.

6.6 Existence of critical mass blow up solutions

As a first consequence of previous lemma, we may integrate the flow backward from the singularity to prove there are critical mass blow up solutions. The following proposition ends the existence part.

Proposition 6.11. (Existence of critical mass blow up solutions) Let

$$\gamma_0 \in \mathbb{R}, \quad E_0 + C_E > 0,$$

and C_0 given by (6.104), then there exists $t_0 < 0$ and a solution $u_c \in C([t_0, 0), H^{3/2})$ to (6.1) which blows up at T = 0 with

$$E(u_c) = E_0$$
 and $||u_c||_{L^2(dg_x)} = ||Q||_{L^2}$

Moreover, the solution admits on $[t_0, 0)$ a geometrical decomposition :

$$u_c(t,x) = \left(\frac{\rho}{\varphi_c(y,\lambda_c,\alpha_c)}\right)^{1/2} \frac{1}{k(\alpha_c(t))^{1/2}} \frac{1}{\lambda_c(t)} \left(\tilde{Q}_{\mathcal{P}_c(t)} + \epsilon_c(t)\right) \left(t, \frac{x - \alpha_c(t)}{\lambda_c(t)}\right) e^{i\gamma_c(t)} = \tilde{Q}_c + \tilde{u}_c \qquad (6.136)$$

where ϵ_c satisfies the orthogonality conditions (6.42)-(6.46), and there holds the bounds

$$\begin{aligned} \|\tilde{u}_{c}\|_{L^{2}} \lesssim \lambda_{c}^{3}, \quad \|\tilde{u}_{c}\|_{H^{1}} \lesssim \lambda_{c}^{2}, \quad \|\tilde{u}_{c}\|_{H^{3/2}} \lesssim \lambda_{c}^{3/4}, \\ \lambda_{c} + \frac{t}{C_{0}} = \mathcal{O}(\lambda_{c}^{2}), \quad \frac{b_{c}}{\lambda_{c}} - \frac{1}{C_{0}} = \mathcal{O}(\lambda_{c}), \quad |\alpha_{c}| + |\beta_{c}| \lesssim \lambda_{c}^{3/2}, \quad \gamma_{c} = -\frac{C_{0}^{2}}{t} + \gamma_{0} + \mathcal{O}(\lambda_{c}) \end{aligned}$$

$$\tag{6.137}$$

Proof of Proposition 6.11

Step 1 Backwards uniform bounds.

Let a sequence $t_n \to 0$ and u_n be the solution to (6.1) with initial data at $t = t_n$ given by :

$$u_n(t_n, x) = \frac{1}{\lambda_n} \tilde{Q}_{\mathcal{P}_n}\left(\frac{x}{\lambda_n}\right) e^{i\gamma_n(t_n)}$$
(6.138)

with $\mathcal{P}_n = (b_n(t_n), \lambda_n(t_n), \beta_n(t_n), \alpha_n(t_n))$ and :

$$b_n(t_n) = -\frac{t_n}{C_0^2}, \quad \lambda_n(t_n) = -\frac{t_n}{C_0}, \quad \alpha_n(t_n) = \beta_n(t_n) = 0, \quad \gamma_n(t_n) = \gamma_0 - \frac{C_0^2}{t_n}.$$

Recall the conservation of mass identity we have computed in (6.32) which here becomes

$$||u_n(t_n)||^2_{L^2(dg_x)} = \int Q^2 \, dy + \mathcal{O}(t_n^4)$$

Furthermore, we have $\tilde{u}_n(t_n) = 0$ by construction. Hence u_n satisfies at $t_1 = t_n$ the assumptions of the backward propagation of smallness lemma, and thus we can find a time t_0 independent of n such that $\forall t \in [t_0, t_n), u_n$ admits a geometrical decomposition

$$u_n(t,x) = \left(\frac{\rho}{\varphi(y,\lambda_n(t),\alpha_n(t))}\right)^{1/2} \frac{1}{k(\alpha_n(t))^{1/2}} \frac{1}{\lambda_n(t)} \tilde{Q}_{\mathcal{P}_n(t)}\left(t,\frac{x-\alpha_n(t)}{\lambda_n(t)}\right) e^{i\gamma_n(t)} + \tilde{u}_n \tag{6.139}$$

with uniform bounds in n:

$$\|\nabla \tilde{u}_n(t)\|_{L^2}^2 + \frac{\|\tilde{u}_n(t)\|_{L^2}^2}{\lambda_n^2(t)} \lesssim \lambda_n^4(t)$$
(6.140)

$$\frac{b_n}{\lambda_n}(t) - \frac{1}{C_0} \bigg| \lesssim \lambda_n(t), \quad \left| \lambda_n(t) + \frac{t}{C_0} \right| \lesssim \lambda_n^2(t), \quad |\alpha_n(t)| + |\beta_n(t)| \lesssim \lambda_n^{3/2}(t)$$
(6.141)

From Strichartz estimate, this implies the uniform $H^{3/2}$ bound :

$$\|\tilde{u}_n(t)\|_{L^{\infty}\left([t,t_n],H^{3/2}\right)} \lesssim \lambda_n^{3/4}(t)$$
 (6.142)

that we will prove in step 2.

Now the H^1 compactness of $(u_n(t_0))$ is a consequence of a standard localization procedure. Indeed let a cut off function $\chi(x) = 0$ for $|x| \le 1$ and $\chi(x) = 1$ for $|x| \ge 2$, and $\chi_R(x) = \chi(\frac{x}{R})$, then since

$$\partial_t u_n = i \left[\Delta_g u_n + k(x) \, |u_n|^2 \, u_n + V(x) \, u_n \right]$$

we have

$$\left| \frac{d}{dt} \int \chi_{R} |u_{n}|^{2} dg_{x} \right| = 2 \left| \mathcal{I}m \int g^{ij} \partial_{i} u_{n} \partial_{j} \chi_{R} \overline{u_{n}} dg_{x} \right| \lesssim \frac{1}{R}$$

and

$$\left| \frac{d}{dt} \int \chi_R \left(\frac{1}{2} g^{ij} \partial_i u_n \partial_j \overline{u_n} - \frac{1}{4} k(x) |u_n|^4 - \frac{1}{2} V(x) |u_n|^2 \right) dg_x \right|$$
$$= \left| \mathcal{I}m \int g^{ij} \partial_i \chi_R \partial_j \overline{u}_n \left(\Delta_g u_n + k(x) |u_n|^2 u_n + V(x) u_n \right) dg_x \right| \lesssim \frac{1}{R}$$

where we have used (6.140), (6.141) and (6.142). Integrating this backwards from t_1 to t_0 we have :

$$\int \chi_R |u_n(t_0)|^2 dg_x \lesssim \frac{1}{R}, \qquad \int \chi_R |\nabla u_n(t_0)|^2 dg_x \lesssim \frac{1}{R}$$
(6.143)

Let's say a bit more on how we got the last estimation. We have used the Gagliardo-Nirenberg inequality as follows

$$\begin{split} \int \chi_{\scriptscriptstyle R} \, |\nabla u_n|^2 \, dg_x &\lesssim \left| \int \chi_{\scriptscriptstyle R} \left[\frac{1}{2} \, g^{ij} \, \partial_i u_n \, \partial_j \overline{u}_n - \frac{1}{4} \, k(x) \, |u_n|^4 - \frac{1}{2} \, V(x) \, |u_n|^2 \right] \, dg_x \\ &+ \int |\chi_{\scriptscriptstyle R}^{1/4} \, u_n|^4 \, dg_x + \int \chi_{\scriptscriptstyle R} \, |u_n|^2 \, dg_x \\ &\lesssim \frac{1}{R} + \int \chi_{\scriptscriptstyle R}^{1/2} \, |u_n|^2 \, dg_x \, \int \chi_{\scriptscriptstyle R}^{1/2} \, |\nabla u_n|^2 \, dg_x + \frac{1}{R} \\ &\lesssim \frac{1}{R} + \frac{1}{R} \, \int \chi_{\scriptscriptstyle R}^{1/2} \, |\nabla u_n|^2 \, dg_x \\ &\lesssim \frac{1}{R} + \frac{1}{R} \, \int \chi_{\scriptscriptstyle R} \, |\nabla u_n|^2 \, dg_x + \frac{1}{R} \, \int (\chi_{\scriptscriptstyle R}^{1/2} - \chi_{\scriptscriptstyle R}) \, |\nabla u_n|^2 \, dg_x \end{split}$$

then

$$\int \left(\chi_R^{1/2} - \chi_R\right) |\nabla u_n|^2 \, dg_x \lesssim \int_{R \le |x| \le 2R} |\nabla u_n|^2 \, dg_x \lesssim 1$$

This last result (6.143) gives us a control at infinity of functions $u_n(t_0)$ that will help us to get H^1 compactness of the $(u_n(t_0))$ sequence. Then using Lemma 8.10, see Appendix B, with $\sqrt{|g|}dx$ in the role of the volume form, we have for any $\delta > 0$

$$H^{1}_{\delta}(\mathbb{R},\sqrt{|g|}dx) = \left\{ u \in H^{1}(\mathbb{R}^{2}), \quad \left(\langle x \rangle^{\delta} + \langle D_{x} \rangle^{\delta}\right) \ \left(|u| + |\nabla u|\right) \in L^{2}(\sqrt{|g|}dx) \right\} \underset{\text{compact}}{\hookrightarrow} H^{1}(\mathbb{R}^{2},\sqrt{|g|}dx)$$

where D_x is the derivative with respect to x, defined by its Fourier transform.

Our next move is to claim (6.143) ensures $(u_n(t_0)) \subset H^1_{\delta}$. Once this is done, we know there is a subsequence of $u_n(t_0)$ which converges to some $u_c(t_0) \in H^1(\mathbb{R}^2)$.

First, from (6.138), (6.139) and (6.140) we have $u_n(t_0) \in H^{3/2}(\mathbb{R}^2)$. Now we prove $u_n(t_0) \in H^1_{\delta}$ whenever $0 \le \delta < \frac{1}{2}$.

Using (6.143) we have for every $j \in \mathbb{N}$

$$\int_{2^{j} \le |x| \le 2^{j+1}} \frac{\langle x \rangle^{2\delta}}{\langle 2^{j+1} \rangle^{2\delta}} |u_{n}(t_{0})|^{2} dg_{x} \lesssim \int \chi_{2^{j-1}} |u_{n}(t_{0})|^{2} dg_{x} \lesssim \frac{1}{2^{j-1}} |u_{n}(t_{0}$$

and since $\delta < \frac{1}{2}$, summing this over *j* finally yields

$$\int \langle x \rangle^{2\delta} |u_n(t_0)|^2 dg_x = \int_{0 \le |x| \le 1} \langle x \rangle^{2\delta} |u_n(t_0)|^2 dg_x + \sum_{j=0}^{+\infty} \int_{2^j \le |x| \le 2^{j+1}} \langle x \rangle^{2\delta} |u_n(t_0)|^2 dg_x$$
$$\lesssim \sum_{j=0}^{+\infty} \frac{1}{2^{(1-2\delta)j}} < +\infty$$

The same preocess applied to $\nabla u_n(t_0)$ instead of $u_n(t_0)$ yields $\int \langle x \rangle^{\delta} |\nabla u_n(t_0)|^2 < +\infty$ \Box

Eventually, we have proved

 $u_n(t_0) \longrightarrow u_c(t_0)$ in $H^1(\mathbb{R}^2)$, as $n \to +\infty$

Let then u_c be the solution to (6.1) with initial data $u_c(t_0)$. From the H^1 continuity of the flow of our Schrödinger operator, we have

$$\forall t \in [t_0, 0), \quad u_n(t) \longrightarrow u_c(t) \text{ in } H^1(\mathbb{R}^2), \quad \text{as } n \to +\infty$$

By the way u_c admits a geometrical decomposition like in (6.136) with

$$\forall t \in [t_0, 0), \quad \mathcal{P}_n(t) \longrightarrow \mathcal{P}_c(t), \quad \text{as } n \to +\infty$$

By passing to the limit in (6.140), (6.141) we obtain the H^1 bound along with the estimates on the parameters in (6.137). This implies in particular that u_c blows up at t = 0. Similarly the conservation of the L^2 norm ensures

$$||u_c||_{L^2(dg_x)} = \lim_{n \to +\infty} ||u_n(t_n)||_{L^2(dg_x)} = ||Q||_{L^2}$$

Recall that from the very method we used to approximate our solution through the geometric decomposition

$$\tilde{Q}_{\mathcal{P}} = \tilde{P}_{\mathcal{P}} e^{-i b \frac{|y|^2}{4} + i \beta \cdot y}, \quad \tilde{P}_{\mathcal{P}} = Q + \mathcal{O}(\lambda^2 e^{-C_2 |y|})$$

which along with (6.136), (6.137), (6.140) and the calculation of the Energy that we have done in the first step of Proposition 6.5 yields

$$\begin{split} E(u_c(t)) &= \frac{1}{2} \int g^{ij} \,\partial_i u_c \,\partial_j \overline{u}_c \,dg_x - \frac{1}{4} \int k(x) \,|u_c|^4 \,dg_x - \frac{1}{2} \int V(x) \,|u_c|^2 \,dg_x \\ &= \frac{1}{k(\alpha_c) \,\lambda_c^2} \left[\frac{1}{2} \int g^{ij} \,\partial_i \tilde{Q}_{\mathcal{P}_c} \,\overline{\partial_j \tilde{Q}_{\mathcal{P}_c}} \,\frac{\rho}{\varphi} \sqrt{|g|} \,dy - \frac{1}{4} \int \frac{\rho}{\varphi} \,\frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \,|\tilde{Q}_{\mathcal{P}_c}|^4 \,\frac{\rho}{\varphi} \sqrt{|g|} \,dy \right] \\ &- \frac{\lambda_c^2}{2} \int V(\lambda_c y + \alpha_c) \,|\tilde{Q}_{\mathcal{P}_c}|^2 \,\frac{\rho}{\varphi} \sqrt{|g|} \,dy \right] + \mathcal{O}(\lambda_c) \\ &= \frac{1}{k(\alpha_c) \,\lambda_c^2} \left[\frac{b_c^2}{8} \int |y|^2 Q^2 \,dy - \lambda_c^2 \,C_E \right] + \mathcal{O}(\lambda_c) \\ &= \frac{1}{8} \left(\left(\frac{b_c}{\lambda_c} \right)^2 - \frac{1}{C_0^2} \right) \|y \,Q\|_{L^2}^2 + E_0 + \mathcal{O}(\lambda_c) \xrightarrow{t \to 0} E_0 \end{split}$$

Now from the conservation of energy, we may conclude $E(u_c) = E_0$.

There is only left to prove the estimate about the phase parameter. This may be done using equation (6.15) along with (6.140) and (6.141) that lead to the rough bound

$$\left| (\tilde{\gamma}_n)_s \right| \lesssim \lambda_n^2$$

so that

$$\left|\frac{d}{dt}\left(\gamma_n + \frac{C_0^2}{t}\right)\right| = \left|\frac{(\gamma_n)_s}{\lambda_n^2} - \frac{C_0^2}{t^2}\right| = \left|\frac{(\tilde{\gamma_n})_s}{\lambda_n^2} + \left(\frac{1}{\lambda_n^2} - \frac{C_0^2}{t^2}\right)\right| \lesssim 1$$

from which we may conclude after integrating

$$\gamma_n(t) + \frac{C_0^2}{t} = \gamma_0 + \mathcal{O}(t)$$

and taking the limit $n \to +\infty$

$$\gamma_c(t) + \frac{C_0^2}{t} = \gamma_0 + \mathcal{O}(t)$$

Step 2 $H^{3/2}$ bound

It remains to prove the bound (6.142). In particular, when done this implies the $H^{3/2}$ bound in (6.137) by taking the weak limit of $H^{3/2}$.

 \tilde{u}_n satisfies

$$i \partial_t \tilde{u}_n + (\Delta_g + V)\tilde{u}_n = -\psi_n - k(x) \,\tilde{u}_n \,|\tilde{u}_n|^2 - V(x) \,\tilde{u}_n - F_n$$

with

$$i \,\partial_t \tilde{Q}_n + \Delta_g \tilde{Q}_n + V(x) \,\tilde{Q}_n + k(x) \,\tilde{Q}_n \,|\tilde{Q}_n|^2 = \psi_n$$

$$F_n = k(x) \,(\tilde{Q}_n + \tilde{u}_n) \,|\tilde{Q}_n + \tilde{u}_n|^2 - k(x) \,\tilde{Q}_n \,|\tilde{Q}_n|^2 - k(x) \tilde{u}_n \,|\tilde{u}_n|^2$$
(6.144)

Given $\tilde{u}_n(t_n) = 0$, and $L_g = \Delta_g + V$ using the Duhamel formula

$$\tilde{u}_n(t) = i \int_{t_n}^t e^{i(t-s)L_g} \left(\psi_n(s) + k(x) \,\tilde{u}_n(s) \,|\tilde{u}_n(s)|^2 + V(x) \,\tilde{u}_n(s) + F_n(s) \right) ds$$

then the Strichartz estimates coupled with the smoothing effect of the linear Schrödinger flow leaves us with three terms to control

$$\|\nabla^{3/2}\tilde{u}_n\|_{L^{\infty}_{[t,t_n]}L^2} \lesssim \|\nabla^{3/2}\psi_n\|_{L^{4/3}_{[t,t_n]}L^{4/3}} + \|(1+|x|^2)F_n\|_{L^2_{[t,t_n]}H^1} + \|\nabla^{3/2}(\tilde{u}_n|\tilde{u}_n|^2)\|_{L^{4/3}_{[t,t_n]}L^{4/3}}$$
(6.145)

See Appendix B 8.3.2 for more details about that last statement.

First, we deal with the error ψ_n which is to be estimated thanks to (6.131)

$$\|\nabla^{3/2}\psi_n\|_{L^{4/3}} \lesssim \frac{1}{\lambda_n^3} \left(|Mod_n(t)| + |\alpha_n|^2 + \lambda_n^5\right) \lesssim 1$$

where of course $Mod_n(t)$ corresponds to the modulation equation of $\mathcal{P}_n(t)$ which is estimated as in (6.62), and where we use (6.140), (6.141) to get $|Mod_n(t)| \leq \lambda_n^4$ and $|\alpha_n|^2 \leq \lambda_n^3$, hence

$$\|\nabla^{3/2}\psi_n\|_{L^{4/3}_{[t,t_n]}L^{4/3}} \lesssim \lambda_n^{3/4}.$$
(6.146)

The F_n term is local in y with linear and quadratic terms in \tilde{u}_n . Expanding (6.144) we see

$$F_n = \mathcal{O}\left[\left(\frac{|\tilde{u}_n|^2}{\lambda_n} + \frac{|\tilde{u}_n|}{\lambda_n^2}\right)e^{-c\frac{|x-\alpha_n|}{\lambda_n}}\right], \quad \nabla F_n = \mathcal{O}\left[\left(\frac{|\tilde{u}_n|^2}{\lambda_n^2} + \frac{|\tilde{u}_n\nabla\tilde{u}_n|}{\lambda_n} + \frac{|\tilde{u}_n|}{\lambda_n^3} + \frac{|\nabla\tilde{u}_n|}{\lambda_n^2}\right)e^{-c\frac{|x-\alpha_n|}{\lambda_n}}\right]$$

so that

$$\|(1+|x|^2)F_n\|_{L^2} + \|(1+|x|^2)\nabla F_n\|_{L^2} \lesssim \frac{1}{\lambda_n^3} \|\tilde{u}_n\|_{L^2} + \frac{1}{\lambda_n^2} \|\tilde{u}_n\|_{L^4}^2 + \frac{1}{\lambda_n^2} \|\nabla \tilde{u}_n\|_{L^2} + \frac{1}{\lambda_n} \|\tilde{u}_n\|_{L^4} \|\nabla \tilde{u}_n\|_{L^4} \|\nabla \tilde{u}_n\|$$

Again by Sobolev embedding result Theorem along with a Gagliardo-Nirenberg estimate

Sobolev
$$H^{1/2}(\mathbb{R}^2) \subseteq L^4(\mathbb{R}^2)$$

GN $\|\tilde{u}_n\|_{L^4} \lesssim \|\tilde{u}_n\|_{L^2}^{1/2} \|\nabla \tilde{u}_n\|_{L^2}^{1/2}$

and (6.140) yields

 $\|(1+|x|^2)F_n\|_{H^1} \lesssim 1+\lambda_n^{3/2} \, \|\nabla^{3/2} \tilde{u}_n\|_{L^2}$

Thus

$$\|(1+|x|^2)F_n\|_{L^2_{[t,t_n]}H^1} \lesssim \lambda_n + \lambda_n^2 \|\nabla^{3/2}\tilde{u}_n\|_{L^\infty_{[t,t_n]}L^2}$$
(6.147)

There only remains to deal with the nonlinear term. A little caution here, remind the symbol $\nabla^{3/2}$ we are working with is a generalized non-integer derivative that stands for

$$\nabla^{3/2} u(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \left(i \, \xi \right)^{3/2} \widehat{u}(\xi) \, d\xi \quad \text{where} \quad \mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} \, u(x) \, dx$$

is the Fourier transform, and $\quad \overline{\mathcal{F}}u(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \, \widehat{u}(\xi) \, d\xi \quad \text{is its inverse}$ (6.148)

Then, standard non linear estimates for paraproducts, see Appendix B 8.2.2, show

$$\|\nabla^{3/2} (\tilde{u}_n \,| \tilde{u}_n |^2)\|_{L^{4/3}} \lesssim \|\nabla^{3/2} \tilde{u}_n\|_{L^2} \,\| \tilde{u}_n \|_{H^1}^2 \lesssim \|\nabla^{3/2} \tilde{u}_n\|_{L^2} \,\lambda_n^{19/4} \tag{6.149}$$

where we have used (6.140). Eventually (6.142) follows from (6.146), (6.147) and (6.149). This ends the proof of our Proposition, existence of solutions for equation (6.1) is now done provided we justify the boot strap is actually valid, that all terms involved are in control as we assumed they should be when moving toward blow up time. It is what we do in the next section.

6.7 Rigidity of the flow around $Q_{\mathcal{P}}$

We cannot delay any longer a more precise study of the geometrical decomposition we have been working with so far. We intend to prove that close enough to the (t, x) = (0, 0) point, a solution to (6.1) that would be decomposed as in (6.38) must satisfy to some precise and restrictive properties, such as dispersive results and the estimate (6.8) we have claimed we could prove since the beginning, which basically was

 $|\alpha| + |\beta| \lesssim \lambda^{3/2}, \quad b \sim \lambda, \quad 0 < \mu \lesssim \lambda^3$

This will mean two things for the study :

the modulation of the geometrical parameters built on the decomposition (6.38) is actually stable enough to ensure the convergence process we have established for the existence proof in Theorem 6.11.

for any two solutions of (6.1) the rigidity of the flow will imply dispersion estimates which oblige them to be so close they will in fact be the same

For the record, recall estimate (6.8) particularly simplifies our parameters laws since they should satisfy (6.22). We now want to prove these assumptions are a result of rigidity in the regime that governs the parameters (\mathcal{P}, ϵ) and that it opens the door to discussions for unicity of the critical mass blow up solution. We then lead a variational study of the decomposition (6.38) in the same fashion than what it done in [RS11].

We let k satisfy assumption (H1). In this section we start proceeding in a perturbation kind of way. In the next section we consider a solution for equation (6.1) which would be given by profile Q and a remaining term ϵ . We then prove the Lemma 6.12 that proves such a solution tends, as $t \to 0$ - that is as we move forward to blow up time - to satisfy all assumptions we made to build approximate solution $\tilde{Q}_{\mathcal{P}}$ in previous sections. Therefore, for some time close enough to blow up time, our solution will enter in a regime where its approximation may be sharpened by considering $\tilde{Q}_{\mathcal{P}}$ the approximate profile instead of Ground State Q. So this will prove, the geometric decomposition (6.158) still provides a good description, with all due estimations on parameters and remaining term ϵ , even when moving towards blow up time.

Then, a localized virial type of estimation, along with a refined dispersion result on the tail of the (6.158) decomposition - the one induced by remaining term ϵ - will get us to a final argument in which we manage to ensure the control of the geometric parameters for (6.158), and so closing the loop on our bootstrap argument.

6.7.1 Variational estimates and convergence of the concentration point.

The existence of a geometrical decomposition for a mass critical blowing up solution, and its consequences, among which is the convergence of the concentration point, are a well known result based on a variational analysis. We begin with adapting this one to our metric situation.

Lemma 6.12. (Variational control of minimal mass blow up solutions).

Let u(t) be a critical mass solution to (6.1) which blows up at T = 0. Then for t < 0 close enough to 0, u(t) admits a geometrical decomposition

$$u(t,x) = \frac{1}{\lambda(t)} \left(Q + \epsilon\right) \left(t, \frac{x - \alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)},\tag{6.150}$$

for some C^1 parameters $(\lambda(t), \alpha(t), \gamma(t)) \in \mathbb{R}^*_+ \times \mathbb{R}^2 \times \mathbb{R}$ with

1. Uniform bound on the decomposition :

$$|g^{ij}(\alpha) - I_{ij}|^{1/2} + |1 - k(\alpha)|^{1/2} + ||\epsilon(t)||_{H^1(\mathbb{R}^2)} \lesssim \lambda(t) \longrightarrow 0, \quad as \quad t \to 0$$
(6.151)

2. Convergence of the concentration point :

$$\alpha(t) \longrightarrow \alpha^* \quad with \quad k(\alpha^*) = 1, \quad g^{ij}(\alpha^*) = I_{ij}, \quad i, j = 1, 2$$
(6.152)

3. Lower bound on the blow up rate :

$$\lambda(t) \le C(u_0) |t| \tag{6.153}$$

Proof of Lemma 6.12

Step 1

Let

$$v_0(t,x) = \lambda_0(t) u(t, \lambda_0(t) x) \quad \text{with} \quad \lambda_0(t) = \frac{\|\nabla Q\|_{L^2}}{\left(\int g^{ij} \partial_i u(t) \partial_j \overline{u(t)} \, dg_x\right)^{1/2}}$$

then

$$\int |v_0(t)|^2 \sqrt{|g|} (\lambda_0 x) \, dx = \int |u(t)|^2 \, dh_z = ||Q||_{L^2}^2, \quad \text{and}$$
$$\int g^{ij} (\lambda_0 x) \, \partial_i v_0(t, x) \, \partial_j \overline{v_0}(t, x) \, \sqrt{|g|} (\lambda_0 x) \, dx = \lambda_0^2 \, \int g^{ij}(z) \, \partial_i u(t, z) \, \partial_j \overline{u}(t, z) \, dh_z = \int |\nabla Q|^2 \, dz$$

so that using assumption (H1), $k \leq 1$, and (6.4) one gets

$$\frac{1}{2} \int g^{ij}(\lambda_0 x) \,\partial_i v_0(t,x) \,\partial_j \overline{v_0}(t,x) \,dg_{\lambda_0 x} - \frac{1}{4} \int k(\lambda_0 x) \,|v_0(t,x)|^4 \,dg_{\lambda_0 x} - \frac{1}{2} \int V(\lambda_0 x) \,|v_0(t,x)|^2 \,dg_{\lambda_0 x} \\
\geq \frac{1}{2} \int |\nabla v_0|^2 \,dx - \frac{1}{4} \int |v_0|^4 \,dx - \frac{V(0)}{2} \int |u|^2 \,dg_x + \mathcal{O}(\lambda_0)$$

and the conservation of energy

$$\begin{split} &\frac{1}{2} \int g^{ij}(\lambda_0 x) \,\partial_i v_0(t,x) \,\partial_j \overline{v_0}(t,x) \,dg_{\lambda_0 x} - \frac{1}{4} \,\int k(\lambda_0 x) \,|v_0(t,x)|^4 \,dg_{\lambda_0 x} - \frac{1}{2} \,\int V(\lambda_0 x) \,|v_0(t,x)|^2 \,dg_{\lambda_0 x} \\ &= \frac{\lambda_0^2}{2} \,\int g^{ij}(z) \,\partial_i u(t,z) \,\partial_j \overline{u}(t,z) \,dg_z - \frac{\lambda_0^2}{4} \,\int k(z) \,|u(t,x)|^4 \,dg_z - \frac{1}{2} \,\int V(z) \,|u(t,z)|^2 \,dg_z \\ &= \lambda_0^2 \,E_0 - \frac{1}{2} \,(1-\lambda_0^2) \,\int V(z) \,|u|^2 \,dg_z \end{split}$$

Eventually using $(H2), V \ge V(0)$, one gets

$$\frac{1}{2} \int |\nabla v_0(t,x)|^2 \, dx - \frac{1}{4} \int |v_0(t,y)|^4 \, dx \le \frac{1}{2} \int \left(V(0) - V(z) \right) |u|^2 \, dx + \mathcal{O}(\lambda_0) \le 0, \quad \text{as} \quad t \to 0$$

From a standard concentration compactness argument along with the variational characterisation of the Ground State Q, this implies we can find $(x_0(t), \gamma_0(t)) \in \mathbb{R}^2 \times \mathbb{R}$ such that

$$v_0(t, .+x_0(t)) e^{i\gamma_0(t)} \longrightarrow Q$$
 in H^1 as $t \to 0$

In another words, u(t) admits near blow up time a decomposition

$$u(t,x) = \frac{1}{\lambda(t)} \left(Q + \epsilon \right) \left(t, \frac{x - \alpha(t)}{\lambda(t)} \right) e^{i \gamma(t)},$$

with $\lambda(t) = \lambda_0(t) \longrightarrow 0$, as $t \to 0$ from blow-up assumption, and since the decomposition is set so that

$$\left\| u(t,x) - \frac{1}{\lambda(t)} Q\left(\frac{x - \alpha(t)}{\lambda(t)}\right) e^{i \,\gamma(t)} \right\|_{H^1(dg_x)} \longrightarrow 0, \quad \text{as } t \to 0$$

one easily deduce from approximate values of mass and energy the smallness conditions

$$\|\epsilon\|_{H^1} + \left|\sqrt{|g|}(\alpha) - 1\right| + \left|\frac{1}{2}Tr(g^{ij}(\alpha)) - k(\alpha)\right| \longrightarrow 0, \quad \text{as } t \to 0$$

$$(6.154)$$

Using the implicit function theorem, the uniqueness of the decomposition (6.150) can be ensured through a suitable choice of orthogonality conditions. We then set the orthogonality conditions on ϵ to be

$$(\epsilon_1, |y|^2 Q) = 0, \quad (\epsilon_1, y Q) = 0, \quad (\epsilon_2, \varrho) = 0$$

where ϵ_1 and ϵ_2 respectively stand for the real and imaginary parts of ϵ .

Let $v = Q + \epsilon$, it almost satisfies (6.15)

$$i\partial_s v + \Delta_{g(\lambda y + \alpha)} v - v + k(\lambda y + \alpha) v |v|^2 + \lambda^2 V(\lambda y + \alpha) v = i\frac{\lambda_s}{\lambda} \Lambda v + i\frac{\alpha_s}{\lambda} \cdot \nabla v + \widetilde{\gamma}_s v$$

where $\tilde{\gamma}_s = \gamma_s - 1$. Note that we are currently considering Q an approximate solution of (6.16) or (6.19). It is a very rough approximation, but since we do not know yet that $\alpha(t)$ converges to α^* , there is no point using our approximation $\tilde{Q}_{\mathcal{P}}$ for it takes sense only for $\alpha(t)$ close enough to α^* .

Now the implicit function theorem leads to the C^1 regularity of our parameters λ and α . Then expanding k and h around α , and using the Ground State equation

$$i\left(\frac{\lambda_s}{\lambda}\Lambda Q + \frac{\alpha_s}{\lambda} \cdot \nabla Q\right) + \widetilde{\gamma}_s Q = \left(g^{ij}(\alpha) - I_{ij}\right)\partial_{ij}^2 Q + \left(k(\alpha) - 1\right)Q^3 + g^{ij}(\alpha)\partial_{ij}^2 \epsilon + \mathcal{O}\left(\lambda + |\epsilon| + |\nabla\epsilon|\right)$$

Taking the scalar product of that last equation, first with ΛQ , then with ∇Q yields

$$-i\frac{\lambda_s}{\lambda}\frac{\|\Lambda Q\|_{L^2}^2}{\|Q\|_{L^2}^2} = \left(\frac{1}{2}Tr(g^{ij}(\alpha)) - k(\alpha)\right) + \mathcal{O}(\lambda + \|\epsilon\|_{H^1}), \quad i\frac{\alpha_s}{\lambda} = \mathcal{O}(\lambda + \|\epsilon\|_{H^1})$$

so that

$$\left|\frac{\lambda_s}{\lambda}\right| + \left|\frac{\alpha_s}{\lambda}\right| \lesssim \left|\frac{1}{2}Tr(g^{ij}(\alpha)) - k(\alpha)\right| + \lambda(t) + \|\epsilon\|_{H^1}$$
(6.155)

Step 2 Expansion of the conservation laws

To get a more precise estimation of our approximation remainings, we expand the conservation laws in the ϵ variables. From the critical mass assumption, we know we have

$$\int Q^2 \, dy = \int |u(t,x)|^2 \, dg_x = \int |v(s,y)|^2 \, dg_y = \sqrt{|g|}(\alpha) \int \left(Q^2 + 2\,\epsilon_1 \, Q + |\epsilon|^2\right) \, dy + \mathcal{O}\left(\lambda^2 + \lambda \, \|\epsilon\|_{L^2}\right)$$

and thus

$$\sqrt{|g|}(\alpha) \left\{ \int \epsilon_1 Q \, dy + \frac{1}{2} \int |\epsilon|^2 \, dy \right\} = \frac{1}{2} \left(1 - \sqrt{|g|}(\alpha) \right) \int Q^2 \, dy + \mathcal{O}\left(\lambda^2 + \lambda \, \|\epsilon\|_{L^2}\right) \tag{6.156}$$

Using the boundedness of the derivatives of $f = \sqrt{|g|}, g^{ii}, k$ up to order 2 we have

$$f(\lambda y + \alpha) = 1 + (f(\alpha) - 1) + \lambda \nabla f(\alpha) \cdot y + \mathcal{O}(\lambda^2)$$

As already said, and very similarly to the calculation which led to (6.66), the conservation of energy simply becomes by rescalling

$$\begin{split} \lambda^{2} E_{0} &= \frac{1}{2} \int g^{ij} (\lambda \, y + \alpha) \, \partial_{i} v \, \partial_{j} \overline{v} \, dg_{y} - \frac{1}{4} \int k(\lambda \, y + \alpha) \, |v|^{4} \, dg_{y} + \mathcal{O}(\lambda^{2}) \\ &= \sqrt{|g|}(\alpha) \left\{ E^{0}(Q) + \frac{1}{2} \left(g^{ij}(\alpha) - I_{ij} \right) \int \partial_{i} Q \partial_{j} Q \, dy + \frac{1}{4} \left(1 - k(\alpha) \right) \int Q^{4} \, dy + \frac{1}{2} \int |\nabla \epsilon|^{2} \, dy \\ &+ \int \nabla \epsilon_{1} \, . \, \nabla Q \, dy - \int \epsilon_{1} \, Q^{3} \, dy - \int Q^{2} \left(\epsilon_{1}^{2} + \frac{1}{2} \, |\epsilon|^{2} \right) \, dy \\ &+ \mathcal{O}\Big(\lambda^{2} + \|\epsilon\|_{H^{1}}^{3} + \left(\lambda + |g^{ij}(\alpha) - I_{ij}| + |1 - k(\alpha)| \right) \|\epsilon\|_{H^{1}} \Big) \Big\} \\ &= \sqrt{|g|}(\alpha) \left\{ \left(\frac{1}{2} \, Tr \left(g^{ij}(\alpha) \right) - 1 \right) \int \frac{Q^{2}}{2} \, dy + \frac{1}{2} \left(1 - k(\alpha) \right) \int Q^{2} \, dy + \frac{1}{2} \int |\nabla \epsilon|^{2} \, dy - \int \epsilon_{1} \, Q \, dy \\ &- \frac{1}{2} \int Q^{2} \left(3 \, \epsilon_{1}^{2} + \epsilon_{2}^{2} \right) \, dy + \mathcal{O}\Big(\lambda^{2} + \|\epsilon\|_{H^{1}}^{3} + \left(\lambda + |g^{ij}(\alpha) - I_{ij}| + |1 - k(\alpha)| \right) \|\epsilon\|_{H^{1}} \Big) \Big\} \end{split}$$

which together with (6.156) leads to

$$\begin{split} \lambda^{2} E_{0} &= \sqrt{|g|}(\alpha) \left\{ \left[\left(\frac{1}{2} Tr(g^{ij}(\alpha)) - k(\alpha) \right) + \left(\sqrt{|g|}(\alpha) - 1 \right) \right] \int \frac{Q^{2}}{2} \, dy + \frac{1}{2} \left[\left(L_{+}\epsilon_{1}, \epsilon_{1} \right) + \left(L_{-}\epsilon_{2}, \epsilon_{2} \right) \right] \right. \\ &+ \mathcal{O} \left(\lambda^{2} + \|\epsilon\|_{H^{1}}^{3} + \left(\lambda + |g^{ij}(\alpha) - I_{ij}| + |\sqrt{|g|}(\alpha) - 1| + |1 - k(\alpha)| \right) \|\epsilon\|_{H^{1}} \right) \bigg\} \end{split}$$

Now using (6.154) and (6.71) of Lemma 6.6, one easily gets the following estimate

$$\left|\frac{1}{2}Tr(g^{ij}(\alpha)) - k(\alpha)\right| + \left|\sqrt{|g|}(\alpha) - 1\right| + \|\epsilon\|_{H^1}^2 \lesssim \lambda^2$$

Step 3 Convergence of the concentration point and upper bound on the blow up rate.

There is only (6.152) left to prove. It essentially follows from (6.155) since

$$|\alpha_t| = \frac{1}{\lambda} \left| \frac{\alpha_s}{\lambda} \right| \lesssim \frac{\|\epsilon\|_{H^1} + \left| \frac{1}{2} Tr(g^{ij}(\alpha)) - k(\alpha) \right| + \lambda}{\lambda} \lesssim 1, \text{ so that } \int_{-1}^0 |\alpha_t| < +\infty$$

This implies $\alpha(t) \to \alpha^*$ as $t \to 0$ and $k(\alpha^*) = 1$, $h^{ij}(\alpha^*) = I_{ij}$ from (6.151). Similarly since the blow up assumption at t = 0 is $\lambda(0) = 0$

$$|\lambda_t| = \frac{1}{\lambda} \left| \frac{\lambda_s}{\lambda} \right| \lesssim \frac{\|\epsilon\|_{H^1} + \left| \frac{1}{2} Tr(g^{ij}(\alpha)) - k(\alpha) \right| + \lambda}{\lambda} \lesssim 1, \text{ so that } \lambda(t) \lesssim \int_t^0 |\lambda_t| \lesssim |t|$$

This ends the proof of our lemma.

6.7.2 Strict lower bound on the energy.

From previous section we have underlined fact that center of mass $\alpha(t)$ must stabilize as $t \to 0$ around some α^* where both k and g reach their maxima. Without loss of gnerality we may naturally assume

$$\alpha^* = 0, \quad k(0) = 1, \quad \nabla k(0) = 0, \quad \nabla^2 k(0) < 0, \quad g(0) = I_{ij}, \quad \nabla g(0) = 0$$
 (6.157)

Once more, notice the last assumption $\nabla g(0) = 0$ is not absolutely necessary. As explained in the beginning it may be obtained when working with a well suited set of geodesic normal coordinates.

Then, for $T_0 < 0$ close enough to blow up time, $|\mathcal{P}(t)|$, $t \in [T_0, 0)$, is small enough for u to be in the regime described in section 6.2. This induces we can sharpen our previous approximation profile by replacing

Q by $\tilde{Q}_{\mathcal{P}}$ which is a small deformation of Q. Now we use the decomposition we have long been working with; for $t \in [T_0, 0)$ let

$$u(t,x) = \left(\frac{\rho}{\varphi(y,\lambda(t),\alpha(t))}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \left(\tilde{Q}_{\mathcal{P}(t)} + \epsilon\right) \left(t, \frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
(6.158)

where ϵ satisfies orthogonaliy conditions (6.42)-(6.46). We also introduced the global rescaled time

$$s(t) = \int_{T_0}^t \frac{d\tau}{\lambda^2(\tau)} \to +\infty, \quad \text{as} \quad t \to 0$$

Thus, for every $s \in [s_0, +\infty)$ applying Proposition 6.5 we get

$$b^{2} + |\beta|^{2} + |\alpha|^{2} + \|\epsilon\|_{H^{1}}^{2} \lesssim \lambda^{2} \left(E_{0} + C_{E} + \phi_{E}(\alpha) - \lambda C_{1} \right) + \mathcal{O}\left(\mathcal{P}^{4} + \mathcal{P} |\alpha|^{2} + |\alpha|^{3}\right)$$
(6.159)

where recall C_E is given by (6.58), and

$$\left| Mod(t) \right| \lesssim \mathcal{P}^4 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2 \right) + \mathcal{P}^2 \, \|\epsilon\|_{H^1} + \|\epsilon\|_{L^2}^2 \tag{6.160}$$

We claim these estimations bring the following strict lower bound on the energy

Lemma 6.13. There holds the strict lower bound on the energy :

$$E_0 > -C_E \tag{6.161}$$

Proof of Lemma 6.13

By contradiction, if $E_0 + C_E < 0$, then let $E_1 = E_0 + C_E + \phi_E(\alpha) - \lambda C_1$, for t close enough to 0, one has $E_1 < 0$, so that from (6.159) we have

 $b^2 + |\beta|^2 + |\alpha|^2 + \|\epsilon\|_{H^1}^2 \lesssim \lambda^2 E_1 + \lambda^4$, hence $b^2 + |\beta|^2 + |\alpha|^2 + \|\epsilon\|_{H^1}^2 \lesssim \lambda^4$

Then, thanks to (6.160) this yields

$$\frac{\lambda_s}{\lambda} = \mathcal{O}(\lambda^2)$$

so that

$$\frac{\lambda_t}{\lambda} = \mathcal{O}(1), \text{ and hence } \ln(\lambda(t)) = \mathcal{O}(1), \text{ as } t \to 0$$

which contradicts the fact $\lambda(t) \to 0$ as $t \to 0$, that is u blows up at t = 0. It ends proof of Lemma 6.13.

6.7.3 The localized virial identity

In order to improve our estimate (6.58) as announced in (6.8) we need to get a more precise estimate on ϵ locally on the singularity that will help us treat the remaining terms of Lemma 6.9.

Keep in mind, since no a priori upper bound holds on blow up speed, we may not rule out the regime for which $b \ll \lambda$ that would make the $\frac{b}{\lambda}$ terms a negligible part. This makes the (6.99) estimate useless for the gain might as well degenerate.

It is a Morawetz type computation in the spirit of the local virial estimate that will here help us.

Lemma 6.14. (Local Virial Control)

Let ϕ be given by (6.96). There exists universal constants $c, c \geq 0$, and a large enough constant A > 0 such that for t close enough to 0, we have :

$$\left\{ -\left(\frac{b}{\lambda}\right) \frac{\|y\,Q\|_{L^{2}}^{2}}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A\,g^{ij}(\lambda y + \alpha)\,\partial_{i}\phi\left(\frac{y}{A}\right)\partial_{j}\epsilon\,\bar{\epsilon}\,dg_{y} - \tilde{c}_{0}\,C_{0}\,\lambda\right\}_{s} \\
\geq \frac{c}{\lambda}\left\{ |\alpha|^{2} + \int |\nabla\epsilon|^{2}\,e^{-\frac{|y|}{\sqrt{A}}}\,dy + \int |\epsilon|^{2}\,dy \right\} + \mathcal{O}\left(\|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3} + |\beta|^{2}\right) \tag{6.162}$$

with

$$\tilde{c}_0 = \frac{c_0}{8} \int [9 + \rho^2 Q^2] \rho Q^2 \, dy$$

Note from (6.58), the terms involved in the left hand side are uniformly bounded in time

$$\left| - \left(\frac{b}{\lambda}\right) \frac{\|y Q\|_{L^2}^2}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A g^{ij} (\lambda y + \alpha) \partial_i \phi\left(\frac{y}{A}\right) \partial_j \epsilon \,\overline{\epsilon} \, dg_y - \tilde{c}_0 \, C_0 \, \lambda \right| \lesssim \frac{|b|}{\lambda} + \frac{\|\epsilon\|_{H^1}^2}{\lambda} + \lambda \lesssim 1$$

From (6.58) again, and the finite time blow up assumption

$$\int_{s_0}^{+\infty} \left(\|\epsilon\|_{H^1}^2 + \mathcal{P}^3 + |\beta|^2 \right) ds \lesssim \int_{s_0}^{+\infty} \lambda^2(s) \, ds = \int_{t_0}^{0} dt \lesssim 1$$

Hence, integrating (6.162) between s_0 and $+\infty$

$$\int_{s_0}^{+\infty} \frac{1}{\lambda} \left\{ |\alpha|^2 + \int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} \, dy + \int |\epsilon|^2 \, dy \right\} ds \lesssim 1 \tag{6.163}$$

while on the other hand, thanks to (6.153)

$$\int_{s_0}^{+\infty} \lambda(s) \, ds = \int_{t_0}^{0} \frac{d\tau}{\lambda(\tau)} = +\infty$$

Proof of Lemma 6.14

The proof relies on an algebraic computation and the specific structure of the quadratic terms in ϵ appearing in (6.162).

Step 1 Computation of $\left(\frac{b}{\lambda}\right)_s$

We have from (6.22)

$$\left(\frac{b}{\lambda}\right)_s = \frac{b_s + b^2 - B_1 - K_b}{\lambda} - \frac{b}{\lambda} \left(\frac{\lambda_s}{\lambda} + b\right) + \frac{B_1 + K_b}{\lambda}$$
$$= \frac{b_s + b^2 - B_1 - K_b}{\lambda} + \lambda^2 K_0 + \frac{K_1}{2\lambda} \nabla^2 k(0).(\alpha, \alpha) + \mathcal{O}(\mathcal{P}^3)$$

where we have made use of both estimates (6.58), (6.62).

We recall that (6.86) along with (6.65) now yields

$$\frac{b_s + b^2 - B_1 - K_b}{\lambda} \int \frac{\rho^2}{4} Q^2 dy + \frac{1}{\lambda} \left[\int |\epsilon|^2 dy - \left(R_1(\epsilon) \,, \Lambda \Sigma \right) - \left(R_2(\epsilon) \,, \Lambda \Theta \right) \right] = \mathcal{O} \left(\|\epsilon\|_{H^1}^2 + \mathcal{P} \,\|\epsilon\|_{L^2} + \mathcal{P}^3 \right)$$

so that from (6.22)

$$\begin{split} &\left(\frac{b}{\lambda}\right)_{s}\left\|\frac{|\underline{y}|}{2}Q\right\|_{L^{2}}^{2} \\ &= -\frac{1}{\lambda}\bigg[\int |\epsilon|^{2} dy - \lambda^{3} \tilde{K}_{0} - \frac{1}{2} \nabla^{2} k(0).(\alpha, \alpha) \int Q^{2} dy - \left(R_{1}(\epsilon), \Lambda \Sigma\right) - \left(R_{2}(\epsilon), \Lambda \Theta\right)\bigg] \\ &+ \mathcal{O}\bigg(\mathcal{P}^{3} + |\alpha|^{2} + |\beta|^{2} + \mathcal{P}^{2} \left(b - \frac{\lambda}{C_{0}}\right) + \mathcal{P} \|\epsilon\|_{L^{2}} + \|\epsilon\|_{H^{1}}^{2}\bigg) \\ &= -\frac{1}{\lambda}\bigg[\int |\epsilon|^{2} dy - \lambda^{3} \tilde{c}_{0} - \frac{1}{2} \nabla^{2} k(0).(\alpha, \alpha) \int Q^{2} dy \\ &- \left(3\Sigma \epsilon_{1}^{2} + 2\Theta \epsilon_{1} \epsilon_{2} + \Sigma \epsilon_{2}^{2}, \Lambda \Sigma\right) - \left(3\theta \epsilon_{2}^{2} + 2\Sigma \epsilon_{1} \epsilon_{2} + \Theta \epsilon_{1}^{2}, \Lambda \Theta\bigg)\bigg] \\ &+ \mathcal{O}\bigg(\mathcal{P}^{3} + |\alpha|^{2} + |\beta|^{2} + \mathcal{P}^{2} \left(b - \frac{\lambda}{C_{0}}\right) + \mathcal{P} \|\epsilon\|_{L^{2}} + \|\epsilon\|_{H^{1}}^{2}\bigg) \end{split}$$
(6.164)

where $\tilde{c}_0 = \frac{c_0}{8} \int [9 + \rho^2 Q^2] \rho Q^2 dy.$

Step 2 Computation of the localized virial identity.

Our goal is now to compute, for A > 0 large enough, the following identity

$$\left(\frac{1}{\lambda}\mathcal{I}m\int A\,g^{ij}(\lambda y+\alpha)\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon\,\bar{\epsilon}\,dg_y\right)_s = -\frac{\lambda_s}{\lambda^2}\mathcal{I}m\int A\,g^{ij}(\lambda y+\alpha)\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon\,\bar{\epsilon}\,dg_y$$

$$+\frac{1}{\lambda}\mathcal{I}m\int \left[\Delta_g\phi\left(\frac{y}{A}\right)\epsilon+2A\,g^{ij}(\lambda y+\alpha)\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon\right]\overline{\partial_s\epsilon}\,dg_y$$

$$+\mathcal{I}m\int A\left\{\left(\frac{\lambda_s}{\lambda}+b\right)y+\left(\frac{\alpha_s}{\lambda}-2\beta\right)+\left(2\beta-b\,y\right)\right\}.\nabla\left(\sqrt{|g|}g^{ij}(\lambda y+\alpha)\right)\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon\,\bar{\epsilon}\,dy$$

$$(6.165)$$

$$+\mathcal{I}m\left(6.58\right) \text{ and } (6.62) \text{ we have}$$

From (6.58) and (6.62) we have

$$\frac{\lambda_s}{\lambda^2} = \frac{1}{\lambda} \left(\frac{\lambda_s}{\lambda} + b \right) - \frac{b}{\lambda} = \mathcal{O}(1)$$

which implies

$$-\frac{\lambda_s}{\lambda^2} \mathcal{I}m \int A g^{ij} (\lambda y + \alpha) \partial_i \phi\left(\frac{y}{A}\right) \partial_j \epsilon \,\overline{\epsilon} \, dh_y = \mathcal{O}\left(A \,\|\epsilon\|_{H^1}^2\right)$$

so that (6.165) may be rewritten as

$$\begin{pmatrix} \frac{1}{\lambda} \mathcal{I}m \int A g^{ij} (\lambda y + \alpha) \partial_i \phi(\frac{y}{A}) \partial_j \epsilon \bar{\epsilon} \, dg_y \end{pmatrix}_s$$

$$= \frac{1}{\lambda} \int \left[\Delta_g \phi(\frac{y}{A}) \left(\partial_s \epsilon_1 \epsilon_2 - \epsilon_1 \partial_s \epsilon_2 \right) + 2 A g^{ij} (\lambda y + \alpha) \partial_i \phi(\frac{y}{A}) \left(\partial_s \epsilon_1 \partial_j \epsilon_2 - \partial_s \epsilon_2 \partial_j \epsilon_1 \right) \right] dg_y$$

$$+ \mathcal{O} \left(A \|\epsilon\|_{H^1}^2 \right)$$

and since

$$\Delta_{g}\phi\left(\frac{y}{A}\right) = \Delta\phi\left(\frac{y}{A}\right) + \mathcal{O}\left(\mathcal{P}^{2}\left(|\Delta\phi| + A |\nabla\phi|\right)\right), \quad A g^{ij}(\lambda y + \alpha) \,\partial_{i}\phi\left(\frac{y}{A}\right) = A \,\partial_{i}\phi\left(\frac{y}{A}\right) + \mathcal{O}\left(A \,\mathcal{P}^{2} |\nabla\phi|\right)$$

we may simply write (6.165) as

we may simply write (6.165) as

$$\begin{split} &\left(\frac{1}{\lambda}\mathcal{I}m\int A\,g^{ij}(\lambda y+\alpha)\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon\,\overline{\epsilon}\,\,dg_y\right)_s\\ &=\frac{1}{\lambda}\int\Delta\phi\left(\frac{y}{A}\right)\left(\partial_s\epsilon_1\,\epsilon_2-\epsilon_1\,\partial_s\epsilon_2\right)dy+\frac{2}{\lambda}\int A\,\nabla\phi\left(\frac{y}{A}\right).\left(\partial_s\epsilon_1\,\nabla\epsilon_2-\partial_s\epsilon_2\,\nabla\epsilon_1\right)dy\\ &+\mathcal{O}\big(A\,\|\epsilon\|_{H^1}^2\big) \end{split}$$

Reintroducing (6.76)-(6.77) along with the estimates of Proposition 6.5 we get

$$\begin{split} &\left(\frac{1}{2\lambda}\mathcal{I}m\int A\,g^{ij}(\lambda y+\alpha)\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon\,\bar{\epsilon}\,dg_y\right)_s \\ &= \frac{1}{2\lambda}\mathcal{I}m\int\Delta\phi\left(\frac{y}{A}\right)\left(M_2(\epsilon)\,\epsilon_2 + M_1(\epsilon)\,\epsilon_1\right)dy + \frac{1}{\lambda}\mathcal{I}m\int A\,\nabla\phi\left(\frac{y}{A}\right).\left(M_2(\epsilon)\,\nabla\epsilon_2 + M_1(\epsilon)\,\nabla\epsilon_1\right)dy \\ &+ \mathcal{O}\Big(A\,\|\epsilon\|_{H^1}^2 + \mathcal{P}^3 + |\alpha|^2 + |\beta|^2 + \mathcal{P}^2\left(b - \frac{\lambda}{C_0}\right) + \mathcal{P}\,\|\epsilon\|_{L^2}\Big) \\ &= \frac{1}{\lambda}\mathcal{R}e\int\nabla^2\phi\left(\frac{y}{A}\right).\left(\nabla\epsilon,\,\nabla\bar{\epsilon}\right)dy - \frac{1}{4A^2\,\lambda}\int\Delta^2\phi\left(\frac{y}{A}\right)|\epsilon|^2\,dy \\ &+ \frac{1}{2\lambda}\int A\,\nabla\phi\left(\frac{y}{A}\right).\left[\nabla\left(Q^2 + 2\,\Sigma^2\right)\epsilon_1^2 + 4\,\nabla\left(\Sigma\,\Theta\right)\epsilon_1\,\epsilon_2 + \nabla\left(Q^2 + 2\,\Theta^2\right)\epsilon_2^2\right]dy \\ &+ \mathcal{O}\Big(A\,\|\epsilon\|_{H^1}^2 + \mathcal{P}^3 + |\alpha|^2 + |\beta|^2 + \mathcal{P}^2\left(b - \frac{\lambda}{C_0}\right) + \mathcal{P}\,\|\epsilon\|_{L^2}\Big) \end{split}$$
(6.166)

Step 3 Conclusion

Summing up (6.164) and (6.166), using (6.84) to expand non-linear terms of (6.164) finally leads to

$$\begin{cases} -\frac{b}{\lambda} \frac{\|yQ\|_{L^{2}}^{2}}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A g^{ij} (\lambda y + \alpha) \partial_{i} \phi\left(\frac{y}{A}\right) \partial_{j} \epsilon \bar{\epsilon} \, dg_{y} \\ \\ = \frac{1}{\lambda} \left[\int \left[1 - \frac{1}{4A^{2}} \Delta^{2} \phi\left(\frac{y}{A}\right) \right] |\epsilon|^{2} \, dy + \mathcal{R}e \int \nabla^{2} \phi\left(\frac{y}{A}\right) . \left(\nabla \epsilon, \, \nabla \bar{\epsilon}\right) \, dy - \int \left(3 \, Q^{2} \, \epsilon_{1}^{2} + Q^{2} \, \epsilon_{2}^{2} \right) \, dy \right] \\ + \frac{1}{\lambda} \int \left(A \, \nabla \phi\left(\frac{y}{A}\right) - y \right) . \left[3 \, Q \, \nabla Q \, \epsilon_{1}^{2} + Q \, \nabla Q \, \epsilon_{2}^{2} \right] \, dy - \lambda^{2} \, \tilde{c}_{0} - \frac{1}{2\lambda} \, \nabla^{2} k(0) . (\alpha, \alpha) \int Q^{2} \, dy \\ + \mathcal{O}\left(A \, \|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3} + |\alpha|^{2} + |\beta|^{2} + \mathcal{P}^{2} \left(b - \frac{\lambda}{C_{0}} \right) + \mathcal{P} \, \|\epsilon\|_{L^{2}} \right) \end{cases}$$
(6.167)

Notice from (6.114) that $-\lambda^2 = C_0 \lambda_s + \mathcal{O}(\lambda^3)$ so the $-\lambda^2 \tilde{c}_0$ term may be put in the left hand side.

Now, the choice for the cut-off function ϕ in (6.96) implies

$$\mathcal{R}e\int\nabla^2\phi\left(\frac{y}{A}\right).\left(\nabla\epsilon,\,\nabla\bar\epsilon\right)dy + \int\left[1 - \frac{1}{4A^2}\,\Delta^2\phi\left(\frac{y}{A}\right)\right]|\epsilon|^2\,dy \ge \int e^{-\frac{|y|}{2A}}\,|\nabla\epsilon|^2\,dy + \int |\epsilon|^2\,dy + \mathcal{O}\left(\frac{1}{A^2}\,\|\epsilon\|_{L^2}^2\right)dy + \mathcal{O}\left(\frac{1}{A^$$

The previous quadratic term is hence to be seen as a small deformation (as always, around the Ground-State) of the linearized energy $(L_{+}\epsilon_{1}, \epsilon_{1}) + (L_{-}\epsilon_{2}, \epsilon_{2})$. It can be estimated thanks to the coercivity property (6.71), which will ensure the existence of some universal constant $\underline{c} > 0$ such that whenever A is chosen large enough

$$\underline{c}\left[\int |\epsilon|^2 + \int e^{-\frac{|y|}{\sqrt{A}}} |\nabla\epsilon|^2\right] \leq \int e^{-\frac{|y|}{2A}} |\nabla\epsilon|^2 + \int |\epsilon|^2 - \int \left(3Q^2\epsilon_1^2 + Q^2\epsilon_2^2\right) + \mathcal{O}\left(\|\epsilon\|_{H^1}^4 + \mathcal{P}^4\right)$$

Altogether, previous statements, the bound (3.21) and assumption (H2) $\nabla^2 k(0) < 0$ shows that for A > 0 large enough, and t(A) < t < 0 close enough to 0, we have

$$\left\{ -\frac{b}{\lambda} \frac{\|yQ\|_{L^{2}}^{2}}{4} + \frac{1}{2\lambda} \mathcal{I}m \int A g^{ij} (\lambda y + \alpha) \partial_{i} \phi\left(\frac{y}{A}\right) \partial_{j} \epsilon \bar{\epsilon} \, dg_{y} - \tilde{c}_{0} C_{0} \lambda \right\}_{s} \qquad (6.168)$$

$$\geq \frac{c}{\lambda} \left[|\alpha|^{2} + \int |\epsilon|^{2} \, dy + \int e^{-\frac{|y|}{\sqrt{A}}} |\nabla \epsilon|^{2} \, dy \right] + \mathcal{O}\left(A \|\epsilon\|_{H^{1}}^{2} + \mathcal{P}^{3}\right)$$

This concludes the proof of Lemma 6.14. Note A > 0 may now be fixed once and for all.

6.7.4 Convergence to 0 of \tilde{u} in H^1 away from the concentration point.

We wish to prove that profile $\tilde{Q}_{\mathcal{P}}$ contains all of the solution's information (in a H^1 sense) when staying away from the concentration point (which, for the record, has been chosen as $x_0 = 0 \in \mathbb{R}^2$). In other words, the difference between the exact solution u and the approximate solution $\tilde{Q}_{\mathcal{P}}$ we have consctructed so far completely vanish in H^1 as $t \to 0$. More precisely, working with the original variables, let

$$\tilde{Q}(t,x) = \left(\frac{\rho}{\varphi}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \tilde{Q}_{\mathcal{P}}\left(\frac{x-\alpha(t)}{\lambda}\right) e^{i\gamma(t)}$$

$$\tilde{u}(t,x) = \left(\frac{\rho}{\varphi}\right)^{1/2} \frac{1}{k(\alpha(t))^{1/2}} \frac{1}{\lambda(t)} \epsilon\left(t, \frac{x-\alpha(t)}{\lambda}\right) e^{i\gamma(t)}$$
(6.169)

Using its very definition, and estimate (6.58) we have

$$\|\tilde{u}(t)\|_{L^2} \lesssim \lambda(t), \quad \|\tilde{u}(t)\|_{H^1} \lesssim 1, \quad \forall \ t_0 \le t < 0 \tag{6.170}$$

The goal is now to improve the energy bound (6.170) for the dispersive bound

$$\tilde{u}(t) \to 0$$
 in H^1 as $t \to 0$

The first step is dispersion away from the blow up point. From now on and in the next two sections, we fix $\eta > 0$ so that

$$\left| \left(1 - \frac{1}{2} x \cdot \nabla \right) g^{ij} \right| = \left| I_{ij} + \mathcal{O}(r^2) \right| \gtrsim 1, \quad \text{on } |x| = r \leq 3\eta$$

$$f \geq d_0 \quad \text{and} \quad \partial_r f \geq \frac{d_0}{r} f^2 \quad \text{on} \quad 2\eta \leq r \leq 4\eta, \quad d_0 > 0$$
(6.171)

which we will use in the next section. Now we prove the following result in the region $\{|x| \ge 3\eta\}$ away from the concentration point.

Lemma 6.15. (H^1 dispersion away from the concentration point).

 $There \ holds :$

$$\lim_{t \to 0} \|\tilde{u}(t)\|_{H^1(|x| \ge 3\eta)} = 0 \tag{6.172}$$

Remark 18. This is the first time, we really focus on what is going on away from the blow up point. We have already explained we are considering the metric g is asymptotically euclidean. So the calculations here will be basically the same as for the euclidean case.

Proof of Lemma 6.15

Let \tilde{Q} , \tilde{u} be given by (6.169), then \tilde{u} satisfies

$$i \partial_t \tilde{u} + \Delta_g \tilde{u} + V(x) \,\tilde{u} = -R - k(x) \,|\tilde{u}|^2 \,\tilde{u} \tag{6.173}$$

with

$$R = k(x) \left[\left(\tilde{Q} + \tilde{u} \right) \left| \tilde{Q} + \tilde{u} \right|^2 - \tilde{Q} \left| \tilde{Q} \right|^2 - \tilde{u} \left| \tilde{u} \right|^2 \right] + i \partial_t \tilde{Q} + (\Delta_g + V) \tilde{Q} + k(x) \left| \tilde{Q} \right|^2 \tilde{Q}$$

$$(6.174)$$

Step 1 $L_t^2 H_{loc}^{3/2}$ bound away from the concentration point.

Here we use some technical results a bit like we already did in the proof for Proposition 6.11. We will also use the smoothing effect of the linear Schrödinger flow to claim the space time bound

$$\int_{t_0}^0 \|\tilde{u}(\tau)\|_{H^{3/2}(2\eta \le r \le 4\eta)}^2 \, d\tau < +\infty \tag{6.175}$$

Starting with the nonlinear term in (6.173), let $\tilde{\zeta}$ be the solution to

$$i \partial_t \tilde{\zeta} + \left(\Delta_g + V(x)\right) \tilde{\zeta} = -k(x) \, |\tilde{u}|^2 \, \tilde{u}, \quad \tilde{\zeta}(0) = 0 \tag{6.176}$$

Now following the procedure explained in Appendix B 8.3, using successively Strichartz bounds and the smoothing effect of the linear Schrödinger flow, we have

$$\begin{split} \|\tilde{\zeta}\|_{L^{2}(t_{0},0)} & H_{loc}^{3/2} \lesssim \left\|k(x) \, |\tilde{u}|^{2} \, \tilde{u}\right\|_{L^{2}(t_{0},0)} H_{loc}^{1} \lesssim \|\tilde{u}\|_{L^{\infty}(t_{0},t)}^{3} H_{loc}^{1} \lesssim 1 \quad \text{(smoothing effect)} \\ \|\tilde{\zeta}\|_{L^{\infty}(t_{0},0)} & H^{1} = \|D\tilde{\zeta}\|_{L^{\infty}(t_{0},0)} L^{2} \lesssim \|D\big(k(x) \, |\tilde{u}|^{2} \, \tilde{u}\big)\|_{L^{4/3}(t_{0},0)} L^{4/3} \quad \text{(strichartz bound)} \end{split}$$

so that using paraproduct and Hölder estimates along with the Sobolev embedding Theorem result 8.2

$$\left\| D(k(x) \, |\tilde{u}|^2 \, \tilde{u}) \right\|_{L^{4/3}} \lesssim \left\| |\tilde{u}|^2 \, D\tilde{u} \right\|_{L^{4/3}} \lesssim \|\tilde{u}\|_{L^4}^2 \, \|D\tilde{u}\|_{L^4} \lesssim \|\tilde{u}\|_{H^1}^3$$

one gets

$$\|\tilde{\zeta}\|_{L^{2}(t_{0},0)H_{loc}^{3/2}} + \|\tilde{\zeta}\|_{L^{\infty}(t_{0},0)H^{1}} \lesssim 1 + \||\tilde{u}|^{2} \tilde{u}\|_{L^{4/3}(t_{0},0)W^{1,4/3}} \lesssim 1 + \||\tilde{u}|^{2} \tilde{u}\|_{L^{\infty}(t_{0},0)H^{1}}^{3} \lesssim 1$$

$$(6.177)$$

where we have used the $L^{\infty}((t_0, 0), H^1)$ bound (6.170) in the last line.

We introduce the Fourier multiplier $D = (1 - \Delta)^{1/2}$. Note that using our definition (6.148) for the Fourier transform, for every $s \in \mathbb{R}$ one has

$$D^{s}u(x) = \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^{2}} e^{ix \cdot \xi} \left(1 + |\xi|^{2}\right)^{s/2} \widehat{u}(\xi) \, d\xi, \quad \text{so that} \quad \|u\|_{H^{s}} = \|D^{s}u\|_{L^{2}} \tag{6.178}$$

In order to get (6.175), let $\tilde{w} = D^{1/2}(\tilde{\zeta} - \tilde{u})$. Indeed, $\tilde{\zeta}$ being controlled as seen in (6.177), a H^1 bound for \tilde{w} , would bring a $H^{3/2}_{loc}$ bound for \tilde{u} as required in (6.175). Then let also χ be a radial smooth cut-off function with

$$\chi'(r) = \int_0^r \chi''(\rho) \, d\rho, \quad \text{with} \quad \chi''(r) = \begin{cases} 0 & \text{for} \quad 0 \le r \le \eta, \\ 1 & \text{for} \quad 2\eta \le r \le 4\eta, \\ \frac{1}{r^2} & \text{for} \quad r \ge 5\eta \end{cases}$$
(6.179)

and such that

$$\forall r \ge 0, \quad \frac{|\chi'|^2}{r^2} \lesssim \chi''(r) \le 1.$$
 (6.180)

We claim (6.175) follows from (6.177) and

$$\int_{t_0}^0 \left\|\nabla \tilde{w}\right\|_{L^2(2\eta \le r \le 4\eta)}^2 \lesssim 1$$
(6.181)

Since $D^{1/2}$ commutes with $\Delta_q + V$, from (6.173) \tilde{w} satisfies

$$i \partial_t \tilde{w} + (\Delta_q + V)\tilde{w} = D^{1/2}R$$

where R is defined by (6.174).

We compute the associated localized virial identity :

$$\frac{1}{2} \frac{d}{dt} \mathcal{I}m \int \nabla \chi \cdot \nabla \tilde{w} \,\overline{\tilde{w}} \, dg_x$$

$$= -\mathcal{R}e \int \left[(\Delta_g + V)\tilde{w} - D^{1/2}R \right] \overline{\left(\frac{1}{2} \left[\Delta \chi + \frac{\nabla(\sqrt{|g|})}{\sqrt{|g|}} \cdot \nabla \chi \right] \tilde{w} + \nabla \chi \cdot \nabla \tilde{w} \right)} \, dg_x$$

$$= \int f \chi'' |\partial_r \tilde{w}|^2 \, dx + \int \frac{\partial_r f}{f^2} \chi' |\partial_\tau \tilde{w}|^2 \, dx - \frac{1}{4} \int \left(\Delta_g + V \right) \left[\Delta \chi + \frac{\chi' \partial_r f}{f} \right] |\tilde{w}|^2 \, f \, dx$$

$$+ \frac{1}{4} \int \partial_r V \chi' |\tilde{w}|^2 \, f \, dx + \mathcal{R}e \left(D^{1/2}R, \frac{1}{2} \left[\Delta \chi + \frac{\chi' \partial_r f}{f} \right] \tilde{w} + \nabla \chi \cdot \nabla \tilde{w} \right)_{L^2(dg_x)}$$
(6.182)

where $\partial_{\tau}\tilde{w} = \frac{1}{r}\partial_{\theta}\tilde{w}$. According to (6.179), $\chi'' = \frac{\chi'}{r} = 1$ on $2\eta \leq r \leq 4\eta$. From (6.3), provided $\eta > 0$ is chosen small enough, one may assume as previously announced in (6.171), $f \geq d_0$ and $\partial_r f \geq \frac{d_0}{r} f^2$ on $2\eta \leq r \leq 4\eta$ for some $d_0 > 0$, we have

$$\begin{split} &\frac{1}{2} \frac{d}{dt} \mathcal{I}m \int \nabla \chi \, . \, \nabla \tilde{w} \, \overline{\tilde{w}} \, dg_x \geq d_0 \bigg(\int_{2\eta \leq r \leq 4\eta} |\partial_r \tilde{w}|^2 \, dx + \int_{2\eta \leq r \leq 4\eta} |\partial_\tau \tilde{w}|^2 \, dx \bigg) \\ &- \frac{1}{4} \int \left(\Delta_g + V \right) \left[\Delta \chi + \frac{\chi' \, \partial_r f}{f} \right] |\tilde{w}|^2 \, f \, dx + \frac{1}{4} \, \int \partial_r V \, \chi' \, |\tilde{w}|^2 \, f \, dx \\ &+ \mathcal{R}e \Big(D^{1/2}R \, , \, \overline{\frac{1}{2} \left[\Delta \chi + \frac{\chi' \, \partial_r f}{f} \right] \tilde{w} + \nabla \chi \, . \, \nabla \tilde{w}} \bigg)_{L^2(dg_x)} \end{split}$$

then, one can easily deduce the following estimate

$$\begin{split} &\int_{t_0}^0 \left\|\nabla \tilde{w}\right\|_{L^2(2\eta \le r \le 4\eta)}^2 = \int_{t_0}^0 \int_{2\eta \le r \le 4\eta} \left(|\partial_r \tilde{w}|^2 + |\partial_\tau \tilde{w}|^2\right) \, dr d\tau \\ &\lesssim \frac{1}{4} \int_{t_0}^0 \int \left|\left(\Delta_g + V\right) \left[\Delta \chi + \frac{\chi' \, \partial_r f}{f}\right] - \partial_r V \, \chi' \, \left|\left|\tilde{w}\right|^2 \, f \, dx \right. \\ &+ \int_{t_0}^0 \left|\mathcal{R}e\left(D^{1/2}R, \frac{1}{2}\left[\Delta \chi + \frac{\chi' \, \partial_r f}{f}\right] \tilde{w} + \nabla \chi \cdot \nabla \tilde{w}\right)_{L^2(dg_x)}\right| + \left|\frac{1}{2} \, \mathcal{I}m \int \nabla \chi \cdot \nabla \tilde{w} \, \overline{\tilde{w}} \, dg_x\right| \end{split}$$

We now have to estimate the terms in (6.182). To begin with, using (6.170) and (6.177), from definition of \tilde{w} the boundary term in time is bounded

$$\left|\mathcal{I}m\int\nabla\chi\,\cdot\nabla\tilde{w}\,\overline{\tilde{w}}\,\,dg_x\right| \lesssim \|\tilde{w}\|_{H^{1/2}}^2 \lesssim \|\tilde{u}\|_{H^1}^2 + \|\tilde{\zeta}\|_{H^1}^2 \lesssim 1 \tag{6.183}$$

and similarly

$$\frac{1}{4} \int_{t_0}^0 \int \left| \left(\Delta_g + V \right) \left[\Delta \chi + \frac{\chi' \,\partial_r f}{f} \right] - \partial_r V \,\chi' \right| |\tilde{w}|^2 \, f \, dx \lesssim \|\tilde{u}\|_{H^{1/2}}^2 + \|\tilde{\zeta}\|_{H^{1/2}}^2 \lesssim 1 \tag{6.184}$$

Now with the *R* term in (6.182), first by Cauchy-Scwharz, from (6.2) and assumption (*H*5)(1), $\frac{\partial_r f}{f} = \mathcal{O}(1)$ so that

$$\begin{split} \int_{t_0}^0 \left| \mathcal{R}e\Big(D^{1/2}R \,,\, \overline{\frac{1}{2} \left[\Delta \chi + \frac{\chi' \,\partial_r f}{f} \right] \tilde{w} + \nabla \chi \,, \nabla \tilde{w}} \Big)_{L^2(dg_x)} \right| \lesssim \int_{t_0}^0 \|x \, D^{1/2}R\|_{L^2(r \ge \eta)} \left[\left\| \frac{2\Delta \chi}{r} \,\, \tilde{w} \right\|_{L^2(r \ge \eta)} + \left\| \frac{\chi'}{r} \,\partial_r \tilde{w} \right\|_{L^2} \right] \\ & \leq \frac{1}{\delta} \int_{t_0}^0 \|x \, D^{1/2}R\|_{L^2(r \ge \eta)}^2 + 2 \,\delta \int_{t_0}^0 \left(\left\| \frac{2\Delta \chi}{r} \,\, \tilde{w} \right\|_{L^2(r \ge \eta)}^2 + \left\| \frac{\chi'}{r} \,\partial_r \tilde{w} \right\|_{L^2}^2 \right) \\ & \leq \frac{1}{\delta} \int_{t_0}^0 \int_{|x|\ge \eta} |x|^2 \, |D^{1/2}R|^2 + 2 \,\int_{t_0}^0 \left(\| \tilde{w} \|_{L^2}^2 + \int \chi'' \,|\partial_r \tilde{w} |^2 \right) \\ & \lesssim \delta \int_{t_0}^0 \int \chi'' \,|\partial_r \tilde{w}|^2 + \frac{1}{\delta} \int_{t_0}^0 \int_{|x|\ge \eta} |x|^2 \, |D^{1/2}R|^2 + 1 \end{split}$$

for some small $\delta > 0$, where we have used (6.180) and the H^1 bound (6.170) in the last line. And there is only left to prove the following

Lemma 6.16.

$$\int_{t_0}^0 \left\| x \, D^{1/2} R \right\|_{L^2}^2 < +\infty \tag{6.185}$$

Proof of Lemma 6.16 :

With use of Pseudo Differential Calculus, see Appendix B 8.4, one may prove

$$D^{1/2} |x|^2 D^{1/2} = x D x + a_{-1}(x, D), \quad a_{-1} \in S^{-1} \text{ is a symbol of order } -1$$
(6.186)

In particular, what will be useful for us now, is that $a_{-1}(x, D)$ is a bounded operator $L^{4/3} \to W^{1,4/3} \subseteq L^4$ thanks to Sobolev embedding result Theorem 8.1. For details about these statements, see Appendix B 8.4. From (6.186) we then deduce

$$\left\| x D^{1/2} R \right\|_{L^{2}}^{2} = \left(D^{1/2} |x|^{2} D^{1/2} R, R \right) \lesssim \left\| D^{1/2} x R \right\|_{L^{2}}^{2} + \left\| R \right\|_{L^{4/3}}^{2}$$
(6.187)

We then change to rescaled variables

$$R(t,x) = \left(\frac{\rho}{\varphi}\right)^{1/2} \frac{1}{k(\alpha)^{1/2} \lambda^3(t)} S\left(\frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$
(6.188)

where, recalling the calculation that led to (6.15), S should quite naturally be defined as

$$S(s,y) = i \partial_s \tilde{Q}_{\mathcal{P}} + \mathcal{L}_g \tilde{Q}_{\mathcal{P}} - \tilde{Q}_{\mathcal{P}} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} |\tilde{Q}_{\mathcal{P}}|^2 Q_{\mathcal{P}} + \left[\lambda^2 V(\lambda y + \alpha) + V_{\varphi}\right] \tilde{Q}_{\mathcal{P}}$$
$$- i \frac{\alpha_s}{\lambda} \cdot \left(\nabla \tilde{Q}_{\mathcal{P}} + \frac{\lambda}{2} \frac{\nabla k(\alpha)}{k(\alpha)} \tilde{Q}_{\mathcal{P}}\right) - \tilde{\gamma}_s \tilde{Q}_{\mathcal{P}} + \frac{i}{2} \left[\frac{\lambda_s}{\lambda} \left(\frac{y \cdot \nabla \varphi}{\varphi} - 1\right) + \frac{\alpha_s}{\lambda} \cdot \left(\frac{\nabla \varphi}{\varphi} - \frac{y}{\rho^2}\right) - \frac{\partial_s \varphi}{\varphi}\right] \tilde{Q}_{\mathcal{P}} \quad (6.189)$$
$$- i \frac{\lambda_s}{\lambda} \Lambda \tilde{Q}_{\mathcal{P}} + \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[(\tilde{Q}_{\mathcal{P}} + \epsilon) |\tilde{Q}_{\mathcal{P}} + \epsilon|^2 - \tilde{Q}_{\mathcal{P}} |\tilde{Q}_{\mathcal{P}}|^2 - \epsilon |\epsilon|^2\right]$$

which is well localized in y. Changing variables $(Z, \eta) = \left(\frac{z-\alpha}{\lambda}, \lambda \xi\right)$, with $dZ \, d\eta = dz \, d\xi$ and from Sobolev embedding result Theorem 8.3

$$D^{1/2}x R = (2\pi)^{-2} \int e^{i(x-z) \cdot \xi} \langle \xi \rangle^{1/2} z R(z) dz d\xi$$

= $\frac{(2\pi)^{-2}}{k(\alpha)^{1/2}} \int e^{i(y-Z) \cdot \eta} \frac{1}{\lambda^{1/2}} (\lambda^2 + |\eta|^2)^{1/4} \left(\frac{|Z|}{\varphi(\lambda, \alpha, Z)}\right)^{1/2} \frac{e^{i\gamma}}{\lambda^3} (\lambda Z + \alpha) S(Z) dZ d\eta$
= $\frac{1}{\lambda^{5/2} k(\alpha)^{1/2}} (\lambda^2 - \Delta)^{1/4} \left[\left(\frac{\rho}{\varphi}\right)^{1/2} \left(y + \frac{\alpha}{\lambda}\right) S(y) \right]$

Furthermore using (6.12) one easily sees

$$\begin{split} \left(\frac{\rho}{\varphi}\right)^{1/2} & \left(y + \frac{\alpha}{\lambda}\right) = y + \frac{\alpha}{\lambda} + p_2(y), \quad p_2 = \mathcal{O}\Big(\left(\lambda^2 + |\alpha|^2\right)|y|^2\Big), \quad \text{and} \quad \left(\lambda^2 - \Delta\right)^{1/4} = D^{1/2} + a_{-3/2}(\lambda, D) \\ \text{with} \quad a_{-3/2}(\lambda, \eta) = \frac{\lambda^2 - 1}{\left[\left(\lambda^2 + |\eta|^2\right)^{1/4} + \langle\eta\rangle^{1/2}\right]\left[\left(\lambda^2 + |\eta|^2\right)^{1/2} + \langle\eta\rangle\right]} = \mathcal{O}\Big(\langle\eta\rangle^{-3/2}\Big) \\ a_{-3/2} : L^{4/3} \to W^{3/2, 4/3} \hookrightarrow H^1 \subset L^2, \quad \text{hence} : \\ & \left(\lambda^2 - \Delta\right)^{1/4}\left[\left(\frac{\rho}{\varphi}\right)^{1/2} \left(y + \frac{\alpha}{\lambda}\right)S(y)\right] = D^{1/2}yS(y) + \mathcal{O}\Big(\lambda^2 |D^{1/2}yS| + |a_{-3/2}(\lambda, D)S|\Big) \end{split}$$

so we have

$$\begin{split} \int \left| D^{1/2} x \, R \right|^2 dx &= \frac{1}{\lambda^3 k(\alpha)} \, \int \left| \left(\lambda^2 - \Delta \right)^{1/4} \left(y + \frac{\alpha}{\lambda} \right) S(y) \right|^2 dy \\ &\lesssim \frac{1}{\lambda^3} \int \left| D^{1/2} \left(y + \frac{\alpha}{\lambda} \right) S(y) \right|^2 dy + \frac{1}{\lambda^3} \left\| a_{-3/2}(\lambda, D) S \right\|_{L^2}^2 \end{split}$$

and thus from (6.58)

$$\left\| D^{1/2} x R \right\|_{L^2} \lesssim \frac{1}{\lambda^{3/2}} \left(\left\| D^{1/2} y S \right\|_{L^2} + \left\| D^{1/2} S \right\|_{L^2} + \left\| S \right\|_{L^{4/3}} \right)$$

Now thanks to (6.187), this yields

$$\left\| x \, D^{1/2} \, R \right\|_{L^2} \lesssim \frac{1}{\lambda^{3/2}} \left(\left\| D^{1/2} y \, S \right\|_{L^2} + \left\| D^{1/2} S \right\|_{L^2} + \left\| S \right\|_{L^{4/3}} \right) \tag{6.190}$$

We now explicitly expand the nonlinear terms in ϵ , in S, from definition (6.174) of R and (6.188), (6.189) of S, together with (6.31) we have

$$\begin{split} S(s,y) &= -\psi_{\mathcal{P}} + i\left(b_s + b^2 - B_1\right)\partial_b\tilde{Q}_{\mathcal{P}} + i\lambda\left(\frac{\lambda_s}{\lambda} + b\right)\partial_\lambda\tilde{Q}_{\mathcal{P}} + i\left(\beta_s + b\beta - B_2\right)\partial_\beta\tilde{Q}_{\mathcal{P}} \\ &+ i\lambda\left(\frac{\alpha_s}{\lambda} - 2\beta\right)\partial_\alpha\tilde{Q}_{\mathcal{P}} - i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\tilde{Q}_{\mathcal{P}} - i\left(\frac{\alpha_s}{\lambda} - 2\beta\right)\left[\nabla\tilde{Q}_{\mathcal{P}} + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)}\tilde{Q}_{\mathcal{P}}\right] \\ &- \left(\tilde{\gamma}_s - |\beta|^2\right)\tilde{Q}_{\mathcal{P}} + \frac{\rho}{\varphi}\frac{k(\lambda y + \alpha)}{k(\alpha)}\left[\tilde{Q}_{\mathcal{P}} |\epsilon|^2 + 2\epsilon\mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right) + |\tilde{Q}_{\mathcal{P}}|^2\left(\epsilon + 2\epsilon_1\right)\right] \end{split}$$

Thus, (6.41), (6.61) and (6.62) implies

$$S(s,y) = \frac{\rho}{\varphi} \frac{k(\lambda y + \alpha)}{k(\alpha)} \left[\tilde{Q}_{\mathcal{P}} |\epsilon|^2 + 2\epsilon \mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right) + |\tilde{Q}_{\mathcal{P}}|^2 \left(\epsilon + 2\epsilon_1\right) \right] + \left(K_b \frac{\rho^2}{4} - K_{\tilde{\gamma}}\right) Q + \mathcal{O}\left(\left[\lambda^4 + \lambda \left(|\alpha|^2 + \mu\right) + \lambda^2 |\epsilon|\right] e^{-c|y|} \right) = \tilde{Q}_{\mathcal{P}} |\epsilon|^2 + 2\epsilon \mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right) + |\tilde{Q}_{\mathcal{P}}|^2 \left(\epsilon + 2\epsilon_1\right) + \left(K_b \frac{\rho^2}{4} - K_{\tilde{\gamma}}\right) Q + \mathcal{O}\left(\left[\lambda^4 + \lambda \left(|\alpha|^2 + \mu\right) + \lambda^2 |\epsilon|\right] e^{-c|y|} \right)$$

$$(6.191)$$

Then using standard commutator estimates, see Appendix B 8.4, together with the good localization in space of S(s, y), the bound on the geometrical parameters (6.58), (6.62) and the $\mathcal{O}(\lambda^3)$ control in the construction of $\tilde{Q}_{\mathcal{P}}$ (6.20), (6.21) to conclude :

$$\|x D^{1/2} R\|_{L^{2}}^{2} \lesssim \frac{1}{\lambda^{3}} \left(\int |\nabla \epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} dy + \|\epsilon\|_{L^{2}}^{2} + \lambda^{3} + |\alpha|^{2} \right)$$

$$\lesssim 1 + \frac{|\alpha|^{2}}{\lambda^{3}} + \frac{1}{\lambda^{3}} \left(\int |\nabla \epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}} dy + \|\epsilon\|_{L^{2}}^{2} \right)$$

$$(6.192)$$

and (6.185) follows from (6.163) (remind $\frac{ds}{dt} = \frac{1}{\lambda^2}$).

Step 2 Strong H^1 convergence outside the blow up point.

The strong convergence (6.172) is now straightforward. Indeed, with space time bound (6.175) we have gained regularity in some region $2\eta \leq r \leq 4\eta$ away from concentration point. Now we see any truncation of a solution that cuts off the region near concentration point may be controlled in some way by that bound, using once more Strichartz estimates and the smoothing effect to deal respectively with nonlinearities and additional remaining terms.

Let a smooth cut off function $\tilde{\psi}$ with $\tilde{\psi} = 1$ on $|x| \ge 3\eta$ and $\tilde{\psi} = 0$ on $|x| \le 2\eta$, then $w = \tilde{\psi} \tilde{u}$ satisfies the following equation

$$i \partial_t w + L_g w + k(x) |w|^2 w = F, \quad w(t) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \to 0$$
(6.193)
with
$$E = \Delta \hat{\psi} \tilde{u} + 2 a^{ij}(x) \partial_x \tilde{\psi} \partial_y \tilde{u} - k(x) \hat{\psi} (1 - \hat{\psi}^2) |\tilde{u}|^2 \tilde{u} - \hat{\psi} B$$

with
$$\Gamma = \Delta_g \psi \, u + 2g \, (x) \, \partial_i \psi \, \partial_j u - \kappa(x) \, \psi \, (1 - \psi) \, |u| \, u - \psi \, R$$

where R is given by (6.174) and $L_g = \Delta_g + V$. To get (6.172) we now only have to show

$$||w||_{L^{\infty}((t_0,0),H^1)} \to 0 \quad \text{as} \quad t_0 \to 0$$
 (6.194)

First we write down Duhamel formula for (6.193)

$$w(t) = e^{i t L_g} w(t_0) - i \int_{t_0}^t e^{i (t-\tau)L_g} [F(\tau)] d\tau, \quad \text{hence} \quad \nabla w(t) = e^{i t L_g} \nabla w(t_0) - i \int_{t_0}^t e^{i (t-\tau)L_g} [\nabla F(\tau)] d\tau$$

then, we use the Strichartz estimates for the 2-dimensional Strichartz pairs (4, 4), $(\infty, 2)$ along with the the smoothing effect for the linear Schrödinger operator (again, see Appendix B 8.3.2 for details) to get

$$\begin{aligned} \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4} \right)} + \left\| \nabla w \right\|_{L^{\infty}\left((t_{0},0),L^{2} \right)} \\ \lesssim \left\| \nabla (k(x) \left| w \right|^{2} w \right) \right\|_{L^{4/3}\left((t_{0},0),L^{4/3} \right)} + \left\| \left| x \right| F \right\|_{L^{2}\left((t_{0},0),H^{1/2} \right)} \end{aligned}$$

$$(6.195)$$

We estimate the nonlinear term as follows, using the Gagliardo-Nirenberg estimate

$$\begin{aligned} \left\| \nabla(k(x) |w|^{2} w) \right\|_{L^{4/3}\left((t_{0},0),L^{4/3}\right)} \\ \lesssim \left\| |w|^{2} w \right\|_{L^{4/3}\left((t_{0},0),L^{4/3}\right)} + \left\| |w|^{2} \nabla w \right\|_{L^{4/3}\left((t_{0},0),L^{4/3}\right)} \\ \lesssim \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \left\| \tilde{u} \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \\ \lesssim \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \left\| \tilde{u} \right\|_{L^{\infty}\left((t_{0},0),H^{1}\right)} \left\| \tilde{u} \right\|_{L^{\infty}\left((t_{0},0),L^{2}\right)} \\ \lesssim \left\| \lambda \right\|_{L^{\infty}(t_{0},0)} \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \lesssim \delta \left\| \nabla w \right\|_{L^{4}\left((t_{0},0),L^{4}\right)} \end{aligned}$$
(6.196)

for a constant $\delta > 0$ small enough, where we have used (6.170) in the last inequality. For the second term, we make use of the compact support property of $\nabla \tilde{\psi}$, $\Delta_G \tilde{\psi}$, $(1 - \tilde{\psi}^2)$ and (6.175) along with (6.185)

$$\| |x| F \|_{L^{2}\left((t_{0},0),H^{1/2}\right)} \lesssim \| |x| R \|_{L^{2}\left((t_{0},0),H^{1/2}(|x|\geq\eta)\right)} + \| \tilde{u} \|_{L^{2}\left((t_{0},0),H^{3/2}(2\eta\leq r\leq 4\eta)\right)}$$

$$\lesssim o(1) \quad \text{as} \quad t_{0} \to 0$$

$$(6.197)$$

Altogether, these last estimations prove

$$\|\nabla w\|_{L^4((t_0,0),L^4)} + \|\nabla w\|_{L^\infty((t_0,0),L^2)} \lesssim o(1) \quad \text{as} \quad t_0 \to 0$$
 (6.198)

which ends the proof of Lemma 6.16.

6.7.5 Convergence to 0 in average of \tilde{u} in H^1 .

The goal now is to propagate the H^1 convergence of \tilde{u} away from the concentration point (6.172) to the blow up region as well. We will now make use of the refined estimate (6.163) which has been deduced from localized virial estimate (6.162) and which now implies

$$\begin{split} \int_{t_0}^0 \frac{1}{\lambda} \bigg(\frac{\|\tilde{u}(t)\|_{L^2(dg_x)}}{\lambda} \bigg)^2 dt &= \int_{s_0}^{+\infty} \frac{1}{\lambda} \bigg(\int |\tilde{u}|^2 \, dg_x \bigg) \, ds = \int_{s_0}^{+\infty} \frac{1}{\lambda} \left(\int |\epsilon|^2 \frac{\rho}{\varphi} \sqrt{|g|} \, dy \right) ds \\ &\lesssim \int_{s_0}^{+\infty} \frac{1}{\lambda} \left(\int |\epsilon|^2 \, dy \right) \, ds < +\infty \end{split}$$

so that

$$\liminf_{t \to 0} \frac{\|\tilde{u}(t)\|_{L^2(dg_x)}}{\lambda(t)} = 0$$

The first step is to obtain a convergence in average in time.

Lemma 6.17. $(H^1 \text{ dispersion in average in time.})$ There holds

$$\lim_{t \to 0} \frac{1}{|t|} \int_{t}^{0} \left(\frac{1}{|\tau|} \int_{\tau}^{0} \|\tilde{u}\|_{H^{1}(\mathbb{R}^{2}, dg_{x})}^{2} d\sigma \right) d\tau = 0$$
(6.199)

Proof of Lemma 6.17 : Thanks to (6.172), we may here restrict the H^1 norm to the region $|x| \leq 3$. Step 1 Morawetz identity.

First, we claim the virial type bound

$$\int_{t}^{0} \int_{|x| \le 3\eta} \left| \nabla \tilde{u}(\tau) \right|^{2} dg_{x} d\tau \lesssim o(|t|) + \|\tilde{u}(t)\|_{L^{2}} + \int_{t}^{0} \frac{|\alpha|^{2} + \|\epsilon(\tau)\|_{L^{2}}^{2} + \int |\nabla\epsilon|^{2} e^{-\frac{|y|}{\sqrt{A}}}}{\lambda^{2}(\tau)} d\tau$$
(6.200)

Let χ a smooth radial cut off function on \mathbb{R}^2 such that

$$supp(\chi) \subset \{ |x| \le 4\eta \}, \quad \nabla \chi(x) = x, \quad \text{on} \quad |x| \le 3\eta$$

$$f_{\chi} := h^{-1} \nabla \ ^{t} \nabla \chi - \frac{1}{2} \nabla \chi . \nabla (h^{-1}) \quad \text{is positive semidefinite}$$
(6.201)

thus $\partial_{j\,l}^2 \chi = 0$ for $j \neq l$, and $\partial_j^2 \chi = 1$ on $|x| \leq 3\eta$, so that the form f_{χ} is given by the matrix

$$(f_{\chi})_{i,j} = (1 - \frac{1}{2} x \cdot \nabla) g^{ij} \gtrsim I_{ij}, \quad i, j = 1, 2, \qquad |x| \le 3\eta$$

thanks to the choice we have made for $\eta > 0$ small enough as announced in (6.171).

Then, to control the H^1 norm of \tilde{u} on region $|x| \leq 3\eta$, one may as before use an estimate on the localized momentum. So, recall \tilde{u} is solution to equation (6.173), one gets

$$\frac{d}{dt}\mathcal{I}m\int\nabla\chi\cdot\nabla\tilde{u}\,\overline{\tilde{u}}\,dg_{x}$$

$$= -2\,\mathcal{R}e\int\left(L_{g}\tilde{u}+k(x)\,|\tilde{u}|^{2}\,\tilde{u}+R\right)\left[\frac{1}{2}\left(\Delta\chi+\frac{\nabla(|g|^{\frac{1}{2}})}{|g|^{\frac{1}{2}}}\cdot\nabla\chi\right)\overline{\tilde{u}}+\nabla\chi\cdot\nabla\overline{\tilde{u}}\right]\,dg_{x}$$

$$= 2\,\mathcal{R}e\int\left[g^{lj}\partial_{l}\partial_{i}\chi-\frac{1}{2}\,\partial_{l}g^{ij}\,\partial_{l}\chi\right]\partial_{i}\overline{\tilde{u}}\,\partial_{j}\tilde{u}\,dg_{x}-\frac{1}{2}\int\left(L_{g}+k(x)\,|\tilde{u}|^{2}\right)\left[\Delta\chi+\frac{\nabla(|g|^{\frac{1}{2}})}{|g|^{\frac{1}{2}}}\cdot\nabla\chi\right]|\tilde{u}|^{2}\,dg_{x}$$

$$+\frac{1}{2}\int\nabla\chi\cdot\left[\nabla V(x)+|\tilde{u}|^{2}\,\nabla k(x)\right]|\tilde{u}|^{2}\,dg_{x}-2\,\mathcal{R}e\int R\left[\frac{1}{2}\left(\Delta\chi+\frac{\nabla(|g|^{\frac{1}{2}})}{|g|^{\frac{1}{2}}}\cdot\nabla\chi\right)\overline{\tilde{u}}+\nabla\chi\cdot\nabla\overline{\tilde{u}}\right]\,dg_{x}$$
(6.202)

where \tilde{u} , \tilde{Q} and R are defined as in (6.169) and (6.174). We estimate the various terms of the right-hand side of (6.202). Using (6.170), and a Gagliardo-Nirenberg estimate, we have

$$\left| \int \left(L_g + k(x) \, |\tilde{u}|^2 \right) \left[\Delta \chi + \frac{\nabla (|g|^{\frac{1}{2}})}{|g|^{\frac{1}{2}}} \, \cdot \nabla \chi \right] |\tilde{u}|^2 \, dg_x - \int \nabla \chi \, \cdot \left[\nabla V(x) + |\tilde{u}|^2 \, \nabla k(x) \right] |\tilde{u}|^2 \, dg_x \right|$$

$$\lesssim \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^4}^4 \lesssim \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 \|\tilde{u}\|_{H^1}^2 \lesssim \lambda^2(t)$$

$$(6.203)$$

Then, recall from (6.191)

$$S = \tilde{Q}_{\mathcal{P}} |\epsilon|^2 + 2 \epsilon \mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right) + |\tilde{Q}_{\mathcal{P}}|^2 \left(\epsilon + 2\epsilon_1\right) + \left(K_b \frac{\rho^2}{4} - K_{\tilde{\gamma}}\right) Q + \mathcal{O}\left(\left[\lambda^4 + \lambda \left(|\alpha|^2 + \mu\right) + \lambda^2 |\epsilon|\right] e^{-c|y|}\right)$$

and hence, making use of property $\nabla \chi(x) \sim x$ near the origin, it is now clear that

$$\begin{aligned} \left| \mathcal{R}e \int R(x) \left[\frac{1}{2} \left(\Delta \chi + \frac{\nabla(|g|^{\frac{1}{2}})}{|g|^{\frac{1}{2}}} \cdot \nabla \chi \right) \overline{\tilde{u}} + \nabla \chi \cdot \nabla \overline{\tilde{u}} \right] dg_x \right| \\ &= \frac{1}{\lambda^2} \left| \int S(y) \left[\frac{1}{2} \left(\Delta \chi + \frac{\nabla(|g|^{\frac{1}{2}})}{|g|^{\frac{1}{2}}} \cdot \nabla \chi \right) (\lambda y + \alpha) \overline{\epsilon} + \frac{1}{\lambda} \nabla \chi (\lambda y + \alpha) \cdot \nabla \overline{\epsilon} \right] \left(\frac{\rho}{\varphi} \right)^{\frac{1}{2}} |g|^{\frac{1}{2}} dy \right| \\ &\lesssim \frac{1}{\lambda^2} \left[|\alpha|^2 + \lambda^3 + \|\epsilon\|_{L^2}^2 + \int |\nabla \epsilon|^2 e^{-\frac{|y|}{\sqrt{A}}} dy \right] \end{aligned}$$
(6.204)

Eventually, we inject (6.203) and (6.204) in (6.202) and integrate in time, then we use property $f_{\chi} \gtrsim 1$ and (6.153) to get

$$\begin{split} &\int_{t}^{0} \int_{|x| \leq 3\eta} |\nabla \tilde{u}(\tau)|^{2} \, d\tau \lesssim \int_{t}^{0} \int f_{\chi} \left(\nabla \tilde{u} \,, \, \overline{\nabla \tilde{u}} \right) dg_{x} \\ &\lesssim \left[\mathcal{I}m \int \nabla \chi \,. \, \nabla \tilde{u} \, \overline{\tilde{u}} \,\, dg_{x} \right]_{t}^{0} + \int_{t}^{0} \frac{1}{\lambda^{2}(\tau)} \Big[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} + \int |\nabla \epsilon|^{2} \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau + \int_{t}^{0} \lambda(\tau)^{2} \, d\tau \\ &\lesssim o\big(|t|\big) + \|\tilde{u}(t)\|_{L^{2}} + \int_{t}^{0} \frac{1}{\lambda^{2}(\tau)} \Big[|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} + \int |\nabla \epsilon|^{2} \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \end{split}$$

where we have used (6.170) which ensures

$$\left| \mathcal{I}m \int \nabla \chi \cdot \nabla \tilde{u} \, \overline{\tilde{u}} \, dg_x \right| \lesssim \| \nabla \tilde{u} \|_{L^2} \| \tilde{u} \|_{L^2} \lesssim \| \tilde{u} \|_{L^2}$$

and (6.200) is proved since $\lim_{t \to 0} \frac{1}{|t|} \int_t^0 \lambda(\tau) \, d\tau = \lim_{t \to 0} \lambda(t) = 0.$

Step 2 Averaged in time dispersion.

We now divide (6.200) by |t| and integrate in time. From the pointwise lower bound (6.153) and the dispersive bound (6.163), once recalled that $\frac{ds}{dt} = \frac{1}{\lambda^2}$, we have

$$\begin{aligned} \frac{1}{|t|} \int_t^0 \frac{1}{\lambda^2(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy \Big] \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \Big[|\alpha|^2 + \|\epsilon\|_{L^2}^2 + \int |\nabla\epsilon|^2 \, d\tau \lesssim \int_t^0 \frac{1}{\lambda^3(\tau)} \, d\tau$$

Similarly, from Cauchy-Schwarz estimate, and then (6.153) together with (6.163) again

$$\begin{aligned} \frac{1}{|t|} \int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}}{|\tau|} d\tau &\lesssim \left(\int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}^{2}}{|t||\tau|^{2}} d\tau\right)^{1/2} \left(\int_{t}^{0} \frac{d\tau}{|t|}\right)^{1/2} \lesssim \left(\int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}^{2}}{|\tau|^{3}} d\tau\right)^{1/2} \lesssim \left(\int_{t}^{0} \frac{\|\tilde{u}(\tau)\|_{L^{2}}^{2}}{\lambda^{3}(\tau)} d\tau\right)^{1/2} \\ &= \left(\int_{s}^{+\infty} \frac{1}{\lambda} \left\|\epsilon\right\|_{L^{2}}^{2} d\sigma\right)^{1/2} \to 0 \text{ as } t \to 0 \end{aligned}$$

Thus, (6.199) is proved, which concludes the proof of Lemma 6.17.

6.7.6 Control of the modulation parameters.

We now claim the H^1 dispersion (6.199) together with the conservation laws imply the sharpen control of the modulation parameters we have announced in (6.8).

Proposition 6.18. (Pointwise dispersive bounds)

There holds the pointwise bounds :

$$|\alpha| + |\beta| + \|\epsilon\|_{H^1} \lesssim \lambda^{3/2}, \quad 0 < \mu \lesssim \lambda^3 \tag{6.205}$$

$$\left|\frac{b(s)}{\lambda(s)} - \frac{1}{C_0}\right| \lesssim \lambda \tag{6.206}$$

$$\lambda(t) = -\frac{t}{C_0} + \mathcal{O}(|t|^2), \quad \text{for } t_0 \le t \le 0.$$
(6.207)

Moreover, there exists $\gamma_0 \in \mathbb{R}$ such that :

$$\gamma(t) = -\frac{C_0^2}{t} + \gamma_0 + \mathcal{O}(|t|)$$
(6.208)

Remark 19. Note that (6.205) imply the zero momentum limit $\lim_{t\to 0} \mathcal{I}m \int \nabla u \,\overline{u} = 0$ so (1.9) is proved, and also sharpens the bound (6.170) by a factor λ . Indeed, let $\mathcal{G} = \nabla + \frac{\lambda}{2} \left(\frac{y}{\rho^2} - \frac{\nabla \varphi}{\varphi} \right)$, so that using (6.41) one easily sees

$$\mathcal{I}m \int \nabla u \,\overline{u} \, dg_x = \frac{1}{\lambda k(\alpha)} \int \left[\left(\Sigma + \epsilon_1 \right) \mathcal{G}(\Theta + \epsilon_2) - \left(\Theta + \epsilon_2 \right) \mathcal{G}\left(\Sigma + \epsilon_1 \right) \right] \frac{\rho}{\varphi} \sqrt{|g|} \, dy$$
$$= \frac{1}{\lambda k(\alpha)} \left\{ \int \left(\Sigma \,\nabla \Theta - \Theta \,\nabla \Sigma \right) dy + \mathcal{O}\left(\lambda^2 + \|\epsilon\|_{H^1} \right) \right\} = \mathcal{O}(\lambda) \to 0, \quad \text{as } t \to 0$$

Proof of Proposition 6.18:

Step 1 Control in average of α and β .

We claim

$$\lim_{t \to 0} \frac{1}{|t|} \int_t^0 \left(\frac{1}{|\tau|} \int_\tau^0 \left(\frac{|\alpha|^2}{\lambda^2} + \frac{|\beta|^2}{\lambda^2} \right) d\sigma \right) d\tau = 0$$
(6.209)

First, we deal with α . From the pointwise bound (6.153) and the dispersive bound (6.163)

$$\frac{1}{|t|} \int_t^0 \frac{|\alpha|^2}{\lambda^2} d\tau \lesssim \int_t^0 \frac{|\alpha|^2}{\lambda^3} d\tau = \int_s^{+\infty} \frac{|\alpha|^2}{\lambda} d\sigma \to 0 \quad \text{as } t \to 0$$

which proves (6.209) as far as α is concerned. Now dealing with β , we have

$$\left(\frac{\alpha \cdot \beta}{\lambda}\right)_{s} = \alpha_{s} \cdot \frac{\beta}{\lambda} + \beta_{s} \cdot \frac{\alpha}{\lambda} - \frac{\alpha \cdot \beta}{\lambda} \frac{\lambda_{s}}{\lambda}$$

$$= 2 \left|\beta\right|^{2} + \left(-b\beta + B_{2}\right) \cdot \frac{\alpha}{\lambda} + b \frac{\alpha \cdot \beta}{\lambda} + \left(\frac{\alpha_{s}}{\lambda} - 2\beta\right) \cdot \beta$$

$$+ \left(\beta_{s} + b\beta - B_{2}\right) \cdot \frac{\alpha}{\lambda} - \left(\frac{\lambda_{s}}{\lambda} + b\right) \frac{\alpha \cdot \beta}{\lambda}$$

$$= 2 \left|\beta\right|^{2} + B_{2} \cdot \frac{\alpha}{\lambda} + \mathcal{O}\left(\left|Mod(t)\right|\right) = 2 \left|\beta\right|^{2} + l_{0}(\alpha) \cdot \alpha + \mathcal{O}\left(\lambda^{3} + \left\|\epsilon\right\|_{L^{2}}^{2}\right)$$

$$(6.210)$$

where we have used (6.22) for definition of B_2 , and (6.62) in the last line. Integrating (6.210) in time between s and $+\infty$, using (6.58) which imply that $\frac{\alpha \cdot \beta}{\lambda} \to 0$ as $s \to +\infty$, we obtain :

$$2\int_{s}^{+\infty} |\beta|^{2} d\sigma = -\frac{\alpha(s) \cdot \beta(s)}{\lambda(s)} - \int_{s}^{+\infty} l_{0}(\alpha) \cdot \alpha \, d\sigma + \int_{s}^{+\infty} \mathcal{O}\left(\lambda^{3}(\sigma) + \|\epsilon\|_{L^{2}}^{2}\right) d\sigma$$
$$\lesssim |\alpha(s)| + \int_{s}^{+\infty} \left(|\alpha(\sigma)|^{2} + \|\epsilon\|_{L^{2}}^{2}\right) d\sigma + \int_{s}^{+\infty} \lambda^{3}(\sigma) \, d\sigma$$

Now dividing that last estimate by $|\tau|$, changing variables - $\frac{ds}{dt} = \frac{1}{\lambda^2}$ - and using (6.163) yields

$$\begin{aligned} \frac{1}{|\tau|} \int_{\tau}^{0} \frac{|\beta|^2}{\lambda^2} \, d\sigma &\lesssim \frac{|\alpha(\tau)|}{|\tau|} + \frac{1}{|\tau|} \int_{\tau}^{0} \left(|\alpha(\sigma)|^2 + \|\epsilon\|_{L^2}^2 \right) \frac{d\sigma}{\lambda^2(\sigma)} + \frac{1}{|\tau|} \int_{\tau}^{0} \lambda(\sigma) \, d\sigma \\ &\lesssim \frac{|\alpha(\tau)|}{|\tau|} + \int_{\tau}^{0} \left(|\alpha(\sigma)|^2 + \|\epsilon\|_{L^2}^2 \right) \frac{d\sigma}{\lambda^3(\sigma)} + o(1) \\ &\lesssim \frac{|\alpha(\tau)|}{|\tau|} + o(1) \end{aligned}$$

Integrating once more, using Cauchy-Scwharz and (6.163) again, we have

$$\frac{1}{|t|} \int_t^0 \frac{1}{|\tau|} \int_\tau^0 \frac{|\beta|^2}{\lambda^2} d\sigma \lesssim \frac{1}{|t|} \int_t^0 \frac{|\alpha(\tau)|}{|\tau|} d\tau + o(1) \lesssim \left(\int_t^0 \frac{|\alpha(\tau)|^2}{\lambda^3(\tau)} d\tau\right)^{1/2} + o(1)$$
$$\lesssim \left(\int_s^{+\infty} \frac{1}{\lambda} |\alpha|^2 d\sigma\right)^{1/2} + o(1) \to 0 \quad \text{as } t \to 0$$

hence (6.209) follows for β as well.

Step 2 Limit of $\frac{b}{\lambda}$ on a subsequence as $t \to 0$.

From (6.209) and (6.199) we have now

$$\lim_{t \to 0} \frac{1}{|t|} \int_t^0 \left(\frac{1}{|\tau|} \int_\tau^0 \left(\frac{|\alpha|^2}{\lambda^2} + \frac{|\beta|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} \right) d\sigma \right) d\tau = 0$$

This obviously imply the existence of a sequence $t_n \to 0$ such that

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$$\lim_{n \to +\infty} \frac{|\alpha(t_n)|}{\lambda(t_n)} + \frac{|\beta(t_n)|}{\lambda(t_n)} + \frac{\|\epsilon(t_n)\|_{H^1}}{\lambda(t_n)} = 0$$
(6.211)

Injecting this into (6.70) yields

$$\left|E_0 + C_E - \left(\frac{b}{\lambda}\right)^2 (t_n) \frac{1}{8} \int |y|^2 Q^2 \right| \lesssim \left(\frac{|\alpha(t_n)|}{\lambda(t_n)}\right)^2 + \left(\frac{|\beta(t_n)|}{\lambda(t_n)}\right)^2 + \left(\frac{\|\epsilon(t_n)\|_{H^1}}{\lambda(t_n)}\right)^2 + \lambda(t_n) + |\alpha(t_n)|$$

so that, recall definition (6.104) of constant C_0

$$\lim_{n \to +\infty} \left(\frac{b(t_n)}{\lambda(t_n)}\right)^2 = \frac{E_0 + C_E}{\frac{1}{8}\int |y|^2 Q^2} = \frac{1}{C_0^2}$$
(6.212)

Then, observe from (6.164) the bound

$$\left| \left(\frac{b}{\lambda}\right)_{s} \right| \lesssim \frac{1}{\lambda} \left(\|\epsilon\|_{L^{2}}^{2} + |\alpha|^{2} \right) + \mathcal{O}(\lambda^{2})$$

$$(6.213)$$

and thus, (6.163) ensures

$$\int_{s}^{+\infty} \left| \left(\frac{b}{\lambda} \right)_{s} \right| d\sigma < +\infty$$

Hence, $\frac{b}{\lambda}$ has a limit as $s \to +\infty$, so that from (6.212)

$$\frac{b}{\lambda} \to \pm \frac{1}{C_0}$$
 as $t \to 0$

Yet that limit cannot be $-\frac{1}{C_0}$, since it would contradict the finite blow up assumption. Indeed, since $\lambda(t) > 0$, and from (6.58), (6.62), one gets

$$\left|\lambda_t + \frac{b}{\lambda}\right| = \frac{1}{\lambda} \left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \lambda \tag{6.214}$$

so that $\frac{b}{\lambda} \to -\frac{1}{C_0}$ would mean at the same time $\lambda \to 0$ and $\lambda_t \ge \frac{1}{2C_0}$ which imply λ can never go below a certain positive constant, thus certainly does not tend to 0. Eventually

$$\lim_{t \to 0} \frac{b(t)}{\lambda(t)} = \frac{1}{C_0} > 0 \tag{6.215}$$

At last, injecting this into (6.214) and then integrating in time yields in particular the pseudo conformal speed :

$$\lambda(t) = \frac{|t|}{C_0} (1 + o(1)), \quad \text{as } t \to 0$$
(6.216)

Step 3 Improved bounds.

We now claim the knowledge of the limit of $\frac{b}{\lambda}$, together with monotonicity result (6.162) and the conservation of energy will bring a spectacular improvement on the bounds of $\|\epsilon\|_{H^1}$.

Indeed, we integrate the local virial identity (6.162) between s and s_n and then let $t_n \to 0$.

First, the boundary term in t_n is estimated using (6.58), (6.215)

$$-\left(\frac{b(s_n)}{\lambda(s_n)}\right)\frac{\|yQ\|_{L^2}^2}{4} + \frac{1}{2\lambda(s_n)}\mathcal{I}m\int A\,g^{ij}(\lambda y + \alpha)\,\partial_i\phi\left(\frac{y}{A}\right)\partial_j\epsilon(s_n)\,\overline{\epsilon(s_n)}\,dg_y - \tilde{c}_0\,C_0\,\lambda(s_n)$$

$$= -\left(\frac{b(t_n)}{\lambda(t_n)}\right)\frac{\|yQ\|_{L^2}^2}{4} + \mathcal{O}\left(\frac{\|\epsilon(t_n)\|_{H^1}^2}{\lambda(t_n)} + \lambda(t_n)\right) \to -\frac{1}{C_0}\frac{\|yQ\|_{L^2}^2}{4} \quad \text{as } t_n \to 0$$
(6.217)

and thus $\forall s > 0$, from definition of K_0 in Lemma 6.10, and of \tilde{c}_0 in Lemma 6.14

$$\left(\frac{b(s)}{\lambda(s)} - \frac{1}{C_0} + K_0 C_0 \lambda(s)\right) \frac{\|yQ\|_{L^2}^2}{4} - \frac{1}{2\lambda(s)} \mathcal{I}m \int A g^{ij}(\lambda y + \alpha) \partial_i \phi\left(\frac{y}{A}\right) \partial_j \epsilon(s) \overline{\epsilon(s)} \, dg_y \\ \geq \int_s^{+\infty} \left(\frac{c}{\lambda} \left(|\alpha|^2 + \|\epsilon\|_{L^2}^2\right) + \mathcal{O}\left(\mathcal{P}^3 + |\beta|^2 + \|\epsilon\|_{H^1}^2\right)\right) d\sigma$$

so that from (6.58)

$$\int_{s}^{+\infty} \frac{1}{\lambda} \left(|\alpha|^{2} + \|\epsilon\|_{L^{2}}^{2} \right) d\sigma \lesssim \left(\frac{b(s)}{\lambda(s)} - \frac{1}{C_{0}} + K_{0} C_{0} \lambda(s) \right) + \|\epsilon\|_{H^{1}} + \int_{s}^{+\infty} \left(\lambda^{3} + |\beta|^{2} + \|\epsilon\|_{H^{1}}^{2} \right) d\sigma \tag{6.218}$$

We now recall (6.70) implies

$$\begin{aligned} |\beta|^2 + |\alpha|^2 + \|\epsilon\|_{H^1}^2 \lesssim \frac{\lambda^2}{C_0^2} - b^2 - \lambda^3 \left(2K_0 - K_2\right) + \lambda^2 l_E(\alpha) + \lambda^4 \\ \text{with} \\ K_2 &= \frac{9c_0}{8} \int \rho Q^2 \, dy \, \left\|\frac{\rho}{2}Q\right\|_{L^2}^{-2}, \quad l_E(\alpha) = 2 \left\|\frac{\rho}{2}Q\right\|_{L^2}^{-2} \phi_E(\alpha) \end{aligned}$$
(6.219)

where $\phi_E(\alpha)$ is defined in (6.34). Dividing by λ^2 , we obtain

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} \lesssim \frac{1}{C_0^2} - \frac{b^2}{\lambda^2} - \lambda \left(2K_0 - K_2\right) + l_E(\alpha) + \lambda^2 \tag{6.220}$$

Now, multiplying (6.218) by $\frac{b(s)}{\lambda(s)} + \frac{1}{C_0}$ and adding to (6.220), noticing thanks to (6.110) that the terms $\frac{b^2}{\lambda^2} - \frac{1}{C_0^2} + 2 K_0 \lambda$ cancel each other yields

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} + \frac{\|\epsilon\|_{H^1}^2}{\lambda^2} + \int_s^{+\infty} \frac{1}{\lambda} \left(|\alpha|^2 + \|\epsilon\|_{L^2}^2 \right) d\sigma$$

$$\lesssim K_2 \lambda + l_E(\alpha) + \lambda^2 + \int_s^{+\infty} \left(\lambda^3 + |\beta|^2 + \|\epsilon\|_{H^1}^2 \right) d\sigma$$

$$\lesssim \lambda + \int_s^{+\infty} \lambda^2 d\sigma \lesssim |t| \lesssim \lambda$$
(6.221)

where we have used both (6.58) and (6.216). This yields the improved bound

$$|\alpha|^2+|\beta|^2+\|\epsilon\|_{H^1}^2\lesssim\lambda^3$$

This proves the pointwise bound (6.205) for α and β . It also proves the property about parameter μ by estimating the integral in definition (6.9).

Now let's turn to the proof of (6.206). Integrating the localized virial identity (6.162) between s and $+\infty$, using (6.215) we may estimate

$$\frac{b(s)}{\lambda(s)} - \frac{1}{C_0} \Big| \quad \lesssim \lambda + \int_s^{+\infty} \frac{1}{\lambda} \left(|\alpha|^2 + \|\epsilon\|_{L^2}^2 \right) d\sigma + \|\epsilon\|_{H^1} + \int_s^{+\infty} \lambda^3 \, d\sigma \lesssim \lambda$$

which proves (6.206). Then, (6.207) simply follows from (6.62), (6.205) and (6.206)

$$\mathcal{O}(\lambda^3) = \frac{\lambda_s}{\lambda} + b = \lambda \left(\lambda_t + \frac{b}{\lambda}\right) = \lambda \left(\lambda_t + \frac{1}{C_0} + \mathcal{O}(\lambda)\right)$$

and thus $\lambda_t + \frac{1}{C_0} = \mathcal{O}(|t|)$ (6.222)

which proves (6.207) by integration in time. Eventually, using also (6.62), (6.205) this implies

$$\begin{aligned} \left| \frac{d}{dt} \left(\gamma + \frac{C_0^2}{t} \right) \right| &= \frac{1}{\lambda^2} \left| \gamma_s - \frac{C_0^2 \lambda^2}{t^2} \right| = \frac{1}{\lambda^2} \left| \tilde{\gamma}_s - \left(\frac{C_0^2 \lambda^2}{t^2} - 1 \right) \right| \\ &= \frac{1}{\lambda^2} \left(|\beta|^2 + \mathcal{O}\left(|t|^2 + |\alpha|^2 + |Mod(t)| \right) \right) \lesssim 1 \end{aligned}$$

which finally proves (6.208).

Now to prove the last refinement of (6.205), we use the result of Lemma 6.9, proving estimate (6.116) with use of the bound we have just proved $\|\epsilon\|_{H^1} \leq \lambda^{3/2}$ and the refined bound (6.59)

$$\|\epsilon\|_{L^2} \lesssim \lambda^4 + |\alpha|^2 + \mu \lesssim \lambda^3$$

Then, since we have just proved $\frac{b}{\lambda^4} = \frac{1}{C_0\lambda^3} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \gtrsim \frac{1}{\lambda^2}$, recall (6.116) proves

$$\frac{d\mathcal{I}}{dt} \ge \frac{b}{\lambda^4} \|\tilde{u}\|_{L^2(dg_x)}^2 + \mathcal{O}\left(\lambda^3 + \|\tilde{u}\|_{H^1(dg_x)}^2\right) \gtrsim \mathcal{O}\left(\lambda^3 + \frac{\|u\|_{L^2(dg_x)}^2}{\lambda^2} + \|\tilde{u}\|_{H^1(dg_x)}^2\right)$$

using also (6.117) to get $\lim_{t\to 0} \mathcal{I}(t) = 0$, integrating $\frac{d\mathcal{I}}{dt}$ between t and 0 yields

$$\mathcal{I}(t) \lesssim \int_{t}^{0} \left(\frac{\|\tilde{u}(\tau)\|_{L^{2}(dg_{x})}^{2}}{\lambda^{2}(\tau)} + \lambda^{3}(\tau) + \|\tilde{u}(\tau)\|_{H^{1}(dg_{x})}^{2} \right) d\tau \lesssim \lambda^{4}(t) + \int_{t}^{0} \left(\frac{\|\tilde{u}(\tau)\|_{L^{2}(dg_{x})}^{2}}{\lambda^{2}(\tau)} + \|\tilde{u}(\tau)\|_{H^{1}(dg_{x})}^{2} \right) d\tau$$

Now using again Lemma 6.6 and (6.59) in the same way that we did to obtain (6.118), one gets

$$\|\nabla \tilde{u}(t)\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} \lesssim \lambda^{4}(t) + \int_{t}^{0} \left(\frac{\|\tilde{u}(\tau)\|_{L^{2}(dg_{x})}^{2}}{\lambda^{2}(\tau)} + \|\tilde{u}(\tau)\|_{H^{1}(dg_{x})}^{2}\right) d\tau$$

From Gronwall Lemma, one may eventually deduce the refined bound

$$\|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{\|\tilde{u}(t)\|_{L^2}^2}{\lambda^2(t)} \lesssim \lambda^4(t)$$

This concludes the proof of Proposition 6.18.

Remark 20. We have finally proved the \mathcal{P} parameters are not of same orders, and that compared to λ , α and β are of order 2, b is of order 1 - that is $b \sim \lambda$ - and $\tilde{\gamma}$ is of order -1.

This also completes the proof of (6.8), and therefore justifies the use we have previously made of it.

6.8 Uniqueness

First of all, we will need to restrain our choice of the metric g as we need some refined estimates about the remaining tail element ϵ of the profile $\tilde{Q}_{\mathcal{P}}$ we have built a NLS solution with, along with refinements about the geometrical parameters we have used, so the uniqueness is ensured following the procedure that was already used in [RS11].

6.8.1 Refined estimates under stronger assumption

From here, we claim that whenever assumption (H4) holds, the bound (6.8) we have used from the start can be improved to

$$|\alpha| + |\beta| \lesssim \lambda^2 \tag{6.223}$$

Therefore the computation of the approximate profile is simplified, since many terms we had to deal with are now negligible : all $\mathcal{O}(\mathcal{P}(|\alpha|^2 + |\beta|^2 + |\alpha| |\beta|))$ disappear in the $\psi_{\mathcal{P}}$ remaining terms, so that we no longer need to introduce parameter μ at all, while $\mathcal{O}(\mathcal{P}^2(|\alpha| + |\beta|))$ terms are hidden in the fourth orders approximation terms. The β law is not noticeably changed, while one should notice now that

$$B_{1} = -\frac{\kappa_{0}}{6} |\alpha|^{2} K_{1} + \mathcal{O}(\lambda^{4}), \quad B_{2} = \lambda L_{0}(\alpha) + \lambda^{3} C_{3}, \quad \text{with} :$$

$$K_{1} = \|Q\|_{L^{2}}^{2} \left\|\frac{|y|}{2} Q\right\|_{L^{2}}^{-2}$$

$$L_{0}(\alpha) = \nabla^{2} k(0).(\alpha, .), \quad C_{3} = \|Q\|_{L^{2}}^{-2} \nabla \kappa(0) \int \rho^{2} |\nabla Q|^{2} dy,$$
and moreover $(L_{+}T_{2}, \Lambda Q) = (L_{+}T_{3}, \Lambda Q) = 0$

$$(6.224)$$

In particular, the conservation of mass for profile $\tilde{Q}_{\mathcal{P}}$, and the computation of the \mathcal{P} laws are improved in the following way

$$\mathcal{R}e\int\epsilon \ \overline{\tilde{Q}_{\mathcal{P}}} \ \frac{\rho}{\varphi}\sqrt{|g|} \ dy + \frac{1}{2}\int|\epsilon|^2 \ dy - \frac{\kappa_0}{12} \ |\alpha|^2 \int Q^2 \ dy = \mathcal{O}\left(\mathcal{P}^4 + |\alpha|^3 + \mathcal{P}^3 \ \|\epsilon\|_{L^2}\right)$$

so that $\left|\left(\epsilon_1, Q\right)\right| \lesssim \lambda^4$ and $K_b = \frac{\kappa_0}{6} \ |\alpha|^2 \ \|Q\|_{L^2}^2 \ \left\|\frac{\rho}{2} \ Q\right\|_{L^2}^{-2}$ (6.225)

The last two bounds prove $\frac{1}{\lambda}(B_1 + K_b) = \mathcal{O}(\lambda^3)$ and may then be used to improve the backward propagation of smallness Lemma 6.10 conclusions that become

$$\|\nabla \tilde{u}(t)\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t)\|_{L^{2}}^{2}}{\lambda^{2}(t)} \lesssim \|\nabla \tilde{u}(t_{1})\|_{L^{2}}^{2} + \frac{\|\tilde{u}(t_{1})\|_{L^{2}}^{2}}{\lambda^{2}(t_{1})} + \lambda^{6}(t)$$
(6.226)

$$\left|\frac{b}{\lambda}(t) - \frac{1}{C_0}\right| \lesssim \lambda^2(t), \quad \left|\lambda(t) + \frac{t}{C_0}\right| \lesssim \lambda^3(t)$$
(6.227)

$$\left|\frac{\alpha}{\lambda}(t)\right| \lesssim \lambda(t), \quad \left|\frac{\beta}{\lambda}(t)\right| \lesssim \lambda(t)$$
(6.228)

notice also the following refinement that allows one to get a λ^6 remaining term in (6.226) and is a consequence of the gain (6.223) on the estimates of α and β :

$$\frac{d\mathcal{I}}{dt} \ge \frac{b}{\lambda^4} \int |\tilde{u}|^2 \, dg_x + \mathcal{O}\left(K^4 \, \lambda^5 + \|\tilde{u}\|_{H^1}^2\right) \tag{6.229}$$

Having done all this, the Proposition 6.11 that proves existence of a NLS solution with profile $\tilde{Q}_{\mathcal{P}_c}$ and tail remaining term ϵ_c may then be improved with the conclusions

$$\|\tilde{u}_{c}\|_{L^{2}} \lesssim \lambda_{c}^{4}, \quad \|\tilde{u}_{c}\|_{H^{1}} \lesssim \lambda_{c}^{3}, \quad \|\tilde{u}_{c}\|_{H^{3/2}} \lesssim \lambda_{c}^{3/2}, \quad \lambda_{c}(t) + \frac{t}{C_{0}} = \mathcal{O}\left(\lambda_{c}^{3}(t)\right)$$

$$\frac{b_{c}}{\lambda_{c}}(t) - \frac{1}{C_{0}} = \mathcal{O}\left(\lambda_{c}^{2}(t)\right), \quad |\alpha_{c}| + |\beta_{c}| \lesssim \lambda_{c}^{2}, \quad \gamma_{c} = -\frac{C_{0}^{2}}{t} + \gamma_{0} + \mathcal{O}(\lambda_{c})$$

$$(6.230)$$

Then there is only left to go back to part 6.7 to 6.7.6 and see what refinements there are. Nothing changes really before 6.7.6 and the third step of the proof of Proposition 6.18 in which we actually get the improved bounds. Now rewrite (6.220) and (6.221) with new improved bounds, knowing there is no longer the $l_E(\alpha)$ which was dismissed earlier for being of higher order and that $K_2 = 0$, one has

$$\frac{|\beta|^2}{\lambda^2} + \frac{|\alpha|^2}{\lambda^2} + \frac{\|\epsilon\|^2_{H^1}}{\lambda^2} + \int_s^{+\infty} \frac{1}{\lambda} \left(|\alpha|^2 + \|\epsilon\|^2_{L^2} \right) d\sigma$$

$$\lesssim \lambda^2 + \int_s^{+\infty} \left(\lambda^3 + |\beta|^2 + \|\epsilon\|^2_{H^1} \right) d\sigma \lesssim \lambda^2$$
(6.231)

so we actually prove any solution built with geometric decomposition (6.136), using the profile $\hat{Q}_{\mathcal{P}}$ that we have constructed in the first sections and with parameters solutions to the laws described initially in (6.18) and then verifying (6.62) and (6.63) are also satisfying (6.230) as proven in Proposition 6.18

$$|\alpha|^2 + |\beta|^2 + \|\epsilon\|_{H^1} \lesssim \lambda^4, \tag{6.232}$$

$$b(t) = -\frac{t}{C_0^2} + \mathcal{O}(|t|^3), \quad \text{for } t_0 \le t \le 0,$$
(6.233)

$$\lambda(t) = -\frac{t}{C_0} + \mathcal{O}(|t|^3), \quad \text{for } t_0 \le t \le 0.$$
(6.234)

Moreover, there exists $\gamma_0 \in \mathbb{R}$ such that :

$$\gamma(t) = -\frac{C_0^2}{t} + \gamma_0 + \mathcal{O}(|t|) \tag{6.235}$$

Eventually, the backward propagation of smallness lemma which conclusions were improved in (6.226)-(6.228) now yields improved bounds on the ϵ remaining term as we claim the previous improvements yield

$$\|\tilde{u}(t)\|_{L^2} \lesssim \lambda^4(t), \quad \|\nabla\tilde{u}(t)\|_{L^2} \lesssim \lambda^3(t) \quad \text{or} \quad \|\epsilon(t)\|_{H^1} \lesssim \lambda^4(t) \tag{6.236}$$

To prove it, first decompose u according to the geometrical decomposition (6.38). Then, let $t_n \to 0$ be an increasing sequence of times. From previous improvements and bound (6.59), one has then $\|\epsilon\|_{L^2} \leq \lambda^4$ hence $\|\tilde{u}\|_{L^2(dg_x)} \leq \lambda^4$. Thanks to Proposition 6.18, it is now clear that assumptions (6.105)-(6.107) modified as above, of Lemma 6.10 are satisfied at any t_n . In particular, there is a time $t_0 < 0$ such that we have (6.109) for any $t_0 \leq t \leq t_n$:

$$\|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{\|\tilde{u}(t)\|_{L^2}^2}{\lambda^2(t)} \lesssim \|\nabla \tilde{u}(t_n)\|_{L^2}^2 + \frac{\|\tilde{u}(t_n)\|_{L^2}^2}{\lambda^2(t_n)} + \lambda^6(t)$$
(6.237)

Finally, let $n \to +\infty$ and (6.232) yields (6.236).

We then look forward to proving the solution we built in Proposition 6.11 is actually the only one blowing up at T = 0, $\alpha^* = 0$, with energy $E_0 > -C_E$ and phase parameter γ_0 given by (6.235). In the sequel, we will call u_c that solution given by Proposition 6.11.

So far, we have proved $b \sim \lambda$ which provides the exact blow up speed, and we have obtained the dispersive behavior for the remains \tilde{u} of the approximation

$$\tilde{u} \to 0$$
, in H^1 as $t \to 0$

We now need to prove that given u a solution of (6.1), we have $u = u_c$. Again, as in [RS11], we will proceed in two steps.

First we will show the refined estimates of Proposition 6.18 together with the Backward Propagation of Smallness achieved in Lemma 6.10 imply the strong H^1 convergence

$$u_c - u \to 0$$
, in H^1 as $t \to 0$

Then, we will have to show the estimates obtained for $u - u_c$ will be strong enough to treat pertubatively the growth induced by the null space of $L^{[g]} = (-L_g + 1 - 3Q^2, -L_g + 1 - Q^2)$, with $L_g = \Delta_g + V$, when linearizing the equation close to u_c and running the estimates of Lemma 6.9.

Both these steps will make clear why the construction of an approximate solutions with an error term of at most $\mathcal{O}(\lambda^5)$ order was needed.

6.8.2 H^1 convergence to the critical element.

We claim the following dispersive property which somehow uses all previous estimates on the solution and is a key result for the proof of uniqueness.

Lemma 6.19. (H^1 convergence to the critical element)

There holds the strong convergence at blow up time :

$$u - u_c \to 0, \quad in \quad H^1 \quad as \quad t \to 0$$

$$(6.238)$$

More precisely

$$\left\|\nabla(u-u_c)\right\|_{L^2} + \frac{\|u-u_c\|_{L^2}}{|t|} \lesssim |t|^3 \quad as \quad t \to 0$$
(6.239)

Proof of Lemma 6.19

Step 1 Comparison between the modulation parameters of u and u_c .

Let $b, \lambda, \alpha, \beta, \gamma$ denote the modulation parameters of u and $b, \lambda_c, \alpha_c, \beta_c, \gamma_c$ denote the modulation parameters of u_c . We claim

$$\frac{|\lambda - \lambda_c|}{|t|} + |b - b_c| + \frac{|\alpha - \alpha_c|}{|t|} + |\beta - \beta_c| + |\gamma - \gamma_c| \lesssim |t|^4$$

$$(6.240)$$

The proof of (6.240) is rather technical, and will be done in Appendix 6.11 according to what is done in Appendix C of [RS11].

Step 2 Comparison between u and u_c .

(6.239) is now a simple consequence of (6.236) and (6.240). Indeed

$$\frac{\|u - u_c\|_{L^2}}{|t|} + \|\nabla(u - u_c)\|_{L^2} \lesssim \frac{\|\tilde{u}\|_{L^2}}{|t|} + \|\nabla\tilde{u}\|_{L^2} + \frac{\|\tilde{u}_c\|_{L^2}}{|t|} + \|\nabla\tilde{u}_c\|_{L^2} + \frac{\|\tilde{Q} - \tilde{Q}_c\|_{L^2}}{|t|} + \|\nabla(\tilde{Q} - \tilde{Q}_c)\|_{L^2}$$

Thus, thanks to (6.240) a simple computation yields

 $\|\tilde{Q} - \tilde{Q}_c\|_{L^2} \lesssim |t|^4, \quad \|\nabla(\tilde{Q} - \tilde{Q}_c)\|_{L^2} \lesssim |t|^3$

and (6.230), (6.236) then imply (6.239) and concludes the proof of Lemma 6.19.

6.9 Energy estimates for the flow near u_c .

Let us now decompose :

$$u = u_c + \tilde{\tilde{u}}, \quad \tilde{\tilde{u}}(t, x) = \left(\frac{\rho}{\varphi_c(\lambda_c, \alpha_c, y)}\right)^{\frac{1}{2}} \frac{1}{k(\alpha_c(t))^{1/2} \lambda_c(t)} \epsilon\left(t, \frac{x - \alpha_c(t)}{\lambda_c(t)}\right) e^{i\gamma_c(t)}$$
(6.241)

Here there is no further orthogonality conditions on ϵ , neither is there modulations equations on the parameters. Indeed, there is no uniform well localized bounds on the \tilde{u}_c part of u_c , and the only control there is on the $\partial_t \tilde{u}_c$ part is given by the $H^{3/2}$ bound (6.230). However, we have from (6.230) and (6.239) that

$$\left\|\tilde{\tilde{u}}\right\|_{L^2} \lesssim \lambda^4, \quad \left\|\tilde{\tilde{u}}\right\|_{H^1} \lesssim \lambda^3 \tag{6.242}$$

and that this is sharp enough a bound to treat pertubatively the instability generated by the null space of $(-\mathcal{L}_g + 1 - 3Q^2, -\mathcal{L}_g + 1 - Q^2)$. Let

$$N(t) := \sup_{t < \tau < 0} \left(\left\| \tilde{\tilde{u}}(\tau) \right\|_{H^1}^2 + \frac{\left\| \tilde{\tilde{u}}(\tau) \right\|_{L^2}^2}{\lambda_c^2(\tau)} \right)$$
(6.243)

and

$$Scal(t) := \left(\epsilon_1, Q\right)^2 + \left(\epsilon_2, \Lambda Q\right)^2 + \left(\epsilon_1, |y|^2 Q\right)^2 + \left(\epsilon_2, \varrho\right)^2 + \left(\epsilon_1, y Q\right)^2 + \left(\epsilon_2, \nabla Q\right)^2$$

$$(6.244)$$

We first claim the following energy bound :

Lemma 6.20. There holds for t close enough to 0:

$$N(t) \lesssim \sup_{t < \tau < 0} \frac{Scal(\tau)}{\lambda_c^2(\tau)} + \int_t^0 \frac{Scal(\tau)}{\lambda_c^3(\tau)} d\tau$$
(6.245)

Proof of Lemma 6.20

It is a consequence of the energy estimate (6.99) together with the a priori bound (6.230).

Step 1 Application of Lemma 6.9.

Let $\mathcal{I}(\tilde{\tilde{u}})$ be given by (6.98), we claim that

$$\frac{\left\|\tilde{\tilde{u}}\right\|_{L^{2}}^{2}}{\lambda_{c}^{3}} + \mathcal{O}\left(N(t) + \frac{Scal(t)}{\lambda_{c}^{3}}\right) \lesssim \frac{d\mathcal{I}}{dt}$$
(6.246)

We are willing to use Lemma 6.9 with $w = u_c = (u_c)_1 + i (u_c)_2$. Note that the bounds (6.92) hold from the proof of existence of u_c in Proposition 6.11, which was refined to the bounds (6.230). Moreover ψ given by (6.91) is now identically zero, since u_c is an exact solution. Hence (6.99) becomes

$$\frac{d\mathcal{I}}{dt} = -\frac{1}{\lambda_c^2} \mathcal{I}m \int k(x) u_c^2 \overline{\tilde{u}^2} \, dg_x - \mathcal{R}e \int k(x) \partial_t u_c \overline{(2 |\tilde{\tilde{u}}|^2 u_c + \tilde{\tilde{u}}^2 \overline{u_c})} \, dg_x \\
+ \frac{b_c}{\lambda_c^2} \left[\int \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} \, dg_x + \mathcal{R}e \int \nabla_g^2 \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) . \left(\nabla \tilde{\tilde{u}}, \nabla \overline{\tilde{\tilde{u}}}\right) \, dg_x - \frac{1}{4A^2} \int \Delta_g^2 \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) \frac{|\tilde{\tilde{u}}|^2}{\lambda^2} \, dg_x \\
+ \lambda_c \mathcal{R}e \int A g^{ij} \, \partial_i \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) k(x) \left(2 \, u_c \, |\tilde{\tilde{u}}|^2 + \overline{u_c} \, \tilde{\tilde{u}}^2\right) \partial_j \overline{u_c} \, dg_x \right] + \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2\right) \tag{6.247}$$

Now we consider the first two terms in the right-hand side of (6.247) and expand $u_c = \tilde{Q}_c + \tilde{u}_c$, with $\tilde{Q}_c = \tilde{\Sigma}_c + i \tilde{\Theta}_c$

$$-\frac{1}{\lambda_c^2} \mathcal{I}m \int k(x) u_c^2 \overline{\tilde{u}^2} \, dg_x - \mathcal{R}e \int k(x) \partial_t u_c \overline{(2 \, |\tilde{\tilde{u}}|^2 \, u_c + \tilde{\tilde{u}}^2 \, \overline{u_c})} \, dg_x$$

$$= -\frac{1}{\lambda_c^2} \mathcal{I}m \int k(x) \tilde{Q}_c^2 \overline{\tilde{u}}^2 \, dg_x - \mathcal{R}e \int k(x) \partial_t \tilde{Q}_c \overline{(2 \, |\tilde{\tilde{u}}|^2 \, \tilde{Q}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{Q}_c})} \, dg_x$$

$$-\frac{1}{\lambda_c^2} \mathcal{I}m \int k(x) \left(2 \, \tilde{u}_c \, \tilde{Q}_c + \tilde{u}_c^2\right) \overline{\tilde{u}}^2 \, dg_x - \mathcal{R}e \int k(x) \partial_t \tilde{Q}_c \overline{(2 \, |\tilde{\tilde{u}}|^2 \, \tilde{u}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{u}_c})} \, dg_x$$

$$-\mathcal{R}e \int k(x) \partial_t \tilde{u}_c \overline{(2 \, |\tilde{\tilde{u}}|^2 \, \tilde{Q}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{Q}_c})} \, dg_x - \mathcal{R}e \int k(x) \partial_t \tilde{u}_c \overline{(2 \, |\tilde{\tilde{u}}|^2 \, \tilde{u}_c + \tilde{\tilde{u}}^2 \, \overline{\tilde{u}_c})} \, dg_x$$
(6.248)

Now, we may rewrite the first two terms in the right-hand side of (6.248) the same way that led to (6.129)

$$-\frac{1}{\lambda_c^2} \mathcal{I}m \int k(x) \tilde{Q}_c^2 \bar{\tilde{u}}^2 dg_x - \mathcal{R}e \int k(x) \partial_t \tilde{Q}_c \overline{(2 |\tilde{\tilde{u}}|^2 \tilde{Q}_c + \tilde{\tilde{u}}^2 \overline{\tilde{Q}_c})} dg_x$$

$$= -\frac{b_c}{\lambda_c^2} \int k(x) \left[\left(|\tilde{Q}_c|^2 + 2\tilde{\Sigma}_c^2 \right) \tilde{\tilde{u}}_1^2 + 4\tilde{\Sigma}_c \tilde{\Theta}_c \tilde{\tilde{u}}_1 \tilde{\tilde{u}}_2 + \left(|\tilde{Q}_c|^2 + 2\tilde{\Theta}_c^2 \right) \tilde{\tilde{u}}_2^2 \right] dg_x$$

$$- \frac{b_c}{\lambda_c} \mathcal{R}e \int \left(\frac{x - \alpha_c}{\lambda_c} \right) k(x) \left(2 |\tilde{\tilde{u}}|^2 \tilde{Q}_c + \tilde{\tilde{u}}^2 \overline{\tilde{Q}_c} \right) \cdot \overline{\nabla \tilde{Q}_c} dg_x + \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2 \right)$$
(6.249)

We then treat the next two terms in the right-hand side of (6.248) by Hölder estimates and (6.230) to get

$$-\frac{1}{\lambda_{c}^{2}} \mathcal{I}m \int k(x) \left(2 \,\tilde{u}_{c} \,\tilde{Q}_{c} + \tilde{u}_{c}^{2} \right) \overline{\tilde{u}}^{2} \, dg_{x} - \mathcal{R}e \int k(x) \,\partial_{t} \tilde{Q}_{c} \,\overline{\left(2 \,|\tilde{\tilde{u}}|^{2} \,\tilde{u}_{c} + \tilde{\tilde{u}}^{2} \,\overline{\tilde{u}_{c}} \right)} \, dg_{x} \bigg|$$

$$\lesssim \frac{1}{\lambda_{c}^{2}} \|\tilde{Q}_{c}\|_{L^{\infty}} \|\tilde{u}_{c}\|_{L^{2}} \|\tilde{\tilde{u}}\|_{L^{4}}^{2} + \frac{1}{\lambda_{c}^{2}} \|\tilde{u}_{c}\|_{L^{4}}^{2} \|\tilde{\tilde{u}}\|_{L^{4}}^{2} + \|\partial_{t} \tilde{Q}_{c}\|_{L^{\infty}} \|\tilde{u}_{c}\|_{L^{2}} \|\tilde{\tilde{u}}\|_{L^{4}}^{2}$$

$$\lesssim \frac{\|\tilde{\tilde{u}}\|_{L^{2}}^{2}}{\lambda_{c}^{2}} + \|\tilde{\tilde{u}}\|_{H^{1}}^{2}$$

$$(6.250)$$

Indeed, recall as in computation (6.127)

$$\partial_t \tilde{Q}_c = -\frac{(\alpha_c)_t \cdot \nabla k(\alpha_c)}{2 k(\alpha_c)} \tilde{Q}_c - \frac{(\lambda_c)_t}{\lambda_c} \tilde{Q}_c - \left(\frac{\rho}{\varphi_c}\right)^{\frac{1}{2}} \frac{\lambda_c (\alpha_c)_t + (\lambda_c)_t (x - \alpha_c)}{k(\alpha_c)^{1/2} \lambda_c^3} \cdot \nabla Q_{\mathcal{P}_c} \left(\frac{x - \alpha_c}{\lambda_c}\right) e^{i \gamma_c} + i (\gamma_c)_t \tilde{Q}_c + \left(\frac{\rho}{\varphi_c}\right)^{\frac{1}{2}} \frac{1}{k(\alpha_c)^{1/2} \lambda_c} (\mathcal{P}_c)_t \frac{\partial \tilde{Q}_{\mathcal{P}_c}}{\partial \mathcal{P}_c} \left(\frac{x - \alpha_c}{\lambda_c}\right) e^{i \gamma_c} - \frac{\partial_t \varphi_c}{2\varphi_c} \tilde{Q}_c$$

$$(6.251)$$

which is to be estimated only with bounds (6.230), and one may check the worst term is generated by $\frac{(\gamma_c)_t}{\lambda_c}$ in the last term so that

$$\|\partial_t \tilde{Q}_c\|_{L^{\infty}} \lesssim \frac{1}{\lambda_c^3}$$

and so one gets (6.250).

To deal with the last two terms in the right-hand side of (6.248), we need to focus a bit more on the equation satisfied by \tilde{u}_c and its remainings

$$i \partial_t \tilde{u}_c = -L_g \tilde{u}_c - k(x) \left(|u_c|^2 u_c - |\tilde{Q}_c|^2 \tilde{Q}_c \right) - \psi_c
\tilde{\psi}_c = i \partial_t \tilde{Q}_c + L_g \tilde{Q}_c + k(x) |\tilde{Q}_c|^2 \tilde{Q}_c$$
(6.252)

Expanding this term like we did with (6.131), then using (6.62) and (6.230)

$$\|\tilde{\psi}_c\|_{L^2} \lesssim \frac{Mod_c(t) + |\alpha_c|^2 + \lambda_c^5}{\lambda_c^2} \lesssim \lambda_c^2$$

Now, using this together with (6.252), integration by parts, Hölder estimates and the $H^{3/2}$ bound in (6.230) yields

$$\left| -\mathcal{R}e \int k(x) \partial_{t}\tilde{u}_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} \tilde{Q}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{Q}_{c}})} \, dg_{x} - \mathcal{R}e \int k(x) \partial_{t}\tilde{u}_{c} \overline{(2 |\tilde{\tilde{u}}|^{2} \tilde{u}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{u}_{c}})} \, dg_{x} \right|$$

$$\leq \|\tilde{u}_{c}\|_{H^{3/2}} \left[\|2 |\tilde{\tilde{u}}|^{2} \tilde{Q}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{Q}_{c}}\|_{H^{1/2}} + \|2 |\tilde{\tilde{u}}|^{2} \tilde{u}_{c} + \tilde{\tilde{u}}^{2} \overline{\tilde{u}_{c}}\|_{H^{1/2}} \right]$$

$$+ \|k(x) \left(|u_{c}|^{2} u_{c} - |\tilde{Q}_{c}|^{2} \tilde{Q}_{c} \right) + \tilde{\psi}_{c} \|_{L^{2}} \left(1 + \|V\|_{L^{\infty}} \right) \left[\|\tilde{Q}_{c}\|_{L^{\infty}} \|\tilde{\tilde{u}}\|_{L^{4}}^{2} + \|\tilde{\tilde{u}}_{c}\|_{L^{6}} \|\tilde{\tilde{u}}\|_{L^{6}}^{2} \right]$$

$$\leq \frac{\|\tilde{\tilde{u}}\|_{L^{2}}^{2}}{\lambda_{c}^{2}} + \|\tilde{\tilde{u}}\|_{H^{1}}^{2}$$

$$(6.253)$$

There is but one last term in the right-hand side of (6.247) we need to compute

$$\mathcal{R}e \int A g^{ij}(x) \,\partial_i \phi\left(\frac{x-\alpha_c}{A\lambda_c}\right) k(x) \left(2 \left|\tilde{\tilde{u}}\right|^2 u_c + \tilde{\tilde{u}}^2 \overline{u_c}\right) \overline{\partial_j u_c} \, dg_x \\ = \mathcal{R}e \int A g^{ij}(x) \,\partial_i \phi\left(\frac{x-\alpha_c}{A\lambda_c}\right) k(x) \left(2 \left|\tilde{\tilde{u}}\right|^2 \tilde{Q}_c + \tilde{\tilde{u}}^2 \overline{\tilde{Q}_c}\right) \overline{\partial_j \tilde{Q}_c} \, dg_x + Error$$
(6.254)

where the remaining Error term may be handled with Hölder estimates and (6.230)

$$\begin{aligned} \left| Error \right| &\lesssim \left| \mathcal{R}e \int A g^{ij}(x) \partial_i \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) k(x) \left(2 \left| \tilde{\tilde{u}} \right|^2 \tilde{u}_c + \tilde{\tilde{u}}^2 \overline{\tilde{u}_c} \right) \overline{\partial_j \tilde{Q}_c} \, dg_x \right| \\ &+ \left| \mathcal{R}e \int A g^{ij}(x) \partial_i \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) k(x) \left(2 \left| \tilde{\tilde{u}} \right|^2 \tilde{Q}_c + \tilde{\tilde{u}}^2 \overline{\tilde{Q}_c} \right) \overline{\partial_j \tilde{u}_c} \, dg_x \right| \\ &+ \left| \mathcal{R}e \int A g^{ij}(x) \partial_i \phi \left(\frac{x - \alpha_c}{A \lambda_c} \right) k(x) \left(2 \left| \tilde{\tilde{u}} \right|^2 \tilde{u}_c + \tilde{\tilde{u}}^2 \overline{\tilde{u}_c} \right) \overline{\partial_j \tilde{u}_c} \, dg_x \right| \\ &\lesssim \left(\| \nabla \tilde{Q}_c \|_{L^\infty} \| \tilde{u}_c \|_{L^2} + \| \tilde{Q}_c \|_{L^2} \| \nabla \tilde{u}_c \|_{L^2} \right) \| \tilde{\tilde{u}} \|_{L^4}^2 + \| \tilde{u}_c \|_{L^6} \| \nabla \tilde{u}_c \|_{L^2} \| \tilde{\tilde{u}} \|_{L^6}^2 \\ &\lesssim \frac{\| \tilde{\tilde{u}} \|_{L^2}^2}{\lambda_c^2} + \| \tilde{\tilde{u}} \|_{H^1}^2 \end{aligned}$$

$$(6.255)$$

Eventually, summing everything up, we obtain the following estimate

$$\frac{d\mathcal{I}}{dt} = \frac{b_c}{\lambda_c^2} \left[\int \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} dg_x + \mathcal{R}e \int \nabla_g^2 \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) \cdot \left(\nabla \tilde{\tilde{u}}, \nabla \overline{\tilde{\tilde{u}}}\right) dg_x - \frac{1}{4A^2} \int \Delta_g^2 \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) \frac{|\tilde{\tilde{u}}|^2}{\lambda^2} dg_x + \lambda_c \mathcal{R}e \int \left\{ A g^{ij} \partial_i \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) - \left(\frac{x - \alpha_c}{\lambda_c}\right)_j \right\} k(x) \left(2 u_c |\tilde{u}|^2 + \overline{u_c} \, \tilde{\tilde{u}}^2\right) \partial_j \overline{u_c} dg_x - \int k(x) \left[\left(|\tilde{Q}_c|^2 + 2 \, \tilde{\Sigma}_c^2\right) \tilde{\tilde{u}}_1^2 + 4 \, \tilde{\Sigma}_c \, \tilde{\Theta}_c \, \tilde{\tilde{u}}_1 \, \tilde{\tilde{u}}_2 + \left(|\tilde{Q}_c|^2 + 2 \, \tilde{\Theta}_c^2\right) \, \tilde{\tilde{u}}_2^2 \right] dg_x \right] + \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2\right) \tag{6.256}$$

changing variables with (6.241), and recall

$$\nabla u = \left(\frac{\rho}{\varphi_c}\right)^{\frac{1}{2}} \frac{1}{k(\alpha_c)^{\frac{1}{2}}} \frac{1}{\lambda_c^2} \left\{ \nabla + \frac{\lambda_c}{2} \Phi_c \right\} \left(\tilde{Q}_{\mathcal{P}_c} + \epsilon\right) \left(t, \frac{x - \alpha_c}{\lambda_c}\right) e^{i\gamma_c}, \quad \Phi_c = \frac{y}{\rho^2} - \frac{\nabla\varphi_c}{\varphi_c} = \mathcal{O}(\lambda_c^2)$$

(6.256) may be rewritten in the same way as we have already used in step 3 of the proof of the backward propagation of smallness Lemma 6.10

$$\begin{split} \frac{d\mathcal{I}}{dt} &= \frac{1}{k(\alpha_c)} \frac{b_c}{\lambda_c^4} \left[\int \left(1 - \frac{\Delta_g^2 \phi\left(\frac{y}{A}\right)}{4A^2} \right) |\epsilon|^2 \frac{\rho}{\varphi_c} \, dg_y + \mathcal{R}e \int \nabla_g^2 \phi\left(\frac{y}{A}\right) \cdot \left(\nabla \epsilon + \frac{\lambda_c}{2} \, \Phi_c \, \epsilon \,, \, \overline{\nabla \epsilon + \frac{\lambda_c}{2} \, \Phi_c \, \epsilon} \right) \frac{\rho}{\varphi_c} \, dg_y \\ &- \int \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \left[\left(|\tilde{Q}_{\mathcal{P}_c}|^2 + 2\Sigma_c^2 \right) \epsilon_1^2 + 4\Sigma_c \, \Theta_c \, \epsilon_1 \, \epsilon_2 + \left(|\tilde{Q}_{\mathcal{P}_c}|^2 + 2\Theta_c^2 \right) \epsilon_2^2 \right] \frac{\rho}{\varphi_c} \, dg_y \\ &+ \mathcal{R}e \int \left(A \, g^{ij} \, \partial_i \phi\left(\frac{y}{A}\right) - y_j \right) \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \left(2 \, |\epsilon|^2 \, \tilde{Q}_{\mathcal{P}_c} + \epsilon^2 \, \overline{\tilde{Q}_{\mathcal{P}_c}} \right) \partial_j \overline{\tilde{Q}_{\mathcal{P}_c}} \frac{\rho}{\varphi_c} \, dg_y \\ &+ \mathcal{O}\left(\frac{\|\tilde{u}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{u}\|_{H^1}^2 \right) \end{split}$$

from (6.242) and since $\frac{\rho}{\varphi_c} dg_y = dy + \mathcal{O}(\lambda^3), \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} = 1 + \mathcal{O}(\lambda^2)$

$$\begin{aligned} \frac{d\mathcal{I}}{dt} &= \frac{b_c}{k(\alpha_c)\,\lambda_c^4} \left[\left(L_+\epsilon_1 \,,\,\epsilon_1 \right) + \left(L_-\epsilon_2 \,,\,\epsilon_2 \right) - \frac{1}{4A^2} \int \Delta_g^2 \phi\left(\frac{y}{A}\right) |\epsilon|^2 \,\,dy + \mathcal{R}e \int \left(\nabla_g^2 \phi\left(\frac{y}{A}\right) - Id \right) . \left(\nabla \epsilon \,,\,\overline{\nabla \epsilon} \right) \,\,dy \\ &+ \mathcal{R}e \,\int \left[2\,|\epsilon|^2 \,\tilde{Q}_{\mathcal{P}_c} + \epsilon^2 \,\overline{\tilde{Q}}_{\mathcal{P}_c} \right] \left\{ A\,g^{ij}(\lambda_c y + \alpha_c) \,\partial_i \phi\left(\frac{y}{A}\right) - y_j \right\} \partial_j \overline{\tilde{Q}}_{\mathcal{P}_c} \,\,dy \right] + \mathcal{O}\left(\frac{\|\tilde{u}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{u}\|_{H^1}^2 \right) \end{aligned}$$

so that from uniform closeness of $\tilde{Q}_{\mathcal{P}_c}$ and Q along with (6.96), the choice we have made for A > 0 so we could get some localized virial control in (6.167) and (6.168), and the coercivity property (6.71) of Lemma 6.6, that we may conclude as we did for the proof of (6.130) which may be refined with a $\mathcal{O}(K^4 \lambda^5)$ rest instead of $\mathcal{O}(K^4 \lambda^3)$ thanks to (6.225)

$$\frac{d\mathcal{I}}{dt} \gtrsim \frac{b_c}{\lambda_c^4} \left[\int |\epsilon|^2 \, dy + \int |\nabla\epsilon|^2 \, e^{-\frac{|y|}{\sqrt{A}}} \, dy + \mathcal{O}\left(Scal(t)\right) \right] + \mathcal{O}\left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2\right) \tag{6.257}$$

This concludes the proof of (6.246).

Step 2 Coercivity of \mathcal{I} .

Recall from (6.98) the formula

$$\begin{split} \mathcal{I} &= \frac{1}{2} \, \int g^{ij}(x) \, \partial_i \tilde{\tilde{u}} \, \overline{\partial_j \tilde{\tilde{u}}} \, dg_x + \frac{1}{2} \, \int \frac{|\tilde{\tilde{u}}|^2}{\lambda_c^2} \, dg_x + \frac{b_c}{2\lambda_c} \int A \, g^{ij} \partial_i \phi \left(\frac{x - \alpha_c}{A\lambda_c}\right) \partial_j \tilde{\tilde{u}} \, \overline{\tilde{\tilde{u}}} \, dg_x \\ &- \frac{1}{4} \, \int k(x) \, |u_c + \tilde{\tilde{u}}|^4 \, dg_x + \frac{1}{4} \, \int k(x) \, |u_c|^4 \, dg_x + \int k(x) \, |u_c|^2 \left[(u_c)_1 \, \tilde{\tilde{u}}_1 + (u_c)_2 \, \tilde{\tilde{u}}_2 \right] \, dg_x \\ &- \frac{1}{2} \, \int V(x) \, |u_c + \tilde{\tilde{u}}|^2 \, dg_x + \frac{1}{2} \, \int V(x) \, |u_c|^4 \, dg_x + \int V(x) \left[(u_c)_1 \, \tilde{\tilde{u}}_1 + (u_c)_2 \, \tilde{\tilde{u}}_2 \right] \, dg_x \end{split}$$

As we did proving (6.246), by expanding $u_c = \tilde{Q}_c + \tilde{u}_c$ one may get thanks to (6.242) the rough upper bound

$$|\mathcal{I}| \lesssim \frac{\|\tilde{\tilde{u}}(t)\|_{L^2}^2}{\lambda_c^2(t)} + \|\tilde{\tilde{u}}(t)\|_{H^1}^2 \to 0 \quad \text{as} \quad t \to 0$$
(6.258)

Then, very similarly to the proof of (6.118), using also control of interaction terms such as those we handled in (6.250), (6.253) and (6.255), we may obtain the lower bound

$$\mathcal{I}(t) \geq \frac{1}{2\lambda_{c}^{2}} \left[\left(L_{+}\epsilon_{1}, \epsilon_{1} \right) + \left(L_{-}\epsilon_{2}, \epsilon_{2} \right) + o\left(\|\epsilon\|_{H^{1}}^{2} \right) \right] \geq \frac{c}{\lambda_{c}^{2}} \left[\|\epsilon\|_{L^{2}}^{2} + \|\nabla\epsilon\|_{L^{2}}^{2} - Scal(t) \right] \\
\geq \underline{C} \left[\frac{\|\tilde{\tilde{u}}\|_{L^{2}}^{2}}{\lambda_{c}^{2}} + \|\nabla\tilde{\tilde{u}}\|_{L^{2}}^{2} - \frac{Scal(t)}{\lambda_{c}^{2}} \right] \geq C \left[\frac{\|\tilde{\tilde{u}}\|_{L^{2}}^{2}}{\lambda_{c}^{2}} + \|\tilde{\tilde{u}}\|_{H^{1}}^{2} - \frac{Scal(t)}{\lambda_{c}^{2}} \right]$$
(6.259)

To finally get (6.245), we now integrate (6.246) between t and 0. Using (6.258) and (6.259) we have

$$\int_t^0 \frac{d\mathcal{I}}{d\tau} \, d\tau = \mathcal{I}(0) - \mathcal{I}(t) \lesssim \frac{Scal(t)}{\lambda_c^2} - \frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} - \|\tilde{\tilde{u}}\|_{H^1}^2$$

Hence (6.257) and (6.258) yield after change of variables

$$\begin{split} \frac{Scal(t)}{\lambda_{c}^{2}(t)} &- \frac{\|\tilde{\tilde{u}}(t)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(t)} - \|\tilde{\tilde{u}}(t)\|_{H^{1}}^{2} \gtrsim \int_{t}^{0} \frac{1}{\lambda_{c}} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^{1}}^{2} \right) d\tau + \mathcal{O}\bigg[\int_{t}^{0} \frac{Scal(\tau)}{\lambda_{c}^{3}(\tau)} d\tau \bigg] \\ &\gtrsim \int_{t}^{0} \frac{1}{\lambda_{c}} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^{1}}^{2} \right) d\tau - \int_{t}^{0} \frac{Scal(\tau)}{\lambda_{c}^{3}(\tau)} d\tau \end{split}$$

which involves

$$\sup_{t < \tau < 0} \left(\frac{\|\tilde{\tilde{u}}\|_{L^2}^2}{\lambda_c^2} + \|\tilde{\tilde{u}}\|_{H^1}^2 \right) + \int_t^0 \frac{1}{\lambda_c} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^2}^2}{\lambda_c^2(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^1}^2 \right) d\tau \lesssim \sup_{t < \tau < 0} \frac{Scal(\tau)}{\lambda_c^2(\tau)} + \int_t^0 \frac{Scal(\tau)}{\lambda_c^3(\tau)} d\tau$$

that eventually yields the wanted estimate (6.245), since from (6.230)

$$\int_{t}^{0} \frac{1}{\lambda_{c}} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^{1}}^{2} \right) d\tau \geq \frac{1}{|t|} \int_{t}^{0} \left(\frac{\|\tilde{\tilde{u}}(\tau)\|_{L^{2}}^{2}}{\lambda_{c}^{2}(\tau)} + \|\tilde{\tilde{u}}(\tau)\|_{H^{1}}^{2} \right) d\tau = \mathcal{O}(N(t))$$

This concludes the proof of Lemma 6.20.

6.10 Control of the scalar products and proof of Theorem 6.2

To achieve the unicity proof, there is now only left to control the possible growth of the scalar products terms involved in estimate (6.245). So, we claim

Lemma 6.21. (A priori control of the null space).

There holds for $t\ close\ enough\ to\ 0$

$$Scal(t) \lesssim |t|^{1/2} |t|^2 N(t)$$
 (6.260)

Let us assume Lemma 6.21 and estimate 6.260 are proved and conclude the proof of Theorem 6.2.

Proof of Theorem 6.2

From (6.245), (6.260) and the law $\lambda_c \sim |t|$ as shown in (6.230), we have for t close enough to 0

$$N(t) \lesssim |t|^{1/2} N(t) + \int_{t}^{0} \frac{N(\tau)}{\sqrt{|\tau|}} d\tau \lesssim |t|^{1/2} N(t)$$

and hence N(t) = 0 for t small enough. From definition (6.243) of N, this yields $u = u_c$ and achieves the proof of the Theorem (6.2).

Proof of Lemma 6.21

The proof follows by deriving the null space close to $\tilde{Q}_{\mathcal{P}_c}$ to sufficiently high order and reintegrating the corresponding modulation equations from blow up time. The worst behavior is on the even terms where the modulation equations are a deformation of the ones for L_+ , L_- , and will roughly correspond to the system of ODE 's:

$$\left(\epsilon_{2}, \Lambda Q\right)_{s} = 2\left(\epsilon_{1}, Q\right), \quad \left(\epsilon_{1}, |y|^{2} Q\right)_{s} = -4\left(\epsilon_{2}, \Lambda Q\right), \quad \left(\epsilon_{2}, \varrho\right)_{s} = -\left(\epsilon_{1}, |y|^{2} Q\right)$$

with initial degeneracy provided by the L^2 norm conservation law (6.65) and the a priori bound (6.242):

$$\left| \left(\epsilon_1 \,, \, Q \right) \right| \lesssim \int |\epsilon|^2 \lesssim \|\epsilon\|_{L^2} \,\lambda_c(t) \,\sqrt{N(t)} \lesssim \lambda_c^4(t) \,\lambda_c(t) \,\sqrt{N(t)}$$

The control of the worst paramater (related to the phase) requires

$$\begin{split} \left| \left(\epsilon_2(s) \,, \, \varrho \right) \right| &\lesssim \int_s^{+\infty} \int_{s_1}^{+\infty} \int_{s_2}^{+\infty} \left| \left(\epsilon_1(s_3) \,, \, Q \right) \right| \, ds_3 \, ds_2 \, ds_1 \\ &\lesssim \lambda_c(s) \, \sqrt{N(s)} \, \int_s^{+\infty} \int_{s_1}^{+\infty} \int_{s_2}^{+\infty} \lambda_c^4(s_3) \, ds_3 \, ds_2 \, ds_1 \\ &\lesssim \frac{\lambda_c(s) \, \sqrt{N(s)}}{s} \end{split}$$

This implies (6.260) and explains why we needed a small enough estimate $\|\epsilon\|_{L^2} \leq \lambda_c^4$ in (6.239), (6.242). Let us now implement the above strategy which requires being careful with respect to polynomial losses, and in particular demands a high order approximation of the null space close to $\tilde{Q}_{\mathcal{P}_c}$ for which we have been working hard computing the approximation up to the fourth order.

Step 1 Approximate equation in conformal variables to the order $\mathcal{O}(\lambda_c^4)$.

Let v, w be defined from solution u as in the ansatz we have used from the start

$$u(t,x) = \left(\frac{\rho}{\varphi_c}\right)^{\frac{1}{2}} \frac{1}{k(\alpha_c(t))^{\frac{1}{2}}} \frac{1}{\lambda_c(t)} v\left(t, \frac{x - \alpha_c(t)}{\lambda_c}\right) e^{i\gamma_c(t)} \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda_c^2}$$

$$w(s,y) = v(s,y) \ e^{i \ b_c \ \frac{|y|^2}{4} - i\beta_c \cdot y}$$
(6.261)

Then we may compute as we did to get equations (6.15), (6.14) and (6.16) and show that here again w is solution to
$$\begin{split} &i \partial_s w + L_{g_c} w - w + \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} |w|^2 w + \left((b_c)_s + b_c^2\right) \frac{\rho^2}{4} w \\ &- \left\{ \left((\beta_c)_s + b_c \beta_c\right) \cdot y + i \left[\lambda_c \beta_c \cdot \frac{\nabla k(\alpha_c)}{k(\alpha_c)} + \frac{\partial_s \varphi_c}{2 \varphi_c} + \frac{b_c}{2} \left(\frac{y \cdot \nabla \varphi_c}{\varphi_c} - 1\right) - \beta \cdot \left(\frac{\nabla \varphi_c}{\varphi_c} - \frac{y}{\rho^2}\right) \right] \right\} w \\ &- \left(\frac{(\lambda_c)_s}{\lambda_c} + b_c\right) \left[\left(b_c \frac{\rho^2}{2} - \beta_c \cdot y\right) w + i \left(\Lambda w - \frac{1}{2} \left(\frac{y \cdot \nabla \varphi_c}{\varphi_c} - 1\right)\right) \right] \\ &- \left(\frac{(\alpha_c)_s}{\lambda_c} - 2 \beta_c\right) \cdot \left[\left(b_c \frac{y}{2} - \beta\right) w + \frac{i}{2} \left(\frac{\lambda_c \nabla k(\alpha_c)}{k(\alpha_c)} - \left(\frac{\nabla \varphi_c}{\varphi_c} - \frac{y}{\rho^2}\right)\right) w \right] \\ &- \left((\tilde{\gamma}_c)_s - |\beta_c|^2\right) w + \left(\frac{(\beta_c \cdot y)^2}{\rho^2} - |\beta_c|^2\right) w = 0 \end{split}$$

with

$$L_{g_c} = \Delta + \lambda_c^2 \left(V(0) + \frac{\kappa_0}{3} \right) + \lambda_c^3 y \cdot \left[\nabla V(0) - \frac{\rho}{2} \nabla \kappa(0) \partial_\rho \right] + \lambda_c^3 R_\Delta^1(\partial_\omega, \partial_{\omega\rho}^2, \partial_\rho^2) + \lambda_c^4 R_\Delta^2$$

$$= \Delta + \lambda_c^2 C_{g_c} + \lambda_c^3 y \cdot \left[\nabla V(0) + \Phi_3(y) \partial_\rho \right] + \mathcal{O}\left(\lambda_c^3 |\nabla_\omega| + \lambda_c^4\right)$$
(6.262)

where we have used hypothesis (H4), noticeably $c_0 = 0$. Doing the same with u_c , we also define v_c and $w_c(s, y) = v_c(s, y) e^{i b_c \frac{|y|^2}{4} - i\beta_c \cdot y}$. We let $u = u_c + \tilde{\tilde{u}}$ and define

$$v = v_c + \epsilon, \quad w = w_c + \tilde{\epsilon}, \quad \text{i.e.} \quad \tilde{\epsilon} = \epsilon \ e^{i b_c \frac{|y|^2}{4} - i\beta_c \cdot y}$$

$$(6.263)$$

Since u_c satisfies (6.1), and therefore w_c is also solution to that same previous equation, hence substracting those, $\tilde{\epsilon}$ is solution to

$$i\partial_{s}\tilde{\epsilon} + L_{g_{c}}\tilde{\epsilon} - \tilde{\epsilon} + \frac{\rho}{\varphi_{c}}\frac{k(\lambda_{c}y + \alpha_{c})}{k(\alpha_{c})}\left(|w|^{2}w - |w_{c}|^{2}w_{c}\right) + \left((b_{c})_{s} + b_{c}^{2}\right)\frac{\rho^{2}}{4}\tilde{\epsilon} - \left\{\left((\beta_{c})_{s} + b_{c}\beta_{c}\right) \cdot y + i\left[\lambda_{c}\beta_{c}\cdot\frac{\nabla k(\alpha_{c})}{k(\alpha_{c})} + \frac{\partial_{s}\varphi_{c}}{2\varphi_{c}} + \frac{b_{c}}{2}\left(\frac{y\cdot\nabla\varphi_{c}}{\varphi_{c}} - 1\right) - \beta\cdot\left(\frac{\nabla\varphi_{c}}{\varphi_{c}} - \frac{y}{\rho^{2}}\right)\right]\right\}\tilde{\epsilon} - \left(\frac{(\lambda_{c})_{s}}{\lambda_{c}} + b_{c}\right)\left[\left(b_{c}\frac{\rho^{2}}{2} - \beta_{c}\cdot y\right)\tilde{\epsilon} + i\left(\Lambda\tilde{\epsilon} - \frac{1}{2}\left(\frac{y\cdot\nabla\varphi_{c}}{\varphi_{c}} - 1\right)\right)\right] - \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2\beta_{c}\right)\cdot\left[\left(b_{c}\frac{y}{2} - \beta\right)\tilde{\epsilon} + \frac{i}{2}\left(\frac{\lambda_{c}\nabla k(\alpha_{c})}{k(\alpha_{c})} - \left(\frac{\nabla\varphi_{c}}{\varphi_{c}} - \frac{y}{\rho^{2}}\right)\right)\tilde{\epsilon}\right] - \left((\tilde{\gamma}_{c})_{s} - |\beta_{c}|^{2}\right)\tilde{\epsilon} + \left(\frac{(\beta_{c}\cdot y)^{2}}{\rho^{2}} - |\beta_{c}|^{2}\right)\tilde{\epsilon} = 0$$

$$(6.264)$$

Then, from (6.62), and (6.232)-(6.234), one has

$$|\alpha_c|^2 + |\beta_c|^2 + |Mod_c(t)| \lesssim \lambda_c^4$$

so that using (6.224), (6.264) may be rewritten as

$$i \partial_s \tilde{\epsilon} + L_{g_c} \tilde{\epsilon} - \tilde{\epsilon} + \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \left(|w|^2 w - |w_c|^2 w_c \right) - \left(\lambda_c L_0(\alpha_c) + \lambda_c^3 C_3 \right) \cdot y \tilde{\epsilon}$$

$$= \mathcal{O} \left(\lambda_c^4 \left(1 + |y|^2 \right) \tilde{\epsilon} + \lambda_c^4 \left(1 + |y|^2 \right) \nabla \tilde{\epsilon} \right)$$
(6.265)

Let $\tilde{\epsilon} = \tilde{\epsilon}_1 + i \tilde{\epsilon}_2$ and $w_c = (w_c)_1 + i (w_c)_2$. We now expand the nonlinear term in (6.265) as well as $w_c = \tilde{P}_{\mathcal{P}_c} + \tilde{\epsilon}_c$ and the expansion of the approximate solution (6.20) to obtain up to order $\mathcal{O}(\lambda_c^4)$

$$-i\partial_s \tilde{\epsilon} + M^{(4)}(\tilde{\epsilon}) + \left(\lambda_c L_0(\alpha_c) + \lambda_c^3 C_3\right) \cdot y \,\tilde{\epsilon} = -\psi_c \tag{6.266}$$

where $M^{(4)}$ is the fourth order expansion of M defined in (6.51) and

$$M^{(4)}(\tilde{\epsilon}) = M_1^{(4)}(\tilde{\epsilon}) + i M_2^{(4)}(\tilde{\epsilon})$$

which, according to their now simplified definitions induced by (6.232) and assumption (H4), are given by

$$M_{1}^{(4)}(\tilde{\epsilon}) = -L_{g_{c}}\tilde{\epsilon}_{1} + \tilde{\epsilon}_{1} - 6Q\left(T_{2} + T_{3}\right)\tilde{\epsilon}_{1} - 2QS_{3}\tilde{\epsilon}_{2} - 3Q^{2}\left[1 + \frac{\lambda_{c}^{2}}{2}\left(\nabla^{2}k(0) + \frac{\kappa_{0}}{3}Id\right).(y, y) + \lambda_{c}\nabla^{2}k(0).(y, \alpha_{c}) + \frac{\lambda_{c}^{3}}{6}\nabla^{3}k(0).(y, y, y)\right]\tilde{\epsilon}_{1}$$
(6.267)

and

$$M_{2}^{(4)}(\tilde{\epsilon}) = -L_{g_{c}}\tilde{\epsilon}_{2} + \tilde{\epsilon}_{2} - 2Q\left(T_{2} + T_{3}\right)\tilde{\epsilon}_{2} - 2QS_{3}\tilde{\epsilon}_{1} - Q^{2}\left[1 + \frac{\lambda_{c}^{2}}{2}\left(\nabla^{2}k(0) + \frac{\kappa_{0}}{3}Id\right).(y, y) + \lambda_{c}\nabla^{2}k(0).(y, \alpha_{c}) + \frac{\lambda_{c}^{3}}{6}\nabla^{3}k(0).(y, y, y)\right]\tilde{\epsilon}_{2}$$
(6.268)

and where the remainder ψ_c satisfies

$$\psi_c = \mathcal{O}\left(\lambda_c^4 \left(1 + |y|^2\right)\tilde{\epsilon} + \lambda_c^4 \left(1 + |y|^2\right)\nabla\tilde{\epsilon} + \tilde{\epsilon}_c\,\tilde{\epsilon} + \tilde{\epsilon}_c^2\,\tilde{\epsilon} + w_c\,\tilde{\epsilon}^2 + \tilde{\epsilon}^3\right)$$

Step 2 Approximate null space.

Let $f(s, y) = \mathcal{O}(e^{-c|y|})$ be a smooth well localized slowly time dependent function. Then from equation (6.266)

$$\frac{d}{ds}\mathcal{I}m\left(\tilde{\epsilon}\,,\,\overline{f}\right) = -\mathcal{R}e\left(\tilde{\epsilon}\,,\,\overline{M^{(4)}(f) - i\,\partial_s f + \left(\lambda_c\,L_0(\alpha_c) + \lambda_c^3\,C_3\right)\,.\,y\,f}\right) - \mathcal{R}e\left(\psi_c\,,\,\overline{f}\right) \tag{6.269}$$

with

$$\left| \left(\psi_{c} \,, \, \overline{f} \right) \right| \lesssim \lambda_{c}^{4} \, \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}_{c}\|_{L^{2}} \, \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}_{c}\|_{L^{4}}^{2} \, \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}_{c}\|_{L^{4}} \, \|\tilde{\epsilon}\|_{L^{4}} \, \|\tilde{\epsilon}\|_{L^{2}} + \|\tilde{\epsilon}\|_{L^{4}}^{2} \, \|\tilde{\epsilon}\|_{L^{2}}$$

$$\lesssim \lambda_{c}^{4} \, \|\tilde{\epsilon}\|_{L^{2}} \tag{6.270}$$

where we have used (6.233) to get $\frac{b_c \lambda_c^2}{2} = \frac{\lambda_c^3}{2C_0} + \mathcal{O}(\lambda_c^4)$.

We now claim that we can find some real valued smooth well localized functions $A_2(y)$, $B_2(y)$, $D_3(s, y)$, $E_2(s, y)$, $E_3(s, y)$, $F_2(s, y)$ with

$$A_{2}, B_{2} = \mathcal{O}(\lambda_{c}^{2} e^{-c|y|}), \quad E_{2}, F_{2} = \mathcal{O}(\lambda_{c}^{2} e^{-c|y|}), \quad E_{3}, D_{3} = \mathcal{O}(\lambda_{c}^{3} e^{-c|y|})$$

such that we have the following approximate null space relations

$$M^{(4)}\left(\nabla Q + \lambda_c^2 A_2\right) - i \,\partial_s \left(\nabla Q + \lambda_c^2 A_2\right) = -a_1 \,\lambda_c^2 \, y \, Q + \mathcal{O}\left(\lambda_c^3 \, e^{-c|y|}\right) \tag{6.271}$$

for some universal constant

$$a_1 > 0$$
 (6.272)

$$M^{(4)}\left[i\left(yQ + \lambda_c^2 B_2\right)\right] - i\partial_s\left[i\left(yQ + \lambda_c^2 B_2\right)\right]$$

= $-2i\left(\nabla Q + \lambda_c^2 A_2\right) + \mathcal{O}\left(\lambda_c^3 e^{-c|y|}\right)$ (6.273)

and for the even part

$$M^{(4)} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3} \right) - i \partial_{s} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3} \right) + \left(\lambda_{c} c_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3} \right) \cdot y \left(\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3} \right) = -2 \tilde{P}_{\mathcal{P}_{c}} + \left(\lambda_{c} a_{2}(\alpha_{c}) + \lambda_{c}^{3} a_{3} \right) \cdot y Q + \mathcal{O} \left(\lambda_{c}^{4} e^{-c|y|} \right)$$
(6.274)

for a linear map a_2 on \mathbb{R}^2 , a vector $a_3 \in \mathbb{R}^2$ to be fixed calculated later in Appendix 6.B 6.12

$$M^{(4)}[i(|y|^{2}\tilde{P}_{\mathcal{P}_{c}}+F_{2})] - i\partial_{s}[i(|y|^{2}\tilde{P}_{\mathcal{P}_{c}}+F_{2})] = -4i(\Lambda\tilde{P}_{\mathcal{P}_{c}}+E_{2}) + \mathcal{O}(\lambda_{c}^{3}e^{-c|y|})$$
(6.275)

and

$$M^{(4)}(\varrho) = |y|^2 Q + \mathcal{O}(\lambda_c^2 e^{-c|y|})$$
(6.276)

The Algebraic derivation of (6.271), (6.273), (6.274), (6.275) and (6.276) is done in Appendix 6.B (6.12).

Step 3 Control of (ϵ_1, yQ) and (ϵ_2, yQ) . Let $A = \nabla Q + \lambda_c^2 A_2$ and $B = yQ + \lambda_c^2 B_2$. We now use (6.269) and (6.270) when f = A on one hand, and when f = iB on the orther hand to get

$$\begin{cases} \left(\tilde{\epsilon}_{2}, A\right)_{s} = a_{1} \lambda_{c}^{2} \left(\tilde{\epsilon}_{1}, y Q\right) + \mathcal{O}\left(\lambda_{c}^{3} e^{-c|y|}\right) \\ \left(\tilde{\epsilon}_{1}, B\right)_{s} = -2 \left(\tilde{\epsilon}_{2}, A\right) + \mathcal{O}\left(\lambda_{c}^{3} e^{-c|y|}\right) \end{cases}$$

$$(6.277)$$

Then, using the fact $\frac{ds}{dt} = \frac{1}{\lambda_c^2}$ together with (6.230) yields

$$\lambda_c(s) = \frac{C_0}{s} + \mathcal{O}\left(\frac{1}{s^2}\right) \tag{6.278}$$

Rewritting (6.277) in view of (6.278) and $B = y Q + \mathcal{O}(\lambda_c^2 e^{-c|y|})$ thus leads to the following 2-dimensional ODE system

$$\begin{cases} \left(\tilde{\epsilon}_{1}, B\right)_{s} = -2\left(\tilde{\epsilon}_{2}, A\right) + F_{1} \\ \left(\tilde{\epsilon}_{2}, A\right)_{s} = a_{1} \lambda_{c}^{2}\left(\tilde{\epsilon}_{1}, y Q\right) + F_{2} \end{cases} \quad \text{with} \quad F_{1}, F_{2} = \mathcal{O}\left(\frac{\|\tilde{\epsilon}\|_{L^{2}}}{s^{3}}\right) \tag{6.279}$$

As we will do later in Appendix 6.A 6.11, we may now apply Lemma 8.10 with $\zeta = a_1 C_0^2 > 0$ thanks to (6.272), hence

$$\left|\left(\tilde{\epsilon}_{1}, B\right)\right| + \left|\left(\tilde{\epsilon}_{2}, A\right)\right| \lesssim \int_{s}^{+\infty} \frac{\ln(\sigma)}{\sigma^{2}} \|\tilde{\epsilon}\|_{L^{2}} \, d\sigma$$

Using the fact that $A = \nabla Q + \mathcal{O}(s^{-2} e^{-c|y|})$ and $B = y Q + \mathcal{O}(s^{-2} e^{-c|y|})$ yields

$$\left| \left(\tilde{\epsilon}_1 \,, \, y \, Q \right) \right| + \left| \left(\tilde{\epsilon}_2 \,, \, \nabla Q \right) \right| \lesssim \frac{\|\tilde{\epsilon}\|_{L^2}}{s^2} + \int_s^{+\infty} \frac{\ln(\sigma)}{\sigma^2} \, \|\tilde{\epsilon}\|_{L^2} \, d\sigma$$

then, the definition (6.243) together with (6.263) and (6.230) imply

$$\|\tilde{\epsilon}(s)\|_{L^2} \le k(\alpha_c) \left\|\tilde{\tilde{u}}(s)\right\|_{L^2} \lesssim \frac{\sqrt{N(s)}}{s} \lesssim |t| \sqrt{N(t)}$$
(6.280)

so that we have

$$\left| \left(\epsilon_1(t) \,, \, y \, Q \right) \right| + \left| \left(\epsilon_2(t) \,, \, \nabla Q \right) \right| \lesssim \frac{\left\| \tilde{\tilde{u}} \right\|_{L^2}}{s^2} + \frac{\sqrt{N(s)}}{s} \int_s^{+\infty} \frac{\ln(\sigma)}{\sigma^2} \, d\sigma \lesssim \frac{\sqrt{N(s)}}{s} \left(\frac{1}{s^2} + \frac{\ln(s)}{s} \right) \\ \lesssim |t|^{1/2} \, |t| \, \sqrt{N(t)}$$

This ends the proof of (6.260) for the odd directions.

Step 4 Control of
$$(\epsilon_1, Q)$$
, $(\epsilon_2, \Lambda Q)$, $(\epsilon_1, |y|^2 Q)$, (ϵ_2, ϱ) .

Notice here we have to keep the $L^2\left(\frac{\rho}{\varphi_c} dg_y\right)$ scalar products until the end so the calculations are precise enough.

We first have from the L^2 conservation law, the critical mass assumption and (6.263)

$$\begin{aligned} k(\alpha_c) \int Q^2 \, dy &= \int |v|^2 \frac{\rho}{\varphi_c} \, dg_y = \int |w|^2 \frac{\rho}{\varphi_c} \, dg_y = \int |v_c|^2 \frac{\rho}{\varphi_c} \, dg_y = \int |w_c|^2 \frac{\rho}{\varphi_c} \, dg_y \\ \int |w|^2 \frac{\rho}{\varphi_c} \, dg_y &= \int |w_c + \tilde{\epsilon}|^2 \frac{\rho}{\varphi_c} \, dg_y = \int |w_c|^2 \frac{\rho}{\varphi_c} \, dg_y + \int |\tilde{\epsilon}|^2 \frac{\rho}{\varphi_c} \, dg_y + 2 \,\mathcal{R}e \int \tilde{\epsilon} \, \overline{w_c} \, \frac{\rho}{\varphi_c} \, dg_y \\ \int |v|^2 \frac{\rho}{\varphi_c} \, dg_y &= \int |v_c + \epsilon|^2 \frac{\rho}{\varphi_c} \, dg_y = \int |v_c|^2 \frac{\rho}{\varphi_c} \, dg_y + \int |\epsilon|^2 \frac{\rho}{\varphi_c} \, dg_y + 2 \,\mathcal{R}e \int \epsilon \, \overline{v_c} \, \frac{\rho}{\varphi_c} \, dg_y \end{aligned}$$

thus from (6.242) and the definition (6.243)

$$\mathcal{R}e\int \tilde{\epsilon} \ \overline{w_c} \ \frac{\rho}{\varphi_c} dg_y = \mathcal{R}e\int \epsilon \ \overline{v_c} \ \frac{\rho}{\varphi_c} dg_y = -\frac{1}{2} \ \int |\epsilon|^2 \ \frac{\rho}{\varphi_c} dg_y = \mathcal{O}\big(\lambda_c^4 \|\epsilon\|_{L^2}\big) = \mathcal{O}\big(\lambda_c^4 \|t\| \sqrt{N(t)}\big)$$

Moreover, from (6.263) again and the fact that $\int |\tilde{P}_{\mathcal{P}_c}|^2 \frac{\rho}{\varphi_c} dg_y = \int |\tilde{Q}_{\mathcal{P}_c}|^2 \frac{\rho}{\varphi_c} dg_y$ we have

$$\int \left|\tilde{P}_{\mathcal{P}_{c}} + \tilde{\epsilon}\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} = \int \left|\tilde{P}_{\mathcal{P}_{c}}\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} + \int \left|\tilde{\epsilon}\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} + 2\mathcal{R}e \int \tilde{\epsilon} \overline{P_{\mathcal{P}_{c}}} \frac{\rho}{\varphi_{c}} dg_{y} \\ = \int \left|\tilde{Q}_{\mathcal{P}_{c}} + \epsilon\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} = \int \left|\tilde{Q}_{\mathcal{P}_{c}}\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} + \int \left|\epsilon\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} + 2\mathcal{R}e \int \epsilon \overline{\tilde{Q}_{\mathcal{P}_{c}}} \frac{\rho}{\varphi_{c}} dg_{y} dg_{y} \\ = \int \left|\tilde{Q}_{\mathcal{P}_{c}} + \epsilon\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} = \int \left|\tilde{Q}_{\mathcal{P}_{c}}\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} + \int \left|\epsilon\right|^{2} \frac{\rho}{\varphi_{c}} dg_{y} + 2\mathcal{R}e \int \epsilon \overline{\tilde{Q}_{\mathcal{P}_{c}}} \frac{\rho}{\varphi_{c}} dg_{y} dg_{y}$$

so that

$$2\mathcal{R}e\int\tilde{\epsilon}\ \overline{\tilde{P}_{\mathcal{P}_c}}\ \frac{\rho}{\varphi_c}\,dg_y = 2\mathcal{R}e\int\epsilon\ \overline{\tilde{Q}_{\mathcal{P}_c}}\ \frac{\rho}{\varphi_c}\,dg_y = -\int|\epsilon|^2\ \frac{\rho}{\varphi_c}\,dg_y = \mathcal{O}\big(\lambda_c^4\,|t|\,\sqrt{N(t)}\big).\tag{6.281}$$

Going back to (6.269), doing the same computation with the $L^2\left(\frac{\rho}{\varphi_c} dg_y\right)$ scalar product, notice from (6.35) the remaining derivative part is negligible since $\partial_s\left(\frac{\rho}{\varphi_c} dg_y\right) = \mathcal{O}\left(\lambda_c^4 dy\right)$

$$\frac{d}{ds}\mathcal{I}m\left(\tilde{\epsilon}\,,\,\overline{f}\right)_{L^2\left(\frac{\rho}{\varphi_c}\,dg_y\right)} = -\mathcal{R}e\int\tilde{\epsilon}\,\,\overline{\left(M^{(4)}(f) - i\,\partial_s f + \left(\lambda_c\,c_0(\alpha_c) + \lambda_c^3\,C_3\right)\,.\,y\,f\right)}\,\,\frac{\rho}{\varphi_c}\,dg_y + \mathcal{O}\left(\lambda_c^4\,\|\tilde{\epsilon}\|_{L^2}\right)$$

Using (6.269), (6.270) and (6.274) where we take $f = \Lambda P_{\mathcal{P}_c} + E_2 + E_3 + i D_3$, we have

$$\begin{split} &\frac{d}{ds}\mathcal{I}m\Big(\tilde{\epsilon},\,\overline{\Lambda\tilde{P}_{\mathcal{P}_{c}}+E_{2}+E_{3}+i\,D_{3}}\Big)_{L^{2}\left(\frac{\rho}{\varphi_{c}}\,dg_{y}\right)} \\ &=-\mathcal{R}e\int\tilde{\epsilon}\,\left[\left(-2\,\tilde{P}_{\mathcal{P}_{c}}+\left(\lambda_{c}\,a_{2}(\alpha_{c})+\lambda_{c}^{3}\,a_{3}\right).\,y\,Q\right)\right]\,\frac{\rho}{\varphi_{c}}\,dg_{y}+\mathcal{O}\big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\big) \\ &=2\,\mathcal{R}e\int\tilde{\epsilon}\,\overline{\tilde{P}_{\mathcal{P}_{c}}}\,\frac{\rho}{\varphi_{c}}\,dg_{y}-\left(\lambda_{c}\,a_{2}(\alpha_{c})+\lambda_{c}^{3}\,a_{3}\right).\,\int\tilde{\epsilon}_{1}\,y\,Q\,\frac{\rho}{\varphi_{c}}\,dg_{y}+\mathcal{O}\big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\big) \\ &=2\,\mathcal{R}e\int\tilde{\epsilon}\,\overline{\tilde{P}_{\mathcal{P}_{c}}}\,\frac{\rho}{\varphi_{c}}\,dg_{y}-\left(\lambda_{c}\,a_{2}(\alpha_{c})+\lambda_{c}^{3}\,a_{3}\right).\,\left(\tilde{\epsilon}_{1}\,,\,y\,Q\right)+\mathcal{O}\big(\lambda_{c}^{4}\,\|\tilde{\epsilon}\|_{L^{2}}\big) \end{split}$$

(6.278), (6.281), the definition (6.243) of N(t) and the result of the previous step now imply after integration

$$\begin{aligned} \left| \mathcal{I}m\left(\tilde{\epsilon}\,,\,\overline{\Lambda P_{\mathcal{P}_{c}} + E_{2} + E_{3} + i\,D_{3}}\right)_{L^{2}\left(\frac{\rho}{\varphi_{c}}\,dg_{y}\right)} \right| \\ &\lesssim \int_{s}^{+\infty} \left(\left| \mathcal{R}e\int \tilde{\epsilon}\,\,\overline{\tilde{P}_{\mathcal{P}_{c}}}\,\,\frac{\rho}{\varphi_{c}}\,dg_{y} \right| + \frac{1}{\sigma^{3}} \left| \left(\tilde{\epsilon}_{1}\,,\,y\,Q\right) \right| + \frac{\|\tilde{\epsilon}\|_{L^{2}}}{\sigma^{4}} \right) d\sigma \\ &\lesssim \int_{s}^{+\infty} \left[\frac{\sqrt{N(\sigma)}}{\sigma} \left(\frac{1}{\sigma^{4}} + \frac{1}{\sigma^{7/2}} \right) + \frac{\|\tilde{\epsilon}(\sigma)\|_{L^{2}}}{\sigma^{4}} \right] d\sigma \\ &\lesssim \frac{\sqrt{N(s)}}{s} \int_{s}^{+\infty} \left(\frac{1}{\sigma^{7/2}} + \frac{1}{\sigma^{4}} \right) d\sigma \lesssim |t|^{5/2} |t| \sqrt{N(t)} \end{aligned}$$
(6.282)

Using (6.269), (6.270) and (6.275) where we take $f = i (|y|^2 \tilde{P}_{\mathcal{P}_c} + F_2)$, we have

$$\frac{d}{ds} \mathcal{R}e\Big(\tilde{\epsilon}\,,\,\overline{|y|^2\,\tilde{P}_{\mathcal{P}c}+F_2}\Big)_{L^2\left(\frac{\rho}{\varphi_c}\,dg_y\right)} = -4\,\mathcal{I}m\Big(\tilde{\epsilon}\,,\,\overline{\Lambda\tilde{P}_{\mathcal{P}c}+E_2}\Big)_{L^2\left(\frac{\rho}{\varphi_c}\,dg_y\right)} + \mathcal{O}\big(\lambda_c^3\,\|\tilde{\epsilon}\|_{L^2}\big)$$
(78) and (6.282) yield

(6.278) and (6.282) yield

$$\left| \mathcal{R}e\left(\tilde{\epsilon}, \overline{|y|^{2} \tilde{P}_{\mathcal{P}_{c}} + F_{2}}\right)_{L^{2}\left(\frac{\rho}{\varphi_{c}} dg_{y}\right)} \right| \lesssim \int_{s}^{+\infty} \left(\left| \mathcal{I}m\left(\tilde{\epsilon}, \overline{\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2}}\right)_{L^{2}\left(\frac{\rho}{\varphi_{c}} dg_{y}\right)} \right| + \frac{\|\tilde{\epsilon}\|_{L^{2}}}{\sigma^{3}} \right) d\sigma$$

$$\lesssim \frac{\sqrt{N(s)}}{s} \int_{s}^{+\infty} \left(\frac{1}{\sigma^{3/2}} + \frac{1}{\sigma^{3}}\right) d\sigma$$

$$\lesssim |t|^{3/2} |t| \sqrt{N(t)}$$
(6.283)

Using (6.269), (6.270) and (6.276) where we take $f = \rho$, we have

$$\left(\tilde{\epsilon}_{2}, \varrho\right)_{s} = -\left(\tilde{\epsilon}_{1}, |y|^{2} Q\right) + \mathcal{O}\left(\lambda_{c}^{2} \|\tilde{\epsilon}\|_{L^{2}}\right)$$

$$(6.284)$$

which eventually yields after integration using (6.283), the fact that $\tilde{P}_{\mathcal{P}_c} = Q + \mathcal{O}(|t|^2 e^{-c|y|})$, and once agin the definition (6.243)

$$\left| \left(\tilde{\epsilon}_{2}, \varrho \right) \right| \lesssim \int_{s}^{+\infty} \left(\left| \left(\tilde{\epsilon}_{1}(\sigma), |y|^{2} Q \right) \right| + \frac{\|\tilde{\epsilon}\|_{L^{2}}}{\sigma^{2}} \right) d\sigma$$

$$\lesssim \frac{\sqrt{N(s)}}{s} \int_{s}^{+\infty} \left(\frac{1}{\sigma} + \frac{1}{\sigma^{2}} \right) d\sigma$$

$$\lesssim |t|^{1/2} |t| \sqrt{N(t)}$$
(6.285)

Notice that (6.283) and (6.282) can now be rewritten in terms of the $L^2(dy)$ scalar product since $\|\tilde{\epsilon}\|_{L^2} = |t| \sqrt{N(t)}$ and $P_{\mathcal{P}_c} = Q + \mathcal{O}(|t|^2 e^{-c|y|})$, so we have

$$\left(\epsilon_2 \,, \, \Lambda Q \right) = \mathcal{I}m\left(\tilde{\epsilon} \,, \, \overline{\Lambda P_{\mathcal{P}_c} + E_2 + E_3 + i \, D_3} \right)_{L^2\left(\frac{\rho}{\varphi_c} \, dg_y\right)} + \mathcal{O}\left(|t|^2 \, \|\tilde{\epsilon}\|_{L^2} \right) \lesssim |t|^2 \, |t| \, \sqrt{N(t)}$$

$$\left(\epsilon_1 \,, \, |y|^2 \, Q \right) = \mathcal{R}e\left(\tilde{\epsilon} \,, \, \overline{|y|^2 \, P_{\mathcal{P}_c} + F_2} \right)_{L^2\left(\frac{\rho}{\varphi_c} \, dg_y\right)} + \mathcal{O}\left(|t|^2 \, \|\tilde{\epsilon}\|_{L^2} \right) \lesssim |t| \, |t| \, \sqrt{N(t)}$$

Finally, putting altogether (6.281), (6.282), (6.283) and (6.285) yields

$$\left| \left(\epsilon_1 \,, \, Q \right) \right| + \left| \left(\epsilon_2 \,, \, \Lambda Q \right) \right| + \left| \left(\epsilon_1 \,, \, |y|^2 \, Q \right) \right| + \left| \left(\epsilon_2 \,, \, \varrho \right) \right| \lesssim |t|^{1/2} \, |t| \, \sqrt{N(t)}$$

This ends the proof of (6.260) for the even directions.

6.11 Appendix 6.A : Proof of (6.240)

Here we now complete the proof of the step 2 of Lemma 6.19. Once again all we do here is to adapt the proof of Appendix C of [RS11] which we follow linearly. We will here prove a sligtly more precise estimate than (6.240), for we will need a refinement to control the phase parameter. So we claim

 $|\lambda - \lambda_c| + |b - b_c| \lesssim |t|^6, \ |\alpha - \alpha_c| + |\beta - \beta_c| \lesssim |t|^5, \ |\gamma - \gamma_c| \lesssim |t|^4$ (6.286)

 ${\bf Step \ 1}$ Improved bound on the modulation equations.

Let Mod(t) be the vector of modulation equations given by (6.60), for which we have proved estimation (6.62). Recall the process we used in the proof of (6.62) had led to a slightly more general result than what we had announced :

$$Mod(t) = \mathcal{R}(\mathcal{P}) + \mathcal{O}\left(\mathcal{P}^{7} + \left(\mathcal{P}^{2} + \left|Mod(t)\right|\right) \|\epsilon\|_{L^{2}} + \|\epsilon\|_{L^{2}}^{2} + \|\epsilon\|_{H^{1}}^{3} + \left|\|u\|_{L^{2}(dg_{x})}^{2} - \|Q\|_{L^{2}}^{2}\right|\right)$$
(6.287)

where $\mathcal{R}(\mathcal{P})$ is polynomial in \mathcal{P} that satisfies

$$\left|\mathcal{R}(\mathcal{P})\right| \lesssim \mathcal{P}^4 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^3\left(b - \frac{\lambda}{C_0}\right)$$

Now using bounds (6.236) and (6.232), there is a polynomial $\mathcal{R}(\mathcal{P})$ in \mathcal{P} such that

$$Mod(t) = \mathcal{R}(\mathcal{P}) + \mathcal{O}\left(\mathcal{P}^7 + \mathcal{P}^2 \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3\right) = \mathcal{R}(\mathcal{P}) + \mathcal{O}\left(|t|^6\right)$$
$$\left|\mathcal{R}(\mathcal{P})\right| \lesssim \mathcal{P}^4 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^3\left(b - \frac{\lambda}{C_0}\right) \lesssim |t|^4$$
(6.288)

Unifortunately, this estimate is not strong enough, and we have to first refine both scalar products with $\nabla \tilde{Q}_{\mathcal{P}}$ and $\Lambda \tilde{Q}_{\mathcal{P}}$ in (6.80) to gain a cancellation on the null space for the (b, β) laws which should be set to satisfy

$$(b_s + b^2 - B_1 - K_b, \beta_s + b\beta - B_2) = \tilde{\mathcal{R}}(\mathcal{P}) + \mathcal{O}(\mathcal{P}^3 \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{H^1}^3)$$

= $\tilde{\mathcal{R}}(\mathcal{P}) + \mathcal{O}(|t|^7)$ (6.289)

For now, we assume (6.289) holds and finish the proof of (6.286).

We wish to compare Mod(t) and $Mod_c(t)$, so writing (6.288) for both, and using the fact \mathcal{R} vanishes at least at order 2 at the origin

$$\left| Mod(t) - Mod_c(t) \right| \lesssim |t|^2 \left| \mathcal{P} - \mathcal{P}_c \right| + \mathcal{O}\left(|t|^6 \right) \tag{6.290}$$

Similarly, using (6.289), the definition (6.61) of K_b , and from (6.232) the degeneracy $\alpha, \alpha_c = \mathcal{O}(|t|^2)$, we have

$$\begin{aligned} \left| \left(b_s + b^2 - B_1 \right) - \left((b_c)_s + b_c^2 - (B_1)_c \right) \right| + \left| \left(\beta_s + b\beta - B_2 \right) - \left((\beta_c)_s + b_c\beta_c - (B_2)_c \right) \right| \\ \lesssim \left| Mod(t) - Mod_c(t) \right| + \left| K_b - K_{b_c} \right| \lesssim |t|^2 \left| \mathcal{P} - \mathcal{P}_c \right| + \mathcal{O}(|t|^7) \left| + |\alpha| \left| \alpha - \alpha_c \right| \\ \lesssim |t|^2 \left| \mathcal{P} - \mathcal{P}_c \right| + \mathcal{O}(|t|^7) \end{aligned}$$

$$(6.291)$$

Step 2 Estimates for $\lambda - \lambda_c$ and $b - b_c$.

Let us define

$$\underline{\mathcal{P}} = \mathcal{P} - \mathcal{P}_c = \mathcal{O}(|t|^2) \tag{6.292}$$

from (6.232), (6.233), (6.234).

We will now prove

$$|b - b_c| + |\lambda - \lambda_c| \lesssim |t|^6 + \int_t^0 |\underline{\mathcal{P}}| \, d\tau + |t| \int_t^0 |\underline{\mathcal{P}}| \, \frac{d\tau}{\tau}$$
(6.293)

First, we have

$$\lambda_t + \frac{b}{\lambda} = \frac{1}{\lambda} \left(\frac{\lambda_s}{\lambda} + b \right), \quad b_t + \frac{b^2}{\lambda^2} = \frac{1}{\lambda^2} \left(b_s + b^2 \right)$$
(6.294)

Then observe from (6.233), (6.234), (6.230)

$$\frac{b}{\lambda} - \frac{b_c}{\lambda_c} = \frac{1}{\lambda} \left(b - b_c \right) + b_c \left(\frac{1}{\lambda} - \frac{1}{\lambda_c} \right) = -\frac{C_0}{t} \underline{b} + \mathcal{O}\left(|t| \underline{\mathcal{P}} \right) - \frac{b_c}{\lambda_c} \frac{1}{\lambda} \underline{\lambda} = -\frac{C_0}{t} \underline{b} + \frac{1}{t} \underline{\lambda} + \mathcal{O}\left(|t| \underline{\mathcal{P}} \right)$$

$$(6.295)$$

Injecting this into (6.294) together with (6.290) for λ and the improved bound (6.291) for b yields

$$\begin{split} \underline{\lambda}_{t} - \frac{C_{0}}{t} \underline{b} + \frac{1}{t} \underline{\lambda} &= \left(\lambda_{t} + \frac{b}{\lambda}\right) - \left((\lambda_{c})_{t} + \frac{b_{c}}{\lambda_{c}}\right) + \mathcal{O}\left(|t|\underline{\mathcal{P}}\right) \\ &= \frac{1}{\lambda} \left(Mod(t) - Mod_{c}(t)\right) + \mathcal{O}\left(|t|\underline{\mathcal{P}}\right) \\ &= F_{1}, \quad F_{1} = \mathcal{O}\left(|t|^{5} + |t|\underline{\mathcal{P}}\right) \end{split}$$
(6.296)
$$\underline{b}_{t} - \frac{2}{t} \underline{b} + \frac{2}{C_{0} t} \underline{\lambda} &= \left(b_{t} + \frac{b^{2}}{\lambda^{2}}\right) - \left((b_{c})_{t} + \frac{b^{2}_{c}}{\lambda^{2}_{c}}\right) + \mathcal{O}\left(|t|\underline{\mathcal{P}}\right) \\ &= \frac{1}{\lambda^{2}} \left(L_{b} - L_{b_{c}} - |K_{b} - K_{b_{c}}|\right) + \frac{|\alpha|}{\lambda^{2}} |\alpha - \alpha_{c}| + \mathcal{O}\left(|t|\underline{\mathcal{P}}\right) \\ &= F_{2}, \quad F_{1} = \mathcal{O}\left(|t|^{5} + |t|\underline{\mathcal{P}}\right) \end{split}$$
(6.297)

We may now rewrite (6.296), (6.297) as

$$Z_t = \frac{1}{t} M Z + F$$

$$Z = \begin{pmatrix} \underline{\lambda} \\ \underline{b} \end{pmatrix}, \quad M = \begin{pmatrix} -1 & C_0 \\ -\frac{2}{C_0} & 2 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$
(6.298)

The eigenvalues of the matrix M are 0 and 1, hence the system may be rewritten in an eigenbasis

$$\tilde{Z}_t = \frac{1}{t} D \tilde{Z} + \tilde{F} \quad \text{with} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
(6.299)

and from (6.233), (6.234), (6.296), (6.297)

$$\left|\tilde{F}\right| \lesssim |t|^5 + \left|\underline{\mathcal{P}}\right|, \quad \frac{\tilde{Z}}{t} \to 0 \quad \text{as} \quad t \to 0$$

the explicit integration of (6.299) implies

$$\tilde{Z}_1 = \int_t^0 \tilde{F}_1(\tau) \, d\tau, \quad \tilde{Z}_2 = t \int_t^0 \frac{\tilde{F}_2(\tau)}{\tau} \, d\tau$$

Thus

$$\left|Z(t)\right| \lesssim \left|\tilde{Z}(t)\right| \lesssim \int_{t}^{0} \left|\tilde{F}(\tau)\right| d\tau + \left|t\right| \int_{t}^{0} \frac{\left|\tilde{F}(\tau)\right|}{\left|\tau\right|} d\tau \lesssim \left|t\right|^{6} + \int_{t}^{0} \left|\underline{\mathcal{P}}(\tau)\right| d\tau + \left|t\right| \int_{t}^{0} \frac{\left|\underline{\mathcal{P}}(\tau)\right|}{\left|\tau\right|} d\tau$$

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which proves (6.293).

Step 3 Estimates for $\alpha - \alpha_c$, $\beta - \beta_c$.

Next, we claim the bound

$$\left|\alpha - \alpha_{c}\right| + \left|\beta - \beta_{c}\right| \lesssim \left|t^{6} \ln(|t|)\right| + \left|t\right| \left(\int_{t}^{0} \frac{\left|log(\tau)\right|}{|\tau|} \left|\underline{\mathcal{P}}(\tau)\right| d\tau\right)$$

$$(6.300)$$

First, we have

 $\alpha_t - 2\frac{\beta}{\lambda} = \frac{1}{\lambda} \left(\frac{\alpha_s}{\lambda} - 2\beta\right), \quad \beta_t + \frac{b\beta}{\lambda^2} - \frac{B_2}{\lambda^2} = \frac{1}{\lambda^2} \left(\beta_s + b\beta - B_2\right)$ (6.301)

then from (6.230)

$$\frac{\beta}{\lambda} - \frac{\beta_c}{\lambda_c} = \frac{1}{\lambda} \left(\beta - \beta_c \right) + \beta_c \left(\frac{1}{\lambda} - \frac{1}{\lambda_c} \right) = -\frac{C_0}{t} \frac{\beta}{\Delta} - \frac{\beta_c}{\lambda \lambda_c} \frac{\lambda}{\Delta} = -\frac{C_0}{t} \frac{\beta}{\Delta} + \mathcal{O}(\underline{\mathcal{P}})$$
(6.302)

hence from both (6.230) and (6.233), using definition (6.22) of $B_2 = \lambda^3 C_3 + \lambda L_0(\alpha)$ and (6.234)

$$\frac{b\,\beta}{\lambda^2} - \frac{B_2}{\lambda^2} - \frac{b_c\,\beta_c}{\lambda_c^2} + \frac{(B_2)_c}{\lambda_c^2} = \frac{b}{\lambda}\frac{\beta}{\lambda} - \frac{b_c}{\lambda_c}\frac{\beta_c}{\lambda_c} - \left(\frac{B_2}{\lambda^2} - \frac{(B_2)_c}{\lambda_c^2}\right) \\
= \left(\frac{1}{C_0} + \mathcal{O}(|t|^2)\right) \left(\frac{\beta}{\lambda} - \frac{\beta_c}{\lambda_c}\right) - \left(\frac{L_0(\alpha) + \lambda^2 C_3}{\lambda} - \frac{L_0(\alpha_c) + \lambda_c^2 C_3}{\lambda_c} + \mathcal{O}(\underline{\mathcal{P}})\right) \\
= -\frac{1}{t}\frac{\beta}{L} - \frac{1}{\lambda}\left(L_0(\alpha) - L_0(\alpha_c)\right) - L_0(\alpha_c)\left(\frac{1}{\lambda} - \frac{1}{\lambda_c}\right) + \mathcal{O}(\underline{\mathcal{P}}) \\
= -\frac{1}{t}\frac{\beta}{L} + \frac{C_0}{t}L_0(\underline{\alpha}) + \mathcal{O}(\underline{\mathcal{P}})$$
(6.303)

Thus, injecting this into (6.301), using bound (6.290) we have

$$\underline{\alpha}_{t} + \frac{2C_{0}}{t} \underline{\beta} = \frac{1}{\lambda} \left[\left(\frac{\alpha_{s}}{\lambda} - 2\beta \right) - \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2\beta_{c} \right) \right] - \frac{\lambda}{\lambda\lambda_{c}} \left(\frac{(\alpha_{c})_{s}}{\lambda_{c}} - 2\beta_{c} \right) + \mathcal{O}(\underline{\mathcal{P}})
= \frac{1}{\lambda} \left(L_{\alpha} - L_{\alpha_{c}} \right) + \mathcal{O}(\underline{\mathcal{P}})
= G_{1}, \quad G_{1} = \mathcal{O}(|t|^{5} + \underline{\mathcal{P}})$$
(6.304)

and thanks to the improved bound (6.291)

$$\underline{\beta}_{t} - \frac{1}{t} \underline{\beta} + \frac{C_{0}}{t} L_{0}(\underline{\alpha}) = \frac{1}{\lambda^{2}} \left(L_{\beta} - L_{\beta_{c}} \right) - \frac{\underline{\lambda}^{2}}{(\lambda\lambda_{c})^{2}} L_{\beta_{c}} + \mathcal{O}(\underline{\mathcal{P}})$$

$$= G_{2}, \quad G_{2} = \mathcal{O}(|t|^{5} + \underline{\mathcal{P}})$$

$$(6.305)$$

Now we set

$$\underline{\eta} = \frac{\beta}{\overline{t}} \tag{6.306}$$

so that we may rewrite the system of equations (6.304)-(6.305) as

$$\underline{\alpha}_t + 2 C_0 \, \underline{\eta} = G_1, \quad \underline{\eta}_t + \frac{C_0}{t^2} \, L_0(\underline{\alpha}) = \frac{1}{t} \, G_2$$

Recall from (6.22) that matrix $L_0 = \nabla^2 k(0)$ is negative definite from assumption (H2), with eigenvalues $r_1, r_2 < 0$. Thus, we can find an eigenbasis, so that for j = 1, 2

$$\begin{cases} \left(\underline{\alpha}_{j}\right)_{t} = -2C_{0}\,\underline{\eta}_{j} + \mathcal{O}\left(t^{5} + \underline{\mathcal{P}}\right) \\ \left(\underline{\eta}_{j}\right)_{t} = -\frac{C_{0}\,r_{j}}{t^{2}}\,\underline{\alpha}_{j} + \mathcal{O}\left(t^{4} + \frac{\underline{\mathcal{P}}}{t}\right) \end{cases}$$
(6.307)

Then, we perform the change of variables $s = \frac{1}{|t|}$, and rewrite (6.307), for j = 1, 2

$$Z_j = \begin{pmatrix} \underline{\eta}_j \\ \frac{r_j C_0}{2} \underline{\alpha}_j \end{pmatrix}, \quad (Z_j)_s = \begin{pmatrix} 0 & -2 \\ \frac{\zeta_j}{s^2} & 0 \end{pmatrix} Z_j + F_j, \quad \zeta_j = -r_j C_0^2 > 0$$
(6.308)

with

$$F_j = ((F_j)_1, (F_j)_2), \quad (F_j)_1 = \mathcal{O}\left(\frac{1}{s^6} + \frac{\mathcal{P}}{s}\right), \quad (F_j)_2 = \mathcal{O}\left(\frac{1}{s^7} + \frac{\mathcal{P}}{s^2}\right)$$

Moreover, from (6.233), (6.234), (6.292)

$$\lim_{s \to +\infty} Z_j = 0 \quad \text{and} \quad F_j(s) = \mathcal{O}\left(\frac{1}{s^3}\right)$$

We then use Lemma 8.10, see Appendix B, that we apply to (α, η) so we get

$$|\underline{\eta}_{j}| + s |\underline{\alpha}_{j}| \lesssim \int_{s}^{+\infty} \left(\frac{1}{\sigma^{6}} + \frac{|\underline{\mathcal{P}}|}{\sigma}\right) \log(\sigma) \, d\sigma \lesssim \left|t^{5} \log|t|\right| + \int_{t}^{0} \frac{|\log(\tau)|}{|\tau|} \, |\underline{\mathcal{P}}(\tau)| \, d\tau$$

and hence (6.300) follows from (6.306).

Step 4 Bound on $\underline{\mathcal{P}}$.

We conclude from (6.293), (6.300) that

$$|\underline{\mathcal{P}}| \lesssim \left| t^6 \log|t| \right| + \int_t^0 |\underline{\mathcal{P}}(\tau)| \, d\tau + |t| \, \int_t^0 \frac{|\log(\tau)|}{|\tau|} \, |\underline{\mathcal{P}}(\tau)| \, d\tau$$

Injecting the a priori bound (6.292) in the previous estimate

$$|\underline{\mathcal{P}}| \lesssim t^{6} |\log|t||^{4} + \int_{t}^{0} |\tau|^{2} d\tau + |t| \int_{t}^{0} |\tau| |\log(\tau)| d\tau \lesssim t^{6} |\log|t||^{4} + |t|^{3} + |t|^{3} \left|\log|t|\right| \lesssim |t|^{3} \left|\log|t|\right|$$

and iterating 3 more times now imply

$$|\underline{\mathcal{P}}| \lesssim t^6 |\log|t||^4$$

Eventually, injecting this into (6.293), (6.300) clearly yields

$$|\lambda - \lambda_c| + |b - b_c| \lesssim |t|^6, \quad |\alpha - \alpha_c| + |\beta - \beta_c| \lesssim |t|^5$$
(6.309)

Step 5 Bound on the phase parameter.

There is only left to get control on the phase parameter, which is why we had to sharpen control on the scalling parameter λ in previous steps. We are willing to prove

$$|\gamma - \gamma_c| \lesssim |t|^4 \tag{6.310}$$

Indeed we have

$$\gamma_t - \frac{1 + |\beta|^2 + K_{\tilde{\gamma}}}{\lambda^2} = \frac{1}{\lambda^2} \left(\tilde{\gamma}_s - |\beta|^2 - K_{\tilde{\gamma}} \right)$$
(6.311)

and from (6.309)

$$\frac{1+|\beta|^2+K_{\tilde{\gamma}}}{\lambda^2}-\frac{1+|\beta_c|^2+K_{\tilde{\gamma}_c}}{\lambda_c^2} = \mathcal{O}\Big(\frac{|\lambda-\lambda_c|}{|t|^3}+|\beta-\beta_c|+|\alpha-\alpha_c|\Big) = \mathcal{O}\big(|t|^3\big)$$
(6.312)

so that (6.290), (6.311) and (6.312) imply

$$\underline{\gamma}_t = \mathcal{O}(|t|^3 + |\underline{\mathcal{P}}|)$$

which after integration yields (6.310) and concludes the proof of (6.286) assuming (6.289).

Step 6 Proof of the improved bound (6.289)

To get the refinment (6.289) we now need to go back to the very definition of the remaining ϵ through the choice of the orthogonality conditions (6.42)-(6.46). Indeed, one way to sharpen the modulation laws estimates, is to pertubatively modify the remain ϵ through the orthogonality conditions (6.42) and (6.44) which respectively govern the law of β and b, and that one should now replace by

$$\int \left(\epsilon_2, \, \nabla\Sigma + A \,\lambda^2\right) - \left(\epsilon_1, \, \nabla\Theta\right) = 0 \tag{6.313}$$

$$\left(\left(\epsilon_{2},\,\Lambda\Sigma+B\,\lambda^{2}\right)-\left(\epsilon_{1},\,\Lambda\Theta\right)=0\tag{6.314}$$

where A and B are well-localized real functions to be chosen. We claim for a suitable choice for A and B, the computation of the modulations equations, like we did in (6.90) and (6.85) will lead us to get (6.289).

Let M_1 , M_2 and M_1 , M_2 given by (6.50) and (6.51), and then let the complex operator for $\epsilon = \epsilon_1 + i \epsilon_2$

$$\mathcal{M}(\epsilon) = \mathcal{M}_1(\epsilon_1, \epsilon_2) + i \,\mathcal{M}_2(\epsilon_1, \epsilon_2) = M_1(\epsilon_1, \epsilon_2) + i \,M_2(\epsilon_1, \epsilon_2) - i \,b \,\Lambda \epsilon + 2 \,i \,\beta \,. \,\nabla \epsilon$$

These refined orthogonality conditions allow one to sharpen the scalar products computations (6.80) such that

$$\mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(\nabla\tilde{Q}_{\mathcal{P}} + \lambda^{2}A) - i\partial_{s}(\nabla\tilde{Q}_{\mathcal{P}} + \lambda^{2}A)}\right) = \mathcal{O}\left(|t|^{3} \|\epsilon\|_{L^{2}}\right)$$
(6.315)

$$\mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(\Lambda\tilde{Q}_{\mathcal{P}}+\lambda^{2}B)-i\partial_{s}(\Lambda\tilde{Q}_{\mathcal{P}}+\lambda^{2}B)}\right)+2\mathcal{R}e\left(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\right)=\mathcal{O}\left(|t|^{3}\|\epsilon\|_{L^{2}}\right)$$
(6.316)

First, note one has the following adjunction formula

$$\mathcal{R}e\left(\mathcal{M}(\epsilon), \overline{f}\right) = \mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(f)}\right)$$

then, recall the equations of real and imaginary parts of ϵ (6.76)-(6.77), and given an orthogonality condition $\mathcal{I}m(\epsilon, \overline{f}) = 0$ up to $\mathcal{O}(|t|^3 ||\epsilon||_{L^2})$, the linear term in the computation of these equations is

$$-\left(\mathcal{M}_1(\epsilon_1,\epsilon_2), f_1\right) + \left(\epsilon_2, \partial_s f_1\right) - \left(\mathcal{M}_2(\epsilon_1,\epsilon_2), f_2\right) - \left(\epsilon_1, \partial_s f_2\right) = -\mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(f) - i\partial_s f}\right)$$

Proof of (6.315)

From equation (6.31), definition (6.13) and expansion (6.41) $\tilde{Q}_{\mathcal{P}}$ satisfies

$$i\left(-b^{2}+B_{1}\right)\partial_{b}\tilde{Q}_{\mathcal{P}}+\Delta\tilde{Q}_{\mathcal{P}}+\lambda^{2}\left(V(0)+\frac{\kappa_{0}}{3}\right)\tilde{Q}_{\mathcal{P}}-\tilde{Q}_{\mathcal{P}}+\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}\tilde{Q}_{\mathcal{P}}|\tilde{Q}_{\mathcal{P}}|^{2}$$
$$+ib\Lambda\tilde{Q}_{\mathcal{P}}-2i\beta\cdot\nabla\tilde{Q}_{\mathcal{P}}=\mathcal{O}\left(|t|^{3}e^{-|y|}\right)$$
(6.317)

By differentiating (6.317), since

$$\nabla \left(|\tilde{Q}_{\mathcal{P}}|^2 \, \tilde{Q}_{\mathcal{P}} \right) = \tilde{M}_1(\nabla \Sigma, \nabla \Theta) + i \, \tilde{M}_2(\nabla \Sigma, \nabla \Theta)$$

we obtain

$$i\left(-b^{2}+B_{1}\right)\partial_{b}\nabla\tilde{Q}_{\mathcal{P}}+\left[\Delta+\lambda^{2}\left(V(0)+\frac{\kappa_{0}}{3}\right)\right]\left(\nabla\tilde{Q}_{\mathcal{P}}\right)-\nabla\tilde{Q}_{\mathcal{P}}$$

+
$$\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}\left(\tilde{M}_{1}(\nabla\tilde{Q}_{\mathcal{P}})+i\tilde{M}_{2}(\nabla\tilde{Q}_{\mathcal{P}})+ib\Lambda\left(\nabla\tilde{Q}_{\mathcal{P}}\right)-2i\beta\cdot\nabla^{2}\tilde{Q}_{\mathcal{P}}$$

+
$$\frac{\rho}{\varphi}\left[\left(\frac{y}{\rho^{2}}-\frac{\nabla\varphi}{\varphi}\right)\frac{k(\lambda y+\alpha)}{k(\alpha)}+\frac{\lambda\nabla k(\lambda y+\alpha)}{k(\alpha)}\right]|\tilde{Q}_{\mathcal{P}}|^{2}\tilde{Q}_{\mathcal{P}}+ib\nabla\tilde{Q}_{\mathcal{P}}=\mathcal{O}\left(|t|^{3}e^{-|y|}\right)$$
(6.318)

Thus

$$\mathcal{M}(\nabla \tilde{Q}_{\mathcal{P}}) - i \partial_s \left(\nabla \tilde{Q}_{\mathcal{P}}\right) = \frac{\rho}{\varphi} \left[\left(\frac{y}{\rho^2} - \frac{\nabla \varphi}{\varphi} \right) \frac{k(\lambda y + \alpha)}{k(\alpha)} + \frac{\lambda \nabla k(\lambda y + \alpha)}{k(\alpha)} \right] |\tilde{Q}_{\mathcal{P}}|^2 \tilde{Q}_{\mathcal{P}} + i b \nabla \tilde{Q}_{\mathcal{P}} + \mathcal{O}\left(|t|^3 e^{-|y|} \right)$$

$$(6.319)$$

moreover thanks to (6.41) and Proposition 6.18 one easily check

$$\tilde{M}_1(\lambda^2 A) = \lambda^2 \left(3 Q^2 A + \mathcal{O}(|t| e^{-|y|}) \right) = 3 \lambda^2 Q^2 A + \mathcal{O}(|t|^3 e^{-|y|}), \quad \tilde{M}_2(\lambda^2 A) = \mathcal{O}(|t|^3 e^{-|y|})$$

then, expanding the remaining terms with (6.17)

$$\mathcal{M}(\nabla \tilde{Q}_{\mathcal{P}} + \lambda^2 A) - i \partial_s (\nabla \tilde{Q}_{\mathcal{P}} + \lambda^2 A) = \lambda^2 \left[\nabla^2 k(0).(y, .) Q^3 + \frac{\kappa_0}{3} y Q^3 + L_+(A) \right] + i b \nabla \tilde{Q}_{\mathcal{P}} + \mathcal{O}(|t|^3 e^{-|y|})$$
(6.320)

From (8.3), (8.1), (8.2), see Appendix B 8.1 we then choose A solution to

$$\begin{split} L_{+}(A) &= -\left[\nabla^{2}k(0).(y, ..) + \frac{\kappa_{0}}{3}y\right]Q^{3} + a\,yQ,\\ a &= \frac{\left(\left[\nabla^{2}k(0).(y, ..) + \frac{\kappa_{0}}{3}y\right]Q^{3}, \nabla Q\right)}{\left(yQ, \nabla Q\right)} = \frac{1}{2}\left(\Delta k(0) + \frac{2\kappa_{0}}{2}\right) \end{split}$$

and thus from (6.320) for $f = \nabla \tilde{Q}_{\mathcal{P}} + \lambda^2 A$, using orthogonality conditions (6.43) and (6.313)

$$\mathcal{R}e\left(\epsilon, \overline{\mathcal{M}(f) - i\partial_s f}\right) = b\mathcal{I}m\left(\epsilon, \nabla \overline{\tilde{Q}_{\mathcal{P}}}\right) + a\lambda^2 \mathcal{R}e\left(\overline{\epsilon}, yQ\right) + \mathcal{O}\left(|t|^3 e^{-|y|}\right) = \mathcal{O}\left(|t|^3 \|\epsilon\|_{L^2}\right)$$

This concludes the proof of (6.315)

This concludes the proof of (6.315)

Proof of (**6.316**)

First, we gather some computational results. We use definition (6.13) of \mathcal{L}_g , and let $L_g = \mathcal{L}_g + \lambda^2 V(\lambda y + \lambda^2 V(\lambda y + \lambda^2 V))$ $\alpha) + V_{\varphi}$

$$\Lambda^{2}\tilde{Q}_{\mathcal{P}} = \Lambda\tilde{Q}_{\mathcal{P}} + y \cdot \nabla\tilde{Q}_{\mathcal{P}} + y \cdot \Lambda(\nabla\tilde{Q}_{\mathcal{P}})$$

$$\beta \cdot \nabla(\Lambda\tilde{Q}_{\mathcal{P}}) = 2\beta \cdot \nabla\tilde{Q}_{\mathcal{P}} + y \cdot \beta \cdot \nabla(\nabla\tilde{Q}_{\mathcal{P}})$$

$$\tilde{M}_{1}(\Lambda\tilde{Q}_{\mathcal{P}}) + i\tilde{M}_{2}(\Lambda\tilde{Q}_{\mathcal{P}}) = 3|\tilde{Q}_{\mathcal{P}}|^{2}\tilde{Q}_{\mathcal{P}} + y \cdot \tilde{M}_{1}(\nabla\tilde{Q}_{\mathcal{P}}) + iy \cdot \tilde{M}_{2}(\nabla\tilde{Q}_{\mathcal{P}})$$

$$L_{g}(\Lambda\tilde{Q}_{\mathcal{P}}) = \left[\Delta + \lambda^{2}\left(V(0) + \frac{\kappa_{0}}{3}\right) + \mathcal{O}(|t|^{3})\right]\Lambda\tilde{Q}_{\mathcal{P}}$$

$$= 3L_{g}\tilde{Q}_{\mathcal{P}} + y \cdot \nabla(L_{g}\tilde{Q}_{\mathcal{P}}) - 2\lambda^{2}\left(V(0) + \frac{\kappa_{0}}{3}\right)\tilde{Q}_{\mathcal{P}} + \mathcal{O}(|t|^{3}e^{-|y|})$$
(6.321)

thanks to which and (6.317), (6.318) we deduce

$$\begin{split} &i(-b^{2}+B_{1})\partial_{b}(\Lambda\tilde{Q}_{\mathcal{P}})+\mathcal{L}_{g}(\Lambda\tilde{Q}_{\mathcal{P}})-\Lambda\tilde{Q}_{\mathcal{P}}+\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}\left[\tilde{M}_{1}(\Lambda\tilde{Q}_{\mathcal{P}})+i\,\tilde{M}_{2}(\Lambda\tilde{Q}_{\mathcal{P}})\right]\\ &+i\,b\,\Lambda^{2}\tilde{Q}_{\mathcal{P}}-2\,i\,\beta\,.\,\nabla(\Lambda\tilde{Q}_{\mathcal{P}})\\ &=3\left[i(-b^{2}+B_{1})\partial_{b}\tilde{Q}_{\mathcal{P}}+\mathcal{L}_{g}\tilde{Q}_{\mathcal{P}}-\tilde{Q}_{\mathcal{P}}+\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}\,\tilde{Q}_{\mathcal{P}}|\tilde{Q}_{\mathcal{P}}|^{2}+i\,b\,\Lambda\tilde{Q}_{\mathcal{P}}-2\,i\,\beta\,.\,\nabla\tilde{Q}_{\mathcal{P}}\right]\\ &+y\,.\left[i(-b^{2}+B_{1})\partial_{b}\nabla\tilde{Q}_{\mathcal{P}}+\mathcal{L}_{g}(\nabla\tilde{Q}_{\mathcal{P}})-\nabla\tilde{Q}_{\mathcal{P}}+\frac{\rho}{\varphi}\frac{k(\lambda y+\alpha)}{k(\alpha)}\left(\tilde{M}_{1}(\nabla\tilde{Q}_{\mathcal{P}})+i\,\tilde{M}_{2}(\nabla\tilde{Q}_{\mathcal{P}})\right)\right.\\ &+i\,b\,\Lambda(\nabla\tilde{Q}_{\mathcal{P}})-2\,i\,\beta\,.\,\nabla^{2}\tilde{Q}_{\mathcal{P}}\right]+2\,i\,(b^{2}-B_{1})\,\partial_{b}\tilde{Q}_{\mathcal{P}}+2\,\tilde{Q}_{\mathcal{P}}+i\,b\,y\,.\,\nabla\tilde{Q}_{\mathcal{P}}\\ &-2\,i\,b\,\Lambda\tilde{Q}_{\mathcal{P}}+2\,i\,\beta\,.\,\nabla\tilde{Q}_{\mathcal{P}}-2\,\lambda^{2}\left(V(0)+\frac{\kappa_{0}}{3}\right)\tilde{Q}_{\mathcal{P}}\\ &=2\,i\,(b^{2}-B_{1})\,\partial_{b}\tilde{Q}_{\mathcal{P}}+2\,\tilde{Q}_{\mathcal{P}}-2\,i\,b\,\Lambda\tilde{Q}_{\mathcal{P}}+2\,i\,\beta\,.\,\nabla\tilde{Q}_{\mathcal{P}}\\ &-\left\{\frac{\rho}{\varphi}\left[\left(1-\frac{y\,.\,\nabla\varphi}{\varphi}\right)\frac{k(\lambda y+\alpha)}{k(\alpha)}+\frac{\lambda\,y\,.\,\nabla k(\lambda y+\alpha)}{k(\alpha)}\right]|\tilde{Q}_{\mathcal{P}}|^{2}\,\tilde{Q}_{\mathcal{P}}+2\,\lambda^{2}\left(V(0)+\frac{\kappa_{0}}{3}\right)\tilde{Q}_{\mathcal{P}}\right\}\\ &+\mathcal{O}(|t|^{3}\,e^{-|y|})\end{split}$$

Eventually, with (6.22) and (6.17) we obtain

$$\mathcal{M}(\Lambda \tilde{Q}_{\mathcal{P}} + \lambda^2 B) - i \partial_s \left(\Lambda \tilde{Q}_{\mathcal{P}} + \lambda^2 B\right)$$

= $\lambda^2 \left[\left(\nabla^2 k(0) + \frac{\kappa_0}{3} Id \right) \cdot (y, y) Q^3 + 2 \left(V(0) + \frac{\kappa_0}{3} \right) Q + L_+(B) \right]$
- $2 i b^2 \partial_b \tilde{Q}_{\mathcal{P}} - 2 \tilde{Q}_{\mathcal{P}} + 2 i b \Lambda \tilde{Q}_{\mathcal{P}} - 2 i \beta \cdot \nabla \tilde{Q}_{\mathcal{P}} + \mathcal{O}(|t|^3 e^{-|y|})$ (6.322)

We hence choose B solution to

$$L_{+}(B) = -\left(\nabla^{2}k(0) + \frac{\kappa_{0}}{3}Id\right).(y,y)Q^{3} - 2\left(V(0) + \frac{\kappa_{0}}{3}\right)Q =: f_{2}$$

which is possible thanks to (8.2), since one easily checks that $(f_2, \nabla Q) = 0$.

Then, observe as in (6.56) that

$$\partial_b \tilde{Q}_{\mathcal{P}} = -i \, \frac{|y|^2}{4} \, Q + \mathcal{O}\big(|t| \, e^{-|y|}\big)$$

so that (6.322) along with refined orthogonality conditions (6.313), (6.314) and usual orthogonality condition (6.45) yields

$$\begin{aligned} &\mathcal{R}e\Big(\epsilon, \overline{\mathcal{M}(\Lambda\tilde{Q}_{\mathcal{P}}+\lambda^{2}B)-i\partial_{s}(\Lambda\tilde{Q}_{\mathcal{P}}+\lambda^{2}B)}\Big)+2\mathcal{R}e\Big(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\Big)\\ &=-\frac{b^{2}}{2}\mathcal{R}e\Big(\epsilon, |y|^{2}\,\overline{\tilde{Q}_{\mathcal{P}}}\Big)-2\mathcal{R}e\Big(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\Big)+2b\mathcal{I}m\Big(\epsilon, \Lambda\overline{\tilde{Q}_{\mathcal{P}}}\Big)-2\beta.\mathcal{I}m\Big(\epsilon, \nabla\overline{\tilde{Q}_{\mathcal{P}}}\Big)+2\mathcal{R}e\Big(\epsilon, \overline{\tilde{Q}_{\mathcal{P}}}\Big)+\mathcal{O}\big(|t|^{3}\,e^{-|y|}\big)\\ &=\mathcal{O}\big(|t|^{3}\,e^{-|y|}\big)\end{aligned}$$

This concludes the proof of (6.316).

6.12 Appendix 6.B : Computation of the approximate null space of the linearized NLS operator around the Ground-State.

Proof of (6.271) and (6.272) We compute thanks to (6.267), (6.268) along with the control of the modulation parameters of (6.232)

$$M^{(4)}\left(\nabla Q + \lambda_c^2 A_2\right) - i \partial_s \left(\nabla Q + \lambda_c^2 A_2\right) = L_+ \left(\nabla Q + \lambda_c^2 A_2\right) - \lambda_c^2 \left(V(0) - \frac{c_3}{3}\right) \nabla Q - 6 Q T_2 \nabla Q - \frac{3\lambda_c^2}{2} \left(\nabla^2 k(0) + \frac{\kappa_0}{3} Id\right) (y, y) Q^2 \nabla Q + \mathcal{O}\left(\lambda_c^3 e^{-c|y|}\right)$$

$$(6.323)$$

Recall $L_+(\nabla Q) = 0$, and then from (6.224), using (6.23) and again (6.230) to discard $\mathcal{O}(|\alpha|^2 + \lambda |\alpha|) = \mathcal{O}(\lambda^3)$ terms, the following identity which defined T_2 , thanks to (8.2), let T_2^0 be the radial solution to

$$L_{+}T_{2}^{0} = \frac{1}{2} \left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} Id \right) (y, y) Q^{3} + \left(V(0) + \frac{\kappa_{0}}{3} \right) Q$$

so that we may rewrite (6.323) as

$$M^{(4)}\left(\nabla Q + \lambda_c^2 A_2\right) - i \partial_s \left(\nabla Q + \lambda_c^2 A_2\right)$$

= $\lambda_c^2 \left[L_+(A_2) - 6 Q T_2^0 \nabla Q - \left(V(0) + \frac{\kappa_0}{3}\right) \nabla Q - \frac{3}{2} \left(\nabla^2 k(0) + \frac{\kappa_0}{3} Id\right) (y, y) Q^2 \nabla Q \right] + \mathcal{O}\left(\lambda_c^3 e^{-c|y|}\right)$

To fulfill (6.271) we may now simply choose A_2 solution to

$$L_{+}(A_{2}) = 6 Q T_{2}^{0} \nabla Q + \left(V(0) + \frac{\kappa_{0}}{3}\right) \nabla Q + \frac{3}{2} \left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} Id\right) (y, y) Q^{2} \nabla Q - a_{1} y Q$$
(6.324)

Q being is radial, T_2^0 is hence a second order polynomial function in y with radially symmetric coefficients (also in variable y), and therefore from (8.2) one should also set constant a_1 as

$$a_1 = \frac{\left(6QT_2^0\nabla Q + \left(V(0) + \frac{\kappa_0}{3}\right)\nabla Q + \frac{3}{2}\left(\nabla^2 k(0) + \frac{\kappa_0}{3}Id\right).(y,y)Q^2\nabla Q, \nabla Q\right)}{\left(yQ, \nabla Q\right)} \tag{6.325}$$

Yet there is left to ensure that $a_1 > 0$. To do so, one needs to compute a_1 more explicitly, which may be done provided one observe by differentiating twice the Ground-State equation $\Delta Q = Q - Q^3$, $L_+(\Delta Q) = 6 Q |\nabla Q|^2$.

$$\begin{split} & \left(6\,Q\,T_2^0\,\nabla Q\,,\,\nabla Q \right) = 6\,\left(T_2^0\,,\,Q\,|\nabla Q|^2\right) = \left(T_2^0\,,\,L_+(\Delta Q)\right) = \left(L_+(T_2^0)\,,\,\Delta Q\right) \\ & = \left(\frac{1}{2}\,\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right).(y,y)\,Q^3 + \left(V(0) + \frac{\kappa_0}{3}\right)Q\,,\,\Delta Q\right) \\ & = -\left(\frac{3}{2}\,\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right).(y,y)\,Q^2\,\nabla Q\,,\,\nabla Q\right) - \left(\frac{1}{2}\,\nabla\Big[\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right).(y,y)\Big]\,Q^3\,,\,\nabla Q\Big) \\ & - \left(V(0) + \frac{\kappa_0}{3}\right)\left(\nabla Q\,,\,\nabla Q\right) \\ & = -\left(\frac{3}{2}\,\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right).(y,y)\,Q^2\,\nabla Q + \left(V(0) + \frac{\kappa_0}{3}\right)\nabla Q\,,\,\nabla Q\right) + \frac{1}{8}\,\int\Delta\Big[\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right).(y,y)\Big]\,Q^4\,dy \\ & = -\left(\frac{3}{2}\,\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right).(y,y)\,Q^2\,\nabla Q + \left(V(0) + \frac{\kappa_0}{3}\right)\nabla Q\,,\,\nabla Q\right) + \frac{1}{2}\,Tr\left(\nabla^2 k(0) + \frac{\kappa_0}{3}\,Id\right)\,\int Q^2 \end{split}$$

Altogether, this yields

$$a_1 = -\frac{1}{2} Tr\left(\nabla^2 k(0) + \frac{\kappa_0}{3} Id\right)$$
(6.326)

Consequently, from hypothesis (H2), $\nabla^2 k(0) + \frac{\kappa_0}{3} < 0$ is negative definite, which now imply $a_1 > 0$, and ends the proof of (6.272).

Proof of (6.273) We compute thanks to (6.267), (6.268) along with the control of the modulation parameters of (6.232)

$$M^{(4)}(i(yQ + \lambda_c^2 B_2)) - i\partial_s(i(yQ + \lambda_c^2 B_2)) = iL_-(yQ + \lambda_c^2 B_2) - i\lambda_c^2 \left[2QT_2^0 + \frac{1}{2}\left(\nabla^2 k(0) + \frac{\kappa_0}{3}Id\right).(y,y)Q^2 + \left(V(0) + \frac{\kappa_0}{3}\right)\right]yQ + \mathcal{O}(\lambda_c^3 e^{-c|y|})$$

so that to ensure (6.273), using $L_{-}(yQ) = -2\nabla Q$, we may simply choose B_{2} solution to

$$L_{-}(B_{2}) = -2A_{2} + \left[2QT_{2}^{0} + \frac{1}{2}\left(\nabla^{2}k(0) + \frac{\kappa_{0}}{3}Id\right).(y,y)Q^{2} + \left(V(0) + \frac{\kappa_{0}}{3}\right)\right]yQ$$

which is solvable from (8.2) since its right-hand side is orthogonal to Q by definition of A_2 (6.324).

Proof of (6.274) We compute thanks to (6.267), (6.268) and the foruth order precision of approximation $\tilde{P}_{\mathcal{P}_c}$ along with the control of the modulation parameters of (6.232)

$$M^{(4)} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3} \right) - i \partial_{s} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3} \right) + \left(\lambda_{c} L_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3} \right) \cdot y \left[\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3} \right] = M^{(4)} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} \right) - i \partial_{s} \Lambda T_{2} + L_{+} (E_{2} + E_{3}) + i L_{-} (D_{3}) - i \partial_{s} E_{2} + \left(\lambda_{c} L_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3} \right) \cdot y \Lambda Q + \mathcal{O} (\lambda_{c}^{4} e^{-c|y|})$$
(6.327)

Then, one easily checks from (6.23), (6.24) that $\partial_{\lambda}\tilde{P}_{\mathcal{P}_c}$, $\partial_{\alpha}\tilde{P}_{\mathcal{P}_c} = \mathcal{O}(\mathcal{P}_c)$ and $\partial_b\tilde{P}_{\mathcal{P}_c}$, $\partial_{\beta}\tilde{P}_{\mathcal{P}_c} = \mathcal{O}(\mathcal{P}_c^2)$. Thus, recall from equation (6.20), which is now simplified thanks to (6.232), definition (6.262) and (6.224), that $\tilde{P}_{\mathcal{P}_c}$ is solution to

$$-i b_c \lambda_c \partial_{\lambda_c} \tilde{P}_{\mathcal{P}_c} + \left(\Delta + \lambda_c^2 C_{g_c} + \lambda_c^3 y \cdot \left[\nabla V(0) + \Phi_3(y) \partial_{\rho} \right] \right) \tilde{P}_{\mathcal{P}_c} - \tilde{P}_{\mathcal{P}_c} + \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \tilde{P}_{\mathcal{P}_c} |\tilde{P}_{\mathcal{P}_c}|^2 - \left(\lambda_c L_0(\alpha_c) + \lambda_c^3 C_3 \right) \cdot y \tilde{P}_{\mathcal{P}_c} = \mathcal{O}\left(\lambda_c^4 e^{-c|y|} \right)$$

$$(6.328)$$

Therefore, as we did in (6.322), we derive (6.328) and use (6.319), a more accurate version of (6.321) to get

$$\begin{split} &-i b_c \lambda_c \, \partial_{\lambda_c} \left(\Lambda \tilde{P}_{\mathcal{P}_c} \right) + L_{g_c} \left(\Lambda \tilde{P}_{\mathcal{P}_c} \right) - \Lambda \tilde{P}_{\mathcal{P}_c} + \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \tilde{M} \left(\Lambda \tilde{P}_{\mathcal{P}_c} \right) - \left(\lambda_c \, L_0(\alpha_c) + \lambda_c^3 \, C_3 \right) \cdot y \, \Lambda \tilde{P}_{\mathcal{P}_c} \\ &= 3 \left[-i b_c \lambda_c \, \partial_{\lambda_c} \tilde{P}_{\mathcal{P}_c} + L_{g_c} \tilde{P}_{\mathcal{P}_c} - \tilde{P}_{\mathcal{P}_c} + \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} |\tilde{P}_{\mathcal{P}_c}|^2 \tilde{P}_{\mathcal{P}_c} - \left(\lambda_c \, L_0(\alpha_c) + \lambda_c^3 \, C_3 \right) \cdot y \, \tilde{P}_{\mathcal{P}_c} \right] \\ &+ y \cdot \left[-i b_c \lambda_c \, \partial_{\lambda_c} \left(\nabla \tilde{P}_{\mathcal{P}_c} \right) + L_{g_c} \nabla \tilde{P}_{\mathcal{P}_c} - \nabla \tilde{P}_{\mathcal{P}_c} + \frac{\rho}{\varphi_c} \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \tilde{M} \left(\nabla \tilde{P}_{\mathcal{P}_c} \right) - \left(\lambda_c \, c_0(\alpha_c) + \lambda_c^3 \, C_3 \right) \cdot y \, \nabla \tilde{P}_{\mathcal{P}_c} \right] \\ &+ 2 \, i \, b_c \lambda_c \, \partial_{\lambda_c} \tilde{P}_{\mathcal{P}_c} - \left(2 \, \lambda_c^2 \, C_{g_c} + \lambda_c^3 \, y \cdot \left[2 \, \nabla V(0) + \Phi_3 \, \partial_{\rho} \right] \right) \tilde{P}_{\mathcal{P}_c} + 2 \, \tilde{P}_{\mathcal{P}_c} + 2 \left(\lambda_c \, L_0(\alpha_c) + \lambda_c^3 \, C_3 \right) \cdot y \, \tilde{P}_{\mathcal{P}_c} + \mathcal{O} \left(\lambda_c^4 \, e^{-c|y|} \right) \\ &= 2 \, \tilde{P}_{\mathcal{P}_c} + 2 \, i \, b_c \lambda_c \, \partial_{\lambda_c} \tilde{P}_{\mathcal{P}_c} - 2 \, \lambda_c^2 \, C_{g_c} \, \tilde{P}_{\mathcal{P}} - 3 \, \lambda_c^3 \, y \cdot \left[\nabla V(0) + \Phi_3 \, \partial_{\rho} \right] \tilde{P}_{\mathcal{P}} + 3 \left(\lambda_c \, L_0(\alpha_c) + \lambda_c^3 \, C_3 \right) \cdot y \, \tilde{P}_{\mathcal{P}_c} \\ &- \frac{\rho}{\varphi_c} \left[\left(1 - \frac{y \cdot \nabla \varphi_c}{\varphi_c} \right) \frac{k(\lambda_c y + \alpha_c)}{k(\alpha_c)} + \frac{\lambda_c \, \nabla k(\lambda_c y + \alpha_c)}{k(\alpha_c)} \right] |\tilde{P}_{\mathcal{P}}|^2 \, \tilde{P}_{\mathcal{P}} + \mathcal{O} \left(\lambda_c^4 \, e^{-c|y|} \right) \\ &= 2 \, \tilde{P}_{\mathcal{P}_c} + 2 \, i \, b_c \lambda_c \, \partial_{\lambda_c} \, \tilde{P}_{\mathcal{P}_c} + 3 \left(\lambda_c \, L_0(\alpha_c) + \lambda_c^3 \, C_3 \right) \cdot y \, \tilde{P}_{\mathcal{P}_c} - 2 \, \lambda_c^2 \left[\frac{1}{2} \left(\nabla^2 k(0) + \frac{\kappa_0}{3} \, Id \right) \cdot (y, y) \, Q^3 + \left(V(0) + \frac{\kappa_0}{3} \right) Q \right] \\ &- \lambda_c \, \nabla^2 k(0) \cdot (y, \alpha_c) \, Q^3 - \frac{\lambda_c^3}{2} \left[\nabla^3 k(0) \cdot (y, y, y) \, Q^3 + 3 \, y \cdot \left(2 \, \nabla V(0) \, Q - \rho \, \nabla \kappa(0) \, \partial_{\rho} Q \right) \right] + \mathcal{O} \left(\lambda_c^4 \, e^{-c|y|} \right) \\ & \text{Eventually, with } (6.233), (6.327) \text{ becomes} \end{aligned}$$

$$M^{(4)} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) - i \partial_{s} \left(\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right) \\ + \left(\lambda_{c} L_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y \left[\Lambda \tilde{P}_{\mathcal{P}_{c}} + E_{2} + E_{3} + i D_{3}\right] \\ = -2 \tilde{P}_{\mathcal{P}_{c}} - 2 i b_{c} \lambda_{c} \partial_{\lambda_{c}} \tilde{P}_{\mathcal{P}_{c}} - 3 \left(\lambda_{c} L_{0}(\alpha_{c}) + \lambda_{c}^{3} C_{3}\right) \cdot y \tilde{P}_{\mathcal{P}_{c}} + \lambda_{c}^{2} \left(\nabla^{2} k(0) + \frac{\kappa_{0}}{3} Id\right) \cdot (y, y) Q^{3} \\ + 2 \lambda_{c}^{2} \left(V(0) + \frac{\kappa_{0}}{3}\right) Q + \lambda_{c} \nabla^{2} k(0) \cdot (y, \alpha_{c}) Q^{3} + L_{+}(E_{2} + E_{3}) + i L_{-}(D_{3}) - i \partial_{s} E_{2} \\ + \frac{\lambda_{c}^{3}}{2} \left[\nabla^{3} k(0) \cdot (y, y, y) Q^{3} + 3 y \cdot \left(2 \nabla V(0) Q - \rho \nabla \kappa(0) \partial_{\rho} Q\right)\right] + \mathcal{O}(\lambda_{c}^{4} e^{-C|y|}) \\ \text{To ensure (6.274) we choose } E_{2}, E_{3} \text{ and } D_{3} \text{ solutions to}$$

$$L_{+}E_{2} = -\lambda_{c}^{2} \left(\nabla^{2}k(0) + \frac{\kappa_{0}}{3} Id \right) (y, y) Q^{3} - 2\lambda_{c}^{2} \left(V(0) + \frac{\kappa_{0}}{3} \right) Q - \lambda_{c} \nabla^{2}k(0) (y, \alpha_{c}) Q^{3} + 3\lambda_{c} L_{0}(\alpha_{c}) \cdot y Q + \lambda_{c} a_{2}(\alpha_{c}) \cdot y Q \right)$$
(6.330)

with

$$a_2(\alpha_c) = \frac{-2\left(\nabla^2 k(0).(y,\alpha_c) Q^3, \nabla Q\right)}{\int Q^2} - 3L_0(\alpha_c),$$
(6.331)

and

$$L_{+}E_{3} = -\frac{\lambda_{c}^{3}}{2} \left[\nabla^{3}k(0).(y, y, y) Q^{3} + 3y. \left(2\nabla V(0) Q - \rho \nabla \kappa(0) \partial_{\rho} Q \right) \right] + 3\lambda_{c}^{3} C_{3}. y Q + \lambda_{c}^{3} a_{3}. y Q, \qquad (6.332)$$

with

$$a_{3} = \frac{3}{4} \|Q\|_{L^{2}}^{-2} \left(\int \nabla^{3} k(0) . (., y, y) Q^{4} dy + 2 \nabla \kappa(0) \int |y|^{2} |\nabla Q|^{2} dy \right) - 3 C_{3}$$
(6.333)

and finally we wish to choose D_3 such as

$$L_{-}D_{3} = \frac{2}{C_{0}} \lambda_{c}^{2} \partial_{\lambda_{c}} T_{2} - \frac{1}{C_{0}} \lambda_{c}^{2} \partial_{\lambda_{c}} E_{2}$$

$$(6.334)$$

From (8.2), (6.330) and (6.332) are solvable since their right-hand side are both orthogonal to ∇Q . So there is only left to prove that the right-hand side of (6.334) is orthogonal to Q.

First remind from (6.224) that

$$(T_2, Q) = -\frac{1}{2} (L_+ T_2, \Lambda Q) = 0$$
 (6.335)

Similarly, using the algebraic cancellation $(y_j y_l Q^3, \Lambda Q) = 0$, for j, l = 1, 2 that we have already encountered, along with expression $(Q, \Lambda Q) = 0$, we also get

$$(E_2, Q) = -\frac{1}{2} (L_+ E_2, \Lambda Q) = 0$$
 (6.336)

(6.335) and (6.336) now proves the existence of D_3 .

Proof of (6.275) We compute thanks to (6.267), (6.268) along with the control of the modulation parameters of (6.230) and the equation (6.328)

$$M^{(4)}\left(i\left(|y|^{2}\tilde{P}_{\mathcal{P}_{c}}+F_{2}\right)\right)-i\partial_{s}\left(i\left(|y|^{2}\tilde{P}_{\mathcal{P}_{c}}+F_{2}\right)\right)$$

$$=-i\left[-ib_{c}\lambda_{c}\partial_{\lambda_{c}}|y|^{2}\tilde{P}_{\mathcal{P}_{c}}+L_{g_{c}}\left(|y|^{2}\tilde{P}_{\mathcal{P}_{c}}\right)-|y|^{2}\tilde{P}_{\mathcal{P}_{c}}+\frac{\rho}{\varphi_{c}}\frac{k(\lambda_{c}y+\alpha_{c})}{k(\alpha_{c})}|y|^{2}\tilde{P}_{\mathcal{P}_{c}}|\tilde{P}_{\mathcal{P}_{c}}|^{2}\right]$$

$$+iL_{-}F_{2}+\mathcal{O}\left(\lambda_{c}^{3}e^{-c|y|}\right)$$

$$=-i|y|^{2}\left[-ib_{c}\lambda_{c}\partial_{\lambda_{c}}\tilde{P}_{\mathcal{P}_{c}}+L_{g_{c}}\tilde{P}_{\mathcal{P}_{c}}-\tilde{P}_{\mathcal{P}_{c}}+\frac{\rho}{\varphi_{c}}\frac{k(\lambda_{c}y+\alpha_{c})}{k(\alpha_{c})}\tilde{P}_{\mathcal{P}_{c}}|\tilde{P}_{\mathcal{P}_{c}}|^{2}\right]-4i\Lambda\tilde{P}_{\mathcal{P}_{c}}$$

$$+iL_{-}F_{2}+\mathcal{O}\left(\lambda_{c}^{3}e^{-c|y|}\right)$$

$$=i\left[L_{-}F_{2}-4\Lambda\tilde{P}_{\mathcal{P}_{c}}\right]+\mathcal{O}\left(\lambda_{c}^{3}e^{-c|y|}\right)$$

$$(6.337)$$

To fulfill (6.275), we choose F_2 solution to

$$L_{-}F_{2} = -4E_{2} \tag{6.338}$$

which is obviously possible from (8.2).

Proof of (6.276) Ultimately, (6.276) simply relies on (6.47) and

$$M^{(4)}(\varrho) = L_+(\varrho) + \mathcal{O}\left(\lambda_c^2 e^{-c|y|}\right) = |y|^2 Q + \mathcal{O}\left(\lambda_c^2 e^{-c|y|}\right)$$

7 Appendix A : Some remarks about the Riemannian Geometric aspects

In the sequel, we denote by (M, g_M) a smooth Riemannian Manifold of dimension 2, equipped with the Levi-Civita connection ∇ .

First, recall the Levi-Civita connection is a smooth section $\nabla \in \Gamma(\otimes^2 T^*M \otimes TM)$ that satisfies

- (i) $\nabla g_M = 0$ (Metric is preserved)
- (*ii*) $\forall X, Y \in \Gamma(TM), \quad \nabla_X Y \nabla_Y X = [X, Y]$ (It is torsion free) where [X, Y] is the Lie-brackets for vector fields X and Y
- (*iii*) $\forall f \in C^{\infty}(M), \forall X \in \Gamma(TM) \quad \nabla_X f = Xf$
- (*iv*) In local coordinates(x^i): $\Gamma_{ij}^k = \frac{1}{2} g_M^{kl} (\partial_j (g_M)_{il} + \partial_i (g_M)_{jl} \partial_l (g_M)_{ij})$ where $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$
- (v) In local coordinates (x^i) , the connection Laplacian is defined by : $\forall f \in C^{\infty}(M)$:

$$\Delta f = (g_M)^{ij} \nabla_{\partial_i} \nabla_{\partial_j} f = (g_M)^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right)$$

(vi) The curvature tensor $R_{\nabla} \in \Gamma(\otimes^3 T^*M \otimes TM)$ is defined by $R_{\nabla}(X,Y)\xi = \nabla_Y(\nabla_X\xi) - \nabla_X(\nabla_Y\xi) + \nabla_{[X,Y]}\xi, \quad \forall X,Y,\xi \in \Gamma(TM)$ The Riemannian Curvature Tensor $R \in \Gamma(\otimes^4 T * M)$ is defined by $R(X,Y,Z,W) = g_M(R_{\nabla}(X,Y)Z,W), \quad \forall X,Y,Z,W \in \Gamma(TM)$

(vii) In local coordinates (x^i) :

$$R_{\nabla} = \left(R_{\nabla}\right)_{ijk}^{l} dx^{i} \otimes dx^{j} \otimes dx^{l} \otimes \partial_{l}, \quad \text{with} \quad R_{\nabla}(\partial_{i}, \partial_{j})\partial_{k} = \left(R_{\nabla}\right)_{ijk}^{l} \partial_{l}$$
$$R = R_{ijkl} dx^{i} \otimes dx^{j} \otimes dx^{l} \otimes dx^{l}, \quad \text{with} \quad R_{ijkl} = (g_{M})_{lp} \left(R_{\nabla}\right)_{ijk}^{p} \partial_{l}$$

and :

$$: (R_{\nabla})_{ijk}^{\ l} = \partial_j \Gamma_{ik}^{\ l} - \partial_i \Gamma_{jk}^{\ l} + \Gamma_{ik}^{\ m} \Gamma_{jm}^{\ l} - \Gamma_{jk}^{\ m} \Gamma_{im}^{\ l}$$
$$R_{ijkl} = \frac{1}{2} \left(\partial_j \partial_k (g_M)_{il} + \partial_i \partial_l (g_M)_{jk} - \partial_i \partial_k (g_M)_{jl} - \partial_j \partial_l (g_M)_{ik} \right) + (g_M)_{lp} \left(\Gamma_{ik}^{\ m} \Gamma_{jm}^{\ p} - \Gamma_{jk}^{\ m} \Gamma_{im}^{\ p} \right)$$

The whole purpose of introducing the metric terms g and G was to make our laplacian operator in the NLS equation to mimic the so-called Laplace-Beltrami operator, which, in a local set of coordinates (x^i) , may be written as

$$\Delta_{g_{M}} = \frac{1}{\sqrt{g_{M}}} \, \partial_{i} \Big(\sqrt{g_{M}} \left(g_{M} \right)^{ij} \partial_{j} \Big)$$

where the metric g_M is locally given by its matrix $((g_M)_{ij}) = (g_M(\partial_i, \partial_j))$, while the inverse matrix of g_M^{-1} is denoted by $((g_M)^{ij})$.

In terms of the connection ∇ , one may now easily check that the operator $\Delta_{G,g} = \frac{1}{g} \partial_i (G_{ij} \partial_j)$ may now be written

$$\Delta_{G,g}f = \frac{1}{g}G_{ij}\nabla_{\partial_i}\nabla_{\partial_j}f + \frac{1}{g}G_{ij}\Gamma_{ij}^{\ k}\nabla_{\partial_k}f + \frac{1}{g}\nabla_{\partial_i}G_{ij}\nabla_{\partial_j}f$$

In other words,

$$\Delta_{G,q}f = \Delta_{g_M}f, \quad \text{with} \quad g_M = g \, G^{-1} \tag{7.1}$$

Now, seeing the euclidean space \mathbb{R}^2 we are working with from the beginning as the riemannian manifold (\mathbb{R}^2, h) , with metric $h \in \Gamma(Sym^2T^*\mathbb{R}^2) \subset \Gamma(\otimes^2 T^*\mathbb{R}^2)$ the smooth section defined by $h(x) = g(x) G(x)^{-1}$ for all $x \in \mathbb{R}^2$.

Then, we focus on what must look like that metric in our region of interest in \mathbb{R}^2 , in our case near the blow up point we have described, that is near the origin the way we built it. A proof to the Gauss Lemma may be found in details in [AH11], as we recall it

Lemma 7.1. Let (M, g_M) be a riemannian manifold. With respect to a geodesic normal coordinates system about $p \in M$, the metric $(g_M)_{ij}$ may be expressed as :

$$(g_M)_{ij}(x^1, x^2) = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + \mathcal{O}(||x||^3)$$

This means the normal mapping provides a coordinates system that describes the neighborhood of some point $p \in M$ and directly link the default of flatness at p with the riemannian curvature tensor. Recall for a riemannian manifold of dimension n the exponential map at p is defined by

$$\exp_n: T_p M \cong \mathbb{R}^n \to M, \quad X \in T_p M \mapsto \gamma_X(1)$$

where γ_X is the unique geodesic passing through p at 0 and whose tangent vector at 0 is X. Thus, $\gamma_X : t \mapsto \gamma_X(t) = \exp_p(tX).$

Consider here $M = \mathbb{R}^2$, $p = 0 \in \mathbb{R}^2$ is the blow-up point, then there are neighborhoods $U_0, V_0 \subset \mathbb{R}^2$, such that

$$(\exp_0)^{-1}: U_0 \subset (\mathbb{R}^2, h) \to V_0 \subset T_0 \mathbb{R}^2 = \mathbb{R}^2, \quad x \in \mathbb{R}^2 \mapsto z = (z^1, z^2) = (\exp_0)^{-1}(x)$$

 $(\exp_0)^{-1}: U_0 \to V_0$ being a diffeormorphism, given a system of cartesian coordinates on V_0 , one may now define the normal coordinates system (z^1, z^2) on U_0 . One may now recall the following lemma for normal coordinates :

Lemma 7.2. In normal coordinates, the Levi-Civita connection equals at the origin the naive connection, *i.e.* the Christoffel symbols vanish at the origin.

Back to our case, Lemma 7.1 together with Lemma 7.2 easily imply

$$\frac{\partial h}{\partial x^i}(0) = \frac{\partial h}{\partial z^i}(0) = 0, \quad i = 1, 2$$

which of course justify the part of hypothesis (H2) and (H3) that

$$\nabla g(0) = 0, \quad \nabla G_{ij}(0) = 0, \quad i, j = 1, 2$$

Our next goal is to go deeper into the riemannian manifold structure idea to rewrite hypothesis (H4)-(H7). This will inolve we look a bit closer the Riemannian curvature in the riemannian manifold (\mathbb{R}^2, h) we consider our situation in - or at least considering a neighborhood of the origin is a piece of that manifold. First, from Lemma 7.1, one can deduce the components of the riemannian curvature tensor in a small neighborhood of the origin are given by

$$h^{ij}(x^1, x^2) = \delta_{ij} + \frac{1}{3} R_{ikjl} + \mathcal{O}(||x||^3), \quad \text{so that} \quad \nabla^2 h^{ij}(0) \cdot (\partial_k, \partial_l) = \frac{2}{3} R_{ikjl}$$

hence
$$\nabla^2 \left(\frac{1}{g} G_{ij}\right)(0) \cdot (\partial_k, \partial_l) = \frac{2}{3} R_{ikjl}$$

Then, using that g(0) = 1, $G_{ij}(0) = \delta_{ij}$ and $\nabla g(0) = \nabla G_{ij}(0) = 0$, i, j = 1, 2, one gets

$$\nabla^2 G_{ij}(0) \cdot (\partial_k, \partial_l) = \frac{2}{3} R_{ikjl} + \partial_{kl}^2 g(0) \,\delta_{ij} \tag{7.2}$$

Recall the symmetries of the riemannian curvature tensor are given by

$$\begin{aligned} R_{ijkl} + R_{jikl} &= 0 = R_{ijkl} + R_{ijlk} \\ R_{ijkl} &= R_{klij} \\ R_{ijkl} + R_{jkil} + R_{kijl} &= 0 \quad (\text{ First Bianchi identity }) \end{aligned}$$

Now, since the metric h is symmetric, so is the riemannian curvature tensor component R_{ikjl} with respect to index i and j, or k and l respectively : $R_{ikjl} = R_{jkil}$ (*). In particular this means one has the equalities :

$$R_{1112} = R_{2221}, \quad R_{1221} = R_{2112} \stackrel{(*)}{=} R_{2211} = R_{1122}, \quad R_{1212} = R_{2122}$$

In terms of curvature components, using (7.2), the (1.42) definitions, (H4) - (H6) become

$$(H4)_R: \quad \Delta g(0) + K_2 R_{1212} = 0$$

$$(H5)_R: \quad \nabla^2 (k-g)(0) + \frac{1}{3} \begin{pmatrix} -R_{1212} & 0 \\ R_{1212} & 0 \end{pmatrix} < 0$$

$$(H6)_R: \quad \Delta (k-g)(0) < \kappa_{22} \Delta g(0) + \frac{2}{3} R_{1212}$$

with constants

$$K_{2} = \frac{4}{3} \frac{1+2K_{22}}{1+2(K_{21}+K_{22})}, \qquad \kappa_{22} = 6 \|Q\|_{L^{2}}^{-2} \left\|\frac{|y|}{2}|\nabla Q|\right\|_{L^{2}}^{-2}$$
$$K_{21} = \frac{1}{8} \int \left[\left(3+\frac{|y|^{2}}{2}\right)Q^{2}-3|y|^{2}|\nabla Q|^{2}\right] \left\|\frac{|y|}{2}Q\right\|_{L^{2}}^{-2}$$
$$K_{22} = \frac{1}{8} \int \left[\left(1-\frac{|y|^{2}}{2}\right)Q^{2}-|y|^{2}|\nabla Q|^{2}\right] \left\|\frac{|y|}{2}Q\right\|_{L^{2}}^{-2}$$

Now consider a more simple situation where the metric h is actually a perturbation of the identity near the origin, where our schrödinger solution was designed to blow up. Recall, in two-dimensional manifolds, every metric is conformal. So let φ be a month function so that $h(x) = e^{\varphi(x)} Id$, then

$$G_{ij}(x) = g(x) e^{-\varphi(x)} \delta_{ij}, \quad i, j = 1, 2$$

and using as before the fact that in geodesic normal coordinates, the first derivatives of the mertic terms g and G_{ij} vanish, one gets

$$\nabla^2 G_{ij}(0) \cdot (\partial_k, \partial_l) = \partial_{kl}^2 (g - \varphi)(0) \,\delta_{ij}, \quad i, j = 1, 2$$

and the (H4)-(H6) hypothesis may then be written as

$$(H4)_{\varphi}: \quad \Delta\varphi(0) = \frac{1+2K_{23}}{2+2K_{23}}\Delta g(0)$$

$$(H5)_{\varphi}: \quad \nabla^2(k-g)(0) + \nabla^2\varphi(0) < 0$$

$$(H6)_{\varphi}: \quad \Delta k(0) < \left(1+\kappa_{22}-\frac{1+2K_{23}}{2+2K_{23}}\right)\Delta g(0)$$

with constants

$$K_{23} = \frac{1}{2} \int \left[Q^2 - |y|^2 |\nabla Q|^2 \right] \left\| \frac{|y|}{2} Q \right\|_{L^2}^{-2}, \qquad \kappa_{22} = 6 \left\| Q \right\|_{L^2}^{-2} \left\| \frac{|y|}{2} |\nabla Q| \right\|_{L^2}^{-2}$$

Eventually, notice that provided $\Delta g(0) > 0$, $(H5)_{\varphi} \Rightarrow (H6)_{\varphi}$.

8 Appendix B : Some remarks about technical arguments

8.1 The kernel of the linearized operator

In order to approximate the inhomogeneous NLS solution of equation (1.1) or (6.1), we have used properties of the NLS operator linearized around the ground state $L = (L_+, L_-)$ which we will recall here. For details about the ground state properties and the previous operator along with introduction of the modulation theory, one should refer to [Wei83], [Wei85], [Cha08].

First, the kernel of the linearized operator close to Q may be computed as

$$Ker(L_{-}) = span(Q), \quad Ker(L_{+}) = span(\partial_{j}Q, j = 1, 2) = span(\nabla Q)$$

$$(8.1)$$

so that L_+ and L_- being self-adjoint operators, it implies the image of L may then be deduced as follows

$$\exists T \in H^2, \quad L_+T = F \Leftrightarrow F \in Im(L_+) = Ker(L_+)^{\perp} \Leftrightarrow \left(F, \nabla Q\right) = 0$$

$$\exists S \in H^2, \quad L_-S = G \Leftrightarrow G \in Im(L_-) = Ker(L_-)^{\perp} \Leftrightarrow \left(G, Q\right) = 0$$
(8.2)

which gives a simple process to decide we ther or not T and S exist.

Exponential estimate of the remaining terms We now look forward to proving (1.41) and (6.21). We recall that the description (8.1) of the kernel of the linearized operator (L_+, L_-) ensures the standard uniform elliptic estimates :

$$\forall f \in (\nabla Q)^{\perp}, \quad \|e^{\delta|y|} L_{+}^{-1} f\|_{H^{2}} \lesssim \|e^{2\delta|y|} f\|_{L^{2}} \forall f \in (Q)^{\perp}, \quad \|e^{\delta|y|} L_{-}^{-1} f\|_{H^{2}} \lesssim \|e^{2\delta|y|} f\|_{L^{2}}$$

$$(8.3)$$

for a universal constant $\delta > 0$. At the end of part 1.4 and the beginning of part 6, we have constructed a fourth order approximate solution of (1.40) and (6.20), which means there is a well-localized function $\Psi_{\mathcal{P}}$ that embodies the polynomials remaining terms of NLS equation and satisfies thanks to (8.3)

$$e^{\delta |y|} \Psi_{\mathcal{P}}(y) = \mathcal{O}\left(\mathcal{P}^5 + \mathcal{P}\left(|\alpha|^2 + |\beta|^2\right) + \mathcal{P}^3\left(b - \frac{\lambda}{C_0}\right)\right) \quad \text{in} \quad H^2(\mathbb{R}^2)$$

so that for any $p \in \mathbb{N}^2$

$$\partial^{p}\Psi_{\mathcal{P}}(y) = \mathcal{O}\left(\left[\mathcal{P}^{5} + \mathcal{P}\left(|\alpha|^{2} + |\beta|^{2}\right) + \mathcal{P}^{3}\left(b - \frac{\lambda}{C_{0}}\right)\right]e^{-\delta_{p}|y|}\right)$$
(8.4)

8.2 Estimates in Sobolev Space.

8.2.1 Sobolev embeddings.

Often in the study of PDEs have we need to use functional analysis tools among which Sobolev Spaces are of most importance. A comprehensive introduction to Sobolev Spaces may be found in [Leo09]. Let us here only recall the Sobolev emedding results theorems we have been using **Theorem 8.1.** (a) For any $u \in W^{1,p}(\mathbb{R}^N)$, and $1 \le p < N$, one has

$$\|u\|_{L^{p^*}} \lesssim \|\nabla u\|_{L^q}$$

In particular, there is a continuous embedding

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \text{ for all } p \leq q \leq p^* = \frac{Np}{N-p}$$

- (b) $W^{1,N}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$, for all $N \leq q < +\infty$
- (c) For any $N , <math>W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $C^{0,1-\frac{N}{p}}_{\to 0}(\mathbb{R}^N)$

The Paley-Littlewood decomposition (or dyadic decomposition) is a key tool to get useful results for PDEs study, such as Sobolev embeddings, paradifferential operators. For a more detailed study of the decomposition and some applications, one may refer to [Xu96]. Let us recall then Sobolev embedding theorem results obtained that way

Theorem 8.2. (a) For any $s \in \mathbb{R}$, we have continuous embedding $H^s(\mathbb{R}^N) \subset C^{s-N/2}(\mathbb{R}^N)$

- (b) For any 0 < s < N/2, we have continuous embedding $H^s(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$, for all $2 \le q \le \frac{2N}{N-2s}$
- (c) We have continuous embedding $H^{N/2}(\mathbb{R}^N) \subset VMO$

where VMO denotes the vanishing mean oscillation functions that satisfy $u \in L^1_{loc}(\mathbb{R}^N)$ and

$$\sup_{B} \frac{1}{|B|} \int_{B} |u - u_{B}| \, dx < +\infty \quad \lim_{diam(B) \to 0} \frac{1}{|B|} \int_{B} |u - u_{B}| \, dx = 0$$

In particular notice

$$H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2), \quad 2 \le q < +\infty$$

$$(8.5)$$

Now one may also find a slightly more general result, for instance, see the lectures notes of Pr. T. Tao on his UCLA webpage http://www.math.ucla.edu/~tao/254a.1.01w/, where the author uses again Paly-Littlewood analysis to get the weak derivatives version of Sobolev embedding

Theorem 8.3. For any $n \in \mathbb{N}$, we have the following

$$W^{s,p}(\mathbb{R}^n) \subset W^{s',q}(\mathbb{R}^n), \quad 1 \le p \le q \le \frac{np}{n - p(s - s')} < \infty, \quad 0 < s' < s < s' + \frac{n}{p}$$

Let us then finish with a weighted Sobolev space embedding type of result that is useful. We consider here a smooth metric g(x)dx which asymptotically resembles the euclidean metric in the sense we have pointed out earlier, that is it has the form $g(x) = 1 + o(|x|^{-1})$, as $|x| \to +\infty$.

Lemma 8.4. For $\delta > 0$, and $x \in \mathbb{R}^2$, let $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let

$$H^1_{\delta}(gdx) = \left\{ u \in H^1(\mathbb{R}^2, gdx), \quad \left(\left\langle x \right\rangle^{\delta} + \left\langle D_x \right\rangle^{\delta} \right) \ \left(|u| + |\nabla u| \right) \in L^2(gdx) \right\}$$

where D_x is the derivative with respect to x, defined by its Fourier transform. Then

$$H^1_{\delta}(\mathbb{R}^2, gdx) \underset{compact}{\hookrightarrow} H^1(\mathbb{R}^2, gdx)$$

Proof of Lemma 8.4

Let $(v_n) \subset H^1_{\delta}$. Our goal is to show there is a subsequence of (v_n) that converges in H^1 . $H^1(\mathbb{R}^2)$ being a Hilbert space, it suffices to prove (v_n) is a Cauchy sequence.

First we use the well known fact that for every N > 0, B_N being the closed ball of radius N which is a compact of \mathbb{R}^2 , from Rellich-Kondrachov theorem we have : $H^1_{\delta}(B_N) \hookrightarrow H^1(B_N)$. Thus, there is a subsequence of (v_n) we still note by v_n that satisfies :

$$v_n|_{B_N} \longrightarrow w_N$$
 in $H^1(B_N)$, as $n \to +\infty$

Now, let $\eta > 0$. Let R > 0 such that $\langle R \rangle^{2\delta} > \frac{1}{\eta}$. This implies

$$\int_{|x| \ge R} \left(|v|^2 + |\nabla v|^2 \right) g \, dx \le \int_{|x| \ge R} \frac{\langle x \rangle^{2\delta}}{\langle R \rangle^{2\delta}} \left(|v|^2 + |\nabla v|^2 \right) g \, dx \le \eta \, \|v\|_{H^1_{\delta}(g \, dx)}$$

Then, since $v_n \longrightarrow w_R$ in $H^1(|x| \le R)$, for p, q big enough we have $||v_p - v_q||_{H^1(|x| \le R)} \le \eta$ and hence

$$\|v_p - v_q\|_{H^1} \leq \|v_p - v_q\|_{H^1(|x| \le R)} + \|v_p - v_q\|_{H^1(|x| \ge R)} \le \eta + \eta \, \|v_p - v_q\|_{H^1_\delta} \le C \, \eta$$

This proves the Lemma.

8.2.2 Paraproducts estimates.

A last tool for estimating in Sobolev spaces that may prove very useful is the so-called Paraproducts, that is a non-commutative multilinear operator which in some sense is acting on functions very much like products. Here we only recall the Coifman-Meyer multilinear theorem, whose proof can be found in [MPTT03a], and that well summarizes the kind of estimates we intend to obtain from its use

Theorem 8.5. Let $m \in L^{\infty}(\mathbb{R}^n)$ be a bounded function which is smooth away from the origin and satisfies the following Marcinkiewicz-Mihlin-Hörmander type condition

 $|\partial^{\alpha}m(\xi)| \lesssim |\xi|^{-|\alpha|}, \quad \text{for sufficiently many multiindices } \alpha$

and for $f_1, ..., f_n$ schwartz functions on the real line, we define the n-linear operator T_m by the formula

$$T_m(f_1, ..., f_n)(x) = \mathcal{F}_{\xi_1, ..., \xi_n}^{-1} \left[m(\xi) \, \hat{f}_1(\xi_1) ... \hat{f}_n(\xi_n) \right]$$

where \mathcal{F} designs the Fourier transform. Then T_m maps $L^{p_1} \times \ldots \times L^{p_n} \to L^p$ as long as $1 < p_i \leq \infty$, for any $1 \leq i \leq n$, and $0 , and <math>1/p = 1/p_1 + \ldots + 1/p_n$.

More details and extensions may be found in [KP], [MPTT03b] or [MPTT04]. Basically, this all work is about generalizing estimates of the type

$$\begin{aligned} \left\|\nabla^{s}(uv)\right\|_{L^{r}} &\lesssim \left\|\nabla^{s}u\right\|_{L^{p}} \|v\|_{L^{q}} + \|u\|_{L^{p}} \left\|\nabla^{s}v\right\|_{L^{q}}, \quad \text{for any } u, v \in \mathcal{S}, \ s > 0, \\ \text{and} \ 1 \le p, q \le \infty, \ 1/r = 1/p + 1/q \end{aligned}$$

We use the result in the following situation to get (3.124) or (6.149). Indeed, since $\frac{3}{4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8}$, using also Sobolev embedding (8.5) we get

$$\left\|\nabla^{3/2}(u|u|^2)\right\|_{L^{4/3}} \lesssim \left\|\nabla^{3/2}u\right\|_{L^2} \|u\|_{L^8}^2 \lesssim \left\|\nabla^{3/2}u\right\|_{L^2} \|u\|_{H^1}^2$$

8.3 The Strichartz Estimates and The Smoothing effect.

8.3.1 The Strichartz estimates for Schrodinger equations with variable coefficients

The Strichartz estimates are a set of $L_t^t L_x^r$ estimates for solutions of homogeneous and inhomogeneous linear dispersive and wave equations. For the Schrödinger equation, it may be proved using a dispersion result, the Hardy-Littlewood-Sobolev Lemma and a duality argument (the so called TT^* method). While the dispersion result is easily obtained in the flat Laplacian case from writing the homogeneous solution in term of the initial datum u_0 with Fourier transform of the Schrödinger kernel. In a general non-flat Laplacian case, an explicit formula is out of the question, and the dispersion result is therefore much harder to get and may be replaced by a microlocalized dispersion result [RZ05] combined with the Keel-Tao Theorem [KT98] and the Christ-Kiselev Lemma [CK]. For even weaker assumptions, one may look into [ST].

For a comprehensive study of local in time Strichatz estimates for the Schrödinger operator, one should read [RZ05] for an operator related to a non trapping asymptotically flat pertubation of the usual Laplacian in \mathbb{R}^n , with a bounded potential, or [Miz12] for an operator with variable coefficients and unbounded electromagnetic potentials. Though it would be possible to refine the assumptions we have made regarding the potential V, for convinience we keep working with a bounded potential.

Let $\mathcal{L} = \frac{1}{g} div(G\nabla) + V$ be the type of operator we have been working with (which satisfies $G_{ij} = G_{ji}$), and let

$$T: (t,f) \in \mathbb{R} \times L^q_t L^r_x \mapsto Tf = e^{it\mathcal{L}} f$$

be the action of its associated semi-group. Let also

$$p(x,\xi) = G_{ij}(x)\,\xi_i\,\xi_j$$

be the principal symbol of operator \mathcal{L} . In order to recall the Theorem I.0.1 from [RZ05], we first introduce its assumptions. Associate to the symbol p the bicharacteristic flow given by the system

$$\begin{cases} \dot{x_j}(t) = \frac{\partial p}{\partial \xi_j} (x(t), \xi(t)), & x_j(0) = x_j \\ \dot{\xi_j}(t) = -\frac{\partial p}{\partial x_j} (x(t), \xi(t)), & \xi_j(0) = \xi \end{cases}$$

Now for any $0 < \sigma_0 < 1$, set the following space

$$B_{\sigma_0} = \left\{ a \in C^{\infty}(\mathbb{R}^n) : \forall \alpha \in \mathbb{N}^n, \exists C_{\alpha} > 0 : |\partial^{\alpha} a(x)| \le \frac{C_{\alpha}}{\langle x \rangle^{1+|\alpha|+\sigma_0}}, \forall x \in \mathbb{R}^n \right\}$$

We shall then assume

The coefficients
$$g(x), G_{ij}(x), V(x)$$
 are real valued, $1 \le i, j \le n$
 $\exists \sigma_0 > 0, \quad G^{ij} - I_{ij} \in B_{\sigma_0}, \quad g - 1 \in B_{\sigma_0}, \quad V \in L^{\infty}(\mathbb{R}^n)$
 $\exists \nu > 0, \ \forall \ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \ p(x,\xi) \ge \nu |\xi|^2$

$$(8.6)$$

Denote by $(x(t, x, \xi), \xi(t, x, \xi))$ the solution of the above system. That flow exists for every $t \in \mathbb{R}$, since from (8.6) and the fact that p is constant along the bicharacteristic, the times derivatives remain bounded

$$\begin{cases} |\dot{x_j}(t)| \le 2 \left| G^{ij}(x) \,\xi_i \right| \le C \left| \xi(t) \right| \le \frac{C}{\nu^{1/2}} \, p\big(x(0), \xi(0)\big)^{1/2} \\ |\dot{\xi_j}(t)| \le \left| \partial_{x_j} G^{ij}(x) \,\xi_i \,\xi_j \right| \le C \left| \xi(t) \right|^2 \le \frac{C}{\nu} \, p\big(x(0), \xi(0)\big) \end{cases}$$

Now the *non-trapping* condition we have been referring to before can be expressed by the fact the above flow is neither trapped backward nor forward in time (the flow goes out of any compact set in finite time), in other words

$$\forall (x,\xi) \in T^* \mathbb{R}^n \setminus \{0\}, \ \lim_{t \to \pm \infty} \left| x(t,x,\xi) \right| = +\infty$$
(8.7)

Theorem I.0.1 of [RZ05] now states

Theorem 8.6. Under assumptions (8.6) and (8.7), for any $u_0 \in L^2(\mathbb{R}^n)$, $\tau > 0$ and (q,r) a couple of real numbers such that q > 2, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$

$$\left\| e^{i t \mathcal{L}} u_0 \right\|_{L^q_{[-\tau,\tau]};L^r_x} \lesssim \| u_0 \|_{L^2}$$

Then denote the hermitian adjoint T^* of previous operator T, one has

$$TT^*: f \in L^{\overline{q}}_t(\mathbb{R}, L^{\overline{r}}_x(\mathbb{R}^n)) \mapsto \int_{\mathbb{R}} e^{i(t-s)\mathcal{L}} \left(f(s)\right) ds \in L^q_t(\mathbb{R}, L^r_x(\mathbb{R}^n))$$

and the Christ-Kiselev Lemma, together with the TT^* -principle yields the inhomogeneous Strichartz estimates

$$\left\|T T^* f\right\|_{L^{q_1}\left(\mathbb{R}, L^{r_1}\left(\mathbb{R}^n\right)\right)} \lesssim \left\|f\right\|_{L^{\overline{q}_2}\left(\mathbb{R}, L^{\overline{r}_2}\left(\mathbb{R}^n\right)\right)}, \quad \forall f \in L^{\overline{q}_2}\left(\mathbb{R}, L^{\overline{r}_2}\left(\mathbb{R}^n\right)\right)$$

for any couples (q_j, r_j) , j = 1, 2 such that $2 < q_j \le +\infty$, and $\frac{1}{q_j} + \frac{1}{r_j} = \frac{1}{2}$, where $\frac{1}{p} + \frac{1}{p} = 1$. Further details and externsions may be found for instance in the thesis of E.Y. Ovcharov, avalable on the website of the University of Edinburgh :

http://www.maths.ed.ac.uk/pg/thesis/ovcharov.pdf.

Then, using the Duhamel formula for the solution of the Schrödinger equation $i \partial_t u + \mathcal{L}u = F$, with initial datum $u_0 \in H^1$, one has

$$u(t) = e^{i t \mathcal{L}} u_0 - i \int_0^t e^{i (t-s) \mathcal{L}} \left[F(s) \right] ds$$

so the solution u satisfies the local in time Strichartz estimates

$$\|u\|_{L^{q_1}\left([-\tau,\tau],L^{r_1}(\mathbb{R}^n)\right)} \lesssim \|u_0\|_{L^2} + \|F\|_{L^{\overline{q}_2}\left([-\tau,\tau],L^{\overline{r}_2}(\mathbb{R}^n)\right)}$$

8.3.2 The Smoothing Effect of the Schrödinger linear flow

The smoothing effect of the Schrödinger linear flow has been extensively studied since the 1980's. For a sharp view of the problem, one could read the complete study of the matter in [CKS], or the works that followed with [Doib], [RZa] or [RZb]. Here we are working with a Laplace-Beltrami type of operator and a potential. The associated metric has been assumed from the beginning to be a perturbation of usual euclidean metric around zero which was our blowing-up point. Also see Doia for complementary statements.

The smoothing effect requires similar assumptions than those we have made to obtain the Strichartz estimates. We begin with recalling some statement from S. Doi [Doib], [Doia]

Theorem 8.7. Assume (M, g_0) is either the euclidean space or the hyperbolic space of constant curvature $-\rho^2$, $\rho > 0$, and $g = g_0$ outside of a compact set, then one has

 $u_0 \in L^2(M) \mapsto e^{i t \Delta_g} u_0 \in L^2_{loc}(\mathbb{R}, H^{1/2}_{loc}(M))$ is continuous

if and only if there is no complete geodesic contained in a compact subset.

which suggests that the existence of a trapped geodesic is a key obstruction to the smoothing effect. With basically the same assumptions, [CKS] proved

Theorem 8.8. Assume $M = \mathbb{R}^n$, and g is equal to the euclidean metric outside of a compact set, then for any $k \in \mathbb{N}$ the two following applications are also continuous

$$u_{0} \in L^{2}(M, (1+|x|^{k}) dx) \mapsto t^{k/2} e^{i t \Delta_{g}} u_{0} \in L^{2}_{loc}([0,\infty), H^{(1+k)/2}_{loc}(M))$$

$$u_{0} \in L^{2}(M, (1+|x|^{k}) dx) \mapsto t^{k/2} e^{i t \Delta_{g}} u_{0} \in C^{0}([0,\infty), H^{k/2}_{loc}(M))$$

giving a higher order smoothing effect. However, these results are homogeneous. To see how to get inhomogeneous local smoothing results, and to somehow comprehend how the local smoothing effect may be reduced to bounds on the cut-off resolvent of the operator, one may refer to [BGTb].

Following [RZb], and [AKR12] denote again by $p(x,\xi)$ the principal symbol of operator \mathcal{L} , we assume 8.6 still holds, and now denote by Φ_t the bicharacteristic flow of p. Recall Φ_t was defined for all $t \in \mathbb{R}$ thanks to 8.6, and $\Phi_t(x,\xi) = (x(t),\xi(t))$ were its geodesics. Then define $S^*(\mathbb{R}^n) = \{(x,\xi) \in T^*(\mathbb{R}^n) : p(x,\xi) = 1\}$, so that $\Phi_t(S^*(\mathbb{R}^n))$ is the set of geodesics with unit velocity. The non trapping assumption may be written as

 $\forall K \text{ compact } \subseteq S^*(\mathbb{R}^n), \ \exists \ t_K > 0, \text{ such that } \forall t \ge t_k, \ \Phi_t(K) \cap K = \emptyset$

and is equivalent to 8.7. Eventually a last assumption regarding the asymptotic behavior of the metric terms is required

$$\begin{aligned} |x| \nabla_x G_{ij}(x) &\to 0, \quad \text{as } |x| \to +\infty, \quad 1 \le i, j \le n \\ V \in L^{\infty}(\mathbb{R}^n), \quad V \ge -C_0, \quad C_0 \ge 0 \end{aligned}$$

$$\tag{8.8}$$

Now Theorem 1.1 of [RZb] or Theorem 1.2 of [AKR12] may be weakened to

Theorem 8.9. Assuming the operator \mathcal{L} satisfies to assumptions 8.6, 8.7, 8.8, and since \mathcal{L} is self-adjoint in $L^2(g \, dx)$, for any $\nu \in (1/2, 1]$, and $s \in (-1/2, 1/2)$, if u(t) is solution of the Schrödinger equation $i \partial_t u + \mathcal{L}u = F$ on [0, T] with initial datum $u_0 \in H^1$, one has

$$\left\| \langle x \rangle^{-\nu} \mathcal{D}^{s+1/2} u \right\|_{L^{2}[0,T];L^{2}(g(x)dx)} \lesssim \| \mathcal{D}^{s} u_{0} \|_{L^{2}} + \left\| \langle x \rangle^{\nu} \mathcal{D}^{s-1/2} F \right\|_{L^{2}[0,T];L^{2}(g(x)dx)}$$

where $\mathcal{D} = ((1 + C_0) Id + \mathcal{L})^{1/2}$ which is well defined by functional calculus of self-adjoint positive operators.

Applying this last result for $\langle x \rangle u$ and s = 0, $\nu = 1$ yields

$$\left\|\mathcal{D}^{1/2}u\right\|_{L^{2}[0,T];L^{2}(g(x)dx)} \lesssim \left\|\langle x\rangle u_{0}\right\|_{L^{2}} + \left\|\langle x\rangle^{2}\mathcal{D}^{-1/2}F\right\|_{L^{2}[0,T];L^{2}(g(x)dx)}$$

In other words the operator $T = e^{i t \mathcal{L}}$ we have introduced in the previous section satisfies

$$T: L^2 \to L^2[0,T]\langle x \rangle^{-1} H^{1/2} \quad \text{and} \quad T^*: L^2[0,T]\langle x \rangle H^{1/2} \to L^2_x \quad \text{are continuous}$$

so that combined with Strichartz estimates of the previous section with admissible pairs (4, 4) and $(\infty, 2)$

$$TT^*: L_t^{4/3} L_x^{4/3} \cap L^2[0,T] \langle x \rangle^2 H^1 \to L_t^\infty L_x^2 \cap L^2[0,T] H^{3/2} \quad \text{is continuous}$$

Altogether with the Duhamel formula, this explains how we got estimates of type (3.120) and (6.145). Indeed, whenever a solution for the Schrödinger equation would satisfy $i\partial_t u + \mathcal{L}u = F_1 + F_2$, one could thus write

$$\|u\|_{L^{\infty}, H^{3/2}} \lesssim \|F_1\|_{L_t^{4/3}, L_x^{4/3}} + \|(1+|x|^2) F_2\|_{L_t^2, H^1}$$

8.4 Elements of pseudodifferential calculus.

We recall now a few basic properties about pseudo-differential calculus. For more details, one may read [Mar02], or [Tay91] for even further details about nonlinear PDE's aspects. Here we mainly refer to [Tay97] and the lectures notes [Tay08] available on Mr Taylor's webpage http://math.unc.edu/Faculty/met/.

Let $\langle p \rangle = (1 + |p|^2)^{1/2}$. Assuming $\rho \ge 0, \, \delta_1, \delta_2, \delta \in [0, 1], \, m \in \mathbb{R}$, we define the space of symbols $S^m_{\rho, \delta_1, \delta_2}$ to consist of $C^{\infty}((\mathbb{R}^2)^3)$ functions satisfying

$$\left|D_x^{\beta_1}D_y^{\beta_2}D_\xi^{\alpha}a(x,y,\xi)\right| \leq C_{\alpha\beta_1\beta_2}\,\langle\xi\rangle^{m-\rho\,|\alpha|+\delta_1\,|\beta_1|+\rho_2\,|\beta_2|}, \quad \forall \ x,y,\xi\in\mathbb{R}^2$$

We then consider the Fourier integral representation of a pseudodifferential operator on \mathbb{R}^2 given by

$$Au(x) = Op(a)u(x) = (2\pi)^{-2} \iint e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) \, dy \, d\xi \tag{8.9}$$

where we have defined the Fourier transform and its inverse of u as

$$\hat{u}(\xi) = \int e^{-i\,y\,\cdot\,\xi} u(y)\,dy, \quad u(x) = (2\pi)^{-2} \int e^{i\,x\,\cdot\,\xi} \hat{u}(\xi)\,d\xi$$

Also define the differential operators

$$d_{\xi} = \frac{1}{i} \frac{\partial}{\partial \xi}, \quad d_x = \frac{1}{i} \frac{\partial}{\partial x}, \quad D = (1 - \Delta)^{1/2}$$

One may then notice an operator of the form (8.9) can be rewritten in the following form

$$Au(x) = Op(p)u(x) = (2\pi)^{-2} \iint e^{i(x-y) \cdot \xi} p(x,\xi) u(y) \, dy \, d\xi$$

with $p(x,\xi) = (2\pi)^{-2} \iint e^{(x-y) \cdot (\eta-\xi)} a(x,y,\xi) \, dy \, d\eta = e^{i \, d_{\xi} \cdot d_{y}} a(x,y,\xi) \big|_{y=x}$ (8.10)

and it is proved in [Tay97] that assuming $0 \leq \delta_2 < \rho \leq 1$, $A \in OPS^m_{\rho,\delta}$, with $\delta = \max(\delta_1, \delta_2)$. So one may reduce the study of operators as in (8.9) to those as in (8.10) with symbols p of $S^m_{\rho,\delta}$ that satisfy

$$\left| D_x^{\beta} D_{\xi}^{\alpha} p(x,\xi) \right| \le C_{\alpha\beta} \left< \xi \right>^{m-\rho |\alpha|+\delta |\beta|}, \quad \forall \ x, y, \xi \in \mathbb{R}^2$$

Furthermore, the above symbol p for operator A satisfy the asymptotic expression

$$p(x,\xi) - \sum_{0 \le |\alpha| \le N} \frac{i^{|\alpha|}}{\alpha!} \, d_{\xi}^{\alpha} d_{y}^{\alpha} a(x,y,\xi) \big|_{y=x} \in S^{m-N(\rho-\delta)}_{\rho,\delta}, \quad \forall \ N \in \mathbb{N}$$

The asymptotic expression is particularly useful when studying compositions, and commutators for such operators. Thus, given two symbols $p_j \in S_{\rho,\delta}^{m_j}$, j = 1, 2, it is proven in [Tay97] that

$$(p_{1} \circ p_{2})(x,\xi) = p_{1}(x,\xi)p_{2}(x,\xi) + \frac{1}{i}\sum_{j=1}^{2}\frac{\partial p_{1}}{\partial\xi_{j}}\frac{\partial p_{2}}{\partial x_{j}} \mod S_{\rho,\delta}^{m_{1}+m_{2}-2(\rho-\delta)}$$

$$[p_{1}(x,d),p_{2}(x,d)] = [p_{1},p_{2}](x,d) \in OPS_{\rho,\delta}^{m_{1}+m_{2}-(\rho-\delta)}, \quad \text{with}:$$

$$[p_{1},p_{2}](x,\xi) = \frac{1}{i}\sum_{j=1}^{2}\left(\frac{\partial p_{1}}{\partial\xi_{j}}\frac{\partial p_{2}}{\partial x_{j}} - \frac{\partial p_{1}}{\partial x_{j}}\frac{\partial p_{2}}{\partial\xi_{j}}\right) \mod S_{\rho,\delta}^{m_{1}+m_{2}-2(\rho-\delta)}$$
(8.11)

Eventually, notice the following useful property

$$p(x,\xi) \in S^m_{\rho,\delta}, \ 0 \le \delta < \rho \le 1, \quad Op(p) : W^{s,p} \longrightarrow W^{s-m,p}, \ \forall \ s \in \mathbb{R}, \ 1 < p < +\infty$$
(8.12)

Now this may be used to prove the decomposition we have used in (4.40) and (6.186):

$$D^{1/2}|x|^2D^{1/2} - xDx = \left[D^{1/2}x, xD^{1/2}\right] \in OPS_{1,0}^{-1}$$

First, let $\{\psi_j, j \in \mathbb{N}\}$ be a partition of unity such that $\sum_j \psi_j = 1$. Then define also (ϕ_j) a collection of compactly supported functions such that $\phi_j(x) = \psi_j(x) x$, $x \in \mathbb{R}^2$. We then use these and (8.11) to decompose

$$D^{1/2}x = Op(\langle\xi\rangle^{\frac{1}{2}}y) = \sum_{j} Op(\langle\xi\rangle^{\frac{1}{2}}\phi_{j}(y)) = \sum_{j} Op(p_{j}(x,\xi)), \quad p_{j}(x,\xi) = \langle\xi\rangle^{\frac{1}{2}}\phi_{j}(x) \mod S_{1,0}^{-1/2}$$
$$xD^{1/2} = Op(x\langle\xi\rangle^{\frac{1}{2}}) = \sum_{l} Op(\phi_{l}(x)\langle\xi\rangle^{\frac{1}{2}}) = \sum_{l} Op(q_{l}(x,\xi)), \quad q_{l}(x,\xi) = \phi_{j}(x)\langle\xi\rangle^{\frac{1}{2}} \mod S_{1,0}^{-1/2}$$
$$D^{1/2}x, xD^{1/2}] = \sum_{j,l} [Op(p_{j}), Op(q_{l})] = \frac{1}{i} \sum_{m=1}^{2} \sum_{j,l} \left(\frac{\partial p_{j}}{\partial\xi_{m}}\frac{\partial q_{l}}{\partial x_{m}} - \frac{\partial p_{j}}{\partial\xi_{m}}\frac{\partial q_{l}}{\partial\xi_{m}}\right) \mod S_{1,0}^{-1} = 0 \mod S_{1,0}^{-1}$$

In particular, using (8.12) $a_{-1}(x,D) = [D^{1/2}x, xD^{1/2}]$ is a bounded operator $L^{4/3} \to W^{1,4/3} \subseteq L^4$ thanks to Sobolev embedding result of Theorem 8.1. Indeed, from (4.40) that proves

$$\begin{split} \left\| x \, D^{1/2} \, R \right\|_{L^2}^2 &= \left(D^{1/2} \, |x|^2 \, D^{1/2} \, R \,, R \right) = \left(\left[x D x + a_{-1}(x, D) \right] R, R \right) \\ &= \left\| D^{1/2} \, x \, R \right\|_{L^2}^2 + \left(a_{-1}(x, D) R, R \right) \\ &\lesssim \left\| D^{1/2} \, x \, R \right\|_{L^2}^2 + \left\| a_{-1}(x, D) R \right\|_{L^4} \| R \|_{L^{4/3}} \\ &\lesssim \left\| D^{1/2} \, x \, R \right\|_{L^2}^2 + \| R \|_{L^{4/3}}^2 \end{split}$$

8.5 An estimating Lemma for an ODE solutions

Here we recall Lemma 5.4 of Appendix A in [RS11], that is needed in the Unicity argument to get bounds on the parameters.

Lemma 8.10. Let $Z = (Z_1, Z_2) : \mathbb{R} \to \mathbb{R}^2$ satisfying the following ordinary differential system

$$Z_s = \begin{pmatrix} 0 & -2\\ \frac{\zeta}{s^2} & 0 \end{pmatrix} Z + F$$

where $\zeta > 0$ is a constant and where $F = (F_1, F_2) : \mathbb{R} \to \mathbb{R}^2$. Assume also that

$$\lim_{s \to +\infty} Z(s) = 0 \quad and \quad \left| F(s) \right| \lesssim \frac{1}{s^3}.$$

Then we have the following estimate for Z for $s \geq 2$

$$|Z_1(s)| + s |Z_2(s)| \lesssim \int_s^{+\infty} (|F_1(\sigma)| + \sigma |F_2(\sigma)|) \log(\sigma) \, d\sigma.$$

The proof is straightforward, and we don't reproduce it here.

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