Sur les dérivées généralisées, les conditions d'optimalité et l'unicité des solutions en optimisation non lisse

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Abstract of the PhD Thesis

ON GENERALIZED DERIVATIVES, OPTIMALITY CONDITIONS AND UNIQUENESS OF SOLUTIONS IN NONSMOOTH OPTIMIZATION

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Introduction

Optimality conditions for nonsmooth optimization have become one of the most important topics in the study of optimization-related problems. Various notions of generalized derivatives have been introduced to establish optimality conditions. Besides establishing optimality conditions, generalized derivatives also is an important tool for studying the local uniqueness of solutions. During the last three decades, these topics have been being developed, generalized and applied to many fields of mathematics by many authors all over the world. The purpose of this thesis is to investigate the above topics. It consists of five chapters. In Chapter 1, we develop elements of calculus of variational sets for set-valued mappings, which were recently introduced in Khanh and Tuan (2008). Most of the usual calculus rules, from chain and sum rules to rules for unions, intersections, products. and other operations on mappings, are established. As applications we provide a direct employment of sum rules to establishing an explicit formula for a variational set of the solution map to a parametrized variational inequality in terms of variational sets of the data. Furthermore, chain rules and sum or product rules are also used to prove optimality conditions for weak solutions of some vector optimization problems. In Chapter 2, we propose notions of higher-order outer and inner radial derivatives of set-valued maps and obtain main calculus rules. Some direct applications of these rules in proving optimality conditions for particular optimization problems are provided. Then, we establish higher-order optimality necessary conditions and sufficient ones for a general set-valued vector optimization problem with inequality constraints. Chapter 3 is devoted to using first and second-order approximations, which were introduced by Jourani and Thibault (1993) and Allali and Amaroq (1997), as generalized derivatives, to establish both necessary and sufficient optimality conditions for various kinds of solutions to nonsmooth vector equilibrium problems with functional constraints. Our first-order conditions are shown to be applicable in many cases, where existing ones cannot be applied. The second-order conditions are new. In Chapter 4, we consider nonsmooth multiobjective fractional programming on normed spaces. Using first and second-order approximations as generalized derivatives, first and second-order optimality conditions are established. For sufficient conditions no convexity is needed. Our results can be applied even in infinite dimensional cases involving infinitely discontinuous maps. In Chapter 5, we establish sufficient conditions for the local uniqueness of solutions to nonsmooth strong and weak vector equilibrium problems. Also by using approximations, our results are valid even in cases where the maps involved in the problems suffer infinite discontinuity at the considered point.

Chapter 1. Variational sets: calculus and applications to nonsmooth vector optimization

1.1. Introduction

In this chapter, we establish elements of calculus for variational sets to ensure that they can be used in practice. Most of the usual rules, from the sum and chain rules to various operations in analysis, are investigated. It turns out that the variational sets possess many fundamental and comprehensive calculus rules. Although this construction is not comparable with objects in the dual approach like Mordukhovich's coderivatives in enjoying rich calculus, it may be better in dealing with higher-order properties. We pay attentions also on relations between the established calculus rules and applications of some rules to get others. (Of course, significant applications should be those in other topics of nonlinear analysis and optimization.) As such applications we provide a direct employment of sum rules to establishing an explicit formula for a variational set of the solution map to a parametrized variational inequality in terms of variational sets of the data. Furthermore, chain rules and sum and product rules are also used to prove optimality conditions for weak solutions of some vector optimization problems.

Let X and Y be real normed spaces, $C \subseteq Y$ a pointed closed convex cone with nonempty interior and $F : X \to 2^Y$. For $A \subseteq X$, int$A$, cl$A$ (or $\bar{A}$), bd$A$ denote its interior, closure and boundary, respectively. X^* is the dual space of X and B_X stands for the closed unit ball in X. For $x_0 \in X$, $U(x_0)$ is used for the set of all neighborhoods of $x_0 \in X$. $\mathbb{R}^k_+$ is the nonnegative orthant of the k-dimensional space. We often use the following cones, for $A \subseteq X$, C above and $u \in X$,

$$\text{cone}A = \{\lambda a \mid \lambda \geq 0,\ a \in A\},\ \text{cone}_+A = \{\lambda a \mid \lambda > 0,\ a \in A\},$$

$$A(u) = \text{cone}(A + u),\ C^* = \{y^* \in Y^* \mid \langle y^*, c\rangle \geq 0,\ \forall c \in C\}\ (\text{polar cone}).$$

A subset S of a linear space is called star-shaped at $x_0 \in S$ if, for all $x \in S$ and $\alpha \in [0, 1], (1-\alpha)x_0 + \alpha x \in S$. A set-valued mapping $H : X \to 2^Y$ between two linear spaces is said to be star-shaped at $x_0 \in S$ on the star-shaped at x_0 subset $S \subseteq \text{dom}H$ if, for all $x \in S$ and $\alpha \in [0, 1]$,

$$(1-\alpha)H(x_0) + \alpha H(x) \subseteq H((1-\alpha)x_0 + \alpha x).$$

If $C \subseteq Y$ is a cone (not necessarily convex) and we have, for all $x \in S$ and $\alpha \in [0, 1]$,

$$(1-\alpha)H(x_0) + \alpha H(x) \subseteq H((1-\alpha)x_0 + \alpha x) + C,$$

we say that H is C-star-shaped at x_0. When X and Y are normed, $F : X \to 2^Y$ is called pseudo-convex at $(x_0, y_0) \in \mathrm{gr}F$ if $\mathrm{epi}F \subseteq (x_0, y_0) + T_{\mathrm{epi}F}(x_0, y_0)$.

1.2. Variational sets

Definition 1.2.1. (Khanh and Tuan 2008) The variational sets of type 1 are defined as follows:

$$V^1(F, x_0, y_0) = \operatorname*{Limsup}_{x \xrightarrow{F} x_0,\ t \to 0^+} \frac{1}{t}(F(x) - y_0), ...,$$

$$V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \operatorname*{Limsup}_{x \xrightarrow{F} x_0,\ t \to 0^+} \frac{1}{t^m}(F(x) - y_0 - tv_1 - \cdots t^{m-1}v_{m-1}).$$

Definition 1.2.2. (Khanh and Tuan 2008) The variational sets of type 2 are defined as follows:

$$W^1(F, x_0, y_0) = \operatorname*{Limsup}_{x \xrightarrow{F} x_0} \mathrm{cone}_+(F(x) - y_0), ...,$$

$$W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) = \operatorname*{Limsup}_{x \xrightarrow{F} x_0\ t \to 0^+} \frac{1}{t^{m-1}}(\mathrm{cone}_+(F(x) - y_0) - v_1 - \cdots - t^{m-2}v_{m-1})$$

By using equivalent formulations for the Painlevé-Kuratowski sequential upper limit we obtain some formulae of the two types of variational sets in Propositions 1.2.1 and 1.2.2.

1.3. Calculus of variational sets

In this section, we develop algebraic and set operations rules and compositions rules of variational sets for set-valued mappings.

Proposition 1.3.1 (Union Rule). *Let $F_i : X \to 2^Y$, $i = 1, ..., k$, $(x_0, y_0) \in \bigcup_{i=1}^{k} \mathrm{gr}F_i$, and $I(x_0, y_0) = \{i \mid (x_0, y_0) \in \mathrm{gr}F_i\}$. Then,*

(i) $V^m(\bigcup_{i=1}^{k} F_i, x_0, y_0, v_1, ..., v_{m-1}) = \bigcup_{i \in I(x_0, y_0)} V^m(F_i, x_0, y_0, v_1, ..., v_{m-1});$

(ii) $W^m(\bigcup_{i=1}^{k} F_i, x_0, y_0, v_1, ..., v_{m-1}) = \bigcup_{i \in I(x_0, y_0)} W^m(F_i, x_0, y_0, v_1, ..., v_{m-1}).$

Proposition 1.3.2 (Intersection Rule). *Let $F_i : X \to 2^Y$, $i = 1, ..., n$, and $(x_0, y_0) \in \bigcap_{i=1}^{n} \mathrm{gr}F_i$. Then,*

(i) $V^m(\bigcap_{i=1}^n F_i, x_0, y_0, v_1, ..., v_{m-1}) \subseteq \bigcap_{i=1}^n V^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$;

(i) $W^m(\bigcap_{i=1}^n F_i, x_0, y_0, v_1, ..., v_{m-1}) \subseteq \bigcap_{i=1}^n W^m(F_i, x_0, y_0, v_1, ..., v_{m-1})$.

Proposition 1.3.3 (Sum Rule for V^m). *Let $F_i : X \to 2^Y$, $x_0 \in \mathrm{dom}F_1 \bigcap \mathrm{int} \bigcap_{i=2}^k \mathrm{dom}F_i$, $y_i \in F_i(x_0)$, and $v_{i,1}, ..., v_{i,m-1} \in Y$, for $i = 1, ..., k$. If $F_i, i = 2, ...k$ have proto-variational sets $V^m(F_i, x_0, y_0, v_{i,1}, ..., v_{i,m-1})$, respectively, then,*

$$\sum_{i=1}^k V^m(F_i, x_0, y_i, v_{i,1}, ..., v_{i,m-1}) \subseteq V^m(\sum_{i=1}^k F_i, x_0, \sum_{i=1}^k y_i, \sum_{i=1}^k v_{i,1}, ..., \sum_{i=1}^k v_{i,m-1}).$$

Proposition 1.3.4 (Sum Rule for W^1). *Let $F_i : X \to 2^Y$, $(x_0, y_i) \in \mathrm{gr}F_i$, and F_i be compact at x_0 for $i = 1, ..., k$. Then,*

$$\sum_{i=1}^k W^1(F_i, x_0, y_i) \supseteq W^1(\sum_{i=1}^k F_i, x_0, \sum_{i=1}^k y_i).$$

Proposition 1.3.7 (Chain Rule for V^m). *Let $F : X \to 2^Y$, $G : Y \to 2^Z$, $(x_0, y_0) \in \mathrm{gr}F$, $(y_0, z_0) \in \mathrm{gr}G$ and $\mathrm{im}F \subseteq \mathrm{dom}G$.*

(i) *If G is Lipschitz around y_0 then, for $u_1 \in V^1(F, x_0, y_0), ..., u_{m-1} \in V^{m-1}(F, x_0, y_0, u_1, ..., u_{m-2})$ and $v_1 \in D^bG(y_0, z_0)(u_1), ..., v_{m-1} \in D^{b(m-1)}G(y_0, z_0, v_1, ..., v_{m-2})(u_{m-1})$, we have*

$$D^{bm}G(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(V^m(F, x_0, y_0, u_1, ..., u_{m-1}))$$
$$\subseteq V^m(G \circ F, x_0, z_0, v_1, ..., v_{m-1}).$$

(ii) *If additionally F has a proto-variational set of order m of type 1 at (x_0, y_0), then*

$$D^mG(y_0, z_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(V^m(F, x_0, y_0, u_1, ..., u_{m-1}))$$
$$\subseteq V^m(G \circ F, x_0, z_0, v_1, ..., v_{m-1}).$$

(iii) *If F is l.s.c. at (x_0, y_0), then*
$V^m(G\Box F, x_0, z_0, v_1, ..., v_{m-1}) \subseteq V^m(G, y_0, z_0, v_1, ..., v_{m-1})$.

Proposition 1.3.8 (Chain Rule for W^m). *Let* $F : X \to 2^Y$, $G : Y \to 2^Z$, $(x_0, y_0) \in \text{gr}F$, $(y_0, z_0) \in \text{gr}G$, *and* $\text{im}F \subseteq \text{dom}G$.

(i) *If F is star-shaped at x_0, and G is Lipschitz around y_0, then*

$$D^bG(y_0, z_0)[W^1(F, x_0, y_0)] \subseteq DG(y_0, z_0)[W^1(F, x_0, y_0)] \subseteq V^1(G \circ F, x_0.z_0).$$

(ii) *If F is l.s.c. at (x_0, y_0), then*

$$W^m(G \square F, x_0, z_0, v_1, ..., v_{m-1}) \subseteq W^m(G, y_0, z_0, v_1, ..., v_{m-1}).$$

(iii) *If F^{-1} is l.s.c. at (y_0, x_0), then*

$$W^1(G, y_0, z_0) \subseteq W^1(G \circ F, x_0, z_0).$$

The calculus rules are also established for other operators as composition with differentiable map, composition with linear continuous map, inner product, outer product, quotient, maximum and minimum. Many examples are given to illustrate properties of the above calculus rules. Example 1.3.1 explains that equality may fail for the intersection rule while Example 1.3.2 shows a case where equality holds for the intersection rule. Example 1.3.3 ensures that the condition $x_0 \in \text{dom}F_1 \bigcap \text{int} \bigcap_{i=2}^{k} \text{dom}F_i$ cannot be reduced to $x_0 \in \bigcap_{i=1}^{k} \text{dom}F_i$ in the sum rule.

Since general chain rules may often encompass sum rules as special cases, we investigate the sum $M + N$ of two multifunctions $M, N : X \to 2^Y$. To express $M + N$ as a composition, define $F : X \to 2^{X \times Y}$ and $G : X \times Y \to 2^Y$ by, for I being the identity map on X and $(x, y) \in X \times Y$,

$$F = I \times M \qquad \text{and} \qquad G(x, y) = y + N(x).$$

Then, clearly $M + N = G \circ F$. Now we present some definitions.

Definition 1.3.3. Let $((x, z), y) \in \text{gr}C$, $u_1, ..., u_{m-1} \in Y$ and $w, w_1, ..., w_{m-1} \in Z$.

(i) The mth-order y-variational set of the multimap $G \circ F$ at (x, z) is the set

$$V^m(G \circ_y F, x, z, w_1, ..., w_{m-1}) := \{w \in Z : \exists t_n \to 0^+, \exists (x_n, y_n, w_n) \to (x, y, w),$$

$$\forall n \in \mathbb{N}, y_n \in C(x_n, z + t_n w_1 + ... + t_n^{m-1} w_{m-1} + t_n^m w_n)\}.$$

(ii) The mth-order quasi-variational set of the multimap C at (x,z) with $w\in Z$ is the set

$$\hat{V}^m(C,(x,z[w]),y,w_1,...,w_{m-1}) := \{\overline{y}\in Y : \exists t_n\to 0^+, \exists(x_n,\overline{y_n},w_n)\to (x,\overline{y},w), y+t_n\overline{y_n}\in C(x_n,z+t_nw_1+...+t_n^{m-1}w_{m-1}+t_n^mw_n)\}.$$

Definition 1.3.4. Given $((x,z),y)\in \mathrm{cl}(\mathrm{gr}S)$ and $v_1,...,v_{m-1}\in Y$, the mth-order y-variational set of $M+N$ at (x,z) is the set

$$V^m(M+_yN,x,z,v_1,...,v_{m-1}) := \{w\in Y : \exists t_n\to 0^+, \exists(x_n,y_n,w_n)\to (x,y,w), y_n\in S(x_n,z+t_nv_1+...+t_n^{m-1}v_{m-1}+t_n^mw_n\}.$$

Observe that

$$V^m(M+_yN,x,z,v_1,...,v_{m-1}) = V^m(G\circ_y F,x,z,v_1,...,v_{m-1}).$$

Using the above definitions, we obtain the composition rules for G and F in Propositions 1.3.14 and 3.15. Then, we apply the preceding composition rules to establish sum rules for $M,N : X\to 2^Y$ in Propositions 1.3.18 and 1.3.19.

1.4. Applications

We apply calculus rules to compute variational sets of solution maps to variational inequalities.

Let $F: W\times X\to 2^Z$ and $N: X\to 2^Z$ be multimaps between normed spaces and K be a subset of X. Let

$$M(w,z) := \{x\in K : z\in F(w,x)+N(x)\}$$

where $F: W\times X\to 2^Z$ and $N: X\to 2^Z$ are multimaps between normed spaces and K is a subset of X. When K is convex, $N(x)$ is the normal cone to K at x and w is a parameter, M is the solution map of a parametrized variational inequality.

Let $N_K : W\times X\to 2^Z$ be the multimap given by $N_K(w,x) := N(x)$ for $(w,x)\in W\times K$, $N_K(w,x) := \emptyset$ for $(w,x)\in W\times (X\setminus K)$. Then, M is related to the sum map $Q := F+N_K$ by

$$x\in M(w,z)\Longleftrightarrow z\in Q(w,x).$$

Thus, $x'\in \hat{V}(M,(w,z[z']),x)$ if and only if $z'\in \hat{V}(Q,(w,x[x']),z)$ for any $(x',z')\in X\times Z$. Let $S: W\times X\times Z\to 2^Z$ be defined by

$$S(w,x,z) := F(w,x)\cap (z-N_K(w,x)).$$

Lemma 1.4.1. *Let Z be finite dimensional and $((w,z),x) \in \mathrm{gr}M$. If for $z' \neq 0_Z$,*

$$\hat{V}(M,(w,z[0]),x) = \{0_X\} \;\;,\;\; \{0_X\} \notin \hat{V}(M,(w,z[z']),x), \qquad (*)$$

then, for every $v \in V(M,(w,z),x)$, there exists $z' \in Z$ such that

$v \in \hat{V}(M,(w,z[z']),x)$.

Proposition 1.4.2. *Let Z be finite dimensional and $((w,z),x) \in \mathrm{gr}(M)$. If S is directionally semi-compact at (w,x,z), $(*)$ holds at $((w,z),x)$ and*

$$\hat{V}(S,(w,x[0],y[0]),\overline{y}) = \{0\}$$

holds for every $\overline{z} \in \mathrm{cl}S(w,x,z)$, then, for $x' \in X$,

$V(M,(w,z),x) \subseteq$

$$\{x' \in X : \bigcup_{\overline{z}\in \mathrm{cl}S(w,x,z)} (\hat{V}(F,(w,x[x']),\overline{z}) + \hat{V}(N_K,(w,x[x']),z-\overline{z})) \neq \emptyset\}. \quad (1)$$

If, additionally,

$$\hat{V}(F,(w,x[x']),\overline{y}) \cap [y' - \hat{V}(N,(w,x[x']),y-\overline{y})] \subseteq \hat{V}(S,(w,x[x'],y[y']),\overline{y})$$

holds for every $\overline{z} \in \mathrm{cl}S(w,x,z)$, then (1) becomes an equality.

In the rest of this section, we employ calculus rules to necessary conditions for weak solutions of several particular optimization problems. Let X and Y be normed spaces, Y is partially ordered by a pointed closed cone C with nonempty interior, $F : X \to 2^Y$ and $G : X \to 2^X$. Consider

$$(\mathrm{P}_1) \qquad \min F(x') \text{ subject to } x \in X \text{ and } x' \in G(x).$$

This problem can be restated as the following unconstrained problem: $\min(F \circ G)(x)$.

Proposition 1.4.3. *Assume for (P_1) that $\mathrm{dom}F \subseteq \mathrm{Im}G$ and G^{-1} is l.s.c. at (z_0,x_0). If (x_0,y_0) is a local weakly efficient pair of (P_1), then*

$$W^1(F_+,z_0,y_0) \cap -\mathrm{intC} = \emptyset.$$

To illustrate sum rules we consider the following problem

$$(P_2) \qquad \min F(x) \text{ subject to } \; g(x) \leq 0,$$

where X, Y are as for problem (P_1), $F : X \to 2^Y$ and $g : X \to Y$. Denote $S = \{x \in X \mid g(x) \leq 0_Y\}$. Define $G : X \to Y$ by

$$G(x) = \begin{cases} 0, \text{ if } x \in S, \\ g(x), \;\; \text{otherwise.} \end{cases}$$

Consider the following unconstrained set-valued optimization problem, for an arbitrary positive s,

$$(P_C) \quad \min(F + sG)(x).$$

In a particular case, when $Y = R$ and F is single-valued, (P_C) is used to approximate (P_2) in penalty methods (see Rockafellar and Wets 1997).

Proposition 1.4.4. *Let $x_0 \in S, y_0 \in F(x_0)$, and F_+ or G_+ have a proto-variational set. If (x_0, y_0) is a local weak efficient pair of (P_C) then*

$$(V^1(F_+, x_0, y_0) + sV^1(G_+, x_0, 0)) \cap -\text{intC} = \emptyset.$$

Examples 1.4.1 and 1.4.2 indicate that Propositions 1.4.3 and 1.4.4 can be applied while calculus rules of contingent epiderivatives in Jahn and Khan (2002) cannot be in use.

Chapter 2. Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization

2.1. Introduction

For a subset A of a normed space X, the contingent cone of A at $\bar{x} \in \text{cl}A$ is

$$T_A(\bar{x}) = \{u \in X : \exists t_n \to 0^+, \exists u_n \to u, \forall n, \bar{x} + t_n u_n \in A\}.$$

However, they capture only the local nature of sets and mappings and are suitable mainly for convex problems. The (closed) radial cone of A at $\bar{x} \in \text{cl}A$ is defined by

$$R_A(\bar{x}) = \overline{\text{cone}}(A - \bar{x}) = \{u \in X : \exists t_n > 0, \exists u_n \to u, \forall n, \bar{x} + t_n u_n \in A\}$$

and carries global information about A. We have $T_A(\bar{x}) \subseteq R_A(\bar{x})$ and this becomes equality if A is convex (in fact, we need A being only star-shape at $\bar{x}$). Hence, the corresponding radial derivative, first proposed in Taa (1997), is proved to be applicable to nonconvex problems and global optimal solutions. The radial epiderivatives were introduced by Flores-Bazan (2001), taking some advantages of other kinds of epiderivatives. In this chapter, we propose notions of higher-order outer and inner radial derivatives of set-valued maps and obtain main calculus rules and their application.

2.2. Higher-order radial derivatives and their calculus rules

Definition 2.2.1. Let $F : X \to 2^Y$ be a set-valued map and $u \in X$.

(i) The mth-order outer radial derivative of F at $(x_0, y_0) \in \text{gr}F$ is $\overline{D}_R^m F(x_0, y_0)(u) = \{v \in Y : \exists t_n > 0, \exists (u_n, v_n) \to (u, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}$.

(ii) The mth-order inner radial derivative of F at $(x_0, y_0) \in \mathrm{gr}F$ is $\underline{D}_R^m F(x_0, y_0)(u) = \{v \in Y : \forall t_n > 0, \forall u_n \to u, \exists v_n \to v, \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}$.

Example 2.2.1 highlights detailed differences between 2th-order outer radial derivatives and second-order contingent derivatives.

Definition 2.2.2. Let $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$. If $\overline{D}_R^m F(x_0, y_0)(u) = \underline{D}_R^m F(x_0, y_0)(u)$ for any $u \in dom[\overline{D}_R^m F(x_0, y_0)]$, then we call $\overline{D}_R^m F(x_0, y_0)$ a mth-order proto-radial derivative of F at (x_0, y_0).

We obtain some main calculus rules of the mth-order radial derivative.

Proposition 2.2.2. *Let $F_1, F_2 : X \to 2^Y$, $x_0 \in \mathrm{int}(\mathrm{dom}F_1) \cap \mathrm{dom}F_2$, and $y_i \in F_i(x_0)$ for $i = 1,2$. Suppose that F_1 has a mth-order proto-radial derivative at (x_0, y_1). Then, for any $u \in X$,*

$$\overline{D}_R^m F_1(x_0, y_1)(u) + \overline{D}_R^m F_2(x_0, y_2)(u) \subseteq \overline{D}_R^m (F_1 + F_2)(x_0, y_1 + y_2)(u).$$

Proposition 2.2.3. *Let $F : X \to 2^Y$, $G : Y \to 2^Z$ with $\mathrm{Im}F \subseteq \mathrm{dom}G$, $(x_0, y_0) \in \mathrm{gr}F$ and $(y_0, z_0) \in \mathrm{gr}G$.*

(i) *Suppose that G has a mth-order proto-radial derivative at (y_0, z_0). Then, for any $u \in X$,*

$$\overline{D}_R^m G(y_0, z_0)(\overline{D}_R^1 F(x_0, y_0)(u)) \subseteq \overline{D}_R^m (G \circ F)(x_0, z_0)(u).$$

(ii) *Suppose that G has a proto-radial derivative of order 1 at (y_0, z_0). Then, for any $u \in X$,*

$$\overline{D}_R^1 G(y_0, z_0)(\overline{D}_R^m F(x_0, y_0)(u)) \subseteq \overline{D}_R^m (G \circ F)(x_0, z_0)(u).$$

We also get some sum rules in Propositions 2.2.5 - 2.2.7 by using the observation that chain rules may often encompass sum rules as special cases.

Examples 2.2.2 - 2.2.4 show that the assumption about the proto-radial derivative cannot be dispensed in Propositions 2.2.2 and 2.2.3.

Applying the above calculus rules, we get some optimality conditions. Let X and Y be normed spaces, Y being partially ordered by a pointed closed convex cone C with nonempty interior, $F : X \to 2^Y$ and $G : X \to 2^X$. Consider the problem

$$(\mathrm{P}_1) \qquad \min F(x') \text{ subject to } x \in X \text{ and } x' \in G(x).$$

Proposition 2.2.9. *Let* $\mathrm{Im}G \subseteq \mathrm{dom}F$*,* $(x_0, z_0) \in \mathrm{gr}G$*, and* $(z_0, y_0) \in \mathrm{gr}F$*. Assume that* (x_0, y_0) *is a* Q*-minimal solution of* (P_1)*.*

(i) *If* F *has a mth-order proto-radial derivative at* (z_0, y_0)*, then, for any* $u \in X$*,*

$$\overline{D}_R^m F(z_0, y_0)(\overline{D}_R^1 G(x_0, z_0)(u)) \cap (-Q) = \emptyset.$$

(ii) *If* F *has a proto-radial derivative of order 1 at* (z_0, y_0)*, then, for any* $u \in X$*,*

$$\overline{D}_R^1 F(z_0, y_0)(\overline{D}_R^m G(x_0, z_0)(u)) \cap (-Q) = \emptyset.$$

Consider the following unconstrained set-valued optimization problem, for an arbitrary positive s,

$$(P_C) \quad \min(F + sG)(x).$$

Proposition 2.2.10. *Let* $\mathrm{dom}F \subseteq \mathrm{dom}G$*,* $x_0 \in S, y_0 \in F(x_0)$ *and either* F *or* G *has a mth-order proto-radial derivative at* (x_0, y_0) *or* $(x_0, 0)$*, respectively. If* (x_0, y_0) *is a* Q*-minimal solution of* (P_C)*, then, for any* $u \in X$*,*

$$(\overline{D}_R^m F(x_0, y_0)(u) + s\overline{D}_R^m G(x_0, 0)(u)) \cap -\mathrm{Q} = \emptyset.$$

Examples 2.2.6 and 2.2.7 show cases where Propositions 2.2.9 and 2.2.10 can be applied while calculus rules of contingent epiderivatives of Jahn and Khan (2002) do not work.

2.3. Optimality conditions

Let X and Y be normed spaces partially ordered by pointed closed convex cones C and D, respectively, with nonempty interior. Let $S \subseteq X, F : X \to 2^Y$ and $G : X \to 2^Z$. In this section, we discuss optimality conditions for the following general set-valued vector optimization problem with inequality constraints

$$(P) \ \ \min F(x), \text{ subject to } x \in S, \ G(x) \cap (-D) \neq \emptyset.$$

Let $A := \{x \in S : G(x) \cap (-D) \neq \emptyset\}$ and $F(A) := \bigcup_{x \in A} F(x)$. We assume that $F(x) \neq \emptyset$ for all $x \in X$.

Definition 2.3.1 (Ha 2010). Let $x_0 \in A$, $y_0 \in F(x_0)$, and $Q \subseteq Y$ be an arbitrary nonempty open cone, different from Y. We say that (x_0, y_0) is a Q-minimal solution for (P) if, for all $x \in A$,

$$(F(x) - y_0) \cap (-Q) = \emptyset.$$

Various kinds of efficient solutions of (P) are in fact Q- minimal solutions with Q being appropriately chosen cones. Hence, the optimality conditions for Q-minimal solutions can imply those for various kinds of efficient solutions of (P).

Proposition 2.3.1. *Let* $\mathrm{dom}F \cup \mathrm{dom}G \subseteq S$ *and* (x_0, y_0) *be a* Q*-minimal solution for* (P)*. Then, for any* $z_0 \in G(x_0) \cap (-D)$ *and* $x \in X$,

$$\overline{D}_R^m(F,G)(x_0,y_0)(x) \cap (-Q \times -\mathrm{int}D) = \emptyset.$$

Proposition 2.3.3. *Let* $\mathrm{dom}F \cup \mathrm{dom}G \subseteq S$, $x_0 \in A$, $y_0 \in F(x_0)$ *and* $z_0 \in G(x_0) \cap (-D)$*. Then* (x_0, y_0) *is a* Q*-minimal solution of* (P) *if the following condition holds*

$$\overline{D}_R^m(F,G)(x_0,y_0)(A - x_0) \cap -(Q \times D(z_0)) = \emptyset.$$

Example 2.3.1. *Let* $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$ *and* F *be defined by*

$$F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{|x|\}, & \text{if } x = -\dfrac{1}{n}, n = 1, 2, ..., \\ \{-1\}, & \text{if } x = \frac{1}{n}, n = 1, 2, ..., \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $(x_0, y_0) = (0, 0)$ *and* $u = 0$*. Then*

$$\overline{D}_R^1 F(x_0,y_0)(u) = \{0\}, \ \overline{D}_R^2 F(x_0,y_0)(u) = \mathbb{R}.$$

Because $\overline{D}_R^2 F(x_0,y_0)(u) \cap (-\mathrm{int}C) \neq \emptyset$, (x_0, y_0) *is not a weak efficient solution. (But* $\overline{D}_R^1 F(x_0,y_0)$ *cannot be used here.)*

Example 2.3.3 shows that we cannot replace D by $D(z_0)$ in the necessary condition given by Proposition 2.3.1 to obtain a smaller gap with the sufficient one, expressed by Proposition 2.3.3. Some advantages of sufficient conditions by using mth-order radial derivative instead of variational sets or contingent epiderivatives are illustrated in Example 2.3.4.

Chapter 3. First and second-order optimality conditions using approximations for vector equilibrium problems with constraints

3.1. Introduction

Beginning with Blum and Oettli (1994), who stated equilibrium problems as a direct generalization of variational inequalities and optimization problems, these general problems have been intensively developed. However, the efforts have been focused on the solution existence, stability and sensitivity, well-posedness, properties of solution sets and numerical methods. Very few contributions to optimality conditions for these models can be found in the literature. Although some similarity can be recognized between minimization and equilibrium problems, detailed investigations of optimality conditions for equilibrium problems seem to be different and interesting. By using first and second-order approximations as generalized derivatives we establish both necessary and sufficient optimality conditions. Our first-order conditions are shown to be applicable in many cases, where existing ones cannot be applied. The second-order conditions are new.

3.2. Preliminaries

Definition 3.2.1. Let X be real normed space, $x_0, v \in X$ and $S \subseteq X$.

(i) The cone of weak feasible directions to S at x_0 is

$$W_f(S, x_0) = \{u \in X \mid \exists t_n \to 0^+, \forall n, x_0 + t_n u \in S\}.$$

(ii) The second-order contingent set of S at (x_0, v) is

$$T^2(S, x_0, v) = \{w \in X \mid \exists t_n \to 0^+, \exists w_n \to w, \forall n \in \mathbb{N}, x_0 + t_n v + \frac{1}{2} t_n^2 w_n \in S\}.$$

(iii) The asymptotic second-order tangent cone (Penot 1998) of S at (x_0, v) is

$$T''(S, x_0, v) = \{w \in X \mid \exists (t_n, r_n) \to (0^+, 0^+) : \tfrac{t_n}{r_n} \to 0, \exists w_n \to w, \\ \forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2} t_n r_n w_n \in S\}.$$

The recession cone of S is defined by

$$S_\infty = \{\lim t_n a_n \mid a_n \in S, t_n > 0 \text{ and } \lim_{n \to \infty} t_n = 0\}.$$

A subset $A \subseteq L(X, Y)$ ($B \subseteq L(X, X, Y)$) is called asymptotically pointwise compact (shortly asymptotically p-compact) if

(a) each bounded net $\{f_\alpha\} \subseteq A$ ($\subseteq B$, respectively) has a subnet $\{f_\beta\}$ and $f \in L(X, Y)$ ($f \in L(X, X, Y)$) such that $f = \text{p}-\lim f_\beta$;

(b) for each net $\{f_\alpha\} \subseteq A$ ($\subseteq B$, respectively) with $\lim \|f_\alpha\| = \infty$, the net $\{f_\alpha / \|f_\alpha\|\}$ has a subnet which pointwise converges to some $f \in L(X, Y) \setminus \{0\}$

$(f \in L(X, X, Y) \setminus \{0\})$.

Definition 3.2.4. (i) A set $A_f(x_0) \subseteq L(X, Y)$ is said to be a first-order approximation (Jourani and Thibault 1993)of $f : X \to Y$ at $x_0 \in X$ if there exists a neighborhood U of x_0 such that, for all $x \in U$,

$$f(x) - f(x_0) \in A_f(x_0)(x - x_0) + o(\|x - x_0\|).$$

(ii) A set $(A_f(x_0), B_f(x_0)) \subseteq L(X, Y) \times L(X, X, Y)$ is called a second-order approximation (Allali and Amaroq 1998) of $f : X \to Y$ at $x_0 \in X$ if

(a) $A_f(x_0)$ is a first-order approximation of f at x_0;

(b) $f(x) - f(x_0) \in A_f(x_0)(x - x_0) + B_f(x_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2)$.

Let X, Y and Z be normed spaces, $S \subseteq X$ be nonempty and $F : X \times X \to Y, g : X \to Z$ be mappings. Let $C \subseteq Y$ and $K \subseteq Z$ be pointed closed convex cones. Denote $\Omega = \{x \in S : g(x) \in -K\}$ (the feasible set), $F(x, \Omega) = \bigcup_{y \in \Omega} F(x, y)$. Define $F_{x_0} : X \to Y$ by $F_{x_0}(y) = F(x_0, y)$ for $y \in X$ and assume that $F_{x_0}(x_0) = 0$ (which is without loss of generality). The vector equilibrium problem (EP) with constraints under our consideration is described depending on kinds of solutions as follows.

Definition 3.2.5.

(i) If $\text{int}C \neq \emptyset$, a vector $x_0 \in \Omega$ is said to be a local weak solution of problem (EP), if there exists a neighborhood U of x_0 such that

$$F(x_0, U \cap \Omega) \not\subseteq -\text{int}C.$$

(ii) A vector $x_0 \in \Omega$ is termed a local Henig-proper solution to (EP) if there exists a neighborhood U of x_0 and a pointed convex cone $H \subseteq Y$ with $C \setminus \{0\} \subseteq \text{int}H$ such that

$$F(x_0, U \cap \Omega) \cap (-H \setminus \{0\}) = \emptyset.$$

(iii) A vector $x_0 \in \Omega$ is determined as a local Benson-proper solution to (EP) if there exists a neighborhood U of x_0 such that

$$\text{clcone}(F(x_0, U \cap \Omega) + C) \cap (-C) = \{0\}.$$

(iv) For $m \geq 1$, a vector $x_0 \in \Omega$ is said to be a local firm (known also as strict/isolated) solution of order m of (EP) if there exists a neighborhood U of x_0 and $\gamma > 0$ such that, for all $x \in U \cap \Omega \setminus \{x_0\}$,

$$(F(x_0, x) + C) \cap B_Y(0, \gamma\|x - x_0\|^m) = \emptyset.$$

3.3. First-order optimality conditions

Now, we establish first order necessary conditions for local weak solution of (EP) and first order sufficient conditions for local Henig, Benson and firm solutions of (EP).

Theorem 3.3.2. *Assume that C and K have nonempty interior and $A_{F_{x_0}}(x_0)$ and $A_g(x_0)$ are asymptotically p-compact first-order approximations of F_{x_0} and g, respectively, at x_0. If x_0 is a local weak solution of (EP), then, $\forall u \in X, \exists P \in$ $\mathrm{p}{-}A_{F_{x_0}}(x_0), \exists Q \in A_g(x_0), \exists (c^*, d^*) \in C^* \times D^* \setminus \{(0,0)\}$ such that*

$$\langle c^*, P(u)\rangle + \langle d^*, Q(u)\rangle \geq 0,\ \langle d^*, g(x_0)\rangle = 0.$$

Furthermore, for u satisfying $0 \in \mathrm{int}(Q(u) + g(x_0) + K)$ for all $Q \in A_g(x_0)$, we have $c^* \neq 0$.

Theorem 3.3.4. *Let $A_{F_{x_0}}(x_0)$ and $A_g(x_0)$ be asymptotically p-compact first-order approximations of F_{x_0} and g, respectively, at $x_0 \in \Omega$. Assume that there exists a pointed cone $H \subseteq Y$ with $C \setminus \{0\} \subseteq \mathrm{int}H$. Then, x_0 is a local Henig-proper solution (relative to H) of (EP) if either of the following conditions holds.*

(i) $(F_{x_0}, g) : X \to Y \times Z$ *is $C \times K$-arcwise-connected at x_0; for all $x \in \Omega$ and $(P,Q) \in \mathrm{p}{-}A_{(F_{x_0},g)}(x_0)_\infty \setminus \{0\}$ one has $(P,Q)(L'_{x_0,x}(0^+)) \notin -(C \times K)$ and, for some $(c^*, d^*) \in H^* \times K^* \setminus \{(0,0)\}$,*

$$\langle c^*, y\rangle + \langle d^*, z\rangle > 0,\ \langle d^*, g(x_0)\rangle = 0, \forall (y,z) \in \mathrm{p} - \mathrm{cl}A_{(F_{x_0},g)}(x_0)(L'_{x_0,x}(0^+)).$$

(ii) (F_{x_0}, g) *is pseudoconvex at x_0; for all $x \in \Omega$ and $(P,Q) \in \mathrm{p}{-}A_{(F_{x_0},g)}(x_0)_\infty \setminus \{0\}$ one has $(P,Q)(x - x_0) \notin -(C \times K)$ and, for some $(c^*, d^*) \in H^* \times K^* \setminus \{(0,0)\}$,*

$$\langle c^*, y\rangle + \langle d^*, z\rangle > 0,\ \langle d^*, g(x_0)\rangle = 0, \forall (y,z) \in \mathrm{p} - \mathrm{cl}A_{(F_{x_0},g)}(x_0)(x - x_0).$$

Theorem 3.3.5. *If $x_0 \in \Omega$ and $A_{F_{x_0}}(x_0)$ and $A_g(x_0)$ are asymptotically p-compact first-order approximations of $F_{x_0}(.)$ and $g(.)$, respectively, at x_0, then x_0 is a Benson-proper solution of (EP) whenever either of the following conditions holds.*

(i) $(F_{x_0}, g) : X \to Y \times Z$ *is $C \times K$-arcwise-connected at x_0 on Ω; for all $x \in \Omega$ and $(P,Q) \in \mathrm{p}{-}A_{(F_{x_0},g)}(x_0)_\infty \setminus \{0\}$ one has $(P,Q)(L'_{x_0,x}) \notin -(C \times K)$ and, for some $(c^*, d^*) \in C^{*i} \times K^* \setminus \{(0,0)\}$,*

$$\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0\ \langle d^*, g(x_0)\rangle = 0,\ \forall (y,z) \in \mathrm{p} - \mathrm{cl}A_{(F_{x_0},g)}(x_0)(L'_{x_0,x}(0^+)).$$

(ii) (F_{x_0}, g) *is pseudoconvex at* x_0*; for all* $x \in \Omega$ *and* $(P, Q) \in \text{p}{-}A_{(F_{x_0},g)}(x_0)_\infty \setminus \{0\}$ *one has* $(P, Q)(x - x_0) \notin -(C \times K)$ *and, for* $(c^*, d^*) \in C^{*i} \times K^* \setminus \{(0, 0)\}$,

$$\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0,\ \langle d^*, g(x_0)\rangle = 0,\ \forall (y, z) \in \text{p} - \text{cl}A_{(F_{x_0},g)}(x_0)(x - x_0).$$

Theorem 3.3.6. *Assume that* X *is finite dimensional,* $x_0 \in \Omega$ *and* $A_{F_{x_0}}(x_0)$ *and* $A_g(x_0)$ *are asymptotically p-compact first-order approximations of* F_{x_0} *and* g*, respectively, at* x_0*. Suppose that, for all* $u \in T(\Omega, x_0)$ *with norm one, all* $P \in \text{p} - \text{cl}A_{F_{x_0}}(x_0) \cup (\text{p}{-}A_{F_{x_0}}(x_0)_\infty \setminus \{0\})$ *and* $Q \in \text{p} - \text{cl}A_g(x_0) \cup (\text{p}{-}A_g(x_0)_\infty \setminus \{0\})$*, there exists* $(c^*, d^*) \in C^* \times K^* \setminus \{(0, 0)\}$ *such that*

$$\langle c^*, Pu\rangle + \langle d^*, Qu\rangle > 0,\ \langle d^*, g(x_0)\rangle = 0.$$

Then, x_0 *is a local firm solution of order 1 of* (EP).

Examples 3.3.1-3.3.3 indicate that our results can be applied in some cases when others previous results are out of use.

3.4. Second-order optimality conditions

In this section, we establish second-order necessary and sufficient conditions for weak and firm solutions of (EP) in both cases : F_{x_0} and g are or are not first-order Fréchet differentiable at x_0.

Theorem 3.4.1. *Let* C *be polyhedral,* $x_0 \in \Omega$ *and* $d^* \in K^*$ *with* $\langle d^*, g(x_0)\rangle = 0$*. Assume that* $(F'_{x_0}(x_0), B_{F_{x_0}}(x_0))$ *and* $(g'(x_0), B_g(x_0))$ *are asymptotically p-compact second-order approximations of* F_{x_0} *and* g*, respectively, at* x_0 *with* $B_g(x_0)$ *being norm-bounded.*

If x_0 *is a local weak solution of (EP) then, for any* $v \in T(G(d^*), x_0)$*, there exists* $c^* \in B$*, where* B *is finite and* $\text{cone}(\text{co}B) = C^*$*, such that* $\langle c^*, F'_{x_0}(x_0)v\rangle + \langle d^*, g'(x_0)v\rangle \geq 0$*. If, furthermore,* $c^* \circ F'_{x_0}(x_0) + d^* \circ g'(x_0) = 0$*, then either, for some* $M \in \text{p} - \text{cl}B_{F_{x_0}}(x_0)$ *and* $N \in \text{p} - \text{cl}B_g(x_0)$*,*

$$\langle c^*, M(v, v)\rangle + \langle d^*, N(v, v)\rangle \geq 0$$

or, for some $M \in \text{p}{-}B_{F_{x_0}}(x_0)_\infty \setminus \{0\}$*,*

$$\langle c^*, M(v, v)\rangle \geq 0.$$

Theorem 3.4.2. *Assume that* X *is finite dimensional,* $x_0 \in \Omega$ *and*

$(F'_{x_0}(x_0), B_{F_{x_0}}(x_0))$ *and* $(g'(x_0), B_g(x_0))$ *are asymptotically p-compact second-order approximations of* F_{x_0} *and* g*, respectively, at* x_0 *with norm-bounded* $B_g(x_0)$*. Impose further the existence of* $(c^*, d^*) \in C_0^* \times K_0^*$ *such that, for all* $v \in T(\Omega, x_0)$ *with* $\|v\| = 1$ *and* $\langle c^*, F'_{x_0}(x_0)v\rangle = \langle d^*, g'(x_0)v\rangle = 0$*,*

(i) *for all* $M \in \mathrm{p}-\mathrm{cl}B_{F_{x_0}}(x_0)$ *and* $N \in \mathrm{p}-\mathrm{cl}B_g(x_0)$, *one has*

$$\langle c^*, M(v,v)\rangle + \langle d^*, N(v,v)\rangle > 0;$$

(ii) *for all* $M \in \mathrm{p}-B_{F_{x_0}}(x_0)_\infty \setminus \{0\}$, *one has*

$$\langle c^*, M(v,v)\rangle > 0.$$

Then, x_0 *is a local firm solution of order 2 to* (EP).

When F_{x_0} and g are not first-order Fréchet differentiable at x_0, using first-order approximations of F_{x_0} and g instead of F'_{x_0} and $g'(x_0)$, we obtain second-order optimality conditions for the nondifferentiable case in Theorems 3.4.3 and 3.4.4. Applications of second-order optimality conditions for (EP) are illustrated in examples 3.4.1-3.4.4.

Chapter 4. First and second-order optimality conditions for multiobjective fractional programming

4.1. Introduction

Fractional programming has been an intensively developed topic in optimization. Along with numerous contributions to vector optimization, a very important area with significant practical applications due to the presence of many criteria in models met in science, economics and engineering, multiple objective fractional programming has also become attractive to many researchers. Increasing efforts of dealing with nonsmooth problems, relying on various generalized derivatives, can be recognized in the literature for fractional programming. Severe convexity requirements, especially in sufficient optimality conditions, have been gradually reduced, using relaxed convexity notions. We observe that almost no contributions to problems in infinite dimensional spaces and very few contributions for second-order optimality conditions. Convexity assumptions have not been completely removed so far. Inspired by these observations, we consider in the present paper a nonsmooth multiobjective fractional programming problem in normed spaces. To avoid completely convexity restrictions we employ first and second-order approximations as generalized derivatives.

4.2. Preliminaries

Let X, Y be normed spaces, $K \subseteq Y$ and $C \subseteq \mathbb{R}^m$ be pointed closed convex cones with nonempty interior. We consider the following multi-objective fractional programming problem

$$(P) \qquad \min \varphi(x) = \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_m(x)}{g_m(x)}\right) \text{ s.t. } h(x) \in -K,$$

with $f_i, g_i : X \to \mathbb{R}, i = 1, 2, ..., m$, $h : X \to Y$ and all g_i being continuous.

For $m \in \mathbb{N}$, $f : X \to Y$ is said to be $m-$calm at x_0 if there exists $L > 0$ and neighborhood U of x_0 such that, for all $x \in U$,

$$\|f(x) - f(x_0)\| \leq L\|x - x_0\|^m.$$

Then, L is called the coefficient of calmness of f. (1-calmness is called simply as calmness.)

Proposition 4.2.1. *Let $f : X \to Y$.*

(i) Suppose that $(A_f(x_0), B_f(x_0))$ is a second-order approximation of f at x_0 with $A_f(x_0) = \{0\}$ and $B_f(x_0)$ is bounded. Then, f is 2-calm at x_0.

(ii) Let $Y = \mathbb{R}$. If the Fréchet derivative f' exists in a convex neighborhood U of x_0, is calm at x_0 with coefficient L and $f'(x_0) = 0$, then f is 2-calm at x_0 with the same coefficient L.

(iii) If f is 2-calm at x_0, then $f'(x_0) = 0$.

4.3. Properties and calculus rules of approximations

In the following propositions some calculus rules, needed in establishing optimality conditions for problem (P), are developed.

Proposition 4.3.1. *Let $f_i : X \to Y$ and $\lambda_i \in \mathbb{R}$, for $i = 1, 2, ..., k$. Let $A_{f_i}(x_0)$ be first-order approximations of f_i at x_0, respectively. Then, the following assertions hold.*

(i) *$\sum_{i=1}^{k} \lambda_i A_{f_i}(x_0)$ is a first-order approximation of $\sum_{i=1}^{k} \lambda_i f_i$ at x_0.*

(ii) *Let Y_i be normed spaces, $f_i : X \to Y_i$, $i = 1, ...k$, $f = (f_1, f_2, ..., f_k)$ and $A_{f_1}(x_0), ..., A_{f_k}(x_0)$ be first-order approximations of $f_1, ..., f_k$, respectively, at x_0. Then, $A_{f_1}(x_0) \times \ldots A_{f_k}(x_0)$ is a first-order approximation of f at that point.*

(iii) *Let Y be a Hilbert space and $f, g : X \to Y$, $\langle f, g\rangle(x) = \langle f(x), g(x)\rangle$. If $A_f(x_0), A_g(x_0)$ are first-order approximations of f and g at x_0 and f, g are calm at x_0, then $\langle g(x_0), A_f(x_0)\rangle + \langle f(x_0), A_g(x_0)\rangle$ is a first-order approximation of $\langle f, g\rangle$ at x_0.*

(iv) *Let $f : X \to Y$ and $g : Y \to Z$. If f is calm at x_0, then $A_g(f(x_0)) \circ A_f(x_0)$ is a first-order approximation of $f \circ g$ at x_0.*

Proposition 4.3.2. *Let $f, g : X \to \mathbb{R}$. Let $A_f(x_0)$, $A_g(x_0)$ be first-order approximations of f and g, respectively, at x_0. Then, the following assertions hold.*

(i) *If f or g is calm at, x_0 and at least one of $f(x_0)$ and $g(x_0)$ is nonzero whenever both $A_g(x_0)$ and $A_f(x_0)$ are unbounded, then $g(x_0)A_f(x_0) + f(x_0)A_g(x_0)$ is a first-order approximation of $f.g$ at x_0.*

(ii) *If g is calm at x_0 and $g(x_0) \neq 0$, then*

$$\frac{g(x_0)A_f(x_0) - f(x_0)A_g(x_0)}{g^2(x_0)}$$

is a first-order approximation of f/g at x_0. When X is finite dimensional, the calmness can be reduced to continuity.

With similar assumptions and notations, we obtain sum rule, Descartes product rule and inner product rule for second-order approximations as follows.

Proposition 4.3.4. *Let $f, g : X \to \mathbb{R}$, g be 2-calm at x_0 and $(A_f(x_0), B_f(x_0))$, $(0, B_g(x_0))$ be second-order approximations of f and g, respectively, at x_0. Then,*

(i) *$(g(x_0)A_f(x_0), g(x_0)B_f(x_0) + f(x_0)B_g(x_0))$ is a second-order approximation of $f.g$ at x_0;*

(ii) *if $g(x_0) \neq 0$, then*

$$\left(\frac{A_f(x_0)}{g(x_0)}, \frac{g(x_0)B_f(x_0) - f(x_0)B_g(x_0)}{g^2(x_0)}\right)$$

is a second-order approximation of f/g at x_0.

The assumed 2-calmness of g in Proposition 4.3.4 is restrictive. But, in Example 4.3.1, we show that this cannot be replaced by the calmness assumption.
Proposition 4.3.5.

(i) *Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq 0$. If $A_1, A_2 \in L(X, Y)$ are asymptotically p-compact sets with $A_2 \in L(X, Y)$ being bounded then $\lambda_1 A_1 + \lambda_2 A_2$ is an asymptotically p-compact set.*

(ii) *For asymptotically p-compact sets $A_i \in L(X, Y), i = 1, ..., k$, $\prod_{i=1}^{k} A_i$ is also asymptotically p-compact.*

4.4. First order optimality conditions

Proposition 4.4.1 (Necessary condition). *For problem (P), let $g_i(x_0) \neq 0$ and $A_{f_i}(x_0)$, $A_{g_i}(x_0), A_h(x_0)$ be asymptotically p-compact first-order approximations of f_i, g_i and h, respectively, at x_0, with $A_{g_i}(x_0)$ being bounded, for $i = 1, ...m$. Denote*

$$A_\varphi(x_0) := \prod_{i=1}^{m} \frac{g_i(x_0)A_{f_i}(x_0) - f_i(x_0)A_{g_i}(x_0)}{g_i^2(x_0)}.$$

If x_0 is a local weak solution of (P), then, $\forall u \in X$, $\exists P \in \mathrm{p}-\mathrm{cl}A_\varphi(x_0) \cup (\mathrm{p}-A_\varphi(x_0)_\infty \setminus \{0\})$, $\exists Q \in \mathrm{cl}A_h(x_0)$, $\exists (c^*, d^*) \in C^* \times D^* \setminus \{(0,0)\}$,

$$\langle c^*, Pu\rangle + \langle d^*, Qu\rangle \geq 0,\ \langle d^*, h(x_0)\rangle = 0.$$

Furthermore, for u satisfying $0 \in \mathrm{int}(Q(u) + h(x_0) + K)$, for all $Q \in A_h(x_0)$, we have $c^* \neq 0$.

Proposition 4.4.2 (Sufficient condition). *Let $X = \mathbb{R}^n$, $x_0 \in h^{-1}(-K)$. Assume that, for $i = 1, ..., m$, $A_{f_i}(x_0), A_{g_i}(x_0), A_h(x_0)$ are asymptotically p-compact first-order approximations of f_i, g_i and h at x_0, with all $A_{g_i}(x_0)$ being bounded. Denote $A_\varphi(x_0) = \prod_{i=1}^{m} \frac{g_i(x_0)A_{f_i}(x_0) - f_i(x_0)A_{g_i}(x_0)}{g_i^2(x_0)}$. Impose further that, for all $u \in T(h^{-1}(K), x_0)$ with norm one, all $P \in \mathrm{cl}A_\varphi(x_0) \cup (A_\varphi(x_0)_\infty \setminus \{0\})$ and all $Q \in \mathrm{p}-\mathrm{cl}A_h(x_0) \cup (\mathrm{p}-A_h(x_0)_\infty \setminus \{0\})$, there exists $(y^*, z^*) \in C^* \times K^* \setminus \{(0,0)\}$ such that*

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle > 0,\ \langle z^*, h(x_0)\rangle = 0.$$

Then, x_0 is a local firm solution of order 1 of (P).

Note that in most of the known optimality conditions for fractional problems, X is assumed to be finite dimensional. Furthermore, when applied to the finite dimensional case, Theorem 4.4.1 is also advantageous, since f is not required to be continuous.

Example 4.4.1. *Let $X = \mathbb{R}, m = 1, Y = \mathbb{R}$, $C = K = \mathbb{R}_+$, $x_0 = 0$,*

$$f(x) = \begin{cases} -\frac{1}{x}, & \textit{if}\, x > 0, \\ -x, & \textit{if}\, x \leq 0, \end{cases}$$

$g(x) = x^2 + 1$, and $h(x) = -\sqrt[3]{x} + x^2$. We can take approximations $A_g(x_0) = \{0\}$ and $A_h(x_0) = (-\infty, \beta)$ with $\beta < 0$ being arbitrary and fixed. Since f is (infinitely) discontinuous at $x_0 = 0$, the mentioned known results are not in use. Since $g(x_0) = 1, f(x_0) = 0$ and, for $-1 < \alpha < 0$, $A_f(x_0) = (-\infty, \alpha)$, one has $A_\varphi(x_0) = A_f(x_0)$, $\mathrm{cl}A_\varphi(x_0) = (-\infty, \alpha]$, $A_\varphi(x_0)_\infty = (-\infty, 0]$. For $u = 1$,

we see that, $\forall P \in \mathrm{cl}A_\varphi(x_0) \cup (A_\varphi(x_0)_\infty \setminus \{0\})$, $\forall Q \in \mathrm{cl}A_h(x_0)$, $\forall (c^*, d^*) \in C^* \times K^* \setminus \{(0,0)\} = \mathbb{R}^2_+ \setminus \{(0,0)\}$ *with* $\langle d^*, h(x_0)\rangle = 0$,

$$\langle c^*, Pu\rangle + \langle d^* Q, u\rangle = c^* P + \beta d^* < 0.$$

According to Theorem 4.4.1, x_0 *is not a local weak solution of (P).*

4.5. Second order optimality conditions

Proposition 4.5.1. *Assume that* C *is polyhedral,* $g_i(x_0) \neq 0$, g_i *is 2-calm at* x_0, *for* $i = 1, ..., m$, *and* $z^* \in K^*$ *with* $\langle z^*, g(x_0)\rangle = 0$. *Impose further that* $(f_i'(x_0), B_{f_i}(x_0))$, $(0, B_{g_i}(x_0))$ *and* $(h'(x_0), B_h(x_0))$ *are asymptotically p-compact second-order approximations of* f_i, g_i *and* h, *respectively, at* x_0 *with* $B_{g_i}(x_0)$ *and* $B_h(x_0)$ *being bounded, for* $i = 1, ..., m$. *Set*

$$A_\varphi(x_0) = \prod_{i=1}^m \frac{f_i'(x_0)}{g_i(x_0)},\ B_\varphi(x_0) = \prod_{i=1}^m \frac{g_i(x_0)B_{f_i}(x_0) - f_i(x_0)B_{g_i}(x_0)}{g_i^2(x_0)}.$$

If x_0 *is a local weak solution of (P), then, for any* $v \in T(H(z^*), x_0)$, *there exists* $y^* \in B$, *where* B *is finite and* $\mathrm{cone}(\mathrm{co}B) = C^*$, *such that* $\langle y^*, A_\varphi(x_0)v\rangle + \langle z^*, h'(x_0)v\rangle \geq 0$. *If, furthermore,* $y^* \circ A_\varphi(x_0) + z^* \circ h'(x_0) = 0$, *we have either* $M \in \mathrm{p} - \mathrm{cl}B_\varphi(x_0)$ *and* $N \in \mathrm{p} - \mathrm{cl}B_h(x_0)$ *such that*

$$\langle y^*, M(v,v)\rangle + \langle z^*, N(v,v)\rangle \geq 0,$$

or $M \in \mathrm{p}{-}B_\varphi(x_0)_\infty \setminus \{0\}$ *such that*

$$\langle y^*, M(v,v)\rangle \geq 0.$$

Proposition 4.5.2. *Assume that* X *is finite dimensional,* $(x_0) \in h^{-1}(-K)$, $g_i(x_0) \neq 0$, g_i *2-calm at* x_0 *and* $(f_i'(x_0), B_{f_i}(x_0))$, $(0, B_{g_i}(x_0))$ *and* $(h'(x_0), B_h(x_0))$ *are asymptotically p-compact second-order approximations of* f_i, g_i *and* h, *respectively, at* x_0, *with* $B_h(x_0)$ *being bounded, for* $i = 1, ..., m$. *Set* $A_\varphi(x_0)$ *and* $B_\varphi(x_0)$ *as in Theorem 4.5.1 and*

$$C_0^* \times K_0^* = \{(y^*, z^*) \in C^* \times K^* \setminus \{(0,0)\} \mid y^* \circ A_1(x_0) + z^* \circ h'(x_0) = 0, \langle z^*, h(x_0)\rangle = 0\}.$$

Impose further the existence of $(y^*, z^*) \in C_0^* \times K_0^*$ *such that, for all*

$v \in T(h^{-1}(-K), x_0)$ *with* $\|v\| = 1$ *and* $\langle y^*, A_\varphi(x_0)v\rangle = \langle z^*, h'(x_0)\rangle = 0$, *one has*

(i) *for each* $M \in \mathrm{cl}B_\varphi(x_0)$ *and* $N \in \mathrm{p} - \mathrm{cl}B_h(x_0)$,

$$\langle y^*, M(v,v)\rangle + \langle z^*, N(v,v)\rangle > 0;$$

(ii) *for each* $M \in \text{p}-B_\varphi(x_0)_\infty \setminus \{0\}$,

$$\langle y^*, M(v,v)\rangle > 0.$$

Then, x_0 *is a local firm solution of order 2.*

We present second-order optimality conditions for the nondifferentiable case in Theorems 4.5.3 and 4.5.4. Example 4.5.2 illustrates that Proposition 4.5.1 can be employed in infinite dimensional spaces while Theorem 4.1 of Reedy and Mukherjee (2002) cannot.

Chapter 5. Local uniqueness of solutions to vector equilibrium problems using approximations

5.1 Introduction

As known from Chapter 3, for equilibrium problems, main efforts have been focussed on the solution existence, stability, well-posedness, algorithms, duality and optimality conditions. Although the uniqueness of solutions is an important theoretical topic, we observe that it was investigated mainly for mathematical programming and variational inequalities, particular cases of equilibrium problems, see Tawid (2002), Luc (2002), Luc and Noor (2003), and for a scalar equilibrium problem only in Khanh et al. (2006). This motivates our consideration of vector equilibrium problems. Furthermore, we will employ the approximation as a generalized derivative. Results based on this derivative may be not only more general but also under weaker assumptions than those using other generalized derivatives. The results in [5] is included in this chapter.

5.2. Preliminaries

Let $H \subseteq R^n$ be nonempty, $K \subseteq R^n$ nonempty closed convex and $C \subseteq R^l$ be a pointed closed convex cone with nonempty interior. Let $f : R^n \to R^m$, $g : R^n \to R^n$ and $\varphi : R^m \times R^n \to R^l$ with the components $(\varphi_1(y,x), \varphi_2(y,x), ..., \varphi_l)$. Setting $\Omega = \{x \in H \text{ with } g(x) \in K\}$, the vector strong equilibrium problem (SEP) (weak equilibrium problem (WEP)) under our consideration is:

find $x_0 \in \Omega$ such that, for every $x \in \Omega$,

$$\varphi(f(x_0), g(x)) - \varphi(f(x_0), g(x_0)) \in C$$

$$(\varphi(f(x_0), g(x)) - \varphi(f(x_0), g(x_0)) \notin -\text{int}C, \text{ respectively}).$$

Notice that if $l = 1, C = \mathbb{R}_+, n = m, H = K$, $f = g \equiv I$ and if $\varphi(x,x) = 0$ for all $x \in K$, the the above problems collapse to the classical equilibrium problem: find $x_0 \in K$ such that, for every $x \in K$,

$$\varphi(x_0, x) \geq 0.$$

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. A closed subset $\partial h(x_0) \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ is called (Jeyakumar and Luc 1998) a pseudo Jacobian (known also as approximate Jacobian) of h at $x_0 \in \mathbb{R}^n$ if, for each $v \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$,

$$(vh)^+(x_0, u) \leq \sup_{M \in \partial h(x_0)} \langle v, Mu \rangle,$$

where $(g)^+$ denotes the upper Dini directional derivative of a scalar function g, i.e.,

$$(vh)^+(x_0, u) := \limsup_{t \downarrow 0} \frac{\langle v, h(x_0 + tu) - h(x_0) \rangle}{t}.$$

A pseudo Jacobian $\partial h(x_0)$ is termed a Fréchet pseudo Jacobian of h at x_0 if there is a neighborhood U of x_0 such that, for each $x \in U$,

$$h(x) - h(x_0) \in \partial h(x_0)(x - x_0) + o(\|x - x_0\|).$$

5.3. Sufficient conditions for the local uniqueness of solutions

We establish now a sufficient condition for the local uniqueness of solutions of a strong equilibrium problem (SEP). By $\hat{f}$ we denote the restriction of f to Ω.

Theorem 5.3.1. *Let $x_0 \in \Omega$, $\hat{f}$ be continuous at x_0 and $A_{\hat{f}}(x_0)$ and $A_{\hat{g}}(x_0)$ be approximations of $\hat{f}$ and $\hat{g}$, respectively, where $A_{\hat{g}}(x_0)$ is bounded. Assume that $g(H) \supseteq K$ and the following conditions hold.*

(i) *For y in a neighborhood of $f(x_0)$, $\varphi(y, .)$ has first and second Fréchet derivatives, denoted by φ'_2 and φ''_{22}, which are jointly continuous (in both variables) at $(f(x_0), g(x_0))$.*

(ii) *$\varphi'_2(., g(x_0))$ and $\varphi''_{22}(., g(x_0))$ have approximations at $f(x_0)$, denoted by $(A_\varphi)_1[\varphi'_2(f(x_0), g(x_0))]$ and $(A_\varphi)_1[\varphi''_{22}(f(x_0), g(x_0))]$, respectively, where $(A_\varphi)_1[\varphi''_{22}(f(x_0), g(x_0))]$ is bounded.*

If x_0 is a solution of (SEP), then each of the following conditions is sufficient for its local uniqueness

(a) *for every* $M \in \mathrm{cl}A_{\hat{f}}(x_0) \cup (A_{\hat{f}}(x_0)_\infty \setminus \{0\})$, $G \in \mathrm{cl}A_{\hat{g}}(x_0)$ *and* $N \in \mathrm{cl}(A_\varphi)_1[\varphi_2'(f(x_0), g(x_0))] \cup ((A_\varphi)_1[\varphi_2'(f(x_0), g(x_0))]_\infty \setminus \{0\})$ *one has*

$$[N(M(v))]G(v) \notin -C$$

for all $v \in T(H, x_0) \setminus \{0\}$ *with* $G(v) \in C^\varphi(K, g(x_0))$;

(b) K *is polyhedral and condition (a) is satisfied for all* $v \in T(H, x_0) \setminus \{0\}$ *with* $G(v) \in C^\varphi(K, g(x_0))$ *and*

$\varphi_2'(f(x_0), x_0) + N(M(v))$
$\in [T(K, g(x_0))]^*_C \cup ([T(K, g(x_0))]^*_C - \varphi_{22}''(f(x_0), g(y_0))G(v))$.

By a similar proof with some changes, we obtain sufficient conditions for the local uniqueness of solutions of weak equilibrium problem in Theorem 5.3.2.

5.4. Special cases and examples

As equilibrium problems encompass many optimization-related problems, we can derive for them consequences from the results obtained in several important particular cases.

If $l = 1, C = \mathbb{R}_+$ and $\varphi(y, x) = \langle y, x\rangle$, our two problems (SVP) and (WEP) come down to the (scalar) generalized variational inequality of

(GVI): finding $x_0 \in \Omega$ such that, for every $x \in \Omega$,

$$\langle f(x_0), g(x) - g(x_0)\rangle \geq 0.$$

More specifically, when g is the identity, (GVI) becomes the classical (Stampacchia) variational inequality, denoted by (VI).

Corollary 5.4.1. *Let* $x_0 \in \Omega$, $\hat{f}$ *be continuous at* x_0, $g(H) \supseteq K$ *and* $A_{\hat{f}}(x_0)$ *and* $A_{\hat{g}}(x_0)$ *be approximations of* $\hat{f}$ *and* $\hat{g}$, *respectively, where* $A_{\hat{g}}(x_0)$ *is bounded. If* x_0 *is a solution of (GVI), then each of the following conditions is sufficient for the local uniqueness of* x_0:

(a) *for every* $M \in \mathrm{cl}A_{\hat{f}}(x_0) \cup (A_{\hat{f}}(x_0)_\infty \setminus \{0\})$, $G \in \mathrm{cl}A_{\hat{g}}(x_0)$ *one has*

$$\langle M(v), G(v)\rangle > 0$$

for all $v \in T(H, x_0)\setminus\{0\}$ *with* $G(v) \in C_{(f,g)}(K, g(x_0)) = \{u \in T(K, g(x_0)) : \langle f(x_0), u\rangle = 0\}$;

(b) K *is polyhedral and condition (a) is satisfied for all* $v \in T(H, x_0) \setminus \{0\}$ *with* $G(v) \in C_{(f,g)}(K, g(x_0))$ *such that*

$f(x_0) + M(v) \in [T(K, g(x_0))]^*$.

Turning now to a result for weak problems as example, we have the following immediate consequence of Theorem 5.3.2 for the classical weak vector equilibrium problem, i.e., the case with $n = m$, $H = K$, $f = g$ being the identity and $\varphi(x, x) = 0$, for all $x \in \mathbb{R}^n$.

Corollary 5.4.2. *Consider the classical weak vector equilibrium problem. Let the following assumptions be fulfilled.*

(i) For each y in a neighborhood of x_0, the map $\varphi(y, .)$ has first and second Fréchet derivatives, denoted by φ'_2 and φ''_{22}, which are jointly continuous (in both variables) at (x_0, x_0).

(ii) $\varphi'_2(., x_0)$ and $\varphi''_{22}(., x_0)$ have approximations at x_0, denoted by $(A_\varphi)_1[\varphi'_2(x_0, x_0)]$ and $(A_\varphi)_1[\varphi''_{22}(x_0, x_0)]$, respectively, with the latter being bounded.

If x_0 is a solution, then each of the following conditions is sufficient for its local uniqueness

(a) for every $N \in \mathrm{cl}(A_\varphi)_1(\varphi'_2(x_0, x_0)) \cup ((A_\varphi)_1(\varphi'_2(x_0, x_0))_\infty \setminus \{0\}$, one has

$$[N(v)](v) \in \mathrm{int}C,$$

for all $v \in C_1^\varphi(K, x_0) \setminus \{0\}$;

(b) K is polyhedral and, for every $N \in \mathrm{cl}(A_\varphi)_1(\varphi'_2(x_0, x_0)) \cup ((A_\varphi)_1(\varphi'_2(x_0, x_0))_\infty \setminus \{0\}$, one has

$$[N(v)](v) \in \mathrm{int}C,$$

for all $v \in C_1^\varphi(K, x_0) \setminus \{0\}$ with

$$\varphi'_2(x_0, x_0) + N(v) \in [T(K, x_0)]^\sharp_C \cup ([T(K, x_0)]^\sharp_C - \varphi''_{22}(x_0, x_0)).$$

Examples 5.5.1-5.5.3 show that Theorem 5.3.1 and its corollaries can be applied while previous results cannot.