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Auteur:
Dino BORIE

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Jury :

M. **Thibault Gajdos** (Directeur de Recherche, CNRS), Rapporteur

M. **Pierre Garouste** (Professeur, Université Nice Sophia Antipolis)

M. **Olivier L'Haridon** (Professeur, Université Rennes 1)

M. **Massimo Marinacci** (Professeur, Université Bocconi), Rapporteur

M. **Alain Raybaut** (Chargé de Recherche, CNRS), Directeur de thèse

M. **Dominique Torre** (Professeur, Université Nice Sophia Antipolis), Directeur de thèse

"A choice of axioms is not a purely objective task. It is usually expected to achieve some definite aim - some specific theorem or theorems are to be derivable from the axiom - and to this extent the problem is exact and objective. But beyond this there are always other important desiderata of a less exact nature : The axioms should not be too numerous, their system is to be as simple and transparent as possible, and each axiom should have an immediate intuitive meaning by which its appropriateness may be judged directly."

von Neumann and Morgenstern (1944, p. 25)

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Introduction

Motivations

In most decisions individuals must to choose between options that involve some uncertainty about their outcomes and their effect on their well-being. Experimental studies suggest that, in making these decisions, individuals often deviate from the paradigm of classical decision theory, even in relatively simple situations. In carefully controlled studies psychologists show, in more complex situations, that individual choices are sensitive to the description of the options, their contextualization and elicitation method. In social context individuals care not just about their outcomes but also about the outcomes and the intentions of those around them.

This thesis is divided in three independent chapters. The first one deals with additively separable preferences on the set of lotteries. This study leads to a non-linear expected utility representation and a weak form of event-separability of preferences. Also, I deduce a simple axiomatic foundation of an entropy modified expected utility. In the second chapter, I provide a general choice model under risk with social interactions. The third chapter of the thesis has to do with the potential of quantum probability theory in the von Neumann and Morgenstern framework. Each chapter focus on a particular generalization of the expected utility model and I am going to present in the next paragraphs the general ideas and theories that are common to most of them.

Decision making under risk

Decision theory has a long history since the emergence [1] of probabilities. The first natural criterion of decision making under risk is the standard expectation value. This naive criterion has been challenged by the St. Petersburg paradox in the 18th century and *solved* by Bernoulli (1738-1954) that postulates the expected utility[2] criterion. This criterion has been axiomatised by von Neumann and Morgenstern (1944) (modern presentations refer usually to Marschak (1950), Herstein and Milnor (1953), Luce and Raiffa (1957), Jensen (1967) or Fishburn (1970) for exogenous probabilities). This formulation is tractable, it defines the attitude toward risk and it is applicable to many academic fields, especially those related to the theory of non-cooperative games. Expected utility is based on the independence axiom [3] and has a normative appeal. However, many experimental results have shown that this decision criterion was questionable. The most popular is definitely the Allais paradox (1953) which leads to the definition of two more general phenomena, common ration effect and common consequence effect. Both phenomena have been reproduced by Kahneman and Tversky (1979) (problems 1,2 and 3,4 respectively). MacCrimmon and Larsson (1979) provide a detailed study of the paradoxes of the independence axiom in the context of risk and radical uncertainty. The descriptive accuracy of expected utility has led to generalizations that we can classify[4] into three non-exhaustive and non-mutually exclusive categories.

The first class of generalizations weakens the independence axiom. The contribution of the first chapter is in this class. The sophistication of the expected utility theories with a clear axiomatic framework, identifying the weakening of the independence axiom, allows to better understand the normative and descriptive aspects of these theories. For example, weighted expected utility [5] proposed by Chew and MacCrimmon (1979) where the independence axiom holds for lotteries

[1] The reader is referred to Hacking (1975).

[2] This is not the value of the random variable that must go into the calculation of expectation but the moral value, utility, that individual assigns to the variable that must go into the calculation of expectation, giving what we call today the expected utility.

[3] Surprisingly absent in von Neumann and Morgenstern , see Fishburn and Wakker (1995) for an historical perspective on the formulation of this axiom.

[4] I do not mention here intransitive preferences, such as the approach of Bell (1982), Fishburn (1982) or Loomes and Sugden (1982).

[5] See Chew (1989) for this class of model.

in the same equivalence class and rank dependent expected utility[6] axiomatised by Quiggin (1982) in the risk where the independence axiom holds for co-monotonic lotteries.

The second class of generalizations rejects the independence axiom. One example is the local expected utility initiated by Machina (1982) and developed by Chew and Nishimura (1992), Chew and Hui (1995) and for a recent contribution Chaterjee and Krishna (2011). This approach preserves the weak order condition and requires a notion of differentiability of the representation of the preferences. The expected utility becomes a local notion (in the topological sense) because differentiable functional can be considered as locally linear. This approach allows great flexibility and a generalization of criteria attitudes towards risk[7]. Another example is the theory of Luce (see Luce (2000) for a compilation of all of his work and Wakker (2000) for a short summary). Luce built an alternative theory from psychological concepts and found the majority of standard models from behavioural axioms with a concatenation operation between lotteries.

The third class consists of generalizations based on experimental approaches focusing on the descriptive aspect of the individual decision-making, identifying utility functionals that reproduce the experimental results. The best known example is the prospect theory of Kahneman and Tversky (1979) and refinement, the cumulative prospect theory (i.e., the rank-dependent prospect theory (1992)). Another example is the TAX model (Transfer of Attention Exchange) developed by Brinbaum and Chavez (1997).

Rank-dependent expected utility is certainly the contribution that has had the most success in decision theory (e.g., see Weber and Kirsner (1997), Diecidue and Wakker (2001) and Mongin (2009) for arguments highlighting the rank dependent expected utility). This theory has, both in the context of risk or uncertainty [8], and in many models [9] to accommodate violations of expected utility theory. In addition, the functional associated with this theory preserves interesting properties such as stochastic dominance. Empirically, the rank-dependent expected utility leads to better results than expected utility or weighted expected utility

[6]See Yaari (1987) for dual expected utility and Segal (1984, 1989) or Green and Jullien (1988) for generalized representations.

[7]See Cohen (1995).

[8]Axiomatized by Schmeidler (1989).

[9]Inter alia, Luce and Fishburn (1991, 1995), Tversky and Kahneman (1992), Wakker and Tversky (1993), Chateauneuf and Wakker (1999), Schmidt and Zank (2001, 2009, 2012) and Abdellaoui (2002).

for choice situations referring to the Allais paradox. In more general situations, rank-dependent expexted utility does not perform better than expected utility or other theories [10].

An important part of axiomatic foundations of rank-dependent expected utility related models[11] requires the result obtained by Wakker (1991, 1993) on additively separable representations for rank-ordered subsets of Cartesian products. This result gives an additively separable representation for all lotteries when they are assimilated as their cumulative distribution functions.

The main contribution of the first chapter is to provide an axiomatic foundation of additively separable functional for lotteries given by their distribution functions, using the orthogonal additivity property of the whole functional. Orthogonal additivity means additive over orthogonal alternatives. Orthogonal alternatives are alternatives with disjoint supports.

In a series of papers, Luce, Ng, Marley and Aczel (2008a, 2008b, 2008, 2009a, 2009b) provide a behavioural axiomatic foundation for the utility of gambling[12], based on the theory of Luce (2000). Their approach is based on the work of Meginniss (1976) who analytically introduced a variational functional [13] represented by the sum of an expected utility term and an entropic term. This functional is a particular case of preferences where the independence axiom hold for lotteries with disjoint supports.

Social interactions and other-regarding preferences

Social Interactions[14] refer to socio-economic phenomena where individuals' choices are not solely mediated by the price mechanism and are influenced in particular by the choices of other individuals in their social reference groups. Rather than

[10] e.g., Wakker *et al.* (1994), Wu (1994) and Birnbaum *et al.* (1999).

[11] Segal (1984, 1989), Green and Jullien (1988), Wakker and Tversky (1993), Chateauneuf and Wakker (1999), Wakker (1994), Schmidt and Zank and Abdellaoui (2002).

[12] See Diecidue *et al.* (2004) and Le Menestrel (2001) for a comprehensive study of utility of gambling.

[13] The term "variational" is used in Maccheroni *et al.* (2006) for preferences including a general cost function.

[14] See Zanella (2004) or Scheinkman (2008) for a short presentation.

existing as isolated[15] entities, individuals are embedded within networks of relationships, e.g. peer groups, families, colleagues, neighbours, or more generally any socio-economic group. In game theory, empirical evidences[16] challenge the paradigm of self-interested agents. In economics, the idea that the well-being of an agent depends on the relative as well as the absolute well-being is usually attributed Veblen (1899). More "recently", Duesenberry (1949) or Leibenstein Leibenstein (1950) had already considered that, in some situations, the relative well-being may be more meaningful than the absolute well-being. In decision theory it is then necessary to introduce interdependent or other-regarding preferences[17]. The introduction of agents' concerns for relative outcomes into economic models are of particular interest and has been shown to carry serious implications in different fields such as demand analysis[18], labour economics[19], growth[20], Asset pricing[21], Attitude toward risk or uncertainty[22].

Pioneering work of Schelling (1971) had persuasively shown that several important aggregate phenomena crucially hinge not only upon the self-interested motives of individuals, but more deeply upon the interactions among them. This was next pointed out in a paper by Föllmer (1974) for non-market interactions, random preferences and random endowments in a general equilibrium model. These effects are due to social multiplier in behaviours.

Blume, Brock and Durlauf's contribution (Brock and Durlauf (2001a, 2001b), Blume and Durlauf (2001)) gives [23] a first analytical treatment of decision making in the presence of social interactions. They propose a model where individuals make binary choices. The utility of a choice depends on a private utility term, a random utility term and a social utility term specified by the social distance and the subjective belief of the agent. From assumptions [24] on the random utility

[15]See Sen (1977), Akerlof et Kranton (2000) or Davis (2013) for a criticism of the "self-centered, self-interested" individual.

[16]In game theory, see Rosenthal Rosenthal (1981), Guth et al. Güth et al. (1982) or Forsythe et al. Robert et al. (1994).

[17]See Karni and Safra (2002) or Maccheroni *et al.* (2012) in the theory of individual decision preferences incorporating equity or interdependence under risk and uncertainty, Rabin (1993), Fehr and Schmidt (1999) or Segal and Sobel (2007) in game theory. See Cooper and Kagel (2009b) for a review of experimental results.

[18]Gaertner (1974), Pollak (1976), Becker (1991), Cowan et al. (1997), Binder et Pesaran (2001).

[19]Akerlof et Yellen (1990), Neumark et Postlewaite (1998) Bowles et Park (2005).

[20]Corneo et Jeanne (2001), Alvarez-Cuadrado et al. (2004), Liu et Turnovsky (2005).

[21]Chan et Kogan (2001), Dupor et Liu (2003).

[22]Rohde et Rohde (2011), Linde et Sonnemans (2012), Charness et al. (2013).

[23]See Manski (1993).

[24]Random variables i.i.d according to the law of extreme values.

term, the authors obtain a law of probabilities (a Boltzmann law) on the choice of agents, analogous than the Curie Weiss model in statistical physics. Assuming that agents have rational expectations, they derive the corresponding mean-field Nash equilibria. Theoretically, their formulation clarifies interactions among agents, and empirically, allows identification of social interactions or " peer effects[25]" between individuals. Manski (1993) defines three peer effects. An endogenous effect, that is, the influence of group behaviour on the behaviour of the individual, an exogenous effect , that is, the influence of the characteristics of the group on the individual's characteristics and the correlation effect, when members of a group acting identically because they have the same characteristics. Empirically, their model allows to test econometric-ally social interactions due to endogenous and exogenous effects.

The Blume, Brock and Durlauf's model separates into a deterministic part and a stochastic part essential for its econometric application. From a theoretical point of view, it is appropriate to focus on the axiomatic foundation of the deterministic part of the model. The single use of the deterministic part in game theory would lead probably to Nash equilibria in mixed strategies. These equilibria, probably, would be different from equilibrium in the individual setting[26]. One solution is to replace the stochastic utility term by an appropriate variational term to obtain the desired result. The appropriate term is the Shannon entropy. This is not surprising, Blume, Brock and Durlauf's model is statistical physics inspired. To recap, the logit choice model is a Boltzmann distribution and the latter is the solution under constraint of expectation of the Shannon entropy.

The second chapter provides an axiomatic basis of interdependent preferences in risk, in the design of Blume, Brock and Durlauf's model. I derive a private utility and a social utility, coupled with a variational term. Such a foundation allows exogenous and endogenous reference groups. In addition, I give axioms for an additively separable, among individuals in the reference group, social utility.

[25] For a detailed analysis the econometric approach and a review of existing literature on the subject, see Blume et al. (2010). For an econometric issue of peer effects, see Manski (1993, 2000).

[26] To be convinced, without terms of social utility, it suffices to compare the standard Nash equilibrium and the logit equilibrium introduced by McKelvey and Palfrey (1995, 1998). The authors extend the approach of McFadden (1974) to the theory of games.

Quantum probabilities

Several studies use quantum formalism in decision theory to explain the various paradoxes. The way that quantum formalism is used varies across studies but it offers new opportunities in the form of new technical capabilities from the same mathematical tool. In a deterministic choice framework (Danilov and Lambert-Mogiliansky (2005, 2010), Gyntelberg and Hansen (2005, 2009), Hansen (2005), La Mura (2009)) or in stochastic choice framework (Aerts and Aerts (1995), Aerts and Gabora (2005a,b),Aerts and D'Hooghe (2009), Aerts et al. (2011), Busemeyer et al. (2006a), Busemeyer et al. (2011), Busemeyer and Bruza (2012), Conte et al. (2009), Khrennikov (1999, 2010), Lambert-Mogiliansky et al. (2009), Pothos and Busemeyer (2009), Pothos and Busemeyer (2013), Yukalov and Sornette (2010)). What is the motivation for employing quantum formalism in decision making ? What is the contribution ?

Foremost, it is necessary to precise that decision theory is not a quantum mechanics phenomenon. But, the probabilistic framework can be used independently. To avoid confusion, I prefer to speak of non-commutative probability rather than quantum probability. Building on all this work, I present a model based on non-commutative probabilities. My work introduces a decision-theoretic framework which extends the expected utility methodology and in addition tries to connect descriptive and normative approaches. The key aspect is to suppose that events are subjective. It becomes natural to embed classical probabilities spaces in an enclosing structure : a non-commutative probability space.

Chapter 1

Additive Utility Under Risk

1.1 Introduction

This chapter studies additive representation theory on simplices which are subsets of Cartesian products. Interest of this subject is its application to non-linear expected utility theory. Expected utility theory was characterized by "independence" axioms. Experimental failures[1] of traditional expected utility has led to an extensive literature in non-linear expected utility theory whose goal is to weaken the independence axiom. Main cases of this literature under risk include utility theory with the betweenness[2] property, which requires independence to hold only within equivalence classes, and rank-dependent[3] utility, which requires independence to hold only on comonotonic lotteries. The present chapter has been motivated by several generalizations of the classical choice criterion under risk where functionals have the property of being additively separable over probabilities. More specifically, examples of such functionals can be found under an analytical framework in Meginniss (1976), who assumes explicitly additive separability, and in Kahneman and Tversky (1979), who establish, from experimental data, an expected utility with a probability weighting function.

The purpose of this chapter is to provide an axiomatic foundation to additively separable functionals, that yields representations given by

[1]See Allais (1953) and Kahneman and Tversky (1979) for the independence axiom or Wu (1994) and Wakker et al. (1994) for the comonotonic independence axiom, this axiom does not perform better than independence except for Allais-type choices.

[2]E.g. Chew and MacCrimmon (1979), Chew (1983) and Chew (1989).

[3]E.g. Quiggin (1982) or Segal (1989).

$$V(l) = \sum_{X} \Phi(x_k, l_k) \tag{1.1}$$

where X is a finite set of outcomes and l is a finite lottery over X. This problem is equivalent to show that preferences are additively decomposable. In finite dimension, additive representations starts in the papers of Debreu (1959) for topological full Cartesian products, Luce and Tukey (1964)[4] for algebraic structures of full Cartesian products, and Scott (1964) for finite subsets of Cartesian products. This last result is extended by Jaffray (1974a,b) for arbitrary countable subsets of Cartesian products. For full Cartesian products Wakker (1991, 1993) provides necessary and sufficient conditions for additive representation over rank-ordered subsets of full Cartesian products. In recent decades, research has focused on monotonic additive representations on connected subsets of Cartesian product sets to justify global[5] additive representability under local additive representability.

I shall show that on simplices additive representation theory is characterized by the properties of the above functional. An additively decomposable functional is orthogonally additive. That is, the functional is additive whenever arguments are orthogonal in terms of disjoint supports. My approach can be applied to the usual case of a full Cartesian products and provides a general framework.

This chapter is organized as follows. In section 2, I introduce the key concept of my work. Section 2.1 describes orthogonally additive functional. Section 2.2 points the differences with other approaches in the literature. Section 3 gives the main results of this chapter. Section 4 applies the main results to variational preferences. Proofs are given in section 5. Section 6 concludes.

As usual, decision maker's *preferences* are represented by a binary relation $\succeq$ over the choice set, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succeq$. $\succeq$ is a weak order on the choice set if it is transitive and complete. A real-valued function V from the choice set to $\mathbb{R}$ is called an utility functional (or functional for short) if it represents $\succeq$ on the choice set or, in mathematical terms, if it is an order homomorphism.

[4] This approach is well explained in Krantz et al. (1971). See e.g. Wakker (1988b) and Luce et al. (1971) for a comparison between the topological and the algebraic framework.

[5] See Segal (1992, 1994) and Chateauneuf and Wakker (1993). These results are used in rank-dependent utility theory.

1.2 Orthogonally additive functional

1.2.1 Orthogonal additivity

Let $(X = \mathbb{R}^n, \succeq)$ be a non-trivial weakly ordered coordinate space[6] over $\mathbb{R}$. I assume that $n \geq 3$. I write $x_{-i}a$ for x with x_i replaced by a. A functional V is additively decomposable on X if $\forall x \in X : V(x) = \sum_1^n V_i(x_i)$ for some functions from $\mathbb{R}$ to $\mathbb{R}$. In this case, then the V_i's are additive value functions. A functional is cardinal if it is unique up to positive linear transformations. Additive value functions $\{V_i\}_i$ are jointly cardinal if they are unique up to similar positive linear transformations. Two alternatives x, y are called orthogonal if they have disjoint supports[7]. As the sum of alternatives is well defined in X, a functional V is said to be *orthogonally additive*[8] on X if $V(x+y) = V(x) + V(y)$ whenever $x, y \in X$ and have disjoint supports.

I suppose first that an additively decomposable functional V represents $\succeq$ on X. Let $x, y \in X$ such that x and y are orthogonal, it is clear that :

$$
\begin{aligned}
V(x+y) &= \sum_i V_i(x_i + y_i) \\
&= \sum_{\text{supp}(x)} V_i(x_i) + \sum_{\text{supp}(y)} V_i(y_i) + \sum_{i \notin \text{supp}(x) \cup \text{supp}(y)} V_i(0) \\
&= \sum_i V_i(x_i) + \sum_i V_i(y_i) - \sum_i V_i(0)
\end{aligned}
$$

[6]There is no loss of generality to work with $\mathbb{R}$. For $X = \prod_{i=1}^n X_i$ if each factors X_i is endowed with a concatenation operation or joint receipt operation $\oplus_i$ such that $(X_i, \oplus_i)$ is a group and if each factors is a connected separable topological space, the remainder of this section applies.

[7]The support of $x \in \mathbb{R}^n$ is defined by $\text{supp}(x) = \{i \in \{1, \dots, n\} \mid x_i \neq 0\}$.

[8]The theory of disjoint additivity is well established and has important applications such as characterization problems of integral operators. It is well known (the reader is referred Rao (1980) for a review of results), in brief words, that a continuous orthogonally additive functional over normed or metric linear $\mathbb{R}$-vector space is representable by

$$F(f) = \int \Phi(x, f(x))\, dx$$

where Φ has to possess certain properties.

As V is cardinal and $\{V_i\}_i$ are jointly cardinal, it follows that there is a class of orthogonally additive functionals that represents $\succeq$. Conversely, if I suppose that $\succeq$ is represented by an orthogonally additive functional then

$$\begin{aligned} V(x) &= V(x_1, \dots, x_n) \\ &= V(\sum_i (0, \dots, 0, x_i, 0, \dots, 0)) \\ &= \sum_i V(0, \dots, 0, x_i, 0, \dots, 0) \end{aligned}$$

Defining $V_i = V(0, \dots, 0, x_i, 0, \dots, 0)$, then V is additively decomposable.

1.2.2 Discussion

Orthogonal additivity is the key aspect of additively decomposable utility in my framework. The main axiom of Debreu's theory is coordinate independence : $\succeq$ is *coordinate independent* (CI) on X if $x_{-i}a \succeq y_{-i}a \Leftrightarrow x_{-i}b \succeq y_{-i}b$, for all i, whenever all alternatives in question are contained in X. (CI) is special case of the *2*nd-order cancellation condition (C_2) : $\succeq$ satisfies (C_2) if for x^1, x^2, y^1 and $y^2 \in X$ be such that (y^1_j, y^2_j) is a permutation of (x^1_j, x^2_j) for every $j = 1, \dots, n$ then $x^1 \succeq y^1 \Leftrightarrow y^2 \succeq x^2$. This last axiom can be stated elegantly in my framework by :

Axiom (Orthogonal Cancellation (OC))**.** For all $x, y, z, z' \in X$ such that z and z' are orthogonal to $\{x, y\}$,

$$x + z \succeq y + z \Leftrightarrow x + z' \succeq y + z'$$

It is clear that (OC) and (C_2) are equivalent. Under structural[9] assumptions, (OC) is necessary and sufficient for an additively decomposable utility on a weakly ordered full Cartesian product set.

Some remarks on the literature about additive representations were already given in introduction. I focus here on the main differences with the other approaches. Analytically, the result of Debreu (1959) is derived from web geometry and more specifically by properties of 3-web. A 3-web is a set consisting of elements of two types, lines and points. The set of lines of the 3-web is divided into three families such that two lines of different families lie on exactly one common point and each point is incident to exactly one line of each family. In the special case where the first family corresponds to straight lines with constant first coordinate, the second family to lines with constant second coordinate, and the last family to equivalence classes of a preference relation, existence of an additively decomposable representation on $X = X_1 \times X_2$ is equivalent to the existence of a topological transformation carrying the three families of lines into three families of parallel straight lines. Thomsen (1927) gave an affirmative answer to parallelization of 3-web under differentiability assumptions and later Blaschke (1928) under continuity. Basic cancellation conditions of web geometry as Thomsen condition, Reidmeister condition, Hexagonal condition can be found in (Krantz et al., 1971, Chapter 6) for a modern description. Debreu, for the case of three[10] dimensional, or more, full Cartesian products showed that (CI) implies the Thomsen condition locally on a two-dimensional subspace and inductively the result is demonstrated for arbitrary dimension. The Algebraic approach can be found in Krantz et al. (1971). Instead of topological assumptions, they use the restrictive assumption of solvability and an Archimedean axiom. Other axioms are identical to Debreu, moreover the general case is also derived from the two-dimensional case. Wakker (1988a) provides derivation distinguishing the case of a two-dimensional and a three-dimensional or more full Cartesian products by treating them separately.

My approach highlights that the full force of (CI) or (C_2) does not apply in the two-dimensional case because there is not enough orthogonal sequences[11]. I emphasize on the fact that my framework may seem more restrictive about the structure of each factors but, in fact, without orthogonal sequences, additive separable utility

[9] In the topological approach, $\succeq$ must be continuous with respect to the product topology of $\mathbb{R}^n$.

[10] The two-dimensional case can be founded in (Fishburn, 1970, Chapter 5)

[11] See Rätz (2001) for numerous results in mathematics where the properties of a vector space V is completely different for $dim V \geq 3$ and $dim V = 2$.

does not make sense. To emphasize this comment, note that (C_2) expressed as (OC), is the translation-invariance[12] axiom ,stated by Aumann (1962), restricted to disjoint sequences. For arbitrary subset X of a linear space, it is always possible, by translation, to find a zero and consequently to apply this approach.

1.3 The case of the simplex

1.3.1 Setup

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of outcomes or certain consequences where $n \geq 2$ (if $n = 1$, the set of lotteries is a singleton) is the cardinal of X. Let $\Delta(X) = \{(l_1, \ldots, l_n) \in [0,1]^n \mid \sum_i l_i = 1\}$ be the space of lotteries over X closed under convex mixture operations with $X \subset \Delta(X)$ consists of all degenerate lotteries (denoted by δ_x for all $x \in X$). I denote by $\text{supp}(l) \subset X$ the support of the lottery l. Two lotteries l, m are called orthogonal if they have disjoint supports. In the remainder of this chapter, $\succeq$ is a weak order on $\Delta(X)$. In most studies, $\succeq$ satisfies an Archimedean axiom when X is finite or $\succeq$ satisfies a continuity axiom when X is countable or uncountable. In the last case, $\Delta(X)$ is endowed with the topology of weak convergence. This topology is the coarsest topology such that for every continuous and bounded real-valued function g on X the map $f \mapsto \int_X fg$ is continuous. As we have seen previously for additively decomposable functional, a continuity assumptions is necessary in the topological approach. At the best of my knowledge, the Archimedean axiom is to weak to derive a continuous additively decomposable representation. Therefore, the appropriate topology when X is finite is the relative topology induced by the product topology of $[0,1]^n$. When X is finite, this topology coincides[13] with the topology of weak convergence. But when X is uncountable these two topologies are not equivalent, in fact, the topology of

[12]See Trockel (1992), Candeal and Induráin (1995) or Chatterjee and Krishna (2008) for results in linear utility.

[13]The product topology is equivalent to the strong topology which is equivalent to the weak topology in finite dimensional spaces.

weak convergence is less interesting[14] for generic non-linear functionals. A representation theorem for X countable or uncountable requires a continuity axiom with a finer [15] topology than the topology of weak convergence and its use would be a mistake. In the sequel of this chapter, I denote by $\mathcal{T}$ the relative topology induced by the product topology of $[0,1]^n$.

1.3.2 Orthogonal additivity in the simplex

It is clear that $\Delta(X) \subset [0,1]^n$ and it is not a full Cartesian product set. In addition, lottery coordinates (probabilities) are dependent, this is the reason that makes the usual theorems useless. However the structure of $\Delta(X)$, that is, the possibility to have a zero in each coordinate and the composition by convex combination allow to bring close orthogonally additive representations and additively decomposable representations. We shall observe that, just as linearity of expected utility is the key aspect of the standard representation under risk, orthogonal additivity on $\Delta(X)$ is the key aspect of the generalized representation. I say that a functional on $\Delta(X)$ is *Orthogonally additive** if and only if extended to $[0,1] \times \Delta(X)$ it satisfies for all $\alpha \in [0,1]$

$$V(\alpha l + (1-\alpha)m) = V(\alpha l) + V((1-\alpha)m) \tag{1.2}$$

whenever $l, m \in \Delta(X)$ and have disjoint supports. As expected utility, which is described analytically by an affine functional on $\Delta(X)$ which may be extended to a linear functional on $\mathbb{R}^n$, orthogonally additive* functional, in the above sense, can be extended to an orthogonally additive functional on $\mathbb{R}^n$ without difficulties. In the sequel, I do not distinguish the two concepts. Suppose that an orthogonally

[14]In general rank-dependent utility model (Green and Jullien (1988) and Segal (1989)) or in the Machina approach (Machina (1982)), the primitive of the preference relation are cumulative distributions. It does not seem to be a problem to use weak convergence with stochastic dominance. Stochastic dominance implies for some functionals continuity with respect to the topology of weak convergence as in (Becker and Sarin, 1987, Lemma 2) for lottery dependent utility or in Delbaen et al. (2011), Spinu and Wakker (2013) for expected utility.

[15]I shall not examine this matter in the present chapter.Note that as in Grodal et al. Grodal and Mertens (1968) or Vind et al. (Vind and Grodal, 2003, Chapter 11), for additive representation over countable or uncountable full Cartesian product, we can extend the definition of the weak convergence to glue the desired result. That is, define the coarsest topology such that for every bounded real-valued function Φ on $X \times \mathbb{R}$ continuous in both variables the map $f \mapsto \int_X \Phi(x, f(x))dx$ is continuous in $\Delta(X)$. Or, as in Chew and Lee (1985), we can use a strongest continuity axiom : for any pointwise convergent sequences (f_n) and (g_n) in $\Delta(X)$, $f_n \succeq g_n$ for all n imply $f \succeq g$.

additive functional represents $\succeq$ on $\Delta(X)$ and that $n \geq 4$. There are $l, m, n, o \in \Delta(X)$ such that $n \perp \{l, m\}$, and $o \perp \{l, m\}$ and for all $\alpha \in (0,1)$ then

$$\begin{aligned}
& V(\alpha l + (1-\alpha)n) \geq V(\alpha m + (1-\alpha)n) \\
\Leftrightarrow \quad & V(\alpha l) + V((1-\alpha)n) \geq V(\alpha m) + V((1-\alpha)n) \\
\Leftrightarrow \quad & V(\alpha l) \geq V(\alpha m) \\
\Leftrightarrow \quad & V(\alpha l) + V((1-\alpha)o) \geq V(\alpha m) + V((1-\alpha)o) \\
\Leftrightarrow \quad & V(\alpha l + (1-\alpha)o) \geq V(\alpha m + (1-\alpha)o)
\end{aligned}$$

which is exactly the *2*nd-order cancellation condition (C_2) on $\Delta(X)$. We can see that if $n \leq 3$ this property of orthogonally additive functionals is meaningless as they do not exist $l, m, n, o \in \Delta(X)$ such that $n \perp \{l, m\}$, and $o \perp \{l, m\}$. Before I state the axioms for $n \geq 4$ I will discuss special cases $n = 2$ and $n = 3$.

1.3.3 $n = 2$ and $n = 3$

The case $n = 2$ is somewhat outside the main interest of the chapter; cancellation conditions are meaningless as lotteries are entirely determined by the probability of one certain consequence. The representation result in this case is an immediate consequence of structural[16] assumptions on $\succeq$ and that $l_1 + l_2 = 1$. Let V be a continuous representation of $\succeq$, we can define $\phi_1(l_1) = V(l_1, 1 - l_1)$ and $\phi_2(l_2) = V(1 - l_2, l_2)$. It is obvious that $\phi_1 + \phi_2$ represents $\succeq$ and that ϕ_i's are continuously ordinal.

The case $n = 3$ is of great interest. In this case, the simplex is a two-dimensional convex space and a representation must involve cancellation conditions similar to Thomsen condition. It was noted that (C_2) is meaningless so we must turn to the *3*th-order cancellation condition (C_3) or double cancellation[17] : $\succeq$ satisfies (C_3) if for $l^k, m^k \in \Delta(X)$, $k = 1, \ldots, 3$ be such that $(m_j^1, \ldots, m_j^3)$ is a permutation of $(l_j^1, \ldots, l_j^3)$ for every $j = 1, \ldots, 3$ then $\forall i \leq 2$, $l^i \succeq m^i \Rightarrow m^3 \succeq l^3$.

[16] Weak order and continuity.

[17] See (Krantz et al., 1971, Definition 3, Chapter 6).

Firstly, for trivial permutations (C_3) implies transitivity[18]. Secondly, consider $l^1, l^2, l^3, m^1, m^2, m^3 \in \Delta(X)$ such that $m_1^1 = l_1^3, m_2^1 = l_2^2, m_3^1 = l_3^1, m_1^2 = l_1^2, m_2^2 = l_2^1, m_3^2 = l_3^3, m_1^3 = l_1^1, m_2^3 = l_2^2, m_3^3 = l_3^2$. Then ($C_3$) can be applied. An illustration is given in figure 1.1.

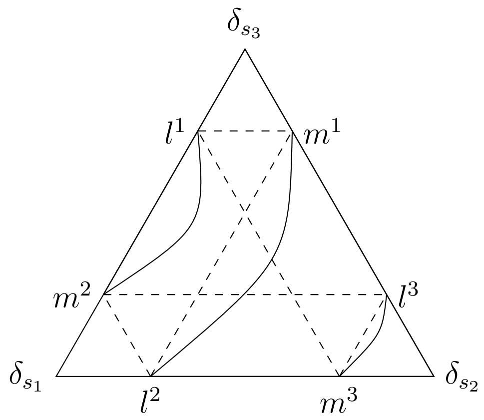

FIGURE 1.1: Graphical illustration in the 2-simplex. an indifference ($\sim$) part C_3 asserts that if l^1 and m^2 lie on the same indifference curve while the same hold for l^2 and m^1, then l^3 and m^3 also lie on the same indifference curve. All these types of conditions are similar to Thomsen condition.

Unfortunately, as (C_2) is meaningless, coordinate are not easily separable and there is no "clear" monotonicity between coordinates. From the point of view of web geometry, suppose that $\succeq$ over $\Delta(X)$ is represented by a continuous functional. The 2-simplex is a two-dimensional convex set (represented by an equilateral triangle), the three families of straight lines in the 2-simplex, parallel to the sides of the 2-simplex and the family of lines defined by the indifference curves[19] of the representation V give a family of four lines (variety of dimension one). The whole forms a 4-web $W(4, 2, 1)$[20]. By definition a 4-web which is topologically equivalent to a parallel 4-web is called parallelizable. In particular, if $W(4, 2, 1)$ is parallelizable then there exist ϕ_i for $i = 1, \ldots, 3$ such that $\phi_1 + \phi_2 + \phi_3$ represent $\succeq$ with one of the ϕ_i's is a linear combination[21] of the two others. A key result is that the

[18]This fact is also valid for full Cartesian product and shown by several authors, see (Krantz et al., 1971, Chapter 6).

[19]By transitivity and continuity the family of indifference curves are continuous and non-intersecting.

[20]See (Goldberg, 1988, Chapter 7) for a comprehensive treatment of such structures. $W(4, 2, 1)$ means that it is a 4-web in a two-dimensional convex set and that the dimension of each line is 1.

[21]To see this,consider the case where the indifference curves in the 2-simplex are given by parallel straight lines. This situation arises when $\succeq$ satisfies the von Neumann and Morgenstern Axioms and $\succeq$ is then represented by an expected utility.

4-web must be linearisable or topologically equivalent to a linear[22] 4-web to obtain a representation U such that $U(l^1)+U(l^2)+U(l^3) = U(m^1)+U(m^2)+U(m^3)$ for all $l^i, m^i \in \Delta(X)$ such that $(m^1_j, \dots, m^3_j)$ is a permutation of $(l^1_j, \dots, l^3_j)$ for every $j = 1, \dots, 3$.

Denote by $1, 2, 3$ the three families of straight lines corresponding to the sides of the 2-simplex and by 4 the family corresponding to the indifference curves. I denote by $[i, j, k]$, with $i, j, k = 1, \dots, 4$ and $i \neq j \neq k$ the 3-subweb formed by the families of curves i, j, k. $[1, 2, 3]$ form a hexagonal (regular) 3-subweb. Under *differentiability* assumptions, Goldberg (2004) gives necessary and sufficient conditions for such results depending on the structure of the subwebs $[1, 2, 4]$, $[1, 3, 4]$ and $[2, 3, 4]$ (Structure defined by (C_3)). Consequently, following Goldberg (2004), a clean proof for a representation in the case $n = 3$ would necessitate, in addition to (C_3), "smooth" preferences.

However, it is possible to consider instead of $\succeq$ the partial order $\succeq_j$ defined for an arbitrary $j = 1, \dots, 3$ by $l \succeq_j m$ if and only if $l_j = m_j$ and $l \succeq m$. If an additively separable representation exit then

$$l \succeq_j m \Rightarrow \phi_i(l_i) + \phi_k(l_k) \geq \phi_i(m_i) + \phi_k(m_k)$$

Embed $\Delta(X)$ in $\mathbb{R}^2$ with δ_j identified with $(0, 0) \in \mathbb{R}^2$. Define $l \perp m$ if and only if l and m have disjoint support in $\mathbb{R}^2$. Let $\Delta = \Delta(X)/\sim_j$ be the set of equivalence classes of $\Delta(X)$ under $\sim_j$ with typical element L. We construct a structure on Δ Letting

$$\mathbb{B} = \{(L, M) \mid \exists l, m \text{ such that } l \in L, m \in M, l \perp m \text{ and } l + m \in \Delta(X)\}$$

Define $\circ$ on $\mathbb{B}$ by letting $L \circ M = [l + m]$. By (C_3), $\circ$ is well defined but without (C_2), it is not possible to fit a known extensive structure (See (Krantz et al., 1971, Definition 3, Chapter 3)). The problem is not to find a representation Φ such that if $(L, M) \in \mathbb{B}$ then $\Phi(L \circ M) = \Phi(L) + \Phi(M)$, it is meaningless for $\succeq_j$. We need to find a representation Φ such that if $(L, M) \in \mathbb{B}$, $(L', M') \in \mathbb{B}$ then

$$\Phi(L \circ M) - \Phi(L' \circ M') = \Phi(L) + \Phi(M) - \Phi(L') + \Phi(M')$$

[22]That is a web formed by straight lines not necessarily parallel.

with respect to $\succeq_j$, that is, $L \circ M$ and $L' \circ M'$ comparable and so on. In fact, derivation of the desired result in the case $n = 3$ is relatively straightforward with some additional structural assumptions but difficult with only (C_3) and continuity. However, it seems to be possible.

1.3.4 The axioms

The axioms are given for the case $n \geq 4$. The first two axioms are standard and the topology used has been discussed before. The third axiom is the translation of (C_2) on $\Delta(X)$, it is discussed bellow.

Axiom A. 1 (Weak Order (WO))**.** $\succeq$ is non trivial, complete and transitive.

Axiom A. 2 (Continuity (C))**.** For all $l \in \Delta(X)$, $\{m \mid m \succeq l\}$ and $\{m \mid l \succeq m\}$ are closed in $\mathcal{T}$.

Axiom A. 3 (Weak Orthogonal Independence (WOI))**.** For all $l, m, n, o \in \Delta(X)$ such that $n \perp \{l, m\}$, and $o \perp \{l, m\}$ and for all $\alpha \in (0, 1)$

$$\alpha l + (1-\alpha)n \succeq \alpha m + (1-\alpha)n \Leftrightarrow \alpha l + (1-\alpha)o \succeq \alpha m + (1-\alpha)o$$

(WOI) has a simple intuitive interpretation : If two lotteries, l and m, coincide for a set of consequences, then the preference between l and m does not depend on the common partial distributions. Common partial distributions can be substituted in l and m by over identical partial distributions without modify preferences. Consequently, (WOI) seems to be the desired axiom for branch-separability over lotteries. This axiom implies in fact that an individual consistently edit and eliminate common components prior to choosing between lotteries as in Kahneman and Tversky (1979).

1.3.5 The theorem

Theorem 1.1.
Let $X = \{x_1, \ldots, x_n\}$ with $n \geq 4$, $\succeq$ on $\Delta(X)$ satisfies (WO), (C) and (WOI) if and only if there are continuous real-valued functions $\phi_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, n$, such that

$$V(l) = \sum_{1}^{n} \phi_i(l_i)$$

represents $\succeq$. Moreover, ϕ_i, $i = 1, \ldots, n$, are unique up to similar positive linear transformations.

For $n \geq 4$, as there is enough orthogonal sequence, it is possible to demonstrate without additional assumptions that an orthogonally additive representation exists, which leads to additively decomposable representations. The sketch of the proof is as follows :

Embed $\Delta(X)$ in $\mathbb{R}^X$ and consider the translation $\Delta_0 = \Delta(X) + m^0$ for an arbitrary m^0 in the interior of $\Delta(X)$. Define $\perp$ on Δ_0, orthogonality given by disjointness of the supports. Let $\Delta = \Delta_0 / \sim$ be the set of equivalence classes of Δ_0 under $\sim$. Construct a structure on Δ Letting

$$\mathbb{B} = \{(L, M) \mid \exists l, m \text{ such that } l \in L, m \in M, l \perp m\}$$

Define $\circ$ on $\mathbb{B}$ by letting $L \circ M = [l + m]$. By (WOI), $L \circ M$ exists and is well-defined as (WOI) implies (C_2). There exist a substructure of $(\Delta, \succeq, \mathbb{B}, \circ)$ that is an extensive structure. Consequently, it is possible to obtain $\Phi(L \circ M) = \Phi(L) + \Phi(M)$ for some $(L, M) \in \mathbb{B}$. The proof is to extend this representation to Δ and note that it is an orthogonally additive representation.

1.3.6 Stochastic dominance

I suppose in this subsection that $V(l) = \sum_1^n \phi_i(l_i)$ represents $\succeq$ over $\Delta(X)$ and that the ϕ_i's are differentiable. Following the approach of Machina (1982) extended by Chew and Nishimura (1992) and Chew and Hui (1995), V is Gâteaux differentiable and

$$\forall l, m \in \Delta(X), \lim_{\alpha \to 0} \frac{V((1-\alpha)l + \alpha m) - V(l)}{\alpha} = \sum_{1}^{n} (m_i - l_i)\phi_i'(l_i) \tag{1.3}$$

An equivalent way of representing this equation is to write

$$\forall l, m \in \Delta(X), V(l + \alpha(m-l)) - V(l) = \sum_{1}^{n} \alpha(m_i - l_i)\phi_i'(l_i) + o(m-l) \tag{1.4}$$

As the Gâteaux derivative is linear in $\alpha(m-l)$, V is also Frèchet differentiable and Machina theorems[23] apply. Consequently, under differentiability, an immediate consequence of Theorem 1 in Machina (1982) is the characterization of stochastic dominance :

Corollary 1.2. *$l \succeq m$ whenever l stochastically dominates m if and only if*

$$\forall i, j, \;\; j > i, \;\; \forall p, q, \;\; \phi_j'(p) > \phi_i'(q)$$

This means that the derivative of ϕ_i must be always inferior to the derivative of ϕ_j if $i < j$.

1.4 Applications

1.4.1 An application to variational preferences

Weak orthogonal independence axiom suggest the study of the following axiom :

Axiom A. 4 (Orthogonal Independence (OI))**.** For all $l, m, n \in \Delta(X)$ such that $n \perp \{l, m\}$ and for all $\alpha \in (0,1)$

$$l \succeq m \Leftrightarrow \alpha l + (1-\alpha)n \succeq \alpha m + (1-\alpha)n$$

[23] There is no difference here, even if the distributions are not cumulative distributions as in Machina (1982).

A first immediate observation is that this axiom is the independence axiom where the composition is restricted to lotteries with disjoint supports. This means that for lotteries with disjoint supports independence is assumed to hold. Of course, the classical independence axiom implies it and for X sufficiently rich[24], it is clear that (OI) implies (WOI). Thus, when an individual evaluates a lottery, he compares and cancels common components at first. Next, he reduces uniformly the remaining distribution. For $l, m \in \Delta(X)$ such that $l_i = m_i \neq 1$ and for all $j \neq i$, $l_j \neq m_j$:

$$\begin{aligned}
l \succeq m &\Leftrightarrow l_i\delta_{x_i} + \sum_{j\neq i} l_j\delta_{x_j} \succeq l_i\delta_{x_i} + \sum_{j\neq i} m_j\delta_{x_j} \\
&\Leftrightarrow l_i\delta_{x_i} + (1-l_i)\sum_{j\neq i} \frac{l_j}{1-l_i}\delta_{x_j} \succeq l_i\delta_{x_i} + (1-l_i)\sum_{j\neq i} \frac{m_j}{1-l_i}\delta_{x_j} \\
&\Leftrightarrow \sum_{j\neq i} \frac{l_j}{\sum_{k\neq i} l_k}\delta_{x_j} \succeq \sum_{j\neq i} \frac{m_j}{\sum_{k\neq i} m_k}\delta_{x_j}
\end{aligned}$$

Consequently, (OI) couples additivity and substitution by orthogonal lotteries. The previous equivalences remain valid if i is replaced by a non-empty proper subset I. The following theorem characterizes the orthgonal independence axiom when $n \geq 4$.

Theorem 1.3.

Let $X = \{x_1, \ldots, x_n\}$ with $n \geq 4$, $\succeq$ on $\Delta(X)$ satisfies (WO), (C) and (OI) if and only if there is a real-valued functions $U : X \to \mathbb{R}$ such that

$$V(l) = \sum_1^n \left(l_i u(x_i) - A l_i \ln l_i\right), \ \textit{if } c = 1 \tag{1.5}$$

or

$$V(l) = \sum_1^n \left(l_i^c u(x_i) - \frac{A}{c-1}(l_i^c - l_i)\right), \ \textit{if } c > 0 \textit{ and } c \neq 1 \tag{1.6}$$

[24] For $n \leq 3$, (WOI) is meaningless.

represents $\succeq$. Moreover, $(\tilde{u}, \tilde{A})$ is another representation of $\succeq$ in the above sense if and only if there exist $\alpha \in \mathbb{R}_+^*$ such that $\tilde{u} = \alpha u$ and $\tilde{A} = \alpha A$. In the case $c = 1$, $\tilde{u} = \alpha u + \beta$ for $\beta \in \mathbb{R}$ is allowed.

This functional is the sum of an expected utility or weighted expected utility and an entropic term (Shannon Entropy for the EU and Tsallis[25] entropy for the weighted EU). This functional appears first in Meginniss (1976) and shortly after in Aczél (1978), Aczél and Daróczy (1978), Aczél and Kannappan (1978) in the mixed theory of information. Both works use analytic assumptions to derive such results. In a series of papers, Luce, Ng, Marley et Aczèl (2008a, 2008b, 2008, 2009a, 2009b) derived the same representation from a theory of joint receipts and call this representation entropy-modified expected utility. Some properties have been studied by Yang and Qiu (2005) that propose an expected utility-entropy measure of risk. This functional is related to variational[26] preferences, that is, preferences represented by an utility function term and a cost function term. In the economic literature, the addition of the entropic term (Shannon entropy specifically) is justified by the cognitive cost of the decision process as in rational inattention theory[27] Sims (1998, 2003). Usually, Authors claims that this kind of functional reflects bounded rationality. In fact, the difference between these functionals and expected utility is quite thin in terms of axioms. Optimisation of equation (1.5) is the most simple entropic optimisation problem under expectation constraint and leads to the multinomial logit introduced[28] by Luce (1959). The multinomial logit model is extensively used in model of discrete choice and it has two canonical foundations[29] involving uncertainty. My derivation is different, if decision maker has preferences described by equation (1.5) and if we ask him to make a random choice between sure consequences then its choice probabilities are given by the multinomial logit

[25]See Tsallis (2009) for an extensive study of this generalization of the classical entropy.

[26]See Maccheroni et al. (2006) for a general result under uncertainty.

[27]This theory suggests that an agent must allocate his limited attention to the available or imperfect information in a choice situation. To implement this idea, it is necessary to quantify information flows. Classical entropy in information theory answers this question in the case of entropy. For a general framework, see de Oliveira et al. (2013).

[28]The logit model was first proposed for binary choices by Bradley and Terry (1952)

[29]According to Luce derivation, if decision maker choices are probabilistic then by independence of irrelevant alternatives axiom, ratios of choice probabilities are independent of the choice set and the multinomial logit model emerges. According to *random utility models* developed by McFadden (1974), decision maker preferences are represented by an utility function with an additively separable random noise. If the random noise is independently distributed by the extreme value distribution, then choice probabilities are given by the multinomial logit model.

model. Although it may seem incongruous as conscious randomization procedure is not a unanimous fact[30]. Nonetheless, my result is more significant in game theory. To the best of my knowledge Stahl (1990) was the first to propose a functional as equation (1.5) into game theory thus highlighting logit equilibrium. As in rational inattention theory, he introduces a entropic cost function. Following McFadden (1974), McKelvey and Palfrey (1995, 1998) uses random utility models to capture randomness in the responses of experimental subjects playing a game. Their work led to the concept of *quantal response equilibrium*[31] and a large literature. Consequently, my work provides a deterministic framework for logit equilibrium and more. In the full generality, orthogonally additive functional provides theoretical foundations to derive such equilibriums.

1.4.2 Data in the literature

Birnbaum and Chavez (1997) reports data on binary choices between gambles that appear not to be accounted for by any of the models satisfying branch-independence or distribution-independence. These properties are implied by weak orthogonal independence axiom. The authors test and reject models as expected utility and original prospect theory. I would be more convinced if the results were testing functionals with not separated utility and probability weighting functions. However, the result is not surprising as a simple weakening of the independence axiom is not sufficient to explain individuals behaviour. In contrast, Luce et al. (2008b) find opposite results for preferences represented by equation (1.5) and consequently that accommodates for weak orthogonal independence axiom.

1.5 Proofs

Proof of theorem 1.1. The necessary part of the theorem is transparent, I focus only on the sufficiency part. Let $X = \{x_1, \dots, x_n\}$ with $n \geq 4$ and $\Delta(X) = \{(l_1, \dots, l_n) \in [0,1]^n \mid \sum_i l_i = 1\}$ the set of lotteries over X with finite supports.

Firstly, for $n \geq 4$, $\succeq$ satisfies (WOI) then restricted solvability and standard sequence are not meaningless. In fact, following Theorem 14 in (Krantz et al., 1971,

[30]See for example Harsanyi (1973), Rubinstein (1991) and Machina (1985).
[31]See Haile et al. (2008) for a review.

Chapitre 6), it is true that restricted solvability and that Archimedean property of additive conjoint measurement is satisfied over supports, that is $(l_1, \dots, l_n) \equiv (l_I, l_{X \setminus I})$ for I non-empty proper subset of X. By lemma 14 in (Krantz et al., 1971, Chapitre 6), it follows that (C_3) is satisfied for $(l_I, l_{X\setminus I})$ with I non-empty proper subset of X.

$$(a_I, x_{X\setminus I}) \succeq (f_I, q_{X\setminus I}) \text{ and } (f_I, p_{X\setminus I}) \succeq (b_I, x_{X\setminus I}) \Rightarrow (a_I, p_{X\setminus I}) \succeq (b_I, q_{X\setminus I})$$

With respect to the measure ($\sum_J l_i = \sum_J m_i$, for all l, m, J).

Secondly, let m^0 be in the interior of $\Delta(X)$, consider the translation

$$\Delta_0 = \Delta(X) + m^0 = \left\{ l - m^0 \mid l \in \Delta(X) \right\}$$

Embed Δ_0 in the linear space $\mathbb{R}^X$ and denote by 0 the zero of Δ_0. For $\tilde{l}, \tilde{m} \in \Delta_0$, define $\perp$ by $\tilde{l} \perp \tilde{m}$ if and only if $\tilde{l}$ and $\tilde{m}$ have disjoint supports. With a slight abuse of notation, consider $\succeq$ over Δ_0 defined by

$$\tilde{l} \succeq \tilde{m} \Leftrightarrow \tilde{l} + m^0 \succeq \tilde{m} + m^0$$

Let $\Delta = \Delta_0 / \sim$ be the set of equivalence classes of Δ_0 under $\sim$ with typical element L. Let

$$\mathbb{B} = \left\{ (L, M) \mid \exists \quad \tilde{l}, \tilde{m} \text{ such that } \tilde{l} \in L, \tilde{m} \in M, \tilde{l} \perp \tilde{m} \right\}$$

Define $\circ$ on $\mathbb{B}$ by letting, for $(L, M) = (\left[\tilde{l}\right], [\tilde{m}])$, $L \circ M = \left[\tilde{l} + \tilde{m}\right]$ if $\tilde{l} \perp \tilde{m}$.

As m^0 is in the interior of $\Delta(X)$, there is m^1, l^0 and l^1 such that (l_j^0, l_j^1) is a permutation of (m_j^0, m_j^1) for every $j = 1, \dots, n$. Trivially, $(l^0 - m^0)$ and $(l^1 - m^0)$ have disjoint supports and consequently the graph of $\perp$ is non-empty. It is clear that $(m^1 - m^0) = (l^0 - m^0) + (l^1 - m^0)$ and in Δ_0, the previous equation can be stated as $\tilde{m}^1 = \tilde{l}^0 + \tilde{l}^1$ with $\tilde{l}^0 \perp \tilde{l}^1$.

Consequently, the binary function $\circ$ from $\mathbb{B}$ to Δ exists and is onto (surjective) as $\tilde{m}^1$ is on Δ_0 without restrictions. Furthermore, by (C_3), If $\tilde{l}, \tilde{l}' \in L$ and $\tilde{m}, \tilde{m}' \in M$

are such that $\tilde{l} \perp \tilde{m}$ and $\tilde{l}' \perp \tilde{m}'$ then $\tilde{l} + \tilde{m} \sim \tilde{l}' + \tilde{m}'$ and $\circ$ is well defined. An immediate observation is that if $(L, M) \in \mathbb{B}$ then $(M, L) \in \mathbb{B}$ and $\circ$ is commutative by definition as $\left[\tilde{l} + \tilde{m}\right] \sim \left[\tilde{m} + \tilde{l}\right]$. It is clear that for all $L \in \Delta$, $(L, [0]) \in \mathbb{B}$ and that $L \circ [0] = [0] \circ L = L$.

Let $\mathbb{B}(L) = \{M \in \Delta \mid (L, M) \in \mathbb{B}\}$ and define $\overline{L}$ such that for all L, $L \succ \overline{L} \succ [0]$ and $\mathbb{B}(L) = \{[0]\}$ and similarly $\underline{L}$ such that for all L, $L \prec \underline{L} \prec [0]$ and $\mathbb{B}(L) = \{[0]\}$. They are respectively the greatest element and the least element in Δ that admit orthogonal elements.

Define Δ^+ and Δ^- by $\{L \in \Delta \mid L \succeq [0]\}$ and $\{L \in \Delta \mid [0] \succeq L\}$ respectively, the positive cone and the negative cone of Δ.

Let $I^+_{\overline{L}}$, $I^-_{\underline{L}}$ be defined by $\left\{L \in \Delta \mid \overline{L} \succeq L \succeq [0]\right\}$, $\{L \in \Delta \mid \underline{L} \preceq L \preceq [0]\}$ respectively.

We shall now prove that $(I^+_{\overline{L}}, \succeq, \mathbb{B}_{|I^+_{\overline{L}}}, \circ)$ where

$$\mathbb{B}_{|I^+_{\overline{L}}} = \left\{(L, M) \mid \exists \quad \tilde{l}, \tilde{m} \text{ such that } \tilde{l} \in L, \tilde{m} \in M, \tilde{l} \perp \tilde{m} \text{ and } L + M \in I^+_{\overline{L}}\right\}$$

is an extensive structure with no essential maximum ((Krantz et al., 1971, Definition 3, Chapter 3)). The six axioms are established in corresponding numbered paragraphs.

1. $(I^+_{\overline{L}}, \succeq)$ is a weak order by axiom (W0); in fact, it is a simple order.
5. In $I^+_{\overline{L}}$, $L, M \succeq [0]$, by weak orthogonal independence $L \circ M \succeq L, M$. Thereby positivity holds.
2. Associativity. Disjoint sum is associative then by construction it is possible to have *naturally* (M, N) and $(L, M \circ N) \in \mathbb{B}_{|I^+_{\overline{L}}}$. If not, suppose that $(L, M) \in \mathbb{B}_{|I^+_{\overline{L}}}$ and $(L \circ M, N) \in \mathbb{B}_{|I^+_{\overline{L}}}$. Let $\tilde{o} \in L \circ M$ and $\tilde{n} \in N$ such that $\tilde{o} \perp \tilde{n}$. By positivity $[\tilde{o}] \succeq L, M$ then by continuity (restricted solvability) there is $\tilde{m} \in M$ such that $\tilde{m}$ belongs to the interval $[[0], L \circ M]$ and consequently $\tilde{m} \perp \tilde{n}$ and then $(M, N) \in \mathbb{B}_{|I^+_{\overline{L}}}$. An identical argument gives $(L, N) \in \mathbb{B}_{|I^+_{\overline{L}}}$. Let $\tilde{m}' \in M$ and $\tilde{l} \in L$ such that $\tilde{m}' \perp \tilde{l}$, then $\tilde{l}$ can be chosen collinear or orthogonal to elements in M, as $M \circ N \preceq \overline{L}$ then $(L, M \circ N) \in \mathbb{B}_{|I^+_{\overline{L}}}$.

3. If $(L, N) \in \mathbb{B}_{|I_L^+}$ and $L \succeq M$ then by continuity there is $\tilde{m} \in M$ such that $\tilde{m}$ belongs to the interval $[[0], L]$ and consequently $(M, N) \in \mathbb{B}_{|I_L^+}$. By weak orthogonal independence $L \circ N \succeq M \circ N$

4. Solvability. If $L \succ M$ as $M \prec \overline{L}$ then $\{N \mid N \perp M\}$ is non-empty and by continuity there exists N such that $L \succeq M + N$.

6. By continuity, $\succeq$ is Archimedean.

By Theorem 3 in (Krantz et al., 1971, Chapter 3) there is a function Φ from I_L^+ to $\mathbb{R}^+$ such that

$$\begin{aligned}
&\text{(i) } L \succeq M \Leftrightarrow \Phi(L) \geq \Phi(M)\\
&\textit{and}\\
&\text{(ii) } (L, M) \in \mathbb{B}_{|I_L^+} \Rightarrow \Phi(L \circ M) = \Phi(L) + \Phi(M)
\end{aligned}$$

and if another function Φ' satisfies (i) and (ii), then there exists $\alpha > 0$ such that, $\Phi'{=}\alpha\Phi$. It remains to extend this representation on Δ.

For all $L \in \Delta^+$ as $\circ$ is onto there is M, N such that $(M, N) \in \mathbb{B}$ and $L = M \circ N$. But as $(M, N) \in \mathbb{B}$ then $M, N \in I_L^+$. It suffices to set $\Phi(L) = \Phi(M) + \Phi(N)$. The prolongation is pasted by continuity. To show that additivity holds throughout Δ^+, it suffices to consider $L \circ M, N \circ O$ whenever $(L, M) \in \mathbb{B}$ and $(N, O) \in \mathbb{B}$. Suppose $L \circ M \sim N \circ O$, by continuity there is R, Q such that $(N, R) \in \mathbb{B}_{|I_L^+}$, $(Q, M) \in \mathbb{B}_{|I_L^+}$, $N \circ R \sim Q \circ M$. So we have already prove associativity : $(L, R) \in \mathbb{B}_{|I_L^+}$, $(Q, O) \in \mathbb{B}_{|I_L^+}$ and by (C_3) $L \circ R \sim Q \circ O$. Consequently, by cancellation : $\Phi(L \circ M) = \Phi(L) + \Phi(M) = \Phi(N \circ O) = \Phi(N) + \Phi(O)$. Similar argument gives an identical result if we suppose $L \circ M \succ N \circ O$.

The same reasoning leads to an orthogonality additive representation over Δ^-.

As a neutral element exist on Δ, we consider whenever it is defined $-L$ given for $L \succ [e]$ by the solvable equation $L \circ -L$ if $(L, -L) \in \mathbb{B}$. By continuity a representation exists over Δ, that is, there is a function Φ from Δ to $\mathbb{R}$ such that

$$\text{(i) } L \succeq M \Leftrightarrow \Phi(L) \geq \Phi(M)$$

and

$$\text{(ii) } (L, M) \in \mathbb{B} \Rightarrow \Phi(L \circ M) = \Phi(L) + \Phi(M)$$

and if another function Φ' satisfies (i) and (ii), then there exists $\alpha > 0$ such that, $\Phi'{=}\alpha\Phi$.

Finally, there is Φ from Δ_0 to $\mathbb{R}$ that represents $\succeq$ and such that $\Phi(\tilde{l}+\tilde{m}) = \Phi(\tilde{l})+\Phi(\tilde{m})$ whenever $\tilde{l}$ and $\tilde{m}$ have disjoint supports therefore orthogonally additive and by translation over $\Delta(X)$: there is a function V from $\Delta(X)$ to $\mathbb{R}$ such that

$$\text{(i) } l \succeq m \Leftrightarrow V(l) \geq V(M)$$

and

$$\text{(ii) } V(\alpha l + (1-\alpha)m) = V(\alpha l) + V((1-\alpha)m)$$

and if another function $\tilde{V}$ satisfies (i) and (ii), then there exists $\alpha > 0$ such that, $\tilde{V}{=}\alpha V$. Adding a constant non equal to zero would eliminate orthogonal additivity. The additively separable representation is given by induction and by letting $\phi_i(l_i) = V(l_i\delta_{s_i})$ for all i :

$$l \succeq m \Leftrightarrow \sum_1^n \phi_i(l_i) \geq \sum_1^n \phi_i(m_i)$$

Obviously, ϕ_i, $i = 1, \ldots, n$, are unique up to similar positive linear transformations.

The above proof means that for α fixed and for $m, n \in \Delta(X)$ with disjoint supports then there is an injective transformation such that orthogonal additivity stands for all l, o such that $l \perp \{n, o\}$ and $o \perp \{l, m\}$. From Eilenberg (1941), axioms (WO) and (C) imply the existence of continuous function V which represents $\succeq$ over $\Delta(X)$ and leaves (WOI) undisturbed. By (WOI), for all l, m, no such that $\{l, m\} \perp \{n, o\}$

$$V(\alpha l + (1-\alpha)n = V(\alpha m + (1-\alpha)n \Leftrightarrow V(\alpha l + (1-\alpha)o = V(\alpha l + (1-\alpha)o$$

Consequently, by the above reasoning and continuity there is $h : \Delta(X) \to \mathbb{R}$ a continuous orthogonally additive map, and $U : \mathbb{R} \to \mathbb{R}$ continuous and injective.

The above proof highlight the necessity of condition (C_2) and the fact that ordering on each component in additive conjoint measurement is not an important feature.

■

Proof of corollary 1.2. Under differentiability,

$$\forall l, m \in \Delta(X), V(l + \alpha(m-l)) - V(l) = \sum_1^n \alpha(m_i - l_i)\phi_i'(l_i) + o(m-l)$$

By Theorem 1 in Machina (1982), $\phi_i'(l_i)$ is non-decreasing in i, for all l_i. the result follow. ■

Proof of theorem 1.3. By Theorem 1.1, a continuous orthogonally additive functional represents $\succeq$. Furthermore, $\succeq$ satisfies (OI) if for all $l, m, n \in \Delta(X)$ such that $n \perp \{l, m\}$ and for all $\alpha \in (0,1)$:

$$l \succeq m \Leftrightarrow \alpha l + (1-\alpha)n \succeq \alpha m + (1-\alpha)n$$

It follows, taking the indifference part, that for all $l, m \in \Delta(X)$ such that $supp(l), supp(m) \subsetneq X$, it is true that $V(l) = V(m)$ if and only if for all $\alpha \in (0,1)$, $V(\alpha l) = V(\alpha m)$ or

$$\sum_{supp(l)} \phi_i(l_i) = \sum_{supp(m)} \phi_i(m_i) \Leftrightarrow \sum_{supp(l)} \phi_i(\alpha l_i) = \sum_{supp(m)} \phi_i(\alpha m_i)$$

Therefore, as the previous equivalence is true for all proper subset of X, V satisfies at least generalized homogeneity, that is, for all $l \in \Delta(X)$

$$\sum_i \phi_i(\alpha l_i) = \sum_{supp(l)} f(\alpha)\phi_i(l_i)$$

with the ϕ_i's and f are the unknows function. It is clear that f, as the ϕ_i's, are continuous over $[0,1]$. By Theorem 1 in (Aczél and Dhombres, 1989, Chapter 20), $f(\alpha) = \alpha^c$ for an arbitrary $c \in \mathbb{R}$ and there exists h_i from $[0,1]^{n-1}$ such that

$$\sum_k \phi_k(l_k) = l_i^c h_i(\frac{l_1}{l_i}, \ldots, \frac{l_{i-1}}{l_i}, \frac{l_{i+1}}{l_i}, \ldots, \frac{l_n}{l_i})$$

Firstly, by continuity, c must be greater than 0. Secondly, the additive structure of V implies that there exist $v_i \in \mathbb{R}$ for $i = 1, \ldots, n$ such that

$$V(l) = \sum_i l_i^c v_i$$

Where for all i, $v_i = h_i(0, \ldots, 0) = \sum_{k \neq i} \phi_k(0) + \phi_i(1)$.

For $c \neq 1$ and an arbitrary constant $A \in \mathbb{R}$, let u_i defined by $u_i = v_i + \frac{A}{c-1}$. It follows that

$$V(l) = \sum_i l_i^c u_i - \frac{A}{c-1} \sum_i l_i^c$$

Adding a constant not modify the preferences then

$$\begin{aligned} \tilde{V}(l) &= \sum_i l_i^c u_i - \frac{A}{c-1} \sum_i l_i^c + \frac{A}{c-1} \\ &= \sum_i l_i^c u_i - \frac{A}{c-1} \sum_i (l_i^c - l_i) \end{aligned}$$

is an representation of $\succeq$. When $c \to 1$, $\tilde{V}$ tends to $\sum_i l_i u_i - A \sum_i l_i \ln(l_i)$ which is also a possible representation of $\succeq$. The uniqueness of these representations and the necessary part are transparent. ∎

1.6 Conclusion

The purpose of this chapter was to find some concept for applied additive representation of preferences on simplices as subsets of Cartesian products. It is more complicated than the usual case but very similar to linear utility theory. In this direction, a work must be undertaken. Limited experimental data and their conflicting aspects does not allow us to make a judgement about the credibility of such functionals even if, in full generality, it is low given typical failures of individuals behaviour as framing effects or intransitive preferences. However, These representations seem to have an analytical substantial advantage over most of the earlier proposals. The most straightforward formal application can take place into game theory as optimization problem with additively separable functional is simple. The obtained result make possible the derivation of further results for entropy-modified expected utility extensively used in economic studies. In this perspective a great care must be imposed to studies that suppose an entropic term. Entropy-modified expected utility reports a very limited bounded rationality as the cognitive impairment is thin.

Chapter 2

An Axiomatic Foundation for "Discrete" Choice with Social Interactions

2.1 Introduction

2.1.1 Motivations

Recent research in behavioural economics[1] has shown how decision makers often fail to maximize their narrow self-interest. For example, peoples do not play the selfish, sub-game perfect equilibrium and often make positive gifts in dictator games (Robert et al. (1994)), or refuse unfair allocations in ultimatum games (Guth et al. (1982)). Thus, a large literature[2] suggests that decision makers have other-regarding preferences[3], that is, preferences that depend on more than their own outcome and that are influenced by her own outcomes and those of others in a absolute or relative way. This is phenomenon whose theoretical properties merits thorough study. In economics, other-regarding preferences refer to particular forms of interactions, in which behaviours (or beliefs about behaviours) of a

[1]See Cooper and Kagel (2009a) for a recent survey of the experimental evidence.

[2]See Fehr and Gächter (2000) or Sobel (2005) for a recent survey.

[3]The idea that individual's welfares depend on *relative* and *absolute* income was first introduced by Veblen (1899). It is not a new subject, see Duesenberry (1949), Leibenstein (1950), Pollak (1976).

neighbourhood (the reference group) affect decision maker's behaviours. The reference group depends on the social context, which is typically family, neighbours, friends or peers.

In game theory, there is a field of research that confirms that people care for particular standards of fairness. Economists have found that anonymously interacting agents frequently agree on rather egalitarian outcomes in bilateral bargaining situations (Bolton and Ockenfels (2000), Camerer and Thaler (1995), Fehr and Schmidt (1999), Falk et al. (2003). An important observation is that reciprocity is also relevant (Fehr and Schmidt (1999)). If individuals signal bad intention, they will receive a lower share. People punish unfair behaviour in fact. Some relevant theoretical works study market equilibrium with generic preferences, Dufwenberg et al. (2011) consider a general-equilibrium model in which agents have separable non relative other-regarding preferences; or game equilibrium, Segal and Sobel (2007) consider a model where players in a strategic environment have preferences over strategies, which can be represented by a weighted average of the utility from outcomes of the individual and his opponents. The weight one player places on an opponent utility depends on the players joint behaviour, thus depicting reciprocity in standard game theoretical way.

Less theoretical but also related are the socio-economic studies. Other-regarding preferences has led to a rich theoretical literature in economics along with interest in social determinants of individual behaviour. Many efforts[4] to measure these influences have been made. One of them is the model developed by Brock and Durlauf (2001b). This model is of great interest. Firstly, because he solved two problems raised by Manski (1993) in the field of social econometrics. The authors give theoretical foundations for identification of exogenous and endogenous peer effects in a context of binary choice and, in the same time, they determine the Nash equilibria of their theoretical model under a suitable random utility assumption. This assumption comes from the individual discrete choice theory proposed by McFadden (1974). It is well known that when the random utility term is independent and identically distributed according to generalized extreme value distributions then the resulting choice is stochastic and follows a Boltzmann-Gibbs distribution[5]. Secondly, because this last distribution is also well known in

[4]See for extensive treatment, Manski (1993, 2000) and Brock and Durlauf (2007) for open problem in econometrics methods and Cox (2004) for identifications of social interactions in game theory.

[5]Logit.

statistical physics and comes from an entropy maximization problem under expectation constraints. So, along with the model of Blume and Durlauf (2001), the model of Brock and Durlauf (2001b) (now denoted BBD for Blume, Brock and Durlauf) is a basis for a part of the econophysics field, for interaction-based model and for economics under social interactions. In this chapter, my motivation is to propose an axiomatic foundation for other-regarding preferences under risk in the framework of BBD.

BBD model is a standard economic model of individual discrete choices with a social influence term. I will derive first an "expected utility" representation with other-regarding preferences but without stochastic term. Work under risk allows to apply the model in game theory. Moreover, this framework enables sources of risk to come from the interaction among lotteries chosen by the agent, lotteries chosen by his peers and beliefs over his peers choices. I discuss the case of exogenous reference groups and also endogenous reference groups. In this two case, I give axioms for separate the social influence term between individuals of the reference group. In this way, social influence between peers can be compared. A feature of my work is to allow agents to interact in a non-anonymous fashion. The anonymity hypothesis is suitable for large reference groups but not for small groups. However, is not a loss of generality.

Formally, I consider preferences of an agent i. Let (l_i, l_J) represents the situation in which agent i evaluates lottery l_i, while l_J is the joint lottery of lotteries l_j, $j \in J$. l_j can be the lottery chosen by j or beliefs about the choice of j. Agent i evaluates this situation according to :

$$V(l_i, l_J) = \int_{X_i} u(x_i) dl_i(x_i) + \sum_{j \in J} \left(\int_{X_i \times X_j} S_j(x_i, x_j) dl_i(x_i) dl_j(x_j) \right) \tag{2.1}$$

The first term of this representation represents the expected utility of the decision maker over the continuous set of outcomes X_i. The effect on i's welfare of the outcome of the other individuals is reported in the second term. S_j represents the social index between i and j. The individual i forms the expected value of S_j over the Cartesian product $X_i \times X_j$ where X_j is the continuous set of outcomes of individual j. Finally, i sums the other-regarding term over individuals. Note that preferences are given over a product set to allow a clean separation between the private and the social utility.

Secondly, to achieve similar derivation of Nash equilibria, I discuss for the case of a simple exogenous reference group the addition of an entropic term as in chapter 1. Adding an entropic term provides a way, into a game theoretic settings, to obtain logit equilibria. As the econometric framework must add a random utility term, this *last* approach is not suitable for empirical investigations.

2.1.2 Related results

Despite the relevance of these contributions, there is little theoretical work on other-regarding preferences, especially under risk and uncertainty. Approaches incorporating other-regarding preferences into a decision theoretical framework include Borah (2009), Gilboa and Schmeidler (2001), Karni and Safra (2002), Maccheroni et al. (2012), Neilson and Stowe (2004), Ok and Kockesen (2000), Saito (2008).

Borah (2009) introduces a concern for possibilities in the decision maker's evaluation of others' outcomes. That is, the decision maker anticipates the welfare of this peers. Gilboa and Schmeidler (2001) develop a model of individual welfare that takes into account cognitive factors. They postulate that individuals compare incomes with aspiration levels determined by past experiences, interpersonal comparison, and reasoning. They give an axiomatic foundation of a measure of welfare given by a linear combination of differences between incomes and aspiration levels, where the aspiration level at each instance is a linear function of past incomes. However, the linearity of their model is not compatible with further extensions under risky situations. Karni and Safra (2002) present an axiomatic model of choice behaviour for a "self-interest seeking moral individual" over random allocation. Individual preferences are decomposed into a classical self-interest component and a "Social" component which represents individual's moral judgement. These authors depict behaviour in terms of fairness in a way that does not allow for distinctions with other motives such as conformism. Maccheroni et al. (2012) generalizes the classic subjective expected utility model by allowing decision maker's preferences to depend on the outcomes of his peers. The choice criterion combines a classical expected utility with expected social externalities anticipated by decision makers. Again, the reference group is not clearly identifiable because these authors suppose anonymous peers outcomes. Nevertheless this article is the closest to the present contribution. In particular, I use their conformistic reference axiom to separate the private and the social utility. Neilson and Stowe (2004) consider preferences over a vector of probability distributions which can be represented by a convex sum of the expected utility of the decision maker and expected utility of his opponents. The resulting specification allows the weights placed on the opponents outcomes to be player-dependent. But, under optimization, their way of formalising player-dependency lead to the vanishing of considerations for others. That is, other-regarding preferences do not take into account relative welfare. Ok and Kockesen (2000) incorporate the widely acknowledged phenomenon of keeping up with the Joneses, i.e., preferences are dependent on both "relative standing" in society and material consumption. The principal ingredient of their analysis is the assumption that individuals desire to occupy a subjectively better position than their peers. They consider negative interdependent preferences over income distributions and provide an axiomatization of the relative income criterion under certainty where they emphasize the distinction between relative and individual income effects. However their starting point - what is actually regarded as "relative standing" in society - is not satisfactory. The main reason is that the reference group is not clearly identifiable. Saito (2008) introduces a model of inequality aversion under risk that extends the model of Fehr and Schmidt (1999).

This chapter is organized as follows. In section 2, I introduce the key concept of my work. Section 2.1 describes orthogonally additive functional. Section 2.2 points the differences with other approaches in the literature. Section 3 gives the main results of this chapter. Section 4 applies the main results to variational preferences. Proofs are given in section 5. Section 6 concludes.

The chapter is organized as follows. Section 2 provides preliminary notions and axiomatizes preferences in the case where the reference group is exogenously given, while Section 3 treats the case where the reference group is endogenously given. Section 4 discusses the addition of an entropic term. Proofs are given in section 5. Section 6 concludes.

2.2 Exogenous Reference Group

2.2.1 Non separable preferences across individuals

Let $N = \{1, \dots, n\}$ be the non-empty finite set of all individuals agents, with agent $i \in N$ making the decisions. Let $X \subseteq \mathbb{R}$ be the space of outcomes with typical element x. I denote by X_j the copy of X which is the space of outcomes to individual j with typical element x_j. Let $\Delta(X_i)$ be the space of lotteries over X_i with typical element l_i, and $\Delta(\prod_{j \neq i} X_j)$ be the space of lotteries over $\prod_{j \neq i} X_j$ with typical element l_{-i} which represents the joint lottery of $n-1$ individuals other than i. Both are endowed with the weak topology. To every l_{-i} there corresponds elements $l_k \in \Delta(X_k)$, for $k \neq i$, which are the marginal lotteries of l_{-i} on the subset X_k. Concretely, l_{-i} is a multivariate probability distribution over all the individuals different from i. l_k represents the lottery chosen by k or beliefs about the choice of k, l_{-i} is the joint lottery, that is, the general lottery for the society.

I denote by $\mathcal{L}_N$ the product set $\Delta(X_i) \times \Delta(\prod_{j \neq i} X_j)$ endowed with the product topology. Following Fishburn (1976), I introduce a binary relation $\mathcal{R}$ on $\mathcal{L}_N$ defined by

$$(l_i, l_{-i})\mathcal{R}(m_i, m_{-i}) \Leftrightarrow l_k \neq m_k \text{ for at most one } k \in \{i, -i\}$$

When individual i is isolated from other individuals, I assume that he has a self-interested preference relation $\succeq^{\emptyset}$ over $\Delta(X_i)$. Since $\Delta(X_i)$ is a mixture space, it lends itself to the expected utility setup. In contrast, when individual i is not isolated from $N \setminus \{i\}$, I assume that he has a preference relation $\succeq$ over $\mathcal{L}_N$ with $(l_i, l_{-i}) \succeq (m_i, m_{-i})$ read as “the individual prefers the 2-tuple (l_i, l_{-i}) to the 2-tuple (m_i, m_{-i})”. I denote by x_N an element $l = (l_i, l_{-i}) \in \mathcal{L}_N$ such that $l_k = x \in X$ for all $k \in N$. I assume that the preference relation $\succeq$ satisfies the following axioms.

Axiom B. 1 (Weak Order)**.** $\succeq$ is complete and transitive.

Axiom B. 2 (Continuity)**.** For every $l \in \mathcal{L}_N$, $\{m \mid m \succeq l\}$ and $\{m \mid l \succeq m\}$ are closed in the product topology.

Axiom B. 3 (Independence)**.** For every $l, m, n, o \in \mathcal{L}_N$ and $\alpha \in (0, 1)$, if $l\mathcal{R}n$, $m\mathcal{R}o$, $n \sim o$

$$l \succeq m \Rightarrow \alpha l + (1 - \alpha)n \succeq \alpha m + (1 - \alpha)o$$

.

These first two axioms are standard. The Continuity axiom states that $\succeq$ is continuous on $\mathcal{L}_N$ with respect to its topology, that is, the product topology. Note that the comparison is between allocation lotteries. The decision maker compares lottery profiles, not just his own lotteries but the group lotteries also. The Independence axiom given here uses convex combinations of elements in $\mathcal{L}_N$ only when the elements being combined differ in at most one coordinate. That is, elements of the form (l_i, l_{-i}) and (l_i, m_{-i}), or (l_i, l_{-i}) and (m_i, l_{-i}). Standard interpretations and critiques of this axiom apply equally well to the classical independence axiom. This axiom allows the application of convex combinations sum to different coordinates which is essential to obtain interconnections between coordinates. For example, this axiom says that if $(l_i, l_{-i}) \succeq (m_i, m_{-i})$, $(n_i, l_{-i}) \sim (m_i, o_{-i})$ and $\alpha \in (0, 1)$ then

$$(\alpha l_i + (1 - \alpha)n_i, l_{-i}) \succeq (m_i, \alpha m_{-i} + (1 - \alpha)o_{-i})$$

.

Remember that when i is isolated from other individuals, I assume that he has a self-interested preference relation $\succeq^{\emptyset}$ over $\Delta(X_i)$ and that I denote by x_N an element $l = (l_i, l_{-i}) \in \mathcal{L}_N$ such that $l_k = x \in X$ for all $k \in N$.

Axiom B. 4 (Conformistic Reference)**.** For every $x, y \in X$, $x_N \succeq y_N$ if, and only if, $x \succeq^{\emptyset} y$

This axiom is slightly identical to the axiom stated by Maccheroni et al. (2012), the decision maker reduces an egalitarian situation to the "self-interested preference" situation. Simply, in a group where everybody has the same certain outcome, the well-being of i in this situation is the same as if he were isolated. However it is possible to choose an other reference. Maccheroni et al. (2012) introduce directly this axiom for varying reference group. However, for now, the reference group is exogenous, it suffices therefore to choose a trade-off in the reference group and not among the reference groups.

The basic axioms B.1-B.4 lead to this multi-affine[6] basic representation.

Theorem 2.1. *A binary relation $\succeq$ on $\mathcal{L}_N$ satisfies Axioms B.1-B.4 if and only if there exist a continuous function $u : X_i \to \mathbb{R}$, and a continuous function $S : X_i \times \prod_{j\neq i} X_j \to \mathbb{R}$, with $S(x_N) = 0$ for all $x \in X$, such that*

$$V(l_i, l_{-i}) = \int_{X_i} u(x_i) dl_i(x_i) + \int_{X_i \times \prod_{j\neq i} X_j} S(x_i, x_{-i}) dl_i(x_i) dl_{-i}(x_{-i}) \qquad (2.2)$$

represents $\succeq$. Moreover, $(\tilde{u}, \tilde{S})$ is another representation of $\succeq$ in the above sense if and only if there exist $(\alpha, \beta) \in \mathbb{R}^*_+ \times \mathbb{R}$ such that $\tilde{u} = \alpha u + \beta$ and $\tilde{S} = \alpha S$.

Theorem 2.1 provides a first representation result for exogenous reference group, the function S captures the comparative outcome concerns of i. As V is multi-affine, S is also, and the expectation of S represents the expected social utility given the risk that the agents face or represents the given expected social utility given the i agent's belief on the choices of the rest of the agents in $N \setminus \{i\}$.

2.2.2 Separable preferences across individuals

The following axiom allows additive representation over $N \setminus \{i\}$. The agent i evaluates the social utility with respect to all individuals separately. Suppose that (l_i, m_{-i}) and (l_i, n_{-i}) are such that $m_k = n_k$ for all $k \neq i$. That is, all marginal lotteries are equal, then the decision maker is indifferent between (l_i, m_{-i}) and (l_i, n_{-i}). It is a usual assumption[7] for separate multivariate expected utility

[6] Affine in each variable

[7] See (Fishburn, 1970, Chapter 11).

Axiom B. 5 (Individual Comparative Preference)**.** For every $l_i \in \Delta(X_i)$,$m_{-i}, n_{-i} \in \Delta(\prod_{j\neq i} X_j)$, if $m_k, n_k \in \Delta(X_k)$ are the $k-th$ marginals respectively of m_{-i}, n_{-i} and if $m_k = n_k$ for $k \in N\backslash\{i\}$, then $(l_i, m_{-i}) \sim (l_i, n_{-i})$

Theorem 2.2. *A binary relation $\succeq$ on $\mathcal{L}_N$ satisfies Axioms B.1-B.5 if and only if there exists a continuous function $u : X_i \to \mathbb{R}$, and continuous functions $S_j : X_i \times X_j \to \mathbb{R}$ for $j \in N \setminus \{i\}$, with $\sum_{j\neq i} S_j(x,x) = 0$ for all $x \in X$, such that*

$$V(l_i, l_{-i}) = \int_{X_i} u(x_i)dl_i(x_i) + \sum_{j\neq i} \left(\int_{X_i\times X_j} S_j(x_i,x_j)dl_i(x_i)dl_j(x_j) \right) \tag{2.3}$$

represents $\succeq$. Moreover, $(\tilde{u}, (\tilde{S}_j)_{j\neq i})$ is another representation of $\succeq$ in the above sense if and only if there exist $(\alpha, \beta, (\beta_j)_{j\neq i}) \in \mathbb{R}^*_+ \times \mathbb{R}^n$ such that $\tilde{u} = \alpha u + \beta$, $\tilde{S}_j = \alpha S_j + \beta_j$ for all $j \in N \setminus \{i\}$ and $\sum_{j\neq i} \beta_j = 0$.

This last representation allows to separate among individuals the social component of the representation. It is very useful to compare effect of each peer over behaviour of the decision maker.

2.3 Endogenous Reference Group

Here, other-regarding preferences are unconstrained because a "selfish" alternative can be preferred to an alternative that takes into account the social aspect and vice versa. I Focus directly to separable preferences among individuals, the non separable case is trivial.

Let $N = \{1, \ldots, n\}$ be the non-empty, finite, set of all individuals, with individual $i \in N$ making the decisions. I denote by $\mathcal{P}(N \setminus \{i\})$ the set of all finite subsets of N with typical element J. Then J is by definition a group of individuals included in $N \setminus \{i\}$; notice that $\emptyset \in \mathcal{P}(N \setminus \{i\})$. That is, i can be isolated.

For all J, I denote by J_i the set $J \cup \{i\}$; similarly, if k does not belong to J, I denote by J_k the set $J \cup \{k\}$. For all J, let $\mathcal{L}_J$ the product set $\Delta(X_i) \times \Delta(\prod_{j \in J} X_j)$ endowed with the product topology.

Following Candeal et al. (2004), let $\mathcal{L} = \bigsqcup_{J \in \mathcal{P}(N \setminus \{i\})} \mathcal{L}_J$ endowed with the disjoint union topology[8]. This assumption is very technical and impose continuity for the preferences among the reference groups.

I assume that individual i has a preference relation $\succeq$ over $\mathcal{L}$ with $(l_i, (l_j)_{j \in J}) \succeq (m_i, (m_k)_{k \in K})$ read as "the individual prefers the 2-tuple $(l_i, (l_j)_{j \in J})$ to the 2-tuple $(m_i, (m_k)_{k \in K})$". I denote by x_{J_i} an element $l = (l_i, (l_j)_{j \in J}) \in \mathcal{L}_J$ such that $l_p = x \in X$ for all $p \in J_i$. I assume that the preference relation $\succeq$ satisfies the following axioms.

Axiom C. 1 (Weak Order)**.** $\succeq$ is complete and transitive.

Axiom C. 2 (Continuity)**.** For every $l \in \mathcal{L}$, $\{m \mid m \succeq l\}$ and $\{m \mid l \succeq m\}$ are closed in the disjoint union topology.

These first two axioms are again standard but note that $\succeq$ is continuous on $\mathcal{L}$ with respect to the disjoint union topology. That is, if X_{α_1} and X_{α_2} are disjoint topological spaces and $X = X_{\alpha_1} \sqcup X_{\alpha_2}$ then X inherits a natural topology called the disjoint union topology. The idea of this topology is that if X_{α_1} and X_{α_2} do not interact in any way, the topology have some basic properties. No sequence in X_{α_1} or subset of X_{α_1} has a limit point in X_{α_2}, and vice-versa. If $S \subset X_{\alpha_1}$ then the closure of S is also a subset of X_{α_1}, the same holds for X_{α_2}.

Here, the decision maker compares lottery profiles, not just his own lotteries but the group lotteries also for all possible groups.

Axiom C. 3 (Independence)**.** For every $J, K \in \mathcal{P}(N \setminus \{i\})$, $l, n \in \mathcal{L}_J$, $m, o \in \mathcal{L}_K$ and $\alpha \in (0, 1)$, if $l \mathcal{R} n$, $m \mathcal{R} o$, $n \sim o$

[8]Given a family X_α of topological spaces, $\alpha \in A$, the topological space $\bigsqcup_\alpha X_\alpha$ is the disjoint union of the spaces X_α endowed with the topology in which U is open if and only if $U \cap X_\alpha$ is open for all α.

$$l \succeq m \Rightarrow \alpha l + (1-\alpha)n \succeq \alpha m + (1-\alpha)o$$

.

This axiom is a natural generalization of the independence axiom to disjoint union of spaces.

Axiom C. 4 (Conformistic Reference)**.** For every $J \in \mathcal{P}(N \setminus \{i\})$, $x \in X$, and $k \notin J$ $x_{J_i} \sim x_{J_i \cup \{k\}}$

All egalitarian situations are equivalent to the "self-interested preference" situation. but in comparison with Axiom B.4, for agent i it does not matter if a group where everybody has the same outcome, is added an another individual, again, with the same outcome. Maccheroni et al. (2012) describe as follows an another reference :" In the representation this axiom translates into the condition that the externality function is zero when all members of the group have the same outcome. Different trade-offs have a similar axiomatization. For example, if individual prefers, for the same outcome c, a smaller society, then a similar axiom would require that, for some improvement over c, he would feel indifferent between the smaller society with a less preferred outcome and a larger one with better common outcome "

Axiom C. 5 (Expansion by individual comparison)**.** For every $J \in \mathcal{P}(N \setminus \{i\})$, $k \notin J$ and $l \in \mathcal{L}_{J \cup \{k\}}$, if l_J and l_k are respectively the J-marginal and the $\{k\}$-marginal of $l_{J \cup \{k\}}$,

$$(l_i, l_{J \cup \{k\}}) \succeq (l_i, l_J) \Rightarrow (l_i, l_{\{k\}}) \succeq (l_i)$$

.

In the representation this axiom translates into the condition that the social utility function depends exclusively of the individual k considered.

Theorem 2.3. *A binary relation $\succeq$ on $\mathcal{L}$ satisfies Axioms C.1-C.5 if and only if there exist a continuous function $u : X_i \to \mathbb{R}$, and continuous functions $S_j : X_i \times X_j \to \mathbb{R}$ for $j \in N \setminus \{i\}$, with $S_j(x,x) = 0$ for all $j \neq i$ and $x \in X$, such that*

$$V(l_i, l_J) = \int_{X_i} u(x_i) dl_i(x_i) + \sum_{j \in J} \left(\int_{X_i \times X_j} S_j(x_i, x_j) dl_i(x_i) dl_j(x_j) \right) \tag{2.4}$$

represents $\succeq$. Moreover, $(\tilde{u}, (\tilde{S}_j)_{j \neq i})$ is another representation of $\succeq$ in the above sense if and only if there exist $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}$ such that $\tilde{u} = \alpha u + \beta$, $\tilde{S}_j = \alpha S_j$ for all $j \in N \setminus \{i\}$.

This theorem allows to assert that the decision maker may have other-regarding preferences but it is not automatic as in Lemma 1.

$$\forall J, K \in \mathcal{P}(N \setminus \{i\}), \forall l, m \in \mathcal{L} \quad (l_i, l_J) \succ (m_i, m_K) \Leftrightarrow V(l_i, l_J) > V(m_i, m_K) \tag{2.5}$$

In this case the reference group is not exogenously given, and memberships are expensive. There is a second constraint during the maximisation process. the decision maker evaluates first, as usually but for each reference groups and then maximises among reference groups. This kind of assumption is maybe acceptable for very small reference groups or in particular situation. However, facts about limited rationality reduce the scope of this type of result.

2.4 Adding an entropic term

To avoid some difficulties, I suppose in this section that X is a finite set with at least 4 elements as in chapter 1. Let $\Delta(X_i)$ be the space of lotteries over X_i with typical element l_i, and $\Delta(\prod_{j \neq i} X_j)$ be the space of lotteries over $\prod_{j \neq i} X_j$ with typical element l_{-i} which represents the joint lottery of $n-1$ individuals other than i. Both are now endowed with the relative topology induced by the product topology of $[0,1]^X$. $\perp$ stands for disjoint supports.

Let $\mathcal{R}_\perp$ be a binary relation on $\mathcal{L}_N$ defined by

$$(l_i, l_{-i})\mathcal{R}_\perp(m_i, m_{-i}) \Leftrightarrow l_k \neq m_k \text{ and } l_k \perp m_k \text{ for at most one } k \in \{i, -i\}$$

Following Fishburn (1976) and theorem 1.3 in chapter 1, the correct axiom for a full multi-linear plus an entropic term is the following.

Axiom B. 6 (Orthogonal Independence)**.** For every $l, m, n, o \in \mathcal{L}_N$ and $\alpha \in (0, 1)$, if $l\mathcal{R}_\perp n$, $m\mathcal{R}_\perp o$, $n \sim o$

$$l \succeq m \Rightarrow \alpha l + (1-\alpha)n \succeq \alpha m + (1-\alpha)o$$

.

I state the following proposition without proof and discuss the problem of this representation. It is possible to consider that the orthogonal restriction can be applied only to the private representation but in this case we loose homogeneity between the expectation of the private and the social utility. In the following proposition, $A_i H(l_i)$ is the sum of the entropic term derived from the private and the social utility part as l_i and l_{-i} are considered independent there is no difficulty to separate the joint entropy. For exogenous reference group, the optimisation problem is not modified and the entropy of l_{-i} cancels. In fact, with identical agent and rational anticipation the following utility representation give the same equilibrium that in Brock and Durlauf (2001a). I have no solutions, for now, to find a representation only with the individual entropic term to extend next the following proposition to endogenous group.

Proposition 2.4. *A binary relation* $\succeq$ *on* $\mathcal{L}_N$ *satisfies Axioms B.1,B.2,B.4 and B.6 if and only if there exist* $u : X_i \to \mathbb{R}$, *and* $S : X_i \times \prod_{j \neq i} X_j \to \mathbb{R}$, *with*

$S(x_N) = 0$ for all $x \in X$, such that

$$V(l_i, l_{-i}) = \sum_{X_i} l_i(x_i)u(x_i) + A_i H(l_i) + \sum_{X_i \times \prod_{j \neq i} X_j} S(x_i, x_{-i}) l_i(x_i) l_{-i}(x_{-i}) + A_{-i} H(l_{-i})$$

represents $\succeq$ with H is the Shannon entropy . Moreover, $(\tilde{u}, \tilde{S})$ is another representation of $\succeq$ in the above sense if and only if there exist $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}$ such that $\tilde{u} = \alpha u + \beta$ and $\tilde{S} = \alpha S$.

2.5 Proofs

Proof of theorem 2.1.

Claim 1. If $\succeq$ satisfies continuity then it is archimedean in the sense that for every $l, m, n \in \mathcal{L}_N$,

$l\mathcal{R}n$ and $l \succ m \succ n \Rightarrow \exists \alpha, \beta \in (0,1)$ such that $\alpha l + (1-\alpha)n \succ m \succ \beta l + (1-\beta)n$.

Proof. Proving this implication is a standard exercise. I report it just for the sake of completeness. Let $\succeq$ be continuous and l, m, n such that $l\mathcal{R}n$ and $l \succ m \succ n$. Remember that $l\mathcal{R}n$ only when l and n differ in at most one coordinate, thus either $l = (l_i, l_{-i})$ and $n = (l_i, n_{-i})$ or $l = (l_i, l_{-i})$ and $n = (n_i, l_{-i})$. To prove that $\succeq$ is Archimedean, we first need to find $\alpha \in (0,1)$ such that $\alpha l + (1-\alpha)n \succ m$. Consider the sequence defined by $(\frac{p}{p+1} l + \frac{1}{p+1} n)$ for $p \geq 1$. As $l\mathcal{R}n$, the sequence is well defined and converges to l. Suppose that, for all p, we have $m \succeq \frac{p}{p+1} l + \frac{1}{p+1} n$. As $\succeq$ is continuous in the product topology, we must have $m \succeq l$, which is a contradiction. Therefore there exists some p_0 for which $\frac{p_0}{p_0+1} l + (1 - \frac{1}{p_0+1})n \succ m$. By letting $\alpha = \frac{p_0}{p_0+1}$ the result follows. The proof that there exists $\beta \in (0,1)$ such that $m \succ \beta l + (1-\beta)n$ is nearly identical and does not require a new proof. $\square$

As $\succeq$ is continuous then it is archimedean, by Theorem 2 in Fishburn and Roberts (1978), Axioms B.1-B.3 imply the existence of a multi-affine functional $V : \mathcal{L}_N \to \mathbb{R}$, which represents $\succeq$. Moreover, V is unique up to positive affine transformation. We must show that with our stronger assumptions, this function V is continuous.

Claim 2. Let V be an multi-affine functional representing $\succeq$, under Axiom B.2 V is continuous in the product topology.

Proof. Let V be an multi-affine functional representing $\succeq$. Consider any convergent sequence in the product topology (l_p) in $\mathcal{L}_N$ whose limit is $\bar{l} \in \mathcal{L}_N$. Suppose first that there exists $l_+ \in \mathcal{L}_N$ for which $l_+ \mathcal{R} \bar{l}$ and $l_+ \succ \bar{l}$. Then Axiom B.2 implies that, for every $\epsilon > 0$, the set

$$\begin{aligned}
&P_\epsilon = \left\{ l \mid \epsilon l_+ + (1-\epsilon)\,\bar{l} \succ l \succ \bar{l} \right\} \\
\Leftrightarrow\ &P_\epsilon = \left\{ l \mid 0 < V(l) - V\left(\bar{l}\right) < \epsilon \left(V(l_+) - V\left(\bar{l}\right) \right) \right\} \text{ as } V \text{ is multi-affine}
\end{aligned}$$

is open. In this case there exists an integer p_+^ϵ for which, if $p > p_+^\epsilon$ and $l_p \succ \bar{l}$, then $l_p \in P_\epsilon$ and so

$$0 < V(l_p) - V\left(\bar{l}\right) < \epsilon \left(V(l_+) - V\left(\bar{l}\right) \right)$$

Alternatively, suppose that there exists $l_- \in \mathcal{L}_N$ for which $l_- \mathcal{R} \bar{l}$ and $l_- \prec \bar{l}$. In this case, reversing the preferences and inequalities in the previous argument shows that there must exists an integer p_-^ϵ for which, if $p > p_-^\epsilon$ and $l_p \prec \bar{l}$, then $l_p \in P_\epsilon$ and so

$$0 > V(l_p) - V\left(\bar{l}\right) > \epsilon \left(V(l_-) - V\left(\bar{l}\right) \right)$$

The last two inequalities together imply that $V(l_p) \to V\left(\bar{l}\right)$ as $p \to \infty$. Therefore V is continuous when $\mathcal{L}_N$ is given the product topology. $\square$

Now suppose that $V : \mathcal{L}_N \to \mathbb{R}$ is multi-affine and continuous. We recall that as X_i (respectively $\prod_{j \neq i} X_j$) is separable, then the finitely supported probabilities over X_i (respectively $\prod_{j \neq i} X_j$) are dense in $\Delta(X_i)$ (respectively $\Delta(\prod_{j \neq i} X_j)$).

For $(x_i, x_{-i}) \in X_i \times \prod_{j \neq i} X_j$ define $v(x_i, x_{-i}) = V(\delta_{x_i}, \delta_{x_{-i}})$. By multi-affinity

$$V(l_i, l_{-i}) = \int_{X_i \times \prod_{j \neq i} X_j} v(x_i, x_{-i}) dl_i(x_i) dl_{-i}(x_{-i})$$

for any finitely supported l_i, l_{-i}. For (m_i, m_{-i}) which is not finitely supported, let $(l_i, l_{-i})_p$ be a sequence of finitely supported probabilities converging to (m_i, m_{-i}). By continuity

$$V(m_i, m_{-i}) = \lim_p V((l_i, l_{-i})_p) = \int_{X_i \times \prod_{j \neq i} X_j} v(x_i, x_{-i}) dl_i(x_i) dl_{-i}(x_{-i})$$

by the Portemanteau Theorem.

Set $u(x) = v(x_N)$ and define $U : \Delta(X_i) \to \mathbb{R}$ by,

$$\forall l_i \in \Delta(X_i), U(l_i) = \int_{X_i} u(x_i) dl_i(x_i)$$

Notice that by Axiom B.4, $V(x_N) = U(x)$ for all $x_N \in X_i \times \prod_{j\neq i} X_j$, then U is a continuous VNM-representation of $\succeq^{\emptyset}$ over $\Delta(X_i)$. Set $S(x_i, x_{-i}) = v(x_i, x_{-i}) - u(x_i)$ for all $(x_i, x_{-i}) \in X_i \times \prod_{j\neq i} X_j$. Then $S(x_N) = v(x_N) - u(x) = 0$ for all $x_N \in X_i \times \prod_{j\neq i} X_j$, and

$$\begin{aligned}
V(l_i, l_{-i}) &= \int_{X_i \times \prod_{j\neq i} X_j} (u(x_i) + S(x_i, x_{-i})))dl_i(x_i)dl_{-i}(x_{-i}) \\
&= \int_{X_i \times \prod_{j\neq i} X_j} u(x_i)dl_i(x_i)dl_{-i}(x_{-i}) + \int_{X_i \times \prod_{j\neq i} X_j} S(x_i, x_{-i})dl_i(x_i)dl_{-i}(x_{-i}) \\
&= \int_{X_i} u(x_i) \left(\int_{\prod_{j\neq i} X_j} dl_{-i}(x_{-i}) \right) dl_i(x_i) + \int_{X_i \times \prod_{j\neq i} X_j} S(x_i, x_{-i})dl_i(x_i)dl_{-i}(x_{-i}) \\
&= \int_{X_i} u(x_i)dl_i(x_i) + \int_{X_i \times \prod_{j\neq i} X_j} S(x_i, x_{-i})dl_i(x_i)dl_{-i}(x_{-i})
\end{aligned}$$

for all $(l_i, l_{-i}) \in \mathcal{L}_N$. Which gives the desired representation. Notice that u and S are continuous by definition.

Conversely, assume that there exist a continuous function $u : X_i \to \mathbb{R}$ and a continuous function $S : X_i \times \prod_{j\neq i} X_j \to \mathbb{R}$ with $S(x_N) = 0$ for all $x \in X$, which represent $\succeq$ then Axioms B.1-B.4 holds.

For the last part of the lemma, let $\tilde{u} : X_i \to \mathbb{R}$ and $\tilde{S} : X_i \times \prod_{j\neq i} X_j \to \mathbb{R}$ with $\tilde{S}(x_N) = 0$ for all $x \in X$ be continuous functions, such that $\tilde{V} : \mathcal{L}_N \to \mathbb{R}$, defined by

$$\tilde{V}(l_i, l_{-i}) = \int_{X_i} \tilde{u}(x_i)dl_i(x_i) + \int_{X_i \times \prod_{j\neq i} X_j} \tilde{S}(x_i, x_{-i})dl_i(x_i)dl_{-i}(x_{-i})$$

represents $\succeq$. As V is unique up to positive affine transformation there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\tilde{V} = \alpha V + \beta$. By Axiom B.4, for all $x \in X$,

$\tilde{U}(x) = \tilde{V}(x_N)$ then $\tilde{u} = \alpha u + \beta$. Consequently,

$$\begin{aligned}\tilde{S} &= \tilde{V} - \tilde{U} \\ &= (\alpha V + \beta) - (\alpha U + \beta) \\ &= \alpha S\end{aligned}$$

Conversely, if there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\tilde{u} = \alpha u + \beta$ and $\tilde{S} = \alpha S$, then $\tilde{S}(x_N) = 0$ for all $x \in X$, then $\tilde{V} = \tilde{U} + \tilde{S}$ represents $\succeq$ over $\mathcal{L}_N$.

■

Proof of theorem 2.2. By Lemma 1, there exist a continuous function $u : X_i \to \mathbb{R}$ and a continuous function $S : X_i \times \prod_{j \neq i} X_j \to \mathbb{R}$ with $S(x_N) = 0$ for all $x \in X$, such that the functional $V : \mathcal{L}_N \to \mathbb{R}$, defined by

$$V(l_i, l_{-i}) = \int_{X_i} u(x_i) dl_i(x_i) + \int_{X_i \times \prod_{j \neq i} X_j} S(x_i, x_{-i}) dl_i(x_i) dl_{-i}(x_{-i})$$

for all $(l_i, l_{-i}) \in \mathcal{L}_N$ and represents $\succeq$. In what follows we denote V by

$$V(l_i, l_{-i}) = U(l_i) + S(l_i, l_{-i})$$

Let $l_i \in \Delta(X_i)$, fix $l_{-i} \in \Delta(\prod_{j \neq i} X_j)$ and define $S_j(l_i, l_j)$ for $j \neq i$, where $l_j \in \Delta(X_j)$ is the $j - th$ marginal of l_{-i}, such that

$$S(l_i, l_{-i}) = \sum_{j \neq i} S_j(l_i, l_j)$$

Let $m_{-i}, (m^j_{-i})_{j \neq i} \in \Delta(\prod_{j \neq i} X_j)$ be such that $(m^j_j) = (m_j)$ and $(m^j_k) = (l_k)$ for all $k \neq j$ and define

$$S_j(l_i, m_j) = S(l_i, m^j_{-i}) - \sum_{k \neq j} S_k(l_i, l_k)$$

Summing over j we get

$$\sum_{j \neq i} S_j(l_i, m_j) = \sum_{j \neq i} S(l_i, m^j_{-i}) - (n-2) S(l_i, l_{-i}) \tag{2.6}$$

As $(l_i, l_{-i}) \mathcal{R} (l_i, m_{-i})$ and $(l_i, m^{j_1}_{-i}) \mathcal{R} (l_i, m^{j_2}_{-i})$ for all $j_1, j_2 \neq i$, the lotteries $\frac{n-2}{n-1}(l_i, l_{-i}) + \frac{1}{n-1}(l_i, m_{-i})$ and $\sum_{j \neq i} \frac{1}{n-1}(l_i, m^j_{-i})$ are well defined, moreover the $k - th$ marginal of $\sum_{j\neq i}\frac{1}{n-1}m^j_{-i}$ is given by

$$\begin{aligned}(\sum_{j\neq i}\frac{1}{n-1}m^j_{-i})_k &= \frac{1}{n-1}(m^1_k+\cdots+m^{i-1}_k)+m^{i+1}_k+\cdots+m^n_k)\\ &= \frac{1}{n-1}((n-2)l_k+m_k)\end{aligned}$$

By Axiom B.5

$$\frac{n-2}{n-1}(l_i,l_{-i})+\frac{1}{n-1}(l_i,m_{-i})\sim\sum_{j\neq i}\frac{1}{n-1}(l_i,m^j_{-i}))$$

Or equivalently as V is multi-affine representation of $\succeq$

$$S(l_i,m_{-i})=\sum_{j\neq i}S(l_i,m^j_{-i}))-(n-2)S(l_i,l_{-i}) \tag{2.7}$$

(2.6) and (2.7), yield $S(l_i,m_{-i})=\sum_{j\neq i}S_j(l_i,m_j)$ and this concludes the proof of the sufficiency part, since $\forall(l_i,l_{-i})\in\mathcal{L}_N$,

$$\begin{aligned}V(l_i,l_{-i}) &= \int_{X_i}u(x_i)dl_i(x_i)+\int_{X_i\times\prod_{j\neq i}X_j}S(x_i,x_{-i})dl_i(x_i)dl_{-i}(x_{-i})\\ &= \int_{X_i}u(x_i)dl_i(x_i)+\int_{X_i\times\prod_{j\neq i}X_j}\sum_{j\neq i}S_j(x_i,x_j)dl_i(x_i)dl_{-i}(x_{-i})\\ &= \int_{X_i}u(x_i)dl_i(x_i)+\sum_{j\neq i}\left(\int_{X_i\times X_j\times\prod_{k\neq i,j}X_k}S_j(x_i,x_j)dl_i(x_i)dl_{-i}(x_{-i})\right)\\ &= \int_{X_i}u(x_i)dl_i(x_i)+\sum_{j\neq i}\left(\int_{X_i\times X_j}S_j(x_i,x_j)\left(\int_{\prod_{k\neq i,j}X_k}dl_{-i}(x_{-i})\right)dl_i(x_i)\right)\\ &= \int_{X_i}u(x_i)dl_i(x_i)+\sum_{j\neq i}\left(\int_{X_i\times X_j}S_j(x_i,x_j)dl_i(x_i)dl_j(x_j)\right)\end{aligned}$$

Notice that for all $j\neq i$, S_j is continuous and that $\sum_{j\neq i}S_j(x,x)=0$ for all $x\in X$ by definition. The necessary part of the lemma is obvious since $m_k=n_k$ for all $k\in N\setminus\{i\}$ gives $V(l_i,m_{-i})=V(l_i,n_{-i})$ from (2.3) and consequently implies that $(l_i,m_{-i})\sim(l_i,n_{-i})$. For the last part of the lemma, let $\tilde{u}:X_i\to\mathbb{R}$ and for $j\neq i$, $\tilde{S}_j:X_i\times X_j\to\mathbb{R}$ with $\sum_{j\neq i}\tilde{S}_j(x,x)=0$ for all $x\in X$ be continuous functions, such that $\tilde{V}:\mathcal{L}_N\to\mathbb{R}$ represents $\succeq$. By Lemma 1 there exist $\alpha,\beta\in\mathbb{R}$ with $\alpha>0$ such that $\tilde{u}=\alpha u+\beta$ and $\tilde{S}=\alpha S$ with $\tilde{S}=\sum_{j\neq i}\tilde{S}_j$ and $S=\sum_{j\neq i}S_j$. Then, with l_{-i} fixed as before, $S(l_i, m^j_{-i}) = \sum_{k\neq i} S_k(l_i, m^j_k$, $\tilde{S}(l_i, m^j_{-i}) = \sum_{k\neq i} \tilde{S}_k(l_i, m^j_k$ and $\tilde{S} = \alpha S$ imply that

$$\begin{aligned}\tilde{S}_j(l_i, m_j) &= \alpha S_j(l_i, m_j) + \left(\alpha \sum_{k\neq i,j} S_k(l_i, l_k) - \sum_{k\neq i,j} \tilde{S}_k(l_i, l_k)\right)\\ &= \alpha S_j(l_i, m_j) + \tilde{S}_j(l_i, l_j) - \alpha S_j(l_i, l_j)\end{aligned}$$

Let $\beta_j = \tilde{S}_j(l_i, l_j) - \alpha S_j(l_i, l_j)$ for $j \neq i$, then for all $j \neq i$, $\tilde{S}_j = \alpha S_j + \beta_j$ with $\sum_{j\neq i} \beta_j = \sum_{j\neq i} \left(\tilde{S}_j(l_i, l_j) - \alpha S_j(l_i, l_j)\right) = \tilde{S}(l_i, l_{-i}) - \alpha S(l_i, l_{-i}) = 0$. Conversely, if there exist $(\alpha, \beta, (\beta_j)_{j\neq i}) \in \mathbb{R}^*_+ \times \mathbb{R}^n$ such that $\tilde{u} = \alpha u + \beta$, $\tilde{S}_j = \alpha S_j + \beta_j$ for all $j \in N \setminus \{i\}$ and $\sum_{j\neq i} \beta_j = 0$, then $\tilde{V} = \tilde{U} + \sum_{j\neq i} \tilde{S}_j$ represents $\succeq$ over $\mathcal{L}_N$. This completes the proof. ∎

Proof of theorem 2.3. We first establish the sufficiency. It can easily be checked that $\Delta(X_i)$ is separable in the weak topology and that for all $J \in \mathcal{P}(N \setminus \{i\}), J \neq \{\emptyset\}$ $\Delta(\prod_{j\in J} X_j)$ is separable in the weak topology. Consequently, $\mathcal{L}_J$ is separable in the product topology for every J as product of separable spaces and $\mathcal{L}$ is separable in the disjoint union topology as finite disjoint union of separable spaces. $\mathcal{L}_J$ is connected and therefore locally connected for every J, since the disjoint union of a family of spaces is locally connected if and only if each is locally connected, then $\mathcal{L}$ is locally connected. By Theorem 1 in Candeal et al. (2004), Axioms C.1-C.2 imply that $\succeq$ is a continuous total preorder on $\mathcal{L}$, a locally connected and separable space; that is, there exists a continuous function $V : \mathcal{L} \to \mathbb{R}$ such that

$$(l_i, l_J) \succeq (m_i, m_K) \Leftrightarrow V(l_i, l_J) \geq V(m_i, m_K) \tag{2.8}$$

for all $(l_i, l_J), (m_i, m_K) \in \mathcal{L}$. For every J, let $\succeq^J \equiv \succeq \cap (\mathcal{L}_J \times \mathcal{L}_J)$ be the restriction of $\succeq$ to $\mathcal{L}_J$.

Claim 3. If $\succeq$ satisfies Axioms C.1-C.4 then for all J, $\succeq^J$ satisfies Axioms B.1-B.4

Proof. Fix a J and suppose that $\succeq$ satisfies Axioms C.1-C.4,

- If $\succeq$ is a weak order then $\succeq^J$ is also a weak order.
- Let $l \in \mathcal{L}_J$ we know that the upper contour set $\{m \mid m \succeq l\}$ is closed in the disjoint union topology and consequently that $\{m \mid m \succeq l\} \cap \mathcal{L}_J$ is closed in the product topology. We conclude that $\succeq^J$ is upper semi-continuous. Similarly, $\succeq^J$ is lower semi-continuous and finally $\succeq^J$ satisfies Axiom B.2.

- Taking $K = J$ shows that $\succeq^J$ satisfies Axiom B.3.
- For every $x, y \in X$, iterated application of Axiom C.4 delivers $x_J \sim x_\emptyset$ and $y_J \sim y_\emptyset$. Therefore $x_J \succeq^J y_J$ if, and only if, $x \succeq^\emptyset y$.

□

The restriction of V to $\mathcal{L}_J$ represents $\succeq^J$. Therefore, by Lemma 1, for every J there exist continuous functions $u^J : X_i \to \mathbb{R}$, and continuous functions $S^J : X_i \times \prod_{j \neq i} X_j \to \mathbb{R}$, with $S^J(x_J) = 0$ for all $x \in X$, such that for all $(l_i, l_J) \in \mathcal{L}_J$

$$V(l_i, l_J) = V_{|\mathcal{L}_J}(l_i, l_J) = \int_{X_i} u^J(x_i) dl_i(x_i) + \int_{X_i \times \prod_{j \in J} X_j} S^J(x_i, x_J) dl_i(x_i) dl_J(x_J) \tag{2.9}$$

By using (2.8) and (2.9), we obtain

$$(l_i, l_J) \succeq (m_i, m_K) \Leftrightarrow V_{|\mathcal{L}_J}(l_i, l_J) \geq V_{|\mathcal{L}_K}(m_i, m_K)$$

for all $(l_i, l_J), (m_i, m_K) \in \mathcal{L}$. But then by Axiom C.4, we must have

$$\forall x \in X, \forall J, K \in \mathcal{P}(N \setminus \{i\}), V_{|\mathcal{L}_J}(x_J) = V_{|\mathcal{L}_K}(x_K)$$

and therefore

$$\forall x \in X, \forall J, K \in \mathcal{P}(N \setminus \{i\}), u^J(x) = u^K(x)$$

Choosing $u = u^\emptyset$ gives

$$V_{|\mathcal{L}_J}(l_i, l_J) = \int_{X_i} u(x_i) dl_i(x_i) + \int_{X_i \times \prod_{j \in J} X_j} S^J(x_i, x_J) dl_i(x_i) dl_J(x_J) \tag{2.10}$$

We will show that Axiom C.5 implies that for all $(l_i, l_{J\cup\{k\}}) \in \mathcal{L}$, $S^{J\cup\{k\}}(l_i, l_{J\cup\{k\}}) = S^J(l_i, l_J) + S^{\{k\}}(l_i, l_{\{k\}})$ where l_J and l_k are respectively the J-marginal and the $\{k\}$-marginal of $l_{J\cup\{k\}}$ by showing the contraposive. In the first place observe that Axiom C.5 implies that for every $J \in \mathcal{P}(N \setminus \{i\})$, $k \notin J$ and $l \in \mathcal{L}_{J\cup\{k\}}$,

$$(l_i, l_{J\cup\{k\}}) \sim (l_i, l_J) \Rightarrow (l_i, l_{\{k\}}) \sim (l_i)$$

Secondly, suppose that there exists $l = (l_i, l_{J\cup\{k\}}) \in \mathcal{L}_{J\cup\{k\}}$, with l_J and l_k are respectively the J-marginal and the $\{k\}$-marginal of $l_{J\cup\{k\}}$, such that $S^{J\cup\{k\}}(l_i, l_{J\cup\{k\}}) \neq$

$S^J(l_i, l_J) + S^{\{k\}}(l_i, l_{\{k\}})$.

$$\begin{aligned}(l_i, l_{J\cup\{k\}}) \sim (l_i, l_J) &\Rightarrow V_{|\mathcal{L}_{J\cup\{k\}}}(l_i, l_{J\cup\{k\}}) = V_{|\mathcal{L}_J}(l_i, l_J) \\ &\Rightarrow S^{J\cup\{k\}}(l_i, l_{J\cup\{k\}}) = S^J(l_i, l_J) \\ &\Rightarrow S^{\{k\}}(l_i, l_{\{k\}}) \neq 0 \text{ by hypothesis} \\ &\Rightarrow U(l_i) + S^{\{k\}}(l_i, l_{\{k\}}) \neq U(l_i) \\ &\Rightarrow (l_i, l_{\{k\}}) \not\sim (l_i)\end{aligned}$$

Which contradicts Axiom C.5, therefore,

$$\forall (l_i, l_{J\cup\{k\}}) \in \mathcal{L}, S^{J\cup\{k\}}(l_i, l_{J\cup\{k\}}) = S^J(l_i, l_J) + S^{\{k\}}(l_i, l_{\{k\}}) \tag{2.11}$$

To complete the proof we will show by induction on the cardinal of J that for all $(l_i, l_J) \in \mathcal{L}$

$$V(l_i, l_J) = \int_{X_i} u(x_i) dl_i(x_i) + \sum_{j\in J} \left(\int_{X_i\times X_j} S_j(x_i, x_j) dl_i(x_i) dl_j(x_j) \right)$$

This is obviously true for $|J| = 0$, in this case $V(l_i) = U(l_i)$. It is also true for all J such that $|J| = 1$ by (2.9), for all $j \neq i$ there exist continuous functions $S^j : X_i \times X_j \to \mathbb{R}$, with $S^j(x, x) = 0$ for all $x \in X$, such that for all $(l_i, l_j) \in \mathcal{L}_j$

$$V(l_i, l_j) = V_{|\mathcal{L}_j}(l_i, l_j) = \int_{X_i} u(x_i) dl_i(x_i) + \int_{X_i\times X_j} S^j(x_i, x_j) dl_i(x_i) dl_j(x_j) \tag{2.12}$$

Define for all $j \neq i$, $S_j : X_i \times X_j \to \mathbb{R}$ by $S_j = S^j$, obviously $S_j(x, x) = 0$ for all $x \in X$ and S_j is continuous. Now assume it for all J such that $|J| = p$, $1 \leq p \leq n-2$ then by using (2.11), it is true for all J such that $|J| = p+1$ and so the result follows by induction. Which gives the desired representation. Conversely, assume that there exist a continuous function $u : X_i \to \mathbb{R}$ and continuous functions $S_j : X_i \times X_j \to \mathbb{R}$ for $j \in N \setminus \{i\}$, with $S_j(x, x) = 0$ for all $j \neq i$ and $x \in X$, which represent $\succeq$ then Axioms C.1-C.5 holds. The necessity part of the uniqueness representation is obvious. To prove the sufficiency part of the uniqueness of the representation, let $\tilde{u} : X_i \to \mathbb{R}$ and for $j \neq i$, $\tilde{S}_j : X_i \times X_j \to \mathbb{R}$ with $\tilde{S}_j(x, x) = 0$ for all $x \in X$ and $j \neq i$ be continuous functions, such that $\tilde{V} : \mathcal{L}_N \to \mathbb{R}$ represents $\succeq$. Let $J \in \mathcal{P}(N \setminus \{i\})$, by Lemma 1 the restriction of $\tilde{V}$ to $\mathcal{L}_J$ represents $\succeq^J$ if there exist $(\alpha_J, \beta_J) \in \mathbb{R}_+^* \times \mathbb{R}$ such that $\tilde{V}_{|\mathcal{L}_J} = \alpha_J V_{|\mathcal{L}_J} + \beta_J$ and therefore $\tilde{u} = \alpha_J u + \beta_J$ and $\tilde{S}^J = \alpha_J S^J$. By Axiom C.4, $(\alpha_J, \beta_J) = (\alpha_K, \beta_K)$ for all J, K.

Choosing $(\alpha, \beta) = (\alpha_\emptyset, \beta_\emptyset)$ gives $\tilde{u} = \alpha u + \beta$ and $\tilde{S}^J = \alpha S^J$. By Axiom C.5 and consequently by (2.11), an induction on the cardinal of J lets prove that $\tilde{S}_j = \alpha S_j$ for all $j \neq i$. ■

2.6 Conclusion

In this chapter I have studied individual preference relations under risk which are possibly other-regarding. My primary aim has been to provide a formal background to other-regarding expected utility representations already used in various applications which covers the case where the other agents make risky choices and the case where the risk is a belief about the other agents choices or both. In addition, I introduced a preference relation over a disjoint union of spaces to account for the selection among groups of individuals that operates the decision maker. This allows us to explain the social anchoring of the decision maker. The addition of an entropic term is partially processed to fit the Blume, Brock and Durlauf's model. Some difficulties arise, principally are due to the homogeneity condition between the expectation of the private utility term and the expectation of the social utility term.

Chapter 3

Expected Utility Theory with Non-Commutative Probability Theory

3.1 Introduction

The theory of expected utility in economics was first proposed by von Neumann and Morgenstern in their seminal work on economic behaviour and games theory (1944). It has become the classical model of decision under a risky environment. Soon after the model was proposed, it was challenged by experimental paradoxes. The Allais paradox (1953) and the Ellsberg paradox (1961), for example, indicate violation of the independence axiom and consequently that the treatment of probabilities by individuals is nonlinear. Since the second half of the 20^{th} century until today, the expected utility model has been generalized by positing nonlinear functional forms for the individual preference function in different ways : prospect theory by Kahneman and Tversky (1979), regret theory by Loomes and Sugden (1982), local expected utility by Machina (1982), rank-dependent utility by Quiggin (1982), quadratic utility by Chew et al. (1991).

Several studies use quantum formalism in decision theory to explain the various paradoxes. The way that quantum formalism is used varies across studies but it offers new opportunities in the form of new technical capabilities from the same mathematical tool. In a deterministic choice framework (Danilov and Lambert-Mogiliansky (2005, 2010), Gyntelberg and Hansen (2005, 2009), Hansen (2005), La Mura (2009)) or in stochastic choice framework (Aerts and Aerts (1995), Aerts and Gabora (2005a,b),Aerts and D'Hooghe (2009), Aerts et al. (2011), Busemeyer et al. (2006a), Busemeyer et al. (2011), Busemeyer and Bruza (2012), Conte et al. (2009), Khrennikov (1999, 2010), Lambert-Mogiliansky et al. (2009), Pothos and Busemeyer (2009), Pothos and Busemeyer (2013), Yukalov and Sornette (2010)). What is the motivation for employing quantum formalism in decision making ? What is the contribution ?

Foremost, it is necessary to precise that decision theory is not a quantum mechanics phenomenon. But, the probabilistic framework[1] can be used independently. To avoid confusion, I prefer to use non-commutative probability theory rather than quantum probability. In this chapter, I am interested in non-commutative probability theory as a mathematical framework for decision theory, its relevance in decision making and its contribution regarding the nature of human rationality.

Building on all this work, I present a model based on non-commutative probabilities, by using density operators which are a generalization of classical probability distributions. My theory introduces a decision-theoretic framework which extends the expected utility methodology and in addition links the descriptive and normative approaches. To enable comparison with previous work, I develop the theory in an algebraic context. That is, I begin with a commutative algebra of bounded random variables and introduce a much larger structure - a non-commutative algebra - to show that what is often presented as irrational can be understood, from a different point of view, as rational.

The chapter is organized as follows. Section 2 presents some "quantum" concepts and reviews the literature. I introduce the hypothesis of "own rationality". Section 3 provides a reformulation of von Neumann and Morgenstern's analysis of non-commutative probability theory and proposes an analogue theorem to represent preference relations. Section 4 explains the representation result for non-commutative probability theory, offers an interpretation of the matrix of utilities in terms of individual behaviour and introduces the concepts of utility under risk. I show that my approach is equivalent to expected utility from the decision maker's

[1] The oldest example of non-commutative theory of integration leading to a theory of non-commutative probabilities is due to Neumann (1932). Reader interested in a full understanding of the quantum probabilistic formalism should consult Birkhoff and Von Neumann (1936), Suppes (1969) or Varadarajan (1968, 1970).

point of view. Section 5 applies my results to prospect theory, the Allais paradox and the Marschak-Machina triangle. Section 6 discusses the contributions of this formalism. Proofs are given in section 7. Section 8 concludes.

3.2 Towards non-commutative probability

3.2.1 Motivations

Classical probability theory has proved satisfactory for almost all scientific purposes. The one outstanding exception is quantum mechanics. In that theory, classical probabilities are meaningless. For two events A and B, both $\mathbb{P}(A)$ and $\mathbb{P}(B)$ may exist and yet $\mathbb{P}(A \cap B)$ need not. As basic example, suppose that A is an event related to the location of a particle and that B is an event related to the momentum of this same particle. The event $A \cap B$ represents both the location and the momentum. One basic feature of quantum mechanics is that A and B are respectively observable but not necessarily both A and B. Consequently, no probability is assignable to $A \cap B$ even when $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are specified. From this observation, it is natural that a specific probability theory should be used. The quantum probability theory is fundamentally different from the classical probability theory. The core of the difference lies in the fact that in quantum probability the measurement process influences the result, while in classical probability all properties are assumed to have a definite value before measurement, and that this value is the outcome of the measurement. Quantum probability are well defined since Neumann (1932). The point is that a classical probabilistic system (or measurable space) is an algebra of random variables that satisfies relevant axioms. One of the restrictions on the classical algebra is commutativity : If X and Y are two real random variables, then XY and YX are the same random variable. In quantum probability, this commutative algebra is replaced by a non-commutative algebra called a von Neumann algebra. The remaining definitions stay as much the same as possible. Non-commutativity is then a key aspect and expresses that "observations" or "measurements" disturb the subject we are measuring.

In decision making, there are a numerous observations which show that the probability framework is questionable. The disjunction effect exhibited by Tversky and Shafir (1992) can be considered as a probabilistic anomaly. According to the sure thing principle, choice over acts are independents from the knowledge of the state of the world. Tversky and Shafir are tested this principle by presenting 98 students with a two stage gamble, that is a gamble which can be played twice. At each stage of the experiment the available choices was whether or not to play a gamble. After finishing the first stage, participants are informed of the outcome of the gamble, gain, loss or no information. The key result is based on the decision for the second play, If they knew they won the first gamble, the majority (69%) chose to play again; if they knew they lost the first gamble, then again the majority (59%) chose to play again; but if they didn't know whether they won or lost, then the majority chose not to play (only 36% wanted to play again). Busemeyer et al. (2006b) originally suggested that this experiment was an example of an interference effect as in quantum mechanics. A most striking example is that Moore (2002) reported that the probability of a response to the questions "Is Gore honest?" and "Is Clinton honest?" depends on the relative order of the questions. In this way, conjunction fallacy (Tversky and Kahneman (1983)) and disjunction fallacy (See for this situation Carlson and Yates (1989)) are effects that show probability judgement error. Student are confronted to the hypothetical Linda who has a "liberal" profile and they must rank the following events : A, "Linda is active in the feminist movement", B, "Linda is a bank teller", and their conjunction $A \cap B$. It appears systematically that individuals report $\mathbb{P}(A \cap B) \leq \mathbb{P}(B)$. This inequality violates classical probability laws but not non-commutative probability laws. The analogy with the quantum mechanics is that there exist a lack of knowledge concerning how context and framing influences individual under consideration. Even if we were to suppose that at the ontological level the interaction between the context and the framing with an individual engenders a change of thought that is deterministic, a lack of knowledge about this interaction gives rise to a probability model which does not satisfy the axioms of Kolmogorov (1950).

3.2.2 Literature review

For stochastic choice framework in a static context, Aerts and Aerts (1995), Aerts and D'Hooghe (2009) and Aerts et al. (2011) modelled incompatibility and interference effects that arise in individual preference judgements. Authors used a contextual axiomatization of quantum theory. Busemeyer et al. (2006a) modelled cognition in a dynamical context. Busemeyer et al. (2011), Pothos and Busemeyer

(2009, 2013) reviewed decision making paradoxes and applied quantum formalism to explain human probability judgement errors including the conjunction and disjunction fallacies, averaging effects, unpacking effects, and order effects on inference. Lambert-Mogiliansky et al. (2009) showed that this kind of choice model does not satisfy weak axiom of revealed preferences. Yukalov and Sornette (2010) described a model where quantum behaviour comes from interferences between intention and action.

For deterministic choice framework, Hansen (2005) formulates a theory of decision making based on algebraic formulations of quantum mechanics. He draws an analogy between an event and a physical observable and develops a general model of a set of events. This more general representation is used in the context of decision theory by Gyntelberg and Hansen (2005, 2009) to adapt expected utility theory in a Savage type formulation (1954), while Danilov and Lambert-Mogiliansky (2005, 2010) formulate a theory of decision making under uncertainty in a non-classical environment. The latter authors propose a non-classical environment, representation for an individual, of a set of events, using a propositional system and allowing the necessary conditions for the set of events to be isomorphic to Hilbert lattice (see Appendix A - Quantum Logic) $\mathcal{P}(\mathcal{H})$. Obviously, the decision problem is not associated with a "quantum" phenomenon: instead of weakening the axioms of the theory of expected utility, here the individual does not represent the set of events by a σ-algebra $\mathcal{F}$ but by an arbitrary structure comparable to $\mathcal{P}(\mathcal{H})$. However in quantum mechanics and mathematics, it is an open problem to find the necessary and sufficient conditions for an abstract propositional system to be isomorphic to $\mathcal{P}(\mathcal{H})$ for some (real, complex, quaternionic or generalized) Hilbert space $\mathcal{H}$. La Mura (2009) defines lotteries as normalized vectors in Hilbert space and obtains a representation theorem that extends the one proposed by von Neumann and Morgenstern.

Whether the choice is deterministic or stochastic, the inclusion of non-commutative probability gives some interesting properties. I focus in the reminder of this chapter to the deterministic case.

3.2.3 Replace the set of events ?

Let $S = \{s_1, \ldots, s_n\}$ be a finite set of outcomes or certain consequences where $n \geq 2$ (if $n = 1$, the set of lotteries is a singleton) is the cardinal of S. Let

$\mathcal{L} = \{(l_1, \ldots, l_n) \in [0,1]^n \mid \sum_i l_i = 1\}$ be the space of lotteries over S closed under convex mixture operations with $S \subset \mathcal{L}$ consists of all degenerate lotteries (denoted by δ_s for all $s \in S$). An implicit assumption of expected utility theory is to identify the set of lotteries with the set of classical probability distributions on S. By definition, a classical finite probability space is a triple $(\Omega, \mathcal{F}, \mu)$ where Ω is a finite sample space, $\mathcal{F}$ is a Boolean σ-algebra or a tribe over Ω, and $\mu : \mathcal{F} \to [0,1]$ is a countable additive map. Elements of $\mathcal{F}$ represent events and the map μ is a probability measure that associates a probability $\mu(\mathcal{E})$ to each of the events $\mathcal{E} \in \mathcal{F}$. Equivalently, consider such a classical finite probability space $(\Omega, \mathcal{F}, \mu)$ and the space $l^\infty(\Omega, \mathcal{F}, \mu)$ of bounded[2] random variables on Ω. Then μ induces a state[3] φ_μ on $l^\infty(\Omega, \mathcal{F}, \mu)$ by

$$\begin{aligned} \varphi_\mu : l^\infty(\Omega, \mathcal{F}, \mu) &\to \mathbb{R} \\ f &\mapsto \int_\Omega f d\mu \end{aligned}$$

An event is a set $\mathcal{E} \in \mathcal{F}$, or equivalently the projection operator $P_\mathcal{E}$ on $l^\infty(\Omega, \mathcal{F}, \mu)$ defined by setting

$$(P_\mathcal{E} f)(\omega) = 1_\mathcal{E}(\omega) f(\omega), \ \omega \in \Omega \tag{3.1}$$

for each f on $l^\infty(\Omega, \mathcal{F}, \mu)$, where $1_\mathcal{E}$ is the characteristic function of $\mathcal{E}$ on Ω. Therefore all information in $(\Omega, \mathcal{F}, \mu)$ is also contained in $(l^\infty(\Omega, \mathcal{F}, \mu), \varphi_\mu)$. More precisely, we have two equivalent descriptions of classical probability theory. A representation of the set of events for an individual is $\mathcal{F}$ or equivalently the set $\{P_\mathcal{E}\}_{\mathcal{E}\in\mathcal{F}}$.

$l^\infty(\Omega, \mathcal{F}, \mu)$ encodes all the information contained in the classical structure and has an appropriate algebraic structure. In Appendix, I characterize the resulting algebra axiomatically. One of the axioms will be commutativity and this axiom will be removed. The core of this chapter uses a transformation of $l^\infty(\Omega, \mathcal{F}, \mu)$ to propose a reformulation of the von Neumann and Morgenstern framework.

[2]Exponent ∞ refers to the fact that the bound of a $\mathcal{F}$-measurable function is its infinity norm.
[3]A positive linear functional of norm 1.

3.2.4 The assumption of "own rationality"

Specific experiments on decision making under risk show the irrelevance of utility theory in its traditional formulation. For instance, the observed choices show that the Allais paradox (1953) exhibits frequent violation of the independence axiom. Work in cognitive psychology, which contributes to economic decision theory, shows that individuals are not fully rational and frequently are subject to cognitive biases. Here rationality captures the idea that an individual makes a choice that is consistent with the theory of expected utility for objective information. Contrasting with the normative approach is Kahneman and Tversky's notion of prospect theory (1979, 1981, 1984) which is motivated by the fact that an individual involved in a decision making process is influenced by the framing of the problem, the context of choice, and individual reasoning. It has been acknowledged that from the perspective of the observer, decision makers (who consider themselves rational) are not rational. I now need to define more precisely this "assumption of own rationality".

When a lottery involves a decision maker, the observer (M) is positioned in the canonical probability space associated with the lottery (which is not a quantum phenomenon). I denote this space $(\Omega_M, \mathcal{F}_M, \mu_M)$. I assume that the decision maker (DM) occupies his own probability space $(\Omega_{DM}, \mathcal{F}_{DM}, \mu_{DM})$. My assumption is equivalent to assuming that, under risk, individuals perceive events differently. Although perceptions may be different, people are rational. In their representations of events, a decision problem under risk can be reduced to the expected utility framework for a particular classical probability space. In the Appendix B, I present the concept of D-algebras which allows us to characterize the representations of the set of events. I use the concept of non-commutative probability space seen as representing the set of all possible classical probability spaces $(\Omega_a, \mathcal{F}_a, \mu_a)$ for all agents "a". A non-commutative probability space $\mathcal{A}$ is a collection of many incompatible classical probability models, each of which coincides with a commutative $*$-subalgebra (see Appendix B1 - $*$-algebras for a definition) of $\mathcal{A}$.

Definition 1. A decision problem is characterized by a pair $\{\mathcal{L}, \mathcal{A}\}$ where : $\mathcal{A}$, the set of agents, is the collection of all the self-adjoint elements of a D-algebra; and $\mathcal{L}$ is the set of lotteries.

In this formalization, I look first at the conventional case. Let $\mathcal{H}$ be a finite-dimensional Hilbert space with the canonical orthonormal basis $(|\delta_{\omega_1}\rangle, \ldots, |\delta_{\omega_n}\rangle)$, let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $\mathcal{H}$ and let ρ be a density operator (see Appendix B2 - Density Operators for a definition) a probability law on $\mathcal{L}(\mathcal{H})$. The canonical representation $\mathcal{R}_c$ of the set of events is given by a map that injects the set of events $\mathcal{F}$ in $\mathcal{L}(\mathcal{H})$ by associating the operator O_i[4] to the event ω_i. The following properties characterize $\mathcal{R}_c$:

$$
\begin{array}{lll}
(P1) & \mathcal{R}_c(\omega) \text{ is an orthogonal projection} & \forall \omega \in \Omega \\
(P2) & \mathcal{R}_c\left(\bigcup \omega_i\right) = \sum \mathcal{R}_c(\omega_i) & \text{for all disjoint sequences } (\omega_i) \in \Omega \\
(P3) & \mathcal{R}_c(\Omega) = I & \text{Identity operator of } \mathcal{L}(\mathcal{H})
\end{array}
$$

Definition 2. A representation $\mathcal{R}$ of the set of events is a map that associates any measurable subset ω of a given measurable space Ω to a projection $\mathcal{R}(\omega)$ on a given Hilbert space $\mathcal{H}$ such that for any partition $\{\omega_i\}$ of Ω, $\sum_i \mathcal{R}(\omega_i) = I$.

This means that for every density operator ρ,

$$
\begin{aligned}
\mu : \mathcal{F}_{DM} &\to [0,1] \\
\omega &\mapsto \mu^{\mathcal{R}}(\omega) = tr(\rho \mathcal{R}(\omega))
\end{aligned}
$$

is a classical probability law. Thus $\mathcal{R}$ characterizes $(\Omega_{DM}, \mathcal{F}_{DM}, \mu_{DM})$.

[4] $O_i = |\delta_{\omega_i}\rangle \langle \delta_{\omega_i}|$ is the operator with 1 in the $(i,i)^{th}$ entry and zero elsewhere in the canonical basis.

3.3 Von Neumann and Morgenstern's approach in $\mathcal{L}(\mathcal{H})$

In this section, I extend expected utility theory to non-commutative probability theory. I explain the notions of prospect for the decision maker and of gamble for the observer. Finally I extend the von Neumann - Morgenstern utility theorem.

3.3.1 Gamble set

Let $S = (s_1, \ldots, s_n)$ be a finite set of outcomes associated with the canonical space of events $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\mathcal{H}$ as a finite-dimensional real Hilbert space. The orthonormal basis $\mathcal{B} = (|\delta_{\omega_1}\rangle, \ldots, |\delta_{\omega_n}\rangle)$ is associated with the canonical space of events. Our knowledge of the decision maker's representation of the set of events for a lottery l can be described by a probability law on the D-algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$. Let ρ be a density operator, if ρ is a pure state (see Appendix C2 - States for a definition) then there is a normalized vector $|\psi\rangle$ of $\mathcal{H}$ such that $\rho = |\psi\rangle\langle\psi|$.

In the canonical basis $\mathcal{B}$ the vector $|\psi\rangle$ can be decomposed into

$$|\psi\rangle = \sum_i \langle\psi|\delta_{\omega_i}\rangle\,|\delta_{\omega_i}\rangle \tag{3.2}$$

where, according to the Born rule (see Appendix C3 - Born rule)

$$\mathbb{P}_{s_i}(l) = |\langle\psi|\delta_{\omega_i}\rangle|^2 \tag{3.3}$$

Equations (3.2) and (3.3) express that the state $|\psi\rangle$ gives the probability of the outcomes of l from the point of view of the observer. From the decision maker's viewpoint $\mathcal{B}_{DM} = (|\pi_{\omega_1}\rangle, \ldots, |\pi_{\omega_n}\rangle)$, the vector $|\psi\rangle$ can also be decomposed into

$$|\psi\rangle = \sum_i \langle\psi|\pi_{\omega_i}\rangle\,|\pi_{\omega_i}\rangle \tag{3.4}$$

where, the Born rule dictates that

$$\mathbb{P}_{s_i}^{DM}(l) = |\langle\psi|\pi_{\omega_i}\rangle|^2 \tag{3.5}$$

As the basis of the decision maker's representation is unknown, we have no details on how he transforms the lottery probabilities. Therefore, I work from the canonical basis. Also, as probability law is quite different in $\mathcal{L}(\mathcal{H})$, I can propose the following definition for a generalized gamble.

Definition 3. A generalized gamble[5] ρ_l associated with a lottery l is a density operator in $\mathcal{L}(\mathcal{H})$.

I denote the set of gambles as $\mathcal{D}(\mathcal{H})$. Note that $\mathcal{D}(\mathcal{H})$ is convex: a convex combination of density operators is still a density operator. Accordingly, I define a mixed gamble as a convex combination of gambles. If ρ_l and ρ_m are gambles and $0 \leq \lambda \leq 1$ then $\lambda\rho_l \oplus (1-\lambda)\rho_m$ is the mixed gamble $\lambda\rho_l + (1-\lambda)\rho_m$.

3.3.2 Interpretation of a gamble

Characteristic of interpretations of non-commutative probability in my model is the central role of decision makers. A lottery is nothing more than information about the future and what matters is knowing how to treat this information. By analogy, a gamble contains global information which does not reduce to the probabilities $\{p_{s_1}, \ldots, p_{s_n}\}$ associated with each outcome s_i of a given lottery but also the probabilities that the decision maker considers and in fact for all decision makers. Thus for a decision maker fixed in his own probability space $(\Omega_{DM}, \mathcal{F}_{DM}, \mu_{DM})$ the gamble becomes a simple lottery and must be described by a new probability law (a classical) μ_{DM} on $\mathcal{L}(\mathcal{H})$, defined as

$$\mu_{DM} = \sum_i |\pi_{\omega_i}\rangle\langle\pi_{\omega_i}|\,\rho_l\,|\pi_{\omega_i}\rangle\langle\pi_{\omega_i}| \tag{3.6}$$

In this framework a mixed gamble corresponds to the possibility of considering mixtures of lotteries from the point of view of the decision maker without knowing $\mathcal{B}_{DM}$. As the decision maker's representation of the set of events is unknown and a representation is associated with a lottery I consider that, in a simple decision problem, a gamble is not a mixed gamble; consequently it is a pure gamble. The available information is given by a normalized vector $|\varphi_l\rangle$ of $\mathcal{H}$ or equivalently by the density operator $\rho_l = |\varphi_l\rangle \langle\varphi_l|$. As I work in $\mathcal{B}$, the simplest forms for $|\varphi_l\rangle$ and ρ_l are :

$$|\varphi_l\rangle = \begin{pmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_n} \end{pmatrix} \tag{3.7}$$

$$\rho_l = \begin{pmatrix} p_1 & \sqrt{p_1 p_2} & \cdots & \sqrt{p_1 p_n} \\ \sqrt{p_1 p_2} & p_2 & \cdots & \sqrt{p_2 p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{p_1 p_n} & \sqrt{p_2 p_n} & \cdots & p_n \end{pmatrix} \tag{3.8}$$

3.3.3 Representation of preference relations

Let $\succeq$ be a binary relation defined on $\mathcal{D}(\mathcal{H})$ to represent a decision-maker's preference. This is not a true preference relation because agents do not deal explicitly with density operators. As their representations of the set of events is unknown, we work in $\mathcal{D}(\mathcal{H})$ and the relation $\rho_l \succeq \rho_m$ is read as : "the decision maker (weakly) prefers gamble ρ_l to gamble ρ_m" or equivalently : "the decision maker (weakly) prefers lottery l to lottery m with respect to $\mathcal{R}$ unidentified".

Similar to von Neumann and Morgenstern's approach, I define the following axioms on the preference relation.

Axiom D. 1 (Weak Order (WO))**.** $\succeq$ is non trivial, complete and transitive.

Axiom D. 2 (Continuity (C))**.** For any $\rho_l \in \mathcal{D}(\mathcal{H})$, $\{\rho_m, \rho_m \succeq \rho_l\}$ and $\{\rho_m, \rho_l \succeq \rho_m\}$ are closed in the weak operator topology.

Axiom D. 3 (Independence (I))**.** If $\rho_l \succeq \rho_m$ and $0 \leq \lambda \leq 1$ then for any gambles ρ_n,

$$\lambda\rho_l \oplus (1-\lambda)\,\rho_n \succeq \lambda\rho_m \oplus (1-\lambda)\,\rho_n$$

These axioms have the same interpretation as the usual axioms. Independence requires that mixed gambles can be computed only in $\mathcal{B}_{DM}$ or that observer know the decision maker's representation of the set of events. In this situation, the classical mixture is valid.

Theorem 3.1 (Analogous of von Neumann and Morgenstern theorem's)**.** *(WO), (C) and (I) are jointly equivalent to the existence of a functional* $U : \mathcal{D}(\mathcal{H}) \to \mathbb{R}$*, which represents* $\succeq$ *and such that*

$$U(\rho_l) = tr(\rho_l \mathcal{M}_u) \tag{3.9}$$

and where $\mathcal{M}_u \in \mathcal{M}_n(\mathbb{R})$*. Moreover,* $\mathcal{M}_u$ *is unique up to positive linear transformations.*

Corollary 3.2. *If* ρ_l *is a pure gamble, then*

$$u(\rho_l) = \langle \varphi_l | \mathcal{M}_u | \varphi_l \rangle \tag{3.10}$$

In the next section I investigate the meaning of the utility matrix. As a corollary, I show that for a pure gamble equation (3.9) simplifies to (3.10), and u becomes a bilinear form on $\mathcal{H}$ of matrix $\mathcal{M}_u$ in the basis $(|\delta_{\omega_1}\rangle, \ldots, |\delta_{\omega_n}\rangle)$.

3.4 Utility matrix

In the classical case, the utility of the outcomes under certainty can set the preferences of the decision maker under risk and the expected utility is a weighted sum of the utilities under certainty. In my model, I will show that the preferences under risk depend on the utility of the outcomes under certainty and interference utilities between the certain outcomes. Thus, I show that u is an expected utility and that the utility of the outcomes under risk is different from the utility of the outcomes under certainty.

3.4.1 Interpretation of an utility matrix

In the previous section, I presented the set of gambles using the density operator $\mathcal{D}(\mathcal{H})$ instead of the set $\mathcal{L}$ of classical lotteries. In this subsection, I analyze the consequences of this change by considering the case where, during an experiment, it is possible to ask a decision maker about the utility that he associates with a gamble and with given outcomes of a gamble. We recall that for a gamble that, with certainty, gives an outcome s_i, the projection operator is $\rho_{s_i} = |\delta_{\omega_i}\rangle \langle\delta_{\omega_i}|$. As $\mathcal{M}_u$ is a matrix, I note $\mathcal{M}_u = (u_{ij})$. The utility of ρ_{s_i} is given by

$$u(\rho_{s_i}) = tr(\rho_{s_i}\mathcal{M}_u) = u_{ii},\ \forall i \tag{3.11}$$

We can then define the utility of the outcome s_i under certainty by u_{ii}, which means that all the information about the decision maker's preferences under certainty are contained in the diagonal entries of $\mathcal{M}_u$. Let $|\varphi_{ij}\rangle$ be the pure gamble in which the outcomes s_i and s_j are associated respectively with the probabilities p_i and p_j $(p_i + p_j = 1)$. The utility of this gamble is given by

$$u\left(\rho_{\varphi_{ij}}\right) = \langle\varphi_{ij}| \mathcal{M}_u |\varphi_{ij}\rangle = p_i u_{ii} + p_j u_{jj} + \sqrt{p_i p_j}\,(u_{ij} + u_{ji}),\ \forall i, j \tag{3.12}$$

We denote the interference utility between the sure outcomes s_i and s_j $(u_{ij} + u_{ji})$.

In the classical model of expected utility, the utility of outcomes (identified with their respective sure events) defines preferences on $\mathcal{L}$. Formally, an agent weights the utility of each outcome by its associated probability. In my model, the matrix of utilities captures the idea that the decision maker has a different representation of the set of events. From the perspective of the observer, the decision maker is located first in the state of the world where the outcome of the gamble is s_i, based on which the decision maker evaluates the utility of the outcome s_i relative to this state of the world, and then evaluates the utility of the outcome s_j relative to the same state of the world. The decision maker repeats the operation for the state of the world where the outcome of the lottery is s_j. Finally, u_{ij} can be read as " the utility of the outcome s_i if gamble's outcome is s_j ". The diagonal of the matrix $\mathcal{M}_u$ represents the outcome's utilities, u_{ii} is the utility of the outcome s_i if gamble's outcome is certain.

More generally for a pure gamble, utility is given by

$$u(\rho) = \langle\varphi| \mathcal{M}_u |\varphi\rangle = \sum_{i,j=1}^{n} u_{ij}\sqrt{p_i p_j} \tag{3.13}$$

3.4.2 Utility under certainty and risk

In section 3.3, it was convenient to work in $\mathcal{B}$ because $\mathcal{B}_{DM}$ were unknown. However, based on "own rationality", I postulate that a decision maker is in his own probability space $(\Omega_{DM}, \mathcal{F}_{DM}, \mu_{DM})$ and for he a gamble becomes a simple lottery. We can conclude, therefore, that in $\mathcal{B}_{DM}$, the matrix is diagonal. The following proposition shows that u is an expected utility in $\mathcal{B}_{DM}$.

Proposition 3.3. *$u(\rho_l)$ is an expected utility in $\mathcal{B}_{DM}$ and* $u(\rho_l) = \sum_i \mathbb{P}_{s_i}^{DM}(l)\, u_{s_i}^{DM}$.

An important feature of my model is that a decision maker changes his utility under risk. The term $u_{s_i}^{DM}$ which I denote u_i^r characterizes the utility of the outcome s_i under risk. u_i^r is different from u_i, his utility without risk. To illustrate the differences between u_i^r and u_i, I can use a simple example for a 2-dimensional $\mathcal{H}$. In this case the most simple utility matrix is a real symmetric matrix

$$\mathcal{M}_u = \begin{pmatrix} u_{11} & \frac{r}{2} \\ \frac{r}{2} & u_{22} \end{pmatrix} \tag{3.14}$$

where $r = u_{12} + u_{21}$. Diagonalization allows us to find the eigenvalues that are the utility of the outcome s_i under risk

$$\begin{aligned} Det\,(\mathcal{M}_u - \lambda I) = 0 &\Leftrightarrow \lambda^2 - (u_{11} + u_{22})\,\lambda + u_{11}u_{22} - \left(\frac{r}{2}\right)^2 = 0 \\ &\Leftrightarrow \lambda = \frac{u_{11} + u_{22}}{2} \pm \sqrt{\left(\frac{u_{11} - u_{22}}{2}\right)^2 + \left(\frac{r}{2}\right)^2} \end{aligned}$$

By convention, I can assume that $s_2 \succ s_1$. Under risk the preferences are unchanged, and outcome s_2 is always preferred to s_1. Accordingly, I can define the utility of the outcomes under risk

$$u_1^r = \frac{u_{11}+u_{22}}{2} - \sqrt{\left(\frac{u_{11}-u_{22}}{2}\right)^2 + \left(\frac{r}{2}\right)^2} \tag{3.15}$$

$$u_2^r = \frac{u_{11}+u_{22}}{2} + \sqrt{\left(\frac{u_{11}-u_{22}}{2}\right)^2 + \left(\frac{r}{2}\right)^2} \tag{3.16}$$

We can see that $u_1^r < u_1 < u_2 < u_2^r$. Thus, under risk, the utility of the preferred outcome increases by the effect of the presence of the undesired outcome and vice versa. Another interpretation of r would be :

$$u_2^r - u_1^r = 2\sqrt{\left(\frac{u_{11}-u_{22}}{2}\right)^2 + \left(\frac{r}{2}\right)^2} \tag{3.17}$$

We note $\Delta_u^r = u_2^r - u_1^r$ and $\Delta_u^c = u_{22} - u_{11}$, the previous expression then becomes

$$r^2 = (\Delta_u^r)^2 - (\Delta_u^c)^2 \tag{3.18}$$

Thus, in the case of two outcomes, r is associated with the range of utility under risk and the range of utility under certainty. The greater the number of utilities that are transformed under risk, the higher is r.

3.5 Applications

3.5.1 Prospect theory

Our model draws on the design proposed in Kahneman and Tversky's prospect theory (1979). Prospect theory divides the decision process into two stages, editing and evaluation. Editing involves the ordering of decision outcomes. Agents set a reference point and then consider lesser outcomes as losses and greater ones as gains. In the evaluation stage, agents weight outcomes and their respective probabilities, and choose the one with the highest utility. The functional is given by

$$\mathrm{V}(l) = \sum_i w\left(\mathbb{P}_{s_i}(l)\right) v\left(s_i\right) \tag{3.19}$$

The function w is a probability weighting function for the tendency for people over-react to low probability and under-react to higher probabilities.

In my framework the function $\mathcal{R}$, which is the decision maker's representation of the set of events, is a black box which characterizes the agent. The decision maker considers a different probability space $(\Omega_{DM}, \mathcal{F}_{DM}, \mu_{DM})$. As in the previous subsection I study the case of a 2-dimensional Hilbert space with a utility matrix given by equation (3.14).

Firstly, equations (3.15) and (3.16) show that outcomes s_1 and s_2 are weighted from the reference point $\left(\frac{u_{11}+u_{22}}{2}\right)$.

Secondly, the probability law of the lottery from the point of view of the agent, is modified. Diagonalization of the utility matrix allows us to construct a unitary matrix which is the matrix base change between $\mathcal{B}$ and $\mathcal{B}_{DM}$. In $\mathcal{B}$, φ decomposes into :

$$|\varphi\rangle = \sqrt{p}\,|\delta_{\omega_1}\rangle + \sqrt{1-p}\,|\delta_{\omega_2}\rangle \tag{3.20}$$

Without loss of generality, I use a rotation matrix U :

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.21}$$

Thus, I can express the probability weighting function for the outcomes s_1 and s_2 :

$$w_{s_1}(p) = \left(\sqrt{p}\cos\theta - \sqrt{1-p}\sin\theta\right)^2 \tag{3.22}$$

$$w_{s_2}(1-p) = \left(\sqrt{p}\sin\theta + \sqrt{1-p}\cos\theta\right)^2 \tag{3.23}$$

Figure 3.1 shows $w_{s_1}(p)$ and $w_{s_2}(p)$ for different values of θ.

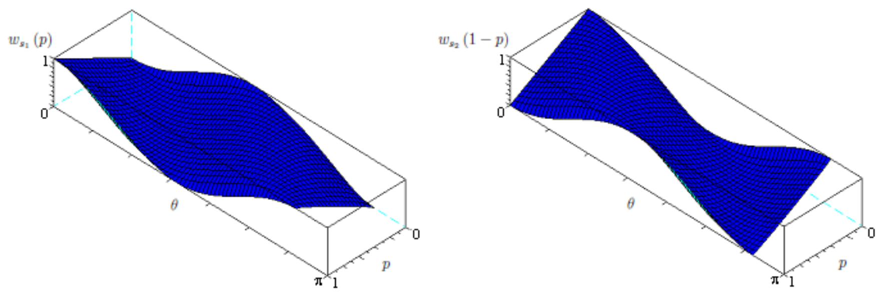

FIGURE 3.1: Probability weighting function for 2-outcome lotteries

3.5.2 The Allais paradox

The so-called Allais paradox refers to experimental results that frequently show violation independence axiom in expected utility theory. It arises from a comparison of the choices made by individuals in two successive experiments, each consisting of a choice between two lotteries. Consider this Allais type paradox. First, the choice between :

A: A chance of winning 10000 euros with certainty

B: A chance of winning 15000 euros with probability 0.9

Next, the choice between :

C: A chance of winning 10000 euros with probability 0.1

D: A chance of winning 15000 euros with probability 0.09

Many subjects report that $A \succ B$ for the first choice and $D \succ C$ for the second. Let E be the gamble "chance of winning 0 euros with certainty", we can verify that $C = 0.1A \oplus 0.9E$ and $D = 0.1B \oplus 0.9E$. By independence axiom $A \succ B$ implies $C \succ D$, hence the contradiction. The mixture of lotteries is not neutral and causes interactions between utilities and probabilities, which change the preferences.

In the Allais paradox, we consider all lotteries are pure gambles. The set of outcomes is given by $S = (0, 10000, 15000)$ and is associated with $\mathcal{H}$ a 3-dimensional Hilbert space with orthonormal basis $(|\delta_0\rangle, |\delta_{10}\rangle, |\delta_{15}\rangle)$. Then our knowledge of the decision maker's representation of the set of events can be described by a probability law on the D-algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$. Accordingly, the normalized vectors associated with gambles A, B, C and D are respectively $|\varphi_A\rangle$, $|\varphi_B\rangle$, $|\varphi_C\rangle$ and $|\varphi_D\rangle$:

$$|\varphi_A\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} |\varphi_B\rangle = \begin{pmatrix} \sqrt{0.1} \\ 0 \\ \sqrt{0.9} \end{pmatrix} |\varphi_C\rangle = \begin{pmatrix} \sqrt{0.9} \\ \sqrt{0.1} \\ 0 \end{pmatrix} |\varphi_D\rangle = \begin{pmatrix} \sqrt{0.91} \\ 0 \\ \sqrt{0.09} \end{pmatrix} |\varphi_E\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Gamble C as a pure gamble is different from convex combination of pure gambles A and E and similarly for D.

$$\rho_C \neq 0.1\rho_A \oplus 0.9\rho_E$$
$$\rho_D \neq 0.1\rho_B \oplus 0.9\rho_E$$

The self-adjoint utility matrix for the experiments is

$$\mathcal{M}_u = \begin{pmatrix} u_{0,0} & \frac{r_{0,10}}{2} & \frac{r_{0,15}}{2} \\ \frac{r_{0,10}}{2} & u_{10,10} & \frac{r_{10,15}}{2} \\ \frac{r_{0,15}}{2} & \frac{r_{10,15}}{2} & u_{15,15} \end{pmatrix}$$

and the utility of pure gambles A, B, C and D is given by

$$u(\rho_A) = u_{10,10}$$
$$u(\rho_B) = 0.9u_{15,15} + 0.3r_{0,15}$$
$$u(\rho_C) = 0.1u_{10,10} + 0.3r_{0,10}$$
$$u(\rho_D) = 0.09u_{15,15} + 0.3\sqrt{0.91}r_{0,15}$$

Since the agent represents the experiences in another probability space for risky situations, interferences appear. Here, all response profiles are allowed for the Allais experiment. The gamble A is certain , the gamble B is risky and the interference $r_{0,15}$ can increase or decrease the value of the gamble compared to the value of his expected utility. Ditto for gamble C and the interference $r_{0,10}$. If I assume that the agent understands the convex combination that leads to gamble D, we can see that there is an additional interference :

$$u(\rho_D) - u(0.1\rho_B \oplus 0.9\rho_E) = 0.3\left(\sqrt{0.91} - 0.1\right) r_{0,15}$$

We assume $A \succ B$, the interference $r_{0,15}$ is negative or positive and small compared to $u_{15,15}$ then the interference $r_{0,10}$ can reverse the decision maker's preferences.

3.5.3 Marschak-Machina triangle

Figure 3.2 depicts the Marschak-Machina triangle for different choices of matrix $\mathcal{M}_u$ in 3-outcome lotteries. The triangle (a) characterizes the von Neumann and Morgenstern expected utility. This case corresponds to the special case of a diagonal matrix $\mathcal{M}_u$. Other triangles show that it is possible to construct indifference curves which are concave (b), fanning out (c) and fanning-in (d).

3.6 Discussion

Objectively, I have shown that the non-commutative or "quantum probability" framework can be used to model the fact that an individual have a particularly subjective representation of the set of events. However, it is a restrictive work as a representation of the set of events can be other structure than a probability space. I have, for prospect theory, exhibited an endogenous reference point and a corresponding probability weighting function which is a probability measure. I would like emphasize the fact that prospect theory is not evident for many outcomes (as noted by the authors) and that consequently, there is, perhaps, many reference points in the editing process. Other features of my framework is that, naturally, density operator are not commutative for the appropriate definition of

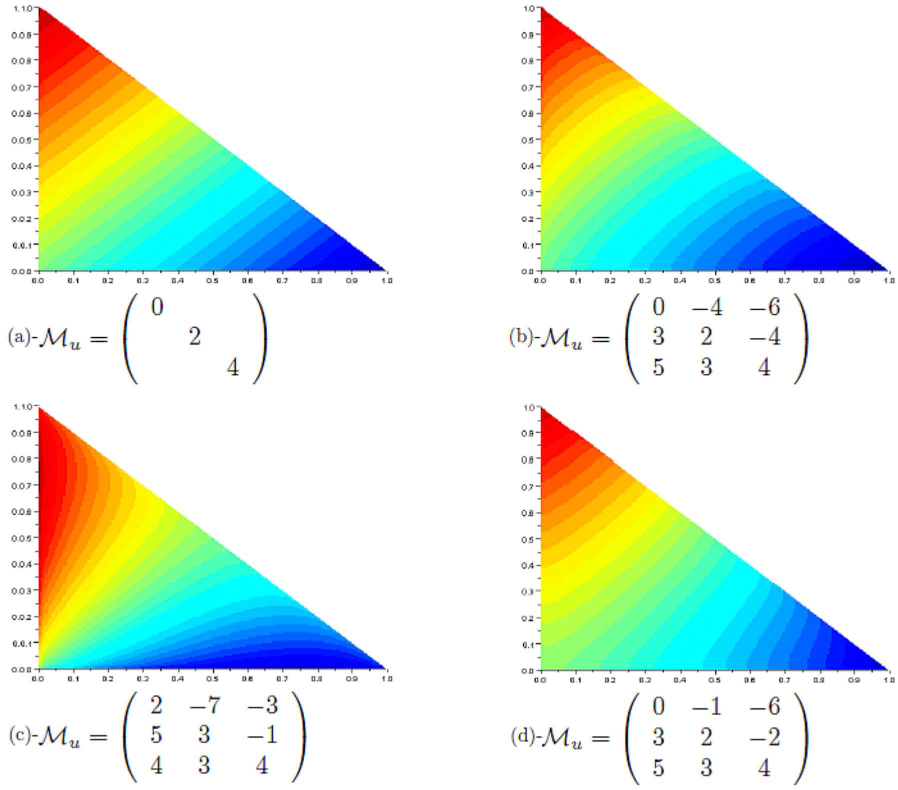

FIGURE 3.2: Examples of indifference curves on the Marschak-Machina triangle for different matrices.

multiplication (unlike classical probability). So, *simple* framing can be accommodated if they arise from products of probability distributions. Considering that a representation of the set of events is valid for only one decision problem, then my approach can accommodated regret theory (intransitive preferences). However, there are *ad hoc* assumptions. Generally, all "quantum" models are poor in the sense that, in comparison with physics, there is no true classical theory of how preferences evolve and how preferences are modified in contextual interaction. And, without discussing the merits of some approaches, a quantum generalization can not be considered without first having a classical, necessarily dynamic, theory. This formalism under-employ available knowledge about the limitations of cognitive processes or contextual interactions. In addition, flexibility in model specification has risks for the use of quantum probability

3.7 Proofs

Proof of theorem 3.1. $\mathcal{D}(\mathcal{H})$, with respect to convex combination, is a mixture space and it satisfies the usual axiom for an affine representation by Theorem 8 in Herstein and Milnor (1953). Since $\mathcal{H}$ is a finite-dimensional vector space, u is a linear form of $\mathcal{M}_n(\mathbb{C}) \supset \mathcal{D}(\mathcal{H})$ to $\mathbb{R}$. Let (E_{ij}) be the canonical basis of $\mathcal{M}_n(\mathbb{C})$

$$\begin{aligned}
\forall \rho_l \in \mathcal{D}(\mathcal{H}), u(\rho_l) &= \sum_{ij} \rho_{ij} \cdot u(E_{ij}) \\
&= \sum_{ij} \rho_{ij} \cdot u_{ji} && \text{, where } u_{ji} = u(E_{ij}) \\
&= \sum_{i} \sum_{j} \rho_{ij} \cdot u_{ji} \\
&= \sum_{i} (\rho_l \mathcal{M}_u)_{ii} && \text{, where } \mathcal{M}_u = (u_{ij}) \\
&= tr(\rho_l \mathcal{M}_u)
\end{aligned}$$

■

Proof of theorem 3.2. If ρ_l is a pure gamble, then:

$$\begin{aligned}
u(\rho_l) &= tr(\rho_l \mathcal{M}_u) \\
&= \sum_i \langle \delta_{s_i} | \rho_l \mathcal{M}_u | \delta_{s_i} \rangle \\
&= \sum_i \langle \delta_{s_i} | \varphi_l \rangle \langle \varphi_l | \mathcal{M}_u | \delta_{s_i} \rangle \\
&= \sum_i \langle \varphi_l | \mathcal{M}_u | \delta_{s_i} \rangle \langle \delta_{s_i} | \varphi_l \rangle \\
&= \langle \varphi_l | \mathcal{M}_u \left(\sum_i (|\delta_{s_i}\rangle \langle \delta_{s_i}|) \right) |\varphi_l \rangle \\
&= \langle \varphi_l | \mathcal{M}_u | \varphi_l \rangle .
\end{aligned}$$

■

Proof of proposition 3.3. The proof of proposition 3.3 is a straightforward application of the spectral theorem. According to proposition 1, for a given preference relation, there is a self-adjoint matrix in $\mathcal{M}_n(\mathbb{C})$ which represents u. According to the spectral theorem $\mathcal{M}_u$ can be diagonalized on an orthonormal basis $(|\emptyset_1\rangle, \ldots, |\emptyset_n\rangle)$, *i.e.* there is an unitary matrix U such that

$$\mathcal{M}_u = \sum_i \lambda_i |\emptyset_i\rangle \langle\emptyset_i| \text{ with } \lambda_i = (U^*\mathcal{M}_u U)_{ii} \text{ and } \lambda_i \in \mathbb{R}$$

We note $\lambda_i = u_{s_i}^{DM}$. The orthonormal basis $(|\emptyset_1\rangle, \ldots, |\emptyset_n\rangle)$ is exactly the basis of the decision maker. For all i, we set $|\emptyset_i\rangle = |\pi_{\omega_i}\rangle$.

$$\begin{aligned}
u(\rho_l) &= \langle\varphi_l|\mathcal{M}_u|\varphi_l\rangle \\
&= \left\langle \varphi_l| \left(\sum_i u_{s_i}^{DM} |\pi_{\omega_i}\rangle\langle\pi_{\omega_i}|\right)|\varphi_l \right\rangle \\
&= \sum_i u_{s_i}^{DM} \langle\varphi_l|\pi_{\omega_i}\rangle\langle\pi_{\omega_i}|\varphi_l\rangle \\
&= \sum_i u_{s_i}^{DM} |\langle\varphi_l|\pi_{\omega_i}\rangle|^2 \\
&= \sum_i \mathbb{P}_{s_i}^{DM}(l)\, u_{s_i}^{DM} \quad \text{by equation (3.5)}
\end{aligned}$$

■

3.8 Conclusion

This chapter investigated the formulation of von Neumann and Morgenstern's theory of decision-making in terms of its algebraic structure, $*$-algebra. I showed that classical theory corresponds to the commutative case and that a non-commutative setting leads to the introduction of fruitful notions for understanding decision theory. My hypothesis is that irrationality can be understood as wrong observation of the probability space in which individuals perceive gambles and therefore that individuals are rational in their representations of the sets of events. I found that from the perspective of the observer, the utilities of outcomes under certainty are not sufficient to describe the behaviour of a decision maker under risk and the observer must consider interference utilities among sure outcomes. I saw that the utility of an outcome differs under risk and under certainty and that some paradoxes and inconsistencies in the classical expected utility theory can be better understood in my framework.

3.9 Appendix

3.9.1 A - Quantum Logic

The quantum logic, often called the logic of subspaces, arises from the set of Hilbert subspaces of the complex Hilbert space $\mathcal{H}$, describing the quantum system of interest, is as follows. Each subspace h is identified with the operator P_h that projects onto the subspace h. The lattice $\mathcal{P}(\mathcal{H})$ of closed linear subspaces of a Hilbert space $\mathcal{H}$ is seen to be equivalent to the lattice of projection operators on $\mathcal{H}$. One can define the two operations $(\wedge)$ (meet) and $(\vee)$ (join) acting pairwise on any two projectors P_1 and P_2 by $P_1 \wedge P_2 = P_1 P_2$, $P_1 \vee P_2 = P_1 + P_2 - P_1 P_2$, and identify the zero as the projector $\mathbb{O}$ onto the zero vector 0 and the identity as the projector $\mathbb{I}$ onto all of $\mathcal{H}$; $\vee$ corresponds to the linear span, and $\wedge$ to the intersection. The rays of $\mathcal{H}$ are considered to be the atomic propositions of $\mathcal{P}(\mathcal{H})$. The complement of the projector P is the operator $P^{\perp} = \mathbb{I} - P$ such that $P \wedge P^{\perp} = \mathbb{O}$ and $P \vee P^{\perp} = \mathbb{I}$. This complement is then unique.

Definition 4. $\mathcal{P}(\mathcal{H})$ is modular if it satisfies the modularity condition :

$$P_1 \leq P_2 \Rightarrow \forall P_3, \quad P_1 \vee (P_2 \wedge P_3) = (P_1 \vee P_2) \wedge P_3$$

Definition 5. $\mathcal{P}(\mathcal{H})$ is orthomodular if it satisfies the orthomodularity condition :

$$P_1 \leq P_2 \Rightarrow P_2 = P_1 \vee \left(P_1^{\perp} \wedge P_2\right)$$

3.9.2 B - D-algebras

B1 - $*$-algebras

We introduce the term D-algebra (D stands for Decision) to reformulate the classical framework of a decision problem. In non-commutative probability theory, a real random variable is represented by a linear self-adjoint operator on a Hilbert space. Accordingly, we need to replace the conventional notion of a classical finite probability space by the notion of a non-commutative finite probability space. The strategy is to replace $l^{\infty}(\Omega, \mathcal{F}, \mu)$, which is a commutative $*$-algebra, by a non-commutative $*$-algebra.

By definition, an algebra is a linear space $\mathcal{A} \neq \{0\}$ over $\mathbb{K}$=$\mathbb{C}$ or $\mathbb{R}$, that is equipped with a multiplication $(A, B) \to AB$ from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{A}$ that is associative, bilinear, and has an identity I. An algebra $\mathcal{A}$ is commutative if the multiplication is commutative.

By definition, a $*$-algebra (pronounce: star-algebra) is an algebra $\mathcal{A}$ that is equipped with an involution on $\mathcal{A}$ which is a map $A \to A^*$ from $\mathcal{A}$ into $\mathcal{A}$ that has the following properties

$$
\begin{array}{rlr}
(i) & (A^*)^* = A & A \in \mathcal{A} \\
(ii) & (aA)^* = a^* A^* & A \in \mathcal{A},\ a \in \mathbb{C} \\
(iii) & (AB)^* = B^* A^* & A, B \in \mathcal{A}
\end{array}
$$

An involution is positive if

$$
\begin{array}{rlr}
(iv) & AA^* = 0 \Rightarrow A = 0 & A \in \mathcal{A}
\end{array}
$$

Definition 6. $\mathcal{A}$ is a D-algebra if $\mathcal{A}$ is a finite-dimensional $*$-algebra over the complex numbers and the involution is positive.

B2 - Density Operators

Definition 7. A probability law on a D-algebra is a linear map $\mu : \mathcal{A} \to \mathbb{R}$ that is positive $\mu(A^*A) \geq 0$, $\forall A \in \mathcal{A}$ and normalized $\mu(I) = 1$.

To characterize a probability law in $\mathcal{A}$, we need the concept of density operators. For a D-algebra $\mathcal{A}$, we consider the usual trace tr. Let ρ be a density operator, *i.e.* a self-adjoint positive operator of trace 1[6]. We can then define a probability law μ by

$$\forall A \in \mathcal{A}, \quad \mu(A) = tr(\rho A)$$

Conversely, every probability on $\mathcal{A}$ arises in this way and ρ is uniquely determined by μ. Indeed, the map $(A, B) \mapsto tr(A^*B)$ defines an inner product on $\mathcal{A}$. A probability law μ on $\mathcal{A}$ is a linear form on $\mathcal{A}$, and there is a unique ρ in $\mathcal{A}$ such that $\mu(A) = tr(\rho^*A)$ for every A in $\mathcal{A}$. Hence, the conditions imposed on μ imply that ρ is a self-adjoint positive operator of trace 1, its diagonal entries are in fact probabilities.

We call $(\mathcal{A}, \mu)$ a (finite dimensional) non-commutative probability space. The above definitions refer to some comments. Firstly, this framework is abstract but it is related to Hilbert formalism. If $\mathcal{H}$ is a finite-dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $\mathcal{H}$, equipped with operator multiplication and adjugation. Then, obviously, $\mathcal{L}(\mathcal{H})$ is a D-algebra. Conversly[7], every D-algebra has a faithful representation, *i.e.* there is an $*$-algebra isomorphism between $\mathcal{A}$ and $\mathcal{L}(\mathcal{H})$ for an appropriate $\mathcal{H}$, finite-dimensional Hilbert space. Secondly, a real random variable is a self-adjoint element of $\mathcal{A}$, $(A^* = A)$. It is not surprising therefore that a commutative probability space is equivalent to a classical one. A formal statement of this assertion is provided by the theorem below.

Theorem 3.4 (Gel'fand theoremDoran (1994))**.** *Let $(\mathcal{A}, \mu)$ be a finite-dimensional commutative probability space. Then there exists a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a finite set, such as $\mathcal{A}$ is isomorphic to $l^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and*

$$\mu(A) = \int A d\mathbb{P} \quad \forall A \in \mathcal{A}$$

[6] $\rho^* = \rho$, $\rho \geq 0$ and $tr(\rho) = 1$

[7] For a proof, we refer the reader to Goodman et al.Goodman et al. (1989).

Proof. As $\mathcal{A}$ is finite-dimensional, we can without loss of generality suppose that $\mathcal{A}$ is a commutative $*$-algebra of operators of a n-dimensional real Hilbert space. As all the elements of $\mathcal{A}$ commute, we can find[8] a unitary operator U such that U^*AU is a diagonal operator for every $A \in \mathcal{A}$. Let $\Omega = \{\omega_1, \dots, \omega_n\}$. Define the map $\iota(A)$ for every $A \in \mathcal{A}$ by

$$\begin{aligned} \iota(A) : \Omega &\to \mathbb{R} \\ i &\mapsto \iota(A)(i) = (U^*AU)_{ii} \end{aligned}$$

Next, define the tribe $\mathcal{F} = \sigma\{\iota(A) | A \in \mathcal{A}\}$ and $\mathbb{P}(\mathcal{E}) = \mu(\iota^{-1}(\chi_{\mathcal{E}}))$ for every $\mathcal{E} \in \mathcal{F}$. This construction give a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a isomorphism ι of $*$-algebra between $\mathcal{A}$ and $l^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Moreover $\mu(A) = \int A d\mathbb{P}$, $\forall A \in \mathcal{A}$. ■

Non-commutative probability theory is more general than classical theory. The classical probability model does not intrinsically contain an entity that corresponds to non-commutative probability law. A non-commutative probability space describes incomplete knowledge about a system in the physical reality. An observation of a given system corresponds to a choice of a commutative $*$-subalgebra $\mathcal{C} \subset \mathcal{A}$ and $\mathcal{C}$ is thus exactly equivalent to a classical probability space. A non-commutative probability space can be seen as representing the set of all possible classical probability spaces $(\Omega_s, \mathcal{F}_s, \mu_s)$ for all experimental settings "s". A non-commutative probability space $\mathcal{A}$ is a collection of many incompatible classical probability models, each of which coincides with a commutative $*$-subalgebra of $\mathcal{A}$. A non-commutative probability law represents the catalogue of all expectations for all possible observations. Obviously an observation needs to give the expectations of an observable with the correct probability and therefore, together with the observation made, to define the probability space uniquely.

3.9.3 C - Quantum mechanics

Let $\mathcal{A}$ be a D-algebra.

[8]The proof is an elementary exercise in linear algebra.

C1 - Observables

In quantum mechanics, self-adjoint operators on $\mathcal{A}$ are called observables. They correspond to real-valued physical quantities and may be regarded as the equivalent of classical random variables by spectral decomposition.

$$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$$

The set $\{P_\lambda : \lambda \in \sigma(A)\}$ is interpreted as an ideal measurement of the observable A. If ρ is a probability law then the expected value of A is

$$\rho(A) = \sum_{\lambda \in \sigma(A)} \lambda \rho(P_\lambda)$$

Definition 8. An ideal measurement is a partition of the identity $\{P_1, \dots, P_n\}$

If an ideal measurement $\{P_1, \dots, P_n\}$ is performed on the system, then our knowledge about the system changes. The new knowledge is described in a new probability law $\tilde{\rho}(A) := \sum \rho(P_i A P_i)$

C2 - States

In quantum mechanics a probability law on $\mathcal{A} = \mathcal{L}(\mathcal{H})$ is called a state.

Definition 9. A pure state is a probability ρ that is not a nontrivial convex combination of other states, i.e., it is not possible to write $\rho = p\rho_1 + (1-p)\rho_2$ with $0 < p < 1$ and $\rho_1 = \rho_2$. A probability that is not a pure state is called a mixed state.

Definition 10. A ket represents a complex column vector. $|\psi\rangle$ denotes the ket vector ψ. The complex conjugate and transpose (*i.e.* the adjoint) of a ket is a *bra*. It represents a complex row vector. $\langle\psi|$ denotes the bra vector ψ.

Definition 11. The inner product of two kets $|\psi\rangle$ and $|\phi\rangle$ returning a scalar is defined by $\langle\psi|\phi\rangle$.

Definition 12. The outer product of two kets $|\psi\rangle$ and $|\phi\rangle$ returning a linear operator is defined by $|\psi\rangle\langle\phi|$.

The following properties are equivalent and characterize pure states.

- There is a normalized vector $|\psi\rangle \in \mathcal{H}$ such that $\rho = |\psi\rangle \langle\psi|$.
- $\left(\rho^2 = \rho\right)$.

C3 - Born rule

Let A be an observable and ψ a pure state. Let $\{|\alpha_i\rangle\}$ the eigenvectors of the observable associated with the eigenvalues $\{\lambda_i\}$, the Born rule states that if A is measured in state ψ, then :

- $\{|\alpha_i\rangle \langle\alpha_i|\}$ is an ideal measurement.
- $|\psi\rangle = \sum_i c_i |\alpha_i\rangle$ where $|c_i|^2 = \mathbb{P}(\lambda = \lambda_i)$.

Clearly, the measured result will be one of the eigenvalues of A, and the probability of measuring a given eigenvalue λ_i will equal to $|c_i|^2$.

Conclusion

This thesis was devoted to define utility functions in different contexts of decision making under risk. This issue affects the validity of a large number of works within many economic fields : behaviour under risk, and specifically investments, insurance, the next move of your opponent in a conflict, and so on. Since the 90s, the emergence in economy of works inspired by sociology or physics or vice versa challenge the traditional foundations of decision making analysis. This dynamic fits experimental failures of classical decision theory[9] and game theory, that is, incapacity to play the sub-game perfect equilibrium[10]. Coordination failures observed in game theory are mainly attributed to the fact that individuals do not take into account other agents' absolute or relative payoffs or in some cases well-beings. Accordingly, even if we ignore interdependent preferences, the remaining problem stems from the extension of individual decision theory within game theory, that is, in context of strategic interactions.

In the referent literature it is now widespread to extend individual rationality to cases of bounded rationality and interdependent preferences. That is why the axiomatic foundations of these evolutions are determinants in order to structure the different types of analytical improvements of individual decision under risk. We can refer for the previously mentioned improvements among others, to Segal (1989) or Chateuneuf and Wakker (1999) for the theory of individual decision making under risk, Karni and Safra (2002) or Maccheroni *et al.* (2012) in the theory of individual decision preferences incorporating equity or interdependence, Rabin (1993), Fehr and Schmidt (1999) or Segal and Sobel (2007) in game theory. Brock and Durlauf (2001a) and Blume and Durlauf (2001) for a theoretical and empirical approach.

[9] See Lichtenstein and Slovic (1971) for reversals preferences and Tversky and Kahneman (1992) for framing effects.

[10] See Rosenthal (1981) for centipede game, Guth et al (1982) for the ultimatum game and Forsythe et al. (1994) for the dictator game.

Blume, Brock and Durlauf's contribution gives us both an empirical framework which highlights peer effects in network, and a theoretical framework allowing to integrate into utility functionals interdependent preferences, that is, the dynamics of individual decisions within networks. This approach is widely used in econophysics that is why my thesis focus on the microeconomic foundations through interdependent preferences.

Contributions

I propose in this thesis an axiomatic foundation of additively separable utility under risk, that is, expected utility with a kernel or Carathéodory functions. I derive then axioms for an entropy modified expected utility. This last result is incorporated with interdependent preferences under risk to fit the Blume, Brock and Durlauf's model. Finally, I introduce quantum probability to taking into account subjective events.

Additively separable utility under risk

The first chapter of this thesis provides a consistent axiomatic foundation of additively separable utility under risk. This problem is equivalent to find an additive representation of preferences on simplices which is subsets of Cartesian products. My approach brings close our object of study and the theory of linear utility developed by Aumann (1962), that is, classical additive independence axioms are translation invariance axiom restricted to alternatives with disjoint supports. Limited experimental data and their conflicting aspects does not allow us to make a judgement about the credibility of such functionals even if, in full generality, it is low given typical failures of individuals behaviour as framing effects or intransitive preferences. However, These representations seem to have an analytical substantial advantage over most of the earlier proposals. The most straightforward formal application can take place into game theory as optimization problem with additively separable functional is simple. The obtained result make possible the derivation of further results for entropy-modified expected utility extensively used in economic studies. In this perspective a great care must be imposed to studies that suppose an entropic term. Entropy-modified expected utility reports a very limited bounded rationality as the cognitive impairment is thin. Concretely, this kind of utility functionals verifies the independence axiom for convex combinations of lotteries restricted to lotteries with disjoint supports.

Interdependent preferences under risk

The second chapter provides an axiomatic foundation for the Blume, Brock and Durlauf's model. I have studied individual deterministic preference relations under risk which are possibly other-regarding. My primary aim has been to provide, following Maccheroni *et al.* (2012), a formal background to other-regarding expected utility representations already used in various applications which covers the case where the other agents make risky choices and the case where the risk is a belief about the other agents choices or both. In addition, I introduced, following Candeal *et al.* (2004), a preference relation over a disjoint union of spaces to account for the selection among groups of individuals that operates the decision maker. This allows us to explain the social anchoring of the decision maker. The addition of an entropic term is partially processed to fit the Blume, Brock and Durlauf's model. Some difficulties arise, principally are due to the homogeneity condition between the expectation of the private utility term and the expectation of the social utility term.

Quantum probabilities

Objectively, I have shown that the non-commutative or "quantum probability" framework can be used to model the fact that an individual have a particularly subjective representation of the set of events in the von Neumann and Morgenstern's framework. My hypothesis is that irrationality can be understood as wrong observation of the probability space in which individuals perceive gambles and therefore that individuals are rational in their representations of the sets of events. I found that from the perspective of the observer, the utilities of outcomes under certainty are not sufficient to describe the behaviour of a decision maker under risk and the observer must consider interference utilities among sure outcomes.

However, my hypothesis is an *ad hoc* assumptions that can be discussed as in all "quantum" models. These models are poor in the sense that, in comparison with physics, there is no true classical theory of how preferences evolve and how preferences are modified in contextual interaction. And, without discussing the merits of some approaches, a quantum generalization can not be considered without first having a classical, necessarily dynamic, theory. This formalism under-employ available knowledge about the limitations of cognitive processes or contextual interactions. In addition, flexibility in model specification has risks for the use of quantum probability

Prospects

Two axes Research seem interesting to continue, based on the results of this thesis.

The first concerns the extension of the theoretical results of the first chapter : the space of lotteries, the simplex, is a very special case of subset of Cartesian products. It seems that an orthogonal additivity utility theory can be build in the continuity of the linear utility theory. In this way, some results can be obtained for arbitrary subset of Cartesian products. Moreover the derivation of variational preferences highlighted the possibility of an entropic term similar to Tsallis entropy with expected utility whose probabilities are weighted. These kinf of functionals could provide more general results compared to simple variational preferences. In game theory for example, or in more applied fields. Finally some contradictory experimental results about additively separable utility should encourage us to study this issue more specifically. At least, it is necessary to compare their explanatory power with the rank-dependent expected utility.

The second axis concerns the theoretical results of the second chapter : a direct application could be undertaken in game theory in order to determine changes in balances and levels of social well-being. Indeed, the proposed foundations are flexible enough to generate positive or negative externalities. Taking into account the Tsallis entropy could give new results for equilibrium.

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Introduction (en Français)

Motivations

Dans certaines situations, les individus doivent faire des choix dans l'incertitude. L'analyse de la décision (individuelle) s'est appliquée depuis l'après guerre à mieux comprendre les modes de décision individuels dans de tels contextes d'incertitude. Bien que sophistiquée, la théorie de la décision en incertitude est toujours sujette à un grand nombre de critiques. L'observation expérimentale établit en particulier que les choix individuels n'en confirment pas les prédictions théoriques. Dans l'expérience, les individus ne respectent pas, en général, les hypothèses faites sur leurs préférences pour fonder les prédictions essentielles de la théorie traditionnelle de la décision, même dans des situations relativement simples. Les psychologues montrent, dans des situations plus complexes, que les choix individuels sont sensibles à la description des options, à leur contextualisation et à la méthode d'élicitation. Ces aspects de la prise de décision sont largement renseignés dans la littérature et des réponses théoriques ont été apportées par des champs disciplinaires différents pour expliquer ces échecs et les dépasser.

Depuis une décennie, on en appelle aussi à d'autres champs disciplinaires tels que la sociologie ou la physique pour traiter des fondamentaux de l'analyse décisionnelle. Les relations de groupes, les perceptions d'appartenance, la référence au social, ont un impact important sur la décision individuelle. D'un point de vue général, les individus sont en effet influencés par l'ensemble des facteurs de leur environnement, aussi bien dans le processus de décision que dans leurs interactions avec d'autres individus. La sociologie constitue donc un corps de connaissances mobilisable en matière d'analyse de la décision. Depuis quelques années, on s'aperçoit aussi que les sciences physiques, autre discipline sollicitée dans ce travail, peuvent apporter également à la compréhension des décisions en incertitude. Par ses méthodes ou ses concepts propres relatifs à l'incertitude, cette discipline participe au renouvellement de l'analyse des attitudes face à l'aléa, qu'il s'agisse de contribuer à la modélisation des interactions sociales ou à l'analyse de strictes décisions individuelles.

Cette thèse se compose de trois chapitres constituant des contributions distinctes mais reliées au même centre d'intérêt, la théorie de la décision dans le risque. Le premier chapitre traite de préférences additivement séparable, en situation de risque, par rapport aux probabilités au sens de Debreu (1959). La théorie des préférences additivement séparable est rapprochée de la théorie linéaire de l'utilité. Ce chapitre présente une forme faible de la séparabilité des évènements pour des préférences sur des loteries objectives. Il en est déduit une axiomatisation simple de préférences variationnelles représentées par une fonctionnelle se décomposant en un terme d'espérance d'utilité et un terme entropique. Cette dernière représentation, déterministe, aboutit, dans certaines conditions, à un modèle de choix discret de type logit. Le second chapitre consiste en une fondation axiomatique de préférences interdépendantes en présence d'interactions sociales, sur la base du modèle initialement élaboré par Blume, Brock et Durlauf. Un modèle de choix discret de type logit en étant un ingrédient essentiel, les résultats du premier chapitre contribuent aussi à cette construction. Le troisième et dernier chapitre pose la question de l'apport du modèle probabiliste de la physique quantique à la théorie de la décision et de son application pour prendre en compte la perception subjective des évènements par les individus.

Les critiques adressées à la théorie de la décision ont remis en question les formes de rationalité usuelles de la microéconomie. Chaque chapitre se concentre donc sur un aspect de cette remise en cause, à partir des développements de l'économie comportementale, de l'économie cognitive, et de l'éconophysique, respectivement. L'objet de cette introduction est d'expliciter la cohérence globale de ces trois chapitres.

La théorie de la décision dans le risque

La théorie de la décision a une longue histoire depuis l'émergence[11] des probabilités. Le premier critère de choix, naturel, à avoir été postulé est celui de l'espérance

[11] Le lecteur est renvoyé à Hacking (1975).

d'une variable aléatoire. Au $18^{ème}$ siècle, le paradoxe de Saint-Petersbourg incite Bernoulli (1738-1954) à postuler le critère de l'utilité espérée[12]. Sa popularité augmente après que von Neumann et Morgenstern (1944) montrent que l'on peut déduire ce critère d'un ensemble d'axiomes élémentaires (les exposés modernes font usuellement référence à la présentation de Marschak (1950), Herstein et Milnor (1953), Luce et Raiffa (1957), Jensen (1967) ou Fishburn (1970) pour une relation de préférence d'un individu sur l'ensemble des Lois de probabilités données de manière exogène). Cette formulation est facile d'utilisation. Elle permet de définir l'attitude face au risque et elle est applicable à de nombreux champs théoriques, notamment ceux qui recourent à la théorie des jeux non-coopératifs. L'utilité espérée repose sur l'axiome d'indépendance[13] et présente l'avantage de disposer d'un fort potentiel normatif. Cependant, de nombreux résultats expérimentaux ont montré que ce critère de décision était contestable. Le plus populaire est certainement le paradoxe d'Allais (1953) qui conduit à la définition de deux phénomènes plus généraux, l'effet de rapport commun et l'effet de conséquence commune. Ces deux phénomènes ont été reproduits par Kahneman et Tversky (1979) (problèmes 1,2 et 3,4 respectivement). MacCrimmon et Larsson (1979) fournissent une étude détaillée des paradoxes de l'axiome d'indépendance, dans un contexte de risque ou en incertitude radicale. Le faible pouvoir descriptif de l'utilité espérée a conduit à des généralisations que nous classerons en trois catégories[14] non exhaustives et non mutuellement exclusives.

La première classe de généralisations repose sur l'affaiblissement systématique de l'axiome d'indépendance. La contribution du premier chapitre s'inscrit dans ce cadre. La sophistication de l'utilité espérée par des théories avec un cadre axiomatique clair, ciblant l'affaiblissement de l'axiome d'indépendance, permet de mieux comprendre les aspects normatifs et descriptifs de ces théories. Par exemple, l'utilité espérée pondérée[15] proposée par Chew et MacCrimmon (1979) où l'axiome d'indépendance est vérifié pour des loteries de la même classe d'équivalence et

[12] Ce n'est pas la valeur, naturelle, de la variable aléatoire qui doit rentrer dans le calcul d'espérance, mais la valeur, morale, que l'individu attribue à cette variable qui doit rentrer dans le calcul de l'espérance, donnant ce que l'on appel aujourd'hui l'utilité espérée.

[13] Étonnement absent chez von Neumann et Morgenstern, voir Fishburn et Wakker (1995) pour une perspective historique sur la formulation de cet axiome.

[14] Je ne fais pas mention ici des théories dont la base est une relation de préférences intransitive, comme par exemple l'approche de Bell (1982), Fishburn (1982) ou Loomes et Sugden (1982).

[15] Cf. Chew (1989) pour cette classe de modèle.

l'utilité espérée dépendante du rang axiomatisée[16] par Quiggin (1982) dans le risque où l'axiome d'indépendance est vérifiée pour des loteries co-monotones.

La seconde classe de généralisations rejette l'axiome d'indépendance. Un exemple en est la théorie de l'utilité espérée locale initiée par Machina (1982) puis développée par Chew et Nishimura (1992), Chew et Hui (1995) et par exemple, pour une contribution récente Chaterjee et Krishna (2011). Cette approche conserve les conditions du pré-ordre et suppose une notion de différentiabilité de la représentation des préférences. L'utilité espérée devient alors une notion locale (au sens topologique du terme) car les fonctionnelles différentiables peuvent être considérées comme localement linéaires. Cette approche permet une grande flexibilité et la généralisation des critères d'attitudes face au risque[17]. Un autre exemple est la théorie de Luce (Cf. Luce (2000) pour une compilation de l'ensemble de ses travaux et Wakker (2000) pour une courte synthèse). Luce construit une théorie alternative, à partir de concepts psychologiques, et retrouvé la majorité des modèles standards à partir d'une opération de concaténation entre les loteries.

La troisième classe de généralisations est constituée des approches s'appuyant sur une démarche expérimentale, mettant l'accent sur l'aspect descriptif de la prise de décision individuelle, pour identifier des fonctionnelles d'utilité capables de reproduire les résultats expérimentaux. L'exemple le plus connu est la théorie des perspectives de Kahneman et Tversky (1979) et son raffinement, la théorie cumulative des perspectives (i.e., la théorie des perspectives dépendantes du rang, (1992)). Un autre exemple est le modèle TAX (Transfer of Attention eXchange) développée par Brinbaum et Chavez (1997).

L'utilité espérée dépendante du rang est certainement l'apport qui a eu le plus de succès en théorie de la décision (par exemple, voir Weber et Kirsner (1997), Diecidue et Wakker (2001) et Mongin (2009) pour des argumentaires mettant en avant l'utilité espérée dépendante du rang). Cette notion a permis, aussi bien dans un contexte de risque que d'incertitude[18], et dans de nombreux modèles[19] de rendre compte des violations de la théorie de l'utilité espérée. En outre, la fonctionnelle associée à cette notion préserve des propriétés intéressantes comme

[16] Cf. Yaari (1987) pour le cas particulier de l'approche duale de l'utilité espérée et Segal (1984, 1989) ou Green et Jullien (1988) pour des représentations généralisées.

[17] Cf. Cohen (1995).

[18] Axiomatisé par Schmeidler (1989).

[19] Entre autres Luce et Fishburn (1991, 1995), Tversky et Kahneman (1992), Wakker et Tversky (1993), Chateauneuf et Wakker (1999), Schmidt et Zank (2001, 2009, 2012) et Abdellaoui (2002).

la dominance stochastique. Sur un plan empirique, l'utilité espérée dépendante du rang conduit à de meilleurs résultats que l'utilité espérée ou l'utilité espérée pondérée pour des situations de choix se référant au paradoxe d'Allais. Dans des situations plus générales, le pouvoir explicatif n'est pas significativement meilleur que celui de l'utilité espérée ou d'autres théories[20].

Une part importante des dérivations axiomatiques de l'utilité espérée dépendante du rang les plus populaires[21] nécessitent le résultat obtenu par Wakker (1991, 1993) sur les représentations additivement séparables pour des sous-ensembles de produits Cartésiens où les coordonnées sont rangées par ordre croissant. Ce résultat permet, par exemple, de donner une représentation additivement séparable pour l'ensemble des loteries lorsque celles-ci sont assimilées à leurs fonctions de répartition.

L'apport principale du premier chapitre est de proposer une fondation axiomatique de fonctionnelles additivement séparables sur l'ensemble des distributions et non des fonctions de répartition, en utilisant la propriété d'additivité orthogonale de l'utilité. Les formes d'indépendance faisant intervenir des loteries à supports disjoints y sont étudiés.

Dans une série de papiers, Luce, Ng, Marley et Aczèl (2008a, 2008b, 2008, 2009a, 2009b) proposent de fonder axiomatiquement l'utilité des jeux d'argents[22], sur la base de la théorie de Luce (2000). Leur approche se base sur les travaux de Meginniss (1976) qui avait introduit, en fondant ce choix analytiquement, un critère de préférences variationnelles[23] représentées par une fonctionnelle se décomposant en un terme d'espérance d'utilité et en un terme entropique. La dérivation de fonctionnelles additivement séparables permet de traduire dans un cadre standard le contenu de ces travaux. Comme il sera montrer dans le premier chapitre, la dérivation de fonctionnelles additivement séparables permet de fournir dans un cadre standard une caractérisation de ces travaux. Cette fonctionnelle est un cas très particulier de préférences où l'axiome d'indépendance est vérifié pour des loteries dont les supports sont disjoints.

[20] Par exemple, Wakker *et al.* (1994), Wu (1994) et Birnbaum *et al.* (1999).

[21] Segal (1984, 1989), Green et Jullien (1988), Wakker et Tversky (1993), Chateauneuf et Wakker (1999), Wakker (1994), Schmidt et Zank (2001, 2009, 2012) et Abdellaoui (2002).

[22] Cf. Diecidue *et al.* (2004) et Le Menestrel (2001) pour une étude exhaustive de l'utilité des jeux d'argents.

[23] Le terme variationnelle est employé dans Maccheroni et al. (2006) pour des préférences incluant une fonction de coût général.

Interactions sociales et préférences interdépendantes

Dans les représentations traditionnelles de la théorie de la décision, les individus sont isolés. Ils maximisent des fonctions d'utilité souvent spécifiées par rapport à un argument strictement individuel et absolu[24]. Les agents sont systématiquement isolés de leur environnement social avant d'interagir, à travers le marché ou d'autres instances de ré-allocation. Depuis quelques années, il y a cependant un nouvel intérêt pour les interactions sociales[25], et plus précisément pour leur appréhension au sein de la théorie de la décision. Au delà des incitations de marché, le contexte social est de plus en plus considéré comme un déterminant des choix microéconomiques. En théorie des jeux, des résultats expérimentaux[26] sont venus affaiblir la notion de joueur isolé qui ne prend pas en compte le gain ou la perte des autres joueurs que ce soit de manière absolu ou relative. Veblen (1899), puis plus près de nous Duesenberry (1949) ou Leibenstein Leibenstein (1950) avaient déjà envisagé que, dans certaines situations, le bien-être relatif puisse avoir plus de sens au niveau individuel que le bien-être absolu. Pour donner substance à une telle option, et pour l'analyse de la décision individuelle, il est nécessaire d'intégrer des facteurs d'interdépendance[27] entre les préférences des agents. La prise en compte du bien-être relatif a un intérêt tout particulier dans des champs de l'analyse économique aussi différents que l'analyse de la demande[28], l'économie du travail[29], la croissance[30], la valorisation des actifs financiers[31], la prise en compte de l'attitude face au risque ou à l'ambiguïté [32].

Les travaux de Schelling (1971) ouvre la voie à l'analyse systématique de l'influence du contexte social sur le comportement individuel. Entre autres choses, Schelling fait la démonstration que la dynamique d'un groupe ne dépend pas uniquement des préférences individuelles mais aussi des interactions à l'intérieur du groupe.

[24] Le lecteur peut se référer à Sen (1977), Akerlof et Kranton (2000) ou Davis (2013) pour une critique de l'individu "self-centered, self-interested".

[25] Cf. Zanella (2004) et Scheinkman (2008) pour une présentation des interactions sociales en théorie économique.

[26] Cf. Rosenthal Rosenthal (1981) pour le jeu du mille-pattes, Guth et al. Güth et al. (1982) pour le jeu de l'ultimatum et Forsythe et al. Robert et al. (1994) pour le jeu du dictateur.

[27] Cf. Cooper and Kagel (2009b) pour l'exposition de résultats expérimentaux sur les préférences interdépendantes.

[28] Gaertner (1974), Pollak (1976), Becker (1991), Cowan et al. (1997), Binder et Pesaran (2001).

[29] Akerlof et Yellen (1990), Neumark et Postlewaite (1998) Bowles et Park (2005).

[30] Corneo et Jeanne (2001), Alvarez-Cuadrado et al. (2004), Liu et Turnovsky (2005).

[31] Chan et Kogan (2001), Dupor et Liu (2003).

[32] Rohde et Rohde (2011), Linde et Sonnemans (2012), Charness et al. (2013).

En outre, l'introduction de préférences interdépendantes peut conduire à des dynamiques inattendues au regard des préférences individuelles. Dans le même esprit, Föllmer (1974) établit que la prise en compte des interactions sociales dans une économie aléatoire modifie radicalement la dynamique macroéconomique à cause des effets de conformisme. Ces effets sont liés au multiplicateur social qui lie la dynamique sociale à la dynamique individuelle.

Les contributions de Blume, Brock et Durlauf (Brock et Durlauf (2001a, 2001b), Blume et Durlauf (2001)) offrent[33] un premier traitement analytique à la prise de décision en présence d'interactions sociales. Ils proposent un modèle où les individus font des choix binaires sur des actions. L'utilité d'une action dépend d'une utilité privée, d'un terme d'utilité aléatoire et d'une utilité sociale spécifiée par la distance entre son action et la moyenne subjective des actions des agents avec qui il interagit socialement. A partir de suppositions[34] sur le terme d'utilité aléatoire, les auteurs obtiennent une loi de probabilités (qui suit une loi de Boltzmann) sur le choix des agents, analogue à celui du modèle de Curie et Weiss en physique statistique. En supposant que les agents ont des anticipations rationnelles, ils dérivent les équilibres de Nash en champ moyen du modèle. Sur un plan microéconomique, leur formulation clarifie l'influence respective qu'exercent mutuellement le bien-être absolu et le bien-être relatif. En spécifiant simplement la fonctionnelle d'utilité, leurs modèle à l'avantage d'être facilement utilisable, sur le plan empirique, pour la mise en évidence d'interactions sociales ou d'"effets de pairs[35]" entre les individus. Manski (1993) définit trois effets de pairs. Un effet endogène, c'est à dire l'influence du comportement du groupe sur le comportement de l'individu, un effet exogène, c'est à dire l'influence des caractéristiques du groupe sur les caractéristiques de l'individu et un effet de corrélation, lorsque les membres d'un même groupe agissent de manière identique parce qu'ils ont les mêmes caractéristiques. Sur le plan empirique, la structure du modèle permet également de tester économétriquement la présence d'interactions sociales en séparant les effets endogènes et exogènes.

L'utilité dans le modèle de Blume, Brock et Durlauf se sépare en une partie déterministe et une partie stochastique essentielle pour son application économétrique.

[33] Cf. Manski (1993).

[34] Variables indépendantes et identiquement distribuées selon la loi des valeurs extrêmes.

[35] Pour une analyse détaillée de l'approche économétrique et une revue de la littérature existante sur le sujet, voir Blume et al. (2010). Pour les enjeux de l'approche économétrique des effets de pairs, voir Manski (1993, 2000).

D'un point de vue théorique, il est pertinent de s'intéresser à la fondation axiomatique de la partie déterministe de ce modèle. L'utilisation de la seule partie déterministe en théorie des jeux conduirait, probablement, à des équilibres de Nash en stratégie mixte. Ces équilibres, selon toute vraisemblance, seraient différents des équilibres du modèle initial[36]. Une solution est de remplacer le terme d'utilité stochastique par un terme variationnel approprié pour recouvrir les bonnes propriétés d'équilibre. Le terme variationnel adéquat est l'entropie de Shannon. Cela n'a rien d'étonnant, compte tenu de la proximité du modèle de Blume, Brock et Durlauf avec la physique statistique. Pour rappel, le modèle de choix logit est équivalent à une distribution de Boltzmann et cette dernière est la solution, sous contrainte d'espérance, de l'entropie.

Le second chapitre propose un fondement axiomatique de préférences interdépendantes dans le risque, se décomposant en une utilité privée et une utilité sociale, couplé à un terme variationnel. Un tel fondement rend compte de la formation des groupes de référence lorsque l'environnement social est donné de manière endogène. De plus, l'utilité sociale peut être axiomatisée comme additivement séparable parmi les individus de l'environnement social de l'agent qui prend la décision.

Les apports du modèle probabiliste de la physique quantique

Au cours la dernière décennie, une littérature non négligeable s'est développée autour de l'utilisation du modèle probabiliste de la physique quantique[37] en théorie de la décision. On peut illustrer cette constatation par les contributions non-exhaustives de, Danilov et Lambert-Mogiliansky (2005, 2010), Lambert-Mogiliansky et al. (2009), Hansen (2005), Gyntelberg et Hansen (2005, 2009), Yukalov et Sornette (2009, 2010), Pothos et Busemeyer (2009, 2013), Busemeyer et al. (2009), Busemeyer et Bruza (2012).

[36]Pour s'en convaincre, sans terme d'utilité sociale, il suffit de comparer les équilibres de Nash standards et les équilibres logit introduits par McKelvey et Palfrey (1995, 1998). Ces auteurs prolongent l'approche de McFadden (1974) à la théorie des jeux.

[37]Cf. Kolmogorov (1950) pour la fondation de la théorie classique des probabilités et Neumann (1932) pour une théorie des probabilités non-commutative.

Le troisième chapitre se propose de prendre en considération cette littérature. Le formalisme sera présenté afin d'en expliciter au mieux son sens. Il sera ensuite appliquer à la théorie standard de la théorie de la décision individuelle sous l'hypothèse qu'un individu a une représentation subjective des évènements. Dans ce cadre, l'agent maximise son utilité espérée relativement à sa représentation subjective des évènements et il est caractérisé par deux fonctions d'utilité et une fonction de transformation des probabilités. Les deux fonctions d'utilité rendent compte, respectivement, des préférences de l'agent dans le certain et dans le risque relativement à sa représentation subjective des évènements. La fonction de transformation des probabilités a pour image une distribution de probabilités. C'est à dire que l'agent réinterprète les probabilités objectives par rapport à sa représentation des évènements. Le chapitre se conclut par une étude critique sur l'apport de ce formalisme.

Conclusion (en Français)

Cette thèse a été consacrée à définir des fonctions d'utilité dans différents contextes constitutifs d'une théorie générale de la décision individuelle dans le risque. Cette question conditionne la validité d'un grand nombre de prédictions relevant de nombreux champs de l'économie : la compréhension du comportement des individus bien évidemment, mais aussi l'ensemble des domaines où intervient les préférences dans le risque comme la finance, l'assurance, l'économie industrielle ou l'économie de l'environnement. Depuis les années 90, l'émergence de travaux d'économie inspirés par la sociologie ou la physique ou inversement sont venues questionner les fondements classiques de l'analyse décisionnelle. Cette dynamique s'inscrit dans la mise en évidence des faiblesses de la théorie de la décision individuelle[38] et de la théorie des jeux[39]. Les défauts de coordination constatées en théorie des jeux sont principalement attribuées au fait que les individus ne prennent pas en compte les gains des autres agents de manière absolue ou relative. Dans ce cas, même si les problèmes observés sont consubstantiels des interactions stratégiques, la nature insatisfaisante de certains résultats de la théorie des jeux renvoie dans une certaine mesure aux insuffisances de la théorie de la décision individuelle.

Dans la littérature, il devient systématique de prendre en compte les limites de la rationalité des individus et l'interdépendance des préférences : il convient donc d'étudier ces options théoriques à partir de la théorie de la décision individuelle. Certaines contributions récentes ont été apportées dans ce champ. Nous pouvons citer, par exemple, Segal (1989) ou Chateuneuf et Wakker (1999) pour la théorie de la décision individuelle dans le risque, Karni et Safra (2002) ou Maccheroni *et al.* (2012) en théorie de la décision individuelle pour des préférences incorporant

[38]Cf. notamment Lichtenstein et Slovic (1971) pour le renversements des préférences ainsi que Tversky et Kahneman (1992) pour les effets de cadrage.

[39]Entre autres Rosenthal (1981) pour le jeu du mille-pattes, Guth et al. (1982) pour le jeu de l'ultimatum et Forsythe et al. (1994) pour le jeu du dictateur.

de l'équité ou de l'interdépendance, Rabin (1993), Fehr et Schmidt (1999) ou Segal et Sobel (2007) en théorie des jeux. Brock et Durlauf (2001a) et Blume et Durlauf (2001) pour une approche théorique et empirique. Cette dernière contribution offre un cadre empirique à la mise en évidence des effets de pairs dans les réseaux et un cadre théorique pour la *dynamique* d'un réseau. L'approche en termes de réseaux est abondamment utilisé en éconophysique et c'est pourquoi il est intéressant d'étudier si les résultats théoriques de cette thèse peuvent, d'une manière ou d'une autre, aider à sa compréhension en microéconomie.

Les contributions de cette thèse

Concernant la théorie de la décision individuelle dans le risque, j'ai proposé des conditions garantissant l'existence d'une utilité additivement séparable et d'une utilité couplée à un terme entropique. Les conditions que j'ai mises en évidence sont assez générales et garantissent l'existence de fonctions d'utilité représentant des préférences interdépendantes, comme formulées dans le modèle de Blume, Brock et Durlauf. Ce modèle offre les premières bases théoriques pour des applications économétrique (Cf. Blume et al. (2010)); ensuite, ce modèle est utilisé en microéconomie ou en éconophysique pour sa simplicité. J'ai mis en avant le formalisme des probabilités quantiques pour construire une représentation des préférences dans le cas où les évènements sont subjectifs.

Espérance d'utilité "additive"

Le premier chapitre de cette thèse propose une fondation axiomatique cohérente d'une utilité additivement séparable par rapport aux probabilités. L'obstacle principal à une généralisation des théorèmes de constructions classiques des fonctions d'utilité additives réside dans les méthodes utilisées pour démontrer ceux-ci. En effet, les deux approches usuelles, l'approche algébrique et l'approche topologique, utilisent des outils se rapportant à la structure des objets étudiés et non aux propriétés globales de l'utilité. J'ai construit une méthode à partir de la notion d'additivité orthogonale pour des séquences à support disjoint. Cette propriété rapproche notre objet d'étude de la théorie de l'utilité linéaire développée par Aumann (1962). Dans leur forme actuelle, les approches topologiques et algébriques ne permettent pas d'étudier les sous-ensembles de produits Cartésiens tels que, par exemple, un ensemble d'alternatives dont les coordonnées ne sont pas indépendantes. Le premier chapitre dépasse ce problème.

Préférences variationnelles

Certains champs théoriques peuvent accueillir directement les résultats théoriques de ce travail. Prenons l'exemple des préférences variationnelles : de nombreux modèles postulent l'existence d'un terme entropique couplé à une terme d'utilité espérée pour rendre compte d'un coût cognitif à la décision ou d'un coût d'incertitude à la décision. Bien qu'il existe des modèles où la forme du coût n'est pas entropique (linéaire, quadratique ou de forme générique), cette forme reste la plus utilisée pour ses propriétés analytiques. Des axiomes, très simples, permettent de démontrer que ce genre de fonctions d'utilité vérifie l'axiome d'indépendance à partir du moment où la formation de combinaisons convexes de loteries est restreinte aux loteries à support disjoints. Ainsi, ce type de modèle peut être considéré comme incorporant une rationalité proche de la rationalité normative de l'axiome d'indépendance, donc une rationalité peu limitée.

Préférences interdépendantes dans le risque

Le second chapitre propose un fondement axiomatique du modèle de Blume, Brock et Durlauf où les individus évaluent une utilité sociale rendant compte de préférences interdépendantes, que ce soit sur les choix ou les gains des individus dans son environnement social. Pour plus de flexibilité, notamment pour pouvoir discuter de la formation des voisinages où prend place l'interaction sociale, l'hypothèse générale d'anonymat a été supprimée. Ce chapitre propose trois formes d'utilité dans un contexte de risque. Ceci permet de considérer soit un risque exogène se référant à la situation des individus de l'environnement social, soit un "risque" objectif sur les croyances que forme l'individu au coeur de l'environnement social à propos des actions des autres individus. La première forme sépare l'utilité privée de l'utilité dite "sociale" pour un groupe de référence exogène. La seconde forme permet des préférences où le groupe de référence est endogène. La dernière forme ajoute un terme entropique pour pouvoir obtenir les résultats théoriques de Blume, Brock et Durlauf sans avoir à ajouter une utilité stochastique pour des applications en microéconomie. Cette étape permet de mieux appréhender les modèles d'éconophysique. Au cours de ce chapitre, je propose des hypothèses pour séparer additivement le terme de préférences sociales par rapport aux individus.

Probabilités quantiques

Je construis dans ce chapitre une approche basée sur le formalisme quantique. Celle-ci permet, il semble, d'étendre les résultats du formalisme classique. Elle permet de plus de mieux comprendre certains problèmes comportementaux. L'hypothèse de base est que l'individu considère des évènements subjectifs, il forme une représentation subjective des évènements, et celle-ci est inaccessible pour l'observateur, avant que la décision ne soit prise. Sur le plan analytique, l'utilité dérivée est semblable à l'utilité quadratique tout en ayant une dimension (au sens physique du terme) d'utilité espérée. Cette approche peut être une base pour expliquer les phénomènes de cadrages. Cependant, ce formalisme a des limites. En dehors de la flexibilité analytique due à la richesse du cadre mathématique, il n'en demeure pas moins que ce formalisme n'apporte pas de solutions satisfaisantes sur les déterminants du choix individuelle. En effet, et c'est une critique générale qu'il est possible d'adresser à ce type de modèle de la littérature, la théorie quantique permet, en physique, une étude dynamique des systèmes et ses résultats expérimentaux suffisent à la justifier. En théorie de la décision, nous ne disposons pas de ce type de justification : son utilisation découle habituellement d'hypothèses *ad hoc* qui peuvent être discutées.

Perspectives

Deux axes de Recherche paraissent intéressants à poursuivre, à partir des résultats de cette thèse. Le premier axe concerne l'extension des résultats théoriques du premier chapitre : l'espace des loteries, le simplexe, reste un cas très particulier de sous-ensemble de produits Cartésiens. Il semble qu'une théorie issue de la propriété d'additivité orthogonale, dans la continuité de l'utilité linéaire, puisse être établie. Ensuite, la dérivation de préférences variationnelles a mis en évidence la possibilité d'un terme entropique semblable à l'entropie de Tsallis avec une espérance d'utilité dont les probabilités sont pondérées. De manière appliquée, l'utilisation de ces fonctionnelles pourrait apporter des résultats bien plus généraux par rapport aux préférences variationnelles simples, en théorie des jeux par exemple ou dans des champs plus appliqués. Enfin, les quelques résultats expérimentaux, contradictoires, sur les fonctionnelles d'utilité additivement séparables, devraient nous inciter à étudier plus particulièrement ce problème, pour comparer leur pouvoir explicatif, du moins, avec celui de l'utilité espérée dépendante du rang. Le second axe concerne les résultats théoriques du second chapitre : une application directe pourrait être entreprise en théorie des jeux pour pouvoir déterminer les modifications des équilibres et des niveaux de bien-être social. En effet, les fondements proposée sont assez flexibles pour générer des externalités négatives ou positives. La prise en compte de l'entropie de Tsallis, ici aussi pourrait apporter de nouveaux résultats.

Résumé

Décision, Risque, Interactions Sociales

par Dino BORIE

Cette thèse se compose de trois chapitres constituant des contributions distinctes mais reliées au même centre d'intérêt, la théorie de la décision dans le risque. Le premier chapitre traite de préférences additivement séparable par rapport aux probabilités. Il en est déduit une axiomatisation simple de préférences variationnelles représentées par une fonctionnelle se décomposant en un terme d'espérance d'utilité et un terme entropique. Le second chapitre consiste en une fondation axiomatique de préférences interdépendantes en présence d'interactions sociales, sur la base du modèle initialement élaboré par Blume, Brock et Durlauf. Le troisième chapitre pose la question de l'apport du modèle probabiliste de la physique quantique à la théorie de la décision et de son application pour prendre en compte la perception subjective des évènements par les individus.

Mots-Clés : Utilité additivement séparable, risque, préférences interdépendantes.

Abstract

Décision, Risque, Interactions Sociales

by Dino BORIE

This thesis consists of three separate chapters related to economic decisions under risk. The first chapter presents axioms for an additively separable representation of preferences over probabilities. A simple axiomatization of variational preferences represented by the sum of an expected utility term and an entropic term is deduced. The second chapter consists of an axiomatic foundation of other-regarding preferences under social interactions, based on the model originally developed by Blume, Brock and Durlauf. The third chapter introduces the probabilistic model of quantum physics to decision theory. In this context, individuals have a private representation of the set of events.

Keywords : Additively separable utility, risk, other-regarding preferences.