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ÉCOLE DOCTORALE DE DAUPHINE

THÈSE DE DOCTORAT

pour obtenir le grade de

Docteur en Sciences de l'Université Paris-Dauphine

présentée par

Khalil CHOUK

Trois chemins contrôlés

Soutenue le 20 Janvier 2014 devant le jury composé de MM. :

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Chapter 1

Aperçu de la thèse

1.1 Introduction générale a la théorie des chemins rugueux

La théorie des chemins rugueux (ou rough path) initiée par Terry Lyons dans [60] il y a maintenant une quinzaine d'années a été introduite afin d'étudier le système différentiel suivant

$$dy_t^i = \sum_{i=1}^n f(y_t^i) dx_t^i, \quad y_0 = a \in \mathbb{R}^d \quad (1.1)$$

où $x \in C([0, T], \mathbb{R}^n)$ est un "bruit irrégulier" et $(f_i)_{i=1, \dots, n}$ un champ de vecteurs lisses. Dans le cas où x est un chemin différentiable ou même Lipschitzien, l'équation (1.1) est assez bien comprise dans le sens où on sait qu'il existe une unique solution globale, de plus le flot $\Phi : x \mapsto y$ est continu. Légitimement on pourrait s'interroger sur ce qui se passe lorsque x n'est plus une fonction assez régulière. Un exemple typique serait une trajectoire d'un mouvement Brownien qui est presque sûrement nul part dérivable et a totale variation infinie. Dans ce cas particulier une réponse a été donnée par le calcul stochastique en interprétant (1.1) comme l'équation intégrale :

$$y_t^i = a + \int_0^t f(y_s^i) dx_s^i \quad (1.2)$$

où l'intégrale qui apparaît ici est dite intégrale d'Itô et est construite en utilisant essentiellement la propriété de martingale du mouvement Brownien. Cependant il faudrait remarques deux choses : premièrement cette approche est inapproprié pour l'étude de la continuité du flot Φ ([57]) et deuxièmement on rencontre des difficultés à étendre cette théorie à des processus Gaussiens plus généraux tels que le mouvement Brownien fractionnaire de paramètre d'Hurst $H \neq 1/2$ qui n'est plus une semimartingale (ie : $H = 1/2$ correspond au cas du Brownien standard). C'est dans ce cadre où entre en jeu la théorie des chemins rugueux. Un développement formel de l'équation intégrale permet de voir que la présumée solution y va s'écrire comme une série d'intégrales itérées en x et plus précisément de la famille \mathbb{X} définie de la manière récursive suivante :

$$\mathbb{X}_{st}^1 = x_t - x_s, \quad \mathbb{X}_{st}^{n+1} = \int_s^t \mathbb{X}_{su}^n \otimes dx_u$$

où \otimes est le produit tensoriel sur \mathbb{R}^n . L'idée de Lyons est alors de porter l'analyse de la dépendance fonctionnel de y par rapport à x à la famille \mathbb{X} . Plus précisément étant donné un chemin x a p -variation

finie tel qu'on peut construire la famille $(\mathbb{X}^n)_{\{n \leq [p]\}}$ de manière convenable à ce qu'elles satisfassent certaines relations algébriques. On peut alors donner un sens à l'équation (1.2) et construire une solution globale \mathbb{Y} de sorte que l'application $\Phi : \mathbb{X} \mapsto \mathbb{Y}$ est continue pour une topologie métrisable.

Cette théorie a été par la suite reprise par Gubinelli dans [36] pour des bruits x qui sont Hölderiens. Un des avantages de cette approche est de pouvoir définir l'intégrale rugueuse donnée par :

$$\int_s^t y_\sigma dx_\sigma$$

pour une classe d'intégrands y dite classe des chemins contrôlés qui est plus large que celle donnée par les compositions de fonctions régulières avec x . De plus cette souplesse a permis l'application de la théorie des chemins rugueux pour des EDPS, tel que les travaux de M.Gubinelli, S.Tindel et A.Deya [29] portant sur l'équation de la chaleur avec un bruit multiplicatif qui est blanc en temps et plus régulier en espace. Une autre application plus exotique est celle de Martin Hairer, donnée en [45] où il utilise la notion de chemin contrôlé en espace afin d'étudier l'équation de Burgers unidimensionnelle en présence d'un bruit blanc dans un domaine périodique. Une majeure contribution de la théorie a été de donner un sens rigoureux au terme non linéaire de l'équation de Kardar-Parisi-Zhang (KPZ) dans le cas périodique [46] et de prouver l'existence d'une solution locale via une méthode de point fixe. Cependant cette théorie présentait la contrainte d'être lié à la dimension une jusqu'aux deux superbes généralisations effectuées pendant ces deux dernières années. La première a été proposée par M.Gubinelli, N.Perkowsky et P.Imkeller dans [38]. Ils usent de l'analyse de Fourier microlocale et plus particulièrement du paraproduit de Bony [5, 6] pour traiter l'équation de Burgers stochastique en dimension supérieure ou encore l'équation parabolique d'Anderson en dimension 2. Une autre approche a été développée par Martin Hairer dans [47] et qui est basée sur l'analyse par ondelettes a permis d'avoir un cadre d'étude pour une large classe d'EDPS parabolique tel que l'équation de KPZ généralisé, l'équation parabolique d'Anderson en dimension 2 ou 3 ou encore l'équation de la quantisation stochastique en dimension 3.

Le but de cette thèse est de donner certaines applications de la théorie des chemins rugueux contrôlés dans diverses problématiques telle que le calcul stochastique à 2 paramètres, l'étude d'EDPs dispersives en milieu non homogène ou encore l'obtention d'une "bonne" notion de solution pour l'équation de la quantisation stochastique en dimension 3.

1.2 État de l'art et résultats du Chapitre 2

Le calcul stochastique à deux paramètres est un sujet assez complexe et ceci pour plusieurs raisons parmi lesquelles, sûrement deux majeures :

1. Perte de la structure temporelle ce qui complique la généralisation de la notion de filtration comme on peut voir dans les travaux de Caroli-Walsh [19]
2. Une formule de changement de variable bien compliqué même pour des draps réguliers :

$$\varphi(x_{st}) = \varphi(x_{00}) + \varphi(x_{0t}) + \varphi(x_{s0}) + \int_0^s \int_0^t \varphi'(x_{uv}) dx_{uv} + \int_0^s \int_0^t \varphi''(x_{uv}) d_1 x_{uv} d_2 x_{uv} \quad (1.3)$$

pour $s, t \in [0, 1]$, avec $x \in C^2([0, 1]^2, \mathbb{R})$ et une fonction $\varphi \in C^2(\mathbb{R})$. Une remarque importante est la présence du terme quadratique en x dans le membre de droite de l'équation (1.3) qui est spécifique au cas multidimensionnel.

Avant d'énoncer les résultats principaux de ce chapitre nous allons introduire les objets probabilistes qui nous intéressent et nous illustrerons certaines approches introduites dans le passé pour étudier ce genre de problème.

1.2.1 Drap Brownien fractionnaire et bruit blanc

À l'instar du mouvement Brownien fractionnaire le drap Brownien fractionnaire à deux paramètres est un processus stochastique intéressant à étudier de par ses propriétés d'autosimilarité et de stationnarité des incréments. Nous donnons d'abord la définition rigoureuse de celui-ci et puis nous présenterons certaines de ces propriétés.

Définition 1.2.1. Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité. Un processus stochastique $(x_{st})_{s,t \geq 0}$ est dit un drap fractionnaire de paramètre d'Hurst $\gamma = (\gamma_1, \gamma_2) \in [0, 1]^2$ si

1. x est Gaussien de moyenne nulle.
2. Sa fonction de covariance est donnée par :

$$R_{s_1 s_2 t_1 t_2} = \mathbb{E}[x_{s_1 t_1} x_{s_2 t_2}] = \frac{1}{4}(s_1^{\gamma_1} + s_2^{\gamma_1} + |s_2 - s_1|^{\gamma_1})(t_1^{\gamma_2} + t_2^{\gamma_2} + |t_2 - t_1|^{\gamma_2})$$

pour tout $s_1, s_2, t_1, t_2 \geq 0$. Dans le cas où $\gamma_1 = \gamma_2 = 1/2$ on dira simplement que x est un Drap Brownien.

Une observation importante est de voir que pour $t > 0$ fixé le processus $(t^{-\gamma_2/2} x_{st})_{s \geq 0}$ est un mouvement Brownien fractionnaire d'indice d'Hurst γ_1 . Nous avons de plus les propriétés suivantes :

Proposition 1.2.2. Soit x un drap Brownien fractionnaire d'indice d'Hurst $\gamma = (\gamma_1, \gamma_2)$ alors on a les propriétés suivantes :

1. x admet une modification à trajectoires presque sûrement continues.
2. $\{x_{as, bt}, s, t \geq 0\} \stackrel{\text{loi}}{=} \{a^{\gamma_1} b^{\gamma_2} x_{st}, s, t \geq 0\}$ pour tout $a, b \geq 0$ (auto-similarité).
3. $\{x_{s+h, t+k} - x_{s+h, t} - x_{s, t+k} + x_{st}, h, k \geq 0\} \stackrel{\text{loi}}{=} \{x_{hk}, h, k \geq 0\}$ pour tout $s, t \geq 0$ (stationnarité des incréments)

Avant de continuer dans l'étude de certaines particularités du drap Brownien fractionnaire nous allons nous concentrer sur le cas particulier $\gamma_1 = \gamma_2 = 1/2$ où x présente une particularité d'indépendance intéressante. Plus précisément on sait que pour un mouvement Brownien B on a indépendance des accroissements au sens où les variables aléatoires $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ sont indépendantes. Cela on peut le montrer avec un simple changement de base sur la matrice de covariance du vecteur Gaussien $(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$. Maintenant il n'est pas difficile de voir que ce résultat se généralise bien au drap Brownien, en effet on a que pour une famille de rectangles $(A_i)_{i \leq n}$, indépendance des incréments du drap Brownien sur ces rectangles et de manière plus explicite on voit que les variables aléatoires $\delta x_{A_1}, \dots, \delta x_{A_n}$ sont indépendantes, où on a introduit ici la notation suivante :

$$\delta x_A = x_{s_2 t_2} - x_{s_1 t_2} - x_{s_1 t_2} + x_{s_1 t_1}$$

pour un rectangle $A = [s_1, s_2] \times [t_1, t_2]$ du plan.

De retour au cas général où $\gamma_1, \gamma_2 \in [0, 1]$ nous allons nous concentrer plus sur la régularité d'un tel processus. En effet si l'on considère le cas unidimensionnel d'un mouvement Brownien fractionnaire

$(B_t^H)_{t \geq 0}$ d'indice d'Hurst H il est bien connue que B a une modification à trajectoire presque sûrement α -Hölderienne pour $\alpha < H$ dû au critère de Kolmogorov (voir [68]) ou plus précisément

$$|B_t^H - B_s^H| \leq C_{T,H,\alpha} |t - s|^\alpha$$

pour tout $T > 0$, $s, t \in [0, T]$ presque sûrement où $C_{T,H,\alpha}$ est une constante aléatoire positive finie (et même intégrable). On pourrait alors se poser la question de ce qu'en est-il avec le drap Brownien fractionnaire, peut-on espérer une régularité du même type ? Le résultat suivant, qui est une application et en même temps une généralisation du Lemme de Garsia-Rodemich-Rumsey (voir [35]) fournit une piste pour répondre à cette question.

Lemma 1.2.3. *Soit z une fonction continue sur $[0, T] \times [0, S]$ à valeur réel alors l'inégalité suivante :*

$$|\delta z_{s_1 s_2 t_1 t_2}| \leq C_{S,T,\alpha,\beta,p} |s_1 - s_1|^\alpha |t_2 - t_1|^\beta \left(\int_0^T \int_0^S \frac{|\delta z_{u_1 u_2 v_1 v_2}|^p}{|u_1 - u_2|^{\alpha p + 2} |v_2 - v_1|^{\alpha p + 2}} du_1 du_2 dv_1 dv_2 \right)^{1/p}$$

avec

$$\delta z_{s_1 s_2 t_1 t_2} := z_{s_2 t_2} - z_{s_1 t_2} - z_{s_2 t_1} + z_{s_1 t_1}$$

est satisfaite pour tout $s_1, s_2 \in [0, S]$, $t_1, t_2 \in [0, T]$ et $p > 1$ où $C_{S,T,\alpha,\beta,p}$ est une constante finie et positive ne dépendant que de S, T, α, β et p .

Démonstration. Voir la preuve du Lemme (2.6.2) dans le deuxième chapitre. □

Ainsi ayant ce résultat un simple calcul nous donne que :

$$\begin{aligned} & \mathbb{E} \left[\sup_{s_1, s_2 \in [0, S]; t_1, t_2 \in [0, T]} \frac{|\delta x_{s_1 s_2 t_1 t_2}|^p}{|s_2 - s_1|^{\beta p} |t_2 - t_1|^{\alpha p}} \right] \\ & \leq C_{S,T,\alpha,\beta,p}^p \int_{[0,S] \times [0,T]} \frac{\mathbb{E}[|\delta x_{u_1 u_2 v_1 v_2}|^p]}{|u_1 - u_2|^{\alpha p + 2} |v_2 - v_1|^{\beta p + 2}} du_1 du_2 dv_1 dv_2 \\ & \leq C_{S,T,\alpha,\beta,p}^p \int_{[0,S] \times [0,T]} |u_2 - u_1|^{(\gamma_1 - \alpha)p - 2} |v_2 - v_1|^{(\gamma_2 - \beta)p - 2} du_1 du_2 dv_1 dv_2 \\ & < +\infty \end{aligned}$$

pour $\alpha < \gamma_1$ et $\beta < \gamma_2$ et pourvue que p soit assez grand. Ce dernier calcul nous prouve en particulier que pour un drap Brownien fractionnaire $(x_{st})_{s \in [0, S], t \in [0, T]}$ d'indice d'Hurst $\gamma = (\gamma_1, \gamma_2)$ on a l'estimation suivante ;

$$|\delta x_{s_1 s_2 t_1 t_2}| \leq C_{\alpha,\beta,\gamma,S,T}(\omega) |s_2 - s_1|^\alpha |t_2 - t_1|^\beta, \quad \text{pour tout } (s_1, s_2, t_1, t_2) \in [0, S]^2 \times [0, T]^2$$

avec probabilité un, où $C_{\alpha,\beta,\gamma,S,T}(\omega)$ une constante aléatoire positive et intégrable. Une remarque est que cette dernière propriété nous implique que $(x_{st})_{st}$ est r -Hölderien au sens usuelle avec $r < \min(\gamma_1, \gamma_2)$. Nous allons maintenant essayer de donner une représentation agréable à manipuler pour le drap Brownien fractionnaire et pour cela on rappelle la proposition suivante :

Proposition 1.2.4. *Soit \mathcal{H} un espace d'Hilbert séparable réel alors il existe un processus Gaussien réel $\{Z(h), h \in \mathcal{H}\}$ de moyenne nul tel que*

1. $h \mapsto Z(h)$ est linéaire.

$$2. \mathbb{E}[Z(h)Z(k)] = \langle h, k \rangle_{\mathcal{H}}$$

Cette proposition va nous permettre de définir un objet important dans la théorie des équations à dérivées partielle stochastique, à savoir le bruit blanc.

Définition 1.2.5. Soit $N \in \mathbb{N}$ et on note par $\mathcal{B}(\mathbb{R}^N)$ la tribu Borélienne sur \mathbb{R}^N alors le processus Gaussien $(W(A))_{A \in \mathcal{B}(\mathbb{R}^N)}$ est appelé un bruit blanc sur \mathbb{R}^N si il est de moyenne nulle et que $\mathbb{E}[W(A)W(B)] = |A \cap B|$ pour tout $A, B \in \mathcal{B}(\mathbb{R}^N)$, où $|A|$ désigne la mesure de Lebesgue de l'ensemble A .

L'existence d'un tel processus est assurée par la Proposition (1.2.4) de plus on peut voir que $W(h)$ est bien définie pour $h \in L^2(\mathbb{R}^N)$. Une propriété importante du bruit blanc W est la suivante : pour deux ensembles $A, B \in \mathcal{B}(\mathbb{R}^N)$ disjoints (ie : $A \cap B = \emptyset$), les variables aléatoires $W(A)$ et $W(B)$ sont indépendantes. De plus $A \mapsto W(A)$ est une mesure signée de $L^2(\mathbb{P})$, plus précisément on a que $W(\emptyset) = 0$ et que

$$W\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_i W(A_i)$$

Dans $L^2(\mathbb{P})$, pour toute famille $(A_i)_i$ de Borelien de \mathbb{R}^N dont les éléments sont deux à deux disjoints et tel que la somme $\sum_i W(A_i)$ soit bien définie. Avant de conclure cette sous-section en donnant une représentation en loi du drap Brownien fractionnaire nous introduisant la transformée de Fourier du bruit blanc.

Définition 1.2.6. Soit W un bruit blanc sur \mathbb{R}^N alors sa transformée de Fourier noté \hat{W} est le processus Gaussien défini par :

$$\hat{W}(h) = W(\hat{h})$$

pour tout $h \in L^2(\mathbb{R}^N)$ où \hat{h} est la transformée de Fourier usuelle de h .

Proposition 1.2.7 (Voire [69]). Soit $\gamma_1, \gamma_2 \in [0, 1]$ et on se donne W un bruit blanc sur \mathbb{R}^2 alors il existe une constante C_{γ_1, γ_2} tel que le processus

$$\left\{ C_{\gamma_1, \gamma_2} \hat{W}(Q_{st}), \quad s, t \geq 0 \right\}, \quad Q_{st}(x, y) = \frac{e^{isx} - 1}{|x|^{\gamma_1+1/2}} \frac{e^{ity} - 1}{|y|^{\gamma_2+1/2}}$$

est un drap Brownien fractionnaire d'indice d'Hurst $\gamma = (\gamma_1, \gamma_2)$

Nous finissons cette partie par remarquer que si W est un bruit blanc sur \mathbb{R}^2 alors le processus $V_{st} = W([0, s] \times [0, t])$ est un drap Brownien.

1.2.2 Théorie des mesures-martingales

Nous présentons ici une méthode introduite par Walsh dans [74] et qui a permis de définir l'intégrale stochastique sur le plan pour un drap Brownien. Bien que ce ne soit pas la seule façon d'introduire le calcul stochastique à deux paramètres (a titre d'exemple on peut citer [19, 44, 65, 73]) on a choisi cette approche pour sa simplicité.

Étant donné un espace de probabilité filtré complet $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ et $\mathcal{B}(\mathbb{R})$ la tribu Borelienne sur \mathbb{R} on a la définition suivante :

Définition 1.2.8. On dit qu'un processus $(M_t(A))_{t \geq 0, A \in \mathcal{B}(\mathbb{R})}$ est une mesure martingale par rapport à la filtration $(\mathcal{F}_t)_{t \geq 0}$ si les trois conditions suivantes :

1. $M_0(A) = 0$ presque sûrement pour tout $A \in \mathcal{B}(\mathbb{R})$
2. Pour tout $t > 0$ M_t est une mesure signée à valeurs dans $L^2(\mathbb{P})$.
3. Pour tout $A \in \mathcal{B}(\mathbb{R})$ le processus $(M_t(A))_{t \geq 0}$ est une $(\mathcal{F}_t)_{t \geq 0}$ -martingale de moyenne nulle.

sont satisfaites.

Un exemple connu de tels processus est donné par :

$$W_t(A) = W([0, t] \times A)$$

où W est un bruit blanc sur \mathbb{R}^2 . Maintenant pour une mesure martingale M nous nous proposons de définir une intégrale du type

$$\int_0^t \int_A f(t, x, \omega) W(ds, dx)$$

sous certaines conditions sur le processus f . Pour cela nous allons procéder d'une manière assez cano- nique à savoir se donner une classe d'intégrands pour lequel l'intégrale s'exprime de manière raisonnable puis l'étendre par des arguments de densité.

Définition 1.2.9. Une fonction $f = \mathbb{R}^+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ est dite élémentaire si

$$f(t, x, \omega) = X(\omega)1_{(a,b]}(t)1_A(x)$$

pour $A \in \mathcal{B}(\mathbb{R})$ et X une variable aléatoire \mathcal{F}_a -mesurable et borné. Une combinaison linéaire finie de fonction élémentaire est dite simple et on dénote par \mathcal{R} l'ensemble des fonctions simples .

Si maintenant $f(t, x, \omega) = X(\omega)1_{(a,b]}1_A(x)$ est une fonction simple et M une mesure martingale on définit alors l'intégral $f \bullet M_t(B)$ par :

$$f \bullet M_t(B) = \int_0^t \int_B f(s, x, \omega) M(ds, dx) := X(\omega)(M_{\min(t,b)}(A \cap B) - M_{\min(t,a)}(A \cap B))$$

et on prolonge aux fonctions de \mathcal{R} par linéarité. Maintenant on peut remarquer que cette définition est bien valide et ne dépend pas de la représentation de f et que $(M_t \bullet f(A))_{t,A}$ définit bien une mesure martingale. Dans la suite on se limitera aux cas où M est la mesure martingale donnée par le bruit blanc et il facile de voir dans ce cas que pour A fixé de mesure non nul que le processus $(|A|^{-1/2}M_t(A))_t$ est un mouvement Brownien et donc on obtient par un simple calcul que

$$\langle M_\bullet(A), M_\bullet(B) \rangle_t = t|A \cap B|$$

où le terme de gauche dans cette équation est tout simplement la variation quadratique du processus, ainsi un calcul direct nous donne que

$$\mathbb{E} \left[\left(\int_0^t \int_B f(s, x, \cdot) W(ds, dx) \right)^2 \right] = \int_0^t \int_B \mathbb{E}[|f(s, x, \cdot)|^2] ds dx$$

pour $f \in \mathcal{R}$. Ainsi on étend la formule d'intégrale aussi tôt par densité à l'ensemble des processus $f(s, x, \omega)$ adapté en temps et tel que :

$$\int_0^t \int_B \mathbb{E}[|f(s, x, \cdot)|^2] ds dx < +\infty$$

à noter que cette approche est sous certaines conditions encore valide pour les mesure martingale (voire par exemple [74]). Ceci dit cette construction de l'intégrale présente la particularité d'être rattaché fortement à la notion de martingale tout comme le calcul stochastique développé dans [19] ce qui empêche l'étude du cas du drap Brownien fractionnaire ou d'autres processus Gaussien qui n'ont pas cette propriété. Dans le prochain paragraphe nous verrons une autre approche qui est basée essentiellement sur le calcul de Malliavin et qui va permettre l'étude de tels processus.

1.2.3 Calcul de Malliavin pour le drap Brownien fractionnaire

Dans cette partie nous exposons brièvement les travaux de C.Tudor et F.Viens [20, 21] qui seront repris dans le deuxième chapitre de cette thèse. Dans [20] les auteurs développent un calcul de Malliavin pour un drap fractionnaire x de paramètres d'Hurst $\gamma_1, \gamma_2 > 1/2$ en interprétant les intégrales

$$\int_0^t \int_0^s f'(x_{uv}) d^\diamond x_{uv}, \quad \int_0^s \int_0^t f''(x_{uv}) d_1^\diamond x_{uv} d_2^\diamond x_{uv} \quad (1.4)$$

aux sens de Skorohod. Plus précisément il montre que $(u, v) \mapsto f'(x_{uv})1_{[0,s] \times [0,t]}$ est bien dans le domaine de l'opérateur de divergence δ^\diamond associé à x , ce qui leur permet de définir l'intégrale de gauche (voire plus loin dans la section (2.7.2)) et aboutir à la représentation en somme de Riemann suivante :

$$\int_0^t \int_0^s f'(x_{uv}) d^\diamond x_{uv} = \lim_{n,m \rightarrow +\infty} \sum_{i,j}^{n,m} f'(x_{s_i t_j}) \diamond \delta x_{s_i s_{i+1} t_j t_{j+1}}$$

pour deux subdivisions $\pi^1 = (s_i)_{i \leq n}$ et $\pi^2 = (t_j)_{j \leq m}$ dont le pas tend vers zéro et où \diamond désigne le produit de Wick. Pour la seconde intégrale de (1.4) ils remarquent le fait suivant :

$$\int_0^s \int_0^t f''(x_{uv}) d_1^\diamond x_{uv} d_2^\diamond x_{uv} = \int_0^s \int_0^t \int_0^u \int_0^v f''(x_{uv}) d^\diamond x_{uv'} d^\diamond x_{u'v}$$

ainsi ils interprètent cette dernière comme $\delta^{2,\diamond}(N(f(x)))$ où $\delta^{2,\diamond}$ étant le second opérateur de divergence et

$$N(f''(x))((u, v'); (u, v)) = f''(x_{uv})1_{[0,s] \times [0,v]}(u, v')1_{[0,u] \times [0,t]}(u', v)$$

On pourra encore se référer à la section (2.7.2) pour une définition plus rigoureuse. Puis ils obtiennent finalement une représentation en somme de Riemann donné par :

$$\int_0^s \int_0^t f''(x_{uv}) d_1^\diamond x_{uv} d_2^\diamond x_{uv} = \lim_{n,m \rightarrow +\infty} \sum_{i,j} f''(x_{s_i t_j}) \diamond (x_{s_{i+1} t_j} - x_{s_i t_j}) \diamond (x_{s_i t_{j+1}} - x_{s_i t_j})$$

Utilisant ces représentations et on prenant en considération les différentes contractions entre les produits de Wick les auteurs de [20] obtiennent la formule suivante :

$$\begin{aligned}
 f(x_{st}) &= f(0) + \int_0^t \int_0^s f^{(1)}(x_{uv}) d^\diamond x_{uv} + 2\gamma_1\gamma_2 \int_0^t \int_0^s f^{(2)}(x_{uv}) u^{2\gamma_1-1} v^{2\gamma_2-1} dudv \\
 &+ \int_0^t \int_0^s f^{(2)}(x_{uv}) d_1^\diamond x_{uv} d_2^\diamond x_{uv} + \gamma_1 \int_0^t \int_0^s f^{(3)}(x_{uv}) u^{2\gamma_1-1} v^{2\gamma_2} dud_2 x_{uv} \\
 &+ \gamma_2 \int_0^t \int_0^s f^{(3)}(x_{uv}) u^{2\gamma_1} v^{2\gamma_2-1} d_1 x_{uv} dv + \gamma_1\gamma_2 \int_0^t \int_0^s f^{(4)}(x_{uv}) u^{4\gamma_1-1} v^{4\gamma_2-1} dudv.
 \end{aligned} \tag{1.5}$$

pour f une fonction suffisamment "régulière" (en fait comme il est indiqué dans le second chapitre il faut une condition supplémentaire qui assure le fait que $f(X)$ ait de bonnes propriétés d'intégrabilité). Dans [21] les auteurs généralisent ce travail au cas ou $\gamma_1, \gamma_2 \in (0, 1)$ mais en interprétant les intégrales au sens de la divergence étendue, qui a des propriétés plus restrictives que la divergence usuelle et qui en revanche a un domaine plus large ce qui facilite sa manipulation.

1.2.4 Résultats du chapitre 2

Dans le second chapitre nous allons présenter le travail que j'ai effectué pendant la première partie de ma thèse conjointement avec Massimiliano Gubinelli puis avec Samy Tindel et qui a aboutit aux deux prépublications [13, 15]. Avant de donner les principaux résultats du second chapitre il seraient judicieux d'exposer une partie du travail de Gubinelli dans [36] pour le cas unidimensionnel.

Etant donné $k \in \mathbb{N}$ et une fonction γ -Hölderienne x sur $[0, T]$ à valeur dans \mathbb{R}^k pour un $\gamma > 1/3$ et x^n une régularisation de x alors on a trivialement le développement suivant :

$$\int_s^t f(x_u^n) \otimes dx_u^n = f(x_s^n) \otimes \delta x_{st}^n + f'(x_s^n) \otimes \int_s^t \int_s^u dx_a^n \otimes dx_u^n + r^n$$

avec cette fois-ci $\delta x_{st}^n = x_t^n - x_s^n$ et \otimes désigne le produit tensoriel sur \mathbb{R}^k . Alors la théorie des rough path a un paramètre nous dit que si $(\delta x^n, \iint dx^n dx^n)$ converge vers un couple $(\delta x, \iint dx dx)$ dans le sens où

$$\sup_{s,t \in [0,T]} \frac{|\delta(x^n - x)_{st}|}{|t-s|^\gamma} + \sup_{s,t \in [0,T]} \frac{|\int_s^t \int_s^u dx_a^n \otimes dx_u^n - \int_s^t \int_s^u dx_a \otimes dx_u|}{|t-s|^{2\gamma}} \rightarrow 0$$

alors il existe r tel que

$$\sup_{s,t \in [0,T]} \frac{|r_{st}^n - r_{st}|}{|t-s|^{3\gamma}} \rightarrow 0$$

et donc on construit l'intégrale rugueuse par

$$\int_s^t f(x_u) \otimes dx_u = f(x_s) \delta x_{st} + f'(x_{st}) \int_s^t \int_s^u dx_a \otimes dx_u + r_{st}$$

et l'objet limite $(\delta x, \iint dx \otimes dx)$ sera appelé rough path. La manière de Gubinelli de trouver ce résultat sera exposé dans le second chapitre à la section (2.2). L'approche de [13] fût alors de tensoriser le travail effectué dans [36] afin d'obtenir une généralisation de l'intégrale pour des draps irréguliers. Plus précisément on se donne un drap x de $[0, T]^2$ valeur scalaire tel que :

$$|\delta x_{s_1 s_2 t_1 t_2}| \leq C |s_2 - s_1|_1^\gamma |t_2 - t_1|_2^\gamma, \quad |\delta_1 x_{s_1 s_2 t_1}| \leq C_1 |s_2 - s_1|_1^\gamma, \quad |\delta_2 x_{s_1 t_1 t_2}| \leq C_2 |t_2 - t_1|_2^\gamma$$

pour tout $s_1, s_2, t_1, t_2 \in [0, T]$ où on a introduit les notations :

$$\delta x_{s_1 s_2 t_1 t_2} = x_{s_2 t_2} - x_{s_1 t_2} - x_{s_2 t_1} + x_{s_1 t_1}$$

$$\delta_1 x_{s_1 s_2 t_1} = x_{s_2 t_1} - x_{s_1 t_1}, \quad \delta_2 x_{s_1 t_1 t_2} = x_{s_1 t_2} - x_{s_1 t_1}$$

où aussi $\gamma_1, \gamma_2 > 1/3$ et $C, C_1, C_2 > 0$ des constantes finies. Maintenant si x^n est une régularisation de x et procédant comme dans le cas à un seul paramètre on a facilement la formule suivante :

$$\begin{aligned} \iint_{(s_1, t_1)}^{(s_2, t_2)} f(x_{st}^n) d_{st} x_{st}^n &= f(x_{s_1 t_1}^n) \delta x_{s_1 s_2 t_1 t_2}^n + \int_{s_1}^{s_2} (f(x_{st_1}^n) - f(x_{s_1 t_1}^n)) \int_{t_1}^{t_2} d_{st} x_{st}^n \\ &+ \int_{t_1}^{t_2} (f(x_{s_1 t}^n) - f(x_{s_1 t_1}^n)) \int_{s_1}^{s_2} d_{st} x_{st}^n + \iint_{(s_1, t_1)}^{(s_2, t_2)} \delta f(x^n)_{s_1 s_2 t_1 t_2} d_{st} x_{st}^n \end{aligned}$$

Ainsi en observant que

$$\delta f(x^n)_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} f'(x_{st}^n) d_{st} x_{st}^n + \iint_{(s_1, t_1)}^{(s_2, t_2)} f''(x_{st}^n) d_s x_{st}^n d_t x_{st}^n$$

on obtient aisément le développement suivant :

$$\begin{aligned} \iint_{(s_1, t_1)}^{(s_2, t_2)} \delta f(x^n)_{s_1 s_2 t_1 t_2} d_{st} x_{st}^n &= f'(x_{s_1 t_1}^n) \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} d_{uv} x_{uv}^n d_{st} x_{st}^n \\ &+ f''(x_{s_1 t_1}^n) \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} d_u x_{uv}^n d_v x_{uv}^n d_{st} x_{st}^n + \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left(\int_{s_1}^s f'(x_{ut_1}^n) - f'(x_{s_1 t_1}^n) \right) \int_{t_1}^t d_{uv} x_{uv}^n d_{st} x_{st}^n \\ &+ \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left(\int_{t_1}^t f'(x_{s_1 v}^n) - f'(x_{s_1 t_1}^n) \right) \int_{s_1}^s d_{uv} x_{uv}^n d_{st} x_{st}^n \\ &+ \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left(\int_{s_1}^s f''(x_{ut_1}^n) - f''(x_{s_1 t_1}^n) \right) \int_{t_1}^t d_u x_{uv}^n d_v x_{uv}^n d_{st} x_{st}^n \\ &+ \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left(\int_{s_1}^s f''(x_{ut_1}^n) - f''(x_{s_1 t_1}^n) \right) \int_{t_1}^t d_u x_{uv}^n d_v x_{uv}^n d_{st} x_{st}^n + r^n \end{aligned}$$

Donc pour continuer notre étude nous avons besoin de contrôler les intégrales de bord suivantes :

$$\begin{aligned} \int_{s_1}^{s_2} (f(x_{st_1}^n) - f(x_{s_1 t_1}^n)) \int_{t_1}^{t_2} d_{st} x_{st}^n, \quad \int_{s_1}^{s_2} (f(x_{st_1}^n) - f(x_{s_1 t_1}^n)) \int_{t_1}^{t_2} d_s x_{st}^n d_t x_{st}^n \\ \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left(\int_{s_1}^s f'(x_{ut_1}^n) - f'(x_{s_1 t_1}^n) \right) \int_{t_1}^t d_{uv} x_{uv}^n d_{st} x_{st}^n \end{aligned}$$

et

$$\int_{s_1}^{s_2} \int_{t_1}^{t_2} \left(\int_{s_1}^s f''(x_{ut_1}^n) - f''(x_{s_1 t_1}^n) \right) \int_{t_1}^t d_u x_{uv}^n d_v x_{uv}^n d_{st} x_{st}^n$$

Pour cela on remarque que seule la théorie des chemins rugueux a un paramètre est nécessaire. Afin de s'en convaincre on va considérer l'exemple suivant :

$$\int_{s_1}^{s_2} (f(x_{st_1}^n) - f(x_{s_1 t_1}^n)) \int_{t_1}^{t_2} d_{st} x_{st}^n = f'(x_{s_1 s_2}^n) \int_{s_1}^{t_1} \delta_1 x_{s_1 s_2 t_1}^n \int_{t_1}^{t_2} d_{st} x_{st}^n + r_1^n$$

où r_1^n un terme de reste dans la première direction, ainsi dès que $(\delta_1 x^n, \int_1 d_1 x^n \int_2 dx^n)$ converge vers $(\delta x, \int_1 d_1 x \int_2 dx)$ dans un espace Hölderien adéquat on aura que r_1^n converge aussi vers un terme de reste r_1 et donc on peut donner la définition suivante :

$$\int_{s_1}^{s_2} f(x_{s_1}) \int_{t_1}^{t_2} d_{st} x_{st} := f(x_{s_1 t_1}) \delta x_{s_1 s_2 t_1 t_2} + f'(x_{s_1 t_1}) \int_{s_1}^{s_2} \delta_1 x_{s_1 s_1} \int_{t_1}^{t_2} d_{st} x_{st} + r_1$$

Donc pour contrôler tous les termes du développement on doit faire l'hypothèse que les suites d'intégrales itérées :

$$\begin{aligned} & \int_{s_1}^{s_2} \int_{t_1}^{t_2} d_s x_{st}^n d_t x_{st}^n, \quad \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 s_1 t_1}^n d_{st} x_{st}^n, \quad \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 s_1 t_1}^n d_s x_{st}^n d_t x_{st}^n \\ & \iint_{(s_1, t_1)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(s, t)} d_{rr'} x_{rr'}^n \right) d_{st} x_{st}^n, \quad \iint_{(s_1, t_1)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(s, t)} d_r x_{rr'}^n d_{r'} x_{rr'}^n \right) d_{st} x_{st}^n, \quad \iint_{(s_3, t_1)}^{(s_4, t_2)} \left(\iint_{(s_1, t_1)}^{(s_2, t)} d_{rr'} x_{rr'}^n \right) d_{st} x_{st}^n \\ & \iint_{(s_3, t_1)}^{(s_4, t_2)} \left(\iint_{(s_1, t_1)}^{(s_2, t)} \delta_1 x_{s_1 r t_1}^n d_{rr'} x_{rr'}^n \right) d_{st} x_{st}^n, \quad \iint_{(s_3, t_2)}^{(s_4, t_3)} \left(\iint_{(s_1, t_2)}^{(s_2, t)} \left(\iint_{(s_1, t_1)}^{(r, t_2)} d_{ab} x_{ab}^n \right) d_{rr'} x_{rr'}^n \right) d_{st} x_{st}^n \end{aligned}$$

(et bien d'autres...) convergent dans un espace topologique $\mathcal{H}_{\gamma_1, \gamma_2}$ liée à leurs régularités Hölderienne respectives, et l'objet limite noté \mathbb{X} sera appelé rough sheet. Ce qui nous donne le premier résultat important énoncé (dans un cadre plus général) et prouver d'une manière rigoureuse dans le second chapitre dans les théorèmes (2.5.6) et (2.5.21) qu'on va ici reformuler d'une manière plus compréhensible pour un lecteur qui n'est pas familier avec la théorie des chemins rugueux contrôlés.

Théoreme 1.2.10. *Soit un drap x et x^n une régularisation de x tel que \mathbb{X}^n converge vers \mathbb{X} dans $\mathcal{H}_{\gamma_1, \gamma_2}$, alors on a que :*

$$\sup_{s_1 \neq s_2, t_1 \neq t_2 \in [0, T]} \frac{\left| \int_{s_1}^{s_2} \int_{t_1}^{t_2} f(x_{uv}^n) dx_{uv}^n - \int_{s_1}^{s_2} \int_{t_1}^{t_2} f(x_{uv}) dx_{uv} \right|}{|s_1 - s_2|^{\gamma_1} |t_2 - t_1|^{\gamma_2}} \rightarrow^{n \rightarrow +\infty} 0$$

pour toute fonction $f \in C^8(\mathbb{R})$ où $\iint f(x) dx$ est définie par la formule

$$\iint f(x) dx := f(x) \delta x + \sum_{a=1,2} \left(\int_a \delta_a f(x) \int_{\hat{a}} dx + \int_a \delta_a f'(x) \int_{\hat{a}} dx dx + \int_a \delta_a f''(x) \int_{\hat{a}} d_1 x d_2 x dx \right) + r$$

avec r un terme de reste et la convention que si $a = 1$ alors $\hat{a} = 2$ et inversement.

Une remarque avant de continuer est qu'on a exactement le même résultat pour une intégrale de type $\iint f(x) d_1 x d_2 x$. Maintenant une application de ce résultat est de construire une intégrale du type Stratonovich pour le drap Brownien fractionnaire et d'obtenir une formule de changement de variable dans ce cas.

Théoreme 1.2.11. *Soit x un drap Brownien fractionnaire d'indice d'Hurst $\gamma_1, \gamma_2 > 1/3$ on rappelle que x a la représentation harmonisable suivante*

$$x_{st} = C_{\gamma_1, \gamma_2} \iint_{\mathbb{R}^2} \mathcal{Q}_{st}(x, y) \hat{W}(dx, dy), \quad \mathcal{Q}_{st}(x, y) = \frac{e^{isx} - 1}{|x|^{\gamma_1+1/2}} \frac{e^{ity} - 1}{|y|^{\gamma_2+1/2}}$$

et on définit le processus x^n par

$$x_{st}^n = C_{\gamma_1, \gamma_2} \iint_{|x|, |y| \leq n} \mathcal{Q}_{st}(x, y) \hat{W}(dx, dy)$$

On a alors que le rough sheet associé à x^n noté par \mathbb{X}^n converge dans $L^p(\Omega, \mathcal{H}_{\alpha, \beta})$ pour tout $\alpha < \gamma_1$ et $\beta < \gamma_2$, ce qui permet de définir le rough sheet associé à x comme la limite \mathbb{X} de \mathbb{X}^n . De plus si $f \in C^{10}(\mathbb{R})$ est une fonction satisfaisant la condition (GC) (voir (2.6.6)) alors on a que

$$\delta f(x) = \iint f'(x) dx + \iint f''(x) d_1 x d_2 x$$

Dans [15] avec S.Tindel nous avons essayé de relayer ce travail avec celui effectué par C.Tudor et F.Viens dans [20] pour cela nous avons pris du recul et essayé de mieux de comprendre le cas où le drap Brownien fractionnaire est d'indices d'Hurst $\gamma_1, \gamma_2 > 1/2$. Dans ce cadre l'intégrale rugueuse à deux paramètres se résume seulement à une intégrale d'Young dans le plan (voire par exemple [70] pour plus d'informations sur ce sujet ou la section (2.4) du second chapitre) et on a remarqué que ces deux notions d'intégration se généralisaient bien à d'autres processus Gaussiens dont la fonction de covariance satisfait une certaine propriété de factorisation. Plus précisément on a le résultat suivant :

Théoreme 1.2.12. . Soit x un processus Gaussian $[0, 1]^2$ de fonction de covariance satisfaisant l'hypothèse (2.7.4), et on considère une fonction $f \in C^4(\mathbb{R})$ satisfaisant la condition (GC). Alors on a que :

$$z_{s_1 s_2 t_1 t_2}^{1, \diamond} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} f'(x_{uv}) d_{12}^{\diamond} x_{uv}, \quad \text{et} \quad z_{s_1 s_2 t_1 t_2}^{2, \diamond} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} f''(x_{uv}) d_1^{\diamond} x d_2^{\diamond} x_{uv}, \quad (1.6)$$

sont bien définies dans le sens de Skorohod pour le calcul de Malliavin. De plus on a :

(i) Convergence de somme de Riemann : si π_n^1 et π_n^2 sont deux partitions de $[s_1, s_2] \times [t_1, t_2]$ dont le pas tend vers 0 quand $n \rightarrow \infty$, alors

$$\lim_{n \rightarrow \infty} \sum_{\pi_n^1, \pi_n^2} f(x_{\sigma_i; \tau_j}) \diamond \delta x_{\sigma_i \sigma_{i+1} \tau_j \tau_{j+1}} = z_{s_1 s_2 t_1 t_2}^{1, \diamond} \quad (1.7)$$

$$\lim_{n \rightarrow \infty} \sum_{\pi_n^1, \pi_n^2} f''(x_{\sigma_i; \tau_j}) \diamond \delta_2 x_{s_i; t_j t_{j+1}} \diamond \delta_1 x_{s_i s_{i+1}; t_j} = z_{s_1 s_2; t_1 t_2}^{2, \diamond}, \quad (1.8)$$

où \diamond dénote le produit de Wick, et où toutes les convergences ont lieu dans $L^2(\Omega)$.

(ii) Une formule de changement de variable pour $y = f(x)$:

$$\begin{aligned} \delta f(x)_{s_1 s_2 t_1 t_2} &= z_{s_1 s_2 t_1 t_2}^{1, \diamond} + z_{s_1 s_2 t_1 t_2}^{2, \diamond} \\ &+ \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} f''(x_{uv}) d_1 R_u^1 d_2 R_v^2 + \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} f^{(3)}(x_{uv}) R_u^1 d_2 R_v^2 d_1^{\diamond} x_{uv} \\ &+ \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} f^{(3)}(x_{uv}) R_v^2 d_1 R_u^1 d_2^{\diamond} x_{uv} + \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} f^{(4)}(x_{uv}) R_u^1 R_v^2 d_1 R_u^1 d_2 R_v^2. \end{aligned} \quad (1.9)$$

(iii) Des termes de corrections explicites entre z^1 , z^2 et $z^{1, \diamond}$, $z^{2, \diamond}$ peuvent être calculés (voire les relations (2.111) et (2.119)) avec

$$z^1 = \iint f'(x) dx, \quad z^2 = \iint f''(x) d_1 x d_2 x$$

où ces intégrales sont aux sens de Young introduit dans la proposition (2.4.1).

On peut voir que le drap Brownien fractionnaire de paramètre d'Hurst $\gamma_1, \gamma_2 > 1/2$ entre bien dans le cadre de ce théorème. Pour généraliser ce résultat au cas rugueux $\gamma_1, \gamma_2 > 1/3$ on procède par régularisation en considérant l'approximation x^n introduite dans le théorème (1.2.10) et en remarquant que x^n satisfait bien les conditions du théorème (1.2.12), un passage à la limite nous donne le résultat suivant :

Théorème 1.2.13. *Soit x un drap Brownien fractionnaire sur $[0, 1]^2$, avec indice d'Hurst $\gamma_j > 1/3$ pour $j = 1, 2$. alors les incréments $z^{1,\diamond}, z^{2,\diamond}$ de l'équation (1.6) sont bien définies aux sens de Skorohod de plus on a que :*

- (i) $z^{1,\diamond}$ and $z^{2,\diamond}$ peuvent être vu comme limite respective de $z^{n,1,\diamond}$ et $z^{n,2,\diamond}$ qui sont donner par le Théorème 2.7.5 pour le processus régulariser x^n .
- (ii) Pour tout $f \in C^{10}(\mathbb{R})$, la formule de changement de variable suivante

$$\begin{aligned} f(x_{s;t}) &= f(0) + z_{0s,0t}^{1,\diamond} + z_{0s,0t}^{2,\diamond} \\ &\quad + 2\gamma_1\gamma_2 \int_0^s \int_0^t f''(x_{uv}) u^{2\gamma_1-1} v^{2\gamma_2-1} dudv + \gamma_2 \int_0^s \int_0^t f^{(3)}(x_{uv}) u^{2\gamma_1} v^{2\gamma_2-1} d_1^\diamond x_{u,v} dv \\ &\quad + \gamma_1 \int_0^s \int_0^t f^{(3)}(x_{uv}) u^{2\gamma_1-1} v^{2\gamma_2} d_2^\diamond x_{u,v} du + \gamma_1\gamma_2 \int_0^s \int_0^t f^{(4)}(x_{uv}) u^{4\gamma_1-1} v^{4\gamma_2-1} dudv. \end{aligned} \quad (1.10)$$

- (iii) Des termes de corrections explicite entre z^1, z^2 et $z^{1,\diamond}, z^{2,\diamond}$ peuvent être calculés (voire les relations (2.136) et (2.144)) avec

$$z^1 = \iint f'(x) dx, \quad z^2 = \iint f''(x) d_1 x d_2 x$$

où ces intégrales sont comprises aux sens des rough-sheet introduit dans le théorème (1.2.10).

1.3 État de l'art et résultats du Chapitre 3

Dans le troisième chapitre de cette thèse on expose un travail que j'ai effectué avec Massimiliano Gubinelli et qui a consisté à étudier une classe d'EDP dispersive perturbées par des modulations non homogènes et irrégulières. Plus précisément on s'est intéresser au problème de Cauchy suivant :

$$\begin{cases} \frac{du_t}{dt} = Au_t \frac{dw_t}{dt} + \mathcal{N}(u_t) \\ u(0, x) = u^0(x) \in H^\alpha(T) \end{cases} \quad (1.11)$$

où $H^\alpha(T)$ étant l'espace de Sobolev réel ou périodique (ie : $T = \mathbb{R}, \mathbb{T}$), A un opérateur qui engendre un groupe $U_t = e^{tA}$, $\mathcal{N}(\phi)$ une non-linéarité polynômiale et w une modulation qui va être une fonction "irrégulière" telle qu'une trajectoire d'un mouvement Brownien fractionnaire. Pour être plus précis on va s'intéresser aux cas suivants :

1. $A = \partial^3$, $\mathcal{N}(\phi) = \partial\phi^2$, $T = \mathbb{R}, \mathbb{T}$ avec \mathbb{T} étant le Tore a une dimension (équation de KdV modulé).
2. $A = \partial^3$, $\mathcal{N}(\phi) = \partial\phi^3$, $T = \mathbb{T}$ avec \mathbb{T} (équation mKdV modulé).
3. $A = i\Delta$, $\mathcal{N}(\phi) = i|\phi|^2\phi$, $T = \mathbb{R}^n, \mathbb{T}$, $n = 1, 2$ (équation NLS cubique modulé)
4. $A = i\partial^2$, $\mathcal{N}(\phi) = i\partial(|\phi|^2 - |\phi|_{H^0}\phi)$, $T = \mathbb{T}$, (équation dNLS modulé)

où ∂ (respectivement Δ) dénote la dérivée en espace (respectivement le Laplacien spatial). Avant de nous pencher sur les équations du type (1.11) pour des modulations "irrégulière" nous rappelons que dans le cas homogène (ie : $w_t = t$) ces équations ont donné naissance à beaucoup de travaux mathématique dont on essayera d'exposer certains dans la sous-section suivante.

1.3.1 EDP dispersive en milieu homogène : théorie d'existence et unicité

La théorie d'existence de solution pour les équations de type KdV ou NLS a été étudiée de manière intensive ces trente dernières années dans de nombreux travaux mathématiques dont on peut citer a titre d'exemple [4, 17, 54]. Nous allons essayer de présenter ici d'une manière abrégée certains outils mathématiques qui ont permis l'étude de telles équations. L'équation qui nous intéresse dans cette section est donnée par :

$$\partial_t \phi_t = A\phi_t + \mathcal{N}(\phi_t)$$

où ∂_t désigne la dérivée temporelle, $A = ih(\partial/i)$ pour un polynôme réel h et $\mathcal{N}(\phi)$ une certaine non-linéarité. L'approche remarquable introduite par Bourgain dans [4] à consisté à regarder en un premier lieu l'équation linéaire donné par :

$$\partial_t \phi_t = A\phi_t$$

et de remarquer que si $\hat{\phi}(\tau, k)$ est la transformée de Fourier en espace de temps de la solution ϕ alors elle est supportée par l'hypersurface $\{(\tau, k) \in \mathbb{R}; \tau = h(k)\}$, de même que si on procède à une localisation de ϕ en temps (en la multipliant par un cut-off η à support compact) alors la fonction $\hat{\eta}\phi$ aura un support contenu dans l'ensemble $\{(\tau, k) \in \mathbb{R}; \tau = h(k) + O(1)\}$. En un second lieu il a été observé de manière surprenante que ce phénomène persiste pour l'équation non-linéaire pour une large classe de donné initiale irrégulière. Une manière de capturer ce phénomène de régularisation par dispersion se fait via les espaces $X^{\alpha,b}(\mathbb{R}^2)$ ou plus communément appelé espace de Bourgain dont la définition est la suivante

Définition 1.3.1. Soit $\alpha, b \in \mathbb{R}$ alors $X^{\alpha,b}(\mathbb{R}^2)$ est définie comme l'espace des distributions de Schwartz $u \in \mathcal{S}'(\mathbb{R}^2)$ tel que :

$$\|u\|_{X^{\alpha,b}}^2 = \iint_{\mathbb{R}^2} (1 + |k|)^{2\alpha} (1 + |\tau - h(k)|)^{2b} |\hat{u}(\tau, k)|^2 d\tau dk < +\infty$$

bien sur dans les cas qui nous intéressent il suffit de prendre $h(k) = k^3$ dans le cas de KdV ($A = \partial^3$) et $h(k) = -k^2$ dans le cas de Schrödinger ($A = i\partial^2$).

L'obtention d'estimées multilinéaires dans ces espaces a permis à plusieurs auteurs (voire par exemple [54]) de fonder une théorie d'existence locale et même globale pour (KdV) et (NLS) assez robuste. À titre d'exemple on peut énoncer le proposition suivante donné dans [54] et qui est due à Kenig, Ponce et Vega

Proposition 1.3.2 (Kenig, Ponce, Vega). Pour tout $\alpha \in (-3/4, 0]$ il existe $b \in (1/2, 1)$ tel que la distribution

$$B(F, F) = 1/2\partial(F^2)$$

bien définie pour tout $F \in X^{\alpha,b}$, de plus on a que

$$\|B(F, F)\|_{X^{\alpha,b-1}(\mathbb{R}^2)} \lesssim \|F\|_{X^{\alpha,b}(\mathbb{R}^2)}^2$$

Ce résultat leur a permis de montrer que l'équation de (KdV) sur \mathbb{R} possédait une unique solution locale via une méthode de point fixe dans l'espace $X^{\alpha,b}$ pour toute condition initiale de $H^\alpha(\mathbb{R})$ pour $\alpha > -3/4$ et que dans ce cas il avait une continuité du flot par rapport à la donnée initiale. L'existence globale pour ce type d'équation dans les espaces de Sobolev d'index positive est due à certaines lois de conservation et à un certain phénomène de persistance de la régularité ce qui n'est pas le cas dans la basse régularité, en effet cette dernière a été démontré dans le cas de (KdV) usant de la I -méthode introduite dans [18]. Cependant cette l'approche basée sur les espaces de Bourgain est reliée profondément à la structure de l'équation linéaire. Cela pose un problème dans notre cas. Pour contourner cette difficulté nous allons présenter une approche plus récente basée sur les chemins contrôlés due à Gubinelli [37] et qui lui a permis d'étudier l'équation KdV dans un domaine périodique. En effet en partant de la formulation mild et en utilisant la propriété de groupe on peut écrire l'équation de KdV dans la forme suivante :

$$v_t = u^0 + \int_0^t U_s^{-1} \partial((U_s v_s)^2) ds \quad (1.12)$$

où $v_t = e^{t\partial^3} u_t$, u étant la solution du problème de Cauchy associé à KdV et u^0 la condition initiale du problème. Ainsi un développement du second ordre nous donne que :

$$v_t - v_s = X_{st}(v_s, v_s) + X_{st}^2(v_s, v_s, v_s) + r_{st} \quad (1.13)$$

avec r un terme de reste et :

$$\begin{aligned} X_{st}(\psi_1, \psi_2) &= \int_s^t U_\sigma^{-1} \partial(U_\sigma \psi_1 U_\sigma \psi_2) d\sigma, & X_{st}^2(\psi_1, \psi_1, \psi_3) \\ &= 2 \int_s^t \int_s^\sigma U_\sigma^{-1} \partial(U_\sigma \psi_1 U_\sigma^{-1} \partial(U_{\sigma_1} \psi_2 U_{\sigma_1} \psi_3)) d\sigma_1 d\sigma \end{aligned}$$

pour des fonction test ψ_1, ψ_2, ψ_3 . Une étude plus approfondie de ces deux derniers opérateurs dans [37] a conduit l'auteur aux estimations suivantes

$$|X_{st}|_{\mathcal{L}^2(H^\alpha(\mathbb{T}))} \lesssim |t-s|^\gamma, \quad |X_{st}^2|_{\mathcal{L}^3(H^\alpha(\mathbb{T}))} \lesssim |t-s|^{2\gamma}$$

pour tout $\alpha > -1/2$ avec $\mathcal{L}^2(H^\alpha(\mathbb{T}))$ (respectivement $\mathcal{L}^3(H^\alpha(\mathbb{T}))$) étant l'espace des opérateurs bilinéaires (respectivement tri-linéaires) de H^α munis de sa norme usuelle et $\gamma > 1/3$ une constante ne dépendant que de α . Ainsi on définit l'espace des chemins contrôlés par les fonctions θ qui ont la forme :

$$\theta_t - \theta_s = X_{st}(\theta'_s) + \theta_s^\sharp$$

pour un certain couple $(\theta', \theta^\sharp) \in \mathcal{C}_1^\gamma(H^\alpha(\mathbb{T})) \times \mathcal{C}_2^{2\gamma}(H^\alpha(\mathbb{T}))$, tel que les espaces \mathcal{C}_1^γ et $\mathcal{C}_2^{2\gamma}$ sont définies de la manière suivante :

$$\mathcal{C}_1^\gamma(H^\alpha) = \left\{ f \in C([0, T], H^\alpha(\mathbb{T})), \quad \|f\|_\gamma = \sup_{s \neq t \in [0, T]} \frac{|f_t - f_s|}{|t - s|^\gamma} < +\infty \right\}$$

et

$$\mathcal{C}_2^{2\gamma}(H^\alpha) = \left\{ g \in C([0, T]^2, H^\alpha(\mathbb{T})), \quad g_{tt} = 0, t \in [0, T]; \quad \|g\|_{2\gamma} = \sup_{s \neq t \in [0, T]} \frac{|g_{st}|}{|t - s|^{2\gamma}} < +\infty \right\}$$

Maintenant la théorie des chemins contrôlés développée par Gubinelli [36] nous dit que si v est un chemin contrôlé et que si de plus v satisfait la relation (1.13) avec un terme de reste $r \in \mathcal{C}_3^{3\gamma}$ alors dans ce cas r sera complètement déterminé par la donnée de X , X^2 , v' et v^\sharp de plus il va s'écrire comme une fonctionnelle multiplicative de ces derniers termes. Par la suite il est possible interpréter l'éq (1.12) comme l'équation de point fixe suivante :

$$v_t - v_s = X_{st}(v_s, v_s) + X_{st}^2(v_s, v_s, v_s) + r_{st}, \quad v_0 = u^0$$

formulée dans l'espace des chemins contrôlés munis d'une structure topologique adéquate. Cette approche a permis à Gubinelli de construire une solution locale v associée à l'équation mild de KdV et d'obtenir le résultat suivant

Théoreme 1.3.3 (Gubinelli). *Soit $u^0 \in H^\alpha(\mathbb{T})$ pour $\alpha > -1/2$ alors il existe $T^* > 0$ et un unique $v \in C_1^\gamma(H^\alpha(\mathbb{T}))$ tel que*

$$v_t = v_s + X_{st}(v_s, v_s) + X_{st}^2(v_s, v_s, v_s) + O(|t - s|^{3\gamma}).$$

Si on défini $u_t = U_t v_t$ et si $\Pi_N u$ désigne le projecteur de Dirichlet sur les mode de Fourier plus petits que N , alors il existe une distribution espace-temps $\mathcal{N}(u) \in \mathcal{S}'([0, T] \times \mathbb{T})$ tel que $1/2\partial(\Pi_N u)^2$ converge vers $\mathcal{N}(u)$ (pour la topologie de $\mathcal{S}'([0, T] \times \mathbb{T})$). De plus u satisfait l'équation

$$\partial_t u_t + \partial^3 u + \mathcal{N}(u) = 0$$

au sens des distributions.

Cette dernière approche sera reprise dans le troisième chapitre de cette thèse pour étudier les équations (KdV), (NLS) et aussi d'autres équations avec une dispersion modulée.

1.3.2 Cadre stochastique

Les équations dispersives ont suscité l'intérêt des probabilistes pendant ces vingt dernières années étant données qu'on pouvait considérer ces mêmes modèles dans des environnements aléatoires, les exemples que nous avons en tête sont les travaux effectués par A. Debussche, A. De Bouard et Y. Tsutsumi dans [27] où ils considèrent l'équation de KdV perturbé par un bruit stochastique de la manière suivante :

$$\partial_t u + \partial_x^3 u + 1/2 \partial_x u^2 = \xi$$

où ξ est un bruit Gaussien qui est blanc en temps et plus régulier en espace, et ils arrivent à donner un sens à cette équation usant du calcul stochastique puis à obtenir l'existence et unicité de solution dans un espace de Bourgain modifié. Dans le même esprit on pourrait aussi citer un travail de même nature effectué dans [25] pour l'équation non linéaire de Schrödinger. Plus récemment les auteurs de [28, 30] étudient (NLS) pour une dispersion stochastique et plus exactement ils s'intéressent à l'équation :

$$\frac{du_t}{dt} = i\Delta u_t \circ dW_t + i|u|^{2\sigma} u$$

où W est un mouvement Brownien d dimensionnel et \circ la multiplication de Stratonovich. Ensuite ils réécrivent cette équation comme une équation aux dérivées partielles stochastique de la manière suivante :

$$idu_t + \Delta^2 u_t dt + \Delta u_t dW_t + |u|^{2\sigma} u dt = 0 \quad (1.14)$$

où le terme $\Delta u dW_t$ est interprété aux sens d'Itô. Obtenant des estimés du type Strichartz ils aboutissent au résultat suivant :

Théoreme 1.3.4 (A.De Bouard, A.DeBussche). *Soit $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ un espace de probabilité filtré associé et W un $(\mathcal{F}_t)_{t \geq 0}$ -mouvement Brownien unidimensionnelle. Soit $u^0 \in L^2(\mathbb{R}^d)$ (presque sûrement) qui soit \mathcal{F}_0 mesurable et on suppose finalement que $\sigma < 2/d$ alors il existe une unique solution u pour le problème de Cauchy (1.14) de donnée initiale u^0 tel que $u \in L^r_{loc}(0, \infty, L^p(\mathbb{R}^d))$ presque sûrement avec $p = 2\sigma + 2 \leq r < 4(\sigma + 1)/(d\sigma)$, de plus $u \in C(\mathbb{R}^+, L^2(\mathbb{R}^d))$ et si $u^0 \in H^1(\mathbb{R}^d)$ alors $u \in C(\mathbb{R}^+, H^1(\mathbb{R}^d))$.*

Une étude plus poussée de ces équations effectuée dans [30] a permis d'affiner les estimées obtenue dans [28] et de traiter le cas de NLS "quintic" ($d = 1, \sigma = 2$) et d'avoir le théorème suivant :

Théoreme 1.3.5 (A.DeBussche, Y.Tsutsumi). *Soit $u^0 \in L^2(\mathbb{R})$ (presque sûrement) qui soit \mathcal{F}_0 mesurable, et $\sigma = 2$ alors il existe une solution u pour l'équation (1.14) tel que $u \in L^5_{loc}(\mathbb{R}^+, L^5(\mathbb{R}))$ (p.s), de plus $u \in C(\mathbb{R}^+, L^2(\mathbb{R}))$. Maintenant si $u^0 \in H^1(\mathbb{R})$ alors $u \in C(\mathbb{R}^+, H^1(\mathbb{R}))$.*

Nous verrons que dans le troisième chapitre que l'étude de ces équations peut se généraliser pour le mouvement Brownien fractionnaire et qu'aucune hypothèse de mesurabilité n'est nécessaire.

1.3.3 Résultats du chapitre 3

Nous présentons ici les principaux résultats du troisième chapitre obtenu en collaboration avec M.Gubinelli dans [14]. Pour cela nous allons commencer par rappeler un résultat d'intégration qui est en fait juste une généralisation de l'intégrale de Young dans un cas non linéaire. Soit V un espace d'Hilbert et on note par $\mathcal{L}^n(V, V)$ l'espace des opérateurs n -linéaire de $V^{\otimes n}$ à valeur dans V dont ont le muni de sa norme d'opérateur. Soit $T > 0$ et $\mathcal{C}^\gamma V = C^\gamma([0, T], V)$ l'espace des fonctions γ -Hölder continues de $[0, T]$ a valeurs dans V équipé de sa semi-norme

$$\|f\|_{\mathcal{C}^\gamma V} = \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_V}{|t - s|^\gamma}.$$

Maintenant si V est un espace de Banach alors $\text{Lip}_M(V)$ dénote l'espace de Banach des fonctions localement Lipschitz sur V de croissance polynomiale d'ordre $M \geq 0$, et plus précisément des fonctions $f : V \rightarrow V$ tel que

$$\|f\|_{\text{Lip}_M(V)} = \sup_{x, y \in V} \frac{\|f(x) - f(y)\|_V}{\|x - y\|_V (1 + \|x\|_V + \|y\|_V)^M} < +\infty.$$

Ainsi on a le résultat suivant :

Proposition 1.3.6. [Young] *Soit $f \in \mathcal{C}^\gamma \text{Lip}_M(V)$ et $g \in \mathcal{C}^\rho V$ tel que $\gamma + \rho > 1$ alors la limite des sommes de Riemann*

$$I_t = \int_0^t f_{du}(g_u) = \lim_{|\Pi| \rightarrow 0} \sum_i f_{t_{i+1}}(g_{t_i}) - f_{t_i}(g_{t_i})$$

existe dans V où Π est une subdivision de $[0, t]$ et $|\Pi|$ son pas, et elle indépendante de la subdivision. De plus on a que

$$\|I_t - I_s - (f_t - f_s)(g_s)\|_V \leq (1 - 2^{1-\gamma-\rho})^{-1} \|g\|_{\mathcal{C}^\rho V} (1 + \|g\|_{\mathcal{C}^0 V})^M |t - s|^{\gamma+\rho}.$$

Démonstration. Une preuve plus directe est donnée dans le troisième chapitre dans la démonstration du Théorème (3.3.1), ici nous proposons une preuve courte qui fait appel à l'existence de l'application de la couturiere (sewing-map) Λ définie dans la Proposition (2.2.1), en effet si f^n est une régularisation de f et g^n celle de g tel que $\sup_n \|g^n\|_\rho \lesssim \|g\|_\rho < +\infty$ et $\sup_n \|f^n\|_{\mathcal{C}^\gamma \text{Lip}_M(V)} \lesssim \|f\|_{\mathcal{C}^\gamma \text{Lip}_M(V)} < +\infty$ alors il est facile de voir que par définition de Λ on a que :

$$I_t^n - I_s^n = \int_s^t f_{d\sigma}^n(g_\sigma^n) = (f_t^n - f_s^n)(g_s) + \Lambda(u^n)_{st}$$

où $u_{sut}^n = (f_t^n - f_u^n)(g_u^n) - (f_t^n - f_s^n)(g_s^n)$ pour $0 \leq s < u < t \leq T$ et ainsi on voit que

$$\sup_n |u_{sut}^n|_V \lesssim_{\rho, \gamma} |t - u|^\rho |u - s|^\gamma \|g\|_\rho \|f\|_{\mathcal{C}^\gamma \text{Lip}_M(V)}$$

de plus on a que u^n converge vers u dans $\mathcal{C}_3^\delta V$ (voir la (2.2) pour la définition de cet espace) avec $u_{sut} = (f_t - f_u)(g_u) - (f_t - f_s)(g_s)$ et ceci pour tout $1 < \delta < \gamma + \rho$ ainsi la continuité du sewing-map Λ (voir la (2.2.1) pour plus de détail) nous assure que $\Lambda(u^n)$ converge vers $\Lambda(u)$ dans $\mathcal{C}_2^\delta V$ d'où la convergence de la suite I^n vers I dans l'espace $\mathcal{C}^{\gamma'}(V)$ pour tout $\gamma' < \gamma$ où I est définie par la formule suivante :

$$I_t = I_s + (f_t - f_s)(g_s) + \Lambda[u]_{st}$$

$I_0 = 0$. Maintenant en tenant compte encore une fois de la continuité du sewing-map Λ et de la régularité de u on obtient que $\Lambda(u) \in \mathcal{C}_2^{\gamma+\rho}$ ce qui nous permet d'avoir la convergence suivante :

$$I_t = \int_0^t f_{du}(g_u) = \lim_{|\Pi| \rightarrow 0} \sum_i f_{t_{i+1}}(g_{t_i}) - f_{t_i}(g_{t_i})$$

L'estimation donnée dans le théorème provient encore de la continuité du sewing map, ce qui finit la preuve. \square

Maintenant en regardant d'un peu plus près la formulation mild des équations (1.11) on voit qu'elles ont la forme :

$$u_t = U_t^w u^0 + \int_0^t U_t^w (U_s^w)^{-1} \mathcal{N}(u_s) ds$$

avec $U_t^w = e^{Aw_t}$, une remarque importante dans ce cas est que cette formulation ne fait plus apparaitre la dérivée de la modulation w et donc celle-ci n'a pas besoin d'être différentiable pour étudier cette dernière équation de plus en tenant compte du fait U^w est un groupe on aboutit à l'équation intégrale suivante :

$$\theta_t = u^0 + \int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w \theta_s) ds \quad (1.15)$$

avec $\theta_t = (U_t^w)^{-1} u_t$. Soit l'opérateur $X_t(\phi) = \int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w \phi) ds$ on voit alors de manière formelle que :

$$\int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w \theta_s) ds = \int_0^t \left[\frac{dX_\sigma}{d\sigma} \right] (\theta_\sigma) = \lim_{|\Pi| \rightarrow 0} \sum_{\Pi} X_{t_i; t_{i+1}}(\theta_{t_i})$$

où la limite est prise sur les subdivisions Π de $[0, t]$ dont le pas tend vers 0, et $X_{s;t} = X_t - X_s$. Une manière de donner un sens à ce calcul formel est de passer par la proposition (1.3.6) et donc sous l'hypothèse que $X \in \mathcal{C}^\gamma \text{Lip}_M(V)$ avec $\gamma > 1/2$ on donne sens à l'équation (1.15) dans $\mathcal{C}^{1/2} V$, puis on

arrive à la résoudre par une méthode de point fixe. Donnons-nous maintenant un exemple concret pour comprendre mieux l'hypothèse faite sur X , en effet dans le cas de KdV (ie : $A = \partial^3$, $\mathcal{N}(u) = \partial u^2$) on a par un simple calcul que :

$$\hat{X}_{st}(\phi_1, \phi_2) = ik \sum_{k_1+k_2=k} \hat{\phi}_1(k_1)\hat{\phi}_2(k_2)\Phi_{st}^w(3kk_1k_2)$$

où $\Phi_{st}^w(a) = \int_s^t e^{iaw_\sigma} d\sigma$ et \hat{X} la transformée de Fourier de l'opérateur X , et donc pour contrôler la norme de l'opérateur X il faut savoir contrôler la quantité Φ^w , une réponse a été donnée par R.Catellier et M.Gubinelli dans [12] pour le mouvement Brownien fractionnaire.

Théoreme 1.3.7 (R.Catellier, M.Gubinelli). *Soit W un mBf d'indice d'Hurst $H \in (0, 1)$ et introduisant la quantité :*

$$\|\Phi^W\|_{\mathcal{W}_T^{\gamma, \rho}} = \sup_{a \in \mathbb{R}} \sup_{0 \leq s < t \leq T} |a|^\rho \frac{|\Phi_{st}^W(a)|}{|t-s|^\gamma}$$

pour $0 < \gamma < 1$ et $\rho > 0$ alors il existe $\delta > 1/2$ tel que pour tout $\rho < 1/(2H)$ on a que :

$$\|\Phi^W\|_{\mathcal{W}_T^{\gamma, \rho}} < +\infty$$

presque surement, pour tout $T > 0$.

En fait le théorème tel qu'il est énoncé dans [12] nous donne même l'intégrabilité exponentielle de Φ^W dans le cas du mBf. Ce résultat cela nous pousse à introduire la définition suivante :

Définition 1.3.8. *Soit $\rho > 0$ et $\gamma > 0$. On dit qu'une fonction $w \in C([0, T]; \mathbb{R})$ est (ρ, γ) -irrégulière si pour tout $T > 0$:*

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} = \sup_{a \in \mathbb{R}} \sup_{0 \leq s < t \leq T} \langle a \rangle^\rho \frac{|\Phi_{s,t}^w(a)|}{|s-t|^\gamma} < +\infty$$

où $\Phi_{s,t}^w(a) = \int_s^t e^{iaw_r} dr$. De plus on dit que w is ρ -irrégulier si il existe $\gamma > 1/2$ tel que w est (ρ, γ) -irregulière.

Maintenant on va introduire l'espace des chemins contrôlés qui va être l'espace sur lequel l'unicité de nos solutions est garantie.

Définition 1.3.9. *On dit qu'une fonction $\phi \in C([0, T], V)$ est contrôlé par w si $\theta_t = (U_t^w)^{-1}\phi_t \in \mathcal{C}^{1/2}V$, et on note $\mathcal{D}^w(V)$ cet espace.*

Alors sous l'hypothèse que w est (ρ, γ) -irrégulier avec $\gamma > 1/2$ on obtient les résultats principaux du troisième chapitre qui sont les suivants :

Théoreme 1.3.10. *Pour tout $\rho > 3/4$ et $\alpha > -\rho$ l'équation de KdV sur le Tore avec modulation w admet une solution locale dans $H^\alpha(\mathbb{T})$, elle est globale si $\alpha \geq -3/2$ et $\alpha > -\rho/(3-2\gamma)$. De plus on a unicité dans l'espace $\mathcal{D}^w(H^\alpha) \subseteq C(\mathbb{R}_+; H^\alpha(\mathbb{T}))$ introduit dans la définition. 1.3.9. Dans le cas non périodique l'équation de KdV avec modulation w admet une solution locale dans $H^\alpha(\mathbb{R})$ pour $\alpha > -\min(\rho, 3/4)$. La solution est globale si $\alpha > -\min(\rho/(3-2\gamma), 3/4)$ et l'unicité a lieu dans $\mathcal{D}^w(H^\alpha)$.*

Une remarque importante est que ce résultat présente un certain phénomène de régularisation par perturbation de la dispersion par des modulations irrégulières. En effet un résultat de [17] nous dit que le flot de l'équation de KdV sans perturbation ne peut être uniformément continu pour $\alpha < -1/2$ ce qui n'est pas le cas si on considère des modulations w assez irrégulières. Pour l'équation de NLS cubique on a le résultat :

Théorème 1.3.11. *Soit $\rho > 1/2$. alors l'équation de NLS cubique sur \mathbb{T} ou sur \mathbb{R} a une solution globale H^α pour tout $\alpha \geq 0$. De plus il'y a unicité dans $\mathcal{D}^w(H^\alpha)$ et le flot est localement continument-Lipschitz dans $\mathcal{D}^w(H^\alpha)$.*

On peut remarquer que ce résultat présente deux faits nouveaux par rapport au théorème (1.3.4) est que il permet de voir que l'équation (NLS) modulé est globalement bien posé sur le Tore ce qui n'est pas évident dû au fait que les estimations de Strichartz sont en générale fausse sur le Tore et aussi l'obtention de l'existence globale pour des conditions initiale $u^0 \in H^\alpha$ avec $\alpha \in [0, 1)$.

Théorème 1.3.12. *Soit $\rho > 1/2$. On a alors les résultats d'existence suivant :*

1. *L'équation NLS cubique avec modulation sur \mathbb{R}^2 a une unique solution locale dans H^α pour $\alpha \geq 1/2$;*
2. *L'équation dNLS avec modulation sur \mathbb{T} a une unique solution locale dans H^α pour $\alpha \geq 1/2$ et $\rho > 1$;*
3. *L'équation mKdV avec modulation sur \mathbb{T} a une unique solution locale dans H^α pour $\alpha \geq 1/2$.*

Au début de ce chapitre on a vus d'une manière formelle comment avoir l'existence local de solution dans ces théorèmes, pour être plus complet on va discuter un peu de la manière d'obtention de certaines existences globales notamment pour le cas de KdV sur le Tore avec modulation. En effet si X est l'opérateur associé a l'équation de KdV modulée sur le Tore alors un simple calcul d'intégrale nous montre que

$$\langle X_{st}(\phi, \phi), \phi \rangle_{L^2(\mathbb{T})} = \int_s^t d\sigma \int_{\mathbb{T}} (U_\sigma^w)^{-1} \partial((U_\sigma^w \phi)^2) \phi = \int_s^t \int_{\mathbb{T}} \partial((U_\sigma^w \phi)^2) U_\sigma^w \phi = 0$$

et donc si θ est la solution locale de KdV modulée sur le Tore avec un temps de vie $T > 0$ alors par un simple calcul on a que :

$$\|\theta_t\|_{L^2}^2 = \|\theta_s\|_{L^2}^2 + 2\langle \theta_s, R_{st} \rangle_{L^2} + \|\theta_t - \theta_s\|_{L^2}^2$$

où on rappel que θ est γ -Hölderienne pour un $\gamma > 1/2$ et que le reste R vérifie $|R_{st}| \lesssim |t - s|^{2\gamma}$, ainsi il s'ensuit que

$$\left| \|\theta_t\|_{L^2}^2 - \|\theta_s\|_{L^2}^2 \right| \lesssim |t - s|^{2\gamma}$$

pour $s, t \leq T$, et donc cela nous donne :

$$\|\theta_t\|_{L^2} = \|u^0\|_{L^2}$$

ce qui nous donne l'existence de solutions globales pour des données initiaux dans L^2 et se généralise à $H^\alpha(\mathbb{T})$ pour $\alpha > 0$ en observant que le terme non linéaire de l'équation de KdV se comporte mieux que la solution elle même ce qui donne une sorte de persistance de la régularité. Le cas d'une donnée initiale de basse régularité est plus fin à gérer et nous pousse à adapter la dite I -méthode introduite dans [18] à notre cas.

Nous remarquons maintenant qu'une limitation de la technique présentée jusqu'à maintenant est de seulement pouvoir traiter les cas de non-linéarité polynomiale due aux simplifications algébriques engendrées par celle-ci dans les calculs. Une autre approche qu'on va aussi exploiter dans le troisième chapitre est une généralisation des travaux effectués dans [30] pour une modulation w qui soit (γ, ρ) -irrégulière (cette fois-ci $\gamma > 0$ seulement), et cette approche nous a permis d'avoir les estimés de Strichartz suivante :

Théoreme 1.3.13. *Soit $A = i\partial_x^2$, $T > 0$, $p \in (2, 5]$, $\rho > \min(\frac{3}{2} - \frac{2}{p}, 1)$ alors il existe une constante $C_{w,T} > 0$ et $\gamma^*(p) > 0$ tel que l'inégalité suivante :*

$$\left\| \int_0^\cdot U^w(U_s^w)^{-1} \psi_s ds \right\|_{L^p([0,T], L^{2p}(\mathbb{R}))} \leq C_w T^{\gamma^*(p)} \|\psi\|_{L^1([0,T], L^2(\mathbb{R}))}$$

soit satisfaite pour tout $\psi \in L^1([0, T], L^2(\mathbb{R}))$.

Une application de ce résultat est l'obtention de solution globale pour l'équation NLS modulé avec une non-linéarité de type $\mathcal{N}(\phi) = |\phi|^\mu \phi$:

Théoreme 1.3.14. *Soit $\mu \in (1, 4]$, $p = \mu + 1$, $\rho > \min(1, 3/2 - \frac{2}{p})$ et $u^0 \in L^2(\mathbb{R})$ alors il existe $T^* = T^*(\|u^0\|_{L^2(\mathbb{R})}) > 0$ et un unique $u \in L^p([0, T], L^{2p}(\mathbb{R}))$ tel qu'on a l'égalité suivante :*

$$u_t = U_t^w u^0 + i \int_0^t U_t^w (U_s^w)^{-1} (|u_s|^\mu u_s) ds$$

pour tout $t \in [0, T^*]$. De plus on a que $\|u_t\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ et donc on a une unique solution global $u \in L_{loc}^p([0, +\infty), L^{2p}(\mathbb{R}))$ et $u \in C([0, +\infty), L^2(\mathbb{R}))$. Si maintenant $u^0 \in H^1(\mathbb{R})$ alors $u \in C([0, \infty), H^1(\mathbb{R}))$.

1.4 État de l'art et résultats du chapitre 4

Nous portant notre intérêt dans le quatrième chapitre de cette thèse à un travail que j'ai effectué en collaboration avec Rémi Catellier et qui consiste dans l'étude de l'équation de la quantisation stochastique et plus précisément du problème de Cauchy suivant :

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - u^3 + \xi \\ u(0, x) = u^0(x) \end{cases} \quad (1.16)$$

où $\Delta_{\mathbb{T}^3}$ désigne le Laplacien sur le Tore tridimensionnelle \mathbb{T}^3 , u^0 une condition initiale à prendre dans un espace adéquat et ξ un bruit blanc en espace-temps. Avant d'entrer dans les détails techniques et les difficultés posés par cette équation nous mettons juste l'accent sur le fait que la présumée solution n'est pas nécessairement une fonction (en fait nous verrons par la suite que c'est une distribution) due à la forte irrégularité du bruit blanc en dimension 3 et que ceci pose problème dans la définition du terme non linéaire u^3 . Maintenant nous rappelons que l'existence d'une solution en dimension 2 a été déjà prouvée dans [23]. Plus récemment une généralisation de la théorie des rough-path appeler théorie des structures de régularité donnée par Martin Hairer dans [47] et qui est basée sur l'analyse par ondelettes a permis l'étude de l'équation en dimension 3 et de prouver l'existence d'une solution locale. Nous nous proposons dans cette partie de retrouver ce résultat par une autre approche basée sur le paraproduit de Bony (voir [5]) et la notion de distributions contrôlées introduite dans [38].

1.4.1 Espace de Besov et Paraproduit de Bony

Dans cette sous-section nous essayerons de présenter dans un cadre simple la théorie des espaces de Besov et nous nous limiterons aux notions qui nous intéressent pour l'étude de l'équation (1.16) (pour plus de lecture sur le sujet voire [5, 6]). Pour cela on commence par introduire quelques notations.

Notation 1.4.1. Soit $d \in \mathbb{N}$, on note par $\mathcal{S}(\mathbb{R}^d)$ l'espace des fonctions test de Schwartz et $\mathcal{S}'(\mathbb{R}^d)$ celui des distributions tempérées. D la dérivé sur \mathbb{R}^d et pour $a = (a_1, a_2, \dots, a_d) \in \mathbb{N}^d$ un multi-indice on introduit aussi la notation suivante

$$\partial^a = \frac{\partial^{a_1+a_2+\dots+a_d}}{\partial^{a_1}x_1\partial^{a_2}x_2\dots\partial^{a_d}x_d}$$

et finalement $\mathcal{F}f$ où \hat{f} désigne la transformée de Fourier de la distribution $f \in \mathcal{S}'(\mathbb{R}^d)$.

Maintenant on se donne deux fonctions de classe $C^\infty(\mathbb{R}^d)$ radial h, θ tel que

1. Le support de h est inclus dans une boule B de \mathbb{R}^d et le support de χ dans un annaux \mathcal{A}
2. $\text{supp}(h) \cap \text{supp}(\theta(2^{-i}\cdot)) = \emptyset$ pour $i \geq 1$ et $\text{supp}(\theta(2^{-j}\cdot)) \cap \text{supp}(\theta(2^{-i}\cdot)) = \emptyset$ pour $|i - j| > 1$
3. $h(x) + \sum_{i \geq 0} \theta(2^{-i}x) = 1$ pour tout $x \in \mathbb{R}^d$

L'existence de telle fonctions est bien connue voir par exemple [6]. Ainsi on peut définir les blocques $(\Delta_i)_{i \geq -1}$ de Littlewood–Paley par :

$$\mathcal{F}(\Delta_{-1}f) = h\hat{f}, \quad \mathcal{F}(\Delta_i f) = \theta(2^{-i}\cdot)\hat{f}; i \geq 0$$

pour $f \in \mathcal{S}'(\mathbb{R}^d)$. On introduit alors les espaces de Besov non homogène par :

$$B_{p,q}^\alpha = \left\{ f \in \mathcal{S}'(\mathbb{R}^d), \|f\|_{B_{p,q}^\alpha} = \left\| (2^{i\alpha} \|\Delta_i f\|_{L^p(\mathbb{R}^d)})_{i \geq -1} \right\|_{l^q} < +\infty \right\}$$

pour $\alpha \in \mathbb{R}$, $1 \leq p, q \leq +\infty$. Une remarque importante est que ces espaces sont des Banach et que leurs topologies en tant qu'espace vectoriel normé ne dépend pas du choix des fonctions h et χ . Une attention particulière sera donnée dans la suite à l'espace $C^\alpha = \mathcal{B}_{\infty,\infty}^\alpha$ qui n'est autre que l'espace des fonctions α -Hölder sur \mathbb{R}^d lorsque $\alpha \in (0, 1)$. L'étude d'un tel espace s'avère assez confortable et pratique due a la bonne localisation de la transformée de Fourier des blocques de Littlewood–Paley. En effet on a le résultat suivant :

Proposition 1.4.2 (voire [6]). *Il existe une constante C telle que si \mathcal{B} est une boule de \mathbb{R}^d , $p, q \in [1, +\infty]$ tel que $p \leq q$, $\lambda > 0$, $k \in \mathbb{N}$ et $f \in \mathcal{S}'(\mathbb{R}^d)$ dont le support de sa transformée de Fourier \hat{f} est contenu dans $\lambda\mathcal{B}$ alors on a que :*

$$\sup_{|a|=k} \|\partial^a f\|_{L^q} \leq C^{k+1} \lambda^{k+d(1/p-1/q)} \|f\|_{L^p}$$

Démonstration. Pour prouver ce résultat il suffit de traiter le cas $\lambda = 1$ et ensuite de généraliser en usant d'une dilatation de taille $\lambda > 0$. Soit ϕ une fonction indéfiniment dérivable sur \mathbb{R}^d tel que $\phi \equiv 1$ dans un voisinage de \mathcal{B} . Ainsi il suffit de remarquer que

$$\partial^a f = \partial^a h \star f; \quad h = \mathcal{F}^{-1}(\phi)$$

D'où une application directe de l'inégalité de Young nous donne que

$$\|\partial^a f\|_{L^q} \leq \|f\|_{L^p} \|\partial h\|_{L^r}$$

avec $1/p + 1/q = 1 + 1/r$. Maintenant en utilisant l'inégalité suivante :

$$\|\partial^a h\|_{L^r} \leq \|\partial^a h\|_{L^1} + \|\partial^a h\|_{L^\infty} \leq C \|(1 + |x|^2)^d \partial^a h\|_{L^\infty}$$

avec $C = 1 + \int_{\mathbb{R}^d} dx (1 + |x|^2)^{-d}$, on obtient que

$$\|\partial^a h\|_{L^r} \leq C \|(Id - \Delta_{\mathbb{R}^d})^d ((\cdot)^a \phi)\|_{L^1} \leq C^{k+1}$$

ce qui finit la preuve. □

Une application directe de ce résultat nous donne immédiatement que

$$\|\Delta_i f\|_\infty \lesssim 2^{-id/q} \|\Delta f\|_{L^q} \lesssim 2^{id/q - i\alpha} \|u\|_{B_{q,q}^\alpha}$$

d'où

$$\|u\|_{C^{\alpha-d/q}} \lesssim \|u\|_{B_{q,q}^\alpha} \quad (1.17)$$

Ce qui donne une inclusion fort utile lorsque qu'on veut travailler avec des processus stochastiques. Nous introduisons a ce stade le parproduit de Bony (voire [5]) qui permet dans un sens de définir le produit fg entre deux distributions de Besov, en fait d'une manière assez formel on voit que :

$$fg = \sum_{i,j} \Delta_i f \Delta_j g = \pi_{<}(f, g) + \pi_0(f, g) + \pi_{>}(f, g) \quad (1.18)$$

où

$$\pi_{<}(f, g) = \sum_{i \geq -1; j \geq i+1} \Delta_i f \Delta_j g = \pi_{>}(g, f), \quad \pi_0(f, g) = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

Le théorème suivant nous précise un cadre dans lequel cas la décomposition (1.18) a un sens

Théorème 1.4.3 (voir [6]). *Soit $f \in L^\infty$ et $g \in C^\alpha$, $\alpha \in \mathbb{R}$ alors on a que $\pi_{<}(f, g)$ est bien définie et de plus*

$$|\pi_{<}(f, g)|_{C^\alpha} \lesssim_{\alpha, \beta} \|f\|_\infty \|g\|_{C^\alpha}$$

maintenant si $f \in C^\beta$ et $g \in C^\alpha$ pour $\beta \in \mathbb{R}$ et $\alpha < 0$ alors $\pi_{>}(f, g)$ est bien définie et on a

$$|\pi_{>}(f, g)|_{C^{\alpha+\beta}} \lesssim_{\alpha, \beta} \|f\|_{C^\beta} \|g\|_{C^\alpha}$$

et finalement si $f \in C^\beta$, $g \in C^\alpha$ avec $\alpha + \beta > 0$ alors $\pi_0(f, g)$ est bien définie de plus on a que :

$$|\pi_0(f, g)|_{C^{\alpha+\beta}} \lesssim_{\alpha, \beta} \|f\|_{C^\beta} \|g\|_{C^\alpha}$$

Pour être plus complet dans notre étude nous rappelons le résultat suivant qui nous donne une sorte de développement de Taylor à l'ordre 1 pour un terme de type $f(u)$ où u est une fonction de Besov et f une fonction assez lisse.

Théorème 1.4.4 (voir[6]). *Soit $\alpha, \rho > 0$ tel que ρ ne soit pas un entier, $u \in \mathcal{C}^\alpha \cap \mathcal{C}^\rho$ et $f \in C_b^2$. alors on a que*

$$f(u) = \pi_{<}(f'(u), u) + u^\sharp$$

avec

$$|u^\sharp|_{\mathcal{C}^{\alpha+\rho}} \lesssim_{f'', \|u\|_\infty} |u|_{\mathcal{C}^\rho} |u|_{\mathcal{C}^\alpha}$$

À ce niveaux nous allons essayer de relayer les notions vues dans cette sous-section avec la théorie des chemins rugueux. Pour cela nous allons nous donner l'exercice très simple de construire $f(x)Dx$ pour une fonction $x \in \mathcal{C}^\gamma(\mathbb{R}^d)$ avec $1/3 \leq \gamma < 1$ et $f \in C_b^2$, bien sur pour faire une telle chose un moyen possible serait de construire les différents paraproducts associés à $f(x)Dx$ ainsi nous avons la décomposition formelle suivante :

$$f(x)Dx = \pi_{<}(f(x), Dx) + \pi_0(f(x), Dx) + \pi_{>}(f(x), Dx)$$

Le théorème (1.4.3) nous indique que le premier et troisième terme de cette équation sont toujours bien définis, et de régularité respective $\gamma - 1$ et $2\gamma - 1$. Nous allons maintenant nous concentrer sur le second terme qui s'avère un peu délicat. Le théorème (1.4.4) nous dit que

$$f(x) = \pi_{<}(f'(x), Dx) + f(x)^\sharp$$

avec $f(x)^\sharp \in \mathcal{C}^{2\gamma}$. Injectant cette identité dans le terme diagonal on trouve que :

$$\pi_0(f(x), Dx) = \pi_0(\pi_{<}(f'(x), x), Dx) + \pi_0(f(x)^\sharp, Dx)$$

et maintenant on remarque que le second terme de cette équation est bien définie dès que $3\gamma - 1 > 0$. Pour étudier le terme $\pi_0(\pi_{<}(f'(x), x), Dx)$ on pourrait essayer de commuter les paraproducts et c'est ce qui a amené les auteurs de [38] à obtenir le résultat suivant :

Proposition 1.4.5 (N.Perkowsky, P.Imkeller, M.Gubinelli). *Soit $\alpha, \beta, \gamma \in \mathbb{R}$ tel que $0 < \alpha < 1$, $\alpha + \beta + \gamma > 0$ et $\beta + \gamma < 0$ alors*

$$R(f, x, y) = \pi_0(\pi_{<}(f, x), y) - f\pi_0(x, y)$$

est bien définie si $f \in \mathcal{C}^\alpha$, $x \in \mathcal{C}^\beta$ and $y \in \mathcal{C}^\gamma$ et plus précisément

$$\|R(f, x, y)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|x\|_\beta \|y\|_\gamma$$

Ainsi en utilisant ce résultat on obtient que

$$\pi_0(\pi_{<}(f'(x), x), Dx) = f'(x)\pi_0(x, Dx) + R(f'(x), x, Dx)$$

on voit ainsi qu'on peut définir $f(x)Dx$ sous la condition qu'on arrive à construire de manière raisonnable "l'aire de Besov" $\pi_0(x, Dx) \in \mathcal{C}^{2\gamma-1}$, ce qui est un résultat similaire a celui de la théorie des chemins rugueux qui dit que l'intégrale :

$$\int f(x)dx$$

est bien définie dès que l'aire de Lévy $\iint dx dx$ est "constructible" et satisfait certaines propriétés de régularité. Nous finirons cette sous-section par les deux lemmes suivants, le premier nous renseigne sur la régularité du bruit blanc vis-à-vis des espaces présentés dans cette partie et le second de la manière dont agit le semi-groupe de la chaleur $(P_t)_{t \geq 0} = (e^{t\Delta})_{t \geq 0}$ sur les paraproducts ce qui sera utile pour la suite .

Lemma 1.4.6. *Soit ξ un bruit blanc sur \mathbb{T}^d alors presque sûrement pour tout $\varepsilon > 0$ on a que $\xi \in \mathcal{C}^{-d/2-\varepsilon}(\mathbb{T}^d)$.*

Démonstration. Rappelons que

$$\xi = \sum_{k \in \mathbb{Z}} \xi(e_k) e_k$$

où (e_k) étant la base de Fourier de $L^2(\mathbb{T}^d)$ et où l'égalité précédente a lieu dans $L^2(\Omega, \mathcal{S}'(\mathbb{T}^d))$. En remarquant que $(\xi(e_k))_k$ est une famille i.i.d de variable aléatoire Gaussienne $\mathcal{N}(0, 1)$, on obtient alors par un simple calcul que pour tout $a \in \mathbb{T}^d$

$$\mathbb{E}[|\Delta_i \xi(a)|^2] = \sum_{|k| \sim 2^i} \theta(2^{-i}k) \lesssim 2^{id}$$

d'où en utilisant l'hypercontractivité Gaussienne (voir [71]) on obtient que

$$\mathbb{E}[|\Delta_i \xi|_{L^p(\mathbb{T}^d)}^p] = \int_{\mathbb{T}^d} \mathbb{E}[|\Delta_i \xi(a)|^2]^{p/2} da \lesssim 2^{ipd/2}$$

et d'où on voit que :

$$\mathbb{E}[|\xi|_{B_{p,p}^{-d/2-\varepsilon/2}}^p] < +\infty$$

pour tout $p > 1$. Donc dès que p est assez grand ($d/p < \varepsilon/2$) et en utilisant l'inclusion (1.17) on arrive au fait que

$$\mathbb{E}[|\xi|_{-d/2-\varepsilon}^p] < +\infty$$

ce qui donne le résultat. □

Lemma 1.4.7. *Soit $\alpha \in \mathbb{R}$ et $\theta \geq 0$. Alors les inégalités suivantes :*

$$\|P_t f\|_{\alpha+2\theta} \lesssim t^{-\theta} \|f\|_{\alpha}; \quad \|(P_{t-s} - 1)\|_{\alpha-2\varepsilon} \lesssim |t-s|^\varepsilon \|f\|_{\alpha}$$

sont satisfaites pour tout $f \in \mathcal{C}^\alpha$. De plus si $\alpha < 1$ et $\beta \in \mathbb{R}$ alors on a que

$$\|P_t \pi_{<}(f, g) - \pi_{<}(f, P_t g)\|_{\alpha+\beta+2\theta} \lesssim t^{-\theta} \|f\|_{\alpha} \|g\|_{\beta}$$

pour tout $f \in \mathcal{C}^\alpha$ et $g \in \mathcal{C}^\beta$.

Nous finiront par remarquer que la commutation entre le paraproduit et le semi-groupe de la chaleur est peu connu pour cela on renvoi au quatrième chapitre pour une preuve de ce fait.

1.4.2 Résultats du chapitre 4

Dans le quatrième et dernier chapitre de cette thèse nous allons nous intéresser au problème de Cauchy suivant :

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - u^3 + \xi \\ u(0, x) = u^0(x) \in \mathcal{C}^\alpha(\mathbb{T}^3) \end{cases} \quad (1.19)$$

où ξ étant un bruit blanc espace-temps, pour simplifier la compréhension nous allons considéré le cas où $u^0 = 0$ et donc dans ce cas la formulation mild de notre équation devient

$$u = X + I(u^3) \quad (1.20)$$

où $\Delta_{\mathbb{T}^3}$ est le Laplacien sur \mathbb{T}^3 , $I(f)(t) = \int_0^t P_{t-s} f_s ds$ avec $P_t = e^{t\Delta}$ le semi-groupe de la chaleur et $X = \int_0^t ds P_{t-s} \xi$ le processus d'Ornstein Uhlenbeck (O.U) qu'on va considérer stationnaire ici, et plus précisément on a que

$$\mathbb{E} \left[\hat{X}_t(k) \overline{\hat{X}_s(k')} \right] = \delta_{k,k'} \frac{e^{-|k|^2|t-s|}}{|k|^2}$$

avec $\hat{X}(0) = 0$ et $\delta_{i,j} = 1$ si $i = j$ et $\delta_{i,j} = 0$ sinon. Ainsi avec cette définition on voit par un simple calcul que $X \in \mathcal{C}_T^{1/2-\delta} := C([0, T], \mathcal{C}^{-1/2-\delta})$ pour tout $\delta > 0$. Maintenant on regardant de plus près notre équation on peut se convaincre que la solution ne va pas être plus régulière que processus X et ceci pose problème même dans la définition du terme u^3 , une approche classique et "naïve" serait de remplacer ξ par

$$\xi^\varepsilon = \sum_k f(\varepsilon k) \hat{\xi}(k) e_k$$

avec f est une fonction régulière à support compact tel que $f(0) = 1$ et (e_k) la base de Fourier de $L^2(\mathbb{T}^3)$ et dans ce cas on sait qu'il existe une unique solution (locale) u^ε de l'équation

$$\partial_t u^\varepsilon = \Delta_{\mathbb{T}^3} u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon$$

Maintenant on pourrait espérer contrôler $(u^\varepsilon)^3$ uniformément en ε dans un espace de Besov de régularité négative et c'est ici où on a un gros problème, en effet intuitivement on pourrait ce dire que pour avoir un tel contrôle il faudrait estimé $(X^\varepsilon)^3$ ou plus modestement $(X^\varepsilon)^2$ mais malheureusement un simple calcul nous montre que :

$$\begin{aligned} \mathbb{E} [(X_t^\varepsilon)^2] &= \sum_{k \in \mathbb{Z}^3} \sum_{k_1+k_2=k} f(\varepsilon k_1) f(\varepsilon k_2) \frac{1}{|k_1|^2} \delta_{k_1+k_2=0} \\ &= \sum_{k \in \mathbb{Z}^3} \frac{|f(\varepsilon k)|^2}{|k|^2} \sim \frac{1}{\varepsilon} \int_{\mathbb{R}^3} f(x) (1+|x|)^{-2} dx \end{aligned}$$

et ceci nous ôte tout espoir d'avoir une limite non triviale pour u^ε . Pour éviter ce genre de problème nous allons considérer l'équation :

$$\partial_t u = \Delta_{\mathbb{T}^3} u^\varepsilon - (u^\varepsilon)^3 + C_\varepsilon u^\varepsilon + \xi^\varepsilon$$

et nous verrons que pour un bon choix de C_ε il est possible de contrôler u^ε uniformément en ε et d'avoir une limite non triviale u qui satisfait une équation au point fixe de type :

$$u = X + I(u^{\diamond 3})$$

où $u^{\diamond 3}$ est une "redéfinition" ou "renormalisation" du cube.

Remark 1.4.8. *Il est judicieux d'observer que \diamond ne désigne pas nécessairement le produit de Wick.*

Maintenant à l'instar de la théorie des rough path la stratégie adoptée pour montrer cette convergence est tout de d'abord de donner un sens à l'équation "abstraite" satisfaite par u sous la condition qu'on peut construire de manière "convenable" une certaine distribution rugueuse \mathbb{X} associer à X et qui va jouer ici le rôle du rough path, puis résoudre cette équation par une méthode de point fixe et ainsi avoir la continuité de la solution par rapport à \mathbb{X} , et finalement prouver que si X est le processus de O.U et X^ε est une régularisation de X alors \mathbb{X}^ε la distribution rugueuse associer à X^ε converge bien vers une limite \mathbb{X} .

Développement de la solution, distribution rugueuse et distributions contrôlés : Nous allons maintenant revenir à notre équation mild initiale donnée par :

$$u = X + I(u^3)$$

ainsi même si le terme $\Phi = I(u^3)$ n'est pas à ce stade bien compris on peut voir formellement que $\Phi \in C([0, T], \mathcal{C}^{1/2-\delta})$ dû au fait que moralement u a pour régularité spatiale $-1/2-$ et à la régularisation faite par le noyau de la chaleur et de plus il satisfait l'équation :

$$\Phi = I(X^3) + 3I(\Phi X^2) + 3I(\Phi^2 X) + I(\Phi^3)$$

Une première remarque est que comme on l'a vu précédemment dans le cas d'un processus de (O U) le terme X^2 est mal définie au sens où $(X^\varepsilon)^2$ ne converge pas à proprement parler mais en revanche nous verrons par la suite que $(X^\varepsilon)^{\diamond 2} = (X^\varepsilon)^2 - \mathbb{E}[(X^\varepsilon)^2]$ converge vers un processus $X^{\diamond 2}$, de même $I((X^\varepsilon)^3)$ ne converge pas mais par contre $I((X^\varepsilon)^{\diamond 3}) = I((X^\varepsilon)^3 - 3\mathbb{E}[(X^\varepsilon)^2])$ converge ce qui nous pousse à "modifier" notre équation de la manière suivante :

$$\Phi = I(X^{\diamond 3}) + 3I(\Phi X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) \quad (1.21)$$

avec $\lim_{\varepsilon \rightarrow 0} I((X^\varepsilon)^{\diamond 3}) = I(X^{\diamond 3})$. À ce stade on remarque que notre stratagème pour donner un sens à l'équation et la résoudre nécessite plus que la donnée du chemin X mais aussi de $X^{\diamond 2}$, $I(X^{\diamond 3})$ et bien d'autres objets, nous allons de manière prématurée préciser la nature supplémentaire que devra satisfaire la distribution X .

Définition 1.4.9. Soit $T > 0$, $\nu, \rho > 0$ et on note par $\overline{C}_T^{\nu, \rho}$ la fermeture de $C^\infty([0, T], \mathbb{R})$ par la semi-norme

$$\|\varphi\|_{\nu, \rho} = \sup_{t \in [0, T]} t^\nu |\varphi_t| + \sup_{t, s \in [0, T]; s \neq t} \frac{s^\nu |\varphi_t - \varphi_s|}{|t - s|^\rho}$$

Maintenant pour $0 < 4\delta' < \delta$ on définit l'espace normé

$$\mathcal{W}_{T, K} = C_T^{\delta', -1/2-\delta} \times C_T^{\delta', -1-\delta} \times C_T^{\delta', 1/2-\delta} \times C_T^{\delta', -\delta} \times C_T^{\delta', -\delta} \times C_T^{\delta', -1/2-\delta} \times \overline{C}_T^{\nu, \rho}$$

avec $K = (\delta, \delta', \nu, \rho)$ équipé de sa topologie produit d'espace métrique. Maintenant on définit pour $(X, \varphi) \in C([0, T], C(\mathbb{T}^3)) \times C^\infty([0, T])$, et $(a, b) \in \mathbb{R}^2$ l'élément $R_{a, b} \mathbf{X} \in \mathcal{W}_{T, K}$ par

$$R_{a, b}^\varphi \mathbf{X} = (X, X^2 - a, I(X^3 - 3aX), \pi_0(I(X^3 - 3aX), X), \\ \pi_0(I(X^2 - a), (X^2 - a)) - b - \varphi, \pi_0(I(X^3 - 3aX), (X^2 - a)) - 3bX - 3\varphi X, \varphi)$$

et on définit ainsi l'ensemble des distributions rugueuses $\mathcal{X}_{T, K}$ comme étant la fermeture de

$$\left\{ R_{a, b}^\varphi \mathbf{X}, \quad (X, \varphi) \in C([0, T], C(\mathbb{T}^3)) \times C^\infty([0, T]), (a, b) \in \mathbb{R}^2 \right\}$$

dans $\mathcal{W}_{T, K}$. Nous dirons dans la suite que X donne lieu à une distribution rugueuse $\mathbb{X} \in \mathcal{X}_{T, K}$ si la première composante de \mathbb{X} est X .

Ainsi on a la notation suivante :

Notation 1.4.10. Soit $\mathbb{X} \in \mathcal{X}_{T, K}$, on note alors ses composantes de la manière suivante :

$$\mathbb{X} = (X, X^{\diamond 2}, I(X^{\diamond 3}), \pi_{0\diamond}(I(X^{\diamond 3}), X), \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi, \pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) - 3\varphi X, \varphi)$$

Or, étant donné $X \in \mathcal{C}([0, T], C^{-1/2-\delta})$ tel qu'il donne lieu à une distribution rugueuse $\mathbb{X} \in \mathcal{X}_{T,K}$ nous allons voir comment se comporte de manière formelle la présumée solution Φ de l'équation (1.21) dont on rappelle la forme :

$$\Phi = I(X^{\diamond 3}) + I(\Phi X^{\diamond 2}) + I(\Phi^2 X) + I(\Phi^3)$$

On peut voir que le premier terme par hypothèse est bien définie et satisfait bien $I(X^{\diamond 3}) \in \mathcal{C}^{1/2-\delta}$, de même que le dernier terme l'est aussi et a pour régularité $5/2 - \delta$ dû au fait que $\Phi \in \mathcal{C}_T^{1/2-\delta}$. En revanche le second et troisième terme sont encore mal définis à ce stade mais devraient avoir pour régularités respectives $1 - \delta$ et $3/2 - \delta$. Nous allons maintenant regarder d'un peu plus près le terme $I(\Phi X^{\diamond 2})$. Une décomposition en paraproduit de ce dernier nous donne que :

$$I(\Phi X^{\diamond 2}) = B_{<}(\Phi, X^{\diamond 2}) + \mathcal{C}_T^{3/2-\delta}$$

avec $B_{<}(f, g) = I(\pi_{<}(f, g))$. Ce qui nous pousse à introduire la définition suivante :

Définition 1.4.11. On définit l'espace des distributions contrôlées $\mathcal{D}_{\mathbb{X},T}^{\delta}$ par

$$\mathcal{D}_{\mathbb{X},T}^{\delta} = \left\{ (\Phi, \Phi') \in C([0, T], C^{1/2-\delta})^2; \quad \Phi^{\sharp} = \Phi - I(X^{\diamond 3}) - B_{<}(\Phi', X^{\diamond 2}) \in \mathcal{C}_T^{3/2-\delta} \right\}$$

une remarque est $\mathcal{D}_{\mathbb{X},T}^{\delta}$ est un espace affine muni de sa semi-norme

$$\|\Phi\|_{\mathcal{D}_{\mathbb{X},T}^{\delta}} = \|\Phi'\|_{\mathcal{C}_T^{1/2-\delta}} + \|\Phi'\|_{\mathcal{C}_T^{\delta, 1/2-3\delta}} + \|\Phi^{\sharp}\|_{\mathcal{C}_T^{3/2-\delta}}$$

et d'une distance d donné par :

$$d_{\delta,T}(\Phi_1, \Phi_2) = \|\Phi_1 - \Phi_2\|_{\mathcal{D}_{\mathbb{X},T}^{\delta}}$$

ce qui le munis d'une structure d'espace métrique complet.

Remark 1.4.12. D'un point de vue analytique cette définition diffère de celle donné dans le quatrième chapitre dû au fait qu'on veut étudier l'équation pour des conditions initiales u^0 assez générales.

Maintenant ce que nous affirmons c'est que si $\mathbb{X} \in \mathcal{X}_{T,K}$ et que si $\Phi \in \mathcal{D}_{\mathbb{X},T}^{\delta}$ pour $\delta > 0$ assez petit alors on peut définir les termes $I(\Phi^2 X)$ et $I(\Phi X^{\diamond 2})$ de manière convenable. En effet nous commençons par voir que

$$I(\Phi^2 X) = I(I(X^{\diamond 3})^2 X) + 2I(\theta^{\sharp} I(X^{\diamond 3}) X) + I((\theta^{\sharp})^2 X)$$

avec $\theta^{\sharp} = B_{<}(\Phi', X^{\diamond 2}) + \Phi^{\sharp} \in \mathcal{C}_T^{1-\delta}$, ainsi le terme $I(\theta^{\sharp} X)$ est bien définie, de plus due à l'hypothèse que $\mathbb{X} \in \mathcal{X}_{T,K}$ on voit que $I(X^{\diamond 3}) X \in \mathcal{C}_T^{-1/2-\delta}$ et donc le second terme de notre équation est aussi bien défini. En revanche le premier terme requière un peu plus d'attention, en effet ce qui pose problème dans ce terme étant sa partie diagonale $\pi_0(I(X^{\diamond 3})^2, X)$ qui peut être décomposé de la manière suivante

$$\pi_0(I(X^{\diamond 3})^2, X) = 2\pi_0(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) + \pi_0(\mathcal{C}_T^{1-\delta}, X)$$

Ainsi due au fait que $X \in \mathcal{C}^{-1/2-\delta}$ on peut voir que le second terme est bien définie et ne pose plus de problème. Pour le premier terme on va utiliser le lemme du commutateur (1.4.5) qui nous donne que :

$$\pi_0(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) = I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), X) + \mathcal{C}_T^{1/2-\delta}$$

Ainsi le premier terme est bien définie grâce au fait que $\pi_0(I(X^{\diamond 3}), X) \in \mathcal{C}^{-\delta}$, ce qui conclut l'étude de $I(\Phi^2 X)$. Nous allons maintenant nous concentrer sur le terme $I(\Phi X^{\diamond 2})$ qui requière un peu plus de vigilance, en effet un simple calcul nous donne que

$$I(\Phi X^{\diamond 2}) = I(I(X^{\diamond 3})X^{\diamond 2}) + I(B_{<}(\Phi', X^{\diamond 2})X^{\diamond 2}) + I(\Phi^\# X^{\diamond 2})$$

on voit qu'ici le troisième terme de cette équation est bien défini dû au fait que $\Phi^\# \in \mathcal{C}_T^{3/2-\delta}$ et $X^{\diamond 2} \in \mathcal{C}_T^{-1-\delta}$. Avant d'étudier le premier nous allons regarder de plus près le second terme de cette équation et pour celui-ci on remarque que seulement le terme diagonal pose problème, en effet :

$$B_0(B_{<}(\Phi', X^{\diamond 2}), X^{\diamond 2}) = I(\pi_0(I(\pi_{<}(\Phi', X^{\diamond 2})), X^{\diamond 2}))$$

ainsi une combinaison des Lemmes de commutation (1.4.7) et (1.4.5) nous permette d'avoir la relation suivante

$$B_0(B_{<}(\Phi', X^{\diamond 2}), X^{\diamond 2}) = I(\Phi' \pi_0(I(X^{\diamond 2}), X^{\diamond 2})) + \mathcal{C}^{3/2-\delta}$$

Dans le cas où X est le processus (O.U) on verra dans la suite que $\pi_0(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2})$ ne converge pas mais par contre $\pi_0(I(X^\varepsilon)^{\diamond 2}, I((X^\varepsilon)^{\diamond 2})) - C_2^\varepsilon$ avec C_2^ε une seconde constante de renormalisation, ce qui justifie l'introduction de la notation $\pi_{0\diamond}(I(X^{2\diamond}), X^{2\diamond})$ au lieu de $\pi_0(I(X^{\diamond 2}), X^{\diamond 2})$ et donc on voit que le terme

$$B_{0\diamond}(B_{<}(\Phi', X^{\diamond 2}), X^{\diamond 2}) = I(\Phi' \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2})) + \mathcal{C}^{3/2-\delta}$$

est bien définie due à la régularité de Φ' et le fait que $\mathbb{X} \in \mathcal{X}_{T,K}$. Maintenant de même que pour $\pi_0(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2})$ le terme $I(I(X^{\diamond 3})X^{\diamond 2})$ nécessite une certaine modification d'où le choix de le noter plus par $I(I(X^{\diamond 3})\diamond X^{\diamond 2})$ ce qui conclut l'étude du terme $I(\Phi X^{\diamond 2})$ qui sera noté par la suite $I(\Phi\diamond X^{2\diamond})$. Ainsi on voit que tous les termes de l'équation :

$$\Phi = X^{\diamond 2} + I(\Phi\diamond X^{2\diamond}) + I(\Phi^2 X) + I(\Phi^3)$$

sont bien définie dès que $\mathbb{X} \in \mathcal{X}_{T,K}$ et $\Phi \in \mathcal{D}_{\mathbb{X},T}^\delta$, ce qui a permis de la résoudre et ainsi de voir que $u := X + \Phi$ sera la limite de u^ε dans le cas où X est le processus de (O.U). Ainsi on a le résultat suivant

Théoreme 1.4.13. *Soit $F : \mathcal{C}^1(\mathbb{T}^3) \times C(\mathbb{R}^+, \mathcal{C}^0(\mathbb{T}^3)) \times \mathbb{R}^2 \rightarrow C(\mathbb{R}^+, \mathcal{C}^1)$ le flot de l'équation*

$$\begin{cases} \partial_t u_t = \Delta u_t - u_t^3 + 3au_t + 9bu_t + \xi_t, & t \in [0, T_C(u^0, X, \varphi, (a, b))] \\ \partial_t u_t = 0, & t \geq T_C(u^0, X, (a, b)) \\ u(0, x) = u^0(x) \in \mathcal{C}^1(\mathbb{T}^3) \end{cases}$$

où $\xi \in C(\mathbb{R}^+, \mathcal{C}^0(\mathbb{T}^3))$ et $T_C(u^0, \xi) > 0$ un temps pour lequel la solution u satisfait l'équation. Soit maintenant $z \in (1/2, 2/3)$ alors il existe $\tilde{T}_{CD} : \mathcal{C}^{-z} \times \mathcal{X} \rightarrow \mathbb{R}^+$ qui soit semi-continu inférieurement, $\tilde{F} : \mathcal{C}^{-z} \times \mathcal{X} \rightarrow C(\mathbb{R}^+, \mathcal{C}^{-z}(\mathbb{T}^3))$ qui est continue en $(u^0, \mathbb{X}) \in \mathcal{C}^{-z}(\mathbb{T}^3) \times \mathcal{X}$ et tel que (\tilde{F}, \tilde{T}) étend (F, T) dans le sens suivant :

$$T_C(u^0, \xi, (a, b)) \geq \tilde{T}_{CD}(u^0, R_{a,b}^\varphi \mathbf{X})$$

et

$$F(u^0, \xi, a, b)(t) = \tilde{F}(u^0, R_{a,b}^\varphi \mathbf{X})(t), \quad \text{pour tout } t \leq \tilde{T}_C(u^0, R_{a,b}^\varphi \mathbf{X})$$

pour tout $(u^0, \xi, \varphi) \in \mathcal{C}^1(\mathbb{T}^3) \times C(\mathbb{R}^+, \mathcal{C}^0(\mathbb{T}^3)) \times C^\infty([0, T])$, $(a, b) \in \mathbb{R}^2$ avec $X_t = \int_0^t ds P_{t-s} \xi$ et $R_{a,b}^\varphi \mathbf{X}$ étant le chemin associé à X dans la définition (1.4.9).

Maintenant on a le résultat suivant pour le processus de (O.U)

Théoreme 1.4.14. *Soit X un processus de O.U et $T > 0$, alors il existe deux constantes C_1^ε et C_2^ε tel que $C_1^\varepsilon, C_2^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty$ et une fonction $\varphi^\varepsilon \in C^\infty(\mathbb{R}^+)$ tel que $(R_{C_1^\varepsilon, C_2^\varepsilon}^{\varphi^\varepsilon} \mathbf{X}^\varepsilon)_\varepsilon$ converge en probabilité dans $\mathcal{X}_{T,K}$ vers un processus $\mathbb{X} \in \mathcal{X}$ de plus la première composante de \mathbb{X} est X .*

ainsi la combinaison de ces deux résultat nous donne que la solution

Corollary 1.4.15. *Soit u^ε la solution de l'équation :*

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon)^3 + C_1^\varepsilon u + C_2^\varepsilon u + \xi^\varepsilon; t \in [0, T^\varepsilon[\\ \partial_t u_t^\varepsilon = 0; t \geq T^\varepsilon \\ u^\varepsilon(0, x) = (u^0)^\varepsilon(x) \end{cases}$$

avec ξ^ε une régularisation du bruit blanc ξ , et T^ε le temps d'existence de u^ε alors u^ε converge en probabilité dans $C(\mathbb{R}^+, \mathcal{C}^{-z})$ vers $u := \tilde{F}(u^0, \mathbb{X})$

Chapter 2

Rough Sheet Vs Malliavin calculus

Résumé

Dans ce chapitre on étudie certains aspects du calcul stochastique à deux paramètres. Plus spécifiquement on se pose pour but d'exposer la manière d'obtenir une formule de changement de variable pour un drap $x : [0, 1] \rightarrow \mathbb{R}$ ayant certaines propriétés Hölderienne d'exposant plus grand que $1/3$ puis de donner une certaine formule de changement de variable pour l'intégrale de Skorohod a deux paramètres pour une large classe de processus Gaussien et enfin de comparer ces deux formules de changement de variable et calculer explicitement lorsque c'est possible les différents termes de contractions entre les deux types d'intégrales.

Abstract

In this chapter we gives an account on various aspects of stochastic calculus in the plane. Specifically, our aim is 3-fold: (i) Derive a pathwise change of variable formula for a path $x : [0, 1]^2 \rightarrow \mathbb{R}$ satisfying some Hölder regularity conditions with a Hölder exponent greater than $1/3$. (ii) Get some Skorohod change of variable formulas for a large class of Gaussian processes defined on $[0, 1]^2$. (iii) Compare the bidimensional integrals obtained with those two methods, computing explicit correction terms whenever possible. As a byproduct, we also give explicit forms of corrections in the respective change of variable formulas.

2.1 Introduction

Stochastic calculus for processes indexed by the plane (or higher order objects) is notoriously a cumbersome topic. In order to get an idea of this fact, let us start from the simplest situation of a smooth function x indexed by $[0, 1]^2$ and a regular function $\varphi \in C^2(\mathbb{R})$. Then some elementary computations show that

$$[\delta\varphi(x)]_{s_1 s_2; t_1 t_2} = \int_{[s_1, s_2] \times [t_1, t_2]} \varphi^{(1)}(x_{u;v}) d_{uv} x_{u;v} + \int_{[s_1, s_2] \times [t_1, t_2]} \varphi^{(2)}(x_{u;v}) d_u x_{u;v} d_v x_{u;v}, \quad (2.1)$$

for all $0 \leq s_1 < s_2 \leq 1$ and $0 \leq t_1 < t_2 \leq 1$, where we have set $[\delta y]_{s_1 s_2; t_1 t_2}$ for the rectangular increment of y in the rectangle $[s_1, s_2] \times [t_1, t_2]$. This simple formula already exhibits the extra term $\int \varphi^{(2)}(x_{u,v}) d_u x d_v x$ with respect to integration in \mathbb{R} , and the mixed differential term $d_u x d_v x$ is one of the main source of complications when one tries to extend (2.1) to more complex situations.

Moving to stochastic calculus in the plane, generalizations of (2.1) to a random process x obviously starts with change of variables formulas involving the Brownian sheet or martingales indexed by the plane. Relevant references include [19, 63, 73], and some common features of the formulas produced in these articles are the following:

- Higher order derivatives of f showing up.
- Mixed differentials involving partial derivatives of x and quadratic variation type elements.
- Huge number of terms in the formula due to boundary effects.

This non compact form of stochastic calculus in the plane has certainly been an obstacle to its development, and we shall go back to this problem later on.

Some recent advances in generalized stochastic calculus have also paved the way to change of variables formulas in the plane beyond the martingale case. One has to distinguish two type of contributions in this direction:

(a) Stochastic Calculus in the plane for a planare Martingale are developed in [19] and an Itô formula it obtained in this case.

(b) Skorohod type formulas for the fractional Brownian sheet (abbreviated as fBs in the sequel) with Hurst parameters greater than $1/2$ have been obtained in [20] thanks to a combination of differential calculus in the plane and stochastic analysis tools inspired by [2]. A subsequent generalization to Hurst parameters smaller than $1/2$ is available in [21], invoking the notion of extended divergence introduced in [58]. Notice however that the extended divergence leads to a rather weak notion of integral, and might not be necessary when the Hurst parameters of the fBs are greater than $1/4$.

Our first goal in this chapter is to show how we can define *rough sheets* which is the basic objects underlying multi-parameter integration suitable to build a theory of path-wise integration over the fractional Brownian sheet and then we construct the 2d rough integral. At this stage our first main result of this chapter can be resumed in the following theorem

Theorem 2.1.1. *Let $\gamma_1, \gamma_2 > 1/3$, then there exist a complete metric space $\mathcal{R}^{\gamma_1, \gamma_2}$ and two continuous application $\mathcal{I}_a : C_b^8(\mathbb{R}) \times \mathcal{R}^{\gamma_1, \gamma_2} \rightarrow \mathcal{C}_{2,2}^{\gamma_1, \gamma_2}$ (see the equation (2.30) for the exact definition of $\mathcal{C}_{2,2}^{\gamma_1, \gamma_2}$), $a = 1, 2$ such that :*

$$\mathcal{I}_1(\varphi, \mathbb{X})_{s_1 s_2; t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) \partial_s \partial_t x_{st} ds dt$$

and

$$\mathcal{I}_2(\varphi, \mathbb{X})_{s_1 s_2; t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) \partial_s x_{st} \partial_t x_{st} ds dt$$

for all $(s_1, s_2, t_1, t_2) \in [0, 1]^4$ and for every smooth sheet x with \mathbb{X} is the rough sheet associated. Moreover if x is a fractional Brownian sheet with Hurst parameter $\gamma_1, \gamma_2 > 1/3$ then it can be enhanced in a rough sheet \mathbb{X} and then we obtain in this case the following Stratonovich formula :

$$\delta \varphi(x) = \mathcal{I}_1(\varphi', \mathbb{X}) + \mathcal{I}_2(\varphi'', \mathbb{X}).$$

And the second aim of this chapter is to point out the link with the work of C.Tudor and F.Viens in [20, 21] where he construct the stochastic integral for the fraction Brownian sheet in the Skorohod meaning and obtain in that case a change of variable formula. The two main result of this part are given in the Theorem (2.7.5) and (2.7.13), for sake of comprehension we don't expose these result in the introduction.

Plan. This note is structured as follows. In Sec. 2.2 we recall the basic setup of [36] which allows to embed the theory of rough paths in a theory of integration of "generalized differentials", called *k-increments*. In Sec. 2.3 we introduce and study a complex of 2d increments (or *biincrements*) suitable to analyze 2d integrals and show the existence of a 2d Λ -map and of an abstract integration theory (in the sense of convergence of Riemann sums of particular biincrements).

In Sec. 2.4 we use the theory outlined in Sec. 2.3 to generalize Young theory of integration to two dimensions. Like in the one-dimensional setting this should be seen as a first (mostly pedagogical) step towards a full theory of rough sheets.

In Sec. 2.5 we proceed to the dissection of a 2d integral with the purpose of exposing the constituent elements of the would-be rough sheet. All the computations will be done in the smooth setting, emphasizing the algebraic aspects and the respective rôles of the various objects.

In Sec. 2.5.1 the definition of the rough sheet will be given, and on 2.5.1 a space of sheets *controlled* by a rough sheet, will be introduced and we will show how to obtain an integration theory for them.

In Sec 2.5.2 we show that for a smooth function φ that $\varphi(x)$ is controlled by x if x is a rough sheet and then obtain some continuity result for the integral constructed in this case using the procedure developed in 2.5.1.

In Sec 2.6 we show that the fractional Brownian motion can be enhanced in a rough sheet and then we obtain a Stratonovich change of variable formula in this case.

In Sec 2.7 we obtain a Skorohod change of variable formula and we describe explicitly the correction term between the different type of Stratonovich-rough integral in the plane and the 2d-Skorohod integral.

2.2 Algebraic integration in one dimension

The integration theory introduced in [36] is based on an algebraic structure, which turns out to be useful for computational purposes, but has also its own interest. Since this setting is quite non-standard, compared with the one developed in [61], and since it lay at the base of our approach to 2d integrals we will recall briefly here its main features.

2.2.1 Increments

Let $T > 0$ be an arbitrary positive real number. For any vector space V we introduce a cochain complex $(\mathcal{C}_*(V), \delta)$ as follows. a *k-increment* with values in V is a function $g : [0, T]^k \rightarrow V$, such that $g_{t_1 \dots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i = 1, \dots, k$. Denote with $\mathcal{C}_k(V)$ the corresponding set. On *k-increments*, define a the following coboundary operator δ :

$$\delta : \mathcal{C}_k(V) \rightarrow \mathcal{C}_{k+1}(V) \quad (\delta g)_{t_1 \dots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i+1} g_{t_1 \dots \hat{t}_i \dots t_{k+1}} \quad (2.2)$$

where \hat{t}_i means that this particular argument is omitted. It is easy to verify that $\delta\delta = 0$. We will denote $\mathcal{Z}\mathcal{C}_k(V) = \mathcal{C}_k(V) \cap \text{Ker}\delta$ and $\mathcal{B}\mathcal{C}_k(V) := \mathcal{C}_k(V) \cap \text{Im}\delta$, respectively the spaces of *k-cocycles* and

of k -coboundaries following standard conventions of homological algebra. We will write \mathcal{C}_k when the underlying vector space is \mathbb{R} .

Some simple examples of actions of δ are obtained by letting $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then, for any $t, u, s \in [0, T]$, we have

$$(\delta g)_{ts} = g_t - g_s, \quad \text{and} \quad (\delta h)_{tus} = h_{ts} - h_{tu} - h_{us}. \quad (2.3)$$

The complex $(\mathcal{C}_*(V), \delta)$ is *acyclic*, i.e. $\mathcal{Z}\mathcal{C}_{k+1}(V) = \mathcal{B}\mathcal{C}_k(V)$ for any $k \geq 0$ or otherwise stated, the sequence

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}_1(V) \xrightarrow{\delta} \mathcal{C}_2(V) \xrightarrow{\delta} \mathcal{C}_3(V) \xrightarrow{\delta} \mathcal{C}_4(V) \rightarrow \dots \quad (2.4)$$

is exact.

This exactness implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_1(V)}$: it measures how much a given 1-increment is far from being an *exact* increment of a function (i.e. a finite difference).

For our discussion only k -increments with $k \leq 3$ will be relevant. When V is a Banach space with norm $|\cdot|$ we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu \equiv \sup_{s,t \in [0,T]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V); \|f\|_\mu < \infty\}.$$

In the same way for $h \in \mathcal{C}_3(V)$ set

$$\begin{aligned} \|h\|_{\gamma,\rho} &= \sup_{s,u,t \in [0,T]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho} \\ \|h\|_\mu &\equiv \inf \left\{ \sum_i \|h_i\|_{\rho_i, \mu - \rho_i}; h = \sum_i h_i, 0 < \rho_i < \mu \right\}, \end{aligned} \quad (2.5)$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, \mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V); \|h\|_\mu < \infty\}.$$

Let $\mathcal{C}_3^{1+}(V) = \cup_{\mu > 1} \mathcal{C}_3^\mu(V)$. Analogous meaning should be given to $\mathcal{Z}\mathcal{C}_2^\mu(V), \dots$

From now on V will be a generic Banach space.

The following proposition is a basic result which is at the core of our approach to path-wise integration:

Proposition 2.2.1 (The Λ -map). *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \rightarrow \mathcal{C}_2^{1+}(V)$ such that $\delta\Lambda = 1_{\mathcal{Z}\mathcal{C}_3(V)}$. Furthermore, for any $\mu > 1$ this map is continuous from $\mathcal{Z}\mathcal{C}_2^\mu(V)$ to $\mathcal{C}_1^\mu(V)$ and we have*

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2} \|h\|_\mu, \quad h \in \mathcal{Z}\mathcal{C}_2^{1+}(V) \quad (2.6)$$

Proof. This result first appeared in [36]. For a substantially simpler proof look at [42]. □

Let $\mathcal{R}^\mu(V) = \{g \in \mathcal{C}_2(V) : \delta g \in \mathcal{C}_2^\mu(V)\}$. When $\mu > 1$ this is the subspace of 1-increments whose coboundary is small enough to be in the domain of Λ . $\mathcal{R}^{1+}(V)$ is defined as the union of all $\mathcal{R}^\mu(V)$ for $\mu > 1$.

An immediate implication of Prop. 2.2.1 is the following algorithm for a canonical decomposition of the elements of $\mathcal{R}^{1+}(V)$:

Corollary 2.2.2. *Take an element $g \in \mathcal{R}^\mu(V)$ for $\mu > 1$. Then g can be decomposed in a unique way as $g = \delta f + \Lambda \delta g$, where $f \in \mathcal{C}_1(V)$. If moreover $g \in \mathcal{C}_2^{1+}(V)$ then $f = 0$ and $g = \Lambda \delta g$: the coboundary δ is invertible (as a linear map) in $\mathcal{C}_1^{1+}(V)$.*

Proof. By assumption there are no ambiguity to define $\Lambda \delta g$ and by definition we have that $\delta(g - \Lambda \delta g) = 0$ and then there exist $f \in \mathcal{C}_1(V)$ such that $\delta f = g - \Lambda \delta g$. In the case when $g \in \mathcal{C}_2^{1+}$ we see that f is a γ -Hölder function for some $\gamma > 1$ and then $f_t = f_0$ which gives our result. \square

At this point the relation of the structure we introduced with the problem of integration of irregular functions can be still quite obscure to the non-initiated reader. However something interesting is already going on and the previous corollary has a very nice consequence which is the subject of the next corollary.

Corollary 2.2.3 (Integration of small increments). *For any 1-increment $g \in \mathcal{C}_2(V)$, such that $\delta g \in \mathcal{C}_1^{1+}$ set $\delta f = (1 - \Lambda \delta)g$, then*

$$(\delta f)_{ts} = \lim_{|\Pi_{ts}| \rightarrow 0} \sum_{i=0}^n g_{t_i t_{i+1}}$$

where the limit is over all partitions $\Pi_{ts} = \{t_0 = t, \dots, t_n = s\}$ of $[t, s]$ as the size of the partition goes to zero. The 1-increment δf is the indefinite integral of the 1-increment g .

Even if the result is already in [36] we would like to repeat the proof since it is quite illuminating and it will be source of inspiration when proving a similar statement in the 2d setting.

Proof. Just consider the equation $g = \delta f + \Lambda \delta g$ and write

$$S_\Pi = \sum_{i=0}^n g_{t_i t_{i+1}} = \sum_{i=0}^n (\delta f)_{t_i t_{i+1}} + \sum_{i=0}^n (\Lambda \delta g)_{t_i t_{i+1}} = (\delta f)_{ts} + \sum_{i=0}^n (\Lambda \delta g)_{t_i t_{i+1}}$$

and observe that, due to the fact that $\Lambda \delta g \in \mathcal{C}_1^{1+}(V)$ the last sum converges to zero. \square

Computations in \mathcal{C}_*

If V is an associative algebra the complex (\mathcal{C}_*, δ) is an (associative, non-commutative) graded algebra once endowed with the following product: for $g \in \mathcal{C}_n(V)$ and $h \in \mathcal{C}_m(V)$ let $gh \in \mathcal{C}_{n+m-1}(V)$ the element defined by

$$(gh)_{t_1, \dots, t_{m+n+1}} = g_{t_1, \dots, t_n} h_{t_n, \dots, t_{m+n+1}}, \quad t_1, \dots, t_{m+n} \in [0, T]. \quad (2.7)$$

The coboundary δ act as a graded derivation with respect to the algebra structure. In particular we have the following useful properties.

Proposition 2.2.4. *The following differentiation rules hold:*

1. Let g, h be two elements of $\mathcal{C}_1(V)$. Then

$$\delta(gh) = \delta g h + g \delta h. \quad (2.8)$$

2. Let $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then

$$\delta(gh) = -\delta g h + g \delta h, \quad \delta(hg) = \delta h g + h \delta g.$$

Proof. We will just prove (2.8), the other relations being equally trivial: if $g, h \in \mathcal{C}_1(V)$, then

$$[\delta(gh)]_{ts} = g_t h_t - g_s h_s = g_t (h_t - h_s) + (g_t - g_s) h_s = g_t (\delta h)_{ts} + (\delta h)_{ts} g_s,$$

which proves our claim. □

The iterated integrals of smooth functions on $[0, T]$ are of course particular cases of elements of \mathcal{C} which will be of interest for us. Let us recall some basic rules for these objects: consider $f, g \in \mathcal{C}_1^\infty$, where \mathcal{C}_1^∞ is the set of smooth functions from $[0, T]$ to \mathbb{R} . Then the integral $\int f dg$, that we will denote by $\mathcal{J}(fdg)$, can be considered as an element of \mathcal{C}_2^∞ . That is, for $s, t \in [0, T]$, we set

$$\mathcal{J}_{ts}(fdg) = \left(\int f dg \right)_{ts} = \int_s^t f_u dg_u.$$

The multiple integrals can also be defined in the following way: given a smooth element $h \in \mathcal{C}_2^\infty$ and $s, t \in [0, T]$, we set

$$\mathcal{J}_{ts}(hdg) \equiv \left(\int hdg \right)_{ts} = \int_s^t h_{us} dg_u.$$

In particular, the double integral $\mathcal{J}_{ts}(f^1 df^2 df^3)$ is defined, for $f^1, f^2, f^3 \in \mathcal{C}_0^\infty$, as

$$\mathcal{J}_{ts}(f^1 df^2 df^3) = \left(\int f^1 df^2 df^3 \right)_{ts} = \int_s^t \mathcal{J}_{us}(f^1 df^2) df_u^3.$$

Now, suppose that the n th order iterated integral of $f^1 df^2 \cdots df^n$, still denoted by the expression $\mathcal{J}(f^1 df^2 \cdots df^n)$, has been defined for $f^1, f^2, \dots, f^n \in \mathcal{C}_1^\infty$. Then, if $f^{n+1} \in \mathcal{C}_1^\infty$, we set

$$\mathcal{J}_{ts}(f^1 df^2 \cdots df^n df^{n+1}) = \int_s^t \mathcal{J}_{us}(f^1 df^2 \cdots df^n) df_u^{n+1}, \quad (2.9)$$

which defines the iterated integrals of smooth functions recursively. Observe that a n th order integral $\mathcal{J}(df^1 df^2 \cdots df^n)$ could be defined along the same lines.

The following relations between multiple integrals and the operator δ will also be useful in the remainder of the paper:

Proposition 2.2.5. *Let f, g be two elements of \mathcal{C}_1^∞ . Then, recalling the convention (2.7); it holds that*

$$\delta f = \mathcal{J}(df), \quad \delta(\mathcal{J}(fdg)) = 0, \quad \delta(\mathcal{J}(dgd f)) = (\delta g)(\delta f) = \mathcal{J}(dg)\mathcal{J}(df),$$

and, in general,

$$\delta(\mathcal{J}(df^n \cdots df^1)) = \sum_{i=1}^{n-1} \mathcal{J}(df^n \cdots df^{i+1}) \mathcal{J}(df^i \cdots df^1). \quad (2.10)$$

Proof. Here again, the proof is elementary, and we will just show the third of these relations: we have, for $s, t \in [0, T]$,

$$\mathcal{J}_{ts}(\mathrm{d}g\mathrm{d}f) = \int_s^t \mathrm{d}g_u(f_u - f_s) = \int_s^t \mathrm{d}g_u f_u - K_{ts},$$

with $K_{ts} = (g_t - g_s)f_s$. The first term of the right hand side is easily seen to be in $\ker \delta_{\Sigma_2^*}$. Thus

$$\delta(\mathcal{J}(\mathrm{d}g\mathrm{d}f))_{tus} = (\delta K)_{tus} = [g_t - g_s][f_u - f_s],$$

which gives the announced result. □

Dissection of an integral

To grasp the algorithm underling the rough-path approach to integrals over irregular functions we will exercise ourselves on the deconstruction of a “classic” integral.

With the notations of Sec. 2.2.1 in mind, we will split the integral $\int \varphi(x)\mathrm{d}x = \mathcal{J}(\varphi(x)\mathrm{d}x)$ for a smooth function $x \in \mathcal{C}_1$ into “more elementary” components. This decomposition suggest the right structure for the 1d rough paths. A similar exercise for 2d integrals will be very important in understanding the correct structure of the rough sheets.

The first idea one can have in mind in order to analyze $\mathcal{J}(\varphi(x)\mathrm{d}x)$ is to perform an expansion around the increment $\mathrm{d}x$. By Taylor expansion we have

$$\mathcal{J}(\varphi(x)\mathrm{d}x) = \varphi(x)\mathcal{J}(\mathrm{d}x) + \mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x). \quad (2.11)$$

The first term in the r.h.s will be considered “elementary” and not elaborated further. Note that it is defined independently of the regularity of x since $\varphi(x)\mathcal{J}(\mathrm{d}x) = \varphi(x)\delta x$.

Moreover, as a 1-increment it is easy to see that the second term in the r.h.s. of eq. (2.11) is smaller than the first but more problematic and we proceed to its *dissection* by the application of δ : invoking Proposition 2.2.5, we get that

$$\delta(\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x)) = \delta(\varphi(x))\delta x, \quad (2.12)$$

Now the r.h.s. is well defined independently of the regularity of x since

$$[\delta(\varphi(x))\delta x]_{tus} = (\varphi(x_t) - \varphi(x_u))(x_u - x_s).$$

Since x is smooth and assuming that φ is differentiable, then $\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x) \in \mathcal{C}_1^{1+}$ (since actually it belongs to \mathcal{C}_1^2 by easy bounds on the iterated integral). Then as a consequence of Corollary 2.2.2 we have that

$$\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x) = \Lambda\delta[\mathcal{J}(\mathrm{d}\varphi(x)\mathrm{d}x)] = -\Lambda[\delta(\varphi(x))\delta x]. \quad (2.13)$$

and, as a result, the following expression for the original integral holds

$$\mathcal{J}(\varphi(x)\mathrm{d}x) = \varphi(x)\delta x - \Lambda(\delta(\varphi(x))\delta x) = (1 - \Lambda\delta)[\varphi(x)\delta x]. \quad (2.14)$$

Eq. (2.14) shows that the ordinary integral on the l.h.s. is equivalent to an expression in the r.h.s which does not depend any more on any differentiability assumptions on x , indeed the r.h.s makes

sense, for example, when $\varphi \in \text{Lip}$ and $x \in \mathcal{C}_0^\gamma$ for any $\gamma > 1/2$: the only thing we have to check is that $\delta(\varphi(x))\delta x \in \mathcal{C}_2^{1+}$ but under these assumptions we have

$$|(\varphi(x_t) - \varphi(x_u))(x_u - x_s)| \leq L_\varphi \|x\|_\gamma^2 \|\varphi(x)\|_\gamma |t - u|^\gamma |u - s|^\gamma$$

where $\|\cdot\|_\gamma$ is the ordinary γ -Hölder norm on functions and L_φ is the Lipschitz norm of φ . In this case we can *define* the integral in the l.h.s. as being equivalent to the well-defined r.h.s. and this new integral is essentially the integral introduced by Young in [75]. What is really relevant to our discussion is to note that the integral can, in this case, be completely recovered from the 1-increment $\varphi(x)\delta x$.

However, the procedure can be continued further on by the next step in the Taylor expansion of the integral (2.11), which reads, for $s, t \in [0, T]$,

$$\int_s^t [d_u \varphi(x_u)] = \int_s^t \varphi'(x_u) dx_u = \varphi'(x_s)[x_t - x_s] + \int_s^t \left(\int_s^u \varphi'(x_v) dx_v \right) dx_u,$$

or according to the notations of Section 2.2.1,

$$\delta \varphi(x) = \mathcal{J}(d\varphi(x)) = \mathcal{J}(\varphi'(x)dx) = \varphi'(x)\mathcal{J}(dx) + \mathcal{J}(d\varphi'(x)dx). \quad (2.15)$$

Injecting this equality in equation (2.11), thanks to (2.9), we obtain

$$\mathcal{J}(\varphi(x)dx) = \varphi(x)\mathcal{J}(dx) + \varphi'(x)\mathcal{J}(dx dx) + \mathcal{J}(d\varphi'(x)dx dx). \quad (2.16)$$

In the two first terms in the r.h.s of eq. (2.16) the function $\varphi(x)$ has “pop out” from the integral, so we consider them elementary (in a sense we will discuss below). Again, the last term in the r.h.s. can be seen to belong to \mathcal{C}_1^{1+} (since actually, in this smooth setting, it belongs to \mathcal{C}_1^3). Then in analogy with eq. (2.13) we can represent it in terms of its image under δ as

$$\mathcal{J}(d\varphi'(x)dx dx) = -\Lambda \delta \mathcal{J}(d\varphi'(x)dx dx) = -\Lambda[\mathcal{J}(d\varphi'(x))\mathcal{J}(dx dx) + \mathcal{J}(d\varphi'(x)dx)\mathcal{J}(dx)] \quad (2.17)$$

where we acted with δ upon the triple iterated integral according to Prop. 2.2.5. Concerning the argument of Λ in this last equation, we note the following two facts: $\mathcal{J}(d\varphi'(x)) = \delta\varphi'(x)$ while the double iterated integral $\mathcal{J}(d\varphi'(x)dx)$ appears in the Taylor expansion for $\delta\varphi(x)$:

$$\delta\varphi(x) = \mathcal{J}(d\varphi(x)) = \mathcal{J}(\varphi'(x)dx) = \varphi'(x)\mathcal{J}(dx) + \mathcal{J}(d\varphi'(x)dx)$$

so

$$\mathcal{J}(d\varphi'(x)dx) = \delta\varphi(x) - \varphi'(x)\delta x. \quad (2.18)$$

Then we can rewrite eq. (2.17) as

$$\mathcal{J}(d\varphi'(x)dx dx) = -\Lambda[\delta\varphi'(x)\mathcal{J}(dx dx) + (\delta\varphi(x) - \varphi'(x)\delta x)\mathcal{J}(dx)] \quad (2.19)$$

and finally we have obtained another expression for the integral $\mathcal{J}(\varphi(x)dx)$:

$$\begin{aligned} \mathcal{J}(\varphi(x)dx) &= \varphi(x)\delta x + \varphi'(x)\mathcal{J}(dx dx) \\ &\quad - \Lambda[\delta\varphi'(x)\mathcal{J}(dx dx) + (\delta\varphi(x) - \varphi'(x)\delta x)\mathcal{J}(dx)] \\ &= (1 - \Lambda\delta)[\varphi(x)\delta x + \varphi'(x)\mathcal{J}(dx dx)] \end{aligned} \quad (2.20)$$

where to go from the first equation to the second we used the algebraic relation

$$\delta\mathcal{J}(dx dx) = \mathcal{J}(dx)\mathcal{J}(dx). \quad (2.21)$$

Up to this point all we got are another equivalent expression for the classic integral in the r.h.s. of eq. (2.19).

It is a very remarkable basic result of rough path theory that the r.h.s. of eq. (2.19) makes sense for paths x which are very irregular like the sample paths of Brownian motion (which a.s. are not Hölder continuous for any index greater than $1/2$), once we have at our disposal also a 1-increment $\mathcal{J}(dx dx)$ which is sufficiently small and satisfy eq. (2.21). Heuristically, in this situation the formula says that the 1-increment $\varphi(x)\delta x$ can be “corrected” or “renormalized” by adding the correction $\varphi'(x)\mathcal{J}(dx dx)$ so that it becomes integrable (in the sense of Corollary 2.2.3).

In the cases where this correction belongs to \mathcal{C}_1^{1+} we have $(1 - \Lambda\delta)[\varphi'(x)\mathcal{J}(dx dx)] = 0$ so eq. (2.19) becomes again eq. (2.13) and we reobtain the Young integral.

It is worth noticing at that point that the integral, as *defined* by eq. (2.19), has now to be understood as an integral over the (step-2) rough path $(x, \mathcal{J}(dx dx))$ [36] and it coincide with the notion of integral over a rough path given by Lyons in [60].

This algorithm has an obvious extension to higher orders if we assume that a reasonable definition of the iterated integrals $\mathcal{J}(dx dx \cdots dx)$ can be given. To proceed further however we need the notion of *geometric* rough path (for more details on this notion see [60]) which must be exploited crucially to show that some terms are small and belongs to the domain of Λ .

Note that we have worked in the scalar setting (i.e. all the object we considered are real-valued). Willing to add some notational burden it is easy to see that this section has an equivalent formulation in the vector case (when x takes values on \mathbb{R}^n and φ is a (smooth) differential form on \mathbb{R}^n). Indeed all the theory is interesting and useful especially in the vector case. This explain the fact that we do not considered techniques like the Doss-Sussmann approach to define one dimensional integrals since they are essentially limited to the scalar setting (where every reasonable differential form φ is exact) and do not have a vectorial counterpart.

2.3 The increment complex in two dimensions

In this paper we are interested in particular two-dimensional integrals which can take two basic forms which in general are not equivalent. If $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are regular enough we can define the two dimensional integral of f wrt. g as

$$\iint_{(s_1, t_1)}^{(s_2, t_2)} f dg := \int_{s_1}^{s_2} ds \int_{t_1}^{t_2} dt f(s, t) \partial_1 \partial_2 g(s, t) \quad (2.22)$$

where ∂_1 and ∂_2 are the partial derivatives wrt. the first and the second coordinate, respectively. Another possible and nonequivalent basic integral in two-dimension is given, for a triple of functions f, g, h , by

$$\iint_{(s_1, t_1)}^{(s_2, t_2)} f d_1 g d_2 h := \iint_{(s_1, t_1)}^{(s_2, t_2)} f(s, t) \partial_1 g(s, t) \partial_2 h(s, t) ds dt$$

Then

$$\iint_{(s_1, t_1)}^{(s_2, t_2)} dg =: (\delta g)(s_1, t_1, s_2, t_2) \quad (2.23)$$

defines the coboundary map δ for functions of two parameters:

$$\begin{aligned} (\delta g)(s_1, t_1, s_2, t_2) &= g(s_2, t_2) - g(s_1, t_2) - g(s_2, t_1) + g(s_1, t_1) \\ &= [g(s_2, t_2) - g(s_1, t_2)] - [g(s_2, t_1) - g(s_1, t_1)] \end{aligned} \quad (2.24)$$

which is just the composition of two finite-difference operator in the two directions. These integrals are to be considered as continuous functions of two points (s_1, t_1) and (s_2, t_2) on the plane which vanishes whenever $s_1 = s_2$ or $t_1 = t_2$. This preliminary observation leads to the following general construction for a 2d cochain complex suitable for the analysis of these two-parameter integrals.

Fix a positive real T and let $\mathcal{C}_{k,l}(V)$ the space of continuous functions from $[0, T]^k \times [0, T]^l \rightarrow V$, V some vector space such that

$$g_{(s_1, \dots, s_k)(t_1, \dots, t_l)} = 0$$

whenever $s_i = s_{i+1}$ or $t_i = t_{i+1}$. We will write $\mathcal{C}_{k,l} = \mathcal{C}_{k,l}(\mathbb{R})$.

For $\mathcal{C}_k(V) = \mathcal{C}_{k,k}(V)$ we will use the natural identification with the space of continuous functions from $([0, T]^2)^k \rightarrow V$. These will play the rôle of 2d k -increments: they are functions of k points in the square $[0, T]^2$ such that they become zero whenever two contiguous arguments have one coordinate in common.

Note that $\mathcal{C}_{k,l} = \mathcal{C}_k \otimes \mathcal{C}_l$ and in general

$$\mathcal{C}_{k,l}(V) = \mathcal{C}_k \otimes \mathcal{C}_l \otimes V \quad (2.25)$$

We will call the elements of $\mathcal{C}_{k,l}(V)$ (k, l) -biincrements and the elements of $\mathcal{C}_k(V)$ k -biincrements. Moreover we introduce one-dimensional coboundaries δ_1, δ_2 which acts as described in Sec. 2.2 on the biincrements view as functions of the first set, or of the second set of arguments, i.e. they acts on the first or second \mathcal{C}_* factor according to factorization of eq. (2.25). To be concrete

$$\delta_1 : \mathcal{C}_{k,l}(V) \rightarrow \mathcal{C}_{k+1,l}(V)$$

$$\delta_2 : \mathcal{C}_{k,l}(V) \rightarrow \mathcal{C}_{k,l+1}(V)$$

and for example, if $g \in \mathcal{C}_{k,l}(V)$ then

$$(\delta_1 g)_{(s_1, \dots, s_{k+1})(t_1, \dots, t_l)} = \sum_{i=1}^{k+1} (-1)^{i+1} g_{(s_1, \dots, \hat{s}_i, \dots, s_{k+1})(t_1, \dots, t_l)}$$

where, as usual, the notation \hat{s}_i means that the corresponding argument is omitted. It is easy to see that δ_1 and δ_2 commute and that

$$\delta = \delta_1 \delta_2 : \mathcal{C}_k(V) \rightarrow \mathcal{C}_{k+1}(V)$$

is a coboundary, i.e. satisfy the equation $\delta \delta = 0$. Moreover, if $g \in \mathcal{C}_{k,l}(V)$ we have

$$(\delta g)_{(s_1, \dots, s_{k+1})(t_1, \dots, t_{l+1})} = \sum_{i=1}^{k+1} \sum_{j=1}^{l+1} (-1)^{i+j} g_{(s_1, \dots, \hat{s}_i, \dots, s_{k+1})(t_1, \dots, \hat{t}_j, \dots, t_{l+1})}$$

Then $(\mathcal{C}_*(V), \delta)$ is a cochain complex. It will be important to note that its cohomology is not trivial and that it will play a rôle in our subsequent results.

2.3.1 Cohomology of (\mathcal{C}_*, δ)

The complex $(\mathcal{C}_*(V), \delta)$ is the diagonal of the following commutative diagram

$$\begin{array}{ccccccc}
 \mathcal{C}_{1,1}(V) & \xrightarrow{\delta_1} & \mathcal{C}_{2,1}(V) & \xrightarrow{\delta_1} & \mathcal{C}_{3,1}(V) & \xrightarrow{\delta_1} & \dots \\
 \delta_2 \downarrow & & \delta_2 \downarrow & & \delta_2 \downarrow & & \\
 \mathcal{C}_{1,2}(V) & \xrightarrow{\delta_1} & \mathcal{C}_{2,2}(V) & \xrightarrow{\delta_1} & \mathcal{C}_{3,2}(V) & \xrightarrow{\delta_1} & \dots \\
 \delta_2 \downarrow & & \delta_2 \downarrow & & \delta_2 \downarrow & & \\
 \mathcal{C}_{1,3}(V) & \xrightarrow{\delta_1} & \mathcal{C}_{2,3}(V) & \xrightarrow{\delta_1} & \mathcal{C}_{3,3}(V) & \xrightarrow{\delta_1} & \dots \\
 \delta_2 \downarrow & & \delta_2 \downarrow & & \delta_2 \downarrow & &
 \end{array} \tag{2.26}$$

We are mainly interested in the first cohomology group

$$H_1(\mathcal{C}, \delta) = \frac{\mathcal{Z}\mathcal{C}_1(V)}{\mathcal{B}\mathcal{C}_1(V)}$$

where as before we denote $\mathcal{Z}\mathcal{C}_k(V) = \text{Ker}\delta|_{\mathcal{C}_k(V)}$ and $\mathcal{B}\mathcal{C}_k(V) = \text{Im}\delta|_{\mathcal{C}_{k-1}(V)}$, the spaces of k -bicycles and $k-1$ -bicoboundaries, respectively.

To compute the cohomology consider applications $\sigma_1 : \mathcal{C}_{k,l}(V) \rightarrow \mathcal{C}_{k-1,l}(V)$ (for $k \geq 1$) and $\sigma_2 : \mathcal{C}_{k,l}(V) \rightarrow \mathcal{C}_{k,l-1}(V)$ (for $l \geq 1$) which fix the first argument on each direction to the (arbitrary) value 0. For example:

$$(\sigma_1 g)_{(s_1, \dots, s_{k-1})(t_1, \dots, t_l)} = g_{(0, s_1, \dots, s_{k-1})(t_1, \dots, t_l)}$$

and a similar equation for σ_2 . Then we have the homotopy formulas

$$\sigma_i \delta_i - \delta_i \sigma_i = 1, \quad \text{for } i=1,2$$

which are at the origin of the exactness of the one-dimensional complexes forming the rows and the columns of the diagram (2.26). Let $k \geq 1$. Take $a \in \mathcal{Z}\mathcal{C}_k(V)$ and let

$$b = a - \sigma_1 \delta_1 a - \sigma_2 \delta_2 a$$

and, using the homotopy formulas, verify that

$$\delta_1 b = \delta_1 a - \delta_1 a - \delta_1 \sigma_2 \delta_2 a = -\sigma_2 \delta_2 a = 0$$

since δ_1 commutes with σ_2 . Similarly $\delta_2 b = 0$. Then $b \in \text{Ker}\delta_1 \cap \text{Ker}\delta_2$ which means that we can write

$$b = \delta_1 s_1 b = \delta_1 \sigma_1 \delta_2 \sigma_2 b$$

but since operators with different indexes commutes we can always rewrite this as

$$b = \delta_1 \delta_2 \sigma_1 \sigma_2 b = \delta \sigma_1 \sigma_2 b$$

so that $b \in \mathcal{B}\mathcal{C}_k(V)$. Next, note that $\sigma_1 \delta_1 a \in \text{Ker}\delta_2$, since $\delta_2 \sigma_1 \delta_1 a = \sigma_1 \delta a = 0$ so $\sigma_1 \delta_1 a = \delta_2 \sigma_2 \sigma_1 \delta_1 a$. Then for any $a \in \mathcal{Z}\mathcal{C}_k(V)$ we have the decomposition

$$a = \delta q + \delta_1 \sigma \delta_2 a + \delta_2 \sigma \delta_1 a \tag{2.27}$$

for some $q \in \mathcal{C}_{k-1}(V)$ where we let $\sigma = \sigma_1 \sigma_2 = \sigma_2 \sigma_1$.

2.3.2 Computations in $\mathcal{C}_{*,*}$

For $a \in \mathcal{C}_{n,m}$, $b \in \mathcal{C}_{k,l}$ we can define the (noncommutative, associative) product $ab \in \mathcal{C}_{n+k-1,m+l-1}$ as

$$ab_{(s_1, \dots, s_{n+k-1})(t_1, \dots, t_{m+l-1})} = a_{(s_1, \dots, s_n)(t_1, \dots, t_m)} b_{(s_{n+1}, \dots, s_{n+k-1})(t_{m+1}, \dots, t_{m+l-1})}.$$

For example for $a \in \mathcal{C}_{2,1}$, $b \in \mathcal{C}_{1,2}$ we have $(ab)_{(s_1, s_2)(t_1, t_2)} = a_{(s_1, s_2)t_1} b_{s_2(t_1, t_2)}$. This definition is suited to work well with the action of δ_1, δ_2 , for example we have that, if $a, b \in \mathcal{C}_{1,*}$:

$$\delta_1(a\delta_1 b) = \delta_1 a \delta_1 b, \quad \delta_1(\delta_1 b a) = -\delta_1 b \delta_1 a$$

If $a, b \in \mathcal{C}_1$:

$$\delta_2(a\delta_1 b) = \delta_2 a \delta_1 b + a \delta b,$$

$$\delta(\delta_1 a \delta_2 b) = \delta_1((\delta_2 \delta_1 a) \delta_2 b) + \delta_1 a (\delta_2 \delta_2 b) = \delta_1(\delta a \delta_2 b) = -\delta a \delta b$$

and

$$\delta(\delta_2 b \delta_1 a) = -\delta b \delta a, \quad \delta(a \delta b) = \delta(\delta a b) = \delta a \delta b$$

as can be easily checked by a direct computation. In the two parameters setting we will improve our algebraic structure by adding a new type of product for $f \in \mathcal{C}_{m,n}$ and $g \in \mathcal{C}_{m,l}$ we define $f \circ_1 g \in \mathcal{C}_{m,n+l}$ by

$$f \circ_1 g_{(s_1 s_2 \dots s_m)(t_1 t_2 \dots t_{n+l-1})} = f_{(s_1 s_2 \dots s_m)(t_1 t_2 \dots t_n)} g_{(s_1 s_2 \dots s_m)(t_{n+1} \dots t_{n+l-1})}$$

and an analogue definition in the second direction .

For a two-dimensional quantity like the basic integral in eq.(2.22) we can write down the following relation

$$\begin{aligned} & \iint_{(s_1, t_1)}^{(s_2, t_2)} \left[\iiint_{(s_1, t_1)}^{(s, t)} df(u, v) \right] dg(s, t) \\ &= \iint_{(s_1, t_1)}^{(s_2, t_2)} [f(s, t) - f(s_1, t) - f(s, t_1) + f(s_1, t_1)] dg(s, t) \\ &= \iint_{(s_1, t_1)}^{(s_2, t_2)} f(s, t) dg(s, t) + f(s_1, t_1) (\delta g)(s_1, t_1, s_2, t_2) \\ & \quad - \int_{t_1}^{t_2} f(s_1, t) d_2 [g(s_2, t) - g(s_1, t)] \\ & \quad - \int_{s_1}^{s_2} f(s, t_1) d_1 [g(s, t_2) - g(s, t_1)] \end{aligned} \tag{2.28}$$

which will play the same rôle as eq. (2.11) in the one-dimensional setting.

As an example of the formalism set up up to now we can consider the decomposition of eq. (2.28) for the (two-dimensional) iterated integral $\iint df dg$ of two smooth elements $f, g \in \mathcal{C}_1$. In compact notation it reads:

$$\iint f dg = -f \delta g + \int f d_1 \delta_2 g + \int f d_2 \delta_1 g + \iint df dg$$

where we understand the integrals as functions of both the extremes of integration. Note that

$$\delta \iint df dg = \delta(f \delta g) = \delta f \delta g$$

so

$$\iint f dg - \int f d_1 \delta_2 g - \int f d_2 \delta_1 g \in \ker \delta. \quad (2.29)$$

Actually the three factors in eq. (2.29) corresponds exactly to the decomposition (2.27) since

$$\iint f dg = \delta \iint_{*,*} f dg, \quad \int f d_1 \delta_2 g = \delta_1 \int_* f d_1 \delta_2 g, \quad \int f d_2 \delta_1 g = \delta_2 \int_* f d_2 \delta_1 g$$

where the star in the integral sign denote that the lower integration point has been fixed arbitrarily (e.g. to $0 \in [0, T]$).

Another relevant remark is to note that the antisymmetric element $\omega^a = \delta_1 f \delta_2 g - \delta_2 f \delta_1 g$ satisfy $\delta \omega^a = 0$. Indeed

$$\delta_1 \omega^a = -\delta_1 f \delta g - \delta f \delta_1 g, \quad \delta \omega^a = -\delta f \delta g + \delta f \delta g = 0$$

according to the rules established above. For the symmetric counterpart $\omega^s = \delta_1 f \delta_2 g + \delta_2 f \delta_1 g$ we have

$$\delta_1 \omega^s = -\delta_1 f \delta g + \delta f \delta_1 g, \quad \delta \omega^s = -\delta f \delta g - \delta f \delta g = -2\delta f \delta g$$

2.3.3 Splitting and other operations

Each of the vector spaces $\mathcal{C}_{k,m}(V)$ is naturally isomorphic to either $\mathcal{C}_k(\mathcal{C}_m(V))$ or to $\mathcal{C}_m(\mathcal{C}_k(V))$: consider each k, m -biincrement either as a k -increment in the first direction with values in m -increments in the other direction or vice-versa. The multiplication in $\mathcal{C}_{*,*}$ is compatible with these isomorphism.

In what follows it will be useful to introduce a one-dimensional *splitting* map S which sends products $ab \in \mathcal{C}_2(V)$ for $a \in \mathcal{C}_1(V)$ and $b \in \mathcal{C}_1(V)$ to the elementary tensor $S(ab) = a \otimes b \in \mathcal{C}_1(V) \overset{V}{\otimes} \mathcal{C}_1(V)$ where the tensor product is over the algebra V . The map is extended by linearity to the subspace \mathcal{M}_2 of \mathcal{C}_2 generated by the linear combinations of products of two elements of \mathcal{C}_1 . Elements of $\mathcal{C}_1(V) \overset{V}{\otimes} \mathcal{C}_1(V)$ are just functions $(t, u, v, s) \mapsto c_{tuvs}$ of four arguments which are 1-increments in the couple (t, u) and in the couple (v, s) but which may be non-zero for $u = v$. The multiplication map $\mu : \mathcal{C}_1(V) \overset{V}{\otimes} \mathcal{C}_1(V) \rightarrow \mathcal{C}_2(V)$ just sends each c to the 2-increment $(t, u, s) \mapsto a_{tuus}$ and is the inverse of S : $\mu \circ S(a) = a$ for any $a \in \mathcal{M}_2$.

We will denote $S_1 : \mathcal{C}_{2,k}(V) \rightarrow \mathcal{C}_1(\mathcal{C}_k(V)) \overset{\mathcal{C}_k(V)}{\otimes}_1 \mathcal{C}_1(\mathcal{C}_k(V))$ and $S_2 : \mathcal{C}_{k,2}(V) \rightarrow \mathcal{C}_1(\mathcal{C}_k(V)) \overset{\mathcal{C}_k(V)}{\otimes}_2 \mathcal{C}_1(\mathcal{C}_k(V))$ the splitting maps according to the first or the second direction. These are understood according to the above isomorphism $\mathcal{C}_{k,m}(V) \simeq \mathcal{C}_k(\mathcal{C}_m(V)) \simeq \mathcal{C}_m(\mathcal{C}_k(V))$ and the index 1,2 on the tensor product remember in which of the two directions the splitting has taken place.

2.3.4 Abstract integration in \mathcal{C}_*

From now on we assume that V is a Banach space with norm $|\cdot|$. When they appears tensor product will be understood according to the projective topology.

Let us introduce the following norms, for any $g \in \mathcal{C}_2(V)$

$$\|g\|_{z_1, z_2} := \sup_{s, t \in [0, T]^2} \frac{|g_{(s_1, t_1)(s_2, t_2)}|}{|s_1 - t_1|^{z_1} |s_2 - t_2|^{z_2}} \quad (2.30)$$

and for $h \in \mathcal{C}_3(V)$

$$\|h\|_{\gamma_1, \gamma_2, \rho_1, \rho_2} := \sup_{s, u, t \in [0, T]^2} \frac{|h_{(s_1, u_1, t_1)(s_2, u_2, t_2)}|}{\prod_{i=1,2} |s_i - u_i|^{\gamma_i} |u_i - t_i|^{\rho_i}}$$

and

$$\|h\|_{z_1, z_2} := \inf \left\{ \sum_i \|h_i\|_{\gamma_{1,i}, \gamma_{2,i}, z_1 - \gamma_{1,i}, z_2 - \gamma_{2,i}} \mid h = \sum_i h_i, \gamma_{j,i} \in (0, z_j), j = 1, 2 \right\}$$

and the corresponding subspaces $\mathcal{C}_2^{z_1, z_2}(V)$, $\mathcal{C}_3^{\gamma_1, \gamma_2, \rho_1, \rho_2}(V)$ and $\mathcal{C}_3^{z_1, z_2}(V)$. Moreover we say that $f \in \mathcal{C}_1^{\rho_1, \rho_2}$ if

$$\mathcal{N}_{\rho_1, \rho_2}(f) = \|\delta f\|_{\rho_1, \rho_2} + \|\delta_1 f\|_{\rho_1, 0} + \|\delta_2 f\|_{0, \rho_2} + \|f\|_{\infty} < +\infty \quad (2.31)$$

with

$$\|\delta_1 f\|_{\rho_1, 0} = \sup_{(s_1, s_2, t_1) \in [0, T]^3} \frac{|\delta_1 f_{s_1 s_2 t_1}|}{|s_2 - s_1|^{\rho_1}}$$

and similar definition in the second direction. The main feature of the space $\mathcal{C}_2^{z_1, z_2}$ is that $\mathcal{C}_2^{z_1, z_2} \cap \ker \delta_1 = \{0\}$ if $z_1 > 1$ and $\mathcal{C}_2^{z_1, z_2} \cap \ker \delta_2 = \{0\}$ if $z_2 > 1$. This implies that the equation $\delta a = 0$ has only a trivial solution $a = 0$ if we require $a \in \mathcal{C}_1^{z_1, z_2}$ with $z_1, z_2 > 1$.

Let $\mathcal{C}_i^{1+}(V) = \cup_{z_1 > 1, z_2 > 1} \mathcal{C}_i^{z_1, z_2}(V)$, $i = 1, 2$.

Note that we have the isomorphism $\mathcal{C}_{a,b}^{z_1, z_2}(V) \simeq \mathcal{C}_a^{z_1}(\mathcal{C}_b^{z_2}(V)) \simeq \mathcal{C}_b^{z_2}(\mathcal{C}_a^{z_1}(V))$ for $a, b = 0, 1, 2$ and $z_1, z_2 \geq 0$.

Before stating the main result of this section we introduce two versions of the one-dimensional Λ map of Prop. 2.2.1, acting on the two different coordinates.

Lemma 2.3.1. $\Lambda_1 : \mathcal{B}_2 \mathcal{C}_{3,a}^{w_1, z_2}(V) \rightarrow \mathcal{C}_{2,a}^{w_1, z_2}(V)$ for $a = 1, 2, 3$ with $w_1 > 1$ such that $\delta_1 \Lambda_1 = 1$ and

$$\|\Lambda_1 h\|_{z_1, w_2} \leq C_{z_1} \|h\|_{z_1, w_2} \quad (2.32)$$

and an analogous bound for Λ_2 .

Proof. If we fix $s_2, u_2, t_2 \in [0, T]$ we can consider $h^{s_2 u_2 t_2} \in \mathcal{C}_2(V)$ such that

$$(s_1, u_1, t_1) \mapsto h_{s_1, u_1, t_1}^{s_2, u_2, t_2} = h_{(s_1, u_1, t_1)}^{(s_2, u_2, t_2)}$$

and note that $h^{s_2 u_2 t_2} \in \mathcal{B} \mathcal{C}_2^{z_1}(V)$ since $\delta h^{s_1 u_1 t_1} = 0$ so that it is in the range of the one-dimensional Λ of Prop. 2.2.1 and

$$|(\Lambda h^{s_2, u_2, t_2})_{s_1 t_1}| \leq C_{z_1} \|h^{s_2, u_2, t_2}\|_{z_1} |s_1 - t_1|^{z_1}. \quad (2.33)$$

Then define Λ_1 as

$$(\Lambda_1 h)_{(t_1, s_1), (t_2, u_2, s_2)} := (\Lambda h^{s_2, u_2, t_2})_{s_1 t_1}$$

and note that the bound (2.33) implies eq. (2.32). Proceeding similarly one can prove a similar statement about Λ_2 . \square

Then it holds the analogous of the one-dimensional result:

Proposition 2.3.2. *There exists a unique map $\Lambda : \mathcal{BC}_2^{1+}(V) \rightarrow \mathcal{C}_1^{1+}(V)$ such that $\delta\Lambda = 1$. Moreover if $z_1, z_2 > 1$, $h \in \mathcal{BC}_2^{z_1, z_2}(V)$ then*

$$\|\Lambda h\|_{z_1, z_2} \leq \frac{1}{2^{z_1} - 2} \frac{1}{2^{z_2} - 2} \|h\|_{z_1, z_2}.$$

Proof. Since $h \in \text{Img}\delta$ we have $\delta_1 h = \delta_2 h = 0$. Then let $\Lambda h = \Lambda_1 \Lambda_2 h$ which is meaningful since the $\delta_1 \Lambda_2 h = \Lambda_2 \delta_1 h = 0$ (by linearity) and the requirement on the regularity is satisfied. Then

$$\delta \Lambda h = \delta_2 \delta_1 \Lambda_1 \Lambda_2 h = \delta_2 \Lambda_2 h = h$$

and

$$\|\Lambda h\|_{z_1, z_2} = \|\Lambda_1 \Lambda_2 h\|_{z_1, z_2} \leq C_{z_1} \|\Lambda_2 h\|_{z_1, z_2} \leq C_{z_1} C_{z_2} \|h\|_{z_1, z_2}.$$

Uniqueness depends on the fact that $z_1, z_2 > 1$. Using the uniqueness it is easy to deduce that $\Lambda = \Lambda_2 \Lambda_1$, i.e. the one-dimensional maps commute (when they can be both applied). \square

We can already state an interesting result about integration of “small” biincrements.

Corollary 2.3.3 (2d integration). *Let $a \in \mathcal{C}_{2,2}(V)$ such that $\delta_1 a \in \mathcal{C}_{3,2}^{z_1, *}$, $\delta_2 a \in \mathcal{C}_{2,3}^{*, z_2}$, $\delta a \in \mathcal{C}_{2,2}^{z_1, z_2}$ with $z_1, z_2 > 1$. There exists $f \in \mathcal{C}_{1,1}(V)$ such that*

$$\delta f = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)a$$

and

$$\lim_{|\Pi| \rightarrow 0} \sum_{i,j} a_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = (\delta f)_{(t^1, s^1), (t^2, s^2)}$$

where the limit is taken over partitions $\Pi = \{(t_i^1, t_j^2)_{i,j}\}$ of the square $[t^1, t^2] \times [s^1, s^2]$ into boxes whose maximum size $|\Pi|$ goes to zero.

Proof. The required conditions on a ensure that the 1-biincrement

$$h = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)a = a - \Lambda_1 \delta_1 a - \Lambda_2 \delta_2 a + \Lambda \delta a$$

is well defined. By direct computation we have that

$$\delta_1 h = \delta_1(1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)a = (\delta_1 - \delta_1)(1 - \Lambda_2 \delta_2)a = 0$$

and $\delta_2 h = 0$. So h must be in the image of δ , i.e. there exists f such that $h = \delta f$. This proves the first claim.

To prove the convergence of the sums consider the above decomposition

$$a = \delta f + \Lambda_2 \delta_2 a + \Lambda_1 \delta_1 a - \Lambda \delta a$$

written as $a = \delta f + r_1 + r_2 + r$ where

$$r_1 = (1 - \Lambda_1 \delta_1) \Lambda_2 \delta_2 a, \quad r_2 = (1 - \Lambda_2 \delta_2) \Lambda_1 \delta_1 a \quad r = \Lambda \delta a.$$

Note that $r_1 \in \mathcal{C}_{2,2}^{*, z_2}$, $\delta_1 r_1 = 0$ and $r_2 \in \mathcal{C}_{2,2}^{z_1, *}$ and $\delta_2 r_2 = 0$.

Then let

$$\begin{aligned}
 S_{\Pi} &= \sum_{i,j} a_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} \\
 &= \sum_{i,j} (\delta f)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} + \sum_{i,j} (r_1)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} \\
 &\quad + \sum_{i,j} (r_2)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} + \sum_{i,j} (r)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)}
 \end{aligned}$$

and note that, using the fact that δf is an exact biincrement

$$\sum_{i,j} (\delta f)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = (\delta f)_{(t^1, s^1)(t^2, s^2)}$$

and

$$\sum_{i,j} (r_1)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = \sum_j (r_1)_{(t^1, s^1)(t_{j+1}^2, t_j^2)}$$

since $\delta_1 r_1 = 0$ (i.e. r_1 is an exact increment in the direction 1). In the same way

$$\sum_{i,j} (r_2)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} = \sum_i (r_2)_{(t_{i+1}^1, t_i^1)(t^2, s^2)}$$

Then

$$\left| \sum_j (r_1)_{(t^1, s^1)(t_{j+1}^2, t_j^2)} \right| \leq \|r_1\| \sum_j |t_{j+1}^2 - t_j^2|^{z_2} \rightarrow 0$$

and

$$\left| \sum_i (r_2)_{(t_{i+1}^1, t_i^1)(t^2, s^2)} \right| \leq \|r_2\| \sum_i |t_{i+1}^1 - t_i^1|^{z_1} \rightarrow 0$$

and finally

$$\left| \sum_{i,j} (r)_{(t_{i+1}^1, t_i^1)(t_{j+1}^2, t_j^2)} \right| \leq \|r\| \sum_{i,j} |t_{i+1}^1 - t_i^1|^{z_1} |t_{j+1}^2 - t_j^2|^{z_2} \rightarrow 0$$

as $|\Pi| \rightarrow 0$ which proves our claim. □

2.4 Two-dimensional Young theory

Proposition 2.4.1. *Let $f \in \mathcal{C}_{1,1}^{\gamma_1, \gamma_2}$, $g \in \mathcal{C}_{1,1}^{\rho_1, \rho_2}$ with $\gamma_1 + \rho_1 = z_1 > 1$, $\gamma_2 + \rho_2 = z_2 > 1$. Then $\delta f \delta g \in \mathcal{C}_2^{z_1, z_2} \cap \text{Img } \delta$ and the integral $\iint f dg$ can be defined as*

$$\begin{aligned}
 \iint f dg &= (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta g) \\
 &= -f \delta g + \Lambda(\delta f \delta g) + \int f d_1 \delta_2 g + \int f d_2 \delta_1 g
 \end{aligned} \tag{2.34}$$

where $\int f d_1 \delta_2 g, \int f d_2 \delta_1 g$ are standard Young integrals. Moreover we can define also the integral $\iint d_1 f d_2 g$ as

$$\iint d_1 f d_2 g = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(\delta_1 f \delta_2 g). \quad (2.35)$$

and for $\rho_1, \rho_2 > 1/2$ we can define

$$\iint f d_1 g d_2 g = (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_1 g \delta_2 g) \quad (2.36)$$

Corollary 2.4.2. *Under the hypothesis of the previous proposition, the two-dimensional sums of the increments $f \delta g$ converge:*

$$\lim_{|\Pi_{z,w}| \rightarrow 0} \sum_{i,j} (f \delta g)_{x_i x_{i+1}; y_j y_{j+1}} = \left(\iint f dg \right)_{z,w}. \quad (2.37)$$

where the partition $\Pi_{z,w}$ is taken on the square (z, w) , $z_i \in \mathbb{R}^2$.

Proposition 2.4.3. *Under the assumption of the Proposition (2.4.1) we have that:*

$$\begin{aligned} \iint f d_1 g d_2 g &= (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_2 g \delta_1 g) \\ &= (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_1 g \bullet \delta_2 g) \end{aligned} \quad (2.38)$$

where $(\delta_1 g \bullet \delta_2 g)_{s_1 s_2 t_1 t_2} = \delta_1 g_{s_1 s_2 t_1} \delta_2 g_{s_1 t_1 t_2}$

Proof. Let $a = f \delta_2 g \bullet \delta_1 g$. By a simple computation we have that $\delta_1 a \in \mathcal{C}_{3,2}^{\gamma_1 + \rho_1, *}$, $\delta_2 a \in \mathcal{C}_{2,3}^{*, \gamma_2, \rho_2}$ and $\delta a \in \mathcal{C}_{3,3}^{\gamma_1 + \rho_1, \gamma_2 + \rho_2}$ then the Corollary (2.3.3) gives

$$\begin{aligned} (1 - \Lambda_1 \delta_1)(1 - \Lambda_2 \delta_2)(f \delta_2 g \bullet \delta_1 g)_{s_1 s_2 t_1 t_2} &= \lim_{|\Pi| \rightarrow 0} \sum_{\Pi} f_{s_i t_j} \delta_2 g_{s_i t_j t_{j+1}} \delta_1 g_{s_i s_{i+1} t_j} \\ &= \lim_{|\Pi| \rightarrow 0} \sum_{\Pi} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta_2 g_{s_{i+1} t_j t_{j+1}} - f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta g_{s_i s_{i+1} t_j t_{j+1}} \end{aligned} \quad (2.39)$$

where $\Pi := \{(s_i, t_j)\}_{i,j}$ is a partition of $[s_1, s_2] \times [t_1, t_2]$ and we denote also by $\Pi_1 = (s_i)_i$ and $\Pi_2 = (t_j)_j$ a respectively partition of $[s_1, s_2]$ and $[t_1, t_2]$. Once again by the Corollary (2.3.3) we have that

$$\lim_{|\Pi| \rightarrow 0} \sum_{\Pi} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta_2 g_{s_{i+1} t_j t_{j+1}} = \left(\iint f d_1 g d_2 g \right)_{s_1 s_2 t_1 t_2}$$

and then the one dimensional Young theory of integration gives

$$\lim_{|\Pi_2| \rightarrow 0} \sum_{\Pi_2} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta g_{s_i s_{i+1} t_j t_{j+1}} = \int_{t_1}^{t_2} f_{s_i t} \delta_1 g_{s_i s_{i+1} t} d_t \delta_1 g_{s_i s_{i+1} t}$$

where $\left| \int_{t_1}^{t_2} f_{s_i t} \delta_1 g_{s_i s_{i+1} t} d_t \delta_1 g_{s_i s_{i+1} t} \right| \lesssim (s_{i+1} - s_i)^{2\rho_1} (t_2 - t_1)^{\rho_2}$. Finally using the fact that $\rho_1, \rho_2 > 1/2$ gives:

$$\lim_{|\Pi_1| \rightarrow 0} \lim_{|\Pi_2| \rightarrow 0} \sum_{|\Pi|} f_{s_i t_j} \delta_1 g_{s_i s_{i+1} t_j} \delta g_{s_i s_{i+1} t_j t_{j+1}} = 0$$

Putting all these last equation together give us the second line of our proposition, the first part is given by the same argument. \square

Proposition 2.4.4. *Let $x \in \mathcal{C}_{1,1}^{\alpha,\beta}$ and $\varphi \in C^4(\mathbb{R})$ then for $\alpha, \beta > 1/2$ the following change of variable formula :*

$$\delta\varphi(x) = \iint \varphi'(x)dx + \iint \varphi''(x)d_1x d_2x$$

hold.

Proof. Let $\Pi_1 = (s_i)_i$ a partition of $[s_1, s_2]$ and $\Pi_2 = (t_j)_j$ of $[t_1, t_2]$ then

$$\begin{aligned} \delta\varphi(x)_{s_1 s_2 t_1 t_2} &= \sum_{\Pi_1 \Pi_2} \delta\varphi(x)_{s_i s_{i+1} t_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} \left(\delta_2 \int_1 \varphi'(x) d_1 x \right)_{s_i s_{i+1} t_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} \int_{s_i}^{s_{i+1}} \delta_2 \varphi'(x)_{s t_j t_{j+1}} d_s x_{s t_{j+1}} + \int_{s_i}^{s_{i+1}} \varphi'(x_{s t_j}) d_s \delta_2 x_{s t_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} a_{ij} + b_{ij} \end{aligned}$$

where we have used the one dimensional change of variable formula for the Young integral. Now let us treat the first term of our sum

$$\begin{aligned} a_{ij} &= \int_{s_i}^{s_{i+1}} \delta_2 \varphi'(x)_{s t_j t_{j+1}} d_s x_{s t_{j+1}} \\ &= \delta_2 \varphi'(x)_{s_i t_j t_{j+1}} \delta_1 x_{s_i s_{i+1} t_{j+1}} + \Lambda_1 (\delta \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} \\ &= \left(\int_{t_j}^{t_{j+1}} \varphi''(x_{s_i t}) d_t x_{s_i t} \right) \delta_1 x_{s_i s_{i+1} t_{j+1}} \\ &\quad + \Lambda_2 \delta_2 (1 - \Lambda_1 \delta_1) (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} - \Lambda \delta (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} \end{aligned}$$

Is not difficult to see that

$$\lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} \Lambda_2 \delta_2 (1 - \Lambda_1 \delta_1) (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} = 0 \quad (2.40)$$

and

$$\lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} \Lambda \delta (\delta_2 \varphi'(x) \delta_1 x)_{s_i s_{i+1} t_j t_{j+1}} = 0 \quad (2.41)$$

then is remind to treat the term $\left(\int_{t_j}^{t_{j+1}} \varphi''(x_{s_i t}) d_t x_{s_i t} \right) \delta_1 x_{s_i s_{i+1} t_{j+1}}$ and for that we have the following expansion :

$$\left(\int_{t_j}^{t_{j+1}} \varphi''(x_{s_i t}) d_t x_{s_i t} \right) \delta_1 x_{s_i s_{i+1} t_{j+1}} = \varphi''(x_{s_i t_j}) \delta_2 x_{s_i t_j t_{j+1}} \delta_1 x_{s_i s_{i+1} t_j} + r_{s_i t_j t_{j+1}}^b \delta_1 x_{s_i s_{i+1} t_j} \quad (2.42)$$

where $r^b = \delta_2 \varphi'(x) - \varphi''(x) \delta_2 x \in \mathcal{C}_{1,2}^{*,2\beta}$ such that $\delta_1 r^b \in \mathcal{C}_{2,2}^{\alpha,2\beta}$ which gives :

$$\lim_{|\Pi_1| \rightarrow 0} \sum_{\Pi_1} r_{s_i t_j t_{j+1}}^b \delta_1 x_{s_i s_{i+1} t_j} = \int_{s_1}^{s_2} r_{s t_j t_{j+1}}^b d_s x_{s t_{j+1}}$$

where the limit satisfy the upper bound $|\int_{s_1}^{s_2} r_{st_j t_{j+1}}^b d_s x_{st_j t_{j+1}}| \lesssim_{\alpha, \beta} (s_2 - s_1)^\alpha (t_2 - t_1)^{2\beta}$ these allow us to conclude that :

$$\lim_{|\Pi_2| \rightarrow 0} \lim_{|\Pi_1| \rightarrow 0} \sum_{\Pi_1 \Pi_2} r_{s_i t_j t_{j+1}}^b \delta_1 x_{s_i s_{i+1} t_j} = 0. \quad (2.43)$$

Then putting together equation (2.40) , (2.41) , (2.42) and (2.43) we obtain that :

$$\left(\iint \varphi''(x) d_1 x d_2 x \right)_{s_1 s_2 t_1 t_2} = \lim_{|\Pi_2| \rightarrow 0} \lim_{|\Pi_1| \rightarrow 0} \sum_{\Pi_1 \Pi_2} a_{ij}$$

the second sum $\sum_{\Pi_1 \Pi_2} b_{ij}$ is more simple to compute indeed :

$$\begin{aligned} \sum_{\Pi_1 \Pi_2} b_{ij} &= \sum_{\Pi_1 \Pi_2} \int_{s_i}^{s_{i+1}} \varphi'(x_{st_j}) d_s \delta_2 x_{st_j t_{j+1}} \\ &= \sum_{\Pi_1 \Pi_2} \varphi'(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} + \Lambda_2 \delta_2 (1 - \Lambda_1 \delta_1) (\varphi'(x) \delta x) - \Lambda \delta (\varphi'(x) \delta x) \end{aligned}$$

and then

$$\begin{aligned} \lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} b_{ij} &= \lim_{|\Pi| \rightarrow 0} \sum_{\Pi_1 \Pi_2} \varphi'(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} \\ &= \left(\iint \varphi'(x) dx \right)_{s_1 s_2 t_1 t_2} \end{aligned}$$

and this gives the needed result. \square

We have also this immediate generalization for the Young integral.

Proposition 2.4.5. *Let $y \in \mathcal{C}_{1,1}^{\gamma_1, \gamma_2}$, $x \in \mathcal{C}_{1,1}^{\rho_1, \rho_2}$ and $z \in \mathcal{C}_{1,1}^{\beta_1, \beta_2}$ such that $\gamma_i + \rho_i > 1, \beta_i + \gamma_i > 1$ and $\beta_i + \rho_i > 1$ for $i = 1, 2$ then we can define*

$$\begin{aligned} \iint y d_1 x d_2 z &= (1 - \Lambda_1 \delta_1) (1 - \Lambda_2 \delta_2) (y \delta_1 x \delta_2 z) \\ \iint y d_2 z d_1 x &= (1 - \Lambda_1 \delta_1) (1 - \Lambda_2 \delta_2) (y \delta_2 z \delta_1 x) \end{aligned}$$

and

$$\iint y d_1 x \bullet d_2 z = (1 - \Lambda_1 \delta_1) (1 - \Lambda_2 \delta_2) (y \delta_1 x \bullet \delta_2 z)$$

and we have the following identity

$$\iint y d_1 x d_2 z = \iint y d_2 z d_1 x = \iint y d_1 x \bullet d_2 z$$

Moreover we have the following Riemann sums representation for our integrals

$$\begin{aligned} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{st} d_s x_{st} d_t z_{st} &= \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta_2 z_{s_{i+1} t_j t_{j+1}} \\ &= \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_1 x_{s_i s_{i+1} t_j} \delta_2 z_{s_i t_j t_{j+1}} \\ &= \lim_{|\pi| \rightarrow 0} \sum_{i,j} y_{s_i t_j} \delta_2 z_{s_i t_j t_{j+1}} \delta_1 x_{s_i s_{i+1} t_j} \end{aligned}$$

Proof. Let $a^1 = y\delta_1x\delta_2z$, $a^2 = y\delta_2z\delta_1x$ and $a^3 = y\delta_1x \bullet \delta_2z$ then by a simple computation we have that

$$\delta_1a^1 = -\delta_1y\delta_1x\delta_2z - y\delta_1x\delta_2z \in \mathcal{C}_{3,2}^{\min(\gamma_1+\rho_1,\rho_1+\beta_1),\beta_2}$$

$$\delta_2a^1 = -\delta_2y\delta_1x\delta_2z - y\delta_1x\delta_2z \in \mathcal{C}_{2,3}^{\rho_1,\min(\gamma_2+\beta_2,\rho_2+\beta_2)}$$

and

$$\delta a^1 = \delta y\delta_1x\delta_2z + \delta_1y\delta_1x\delta_2z + \delta_2y\delta_1x\delta_2z + y\delta_1x\delta_2z \in \mathcal{C}_{3,3}^{\min(\gamma_1+\rho_1,\rho_1+\beta_1),\min(\gamma_2+\beta_2,\rho_2+\beta_2)}$$

then a^1 satisfies the assumption of the Corollary (2.3.3) which also true for a^2 and a^3 by a similar computation. Then the integral $\iint yd_1xd_2z$, $\iint yd_2zd_1x$ and $\iint yd_1x \bullet d_2z$ are well defined and we have that

$$\left(\iint yd_2zd_1x\right)_{s_1s_2t_1t_2} = \lim_{|\pi|\rightarrow 0} \sum_{i,j} y_{s_it_j} \delta_1x_{s_is_{i+1}t_j} \delta_2z_{s_{i+1}t_jt_{j+1}}$$

$$\left(\iint yd_2zd_1x\right)_{s_1s_2t_1t_2} = \lim_{|\pi|\rightarrow 0} \sum_{i,j} y_{s_it_j} \delta_2z_{s_it_jt_{j+1}} \delta_1x_{s_is_{i+1}t_{j+1}}$$

and

$$\left(\iint yd_1x \bullet d_2z\right)_{s_1s_2t_1t_2} = \lim_{|\pi|\rightarrow 0} \sum_{i,j} y_{s_it_j} \delta_1x_{s_is_{i+1}t_j} \delta_2z_{s_it_jt_{j+1}}$$

To prove that this last integral coincide is suffices to show that the difference between the Riemann sum vanish when the mesh of the partition go to zero. In fact we have that

$$\begin{aligned} \lim_{|\pi_2|\rightarrow 0} \sum_{i,j} (a^1 - a^3)_{s_is_{i+1}t_jt_{j+1}} &= \lim_{|\pi_2|\rightarrow 0} \sum_{i,j} y_{s_it_j} \delta_1x_{s_is_{i+1}t_j} \delta z_{s_{i+1}t_jt_{j+1}} \\ &= \sum_i \int_{t_1}^{t_2} y_{s_it} \delta_1x_{s_is_{i+1}t} dt \delta_1z_{s_{i+1}t} \end{aligned} \quad (2.44)$$

Using the fact that $|\int_{t_1}^{t_2} y_{s_it} \delta_1x_{s_is_{i+1}t} dt \delta_1z_{s_{i+1}t}| \lesssim (s_{i+1} - s_i)^{\rho_1+\beta_1}$ we obtain that :

$$\lim_{|\pi_1|\rightarrow 0} \lim_{|\pi_2|\rightarrow 0} \sum_{i,j} (a^1 - a^3)_{s_is_{i+1}t_jt_{j+1}} = 0$$

which gives the following equality $\iint yd_1xd_2z = \iint yd_1x \bullet d_2z$. The other identity is obtained by exactly the same computations. \square

2.5 Analysis of a two-parameter integral

Following the informal approach of sec. 2.2.1 we would like to get inspired for a general construction from the *analysis* of a concrete two-dimensional integral. Then consider $\iint \varphi(x)dx$ for a smooth surface $x : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a smooth function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. The decomposition of eq.(2.28) gives

$$\iint \varphi(x)dx = -\varphi(x) \int dx + \int_1 \varphi(x) \int_2 dx + \int_2 \varphi(x) \int_1 dx + \iint d\varphi(x)dx$$

where, for example, we used the notation $\int_1 \varphi(x) \int_2 dx$ to mean

$$\left(\int_1 \varphi(x) \int_2 dx \right)_{(s_1, s_2; t_1, t_2)} := \int_{s_1}^{t_1} \varphi(x_{u_1, s_2}) \int_{s_2}^{t_2} \partial_1 \partial_2 x(u_1, u_2) du_1 du_2$$

and the others analogous expressions.

This decomposition is at the origin of Prop.2.4.1 when the demanded regularity is satisfied since the iterated integral is given by the formula

$$\iint d\varphi(x) dx = -\Lambda [\delta\varphi(x)\delta x].$$

To proceed further we note that

$$\iint d\varphi(x) = \iint \varphi'(x) dx + \iint \varphi''(x) (d_1 x d_2 x) \quad (2.45)$$

(recall that $d = d_1 d_2$ is not a derivation). Take the first term in the r.h.s. and use again eq.(2.28) to write

$$\iint \varphi'(x) dx = -\varphi'(x) \int dx + \int_1 \varphi'(x) \int_2 dx + \int_2 \varphi'(x) \int_1 dx + \iint d\varphi'(x) dx \quad (2.46)$$

and a similar equation for the term $\iint \varphi''(x) d_1 x d_2 x$. Then

$$\iint d\varphi(x) dx = \iint \varphi'(x) dx dx + \iint \varphi''(x) (d_1 x d_2 x) dx. \quad (2.47)$$

For simplicity we treat explicitly the first term in the r.h.s, the analysis of the second being similar.

Using eq.(2.46) and the definition of iterated integral, we have

$$\iint \varphi'(x) dx dx = -\varphi'(x) \int dx dx + \int_1 \varphi'(x) \int_2 dx dx + \int_2 \varphi'(x) \int_1 dx dx + \iint d\varphi'(x) dx dx \quad (2.48)$$

This expression seems complicated, however it shows that, in order to control the l.h.s. we need two ingredients:

- 1) Being able to define *essentially one-dimensional integrals* like

$$\int_1 \varphi(x) \int_2 dx, \int_1 \varphi'(x) \int_2 dx dx, \int_1 \varphi''(x) \int_2 d_1 x d_2 dx, \dots \quad (2.49)$$

- 2) Control the *remainders* given by the three-fold iterated integrals

$$\mathcal{R} := \iint d\varphi'(x) dx dx \quad \tilde{\mathcal{R}} := \iint d\varphi''(x) d_1 x d_2 dx.$$

The boundary integrals

We call integrals like those appearing in eq.(2.49) *boundary integrals* to emphasize their one-dimensional nature (which, we hope, will be clear from what follows). Their appearance is characteristic of the multidimensional setting and it is linked to the cohomological structure of the complex (\mathcal{C}_*, δ) studied in Sec. 2.3. Take the first of them and expand it according to the (one-dimensional) eq.(2.11):

$$\int_1 \varphi'(x) \int_2 dx dx = \varphi'(x) \iint dx dx + \int_1 d_1 \varphi'(x) \int_2 dx dx \quad (2.50)$$

Next, apply δ_1 to the second term in the r.h.s.:

$$-\delta_1 \int_1 d_1 \varphi'(x) \int_2 dx dx = \int_1 d_1 \varphi'(x) \int_1 \int_2 dx dx + \int_1 d_1 \varphi'(x) \int_2 dx \int_1 dx \quad (2.51)$$

The first term in the r.h.s. is *simple* since it is equivalent to $\delta_1 \varphi'(x) \iint dx dx$, and can be controlled with some regularity of φ' and the a-priori knowledge of $\iint dx dx$. The second needs to be further expanded as

$$\begin{aligned} \int_1 d_1 \varphi'(x) \int_2 dx \int_1 dx &= \int_1 \varphi''(x) d_1 x \int_2 dx \int_1 dx \\ &= \varphi''(x) \int_1 d_1 x \int_2 dx \int_1 dx + \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \int_1 dx \end{aligned} \quad (2.52)$$

Again, the first term in the r.h.s do not pose any further problem so we continue to study only the last one. Set

$$\mathcal{A}_1 := \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \int_1 dx$$

We apply to \mathcal{A}_1 the splitting operator S_1 obtaining

$$S_1 \mathcal{A}_1 = \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \otimes_1 \int_1 dx \quad (2.53)$$

Let us clarify the meaning of this last term: in the first direction is the tensor product of two 1-increments, while in the second direction is a 1-increment. Taking four points (u_1, u_2, u_3, u_4) in direction 1 and two (v_1, v_2) in direction 2, its value is given by the expression

$$\int_{u_3}^{u_4} \int_{v_1}^{v_2} \left(\int_{u_1}^{u_2} \int_{v_1}^v \left(\int_{u_1}^u \int_{u_1}^a d_b \varphi''(x_{bv_1}) d_a x_{av_1} \right) d_{uv} x_{uv} \right) d_{st} x_{st}$$

As the reader can easily check, by setting $u_2 = u_3$ we reobtain \mathcal{A}_1 .

Denote with $\delta_1 \otimes_1 1$ the action of the δ_1 operator on the first factor of an element of $\mathcal{C}_1 \otimes \mathcal{C}_1$, where again the 1 as index of the tensor operation denote that it acts on tensor products according to the first direction. Apply $\delta_1 \otimes_1 1$ to the r.h.s of eq. (2.53) last term in the r.h.s to obtain

$$\begin{aligned} -(\delta_1 \otimes_1 1) \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \otimes_1 \int_1 dx &= - \left[\delta_1 \int_1 d_1 \varphi''(x) d_1 x \int_2 dx \right] \otimes_1 \int_1 dx \\ &= \int_1 d_1 \varphi''(x) \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx + \int_1 d_1 \varphi''(x) d_1 x \int_1 \int_2 dx \otimes_1 \int_1 dx \end{aligned}$$

Hence let

$$\mathcal{A} \mathcal{A}_1 := \int_1 d_1 \varphi''(x) \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx = \delta_1 \varphi''(x) \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx$$

and (cfr. eq. (2.18))

$$\mathcal{A} \mathcal{B}_1 := \int_1 d_1 \varphi''(x) d_1 x \int_1 \int_2 dx \otimes_1 \int_1 dx = (\delta_1 \varphi''(x) - \varphi''(x) \delta_1 x) \int_1 \int_2 dx \otimes_1 \int_1 dx$$

which depend only on the function $\varphi''(x)$ and on the “splitted” iterated integrals

$$\int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx, \quad \int_1 \int_2 dx \otimes_1 \int_1 dx$$

Since we are working under smoothness conditions is possible to recover \mathcal{A}_1 by applying $\Lambda_1 \otimes_1 1$: $\mathcal{A}_1 = -(\Lambda_1 \otimes_1 1)(\mathcal{A}\mathcal{A}_1 + \mathcal{A}\mathcal{B}_1)$. Moreover

$$\mathcal{C}_1 := \int_1 d_1 \varphi''(x) d_1 x = \varphi''(x) \int_1 d_1 x \int_2 dx \int_1 dx + \mu_1 \mathcal{A}_1$$

where recall that μ_1 is the multiplication in the first direction which is the inverse operation of the splitting S_1 . Then we can recover also the term $\int_1 d_1 \varphi'(x) \int_2 dx \int_1 dx$ by eq.(2.50) and an application of Λ_1 . Finally we have obtained the following expression for the boundary term:

$$\int_1 \varphi'(x) \int_2 dx dx = \varphi'(x) \iint dx dx - \Lambda_1 \left[\delta_1 \varphi'(x) \int_1 \int_2 dx dx + \mathcal{C}_1 \right]$$

where the r.h.s. depends only on a finite number of iterated integrals of x .

The remainders $\mathcal{R}, \tilde{\mathcal{R}}$

The three-fold iterated integral $\iint d\varphi'(x) dx dx$ will be analyzed in terms of its image under δ :

$$\begin{aligned} \delta \iint d\varphi'(x) dx dx &= \iint d\varphi'(x) \iint dx dx + \iint d\varphi'(x) dx \iint dx \\ &+ \iint d\varphi'(x) \int_1 dx \int_2 dx + \iint d\varphi'(x) \int_2 dx \int_1 dx \end{aligned} \quad (2.54)$$

The first term is readily controlled in the same spirit as above. Consider the fourth (the third is similar) which we will write as

$$\begin{aligned} \iint d\varphi'(x) \int_2 dx \int_1 dx &= \int_1 d_1 [\delta_2 \varphi'(x)] \int_2 dx \int_1 dx \\ &= \int_1 \delta_2 [\varphi''(x) d_1 x] \int_2 dx \int_1 dx \\ &= \int_1 \delta_2 [\varphi''(x)] d_1 x \int_2 dx \int_1 dx + \int_1 \varphi''(x) d_1 \delta_2 x \int_2 dx \int_1 dx \\ &= \mathcal{D}_1 + \mathcal{E}_1 \end{aligned} \quad (2.55)$$

Expanding the integral of $\delta_2 [\varphi''(x)] d_1 x$ we get

$$\begin{aligned} \mathcal{D}_1 &= \int_1 \delta_2 [\varphi''(x)] d_1 x \int_2 dx \int_1 dx \\ &= \delta_2 [\varphi''(x)] \int_1 d_1 x \int_2 dx \int_1 dx + \int_1 d_1 \delta_2 [\varphi''(x)] d_1 x \int_2 dx \int_1 dx \end{aligned} \quad (2.56)$$

Again we apply $(\delta_1 \otimes_1 1)S_1$ to the second term:

$$\begin{aligned} &-(\delta_1 \otimes_1 1)S_1 \int_1 d_1 \delta_2 [\varphi''(x)] d_1 x \int_2 dx \int_1 dx \\ &= \int_1 d_1 \delta_2 [\varphi''(x)] \int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx + \int_1 d_1 \delta_2 [\varphi''(x)] d_1 x \int_1 \int_2 dx \otimes_1 \int_1 dx \\ &=: \mathcal{D}\mathcal{A}_1 + \mathcal{D}\mathcal{B}_1 \end{aligned} \quad (2.57)$$

From which we get:

$$\mathcal{D}_1 = \delta_2[\varphi''(x)] \int_1 d_1 x \int_2 dx \int_1 dx - \mu_1[\Lambda_1 \otimes_1 1](\mathcal{D}\mathcal{A}_1 + \mathcal{D}\mathcal{B}_1) \quad (2.58)$$

And in the same way:

$$\mathcal{E}_1 = \varphi''(x) \int_1 d_1 \delta_2 x \int_2 dx \int_1 dx - \mu_1[\Lambda_1 \otimes_1 1](\mathcal{E}\mathcal{A}_1 + \mathcal{E}\mathcal{B}_1) \quad (2.59)$$

with

$$\mathcal{E}\mathcal{A}_1 := \delta_1[\varphi''(x)] \int_1 d_1 \delta_2 x \int_2 dx \otimes_1 \int_1 dx$$

and

$$\mathcal{E}\mathcal{B}_1 := \int_1 d_1 \varphi''(x) d_1 \delta_2 x \int_1 \int_2 dx \otimes_1 \int_1 dx.$$

Note that

$$\begin{aligned} \mathcal{F}_1 &:= \mathcal{D}\mathcal{B}_1 + \mathcal{E}\mathcal{B}_1 \\ &= \delta_2 \left[\int_1 d_1 \varphi''(x) d_1 x \right] \int_1 \int_2 dx \otimes_1 \int_1 dx \\ &= \delta_2 [\delta_1 \varphi'(x) - \varphi''(x) \delta_1 x] \int_1 \int_2 dx \otimes_1 \int_1 dx. \end{aligned} \quad (2.60)$$

Together eq.(2.55), (2.60) (2.61) (2.60) imply the representation:

$$\begin{aligned} \iint d\varphi'(x) \int_2 dx \int_1 dx &= \delta_2[\varphi''(x)] \int_1 d_1 x \int_2 dx \int_1 dx + \varphi''(x) \int_1 d_1 \delta_2 x \int_2 dx \int_1 dx \\ &\quad - \mu_1[\Lambda_1 \otimes_1 1](\mathcal{D}\mathcal{A}_1 + \mathcal{E}\mathcal{A}_1 + \mathcal{F}_1). \end{aligned} \quad (2.61)$$

The only term, appearing in \mathcal{R} which is left is the second term in eq.(2.54): $\iint d\varphi'(x) dx \iint dx$. This term can be handled together a similar term appearing in the expansion of $\tilde{\mathcal{R}}$ which reads:

$$\iint d\varphi''(x) d_1 x d_2 x \iint dx$$

(all the other terms in $\tilde{\mathcal{R}}$ are handled as above). Indeed we have that the sum of $\iint d\varphi'(x) dx$ and $\iint d\varphi''(x) d_1 x d_2 x$ which are nontrivial two-dimensional iterated integrals, appear in the expansion for $\delta\varphi(x)$:

$$\begin{aligned} \delta\varphi(x) &= -\varphi'(x) \iint dx + \int_1 \varphi'(x) \int_2 dx + \int_2 \varphi'(x) \int_1 dx \\ &\quad - \varphi''(x) \iint d_1 x d_2 x + \int_1 \varphi''(x) \int_2 d_1 x d_2 x + \int_2 \varphi''(x) \int_1 d_1 x d_2 x \\ &\quad + \iint d\varphi'(x) dx + \iint d\varphi''(x) d_1 x d_2 x \end{aligned} \quad (2.62)$$

In this expression all the terms, except the last two can be expressed, following the approach we used for the boundary integrals above, as functional of a small number of integrals of x .

2.5.1 Rough sheet

We have shown in this section how the 2-dimensional integral $\iint \varphi(x)dx$ admits to be expressed as a well behaved functional F of a family \mathbb{X} of iterated integrals of x and of $\varphi(x)$. This functional can then be extended to more irregular functions x , not necessarily smooth, in two ways:

- a) *Algebraic approach.* We are given a family \mathbb{X} of biincrements which satisfy algebraic conditions analogous to that which allowed us to perform the computations in this section. In this case, the integral can be defined by the same functional F . The algebraic relations are then needed to show that such a definition is consistent with our notion of integral (e.g. that this integral is in the kernel of both δ_1 and δ_2).
- b) *Geometric approach.* We are able to show that, there exists a sequence of families \mathbb{X}^n obtained by iterated integrals over smooth 2-dimensional functions x^n , which converges, under suitable topologies on the biincrements, to a limiting family \mathbb{X} . Then by the continuity of the map F we are able to identify the limit of $\iint \varphi(x^n)dx^n$ and to consider it as an extension of the integral over smooth 2-parameter functions. This is analogous to the *geometric* theory of rough paths. Already in the one-dimensional theory the two approaches can give different notions of integrals.

Algebraic assumption and Boundary integrals

Hypothesis 2.5.1. *If $a = 1, 2$ then $\hat{a} = 2, 1$. Assume that $\alpha, \beta > 1/3$. Now as in the one parameter case we assume the existence of some algebraic object*

$$\begin{aligned}
 A^x, A^\omega &\in \mathcal{C}_{2,2}^{\alpha,\beta}; & B_1^{xx}, B_1^{x\omega} &\in \mathcal{C}_{2,2}^{2\alpha,\beta}; & B_2^{xx}, B_2^{x\omega} &\in \mathcal{C}_{2,2}^{\alpha,2\beta} \\
 C^{xx}, C^{\omega x}, C^{x\omega}, C^{\omega\omega} &\in \mathcal{C}_{1,1}^{2\alpha,2\beta}; & D_1^{xx}, D_1^{\omega x}, D_1^{\omega\omega} &\in (\mathcal{C}_2^\alpha \otimes_1 \mathcal{C}_2^\alpha)(\mathcal{C}_1^{2\beta}), & D_2^{xx}, D_2^{\omega x}, D_2^{\omega\omega} &\in (\mathcal{C}_2^\beta \otimes_2 \mathcal{C}_2^\beta)(\mathcal{C}_1^{2\alpha}) \\
 E_1^{xxx}, E_1^{x\omega x}, E_1^{x\omega\omega}, E_1^{\omega\omega\omega} &\in (\mathcal{C}_2^{2\alpha} \otimes_1 \mathcal{C}_1^\alpha)(\mathcal{C}_1^{2\beta}); & F_1^{xxx}, F_1^{x\omega x}, F_1^{x\omega\omega}, F_1^{\omega\omega\omega} &\in (\mathcal{C}_2^{2\alpha} \otimes_1 \mathcal{C}_2^\alpha)(\mathcal{C}_1^{3\beta}) \\
 E_2^{xxx}, E_2^{x\omega x}, E_2^{x\omega\omega}, E_2^{\omega\omega\omega} &\in (\mathcal{C}_2^{2\beta} \otimes_2 \mathcal{C}_1^\beta)(\mathcal{C}_1^{2\alpha}); & F_2^{xxx}, F_2^{x\omega x}, F_2^{x\omega\omega}, F_2^{\omega\omega\omega} &\in (\mathcal{C}_2^{2\beta} \otimes_2 \mathcal{C}_2^\beta)(\mathcal{C}_1^{3\alpha})
 \end{aligned}$$

satisfying the following equations

1. $A^x = \delta x$,
2. $\delta_a B_a^{xx} = (\delta_a x)A^x$, $\delta_{\hat{a}} B_a^{xx} = -\mu_{\hat{a}} D_a^{xx}$
3. $\delta_a C^{xx} = \mu_a D_a^{xx}$
4. $(1 \otimes_a \delta_a) D_a^{xx} = 0$, $\delta_{\hat{a}} D_a^{xx} = A^x \otimes_a A^x$,
5. $(1 \otimes_a \delta_a) E_a^{xxx} = 0$, $(\delta_a \otimes_a 1) E^{xxx} = \delta_a x \otimes D_a^{xx}$, $\delta_{\hat{a}} E_a^{xxx} = F_a^{xxx} + B_a^{xx} \otimes_a A^x$,
6. $\delta_a A^\omega = 0$
7. $\delta_a B_a^{x\omega} = (\delta_a x)A^\omega$, $\delta_{\hat{a}} B_a^{x\omega} = \mu_{\hat{a}} D_a^{x\omega}$
8. $\delta_a C^{\omega x} = \mu_a D_a^{\omega x}$
9. $(1 \otimes_a \delta_a) D_a^{\omega x} = 0$, $\delta_{\hat{a}} D_a^{\omega x} = A^\omega \otimes_a A^x$
10. $(1 \otimes_a \delta_a) E_a^{x\omega x} = 0$, $(\delta_a \otimes_a 1) E^{x\omega x} = \delta_a x \otimes_a D_a^{\omega x}$, $\delta_{\hat{a}} E_a^{x\omega x} = F_a^{x\omega x} + B_a^{x\omega} \otimes_a A^\omega$
11. $\delta_a C^{x\omega} = \mu_a D_a^{x\omega}$
12. $(1 \otimes_a \delta_a) D_a^{x\omega} = 0$, $\delta_{\hat{a}} D_a^{x\omega} = A^x \otimes_a A^\omega$,
13. $(1 \otimes_a \delta_a) E_a^{x\omega\omega} = 0$, $(\delta_a \otimes_a 1) E^{x\omega\omega} = \delta_a x \otimes D_a^{x\omega}$, $\delta_{\hat{a}} E_a^{x\omega\omega} = F_a^{x\omega\omega} + B_a^{x\omega} \otimes_a A^\omega$,

14. $\delta_a C^{\omega\omega} = \mu_a D_a^{\omega\omega}$
15. $(1 \otimes_a \delta_a) D_a^{\omega\omega} = 0, \delta_{\hat{a}} D_a^{\omega\omega} = A^\omega \otimes_a A^\omega,$
16. $(1 \otimes_a \delta_a) E_a^{x\omega\omega} = 0, (\delta_a \otimes_a 1) E^{x\omega\omega} = \delta_a x \otimes D_a^{\omega\omega}, \delta_{\hat{a}} E_a^{x\omega\omega} = F_a^{x\omega\omega} + B_a^{x\omega} \otimes_a A^\omega,$

Remark 2.5.2. When x is a smooth sheet we can choose this algebraic object as the following iterated integral :

1. $(A^x)_{s_1 s_2 t_1 t_2} = (\iint dx)_{s_1 s_2 t_1 t_2} = \delta x_{s_1 s_2 t_1 t_2}$
2. $(A^\omega)_{s_1 s_2 t_1 t_2} = (\iint d\omega)_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} d_s x_{st} d_t x_{st}$
3. $(B_1^{xx})_{s_1 s_2 t_1 t_2} = (\int_1 d_1 x \int_2 dx)_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 t_1} d_{st} x_{st}$
4. $(B_1^{x\omega})_{s_1 s_2 t_1 t_2} = (\int_1 d_1 x \int_2 d\omega)_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 t_1} d_s x_{st} d_t x_{st}$
5. $(C^{xx})_{s_1 s_2 t_1 t_2} = (\iint dx dx)_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(s, t)} d_{rr'} x_{rr'} \right) d_{st} x_{st}$
6. $(C^{\omega x})_{s_1 s_2 t_1 t_2} = (\iint d\omega dx)_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(s, t)} d_r x_{rr'} d_{r'} x_{rr'} \right) d_{st} x_{st}$
7. $(D_1^{xx})_{s_1 s_2 s_3 s_4 t_1 t_2} = (\int_2 \int_1 dx \otimes_1 \int_1 dx)_{s_1 s_2 s_3 s_4 t_1 t_2} = \iint_{(s_3, t_1)}^{(s_4, t_2)} \left(\iint_{(s_1, t_1)}^{(s_2, t_2)} d_{rr'} x_{rr'} \right) d_{st} x_{st}$
8. $(E_1^{xxx})_{s_1 s_2 s_3 s_4 t_1 t_2} = (\int_1 d_1 x \int_2 dx \otimes_1 \int_1 dx)_{s_1 s_2 s_3 s_4 t_1 t_2} = \iint_{(s_3, t_1)}^{(s_4, t_2)} \left(\iint_{(s_1, t_1)}^{(s_2, t_2)} \delta_1 x_{s_1 r t_1} d_{rr'} x_{rr'} \right) d_{st} x_{st}$
- 9.

$$\begin{aligned} (F_1^{xxx})_{s_1 s_2 s_3 s_4 t_1 t_2 t_3} &= \left(\iint dx \int_2 dx \otimes_1 \int_1 dx \right)_{s_1 s_2 s_3 s_4 t_1 t_2 t_3} \\ &= - \iint_{(s_3, t_2)}^{(s_4, t_3)} \left(\iint_{(s_1, t_2)}^{(s_2, t_2)} \left(\iint_{(s_1, t_1)}^{(r, t_2)} d_{ab} x_{ab} \right) d_{rr'} x_{rr'} \right) d_{st} x_{st} \end{aligned}$$

In this section we assume the previous hypothesis to be true and we will give a "reasonable" construction of the following boundary integrals :

$$\begin{aligned} \int_a y \int_{\hat{a}} dx, \quad \int_a y \int_{\hat{a}} dx dx, \\ \int_a y \int_{\hat{a}} d\omega, \quad \int_a y \int_{\hat{a}} d\omega dx, \end{aligned}$$

which allow us to construct the space of two parameters controlled sheet and integrate them. We begin by recalling the notion of a one dimensional controlled path which the space of the sheet y satisfying the following assumption :

$$\delta_a y = y^{x_a} \delta_a x + y^{\#a}, \quad y^{x_a} \in \mathcal{C}_{1,1}^{\alpha,\beta}, \quad (y^{\#1}, y^{\#2}) \in \mathcal{C}_{2,1}^{2\alpha,\beta} \times \mathcal{C}_{1,2}^{\alpha,2\beta}$$

where $x \in \mathcal{C}_{1,1}^{\alpha,\beta}$, $a \in \{1, 2\}$ and we denoted by $\mathcal{Q}_x^{\alpha,\beta}$ this space. Now we will set out a permutation lemma that is useful to conduct the computation in the following.

Lemma 2.5.3. We have for $h \in (\mathcal{C}_2 \otimes_a \mathcal{C}_2)(\mathcal{C}_2)$ the following identity:

$$\delta_a \mu_a h = \mu_a (\delta_a \otimes_a 1) h - \mu_a (1 \otimes_a \delta_a) h. \quad (2.63)$$

And then we have the following construction for the Boundary integral :

Proposition 2.5.4. *Assume that the hypothesis 2.5.1 to be true, and let $y \in \mathcal{Q}_x^{\alpha,\beta}$. Then we define the boundary integral by :*

1. $\int_a y \int_{\hat{a}} dx := yA^x + y^{x_a} B_a^{xx} + \Lambda_a[y^{\#a} A^x + \delta_a y^{x_a} B_a^{xx}]$,
2. $\int_a y \int_{\hat{a}} dx dx := yC^{xx} + \Lambda_a[\delta_a y C^{xx} + y^x \mu_a E_a^{xxx} + \mu_a(\Lambda_a \otimes_a 1)(y^{\#a} D_a^{xx} + \delta_a y^{x_a} E_a^{xxx})]$,
3. $\int_a y \int_{\hat{a}} d\omega := yA^\omega + y^{x_a} B_a^{x\omega} + \Lambda_a[y^{\#a} A^\omega + y^x B_a^{x\omega}]$
4. $\int_a y \int_{\hat{a}} d\omega dx := yC^{\omega x} + \Lambda_a[\delta_a y C^{\omega x} + y^x \mu_a E_a^{x\omega x} + \mu_a(\Lambda_a \otimes_a 1)(y^{\#a} D_a^{\omega x} + \delta_a y^{x_a} E_a^{x\omega x})]$
5. $\int_a y \int_{\hat{a}} dx d\omega := yC^{x\omega} + \Lambda_a[\delta_a y C^{x\omega} + y^x \mu_a E_a^{x\omega} + \mu_a(\Lambda_a \otimes_a 1)(y^{\#a} D_a^{x\omega} + \delta_a y^{x_a} E_a^{x\omega})]$
6. $\int_a y \int_{\hat{a}} d\omega d\omega := yC^{\omega\omega} + \Lambda_a[\delta_a y C^{\omega\omega} + y^x \mu_a E_a^{x\omega\omega} + \mu_a(\Lambda_a \otimes_a 1)(y^{\#a} D_a^{\omega\omega} + \delta_a y^{x_a} E_a^{x\omega\omega})]$

Moreover all these formulas have meaning and when x is differentiable we can choose the rough sheet so that they coincide well with their definition in the Riemann-Stieltjes case, which justifies the notation.

Proof. We will only prove the first two formula, for the others we have identical proofs. Let now x a smooth sheet and $y \in \mathcal{Q}_x^{\alpha,\beta}$, then we have easily the following expansion

$$\begin{aligned} \left(\int_1 y \int_2 dx \right)_{s_1 s_2 t_1 t_2} &:= \iint_{(s_1, t_1)}^{(s_2, t_2)} y_{st_1} d_{st} x_{st} = \int_{s_1}^{s_2} y_{st_1} d_1 \delta_2 x_{st_1 t_2} \\ &= y_{s_1 t_1} (A^x)_{s_1 s_2 t_1 t_2} + y_{s_1 t_1}^x (B_1^{xx})_{s_1 s_2 t_1 t_2} + \int_{s_1}^{s_2} y_{s_1 t_1}^{\#1} d_s \delta_2 x_{st_1 t_2} \end{aligned}$$

where $A^x = \delta x$ and $(B_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \delta_1 x_{s_1 t_1} d_{st} x_{st}$. Now is easy to check that $r_{s_1 s_2 t_1 t_2} := \int_{s_1}^{s_2} y_{s_1 t_1}^{\#1} d_s \delta_2 x_{st_1 t_2} \in \mathcal{C}_{2,2}^{3\alpha,\beta}$ and $\delta_1 r = y^{\#1} \delta x + \delta_1 y^{x_1} B_1^{xx} \in \mathcal{C}_{3,2}^{3\alpha,\beta}$. So finally we obtain :

$$r = \Lambda_1[y^{\#1} \delta x + \delta_1 y^{x_1} B_1^{xx}]$$

and thus

$$\int_1 y \int_2 dx := yA^x + y^{x_1} B_1^{xx} + \Lambda_1[y^{\#1} A^x + \delta_1 y^{x_1} B_1^{xx}]$$

Now we remark that this formula is still valid when x satisfy only the assumption (2.5.1). Now we will focus on the second equation which requires a different work. then a quick computation gives :

$$\begin{aligned} \left(\int_1 y \int_2 dx dx \right)_{s_1 s_2 t_1 t_2} &:= \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} y_{rt_1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &= y_{s_1 t_1} (C^{xx})_{s_1 s_2 t_1 t_2} + \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} \delta_1 y_{s_1 r t_1} d_{rr'} x_{rr'} d_{st} x_{st} \end{aligned}$$

The first term of this last equation is well understood, we will focus on the second term denoted by \mathcal{H} in the sequel. Now we observed that $\mathcal{H} \in \mathcal{C}_{2,2}^{3\alpha,2\beta}$ and then:

$$\begin{aligned} \delta_1 \mathcal{H}_{s_1 s_2 s_3 t_1 t_2} &= \delta_1 y_{s_1 s_2 t_1} C_{s_2 t_1 s_3 t_2}^{xx} + y_{s_1 t_1}^x \iint_{(s_2, t_1)}^{(s_3, t_2)} \iint_{(s_1, t_1)}^{(s_2, t)} \delta_1 x_{s_1 r t_1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &\quad + \iint_{(s_2, t_1)}^{(s_3, t)} \iint_{(s_1, t_1)}^{(s_2, t)} y_{s_1 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st} \end{aligned}$$

now if we put

$$\mathcal{A}_{s_1 s_2 s_3 t_1 t_2} := \iint_{(s_2, t_1)}^{(s_3, t)} \iint_{(s_1, t_1)}^{(s_2, t)} y_{s_1 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st}$$

we obtain

$$\begin{aligned} (\delta_1 \otimes 1) \mathcal{S}_1(\mathcal{A}_1)_{s_1 s_2 v s_3 s_4 t_1 t_2} &= \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} y_{s_1 r t_1}^{\#1} - y_{s_2 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &= y_{s_1 s_2 t_1}^{\#1} \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} d_{rr'} x_{rr'} d_{st} x_{st} \\ &\quad + \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} \delta_1 y_{s_1 s_2 r t_1}^{\#1} d_{rr'} x_{rr'} d_{st} x_{st} \\ &= y_{s_1 s_2 t_1}^{\#1} \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} d_{rr'} x_{rr'} d_{st} x_{st} \\ &\quad + \delta_1 y_{s_1 s_2 t_1}^x \iint_{(s_3, t_1)}^{(s_4, t_2)} \iint_{(s_2, t_1)}^{(v, t)} \delta_1 x_{s_2 r t_1} d_{rr'} x_{rr'} d_{st} x_{st} \end{aligned}$$

Then if we recall that the two last iterated integrals are denoted respectively by D_1^{xx} and E_1^{xxx} we obtain:

$$\mathcal{S}_1 \mathcal{A}_1 = (\Lambda_1 \otimes 1) [y^{\#1} D_1^{xx} + \delta_1 y^{x1} E_1^{xxx}]$$

then

$$\mathcal{H} = \Lambda_1 [\delta_1 y C^{xx} + y^{x1} \mu_1 E_1^{xxx} + \mu_1 (\Lambda_1 \otimes 1) (y^{\#1} D_1^{xx} + \delta_1 y_1^x E_1^{xxx})]$$

This last equation give us the second formula when x is a smooth sheet, in the general case when x satisfies only the Hypothesis 2.5.1. It's easy to see that all terms or we apply Λ_1 enjoy the regularity and thanks to the Lemma 2.5.3 it also satisfy the required algebraic conditions (i.e. : $\delta_1 [y C^{xx} + y^{x1} \mu_1 E_1^{xxx} + \mu_1 (\Lambda_1 \otimes 1) (y^{\#1} D_1^{xx} + \delta_1 y_1^x E_1^{xxx})] = 0$) and this finishes the proof \square

Controlled Sheet

Definition 2.5.5. Let $x \in \mathcal{C}_{1,1}$ such that $\delta x \in \mathcal{C}_{2,2}^{\alpha,\beta}$, and assume the algebraic hypothesis 2.5.1 to be true then we define the space of the two parameter controlled sheet $\mathcal{K}_x^{\alpha,\beta}$ by $y \in \mathcal{K}_x^{\alpha,\beta}$ if :

1. $\delta y = -y^x A^x - y^\omega A^\omega + \sum_{a=1,2} (\int_a y^x \int_{\hat{a}} dx + \int_a y^\omega \int_{\hat{a}} d\omega) + y^\#$
2. $y, y^x, y^\omega \in \mathcal{Q}_x^{\alpha,\beta}$

Theorem 2.5.6. For $\alpha, \beta > 1/3$, $x \in \mathcal{C}_{1,1}$ such that $\delta x \in \mathcal{C}_{2,2}^{\alpha,\beta}$ and assume that the hypothesis (2.5.1) is true then for $y \in \mathcal{K}_x^{\alpha,\beta}$ we define the increment $\iint y dx \in \mathcal{C}_{2,2}^{\alpha,\beta}$ and $\iint y d\omega \in \mathcal{C}_{2,2}^{\alpha,\beta}$ by:

$$\iint y dx := -y A^x + y^x C^{xx} + y^\omega C^{\omega x} + \sum_{a=1,2} \left(\int_a y \int_{\hat{a}} dx + \int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) + r^b$$

where

$$r^{b,\omega} = \Lambda \left[\delta y A^x - \delta \left(-y^x C^{xx} - y^\omega C^{\omega x} + \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) \right) \right]$$

and

$$\iint y d\omega := -yA^\omega + y^x C^{x\omega} + y^\omega C^{\omega\omega} + \sum_{a=1,2} \left(\int_a y \int_{\hat{a}} d\omega + \int_a y^x \int_{\hat{a}} dx d\omega + \int_a y^\omega \int_{\hat{a}} d\omega d\omega \right) + r^{b,\omega}$$

with

$$r^b = \Lambda \left[\delta y A^\omega - \delta \left(-y^x C^{x\omega} - y^\omega C^{\omega\omega} + \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx d\omega + \int_a y^\omega \int_{\hat{a}} d\omega d\omega \right) \right) \right]$$

Theses two formula are well defined moreover if x is a smooth sheet then this two definition coincide with that given by the Riemann-Stieltjes theory of integration.

Proof. Let x a differentiable sheet then

$$\iint y dx = -y\delta x + \sum_{a=1,2} \int_a y \int_{\hat{a}} dx + \iint dy dx.$$

Now using the fact that $y \in \mathcal{K}_x^{\alpha,\beta}$ we have

$$\iint dy dx = -y^x C^{xx} - y^\omega C^{\omega x} \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) + \iint y^\# dx.$$

Finally if we remark that $\iint y^\# dx \in \mathcal{C}_{2,2}^{3\alpha,3\beta}$ we obtain

$$\iint y^\# dx := \Lambda \left[\delta \iint dy dx - \delta \left(-y^x C^{xx} - y^\omega C^{\omega x} \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx \right) \right) \right]$$

This give us the formula when x is smooth. Now we have to check that last formula have meaning in general case in other word we must show that we can apply Λ for $r := \delta \iint dy dx - \delta(-y^x C^{xx} - y^\omega C^{\omega x} \sum_{a=1,2} (\int_a y^x \int_{\hat{a}} dx dx + \int_a y^\omega \int_{\hat{a}} d\omega dx))$, for this we will make some preliminary computation.

$$\begin{aligned} \delta \iint dy dx &= \delta y \delta x \\ \delta (y^x C^{xx}) &= \delta_2(-\delta_1 y^x C^{xx} + y^x \delta_1 C^{xx}) = \delta y^x C^{xx} + y^x \delta x \delta x - \delta_1 y^x \delta_2 C^{xx} - \delta_2 y^x \delta_1 C^{xx} \\ \delta \int_a d_a y^x \int_{\hat{a}} dx dx &= \delta_{\hat{a}} \left(\delta_a y^x C^{xx} + y^{xx_a} \mu_a E_a^{xxx} + (\Lambda_a \otimes_a 1) \left[\delta_a y^{xx_a} E_a^{xxx} + y^{x\#a} D_a^{xx} \right] \right) \\ &= -\delta y^x C^{xx} + \delta_a y^x \delta_a C^{xx} - \delta_{\hat{a}} y^{xx_a} \mu_a E_a^{xxx} + y^{xx_a} (B_a^{xx} \delta x + \mu_a F_a^{xxx}) \\ &\quad + \mu_a (\Lambda_a \otimes_a 1) [y^{x\#a} \delta x \otimes_a \delta x + \delta_a y^{xx_a} B_a^{xx} \otimes_a \delta x - \delta_{\hat{a}} y^{x\#a} D_a^{xx} + \delta_a y^{xx_a} F_a^{xxx} \\ &\quad - \delta y^{xx_a} E_a^{xxx}] \\ &= -\delta y^x C^{xx} + \delta_a y^x \delta_a C^{xx} - \delta_{\hat{a}} y^{xx_a} \mu_a E_a^{xxx} + y^{xx_a} \mu_a F_a^{xxx} + \left(\int_a d_a y^x \int_{\hat{a}} dx \right) \delta x \\ &\quad + (\Lambda_a \otimes_a 1) \left[-\delta_{\hat{a}} y^{x\#a} D_a^{xx} + \delta_a y^{xx_a} F_a^{xxx} - \delta y^{xx_a} E_a^{xxx} \right] \end{aligned}$$

Is easy to see that we have a similar equation if we replace dx by $d\omega := d_1x d_2x$. Finally we obtain

$$\begin{aligned} r = & y^\sharp \delta x + \delta y^x C^{xx} + \delta y^\omega C^{\omega x} + \sum_{a=1,2} (\delta_{\hat{a}} y^{xxa} \mu_a E_a^{xxx} - y^{xxa} \mu_a F_a^{xxx} + \delta_{\hat{a}} y^{\omega xa} \mu_a E_a^{x\omega x} - y^{\omega xa} \mu_a F_a^{x\omega x} \\ & + (\Lambda_a \otimes_a 1) [\delta_{\hat{a}} y^{x\sharp a} D_a^{xx} - \delta_a y^{xxa} F_a^{xxx} + \delta y^{xxa} E_a^{xxx} + \delta_{\hat{a}} y^{\omega\sharp a} D_a^{\omega x} - \delta_a y^{\omega xa} F_a^{x\omega x} + \delta y^{\omega xa} E_a^{x\omega x}]) \end{aligned}$$

and this allow us to say that $r \in \mathcal{C}_{3,3}^{3\alpha,3\beta}$ which finishes the proof. \square

Remark 2.5.7. We observe that this definition of the two parametric integral are not consistent with the definition of the controlled sheet, indeed if $y \in \mathcal{K}_x^{\alpha,\beta}$ then the element $z \in \mathcal{C}_{1,1}$ defined by $z_{0t} = z_{s0} = 0$ and $\delta z = \iint y dx$ is not in general a controlled sheet.

2.5.2 Stability under mapping by regular functions

In this section we show that $\varphi(x) \in \mathcal{K}_x^{\alpha,\beta}$ under more algebraic and geometric assumptions. To prove this result we will proceed by linear approximation the problem is that the terms which contain $d_1x d_2x$ does not approximate well to bypass this difficulty we will start by giving an alternative expression for the space $\mathcal{K}_x^{\alpha,\beta}$

Hypothesis 2.5.8. Let $\alpha, \beta > 1/3$ and $a = 1, 2$. Assume that there exist

$$G_a^{xx} \in \mathcal{C}_{2,2}^{\alpha,\beta}, \quad H_a^{xx} \in \mathcal{C}_{2,2}^{\hat{a}\alpha,a\beta}, \quad I_a^{xx} \in \mathcal{C}_{2,2}^{\hat{a}\alpha,a\beta}, \quad J_a^{xx} \in \mathcal{C}_{2,2}^{2\alpha,2\beta},$$

with the convention if $a = 1$ then $\hat{a} = 2$ and conversely. And we assume that this object satisfy the following relation:

1. $\delta_a G_a^{xx} = \delta_a I_a^{xx} = 0$
2. $\delta_a H_a^{xx} = G_a^{xx} \delta_a x$
3. $\delta_a J_a^{xx} = I_a^{xx} \delta_a x$
4. $I_a^{xx} = B_a^{xx} + C^{xx}$
5. $A^\omega = 1/2 \delta x^2 - \iint x dx = G_a^{xx} - I_a^{xx}$
6. $B_a^{x\omega} = H_a^{xx} - J_a^{xx}$

Remark 2.5.9. In regular case this last iterated integral are given by :

1. $(G_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \delta_2 x_{st_1 t_2} d_s x_{st_2}$
2. $(H_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \delta_1 x_{s_1 s t_1} \delta_2 x_{st_1 t_2} d_s x_{st_2}$
3. $(I_2^{xx})_{s_1 s_2 t_1 t_2} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \delta_2 x_{st_1 t} d_{st} x_{st}$
4. $(J_1^{xx})_{s_1 s_2 t_1 t_2} = \int_{s_1}^{s_2} \delta_1 x_{s_1 s t_1} \int_{t_1}^{t_2} \delta_2 x_{st_1 t} d_{st} x_{st}$

Now under these assumption we give an alternative expression for the space $\mathcal{K}_x^{\alpha,\beta}$.

Proposition 2.5.10. Assume the Hypothesis 2.5.8 and 2.5.1 are true then for $y \in \mathcal{Q}_x^{\alpha,\beta}$ we have :

1. $\int_a y \delta_{\hat{a}} x d_a x := y G_a^{xx} + y^{xa} H_a^{xx} + \Lambda_a [y^{\sharp a} G_a^{xx} + \delta_a y^{xa} H_a^{xx}]$
2. $\int_a y \int_{\hat{a}} \delta_{\hat{a}} x dx := y I_{\hat{a}}^{xx} + y^{xa} J_a^{xx} + \Lambda_a [y^{\sharp a} I_{\hat{a}}^{xx} + \delta_a y^{xa} J_a^{xx}]$
3. $\int_a y \int_{\hat{a}} d\omega = \int_a y \delta_{\hat{a}} x d_a x - \int_a y \int_{\hat{a}} \delta_{\hat{a}} x dx$

where 1 and 2 are well defined, moreover we can choose G_a , H_a and J_a such that the rough-integral 1 and 2 coincide with their definition in the Riemann-Stieltjes case.

Proof. We will only proof the first assertion (the proof of second assertion is similar). We assume that x is smooth then we have :

$$\left(\int_1 y \delta_2 x d_a x \right)_{s_1 s_2 t_1 t_2} := \int_{s_1}^{s_2} y_{st_1} \delta_2 x_{st_1 t_2} d_s x_{st} = y_{s_1 t_1} (G_1^{xx})_{s_1 s_2 t_1 t_2} + y^{x^1} (H_1^{xx})_{s_1 s_2 t_1 t_2} + r_{s_1 s_2 t_1 t_2}$$

where

$$r_{s_1 s_2 t_1 t_2} := \int_{s_1}^{s_2} y_{st_1 t_2}^{\#1} \delta_2 x_{st_1 t_2} d_s x_{st_2}$$

Now is clear that $r \in \mathcal{C}_{2,2}^{3\alpha,\beta}$ and :

$$\delta_1 r = y^{\#1} G_1^{xx} + \delta_1 y^{x^1} H_1^{xx} \in \mathcal{C}_{3,2}^{3\alpha,\beta}$$

and we get

$$r = \Lambda_1 [y^{\#1} G_1^{xx} + \delta_1 y^{x^1} H_1^{xx}]$$

This give us the proof of the first assertion. The proof of the last assertion is immediate consequence of the assumption five and six of the hypothesis 2.5.8. \square

This proposition allow us to tell that $y \in \mathcal{K}_x^{\alpha,\beta}$ if and only if :

1. $y \in \mathcal{Q}_x^{\alpha,\beta}$
2. $\delta y = -y^x \delta x - y^\omega (1/2 \delta x^2 - x \delta x) + \sum_{a=1,2} \left(\int_a y^x \int_{\hat{a}} dx + \int_a y^\omega \delta_{\hat{a}} x d_a x \right) + y^\#$

where $y^x, y^\omega \in \mathcal{Q}_x^{\alpha,\beta}$, $y^\# \in \mathcal{C}_{2,2}^{\alpha,\beta}$ Now let us describe our strategy to prove that $\varphi(x) \in \mathcal{K}_x^{\alpha,\beta}$. Let introduce the approximation:

$$x_{st}^1 = x_{0t} + s(x_{1t} - x_{0t}) \quad , \quad x_{st}^2 = x_{s0} + t(x_{s1} - x_{s0})$$

$$x_{st}^{12} = x_{00} + s(x_{10} - x_{00}) + t(x_{01} - x_{00}) + st \delta x \square$$

where the \square is the unit square. Now is clear if $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ then we have

$$\begin{aligned} \delta \varphi(x^{12}) &= -\varphi'(x^{12}) \delta x - \varphi''(x^{12}) (1/2 \delta(x^{12})^2 - x^{12} \delta x^{12}) \\ &+ \sum_{a=1,2} \left(\int_a \varphi'(x^{12}) \int_{\hat{a}} dx^{12} + \int_a \varphi''(x^{12}) \delta_{\hat{a}} x^{12} d_a x^{12} \right) + R^{12} \end{aligned} \quad (2.64)$$

where

$$\begin{aligned} R^{12} &= \iint \varphi''(x^{12}) dx^{12} dx^{12} + \iint \varphi'''(x^{12}) d_1 x^{12} d_2 x^{12} dx^{12} + \iint \varphi'''(x^{12}) dx^{12} d_1 x^{12} d_2 x^{12} \\ &+ \iint \varphi^{(iv)}(x^{12}) d_1 x^{12} d_2 x^{12} d_1 x^{12} d_2 x^{12} - \varphi''(x^{12}) \iint dx^{12} dx^{12} - \sum_{a \in \{1,2\}} \int_a d_a \varphi''(x^{12}) \int_{\hat{a}} \delta_{\hat{a}} x^{12} dx^{12} \end{aligned} \quad (2.65)$$

Now our goal is to give similar formula for x^a and to compare them with the same expansion for x . To do that we need to give a meaning for the boundary integral appearing in the expansion of x^a and for that we construct $B_a^{x^{\hat{a}}x^{\hat{a}}}$, $G_a^{x^{\hat{a}}x^{\hat{a}}}$ and $H_a^{x^{\hat{a}}x^{\hat{a}}}$. But by formal computation we see that

$$(B_2^{x^1x^1})_{s_1s_2t_1t_2} = (s_2 - s_1) \left(\int_{t_1}^{t_2} \delta_2 x_{0t_1t} d_t \delta_1 x_{01t} + s_1 \int_{t_1}^{t_2} \delta x_{01t_1t} d_t \delta_1 x_{01t} \right)$$

Then we define $B_2^{x^1x^1}$ by the following formula :

$$(B_2^{x^1x^1})_{s_1s_2t_1t_2} := (s_2 - s_1) (B_2^{xx} + s_1 (K_2^{xx})_{01t_1t_2})$$

where

$$(K_2^{xx})_{s_1s_2t_1t_2} := \int_{t_1}^{t_2} \delta x_{s_1s_2t_1t} d_t \delta_1 x_{s_1s_2t}$$

$$\delta_2 (K_2^{xx})_{s_1s_2t_1t_2t_3} = \delta x_{s_1s_2t_1t_2} \delta x_{s_1s_2t_1t_3}$$

Similar computation allow us to define $(G_2^{x^1,x^1})$ and $H_2^{x^1x^1}$ in the following way :

$$(G_2^{x^1x^1})_{s_1s_2t_1t_2} = (s_2 - s_1) ((G_2^{xx})_{01t_1t_2} + (s_2 - 1) (L_2^{xx})_{01t_1t_2})$$

$$(H_2^{x^1x^1})_{s_1s_2t_1t_2} = (s_2 - s_1) (H_2^{xx})_{01t_1t_2} + (s_2 - 1) (M_2^{xx})_{01t_1t_2} + s_1 ((N_2^{xx})_{01t_1t_2} + (s_2 - 1) (O_2^{xx})_{01t_1t_2})$$

where

$$(L_2^{xx})_{s_1s_2t_1t_2} := \int_{t_1}^{t_2} \delta_1 x_{s_1s_2t} d_t \delta_1 x_{s_1s_2t}, \quad (M_2^{xx})_{s_1s_2t_1t_2} := \int_{t_1}^{t_2} \delta x_{s_1s_2t_1t} \delta_1 x_{s_1s_2t} d_t x_{s_2t}$$

and

$$(O_2^{xxx})_{s_1s_2t_1t_2} := \int_{t_1}^{t_2} \delta x_{s_1s_2t_1t} \delta_1 x_{s_1s_2t} d_t \delta_1 x_{s_1s_2t}$$

and this of course push us to give a more algebraic assumption on the sheet x :

Hypothesis 2.5.11. *Let $\alpha, \beta > 1/3$, $a = 1, 2$ and and assume the existence of :*

$$K_a^{xx}, M_a^{xx} \in \mathcal{C}_{2,2}^{2\alpha,2\beta}, \quad L_a^{xx} \in \mathcal{C}_{2,2}^{a\alpha,\hat{a}\beta}, \quad O_2^{xxx} \in \mathcal{C}_{2,2}^{3\alpha,2\beta}, \quad O_1^{xxx} \in \mathcal{C}_{2,2}^{2\alpha,3\beta}$$

which satisfies the algebraic relation

1. $\delta_2 (K_2^{xx})_{s_1s_2t_1t_2t_3} = \delta x_{s_1s_2t_1t_2} \delta x_{s_1s_2t_1t_3}$
2. $\delta_2 L_2^{xx} = 0$
3. $\delta_2 (M_2^{xx})_{s_1s_2t_1t_2t_3} = \delta_2 x_{s_1t_1t_2} (L_2^{xx})_{s_1s_2t_2t_3}$
4. $\delta_2 (N_2^{xx})_{s_1s_2t_1t_2t_3} = \delta x_{s_1s_2t_1t_2} (G_2^{xx})_{s_1s_2t_1t_2}$
5. $\delta_2 (O_2^{xxx})_{s_1s_2t_1t_2} = \delta x_{s_1s_2t_1t_2} (L_2^{xx})_{s_1s_2t_2t_3}$

and same relation for K_1^{xx} , M_1^{xx} , N_1^{xx} and N_1^{xx} .

Remark 2.5.12. *Now under this new hypothesis we have some fact :*

1. $\delta_2 B_2^{x^1x^1} = \delta x^1 \delta_2 x^2$
2. $\delta_2 G_2^{x^1x^1} = 0$

$$3. \delta_2 H_2^{x^1 x^1} = G_2^{x^1 x^1} \delta_2 x^2$$

Now under this new hypothesis we are able to define

$$R^1 = \delta\varphi(x^1) - \left(-\varphi'(x^1)\delta x^1 - \varphi''(x^1)(1/2\delta(x^1)^2 - x^1\delta x^1) + \sum_{a=1,2} \left(\int_a \varphi'(x^1) \int_{\hat{a}} dx^1 + \int_a \varphi''(x^1)\delta_{\hat{a}}x^1 d_a x^1 \right) \right) \quad (2.66)$$

and an analogue formula for R^2 . Then we have the following relation between the remainder terms

Proposition 2.5.13. *Let R^1, R^2 and R^{12} given respectively by the equations (2.66) and (2.64) and assume that hypothesis 2.5.11, 2.5.8 and 2.5.1 are true then we have $R_{\square} = R_{\square}^1 + R_{\square}^2 - R_{\square}^{12}$ where*

$$R = \varphi(x) - \left(-\varphi'(x)\delta x - \varphi''(x)(1/2\delta x^2 - x\delta x) + \sum_{a \in \{1,2\}} \int_a \varphi'(x) \int_{\hat{a}} dx + \int_a \varphi''(x)\delta_{\hat{a}}x d_a x \right)$$

Proof. The fact that x^1 and x^2 are respectively smooth in the first and second direction gives

$$\begin{aligned} \left(\int_1 \varphi'(x^1) \int_2 dx^1 \right)_{\square} &= \left(\int_1 \varphi'(x^{12}) \int_2 dx^{12} \right)_{\square}, \quad x_{ab}^l = x_{ab}^{12} \\ \left(\int_2 \varphi'(x^2) \int_1 dx \right)_{\square} &= \left(\int_2 \varphi'(x^{12}) \int_1 dx^{12} \right)_{\square} \end{aligned}$$

For $(a, b) \in \{0, 1\}^2$ and $l = 1, 2$. And of course similar equation for the boundary integrals given by $\int_1 \varphi''(x^1)\delta_2 x^1 d_2 x^1$. On the other side we have by the definition of the functional Λ_1 that :

$$\begin{aligned} \left(\int_2 \varphi'(x) \int_1 dx \right)_{\square} &= (1 - \Lambda_1 \delta_1)(\varphi'(x)\delta x + \varphi''(x)B_2^{xx})_{\square} = (1 - \Lambda^{1d}\delta^{1d})(\varphi'(x_0.)\delta x_{01..} + \varphi''(x_0.)(B_2^{xx})_{01..})_{01} \\ &= (1 - \Lambda^{1d}\delta^{1d})(\varphi'(x_0^1.)\delta x_{01..} + \varphi''(x_0^1.)(B_2^{x^1 \tilde{x}^1})_{01..})_{01} \\ &= \left(\int_2 \varphi'(x^1) \int_1 dx^1 \right)_{\square} \end{aligned}$$

and by similar argument we have also that

$$\left(\int_2 \varphi''(x)\delta_1 x d_2 x \right)_{\square} = \left(\int_2 \varphi''(x^1)\delta_1 x^1 d_2 \tilde{x}^1 \right)_{\square}$$

Then putting these equation together we obtain the needed identity. \square

Now to show that $R \in \mathcal{C}_{2,2}^{2\alpha, 2\beta}$ we have to give estimates for the remainder terms R^1 and R^2 . At this point we give three technical lemma which help us to do this.

Lemma 2.5.14. *Let $x \in \mathcal{C}_{2,2}$ and $\varphi \in C^3(\mathbb{R})$, and we define $\nu_1(x)$ by :*

$$\begin{aligned} \nu_1(x) &= -\varphi'(x_{00})\delta x_{0101} - \varphi''(x_{00})(1/2\delta x_{0101}^2 - x_{00}\delta x_{0101}) \\ &\quad + \left(\int_0^1 \varphi'(x_{00} + s\delta_1 x_{010}) ds \right) \delta x_{0101} + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) \delta_2 x_{001} \delta_1 x_{011} \\ &\quad + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) s ds \right) \delta x_{0101} \delta_1 x_{011} \\ &\quad - (\varphi'(x_{10}) - \varphi'(x_{00}) - \varphi''(x_{00})\delta_1 x_{010}) \delta_2 x_{101} \end{aligned}$$

then the following inequality hold

$$|\nu_1(x)| \lesssim \sup_{s \in [0,1]} |\varphi''(x_{s0})| (\delta x_{0101})^2 + \sup_{s \in [0,1]} |\varphi'''(x_{s0})| |\delta_1 x_{010} \delta_2 x_{001} \delta x_{0101}|$$

Proof. We begin by remark that

$$\begin{aligned} & \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) \delta x_{0101} \delta_1 x_{011} = \\ & \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) (\delta x_{0101})^2 + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) \delta_1 x_{010} \delta x_{0101} \\ & = \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) (\delta x_{0101})^2 + \varphi'(x_{10}) \delta x_{0101} - \left(\int_0^1 \varphi'(x_{00} + s\delta_1 x_{010}) ds \right) \delta x_{0101} \end{aligned}$$

and

$$\delta x_{0101}^2 = 2x_{00} \delta x_{0101} + 2\delta_1 x_{010} \delta_2 x_{101} + 2\delta_2 x_{001} \delta x_{0101} + (\delta x_{0101})^2$$

injecting these two equality in the definition of $\nu_1(x)$ gives :

$$\begin{aligned} \nu_1(x) &= -(\varphi'(x_{10}) - \varphi'(x_{00})) \delta_2 x_{001} + \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) \delta_1 x_{011} \delta_2 x_{001} - \varphi''(x_{00}) \delta_2 x_{001} \delta y_{0101} \\ &+ \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) (\delta x_{0101})^2 - 1/2 \varphi''(x_{00}) (\delta y_{0101})^2 \\ &= \left(\int_0^1 \int_0^1 \varphi'''(x_{00} + ss'\delta_1 x_{010}) ds ds' \right) \delta_1 x_{010} \delta_2 x_{001} \delta x_{0101} \\ &+ \left(\int_0^1 \varphi''(x_{00} + s\delta_1 x_{010}) ds \right) (\delta x_{0101})^2 - 1/2 \varphi''(x_{00}) (\delta x_{0101})^2 \end{aligned}$$

Then the desired inequality is a simple consequence of this equality. \square

Lemma 2.5.15. *Let $y \in \mathcal{C}_{1,1}$ and $\alpha, \beta > 1/3$ such that $\delta_2 y \in \mathcal{C}_{1,2}^{\alpha,\beta}$, $\delta y \in \mathcal{C}_{2,2}^{\alpha,\beta}$ moreover we assume that y is smooth in the first direction and that there exists $\int_2 \int_2 d_2 y d_2 y \in \mathcal{C}_{1,2}^{\alpha,2\beta}$, $\int_2 \int dy d_2 y \in \mathcal{C}_{2,2}^{\alpha,2\beta}$, $H_2^{yy} = \int_2 \delta_2 y \delta_1 y d_2 y \in \mathcal{C}_{2,2}^{1,2\beta}$, $G_2^{yy} = \int_2 \delta_1 y d_2 y \in \mathcal{C}_{2,2}^{1,\beta}$ and $B_2^{yy} = \int_2 d_2 y \int_1 dy \in \mathcal{C}_{2,2}^{1,2\beta}$ satisfying the algebraic relation :*

1. $\delta_2 G_2^{yy} = 0$
2. $\delta H_2^{yy} = G_2^{yy} \delta_2 y$
3. $\delta_1 \int_2 \int_2 d_2 y d_2 y = B_2^{yy} + \int_2 \int dy d_2 y$
4. $\delta_2 B_2^{yy} = \delta_2 y \delta y$
5. $\delta_2 \int_2 d_2 y d_2 y = \delta_2 y \delta_2 y$
6. $\delta_2 \int_2 \int_2 dy d_2 y = \delta y \delta_2 y$
7. $\delta_1 y \int_2 \int_2 d_2 y d_2 y = \delta_1 y \circ_1 \int_2 \int dy d_2 y - \int_2 \delta_2 y \circ_2 \delta y d_2 y + H_2^{yy}$
8. $\delta_1 y \delta_2 y = G_2^{yy} - \int_2 \int dy d_2 y.$

Then for $\varphi \in C^5(\mathbb{R})$ the following equality holds :

$$\delta\varphi'(y) = \int_2 \varphi'(y) \int_1 dy + \int_2 \varphi''(y) \delta_1 y d_2 y + \varphi'(y)^{\#1} \delta_2 y + r_1(y)$$

where

$$\begin{aligned} r_1(y) = & \varphi''(y)^{\#1} \int_2 \int_2 d_2 y d_2 y + \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y) \\ & + \Lambda_2 [(\delta_2 \varphi'(y))^{\#1} - \varphi''(y)^{\#1} \delta_2 y - \varphi'''(y) \delta_1 y \circ_1 \delta y] \delta_2 y + \delta_2 \varphi''(y)^{\#1} \int_2 \int_2 d_2 y d_2 y \\ & + \varphi'''(y) (\delta y \circ_1 \int_2 \int dy d_2 y) + \delta_2 \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y) \end{aligned}$$

Proof. By the one dimensional change of variable formula we have

$$\delta_2 \varphi(y) = \int_2 \varphi'(y) d_2 y = (1 - \Lambda_2 \delta_2) (\varphi(y) \delta_2 y + \varphi''(y) \int_2 \int_2 d_2 y d_2 y) \quad (2.67)$$

Then if we apply δ_1 to this equation we get

$$\begin{aligned} \delta\varphi(y) = & (1 - \Lambda_2 \delta_2) (\varphi'(y) \delta y + \varphi''(y) B_2^{yy} + \delta_1 \varphi'(y) \delta_2 y + \delta_1 \varphi''(y) \int_2 \int_2 d_2 y d_2 y + \varphi''(y) \int_2 \int dy d_2 y) \\ = & \int_2 \varphi'(y) \int_1 dy + (1 - \Lambda_2 \delta_2) (\delta_1 \varphi'(y) \delta_2 y + \delta_1 \varphi''(y) \int_2 \int_2 d_2 y d_2 y + \varphi''(y) \int_2 \int dy d_2 y) \end{aligned} \quad (2.68)$$

Expanding the two terms $\delta_1 \varphi'(y) \delta_2 y$, $\delta_1 \varphi''(y) \int_2 \int_2 d_2 y d_2 y$ and using the algebraic assumption gives

$$\begin{aligned} \delta_1 \varphi''(y) \delta_2 y + \varphi''(y) \int_2 \int_2 dy d_2 y + \delta_1 \varphi''(y) \int_2 dy d_2 y = & \varphi''(y) G_2^{yy} + \varphi'''(y) H_2^{yy} - \varphi'''(y) \int_2 \delta_2 y \circ_2 \delta y d_2 y \\ & + (\varphi'(y)^{\#1} \delta_2 y + \varphi''(y)^{\#1} \int_2 \int_2 d_2 y d_2 y + \\ & \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y) \end{aligned} \quad (2.69)$$

Now if we combine the fact that $(1 - \Lambda_2 \delta_2) \int_2 \delta_2 y \circ_2 \delta y d_2 y = 0$ (ie: $\int_2 \delta_2 y \circ_2 \delta y d_2 y \in \mathcal{C}_{2,2}^{1,3\beta}$, $\delta_2 \int_2 \delta_2 y \circ_2 \delta y d_2 y \in \mathcal{C}_{2,3}^{1,3\beta}$) with equation (2.68) and (2.69) we obtain that

$$\begin{aligned} \delta\varphi(y) = & \int_2 \varphi'(y) \int_1 dy + \int_2 \varphi''(y) \delta_1 y d_2 y + (1 - \Lambda_2 \delta_2) (\varphi'(y)^{\#1} \delta_2 y \\ & + \varphi''(y)^{\#1} \int_2 \int_2 d_2 y d_2 y + \varphi'''(y) (\delta_1 y \circ_1 \int_2 \int dy d_2 y) \end{aligned}$$

then to obtain the needed result it suffice to expand the last term of this equality □

Lemma 2.5.16. *Let $y \in \mathcal{C}_{1,1}$ satisfying the assumption of lemma 2.5.15 and $\varphi \in C^5(\mathbb{R})$ then we have the formula :*

$$\begin{aligned}
 & (\delta_2 \varphi'(y)^{\#1} - \varphi''(y)^{\#1} \delta_2 y - \varphi'''(y) \delta_1 y \circ_1 \delta y)_{s_1 s_2 t_1 t_2} = \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta y_{s_1 s_2 t_1 t_2})^2 \\
 & + \int_{[0,1]^2} k \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2}) dk dk' (2 \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} + (\delta y_{s_1 s_2 t_1 t_2})^2) \\
 & + \int_{[0,1]^3} \tilde{k} \varphi^{(v)}(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1} + k'' \delta_2 y_{s_1 t_1 t_2} + \bar{k} k'' \delta y_{s_1 s_2 t_1 t_2}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2})^2 dk dk' dk'' (\delta_1 y_{s_1 s_2 t_2})^2
 \end{aligned} \tag{2.70}$$

where $\bar{k} = k k'$ and $\tilde{k} = k(1 - k'')$ in particular this give us that $r_1(y) \in \mathcal{C}_{2,2}^{2\alpha, \beta}$

Proof. By the usual Taylor formula we have that

$$\begin{aligned}
 \delta_2 \varphi'(y)_{s_1 s_2 t_1 t_2}^{\#1} &= \int_{[0,1]^2} k (\varphi'''(y_{s_1 t_2} + \bar{k} \delta_1 y_{s_1 s_2 t_2}) - \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1})) dk dk' (\delta_1 y_{s_1 s_2 t_2})^2 \\
 &+ 2 \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \\
 &+ \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta y_{s_1 s_2 t_1 t_2})^2
 \end{aligned} \tag{2.71}$$

Let denoted by $a_{s_1 s_2 t_1 t_2}$ the first term in the r.h.s of this equation. Then if we remark that $(\delta_1 y_{s_1 s_2 t_2})^2 = (\delta_1 y_{s_1 s_2 t_1})^2 + 2 \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} + (\delta y_{s_1 s_2 t_1 t_2})^2$ and using Taylor formula once again we obtain

$$\begin{aligned}
 a_{s_1 t_1 s_2 t_2} &= \int_{[0,1]^2} k \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk dk' \delta_2 y_{s_1 t_1 t_2} (\delta_1 y_{s_1 s_2 t_1})^2 \\
 &+ \int_{[0,1]^2} k^2 k' \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta_1 y_{s_1 s_2 t_1})^2 \delta y_{s_1 s_2 t_1 t_2} \\
 &+ \int_{[0,1]^3} \tilde{k} \varphi^{(v)}(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1} + k'' \delta_2 y_{s_1 t_1 t_2} + \bar{k} k'' \delta y_{s_1 s_2 t_1 t_2}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2})^2 dk dk' dk'' \\
 &\times (\delta_1 y_{s_1 s_2 t_2})^2 \\
 &+ \int_{[0,1]^2} k \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) (\delta_2 y_{s_1 s_2 t_1} + k k' \delta y_{s_1 s_2 t_1 t_2}) dk dk' (2 \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} + (\delta y_{s_1 s_2 t_1 t_2})^2)
 \end{aligned} \tag{2.72}$$

The two last terms in the r.h.s of this equation lie in the space $\mathcal{C}_{2,2}^{2\alpha, 2\beta}$ then we will focus on the two first denoted respectively by $a_{s_1 s_2 t_1 t_2}^1$ and $a_{s_1 s_2 t_1 t_2}^2$. By integration by part formula we get

$$\int_0^1 k^2 k' \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk \delta_1 y_{s_1 s_2 t_1} = \varphi'''(y_{s_1 t_1} + k' \delta_1 y_{s_1 s_2 t_1}) - 2 \int_0^1 k \varphi'''(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk$$

multiplying this equation by $\delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2}$ and integrating over k' give us

$$\begin{aligned}
 a_{s_1 s_2 t_1 t_2}^2 &= \int_0^1 \varphi'''(y_{s_1 t_1} + k' \delta_1 y_{s_1 s_2 t_1}) dk' \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \\
 &- 2 \int_{[0,1]^2} k \varphi'''(y_{s_1 t_1} + \bar{k} \delta_1 y_{s_1 s_2 t_1}) dk dk' \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2}
 \end{aligned} \tag{2.73}$$

on the other hand

$$\begin{aligned}
 a_{s_1 s_2 t_1 t_2}^1 &= \int_{[0,1]^2} \varphi^{(iv)}(y_{s_1 t_1} + k k' \delta_1 y_{s_1 s_2 t_1}) dk dk' (\delta_1 y_{s_1 s_2 t_1})^2 \delta_2 y_{s_2 t_1 t_2} \\
 &\quad - \int_{[0,1]} \varphi'''(y_{s_1 t_1} + k \delta_1 y_{s_1 s_2 t_1}) dk \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2} \\
 &= (\varphi''(y))^{\sharp 1} \delta_2 y_{s_1 s_2 t_1 t_2} - \int_{[0,1]} (\varphi'''(y_{s_1 t_1} + k \delta_1 y_{s_1 s_2 t_1}) - \varphi'''(y_{s_1 t_1})) dk \delta_1 y_{s_1 s_2 t_1} \delta y_{s_1 s_2 t_1 t_2}
 \end{aligned} \tag{2.74}$$

Then if we combine equations (2.71) (2.72) (2.73) and (2.74) we obtain the needed result. \square

Now what we have in mind is to take $y = x^1$ in this last two lemma but for this we need to construct $\int_2 \int_2 d_2 x^1 d_2 x^1$ and $\int_2 \int dx^1 dx^1$ then as usual we must add some algebraic conditions

Hypothesis 2.5.17. Let $\alpha, \beta > 1/3$, $x \in \mathcal{C}_{1,1}$, $a = 1, 2$ and we assume the existence of $P_a^{xx} = \int_a \int_a d_a x d_a x \in \mathcal{C}_{1,2}^{*,2\beta}$ and $Q_a^{xx} = \int_a \int dx dx_a x \in \mathcal{C}_{2,2}^{\alpha,2\beta}$ satisfying the algebraic relation

1. $\delta_a P_a^{xx} = \delta_a x \delta_a x$
2. $\delta_{\hat{a}} P_a^{xx} = B_a^{xx} + Q_a^{xx}$
3. $\delta_2 Q_a^{xx} = \delta x \delta_a x$
4. $\delta_{\hat{a}} x P_a^{xx} = \delta_{\hat{a}} x Q_a^{xx} + H_a^{xx} - \int_a \delta_a x \circ_a \delta x d_a x$
5. $\delta_{\hat{a}} x \delta_a x = G_a^{xx} - Q_a^{xx}$

where $\int_a \delta_a x \circ_a \delta x d_a x = \Lambda_a((\delta_a x \circ_a \delta x) \delta_a x + \delta_a x Q_a^{xx} + \delta x P_a^{xx} - \delta x \circ_{\hat{a}} Q_a^{xx})$ and B_a^{xx} , H_a^{xx} are the iterated integrals given respectively in the hypothesis 2.5.1 and 2.5.11.

With this hypothesis we define

$$\left(\int_2 \int_2 d_2 x^1 d_2 x^1 \right)_{s_1 s_2 t_1 t_2} = (P_2^{xx})_{0t_1 t_2} + s((B_2^{xx})_{01t_1 t_2} + (Q_2^{xx})_{01t_1 t_2} + (s-1)(K_2^{xx})_{01t_1 t_2})$$

and

$$\left(\int_2 \int dx^1 d_2 x^1 \right)_{s_1 s_2 t_1 t_2} = (s_2 - s_1)((P_2^{xx})_{01t_1 t_2} + (s_2 - 1)(K_2^{xx})_{01t_1 t_2})$$

Now all the ingredients are ready to prove our main result

Theorem 2.5.18. Let $x \in \mathcal{C}_{1,1}^{\alpha,\beta}$ and $\varphi \in C^5(\mathbb{R})$ such that hypothesis 2.5.1, 2.5.8, 2.5.11 and 2.5.17 are satisfied then $\varphi(x) \in \mathcal{K}_x^{\alpha,\beta}$

Proof. Let R^1 the remainder terms given by 2.66 then if we put $y = x^1$ in the lemma 2.5.15 we obtain

$$R^1 = \int_2 \varphi'(x^1) \int_1 dx^1 + \int_2 \varphi''(x^1) \delta_1 x^1 d_2 x^1 + \varphi'(x^1)^{\sharp 1} \delta_2 x^1 + r_1(x^1)$$

Now if we observe that :

$$(\varphi'(x^1) \delta_2 x^1)_{\square} = \left(\int_1 \varphi'(x^1) \int_2 dx^1 + \int_1 \varphi''(x^1) \delta_2 x^1 dx^1 \right)_{\square} - \nu_1(x)$$

where $\nu_1(x)$ is given in the lemma 2.5.14 we get that $R_{\square}^1 = r_1(x^1)_{\square} - \nu_1(x)_{\square}$ and $R_{\square}^2 = (r_2(x^2) - \nu_2(x))_{\square}$ then by the proposition we obtain that $R_{\square} = r_1(x^1)_{\square} - \nu_1(x)_{\square} + r_2(x^2)_{\square} - \nu_2(x) - R_{\square}^{12}$ of course this relation give us the needed regularity of R on the unite square but if we take $X_{st} = x_{s_1+s(s_2-s_1);t_1+t(t_2-t_1)}$ for $(s, t) \in [0, 1]^2$ then is easy to see that X satisfy all the algebraic assumption and that $R_{s_1s_2t_1t_2}^x = R_{\square}^X$ which give us the result for any rectangle. \square

Now to simplify the notation we introduce the following definition.

Definition 2.5.19. Let $x \in \mathcal{C}_{1,1}$ a sheet satisfying the Hypothesis 2.5.1, 2.5.8, 2.5.11 and 2.5.17 then we denote by \mathbb{X} the collection of all iterated integrals giving in these Hypothesis and we call it Rough-Sheet associated to x and then we define $\mathcal{H}_{\alpha,\beta}$ the space which contain the rough sheet as the product of the Hölder space giving in these Hypothesis equipped with the product topology.

Now we have the following lemma

Lemma 2.5.20. Let $\rho_1, \rho_2 \in (0, 1)$, x^1, x^2 two increments lying in $\mathcal{C}_{1,1}^{\rho_1, \rho_2}$, and $\varphi \in C^3(\mathbb{R})$. Then we have:

$$\|\delta_1 \varphi(x^1)\|_{\rho_1, 0} \lesssim \left(\sup_{s,t \in [0,1]^2} |\varphi(x_{st}^1)| \right) \|\delta x^1\|_{\rho_1, 0} \quad (2.75)$$

and

$$\mathcal{N}_{\rho_1, \rho_2}(\varphi(x^1) - \varphi(x^2)) \lesssim c_{x^1, x^2} \mathcal{N}_{\rho_1, \rho_2}(x^1 - x^2) [1 + \mathcal{N}_{\rho_1, \rho_2}(x^1) + \mathcal{N}_{\rho_1, \rho_2}(x^2)]^2 \quad (2.76)$$

where we recall that $\mathcal{N}_{\rho_1, \rho_2}(\cdot)$ has been defined at equation (2.31). In the relation above we have also set

$$c_{x^1, x^2} = \sum_{i=1}^3 \sup_{(s,t) \in [0,1]^2} |\varphi^{(i)}(x_{st}^1)| + \sup_{(s,t) \in [0,1]^2} |\varphi^{(i)}(x_{st}^2)|$$

Now using the concrete expression of the remainder term obtained previously, the continuity of the sewing map and this lemma we get easily the following continuity theorem.

Theorem 2.5.21. Let $x \in \mathcal{C}_{1,1}^{\alpha, \beta}$ and $\bar{x} \in \mathcal{C}_{1,1}^{\alpha, \beta}$ satisfying the hypothesis (2.5.1), (2.5.8), (2.5.11) and (2.5.17) and $\varphi \in C^8(\mathbb{R})$ then there exist a polynomial function $K \in C([0, +\infty[, [0, +\infty[)$ such that

$$\|\varphi(x)^{\sharp} - \varphi(\bar{x})^{\sharp}\|_{2\alpha, 2\beta} \lesssim_{\alpha, \beta} CK(\|\mathbb{X}\|_{\mathcal{H}_{\alpha, \beta}} + \|\bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}) \|\mathbb{X} - \bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}$$

and then

$$\left\| \iint \varphi(x) dx - \iint \varphi(\bar{x}) d\bar{x} \right\|_{\alpha, \beta} + \left\| \iint \varphi(x) d\omega(x) - \iint \varphi(\bar{x}) d\omega(\bar{x}) \right\|_{\alpha, \beta} \lesssim_{\alpha, \beta} CK(\|\mathbb{X}\|_{\mathcal{H}_{\alpha, \beta}} + \|\bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}) \times \|\mathbb{X} - \bar{\mathbb{X}}\|_{\mathcal{H}_{\alpha, \beta}}$$

where $d\omega(x) := d_1 x d_2 x$, $\omega(\bar{x}) := d_1 \bar{x} d_2 \bar{x}$ and $C = \sum_{k \in \{1, \dots, 8\}} \|\varphi^{(k)}\|_{\infty, M}$ and $M = \|x\|_{\infty} + \|\bar{x}\|_{\infty}$

2.6 Enhancement of the fractional Brownian Sheet and Stratonovich formula

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, in this section we construct the rough-sheet associated to the fractional Brownian sheet x . Before starting with probabilistic computation let us recall the definition of such process.

Definition 2.6.1. *The process $(x_{st})_{(s,t) \in [0,1]^2}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called fractional Brownian sheet with hurst parameter $\alpha, \beta \in [0, 1]$ if x is a Gaussian process with covariance function*

$$R_{s_1 s_2 t_1 t_2} = 1/4(|s_1|^\alpha + |s_2|^\alpha + |s_2 - s_1|^\alpha)(|t_1|^\beta + |t_2|^\beta + |t_2 - t_1|^\beta) \quad (2.77)$$

With this definition we recall the following harmonisable representation for the fractional Brownian sheet

$$x_{st} \stackrel{law}{=} \int_{\mathbb{R}^2} \frac{e^{is\xi} - 1}{|\xi|^{\alpha+1/2}} \frac{e^{it\eta} - 1}{|\eta|^{\beta+1/2}} \hat{W}(d\xi, d\eta) \quad (2.78)$$

where \hat{W} is the Fourier transform of the white noise W . Let us now state some extension of Garsia-Rodemich-Rumsey Lemma (see [35]) which will be useful to estimate the Hölder norm of our subject.

Lemma 2.6.2. *For $p > 1$ and $\alpha, \beta \in (\frac{1}{3}, \frac{1}{2}]$ there exist two non negative constants $C_1 = C_1(\alpha, \beta, p)$ and $C_2 = C_2(\alpha, p)$ such as for every $y \in \mathcal{C}_2$ and $R \in \mathcal{C}_2$ we have :*

$$\|\delta y\|_{\alpha, \beta} \leq C_1 U_{\alpha+\frac{2}{p}, \beta+\frac{2}{p}, p}^2(\delta y)$$

and

$$\|R\|_\alpha \leq C_2 (U_{\alpha+\frac{2}{p}, p}^1(R) + \|\delta^{1d} R\|_\alpha)$$

where

$$U_{\alpha_1, \alpha_2, \dots, \alpha_n, p}^n(V) := \left(\int_{[0,1]^{2n}} \frac{|V_{s_1^1 s_2^1 \dots s_1^n s_2^n}|^p}{\prod_{i=1}^n |s_2^i - s_1^i|^{\alpha_i}} ds_1^1 ds_2^1 \dots ds_1^n ds_2^n \right)^{\frac{1}{p}}$$

for $V \in \mathcal{C}_2^{\otimes n}$

Proof. Let $(s_1, s_2, t_1, t_2) \in [0, 1]^4$ then by the Garsia-Rodemich-Rumsey Lemma we have that :

$$|\delta y_{(s_1, s_2), (t_1, t_2)}|^p = |\delta_1 y_{(s_1, s_2)t_2} - \delta_1 y_{(s_1, s_2)t_1}|^p \lesssim_{\alpha, \beta, p} |t_2 - t_1|^{\beta p} \iint_{[0,1]^2} \frac{|\delta_1 y_{(s_1, s_2)v_2} - y_{(s_1, s_2)v_1}|^p}{|v_2 - v_1|^{\beta p + 2}} dv_1 dv_2$$

Now we remark that :

$$|\delta y_{(s_1, s_2), (t_1, t_2)}| = |\delta_1 y_{(s_1, s_2)t_2} - \delta_1 y_{(s_1, s_2)t_1}| = |\delta_2 y_{s_2(t_1, t_2)} - \delta_2 y_{s_1(t_1, t_2)}|$$

Then if we apply the Garsia-Rodemich-Rumsey again we obtain:

$$|\delta y_{(s_1, s_2), (t_1, t_2)}|^p \lesssim_{\alpha, \beta, p} |s_1 - s_2|^{\alpha p} |t_2 - t_1|^{\beta p} \iiint_{[0,1]^4} \frac{|\delta y_{(u_1, u_2)(v_1, v_2)}|^p}{|u_1 - u_2|^{\alpha p + 2} |v_2 - v_1|^{\beta p + 2}} du_1 du_2 dv_1 dv_2$$

For the proof of second inequality we refer the reader to [36]. \square

Now our strategy is to regularize x in the following way: $\forall N \in \mathbb{N}$ we put

$$x_{st}^N := K_{\alpha,\beta} \iint_{\{|\xi|,|\eta| \leq N\}} \frac{e^{is\xi} - 1}{|\xi|^{\alpha+1/2}} \frac{e^{it\eta} - 1}{|\eta|^{\beta+1/2}} \hat{W}(d\xi, d\eta) \quad (2.79)$$

so we are able to define

$$\partial_1 \partial_2 x_{st}^N := K_{\alpha,\beta} \iint_{\{|\xi|,|\eta| \leq N\}} \frac{i\xi e^{is\xi}}{|\xi|^{\alpha+1/2}} \frac{i\eta e^{it\eta}}{|\eta|^{\beta+1/2}} \hat{W}(d\xi, d\eta)$$

And this allows us to construct rough sheet associated to x^N denoted in the following by \mathbb{X}^N Now we will set out the main theorem of this section which will allow us to say that the fractional Brownian sheet can be enhanced in a rough sheet.

Theorem 2.6.3. *Let x^N the process given by the equation (2.79) and \mathbb{X}^N the associated rough sheet then there exists a random variable $\mathbb{X} \in \mathcal{H}_{h,h'}$ such that \mathbb{X}^N converges to \mathbb{X} in $L^p(\Omega, \mathcal{H}_{h,h'})$ for all $(h, h', p) \in (\frac{1}{3}, \alpha) \times (\frac{1}{3}, \beta) \times [1, +\infty)$.*

To prove the theorem 2.6.3 we will need the following lemma

Lemma 2.6.4. *Let $\alpha > 1/3$ and the function defined on \mathbb{R}^2 by :*

$$\mathcal{Q}(\xi, \eta) := \int_0^1 ds e^{is\xi} \int_0^s dv e^{iv\eta}$$

then \mathcal{Q} satisfies

1. $\mathcal{Q}(\xi, -\xi) = \frac{(1-\cos(\xi))+i(\xi-\sin(\xi))}{|\xi|^2}$.
2. $\mathcal{Q}(\xi_1, \xi_2) \lesssim \frac{1}{|x_i|}$.
3. $\mathcal{Q}(\xi_1, \xi_2) \lesssim \frac{1}{|x_1||x_2|} + \frac{1}{|x_1+x_2||x_i|}$.
4. $\mathcal{Q}(\xi_1, \xi_2) \lesssim 1$.
5. $\iint_{\mathbb{R}^2} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} d\xi d\eta < +\infty$

where $i \in \{1, 2\}$.

Proof. The properties 1,2,3,4 are easy to establish by a direct computation only the prove of the last assertion claim a bit more work. Indeed we begin by decomposing the plane in three region D, U and V given by :

1. $D = \left\{ (\xi, \eta) \in \mathbb{R}^2; |\xi + \eta| \leq \frac{\min(|\xi|, |\eta|)}{2} \right\}$
2. $U = \left\{ (\xi, \eta) \in \mathbb{R}^2; |\xi + \eta| \geq \frac{\max(|\xi|, |\eta|)}{2} \right\}$
3. $V = (D \cup U)^c$

Now when $(\xi, \eta) \in D$ we have that $2/3|\xi| \leq |\eta| \leq 3/2|\xi|$ which leads us by the third property given in Lemma to obtain the following bound :

$$|\mathcal{Q}(\xi, \eta)| \lesssim \frac{1}{|\xi||\xi + \eta|}$$

and then

$$\begin{aligned} \iint_{D \cap \{|\xi+\eta| \geq 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} d\xi d\eta &\lesssim \iint_{\{|\xi+\eta| \geq 1; |\xi| \geq 2|\xi+\eta|\}} \frac{1}{|\xi|^{4\alpha} |\xi + \eta|^2} d\xi d\eta \\ &\lesssim \int_1^{+\infty} dv \frac{1}{v^2} \int_{2v}^{+\infty} du \frac{1}{u^{4\alpha}} < +\infty \end{aligned}$$

Now if $|\xi + \eta| \leq 1$ we can estimate the integrand in the following way :

$$\frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} \lesssim \frac{1}{|\xi|^{4\alpha-2\gamma}}$$

where $\gamma \in [0, 1]$ and then we get :

$$\iint_{D \cap \{|\xi+\eta| \leq 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} d\xi d\eta \lesssim \int_0^1 dv \int_{2v}^{+\infty} du \frac{1}{u^{4\alpha-2\gamma}} < +\infty$$

as soon as $\gamma \in (2\alpha - 1, 2\alpha - 1/2)$ which shows that the integral is finite on D . Now on U and V we have that $|\xi|, |\eta| \lesssim |\xi + \eta|$ and hence we can estimate \mathcal{Q} by:

$$|\mathcal{Q}(\xi, \eta)| \lesssim \frac{1}{|\xi||\eta|}$$

then we obtain

$$\iint_{(U \cup V) \cap \{|\xi| > 1, |\eta| > 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} d\xi d\eta \lesssim \iint_{[1, +\infty)^2} |\xi\eta|^{-4\alpha-1} d\xi d\eta < +\infty$$

In the region of U and V where $|\xi|, |\eta| \leq 1$ we bound the integrand in the following manner.

$$\frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} \lesssim |\xi\eta|^{1-2\alpha}$$

then

$$\iint_{(U \cup V) \cap \{|\xi| \leq 1, |\eta| \leq 1\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} d\xi d\eta \lesssim \left(\int_{\{|\xi| \leq 1\}} |\xi|^{1-2\alpha} d\xi \right)^2$$

The same bound for the region $(U \cup V) \cap \{|y| \leq 1, |x| \geq 1\}$ combined with the fact that $\mathcal{Q}(\xi, \eta) \lesssim \min(|\xi|^{-1}, |\eta|^{-1})$ gives

$$\frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi|^{2\alpha-1} |\eta|^{2\alpha-1}} \lesssim |\xi|^{-1-2\alpha} |\eta|^{-2\alpha+1}$$

This shows that our kernel is integrable on $(U \cup V) \cap \{|\xi| \leq 1, |\eta| \geq 1\}$ and by symmetry we obtain the integrability in the remaining area which completes the proof. \square

2.6.1 Proof of theorem (2.6.3)

Proof. We will decompose the proof of the theorem in two step. In a first step we give the bound for the rough sheet \mathbb{X}^N in $L^2(\Omega)$ for fixed parameters and in the second step we will use a variant of Garsia-Rodemich-Rumsey inequality to proof that our sheet is a Cauchy sequence in $\mathbb{L}^p(\Omega, \mathcal{H}_{h,h'})$.

Step 1: Estimation. Let $A^{NMx} := A^{Nx} - A^{Mx} = \delta(x^N - x^M)$ for $M \leq N$ and similar notation for all other terms of the rough sheet. Now is not difficult to see that

$$\begin{aligned} \mathbb{E}[|A_{(s_1, s_2)(t_1, t_2)}^{NMx}|^2] &= \int_{\{||(\xi, \eta)||_\infty \in [M, N]\}} \frac{|e^{is_2\xi} - e^{is_1\xi}|^2 |e^{it_2\eta} - e^{it_1\eta}|^2}{|\xi|^{1+2\alpha} |\eta|^{1+2\beta}} d\xi d\eta \\ &\lesssim (s_2 - s_1)^{2\alpha} (t_2 - t_1)^{2\beta} (I_A^M)_{(s_1, s_2)(t_1, t_2)} \end{aligned}$$

where

$$(I_A^M)_{(s_1, s_2)(t_1, t_2)} = \int_{\{||((t_2 - t_1)x, (s_2 - s_1)y)||_\infty \geq M(s_2 - s_1)(t_2 - t_1)\}} |x|^{-1-2\alpha} |e^{ix} - 1|^2 |y|^{-1-2\beta} |e^{iy} - 1|^2 dx dy$$

And let us remark that for $(\alpha, \beta) \in (1/3, 1/2]$ we have :

$$\int_{\mathbb{R}^2} |\xi|^{-1-2\alpha} |e^{i\xi} - 1|^2 |\eta|^{-1-2\beta} |e^{i\eta} - 1|^2 d\xi d\eta < +\infty$$

these imply

$$\sup_{M \in \mathbb{N}, (s_1, s_2, t_1, t_2) \in [0, 1]^4} |(I_A^M)_{(s_1, s_2)(t_1, t_2)}| < +\infty$$

and

$$\lim_{M \rightarrow \infty} (I_A^M)_{(s_1, s_2)(t_1, t_2)} = 0$$

for $s_1 \neq s_2$ and $t_1 \neq t_2$. Now in what follows we prove similar bound for the other component of the rough sheet. By Wick theorem

$$\begin{aligned} \mathbb{E}[|C_{(s_1, s_2)(t_1, t_2)}^{Nxx}|^2] &= \left| \int_{\mathbb{R}^2} (K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, -\xi) d\xi \right|^2 + \int_{\mathbb{R}^4} |(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta)|^2 d\xi d\eta \\ &\quad + \left| \int_{\mathbb{R}^4} (K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta) \overline{(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\eta, \xi)} d\xi d\eta \right| \end{aligned} \quad (2.80)$$

where

$$(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta) := \frac{i\xi\eta}{|\xi|^{\alpha+1/2} |\eta|^{\alpha+1/2}} \mathcal{Q}_{s_1 s_2}(\xi, \eta) \frac{i\xi'\eta'}{|\xi'|^{\beta+1/2} |\eta'|^{\beta+1/2}} \mathcal{Q}_{t_1 t_2}(\xi', \eta') \chi_{\{||(\xi, \xi', \eta, \eta')||_\infty \in [M, N]\}}$$

for $\xi := (\xi, \xi') \in \mathbb{R}^2$ et $\eta = (\eta, \eta') \in \mathbb{R}^2$ and $\mathcal{Q}_{s_1 s_2}(\xi, \eta) := e^{is_1(\xi+\eta)} (s_2 - s_1)^2 \mathcal{Q}((s_2 - s_1)\xi, (s_2 - s_1)\eta)$ with \mathcal{Q} is the function defined in the Lemma (2.6.4), which gives us by change of variable formula that

$$\int_{\mathbb{R}^4} |(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta)|^2 d\xi d\eta \lesssim (s_2 - s_1)^{4\alpha} (t_2 - t_1)^{4\beta} (I_C^{1, M})_{(s_1, s_2)(t_1, t_2)}$$

where

$$(I_C^{1, M})_{(s_1, s_2)(t_1, t_2)} := \int_{\mathbb{R}^4} \chi_{\{||((s_2 - s_1)\xi, (s_2 - s_1)\eta, (t_2 - t_1)\xi', (t_2 - t_1)\eta')||_\infty \geq M\}} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi\eta|^{2\alpha-1}} \frac{|\mathcal{Q}(\xi', \eta')|^2}{|\xi'\eta'|^{2\beta-1}} d\xi d\eta$$

Now the Lemma 2.6.4 gives :

$$\int_{\mathbb{R}^4} \frac{|\mathcal{Q}(\xi, \eta)|^2}{|\xi\eta|^{2\alpha-1}} \frac{|\mathcal{Q}(\xi', \eta')|^2}{|\xi'\eta'|^{2\beta-1}} d\xi d\eta < +\infty$$

and then

$$\sup_{M \in \mathbb{N}, (s_1, s_2, t_1, t_2) \in [0, 1]^4} (I_C^{1, M})_{(s_1, s_2)(t_1, t_2)} < +\infty$$

and

$$\lim_{M \rightarrow \infty} (I_C^{1, M})_{(s_1, s_2)(t_1, t_2)} = 0$$

for $s_2 \neq s_1$ and $t_2 \neq t_1$ and by Cauchy-Schwartz inequality we have :

$$\left| \int_{\mathbb{R}^4} (K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta) \overline{(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\eta, \xi)} d\xi d\eta \right| \leq \int_{\mathbb{R}^4} |(K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, \eta)|^2 d\xi d\eta$$

Then is remind to bound the first term appearing in the sum of the equation (2.80), indeed

$$\left| \int_{\mathbb{R}^2} (K_{(s_1, s_2)(t_1, t_2)}^{NM})(\xi, -\xi) d\xi \right|^2 \lesssim (s_2 - s_1)^{2\alpha} (t_2 - t_1)^{2\beta} (I_C^{2, M})_{(s_1, s_2)(t_1, t_2)}$$

where

$$(I_C^{2, M})_{(s_1, s_2)(t_1, t_2)} = \int_{\mathbb{R}^2} \chi_{\{|\|(s_2 - s_1)\xi, (t_2 - t_1)\xi'\|_\infty \geq M\}} |\xi|^{-1-2\alpha} |\xi'|^{1-2\beta} (1 - \cos(\xi))(1 - \cos(\xi')) d\xi d\xi'$$

then is no difficult to see that $I_C^{2, M}$ satisfies the same property that $I_C^{1, M}$. Now if we put $I_C^M := I_C^{2, M} + I_C^{1, M}$ is easy to see that :

$$\mathbb{E}[|C_{(s_1, s_2)(t_1, t_2)}^{NMxx}|^2] \lesssim (s_2 - s_1)^{2\alpha} (t_2 - t_1)^{2\beta} (I_C^M)_{(s_1, s_2)(t_1, t_2)}$$

when (I_C^M) satisfy :

$$\begin{aligned} \sup_{M \in \mathbb{N}, (s_1, s_2, t_1, t_2) \in [0, 1]^4} (I_C^M)_{(s_1, s_2)(t_1, t_2)} &< +\infty \\ \lim_{M \rightarrow \infty} (I_C^M)_{(s_1, s_2)(t_1, t_2)} &= 0 \end{aligned}$$

for $s_2 \neq s_1$ and $t_2 \neq t_1$. All the other term of \mathbb{X}^N can be estimate by the same argument and satisfy the same type of bound to see these let us treat a more complex term of the rough sheet.

$$\mathbb{E}[|(E_1^{NMxxx})_{(s_1, s_2, s_3, s_3)(t_1, t_2)}|^2] = \sum_{l=1}^{15} \mathcal{I}_l$$

where $(\mathcal{I}_l)_l$ is the different Wick-contraction given by:

1. $\mathcal{I}_1 = \int_{\mathbb{R}^6} |(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, c)|^2 da db dc$
2. $\mathcal{I}_2 = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -a) da \right|^2 db$
3. $\mathcal{I}_3 = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, -a, b) da \right|^2 db$
4. $\mathcal{I}_4 = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -b) db \right|^2 da$
5. $\mathcal{I}_5 = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -a) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(b, c, -c)} da db dc$
6. $\mathcal{I}_6 = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, -a) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(c, -c, b)} da db dc$
7. $\mathcal{I}_7 = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, -a, b) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, c, -c)} da db dc$

$$8. \mathcal{I}_8 = \overline{\mathcal{I}_5}, \mathcal{I}_9 = \overline{\mathcal{I}_6}, \mathcal{I}_{10} = \overline{\mathcal{I}_7}$$

$$9. \mathcal{I}_k = \int_{\mathbb{R}^6} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, c) \overline{(G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(\sigma_k(a), \sigma_k(b), \sigma_k(c))} da db dc$$

where σ_k , $k \in \{11, 12, 13, 14, 15\}$ is a permutation of three elements different of identity and for $a = (a, a'), b = (b, b'), c = (c, c') \in \mathbb{R}^2$ the kernel is given by :

$$\begin{aligned} (G_{(s_1, s_2, s_3, s_4)(t_1, t_2)}^{NM})(a, b, c) &:= \left(\frac{iaib}{|a|^{\alpha+1/2}|b|^{\alpha+1/2}} \mathcal{Q}_{s_1 s_2}(a, b) \right) \left(\frac{ia'ic'}{|b'|^{\beta+1/2}|z'|^{\beta+1/2}} \mathcal{Q}_{t_1 t_2}(b', c') \right) \\ &\times \left(\frac{e^{is_4 c} - e^{is_3 c}}{|c|^{\alpha+1/2}} \right) \left(\frac{e^{it_1 a'} - 1}{|a'|^{\beta+1/2}} \right) \chi_{\{\|(a, a', b, b', c, c')\|_\infty \in [M, N]\}} \end{aligned}$$

The study of these integral is confined to study the first four term in fact if $k \in \{11, 12, 13, 14, 15\}$ then by Cauchy-Schwartz we have $|\mathcal{I}_k| \leq \mathcal{I}_1$, $|\mathcal{I}_5|^2 \leq \mathcal{I}_2 \mathcal{I}_4$, $|\mathcal{I}_6|^2 \leq \mathcal{I}_2 \mathcal{I}_3$ and $|\mathcal{I}_7|^2 \leq \mathcal{I}_3 \mathcal{I}_4$. By variable change formula we have that

$$|\mathcal{I}_1| \lesssim (s_2 - s_1)^{4\alpha} (s_4 - s_3)^{2\alpha} (t_2 - t_1)^{4\beta} t_1^{2\beta} (I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)}$$

where

$$(I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)} := \int_{\mathbb{R}^6} \chi_{\mathcal{D}_{s_1 s_2 s_3 s_4 t_1 t_2}^M} \frac{|\mathcal{Q}(a, b)|^2}{|ab|^{2\alpha-1}} \frac{|\mathcal{Q}(b', c')|^2}{|a'b'|^{2\beta-1}} \frac{|e^{ic} - 1|^2}{|c|^{2\alpha+1}} \frac{|e^{ic} - 1|^2}{|c|^{2\beta+1}} da da' db db' dc dc'$$

where \mathcal{X} is the Indicator function and $\mathcal{D}_{s_1 s_2 s_3 s_4 t_1 t_2}^M$ is decreasing to the empty set when M tend to infinity for $s_1 \neq s_2, t_1 \neq t_2$ and $s_3 \neq s_4$ these statement with the fact that :

$$\int_{\mathbb{R}^6} \frac{|\mathcal{Q}(a, b)|^2}{|ab|^{2\alpha-1}} \frac{|\mathcal{Q}(b', c')|^2}{|a'b'|^{2\beta-1}} \frac{|e^{ic} - 1|^2}{|c|^{2\alpha-1}} \frac{|e^{iz} - 1|^2}{|z|^{2\beta-1}} da da' db db' dc dc' < +\infty$$

give us that

$$\sup_{M \in \mathbb{N}, (s_1, s_2, s_3, s_4, t_1, t_2) \in [0, 1]^6} (I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)} < +\infty$$

and

$$\lim_{M \rightarrow +\infty} (I_{E_1}^M)_{(s_1, s_2, s_3, s_4)(t_1, t_2)} = 0$$

Now by Cauchy-Schwartz we have :

$$\begin{aligned} \mathcal{I}_2 &\lesssim \left(\int_{\mathbb{R}^4} |(K_{(s_1, s_2)(t_1, t_2)}^{NM})(a, b)|^2 da db \right) \int_{\mathbb{R}} |a|^{-1-2\alpha} |e^{-is_4 a} - e^{-is_3 a}|^2 da \\ &\left(\int_{\mathbb{R}} |a'|^{-1-2\beta} |e^{it_1 a'} - 1|^2 da' \right) \end{aligned}$$

all these integrals have already studied and then we obtain the good estimate for \mathcal{I}_2 . The estimation of \mathcal{I}_3 and \mathcal{I}_4 is obtained by the same technique.

Step 2: Convergence of \mathbb{X}^N . We prove some Garsia-Rodemich-Rumsey inequalities for our objects and then we obtain the convergence of the sheet X^N . To give an idea we will first show the convergence of our first term in fact by the Lemma 2.6.2 and the Gaussian hypercontractivity we have that :

$$\begin{aligned} \mathbb{E}[|A^{NMx}|_{h,h'}^p] &\lesssim_{h,h',p,\alpha,\beta} \iint_{[0,1]^4} \frac{\mathbb{E}[|A_{s_1 s_2 t_1 t_2}^{NMx}|^p]}{|s_2 - s_1|^{hp+2} |t_2 - t_1|^{h'p+2}} ds_1 ds_2 dt_1 dt_2 \\ &\lesssim \iint_{[0,1]^4} |s_2 - s_1|^{(\alpha-h)p-2} |t_2 - t_1|^{(\beta-h')p-2} ((I_A^M)_{(s_1, s_2)(t_1, t_2)})^{\frac{p}{2}} ds_1 ds_2 dt_1 dt_2 \end{aligned}$$

where $h < \alpha$ and $h' < \beta$. Then if p is large enough these last Integral go to zero when M go to infinity by dominate convergence and this give us the convergence of A^{NMx} . Now we will do the same work for the other terms and for this we must establish the following estimate :

Lemma 2.6.5. *let z and y two smooth sheet, $(h, h') \in (0, \infty)^2$ and $p > 1$ then the following inequalities :*

$$\begin{aligned} \|B_1^{zz} - B_1^{yy}\|_{2h,h'}^p &\lesssim_{h,h',p} (U_{2h,h',p}^2 (B_1^{zz} - B_1^{yy}))^p + (U_{2h,h',p}^3 (D_2^{zz} - D_2^{yy}))^p + \|\delta_1 z \delta z - \delta_1 y \delta y\|_{2h,h'}^p \\ &\quad + \|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}})^p \end{aligned}$$

$$\begin{aligned} \|C^{zz} - C^{yy}\|_{2h,2h'}^p &\lesssim_{h,h',p} (U_{2h,2h',p}^2 (C^{zz} - C^{yy}))^p + (U_{2h+\frac{2}{p}, h', h', p}^3 (D_2^{zz} - D_2^{yy}))^p \\ &\quad + (U_{h, h, 2h'+\frac{2}{p}, p}^3 (D_1^{zz} - D_1^{yy}))^p + (\|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}} (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}))^p \end{aligned}$$

$$\|D_1^{zz} - D_1^{yy}\|_{2h,2h'}^p \lesssim_{h,h',p} (U_{h,h,2h'}^3 (D_1^{zz} - D_1^{yy}))^p + (\|\delta z - \delta y\|_{h+\frac{1}{p}, h'} (\|\delta z\|_{h+\frac{1}{p}, h'} + \|\delta y\|_{h+\frac{1}{p}, h'}))^p$$

hold.

Proof. Let us prove the three first inequalities because all the others are obtained by the same techniques. Lemma 2.6.2 applied in the first direction and using the fact that $\delta_1 B_1^{zz} = \delta_1 z \delta z$ we obtain that

$$\begin{aligned} |(B_1^{zz} - B_1^{yy})_{s_1 s_2 t_1 t_2}|^p &\lesssim_{h,h',p} |s_2 - s_1|^{2hp} \left(\int_{[0,1]^2} \frac{|(B_1^{zz} - B_1^{yy})_{u_1 u_2 t_1 t_2}|^p}{|u_2 - u_1|^{2hp+2}} du_1 du_2 \right. \\ &\quad \left. + \|(\delta_1 z \delta z - \delta_1 y \delta y)_{t_1 t_2}\|_{2h}^p \right) \end{aligned} \quad (2.81)$$

Now we have to deal with the other direction. The second term appearing in the right side of this inequality can be estimated by $|t_2 - t_1|^{h'p} \|\delta_1 x \delta x - \delta_1 y \delta y\|_{2h,h'}^p$ for the first term we need to apply Lemma 2.6.2 once again

$$\begin{aligned} \int_{[0,1]^2} \frac{|(B_1^{zz} - B_1^{yy})_{u_1 u_2 t_1 t_2}|^p}{|u_2 - u_1|^{2hp+2}} du_1 du_2 &\lesssim_{h,h',p} |t_2 - t_1|^{h'p} ((U_{2h,h',p}^2 (B_1^{zz} - B_1^{yy}))^p \\ &\quad + \int_{[0,1]^2} \frac{\|\delta_2 (B_1^{zz} - B_1^{yy})_{u_1 u_2}\|_{h'}^p}{|u_2 - u_1|^{hp+2}} du_1 du_2) \end{aligned} \quad (2.82)$$

then it suffices to note that

$$\begin{aligned}
 \|\delta_2(B_1^{zz} - B_1^{yy})_{u_1 u_2}\|_{h'}^p &= \|(\mu_2 D_2^{zz} - \mu_2 D_2^{yy})_{u_1 u_2}\|_{h'}^p \\
 &\leq \|(D_2^{zz} - D_2^{yy})_{u_1 u_2}\|_{C_2^{h'} \otimes C_2^{h'}}^p \\
 &\lesssim_{h, h', p} \int_{[0,1]^4} \frac{|(D_2^{zz} - D_2^{yy})_{u_1 u_2 v_1 v_2 v_3 v_4}|^p}{|v_4 - v_3|^{h'p+2} |v_2 - v_1|^{h'p+2}} dv_1 dv_2 dv_3 dv_4 \\
 &\lesssim_{h, h', p} |u_2 - u_1|^{hp+2} ((U_{h+\frac{2}{p}, h', h', p}^3 (D_2^{zz} - D_2^{yy}))^p \\
 &\quad + \int_{[0,1]^4} \frac{\|(\delta z \otimes_2 \delta z - \delta y \otimes_2 \delta y)_{v_1 v_2 v_3 v_4}\|_{h+2/p}^p}{|v_4 - v_3|^{h'p+2} |v_2 - v_1|^{h'p+2}} dv_1 dv_2 dv_3 dv_4).
 \end{aligned} \tag{2.83}$$

Now the last term in the right side of this inequality can be bounded by $\|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}})^p$ then putting all these bound together gives

$$\begin{aligned}
 \|B_1^{zz} - B_1^{yy}\|_{2h, h'}^p &\lesssim_{h, h', p} (U_{2h, h', p}^2 (B_1^{zz} - B_1^{yy}))^p + (U_{2h, h', h'}^3 (D_2^{zz} - D_2^{yy}))^p + \|\delta_1 z \delta z - \delta_1 y \delta y\|_{2h, h'}^p \\
 &\quad + \|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}})^p
 \end{aligned}$$

then we have prove the first inequality. Now by the Lemma 2.6.2 once again we have that

$$|C_{s_1 s_2 t_1 t_2}^{zz} - C_{s_1 s_2 t_1 t_2}^{yy}| \lesssim_{h, h', p} |s_2 - s_1|^{2hp} ((U_{2h, p}^1 (C_{.t_1 t_2}^{zz} - C_{.t_1 t_2}^{yy}))^p + \|\delta_1 (C^{zz} - C^{yy})_{.t_1 t_2}\|_{2h}^p)$$

with $\delta_1 C_{zz} = \mu_1 D_1^{zz}$ then by same argument has before we have

$$\begin{aligned}
 \|\delta_1 (C^{zz} - C^{yy})_{.t_1 t_2}\|_{2h}^p &\lesssim_{h, h', p} |t_2 - t_1|^{2h'p} (((U_{h, h, 2h', p}^3 (D_1^{zz} - D_1^{yy}))^p \\
 &\quad + \|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}})^p)
 \end{aligned} \tag{2.84}$$

and

$$\begin{aligned}
 (U_{2h, p}^1 (C_{.t_1 t_2}^{zz} - C_{.t_1 t_2}^{yy}))^p &\lesssim_{h, h', p} |t_2 - t_1|^{2h'p} ((U_{2h, 2h', p}^2 (C^{zz} - C^{yy}))^p \\
 &\quad + \int_{[0,1]^2} \frac{\|\delta_2 (C^{zz} - C^{yy})_{u_1 u_2}\|_{2h'}^p}{|u_2 - u_1|^{hp+2}} du_1 du_2) \\
 &\lesssim_{h, h', p} |t_2 - t_1|^{2h'p} ((U_{2h, 2h', p}^2 (C^{zz} - C^{yy}))^p + (U_{2h+\frac{2}{p}, h', h', p}^3 (D_2^{zz} - D_2^{zz}))^p \\
 &\quad + \|\delta z - \delta y\|_{h+\frac{2}{p}, h'+\frac{2}{p}}^p (\|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}} + \|\delta z\|_{h+\frac{2}{p}, h'+\frac{2}{p}})^p)
 \end{aligned} \tag{2.85}$$

Putting these two bound together we obtain the second inequality. This concludes the proof of the Lemma \square

To obtain convergence of all the term of the rough sheet is suffices to take $x = X^N$ and $y = X^M$

in the lemma (2.6.5), indeed let us give an example :

$$\begin{aligned}
 \mathbb{E}[\|C^{NMxx}\|_{2h,2h'}^p] &\lesssim_{h,h',p} \int_{[0,1]^4} |u_2 - u_1|^{2(\alpha-h)p-2} |v_2 - v_1|^{2(\beta-h')p-2} ((I_C^M)_{(u_1,u_2)(v_1,v_2)})^{\frac{p}{2}} du_1 du_2 dv_1 dv_2 \\
 &+ \int_{[0,1]^6} |u_2 - u_1|^{2(\alpha-h)p-4} |v_2 - v_1|^{(\beta-h')p-2} |v_4 - v_3|^{(\beta-h')p-2} ((I_{D_2}^M)_{(u_1,u_2)(v_1,v_2,v_3,v_4)})^{\frac{p}{2}} du_1 du_2 dv_1 dv_2 dv_3 dv_4 \\
 &+ \int_{[0,1]^6} |v_2 - v_1|^{2(\beta-h')p-4} |u_2 - u_1|^{(\alpha-h')p-2} |u_4 - u_3|^{(\alpha-h')p-2} ((I_{D_1}^M)_{(u_1,u_2,u_3,u_4)(v_1,v_2)})^{\frac{p}{2}} dv_1 dv_2 du_1 du_2 du_3 du_4 \\
 &+ \mathbb{E}[\|A^{NMx}\|_{h+2/p,h'+2/p}^{2p}]^{1/2} \mathbb{E}[(\|\delta x^N\|_{h+2/p,h'+2/p} + \|\delta x^M\|_{h+2/p,h'+2/p})^{2p}]^{1/2}
 \end{aligned}$$

where we used that our term is in the second chaos of \hat{W} and the Gaussian hypercontractivity (see for example [71]). Then if p is large enough the three first terms of right side go to zero when M tend to infinity by dominate convergence and the last term was already studied this allow us to conclude that C^{NXX} is a Cauchy sequence in $L^p(\Omega, CC_{2,2}^{2h,2h'})$. The other terms can be treat similarly . \square

Do to this construction and theorem (2.5.6) we are able to define the two integrals

$$\iint f'(x)dx, \quad \iint f''(x)d_1x d_2x$$

for a function $f \in C^{10}(\mathbb{R})$ and then our goal is to use the continuity result (2.5.21) to obtain the Stratonovich Change of variable formula, for that we need an another assumption on the function f which allow us to control the constant C appearing in (2.5.21).

Definition 2.6.6. Let $k \in \mathbb{N}$, we will say that a function $f \in C^k(\mathbb{R})$ satisfies the growth condition (GC) if there exist positive constants c and λ such that

$$\lambda < \frac{1}{4 \max_{s,t \in [0,1]} (R_s^1 R_t^2)}, \quad \text{and} \quad \max_{l=0,\dots,k} |f^{(l)}(\xi)| \leq c e^{\lambda|\xi|^2} \quad \text{for all } \xi \in \mathbb{R}. \quad (2.86)$$

Remark 2.6.7. This last definition depend on the covariance function of the Gaussian process x where here is given by the fBs.

And a preliminary result ensures that x^N satisfy some uniform exponential integrability.

Proposition 2.6.8. There exist $\lambda > 0$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[e^{\lambda \sup_{(s,t) \in [0,1]^2} |x_{s,t}^N|^2} \right] < +\infty. \quad (2.87)$$

Proof. We now by the Theorem(2.6.3) that $\lim_{N \rightarrow \infty} \mathbb{E}[\|x^N - x\|_{\alpha-\epsilon, \beta-\epsilon}^p] = 0$, which imply that

$$\sup_n \mathbb{E} \left[\|x^N\|_{\alpha-\epsilon, \beta-\epsilon}^p \right] < +\infty$$

We now focus on the exponential integrability of $\sup X^N$. Notice that for a fixed N one can easily get those exponential estimates thanks to Fernique's lemma. However, we claim some uniformity in N here, and we thus come back to uniform estimates of moments in order to prove (2.87). Let then

$r = \max(\lfloor \frac{1}{\alpha - \epsilon} \rfloor, \lfloor \frac{1}{\beta_2 - \epsilon} \rfloor) + 1$ and remark that $\|X^N\|_\infty \lesssim_\epsilon \|X^N\|_{\epsilon, \epsilon}$. We thus use a decomposition of the form $\mathbb{E}[e^{\lambda \sup_{(s,t) \in [0,1]^2} |x_{s;t}^N|^2}] = I^{1,N}(\lambda) + I^{2,N}(\lambda)$, where

$$I^{1,N}(\lambda) = \sum_{l=0}^{r-1} \frac{\lambda^l}{l!} \mathbb{E}[\|x^N\|_\infty^{2l}], \quad \text{and} \quad I^{2,N}(\lambda) = \sum_{l=r}^{+\infty} \frac{\lambda^l}{l!} \mathbb{E}[\|x^N\|_\infty^{2l}].$$

We now bound those 2 terms separately: on the one hand, it is readily checked that

$$I^{1,n}(\lambda) \leq \max_{i=0, \dots, r} \sup_{N \in \mathbb{N}} \mathbb{E}[\|x^N\|_{\epsilon, \epsilon}^{2i}] < +\infty,$$

for $\epsilon < \min(\alpha, \beta)$. On the other hand, the bound on $I^{2,n}(\lambda)$ is obtained invoking Lemma (2.6.2) again. Indeed, starting from expression appearing in (2.6.2) and introducing a standard Gaussian random variable \mathcal{N} , it is easily seen that

$$\mathbb{E}[\|x^N\|_{\epsilon, \epsilon}^{2l}] \leq C_{2l, \epsilon, \epsilon} \mathbb{E}[\mathcal{N}^{2l}] \int_{[0,1]^2 \times [0,1]^2} \frac{\mathbb{E}[|\delta x_{s_1 s_2 t_1 t_2}^N|^2]^l}{|s_2 - s_1|^{2l\epsilon+2} |t_2 - t_1|^{2l\epsilon+2}} ds_1 ds_2 dt_1 dt_2$$

with \mathcal{N} is a Gaussian random variable $\mathcal{N}(0, 1)$. Now we have

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{E} \left[|\delta x_{s_1 s_2 t_1 t_2}^N|^2 \right] &\leq \int_{\mathbb{R}^2} \frac{|e^{ts_2 x} - e^{ts_1 \xi}|^2 |e^{t_2 \eta} - e^{t_1 \eta}|^2}{|\xi|^{2\gamma_1+1} |\eta|^{\gamma_2+1}} d\xi d\eta \\ &\leq |s_2 - s_1|^{2\gamma_1} |t_2 - t_1|^{2\gamma_2} \int_{\mathbb{R}^2} \frac{|e^{t\xi} - 1|^2 |e^{t\eta} - 1|^2}{|\xi|^{2\gamma_1+1} |\eta|^{\gamma_2+1}} d\xi d\eta \lesssim |s_2 - s_1|^{2\gamma_1} |t_2 - t_1|^{2\gamma_2}. \end{aligned}$$

Furthermore, it can be shown that $C_{2l, \epsilon, \epsilon}$ is of the form M^l for a given $M > 1$. Thus

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\|x^N\|_{\epsilon, \epsilon}^{2l}] \lesssim M^l \mathbb{E}[\mathcal{N}^{2l}],$$

from which the relation $I^{2,n}(\lambda) < \infty$ is easily obtained. This finishes the proof of (2.87). \square

Now is not difficult to obtain the following result

Theorem 2.6.9. *Let $f \in C^{10}(\mathbb{R})$ function satisfying the (GC) with a small parameter $\lambda > 0$ then*

$$\delta f(x) = \iint f'(x) dx + \iint f''(x) d_1 x d_2 x \quad (2.88)$$

moreover the following convergence Hold

$$\iint f'(x^N) dx^N \xrightarrow{N \rightarrow +\infty} \iint f'(x) dx, \quad \iint f''(x^N) d_1 x^N d_2 x^N \xrightarrow{N \rightarrow +\infty} \iint f''(x) d_1 x d_2 x \quad (2.89)$$

in $L^p(\Omega, \mathcal{C}_{2,2}^{\alpha-\epsilon, \beta-\epsilon})$ for $\epsilon > 0$ and $0 < \lambda < \lambda(p)$.

Proof. Du to the Theorem (2.5.21) we have that

$$\left\| \iint f'(x^N) dx^N - \iint f'(x) dx \right\|_{\alpha-\varepsilon, \beta-\varepsilon} \lesssim C^N K(\|\mathbb{X}^N\| + \|\mathbb{X}\|) \|\mathbb{X}^N - \mathbb{X}\|_{\mathcal{H}_{\alpha-\varepsilon, \beta-\varepsilon}}$$

Then using the Hölder inequality we can see that the needed convergence is only due to the fact that

$$\mathbb{E}[\|\mathbb{X}^N - \mathbb{X}\|_{\alpha-\varepsilon, \beta-\varepsilon}^a] \rightarrow^{N \rightarrow +\infty} 0, \quad \sup_N \mathbb{E}[K(\|\mathbb{X}^N\| + \|\mathbb{X}\|)^b] < +\infty, \quad \sup_N \mathbb{E}[(C^N)^c] < \infty$$

for some $a, b, c > 0$, Now the two first affirmation are given by the theorem (2.6.3) and for the third is suffice to recall that

$$C = \sum_{i=1}^{10} \sup_{|\xi| \leq \|x\|_\infty + \|x^N\|_\infty} |f^{(i)}(\xi)| \lesssim e^{\lambda \|x\|_\infty^2} e^{\lambda \|x^N\|_\infty^2}$$

where we have used that f satisfy the (GC) and then using Hölder inequality Fernique Theorem and (2.87) we can see that $\sup_N \mathbb{E}[(C^N)^c] < +\infty$ which gives the convergence (2.89). Now to obtain the formula (2.88) is suffice to use the fact that

$$\mathcal{N}_{\alpha-\varepsilon, \beta-\varepsilon}(f(x^N) - f(x)) \rightarrow^{N \rightarrow +\infty} 0$$

in $L^p(\Omega)$ for $p > 1$ du to the Lemma (2.5.20) and the fact that $\sup_N \mathbb{E}[(C^N)^p] < +\infty$, and this end the proof. \square

2.6.2 The Brownian case

In [19] the authors give a definition a la Itô for the multidimensional integral in the case of the Brownian sheet, the aim of this section is to compare their notion of integration with the integral previously defined. More precisely we will show that the two concepts coincide when the rough sheet is understood as Itô meaning. In all this subsection x is a Brownian sheet and $\mathbb{X}^{It\hat{o}}$ the rough sheet be associated where all the iterated integrals are understood in the sense given in [19].

Of course using the same argument as in the lemma 2.6.5 we can see that our objects satisfy the regularity expected. For example

$$\begin{aligned} \mathbb{E}[\|B^{It\hat{o}xx}\|_{2h, h'}^p] &\lesssim_{h, h', p} \int_{[0,1]^4} \frac{\mathbb{E}[(B_1^{It\hat{o}xx})_{u_1 u_2 v_1 v_2}]^p}{|u_2 - u_1|^{2hp+2} |v_2 - v_1|^{h'p+2}} du_1 du_2 dv_1 dv_2 \\ &+ \int_{[0,1]^6} \frac{\mathbb{E}[(D_2^{It\hat{o}xx})_{u_1 u_2 v_1 v_2 v_3 v_4}]^p}{|u_2 - u_1|^{2hp+2} |v_4 - v_3|^{h'p+2} |v_2 - v_1|^{h'p+2}} du_1 du_2 du_3 du_4 dv_1 dv_2 \\ &+ \mathbb{E}[\|\delta_1 x \delta x\|_{2h, h'}^p] + \mathbb{E}[\|\delta x\|_{h+2/p, h'+2/p}^{2p}] \end{aligned} \quad (2.90)$$

but by a simple computation we have that

$$\mathbb{E}[(B_1^{It\hat{o}xx})_{u_1 u_2 v_1 v_2}]^p = c_p^1 u_1^{p/2} (u_2 - u_1)^p (v_2 - v_1)^{p/2}$$

and

$$\mathbb{E}[(D_2^{It\hat{o}xx})_{u_1 u_2 v_1 v_2 v_3 v_4}]^p = c_p^2 (u_2 - u_1)^p (v_2 - v_1)^{p/2} (v_4 - v_3)^{p/2}$$

then if $h, h' < 1/2$ and p large enough then the r.h.s of the equation is finite and this gives that $B_1^{It\hat{o}XX} \in \mathcal{C}_{2,2}^{2h, h'}$. To compare the two definition of integration we will show first that the two definition of boundary integral coincide.

Proposition 2.6.10. *Let $\varphi \in C^2(\mathbb{R})$ satisfying the (GC) then the integral*

$$I_{s_1 s_2 t_1 t_2}^{It\hat{o}, b_1} = \int_{s_1}^{s_2} \varphi(x_{s_{t_1}}) \int_{t_1}^{t_2} \hat{d}_{st} x_{st}$$

with \hat{d} is the Itô differential admit a continuous version which coincide with

$$I_{s_1 s_2 t_1 t_2}^{Rough, b_1} = \int_{s_1}^{s_2} \varphi(x_{s_{t_1}}) \int_{t_1}^{t_2} d_{st} x_{st}$$

where I^{rough, X, b_1} are given by the Proposition 2.5.4

Proof. Let $\pi = (s_i)_i$ a dissection of the interval $[s_1, s_2]$. Now by definition we have that

$$I_{s_1 s_2 t_1 t_2}^{It\hat{o}x, b_1} = \mathbb{P} - \lim_{|\pi| \rightarrow 0} \sum_i \varphi(x_{s_i t_1}) \delta x_{s_i s_{i+1} t_1 t_2}$$

and

$$I_{s_1 s_2 t_1 t_2}^{Rough, x, b_1} = a.s - \lim_{|\pi| \rightarrow 0} \sum_i \varphi(x_{s_i t_1}) \delta x_{s_i s_{i+1} t_1 t_2} + \varphi'(x_{s_i t_1}) (B_1^{It\hat{o}, xx})_{s_i s_{i+1} t_1 t_2}$$

but

$$\begin{aligned} \mathbb{E} \left[\left| \sum_i \varphi'(x_{s_i t_1}) (B_1^{It\hat{o}xx})_{s_i s_{i+1} t_1 t_2} \right|^2 \right] &= \sum_i \mathbb{E} [\varphi'(x_{s_i t_1})^2] \mathbb{E} \left[|(B_1^{It\hat{o}, xx})_{s_i s_{i+1} t_1 t_2}|^2 \right] \\ &= 1/2 t_1 (t_2 - t_1) \sum_i \mathbb{E} [\varphi'(x_{s_i t_1})^2] (s_{i+1} - s_i)^2 \\ &\leq t_1 (t_2 - t_1) (s_2 - s_1) |\pi| \sup_{s \in [0, 1]^2} \mathbb{E} [\varphi'(x_{s t_1})^2] \end{aligned}$$

Then is suffice to remark that the term appearing in the r.h.s vanish when $|\pi|$ go to zero and this finishes the proof. \square

Proposition 2.6.11. *Let $\varphi \in C^3(\mathbb{R})$ satisfying the (GC) and $\Pi = \{(s_i, t_j)\}_{i,j}$ a dissection of the rectangle $[s_1, s_2] \times [t_1, t_2]$ then we have :*

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} \varphi(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, xx} = 0$$

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} \varphi(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, \omega x} = 0$$

and

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_a} = 0$$

$$L^2(\Omega) - \lim_{|\Pi| \rightarrow 0} \sum_{i,j} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, \omega x, b_a} = 0$$

where $J^{Rough, xx, b_a} = \int_a \varphi'(x) d_a x \int_{\hat{a}} d_a x d_a x$ and $J^{Rough, \omega x, b_a} = \int_a \varphi'(x) d_a x \int_{\hat{a}} d_a \omega d_a x$ are given by the Proposition 2.5.4

Proof. We will prove only the first and third statement, for the two other we have exactly the same proof. Now definition we have that $C_{s_1 s_2 t_1 t_2}^{Ito, XX} = \iint_{(s_1, t_1)}^{(s_2, t_2)} \iint_{(s_1, t_1)}^{(s, t)} \hat{d}_{uv} x_{uv} \hat{d}_{st} x_{st}$ and then by independence of the increment of the Brownian sheet we have that

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i,j} \varphi(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, xx} \right|^2 \right] &= \sum_{i,j} 1/4 \mathbb{E}[\varphi(x_{s_i t_j})^2] (s_{i+1} - s_i)^2 (t_{j+1} - t_j)^2 \\ &\lesssim |\Pi| \sup_{(s,t) \in [0,1]^2} \mathbb{E}[\varphi(x_{st})^2] \end{aligned}$$

This give us the first convergence. Now we will focus on the third convergence which require more work. In fact by definition

$$\begin{aligned} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_1} &= \Lambda_1 [\delta_1 \varphi(x) C^{It\hat{o}, xx} + \varphi'(X) \mu_1 E_1^{It\hat{o}, xxx}] \\ &\quad + \mu_1 (\Lambda_1 \otimes 1) (\varphi(x) \#^1 D_1^{It\hat{o}, xx} + \delta_1 \varphi'(x) E_1^{It\hat{o}, xxx})_{s_i s_{i+1} t_j t_{j+1}} \\ &= \Lambda^{1d} [\delta_1 \varphi(x)_{.t_j} C_{.t_j t_{j+1}}^{It\hat{o}, xx} + \varphi'(x)_{.t_j} (\mu_1 E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}} \\ &\quad + \mu_1 (\Lambda_1 \otimes 1) (\varphi(X)_{.t_j} (D_1^{It\hat{o}, xx})_{.t_j t_{j+1}} + \delta_1 \varphi'(X)_{.t_j} (E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}})]_{s_i s_{i+1}} \end{aligned}$$

and then we have the bound

$$\begin{aligned} |J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_1}| &\lesssim_h (s_{i+1} - s_i)^{3h} (\|\delta_1 \varphi(X)_{.t_j}\|_h \|C_{.t_j t_{j+1}}^{It\hat{o}, xx}\|_{2h} + (\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|) \| (E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}} \|_{\mathcal{C}_2^{2h} \otimes \mathcal{C}_2^h} \\ &\quad + \|\varphi(x)_{.t_j} \#^1\|_{2h} \| (D_1^{It\hat{o}, xx})_{.t_j t_{j+1}} \|_{\mathcal{C}^h \otimes \mathcal{C}^h} + \|\delta_1 \varphi'(x)_{.t_j}\|_h \| (E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}} \|_{\mathcal{C}^{2h} \otimes \mathcal{C}^h}) \end{aligned} \quad (2.91)$$

By independence of increments in the second direction, this gives:

$$\mathbb{E} \left[\left| \sum_{i,j} J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_a} \right|^2 \right] \lesssim_h \left| \sum_i (s_{i+1} - s_i)^{3h} \right|^2 (a_1 + a_2 + a_3 + a_4) \quad (2.92)$$

where

1. $a_1 = \sum_j \mathbb{E}[\|\delta_1 \varphi(x)_{.t_j}\|_h^2] \mathbb{E}[\|C_{.t_j t_{j+1}}^{It\hat{o}, xx}\|_{2h}^2]$
2. $a_2 = \mathbb{E}[\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|^2] \sum_j \mathbb{E}[\| (E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}} \|_{\mathcal{C}_2^{2h} \otimes \mathcal{C}_2^h}^2]$
3. $a_3 = \sum_j \mathbb{E}[\|\varphi(x)_{.t_j} \#^1\|_{2h}^2] \mathbb{E}[\| (D_1^{It\hat{o}, xx})_{.t_j t_{j+1}} \|_{\mathcal{C}^h \otimes \mathcal{C}^h}^2]$
4. $a_4 = \sum_j \mathbb{E}[\|\delta_1 \varphi'(x)_{.t_j}\|_h^2] \mathbb{E}[\| (E_1^{It\hat{o}, xxx})_{.t_j t_{j+1}} \|_{\mathcal{C}^{2h} \otimes \mathcal{C}^h}^2]$

To obtain our convergence it suffices to show that all these terms are bounded. A simple computation gives :

$$\mathbb{E}[\|\delta_1 \varphi(x)_{.t_j}\|_h^2] \lesssim \mathbb{E} \left[\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|^4 \right]^{1/2} \mathbb{E}[\|\delta_1 x\|_h^4]^{1/2}$$

where the r.h.s is finite. Now the Lemma 2.6.2 gives

$$\|C_{.t_j t_{j+1}}^{Ito, xx}\|_{2h}^p \lesssim_{h,p} \int_{[0,1]^2} \frac{|C_{u_1 u_2 t_j t_{j+1}}^{It\hat{o}, xx}|^p}{|u_2 - u_1|^{2hp+2}} du_1 du_2 + \int_{[0,1]^4} \frac{|(D_1^{It\hat{o}, xx})_{u_1 u_2 u_3 u_4 t_j t_{j+1}}|^p}{|u_4 - u_3|^{hp+2} |u_2 - u_1|^{hp+2}} du_1 du_2 du_3 du_4$$

then taking the expectation in this last equality and using Jensen inequality we obtain that

$$\mathbb{E}[\|C_{.t_j t_{j+1}}^{Ito, XX}\|_{2h}^2] \lesssim_h c_p (t_{j+1} - t_j)^2 < \infty$$

where $c_p < +\infty$ for p large enough and then

$$a_1 \lesssim_{h,p} \mathbb{E} \left[\sup_{(s,t) \in [0,1]^2} |\varphi'(x_{st})|^4 \right]^{1/2} \mathbb{E} [|\delta_1 x|_h^4]^{1/2} \sum_j (t_{j+1} - t_j)^2$$

the terms a_2, a_3 and a_4 can be treated similarly and then we obtain the wanted convergence of the boundary integral. \square

Corollary 2.6.12. *Let $\varphi \in C^5(\mathbb{R})$ satisfying the (GC) then the integral*

$$\mathcal{J}_{s_1 s_2 t_1 t_2}^{It\hat{o}} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) \hat{d}_{st} x_{st}$$

where \hat{d} is a Ito differential admit a continuous version with coincide with

$$\mathcal{J}_{s_1 s_2 t_1 t_2}^{Rough} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \varphi(x_{st}) d_{st} x_{st}$$

where this last integral are given by Theorem 2.5.6

Proof. let $\Pi = \{(s_i, t_j)\}_{ij}$ a dissection of the rectangle $[s_1, s_2] \times [t_1, t_2]$ then by definition we have that

$$\begin{aligned} \mathcal{J}_{s_1 s_2 t_1 t_2}^{Rough} &= \sum_{ij} (-\varphi(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} + \varphi'(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, xx} + \varphi'(x_{s_i t_j}) C_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, \omega x}) \\ &\quad + \sum_{a=1,2} (I_{s_i s_{i+1} t_j t_{j+1}}^{Rough, x, b_a} + J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, xx, b_a} + J_{s_i s_{i+1} t_j t_{j+1}}^{Rough, \omega x, b_a}) + r_{s_i s_{i+1} t_j t_{j+1}}^b \end{aligned} \quad (2.93)$$

where $r^b \in \mathcal{C}_{2,2}^{1+,1+}$ this fact combined with the proposition (2.6.11) and (2.6.10) give us

$$\mathcal{J}_{s_1 s_2 t_1 t_2}^{Rough} = \mathbb{P} - \lim_{|\Pi| \rightarrow 0} \sum_{ij} -\varphi(x_{s_i t_j}) \delta x_{s_i s_{i+1} t_j t_{j+1}} + I_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, x, b_1} + I_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, x, b_2}$$

Now this last converge to the Ito integral in fact

$$\mathbb{E} [|\mathcal{J}_{s_1 s_2 t_1 t_2}^{It\hat{o}} - \sum_{ij} I_{s_i s_{i+1} t_j t_{j+1}}^{It\hat{o}, x, b_1}|^2] \lesssim \mathbb{E} [\sup_{(s,t) \in [0,1]^2} |\varphi(x_{st})|^4]^{1/2} \mathbb{E} [\sup_{|t-t'|+|s-s'| \leq |\Pi|} |x_{st} - x_{s't'}|^4]^{1/2}$$

which the r.h.s vanish when the mesh of the partition go to zero then we have the result. \square

2.7 Stratanovich formula Vs Skorohod formula

2.7.1 Introduction

In [20] the authors use the Malliavin calculus to give a meaning for the Skorohod integral in the plane for a fractional Brownian sheet x of hurst parameter $\gamma_1, \gamma_2 > 1/2$ and then he obtain the following change of variable formula

$$\begin{aligned} f(x_{st}) &= f(0) + \int_{[0,s] \times [0,t]} f'(x_{uv}) d^\diamond x_{uv} + \int_{[0,s] \times [0,t]} f''(x_{uv}) d_1^\diamond x_{uv} d_2^\diamond x_{uv} \\ &\quad + 2\gamma_1 \gamma_2 \int_1^s \int_2^t f''(x_{uv}) u^{2\gamma_1-1} v^{2\gamma_2-1} du dv + \gamma_2 \int_1^s \int_2^t f^{(3)}(x_{uv}) u^{2\gamma_1} v^{2\gamma_2-1} d_1^\diamond x_{u,v} dv \\ &\quad + \gamma_1 \int_0^s \int_0^t f^{(3)}(x_{uv}) u^{2\gamma_1-1} v^{2\gamma_2} d_2^\diamond x_{u,v} du + 2\gamma_1 \gamma_2 \int_1^s \int_2^t f^{(4)}(x_{uv}) u^{4\gamma_1-1} v^{4\gamma_2-1} du dv. \end{aligned} \quad (2.94)$$

for a smooth function f satisfying an Gaussian bound, and in [21] he give an extension for this formula in the case of the extended divergence which more artificial than the usual divergence operator. Our goal in this section is threefold, first we generalize this formula for a Gaussian process with smooth covariance function which satisfy some factorization property using the two parameters Young integration theory, secondly we will focus on the case of fractional Brownian sheet with hurst parameters $\gamma_1, \gamma_2 > 1/3$ and using the construction given in the previous section for the rough sheet we show that the formula still hold true for the usual divergence, and thirdly we point out the link between the path by path integration given by the notion of Rough-Sheet or the bidimensional Young theory and the different type of the Skorohod integrals in the plane. Now we introduce some notations which are specific to this section.

Notation 2.7.1. *We shall drop the index n of approximations in x^n , which means that x will stand for a generic smooth path defined on $[0, 1]^2$. For a smooth function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, we also write y for the path $\varphi(x)$ and for all $j \geq 1$ we set $y^j = \varphi^{(j)}(x)$ and $y^{j,n} = \varphi^{(j)}(x^n)$.*

Before starting to give our main result we recall in the following part briefly some tools of the Malliavin calculus.

2.7.2 Malliavin calculus framework

We consider in this section a centered Gaussian process $\{x_{s,t}; (s,t) \in [0, 1]^2\}$ with covariance function $\mathbb{E}[x_{s_1,t_1}x_{s_2,t_2}] = R_{s_1s_2;t_1t_2}$. We now briefly define the basic elements of Malliavin calculus with respect to x and then specify a little the setting under which we shall work.

Malliavin calculus with respect to x

We first relate a Hilbert space \mathcal{H} to our process x , defined as the closure of the linear space generated by the functions $\{1_{[0,s] \times [0,t]}, (s,t) \in [0, 1]^2\}$ with respect to the semi define positive form $\langle 1_{[0,s_1] \times [0,t_1]}, 1_{[0,s_2] \times [0,t_2]} \rangle = R_{s_1s_2;t_1t_2}$. Then the map $I_1 : 1_{[0,s] \times [0,t]} \rightarrow x_{s,t}$ can be extended to an isometry between \mathcal{H} and the first chaos generated by $\{x_{s,t}; (s,t) \in [0, 1]^2\}$.

Starting from the space \mathcal{H} , a Malliavin calculus with respect to x can now be developed in the usual way (see [53, 64] for further details). Namely, we first define a set of smooth functionals of x by

$$\mathcal{S} := \{f(I_1(\psi_1), \dots, I_1(\psi_n)); n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), \psi_1, \dots, \psi_n \in \mathcal{H}\}$$

and for $F = f(I_1(\psi_1), \dots, I_1(\psi_n)) \in \mathcal{S}$ we define

$$DF = \sum_{i=1}^n \partial_i f(I_1(\psi_1), \dots, I_1(\psi_n)) \psi_i.$$

Then D is a closable operator from $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H})$. Therefore we can extend D to the closure of smooth functionals under the norm

$$\|F\|_{1,p} = (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathcal{H}}^p])^{\frac{1}{p}}$$

The iteration of the operator D is defined in such a way that for a smooth random variable $F \in \mathcal{S}$ the iterated derivative $D^k F$ is a random variable with values in $\mathcal{H}^{\otimes k}$. The domain $\mathbb{D}^{k,p}$ of D^k is the

completion of the family of smooth random variables $F \in \mathcal{S}$ with respect to the semi-norm :

$$\|F\|_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E}[\|D^j F\|_{\mathcal{H}^{\otimes j}}^p] \right)^{\frac{1}{p}}.$$

Similarly, for a given Hilbert space V we can define the space $\mathbb{D}^{k,p}(V)$ of V -valued random variables, and $\mathbb{D}^\infty(V) = \bigcap_{k,p \geq 1} \mathbb{D}^{k,p}$.

Consider now the adjoint δ^\diamond of D . The domain of this operator is defined as the set of $u \in L^2(\Omega, \mathcal{H})$ such that $\mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}] \lesssim \|F\|_{1,2}$, and for this kind of process $\delta^\diamond(u)$ (called Skorohod integral of u) is the unique element of $L^2(\Omega)$ such that

$$\mathbb{E}[\delta^\diamond(u)F] = \mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}].$$

Note that $\mathbb{E}[\delta^\diamond(u)] = 0$ and

$$\mathbb{E}[\delta^\diamond(u)^2] \leq \mathbb{E}[\|u\|_{\mathcal{H}}^2] + \mathbb{E}[\|Du\|_{\mathcal{H} \otimes \mathcal{H}}^2].$$

The following divergence type property of δ^\diamond will be useful in the sequel:

$$\delta^\diamond(Fu) = F\delta^\diamond(u) - \langle u, DF \rangle_{\mathcal{H}}, \quad (2.95)$$

and we also recall the following compatibility of δ^\diamond with limiting procedures:

Lemma 2.7.2. *let u_n be a sequence of elements in $\text{Dom}(\delta^\diamond)$, which converges to u in $L^2(\Omega, \mathcal{H})$. We further assume that $\delta^\diamond(u_n)$ converges in $L^2(\Omega)$ to some random variable $F \in L^2(\Omega)$. Then $u \in \text{Dom}(\delta^\diamond)$ and $\delta^\diamond(u) = F$.*

Wick's products

Some of our results below will be expressed in terms of Riemann-Wick sums. We give a brief account on these objects, mainly borrowed from [52, 53].

Among functionals F of x such that $F \in \mathbb{D}^\infty$, the set of multiple integrals plays a special role. In order to introduce it in the context of a general process x indexed by the plane, consider an orthonormal basis $\{e_n; n \geq 1\}$ of \mathcal{H} and let $\hat{\otimes}$ denote the symmetric tensor product. Then

$$f_n = \sum_{\text{finite}} f_{i_1, \dots, i_n} e_{i_1} \hat{\otimes} \dots \hat{\otimes} e_{i_n}, \quad f_{i_1, \dots, i_n} \in \mathbb{R} \quad (2.96)$$

is an element of $\mathcal{H}^{\hat{\otimes} n}$ satisfying the relation:

$$\|f_n\|_{\mathcal{H}^{\hat{\otimes} n}}^2 = \sum_{\text{finite}} |f_{i_1, \dots, i_n}|^2. \quad (2.97)$$

Moreover, $\mathcal{H}^{\hat{\otimes} n}$ is the completion of the set of elements like (2.96) with respect to the norm (2.97).

For an element $f_n \in \mathcal{H}^{\hat{\otimes} n}$, the multiple Itô integral of order n is well-defined. First, any element of the form given by (2.96) can be rewritten as

$$f_n = \sum_{\text{finite}} f_{j_1 \dots j_m} e_{j_1}^{\hat{\otimes} k_1} \hat{\otimes} \dots \hat{\otimes} e_{j_m}^{\hat{\otimes} k_m}, \quad (2.98)$$

where the j_1, \dots, j_m are different and $k_1 + \dots + k_m = n$. Then, if $f_n \in \mathcal{H}^{\hat{\otimes} n}$ is given under the form (2.98), define its multiple integral as:

$$I_n(f_n) = \sum_{\text{finite}} f_{j_1, \dots, j_m} H_{k_1}(I_1(e_{j_1})) \cdots H_{k_m}(I_1(e_{j_m})), \quad (2.99)$$

where H_k denotes the k -th normalized Hermite polynomial given by

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} = \sum_{j \leq k/2} \frac{(-1)^j k!}{2^j j! (k-2j)!} x^{k-2j}.$$

It holds that the multiple integrals of different order are orthogonal and that

$$\mathbb{E} [|I_n(f_n)|^2] = n! \|f_n\|_{\mathcal{H}^{\hat{\otimes} n}}^2.$$

This last isometric property allows to extend the multiple integral for a general $f_n \in \mathcal{H}^{\hat{\otimes} n}$ by $L^2(\Omega)$ convergence. Finally, one can define the integral of $f_n \in \mathcal{H}^{\hat{\otimes} n}$ by putting $I_n(f_n) := I_n(\tilde{f}_n)$, where $\tilde{f}_n \in \mathcal{H}^{\hat{\otimes} n}$ denotes the symmetrized version of f_n . Moreover, the chaos expansion theorem states that any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$, where \mathcal{G} is the σ -field generated by x , can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{with} \quad \mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|_{\mathcal{H}^{\hat{\otimes} n}}^2. \quad (2.100)$$

With these notations in mind, one way to introduce Wick products on a Wiener space is to impose the relation

$$I_n(f_n) \diamond I_m(g_m) = I_{n+m}(f_n \hat{\otimes} g_m) \quad (2.101)$$

for any $f_n \in \mathcal{H}^{\hat{\otimes} n}$ and $g_m \in \mathcal{H}^{\hat{\otimes} m}$, where the multiple integrals $I_n(f_n)$ and $I_m(g_m)$ are defined by (2.99). If $F = \sum_{n=1}^{N_1} I_n(f_n)$ and $G = \sum_{m=1}^{N_2} I_m(g_m)$, we define $F \diamond G$ by

$$F \diamond G = \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} I_{n+m}(f_n \hat{\otimes} g_m).$$

By a limit argument, we can then extend the Wick product to more general random variables (see [52] for further details). In this paper, we will take the limits in the $L^2(\Omega)$ topology.

Some corrections between ordinary and Wick products will be computed below. A simple example occurs for products of $f(x)$ by a Gaussian increment. Indeed, for a smooth function f and $g_1, g_2 \in \mathcal{H}$, it is shown in [52] that

$$f(I_1(g_1)) \diamond I_1(g_2) = f(I_1(g_1)) I_1(g_2) - f'(I_1(g_1)) \langle g_1, g_2 \rangle_{\mathcal{H}}. \quad (2.102)$$

We now state a result which is proven in [53, Proposition 4.1].

Proposition 2.7.3. *Let $F \in \mathbb{D}^{k,2}$ and $g \in \mathcal{H}^{\otimes k}$. Then*

1. $F \diamond I_k(g)$ is well defined in $L^2(\Omega)$.
2. $Fg \in \text{Dom } \delta^{\circ k}$.
3. $F \diamond I_k(g) = \delta^{\circ k}(Fg)$.

2.7.3 Young Vs Skorohod

We define the bidimensional integral for a general Gaussian process which have a "Young-Hölder" regularity and obtain a change of variable formula in this case which generalize the result obtained in [20]. Now let us specify what doesn't mean by the Young-Hölder regularity

Hypothesis 2.7.4. *The covariance R of our centered Gaussian process x belongs to the space $\mathcal{C}^{1-var}([0, 1]^4)$, and satisfies a factorization property of the form*

$$\mathbb{E}[x_{s_1;t_1}x_{s_2;t_2}] = R_{s_1s_2;t_1t_2} = R_{s_1s_2}^1 R_{t_1t_2}^2,$$

for two covariance functions R^1, R^2 on $[0, 1]$. In addition, setting $R_a^i = R_{aa}^i$ for $a \in [0, 1]$ and $i = 1, 2$, we assume that $a \mapsto R_a^i$ is differentiable and we suppose that

$$|2R_{ab}^i - R_{aa}^i - R_{bb}^i| \lesssim |a - b|^{\gamma_i} \quad (2.103)$$

for all $a, b \in [0, 1]$, with $\gamma_i > 1$. Finally we suppose that $(R^i)'_a = \partial_a R_{aa}^i \in L^\infty([0, 1])$.

The first consequence of our Hypothesis 2.7.4 is that the regularity of x corresponds to the Young type regularity of Section (2.4). Indeed, it is readily checked that relation (2.103) yields

$$\mathbb{E}[(\delta x_{s_1s_2;t_1t_2})^2] \lesssim |s - s'|^{\gamma_1} |t - t'|^{\gamma_2}.$$

Since x is Gaussian, an easy application of Kolmogorov's criterion ensures that

$$x \in \mathcal{C}_{1,1}^{\alpha_1, \alpha_2}, \quad \text{with} \quad \alpha_1 = \frac{\gamma_1}{2} - \epsilon_1 > \frac{1}{2}, \quad \alpha_2 = \frac{\gamma_2}{2} - \epsilon_2 > \frac{1}{2}, \quad (2.104)$$

for arbitrarily small $\epsilon_1, \epsilon_2 > 0$. This enables us to appeal to Young's integration theory given in the section (2.4) in order to define the integrals

$$z^1 = \iint y^1 dx, \quad z^2 = \iint y^2 d_1 x d_2 x$$

Theorem 2.7.5. *Assume x is a centered Gaussian process on $[0, 1]^2$ with a covariance function satisfying (2.7.4), and consider a function $f \in C^4(\mathbb{R})$ satisfying condition (GC). Then the increments*

$$z_{s_1s_2t_1t_2}^{1,\diamond} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^1 d_{12}^\diamond x_{uv}, \quad \text{and} \quad z_{s_1s_2t_1t_2}^{2,\diamond} = \int_{s_1}^{s_2} \int_{t_1}^{t_2} y^2 d_1^\diamond x d_2^\diamond x_{uv}, \quad (2.105)$$

are well defined in the Skorohod sense of Malliavin calculus. Moreover:

(i) *Some Riemann convergences hold true: if π_n^1 and π_n^2 are 2 partitions of $[s_1, s_2] \times [t_1, t_2]$ whose mesh goes to 0 as $n \rightarrow \infty$, then*

$$\lim_{n \rightarrow \infty} \sum_{\pi_n^1, \pi_n^2} y_{\sigma_i; \tau_j}^1 \diamond \delta x_{\sigma_i \sigma_{i+1} \tau_j \tau_{j+1}} = z_{s_1s_2t_1t_2}^{1,\diamond} \quad (2.106)$$

$$\lim_{n \rightarrow \infty} \sum_{\pi_n^1, \pi_n^2} y_{\sigma_i; \tau_j}^2 \diamond \delta_2 x_{s_i; t_j t_{j+1}} \diamond \delta_1 x_{s_i s_{i+1}; t_j} = z_{s_1s_2t_1t_2}^{2,\diamond}, \quad (2.107)$$

where \diamond stands for the Wick product in the left hand side of the relations above, and where the convergence holds in both a.s and $L^2(\Omega)$ sense.

(ii) The change of variables formula for $y = f(x)$ becomes

$$\begin{aligned} \delta y_{s_1 s_2 t_1 t_2} &= z_{s_1 s_2 t_1 t_2}^{1,\diamond} + z_{s_1 s_2 t_1 t_2}^{2,\diamond} + \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^2 d_1 R_u^1 d_2 R_v^2 + \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^3 R_u^1 d_2 R_v^2 d_1^\diamond x_{uv} \\ &\quad + \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^3 R_v^2 d_1 R_u^1 d_2^\diamond x_{uv} + \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^4 R_u^1 R_v^2 d_1 R_u^1 d_2 R_v^2. \end{aligned} \quad (2.108)$$

(iii) Explicit corrections between z^1 , z^2 and $z^{1,\diamond}$, $z^{2,\diamond}$ can be computed (see relations (2.111) and (2.119)).

Now we will give a crucial property of the Hilbert space \mathcal{H} which allow us to do our computations.

Lemma 2.7.6. Under Hypothesis 2.7.4, we have $\|f\|_{\mathcal{H}} \leq \|f\|_{\infty} \|R\|_{1\text{-var};[0,1]^4}$.

Proof. Consider a step function $f = \sum_{i,j} a_{ij} 1_{\Delta_{ij}}$ related to a partition $(\Delta_{ij})_{i,j}$ of $[0, 1]^2$. We have

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= \sum_{i,j,l,k} a_{ij} a_{lk} R_{s_i s_l}^1 R_{t_j t_k}^2 = \sum_{i,j,k,l} a_{ij} a_{kl} \int_0^{s_i} \int_0^{s_l} \int_0^{t_j} \int_0^{t_k} d_{12} R_{s_1 s_2}^1 d_{12} R_{t_1 t_2}^2 \\ &= \int_{[0,1]^4} f_{s_1;t_1} f_{s_2;t_2} d_{12} R_{s_1 s_2}^1 d_{12} R_{t_1 t_2}^2 \leq \|f\|_{\infty}^2 \|R\|_{1\text{-var};[0,1]^4}. \end{aligned} \quad (2.109)$$

The general case now easily follows by density of the step functions in \mathcal{H} . \square

Let us now recall that we work under the usual assumptions for Skorohod type change of variables formulae given at Definition 2.6.6 and referred to as (GC) condition in the sequel. Notice that $\max_{s,t \in [0,1]} (R_s^1 R_t^2) = \max_{s,t \in [0,1]} \mathbb{E}[|x_{s;t}|^2]$. Thus condition (GC) implies that

$$\mathbb{E} \left[\sup_{s,t \in [0,1]} |f(x_{s;t})|^r \right] < \infty, \quad \text{for all } r \geq 1. \quad (2.110)$$

We now state an approximation result in \mathcal{H} which proves to be useful in order to get our Itô type formula.

Proposition 2.7.7. Let x be a centered Gaussian process on $[0, 1]$ satisfying Hypothesis 2.7.4 and $f \in C^1(\mathbb{R})$ such that the growth condition (GC) is fulfilled for φ and $\varphi^{(1)}$. Consider a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$, $\pi_1 = (s_i)_i$, $\pi_2 = (t_j)_j$ two respective dissections of the intervals $[s_1, s_2]$ and $[t_1, t_2]$ and denote by $\Delta_{i,j} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$. Then

$$\lim_{|\pi_1|, |\pi_2| \rightarrow 0} \mathbb{E} \left[\left\| y \cdot 1_{\Delta} - \sum_{i,j} y_{s_i; t_j} 1_{\Delta_{i,j}} \right\|_{\mathcal{H}}^2 \right] = 0,$$

Proof. Observe first that

$$y_{s;t} 1_{\Delta}(s, t) - \sum_{i,j} y_{s_i; t_j} 1_{\Delta_{i,j}}(s, t) = \sum_{i,j} (y_{s;t} - y_{s_i; t_j}) 1_{\Delta_{i,j}}(s, t)$$

from which the following estimation is easily obtained:

$$|(y_{s;t} - y_{s_i;t_j})1_{\Delta_{i,j}}(s,t)| \leq \left(\sup_{(s,t) \in \Delta} |y_{s;t}^1| \max_{|s_1-s_2| \leq |\pi_1|, |t_1-t_2| \leq |\pi_2|} |x_{s_1;t_1} - x_{s_2;t_2}| \right) 1_{\Delta_{i,j}}.$$

Hence if we take expectations in this last estimation and resort to Hölder's inequality, we obtain that

$$\begin{aligned} \mathbb{E} \left[\left\| y \cdot 1_{\Delta} - \sum_{i,j} y_{s_i;t_j} 1_{\Delta_{i,j}} \right\|_{\infty}^2 \right] \\ \leq \mathbb{E}^{1/2} \left[\sup_{(s,t) \in \Delta} |y_{s;t}^1|^4 \right] \mathbb{E}^{1/2} \left[\max_{|s-s'| \leq |\pi_1|, |t-t'| \leq |\pi_2|} |x_{s;t} - x_{s';t'}|^4 \right] \left\| \sum_{i,j} 1_{\Delta_{i,j}} \right\|_{\infty}. \end{aligned}$$

Now the r.h.s of this inequality goes to zero when the mesh of the partitions π_1, π_2 goes to zero. Our claim thus easily stems from the embedding (2.109). \square

Proposition 2.7.8. *Assume x is a centered Gaussian process on $[0, 1]^2$ with a covariance function satisfying (2.7.4). Consider a function $f \in C^2(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $y^1 1_{\Delta} \in \text{Dom}(\delta^{\diamond})$, and if we define the increment $z^{1,\diamond} \equiv \delta^{\diamond}(y^1 1_{\Delta})$ the following relation holds true:*

$$z_{s_1 s_2; t_1 t_2}^{1,\diamond} = z_{s_1 s_2; t_1 t_2}^1 - \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^2 d_1 R_s^1 d_2 R_t^2, \quad (2.111)$$

where z^1 is given by Proposition (2.4.1) and the second one is of Riemann-Stieltjes type. Moreover, relation (2.106) holds true in the $L^2(\Omega)$ and almost sure sense.

Proof. Consider a sequence of partitions $\pi_n = (\pi_n^1, \pi_n^2)$ whose mesh go to 0 as $n \rightarrow \infty$. The generic elements of π_n will be denoted by (s_i, t_j) . Owing to formula (2.95), we have that

$$\begin{aligned} \sum_{\pi_n} \delta^{\diamond}(y_{s_i;t_j}^1 1_{\Delta_{i,j}}) &= \sum_{\pi_n} y_{s_i;t_j}^1 \delta x_{s_i s_{i+1}; t_j t_{j+1}} - \sum_{\pi_n} y_{s_i;t_j}^2 \mathbb{E}[x_{s_i;t_j} \delta x_{s_i s_{i+1}; t_j t_{j+1}}] \\ &= \sum_{\pi_n} y_{s_i;t_j}^1 \delta x_{s_i s_{i+1}; t_j t_{j+1}} - \sum_{\pi_n} y_{s_i;t_j}^2 (R_{s_i s_{i+1}}^1 - R_{s_i s_i}^1)(R_{t_j t_{j+1}}^2 - R_{t_j t_j}^2) \equiv A_1^n - A_2^n. \end{aligned}$$

We now treat those two terms separately.

Step 1: Estimation of A_1^n . The term $A_1^n = \sum_{\pi_n} y_{s_i;t_j}^1 \delta x_{s_i s_{i+1}; t_j t_{j+1}}$ is a Riemann type sum. Since $x \in \mathcal{C}_{1,1}^{\alpha_1, \alpha_2}$ with $\alpha_1, \alpha_2 > 1/2$, it converges a.s to $\iint y^1 d_{12}x$ as n goes to ∞ , according to Proposition ??.

The $L^2(\Omega)$ convergence of the Riemann sums defining A_1^n is more cumbersome, and we have to go back to the definition of the Young integral given by Proposition 2.4.1. Indeed one can write $\int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^1 d_{12}x_{s;t} = \sum_{\pi_n} \int_{\Delta_{i,j}} y_{s_i;t_j}^1 d_{12}x_{s;t}$, and then using the expansion given in the Proposition (2.4.1) for this integral we get that

$$\begin{aligned} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^1 d_{12}x_{s;t} &= A_1^n + \sum_{\pi_n} (\text{Id} - \Lambda_2 \delta_2)(\Lambda_1 \delta_1)(y^1 \delta x)_{s_i s_{i+1}; t_j t_{j+1}} \\ &\quad + \sum_{\pi_n} (\text{Id} - \Lambda_1 \delta_1)(\Lambda_2 \delta_2)(y^1 \delta x)_{s_i s_{i+1}; t_j t_{j+1}} - \sum_{\pi_n} \Lambda \delta(y^1 \delta x)_{s_i s_{i+1}; t_j t_{j+1}}. \end{aligned}$$

Furthermore, some partial summations can be performed on the terms $\text{Id} - \Lambda_i \delta_i$ for $i = 1, 2$ and setting $D_1^n \equiv \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^1 d_{12}x_{s;t} - A_1^n$ we deduce that

$$D_1^n = \sum_{\pi_n^2} [(\text{Id} - \Lambda_1 \delta_1)(\Lambda_2 \delta_2)] (y^1 \delta x)_{s_1 s_2; t_j t_{j+1}} \\ + \sum_{\pi_n^1} [(\text{Id} - \Lambda_2 \delta_2)(\Lambda_1 \delta_1)] (y^1 \delta x)_{s_i s_{i+1}; t_1 t_2} - \sum_{\pi_n} \Lambda \delta (y^1 \delta x)_{s_i s_{i+1}; t_j t_{j+1}}.$$

Recalling the Hölder regularity (2.104) of our process x , we thus obtain

$$|D_1^n| \lesssim |s_2 - s_1|^{\alpha_1} \|\delta_2 y^1\|_{0, \alpha_1} \|\delta x\|_{\alpha_1, \alpha_2} \sum_j |t_{j+1} - t_j|^{2\alpha_2} \\ + |t_2 - t_1|^{\alpha_2} \|\delta_1 y^1\|_{\alpha_1, 0} \|\delta x\|_{\alpha_1, \alpha_2} \sum_{\pi_n^1} |s_{i+1} - s_i|^{2\alpha_1} \\ + \|\delta y^1\|_{\alpha_1, \alpha_2} \|\delta x\|_{\alpha_1, \alpha_2} \sum_{\pi_n} |s_{i+1} - s_i|^{2\alpha_1} |t_{j+1} - t_j|^{2\alpha_2}.$$

Now taking expectations in the last relation and using Hölder's inequality we end up with

$$\mathbb{E}[|D_1^n|^2] \lesssim_{s_1, s_2, t_1, t_2, \alpha_1, \alpha_2} (|\pi_1|^{4\alpha_1 - 2} + |\pi_2|^{4\alpha_2 - 2}) \mathbb{E}^{1/2}[\|\delta x\|_{\alpha_1, \alpha_2}^4] \\ \times \left(\mathbb{E}^{1/2}[\|\delta_1 y^1\|_{0, \alpha_1}^4] + \mathbb{E}^{1/2}[\|\delta_2 y^1\|_{\alpha_2}^4] + \mathbb{E}^{1/2}[\|\delta y^1\|_{\alpha_1, \alpha_2}^4] \right), \quad (2.112)$$

for n large enough.

We are now going to prove that one can recast (2.112) into

$$\mathbb{E}[|D_1^n|^2] \lesssim_{s_1, s_2, t_1, t_2, \alpha_1, \alpha_2} (|\pi_1|^{4\alpha_1 - 2} + |\pi_2|^{4\alpha_2 - 2}). \quad (2.113)$$

In addition combining the fact that $\mathbb{E}[\|\delta x\|_{\alpha_1, \alpha_2}^4] < +\infty$ with Lemma (2.5.20) plus condition (GC) on the function f , we obtain that

$$E[\|\delta x\|_{\alpha_1, \alpha_2}^4]^{1/2} \left(\mathbb{E}[\|\delta_1 y^1\|_{0, \alpha_1}^4]^{1/2} + \mathbb{E}[\|\delta_2 y^1\|_{\alpha_2}^4]^{1/2} + \mathbb{E}[\|\delta y^1\|_{\alpha_1, \alpha_2}^4]^{1/2} \right) < +\infty.$$

Hence inequality (2.113) is easily deduced from (2.112), and this proves that A_1^n converges in $L^2(\Omega)$ to $\int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^1 d_{12}x_{s;t}$ as $n \rightarrow \infty$.

Step 2: Estimation of A_2^n . Recall that A_2^n is defined by

$$A_2^n = \sum_{\pi_n} y_{s_i; t_j}^2 (R_{s_{i+1} s_i}^1 - R_{s_i s_i}^1) (R_{t_{j+1} t_j}^2 - R_{t_j t_j}^2).$$

In order to treat this term, first remark that for $k = 1, 2$ we have

$$R_{s_i s_{i+1}}^k - R_{s_i s_i}^k = \frac{1}{2} (R_{s_{i+1} s_{i+1}}^k - R_{s_i s_i}^k) + \rho_{s_i s_{i+1}}^k,$$

where $\rho_{s_i s_{i+1}}^k = \frac{1}{2}(2R_{s_i s_{i+1}}^k - R_{s_i s_i}^k - R_{s_{i+1} s_{i+1}}^k)$. Injecting this relation in the definition of the term A_2^n and recalling that we have set $R_a^k \equiv R_{aa}^k$, we obtain

$$\begin{aligned} A_2^n &= 1/4 \sum_{\pi_n} y_{s_i; t_j}^2 (R_{s_{i+1}}^1 - R_{s_i}^1) (R_{t_{j+1}}^2 - R_{t_j}^2) \\ &\quad + 1/2 \sum_{\pi_n} y_{s_i; t_j}^2 \left[(R_{t_{j+1}}^2 - R_{t_j}^2) \rho_{s_i s_{i+1}}^1 + (R_{s_{i+1}}^1 - R_{s_i}^1) \rho_{t_j t_{j+1}}^2 + \rho_{s_i s_{i+1}}^1 \rho_{t_j t_{j+1}}^2 \right] \\ &\equiv A_{21}^n + A_{22}^n + A_{23}^n + A_{24}^n. \end{aligned}$$

We will now show that

$$\lim_{n \rightarrow \infty} A_{21}^n = \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^2 d_1 R_s^1 d_2 R_t^2, \quad \text{and} \quad \lim_{n \rightarrow \infty} \sum_{j=2}^4 A_{2j}^n = 0, \quad (2.114)$$

where the limits are understood in the almost sure and L^2 sense.

Indeed, it is easily understood that the terms $A_{22}^n, A_{23}^n, A_{24}^n$ are remainder terms: according to Hypothesis 2.7.4 we have that $|\rho_{ab}^i| \lesssim |a-b|^{\gamma_i}$, and we get the following inequality for A_{22}^n :

$$\begin{aligned} A_{22}^n &\lesssim |\pi_1|^{\gamma_1-1} \sup_{(s,t) \in \Delta} |y_{s;t}^2| \sum_{\pi_n} (s_{i+1} - s_i) |R_{t_{j+1}}^2 - R_{t_j}^2| \\ &\leq |\pi_1|^{\gamma_1-1} \sup_{(s,t) \in \Delta} |y_{s;t}^2| (s_2 - s_1) \int_{t_1}^{t_2} |d_2 R_t^2|. \end{aligned}$$

This relation, plus the condition (GC) on f , obviously entails that $\lim_{n \rightarrow \infty} A_{22}^n = 0$ in the almost sure and $L^2(\Omega)$ sense. The case of A_{23}^n, A_{24}^n follow exactly along the same lines.

We now focus on the term A_{21}^n : observe that

$$\begin{aligned} \left| a_1 - 1/4 \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^2 d_1 R_s^1 d_2 R_t^2 \right| \\ \lesssim \sup_{(s,t) \in \Delta} |y_{s;t}^2| \max_{|s-s'| \leq |\pi_1|, |t-t'| \leq |\pi_2|} |x_{s;t} - x_{s_2;t_2}| \int_{s_1}^{s_2} \int_{t_1}^{t_2} |d_1 R_s^1| |d_2 R_t^2|. \end{aligned}$$

Invoking the same estimates as before for the Hölder norm of x and condition (GC) on f , the proof of our assertion (2.114) is now completed.

Step 3: Conclusion. Let us summarize the results obtained in the last two steps: plugging relation (2.114) into the definition of A_2^n and recalling the limiting behavior of A_1^n established at Step 1, we have obtained that

$$\lim_{n \rightarrow \infty} \sum_{\pi_n} \delta^\diamond(y_{s_i; t_j}^1 1_{\Delta_{ij}}) = \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^1 d_1 x_{s;t} - \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{s;t}^2 d_1 R_s^1 d_2 R_t^2.$$

where the convergence is understood in both a.s and $L^2(\Omega)$ sense. Furthermore, Proposition 2.7.7 asserts that $\sum_{\pi_n} y_{s_i; t_j}^1 1_{\Delta_{ij}}$ converges in $L^2(\Omega, \mathcal{H})$ to $y^1 1_\Delta$. This finishes our proof of relation (2.111) thanks to a direct application of Lemma 2.7.2.

As far as expression (2.106) with Wick-Riemann sums is concerned, recall that we have proved that

$$\delta^\diamond(y^1 1_\Delta) = \lim_{|\pi_1|, |\pi_2| \rightarrow 0} \sum_{\pi_n} \delta^\diamond(y_{s_i; t_j}^1 1_{\Delta_{ij}}).$$

Now invoke Proposition 2.7.3 for $k = 1$ in order to state that

$$\delta^\diamond(y_{s_i; t_j}^1 1_{\Delta_{ij}}) = y_{s_i; t_j}^1 \diamond \delta^\diamond(1_{\Delta_{ij}}) = y_{s_i; t_j}^1 \diamond \delta x_{s_i s_{i+1}; t_j t_{j+1}},$$

which ends the proof. \square

Proposition 2.7.8 gives a meaning to the increment $z^{1, \diamond}$ and compares them to the corresponding Stratonovich increment z^1 . In order to compare change of variables formulae, we still have to define Skorohod integrals of the form $z^{2, \diamond}$, which is what we proceed to do now.

To this aim, let us start by some formal considerations: it is easily conceived that

$$\int_0^s \int_0^t y_{uv}^2 d_1^\diamond x_{uv} d_2^\diamond x_{u;v} = \int_0^s \int_0^t \int_0^u \int_0^v y_{uv}^2 d_{12}^\diamond x_{u'v} d_{12}^\diamond x_{uv'} = \delta^{2, \diamond}(N(y)) \quad (2.115)$$

where, similarly to [20], we set

$$N(y)_{u'v; uv'} := y 1_{[0, s] \times [0, v]}(u, v') 1_{[0, u] \times [0, t]}(u', v),$$

and where we integrate firstly in (u', v) and then in (u, v') . Our objective in what follows is to give a rigorous meaning to equation (2.115).

Lemma 2.7.9. *Take up the notation of Proposition 2.7.8, and consider $f \in C^3(\mathbb{R})$ satisfying condition (GC). For a sequence of partitions $(\pi_n)_{n \geq 1}$ whose mesh goes to 0 define*

$$a_{u'v; vv'}^{\pi_n} = \sum_{i, j} y_{s_i t_j}^2 1_{[0, s_i] \times [t_j, t_{j+1}]}(u', v) 1_{[s_i, s_{i+1}] \times [0, t_j]}(u, v'). \quad (2.116)$$

Then a^{π_n} converges to $N(y)$ in $L^2(\Omega, \mathcal{H}^{\otimes 2})$ as n goes to infinity.

Proof. First notice that the tensor norm of an element $K \in \mathcal{H}^{\otimes 2}$ can be bounded as:

$$\begin{aligned} \|K\|_{\mathcal{H}^{\otimes 2}} &= \int_{[0, 1]^8} K_{a_1 a'_1; b_1 b'_1} K_{a_2 a'_2; b_2 b'_2} d_{12} R_{a_1 a'_1}^1 d_{12} R_{a_2 a'_2}^1 d_{12} R_{b_1 b'_1}^2 d_{12} R_{b_2 b'_2}^1 \\ &\leq \int_{[0, 1]^8} |K_{a_1 a'_1; b_1 b'_1} K_{a_2 a'_2; b_2 b'_2}| |d_{12} R_{a_1 a'_1}^1| |d_{12} R_{a_2 a'_2}^1| |d_{12} R_{b_1 b'_1}^2| |d_{12} R_{b_2 b'_2}^1|. \end{aligned} \quad (2.117)$$

Furthermore, a simple computation shows that

$$\begin{aligned} a_{u'v; vv'}^{\pi_n} - N(y)_{u'v; vv'} &= \sum_{\pi_n} \left[y_{s_i t_j}^2 - y_{uv}^2 \right] \left[1_{[0, u] \times [t_j, t_{j+1}]}(u', v) 1_{[s_i, s_{i+1}] \times [0, v]}(u, v') \right] \\ &\quad + \sum_{\pi_n} y_{s_i t_j}^2 \left[\left(1_{[0, s_i] \times [0, t_j]}(u', v') - 1_{[0, u] \times [0, v]}(u', v') \right) 1_{[s_i, s_{i+1}] \times [t_j, t_{j+1}]}(u, v) \right], \end{aligned}$$

and thus,

$$\begin{aligned} \left| a_{u'v'}^{\pi_n} - N(y)_{u'v'} \right| \leq & \left(\sup_{(a,b) \in [0,s] \times [0,t]} |y_{ab}^3| \sup_{|a_2-a_1| \leq |\pi_1|, |b_2-b_1| \leq |\pi_2|} |x_{a_2;b_2} - x_{a_1;b_1}| \right. \\ & \left. + \max_{i,j} (1_{[s_i, s_{i+1}]}(u') + 1_{[t_j, t_{j+1}]}(v')) \sup_{(a,b) \in [0,s] \times [0,t]} |y_{ab}^2| \right). \end{aligned} \quad (2.118)$$

Our claims are now easily derived: on the one hand the right hand side of (2.118) converges to zero when $n \rightarrow \infty$ if $u' \neq s_i$ and $v' \neq t_j$ for all i, j . Then using inequality (2.117) and dominated convergence we obtain that a^{π_n} converges a.s to $N(y)$ in $\mathcal{H}^{\otimes 2}$. On the other hand, in order to obtain the convergence in $L^2(\Omega, \mathcal{H}^{\otimes 2})$ it suffices to use the fact that f satisfies condition (GC) and apply once again dominated convergence. \square

Now we are able to define our mixed integral in the Skorohod sense and connect it to the equivalent integral in the Young theory:

Proposition 2.7.10. *Assume x is a centered Gaussian process on $[0, 1]^2$ with a covariance function satisfying (2.7.4). Consider a function $f \in C^4(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $N(y) \in \text{Dom}(\delta^{\circ, 2})$, and if we define $z^{2, \diamond} = \delta^{\circ, 2}(N(y))$ the following relation holds:*

$$\begin{aligned} z_{s_1 s_2; t_1 t_2}^{2, \diamond} = & z_{s_1 s_2; t_1 t_2}^2 - \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^2 d_1 R_u^1 d_2 R_v^2 - \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^3 R_u^1 d_2 R_v^2 d_1 x_{uv} \\ & - \frac{1}{2} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^3 R_v^2 d_1 R_u^1 d_2 x_{uv} + \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^4 R_u^1 R_v^2 d_1 R_u^1 d_2 R_v^2, \end{aligned} \quad (2.119)$$

where $\int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^3 R_u^1 d_2 R_v^2 d_1 x_{uv}$ and $\int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{uv}^3 R_v^2 d_1 R_u^1 d_2 x_{uv}$ are defined according to Proposition 2.4.5. Moreover, relation (2.107) holds true in the $L^2(\Omega)$ and almost sure sense.

Proof. Like for Proposition 2.7.8, our strategy is as follows: consider a sequence $\pi_n = (\pi_n^1, \pi_n^2)$ whose mesh go to 0 as $n \rightarrow \infty$ and set $a^n \equiv a^{\pi_n}$ defined by (2.116). We have seen at Lemma 2.7.9 that $\lim_{n \rightarrow \infty} a^n = N(y)$ in $L^2(\Omega, \mathcal{H}^{\otimes 2})$. We shall now study the convergence of $\delta^{\circ}(a^n)$ by means of Wick-Stratonovich corrections. Then we will conclude by invoking Proposition 2.7.2.

Step 1: Wick-Stratonovich corrections. According to relation (2.101) and Proposition 2.7.3 for $k = 2$ we obtain

$$\begin{aligned} \delta^{\circ, 2}(a^n) &= \sum_{\pi_n} \delta^{\circ, 2}(y_{s_i; t_j}^2 1_{[0, s_i] \times [t_j, t_{j+1}]} \otimes 1_{[s_i, s_{i+1}] \times [0, t_j]}) \\ &= \sum_{\pi_n} y_{s_i; t_j}^2 \diamond \delta_2 x_{s_i; t_j t_{j+1}} \diamond \delta_1 x_{s_i s_{i+1}; t_j}. \end{aligned} \quad (2.120)$$

We now use Theorem 4.10 in [52] in order to get that $\delta^{\circ, 2}(a^n)$ can be decomposed as:

$$\begin{aligned} & \sum_{\pi_n} y_{s_i; t_j}^2 (\delta_2 x_{s_i; t_j t_{j+1}} \diamond \delta_1 x_{s_i s_{i+1}; t_j}) - \sum_{\pi_n} y_{s_i; t_j}^3 R_{s_i}^1 (R_{t_j t_{j+1}}^2 - R_{t_j t_j}^2) \delta_1 x_{s_i s_{i+1}; t_j} \\ & - \sum_{\pi_n} y_{s_i; t_j}^3 R_{t_j}^2 (R_{s_i s_{i+1}}^1 - R_{s_i, s_i}^1) \delta_2 x_{s_i; t_j t_{j+1}} \\ & + \sum_{\pi_n} y_{s_i; t_j}^4 R_{s_i}^1 R_{t_j}^2 (R_{s_i s_{i+1}}^1 - R_{s_i, s_i}^1) (R_{t_j t_{j+1}}^2 - R_{t_j t_j}^2) \equiv B_1^n - B_2^n - B_3^n + B_4^n. \end{aligned} \quad (2.121)$$

Like in the proof of Proposition 2.7.8, we treat those 4 terms separately.

Step 2: Estimation of B_1^n, \dots, B_4^n . The term B_1^n can be decomposed as

$$B_1^n = \sum_{\pi_n} y_{s_i t_j}^2 \delta_1 x_{s_i s_{i+1} t_j} \delta_2 x_{s_i t_j t_{j+1}} - \sum_{\pi_n} y_{s_i t_j}^2 (R_{s_i s_{i+1}}^1 - R_{s_i s_i}^1) (R_{t_j t_{j+1}}^2 - R_{t_j t_j}^2).$$

Moreover, the second term in the r.h.s is the same as A_2^n in the proof of Proposition 2.7.8, while the convergence for $\sum_{\pi_n} y_{s_i t_j}^2 \delta_1 x_{s_i s_{i+1} t_j} \delta_2 x_{s_i t_j t_{j+1}}$ follows exactly along the same lines as A_1^n in the same proof. We thus leave to the patient reader the task of showing that

$$\lim_{n \rightarrow \infty} B_1^n = \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{st}^2 d_1 x_{st} d_2 x_{s;t} - \frac{1}{4} \int_{s_1}^{s_2} \int_{t_1}^{t_2} y_{st}^2 d_1 R_s^1 d_2 R_t^2, \quad (2.122)$$

and we concentrate now on the other terms in (2.121).

The term $B_2^n = \sum_{\pi_n} y_{s_i t_j}^3 R_{s_i}^1 (R_{t_j t_{j+1}}^2 - R_{t_j t_j}^2) \delta_1 x_{s_i s_{i+1} t_j}$ can be decomposed as $B_2^n = B_{21}^n + B_{22}^n$, with

$$B_{21}^n = \frac{1}{2} \sum_{\pi_n} y_{s_i t_j}^3 R_{s_i}^1 (R_{t_{j+1}}^2 - R_{t_j}^2) \delta_1 x_{s_i s_{i+1} t_j}, \quad B_{22}^n = \sum_{\pi_n} y_{s_i t_j}^3 R_{s_i}^1 \rho_{t_j t_{j+1}}^2 \delta_1 x_{s_i s_{i+1} t_j},$$

where we recall that we have set $\rho_{t_j t_{j+1}}^k = \frac{1}{2} (2R_{t_j t_{j+1}}^k - R_{t_j t_j}^k - R_{t_{j+1} t_{j+1}}^k)$ for $k = 1, 2$.

It is now easily seen that the almost sure and L^2 convergence of B_2^n are obtained with the same kind of considerations as for A_2^n in the proof of Proposition 2.7.8. We get that

$$\lim_{n \rightarrow \infty} B_2^n = \frac{1}{2} \int_0^s \int_0^t y_{u,v}^3 R_u^1 d_2 R_v^2 d_1 x_{u,v},$$

and B_3^n, B_4^n are also handled in the same way.

Step 3: Conclusion. Thanks to Step 1 and Step 2, we have obtained that $\delta^\circ(a^n)$ converges to the right hand side of relation (2.119) as $n \rightarrow \infty$, in both almost sure and L^2 senses. As mentioned before, this limiting behavior plus the convergence of a^n to $N(y)$ established at Lemma (2.7.9) yield relation (2.119) by a direct application of Proposition (2.7.2). Furthermore, relation (2.107) is also a direct consequence of relation (2.120). □

Notice that our formula (2.119) involves some mixed integrals of the form $\int_0^s \int_0^t y_{u,v}^3 R_v^2 d_1 R_u^1 d_2 x_{u,v}$, which are defined as Young type integrals. The following proposition, whose proof is similar to Propositions (2.7.8) and (2.7.10) and is left to the reader for sake of conciseness, gives a meaning to the analogue integrals in the Skorohod setting.

Proposition 2.7.11. *Let $f \in C^4(\mathbb{R})$ be a function satisfying condition (GC). Then for every fixed $u \in [0, s]$ we have that $v \mapsto y_{u,v}^3 R_v^2 \in \text{Dom}(\delta^{\circ,u})$ where $\delta^{\circ,u}$ is the divergence operator associated to the process $(x_{uv})_{v \in [0,t]}$. We can thus define $\int_0^s \int_0^t y_{uv}^3 R_v^2 d_1 R_u^1 d_2^\circ x_{uv}$ by :*

$$\int_0^s \int_0^t y_{u,v}^3 R_v^2 d_1 R_u^1 d_2^\circ x_{u,v} := \lim_{|\pi| \rightarrow 0} \sum_{\pi_n} \delta^{\circ, s_i} (y_{s_i t_j}^3 1_{[t_j, t_{j+1}]}) R_{t_j}^2 (R_{s_{i+1}}^1 - R_{s_i}^1)$$

where the convergence holds in both $L^2(\Omega)$ and almost sure senses. In addition, we have the following identity:

$$\int_0^s \int_0^t y_{uv}^3 R_v^2 d_1 R_u^1 d_2^\diamond x_{u,v} = \int_0^s \int_0^t y_{uv}^3 R_v^2 d_1 R_u^1 d_2 x_{u,v} - \frac{1}{2} \int_0^s \int_0^t y_{u,v}^4 R_u^1 R_v^2 d_1 R_u^1 d_2 R_v^2.$$

Finally, the integral $\int_0^s \int_0^s y_{uv}^3 R_u^1 d_2 R_v^2 d_1^\diamond x_{uv}$ is defined similarly.

Remark 2.7.12. We have defined all the integrals we needed in order to prove our Skorohod change of variable formula (2.108). Indeed, start from the Stratonovich type formula (2.4.4). The proof of formula (2.108) is now easily deduced by injecting the identities of Propositions 2.7.8, 2.7.10 and 2.7.11 in the equality above.

2.7.4 Skorohod's calculus in the rough case

Our goal in this section is to extend the formulae given in Propositions 2.7.8 and 2.7.10 to rougher situations, namely for Gaussian processes in the plane with Hölder regularities smaller than $1/2$. This is however a harder task than in the Young case, and this is why we introduce 2 simplifications in our considerations:

- (1) Instead of dealing with a general centered Gaussian process whose covariance admits the factorization property of Hypothesis 2.7.4, we handle here the case of a fractional Brownian sheet $(x_{s;t})_{(s,t) \in [0,1]^2}$ with Hurst parameters $\gamma_1, \gamma_2 \in (1/3, 1/2]$.
- (2) The definition of our Skorohod integrals with respect to x is obtained in the following way: we first regularize x as a smooth process x^n . For this process we can still use the formulae of Propositions 2.7.8 and 2.7.10 like in the Young case. We shall then perform a limiting procedure on these formulae (this is where the specification of a concrete approximation is important), which will give our Stratonovich-Skorohod corrections. Notice however that the interpretation in terms of Riemann-Wick sums will be lost with this strategy.

Our main result for this part is the following

Theorem 2.7.13. Assume x is a fractional Brownian sheet on $[0, 1]^2$, with $\gamma_j > 1/3$ for $j = 1, 2$. Then the increments

$$z^{1,\diamond} = \iint y^1 d_{12}^\diamond x, \quad z^{2,\diamond} = \iint y^2 d_1^\diamond x d_2^\diamond x$$

are well defined in the Skorohod sense of Malliavin calculus. Moreover:

- (i) Both $z^{1,\diamond}$ and $z^{2,\diamond}$ can be seen as respective limits of $z^{n,1,\diamond}$ and $z^{n,2,\diamond}$, computed as in Theorem 2.7.5 for the regularized process x^n .
- (ii) For all $f \in C^{10}(\mathbb{R})$, the following change of variables formula

$$\begin{aligned} \delta y_{s;t} = & z^{1,\diamond} + z^{2,\diamond} + 2\gamma_1\gamma_2 \int_1 \int_2 y_{u,v}^2 u^{2\gamma_1-1} v^{2\gamma_2-1} dudv + \gamma_2 \int_1 \int_2 y_{u,v}^3 u^{2\gamma_1} v^{2\gamma_2-1} d_1^\diamond x_{u,v} dv \\ & + \gamma_1 \int_1 \int_2 y_{u,v}^3 u^{2\gamma_1-1} v^{2\gamma_2} d_2^\diamond x_{u,v} du + \gamma_1\gamma_2 \int_1 \int_2 y_{u,v}^4 u^{4\gamma_1-1} v^{4\gamma_2-1} dudv. \end{aligned} \quad (2.123)$$

- (iii) Explicit corrections between z^1, z^2 and $z^{1,\diamond}, z^{2,\diamond}$ can be computed (see relations (2.136) and (2.144)).

As in the previous section, we start our considerations by specifying the Malliavin framework in which we are working.

Malliavin calculus preliminaries

Recall that the covariance function of our fractional Brownian sheet x is given by (2.77). We can thus consider a Hilbert space \mathcal{H}^x related to x exactly as in Section 2.7.2. In particular we denote by I_1^x the isometry between \mathcal{H}^x and the first chaos generated by x .

However, the Malliavin structure related to the harmonizable representation of x will also play a prominent role in the sequel. Namely, it is well known (see e.g. [69]) that for $s, t \in [0, 1]$, x can be represented as

$$x_{s,t} = c_{\gamma_1, \gamma_2} \hat{W}(Q_{s;t}) = c_{\gamma_1, \gamma_2} \int_{\mathbb{R}^2} Q_{s;t}(\xi, \eta) \hat{W}(d\xi, d\eta), \quad (2.124)$$

where c_{γ_1, γ_2} is a normalization constant whose exact value is irrelevant for our computations, W is the Fourier transform of the white noise on \mathbb{R}^2 , and $Q_{s;t}$ is a kernel defined by

$$Q_{s;t}(\xi, \eta) = \frac{e^{is\xi} - 1}{|\xi|^{\gamma_1 + \frac{1}{2}}} \frac{e^{it\eta} - 1}{|\eta|^{\gamma_2 + \frac{1}{2}}}. \quad (2.125)$$

This induces us to consider the canonical Hilbert space related to \hat{W} , that is $\mathcal{H}^{\hat{W}} = L^2(\mathbb{R}^2)$. The relations between Malliavin calculus with respect to \hat{W} and x are then summarized in the next lemma:

Lemma 2.7.14. *Denote by $\mathbb{D}^{x,k,p}$ (resp. $\mathbb{D}^{\hat{W},k,p}$) the Sobolev spaces related to x (resp. \hat{W}), and recall the notation $\mathbb{L}^{1,2} = \mathbb{D}^{\hat{W},1,2}(L^2(\mathbb{R}^2))$ borrowed from [64]. For $\phi : [0, 1]^2 \rightarrow \mathbb{R}$, set*

$$K\phi(\xi, \eta) = \int_{[0,1]^2} \phi_{s;t} \partial_s \partial_t Q_{s;t}(\xi, \eta) ds dt, \quad (2.126)$$

where we recall that Q is defined by (2.125). Then the following holds true:

- (i) We can represent the space \mathcal{H}^x as the closure of the set of step functions under the norm $\|\phi\|_{\mathcal{H}^x} = \|K\phi\|_{L^2(\mathbb{R}^2)}$.
- (ii) We have $\mathbb{D}^{x,1,2}(\mathcal{H}^x) = K^{-1}(\mathbb{L}^{1,2})$. In addition, for any smooth function F and any \mathcal{H}^x -valued square integrable random variable u the following identity holds:

$$\langle u, D^x F \rangle_{\mathcal{H}^x} = \langle Ku, D^{\hat{W}} F \rangle_{L^2(\mathbb{R}^2)}.$$

- (iii) As far as divergence operators are concerned, the relation is

$$\text{Dom}(\delta^{x,\diamond}) = K^{-1} \text{Dom}(\delta^{\hat{W},\diamond}), \quad \text{and} \quad \delta^{x,\diamond}(u) = \delta^{\hat{W},\diamond}(Ku).$$

Proof. Let $\phi = \sum_{i,j} \phi_{i,j} 1_{[s_i, s_{i+1}] \times [t_j, t_{j+1}]}$ be a step function. We have that:

$$\begin{aligned} I_1^x(\phi) &= \sum_{i,j} \phi_{i,j} \delta x_{s_i s_{i+1}; t_j t_{j+1}} = \sum_{i,j} \phi_{i,j} \hat{W}(\delta Q_{s_i s_{i+1} t_j t_{j+1}}) \\ &= \hat{W} \left(\sum_{i,j} \phi_{i,j} \delta Q_{s_i s_{i+1} t_j t_{j+1}} \right) = \hat{W}(K\phi), \end{aligned} \quad (2.127)$$

which easily yields our first claim (i).

Let now F be a smooth functional of x of the form $F = f(x_{s_1;t_1}, \dots, x_{s_n;t_n})$. Then

$$\begin{aligned} \mathbb{E}[\langle u, D^x F \rangle_{\mathcal{H}^x}] &= \mathbb{E}\left[\sum_{l \in \{1, \dots, n\}} \partial_l f(x_{s_1 t_1}, \dots, x_{s_n t_n}) \langle u, 1_{[0, s_l] \times [0, t_l]} \rangle_{\mathcal{H}^x}\right] \\ &= \mathbb{E}\left[\sum_{l \in \{1, \dots, n\}} \partial_l f(x_{s_1 t_1}, \dots, x_{s_n t_n}) \langle Ku, K1_{[0, s_l] \times [0, t_l]} \rangle_{L^2(\mathbb{R})}\right], \end{aligned} \quad (2.128)$$

and since $K1_{[0, s_l] \times [0, t_l]} = Q_{s_l, t_l}$ we end up with

$$\begin{aligned} \mathbb{E}[\langle u, D^x F \rangle_{\mathcal{H}^x}] &= \mathbb{E}\left[\langle Ku, \sum_{l \in \{1, \dots, n\}} \partial_l f(\hat{W}(Q_{s_1 t_1}), \dots, \hat{W}(Q_{s_n t_n})) Q_{s_l, t_l} \rangle_{L^2(\mathbb{R})}\right] \\ &= \mathbb{E}\left[\langle Ku, D^{\hat{W}} F \rangle_{L^2(\mathbb{R})}\right], \end{aligned}$$

which gives our assertion (ii) by density of smooth functionals. Relation (iii) is easily derived from (ii) by duality. \square

Notice that the preceding result can be extended to second order derivatives thanks to a simple tensorization trick. We label here the result for further use:

Lemma 2.7.15. *Under the conditions of Lemma 2.7.14, set*

$$[K^{\otimes 2} \phi](\xi_1 \xi_2; \eta_1 \eta_2) = \int_{[0, 1]^4} \phi_{s_1 s_2; t_1 t_2} \partial_{st} Q_{s_1; t_1}(\xi_1, \eta_1) \partial_{st} Q_{s_2; t_2}(\xi_2, \eta_2) ds_1 ds_2 dt_1 dt_2. \quad (2.129)$$

Then for any smooth functional F and any $(\mathcal{H}^x)^{\otimes 2}$ -valued square integrable random variable u we have:

$$\langle u, D^{2, x} F \rangle_{\mathcal{H}^x} = \langle K^{\otimes 2} u, D^{2, \hat{W}} F \rangle_{L^2(\mathbb{R}^4)}.$$

Embedding results

Similarly to [8], we now give an embedding result for the space \mathcal{H}^x which proves to be useful for further computations.

Lemma 2.7.16. *Let $\gamma_1, \gamma_2 \in (0, \frac{1}{2}]$. Then the following inequality is satisfied:*

$$\begin{aligned} \|u\|_{\mathcal{H}}^2 &\lesssim \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\delta \tilde{u}_{s_1 s_2; t_1 t_2}|^2}{|s_2 - s_1|^{2-2\gamma_1} |t_2 - t_1|^{2-2\gamma_2}} ds_1 dt_1 \right) ds_2 dt_2 \\ &+ \int_{\mathbb{R}^2} \frac{1}{|s_2 - s_1|^{2-2\gamma_1}} \left(\int_0^1 |u_{s_2 t} 1_{[0, 1]}(s_2) - u_{s_1 t} 1_{[0, 1]}(s_1)|^2 dt \right) ds_1 ds_2 \\ &+ \int_{\mathbb{R}^2} \frac{1}{|t_2 - t_1|^{2-2\gamma_2}} \left(\int_0^1 |u_{s t_2} 1_{[0, 1]}(t_2) - u_{s t_1} 1_{[0, 1]}(t_1)|^2 ds \right) dt_1 dt_2 + \int_{[0, 1]^2} |u_{st}|^2 ds dt, \end{aligned} \quad (2.130)$$

where we have set $\tilde{u}_{s;t} = u_{s;t} 1_{[0, 1]^2}(s, t)$.

Proof. In this proof we only consider the case $\gamma_1, \gamma_2 < 1/2$. Indeed, if $\gamma_1 = 1/2$ or $\gamma_2 = 1/2$ then our process x is simply a Brownian motion in the first or in the second direction, and this situation is handled by L^2 norms.

Set now $w_s^\eta \equiv (\int_0^1 u_{s;t} e^{t\eta} dt) 1_{[0,1]}(s)$. Then for $\gamma_1, \gamma_2 < 1/2$, definitions (2.125) and (2.126) entail that:

$$\begin{aligned} \|u\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}^2} |\xi|^{1-2\gamma_1} |\eta|^{1-2\gamma_2} \left| \int_{[0,1]^2} u_{s;t} e^{is\xi + t\eta} ds dt \right|^2 d\xi d\eta \\ &= \int_{\mathbb{R}} |\eta|^{1-2\gamma_2} \left(\int_{\mathbb{R}} |\xi|^{1-2\gamma_1} |\hat{w}_\xi^\eta|^2 d\xi \right) d\eta = \int_{\mathbb{R}} |\eta|^{1-2\gamma_2} \|\hat{w}^\eta\|_{H^{1/2-\gamma_1}}^2 d\eta \end{aligned}$$

where $H^{1/2-\gamma_1}$ stands for the Sobolev space $W^{1/2-\gamma_1,2}$. Now we use the fact that $1/2 - \gamma_1 \in (0, 1/2)$, and recall that the norm defined by

$$\mathcal{N}_{1/2-\gamma_1}^2(\phi) = \int_{\mathbb{R}^2} \frac{|\phi_{s_1} - \phi_{s_2}|^2}{|s_2 - s_1|^{2-2\gamma_1}} ds_1 ds_2 + \int_{\mathbb{R}} |\phi_s|^2 ds$$

for all $\phi \in H^{1/2-\gamma_1}$ is equivalent to the usual norm in $H^{1/2-\gamma_1}$. This yields

$$\begin{aligned} \|u\|_{\mathcal{H}}^2 &\lesssim \int_0^1 \left(\int_{\mathbb{R}} |\eta|^{1-2\gamma_2} \left| \int_0^1 u_{s;t} e^{t\eta} dt \right|^2 d\eta \right) ds \\ &\quad + \int_{\mathbb{R}^2} \frac{1}{|s_2 - s_1|^{2-2\gamma_1}} \left(\int_{\mathbb{R}} |\eta|^{1-2\gamma_2} \left| \int_0^1 (u_{s_2;t} 1_{[0,1]}(s_2) - u_{s_1;t} 1_{[0,1]}(s_1)) e^{t\eta} dt \right|^2 d\eta \right) ds_2 ds_1. \end{aligned}$$

Now it suffices to repeat the same procedure in the second variable in order to obtain inequality (2.130). \square

Let us now recall the notations :

$$\|f\|_{\alpha,1} = \sup_{(s_1, s_2, t) \in [0,1]^2 \times [0,1]} \frac{|\delta_1 f_{s_1 s_2 t}|}{|s_2 - s_1|^\alpha}, \quad \text{and} \quad \|f\|_{\beta,2} = \sup_{(s, t_1, t_2) \in [0,1]^2} \frac{|\delta_2 f_{s t_1 t_2}|}{|t_2 - t_1|^\beta}.$$

With these notations in hand, the following embedding result is easily deduced from Lemma 2.7.16.

Corollary 2.7.17. *Let $\gamma_1, \gamma_2 \in (0, 1/2]$ and $u \in \mathcal{C}_{1,1}^{\alpha_1, \alpha_2}$ such that $0 < \frac{1}{2} - \alpha_i < \gamma_i$. Then we have the following embedding:*

$$\|u\|_{\mathcal{H}} \lesssim \mathcal{N}_{\alpha_1, \alpha_2}(u), \quad \text{where we recall that } \mathcal{N}_{\alpha, \beta}(f) = \|f\|_{\alpha, \beta} + \|f\|_{\alpha, 1} + \|f\|_{\beta, 2} + \|f\|_{\infty}. \quad (2.131)$$

Strategy and preliminary results

The strategy we shall develop in order to extend Proposition 2.7.8 (and also Proposition 2.7.10) to the rough case is based on a regularization of x . Specifically, for a strictly positive integer n , set

$$x_{s;t}^n = c_{\gamma_1, \gamma_2} \int_{|x|, |y| \leq n} Q_{s;t}(\xi, \eta) \hat{W}(d\xi, d\eta), \quad (2.132)$$

where we recall that x and Q are respectively defined by (2.124) and (2.125). For fixed n , it is readily checked that x^n is a regular Gaussian process. Its covariance function is given by $R_{s_1 s_2; t_1 t_2}^n = R_{s_1 s_2}^{1,n} R_{t_1 t_2}^{2,n}$, where

$$R_{ab}^{i,n} = c_{\gamma_i} \int_{|\xi| \leq n} \frac{(e^{ia\xi} - 1)(e^{-ib\xi} - 1)}{|\xi|^{2\gamma_i+1}} d\xi, \quad \text{for } i = 1, 2,$$

and hence R^n is a regular function which satisfies Hypothesis 2.7.4. One can thus apply Proposition 2.7.8 and obtains the following Skorohod-Stratonovich comparison:

$$z_{s_1 s_2 t_1 t_2}^{1,n,\diamond} \equiv \delta^{x^n, \diamond}(y^{n,1} 1_\Delta) = \int_\Delta y_{st}^{n,1} d_{12} x_{st}^n - \frac{1}{4} \int_\Delta y_{st}^{n,2} d_1 R_s^{1,n} d_2 R_t^{2,n}, \quad (2.133)$$

for $f \in C^6(\mathbb{R})$ satisfying condition (GC), and where we set again $\Delta = [s_1, s_2] \times [t_1, t_2]$. Our goal is now to take limits in equation (2.133).

A first observation in this direction is that equation (2.133) involves Skorohod integrals with respect to x^n . The fact that a different integral has to be defined for each n is somehow clumsy, and this is why we have decided to express all integrals with respect to \hat{W} in the remainder of our computations. Namely, the same computations as for equations (2.127) and (2.128) entail that $\delta^{x^n, \diamond}(y^n) = \delta^{\hat{W}, \diamond}(K^n y^n)$, where K^n is the operator defined by

$$K^n \phi(x, y) = 1_{(|x|, |y| \leq n)} \int_{[0,1]^2} \phi_{s;t} \partial_s \partial_t Q_{s;t}(x, y) ds dt. \quad (2.134)$$

With this representation in hand, our limiting procedure can be decomposed as follows:

- Take L^2 limits in the right hand side of equation (2.133) by means of rough paths techniques.
- Show that $K^n y^n$ converges in $L^2(\Omega, L^2(\mathbb{R}))$ to Ky .

Thanks to the closability of $\delta^{\hat{W}, \diamond}$, this will show the convergence of $\delta^{x^n, \diamond}(y^n 1_{[0,1]^2})$ to $\delta^{x, \diamond}(y 1_{[0,1]^2})$ and our Skorohod-Stratonovich correction formula will be obtained in this way.

We now state and prove 3 useful lemmas for our future computations. The first one deals with convergence of covariance functions:

Lemma 2.7.18. *For $i = 1, 2$, set $R_u^i = u^{2\gamma_i}$. Then for all $\varepsilon > 0$ we have*

$$\lim_{n \rightarrow +\infty} \|R^{i,n} - R^i\|_{2\gamma_i - \varepsilon} = 0.$$

Proof. We recall that $c_{\gamma_i} \int_{\mathbb{R}} \frac{|e^{ia\xi} - 1|^2}{|\xi|^{2\gamma_i+1}} d\xi = a^{2\gamma_i}$. Then an elementary computation shows that

$$|\delta_i(R^{i,n} - R^i)_{ab}| = \left| c_{\gamma_i} \int_{|\xi| \geq n} \frac{\cos(a\xi) - \cos(b\xi)}{|\xi|^{2\gamma_i+1}} d\xi \right| \lesssim_{\gamma_i, \varepsilon} |a - b|^{2\gamma_i - \varepsilon} \int_{|x| \geq n} |x|^{-1-\varepsilon} dx,$$

which gives

$$\|R^{n,i} - R^u\|_{2\gamma_i - \varepsilon} \lesssim_{\gamma_i, \varepsilon} \int_{|\xi| \geq n} |\xi|^{-1-\varepsilon} d\xi,$$

and this finishes the proof. □

Itô-Skorohod type formula

We now turn to the limiting procedure in equation (2.133), beginning with the term involving covariances only:

Proposition 2.7.19. *Let $f \in C^6(\mathbb{R})$ be a function satisfying condition (GC) with a small parameter $\lambda > 0$, and x^n be the regularized version of x defined by (2.132). Then the following convergence:*

$$\lim_{n \rightarrow +\infty} \int_{[0,1]^2} y_{s;t}^n d_1 R_s^{1,n} d_2 R_t^{2,n} = \gamma_1 \gamma_2 \int_{[0,1]^2} y_{s;t} s^{2\gamma_1-1} t^{2\gamma_2-1} ds dt \quad (2.135)$$

holds in $L^2(\Omega)$.

Proof. The integrals involved in (2.135) are all of Young type. Owing to Proposition 2.4.5, we thus have:

$$\int_{[0,1]^2} y_{s;t}^n d_1 R_s^{1,n} d_2 R_t^{2,n} = [(\text{Id} - \Lambda_1 \delta_1)(\text{Id} - \Lambda_2 \delta_2)] (y^n \delta_1 R^{1,n} \delta_2 R^{2,n}).$$

By continuity of the sewing map, the desired convergence will thus stem from the relations $\lim_{n \rightarrow 0} A^{1,n} = 0$ and $\lim_{n \rightarrow 0} A^{2,n} = 0$, where for $\epsilon > 0$ we set:

$$A^{1,n} := \sum_{i=1}^2 \|\delta_i R^{i,n} - \delta_i R^i\|_{2\gamma_i - \epsilon}, \quad \text{and} \quad A^{2,n} := \mathcal{N}_{\gamma_1 - \epsilon, \gamma_2 - \epsilon}(y^n - y).$$

Now the relation $\lim_{n \rightarrow 0} A^{1,n} = 0$ is obviously a direct consequence of Lemma 2.7.18. As far as $A^{2,n}$ is concerned, we start from relation (2.76) and apply Hölder's inequality. This yields

$$\begin{aligned} \mathbb{E}[(\mathcal{N}_{\gamma_1 - \epsilon, \gamma_2 - \epsilon}(y^n - y))^2] \\ \lesssim \mathbb{E}^{1/4}[(c_{x^1, x^2})^8] \mathbb{E}^{1/2}[\|x^n - x\|_{\gamma_1 - \epsilon, \gamma_2 - \epsilon}^4] \mathbb{E}^{1/4}[(1 + \mathcal{N}_{\rho_1, \rho_2}(x^n) + \mathcal{N}_{\rho_1, \rho_2}(x))^{16}]. \end{aligned}$$

Then according to Proposition 2.6.8 we see that the r.h.s of this last equation vanishes when n goes to infinity, which proves our claim. \square

We now compute the correction terms in z^1 , that is the equivalent of Proposition 2.7.8.

Proposition 2.7.20. *Let x be a fBs with Hurst parameters $\gamma_1, \gamma_2 > 1/3$. Consider a function $f \in C^{10}(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $y^1 1_\Delta \in \text{Dom}(\delta^\diamond)$, and if we define the increment $z^{1,\diamond} \equiv \delta^\diamond(y^1 1_\Delta)$ the following relation holds true:*

$$z_{s_1 s_2; t_1 t_2}^{1,\diamond} = z_{s_1 s_2; t_1 t_2}^1 - \gamma_1 \gamma_2 \int_{\Delta} y_{s;t}^2 s^{2\gamma_1-1} t^{2\gamma_2-1} ds dt, \quad (2.136)$$

where $z^1 = \iint f'(x) dx$ is the rough integral given by Theorem 2.6.9.

Proof. Let us start from the corrections for the regularized process x^n , for which we can appeal to Proposition 2.7.8. We obtain relation (2.133), written here again for convenience:

$$z_{s_1 s_2; t_1 t_2}^{1,n,\diamond} \equiv \delta^{x^n, \diamond}(y^{n,1} 1_\Delta) = \int_{\Delta} y_{s;t}^{n,1} d_{12} x_{s;t}^n - \frac{1}{4} \int_{\Delta} y_{s;t}^{n,2} d_1 R_s^{1,n} d_2 R_t^{2,n}. \quad (2.137)$$

Now putting together Proposition 2.7.19 and the continuity of the rough integral, we get convergence of the r.h.s of (2.137) in $L^2(\Omega)$. Thus one can write, in the p.s and $L^2(\Omega)$ sense:

$$\lim_{n \rightarrow +\infty} \delta^{x^n, \diamond}(y^n) = \int_{\Delta} y_{s;t}^1 d_{12}x_{s;t} - \gamma_1 \gamma_2 \int_{\Delta} y_{s;t}^2 s^{\gamma_1-1} t^{\gamma_2-1} dt ds,$$

where the integral with respect to x is interpreted in the sense of Theorem 2.6.9.

Let us further analyze the convergence of $\delta^{x^n, \diamond}(y^n)$: recall that this quantity can be written as $\delta^{\hat{W}, \diamond}(K^n y^n)$, where K^n is defined by (2.134) or specifically as

$$(K^n y^n)(\xi, \eta) = \frac{i\xi \eta}{|\xi|^{\gamma_1+1/2} |\eta|^{\gamma_2+1/2}} \left(\int_{\Delta} y_{uv}^n e^{i\xi u + i\eta v} du dv \right) 1_{(|\xi|, |\eta| \leq n)} \quad (2.138)$$

Hence, owing to closability of the operator $\delta^{\hat{W}, \diamond}$, the proof of (2.136) is reduced to show that $K^n y^{n,1}$ converges in $L^2(\Omega; L^2(\Delta))$ to Ky^1 . Now expression (2.138) easily entails that

$$\begin{aligned} \|K^n y^{n,1} - Ky^1\|_{L^2(\mathbb{R})} &\leq \|y^{n,1} - y^1\|_{\mathcal{H}} \\ &\quad + \int_{|\xi|, |\eta| \geq n} |\xi|^{1-2\gamma_1} |\eta|^{1-2\gamma_2} \left| \int_{\Delta} y_{u,v}^1 e^{i\xi u + i\eta v} du dv \right|^2 d\xi d\eta, \end{aligned}$$

and we shall bound the 2 terms on the r.h.s of this inequality.

Indeed, on the one hand we consider $\gamma_1, \gamma_2 > 1/4$ and $\epsilon > 0$ small enough. This gives

$$\begin{aligned} \mathbb{E} \left[\int_{\|(\xi, \eta)\|_{\infty} \geq n} |\xi|^{1-2\gamma_1} |\eta|^{1-2\gamma_2} \left| \int_{\Delta} y_{u,v} e^{i\xi u + i\eta v} du dv \right|^2 d\xi d\eta \right] \\ \lesssim n^{-\epsilon} \mathbb{E} [(\mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(y))^2] \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

On the other hand, Corollary 2.7.17 asserts that $\|y - y^n\|_{\mathcal{H}} \lesssim \mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(y^n - y)$, and the r.h.s of this relation vanishes as $n \rightarrow \infty$ thanks to Proposition 2.6.8. This concludes our proof. \square

In order to complete our comparison between Itô and Stratonovich formulae, we still have to compare the Skorohod type increment $z^{2, \diamond}$ and the rough integral z^2 . As a previous step, let us give an intermediate result concerning some mixed integrals in R, x :

Proposition 2.7.21. *Let $f \in C^{10}(\mathbb{R})$ and recall that for the fractional Brownian sheet x we have $R_u^i = u^{2\gamma_i}$ for $i = 1, 2$. Then the integral*

$$\int y^1 R^1 d_1 x d_2 R^2 = [(Id - \Lambda_1 \delta_1)(Id - \Lambda_2 \delta_2)](y^1 R^1 \delta_1 x \delta_2 R^2 + 1/2 y^2 R^1 (\delta_1 x)^2 \delta_2 R^2) \quad (2.139)$$

is well defined a.s, in the sense of Corollary (2.3.3). Moreover the following convergence takes place in $L^2(\Omega)$:

$$\lim_{n \rightarrow +\infty} \int_{\Delta} y_{uv}^{1,n} R_u^{1,n} d_2 x_{uv}^n d_2 R_v^{2,n} = \int_{\Delta} y_{u,v}^1 R_u^1 d_2 x_{u,v} d_2 R_v^2. \quad (2.140)$$

Finally, the same kind of result is still verified when one interchanges directions 1 and 2 in relation (2.139).

Proof. Let us first check that the integral in (2.139) is well-defined in the sense of Proposition 2.4.5. To this aim, set $A = y^1 R^1 \delta_1 x \delta_2 R^2 + 1/2 y^2 R^1 (\delta_1 x)^2 \delta_2 R^2$. Then a simple computations yields $\delta_1 A = z \delta_2 R^2$, with

$$z = (y^1 \delta_1 x - \delta_1 y) R^1 \delta_1 x - 1/2 \delta_1 (y^2 R^1) (\delta_1 x)^2 - y^1 \delta_1 R^1 \delta_1 x.$$

It is thus easily seen that $\delta_1 A \in \mathcal{C}_{3,2}^{3\gamma_1-\epsilon, 2\gamma_2}$ for an arbitrary small ϵ , thanks to the fact that $x \in \mathcal{C}_{1,1}^{\gamma_1-\epsilon, \gamma_2-\epsilon}$ and $1/2 y^2 \delta_1 x - \delta_1 y \in \mathcal{C}_{2,1}^{2\gamma_1-\epsilon, \gamma_2}$ almost surely. Notice that with the same kind of considerations we also have that $\delta_2 A \in \mathcal{C}_{2,3}^{\gamma_1, 3\gamma_2-\epsilon}$.

Let us now compute δA : we have

$$\delta A = -\delta_2 z \delta_2 R^2 = -(A_1 + A_2) \delta_2 R^2,$$

where

$$A_1 = g R^1 \delta_1 x, \quad \text{with} \quad g = \delta y^1 - \delta_2 y^2 \delta_1 x - y^2 \delta x,$$

and where setting $(\delta_1 x \circ_1 \delta x)_{s_1 s_2; t_1 t_2} = \delta_1 x_{s_1 s_2 t_1} \delta x_{s_1 s_2; t_1 t_2}$, we have

$$A_2 = \delta_2 y^1 \delta_1 R^1 \delta_1 x - \{\delta_1 y^1 - 1/2 y^2 \delta_1 x\} R^1 \delta x - 1/2 \delta y^2 (\delta_1 x)^2 - 1/2 \delta_1 y^2 \{\delta x \circ_1 \delta_1 x + \delta_1 x \circ_1 \delta x\}.$$

The reader can now easily check that $A_2 \in \mathcal{C}_{2,2}^{3\gamma_1-\epsilon, \gamma_2-\epsilon}$. In order to check the regularity of A_1 , observe that g is of the form $g = \delta_2 h$, with

$$\begin{aligned} h_{s_1 s_2; t} &:= (\delta_1 y_{s_1 s_2; t}^1 - y_{s_1; t}^2) \delta_1 x_{s_2 s_2; t} \\ &= \left(\int_0^1 d\theta \theta \int_0^1 d\theta' y^2(x_{s_1; t} + \theta\theta' \delta_1 x_{s_1 s_2; t}) \right) (\delta_1 x_{s_1 s_2; t})^2. \end{aligned} \quad (2.141)$$

Computing $\delta_2 h$ with formula (2.141), one obtains that $A^1 = \delta_2 h R^1 \delta_1 x \in \mathcal{C}_{2,2}^{3\gamma_1-\epsilon, \gamma_2-\epsilon}$.

Let us summarize our last considerations: we have seen that both A_1 and A_2 lye into $\mathcal{C}_{2,2}^{3\gamma_1-\epsilon, \gamma_2-\epsilon}$, and recalling that $\delta A = -(A_1 + A_2) \delta_2 R^2$, we obtain $\delta A \in \mathcal{C}_{3,3}^{3\gamma_1-\epsilon, 3\gamma_2-\epsilon}$. We have also checked that $\delta_1 A \in \mathcal{C}_{3,2}^{3\gamma_1-\epsilon, 2\gamma_2}$ and $\delta_2 A \in \mathcal{C}_{2,3}^{2\gamma_1, 3\gamma_2-\epsilon}$. Gathering all this information, we have checked the assumptions of Corollary 2.3.3 for the increment A , which justifies expression (2.139).

Now we focus on the convergence formula (2.140). We start by observing that for all $n \geq 1$ the following representation holds true:

$$\int_{\Delta} y_{uv}^{n,1} R_u^{1,n} d_2 x_{uv}^n d_2 R_v^{2,n} = [(\text{Id} - \Lambda_1 \delta_1)(\text{Id} - \Lambda_2 \delta_2)] A^n$$

with $A^n = y^{n,1} R^{1,n} \delta_2 R^{2,n} + 1/2 y^{n,1} R^{1,n} (\delta_1 x^n)^2 \delta_2 R^{2,n}$. Hence, owing to the continuity of the planar sewing-maps $(\Lambda_i)_{i=1,2}$ and Λ , our claim (2.140) is reduced to prove that the sequences $\|A^n - A\|_{\gamma_1-\epsilon, 2\gamma_2-\epsilon}$, $\|\delta_1(A^n - A)\|_{3\gamma_1-\epsilon, 2\gamma_2-\epsilon}$, $\|\delta_2(A^n - A)\|_{\gamma_1-\epsilon, 3\gamma_2-\epsilon}$ and $\|\delta(A^n - A)\|_{3\gamma_1-\epsilon, 3\gamma_2-\epsilon}$ converge in $L^2(\Omega)$ and almost surely to 0. Furthermore, it is readily checked that those convergences all stem from the relations

$$\lim_{n \rightarrow 0} \|R^{i,n} - R\|_{2\gamma_i-\epsilon} + \sum_{i=0}^5 \mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(y^{n,i} - y^i) + \|(\delta_1 x^n)^2 - (\delta_1 x)^2\|_{2\gamma_1-\epsilon, 1} = 0 \quad (2.142)$$

and

$$\lim_{n \rightarrow 0} \|\delta_2(h - h^n)\|_{2\gamma_1 - \epsilon, \gamma_2 - \epsilon} = 0, \quad \text{with } h^n = \delta_1 y^n - y^{n,1} \delta_1 x^n, \quad (2.143)$$

where the limits take place in some $L^p(\Omega)$ with a sufficiently large p , and where we recall that h is defined by (2.141). We now turn to the proof of those two relations.

To begin with, note that the convergence (2.142) has already been established in Lemma 2.7.18. In order to prove (2.143), we invoke again the integral representation (2.141) for both h^n and h . Then some elementary considerations (omitted here for sake of conciseness) allow to reduce the problem to the following relation:

$$L^p(\Omega) - \lim_{n \rightarrow \infty} \left(\mathbb{E}[\mathcal{N}_{\gamma_1 - \epsilon, \gamma_1 - \epsilon}^p(x^n - x)] + \sum_{i=0}^3 \mathbb{E}[\mathcal{N}_{\gamma_1 - \epsilon, \gamma_2 - \epsilon}^p(y^{n,i} - y^i)] \right) = 0.$$

This last relation is a direct consequence of Proposition 2.6.8 and composition with non linearities, whenever f satisfies the growth condition (GC) with a small parameter $\lambda > 0$. The proof is now finished. \square

We can now state our result concerning the Itô-Stratonovich correction for the mixed stochastic integral $\iint y^2 d_1^\diamond x d_2^\diamond x$:

Theorem 2.7.22. *Let x be a fBs with Hurst parameters $\gamma_1, \gamma_2 > 1/3$. Consider a function $f \in C^{10}(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $N(y^2) \in \text{Dom}(\delta^{\diamond, 2})$, and if we define the Skorohod integral $z^{2, \diamond}$ as $\delta^{\diamond, 2}(N(y))$, the following particular case of relation (2.119) holds:*

$$\begin{aligned} z_{s_1 s_2; t_1 t_2}^{2, \diamond} &= z_{s_1 s_2; t_1 t_2}^2 - \gamma_1 \gamma_2 \int_{\Delta} y_{u;v}^2 u^{2\gamma_1 - 1} v^{2\gamma_2 - 1} dudv - \gamma_2 \int_{\Delta} y_{u;v}^3 u^{2\gamma_1} v^{2\gamma_2 - 1} dvd_1 x_{u;v} \\ &\quad - \gamma_1 \int_{\Delta} y_{u;v}^3 u^{2\gamma_1 - 1} v^{2\gamma_2} dud_2 x_{u;v} + \gamma_1 \gamma_2 \int_{\Delta} y_{u;v}^4 u^{4\gamma_1 - 1} v^{4\gamma_2 - 1} dudv. \end{aligned} \quad (2.144)$$

Proof. We follow the same strategy as for Theorem 2.7.20: apply first Proposition 2.7.10 for the regularized process x^n , which yields:

$$\begin{aligned} \int_{\Delta} y_{u;v}^{n,2} d_1^\diamond x_{uv}^n d_2^\diamond x_{uv}^n &= \int_{\Delta} y_{u;v}^{n,2} d_1 x_{u;v}^n d_2 x_{uv}^n - 1/4 \int_{\Delta} y_{u;v}^{n,2} d_1 R_u^{1,n} d_2 R_v^{2,n} \\ &\quad - 1/2 \int_{\Delta} y_{u;v}^{n,3} R_u^{1,n} d_2 R_v^{2,n} d_1 x_{u;v} - 1/2 \int_{\Delta} y_{u;v}^{n,3} R_v^{2,n} d_1 R_u^{1,n} d_2 x_{uv}^n \\ &\quad + 1/4 \int_{\Delta} y_{u;v}^{n,4} R_u^{1,n} R_v^{2,n} d_1 R_u^{1,n} d_2 R_v^{2,n}. \end{aligned} \quad (2.145)$$

Now our preliminary results allow to take limits in relation (2.145). Indeed, owing to Propositions 2.7.19 and 2.7.21 plus the continuity of the rough increment z^2 given at the Proposition (2.6.9), we obtain the convergence in $L^2(\Omega)$ for the four first terms in the r.h.s of equation (2.145). Moreover the last term also converges in $L^2(\Omega)$, thanks to the same arguments as in the proof of the Proposition 2.7.19. We thus get the convergence of the r.h.s of equation (2.145) to the r.h.s of equation (2.144), and also the fact that $z^{2, \diamond}$ converges in $L^2(\Omega)$. Like in the proof of Theorem 2.7.20, the proof of (2.144) is thus reduced to show the L^2 convergence of the integrand defining $z^{2, \diamond}$.

However, mimicking again the proof of Theorem 2.7.20, it is easily seen that

$$\int_0^1 \int_0^1 y_{u;v}^{n,2} d_1^\diamond x_{uv}^n d_2^\diamond x_{uv}^n := \delta^{x^n, 2, \diamond} (N(y^{n,2})) = \delta^{\hat{W}, 2, \diamond} (K^{n, \otimes 2} N(y^{n,2})),$$

where we recall that $K^{\otimes 2}$ is defined by (2.129) and

$$[K^{n, \otimes 2} \phi](x_1 x_2; y_1 y_2) = 1_{\{|x_1|, |x_2|, |y_1|, |y_2| \leq n\}} [K^{\otimes 2} \phi](x_1 x_2; y_1 y_2).$$

It thus remains to show that $K^{n, \otimes 2}(N(y^{n,2}))$ converges to $K^{\otimes 2}(N(y^2))$ in $L^2(\Omega, L^2(\mathbb{R}^4))$. Towards this aim, we introduce the further notation $u_{s;t}(\xi, \eta) = y_{s;t}^{n,2}(e^{is\xi} - 1)(e^{it\eta} - 1)$, $u_{s;t}^n(\xi, \eta) = y_{s;t}^{n,2}(e^{is\xi} - 1)(e^{it\eta} - 1)$ and

$$\begin{aligned} \hat{u}(\xi_1, \xi_2; \eta_1, \eta_2) &= \int_{\Delta} u_{s;t}(\xi_1, \eta_1) e^{is\xi_2 + it\eta_2} ds dt \\ \hat{u}^n(\xi_1, \xi_2; \eta_1, \eta_2) &= \int_{\Delta} u_{s;t}^n(\xi_1, \eta_1) e^{is\xi_2 + it\eta_2} ds dt. \end{aligned}$$

Then note that

$$\|(K^n)^{\otimes 2}(N(y^n)) - K^{\otimes 2}(N(y))\|_{L^2(\mathbb{R}^4)}^2 \leq I_1^n + I_2^n + I_3^n,$$

where

$$\begin{aligned} I_1^n &= \int_{|\xi_1|, |\eta_1| \geq n} \left(\int_{\mathbb{R}^2} |\xi_2|^{1-2\gamma_1} |\eta_2|^{1-2\gamma_2} |\hat{u}(\xi_1, \xi_2; \eta_1, \eta_2)|^2 d\xi_2 d\eta_2 \right) \frac{d\xi_1 d\eta_1}{|\xi_1|^{2\gamma_1+1} |\eta_1|^{2\gamma_2+1}} \\ I_2^n &= \int_{\mathbb{R}^2} \left(\int_{|\xi_2|, |\eta_2| \geq n} |\xi_2|^{1-2\gamma_1} |\eta_2|^{1-2\gamma_2} |\hat{u}(\xi_1, \xi_2; \eta_1, \eta_2)|^2 d\xi_2 d\eta_2 \right) \frac{d\xi_1 d\eta_1}{|\xi_1|^{2\gamma_1+1} |\eta_1|^{2\gamma_2+1}} \end{aligned}$$

and

$$\begin{aligned} I_3^n &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |\xi_2|^{1-2\gamma_1} |\eta_2|^{1-2\gamma_2} |\hat{u}(\xi_1, \xi_2; \eta_1, \eta_2) - \hat{u}^n(\xi_1, \xi_2; \eta_1, \eta_2)|^2 d\xi_2 d\eta_2 \right) \\ &\quad \times \frac{d\xi_1 d\eta_1}{|\xi_1|^{2\gamma_1+1} |\eta_1|^{2\gamma_2+1}}. \end{aligned}$$

In order to bound those 3 terms, observe that

$$\mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(u(\xi, \eta)) \lesssim \mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(y)(1 + |\xi|^{\gamma_1-\epsilon} + |\eta|^{\gamma_2-\epsilon} + |\xi|^{\gamma_1-\epsilon} |\eta|^{\gamma_2-\epsilon})$$

and thus Corollary 2.7.17 entails that

$$\mathbb{E}[I_1^n] \lesssim \mathbb{E}[\mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}^2(y^2)] \int_{|\xi|, |\eta| \geq n} \frac{1}{|\xi|^{\epsilon+1} |\eta|^{\epsilon+1}} d\xi d\eta \xrightarrow{n \rightarrow +\infty} 0,$$

and

$$\mathbb{E}[I_2^n] \lesssim n^{-\epsilon} \mathbb{E}[\mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}^2(y^2)] \xrightarrow{n \rightarrow +\infty} 0.$$

As far as I_3^n is concerned, we remark that

$$\mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(u(\xi, \eta) - u^n(\xi, \eta)) \lesssim \mathcal{N}_{\gamma_1-\epsilon, \gamma_2-\epsilon}(y^2 - y^{n,2})(1 + |\xi|^{\gamma_1-\epsilon} + |\eta|^{\gamma_2-\epsilon} + |\xi|^{\gamma_1-\epsilon} |\eta|^{\gamma_2-\epsilon})$$

and then we can conclude along the same lines as in Theorem 2.7.22 that $\mathbb{E}[I_3^n]$ vanishes as n goes to infinity. This finishes the proof. \square

Now to obtain the Theorem (2.7.13) we only need the following well known one parameters result

Proposition 2.7.23. *Let B a fractional brownian motion with hurst parameters $1/2 \geq \gamma > 1/3$ then we have that $u \mapsto f(B_u)u^{2\gamma} \in \text{Dom}(\delta^{\diamond, B})$ and we have that*

$$\int_{[0,1]} \varphi(B_u)u^{2\gamma} d^{\diamond} B_u = \int_{[0,1]} \varphi(B_u)u^{2\gamma} dB_u - \gamma \int_{[0,1]} \varphi'(B_u)u^{4\gamma-1} du$$

Proof. Use exactly the same arguments of the Proposition (2.7.20) for the one parameters setting. \square

And then we have immediately the following result

Corollary 2.7.24. *For $\gamma_i > 1/3$ and $\varphi \in C^6(\mathbb{R})$ then for every $v \in [0, 1]$ $u \mapsto y_{u,v}^3 u^{2\gamma_1} \in \text{Dom}(\delta^{\diamond, x_{\cdot, v}})$ and the following formula hold true*

$$\int_{\Delta} y_{u,v}^3 u^{2\gamma_1} v^{2\gamma_2-1} d_1^{\diamond} x_{u,v} dv = \int_{\delta} y_{u,v}^3 u^{2\gamma_1} v^{2\gamma_2-1} d_1 x_{u,v} d_2 v - \gamma_1 \int_{\Delta} y_{u,v}^4 u^{4\gamma_1-1} v^{4\gamma_2-1} dudv$$

Proof. we recall that $x_{u,v} \stackrel{\text{law}}{=} v^{\gamma_2} B_u$ with B is a fBm with hurst parameter γ_1 and then it suffice to use the Proposition (2.7.23) \square

Chapter 3

Modulated Dispersive PDEs

Résumé

Dans ce chapitre on étudie une large classe d'EDP qui présente une modulation temporelle non homogène dans le terme de dispersion. En particulier un exemple est celui de l'équation de Korteweg–de Vries (KdV), l'équation de KdV modifier, L'équation de Schrödinger non linéaire (NLS) ou encore l'équation de Schrödinger non linéaire avec dérivée. On introduit pour cela une notion d'"irrégularité" pour la modulation et on obtient des résultats d'existence locale et globale qui sont similaires à ceux obtenue pour ces équation sans modulation. Dans certain cas on remarquera que la présence de modulations irrégulières entrainera une amélioration dans la théorie d'existence de solutions pour ces équations. Une première approche est basé sur l'effet régularisant de la modulation sur le terme non linéaire en utilisant la théorie des chemins contrôler et des estimations provenant de la théorie de l'intégral d'Young. Une seconde approche est une extension des estimée de Strichartz obtenue par Debussche et Tsutsumi dans le cas d'une modulation Brownienne pour l'équation NLS quintique.

Abstract

We study various nonlinear PDEs under the effect of a time–inhomogeneous and irregular modulation of the dispersive term. In particular the modulated 1d periodic or non-periodic versions of the Korteweg–de Vries (KdV) equation, of the modified KdV equation, of the non-linear Schrödinger equation (NLS) and of the derivative NLS. We introduce a deterministic notion of "irregularity" for the modulation and obtain local and global results similar to those valid without modulation. In some cases the irregularity of the modulation improves the well-posedness theory of the equations. A first approach is based on novel estimates for the regularising effect of the modulated dispersion on the non-linear term using the theory of controlled paths and estimates stemming from Young's theory of integration. A second approach is an extension of a Strichartz estimated first obtained by Debussche and Tsutsumi in the case of the Brownian modulation for the quintic NLS.

3.1 Introduction

In this chapter we consider nonlinear PDEs of the form

$$\frac{d}{dt}\varphi_t = A\varphi_t \frac{dw_t}{dt} + \mathcal{N}(\varphi_t), \quad t \geq 0 \tag{3.1}$$

where $w : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an arbitrary continuous function, A is an unbounded linear operator and \mathcal{N} some nonlinear function. The situation we have in mind is where A is a dispersive operator like the Schrödinger operator $i\partial^2$ or the Airy operator ∂^3 acting on periodic or non-periodic functions on \mathbb{R}^n and where \mathcal{N} is some polynomial non-linearity with possibly derivative terms. Our analysis will be mainly devoted to the following cases:

1. (KdV) Korteweg-de Vries equation in \mathbb{T} or \mathbb{R} , $A = \partial^3$, $\mathcal{N}(\phi) = \partial\phi^2$;
2. (NLS) Non-linear cubic Schrödinger equation in $\mathbb{T}^n, \mathbb{R}^n$, $n = 1, 2$, $A = i\partial^2$, $\mathcal{N}(\phi) = i|\phi|^2\phi$;
3. (mKdV) Modified Korteweg-de Vries equation in \mathbb{T} , $A = \partial^3$, $\mathcal{N}(\phi) = \partial(\phi^2 - 3\|\phi\|_{H^0}^2)\phi$;
4. (dNLS) Non-linear (Wick-ordered) derivative cubic Schrödinger equation in \mathbb{T} , $A = i\partial^2$, $\mathcal{N}(\phi) = i\partial^\theta(|\phi|^2 - \|\phi\|_{H^0}^2)\phi$ with $\theta > 0$;

in all these cases the Banach space V will be taken as belonging to the scale of Sobolev spaces H^α , $\alpha \in \mathbb{R}$ defined as the completion of smooth functions with respect to the norm

$$|\phi|_\alpha = \|\phi\|_{H^\alpha} = \|\langle \xi \rangle^\alpha \hat{\phi}(\xi)\|_{L^2(\mathbb{R}^n)} \tag{3.2}$$

where $\hat{\phi}$ is the Fourier transform of $\phi : \mathbb{R}^n \rightarrow \mathcal{C}$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Similar definition holds in the periodic case where \mathbb{R}^n is replaced by \mathbb{T}^n with $\mathbb{T} = [0, 2\pi[$ with periodic boundary conditions.

The (randomly) modulated NLS equation has been subject of interest in recent literature (for example [1, 28, 30, 48, 49, 56, 62, 76]), especially due to the applications to soliton management in optical wave-guides. The authors do not know of any relevant application of the other models, apart from the work of Clarke et al. [16] on dispersion management for KdV.

Aside of specific applications we are motivated by the general problem of understanding the properties of PDEs in non-homogeneous environments and what can be expected as far as "generic" properties of the equation are concerned. Modulated equations rule out classical techniques of Fourier analysis (e.g. Bourgain spaces in the case of KdV) and other important tools like Strichartz estimates. Many conservation laws are also not available in the modulated context affecting the analysis of global solutions.

Another of our motivations has been the study of the regularisation effect of a non-homogeneous time modulation in the spirit of the recent work of Flandoli, Priola and one of the authors [31] on the stochastic transport equation.

Eq. (3.1) is only formal since the derivative of w does not exist in general. If w is a Brownian motion then the differential equation can be understood via stochastic calculus. Interpreting the differential in Stratonovich sense seems the most natural choice in this context since it preserves the mild formulation of the equation (see below). De Bouard and Debussche [28] study the Nonlinear Schrödinger equation with Brownian modulation and they show that it describes the homogenisation of the deterministic Nonlinear Schrödinger Equation with time dependent dispersion satisfying some ergodicity properties. In the more general situation the interpretation of eq. (3.1) as an Itô or Stratonovich SPDE is not

possible and we prefer to describe solutions via a mild formulation. If we denote by $(e^{tA})_{t \in \mathbb{R}}$ the group of isometries of $V = H^\alpha$ generated by A , the mild solution of eq. (3.1) is formally given by

$$\varphi_t = U_t^w \varphi_0 + U_t^w \int_0^t (U_s^w)^{-1} \mathcal{N}(\varphi_s) ds \quad (3.3)$$

where $U_t^w = e^{Aw_t}$ is the operator obtained by a time-change of the linear evolution associated to A using the function w . In this form the equation make sense for arbitrary continuous function w .

The aim of this paper is to analyse eq. (3.3) under some hypothesis on the "irregularity" of the perturbation w . In particular if w is sufficiently irregular (in a precise sense to be specified below) then we will be able to show that the above nonlinear PDE can be solved in spaces which are comparable to those allowed by the unmodulated equation

$$\frac{d}{dt} \varphi_t = A\varphi_t + \mathcal{N}(\varphi_t), \quad t \geq 0 \quad (3.4)$$

and that in some situations the combination of the irregularity of the perturbation and the non-linear interaction provides a strong regularizing effect on the equation.

Let us now be more specific about the kind of solutions we are looking for. Let $\Pi_N : H^\alpha \rightarrow H^\alpha$ be the projector over Fourier modes $|\xi| \leq N$: $\widehat{\Pi_N f}(\xi) = \mathbb{1}_{|\xi| \leq N} \hat{f}(\xi)$ where \hat{f} denotes the Fourier transform of $f \in H^\alpha$ and let $\mathcal{N}_N(\phi) = \Pi_N \mathcal{N}(\Pi_N \phi)$ be the Galerkin regularization of the non-linearity.

Definition 3.1.1. *The function $\varphi \in C(\mathbb{R}_+; V)$ is a local solution to (3.3) in V with initial condition $\phi \in V$ if there exists $T > 0$ and such that*

$$\lim_{N \rightarrow \infty} \int_0^t (U_s^w)^{-1} \mathcal{N}_N(\varphi_s) ds = Q_t(\varphi)$$

exists in V for any $t \in [0, T]$ and the equality

$$\varphi_t = (U_t^w)[\phi + Q_t(\varphi)]$$

holds in V for any $t \in [0, T]$. We say that the solution is global if the equality holds of any $t \geq 0$.

Whenever the limit exists we write

$$\lim_{N \rightarrow \infty} \int_0^t (U_s^w)^{-1} \mathcal{N}_N(\varphi_s) ds = \int_0^t (U_s^w)^{-1} \mathcal{N}(\varphi_s) ds.$$

It should be noted that the quantity on the r.h.s. is not a usual integral but only a convenient notation for the limit procedure. Indeed $\mathcal{N}(\varphi_s)$ will exist only as a space-time distribution and not as a continuous function with values in V .

The next definition concerns the particular notion of "irregularity" of the perturbation that will be relevant in our analysis.

Definition 3.1.2. *Let $\rho > 0$ and $\gamma > 0$. We say that a function $w \in C([0, T]; \mathbb{R})$ is (ρ, γ) -irregular if for any $T > 0$:*

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} = \sup_{a \in \mathbb{R}} \sup_{0 \leq s < t \leq T} \langle a \rangle^\rho \frac{|\Phi_{s,t}^w(a)|}{|s-t|^\gamma} < +\infty$$

where $\Phi_{s,t}^w(a) = \int_s^t e^{iaw_r} dr$. Moreover we say that w is ρ -irregular if there exists $\gamma > 1/2$ such that w is (ρ, γ) -irregular.

As it is apparent from this definition the notion of irregularity that we need is related to the *occupation measure* of the function w (see for example the review of Geman and Horowitz on occupation densities for deterministic and random processes [34]), in particular to the decay of its Fourier transform at large wave-vectors as measured by the exponent ρ . The time regularity of this Fourier transform, measured by the Hölder exponent γ , will also play an important rôle.

Existence of (plenty of) perturbations w which are ρ -irregular is guaranteed by

Theorem 3.1.3. *Let $(W_t)_{t \geq 0}$ be a fractional Brownian motion of Hurst index $H \in (0, 1)$ then for any $\rho < 1/2H$ there exist $\gamma > 1/2$ so that with probability one the sample paths of W are (ρ, γ) -irregular.*

In particular there exists continuous paths which are ρ -irregular for arbitrarily large ρ . Using well known properties of support of the law of the fractional Brownian motion it is also possible to show that there exists ρ -irregular trajectories which are arbitrarily close in the supremum norm to any smooth path. It would be interesting to study more deeply the irregularity of continuous paths “generically”.

In our opinion an important general contribution of our work is the observation that the regularity of the occupation measure of w seems to play a major rôle in the understanding of the regularizing properties of w in a non-linear context and it would be desirable to understand more deeply the link of the notion of ρ -irregularity with the path-wise properties of w .

Apart from the classic contribution of Geman and Horowitz [34], the authors are not aware of any systematic study of occupation measures from the point of view of their action on spaces of functions, topic which is central to our analysis. Let us explain this better: let

$$T_{s,t}^w f(x) = \int_s^t f(x + w_r) dr$$

for measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Then $T_{s,t}^w(e^{ia \cdot})(x) = \Phi_{s,t}^w(a)e^{iax}$ which shows for example that if w is (ρ, γ) -irregular then

$$\|T_{s,t}^w f\|_{H^\rho(\mathbb{R})} \lesssim |t - s|^\gamma \|f\|_{H^0(\mathbb{R})}$$

meaning that T is a regularizing operator. This point of view links our research to the topic of improving bounds for averages along curves (see for example the paper of Tao and Wright [72]). Inspired by the work of Davie [22] on pathwise uniqueness for SDEs, Catellier and Gubinelli [12] provide some analysis of the regularizing properties of random paths but much is still not very well understood.

An open problem is, for example, what happens if we replace w with a regularised version w^ε or with a function which could depend on the solution itself. In this respect we conjecture that if w is (ρ, γ) -irregular then for any smooth function φ the perturbed path $w^\varphi = w + \varphi$ is still (ρ, γ) -irregular but we are only able to prove this in the specific situation where w is a fractional Brownian motion and φ is a deterministic perturbation, or more generally but with a loss of $1/2$ in the ρ irregularity of w^φ : both results (with precise statements) are obtained in [12]. In the case of a smooth w we have the following straightforward result:

Proposition 3.1.4. *Let $w : [0, T] \rightarrow \mathbb{R}$ a twice differentiable function such that $c_T = \inf_{t \in [0, T]} |w'_t| > 0$ for any $T > 0$ and $\frac{w''}{(w')^2} \in L^1_{loc}(0, +\infty)$ then w is $(1 - \gamma, \gamma)$ irregular for all $\gamma \in (0, 1)$.*

Proof. Integration by parts gives

$$ia\Phi_{st}^w(a) = \frac{e^{iaw_t} - e^{iaw_s}}{w'_t} + \int_s^t (e^{iaw_\sigma} - e^{iaw_s}) \frac{w''_\sigma}{(w'_\sigma)^2} d\sigma$$

and the result follow immediately from the hypothesis. □

To deal with irregular modulations in the sense of Def. 3.1.2 we develop two different techniques:

1. The first uses the controlled path approach and Young's integral and it is inspired by the work of one of the authors [37] where the periodic KdV equation in negative Sobolev spaces (and more general Fourier-Lebesgue spaces) is studied without relying on Bourgain spaces and the time-homogeneity of the equation. This work has connection to the normal form analysis of Babin, Ilyin and Titi of the same equation [3].
2. The second is based on a novel deterministic Strichartz estimate for the modulated linear equation which is a generalization of the probabilistic results of Debussche and Tsutsumi [30].

Let us summarise the main contributions of this paper, all along which we are going to make the following basic assumption:

Hypothesis 3.1.5. *The function w is (ρ, γ) -irregular for some $\rho > 0$ and $\gamma > 1/2$.*

Our first result is about the modulated Korteweg-de Vries (KdV) equation.

Theorem 3.1.6. *For any $\rho > 3/4$ and $\alpha > -\rho$ the 1d periodic modulated KdV equation has a local solution in $H^\alpha(\mathbb{T})$. The solution is global if $\alpha \geq -3/2$ and $\alpha > -\rho/(3 - 2\gamma)$. Uniqueness holds in the space $\mathcal{D}^w(H^\alpha) \subseteq C(\mathbb{R}_+; H^\alpha(\mathbb{T}))$ introduced in Def. 3.2.2 below. In the non-periodic setting the 1d modulated KdV equation has local solutions in $H^\alpha(\mathbb{R})$ for $\alpha > -\min(\rho, 3/4)$. The solution is global if $\alpha > -\min(\rho/(3 - 2\gamma), 3/4)$. Uniqueness holds in the same space $\mathcal{D}^w(H^\alpha)$.*

This theorem shows that an irregular modulation provides a *regularisation effect* on the KdV equation. Indeed the unmodulated equation allows for a uniformly continuous flow only if $\alpha \geq -1/2$ in the periodic setting and only if $\alpha \geq -3/4$ in the non-periodic one [17]. Recall that exploiting the complete integrability of the unperturbed model it is possible to show existence of solutions up to $\alpha \geq -1$ [55].

As far as we know there are no existence results for $\alpha < -1$ for the unmodulated equation and since we obtain solutions with standard fixed point methods we have also the existence of continuous flow in situation were it is known to be false for the unmodulated equation.

Ours are the first results of regularization by noise in non-linear dispersive equations with rough initial conditions. It is known that noise can act as to worsen the behavior of the equation, for example blow-up in NLS with multiplicative noise [25, 26].

For the cubic NLS equation we have the following theorem.

Theorem 3.1.7. *Assume that $\rho > 1/2$. Then the modulated cubic NLS equation on \mathbb{T} and \mathbb{R} has a global solution in H^α for any $\alpha \geq 0$. Uniqueness holds in $\mathcal{D}^w(H^\alpha)$ and the flow is locally Lipschitz continuous in $\mathcal{D}^w(H^\alpha)$.*

In the case of Brownian modulation the global solution for $\alpha = 0$ have already been constructed by de Bouard et Debussche [28]. Here we extend their result to any $\alpha \geq 0$ and any sufficiently irregular modulation. Global solutions for any $\alpha \geq 0$ are the result of the L^2 conservation law and some regularity preservation estimates for the non-linear term.

For the other models we considered we obtained the partial results listed in the following theorem.

Theorem 3.1.8. *Assuming $\rho > 1/2$. We have the following results:*

1. *The modulated cubic NLS equation on \mathbb{R}^2 has a unique local solution in H^α if $\alpha \geq 1/2$;*
2. *The modulated dNLS equation on \mathbb{T} has a unique local solution in H^α if $\alpha \geq 1/2$ and $\rho > 1$;*

3. The modulated mKdV equation on \mathbb{T} has a unique local solution in H^α if $\alpha \geq 1/2$.

A key argument in the proof of all these results is the use of explicit computations allowed by the polynomial character of the non-linearity. These results are however limited to modulations irregular enough. Indeed, a bit surprisingly, in the modulated context the application of controlled path techniques is easier if the modulation is very irregular. This has allowed us not to have to deal with second order controlled expansions as has been necessary in [37]. An open problem is to fill the gap between regular and irregular modulations.

A different line of attack to the modulated Schrödinger equation comes from the application of the following Strichartz type estimate which can be proved under the same ρ -irregularity assumption of Hyp. 3.1.5:

Theorem 3.1.9. *Let $A = i\partial_x^2$, $T > 0$, $p \in (2, 5]$, $\rho > \min(\frac{3}{2} - \frac{2}{p}, 1)$ then there exists a finite constant $C_{w,T} > 0$ and $\gamma^*(p) > 0$ such that the following inequality holds:*

$$\left\| \int_0^\cdot U^\cdot(U_s^w)^{-1} \psi_s ds \right\|_{L^p([0,T], L^{2p}(\mathbb{R}))} \leq C_w T^{\gamma^*(p)} \|\psi\|_{L^1([0,T], L^2(\mathbb{R}))}$$

for all $\psi \in L^1([0, T], L^2(\mathbb{R}))$.

As an application we obtain global well-posedness for the modulated NLS equation with generic power nonlinearity ie : $\mathcal{N}(\phi) = |\phi|^\mu \phi$:

Theorem 3.1.10. *Let $\mu \in (1, 4]$, $p = \mu + 1$, $\rho > \min(1, 3/2 - \frac{2}{p})$ and $u^0 \in L^2(\mathbb{R})$ then there exists $T^* > 0$ and a unique $u \in L^p([0, T], L^{2p}(\mathbb{R}))$ such that the following equality holds:*

$$u_t = U_t^w u^0 + i \int_0^t U_t^w (U_s^w)^{-1} (|u_s|^\mu u_s) ds$$

for all $t \in [0, T^*]$. Moreover we have that $\|u_t\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ and then we have a global unique solution $u \in L_{loc}^p([0, +\infty), L^{2p}(\mathbb{R}))$ and $u \in C([0, +\infty), L^2(\mathbb{R}))$. If $u^0 \in H^1(\mathbb{R})$ then $u \in C([0, \infty), H^1(\mathbb{R}))$.

We point out that all our techniques are deterministic and that they provide novel results even in the stochastic context, for example when w is taken to be the sample path of a fractional Brownian motion. In the Brownian case it is not difficult to show that our solutions corresponds to limits of solutions of Stratonovich type SPDEs. Even in the Brownian setting our results on KdV, mKdV and DNLS are, to our knowledge, novel. In the case of NLS we recover the known results of Debusche and Tsustumi adding to that existence of a continuous flow map for the SPDE, result which is usually difficult to obtain in the stochastic framework.

Plan. In Sect. 3.2 we illustrate the controlled path approach to solution to modulated semilinear PDEs. This approach relies on a non-linear generalisation of the Young integral [36, 60, 75] for which we provide complete proofs in Sect. 3.3. Using the non-linear Young integral we define and solve Young-type differential equations in Sect. 3.4. This will provide a general theory for the constructions and approximation of the controlled solutions. In Sect. 3.5 we verify that all our models satisfy the hypothesis to apply the general theory we outlined in the previous section. In Sect. 3.6 we study global solutions in different H^α spaces: above L^2 by seeking suitable preservation of regularity estimates and for KdV below L^2 by an adaptation of the I -method to our context. Finally in Sect. 3.8 we prove the

Strichartz estimate of Thm. 3.1.9 and apply it to the study the modulated NLS equation with general non-linearity without relying on controlled solutions.

Notations. If V, W are two Hilbert spaces we let $\mathcal{L}_n(V, W)$ be the Banach space of bounded operators on $V^{\otimes n}$ (considered with the Hilbert tensor product) with values in W and endowed with the operator norm and set $\mathcal{L}_n(V) = \mathcal{L}_n(V, V)$. We let $T > 0$ denote a fixed time and $\mathcal{C}^\gamma V = C^\gamma([0, T], V)$ the space of γ -Hölder continuous functions form $[0, T]$ to V endowed with the semi-norm

$$\|f\|_{\mathcal{C}^\gamma V} = \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|_V}{|t - s|^\gamma}.$$

If V is a Banach space then $\text{Lip}_M(V)$ will denote the Banach space of locally Lipschitz map on V with polynomial growth of order $M \geq 0$, that is maps $f : V \rightarrow V$ such that

$$\|f\|_{\text{Lip}_M(V)} = \sup_{x, y \in V} \frac{\|f(x) - f(y)\|_V}{\|x - y\|_V (1 + \|x\|_V + \|y\|_V)^M} < +\infty.$$

3.2 Controlled paths

The approach we will use in proving Thms. 3.1.6, 3.1.7 and 3.1.8 is based on ideas coming from the theory of controlled rough paths [36, 66] which have been already used in a variety of contexts:

1. alternative formulation of rough path theory with the related applications to stochastic differential equations and in general to differential equations driven by non-semimartingale noises [29, 40, 42];
2. approximate evolution of three dimensional vortex lines in incompressible fluids where the initial condition is a non-smooth curve [7, 10]
3. study of the stochastic Burgers equation (multi-dimensional target space and various kind of robust approximation results) [45, 66];
4. definition of controlled (or energy, or martingale) solutions for a class of SPDEs including the Kardar-Parisi-Zhang (KPZ) equations [39];
5. Hairer's work on the well-posedness and uniqueness theory for the KPZ equation [46];

Recently the controlled path approach has also been used to highlight the regularisation by noise phenomenon in ODE with irregular additive perturbations [12] where techniques very similar to those used in this paper are exploited (in particular the notion of ρ -irregularity and the non-linear Young integral).

Controlled paths are functions which "looks like" some given reference object. In the case of eq. (3.3) it looks quite clear that the solution should have the form $\varphi_t = U_t^w \psi_t$ for ψ_t another continuous path in V such that $\varphi_0 = \psi_0$. If we stipulate that ψ has a nice time behavior then φ is somehow "following" the flow of a free solution of the linear equation, modulo a time-dependent slowly varying modulation. The space of controlled paths \mathcal{D}^w (to be defined below) in which we will set up the equation will then be given by functions φ such that an Hölder condition holds for $\psi_t = (U_t^w)^{-1} \varphi_t$. Note that this space depends on the modulation and that different driving functions w and w' would give rise a priori to different spaces \mathcal{D}^w and $\mathcal{D}^{w'}$ of controlled functions. This difference is somehow crucial and make the spaces of controlled paths to be more effective in the analysis of the non-linearities. Let us try to explain why. Assume that φ is the simplest path controlled by w , that is the solution of the free

evolution $\varphi_t = U_t^w \phi$ for some fixed $\phi \in V$ (i.e. not depending on time). In this case the non-linear term in eq. (3.3) takes the form

$$\Phi_t = U_t \int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w \phi) ds = U_t X_t(\phi)$$

where $X_t : V \rightarrow V$ is the time-inhomogeneous map given by

$$X_t(\phi) = \int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w \phi) ds \quad (3.5)$$

We will show that, in the specific settings we will consider, it is possible to actually prove the following regularity requirement:

Hypothesis 3.2.1. *The map $X_{st} = X_t - X_s$ is almost surely a locally Lipschitz map on V satisfying the Hölder estimate*

$$\|X_{st}(\phi) - X_{st}(\phi')\|_V \lesssim |t - s|^\gamma (1 + \|\phi\|_V + \|\phi'\|_V)^M \|\phi - \phi'\|_V$$

for some $\gamma > 1/2$ and $M \geq 0$.

In this situation we see that Φ_t is a controlled path such that $\Psi_t = (U_t^w)^{-1} \Phi_t$ belongs at least to $C^{1/2}(V)$. If we want a space of controlled paths stable under the fixed point map

$$\Gamma(\varphi)_t = U_t^w \varphi_0 + U_t^w \int_0^t (U_s^w)^{-1} \mathcal{N}(\varphi_s) ds$$

we have to require $t \mapsto (U_t^w)^{-1} \Gamma(\varphi)_t$ to be at most in $C^{1/2}(V)$ since otherwise even the first step of the Picard iterations will get us out of the space. These considerations suggest us a definition of controlled paths:

Definition 3.2.2. *The space of paths $\mathcal{D}^w(V)$ controlled by w is given by all the paths φ in $C([0, T], V)$ such that $t \mapsto \varphi_t^w = (U_t^w)^{-1} \varphi_t$ belongs to $C^{1/2}(V)$.*

At this point it is still not clear that the non-linear term is well defined for every controlled paths. Hypothesis 3.2.1 ensure that the non-linearity is well defined when the controlled path φ is such that φ^w is constant in time. To allow for more general controlled paths we consider a smooth (in space and time) path f : in this case the following computations can be easily justified in all the models we will consider:

$$\int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w f_s) ds = \int_0^t \left[\frac{d}{ds} X_s \right] (f_s) ds = \int_0^t X_{ds}(f_s).$$

where the last integral in the r.h.s. should be interpreted as the limit of suitable Riemman sums:

$$\int_0^t X_{ds}(f_s) := \lim_{|\Pi_{0,t}| \rightarrow 0} \sum_i X_{t_i t_{i+1}}(f_{t_i}).$$

A key observation is that the map $f \mapsto \int_0^t X_{ds}(f_s)$ can be extended by continuity to all the functions $f \in C^{1/2}(V)$ using the theory of Young integrals, indeed note that X is a path of Lipschitz maps with Hölder regularity $\gamma > 1/2$ and that this is enough to integrate functions of Hölder regularity $1/2$ since the sum of these two regularities exceed 1. Since the kind of Young integral we use is not standard we will provide proofs and estimates in a self-contained fashion below. This allows us to give a natural definition of the nonlinear term for all controlled paths φ , indeed it is now easy to prove the following claim:

Lemma 3.2.3. *Let $\varphi \in \mathcal{D}^w$ and let $(\varphi_n)_{n \geq 0}$ a sequence of elements of $\varphi \in \mathcal{D}^w$ which are smooth and compactly supported in space and such that $\varphi_n \rightarrow \varphi$ in \mathcal{D}^w . Then*

$$\int_0^t (U_s^w)^{-1} \mathcal{N}((\varphi_n)_s) ds \rightarrow \int_0^t X_{ds}(\varphi_s^w)$$

in V uniformly in t .

As it should be clear by now, the time-integral of the non-linearity (even if not the non-linearity itself) is a well defined space distribution for all controlled paths and it is explicitly given by a Young integral involving the modulated operator X . We can then recast the mild equation (3.3) as a Young-type differential equation for controlled paths:

$$\varphi_t^w = \varphi_0 + \int_0^t X_{ds}(\varphi_s^w). \quad (3.6)$$

Any solution of this equation corresponds to a controlled path $\varphi_t = U_t^w \varphi_t^w$ which solves (3.3) where the r.h.s. should be understood according to Lemma 3.2.3.

The Young equation (3.6) can then be solved, at least locally in time and in a unique way, in $C^{1/2}(\mathbb{R}_+, V)$ by a standard fixed point argument. In some cases it is also possible to prove the existence of a conservation law which imply $\|\varphi_t\|_V = \|\varphi_0\|_V$ and obtain global solutions. Another byproduct of this approach is the existence of a Lipschitz flow map on V .

3.3 The nonlinear Young integral

Young theory of integration is well known [33, 60, 61, 75]. Here we introduce a non-linear variant which is not covered by the standard assumptions. For the sake of completeness we derive here the main estimates in our specific context.

Theorem 3.3.1 (Young). *Let $f \in C^\gamma \text{Lip}_M(V)$ and $g \in C^\rho V$ with $\gamma + \rho > 1$ then the limit of Riemann sums*

$$I_t = \int_0^t f_{du}(g_u) = \lim_{|\Pi| \rightarrow 0} \sum_i f_{t_{i+1}}(g_{t_i}) - f_{t_i}(g_{t_i})$$

exists in V as the partition Π of $[0, t]$ is refined, it is independent of the partition, and we have

$$\|I_t - I_s - (f_t - f_s)(g_s)\|_V \leq (1 - 2^{1-\gamma-\rho})^{-1} \|f\|_{C^\gamma \text{Lip}_M(V)} \|g\|_{C^\rho V} (1 + \|g\|_{C^0 V})^M |t - s|^{\gamma+\rho}.$$

Proof. We give a new proof of this fact. Let f, g be smooth functions in $\text{Lip}_M(V)$ and V respectively. Define the bilinear forms $I_{s,t}(f, g) = \int_s^t (d_u f_u)(g_u)$ and $J_{s,t}(f, g) = I_{s,t}(f, g) - f_{s,t}(g_s)$ where $f_{s,t} = f_t - f_s$ and note that this last satisfy $J_{s,t}(f, g) = J_{s,u}(f, g) + J_{u,t}(f, g) + (f_{u,t}(g_u) - f_{u,t}(g_s))$ for all $s \leq u \leq t$. Let $t_k^n = s + (t - s)k2^{-n}$ for $k = 0, \dots, 2^n$. By induction we have that

$$J_{s,t}(f, g) = \sum_{i=0}^{2^n-1} J_{t_i^n, t_{i+1}^n}(f, g) + \sum_{k=0}^n \sum_{i=0}^{2^k-1} (f_{t_{2i+1}^k, t_{2i+2}^k}^{t_{2i+1}^k}(g_{t_{2i+1}^k}) - f_{t_{2i+1}^k, t_{2i+2}^k}^{t_{2i+2}^k}(g_{t_{2i}^k}))$$

Since f, g are smooth we have $\|J_{t_i^n, t_{i+1}^n}(f, g)\|_V \lesssim_{f,g} |t_{i+1}^n - t_i^n|^2 \Omega \lesssim_{f,g} 2^{-2n}$ so that

$$\left\| \sum_{i=0}^{2^n-1} J_{t_i^n, t_{i+1}^n}(f, g) \right\|_V \lesssim_{f,g} 2^{-n} \rightarrow 0$$

as $n \rightarrow \infty$. Then we can estimate

$$\begin{aligned} \|J_{s,t}(f, g)\|_V &\leq \sum_{k=0}^{\infty} \sum_{i=0}^{2^k-1} \|(f_{t_{2i+1}^k, t_{2i+2}^k}(g_{t_{2i+1}^k}) - f_{t_{2i+1}^k, t_{2i+2}^k}(g_{t_{2i}^k}))\|_V \\ &\leq \sum_{k=0}^{\infty} 2^{k(1-\gamma-\rho)} \|f\|_{\mathcal{C}^\gamma \text{Lip}_M(V)} \|g\|_{\mathcal{C}^\rho V} (1 + \|g\|_{\mathcal{C}^0 V})^M \\ &\leq (1 - 2^{1-\gamma-\rho})^{-1} \|f\|_{\mathcal{C}^\gamma \text{Lip}_M(V)} \|g\|_{\mathcal{C}^\rho V} (1 + \|g\|_{\mathcal{C}^0 V})^M \end{aligned}$$

Now assume that $f \in \mathcal{C}^\gamma \text{Lip}_M(V)$ and $g \in \mathcal{C}^\rho V$. Then there exists sequences of smooth function f_n and g_n such that $f_n \rightarrow f$ in $\mathcal{C}^{\gamma'} \text{Lip}_M(V)$ and $g_n \rightarrow g$ in $\mathcal{C}^{\rho'} V$ for all $\gamma' < \gamma$ and all $\rho' < \rho$ and moreover such that $\|f_n\|_{\mathcal{C}^\gamma \text{Lip}_M(V)} \leq \|f\|_{\mathcal{C}^\gamma \text{Lip}_M(V)}$ and $\|g_n\|_{\mathcal{C}^\rho V} \leq \|g\|_{\mathcal{C}^\rho V}$. Then the above estimate implies the convergence of $J_{s,t}(f_n, g_n) \rightarrow J_{s,t}(f, g)$ in V for all s, t . In turn this implies that, by passing to the limit in the estimate we have also $\|J_{s,t}(f, g)\|_V \leq (1 - 2^{1-\gamma-\rho})^{-1} \|f\|_{\mathcal{C}^\gamma \text{Lip}_M(V)} \|g\|_{\mathcal{C}^\rho V} (1 + \|g\|_{\mathcal{C}^0 V})^M$. Which means that we can define

$$I_{s,t}(f, g) = \int_s^t f_{du}(g_u) = f_{s,t}(g_s) + J_{s,t}(f, g)$$

for any $f \in \mathcal{C}^\gamma \text{Lip}_M(V)$ and $g \in \mathcal{C}^\rho V$. Now assume that $\Pi = \{s \leq t_0 < t_1 < \dots < t_n \leq t\}$ is a partition of $[s, t]$ and denote with $S_\Pi = \sum_{i=0}^{n-1} f_{t_i, t_{i+1}}(g_{t_i})$ the associate Riemman sum. By the above construction we have $f_{t_i, t_{i+1}}(g_{t_i}) = I_{t_i, t_{i+1}}(f, g) - J_{t_i, t_{i+1}}(f, g)$ with $\|J_{t_i, t_{i+1}}(f, g)\| \lesssim_{f,g} |t_{i+1} - t_i|^{\gamma+\rho}$ and so

$$S_\Pi = \sum_{i=0}^{n-1} I_{t_i, t_{i+1}}(f, g) + \sum_{i=0}^{n-1} J_{t_i, t_{i+1}}(f, g) = I_{s,t}(f, g) + \sum_{i=0}^{n-1} J_{t_i, t_{i+1}}(f, g)$$

moreover $\left\| \sum_{i=0}^{n-1} J_{t_i, t_{i+1}}(f, g) \right\|_V \lesssim_{f,g} \sum_{i=0}^{n-1} |t_{i+1} - t_i|^{\gamma+\rho} \lesssim_{f,g} |\Pi|^{\gamma+\rho-1} |t - s|$ which implies that $S_\Pi \rightarrow I_{s,t}(f, g)$ as $|\Pi| \rightarrow 0$ and the integral which we defined above by the continuous extension of the bilinear form $I_{s,t}(f, g)$ coincide indeed with the limit of Riemann sums on arbitrary partitions. \square

3.4 Young solutions

With the estimates of Young integral we can set up a standard fixed point procedure to prove existence of local solution and their uniqueness assuming suitable regularity of X . We assume that $X_t(0) = 0$ for simplicity. Define standard Picard's iterations by

$$\psi_t^{(n+1)} = \psi_0 + \int_0^t X_{ds}(\psi_s^{(n)})$$

with $\psi_t^{(0)} = \psi_0$. Now

$$\begin{aligned} \left\| \int_0^t X_{ds}(\psi_s^{(n)}) - X_t(\psi_0) \right\|_V &\lesssim T^{\gamma+1/2} \|X\| (1 + \|\psi^{(n)}\|_{\mathcal{C}^0 V})^M \|\psi^{(n)}\|_{\mathcal{C}^{1/2} V} \\ &\lesssim T^\gamma \|X\| (1 + \|\psi_0\|_V + T^{1/2} \|\psi^{(n)}\|_{\mathcal{C}^{1/2} V})^{M+1} \end{aligned}$$

and

$$\|\psi^{(n+1)}\|_{\mathcal{C}^{1/2}V} \lesssim \|X\|T^\gamma(1 + \|\psi_0\|_V + T^{1/2}\|\psi^{(n)}\|_{\mathcal{C}^{1/2}V})^{M+1}$$

which means that for sufficiently small T (depending only on $\|\psi_0\|_V$) we can have $T^{1/2}\|\psi^{(n)}\|_{\mathcal{C}^{1/2}V} \leq 1$ for all $n \geq 0$. Moreover in this case

$$\|\psi^{(n+2)} - \psi^{(n+1)}\|_{\mathcal{C}^{1/2}V} \lesssim_{\|\psi_0\|_V} \|X\|T^{\gamma-1/2}\|\psi^{(n+1)} - \psi^{(n)}\|_{\mathcal{C}^{1/2}V}$$

which for $\|X\|T^{\gamma-1/2} \lesssim_{\|\psi_0\|_V} 1/2$ implies that $(\psi^{(n)})_{n \geq 0}$ converges in $\mathcal{C}^{1/2}V$ to a limit ψ which by continuity of the Young integral and of the operator X satisfies

$$\psi_t = \psi_0 + \int_0^t X_{ds}(\psi_s).$$

This solution exists at least until $t \leq T$ where T depends only on the norm of X and $\|\psi_0\|_V$. Note that a posteriori ψ belongs to $\mathcal{C}^\gamma V$ and not only to $\mathcal{C}^{1/2}V$. Uniqueness in $\mathcal{C}^{1/2}V$ is now obvious.

Of course if $M = 0$ it is easy to prove that the existence time T of the local solution does not depend on $\|\phi_0\|_V$ and this imply existence of solution on arbitrary intervals. In the general case we need further assumptions on the properties of X :

Lemma 3.4.1. *Assume that for all $\phi \in V$ such that $\|\phi\|_V \leq R$ we have*

$$\| \|\phi + X_{s,t}(\phi)\|_V - \|\phi\|_V | \lesssim C_R |t - s|^\rho$$

where $\rho > 1$, then $\|\psi_t\|_V = \|\psi_0\|_V$ and there exists a unique global solution of the Young equation.

Proof. Consider $M_t = \|\psi_t\|_V$ which satisfy

$$\begin{aligned} |M_t - M_s| &= \| \|\psi_s + X_{s,t}(\psi_s) + R_{s,t}\|_V - \|\psi_s\|_V | \\ &\leq \| \|\psi_s + X_{s,t}(\psi_s)\|_V - \|\psi_s\|_V | + \|R_{s,t}\|_V \end{aligned}$$

by assumption and by the Young estimates on R we have

$$\lesssim_{\|\psi_0\|_V} |t - s|^\rho + |t - s|^{1/2+\gamma}.$$

This relation implies that M_t must be a constant function since both $\rho > 1$ and $1/2 + \gamma > 1$. Then $M_t = M_0$ for all $t < T$. The conservation of the V norm allows then to extend the solution to an arbitrary interval and obtain a global solution. \square

3.4.1 Euler Scheme

Young equations allow for a straightforward Euler approximation scheme. Let $\psi \in \mathcal{C}^\gamma V$ the solution of the Young equation defined before and T the life time of this solution. For any $n \geq 0$ let $\psi_0^n = \psi_0 \in V$ and define recursively

$$\psi_i^n = \psi_{i-1}^n + X_{\frac{i-1}{n}, \frac{i}{n}}(\psi_{i-1}^n).$$

Theorem 3.4.2. *Let for $n \geq 0$ and $0 \leq i \leq nT$, $\Delta_i^n = \psi_i^n - \psi_{\frac{i}{n}}$ then*

$$\max_{0 \leq i < j \leq [nT]} \frac{|\Delta_j^n - \Delta_i^n|}{|i - j|^\gamma} = O(n^{1-2\gamma})$$

Proof. We remark that $\psi_j^n - \psi_i^n = \sum_{l=i}^{j-1} X_{\frac{l}{n} \frac{l+1}{n}}(\psi_l^n)$ and for $0 \leq i < j \leq [nT]$ and define the partition of $[i/n, j/n]$ by $\pi^{j-i+1} = (t_k^n)_{i \leq k \leq j}$ with $t_k^n = \frac{k}{n}$. Denote by $M_{ij}^{\pi^{j-i+1}} = \sum_{l=i}^{j-1} X_{t_l^n t_{l+1}^n}(\psi_l^n)$, now consider the partition $\pi^{j-i} = \pi^{j-i+1} - \{t_k^n\}$ for $i < k < j$ and then

$$M_{ij}^{\pi^{j-i+1}} = M_{ij}^{\pi^{j-i}} + X_{t_k^n t_{k+1}^n}(\psi_k^n) - X_{t_k^n t_{k+1}^n}(\psi_{k-1}^n)$$

and by induction we obtain immediately that

$$\psi_j^n - \psi_i^n = X_{t_i^n t_j^n}(\psi_i^n) + \sum_{k=i+1}^{j-1} X_{t_k^n t_{k+1}^n}(\psi_k^n) - X_{t_k^n t_{k+1}^n}(\psi_{k-1}^n)$$

now for some convenience we denote by $p_{ik}^n = X_{t_i^n t_k^n}(\psi_q^n)$ and $q_{lk}^n = X_{t_l^n t_k^n}(\psi_{q/n})$ then using that ψ satisfies the Young equation we obtain

$$\psi_{j/n} - \psi_{i/n} = q_{ij}^n + \sum_{k=i+1}^{j-1} X_{t_k^n t_{k+1}^n}(\psi_{k/n}) - X_{t_k^n t_{k+1}^n}(\psi_{(k-1)/n}) + R_{ij}^n$$

where

$$R_{ij}^n = \sum_{k=i}^{j-1} \int_{t_k^n}^{t_{k+1}^n} X_{d\sigma}(\psi_\sigma) - X_{t_k^n t_{k+1}^n}(\psi_{t_k^n})$$

For this term we have the bound

$$|R_{ij}^n| \lesssim |\psi|_{|\gamma+|\psi|_0, \|X\|} (j-i)n^{-2\gamma}.$$

Now consider

$$\Delta_j^n - \Delta_i^n = p_{ij}^n - q_{ij} - R_{ij}^n + \sum_{k=i+1}^{j-1} X_{t_k^n t_{k+1}^n}(\psi_k^n) - X_{t_k^n t_{k+1}^n}(\psi_{k-1}^n) - X_{t_k^n t_{k+1}^n}(\psi_{k/n}) + X_{t_k^n t_{k+1}^n}(\psi_{(k-1)/n})$$

and let

$$B_l^n = \max_{0 \leq i < j \leq l} \left(\frac{j-i}{n} \right)^{-1} |\Delta_j^n - \Delta_i^n - p_{ij}^n - q_{ij} + R_{ij}^n|.$$

To prove our result is suffices to show that $B_{[nT]}^n = O(n^{1-2\gamma})$. Observe that when $|i-j| < l$ the sum appearing in the expression of $\Delta_i^n - \Delta_j^n$ can be bounded by B_{l-1}^n : in fact we have that

$$|X_{t_k^n t_{k+1}^n}(\psi_k^n) - X_{t_k^n t_{k+1}^n}(\psi_{k-1}^n) - X_{t_k^n t_{k+1}^n}(\psi_{k/n}) + X_{t_k^n t_{k+1}^n}(\psi_{(k-1)/n})| \leq C \left(\frac{j-i}{n} \right)^{2\gamma} (1 + B_{l-1}^n)^M (n^{1-2\gamma} + B_{l-1}^n)$$

where $C = C(\psi^0, \|X\|)$ and

$$B_l^n \leq C(1 + B_{l-1}^n)^M (B_{l-1}^n + n^{1-2\gamma})(l/n)^{2\gamma-1}.$$

When $l = 1$ we have that $B_1^n = 0$ and the result is clearly true. Now assume that for some l we have that $B_{l-1}^n \leq A$ and define the increasing map $\theta(x) = (l/n)^{2\gamma-1}(1+x)^{M+1}$. Remark that $\theta(n^{2\gamma-1}B_l^n) \leq n^{2\gamma-1}B_{l-1}^n$. Then if l/n is small enough we have that θ admits a fixed point and that $n^{1-2\gamma}B_l^n \leq A < +\infty$ where we take A is the limit of the sequence (x_i) defined by $x_{i+1} = \theta(x_i)$ and $x_0 = 0$. Now is suffice to iterate this argument to prove that bound hold for $l \leq [nT]$. \square

3.4.2 Regular equation

In this section we study the convergence of approximations given by a standard PDE to the solution of the Young equations. Consider the following regularized problem

$$\begin{cases} \partial_t \varphi_t = A\varphi_t \partial_t n_t + \Pi_L \mathcal{N}(\Pi_L \varphi_t), & t \geq 0 \\ \varphi(0, x) = \Pi_L \phi(x) \in C^\infty(\mathbb{T}) \end{cases} \quad (3.7)$$

with n is a differentiable function, $\phi \in L^2(\mathbb{T})$, $A = \partial_x^3$ or $i\partial_x^2$ and \mathcal{N} is the non linearity given in the previous section, of course this Cauchy problem is equivalent to the mild formulation

$$\varphi_t = U_t^n \Pi_L \phi + \int_0^t U_t^n (U_s^n)^{-1} \Pi_L \mathcal{N}(\Pi_L \varphi_s) ds \quad (3.8)$$

or equivalently

$$\psi_t = \Pi_L \phi + \int_0^t (U_s^n)^{-1} \Pi_L \mathcal{N}(\Pi_L U_s^n \psi_s) ds \quad (3.9)$$

with $U_t^n = e^{An_t}$ and $\psi_t = (U_t^n)^{-1} \varphi_t$, In the rest of this section we take $A = \partial_x^3$ and $\mathcal{N}(\phi) = \partial_x \phi^2$ for the case of the Schrödinger equation we can adapt exactly the same argument. Now we can check easily that the modulated operator $X^{n,L}$ associated to the equation (3.9) is well defined and satisfy

$$\|X_{st}^{n,L}\|_{\mathcal{L}^2(H_{\alpha_1}, H_{\alpha_2})} \lesssim_{n,L} |t - s|$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$, and then by a fixed point argument we obtain the existence of a unique Young local solution $\varphi^{n,L} \in C([0, T^*], L^2)$ such that $\psi_t^{n,L} = (U_t^n)^{-1} \varphi_t^{n,L} \in C^1([0, T^*], L^2)$ moreover we have that $\psi^{n,L} \in \cap_{\beta \geq 0} C^1([0, T^*], H_\beta)$ and then clearly

$$\partial_t \varphi_t = A\varphi_t \partial_t n_t + \Pi_L \mathcal{N}(\Pi_L \varphi_t)$$

in the weak sense. To obtain a global solution is sufficient to remark that for all $v \in L^2$

$$\begin{aligned} \langle v, X_{st}^{n,L}(v, v) \rangle &= \int_s^t d\sigma \int_{\mathbb{T}} dx U_\sigma^n v(x) \Pi_L \partial_x (U_\sigma^n v(x))^2 \\ &= - \int_s^t d\sigma \int_{\mathbb{T}} dx \Pi_L (U_\sigma^n v(x))^2 \partial_x (\Pi_L U_\sigma^n v(x))^2 = 0 \end{aligned}$$

and then we obtain that

$$\begin{aligned} \|\psi_t^{n,L}\|_{L^2}^2 &= \|\psi_s^{n,L}\|_{L^2}^2 + \|\psi_t^{n,L} - \psi_s^{n,L}\|_{L^2}^2 + \langle \psi_s^{n,L}, X_{st}^{n,L}(v_s, v_s) \rangle + R_{st} \\ &= \|\psi_s^{n,L}\|_{L^2}^2 + \|\psi_t^{n,L} - \psi_s^{n,L}\|_{L^2}^2 + R_{st} \end{aligned}$$

for all $s, t \in [0, T^*]$ with $|R_{st}| \lesssim |t - s|^2$, then we obtain that $|\|\psi_t^{n,L}\|_{L^2}^2 - \|\psi_s^{n,L}\|_{L^2}^2| \lesssim |t - s|^2$ and this give us $\|\psi_t^{n,L}\| = \|\Pi_L \phi\|_{L^2}$. Using this conservation law we can extend our local solution to a global one. The mild eq. (3.9) has a meaning even when n is only continuous function. Let $R > 0$, $T > 0$ and assume that $\sup_{\sigma \in [0, T]} |n_\sigma| \leq R$ then we obtain

$$\|\psi^{n,L}\|_{C^{1-\varepsilon}([0, T], L^2)} \lesssim_L T_1^\varepsilon \|X^{n,L}\|_{C^1([0, T], \mathcal{L}^2)} (\|\psi^{n,L}\|_{C^{1-\varepsilon}([0, T_1], L^2)} + \|\Pi_L \phi\|_{L^2})^2$$

for all $T_1 < \min(1, T)$, using the fact that $\|X^{n,L}\|_{C^1([0,T],\mathcal{L}^2)} \lesssim_L \sup_{\sigma \in [0,T]} |n_\sigma| \lesssim_L R$ and taking $T_1 = T_1(\|\Pi_L \phi\|_{L^2})$ small enough we can see that $\|\psi^{n,L}\|_{C^{1-\varepsilon}([0,T_1],L^2)} \lesssim_L R$. Finally using the conservation law and iterating these results gives us that $\|\psi^{n,L}\|_{C^{1-\varepsilon}([0,T],L^2)} \lesssim_L R$. By a similar argument we obtain easily $\|\psi^{n^2,L} - \psi^{n^1,L}\|_{C^{1-\varepsilon}([0,T],L^2)} \lesssim_{L,R} \sup_{\sigma \in [0,T]} |n_\sigma^1 - n_\sigma^2|$ for all $n_1, n_2 \in C([0,T],L^2)$ such that $\sup_{\sigma \in [0,T]} |n_\sigma^i| \leq R$ for $i = 1, 2$ where $\psi^{n^1,L}, \psi^{n^2,L}$ are respectively the global solution of the eq. (3.9) associated to the dispersion n^1 and n^2 . Now let w^N a regularization of the continuous ρ -irregular function w and assume that $\sup_{\sigma \in [0,T]} |w_\sigma^N - w_\sigma| \rightarrow_{N \rightarrow +\infty} 0$ for all $T > 0$. Then the solutions $(\varphi^{N,L})_{N \in \mathbb{N}}$ of the regularized problem (3.7) with dispersion w^N converge in $C([0,T],L^2)$ to φ^L which is the solution of the mild equation (3.8) with dispersion w :

$$\varphi_t^L = U^w \Pi_L \phi + \int_0^t (U_t^w)(U_s^w)^{-1} \Pi_L \mathcal{N}(\Pi_L \varphi_s) ds. \quad (3.10)$$

Finally we have

Theorem 3.4.3. *Let $\rho > 3/4$, $T > 0$ and φ^L, φ respectively the solution of the mild eq. (3.10) on $[0, T]$ and the modulated KdV equation then*

$$\|\psi^L - \varphi\|_{C^{1/2}([0,T],L^2)} \rightarrow_{L \rightarrow +\infty} 0$$

with $\psi_t^L = (U_t^w)^{-1} \varphi_t^L$ and $\psi_t = (U_t^w)^{-1} \varphi_t$

Proof. Using the equation

$$\psi_t^L = \Pi_L \phi + \int_0^t X_{d\sigma}^L(\psi_\sigma)$$

we obtain that

$$\|\psi^L\|_{C^{1/2}([0,T_1],L^2)} \lesssim T_1^{\gamma-1/2} \sup_L \|X^L\|_{C^\gamma([0,T],\mathcal{L}^2(L^2))} (\|\psi^L\|_{C^{1/2}([0,T_1],L^2)} + \|\phi\|_{L^2})^2$$

and then taking $T_1 = T_1(\|\phi\|_{L^2})$ small enough we obtain that $\sup_L \|\psi^L\|_{C^{1/2}([0,T_1],L^2)} \lesssim \sup_L \|X^L\|_{C^\gamma([0,T],\mathcal{L}^2(L^2))} + \infty$ using the conservation law we can proceed by induction to recover the interval $[0, T]$ and then $\sup_L \|\psi^L\|_{C^{1/2}([0,T],L^2)} < +\infty$. Now the same argument shows that

$$\|\psi^L - \varphi\|_{C^{1/2}([0,T],L^2)} \lesssim_{\|\phi\|_{L^2}} \|X^L - X\|_{C^{1/2}([0,T],L^2)}$$

and then suffices to use the fact that $\|X - X^L\| \rightarrow_{L \rightarrow \infty} 0$ (proven in Lemma 3.5.2 below) to deduce the needed convergence. □

3.5 Regularity of X

Let w a ρ -irregular path, the aim of this section is to provide the necessary pathwise estimates on the modulated operator X^w in the various models we consider.

Definition 3.5.1. *We say that a n -linear operator X on the Banach space V belongs to $\mathcal{X}_{n,V}^w$ if*

1. For all $T > 0$ we have

$$|X_{st}|_{\mathcal{L}^n V} \leq C \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^\gamma$$

for $s, t \in [0, T]$ and for some finite constant $C > 0$ which does not depend on w .

2. If we let $X_{s,t}^L(\varphi_1, \dots, \varphi_n) = \Pi_L X_{s,t}(\Pi_L \varphi_1, \dots, \Pi_L \varphi_n)$ then $X_L \rightarrow X$ in $\mathcal{C}_T^{1/2} \mathcal{L}^n V$.

Once appropriate bounds are obtained for the relevant X operators, the Young theory of Section 3.4 gives a complete local well-posedness theory for the equation (including convergence of approximations and the Euler scheme). For the KdV equation and the non linear cubic Schrödinger equation we will see in the next section how we can obtain a global solution for an initial data $\phi \in H^\alpha$ with $\alpha \geq 0$ using some smoothing estimates. For the following we will assume that w is ρ -irregular.

3.5.1 Periodic KdV

Here we will bound the modulated operator associated to the periodic KdV equation (ie: $A = \partial^3$ and $\mathcal{N}(\varphi) = \partial\varphi^2$) on $H^\alpha(\mathbb{T})$. We recall that in this case the operator X is given by

$$X_{s,t}(\psi) = X_{s,t}(\psi, \psi) = \int_s^t (U_\sigma^w)^{-1} \partial_x (U_\sigma^w \psi)^2 d\sigma$$

Lemma 3.5.2. *Let $\alpha \geq -\rho$ and $\rho > 3/4$ then $X \in \mathcal{X}_{2,H^\alpha}^w$.*

Proof. Let $\psi_1, \psi_2 \in H^\alpha$. The Fourier transform gives

$$\hat{X}_{st}(\psi_1, \psi_2) = ik \sum_{k_1+k_2=k} \mathbb{I}_{kk_1k_2 \neq 0} \Phi_{st}^w(kk_1k_2) \hat{\psi}_1(k_1) \hat{\psi}_2(k_2).$$

From an application of Cauchy-Schwarz we obtain that

$$\begin{aligned} \left| \sum_{k_1+k_2=k} \mathbb{I}_{kk_1k_2 \neq 0} \Phi_{st}^w(kk_1k_2) \hat{\psi}_1(k_1) \hat{\psi}_2(k_2) \right|^2 &\leq \left(\sum_{k_1+k_2=k} \mathbb{I}_{kk_1k_2 \neq 0} |k_2|^{-2\alpha} |\Phi_{st}^w(kk_1k_2)|^2 |\hat{\psi}_1(k_1)|^2 \right) |\psi_2|_\alpha^2 \\ &\leq \left(\sup_{k_1; kk_1k_2 \neq 0, k_1+k_2=k} \frac{|\Phi_{st}^w(kk_1k_2)|^2}{|k_1|^{2\alpha} |k_2|^{2\alpha}} \right) |\psi_1|_\alpha^2 |\psi_2|_\alpha^2 \end{aligned} \quad (3.11)$$

where the supremum is taken over k_1 . And we obtain

$$|X_{st}|_{\mathcal{L}^2 H^\alpha}^2 \leq \left(\sum_k |k|^{2\alpha+2} \sup_{k_1; kk_1k_2 \neq 0, k_1+k_2=k} \frac{|\Phi_{st}^w(kk_1k_2)|^2}{|k_1|^{2\alpha} |k_2|^{2\alpha}} \right)^{1/2} \quad (3.12)$$

The ρ -irregularity of w allows to estimate this bound by

$$\begin{aligned} |X_{st}|_{\mathcal{L}^2 H^\alpha}^2 &\leq C_{\rho,T} \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^{2\gamma} \sum_k |k|^{2\alpha+2-2\rho} \sup_{k_1; kk_1k_2 \neq 0, k_1+k_2=k} \frac{1}{|k_1|^{2\alpha+2\rho} |k_2|^{2\alpha+2\rho}} \\ &\lesssim_{w,\rho,\gamma} |t-s|^{2\gamma} \sum_k |k|^{2-4\rho} \sup_{k_1; kk_1k_2 \neq 0, k_1+k_2=k} \left(\frac{|k|}{|k_1||k_2|} \right)^{2\alpha+2\rho}. \end{aligned}$$

Now if we remark that $\frac{|k|}{|k_1||k_2|} \leq \frac{1}{|k_1|} + \frac{1}{|k_2|} \leq 2$ and if we take $\alpha \geq -\rho$ and $\rho > 3/4$ we obtain that

$$\sum_k |k|^{2-4\rho} \sup_{k_1; k_1 k_2 \neq 0, k_1+k_2=k} \left(\frac{|k|}{|k_1||k_2|} \right)^{2\alpha+2-2\rho} \leq 2^{2\alpha+2\rho} \sum_k \frac{1}{|k|^{2-4\rho}} < +\infty$$

which gives the claimed regularity for X . As far as the convergence of X^L is concerned we let $\phi_1, \phi_2 \in L^2$ and observe that

$$\begin{aligned} & \|X_{st}^L(\phi_1, \phi_2) - X_{st}(\phi_1, \phi_2)\|_2^2 \\ &= \sum_{|k| < L} |k|^2 \left| \sum_{k_1+k_2=k, k_1 k_2 \neq 0} (\mathbb{I}_{|k_1|, |k_2| \leq L} - 1) \hat{\phi}_1(k_1) \hat{\phi}_2(k_2) \Phi_{st}(kk_1 k_2) \right|^2 \\ &+ \sum_{|k| \geq L} |k|^2 \left| \sum_{k_1+k_2=k, k_1 k_2 \neq 0} \hat{\phi}_1(k_1) \hat{\phi}_2(k_2) \Phi_{st}^w(kk_1 k_2) \right|^2 \\ &\lesssim |\phi_1|_2^2 |\phi_2|_2^2 \left(\sum_k |k|^2 \sup_{|k_1| \geq L; k_1+k_2=k} |\Phi_{st}^w(kk_1 k_2)|^2 + \sum_{|k| \geq L} |k|^2 \sup_{k_1, k_1+k_2=k} |\Phi_{st}(kk_1 k_2)|^2 \right). \end{aligned}$$

Using this bound with the fact that w is ρ -irregular gives us

$$\|X^L - X\|_{C^\gamma([0, T], \mathcal{L}^2(L^2))} \lesssim_{w, T} \sum_k |k|^{2-2\rho} \sup_{|k_1| \geq L; k_1+k_2=k} |k_1|^{-2\rho} |k_2|^{-2\rho} + \sum_{|k| \geq L} |k|^{2-2\rho} \sup_{k_1} |k_1 k_2|^{-2\rho}$$

for some $\gamma > 1/2$ and $\rho > 4/3$. Now the r.h.s of this inequality vanish when L goes to the infinity, in fact choosing $\theta > 0$ small enough we have

$$\sum_k |k|^{2-2\rho} \sup_{|k_1| \geq L; k_1+k_2=k} |k_1|^{-2\rho} |k_2|^{-2\rho} \lesssim_{\theta, \rho} L^{-\theta} \sum_k |k|^{2-4\rho+\theta} \xrightarrow{L \rightarrow +\infty} 0$$

and

$$\sum_{|k| \geq L} |k|^{2-2\rho} \sup_{k_1} |k_1 k_2|^{-2\rho} \lesssim_\rho \sum_{|k| \geq L} |k|^{2-4\rho} \xrightarrow{L \rightarrow +\infty} 0$$

and this finishes the proof. \square

Now we will give an improvement of the Lemma 3.5.2.

Lemma 3.5.3. *Let $\rho > 4/3$, $\alpha > -\rho$ and $\beta < \alpha + 2\rho - \frac{3}{2}$ then there exists $\gamma > 1/2$ such that for all $T > 0$ the following inequality holds*

$$\|X_{st}(\phi_1, \phi_2)\|_{H^\beta} \leq C_{T, \alpha, \beta} \|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} |t-s|^\gamma |\phi_1|_{H^\alpha} |\phi_2|_{H^\alpha}$$

for all $\phi_1, \phi_2 \in H^\alpha$ where $C_{T, \alpha, \beta} < +\infty$.

Proof. Eq. (3.12) can be modified to give

$$\|X_{st}(\phi_1, \phi_2)\|_{H^\beta}^2 \leq |\phi_1|_{H^\alpha}^2 |\phi_2|_{H^\alpha}^2 \sum_k |k|^{2+2\beta} \sup_{k_1} \left(\frac{|\Phi_{st}^w(3kk_1 k_2)|^2}{|k_1|^\alpha |k_2|^\alpha} \right)$$

an

$$\begin{aligned} \sum_k |k|^{2+2\beta} \sup_{k_1} \left(\frac{|\Phi_{st}^w(3kk_1k_2)|^2}{|k_1|^\alpha |k_2|^\alpha} \right) &\leq |t-s|^{2\gamma} \sum_k |k|^{2+2\beta-2\rho} \sup_{k_1} |k_1k_2|^{-2\alpha-2\rho} \\ &\lesssim_{\alpha,\beta} |t-s|^{2\gamma} \sum_k |k|^{2-4\rho+2\beta-2\alpha} < +\infty \end{aligned}$$

if $\beta < \alpha + 2\rho - 3/2$ which finishes the proof. \square

3.5.2 Periodic modified KdV

In the case of the periodic modified KdV equation we have $A = \partial^3$ and $\mathcal{N}(u) = \partial u(u^2 - \|u\|_2^2)$ and the Fourier transform of the modulated operator X reads

$$\hat{X}_{st}(\psi_1, \psi_2, \psi_3) = ik \sum_{*} \hat{\psi}_1(k_1) \hat{\psi}_2(k_2) \hat{\psi}_3(k_3) \Phi_{st}^w(2(k-k_2)(k-k_1)(k-k_3))$$

where the star under the sum mean that $k_1 + k_2 + k_3 = k, k_1k_2k_3 \neq 0$ and $k_2, k_3 \neq k$ and we have used the algebraic relation $k^3 - k_1^3 - k_2^3 - k_3^3 = (k-k_1)(k-k_2)(k-k_3)$. By Cauchy-Schwarz

$$\begin{aligned} |X_{st}(\psi_1, \psi_2, \psi_3)|_{H^\alpha}^2 &= \sum_k |k|^{2\alpha+2} \left| \sum_{*} \hat{\psi}_1(k_1) \hat{\psi}_2(k_2) \hat{\psi}_3(k_3) \Phi_{st}^w(2(k-k_2)(k-k_1)(k-k_3)) \right|^2 \\ &\leq \sum_k |k|^{2\alpha+2} \left(\sum_{*} |k_1k_2k_3|^{-2\alpha} |\Phi_{st}^w(2(k-k_2)(k-k_3)(k-k_1))|^2 \right) \\ &\quad \times \left(\sum_{*} |k_1|^{2\alpha} |k_2|^{2\alpha} |k_3|^{2\alpha} |\hat{\psi}_1(k_1)|^2 |\hat{\psi}_2(k_2)|^2 |\hat{\psi}_3(k_3)|^2 \right) \\ &\leq \left(\sup_{k \neq 0} |k|^{2\alpha+2} \sum_{*} |k_1k_2k_3|^{-2\alpha} |\Phi_{st}^w(2(k-k_2)(k-k_3)(k-k_1))|^2 \right) \|\psi_1\|_{H^\alpha} \|\psi_2\|_{H^\alpha}^2 \|\psi_3\|_{H^\alpha}^2 \end{aligned}$$

from which we obtain that

$$|X_{st}|_{\mathcal{L}^3 H^\alpha}^2 \leq \sup_{k \neq 0} |k|^{2\alpha+2} \sum_{*} |k_1k_2k_3|^{-2\alpha} |\Phi_{st}^w(2(k-k_2)(k-k_3)(k-k_1))|^2. \quad (3.13)$$

Now we will give a lemma which help us to bound our operator

Lemma 3.5.4. *Let $\alpha \geq 1/2$ and $\rho > 1/2$ then we have*

$$\sum_{l \neq 0, k} |l|^{-2\alpha} |l-k|^{-2\rho} \lesssim_{\varepsilon, \rho, \alpha} |k|^{-\min(2\alpha, 2\rho-\varepsilon)}$$

for all $\varepsilon > 0$ small enough.

Proof. We begin by decomposing our sum in two region in the following manner

$$\sum_{k_2 \neq 0, k} \frac{1}{|k_2|^{2\alpha} |k-k_2|^{2\rho}} = I_1 + I_2$$

where

$$I_1 = \sum_{k_2 \neq 0, k; |k-k_2| \leq 2|k_2|} \frac{1}{|k_2|^{2\alpha} |k-k_2|^{2\rho}}, \quad I_2 = \sum_{k_2 \neq 0, k; |k-k_2| \geq 2|k_2|} \frac{1}{|k_2|^{2\alpha} |k-k_2|^{2\rho}}.$$

Remark that if $|k - k_2| \leq 2|k_2|$ then $|k| \leq 3|k_2|$ then we have

$$I_1 \lesssim \frac{1}{|k|^{2\alpha}} \sum_{k_2 \neq 0, k; |k-k_2| \leq 2|k_2|} \frac{1}{|k-k_2|^{2\rho}} \lesssim \frac{1}{|k|^{2\alpha}} \sum_{k_2 \neq k} \frac{1}{|k-k_2|^{2\rho}} = \frac{1}{|k|^{2\alpha}} \sum_{k_2 \neq 0} \frac{1}{|k_2|^{2\rho}} < +\infty.$$

For the second term I_2 , we begin by noting that if $|k - k_2| \geq 2|k_2|$ then $|k| \lesssim |k - k_2|$ so

$$I_2 \lesssim \frac{1}{|k|^{2\rho-\varepsilon}} \sum_{k_2 \neq 0, k; |k-k_2| \geq 2|k_2|} \frac{1}{|k_2|^{2\alpha+\varepsilon}} < +\infty.$$

□

Now using the inequality (3.13) and the (ρ, γ) -irregularity if w we have

$$\begin{aligned} |X_{st}|_{\mathcal{L}^3 H^\alpha} &\leq \sup_{k \neq 0} |k|^{2\alpha+2} \sum_* |k_1 k_2 k_3|^{-2\alpha} |\Phi_{st}^w(2(k-k_2)(k-k_3)(k-k_1))|^2 \\ &\leq C_{w,\rho} |t-s|^\gamma \sup_{k \neq 0} |k|^{2+2\alpha} \sum_* |k_1 k_2 k_3|^{-2\alpha} \frac{1}{|k-k_2|^{2\rho} |k-k_3|^{2\rho} |k-k_1|^{2\rho}} \end{aligned}$$

where $C_{w,\varepsilon,T}$ is a finite constant.

Lemma 3.5.5. *For all $\alpha \geq 1/2$ and $\rho > 1/2$ we have that*

$$I = \sup_{k \neq 0} |k|^{2+2\alpha} \sum_* |k_1 k_2 k_3|^{-2\alpha} \frac{1}{|k-k_2|^{2\rho} |k-k_3|^{2\rho} |k-k_1|^{2\rho}} < +\infty$$

Proof. Now the inequality $|k|^{2\alpha} = |-k_1 + k_2 + k_3|^{2\alpha} \lesssim |k_1|^{2\alpha} + |k_2|^{2\alpha} + |k_3|^{2\alpha}$ gives

$$I \lesssim \sup_k |k|^2 \sum_{k_2, k_3 \neq 0, k} |k_2 k_3|^{-2\alpha} |k-k_2|^{-2\rho} |k-k_3|^{-2\rho} = \sup_k |k|^2 \left(\sum_{k_2 \neq 0, k} |k|^{-2\alpha} |k-k_2|^{-2\rho} \right)^2$$

Then using the Lemma 3.5.4 we conclude that $I < +\infty$ when $\alpha \geq 1/2$. □

Theorem 3.5.6. *Let $\rho > 1/2$ then there exists $\gamma > 1/2$ such that $X \in \mathcal{C}^\gamma([0, T], H^\alpha)$ for all $\alpha \geq 1/2$ and $T > 0$. Moreover if $\alpha > 1/2$ we have that $X \in \mathcal{X}_{3, H^\alpha}^w$.*

3.5.3 KdV on \mathbb{R}

Here we treat the operator X associated to the KdV equation on the non-periodic case. By a simple computation we see that the Fourier transform of X is given by the convolution formula

$$\hat{X}_{st}(\psi_1, \psi_2)(x) = ix \int_{\mathbb{R}} \Phi_{st}^w(xy(x-y)) \hat{\psi}_1(y) \hat{\psi}_2(x-y) dy$$

We begin by treating the case $\alpha \geq 0$. Cauchy-Schwarz inequality gives

$$\begin{aligned} \|X_{st}\|_{H^\alpha}^2 &\leq \sup_{x \in \mathbb{R}} \langle x \rangle^{2\alpha} |x|^2 \int_{\mathbb{R}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha}} dy \\ &\leq \|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} |t-s|^\gamma \sup_{x \in \mathbb{R}} \langle x \rangle^{2\alpha} |x|^2 \int_{\mathbb{R}} \frac{dy}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha} (1 + |xy(x-y)|)^{2\rho}} \end{aligned}$$

and then we have to check that

$$I(\alpha) = \sup_{x \in \mathbb{R}} \langle x \rangle^{2\alpha} |x|^2 \int_{\mathbb{R}} \frac{dy}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha} (1 + |xy(x-y)|)^{2\rho}} < +\infty$$

with $\alpha \geq 0$. Using the fact that $\langle x \rangle^{2\alpha} \lesssim \langle y \rangle^{2\alpha} + \langle x-y \rangle^{2\alpha}$ we obtain $I(\alpha) \lesssim I(0)$ and then is sufficient to prove that $I(0)$ is finite. We will decompose this quantity as $I(0) = I^1 + I^2 + I^3 + I^4$ where

$$I^1 = \sup_{|x| \geq 1} |x|^2 \int_{\{|y| \geq 1/2; |y-x| \geq 1/2\}} \frac{dy}{(1 + |xy(x-y)|)^{2\rho}} \lesssim \sup_{|x| \geq 1} |x|^{2-4\rho} \int_{\{|y| \geq 1/2\}} |z|^{-2\rho} dy < +\infty$$

when $\rho > 1/2$.

$$\begin{aligned} I^2 &= 2 \sup_{|x| \geq 1} |x|^2 \int_{|y| \leq 1/2} \frac{dy}{(1 + |xy(x-y)|)^{2\rho}} \\ &\lesssim \sup_{|x| \geq 1} |x|^2 \int_{|y| \leq 1/2} \frac{dy}{(1 + x^2|y|)^{2\rho}} \lesssim \int_{\mathbb{R}} (1 + |z|)^{-2\rho} dz < +\infty \end{aligned}$$

$$\begin{aligned} I^3 &= \sup_{|x| \leq 1} |x|^2 \int_{|y| \geq 2} \frac{dy}{(1 + |xy(x-y)|)^{2\rho}} \\ &\lesssim \sup_{|x| \leq 1} |x|^2 \int_{|y| \geq 2} \frac{dy}{(1 + |xy^2|)^{2\rho}} \lesssim \int_{\mathbb{R}} (1 + |z|^2)^{-2\rho} dz < +\infty \end{aligned}$$

and finally the last term can easily bounded by

$$I^4 = \sup_{|x| \leq 1} |x|^2 \int_{|y| \leq 2} (1 + |yx(x-y)|)^{-2\rho} dy \leq 4.$$

In the case $\alpha < 0$ we will bound our operators by

$$\begin{aligned} \|X_{st}\|^2 &\leq \int_{|x| \geq 1} |x|^2 \langle x \rangle^{2\alpha} \sup_{|y| \geq 1/2; |x-y| \geq 1/2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha}} dx \\ &\quad + 2 \sup_{|x| \geq 1} \langle x \rangle^{2\alpha} |x|^2 \int_{|y| < 1/2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha}} \\ &\quad + \sup_{|x| < 1} \langle x \rangle^{2\alpha} |x|^2 \int_{\mathbb{R}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha}} = J^1 + J^2 + J^3 \end{aligned}$$

The integral J_1 correspond to the high-high-high part and cant by treated by similar argument used in the periodic setting in fact

$$J^1 \lesssim_{\gamma, \alpha} \|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} |t-s|^\gamma \int_{|x| \geq 1} |x|^{2-4\rho} \left(\sup_{|y|, |x-y| \geq 1/2} \frac{|x|}{|x-y||y|} \right)^{2\alpha+2\rho} < +\infty$$

when $\rho > 3/4$ and $\alpha > -\rho$. For the term J_2 we remark that if $|x| \geq 1$ and $|y| < 1/2$ then $|x-y| \sim |x|$ and

$$J^2 \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} |t-s|^\gamma \sup_{|x| \geq 1} |x|^2 \int_{|y| \leq 1/2} \frac{dy}{(1 + |x^2y|)^{2\rho}} \lesssim \int_{\mathbb{R}} (1 + |z|)^{-2\rho} dz < +\infty.$$

Now split $J^3 = J^{31} + J^{32}$ with

$$J^{31} = \sup_{|x| < 1} |x|^2 \langle x \rangle^{2\alpha} \int_{|y| < 2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{\langle y \rangle^{2\alpha} \langle x-y \rangle^{2\alpha}} \lesssim_{\alpha} |t-s|.$$

If $|x| < 1$ and $|y| \geq 2$ then $|x-y| \sim |y|$ and

$$J^{32} \lesssim |t-s|^{\gamma} \sup_{|x| \leq 1} |x|^2 \int_{|y| \geq 2} \frac{dy}{(1+|y|^2|x|)^{4\rho} \langle y \rangle^{4\alpha}} \lesssim \sup_{|x| \leq 1} |x|^{3/2+2\alpha} \int_{\mathbb{R}} |y|^{-4\alpha} (1+|y|^2)^{2\rho} < +\infty$$

when $\alpha \in (-3/4, 0]$ and $\alpha > -\rho$. These considerations results in the following regularity for X :

Proposition 3.5.7. *Let $\rho > 3/4$ then there exist $\gamma > 1/2$ such that $X \in \mathcal{C}^{\gamma}([0, T], H^{\alpha})$ for all $T > 0$ and $\alpha > -\min(3/4, \rho)$.*

The restriction of the regularity at $-3/4$ is imposed by the low-highy frequency term in the proof above. To bypass this difficulty we will consider distribution spaces given by the following definition.

Definition 3.5.8. *We say that $f \in \mathcal{H}_{\alpha, \beta}$ if $f \in S'(\mathbb{R})$ and $\int_{\mathbb{R}} |\theta_{\alpha, \beta}(x)|^2 |\hat{f}(x)|^2 dx < +\infty$ where $\theta_{\alpha, \beta}(x) = \frac{|x|^{\alpha+\beta}}{(1+|x|)^{\beta}}$*

Observe that $\mathcal{H}^{\alpha} = \mathcal{H}_{\alpha, 0}$ is the homogenous Sobolev space. Now as in periodic case by simple computation we have that

$$|X_{st}|_{\mathcal{L}^2 \mathcal{H}^{\alpha}}^2 \leq \int_{\mathbb{R}} |x|^{2+2\alpha} \sup_{y \in \mathbb{R}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|y|^{2\alpha} |x-y|^{2\alpha}} dx.$$

Now the problem with this bound is that the terms in r.h.s admit a singularity at the origin which not exist in the periodic case to bypass this difficulty we will give another bound of our operator in the region which poses a problem.

Lemma 3.5.9. *There exist a universal constant C such that the following inequality holds*

$$\begin{aligned} |X_{st}|_{\mathcal{L}^2 \mathcal{H}_{\alpha, \beta}}^2 &\leq C \left(\sup_{|x| \leq 1} |x|^2 |\theta_{\alpha, \beta}(x)|^2 \int_{|y| \leq 2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \right. \\ &+ \int_{|x| \leq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \sup_{|y| \geq 2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \\ &+ \sup_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \int_{|y| \leq \frac{1}{2}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dy \\ &+ \sup_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \int_{\{|y| \geq \frac{1}{2}; |y-x| \leq \frac{1}{2}\}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dy \\ &+ \left. \int_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \sup_{\{|y| \geq \frac{1}{2}; |y-x| \geq \frac{1}{2}\}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \right) \end{aligned} \quad (3.14)$$

Proof. Let $\psi_1, \psi_2 \in \mathcal{H}_{\alpha, \beta}$ then by definition we have

$$\begin{aligned} |\hat{X}_{st}(\psi_1, \psi_2)|_{\alpha, \beta}^2 &= \int_{\mathbb{R}} |x|^2 \theta_{\alpha, \beta}^2(x) \left| \int_{\mathbb{R}} \Phi_{st}^w(xy(x-y)) \hat{\psi}_1(y) \hat{\psi}_2(x-y) dy \right|^2 dx \\ &= \int_{|x| \leq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \left| \int_{\mathbb{R}} \Phi_{st}^w(xy(x-y)) \hat{\psi}_1(y) \hat{\psi}_2(x-y) dy \right|^2 dx \\ &\quad + \int_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \left| \int_{\mathbb{R}} \Phi_{st}^w(xy(x-y)) \hat{\psi}_1(y) \hat{\psi}_2(x-y) dy \right|^2 dx \\ &= I_1 + I_2 \end{aligned}$$

Now we begin by study the term I_1 , then by Cauchy-Schwarz we have:

$$\begin{aligned} I_1 &\leq 2 \left(\int_{|x| \leq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \left| \int_{|y| \leq 2} \Phi_{st}^w(xy(x-y)) \hat{\psi}_1(y) \hat{\psi}_2(x-y) dy \right|^2 dx \right. \\ &\quad \left. + \int_{|x| \leq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \left| \int_{|y| \geq 2} \Phi_{st}^w(xy(x-y)) \hat{\psi}_1(y) \hat{\psi}_2(x-y) dy \right|^2 dx \right) \\ &\leq 2 \left(\sup_{|x| \leq 1} |x|^2 |\theta_{\alpha, \beta}(x)|^2 \int_{|y| \leq 2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \right. \\ &\quad \left. + \int_{|x| \leq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \sup_{|y| \geq 2} \left(\frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} \right) dx \right) |\psi_1|_{\alpha, \beta}^2 |\psi_2|_{\alpha, \beta}^2 \end{aligned}$$

By the same argument we can show that I_2 satisfy the following inequality :

$$\begin{aligned} I_2 &\leq 3 \left(\sup_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \int_{|y| \leq \frac{1}{2}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dy \right. \\ &\quad \left. + \sup_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \int_{\{|y| \geq \frac{1}{2}; |y-x| \leq \frac{1}{2}\}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dy \right. \\ &\quad \left. + \int_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \sup_{\{|y| \geq \frac{1}{2}; |y-x| \geq \frac{1}{2}\}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \right) \end{aligned}$$

which finishes the proof. \square

Now to obtain the Young regularity we have to bound this five kernel

$$\begin{aligned} I_{st}^{hhh} &= \int_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \sup_{\{|y| \geq \frac{1}{2}; |y-x| \geq \frac{1}{2}\}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \\ I_{st}^{ll} &= \sup_{|x| \leq 1} |x|^2 |\theta_{\alpha, \beta}(x)|^2 \int_{|y| \leq 2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \\ I_{st}^{lh} &= \int_{|x| \leq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \sup_{|y| \geq 2} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dx \\ I_{st}^{hll} &= \sup_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \int_{|y| \leq \frac{1}{2}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dy \end{aligned}$$

$$I_{st}^{hhh} = \sup_{|x| \geq 1} |x|^2 \theta_{\alpha, \beta}^2(x) \int_{\{|y| \geq \frac{1}{2}; |y-x| \leq \frac{1}{2}\}} \frac{|\Phi_{st}^w(xy(x-y))|^2}{|\theta_{\alpha, \beta}(y)|^2 |\theta_{\alpha, \beta}(x-y)|^2} dy$$

Now we will begin by the term which contain the high-high-high frequency :

$$\begin{aligned} I_{st}^{hhh} &\lesssim |t-s|^\gamma \int_{|x| \geq 1} |x|^{2\alpha+2-2\rho} \sup_{\{|y| \geq 1/2; |y-x| \geq 1/2\}} \frac{1}{|y(x-y)|^{2\alpha+2\rho}} dx \\ &\lesssim |t-s|^\gamma \int_{|x| \geq 1} |x|^{2-4\rho} \sup_{\{|y| \geq 1/2; |y-x| \geq 1/2\}} \left(\frac{|x|}{|y(x-y)|}\right)^{2\alpha+2\rho} dx < +\infty \end{aligned}$$

if $\alpha > -1$ and $\rho > 3/4$ small enough. Now for the term which contain the low frequency we use the inequality $|\Phi_{st}^w(a)| \leq |t-s|$ and then :

$$I_{st}^{ll} \lesssim |t-s| \sup_{|x| \leq 1/2} |x|^{2(\alpha+\beta)+2} \int_{|y| \leq 2} \frac{1}{|y(x-y)|^{2\alpha+2\beta}} dy < +\infty$$

if $-1 \leq \alpha + \beta < 1/2$. Now we will focus on the low-high frequency term, and we begin by remark that by interpolation we have that $|\Phi_{st}^w(a)| \leq C \frac{|t-s|^\gamma}{|a|^{\rho'}}$ for one $\gamma > 1/2$ and all $\rho' \in (0, \rho]$, then using this inequality we obtain that :

$$I_{st}^{lh} \lesssim_{\alpha, \beta} |t-s|^\gamma \int_{|x| \leq 1} |x|^{2(\alpha+\beta)+2-2\rho'} \sup_{|y| \geq 2} \frac{1}{|y(x-y)|^{2\alpha+2\rho'}} dx < +\infty$$

when we can choose $\rho' \in (0, \rho) \cap (-\alpha, \alpha + \beta + 3/2)$ and this is possible if and only if $2\alpha + \beta > -3/2$, $\alpha > -\rho$ and $\alpha + \beta > -3/2$. Now it remains to study the two terms I_{st}^{hlh} and I_{st}^{hhl} but by symmetry these terms are essentially equivalent then it suffices to treat only one of them. Let us for example treat the term I_{st}^{hlh} then we begin by noting that if $|x| \geq 1$ and $|y| \leq 1/2$ then $|x| - 1/2 \leq |x-y| \leq |x| + 1/2$ using this fact we have :

$$\begin{aligned} I_{st}^{hlh} &\lesssim |t-s|^\gamma \sup_{|x| \geq 1} |x|^{2\alpha+2-2\rho'} \int_{|y| \leq 1/2} \frac{1}{|y|^{2\alpha+2\beta+2\rho'} |x-y|^{2\alpha+2\rho'}} dy \\ &\lesssim |t-s|^\gamma \sup_{|x| \geq 1} |x|^{2-4\rho'} \int_{|y| \leq 1/2} \frac{1}{|y|^{2\alpha+2\beta+2\rho'}} dy < +\infty \end{aligned}$$

when we choose $\rho' \in (0, \rho) \cap (1/2, 1/2 - \alpha - \beta)$ and this is possible if and only if $\alpha + \beta < 0$ Then we have the following lemma.

Lemma 3.5.10. *Let $\rho > 3/4$, $\alpha > -\rho$ and $-\alpha > \beta \geq 0$ with $\beta + 2\alpha > -3/2$ then there exist $\gamma^* > 1/2$ such that for all $T > 0$ the following inequality holds*

$$|X_{st}|_{\mathcal{L}^2 \mathcal{H}_{\alpha; \beta}} \leq C \|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} |t-s|^{\gamma^*}$$

for all $(s, t) \in [0, T]^2$ where $C = C(T, \beta, \alpha) > 0$.

Corollary 3.5.11. *Let $\rho > 3/4$ and $0 > \alpha > \max(-3/4, -\rho)$ then there exist $\gamma > 1/2$ such that for all $T > 0$ the following inequality holds:*

$$|X_{st}|_{\mathcal{L}^2 \mathcal{H}^\alpha} \leq C \|\Phi^w\|_{\mathcal{W}_T^{\rho, \gamma}} |t-s|^\gamma$$

for all $(s, t) \in [0, T]$ where $C = C(T, \alpha, \rho) > 0$.

Proof. The condition $\alpha > -3/4$ ensures that you can take $\beta = 0$ in the Lemma 3.5.10 □

3.5.4 Periodic cubic NLS equation

Proposition 3.5.12. *Let $\rho > 1/2$ then there exist $\gamma > 1/2$ such that for all $T > 0$ and $\alpha \geq 0$ we have $X \in C^\gamma([0, T], H^\alpha)$. Moreover if $\alpha > 0$ then $X \in \mathcal{X}_{3, H^\alpha}^w$.*

Proof. By definition $\dot{X}_s(\psi_1, \psi_2, \psi_3)$ is a trilinear operator with Fourier transform given by

$$\begin{aligned} \mathcal{F}\dot{X}_s(\psi_1, \psi_2, \psi_3)(\xi) &= \mathcal{F}X^1(\psi_1, \psi_2, \psi_3) + \mathcal{F}X_s^2(\psi_1\psi_2, \psi_3) \\ &= \psi_3\langle\psi_1, \psi_2\rangle + \psi_2\langle\psi_1, \psi_3\rangle + \sum_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{Z}_0 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} \mathbb{I}_{\xi \neq \xi_2, \xi_3} e^{iws(\xi^2 + \xi_1^2 - \xi_2^2 - \xi_3^2)} \hat{\psi}_1(\xi_1)^* \hat{\psi}_2(\xi_2) \hat{\psi}_3(\xi_3) \end{aligned}$$

where $\hat{\psi}_i = \mathcal{F}\psi_i$. Note that $\xi^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 = 2(\xi - \xi_2)(\xi - \xi_3)$ under the condition that $\xi = -\xi_1 + \xi_2 + \xi_3$. Setting $\Xi = 2(\xi - \xi_2)(\xi - \xi_3)$ we get

$$|\langle\psi, X_{st}^2(\psi_1, \psi_2, \psi_3)\rangle| \leq \sum_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{Z}_0 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} \mathbb{I}_{\xi \neq \xi_2, \xi_3} |\Phi_{s,t}^w(\Xi)| \hat{\psi}(\xi)^* \hat{\psi}_1(\xi_1)^* \hat{\psi}_2(\xi_2) \hat{\psi}_3(\xi_3)$$

By by now standard application of Cauchy-Schwarz we get

$$|\langle\psi, X_{st}^2(\psi_1, \psi_2, \psi_3)\rangle| \leq (I_{\alpha, \rho})^{1/2} \sup_{a \in \mathbb{Z}_0} |a|^{-2\rho} |\Phi_{st}^w(a)| \|\psi\|_\alpha \|\psi_1\|_\alpha \|\psi_2\|_\alpha \|\psi_3\|_\alpha$$

with

$$I_{\alpha, \eta} = \sup_{\xi \in \mathbb{Z}_0} \sum_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{Z}_0 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} \mathbb{I}_{\xi \neq \xi_2, \xi_3} |\xi|^{2\alpha} |\xi_1 \xi_2 \xi_3|^{-2\alpha} |\Xi|^{-2\rho}.$$

Some condition on the finiteness of the constant $I_{\alpha, \beta, \rho}$ are enough to control the regularity of the operator X^2 . Since $\alpha \geq 0$, by using that $|\xi|^{2\alpha} \lesssim |\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha}$ we have $I_{\alpha, \rho} \lesssim I_{0, \rho}$ moreover

$$I_{0, \eta} = \sup_{\xi \in \mathbb{Z}_0} \sum_{\xi_2 \in \mathbb{Z}_0} \mathbb{I}_{\xi \neq \xi_2} |\xi - \xi_2|^{-2\rho} \sum_{\xi_3 \in \mathbb{Z}_0} \mathbb{I}_{\xi \neq \xi_3} |\xi - \xi_3|^{-2\rho}$$

Is then easy to see that all these sums are finite provided $1 < 2\rho$ which means that we can take any $\rho > 1/2$. Now for the operator X^1 we have the following bound

$$\|X_{st}^1(\psi_1, \psi_2, \psi_3)\|_{H^\alpha} \leq 2(t-s) \|\psi_1\|_\alpha \|\psi_2\|_\alpha \|\psi_3\|_\alpha$$

and this finishes the bound of the operator X . \square

3.5.5 Cubic NLS equation on \mathbb{R}

Lemma 3.5.13. *Let $(\psi_i)_{i=1, \dots, 4} \in L^2(\mathbb{R})$ and define the following integral*

$$\begin{aligned} \mathcal{I}(\alpha) &:= \int_{\mathbb{R}^3} dx_1 dx_2 dx_3 |x_2 - x_1|^\alpha |\Phi_{s_1 s_2}^w(2(x_2 - x_1)(x_3 - x_1))| \\ &\quad \times |\hat{\psi}_1(x_1)| |\hat{\psi}_2^*(x_2)| |(\hat{\psi}_3)^*(x_3)| |(\hat{\psi}_4^*)^*(-x_1 + x_2 + x_3)| \end{aligned}$$

then we have the following bound

$$\mathcal{I}(\alpha) < |s_2 - s_1|^\gamma \prod_{i=1, \dots, 4} \|\psi_i\|_{L^2(\mathbb{R})}$$

when $\alpha \in [0, 1)$ and $\rho > 1/2 + \alpha$ or $\alpha = 1$ and $\rho > 1$

Proof. In the case $\alpha < 1$ let us split $\mathbb{R}^3 = \cup_{i=1,\dots,4} D_i$ with

$$D_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; |x_2 - x_1| \geq 1, |x_3 - x_1| \geq 1\},$$

$$D_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; |x_2 - x_1| \leq 1, |x_3 - x_1| \leq 1\},$$

$$D_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; |x_2 - x_1| \leq 1, |x_3 - x_1| \geq 1\},$$

$$D_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; |x_2 - x_1| \geq 1, |x_3 - x_1| \leq 1\}.$$

According to this split $\mathcal{I}(\alpha) = \sum_{i=1,\dots,4} I_i$. By Cauchy-Schwarz we have $I_l \leq J_l \Pi_{i=1}^4 |\psi_i|_{L^2(\mathbb{R})}$ for $l \in \{1, 2, 3\}$ and using the (ρ, γ) -irregularity of w we have

$$\begin{aligned} J_1^2 &= \sup_{x_1} \int_{\mathbb{R}^2} dx_2 dx_3 \mathbb{I}_{\{|x_2 - x_1| \geq 1; |x_3 - x_1| \geq 1\}} |x_2 - x_1|^{2\alpha} |\Phi_{s_1 s_2}^w(2(x_2 - x_1)(x_3 - x_1))|^2 \\ &= \int_{\mathbb{R}^2} dy_2 dy_3 \mathbb{I}_{\{|y_2| \geq 1; |y_3| \geq 1\}} |y_2|^{2\alpha} |\Phi_{s_1 s_2}^w(2y_2 y_3)|^2 \\ &\lesssim |s_2 - s_1|^{2\gamma} \left(\int_{|y_2| \geq 1} \frac{1}{|y_2|^{2\rho - 2\alpha}} dy_2 \right) \left(\int_{|y_3| \geq 1} \frac{1}{|y_3|^{2\rho}} dy_3 \right) < +\infty \end{aligned}$$

when $\rho > \alpha + 1/2$. To bound the term J_3 we use again the (ρ, γ) irregularity of w and we obtain

$$\begin{aligned} J_3^2 &= \sup_{x_1} \int_{\mathbb{R}^2} dx_2 dx_3 \mathbb{I}_{\{|x_2 - x_1| \leq 1; |x_3 - x_1| \geq 1\}} |x_2 - x_1|^{2\alpha} |\Phi_{s_1 T}^w(2(x_2 - x_1)(x_3 - x_1))|^2 \\ &= \int_{\mathbb{R}^2} dy_2 dy_3 \mathbb{I}_{\{|y_2| \leq 1; |y_3| \geq 1\}} |y_2|^{2\alpha} |\Phi_{s_1 s_2}^w(2y_2 y_3)|^2 \\ &\lesssim |s_2 - s_1|^{2\gamma} \int_{|y_2| \leq 1} |y_2|^{2\alpha} \left(\int_{|y_3| \geq 1} \frac{1}{(1 + |y_2 y_3|)^{2\rho}} dy_3 \right) dy_2 \\ &\lesssim |s_2 - s_1|^{2\gamma} \left(\int_{|y_2| \leq 1} \frac{1}{|y_2|^{1 - 2\alpha}} dy_2 \right) \left(\int_{\mathbb{R}} \frac{1}{(1 + |z_3|)^{2\rho}} dz_3 \right) < +\infty \end{aligned}$$

when $\rho > 1/2$, $\alpha > 0$ and this give us the bound for I_3 , we remark also in the case $\alpha = 0$ the integral I_3 and I_4 are essentially the same by symmetry and can be be bounded using the same argument. Now we will focus to bound the term J_2 for that we use only the bound $|\Phi_{s_1 s_2}^w(a)| \leq |s_2 - s_1|$ which is valid for all $a \in \mathbb{R}$, in fact we have

$$\begin{aligned} J_2^2 &= \sup_{x_1} \int_{\mathbb{R}^2} dx_2 dx_3 \mathbb{I}_{\{|x_2 - x_1| \leq 1; |x_3 - x_1| \leq 1\}} |x_2 - x_1|^{2\alpha} |\Phi_{s_1 s_2}^w(2(x_2 - x_1)(x_3 - x_1))|^2 \\ &= \int_{\mathbb{R}^2} dy_2 dy_3 \mathbb{I}_{\{|y_2| \leq 1; |y_3| \leq 1\}} |y_2|^{2\alpha} |\Phi_{s_1 T}^w(2y_2 y_3)|^2 \\ &\leq |s_2 - s_1|^2 \end{aligned}$$

all these bounds give us the estimates for (I_l) , $l \in \{1, 2, 3\}$ then we will focus on the last integrals. To bound the integral I_4 we proceed in a different way, to simplify the notation let $\eta = 2(x_2 - x_1)(x_3 - x_1)$ and then using the Cauchy-Schwarz inequality we have :

$$\int_{\mathbb{R}} dx_2 \mathbb{I}_{|x_2 - x_1| \geq 1} |x_2 - x_1|^\alpha |\Phi_{s_1 T}^w(\eta)| |\hat{\psi}_2(x_2)| |\hat{\psi}_4(x)| \leq \sup_{x_2} (\mathbb{I}_{|x_2 - x_1| \geq 1} |x_2 - x_1|^\alpha |\Phi_{s_1 s_2}^w(\eta)|) |\psi_2|_{L^2(\mathbb{R})} |\psi_4|_{L^2(\mathbb{R})}$$

now injecting this inequality in I_4 and using Cauchy-Schwarz and Young inequality we obtain that

$$\begin{aligned}
 I_4 &\leq \left(\int_{\mathbb{R}^2} dx_3 dx_1 \mathbb{I}_{|x_3-x_1| \leq 1} \sup_{x_2} (\mathbb{I}_{|x_2-x_1| \geq 1} |x_2 - x_1|^\alpha |\Phi_{s_1 s_2}^w(\eta)|) |\hat{\psi}_1(x_1)| |\hat{\psi}_3(x_3)| \right) |\psi_2|_{L^2(\mathbb{R})} |\psi_4|_{L^2(\mathbb{R})} \\
 &= \left(\int_{\mathbb{R}} |\hat{\psi}_{s_1}(x_1)| \left(\int_{\mathbb{R}} \mathbb{I}_{|x_3-x_1| \leq 1} \sup_{x_2} (\mathbb{I}_{|x_2-x_1| \geq 1} |x_2 - x_1|^\alpha |\Phi_{s_1 s_2}^w(\eta)|) |\hat{\psi}_3(x_3)| dx_3 \right) dx_1 \right) |\psi_2|_{L^2(\mathbb{R})} |\psi_4|_{L^2(\mathbb{R})} \\
 &\leq \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathbb{I}_{|x_3-x_1| \leq 1} \sup_{x_2} (\mathbb{I}_{|x_2-x_1| \geq 1} |x_2 - x_1|^\alpha |\Phi_{s_1 s_2}^w(\eta)|) |\hat{\psi}_3(x_3)| dx_3 \right|^2 dx_1 \right)^{1/2} |\psi_2|_{L^2(\mathbb{R})} |\psi_4|_{L^2(\mathbb{R})} |\psi_1|_{L^2(\mathbb{R})} \\
 &\leq \int_{|y_3| \leq 1} \sup_{|y_2| \geq 1} (|y_2|^\alpha |\Phi_{s_1 s_2}^w(2y_2 y_3)|) dy_3 \Pi_{i=1}^4 |\psi_i|_{L^2(\mathbb{R})} \\
 &\lesssim |T - s_1|^\gamma \Pi_{i=1}^4 |\psi_i|_{L^2(\mathbb{R})} \sup_{z_2} (|z_2|^\alpha (1 + |z_2|)^{-\rho}) \int_{|y_3| \leq 1} |y|^{-\alpha} dy_3 < +\infty
 \end{aligned}$$

when $\alpha < 1$ and $\rho > \alpha$. As was noted previously this gives us also a bound for I_3 when $\alpha = 0$. Now to treat the case $\alpha = 1$ we proceed as in [30]. Indeed after change of variable we can rewrite our integral as :

$$I(1) = \int_{\mathbb{R}} |x| \left(\int_{\mathbb{R}} |\hat{\psi}_1(y_1)| |\hat{\psi}_2(x - y_1)| \left(\int_{\mathbb{R}} |\hat{\psi}_3(y_2)| |\hat{\psi}_4(x - y_2)| |\Phi_{s_1 s_2}^w(2x(y_2 - y_1))| dy_2 \right) dy_1 \right) dx$$

and then by Cauchy-Schwarz and Young inequality we have

$$I(1) \leq \left(\sup_x |x| \int_{\mathbb{R}} |\Phi_{s_1 T}^w(2xz)| dz \right) \Pi_{i=1}^4 |\psi_i|_{L^2(\mathbb{R})} \leq |s_2 - s_1|^\gamma \left(\int_{\mathbb{R}} (1 + |z|)^{-\rho} dz \right) \Pi_{i=1}^4 |\psi_i|_{L^2(\mathbb{R})}$$

and the r.h.s is finite if $\rho > 1$ which finish the proof. \square

Proposition 3.5.14. *Let $\rho > 1/2$ and X the modulated operator associated to the NLS on \mathbb{R} then there exist $\gamma > 1/2$ such that $X \in C^\gamma([0, T], H^\alpha(\mathbb{R}))$ for all $\alpha \geq 0$*

Proof. Let $(\psi_i)_{i=1,2,3} \in H^\alpha$ and $\psi_4 \in H^{-\alpha}$ then by a simple computation we have that

$$|\langle \psi_4, X_{st}(\psi_1, \psi_2, \psi_3) \rangle| \leq \int_{\mathbb{R}} (|x|^{-\alpha} |\hat{\psi}_4(x)|) |x|^\alpha |x_1 x_2 x_3|^{-4\alpha} |-x_1 + x_2 + x_3|^\alpha \Pi_{i=1, \dots, 3} |x_i|^\alpha |\hat{\psi}_i(x_i)| dx_1 dx_2 dx_3 \quad (3.15)$$

with $x = -x_1 + x_2 + x_3$ now using the fact that $|x_i|^\alpha \lesssim |x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha$ and the lemma 3.5.13 we obtain immediately $|\langle \psi_4, X_{st}(\psi_1, \psi_2, \psi_3) \rangle| \lesssim |t - s|^\gamma |\psi_4|_{-\alpha} \Pi_{i=1, \dots, 3} |\psi_i|_\alpha$. \square

3.5.6 Cubic non linear Schrödinger equation on \mathbb{R}^2

To extend the previous results to the modulated Schrödinger equation on \mathbb{R}^2 we need to obtain regularity estimates for the appropriate X operators. Here $\dot{X}_s(\psi_1, \psi_2, \psi_3)$ is a trilinear operator with Fourier transform given by

$$\mathcal{F} \dot{X}_s(\psi_1, \psi_2, \psi_3)(\xi) = \int_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{R}^2 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} e^{i w_s (|\xi|^2 + |\xi_1|^2 - |\xi_2|^2 - |\xi_3|^2)} \hat{\psi}_1(\xi_1)^* \hat{\psi}_2(\xi_2) \hat{\psi}_3(\xi_3) d\xi_2 d\xi_3.$$

Note that $|\xi|^2 + |\xi_1|^2 - |\xi_2|^2 - |\xi_3|^2 = 2\langle \xi - \xi_2, \xi - \xi_3 \rangle_{\mathbb{R}^2} = \Xi$ under the condition that $\xi = -\xi_1 + \xi_2 + \xi_3$. Then X has the expression

$$\mathcal{F}X_{s,t}(\psi_1, \psi_2, \psi_3)(\xi) = \int_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{R}^d \\ \xi = -\xi_1 + \xi_2 + \xi_3}} \Phi_{s,t}^w(\Xi) \hat{\psi}_1(\xi_1)^* \hat{\psi}_2(\xi_2) \hat{\psi}_3(\xi_3) d\xi_2 d\xi_3.$$

Using the (ρ, γ) -irregularity of w we can easily obtain that

$$\begin{aligned} |\langle \psi, X_{ts}(\psi_1, \psi_2, \psi_3) \rangle| &\leq \int_{\substack{\xi, \xi_1, \xi_2, \xi_3 \in \mathbb{R}^2 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} |\Phi_{s,t}^w(\Xi)| |\hat{\psi}(\xi)| |\hat{\psi}_1(\xi_1)| |\hat{\psi}_2(\xi_2)| |\hat{\psi}_3(\xi_3)| d\xi_1 d\xi_2 d\xi_3 \\ &\leq J^{1/2} |t - s|^\gamma \|\psi\|_{-\alpha} \|\psi_1\|_\alpha \|\psi_2\|_\alpha \|\psi_3\|_\alpha \end{aligned}$$

with

$$J = \sup_{\xi \in \mathbb{R}^2} \int_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{R}^2 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} (1 + 2|\langle \xi - \xi_2, \xi - \xi_3 \rangle|)^{-2\rho} (1 + |\xi|^2)^\alpha \prod_{i=1,2,3} (1 + |\xi_i|^2)^{-\alpha} d\xi_2 d\xi_3$$

Lemma 3.5.15. *The quantity J is finite when $\alpha > 1/2$ and $\rho > 1/2$.*

Proof. Inserting the estimate $(1 + |\xi|^2)^\alpha \lesssim \sum_{i=1}^3 (1 + |\xi_i|^2)^\alpha$ we obtain that $J = J_1 + J_2$ where

$$J_1 = \sup_{\xi \in \mathbb{R}^2} \int_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{R}^2 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} (1 + 2|\langle \xi - \xi_2, \xi - \xi_3 \rangle|)^{-2\rho} (1 + |\xi_2|^2)^{-\alpha} (1 + |\xi_3|^2)^{-\alpha} d\xi_2 d\xi_3$$

and

$$J_2 = \sup_{\xi \in \mathbb{R}^2} \int_{\substack{\xi_1, \xi_2, \xi_3 \in \mathbb{R}^2 \\ \xi = -\xi_1 + \xi_2 + \xi_3}} (1 + 2|\langle \xi - \xi_2, \xi - \xi_3 \rangle|)^{-2\rho} (1 + |\xi_2|^2)^{-\alpha} (1 + |\xi_1|^2)^{-\alpha} d\xi_2 d\xi_3$$

Let us consider first the J_1 contribution. Let $q_i = \xi - \xi_i$, $i = 2, 3$

$$J_1 = \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi + q_2|^2)^\alpha} \int_{\mathbb{R}^2} \frac{dq_3}{(1 + 2|\langle q_2, q_3 \rangle|)^{2\rho} (1 + |\xi + q_3|^2)^\alpha}$$

Write $q_3^\perp, q_3^\parallel \in \mathbb{R}$ for the perpendicular and parallel components of $q_3 \in \mathbb{R}^2$ with respect to q_2 and similarly for ξ and bound

$$\begin{aligned} J_1 &\leq \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi + q_2|^2)^\alpha} \int_{\mathbb{R}} \frac{dq_3^\perp}{(1 + |\xi^\perp + q_3^\perp|^2)^\alpha} \int_{\mathbb{R}} \frac{dq_3^\parallel}{(1 + 2|q_2||q_3^\parallel|)^{2\rho}} \\ &= \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi + q_2|^2)^\alpha} \int_{\mathbb{R}} \frac{dq_3^\perp}{(1 + |q_3^\perp|^2)^\alpha} \int_{\mathbb{R}} \frac{dq_3^\parallel}{(1 + 2|q_2||q_3^\parallel|)^{2\rho}} \end{aligned}$$

now note that for $\alpha > 1/2$ and $\rho > 1/2$ we have

$$\int_{\mathbb{R}} \frac{dq_3^\perp}{(1 + |q_3^\perp|^2)^\alpha} \int_{\mathbb{R}} \frac{dq_3^\parallel}{(1 + 2|q_2||q_3^\parallel|)^{2\rho}} \lesssim |q_2|^{-1}$$

so that

$$J_1 \lesssim \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi + q_2|^2)^\alpha |q_2|} < +\infty$$

for $\alpha > 1/2$. To estimate the J_2 integral we rewrite it as

$$J_2 = \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (1 + 2|\langle \xi_1 - \xi_2, \xi - \xi_2 \rangle|)^{-2\rho} (1 + |\xi_2|^2)^{-\alpha} (1 + |\xi_1|^2)^{-\alpha} d\xi_2 d\xi_1$$

where we used that $\xi - \xi_3 = \xi_2 - \xi_1$. By writing $q_1 = \xi_1 - \xi_2$ and $q_2 = \xi - \xi_2$ we get

$$J_2 = \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi - q_2|^2)^\alpha} \int_{\mathbb{R}^2} \frac{dq_1}{(1 + 2|\langle q_1, q_2 \rangle|)^{2\rho} (1 + |q_1 + \xi - q_2|^2)^\alpha}$$

Write q_1^\perp, q_1^\parallel for the perpendicular and parallel components of q_1 with respect to q_2 to get the estimate

$$J_2 \leq \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi - q_2|^2)^\alpha} \int_{\mathbb{R}^2} \frac{dq_1^\perp dq_1^\parallel}{(1 + 2|q_1^\parallel||q_2|)^{2\rho} (1 + |q_1^\perp + \xi^\perp|^2)^\alpha}$$

again the condition $\alpha > 1/2$ allows to bound this last quantity as

$$\lesssim \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi - q_2|^2)^\alpha} \int_{\mathbb{R}} \frac{dq_1^\parallel}{(1 + 2|q_1^\parallel||q_2|)^{2\rho}}$$

and $\rho > 1/2$ subsequently by

$$\lesssim \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dq_2}{(1 + |\xi - q_2|^2)^\alpha |q_2|}$$

which is finite when $\alpha > 1/2$. \square

Theorem 3.5.16. *For all $\rho > 1/2$ there exists $\gamma > 1/2$ such that for all $T > 0$ the operator X belongs to $\mathcal{C}^\gamma([0, T], H^\alpha(\mathbb{R}^2))$ for all $\alpha > 1/2$.*

3.5.7 The derivative NLS equation

Here we will focus on the modulated Derivative non linear Schrödinger equation (ie: $A = i\partial^2$ and $\mathcal{N}(u) = \partial^\theta(|u|^2 - \|u\|_2^2)u$ for $\theta > 0$). Now the Fourier transform of the operator associated to this equation is given by

$$\hat{X}_{st}(\psi_1, \psi_2, \psi_3) = (ik)^\theta \sum_{\star} \psi_1(k_1)^* \psi_2(k_2) \psi_3(k_3) \Phi_{st}^w(2(k - k_2)(k - k_3))$$

where the star under the sum means that we have $-k_1 + k_2 + k_3 = k$ and $k_2 \neq k, k_3 \neq k, k_1 k_2 k_3 \neq 0$. Standard application of Cauchy-Schwartz gives

$$\|X_{st}\|_{H^\alpha}^2 \leq \sup_{k \neq 0} |k|^{2\alpha+2\theta} \sum_{\star} |k_1 k_2 k_3|^{-2\alpha} |\Phi_{st}^w(2(k - k_2)(k - k_3))|^2$$

then using the fact that w is (γ, ρ) irregular we obtain

$$\|X_{st}\|_{H^\alpha}^2 \lesssim_{w, \varepsilon, T} |t - s|^\gamma \sup_k |k|^{2+2\alpha} \sum_{\star} |k_1 k_2 k_3|^{-2\alpha} |k - k_2|^{-2\rho} |k - k_3|^{-2\rho}$$

with $s, t \in [0, T]$, then is sufficient to prove that

$$I = \sup_k |k|^{2\alpha+2\theta} \sum_{\star} |k_1 k_2 k_3|^{-2\alpha} |k - k_3|^{-2\rho} |k - k_2|^{-2\rho} < +\infty$$

for that we will need the following lemma.

Lemma 3.5.17. For $\rho > \max(1/2, \theta/2)$ and $\alpha \geq \frac{1}{2}\theta$ the following inequality holds:

$$\sum_{l \neq 0, k} |l|^{-2\alpha} |k-l|^{-2\rho} \lesssim |k|^{-\theta}$$

Proof. The proof of this lemma is very similar to the proof of the Lemma 3.5.4. We begin by decomposing our sum in the following way :

$$\begin{aligned} \sum_{l \neq k, 0} |l|^{-2\alpha} |k-l|^{-2\rho} &= \sum_{l \neq 0, k; |l-k| \leq |l|} |l|^{-2\alpha} |k-l|^{-2\rho} + \sum_{l \neq 0, k; |k-l| \geq |l|} |l|^{-2\alpha} |k-l|^{-2\rho} \\ &\leq |k|^{-\theta} \sum_{l \neq 0} \frac{1}{|l|^{2\rho}} + |k|^{-\theta} \sum_l \frac{1}{|l|^{2\alpha+2\rho-\theta}} \lesssim_{\theta, \alpha, \rho} \frac{1}{|k|^\theta} \end{aligned}$$

□

Lemma 3.5.18. Let $\rho > \max(\theta, 1/2)$ and $\alpha \geq \frac{1}{2}\theta$ then $I < +\infty$.

Proof. If we use the fact that $|k|^{2\alpha} \lesssim |k_1|^{2\alpha} + |k_2|^{2\alpha} + |k_3|^{2\alpha}$ we obtain

$$\begin{aligned} I &\lesssim \sup_k |k|^{2\theta} \left(\sum_* |k_2 k_3|^{2\alpha} |k-k_2|^{-2\rho} |k-k_3|^{-2\rho} + \sum_* |k_1 k_2|^{-2\alpha} |k-k_2|^{-2\rho} |k_2-k_1|^{-2\rho} \right) \\ &\lesssim \sup_k |k|^{2\theta} \left(\sum_{k_2 \neq 0, k} |k_2|^{-2\alpha} |k-k_2|^{-2\rho} \right)^2 + \sup_k |k|^{2\theta} \sum_* |k_1 k_2|^{-2\alpha} |k-k_2|^{-2\rho} |k_2-k_1|^{-2\rho} \\ &= I_1 + I_2 \end{aligned}$$

Now by the Lemma 3.5.17 we have

$$I_1 = \sup_k |k|^{2\theta} \left(\sum_{k_2 \neq 0, k} |k_2|^{-2\alpha} |k-k_2|^{-2\rho} \right)^2 < +\infty$$

for $\rho > \max(\theta, 1/2)$, $\alpha > \frac{1}{2}\theta$. It remains to treat the second term which requires a bit more work:

$$\begin{aligned} I_2 &= \sup_k |k|^{2\theta} \sum_{k_1, k_2} |k_2 k_1|^{-2\alpha} |k_2-k|^{-2\rho} |k_2-k_1|^{-2\rho} \\ &= \sup_k |k|^{2\theta} \sum_{k_2} |k_2|^{-2\alpha} |k-k_2|^{-2\rho} \sum_{k_1} |k_1|^{-2\alpha} |k_2-k_1|^{-2\rho} \lesssim \sup_k |k|^{2\theta} \sum_{k_2} |k_2|^{-2\alpha-\theta} |k-k_2|^{-2\rho} \\ &\lesssim \sup_k |k|^{2\theta} \left(\sum_{k_2; |k-k_2| \leq |k_2|} |k_2|^{-2\alpha-\theta} |k-k_2|^{-2\rho} + \sum_{k_2; |k-k_2| \geq |k_2|} |k_2|^{-2\alpha-\theta} |k-k_2|^{-2\rho} \right) \\ &\lesssim \sup_k (|k|^{\theta-2\alpha}) + \sup_k \sum_{k_2; |k-k_2| \geq |k_2|} |k_2|^{-2\alpha-\theta} |k-k_2|^{-2\rho+2\theta} \\ &\lesssim \sup_k (|k|^{\theta-2\alpha}) + \sum_{k_2 \neq 0} |k_2|^{-2\alpha-2\rho+\theta} < +\infty \end{aligned}$$

if $\rho > \max(1/2, \theta)$, $\alpha \geq \frac{1}{2}\theta$. □

Theorem 3.5.19. Let $\rho > \max(1/2, \theta)$ then there exist $\gamma > 1/2$ such that $X \in C^\gamma([0, T], H^\alpha)$ for all $T > 0$ and $\alpha \geq \frac{1}{2}\theta$.

3.6 Global existence for the modulated KdV in Sobolev spaces with non-negative index

In this section we will concentrate on the periodic modulated KdV equation on \mathbb{T} and \mathbb{R} and on the NLS equation on \mathbb{T} . We prove the existence of a global solution for an initial data $\phi \in H^\alpha(\mathbb{T})$ for any $\alpha \geq 0$ in spite of the fact that the modulation breaks all conservation law apart from that associated to the L^2 norm.

Let us recall how we can establish the L^2 -norm conservation in this case.

Proposition 3.6.1. *Let u the local solution of the periodic modulated KdV equation with initial data $\phi \in L^2(\mathbb{T})$ and $v_t = (U_t^w)^{-1}u_t$ for $t \in [0, T]$ where $T = T(\|\phi\|_{L^2})$ is the life-time of the local solution then we have $\|v_t\|_2 = \|\phi\|_2$ for all $t \in [0, T]$ and we can extend the local solution into a global one.*

Proof. Let ψ a smooth function then we have by integration by part formula

$$\begin{aligned} \langle \psi, X_{st}(\psi, \psi) \rangle_{L^2} &= \int_s^t d\sigma \left(\int_{\mathbb{T}} \psi (U_\sigma^w)^{-1} \partial_x (U_\sigma^w \psi)^2 \right) = \int_s^t d\sigma \int_{\mathbb{T}} U_\sigma^w \psi \partial_x (U_\sigma^w \psi)^2 \\ &= - \int_s^t d\sigma \int_{\mathbb{T}} \partial_x (U_\sigma^w \psi) (U_\sigma^w \psi)^2 = 0 \end{aligned}$$

and then we have that

$$\|v_t\|_2^2 = \|v_s\|_2^2 + \langle v_s, X_{st}(v_s, v_s) \rangle + \|v_t - v_s\|_2^2 + R_{st} = \|v_s\|_2^2 + \|v_t - v_s\|_2^2 + R_{st}$$

where $|R_{st}| \lesssim |t - s|^{2\gamma}$ with $\gamma > 1/2$ and then we can see that $|\|v_t\|_2^2 - \|v_s\|_2^2| \lesssim |t - s|^{\gamma+1/2}$ which give us our result. \square

Now using this proposition and the smoothing effect for X we obtain the global existence for the equation in the Sobolev space with non negative index.

Proposition 3.6.2. *Let $\alpha \geq 0$, $\phi \in H^\alpha$ and $T > 0$ then there exist a unique $v \in C^{1/2}([0, T], H^\alpha)$ such that $v_t = \phi + \int_0^t X_{d\sigma}(v_\sigma, v_\sigma) d\sigma$ holds for all $t \in [0, T]$*

Proof. Let $\phi \in L^2$ then we using the local existence result we know that there exists $\kappa = \kappa(\|\phi\|_{L^2}) > 0$ and a unique $v \in C^{1/2}([0, \kappa], H^\alpha)$ solution of the Young equation associated to X in $[0, \kappa]$. Moreover we have the conservation law $\|v_t\|_{L^2} = \|v_0\|_{L^2}$ and this allow us to iterate our local result to obtain global a solution defined on $[0, T]$ for arbitrary $T > 0$. Now to extend this local we use the lemma 3.5.3 in fact let $\alpha > 0$ and $\phi \in H^\alpha$ then is obvious that $\phi \in L^2$ and using the Lemma 3.5.3 and the fact that v satisfy the Young equation we have easily that

$$\|v_t - v_s\|_{H_\beta} \lesssim_{T, \|X\|_{C^\gamma([0, T], \mathcal{L}(L^2, H_\beta))}} |t - s|^\gamma (\|v\|_{C^{1/2}([0, T], L^2)} + \|\phi\|_{L^2})^2$$

for all $0 < \beta < 2\rho - 3/2$ and then $v \in C^{1/2}([0, T], H_\beta)$. By iterating this result we see that $v \in C^{1/2}(H^\alpha, [0, T])$ and this finishes the proof. \square

Remark 3.6.3. *The bound of the operator X allows us to construct a local solution even when the initial data is in a negative Sobolev space ($\alpha > -\rho$). The method presented in this section gives the possibility to construct a global solution only in the case when we deal with initial data in a positive regularity space. In the next section we present an adaptation of the almost conservation law method developed in [18] which will allow to control global solutions in negative regularity spaces.*

3.6.1 Cubic NLS equation

Here we obtain global solution of positive regularity for the modulated cubic NLS equation.

Lemma 3.6.4. *We have that for any $\phi \in H^0$ and any $0 \leq s \leq t$:*

$$\langle \phi, X_{st}(\phi, \phi, \phi) \rangle \in \mathbb{R}$$

and there exists a constant C_R such that for all $\phi \in H^0$ with $\|\phi\|_{H^0} \leq R$ we have

$$\| \|\phi + X_{s,t}(\phi, \phi, \phi)\|_{H^0} - \|\phi\|_{H^0} \| \leq C_R |t - s|^{2\gamma}.$$

Proof. We start observing that for smooth ϕ :

$$\langle \phi, \dot{X}_s(\phi, \phi, \phi) \rangle = \langle \phi, U_{-s}^w(|U_s \phi|^2 U_s^w \phi) \rangle = \langle U_s^w \phi, |U_s^w \phi|^2 U_s^w \phi \rangle \in \mathbb{R}$$

Integrating in s and extending to arbitrary $\phi \in H^0$ we get the claim. Then if $\phi \in H^0$ we have

$$\|\phi + X_{s,t}(\phi, \phi, \phi)\|_{H^0}^2 = \|\phi\|_{H^0}^2 + \|X_{st}(\phi, \phi, \phi)\|_{H^0}^2$$

so

$$\| \|\phi + X_{s,t}(\phi, \phi, \phi)\|_{H^0} - \|\phi\|_{H^0} \| \leq \frac{\|X_{st}(\phi, \phi, \phi)\|_{H^0}^2}{\|\phi\|_{H^0}} \lesssim |t - s|^{2\gamma} \|\phi\|_{H^0}^5.$$

□

with $\gamma > 1/2$. At this point the Lemma (3.4.1) allow us to obtain the conservation law for our equation in H^0 and extend in this space the local solution in a global solution. Now we prove the existence of a global solution for an initial data in H^α with $\alpha > 0$.

Proposition 3.6.5. *Let $\phi \in H^\alpha$ and $T > 0$ then there exist $v \in \mathcal{C}_T^{1/2} H^\alpha$ such that the following equality holds*

$$v_t = \phi + \int_0^t X_{d\sigma}(v_\sigma, v_\sigma, v_\sigma)$$

for all $t \in [0, T]$.

Proof. The existence of the global solution for an initial data in $\phi \in L^2$ is given by the conservation law $\|v_t\|_{L^2} = \|\phi\|_{L^2}$. Now we will focus on the case when $\phi \in H^\alpha$ and we decompose our modulated Schrödinger operator X as

$$X^2 = X^{21} + X^{22} + X^{23}$$

with

$$\mathcal{F}X_{st}^{2j}(\psi_1, \psi_2, \psi_3)(k) = \sum_{(k_1, k_2, k_3) \in D_j^k} \hat{\psi}_1(k_1)^* \hat{\psi}_2(k_2) \hat{\psi}_3(k_3) \Phi_{st}^w((k - k_2)(k - k_3))$$

for $j \in \{1, 2, 3\}$ and where $D_1^k = \{-k_1 + k_2 + k_3 = k, k_2 \neq k, k_3 \neq k\} \cap \{|k_1| \geq |k|/3\}$, $D_2^k = \{-k_1 + k_2 + k_3 = k, k_2 \neq k, k_3 \neq k\} \cap \{|k_1| < |k|/3, |k_2| \geq |k|/3\}$ and $D_3^k = \{-k_1 + k_2 + k_3 = k, k_2 \neq$

$k, k_3 \neq k\} \cap \{|k_1| < |k|/3, |k_2| < |k|/3, |k_3| \geq |k|/3\}$. Using Cauchy-Schwarz inequality we have the following bound

$$\begin{aligned} \|X_{st}^{2j}(\psi_1, \psi_2, \psi_3)\|_{H^{\beta+\varepsilon}}^2 &\leq \|\psi_j\|_{H^{\beta+\varepsilon}} \Pi_{i \neq j} \|\psi_i\|_{H^\beta} \sup_k |k|^{2\beta+2\varepsilon} \\ &\quad \sum_{D_j^k} |k_j|^{-2\beta-2\varepsilon} \left(\Pi_{i \neq j} |k_i|^{-2\beta} \right) |\Phi_{st}^w(2(k-k_2)(k-k_3))|^2 \end{aligned} \quad (3.16)$$

for $\beta, \varepsilon \geq 0$ then using the fact that $|k| \lesssim |k_j|$ on D_j^k and using the ρ -irregularity of w we obtain

$$\|X_{st}^{2j}(\psi_1, \psi_2, \psi_3)\|_{H^{\beta+\varepsilon}}^2 \lesssim_{\alpha, \varepsilon} \|\Phi^w\|_{W_T^{\rho, \gamma}} \|t-s\|^\gamma \|\psi_j\|_{H^{\beta+\varepsilon}} \Pi_{i \neq j} \|\psi_i\|_{H^\alpha} \left(\sum_{l \neq 0} |l|^{-2\rho} \right)^2 < +\infty$$

when $\rho > 1/2$ and then we have that for all $T > 0$ there exist $\gamma > 1/2$ such that

$$X^{21} \in C^\gamma([0, T], \mathcal{L}^3(H^{\alpha+\varepsilon} \times H^\alpha \times H^\alpha, H^{\alpha+\varepsilon}))$$

for all $\beta, \varepsilon \geq 0$, of course the same statement holds for the other operators. Now let us define the norm on $C^{1/2}([0, T], H^\beta)$ by $\|\psi\|_\beta = \|\psi\|_{C^{1/2}([0, T], H^\beta)} + \|\psi\|_{C^0([0, T], H^\beta)}$ for $\beta \in \mathbb{R}$ and the map Γ by :

$$\Gamma(\psi) := \phi + \int_0^t X_{d\sigma}(\psi_\sigma)$$

for $\psi \in C^{1/2}([0, T], H^\alpha)$. By a simple computation we see that

$$\|\Gamma(\psi)\|_0 \lesssim_{\gamma, w} \|\phi\|_L^2 + T^{\gamma-1/2} \|\psi\|_0^3$$

Now if $0 < T \leq T_1$ is sufficiently small then the equation $r = \|\phi\|_{L^2} + T^{\gamma-1/2} r^3$ admits a positive solution $r^* > 0$ and the closed ball $B_{r^*} =: \{\psi \in C^{1/2}([0, T], L^2); \|\psi\|_0 \leq r^*\}$ is invariant by Γ . Moreover we have that

$$\|\Gamma(\psi_1) - \Gamma(\psi_2)\|_0 \lesssim_{\gamma, w} T^{\gamma-1/2} \|\psi_1 - \psi_2\|_0 (1 + (r^*)^2)$$

and then if $T \leq T_2 \leq T_1$ sufficiently small, Γ is a strict contraction on B_{r^*} which admits a unique fixed point v . Let $\Gamma_{B_{r^*}}$ the restriction of Γ on B_{r^*} and use the fact that

$$\Gamma(\psi)_t = \phi + 2 \int_0^t v_\sigma \|v_\sigma\|_{L^2}^2 d\sigma + \sum_{j \in \{1, 2, 3\}} \int_0^t X_{d\sigma}^{2j}(\psi_\sigma)$$

and the regularity of X^{2j} to deduce that

$$\|\Gamma_{B_{r^*}}(\psi)\|_\alpha \lesssim \|\phi\|_{H^\alpha} + T^{\gamma-1/2} (r^*)^2 \|\psi\|_\alpha.$$

Then $B(0, R) := \{\psi \in C([0, T], H^\beta); \|\psi\| \leq R\}$ is invariant by $\Gamma_{B_{r^*}}$ for $T^* = T^*(\|\phi\|_{L^2})$ small enough depending only on $r^* > 0$. Being the ball closed in $C^{1/2}([0, T], L^2)$ we have that $v \in C^{1/2}([0, T^*], H^\alpha)$ and now is suffice to iterate this result to obtain a global solution in H^α . \square

3.6.2 KdV on \mathbb{R}

Here we go back to the KdV equation to prove the global existence of the modulated KdV equation in non-negative Sobolev space. Now as in the proposition 3.6.5 we will decompose the modulated operator X of the KdV equation in the following way

$$X = X^1 + 2X^2 + X^3 + X^4$$

where

$$\mathcal{F}X_{st}^1(\psi_1, \psi_2) = ix\mathbb{I}_{|x|\geq 1} \int_{|y|, |x-y|\geq 1/2} \hat{\psi}_1(y)\hat{\psi}_2(x-y)\Phi_{st}^w(xy(x-y))dy$$

$$\mathcal{F}X_{st}^2(\psi_1, \psi_2) = ix\mathbb{I}_{|x|\geq 1} \int_{|y|<1/2} \hat{\psi}_1(y)\hat{\psi}_2(x-y)\Phi_{st}^w(xy(x-y))dy$$

$$\mathcal{F}X_{st}^3(\psi_1, \psi_2) = ix\mathbb{I}_{|x|<1} \int_{|y|\geq 2} \hat{\psi}_1(y)\hat{\psi}_2(x-y)\Phi_{st}^w(xy(x-y))dy$$

and

$$\mathcal{F}X_{st}^4(\psi_1, \psi_2) = ix\mathbb{I}_{|x|<1} \int_{|y|<2} \hat{\psi}_1(y)\hat{\psi}_2(x-y)\Phi_{st}^w(xy(x-y))dy$$

As in the periodic case the operator X^1 have some smoothing effect more precisely

$$\|X_{st}^1(\psi_1, \psi_2)\|_{\alpha+\varepsilon} \leq |t-s|^\gamma \|\phi\|_\alpha \|\phi_2\|_\alpha$$

for $\alpha, \varepsilon > 0$ and $\varepsilon > 0$ small enough moreover we have the following bound

$$\|X_{st}^1(\psi_1, \psi_2)\|_\beta \lesssim |t-s|^\gamma (\|\psi_1\|_\alpha \|\psi_2\|_\beta + \|\psi_1\|_\beta \|\psi_1\|_\alpha)$$

for all $\alpha, \beta \geq 0$. We will focus on the operator X^2 . By a usual argument we have that

$$\|X_{st}^2(\psi_1, \psi_2)\|_{\alpha+\varepsilon} \lesssim |t-s|^\gamma \|\psi_1\|_\alpha \|\psi_2\|_{\alpha+\varepsilon} \sup_{|x|\geq 1} |x|^2 \int_{|y|\leq 1/2} (1+|yx^2|)^{-2\rho} < +\infty$$

when $\rho > 1/2$ then for all $\alpha, \varepsilon > 0$ and $T > 0$ we have $X^2 \in C^\gamma([0, T], \mathcal{L}^2(H^\alpha \times H^{\alpha+\varepsilon}, H^{\alpha+\varepsilon}))$. For the third operator X^3 we have the bound

$$\|X_{st}^3(\psi_1, \psi_2)\|_{\alpha+\varepsilon} \lesssim |t-s|^\gamma \|\psi_1\|_\alpha \|\psi_2\|_\alpha \sup_{|x|<1} |x|^2 \int_{|y|>2} (1+y^2|x|)^{-2\rho} < +\infty$$

for $\alpha, \varepsilon > 0$ and then $X^3 \in C^\gamma([0, T], \mathcal{L}^2(H^\alpha \times H^\alpha, H^{\alpha+\varepsilon}))$ for all $T > 0$. Of course we have the same regularity for the operator X^4 and the global existence for the KdV equation follow by the same argument used in the proof of Proposition 3.6.5.

3.7 Global existence for the modulated KdV equation in negative Sobolev spaces

In this section we prove the global existence for the KdV equation with rough initial condition $\phi \in H^\alpha(\mathbb{T})$ with negative α . For the unmodulated equation with initial condition in negative Sobolev spaces [18] proves global existence using the so called ‘‘I-method’’. In this section we try to adapt this technique to our context. To do so we have to study the rescaled Cauchy problem associated to the modulated equation and then give an almost conservation law for the rescaled local solution.

3.7.1 Rescaled equation

Here we study the rescaled solution of our equation we know in the deterministic case if u is a local solution of KdV equation on $[0, T]$ with initial data $\phi \in H^\alpha(\mathbb{T})$ then the function defined by $u^\lambda(t, x) = \lambda^{-2}u(\lambda^{-3}t, \lambda^{-1}x)$ is once again a solution of the KdV equation on $[0, \lambda^3T]$ with initial data $\phi^\lambda(x) = \lambda^{-2}\phi(\lambda^{-1}x)$ and vice versa. We proceed along the same lines in our setting. By a formal computation we see that if u is a local solution for the modulated KdV equation on the torus then the rescaled function satisfies formally the equation

$$\frac{d}{dt}u_t^\lambda = \partial_x^3 u^\lambda \frac{dw_t^\lambda}{dt} + \partial_x(u_t^\lambda)^2$$

with $w_t^\lambda = \lambda^3 w_{\lambda^{-3}t}$ and $u^\lambda(0, x) = \lambda^{-2}\phi(\lambda^{-1}x)$. We must also pay attention to the fact that space has changed because the new solution is λ -periodic function and not a 1-periodic function. Let us introduce some definition and conventions that will be used later. We begin by define the Fourier transform of function on $\mathbb{T}_\lambda = [0, \lambda]$ by

$$\hat{f}(k) = \int_0^\lambda f(x) e^{-2i\pi kx} dx$$

for $k \in \mathbb{Z}/\lambda$ then the usual properties of the Fourier transform holds:

1. $\int_0^\lambda |f(x)|^2 dx = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} |\hat{f}(k)|^2$
2. $f(x) = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} \hat{f}(k) e^{2i\pi kx}$
3. $\mathcal{F}(fg)(k) = \frac{1}{\lambda} \sum_{k_1, k_2 \in \mathbb{Z}/\lambda; k_1+k_2=k} \hat{f}(k_1) \hat{g}(k_2)$

and then we define the Sobolev space $H^\alpha(\mathbb{T}_\lambda)$ by the set of the distribution $f \in \mathcal{S}'(\mathbb{T}_\lambda)$ such that $\hat{f}(0) = 0$ and

$$\|f\|_{H^\alpha(0, \lambda)}^2 = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} |k|^{2\alpha} |\hat{f}(k)|^2 < +\infty.$$

Now we are able to study our rescaled Cauchy problem given by

$$\begin{cases} \frac{d}{dt}u_t^\lambda = \partial_x^3 u_t^\lambda \frac{dw_t^\lambda}{dt} + \partial_x(u_t^\lambda)^2 \\ u(0, x)^\lambda = \psi(x) \in H^\alpha(0, \lambda) \end{cases} \quad (3.17)$$

As usual we write this last equation as

$$v_t^\lambda = \psi + \int_0^t X_{d\sigma}^\lambda(v_\sigma, v_\sigma)$$

with $v_t^\lambda = (U_t^{w^\lambda})^{-1}u_t^\lambda$. Now to solve this last equation by the fixed point method we have to estimate the Hölder norm of the modulated operator X^λ given by :

$$X_{st}^\lambda(\psi_1, \psi_2) = \int_s^t (U_\sigma^{w^\lambda})^{-1} \partial_x (U_\sigma^{w^\lambda} \psi_1 U_\sigma^{w^\lambda} \psi_2) d\sigma$$

for $\psi_1, \psi_2 \in H^\alpha(0, \lambda)$.

Proposition 3.7.1. *Let $\alpha > -\rho$ and $\rho > 3/4$ then there exist $\gamma > 1/2$ such that for all $T > 0$ the following inequality holds.*

$$\|X_{st}^\lambda\|_{\mathcal{L}^2 H^\alpha(0,\lambda)} \leq C_T \|\Phi^w\|_{\mathcal{W}_{\rho,T}^\gamma} \lambda^{\alpha+3/2-3\gamma} |t-s|^\gamma$$

for all $(s, t) \in [0, \lambda^3 T]$, with $C_T > 0$ is a finite positive constant.

Proof. Let $\psi_1, \psi_2 \in H^\alpha(0, \lambda)$ then by a simple computation we have that

$$|X_{st}^\lambda(\psi_1, \psi_2)|_{H^\alpha(0,\lambda)}^2 = \lambda^{-3} \sum_{k \in \mathbb{Z}/\lambda} |k|^{2\alpha+2} \left| \sum_{k_1+k_2=k} \phi_1(k_1) \phi_2(k_2) \Phi_{st}^\lambda(k_1 k_2 k) \right|^2$$

with $\Phi_{st}^\lambda(a) = \int_s^t e^{iaw_\sigma^\lambda} d\sigma$ and then using Cauchy-Schwarz inequality we obtain

$$|X_{st}^\lambda(\psi_1, \psi_2)|_{H^\alpha(0,\lambda)}^2 \leq \lambda^{-1} \sum_{k \in \mathbb{Z}/\lambda} |k|^{2\alpha+2} \sup_{k_1+k_2=k} \frac{|\Phi_{st}^\lambda(k k_1 k_2)|^2}{|k_1|^{2\alpha} |k_2|^{2\alpha}} \|\psi_1\|_{H^\alpha(0,\lambda)} \|\psi_2\|_{H^\alpha(0,\lambda)}$$

Now using the (ρ, γ) irregularity of w we can see that

$$\left| \int_s^t e^{i k k_1 k_2 w_\sigma^\lambda} d\sigma \right| = \lambda^3 \left| \int_{\lambda^{-3}s}^{\lambda^{-3}t} e^{i \lambda^3 k k_1 k_2 w_\sigma} d\sigma \right| \leq C_{w,T} \lambda^{3-3(\gamma+\rho)} |t-s|^\gamma |k k_1 k_2|^{-2\rho+\varepsilon}$$

and then we have

$$\begin{aligned} |X_{st}^\lambda|_{\mathcal{L}^2 H^\alpha(0,\lambda)}^2 &\leq C_{w,T}^2 \lambda^{5-6(\gamma+\rho)} |t-s|^{2\gamma} \sum_{k \in \mathbb{Z}/\lambda} |k|^{2-4\rho} \sup_k \left(\frac{|k|}{|k_1 k_2|} \right)^{2\alpha+2\rho} \\ &\leq C_{w,T}^2 \lambda^{3-6\gamma+2\alpha} |t-s|^{2\gamma} \sum_{k \in \mathbb{Z}^*} |k|^{2-4\rho} < +\infty \end{aligned}$$

and this finishes the proof. \square

Corollary 3.7.2. *Let $\lambda > 0$ then u is a local solution of the modulated KdV equation on the Torus with initial data $\phi \in H^\alpha(\mathbb{T})$ and life time $T > 0$ if and only if $u^\lambda(t, x) = \lambda^{-2} u(\lambda^{-3}t, \lambda^{-1}x)$ is a local solution of the rescaled equation with initial data $\phi^\lambda(x) = \lambda^{-2} \phi(\lambda^{-1}x)$ with life time $\lambda^3 T$*

Proof. Let u a solution of the modulated KdV equation on the Torus then by definition we have that $v_t = U_t^{-1} u_t \in C^{1/2}([0, T], H^\alpha(\mathbb{T}))$ and by a simple computation we have that $v_t^\lambda = (U_t^{w^\lambda})^{-1} u_t^\lambda \in C^{1/2}([0, \lambda^3 T], H^\alpha(\mathbb{T}_\lambda))$. Now we have to check that the rescaled function $u^\lambda(t, x) = \lambda^{-2} u(\lambda^{-3}t, \lambda^{-1}x)$ satisfy the equation. but by a simple computation we have that $\hat{v}_t^\lambda(k) = \frac{1}{\lambda} \hat{v}_{\lambda^{-3}t}(\lambda k)$ and

$$\begin{aligned} \hat{X}_{st}^\lambda(\psi_1, \psi_2)(k) &= ik \lambda^{-1} \sum_{k_1+k_2=k; k_1, k_2 \in \mathbb{Z}/\lambda} \hat{\psi}_1(k_1) \hat{\psi}_2(k_2) \int_s^t e^{i k k_1 k_2 \lambda^3 w_{\lambda^{-3}\sigma}} d\sigma \\ &= ik \lambda^2 \sum_{l_1+l_2=\lambda k; l_1, l_2 \in \mathbb{Z}} \hat{\psi}_1\left(\frac{l_1}{\lambda}\right) \hat{\psi}_2\left(\frac{l_2}{\lambda}\right) \Phi_{\lambda^{-3}s, \lambda^{-3}t}^w(l_1 l_2 \lambda k) \\ &= \lambda^{-1} \hat{X}_{\frac{s}{\lambda^3}, \frac{t}{\lambda^3}}(\psi_1(\lambda \cdot), \psi_2(\lambda \cdot))(\lambda k) \end{aligned}$$

for all $k \in \mathbb{Z}/\lambda$ and all $\psi_1, \psi_2 \in H^\alpha(0, \lambda)$. Let $\lambda^3 t \in [0, \lambda^3 T]$ and $\Pi = (t_i)_i$ a partition of the interval $[0, \lambda^3 t]$ then of course $\Pi^\lambda = (\lambda^{-3} t_i)$ is a dissection of $[0, t]$ and using the relation given above we can easily see that

$$\hat{X}_{t_i t_{i+1}}^\lambda(v_{t_i}^\lambda, v_{t_i}^\lambda) = \lambda^{-1} \hat{X}_{\lambda^{-3} t_{i+1}, \lambda^{-3} t_i}(v_{\lambda^{-3} t_i}, v_{\lambda^{-3} t_i})(\lambda k)$$

and using the fact that u is a solution of the 1-periodic equation we can easily see that

$$v_t = \psi + \lim_{|\Pi^\lambda| \rightarrow 0} \sum_{t_i} X_{\lambda^{-3} t_{i+1}, \lambda^{-3} t_i}(v_{\lambda^{-3} t_i}, v_{\lambda^{-3} t_i})$$

in $H^\alpha(\mathbb{T})$ and then

$$v_t^\lambda = \phi^\lambda + \lim_{|\Pi| \rightarrow 0} \sum_i X_{t_i t_{i+1}}^\lambda(v_{t_i}^\lambda, v_{t_i}^\lambda)$$

in $H^\alpha(0, \lambda)$ with $\phi^\lambda(x) = \lambda^{-2} \phi(\lambda^{-1} x)$ of course this give us our result by the convergence of Riemann sum to the Young integral. \square

3.7.2 Commutator estimates and almost conservation law

The previous section tell us if we want to construct a global solution to the 1-periodic KdV equation is sufficient to prove that for every $T > 0$ and a suitable $\lambda > 1$ we are able to construct a global solution to the rescaled equation. For that let us introduce the spatial Fourier multiplier operator I which act like the identity on the low frequencies and like a smoothing operator of order $|\alpha|$ on the high frequencies more precisely we choose a smooth function m such that

$$m(\xi) = \begin{cases} 1, & |\xi| < 1 \\ |\xi|^\alpha, & |\xi| \geq 10 \end{cases}$$

and for $N \gg 1$ we define I by $\mathcal{F}(I\phi)(k) = m(\frac{k}{N}) \hat{\phi}(k)$ for every $\phi \in H^\alpha(\mathbb{T}_\lambda)$. Now the so called I method to proof the global solution is based on some estimation of the modified energy $\|Iu_t\|_{L^2}$. Let us begin by expand our modified energy

$$\|Iv_t\|_2^2 - \|Iv_s\|_2^2 = \langle Iv_s, IX_{st}^\lambda(v_s, v_s) - X_{st}^\lambda(Iv_s, Iv_s) \rangle + R_{st}$$

with $R_{st} \lesssim |t-s|^{\gamma+1/2}$ then to control R is sufficient to control the first order term of our expansion and for that we have the following commutator estimates. To simplify the notation let $m_N(k) = m(k/N)$.

Proposition 3.7.3. *Let $\alpha \in (-\rho, 0)$, $\rho > 3/4$ there exist $\gamma > 1/2$ such that for all $T > 0$ the following inequality holds a*

$$\|IX_{st}^\lambda(\psi_1, \psi_2) - X_{st}^\lambda(I\psi_1, I\psi_2)\|_2 \leq C_T \|\Phi^w\|_{W_{\rho,T}^\gamma} |t-s|^\gamma N^{-\rho} \lambda^{-\rho+3/2-3\gamma} \|I\psi_1\|_2 \|I\psi_2\|_2$$

for all $s, t \in [0, \lambda^3 T]$, $\lambda > 0$ and $\psi_1, \psi_2 \in H^\alpha(0, \lambda)$, with $C_{w,T} > 0$ a finite constant.

Proof. By a simple computation we have

$$\|IX_{st}^\lambda(\psi_1, \psi_2) - X_{st}^\lambda(I\psi_1, I\psi_2)\|_2^2 = \frac{1}{\lambda^3} \sum_{k \in \mathbb{Z}/\lambda} |k|^2 \left| \sum_{k_1+k_2=k} \Phi_{st}^\lambda(k k_1 k_2) \hat{\psi}_1(k_1) \hat{\psi}_2(k_2) (m_N(k) - m_N(k_1) m_N(k_2)) \right|^2$$

and then we split $\{k_1 + k_2 = k\} = \cup_{i=0,1,2,3} D_i$ with $D_0 = \{k_1 + k_2 = k; |k_1| \leq N/2, |k_2| \leq N/2\}$, $D_1 = \{k_1 + k_2 = k; |k_1| \geq N/2, |k_2| \leq N/2, |k| \leq N/4\}$, $D_2 = \{k_1 + k_2 = k; |k_1| \geq N/2, |k_2| \leq N/2, |k| \geq N/4\}$ and $D_3 = \{k_1 + k_2 = k; |k_1| \geq N/2, |k_2| \geq N/2\}$. Is not difficult to see that the region D_0 give a zero contribution. Using the Cauchy-Schwarz inequality we can see that

$$\lambda^{-3} \sum_{k \in \mathbb{Z}/\lambda} |k|^2 \left| \sum_{D_i} \Phi_{st}^\lambda(kk_1k_2) \hat{\psi}_1(k_1) \hat{\psi}_2(k_2) (m_N(k) - m_N(k_1)m_N(k_2)) \right|^2 \leq h_N^{\lambda,i} \|I\psi_1\|_2^2 \|I\psi_2\|_2^2$$

with

$$h_N^{\lambda,i} = \lambda^{-1} \sum_{|k| \leq N/4} |k|^2 \sup_{D_i} \frac{|\Phi_{st}^\lambda(kk_1k_2)|^2 |m_N(k) - m_N(k_2)m_N(k_1)|^2}{|m_N(k_1)|^2 |m_N(k_2)|^2}$$

for $i = 1, 3$ and

$$h_N^{\lambda,2} = \lambda^{-1} \sup_{|k| \geq N/4} |k|^2 \sum_{D_2} \frac{|\Phi_{st}^\lambda(kk_1k_2)|^2 |m_N(k) - m_N(k_2)m_N(k_1)|^2}{|m_N(k_1)|^2 |m_N(k_2)|^2}.$$

We begin by bounding the term $h_N^{\lambda,2}$:

$$h_N^{\lambda,2} \leq C_{w,T} |t - s|^{2\gamma} \lambda^{5-6(\gamma+\rho)} N^{2\alpha} \sup_{|k| \geq N/4} |k|^{2-2\rho} \sum_{D_2} \frac{|m_N(k) - m_N(k_1)|^2}{|k_1|^{2\alpha+2\rho} |k_2|^{2\rho}},$$

then by the mean value theorem we have $|m_N(k) - m_N(k_1)| \lesssim |k_2|/N$ and if we interpolate this bound with the trivial bound $|m_N(k) - m_N(k_1)| \lesssim 1$ we obtain

$$|m_N(k) - m_N(k_1)| \lesssim N^{-2\alpha(1-\theta)-2\theta} |k|^{2\alpha(1-\theta)} |k_2|^{2\theta}.$$

If $\rho \in (3/4, 3/2)$ we can choose $\theta = \rho - 1/2 - \varepsilon \in [0, 1]$ for $\varepsilon > 0$ small enough to obtain

$$\begin{aligned} h_N^{\lambda,2} &\lesssim C_{w,T} |t - s|^{2\gamma} \lambda^{5-6(\gamma+\rho)} N^{2\alpha\theta-2\theta} \sup_{|k| \geq N/4} |k|^{2-4\rho-2\alpha\theta} \sum_{D_2} (|k||k_1|^{-1})^{2\alpha+2\rho} |k_2|^{-1-\varepsilon} \\ &\lesssim C_{w,T} |t - s|^{2\gamma} N^{3-6\rho+\varepsilon} \lambda^{6-6(\gamma+\rho)+\varepsilon} \\ &\lesssim C_{w,T} |t - s|^{2\gamma} N^{-2\rho} \lambda^{-2\rho+3-6\gamma}. \end{aligned}$$

If $\rho > 3/2$ we use only the trivial bound to get

$$\begin{aligned} h_N^{\lambda,2} &\lesssim C_{w,T} |t - s|^{2\gamma} \lambda^{5-6(\gamma+\rho)} \sup_{|k| \geq N/4} |k|^{2-4\rho} \sum_{D_2} (|k||k_1|^{-1})^{2\alpha+2\rho} |k_2|^{-2\rho} \\ &\lesssim C_{w,T} |t - s|^{2\gamma} N^{2-4\rho} \lambda^{5-6\gamma-4\rho} \\ &\lesssim C_{w,T} |t - s|^\gamma N^{-2\rho} \lambda^{3-6\gamma-2\rho}. \end{aligned}$$

Now we will focus on the term $h_N^{\lambda,1}$ in fact by a simple computation we can see that in this region we have $|k_2| \in [N/4, N/2]$ and then

$$\begin{aligned} h_N^{\lambda,1} &\lesssim C_{w,T} |t - s|^\gamma \lambda^{5-6(\gamma+\rho)} \sum_{|k| \leq N/4} |k|^{2-4\rho} \sup_{D_1} |k|^{2\alpha+2\rho} |k_1|^{-2\alpha-2\rho} |k_2|^{-2\rho} \\ &\lesssim C_{w,T} |t - s|^\gamma \lambda^{3-6\gamma-2\rho} N^{-2\rho}. \end{aligned}$$

It remains to bound $h_N^{\lambda,3}$. We begin by noting that in this region we have $|m_N(k) - m_N(k_1)m_N(k_2)|^2 \lesssim |m_N(k)|^2 + N^{-4\alpha}|k_1k_2|^{2\alpha}$ and then

$$\begin{aligned} h_N^{\lambda,3} &\lesssim C_{w,T}|t-s|^{2\gamma}\lambda^{5-6(\gamma+\rho)}(N^{4\alpha}\sum_k|k|^{2-4\rho-2\alpha}|m_N(k)|^2\sup_{D_3}|k|^{2\alpha+2\rho}|k_1k_2|^{-2\alpha-2\rho} \\ &\quad + \sum_k|k|^{2-4\rho}\sup_{D_3}|k|^{2\rho}|k_1k_2|^{-2\rho}) \\ &\lesssim C_{w,T}|t-s|^{2\gamma}\lambda^{5-6(\gamma+\rho)}N^{-2\rho}(\lambda^{4\rho-2} + N^{2\alpha}\sum_k|k|^{2-4\rho-2\alpha}|m_N(k)|^2) \end{aligned}$$

Now it is not difficult to see that $N^{2\alpha}\sum_k|k|^{2-4\rho-2\alpha}|m_N(k)|^2 \lesssim \lambda^{4\rho-2}$ and that we have

$$h_N^{\lambda,3} \lesssim C_{w,T}|t-s|^{2\gamma}\lambda^{3-6\gamma-2\rho}N^{-2\rho}.$$

This ends the proof. \square

Now we have a useful Corollary which be used to prove a variant of the local existence result.

Corollary 3.7.4. *Let $\alpha \in (-\rho, 0)$, $\rho > 3/4$ then there exist $\gamma > 1/2$ such that for all $T > 0$ there exists a constant such that*

$$\|IX_{st}^\lambda(\psi_1, \psi_2)\|_{L^2} \lesssim_{w,T} |t-s|^\gamma \lambda^{3/2-3\gamma+\alpha} \|I\psi_1\|_2 \|I\psi_2\|_2$$

for all $s, t \in [0, \lambda^3 T]$ and $\psi_1, \psi_2 \in H^\alpha(0, \lambda)$

Let us define $N_I(\psi) = \|I\psi\|_{L^2}$ for all $\psi \in H^\alpha(0, \lambda)$ of course $H^\alpha(0, \lambda)$ equipped with the norm N_I is a Banach space. Now we have the following local existence result.

Proposition 3.7.5. *Let $\psi \in H^\alpha(0, \lambda)$ then there exist a life time $\kappa > 0$ and a solution u of the rescaled problem such that $Iv^\lambda \in C^{1/2}([0, \kappa], L^2)$ moreover we have that*

$$\kappa \sim \min(5, \|I\psi\|^{-\theta})$$

for some $\theta > 0$ and we also have

$$\|Iv\|_{C^0(L^2)} + \|Iv\|_{C^{1/2}(L^2)} \lesssim \|I\psi\|_{L^2}.$$

Proof. Let $v \in C^{1/2}([0, \kappa], H^\alpha)$, $0 < \kappa < 5$ then we introduce the norm $\|v\| = \|Iv\|_{C^{1/2}L^2} + \|Iv\|_{C^0L^2}$ and we define the fixpoint map

$$\Gamma_\kappa(v) = \psi + \int_0^t X_{d\sigma}^\lambda(v_\sigma, v_\sigma).$$

Of course Γ_κ is well defined and if we let $B_C := \{v \in C^{1/2}([0, \kappa], H^\alpha); \|v\| \leq c\|I\psi\|_{L^2}\}$ then if $v \in B_C$ we have by a simple computation that

$$\|I\Gamma_\kappa(v)\|_{C^{1/2}L^2} \lesssim C_{W,\lambda-3\kappa} \lambda^{\alpha+3/2-3\gamma} \|I\psi\|_{L^2}^2 \kappa^{\gamma-1/2}$$

and then for $\lambda > 1$, we have that

$$\|\Gamma_\kappa(v)\| \lesssim C_{W,\kappa} c^2 \|I\psi\|_{L^2}^2 \kappa^{\gamma-1/2}.$$

Now is sufficient to take $\kappa^* \sim \min(5, \|I\psi\|^{\frac{1}{1/2-\gamma}})$ small enough and then there exist $c > 1$ such that $\|\Gamma(v)\| \leq c\|I\psi\|_{L^2}$. Now Γ_κ is a contraction in B_c in fact we have by a simple computation

$$\begin{aligned} \|\Gamma_\kappa(v^1) - \Gamma_\kappa(v^2)\| &\lesssim c\kappa^{\gamma-1/2}\|I\psi\|_{L^2}\|v^1 - v^2\| \\ &\lesssim c\|v^1 - v^2\| \end{aligned}$$

and then if we take $\kappa \sim \min(5, \|I\psi\|^{\frac{1}{1/2-\gamma}}) \leq \kappa^*$ small enough, Γ_κ in this case is a strict contractions in B_c and then it have a unique fixed point in this ball, the proof of the uniqueness is standard. \square

We have now all the ingredients to prove the global existence result.

3.7.3 Global existence

To exhibit a global solution for 1-periodic Cauchy problem with initial data $\phi \in H^\alpha(\mathbb{T})$ it suffices to prove that for every $T > 0$ the rescaled equation admits a solution in $[0, \lambda^3 T]$ with initial condition $\psi^\lambda(x) = \lambda^{-2}\phi(\lambda^{-1}x)$. We begin by noting that

$$\|I\psi^\lambda\|_{L^2} \lesssim \lambda^{-\alpha-3/2}N^{-\alpha}\|\phi\|_{H^\alpha(\mathbb{T})}$$

and then we choose $\lambda \sim_{\|\phi\|_{H^\alpha}} N^{-\frac{\alpha}{3/2+\alpha}}$ such that $\|I\psi^\lambda\|_{L^2} = \varepsilon_0 \ll 1$. Using the local result we know that there exists a solution $v^\lambda = v$ of the rescaled problem with lifetime $\kappa > 1$ now by a simple computation we have

$$\|Iv_t\|_{L^2} - \|Iv_s\|_{L^2} = \langle v_s; IX(v_s, v_s) - X(Iv_s, Iv_s) \rangle + R_{st}$$

where $|R_{st}| \lesssim |t-s|^{2\gamma}$ then for $\rho < 3/2$ and using this last equation, the Young estimation given in the Theorem 3.3.1 and commutator estimate we can see that

$$\|Iv_1\| \leq \varepsilon_0^2 + N^{-\rho}\lambda^{-\rho+3/2-3\gamma}$$

then if we iterate our local result given by the Proposition 3.7.5 we can construct a solution with life time $\sim N^\rho\lambda^{\rho+3/2-3\gamma}$ and then we have to choose N such that

$$\lambda^3 T \lesssim N^\rho\lambda^{\rho-3/2+3\gamma}.$$

This is possible if $\alpha > -\frac{\rho}{3-2\gamma}$ and N large enough.

3.8 Strichartz estimate and the modulated NLS

In this section we study the Schrödinger equation with quintic non linearity (ie : $A = i\partial^2$ and $\mathcal{N}(u) = |u|^\mu u$, $\mu \in (1, 4)$). The problem here is to manage the algebraic difficulty given by the non linearity for this we will use a different strategy than that used in the previous sections. We recall for the case of Brownian motion that this equation has already been studied in [30] using a strong Strichartz type estimates, the goal of this section is to observe that their result generalizes easily to an arbitrary ρ -irregular path. In order to prove Th. 3.1.9 we will follow their strategy and obtain a preliminary estimate which involves computations similar to those used in the study of the X operator for the cubic NLS in Lemma 3.5.12.

Proposition 3.8.1. *Let $\alpha \in [0, 1]$ and $\rho > 0$ such that $0 \leq \alpha < 1$ and $\rho > \alpha + 1/2$ or $\alpha = 1$ and $\rho > 1$ then for all $T > 0$ there exist $\gamma > 1/2$ such that :*

$$\int_0^T dt \left\| D^{\frac{\alpha}{2}} \left| \int_0^t U_t^w (U_s^w)^{-1} \psi_s ds \right|^2 \right\|_{L^2(\mathbb{R})}^2 \lesssim \|\Phi^w\|_{\mathcal{W}_{\rho, T}^\gamma} T^\gamma \|\psi\|_{L^1([0, T], L^2(\mathbb{R}))}^4$$

for every $\psi \in L^1([0, T], L^2(\mathbb{R}))$.

Proof. By going in Fourier variables we can see that :

$$\begin{aligned} \int_0^T dt \left\| D^{\frac{\alpha}{2}} \left| \int_0^t U_t^w (U_s^w)^{-1} \psi_s ds \right|^2 \right\|_{L^2(\mathbb{R})}^2 &= \int_0^T dt \int_{[0, t]^4} ds_1 ds_2 ds_3 ds_4 \int_{\mathbb{R}^3} dx_1 dx_2 dx_3 \times \\ &\quad \times (e^{-i\phi} |x_2 - x_1|^\alpha |\hat{\psi}_{s_1}(x_1)| |\hat{\psi}_{s_2}^*(x_2)| |(\hat{\psi}_{s_3})^*(x_3)| |(\hat{\psi}_{s_4})^*(x_4)|) \end{aligned}$$

where $x_4 = -x_1 + x_2 + x_3$, $\phi = x_1^2(w_t - w_{s_1}) - x_2^2(w_t - w_{s_2}) - x_3^2(w_t - w_{s_3}) + x_4^2(w_t - w_{s_4})$. Now we split the integral over (s_1, s_2, s_3, s_4) in four region where $s_i = \max(s_1, s_2, s_3, s_4)$ for $i = 1, \dots, 4$. Consider for example the first region where $s_1 > s_2, s_3, s_4$. Then using Fubini we can see that this integral is given by :

$$\mathcal{I} = \int_0^T ds_1 \int_{[0, s_1]^3} dx_1 dx_2 dx_3 \left(\int_{s_1}^T e^{-i\phi} dt \right) |x_2 - x_1|^\alpha |\hat{\psi}_{s_1}(x_1)| |\hat{\psi}_{s_2}^*(x_2)| |(\hat{\psi}_{s_3})^*(x_3)| |(\hat{\psi}_{s_4})^*(x_4)|$$

and

$$\left| \int_{s_1}^T e^{-i\phi} dt \right| = \left| \int_{s_1}^T e^{2i(x_2 - x_1)(x_3 - x_1)(w_t - w_{s_1})} dt \right| = |\Phi_{s_1 T}^w(2(x_2 - x_1)(x_3 - x_1))|.$$

Then we have to bound the following integral

$$I(\alpha) = \int_{\mathbb{R}^3} dx_1 dx_2 dx_3 |x_2 - x_1|^\alpha |\Phi_{s_1 T}^w(2(x_2 - x_1)(x_3 - x_1))| |\hat{\psi}_{s_1}(x_1)| |\hat{\psi}_{s_2}^*(x_2)| |(\hat{\psi}_{s_3})^*(x_3)| |(\hat{\psi}_{s_4})^*(x_4)|.$$

An application of Lemma 3.5.13 shows that

$$\mathcal{I} \lesssim T^\gamma \left(\int_0^T |\psi_s|_{L^2(\mathbb{R})} ds \right)^4$$

and concludes the proof. \square

Now we obtain a Gagliardo-Nirenberg type inequality to transform the regularity gain of the previous proposition into an integrability result of Strichartz's type.

Lemma 3.8.2. *Let $p > 2$ and $\varepsilon > 0$ then there exist $C = C(\varepsilon, p)$ such that for all $f \in L^1(\mathbb{R}) \cap \mathcal{H}_s$ the following inequality holds :*

$$\|f\|_{L^p(\mathbb{R})} \leq c \|f\|_{L^1(\mathbb{R})}^{1-\theta} \|f\|_{\mathcal{H}_s}^\theta$$

where \mathcal{H}_s is the homogenous Sobolev space on \mathbb{R} , $s = \frac{1}{2} - \frac{1}{p} + \frac{\varepsilon}{2}$ and $\theta = \frac{2p-2}{(2+\varepsilon)p-2} \in (0, 1)$

Proof. We begin by decomposing f in here Littlewood-Paley block $f = \sum_{i > -1} \Delta_i f$ and then

$$\|f\|_{L^p(\mathbb{R})} \leq \|\Delta_{-1}f\|_{L^p(\mathbb{R})} + \sum_{i \geq 0} \|\Delta_i f\|_{L^p(\mathbb{R})} \quad (3.18)$$

Now we recall that $\Delta_{-1}f = h \star f$ and $\Delta_i f = h_i \star f$ with $h = \mathcal{F}^{-1}\chi_1$ and $h_i = 2^i \mathcal{F}^{-1}(\chi_2(2^i \cdot))$ where χ_1 and χ_2 are radial infinitely differentiable functions such that

- the support of χ_1 is contained in a ball and the support of χ_2 is contained in an annulus.
 - $\chi_1 + \sum_{j \geq 0} \chi_2(2^{-j} \cdot) = 1$
 - $\text{supp}(\chi_1) \cap \text{supp}(\chi_2(2^{-i} \cdot)) = \emptyset$ for $i \geq 1$ and if $|i - j| > 1$ then $\text{supp}(\chi_2(2^{-i} \cdot)) \cap \text{supp}(\chi_2(2^{-j} \cdot)) = \emptyset$
- then using the Young inequality we get :

$$\|\Delta_{-1}f\| \leq \|\mathcal{F}^{-1}\chi_1\|_{L^p(\mathbb{R})} \|f\|_{L^1(\mathbb{R})}$$

this give us the bound for the first term of the inequality 3.18 for the second term we remark that $\Delta_i f$ is a function which the support of here Fourier transform is contained in a ball $2^i B$ then using Bernstein's inequality we obtain that :

$$\|\Delta_i f\|_{L^p(\mathbb{R})} \lesssim 2^{i(\frac{1}{2} - \frac{1}{p})} \|\Delta_i f\|_{L^2(\mathbb{R})}$$

then summing this last equation over $i \geq 0$ and using Jensen inequality we can see that

$$\begin{aligned} \sum_{i \geq 0} \|\Delta_i f\|_{L^p(\mathbb{R})} &\lesssim \sum_{i \geq 0} 2^{i(\frac{1}{2} - \frac{1}{p})} \|\Delta_i f\|_{L^2(\mathbb{R})} \\ &\lesssim \sum_{i \geq 0} (2^{i(\frac{1}{2} - \frac{1}{p} + \frac{\varepsilon}{2})} \|\Delta_i f\|_{L^2(\mathbb{R})}) 2^{-i\frac{\varepsilon}{2}} \\ &\lesssim \varepsilon \left(\sum_{i \geq 0} 2^{2i(\frac{1}{2} - \frac{1}{p} + \frac{\varepsilon}{2})} \|\Delta_i f\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \lesssim \varepsilon \|f\|_{\mathcal{H}_s} \end{aligned}$$

and then we have

$$\|f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})} + \|f\|_{\mathcal{H}_s}$$

now if we put $f_\lambda(x) = f(\lambda x)$ in this inequality we can see that

$$\|f\|_{L^p(\mathbb{R})} \lesssim \lambda^{-1+1/p} \|f\|_{L^1(\mathbb{R})} + \lambda^{\varepsilon/2} \|f\|_{\mathcal{H}_s}$$

then to have our result is suffice to take $\lambda = (\|f\|_{L^1(\mathbb{R})}^{-1} \|f\|_{\mathcal{H}_s})^{\frac{2p}{2-(2+\varepsilon)p}}$ \square

Proof of Proposition 3.1.9. Starting with Lemma 3.8.2 and taking $\alpha = 1$ and $\rho > 1$ in Prop. 3.8.1 we obtain :

$$\begin{aligned} \left\| \int_0^\cdot U^\cdot(U_s^w)^{-1} \psi_s ds \right\|_{L^p([0,T], L^{2p}(\mathbb{R}))}^p &= \int_0^T dt \left\| \int_0^t U_t^w(U_s^w)^{-1} \psi_s ds \right\|_{L^p(\mathbb{R})}^{2p/2} \\ &\lesssim \int_0^T dt \left\| \int_0^t U_t^w(U_s^w)^{-1} \psi_s ds \right\|_{L^1(\mathbb{R})}^{1/2} \left\| D^{\frac{1}{2}} \int_0^t U_t^w(U_s^w)^{-1} \psi_s ds \right\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} \\ &\lesssim C_w T^{\frac{\gamma(p-1)}{4} + \frac{5-p}{4}} \left\| \int_0^\cdot U^\cdot(U_s^w)^{-1} \psi_s ds \right\|_{L^\infty([0,T], L^2(\mathbb{R}))} \left(\int_0^T \|\psi_s\|_{L^2(\mathbb{R})} ds \right)^{p-1}. \end{aligned}$$

Now is suffice to remark that

$$\left\| \int_0^\cdot U_t^w (U_s^w)^{-1} \psi_s ds \right\|_{L^\infty([0,T], L^2(\mathbb{R}))} \leq \sup_{0 \leq t \leq T} \int_0^t \|U_t^w (U_s^w)^{-1} \psi_s\|_{L^2(\mathbb{R})} ds \leq \int_0^T \|\psi_s\|_{L^2(\mathbb{R})} ds$$

and then

$$\left\| \int_0^\cdot U_t^w (U_s^w)^{-1} \psi_s ds \right\|_{L^p([0,T], L^{2p}(\mathbb{R}))}^p \leq C_w T^{\frac{\gamma(p-1)}{4} + \frac{5-p}{4}} \left(\int_0^T \|\psi_s\|_{L^2(\mathbb{R})} ds \right)^p$$

when $p \in (2, 5]$. Now in the case $p \in [2, 4)$ we obtain the same result if we use the Lemma 3.8.2 and take $\alpha = 1 - \frac{2}{p} + \varepsilon \in (0, 1/2]$, $\rho > \alpha + \frac{1}{2}$ in Proposition 3.8.1. \square

To have all the ingredients needed for the fixed point argument we have to estimates the action of the operator U^w on the initial condition.

Proposition 3.8.3. *Let $T > 0$, $p = \mu + 1 \in (4, 5]$, $\rho > \min(\frac{3}{2} - \frac{2}{p})$ then then there exist constant C_p and $\gamma^*(p) > 0$ such that the following inequality holds :*

$$\|U_t^w \psi\|_{L^p([0,T], L^{2p}(\mathbb{R}))} \leq C_p \|\Phi\|_{W_T^{\rho, \gamma}} T^{\gamma^*(p)} \|\psi\|_{L^2(\mathbb{R})}$$

for all $\psi \in L^2(\mathbb{R})$.

Proof. Let us begin by using the Lemma 3.8.2 and then

$$\begin{aligned} \|U_t^w \psi\|_{L^p([0,T], L^{2p}(\mathbb{R}))}^p &= \int_0^T dt \left\| |U_t^w \psi|^2 \right\|_{L^p(\mathbb{R})}^{p/2} \\ &\lesssim \|U^w \psi\|_{L^\infty([0,T], L^2(\mathbb{R}))}^{(1-\theta)p} \int_0^T dt \left\| D^{\alpha/2} |U_t^w \psi|^2 \right\|_{L^2(\mathbb{R})}^{\theta \frac{p}{2}} \end{aligned}$$

where $\theta = \frac{2p-2}{(2+\varepsilon)p-2}$ then is suffice to bound the quantity $\left\| D^{\alpha/2} |U_t^w \psi|^2 \right\|_{L^2(\mathbb{R})}^2$ and to proceed as the in the Proposition 3.1.9. By a simple computation we have

$$\left\| D^{\alpha/2} |U_t^w \psi|^2 \right\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}^3} dx_1 dx_2 dx_3 |x_2 - x_1|^\alpha |\Phi_{0T}^w(\eta)| |\hat{\psi}(x_1)| |\hat{\psi}(x_2)| |\hat{\psi}(x_3)| |\hat{\psi}(-x_1 + x_2 + x_3)|$$

where $\eta = 2(x_2 - x_1)(x_3 - x_2)$. Now applying again Proposition 3.8.1 we concludes the proof. \square

We are now ready to prove Th. 3.1.10 about existence of local solution to the modulated NLS with general non-linearity.

Proof of 3.1.10. Let us define for $\psi \in L^p([0, T], L^{2p})$ the following map :

$$\Gamma(\psi)_t = U_t^w u^0 + i \int_0^t U_t^w (U_s^w)^{-1} (|\psi_s|^\mu \psi_s) ds$$

then we can easily see by proposition $\Gamma(\psi) \in L^p([0, T], L^{2p}(\mathbb{R}))$. Now we will prove that Γ is a strict contraction in a adequate ball of our space. In fact let

$$B_r = \{ \psi \in L^p([0, T], L^{2p}(\mathbb{R})), \|\psi\|_{L^p([0,T], L^{2p}(\mathbb{R}))} \leq r \}$$

then using the Proposition 3.8.3 and Proposition 3.1.9 we have

$$\begin{aligned}
 \|\Gamma(\psi)\|_{L^p([0,T],L^{2p}(\mathbb{R}))} &\leq \|U^w u^0\|_{L^p([0,T],L^{2p}(\mathbb{R}))} + \left\| \int_0^\cdot U^w (U_s^w)^{-1} (|\psi_s|^\mu \psi_s) ds \right\|_{L^p([0,T],L^{2p}(\mathbb{R}))} \\
 &\leq C_{w,T} T^{\gamma^*(p)} \left(\|u^0\|_{L^2(\mathbb{R})} + \int_0^T \| |\psi_s|^\mu \psi_s \|_{L^2(\mathbb{R})} ds \right) \\
 &\leq C_{w,T} T^{\gamma^*(p)} (\|u^0\|_{L^2(\mathbb{R})} + \|\psi\|_{L^p([0,T],L^{2p}(\mathbb{R}))}^p) \\
 &\leq C_{w,T} T^{\gamma^*(p)} (\|u^0\|_{L^2(\mathbb{R})} + r^p)
 \end{aligned}$$

then we can choose T_1 small enough such for all $T \leq T_1$ that the equation $r_T = C_w T^{\gamma^*(p)} (\|u^0\|_{L^2(\mathbb{R})} + r_T^p)$ admit a positive solution $r = r_T$. Now for $T < T_1$ and $\psi_1, \psi_2 \in L^p([0, T], L^{2p}(\mathbb{R})) \cap B_r$ we see by the same argument using previously we have

$$\begin{aligned}
 \|\Gamma(\psi_1) - \Gamma(\psi_2)\|_{L^p([0,T],L^{2p}(\mathbb{R}))} &= \left\| \int_0^\cdot U^w (U_s^w)^{-1} (|\psi_1|^\mu \psi_1 - |\psi_2|^\mu \psi_2) ds \right\|_{L^p([0,T],L^{2p}(\mathbb{R}))} \\
 &\leq C_{w,T} T^{\gamma^*(p)} r^{p-1} \|\psi_1 - \psi_2\|_{L^p([0,T],L^{2p}(\mathbb{R}))}
 \end{aligned}$$

then if we choose $T_2 < T_1$ such that for all $T < T_2$ we have $\|\Phi\|_{\mathcal{W}_T^{\gamma,p}} T^{\gamma^*(p)} r^{p-1} < 1$ then in this case Γ is a strict contraction of the ball $L^p([0, T_2], L^{2p}(\mathbb{R})) \cap B_r$ and then it has a unique fixed point in this ball. The proof of uniqueness is standard. Now the fact that $u \in C([0, T_2], L^2(\mathbb{R}))$ is simply given by the inequality

$$\|u_t - u_s\|_{L^2(\mathbb{R})} \leq \|(U_t^w - U_s^w)u^0\| + \int_s^t \|u_\sigma\|_{L^{2p}(\mathbb{R})}^p d\sigma.$$

Now we will focus on the proof of the conservation law in the quintic case (i.e.: $p = 5$), for simplicity. For the other value of p the argument is similar. Let now $M \in \mathbb{N}$. By the same argument used in the beginning of the proof we can construct a local solution $u^M \in L^5([0, T], L^{10}(\mathbb{R})) \cap C([0, T], L^2(\mathbb{R}))$ of the regularized equation. More precisely we have that

$$u_t^M = U_t^w \Pi_M u^0 + i \int_0^t U_t^w (U_s^w)^{-1} \Pi_M (|\Pi_M u^M|^4 \Pi_M u^M) ds$$

for all $t \in [0, T]$ and some $T = T(\|u^0\|_{L^2(\mathbb{R})})$. Let $v^M = (U_t^w)^{-1} u_t^M$. A simple computation shows that

$$\|v_t^M - v_s^M\|_{L^2(\mathbb{R})} \leq \int_s^t \|\Pi_M u_\sigma^M\|_{L^{10}}^5 d\sigma \lesssim_M (t-s) \|u^M\|_{L^\infty([0,T],L^2(\mathbb{R}))}$$

from which we obtain that

$$\|v_t^M\|_2^2 = \|v_s^M\|_2^2 + 2 \int_s^t \mathcal{I}m \langle v_s, (U_\sigma^w)^{-1} (|U_\sigma^w v_s^M|^4 U_\sigma v_s^M) \rangle d\sigma + O(|t-s|^2).$$

It is not difficult to see that $\langle v_s, (U_\sigma^w)^{-1} (|U_\sigma^w v_s^M|^4 U_\sigma v_s^M) \rangle \in \mathbb{R}$ and then $\|v_t^M\|_2^2 = \|v_s^M\|_2^2 + O(|t-s|^2)$ and then we obtain immediately that $\|u_t^M\|_2 = \|u^0\|_2$. Moreover we have

- $\Pi_M u^M = u^M$;
- for every $T > 0$, $\sup_M \|u^M\|_{L^5([0,T],L^{10}(\mathbb{R}))} < +\infty$.

Using that we have easily

$$\|u^M - u\|_{L^5([0,T],L^{10}(\mathbb{R}))} \lesssim T^\gamma (\|u^0 - \Pi_M u^0\|_2 + \|u^M - u\|_{L^5([0,T],L^{10}(\mathbb{R}))})$$

and for $T < \min(T_2, 1/2)$ small enough $\|u^M - u\|_{L^5([0,T],L^{10}(\mathbb{R}))} \xrightarrow{M \rightarrow +\infty} 0$. It is then sufficient to iterate this procedure to extend it to the interval $[0, T_2]$. Now by a simple computation we can see that

$$\|u^M - u\|_{L^\infty([0,T_2],L^2(\mathbb{R}))} \lesssim \|\Pi_M u^0 - u^0\|_2 + \|u^M - u\|_{L^5([0,T],L^{10}(\mathbb{R}))}$$

and then $\|u_t\|_{L^2(\mathbb{R})} = \|u^0\|_{L^2(\mathbb{R})}$ which gives the conservation law and allow us to extend our local solution in a global solution. Now let $u^0 \in H^1$ and using the Strichartz estimates after taking the first derivative of the function $\Gamma(\psi)$ we obtain that

$$\|\Gamma(\psi)\|_{L^5([0,T],W^{1,10}(\mathbb{R}))} \lesssim_w T^\gamma (\|u^0\|_{H^1} + r^4 \|\psi\|_{L^5([0,T],W^{1,10}(\mathbb{R}))})$$

with $\psi \in B_r$ where B_r is the ball in which we have setup our point fix argument at the beginning of the proof. Then $B(0, R)$, the ball of radius R in $L^5([0, T], W^{1,10}(\mathbb{R}))$ is invariant by Γ_{B_r} the restriction of Γ on B_r for $T_3 = T$ depending only on r and not R . Since closed balls of $L^5([0, T], W^{1,10}(\mathbb{R}))$ are closed also in $L^5([0, T], L^{10}(\mathbb{R}))$ the fixed point of Γ_{B_r} is in $L^5([0, T], W^{1,10}(\mathbb{R}))$ and we obtain that $u \in L^5([0, T_3], W^{1,10}(\mathbb{R}))$. Now by a standard argument we obtain the needed regularity for u . \square

Chapter 4

Stochastic quantization in \mathbb{T}^3 and paracontrolled distribution

Résumé

Nous prouvons l'existence de solution local en temps pour l'équation de Φ_3^4 dans un domaine périodique en utilisant l'approche des distributions contrôlées qui est inspirée de la théorie des chemins rugueux et de la décomposition en paraproduct de Bony.

Abstract

We prove the existence and uniqueness of a local solution to the periodic renormalized Φ_3^4 model of stochastic quantization using the method of controlled distributions introduced recently by Imkeller, Gubinelli and Perkowski ("Paraproducts, rough paths and controlled distributions", [arXiv:1210.2684](#)).

4.1 Introduction

We study here the following Cauchy problem

$$\begin{cases} \partial_t u = \Delta_{\mathbb{T}^3} u - u^3 + \xi \\ u(0, x) = u^0(x) \quad x \in \mathbb{T}^3 \end{cases} \quad (4.1)$$

where ξ is a space-time with noise such that $\int_{\mathbb{T}^3} \xi(x) dx = 0$ i.e. it is a centered Gaussian space-time distribution such that

$$\mathbb{E}[\xi(s, x)\xi(t, y)] = \delta(t - s)\delta(x - y)$$

and $u : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$ is a space-time distribution which is continuous in time. We write this equation in its mild formulation

$$u = P_t u^0 - \int_0^t P_{t-s}(u_s)^3 ds + X_t \quad (4.2)$$

where $P_t = e^{t\Delta}$ is the Heat flow and $X_t = \int_0^t P_{t-s}\xi_s ds$ is a the solution of the linear equation :

$$\partial_t X_t = \Delta_{\mathbb{T}^3} X_t + \xi; \quad X_0 = 0. \quad (4.3)$$

Moreover X is a Gaussian process and as we see below $X \in \mathcal{C}([0, T], \mathcal{C}^{-1/2-\varepsilon}(\mathbb{T}^3))$ for every $\varepsilon > 0$ with $\mathcal{C}^\alpha = B_{\infty, \infty}^\alpha$ is the Besov-Hölder space. The main difficulty of the equation (4.1) comes from the fact that for any fixed time t the space regularity of the solution $u(t, x)$ cannot be better than the one of X_t . If we measure spatial regularity in the scale of Hölder spaces \mathcal{C}^α we should expect that $u(t, x) \in \mathcal{C}^\alpha(\mathbb{T}^3)$ for any $\alpha < -1/2$ but not better. In particular the term u^3 is not well defined. A natural approach to give a well defined meaning to the equation would consist in regularizing the noise in $\xi^\varepsilon = \xi \star \rho^\varepsilon$ with $\rho^\varepsilon = \varepsilon^{-3} \rho(\frac{\cdot}{\varepsilon})$ a smooth kernel and taking the limit of the solution u^ε of the approximate equation

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon)^3 + \xi^\varepsilon. \quad (4.4)$$

Since the non-linear term diverges when ε goes to zero, an a priori estimate for the wanted solution is hard to find. To overcome this problem we have to focus on the following modified equation

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - ((u^\varepsilon)^3 - C_\varepsilon u^\varepsilon) + \xi^\varepsilon \quad (4.5)$$

where $C_\varepsilon > 0$ is a renormalization constant which diverges when ε goes to 0. We will show that we have to take $C_\varepsilon \sim \frac{a}{\varepsilon} + b \log(\varepsilon) + c$ to obtain a non trivial limit for $(u^\varepsilon)^2 - C_\varepsilon$.

Therefore this paper aims at giving a meaning of the equation (4.2) and at obtaining a (local in time) solution. The method developed here uses some ideas of [46] where the author deals with the KPZ equation. More precisely we use the partial series expansion of the solution to define the reminder term using the notion of paracontrolled distributions introduced in [38]. A solution of this equation has already been constructed in the remarkable paper of Hairer [47] where the author shows the convergence of the solution of the mollified equation (4.5).

The stochastic quantization problem has been studied since the eighties in theoretical physics (see for example [50] and [51] In [9] and the references about it in [47]).

From a mathematical point of view, several articles deals with the 2-dimensional case. Weak probabilistic solutions were found by Jona-Lasinio and Mitter in [50] and [51]. Some other probabilistic results are obtained thanks to non perturbative methods by Bertini, Jona-Lasinio and Parrinello in [9]. In [23] Da Prato and Debussche found a strong (in the probabilistic sense) formulation for this 2d problem.

In a recent paper, Hairer [47] gives a fixed point solution to the 3-dimensional case thanks to his theory of regularity structures. Like the theory of paracontrolled distributions, Hairer's theory of regularity structures is a generalization of rough path theory. Hairer gets his result by giving a generalization of the notion of pointwise Hölder regularity. With this extended notion, it is possible to work on a more abstract space where the solutions are constructed thanks to a fixed point argument, and then project the abstract solution into a space of distributions via a reconstruction map. The regularity structures approach is quite general and can treat more singular models.

In the approach of the paracontrolled distribution developed in [38] by Gubinelli, Imkeller and Perkowski, on the other hand, it is the notion of controlled path which is generalized. This allows us to give a reasonable notion of product of distributions. Since all the problems treated by the theory of paracontrolled distributions can be solved by using the theory of regularity structures, asking whether or not the opposite is true is a legitimate (and reasonable) question. The following theorems are a piece of the answer.

We will proceed in two steps. In an analytic part we will extend the flow of the regular equation,

$$\partial_t u_t = \Delta u_t - u^3 + 3au + 9bu + \xi$$

with $(a, b) \in \mathbb{R}^2$ and $\xi \in C([0, T], C^0(\mathbb{T}^3))$ to the situation of more irregular driving noise ξ . More precisely we will prove that the solution u is a continuous function of $(u^0, R_{a,b}^\varphi \mathbf{X})$ with

$$\begin{aligned} R_{a,b}^\varphi \mathbf{X} = & (X, X^2 - a, I(X^3 - 3aX), \pi_0(I(X^3 - 3aX), X), \\ & \pi_0(I(X^2 - a), (X^2 - a)) - b - \varphi, \pi_0(I(X^3 - 3aX), (X^2 - a)) - 3bX - 3\varphi X, \varphi) \end{aligned} \quad (4.6)$$

where $X_t = \int_0^t P_{t-s} \xi ds$, $\pi_0(\cdot, \cdot)$ denotes the remainder term of the paraproduct decomposition given in (4.2.3) and $I(f)_t = \int_0^t P_{t-s} f ds$. This extension is given in the following theorem.

Theorem 4.1.1. *Let $F : \mathcal{C}^1(\mathbb{T}^3) \times C(\mathbb{R}^+, \mathcal{C}^0(\mathbb{T}^3)) \times \mathbb{R} \times \mathbb{R} \rightarrow C(\mathbb{R}^+, \mathcal{C}^1(\mathbb{T}^3))$ the flow of the equation*

$$\begin{cases} \partial_t u_t = \Delta u_t - u_t^3 + 3au_t + 9bu_t + \xi_t, & t \in [0, T_C(u^0, X, (a, b))] \\ \partial_t u_t = 0, & t \geq T_C(u^0, X, (a, b)) \\ u(0, x) = u^0(x) \in \mathcal{C}^1(\mathbb{T}^3) \end{cases}$$

where $\xi \in C(\mathbb{R}^+, \mathcal{C}^0(\mathbb{T}^3))$ and $T_C(u^0, \xi, (a, b))$ is a time such that the equation holds for $t \leq T_C$. Now let $z \in (1/2, 2/3)$, then there exists a Polish space \mathcal{X} , called the space of rough distribution, $\tilde{T}_C : \mathcal{C}^{-z} \times \mathcal{X} \rightarrow \mathbb{R}^+$ a lower semi-continuous function and $\tilde{F} : \mathcal{C}^{-z} \times \mathcal{X} \rightarrow C(\mathbb{R}^+, \mathcal{C}^{-z}(\mathbb{T}^3))$ continuous in $(u^0, \mathbb{X}) \in \mathcal{C}^{-z}(\mathbb{T}^3) \times \mathcal{X}$ such that (\tilde{F}, \tilde{T}) extends (F, T) in the following sense :

$$T_C(u^0, \xi, (a, b)) \geq \tilde{T}_C(u^0, R_{a,b}^\varphi \mathbf{X}) > 0$$

and

$$F(u^0, \xi, a, b)(t) = \tilde{F}(u^0, R_{a,b}^\varphi \mathbf{X})(t), \text{ for all } t \leq \tilde{T}_C(u^0, R_{a,b}^\varphi \mathbf{X})$$

for all $(u^0, \xi, \varphi) \in \mathcal{C}^1(\mathbb{T}^3) \times C(\mathbb{R}^+, \mathcal{C}^0(\mathbb{T}^3)) \times C^\infty([0, T])$, $(a, b) \in \mathbb{R}^2$ with $X_t = \int_0^t ds P_{t-s} \xi$ and where $R_{a,b}^\varphi$ is given in the equation (4.6).

In a second part we obtain a probabilistic estimate for the stationary Ornstein Uhlenbeck (O.U.) process which is the solution if the linear equation (4.3) and this allows us to construct the rough distribution in this case.

Theorem 4.1.2. *Let X be the stationary (O.U.) process and X^ε a space mollification of X . There exists two constants $C_1^\varepsilon, C_2^\varepsilon \rightarrow^{\varepsilon \rightarrow 0} +\infty$ and a function $\varphi^\varepsilon \in C^\infty(\mathbb{R}^+)$ such that $R_{C_1^\varepsilon, C_2^\varepsilon}^{\varphi^\varepsilon} \mathbf{X}^\varepsilon$ converge in $L^p(\Omega, \mathcal{X})$ to some $\mathbb{X} \in \mathcal{X}$. Furthermore the first component of \mathbb{X} is X .*

In the setting, the Corollary below follows immediately.

Corollary 4.1.3. *Let ξ a space time white noise, and ξ^ε is a space mollification of ξ such that :*

$$\xi^\varepsilon = \sum_{k \neq 0} f(\varepsilon k) \hat{\xi}(k) e_k$$

with f a smooth radial function with compact support satisfying $f(0) = 0$, let X the stationary (O.U.) process associated to ξ , \mathbb{X} the element of \mathcal{X} given in the Theorem (4.1.2) and $u^0 \in \mathcal{C}^{-z}$ for $z \in (1/2, 2/3)$ then if u^ε is the solution of the mollified equation :

$$\begin{cases} \partial_t u_t^\varepsilon = \Delta u_t^\varepsilon - (u_t^\varepsilon)^3 + 3C_1^\varepsilon u_t + 9C_2^\varepsilon u_t + \xi_t^\varepsilon, & t \in [0, T^\varepsilon[\\ \partial_t u_t = 0, & t \geq T^\varepsilon \\ u(0, x) = (u^0)^\varepsilon(x) \end{cases}$$

We have the following convergence :

$$\lim_{\varepsilon} u^\varepsilon = \tilde{F}(u^0, \mathbb{X})$$

where the limit is understood in the probability sense in the space $C(\mathbb{R}^+, \mathcal{C}^{-z})$.

The proofs of those two theorems are almost independent, but we need the existence and the properties of the rough distribution, specified in the Definition 4.2.9, to prove the first theorem.

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Plan of the paper. It is the aim of Section 4.2 to introduce the notion spaces of paracontrolled distributions where the renormalized equation will be solved. In Section 4.3 we prove that for a small time the application associated to the renormalized equation is a contraction, which, by a fixed point argument, gives the existence and uniqueness of the solution, but also the continuity with respect to the rough distribution and the initial condition. The last Section 4.4 is devoted to the existence of the rough distribution for the (O.U.) process.

4.2 Paracontrolled distributions

4.2.1 Besov spaces and paradifferential calculus

The results given in this Subsection can be found in [6] and [38]. Let us start by recalling the definition of Besov spaces via the Littelwood-Paley projectors.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions such that

1. The support of χ is contained in a ball and the support of θ is contained in an annulus;
2. $\chi(\xi) + \sum_{j \geq 0} \theta(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;
3. $\text{supp}(\chi) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $i \geq 1$ and $\text{supp}(\theta(2^{-i}\cdot)) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ when $|i - j| > 1$.

For the existence of χ and θ see [6], Proposition 2.10. The Littlewood-Paley blocks are defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \mathcal{F}u) \text{ and for } j \geq 0, \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot) \mathcal{F}u).$$

We define the Besov space of distribution by

$$B_{p,q}^\alpha = \left\{ u \in S'(\mathbb{R}^d); \quad \|u\|_{B_{p,q}^\alpha}^q = \sum_{j \geq -1} 2^{jq\alpha} \|\Delta_j u\|_{L^p}^q < +\infty \right\}.$$

In the sequel we will deal with the special case of $\mathcal{C}^\alpha := B_{\infty,\infty}^\alpha$ and write $\|u\|_\alpha = \|u\|_{B_{\infty,\infty}^\alpha}$. We hold the following result for the convergence of localized series in the Besov spaces, which will prove itself useful.

Proposition 4.2.1. *Let $(p, q, s) \in [1, +\infty]^2 \times \mathbb{R}$, B a ball in \mathbb{R}^d and $(u_j)_{j \geq -1}$ a sequence of functions such that $\text{supp}(u_j)$ is contained in $2^j B$ moreover we assume that*

$$\Xi_{p,q,s} = \left\| (2^{js} \|u_j\|_{L^p})_{j \geq -1} \right\|_{l^q} < +\infty$$

then $u = \sum_{j \geq -1} u_j \in B_{p,q}^s$ and $\|u\|_{B_{p,q}^s} \lesssim \Xi_{p,q,s}$.

The trick to manipulate stochastic objects is to deal with Besov spaces with finite indexes and then go back to space \mathcal{C}^α . For that we have the following useful Besov embedding.

Proposition 4.2.2. *Let $1 \leq p_1 \leq p_2 \leq +\infty$ and $1 \leq q_1 \leq q_2 \leq +\infty$. For all $s \in \mathbb{R}$ the space B_{p_1,q_1}^s is continuously embedded in $B_{p_2,q_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}$, in particular we have $\|u\|_{\alpha-\frac{d}{p}} \lesssim \|u\|_{B_{p,p}^\alpha}$.*

Taking $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ we can formally decompose the product as

$$fg = \pi_{<}(f, g) + \pi_0(f, g) + \pi_{>}(f, g)$$

with

$$\pi_{<}(f, g) = \pi_{>}(g, f) = \sum_{j \geq -1} \sum_{i < j-1} \Delta_i f \Delta_j g; \quad \pi_0(f, g) = \sum_{j \geq -1} \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.$$

With these notations the following results hold.

Proposition 4.2.3. *Let $\alpha, \beta \in \mathbb{R}$*

- $\|\pi_{<}(f, g)\|_\beta \lesssim \|f\|_\infty \|g\|_\beta$ for $f \in L^\infty$ and $g \in \mathcal{C}^\beta$
- $\|\pi_{>}(f, g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta$ for $\beta < 0$, $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$
- $\|\pi_0(f, g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta$ for $\alpha + \beta > 0$ and $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$

On of the key result of [38] is a commutation result for the operator $\pi_{<}$ and π_0 .

Proposition 4.2.4. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha < 1$, $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$ then*

$$R(f, x, y) = \pi_0(\pi_{<}(f, x), y) - f\pi_0(x, y)$$

is well-defined when $f \in \mathcal{C}^\alpha$, $x \in \mathcal{C}^\beta$ and $y \in \mathcal{C}^\gamma$ and more precisely

$$\|R(f, x, y)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|x\|_\beta \|y\|_\gamma$$

We finish this Section by describing the action of the Heat flow on the Besov spaces and a commutation property with the paraproduct . See the appendix for a proof.

Lemma 4.2.5. *Let $\theta \geq 0$ and $\alpha \in \mathbb{R}$ then the following inequality holds*

$$\|P_t f\|_{\alpha+2\theta} \lesssim \frac{1}{t^\theta} \|f\|_\alpha, \quad \|(P_{t-s} - 1)f\|_{\alpha-2\varepsilon} \lesssim |t-s|^\varepsilon \|f\|_\alpha$$

for $f \in \mathcal{C}^\alpha$. Moreover if $\alpha < 1$ and $\beta \in \mathbb{R}$ we have

$$\|P_t \pi_{<}(f, g) - \pi_{<}(f, P_t g)\|_{\alpha+\beta+2\theta} \lesssim \frac{1}{t^\theta} \|f\|_\alpha \|g\|_\beta$$

for all $g \in \mathcal{C}^\beta$.

In the following, we will extensively use some space-time function spaces. Let us introduce the notation

Notation 4.2.6.

$$C_T^\beta = C([0, T], \mathcal{C}^\beta)$$

For $f \in C_T^\beta$ we introduce the norm

$$\|f\|_\beta = \sup_{t \in [0, T]} \|f_t\|_{\mathcal{C}^\beta} = \sup_{t \in [0, T]} \|f_t\|_\beta$$

and by

$$C_T^{\alpha, \beta} := C^\alpha([0, T], \mathcal{C}^\beta(\mathbb{T}^3)).$$

Furthermore, we endow this space with the following distance

$$d_{\alpha, \beta}(f, g) = \sup_{t \neq s \in [0, T]} \frac{\|(f - g)_t - (f - g)_s\|_\beta}{|t - s|^\alpha} + \sup_{t \in [0, T]} \|f_t - g_t\|_\beta.$$

4.2.2 Renormalized equation and rough distribution

Let us focus on the mild formulation of the equation (4.1)

$$u = \Psi + X + I(u^3) = X + \Phi \quad (4.7)$$

where we remind the notation $I(f)(t) = -\int_0^t P_{t-s} f_s ds$, $X = -I(\xi)$ and $\Psi_t = P_t u^0$ for $u^0 \in \mathcal{C}^{-z}(\mathbb{T}^3)$. We can see that a solution u must have at least the same regularity as X . Yet thanks to the definition of I , as $\xi \in C([0, T], \mathcal{C}^{-5/2-\varepsilon})$, for all $\varepsilon > 0$, we have $X \in C([0, T], \mathcal{C}^{-1/2-\varepsilon})$. But in that case the non-linear term u^3 is not well-defined, as there is no universal notion for the product of distributions. A first idea is to proceed by regularization of X , such that products of the regularized quantities are well-defined, and then try to pass to the limit. Let us recall that the stationary O.U process is defined by the fact that $(\hat{X}_t(k))_{t \in \mathbb{R}, k \in \mathbb{Z}^3}$ is a centered Gaussian process with covariance function given by

$$\mathbb{E} \left[\hat{X}_t(k) \hat{X}_s(k') \right] = \delta_{k+k'=0} \frac{e^{-|k|^2|t-s|}}{|k|^2}$$

and $\hat{X}_t(0) = 0$. Let $X_t^\varepsilon = \int_0^t P_{t-s} \xi^\varepsilon ds$ more precisely $\hat{\xi}^\varepsilon = f(\varepsilon k) \hat{\xi}(k)$ where f is a smooth radial function with bounded support such that $f(0) = 1$. Then we have the following approximated equation

$$\Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon (X^\varepsilon)^2) + I((\Phi^\varepsilon)^3)$$

for $\Phi^\varepsilon = I((u^\varepsilon)^3) + \Psi^\varepsilon$ which is well-posed. Then an easy computation gives for $(X^\varepsilon)^2$

$$\begin{aligned} \mathbb{E} [(X_t^\varepsilon)^2] &= \sum_{k \in \mathbb{Z}^3} \sum_{k_1+k_2=k} f(\varepsilon k_1) f(\varepsilon k_2) \frac{1}{|k_1|^2} \delta_{k_1+k_2=0} \\ &= \sum_{k \in \mathbb{Z}^3} \frac{|f(\varepsilon k)|^2}{|k|^2} \sim_0 \frac{1}{\varepsilon} \int_{\mathbb{R}^3} f(x) (1 + |x|)^{-2} dx \end{aligned}$$

and there is no hope to obtain a finite limit for this term when ε goes to zero. This difficulty has to be solved by subtracting to the original equation these problematic contributions. In order to do so consistently we will introduce a renormalized product. Formally we would like to define

$$X^{\diamond 2} = X^2 - \mathbb{E}[X^2]$$

and show that it is well-defined and that $X^{\diamond 2} \in \mathcal{C}_T^{-1-\delta}$ for $\delta > 0$. More precisely we will defined

$$(X^\varepsilon)^{\diamond 2} = (X^\varepsilon)^2 - \mathbb{E}[(X^\varepsilon)^2]$$

and we will show that it converges to some finite limit. The same phenomenon happens for X^3 and other terms, and we have to renormalize them too. This is the meaning of Theorem 4.1.2. We remind the notation of that theorem

Notation 4.2.7. Let C_1^ε and C_2^ε two positives constants (to be specified later). We denote by

$$\begin{aligned} (X^\varepsilon)^{\diamond 2} &:= (X^\varepsilon)^2 - C_1^\varepsilon \\ I((X^\varepsilon)^{\diamond 3}) &:= I((X^\varepsilon)^3 - 3C_1^\varepsilon X^\varepsilon) \\ \pi_{0\diamond}(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) &= I((X^\varepsilon)^{\diamond 2})(X^\varepsilon)^{\diamond 2} - C_2^\varepsilon \\ \pi_{0\diamond}(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) &= I((X^\varepsilon)^{\diamond 3})(X^\varepsilon)^{\diamond 2} - 3C_2^\varepsilon X^\varepsilon. \end{aligned}$$

Remark 4.2.8. In all the sequel the symbol \diamond does not stand for the usual Wick product, also it looks like it, but for renormalized product, where we have subtracted only the diverging quantity in the expression of the stochastic processes. It can be seen as a product between the usual one and the Wick one. When in Section 4.4 we use the usual Wick product (see [71] for its definition and its properties) we use the usual notation $::$.

To include such considerations and notations in the approximated equation, we need to add a renormalized term

$$\begin{aligned} \Phi^\varepsilon &= \Psi^\varepsilon + I((X^\varepsilon)^3) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon (X^\varepsilon)^2) + I((\Phi^\varepsilon)^3) - C^\varepsilon I(\Phi^\varepsilon + X^\varepsilon) \\ &= \Psi^\varepsilon + I((X^\varepsilon)^3 - 3C_1^\varepsilon X^\varepsilon) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon ((X^\varepsilon)^2 - C_1^\varepsilon)) - 3C_2^\varepsilon (\Phi^\varepsilon + X^\varepsilon) + I((\Phi^\varepsilon)^3) \end{aligned}$$

with $C^\varepsilon = 3(C_1^\varepsilon - 3C_2^\varepsilon)$. Then the approximated equation is given by

$$\Phi^\varepsilon = \Psi^\varepsilon + I((X^\varepsilon)^{\diamond 3}) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + 3I(\Phi^\varepsilon \diamond (X^\varepsilon)^{\diamond 2}) + I((\Phi^\varepsilon)^3) \quad (4.8)$$

where

$$I(\Phi^\varepsilon \diamond (X^\varepsilon)^{\diamond 2}) := 3I(\Phi^\varepsilon ((X^\varepsilon)^2 - C_1^\varepsilon)) + 9C_2^\varepsilon I(\Phi^\varepsilon + X^\varepsilon)$$

Then our goal is obtain a uniform bound for the solution Φ^ε . For that we proceed in two steps

1. In a first analytic step we build an abstract fix point equation which allows us to extend continuously the flow of the regular equation given by

$$\begin{cases} \Phi = I(X^3 - 3aX) + 3I(\Phi^2 X) + 3\{I(X^2 - a) - 3bI(\Phi + X)\} + I(\Phi^3) + \Psi \\ (X, \Psi, (a, b)) \in \mathcal{C}_T^1(\mathbb{T}^3) \times \mathcal{C}_T^1(\mathbb{T}^3) \times \mathbb{R}^2 \end{cases} \quad (4.9)$$

to a space \mathcal{X} of a more rough signal X which satisfies some algebraic and analytic assumptions (see (4.2.9) for the exact definition of \mathcal{X}).

2. In a second probabilistic step we show that the stationary (O.U) process can be enhanced in a canonical way in an element \mathbb{X} of \mathcal{X} .

We will give the exact definition of the space \mathcal{X}

Definition 4.2.9. Let $T > 1$, $\nu, \rho > 0$. We denote by $\overline{C}_T^{\nu, \rho}$ the closure of the set of smooth functions $C^\infty([0, T], \mathbb{R})$ by the semi-norm :

$$\|\varphi\|_{\nu, \rho} = \sup_{t \in [0, T]} t^\nu |\varphi_t| + \sup_{t, s \in [0, T]; s \neq t} \frac{s^\nu |\varphi_t - \varphi_s|}{|t - s|^\rho}.$$

For $0 < 4\delta' < \delta$ we define the normed space $\mathcal{W}_{T, K}$

$$\mathcal{W}_{T, K} = C_T^{\delta', -1/2-\delta} \times C_T^{\delta', -1-\delta} \times C_T^{\delta', 1/2-\delta} \times C_T^{\delta', -\delta} \times C_T^{\delta', -\delta} \times C_T^{\delta', -1/2-\delta} \times \overline{C}_T^{\nu, \rho}$$

with $K = (\delta, \delta', \nu, \rho)$ equipped with the product topology . For $(X, \varphi) \in C([0, T], C(\mathbb{T}^3)) \times C^\infty([0, T])$, and $(a, b) \in \mathbb{R}^2$ we define $R_{a, b}^\varphi \mathbf{X} \in \mathcal{W}_{T, K}$ by

$$\begin{aligned} R_{a, b}^\varphi \mathbf{X} = & (X, X^2 - a, I(X^3 - 3aX), \pi_0(I(X^3 - 3aX), X), \\ & \pi_0(I(X^2 - a), (X^2 - a)) - b - \varphi, \pi_0(I(X^3 - 3aX), (X^2 - a)) - 3bX - 3\varphi X, \varphi). \end{aligned}$$

The space of the rough distribution $\mathcal{X}_{T, K}$ is defined as the closure of the set

$$\left\{ R_{a, b}^\varphi \mathbf{X}, \quad (X, \varphi) \in C([0, T], C(\mathbb{T}^3)) \times C^\infty([0, T]), (a, b) \in \mathbb{R}^2 \right\}$$

in $\mathcal{W}_{T, K}$. For $\mathbb{X} \in \mathcal{X}$ we denote its components by

$$\mathbb{X} = (X, X^{\diamond 2}, I(X^{\diamond 3}), \pi_{0\diamond}(I(X^{\diamond 3}), X), \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X, \pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) - 3\varphi^X X, \varphi^X).$$

For two rough distributions $\mathbb{X} \in \mathcal{X}_{T, K}$ and $\mathbb{Y} \in \mathcal{X}_{T, K}$ we introduce the distance :

$$\begin{aligned} \mathbf{d}_{T, K}(\mathbb{Y}, \mathbb{X}) = & d_{\delta', -1/2-\delta}(Y, X) + d_{\delta', -1-\delta}(Y^{\diamond 2}, X^{\diamond 2}) + d_{\delta', 1/2-\delta}(I(Y^{\diamond 3}), I(X^{\diamond 3})) \\ & + d_{\delta', -1/2-\delta}(\pi_0(I(Y^{\diamond 3}), Y), \pi_0(I(X^{\diamond 3}), X)) \\ & + d_{\delta', -1-\delta}(\pi_{0\diamond}(I(Y^{\diamond 2}), \diamond Y^{\diamond 2}) - \varphi^Y, \pi_{0\diamond}(I(X^{\diamond 3}), \diamond X^2) - \varphi^X) \\ & + d_{\delta', -1-\delta}(\pi_{0\diamond}(I(Y^{\diamond 3}), \diamond Y^{\diamond 2}) - 3\varphi^Y Y, \pi_{0\diamond}(I(X^{\diamond 3}), \diamond X^2) - 3\varphi^X X) \\ & + \|\varphi^Y - \varphi^X\|_{\nu, \rho}. \end{aligned} \tag{4.10}$$

with $K = (\delta, \delta', \rho, \nu) \in [0, 1]^4$ and we denote by $\|\mathbb{X}\|_{T, K} = \mathbf{d}_{T, K}(\mathbb{X}, 0)$.

Remark 4.2.10. As we see in the Section (4.4), the term $\pi_0(I((X^\varepsilon)^2), (X^\varepsilon)^2) - C_2^\varepsilon$ where X^ε is a mollification of the O.U process does not converge in the space $\mathcal{C}_T^{\delta', -\delta}$. On the other hand it converge in a explosive norm and more precisely there exist a function $\varphi^\varepsilon \in C^\infty([0, T])$ such that $\varphi^\varepsilon \rightarrow^{\varepsilon \rightarrow 0} \varphi$ in $\overline{C}_T^{\nu, \rho}$ and $\pi_0(I((X^\varepsilon)^2), (X^\varepsilon)^2) - C_2^\varepsilon - \varphi^\varepsilon$ converge in $\mathcal{C}_T^{\delta', -\delta}$ for all $0 < \delta' < \delta/4$.

For $\mathbb{X} \in \mathcal{X}$ we can obviously construct $I(X^{\diamond 2}) \diamond X^{\diamond 2}$ using the Bony paraproduct in the following way

$$I(X^{\diamond 2}) \diamond X^{\diamond 2} = \pi_{<}(I(X^{\diamond 2}), X^{\diamond 2}) + \pi_{>}(I(X^{\diamond 2}), X^{\diamond 2}) + \pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2})$$

and a similar definition for $I(X^{\diamond 3}) \diamond X^{\diamond 2}$. In the sequel we might abusively denote \mathbb{X} by X if there is no confusion, and the rough path terminology we denote the other components of \mathbb{X} by the area components of \mathbb{X} .

4.2.3 Paracontrolled distributions and fixed point equation

The aim of this Section is to define a suitable space in which it is possible to formulate a fix point for the eventual limit of the mollified solution, to be more precise let $\mathbb{X} \in \mathcal{X}$ then we know that there exist $X^\varepsilon \in \mathcal{C}_T^1(\mathbb{T}^3)$, $a^\varepsilon, b^\varepsilon \in \mathbb{R}$ and $\varphi^\varepsilon \in C^\infty([0, T])$ such that $\lim_{\varepsilon \rightarrow 0} R_{a^\varepsilon, b^\varepsilon}^{\varphi^\varepsilon} \mathbf{X}^\varepsilon = \mathbb{X}$. Let us focus more intently on the regular equation given by :

$$\Phi^\varepsilon = I((X^\varepsilon)^3 - 3a^\varepsilon X^\varepsilon) + 3 \{ I(\Phi^\varepsilon((X^\varepsilon)^2 - a^\varepsilon)) - 3b^\varepsilon I(X^\varepsilon + \Phi^\varepsilon) \} + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + I((\Phi^\varepsilon)^3)$$

where we have omitted temporarily the dependence on the initial condition. If we assume that Φ^ε converge to some Φ in $\mathcal{C}^{1/2-\delta}$ we see that the regularity of \mathbb{X} is not sufficient to define $I(\Phi^2 X) := \lim_{\varepsilon \rightarrow 0} I((\Phi^\varepsilon)^2 X^\varepsilon)$ and $I(\Phi \diamond X^{\diamond 2}) := \lim_{\varepsilon \rightarrow 0} I(\Phi^\varepsilon(X^\varepsilon - a^\varepsilon)) + 3b^\varepsilon I(X^\varepsilon + \Phi^\varepsilon)$. To bypass this problem we remark that

$$\Phi^\varepsilon = I((X^\varepsilon)^3 - 3a^\varepsilon X^\varepsilon) + I(\pi_{<}(\Phi^\varepsilon, (X^\varepsilon)^2 - a^\varepsilon)) + (\Phi^\varepsilon)^\sharp$$

then if we impose the convergence of $(\Phi^\varepsilon)^\sharp$ to some Φ^\sharp in $\mathcal{C}_T^{3/2-\delta}$ we see that the limit Φ should satisfy the following relation

$$\Phi = I(X^{\diamond 3}) + I(\pi_{<}(\Phi, X^{\diamond 2})) + \Phi^\sharp.$$

This is the missing ingredient which allows to construct the quantity $I(\Phi^2 X)$ and $I(\Phi \diamond X^{\diamond 2})$ and to solve the equation

$$\Phi = X^{\diamond 2} + I(\Phi^2 X) + I(\Phi \diamond X^{\diamond 2}) + \Phi^\sharp \quad (4.11)$$

Notation 4.2.11. *Let us introduce some useful notations for the sequel*

$$B_{>}(f, g) = I(\pi_{>}(f, g)), \quad B_0(f, g) = I(\pi_0(f, g)) \quad \text{and} \quad B_{<}(f, g) = I(\pi_{<}(f, g)).$$

As we observed in the beginning of this Section to deal with the difficulty of defining the products of distributions, we use the notion of controlled distribution introduced in [38].

Definition 4.2.12. *Let $\mathbb{X} \in \mathcal{X}$ and $z \in (1/2, 2/3)$. We said that a distribution $\Phi \in \mathcal{C}_T^{-z}$ is controlled by \mathbb{X} if*

$$\Phi = I(X^{\diamond 3}) + B_{<}(\Phi', X^{\diamond 2}) + \Phi^\sharp$$

such that

$$\begin{aligned} \|\Phi^\sharp\|_{\star, 1, L, T} &= \sup_{t \in [0, T]} \left(t^{\frac{1+\delta+z}{2}} \|\Phi_t^\sharp\|_{1+\delta} + t^{1/4+\frac{\gamma+z}{2}} \|\Phi_t^\sharp\|_{1/2+\gamma} + t^{\frac{\kappa+z}{2}} \|\Phi_t^\sharp\|_{\kappa} \right) \\ &+ \sup_{(s, t) \in [0, T]^2} s^{\frac{z+a}{2}} \frac{\|\Phi_t^\sharp - \Phi_s^\sharp\|_{a-2b}}{|t-s|^b} < +\infty \end{aligned}$$

and

$$\|\Phi'\|_{\star, 2, L, T} = \sup_{(s, t) \in [0, T]^2} s^{\frac{z+c}{2}} \frac{\|\Phi_t' - \Phi_s'\|_{c-2d}}{|t-s|^d} + \sup_{t \in [0, T]} t^{\frac{\eta+z}{2}} \|\Phi_t'\|_{\eta} < +\infty$$

with $L := (\delta, \gamma, \kappa, a, b, c, d, \eta) \in [0, 1]^8$, $z \in (1/2, 2/3)$ and $2d \leq c$, $2b \leq a$. Let us denote by $\mathcal{D}_{T, \mathbb{X}}^L$ the space of controlled distributions, endowed with the following metric

$$d_{L, T}(\Phi_1, \Phi_2) = \|\Phi_1' - \Phi_2'\|_{\star, T} + \|\Phi_1^\sharp - \Phi_2^\sharp\|_{\star, T}$$

for $\Phi_1, \Phi_2 \in \mathcal{D}_{\mathbb{X}}^L$ and the quantity

$$\|\Phi\|_{\star, T, L} = \|\Phi_1\|_{\mathcal{D}_{T, \mathbb{X}}^L} = d_{L, T}(\Phi_1, I(X^{\diamond 3})).$$

We notice that the distance and the metric introduced in this last definition do not depend on \mathbb{X} . More generally for $\Phi \in \mathcal{D}_{T_1, X}^L$ and $\Psi \in \mathcal{D}_{T_2, Y}^G$ we denote by $d_{\min(L, G), \min(T_1, T_2)}(\Phi, \Psi)$ the same quantity. We claim that if $\Phi \in \mathcal{D}_X^L$ for a suitable choice of L then we are able to define $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ modulo the use of \mathbb{X} .

Let us decompose the end of this Section into two parts, namely we show that $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ are well-defined when Φ is a controlled distribution. We also have to prove that when Φ is a controlled distribution, $\Psi + I(X^{\diamond 3}) + 3I(\Phi^2 X) + 3I(\Phi \diamond X^{\diamond 2}) + I(\Phi^3)$ is also a controlled distribution. All those verifications being made, the only remaining point will be to show that we can apply a fixed point argument to find a solution to the renormalized equation. This is the aim of Section 4.3.

4.2.4 Decomposition of $I(\Phi^2 X)$

Let $\mathbb{X} \in \mathcal{X}$ and $\Phi \in \mathcal{D}_{\mathbb{X}, T}^L$, a quick computation gives :

$$I(\Phi^2 X) = I(I(X^{\diamond 3})^2 X) + I((\theta^\#)^2 X) + 2I(\theta^\# I(X^{\diamond 3}) X)$$

with

$$\theta^\# = B_{<}(\Phi', X^{\diamond 2}) + \Phi^\#.$$

Using the fact that $\Phi \in \mathcal{D}_{\mathbb{X}, T}^L$ and that $I(X^{\diamond 3}) \in \mathcal{C}_T^{1/2-\delta}$ we can see that the two terms $I((\theta^\#)^2 X)$ and $I(\theta^\# I(X^{\diamond 3}) X)$ are well defined. Let us focus on the term $I(X^{\diamond 3})^2 X$ which is at this stage is not well understood, then a paraproduct decomposition of this term give us that

$$\begin{aligned} I(X^{\diamond 3})^2 X &= 2\pi_0(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) + \pi_0(\pi_0(I(X^{\diamond 3}), I(X^{\diamond 3})), X) \\ &\quad + \pi_{<}(I(X^{\diamond 3})^2, X) + \pi_{>}(I(X^{\diamond 3})^2, X) \end{aligned}$$

We remark that only the first term of this expansion is not well understood and to overcome this problem we use the Proposition (4.2.4), indeed we know that

$$R(I(X^{\diamond 3}), I(X^{\diamond 3}), X) = \pi_0(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) - I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), X)$$

is well defined and it lies in the space $\mathcal{C}_T^{1/2-3\delta}$ due to the fact that $\mathbb{X} \in \mathcal{X}$

Remark 4.2.13. *We remark that the "extension" of the term $I(\Phi^2 X)$ is a functional of " $(\Phi, \mathbb{X}) \in \mathcal{D}_{\mathbb{X}, T}^L \times \mathcal{X}$ " and then we use sometimes the notation $I(\Phi^2 \mathbb{X})[\Phi, \mathbb{X}]$ to underline this fact.*

Proposition 4.2.14. *Let $z \in (1/2, 2/3)$, $\Phi \in \mathcal{D}_{\mathbb{X}}^L$, and assume that $\mathbb{X} \in \mathcal{X}$. Then the quantity $I(\Phi^2 X)[\Phi, \mathbb{X}]$ is well-defined via the following expansion*

$$I(\Phi^2 X)[\Phi, \mathbb{X}] := I(I(X^{\diamond 3})^2 X) + I((\theta^\#)^2 X) + 2I(\theta^\# I(X^{\diamond 3}) X)$$

with

$$\theta^\# = B_{<}(\Phi', X^2) + \Phi^\#$$

and

$$\begin{aligned} I(X^{\diamond 3})^2 X &:= \pi_0(I(X^{\diamond 3}), I(X^{\diamond 3}))X + 2\pi_{<}(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) + 2\pi_{>}(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) \\ &\quad + 2I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), X) + R(I(X^{\diamond 3}), I(X^{\diamond 3}), X) \end{aligned} \tag{4.12}$$

where

$$R(I(X^{\diamond 3}), I(X^{\diamond 3}), X) = \pi_0(\pi_{<}(I(X^{\diamond 3}), I(X^{\diamond 3})), X) - I(X^{\diamond 3})\pi_0(I(X^{\diamond 3}), I(X^{\diamond 3}))$$

is well-defined by the Proposition 4.2.4. And there exists a choice of L such that the following bound holds

$$\|I(\Phi^2 X)[\Phi, \mathbb{X}]\|_{\star, 1, T} \lesssim T^\theta \left(\|\Phi\|_{\mathcal{D}_X^L} + 1 \right)^2 (1 + \|\mathbb{X}\|_{T, \nu, \rho, \delta, \delta'})^3$$

for $\theta > 0$ and $\delta, \delta', \rho, \nu > 0$ small enough depending on L and z . Moreover if $X \in \mathcal{C}_T^1(\mathbb{T}^3)$ then

$$I(\Phi^2 X)[\Phi, R_{a,b}^\varphi \mathbf{X}] = I(\Phi^2 X)$$

Proof. By a simple computation it is easy to see that

$$\|B_{<}(\Phi', X^{\diamond 2})(t)\|_\kappa \lesssim \int_0^t ds (t-s)^{-(\kappa+1+r)/2} \|\Phi'_s\|_\kappa \|X_s^{\diamond 2}\|_{-1-r} \lesssim_{r, \kappa} T^{1/2-r/2-\kappa/2-z/2} \|\Phi'\|_{\star, 2, T} \|X^{\diamond 2}\|_{-1-r}$$

for $r, \kappa > 0$ small enough and $1/2 < z < 2/3$. A similar computation gives

$$\begin{aligned} \|B_{<}(\Phi', X^{\diamond 2})(t)\|_{1/2+\gamma} &\lesssim \int_0^t ds (t-s)^{-(3/2+\gamma+r)/2} \|\Phi'_s\|_\kappa \|X_s^{\diamond 2}\|_{-1-r} \\ &\lesssim_{\kappa, r, z} \|\Phi'\|_{\star, 2, T} \|X^{\diamond 2}\|_{-1-r} \int_0^t ds (t-s)^{-(3/2+\gamma+r)/2} s^{-(\kappa+z)/2} \\ &\lesssim t^{1/4-(\gamma+\kappa+z+r)/2} \|\Phi'\|_{\star, 2, L, T} \|X^{\diamond 2}\|_{-1-r} \end{aligned}$$

for $\gamma, r, \kappa > 0$ small enough. Using this bound we can deduce that

$$\begin{aligned} \|I((\theta^\#)^2 X)(t)\|_{1+\delta} &\lesssim \int_0^t ds (t-s)^{-(3/2+\delta+\beta)/2} \|(\theta^\#_s)^2 X_s\|_{-1/2-\beta} \\ &\lesssim_{\beta, \delta} \int_0^t ds (t-s)^{-(3/2+\delta+\beta)/2} \|\theta^\#_s\|_\kappa \|\theta^\#_s\|_{1/2+\gamma} \|X_s\|_{-1/2-\beta} \\ &\lesssim_{L, z} \|\Phi\|_{\star, L, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2 \\ &\quad \times \int_0^t ds (t-s)^{-(3/2+\delta+\beta)} s^{-(1/2+\kappa+\gamma+2z)/2} \\ &\lesssim_{L, z} t^{-(\delta+\kappa+\gamma+\beta+2z)} \|\Phi\|_{\star, L, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2 \end{aligned}$$

for $\gamma, \beta, \delta > 0$ small enough and $2/3 > z > 1/2$. Hence we obtain that

$$\sup_{t \in [0, T]} t^{(1+\delta+z)/2} \|I((\theta^\#)^2 X)(t)\|_{1+\delta} \lesssim_L T^{\theta_1} \|\Phi\|_{\star, L, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2$$

for some $\theta_1 > 0$ depending on L and z . The same type of computation gives

$$\sup_{t \in [0, T]} t^{(\kappa+z)/2} \|I((\theta^\#)^2 X)(t)\|_\kappa \lesssim_{L, z} T^{\theta_2} \|\Phi\|_{\star, L, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2$$

and

$$\sup_{t \in [0, T]} t^{(1/2+\gamma+z)/2} \|I((\theta^\#)^2 X)(t)\|_{1/2+\gamma} \lesssim_{L, z} T^{\theta_3} \|\Phi\|_{\star, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2$$

with θ_2 and θ_3 two non negative constants depending only on L and z . To complete our study for this term, we have also

$$\|I((\theta^\sharp)^2 X)(t) - I((\theta^\sharp)^2 X)(s)\|_{a-2b} \lesssim I_{st}^1 + I_{st}^2$$

with

$$I_{st}^1 = \left\| \int_0^s du (P_{t-u} - P_{s-u})(\theta_u^\sharp)^2 X_u \right\|_{a-2b}, \quad I_{st}^2 = \left\| \int_s^t du P_{t-u}(\theta_u^\sharp)^2 X_u \right\|_{a-2b}.$$

Let us begin by bounding I^1 :

$$\begin{aligned} I_{st}^1 &\lesssim (t-s)^b \int_0^s du \|P_{s-u}(\theta_u^\sharp)^2 X_u\|_a \\ &\lesssim (t-s)^b \int_0^t du (s-u)^{-(1/2+a+\beta)} \|(\theta_u^\sharp)^2 X_u\|_{-1/2-\beta} \\ &\lesssim T^{\theta_4} |t-s|^b \|\Phi\|_{\star, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2 \end{aligned}$$

with $\theta_4 > 0$ depending on L and z . Let us focus on the bound for I^2 ,

$$\begin{aligned} I_{st}^2 &\lesssim \int_s^t (t-u)^{-(1/2+a-2b+\beta)/2} \|(\theta_u^\sharp)^2 X_u\|_{-1/2-\beta} \\ &\lesssim_{L,z} \|\Phi\|_{\star, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2 \int_s^t du (t-u)^{-(1/2+a-2b+\beta)/2} u^{-(1/2+\kappa+\gamma+2z)/2} \end{aligned}$$

and

$$\begin{aligned} &\int_s^t du (t-u)^{-(1/2+a-2b+\beta)/2} u^{-(1/2+\kappa+\gamma+2z)/2} \\ &= (t-s)^{3/4-(a-2b+\beta)} \int_0^1 dx (1-x)^{-(1/2+a-2b+\beta)} (s+x(t-s))^{-(1/2+\kappa+\gamma+2z)/2} \\ &\lesssim_{l,\kappa,\gamma,a,b} (t-s)^{l-(a-2b+\beta)/2} s^{1/2-z+(\kappa+\gamma)/2} \int_0^1 dx (1-x)^{-(1/2+a-2b+\beta)/2} x^{-3/4+l}. \end{aligned}$$

Then using the fact $z < 1$ and choosing $l, \kappa, \gamma, b > 0$ small enough we can deduce that

$$\int_s^t du (t-u)^{-(1/2+a-2b+\beta)/2} u^{-(1/2+\kappa+\gamma+2z)/2} \lesssim_L T^{\theta_5} (t-s)^b s^{-(z+a)/2}$$

with $\theta_5 > 0$. This gives the needed bound for I_2 . Finally we have

$$\sup_{(s,t) \in [0,T]} s^{(z+a)/2} \frac{\|I((\theta^\sharp)^2 X)(t) - I((\theta^\sharp)^2 X)(s)\|_{a-2b}}{|t-s|^b} \lesssim T^{\theta_5} \|\Phi\|_{\star, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2$$

hence

$$\|I((\theta^\sharp)^2 X)\|_{\star, 1, T} \lesssim_L T^\theta \|\Phi\|_{\star, T}^2 (\|X^{\diamond 2}\|_{-1-r} + \|X\|_{-1/2-\beta} + 1)^2.$$

The bound for $\|I(\theta^\sharp I(X^3)X)\|_{\star, 1, T}$ can be obtained by a similar way and then, according to the hypothesis given on the area $I(X^{\diamond 3})X$ and the decomposition of $I(I(X^{\diamond 3})^2 X)$, we obtain easily from the Proposition 4.2.4 and the Proposition 4.2.3 that

$$\|I(I(X^{\diamond 3})^2) \diamond X\|_{\star, 1, T} \lesssim T^\theta (1 + \|\pi_0(I(X^{\diamond 3}), X)\|_{\delta', -1/2-\rho} + \|I(X^{\diamond 3})\|_{\delta', 1/2-\rho} + \|X\|_{\delta', -1/2-\rho})^3$$

for $3\rho < \delta'$ small enough, which gives the wanted result. \square

4.2.5 Decomposition of $I(\Phi \diamond X^{\diamond 2})$

Let us apply the controlled structure to the mollified equation. As in that case the equation is well-posed, we have

$$\tilde{\Gamma}(\Phi^\varepsilon) = \Phi^\varepsilon \text{ where } \tilde{\Gamma}(\Phi^\varepsilon) = I((X^\varepsilon)^{\diamond 3}) + 3I(\Phi^\varepsilon(X^\varepsilon)^{\diamond 2}) + 3I((\Phi^\varepsilon)^2 X^\varepsilon) + I((\Phi^\varepsilon)^3)$$

with of course Φ^ε controlled by X^ε

$$\Phi^\varepsilon = I((X^\varepsilon)^{\diamond 3}) + B_{<}((\Phi^\varepsilon)', (X^\varepsilon)^{\diamond 2}) + (\Phi^\varepsilon)^\sharp.$$

By a direct computation we also have

$$\begin{aligned} I(\Phi^\varepsilon(X^\varepsilon)^{\diamond 2}) &= B_{<}(\Phi^\varepsilon, (X^\varepsilon)^{\diamond 2}) + B_0(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) + B_0(B_{<}((\Phi^\varepsilon)', (X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) \\ &\quad + B_0((\Phi^\varepsilon)^\sharp, (X^\varepsilon)^{\diamond 2}) + B_{>}(\Phi^\varepsilon, (X^\varepsilon)^{\diamond 2}). \end{aligned}$$

Indeed, thanks to the Bony paraproduct, the first and the last terms in the r.h.s are well defined. The only problem is to define $B_0(\cdot)$. By an analysis of the regularity, the structure of controlled distribution for $\tilde{\Gamma}^\varepsilon(\Phi^\varepsilon)$ appears, and we have $\tilde{\Gamma}^\varepsilon(\Phi^\varepsilon)' = 3\Phi^\varepsilon$ hence $(\Phi^\varepsilon)' = 3\Phi^\varepsilon$. Furthermore, $B_0(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2})$ does not converge, and we need to renormalize it by subtracting $3C_2^\varepsilon I(X^\varepsilon)$. We have to deal with the (ill-defined) diagonal term.

$$X^{\varepsilon, \diamond}(\Phi')(t) = B_0(B_{<}((\Phi^\varepsilon)', (X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2})(t) = \int_0^t ds P_{t-s} \pi_0 \left(\int_0^s d\sigma P_{s-\sigma} \pi_{<}((\Phi^\varepsilon)'_\sigma, (X^\varepsilon)_\sigma^{\diamond 2}), (X^\varepsilon)_s^{\diamond 2} \right)$$

Thanks to the properties of the paraproduct, we decompose this term in the following way

$$\begin{aligned} (X^\varepsilon)^{\varepsilon, \diamond}((\Phi^\varepsilon)')(t) &= \int_0^t ds P_{t-s} (\Phi^\varepsilon)'_s \pi_0(I((X^\varepsilon)^{\diamond 2})(s), (X^\varepsilon)_s^{\diamond 2}) \\ &\quad + \int_0^t ds P_{t-s} \int_0^s d\sigma ((\Phi^\varepsilon)'_\sigma - (\Phi^\varepsilon)'_s) \pi_0((X^\varepsilon)_s^{\diamond 2}, P_{s-\sigma}(X^\varepsilon)_\sigma^{\diamond 2}) \\ &\quad + \int_0^t ds P_{t-s} \int_0^s d\sigma \pi_0(R_{s-\sigma}^1((\Phi^\varepsilon)'_\sigma, (X^\varepsilon)_\sigma^{\diamond 2}), (X^\varepsilon)_s^{\diamond 2}) \\ &\quad + \int_0^t ds P_{t-s} \int_0^s R^2((\Phi^\varepsilon)'_\sigma, P_{s-\sigma}(X^\varepsilon)_\sigma^{\diamond 2}, (X^\varepsilon)_s^{\diamond 2}) \\ &\equiv \sum_{i=1}^4 (X^\varepsilon)^{\diamond, i}(t) \end{aligned}$$

with

$$R_{s-\sigma}^1(f, g) = P_{s-\sigma} \pi_{<}(f, g) - \pi_{<}(f, P_{s-\sigma} g), \quad R^2(f, g, h) = \pi_0(\pi_{<}(f, g), h) - f \pi_0(g, h)$$

Here again, to have a convergent quantity we need to renormalize $(X^\varepsilon)^{\diamond, 1}$ by $C_2^\varepsilon \int_0^t P_{t-s} (\Phi^\varepsilon)'_s = C_2^\varepsilon I(\Phi^\varepsilon)$. Hence, the approximated equation must be

$$\tilde{\Gamma}(\Phi^\varepsilon) = \Phi^\varepsilon$$

with

$$\Gamma^\varepsilon(\Phi^\varepsilon) = \tilde{\Gamma}^\varepsilon(\Phi^\varepsilon) + 9C_2^\varepsilon(\Phi^\varepsilon + X^\varepsilon).$$

The same computation holds for the renormalized equation, and we have

$$I(\Phi \diamond X^{\diamond 2}) = B_{<}(\Phi, X^{\diamond 2}) + B_{0, \diamond}(I(X^{\diamond 3}), X^{\diamond 2}) + B_{0, \diamond}(B_{<}(\Phi', X^{\diamond 2}), X^{\diamond 2}) + B_0(\Phi^\sharp, X^{\diamond 2}) + B_{>}(\Phi, X^{\diamond 2}).$$

Indeed, thanks to the Bony paraproduct, the first and the last terms in the r.h.s are well-defined. The only problem is to define $B_0(\cdot)$. The term in $\Phi^\sharp X^{\diamond 2}$ is also well-defined as $\Phi^\sharp \in \mathcal{C}_T^{1+\delta}$. The term $B_{0, \diamond}(I(X^{\diamond 3}), X^{\diamond 2})$ is also well-defined by Definition 4.2.9. So we only have to deal with the diagonal term

$$X^\diamond(\Phi')(t) = B_{0, \diamond}(B_{<}(\Phi', X^{\diamond 2}), X^{\diamond 2})(t) = \int_0^t ds P_{t-s} \pi_{0, \diamond} \left(\int_0^s d\sigma P_{s-\sigma} \pi_{<}(\Phi'_\sigma, X_\sigma^{\diamond 2}), X_s^{\diamond 2} \right)$$

Thanks to the properties of the paraproduct, we decompose this term in the following way

$$\begin{aligned} X^\diamond(\Phi')(t) &= \int_0^t ds P_{t-s} \Phi'_s \pi_{0, \diamond}(I(X^{\diamond 2})_s, X_s^{\diamond 2}) + \int_0^t ds P_{t-s} \int_0^s d\sigma (\Phi'_\sigma - \Phi'_s) \pi_0(X_s^{\diamond 2}, P_{s-\sigma} X_\sigma^{\diamond 2}) \\ &\quad + \int_0^t ds P_{t-s} \int_0^s d\sigma \pi_0(R_{s-\sigma}^1(\Phi'_\sigma, X_\sigma^{\diamond 2}), X_s^{\diamond 2}) + \int_0^t ds P_{t-s} \int_0^s R^2(\Phi'_\sigma, P_{s-\sigma} X_\sigma^{\diamond 2}, X_s^{\diamond 2}) \\ &\equiv \sum_{i=1}^4 X^{\diamond, i}(t) \end{aligned}$$

where

$$R_{s-\sigma}^1(f, g) = P_{s-\sigma} \pi_{<}(f, g) - \pi_{<}(f, P_{s-\sigma} g), \quad R^2(f, g, h) = \pi_0(\pi_{<}(f, g), h) - f \pi_0(g, h)$$

and f, g, h are distributions lying in the suitable Besov spaces for R^1 and R^2 to be defined. Before starting to bound the term X^\diamond , let us give a useful lemma to deal with the explosive Hölder type norm

Lemma 4.2.15. *Let f a space time distribution such that $\sup_{t \in [0, T]} t^{(r+z)/2} \|f_t\|_r < +\infty$ then the following bound holds*

$$\sup_{s, t \in [0, T]} \frac{\|I(f)(t) - I(f)(s)\|_{a-2b}}{|t-s|^b} \lesssim_{b, a, z, r} T^\theta \sup_{t \in [0, T]} t^{(r+z)/2} \|f_t\|_r$$

with $a+z < 2$, $z+r < 2$, $a-r < 2$, $0 < a, b < 1$ and $\theta > 0$ is a constant depending only on a, r, b, z .

Proof. By a simple computation we have

$$I(f)(t) - I(f)(s) = I_{st}^1 + I_{st}^2$$

with

$$I_{st}^1 = (P_{t-s} - 1) \int_0^s du P_{s-u} f_u \quad \text{and} \quad I_{st}^2 = \int_s^t du P_{t-u} f_u.$$

Using the lemma (4.2.5) the following bound holds

$$\|I_{st}^1\|_{a-2b} \lesssim |t-s|^b \int_0^t du (t-u)^{-(a-r)/2} u^{-(r+z)/2} \sup_{t \in [0, T]} t^{(r+z)/2} \|f_t\|_r < +\infty.$$

To handle the second we use Hölder inequality,

$$\|I_{st}^2\|_{a-2b} \lesssim |t-s|^b \left(\int_s^t du (t-u)^{\frac{-(a-2b-r)}{2(1-b)}} u^{-\frac{(z+r)}{2(1-b)}} \right)^{1-b} \sup_{t \in [0, T]} t^{(r+z)/2} \|f_t\|_r < +\infty$$

which ends the proof. \square

The following proposition gives us the regularity for our terms.

Proposition 4.2.16. *Assume that $\mathbb{X} \in \mathcal{X}$ then there exists a choice of L such that for all $z \in (1/2, 2/3)$ the following bound holds*

$$\|X^\diamond(\Phi')\|_{\star, 1, T} \lesssim T^\theta (1 + \|\mathbb{X}\|_{T, K})^2 \|\Phi'\|_{\star, 2, T}$$

with $K \in [0, 1]^4, \theta > 0$ are two small parameters depending only on L and z .

Proof. We begin by estimate the first term of the expansion (4.2.5)

$$\begin{aligned} \|X^{\diamond, 1}(\Phi')(t)\|_{1+\delta} &\lesssim \int_0^t ds (t-s)^{-(1+\delta+\eta/2)/2} \|\Phi'_s \pi_{0, \diamond}(I(X^{\diamond 2})(s), X_s^{\diamond 2}) - \varphi_s^X\|_{-\eta/2} \\ &\quad + \left(\sup_{\sigma \in [0, T]} \sigma^\nu |\varphi_\sigma^X| \right) \int_0^t ds (t-s)^{-(1+\delta-\eta)/2} \|\Phi'_s\|_\eta \\ &\lesssim \|\Phi'\|_{\star, 2, T} (I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{-\eta/2} + \sup_{\sigma \in [0, T]} \sigma^\nu |\varphi_\sigma^X| + 1 \\ &\quad \times \int_0^t ds (t-s)^{-(1+\delta+\eta/2)/2} s^{-(\eta+\nu+z)/2} \\ &\lesssim_{\beta, L} T^{\theta_1} \|\Phi'\|_{\star, 2, T} (\|\pi_{0, \diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{-\eta/2} + \sup_{\sigma \in [0, T]} \sigma^\nu |\varphi_\sigma^X| + 1) \end{aligned}$$

for $\nu, \eta, \delta > 0$ small enough and with $\theta_1 > 0$ depending on L . Hence

$$\sup_{t \in [0, T]} t^{(1+\delta+z)/2} \|X^{\diamond, 1}(\Phi')(t)\|_{1+\delta} \lesssim_{L, z} T^{\theta_1} (\|\pi_{0, \diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{-\eta/2} + \sup_{\sigma \in [0, T]} \sigma^\nu |\varphi_\sigma^X| + 1)$$

Let us focus on the second term. We have

$$\begin{aligned} \|X^{\diamond, 2}(\Phi')(t)\|_{1+\delta} &\lesssim \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} \int_0^s d\sigma \|(\Phi'_\sigma - \Phi'_s) \pi_0(P_{s-\sigma} X_\sigma^{\diamond 2}, X_s^{\diamond 2})\|_\beta \\ &\lesssim_{\beta, \rho} \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} \int_0^s d\sigma (s-\sigma)^{-(2+\rho)/2} \|\Phi'_\sigma - \Phi'_s\|_{c-2d} \|X_s^{\diamond 2}\|_{-1-\rho}^2 \\ &\lesssim_{L, \beta, \rho} \|\Phi'\|_{\star, 2, T} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\rho}}^2 \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} \int_0^s d\sigma (s-\sigma)^{-1-\rho/2+d} \sigma^{-(c+z)/2} \\ &\lesssim_{L, \beta, \rho} \|\Phi'\|_{\star, 2, T} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\rho}}^2 \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} s^{-(\rho+c-2d+z)/2} \\ &\lesssim_{L, \beta, \rho} T^{\theta_2} \|\Phi'\|_{\star, 2, T} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\rho}}^2 \end{aligned}$$

for $\beta = \min(c-2d, \rho) \geq 0$ and all $c, d, \rho > 0$ small enough, $z < 1$ and $\theta_2 > 0$ is a constant depending only on L and z . Using the Lemma 4.2.5 we see

$$\|R_{s-\sigma}^1(\Phi'_\sigma, X_\sigma^{\diamond 2})\|_{1+2\beta} \lesssim (s-\sigma)^{-(2+3\beta-\eta)/2} \|\Phi'_\sigma\|_\eta \|X_\sigma^{\diamond 2}\|_{-1-\beta}$$

for all $\beta > 0$, $\beta < \eta/3$ small enough. By a straightforward computation we have

$$\begin{aligned}
 \|X^{\diamond,3}(\Phi')(t)\|_{1+\delta} &\lesssim \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} \int_0^s d\sigma \|\pi_0(R_{s-\sigma}^1(\Phi'_\sigma, X_\sigma^{\diamond 2}), X_s^{\diamond 2})\|_\beta \\
 &\lesssim \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} \int_0^s d\sigma \|R_{s-\sigma}^1(\Phi'_\sigma, X_\sigma^{\diamond 2})\|_{1+2\beta} \|X_s^{\diamond 2}\|_{-1-\beta} \\
 &\lesssim \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T} \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} \int_0^s d\sigma (s-\sigma)^{-(2+3\beta-\kappa)/2} \sigma^{-(\eta+z)/2} \\
 &\lesssim \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T} \int_0^t ds (t-s)^{-(1+\delta-\beta)/2} s^{-(3\beta-\kappa+\eta+z)/2} \\
 &\lesssim T^{\theta_3} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T}
 \end{aligned}$$

where $\theta_3 > 0$ is a constant depending on L and z , $0 < \beta < \eta/3$ small enough and $z < 1$. To treat the last term it is sufficient to use the commutation result given in the Proposition (4.2.4), indeed we have

$$\|R^2(\Phi'_\sigma, P_{s-\sigma} X_\sigma^{\diamond 2}, X_s^{\diamond 2})\|_{\eta-3\beta} \lesssim_{\eta,\beta} s^{-(\eta+z)/2} (s-\sigma)^{-(2-\beta)/2} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T}$$

for $0 < \beta < \eta/3$ small enough and then

$$\begin{aligned}
 \|X^{\diamond,4}(\Phi')(t)\|_{1+\delta} &\lesssim_{\eta,\beta} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T} \int_0^t ds (t-s)^{-(1+\delta-\eta+3\beta)/2} \int_0^s d\sigma s^{-(\eta+z)/2} (s-\sigma)^{-(2-\beta)/2} \\
 &\lesssim_{\eta,\beta} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T} \int_0^t ds (t-s)^{-(1+\delta-\eta+3\beta)/2} s^{-(\eta+z+\beta)/2} \\
 &\lesssim T^{\theta_4} \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\beta}}^2 \|\Phi'\|_{\star,2,T}
 \end{aligned}$$

for $\theta_4 > 0$ depending on L and $z < 1$ and $\beta, \eta, \delta > 0$ small enough. Binding all these bounds together we conclude that

$$\begin{aligned}
 \sup_{t \in [0, T]} t^{(1+\delta+z)/2} \|X^\diamond(\Phi')(t)\|_{1+\delta} &\lesssim_{L,z} T^\theta (1 + \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\rho}} + \|\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{\mathcal{C}_T^{-1-\rho}} \\
 &\quad + \sup_{t \in [0, T]} t^\nu |\varphi_t^X|^2) \|\Phi'\|_{\star,2,T}
 \end{aligned}$$

for $\theta > 0$ depending on L and z . The same arguments gives

$$\begin{aligned}
 \sup_{t \in [0, T]} t^{(1/2+\gamma+z)/2} \|X^\diamond(\Phi')(t)\|_{1/2+\gamma} &\lesssim_{L,z} T^\theta (1 + \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\rho}} + \|\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{\mathcal{C}_T^{-1-\rho}} \\
 &\quad + \sup_{t \in [0, T]} t^\nu |\varphi_t^X|^2) \|\Phi'\|_{\star,2,T}
 \end{aligned}$$

and

$$\begin{aligned}
 \sup_{t \in [0, T]} t^{(\kappa+z)/2} \|X^\diamond(\Phi')(t)\|_\kappa &\lesssim_{L,z} T^\theta (1 + \|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\rho}} + \|\pi_{0\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{\mathcal{C}_T^{-1-\rho}} \\
 &\quad + \sup_{t \in [0, T]} t^\nu |\varphi_t^X|^2) \|\Phi'\|_{\star,2,T}
 \end{aligned}$$

To obtain the needed bound we still need to estimate the following quantity

$$\sup_{(s,t) \in [0,T]^2} s^{\frac{z+a}{2}} \frac{\|X^\diamond(\Phi')(t) - X^\diamond(\Phi')(s)\|_{a-2b}}{|t-s|^b}.$$

To deal with is we use the fact that $X^{\diamond,i}(\Phi') = I(f^i)$ with

$$f^1(s) = \Phi'_s \pi_{0,\diamond}(I(X^{\diamond 2})(s), X_s^{\diamond 2}), \quad f^2(s) = \int_0^s d\sigma (\Phi'_\sigma - \Phi'_s) \pi_0(X_s^{\diamond 2}, P_{s-\sigma} X_\sigma^{\diamond 2})$$

and

$$f^3(s) = \int_0^s d\sigma \pi_0(R_{s-\sigma}^1(\Phi'_\sigma, X_\sigma^{\diamond 2}), X_s^{\diamond 2}), \quad f^4(s) = \int_0^s R^2(\Phi'_\sigma, P_{s-\sigma} X_\sigma^{\diamond 2}, X_s^{\diamond 2}).$$

By a easy computation we have

$$\|f^1(t)\|_{\eta/2} \lesssim_\eta s^{-(\eta+z)/2} \|\Phi'\|_{\star,2,T} (1 + \|\pi_{0,\diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi^X\|_{-\eta/4} + \sup_{t \in [0,T]} t^\nu |\varphi_t^X|)^2$$

$$\|f^2(s)\|_{-d} \lesssim \|\Phi'\|_{\star,2,T} \|X^{\diamond 2}\|_{-1-d/4}^2 \int_0^s d\sigma (s-\sigma)^{-1+d/2} \sigma^{-(c+z)/2} \lesssim_{z,c,d} s^{d/2-(c+z)/2} \|\Phi'\|_{\star,2,T} \|X^{\diamond 2}\|_{-1-d/4}$$

$$\|f^3(s)\|_{2\eta/3} \lesssim \|\Phi'\|_{\star,2,T} \|X^{\diamond 2}\|_{-1-\eta/9}^2 \int_0^s ds (s-\sigma)^{-1+\eta/9} s^{-(\eta+z)/2} \lesssim s^{-(11\eta+9z)/2} \|\Phi'\|_{\star,2,T} \|X^{\diamond 2}\|_{-1-\eta/9}^2$$

with $\nu > 0$ depending only on L , and a similar bound for f^4 which allows us to conclude by the Lemma (4.2.15) that we have

$$\sup_{(s,t) \in [0,T]^2} s^{\frac{z+a}{2}} \frac{\|X^\diamond(\Phi')(t) - X^\diamond(\Phi')(s)\|_{a-2b}}{|t-s|^b} \lesssim T^\theta \|\Phi'\|_{\star,2,T} \|X^{\diamond 2}\|_{-1-\rho}^2$$

for some $\rho > 0$, $\theta > 0$ and $\eta, c, d > 0$ small enough and $z \in (1/2, 2/3)$. \square

We are now able to give the meaning of $I(\Phi \diamond X^{\diamond 2})$ for a $\Phi \in \mathcal{D}_{\mathbb{X}}^L$.

Corollary 4.2.17. *Assume that $\mathbb{X} \in \mathcal{X}$ and let $\Phi \in \mathcal{D}_{\mathbb{X}}^L$ then for $z \in (1/2, 2/3)$ and for a suitable choice of L the term $I(\Phi \diamond X^{\diamond 2})[\Phi, \mathbb{X}]$ is defined via the following expansion*

$$I(\Phi \diamond X^{\diamond 2})[\Phi, \mathbb{X}] := B_{<}(\Phi, X^{\diamond 2}) + B_{>}(\Phi, X^{\diamond 2}) + B_{0,\diamond}(I(X^{\diamond 3}), X^{\diamond 2}) + X^\diamond(\Phi') + B_0(\Phi^\sharp, X^{\diamond 2})$$

And we have the following bound

$$\|B_0(\Phi^\sharp, X^{\diamond 2})\|_{\star,1,T} + \|B_{>}(\Phi, X^{\diamond 2})\|_{\star,1,T} \lesssim T^\theta \|\Phi\|_{\star,T} \|X^{\diamond 2}\|_{C_T^{-1-\rho}}$$

for some $\theta, \rho > 0$ being a non-negative constant depending on L and z . Moreover if $a, b \in \mathbb{R}$, $X \in \mathcal{C}_T^1(\mathbb{T}^3)$ and $\varphi \in C^\infty([0, T])$ then we have that

$$I(\Phi \diamond X^{\diamond 2})[\Phi, R_{a,b}^\varphi \mathbf{X}] = I(\Phi(X^2 - a)) + 3bI(X + \Phi)$$

for every $\Phi \in \mathcal{D}_{R_{a,b}^\varphi \mathbf{X}}$.

Proof. We remark that all the term in the definition of $I(\Phi \diamond X^{\diamond 2})$ are well-defined due to the Proposition 4.2.16 and the definition of the paraproduct, and we also notice that

$$\begin{aligned} \|B_0(\Phi^\sharp, X^{\diamond 2})(t)\|_{1+\delta} &\lesssim \int_0^t ds (t-s)^{-(1+\delta/2)/2} \|\Phi_s^\sharp\|_{1+\delta} \|X^{\diamond 2}\|_{-1-\delta/2} \\ &\lesssim \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{-1-\delta/2}} \int_0^t ds (t-s)^{-(1+\delta/2)/2} s^{-(1+\delta+z)/2} \\ &\lesssim s^{-(3/2\delta+z)/2} \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{-1-\delta/2}} \end{aligned}$$

which gives easily

$$\sup_{t \in [0, T]} t^{(1+\delta+z)/2} \|B_0(\Phi^\sharp, X^{\diamond 2})(t)\|_{1+\delta} \lesssim T^{1/2-\delta} \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{-1-\delta/2}}$$

for $\delta < 1/2$. By a similar computation we obtain that there exists $\theta > 0$ depending on L and z such that

$$\sup_{t \in [0, T]} t^{(1/2+\gamma+z)/2} \|B_0(\Phi^\sharp, X^{\diamond 2})(t)\|_{1/2+\gamma} + \sup_{t \in [0, T]} t^{(\kappa+z)/2} \|B_0(\Phi^\sharp, X^{\diamond 2})(t)\|_{\kappa} \lesssim T^\theta \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{-1-\delta/2}}.$$

To obtain the needed bound for this term we still need to estimate the Hölder type norm for it. We remark that

$$\|\pi_0(\Phi_s^\sharp, X_s^{\diamond 2})\|_{\delta/2} \lesssim s^{-(1+\delta+z)/2} \|\Phi_s^\sharp\|_{1+\delta} \|X_s^{\diamond 2}\|_{-1-\delta/2}$$

and then as usual we decompose the norm in the following way

$$B_0(\Phi_t^\sharp, X^{\diamond 2})(t) - B_0(\Phi_s^\sharp, X_s^{\diamond 3}) = I_{st}^1 + I_{st}^2$$

with

$$I_{st}^1 = (P_{t-s} - 1) \int_0^t du P_{t-u} \pi_0(\Phi_u^\sharp, X_u^{\diamond 2}), \quad I_{st}^2 = \int_s^t du P_{t-u} \pi_0(\Phi_u^\sharp, X_u^{\diamond 2}).$$

A straightforward computation gives

$$\begin{aligned} \|I_{st}^1\|_{a-2b} &\lesssim \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{1-\delta/2}} |t-s|^b \int_0^t du (t-u)^{-(a-\delta/2)/2} u^{-(1+\delta+z)/2} \\ &\lesssim T^{(1-a-\delta/2-z)/2} |t-s|^b \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{1-\delta/2}}. \end{aligned}$$

For I^2 we use Hölder inequality which gives

$$\begin{aligned} \|I_{st}^2\|_{a-2b} &\lesssim |t-s|^b \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{1-\delta/2}} \left(\int_s^t du (t-u)^{-\frac{a-2b-\delta/2}{2(1-b)}} u^{-\frac{1+\delta+z}{2(1-b)}} \right)^{1-b} \\ &\lesssim T^{(1-a-\delta/2-z)/2} |t-s|^b \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{1-\delta/2}} \end{aligned}$$

for $a, \delta > 0$ small enough and $z < 1$. We have obtained that

$$\|B_0(\Phi^\sharp, X^{\diamond 2})\|_{\star,1,T} \lesssim T^\theta \|\Phi^\sharp\|_{\star,1,T} \|X^{\diamond 2}\|_{C_T^{-1-\delta/2}}$$

for some $\theta > 0$ depending on L and z . The bound for the term $B_{>}(\Phi, X^{\diamond 2})$ is obtained by a similar argument and this ends the proof. \square

Remark 4.2.18. When there are no ambiguity we use the notation $I(\Phi \diamond X^{\diamond 2})$ instead of $I(\Phi \diamond X^{\diamond 2})[\Phi, \mathbb{X}]$.

4.3 Fixed point procedure

Using the analysis of $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ developed in the previous Section, we can now show that the equation

$$\Phi = I(X^{\diamond 3}) + 3I(\Phi \diamond X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi$$

admits a unique solution $\Phi \in \mathcal{D}_{\mathbb{X}}^L$ for a suitable choice of L and $z \in (1/2, 2/3)$ via the fixed point method. We also show that if u^ε is the solution of the regularized equation and Φ^ε is such that $u^\varepsilon = X^\varepsilon + \Phi^\varepsilon$ then $d(\Phi^\varepsilon, \Phi)$ goes to 0 as ε . Hence, by the convergence of X^ε to X we have the convergence of u^ε to $u = \Phi + X$. Let us begin by giving our fixed point result.

Theorem 4.3.1. *Assume that $\mathbb{X} \in \mathcal{X}$ and $u^0 \in \mathcal{C}^{-z}(\mathbb{T}^3)$ with $z \in (1/2, 2/3)$. Let $\Phi \in \mathcal{D}_{\mathbb{X}}^L$ and $\Psi = Pu^0$ then we define the application $\Gamma : \mathcal{D}_{\mathbb{X},T}^L \rightarrow \mathcal{C}_T^{-z}(\mathbb{T}^3)$ by*

$$\Gamma(\Phi) = I(X^{\diamond 3}) + 3I(\Phi \diamond X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi$$

where $I(\Phi \diamond X^{\diamond 2})$ and $I(\Phi^2 X)$ are given by the Corollary (4.2.17) and the Proposition (4.2.14). Then $\Gamma(\Phi) \in \mathcal{D}_{\mathbb{X}}^L$ for a suitable choice of L and it satisfies the following bound

$$\|\Gamma(\Phi)\|_{\star,T} \lesssim (T^\theta \|\Phi\|_{\star,L,T} + 1)^3 (1 + \|\mathbb{X}\|_{T,K} + \|u^0\|_{-z})^3. \quad (4.13)$$

Moreover for $\Phi_1, \Phi_2 \in \mathcal{D}_{\mathbb{X}}^L$ the following bound hold

$$d_{T,L}(\Gamma(\Phi_1), \Gamma(\Phi_2)) \lesssim T^\theta d_{T,L}(\Phi_1, \Phi_2) (\|\Phi_1\|_{\star,L,T} + \|\Phi_2\|_{\star,L,T} + 1)^2 (1 + \|\mathbb{X}\|_{T,K} + \|u^0\|_{-z})^3 \quad (4.14)$$

for some $\theta > 0$ and $K \in [0, 1]^8$ depending on L and z . We can conclude that for this choice of L there exist $T > 0$ and a unique $\Phi \in \mathcal{D}_{\mathbb{X},T}^L$ such that

$$\Phi = \Gamma(\Phi) = I(X^{\diamond 3}) + 3I(\Phi \diamond X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi. \quad (4.15)$$

Proof. By the the Corollary (4.2.17) and the Proposition (4.2.14) we see that $\Gamma(\Phi)$ has the needed algebraic structure of the controlled distribution more precisely

$$\Gamma(\Phi)' = 3\Phi, \quad \Gamma(\Phi)^\sharp = 3B_{>}(\Phi, X^{\diamond 2}) + X^{\diamond}(\Phi') + 3B_0(\Phi^\sharp, X^{\diamond 2}) + 3I(\Phi^2 X) + I(\Phi^3) + \Psi$$

and $\Gamma(\Phi) \in \mathcal{C}_T^{-z}$. To show that $\Gamma(\Phi) \in \mathcal{D}_{\mathbb{X}}^L$ and obtain the first bound it remains to estimate $\|\Phi\|_{\star,2,L,T}$ and $\|\Gamma(\Phi)^\sharp\|_{\star,1,L,T}$. A straightforward computation gives

$$\begin{aligned} \|\Phi_t\|_\eta &\lesssim \|I(X^{\diamond 3})(t)\|_\eta + \|B_{<}(\Phi', X^{\diamond 2})(t)\|_\eta + \|\Phi_t^\sharp\|_\eta \\ &\lesssim \|I(X^{\diamond 3})\|_\eta + \|\Phi'\|_{\star,2,T} \|X^{\diamond 2}\|_{-1-\eta} \int_0^t ds (t-s)^{-(1+2\eta/2)/2} s^{-(\eta+z)/2} + t^{-(\kappa+z)} \|\Phi^\sharp\|_{\star,1,T} \\ &\lesssim (\|\Phi\|_{\star,L,T} + 1) (\|X^{\diamond 2}\|_{-1-\eta} + \|I(X^{\diamond 3})\|_\eta + 1) t^{\min(1/2-(3\eta+z)/2, -(\kappa+z)/2)}. \end{aligned}$$

Then for $0 < \eta < \kappa$ and $\eta < 1/2$ and $z \in (1/2, 2/3)$ small enough we see that

$$\sup_{t \in [0, T]} t^{(\eta+z)/2} \|\Phi\|_\eta \lesssim T^{\kappa-\eta} (\|\Phi\|_{\star,T} + 1) (\|X^{\diamond 2}\|_{\mathcal{C}_T^{-1-\eta}} + \|I(X^{\diamond 3})\|_{\mathcal{C}_T^\eta} + 1).$$

We focus on the explosive Hölder type norm for this term, indeed a quick computation gives

$$\|\Phi_t - \Phi_s\|_{c-2d} \lesssim \|I(X^{\diamond 3})(t) - I(X^{\diamond 3})(s)\|_{c-2d} + \|B_{<}(\Phi', X^{\diamond 2})(t) - B_{<}(\Phi', X^{\diamond 2})(s)\|_{c-2d} + \|\Phi_t^\sharp - \Phi_s^\sharp\|_{c-2d}.$$

Let us estimate the first term in the right hand side. Using the regularity for $I(X^{\diamond 3})$ we obtain that for $d > 0$ small enough and $c < 1/2$

$$\|I(X^{\diamond 3})(t) - I(X^{\diamond 3})(s)\|_{c-2d} \lesssim |t - s|^d \|I(X^{\diamond 3})\|_{d, c-2d}.$$

Then we notice that the increment appearing in second term has the following representation

$$B_{<}(\Phi', X^{\diamond 2}) = I(f)$$

with $f = \pi_{<}(\Phi', X^{\diamond 2})$. To treat this term it is sufficient to notice that

$$\|f_t\|_{-1-\delta} \lesssim \|\Phi_t'\|_{\eta} \|X^{\diamond 2}\|_{-1-\delta} \lesssim t^{-(\eta+z)/2} \|\Phi\|_{\star, L, T} \|X^{\diamond 2}\|_{-1-\delta}$$

and then a usual argument gives

$$\|B_{<}(\Phi', X^{\diamond 2})(t) - B_{<}(\Phi', X^{\diamond 2})(s)\|_{c-2d} \lesssim T^\theta |t - s|^d t^{-(c+z)/2} \|\Phi\|_{\star, L, T} \|X^{\diamond 2}\|_{-1-\delta}$$

for some $\theta > 0$ and $c, \delta > 0$. For the last term we use that

$$\|\Phi_t^\sharp - \Phi_s^\sharp\|_{c-2d} \lesssim |t - s|^b t^{-(a+z)/2} \|\Phi\|_{\star, T} \lesssim T^{b-d+a-c} |t - s|^d t^{-(c+z)/2} \|\Phi\|_{\star, L, T}$$

for $c - 2d < a - 2b$, $d < b$ and then $c < a$ which gives :

$$\sup_{s, t \in [0, T]} s^{-(c+z)/2} \frac{\|\Phi_t - \Phi_s\|_{c-2d}}{|t - s|^d} \lesssim T^\theta (1 + \|I(X^{\diamond 3})\|_{d, c-2d} + \|X^{\diamond 2}\|_{-1-\delta}) (1 + \|\Phi\|_{\star, L, T}).$$

Hence the following bound holds

$$\|\Gamma(\Phi)'\|_{\star, 2, L, T} \lesssim T^\theta (1 + \|I(X^{\diamond 3})\|_{d, c-2d} + \|X^{\diamond 2}\|_{-1-\delta}) (1 + \|\Phi\|_{\star, T}). \quad (4.16)$$

We need to estimate the remaining term $\Gamma(\Phi)^\sharp$. Due to the propositions (4.2.14), (4.2.16) and the corollary (4.2.17) it only remains to estimate the following terms $I(\Phi^3)$ and Ψ . In fact a simple computation gives

$$\|\Psi\|_{\star, 1, L, T} \lesssim \|u^0\|_{-z}.$$

Let us focus to the term $I(\Phi^3)$. We notice that

$$\|I(\Phi^3)(t)\|_{1+\delta} \lesssim \int_0^t ds (t-s)^{-(1+\delta-\eta)/2} s^{-3/2(\eta+z)} \|\Phi\|_{\star, T}^3 (\|X^{\diamond 2}\|_{-1-\rho} + 1)^3$$

for $\delta, \kappa > 0$ small enough and $z < 2/3$ and we obtain the existence of some $\theta > 0$ such that

$$\sup_{t \in [0, T]} t^{(1+\delta+z)/2} \|I(\Phi^3)(t)\|_{1+\delta} \lesssim T^\theta \|\Phi\|_{\star, T}.$$

A similar argument gives

$$\sup_{t \in [0, T]} t^{(1/2+\gamma+z)/2} \|I(\Phi^3)(t)\|_{1/2+\gamma} + \sup_{t \in [0, T]} t^{(\kappa+z)/2} \|I(\Phi^3)(t)\|_{\kappa} \lesssim T^\theta \|\Phi\|_{\star, L, T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3.$$

Let us remark that

$$\|\Phi^3\|_\eta \lesssim t^{-3(\eta+z)/2} \|\Phi\|_{\star,L,T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3$$

and then as usual to deal with the Hölder norms we begin by writing the following decomposition

$$\|I(\Phi^3)(t) - I(\Phi^3)(s)\|_{c-2d} \lesssim I_{st}^1 + I_{st}^2$$

with

$$I_{st}^1 = (P_{t-s} - 1) \int_0^s du P_{s-u} \Phi_u^3, \text{ and } I_{st}^2 = \int_s^t du P_{t-u} \Phi_u^3.$$

For I^1 is suffice to observe that

$$\begin{aligned} \|I_{st}^1\|_{c-2d} &\lesssim |t-s|^d \int_0^s du (s-u)^{-(c-\eta)/2} u^{-3/2(z+\eta)} \|\Phi\|_{\star,T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3 \\ &\lesssim T^{1-(c-\eta)-3/2(z+\eta)} |t-s|^d \|\Phi\|_{\star,L,T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3 \end{aligned}$$

for $\eta, c > 0$ small enough, $z < 2/3$. To obtain the second bound we use the Hölder inequality and then

$$\begin{aligned} \|I_{st}^2\|_{c-2d} &\lesssim |t-s|^d \left(\int_s^t du \|P_{t-u} \Phi_u^3\|_{c-2d}^{1/(1-d)} \right)^{1-d} \\ &\lesssim |t-s|^d \left(\int_s^t du (t-u)^{-\frac{c-2d-\eta}{2-2d}} u^{-\frac{3(z+\eta)}{(2-2d)}} \right)^{1-d} \|\Phi\|_{\star,T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3 \\ &\lesssim |t-s|^d T^{1-(c-2\eta+3z)/2} \|\Phi\|_{\star,T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3 \end{aligned}$$

for $c, \eta, d > 0$ small enough and $z < 2/3$. We can conclude that there exists $\theta > 0$ such that

$$\sup_{s,t} s^{(z+c)/2} \frac{\|I(\Phi^3)(t) - I(\Phi^3)(s)\|_{c-2d}}{|t-s|^d} \lesssim T^\theta \|\Phi\|_{\star,T}^3 (1 + \|X^{\diamond 2}\|_{-1-\rho})^3$$

and then we obtain all needed bounds for the remaining term and we can state that

$$\|\Gamma(\Phi)^\sharp\|_{\star,2,L,T} \lesssim (T^\theta \|\Phi\|_{\star,T} + 1)^3 (1 + \|\mathbb{X}\|_{T,K} + \|u^0\|_{-z})^3$$

for some $K \in [0, 1]^4$ depending on L and this gives the first bound (4.13). The second estimate (4.14) is obtained by the same manner.

Due to the bound (4.13) for $T_1 > T > 0$ small enough, there exists $R_T > 0$ such that $B_{R_T} := \{\Phi \in \mathcal{D}_{\mathbb{X},T}^L; \|\Phi\|_{\star,T} \leq R_T\}$ is invariant by the map Γ . The bound (4.14) tells us that Γ is a contraction on $B_{R_{T_2}}$ for $0 < T_2 < T_1$ small enough. Then by the usual fixed point theorem there exists $\Phi \in \mathcal{D}_{\mathbb{X},T_2}^L$ such that $\Gamma(\Phi) = \Phi$. The uniqueness is obtained by a standard argument. \square

A quick adaptation of the last proof gives a better result (see for example [36] and the continuity result theorem). In fact the flow is continuous with respect to the rough distribution \mathbb{X} and with respect to the initial condition ψ (or u^0).

Proposition 4.3.2. *Let \mathbb{X} and \mathbb{Y} two rough distributions such that $\|\mathbb{X}\|_{T,K}, \|\mathbb{Y}\|_{T,K} \leq R$, $z \in (-2/3, -1/2)$, u_X^0 and u_Y^0 two initial conditions and $\Phi^X \in \mathcal{D}_{T^X, X}^L$ and $\Phi^Y \in \mathcal{D}_{T^Y, Y}^L$ the two unique solutions of the equations associating to \mathbb{X} and \mathbb{Y} , and T_X and T_Y there respective living times. For $T^* = \inf T_X, T_Y$ the following bound hold*

$$\|\Phi^X - \Phi^Y\|_{C([0,T], C^{-z}(\mathbb{T}^3))} \lesssim d_{T,L}(\Phi^X, \Phi^Y) \lesssim_R \mathbf{d}_{T,K}(\mathbb{X}, \mathbb{Y}) + \|u_X^0 - u_Y^0\|_{-z}$$

for every $T \leq T^*$, where d is defined in Definition 4.2.12 and \mathbf{d} is defined in Definition 4.2.9.

Hence, using this result and combining it with the convergence Theorem (4.4.3), we have this second corollary, where the convergence of the approximated equation is proved.

Corollary 4.3.3. *Let $z \in (1/2, 2/3)$, $u^0 \in C^{-z}$ and denote u^ε the unique solutions (with life times T^ε) of the equation*

$$\partial_t u^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon)^3 + C^\varepsilon u^\varepsilon + \xi^\varepsilon$$

where ξ^ε is a mollification of the space-time white noise ξ and $C^\varepsilon = 3(C_1^\varepsilon - 3C_2^\varepsilon)$ with (C_1^ε) and C_2^ε are the constant given by the Definition 4.4.2. Let us introduce $u = X + \Phi$ where Φ is the local solution with life-time $T > 0$ for the fixed point equation given in the Theorem 4.3.1. Then we have the following convergence result

$$\mathbb{P}(d_{T^*,L}(\Phi^\varepsilon, \Phi) > \lambda) \xrightarrow{\varepsilon \rightarrow 0} 0$$

for all $\lambda > 0$ with $T^* = \inf(T, T^\varepsilon)$ and $\Phi^\varepsilon = u^\varepsilon - X^\varepsilon \in \mathcal{D}_{X^\varepsilon, T}^L$.

4.4 Renormalization and construction of the rough distribution

To end the proof of existence and uniqueness for the renormalized equation, we need to prove that the O.U. process associated to the white noise can be extend to a rough distribution of \mathcal{X} . (see Definition 4.2.9). As explained above, to define the appropriate process we proceed by regularization and renormalization. Let us take a smooth radial function f with compact support and such that $f(0) = 1$. We regularize X in the following way

$$X_t^\varepsilon = \sum_{k \neq 0} f(\varepsilon k) \hat{X}_t(k) e_k$$

and then we show that we can choose two divergent constants $C_1^\varepsilon, C_2^\varepsilon \in \mathbb{R}^+$ and a smooth function φ^ε such that $R_{C_1^\varepsilon, C_2^\varepsilon}^{\varphi^\varepsilon} \mathbf{X}^\varepsilon := \mathbb{X}^\varepsilon$ converge in \mathcal{X} . As it has been noticed in the previous Sections, without a renormalization procedure there is no finite limit for such a process.

Notation 4.4.1. *Let $k_1, \dots, k_n \in \mathbb{Z}^3$ we denote by $k_{1, \dots, n} = \sum_{i=1}^n k_i$, and for a function f we denote by δf the increment of the function given by $\delta f_{st} = f_t - f_s$*

Definition 4.4.2. *Let*

$$C_1^\varepsilon = \mathbb{E}[(X^\varepsilon)^2]$$

and

$$C_2^\varepsilon = 2 \sum_{k_1 \neq 0, k_2 \neq 0} \frac{|f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{1,2}|^2)}.$$

Notice that thanks to the definition of the Littlewood-Paley blocs, we can also choose to write C_2^ε as

$$C_2^\varepsilon = 2 \sum_{|i-j| \leq 1} \sum_{k_1 \neq 0, k_2 \neq 0} \theta(2^{-i}|k_{1,2}|) \theta(2^{-j}|k_{1,2}|) \frac{f(\varepsilon k_1) f(\varepsilon k_2)}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{1,2}|^2)}.$$

Let us define the following renormalized quantities

$$\begin{aligned} (X^\varepsilon)^{\diamond 2} &:= (X^\varepsilon)^2 - C_1^\varepsilon \\ I((X^\varepsilon)^{\diamond 3}) &:= I((X^\varepsilon)^3 - 3C_1^\varepsilon X^\varepsilon) \\ \pi_{0\diamond}(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) &= \pi_0(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) - C_2^\varepsilon \\ \pi_{0\diamond}(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) &= \pi_0(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) - 3C_2^\varepsilon X^\varepsilon. \end{aligned}$$

Then the following theorem holds.

Theorem 4.4.3. *For $T > 0$, there exists a deterministic sequence $\varphi^\varepsilon : [0, T] \rightarrow \mathbb{R}$, a deterministic distribution $\varphi : [0, T] \rightarrow \mathbb{R}$ such that for all $\delta, \delta', \nu > 0$ small enough we have*

$$\|\varphi\|_{\nu, \rho, T} = \sup_t t^\nu |\varphi_t| + \sup_{t \neq s} s^\nu \frac{|\varphi_t - \varphi_s|}{|t - s|^\rho} < +\infty$$

and the sequence φ^ε converges to φ for that norm, that is

$$\|\varphi^\varepsilon - \varphi\|_{1, \star, T} \rightarrow 0.$$

Furthermore there exists some stochastic processes

$$\begin{aligned} X^{\diamond 2} &\in \mathcal{C}([0, T], \mathcal{C}^{-1-\delta}) \\ I(X^{\diamond 3}) &\in \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{1/2-\delta-2\delta'}) \\ \pi_0(I(X^{\diamond 3}), X) &\in \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-\delta-2\delta'}) \\ \pi_{0\Diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi &\in \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-\delta-2\delta'}) \\ \pi_{0\Diamond}(I(X^{\diamond 3}), X^{\diamond 2}) - 3\varphi X &\in \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-1/2-\delta-2\delta'}). \end{aligned}$$

Moreover each component of the sequence \mathbb{X}^ε converges respectively to the corresponding component of the rough distribution \mathbb{X} in the good topology, that is for all $\delta, \delta' > 0$ small enough, and all $p > 1$,

$$X^\varepsilon \rightarrow X \in L^p(\Omega, \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-1-\delta-3\delta'-3/2p})) \quad (4.17)$$

$$(X^\varepsilon)^{\diamond 2} \rightarrow X^{\diamond 2} \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-1-\delta-3\delta'-3/2p})) \quad (4.18)$$

$$I((X^\varepsilon)^{\diamond 3}) \rightarrow I(X^{\diamond 3}) \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{1/2-\delta-3\delta'-3/2p})) \quad (4.19)$$

$$\pi_0(I((X^\varepsilon)^{\diamond 3}), X^\varepsilon) \rightarrow \pi_0(I(X^{\diamond 3}), X) \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-\delta-3\delta'-3/2p})) \quad (4.20)$$

$$\pi_{0\Diamond}(I((X^\varepsilon)^{\diamond 2}), (X^\varepsilon)^{\diamond 2}) - \varphi^\varepsilon \rightarrow \pi_{0\Diamond}(I(X^{\diamond 2}), X^{\diamond 2}) - \varphi \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-\delta-3\delta'-3/2p})) \quad (4.21)$$

$$\pi_{0\Diamond}(I((X^\varepsilon)^{\diamond 3}), (X^\varepsilon)^{\diamond 2}) - 3\varphi^\varepsilon X \rightarrow \pi_{0\Diamond}(I(X^{\diamond 3}), X^{\diamond 2}) \text{ in } L^p(\Omega, \mathcal{C}^{\delta'}([0, T], \mathcal{C}^{-1/2-\delta-3\delta'-3/2p})) \quad (4.22)$$

Remark 4.4.4. *Thanks to the proof below (especially in Subsection 4.4.5 and 4.4.6) we have the following expressions for φ^ε and φ .*

$$\varphi_t^\varepsilon = - \sum_{|i-j| \leq 1} \sum_{k_1 \neq 0, k_2 \neq 0} \frac{|\theta(2^{-i}|k_{12})| |\theta(2^{-j}|k_{12})| |f(\varepsilon k_1) f(\varepsilon k_2)|}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)} \exp(-t(|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2))$$

and

$$\varphi_t = - \sum_{|i-j| \leq 1} \sum_{k_1 \neq 0, k_2 \neq 0} \frac{|\theta(2^{-i}|k_{12})| |\theta(2^{-j}|k_{12})|}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)} \exp(-t(|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)).$$

We split the proof of this theorem according to the various components. We start by the convergence of X^ε to X . Then we also give a full proof for $X^{\diamond 2}$. For the other components we only prove the crucial estimates.

4.4.1 Convergence for X

We start by an easy computation for the convergence of X

Proof of (4.17). By a quick computation we have that

$$\delta(X - X^\varepsilon)_{st} = \sum_k (f(\varepsilon k) - 1) \delta \hat{X}_{st}(k) e_k$$

and then

$$\mathbb{E} [|\Delta_q \delta(X - X^\varepsilon)_{st}|^2] = 2 \sum_{k \neq 0; |k| \sim 2^q} |f(\varepsilon k) - 1|^2 \frac{1 - e^{-|k|^2|t-s|}}{|k|^2} \lesssim_{h,\rho} c(\varepsilon) 2^{q(1+2h+\rho)} |t-s|^h$$

for $h, \rho > 0$ small enough, and $c(\varepsilon) = \sum_{k \neq 0} |k|^{-3-\rho} |f(\varepsilon k) - 1|^2$. The Gaussian Hypercontractivity (see [71]) gives

$$\mathbb{E} [|\Delta_q \delta(X - X^\varepsilon)_{st}|_{L^p}^p] \lesssim_p \int_{\mathbb{T}^3} \mathbb{E} [|\Delta_q \delta(X - X^\varepsilon)_{st}(x)|^2]^{p/2} dx \lesssim_{\rho,h} c(\varepsilon)^p |t-s|^{hp/2} 2^{qp/2(2h+\rho+1)}.$$

for $p > 1$. We obtain that

$$\mathbb{E} \left[\|\delta(X - X^\varepsilon)_{st}\|_{B_{p,p}^{-1/2-\rho-h}}^p \right] \lesssim c(\varepsilon)^{p/2} |t-s|^{hp/2}$$

Using the Besov embedding (Proposition 4.2.2) we get

$$\mathbb{E} \left[\|\delta(X - X^\varepsilon)_{st}\|_{C^{-1/2-\rho-h-3/p}}^p \right] \lesssim c(\varepsilon)^{p/2} |t-s|^{hp/2}$$

and by the standard Garsia-Rodemich-Rumsey Lemma (see [35]) we finally obtain :

$$\mathbb{E} \left[\|X - X^\varepsilon\|_{C^{h-\theta}([0,T], C^{-1/2-h-\rho-3/p})} \right] \lesssim c(\varepsilon)^p$$

for all $h > \theta > 0$, $\rho > 0$ small enough and $p > 1$. Moreover we have $X_0 = X_0^\varepsilon = 0$ and then using the fact that $c(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ we obtain that

$$\lim_{\varepsilon \rightarrow 0} \|X^\varepsilon - X\|_{L^p(\Omega, C_T^{\delta', -1/2-\delta-3/p})} = 0$$

for all $0 < \delta' < \delta/3$ and $T > 0$. □

4.4.2 Renormalization for X^2

To prove the theorem for $X^{\diamond 2}$ we first prove the following estimate, and we use the Garsia-Rodemich-Rumsey lemma to conclude.

Proposition 4.4.5. *Let $p > 1$, $\theta > 0$ small enough, then the following bound hold*

$$\sup_\varepsilon \mathbb{E} \left[\|\Delta_q \delta(X^\varepsilon)_{st}^{\diamond 2}\|_{L^{2p}}^{2p} \right] \lesssim_{p,\theta} |t-s|^{p\theta} 2^{2qp(1+2\theta)}$$

and

$$\mathbb{E} \left[\|\Delta_q \left(\delta(X^\varepsilon)_{st}^{\diamond 2} - \delta(X^{\varepsilon'})_{st}^{\diamond 2} \right)\|_{L^{2p}}^{2p} \right] \lesssim_{p,\theta} C(\varepsilon, \varepsilon')^p |t-s|^{2p\theta} 2^{2qp(1+\theta)}$$

with $C(\varepsilon, \varepsilon') \rightarrow 0$ when $|\varepsilon - \varepsilon'| \rightarrow 0$.

Proof. By a straightforward computation we have

$$\begin{aligned} \text{Var}(\Delta_q((X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon)) &= \sum_{k, k' \in \mathbb{Z}^3} \theta(2^{-q}k)\theta(2^{-q}k') \sum_{k_{12}=k; k'_{12}=k'} f(\varepsilon k_1)f(\varepsilon k_2)f(\varepsilon k'_1)f(\varepsilon k'_2) \\ &\quad \times (I_{st}^1 + I_{st}^2)e_k e_{-k'} \end{aligned} \quad (4.23)$$

where (e_k) denotes the Fourier basis of $L^2(\mathbb{T}^3)$ and

$$\begin{aligned} I_{st}^1 &= \mathbb{E} \left[(\hat{X}_t(k_1) - \hat{X}_s(k_1)) \overline{(\hat{X}_t(k'_1) - \hat{X}_s(k'_1))} \right] \mathbb{E} \left[\hat{X}_s(k_2) \overline{\hat{X}_s(k'_2)} \right] = 2\delta_{k_1=k'_1} \delta_{k_2=k'_2} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} \\ I_{st}^2 &= \mathbb{E} \left[(\hat{X}_t(k_1) - \hat{X}_s(k_1)) \overline{\hat{X}_s(k'_2)} \right] \mathbb{E} \left[\overline{(\hat{X}_t(k'_1) - \hat{X}_s(k'_1))} \hat{X}_s(k_2) \right] \\ &= \delta_{k_1=k'_2} \delta_{k'_1=k_2} \frac{(1 - e^{-|k_1|^2|t-s|})(1 - e^{-|k_2|^2|t-s|})}{|k_1|^2|k_2|^2}. \end{aligned}$$

Injecting these two identities in the equation (4.23) we obtain that

$$\begin{aligned} \text{Var}(\Delta_q((X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon)) &\lesssim \sum_{\substack{|k| \sim 2^q \\ k_{12}=k}} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} + \sum_{\substack{|k| \sim 2^q \\ k_{12}=k}} \frac{(1 - e^{-|k_1|^2|t-s|})(1 - e^{-|k_2|^2|t-s|})}{|k_1|^2|k_2|^2} \\ &\lesssim \sum_{\substack{|k| \sim 2^q \\ k_{12}=k}} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2}. \end{aligned} \quad (4.24)$$

We have

$$\begin{aligned} \sum_{\substack{|k| \sim 2^q \\ k_{12}=k, |k_1| \leq |k_2|}} \frac{1 - e^{-|k_1|^2|t-s|}}{|k_1|^2|k_2|^2} &\lesssim |t-s|^\theta \sum_{k \in \mathbb{Z}^3; |k| \sim 2^q, k_{12}=k} |k_1|^{-2+2\theta} |k_2|^{-2} \\ &\lesssim |t-s|^\theta \left\{ \sum_{\substack{|k| \sim 2^q, k_{12}=k, \\ |k_1| \leq |k_2|}} |k_1|^{-2+2\theta} |k_2|^{-2} + \sum_{\substack{|k| \sim 2^q, k_{12}=k \\ |k_1| \geq |k_2|}} |k_1|^{-2+2\theta} |k_2|^{-2} \right\} \\ &\lesssim |t-s|^\theta 2^{2q(1+2\theta)} \left(\sum_{k_1} |k_1|^{-3-2\theta} + \sum_{k_1} |k_2|^{-3-4\theta} \right) < +\infty \end{aligned}$$

and then by the Gaussian hypercontractivity we have

$$\mathbb{E} \left[\|\Delta_q \delta(X^\varepsilon)_{st}^{\diamond 2}\|_{L^{2p}}^2 \right] = \int_{\mathbb{T}^3} (\text{Var}(\delta(X^\varepsilon)_{st}^{\diamond 2})) (\xi)^p d\xi \lesssim |t-s|^{p\theta} 2^{2qp(1+2\theta)}.$$

For the second assertion we see that the computation of the beginning gives

$$\text{Var}((\Delta_q((X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon) - (X_t^\varepsilon - X_s^\varepsilon)X_s^\varepsilon)) \lesssim |t-s|^\theta 2^{2q(1+3\theta)} C(\varepsilon, \varepsilon')$$

where

$$C(\varepsilon, \varepsilon') = \sum_{k_{12}=k} (|f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 - |f(\varepsilon' k_1)|^2 |f(\varepsilon' k_2)|^2) |k|^{-3-\theta} |k_1|^{-3-2\theta} \xrightarrow{|\varepsilon-\varepsilon'| \rightarrow 0} 0$$

by the dominated convergence theorem. Once again the Gaussian hypercontractivity gives us the needed bound. \square

Using the Besov embedding 4.2.2 combined with the standard Garsia-Rodemich-Rumsey lemma (see [35]) the following convergence result holds.

Proposition 4.4.6. *Let $\theta, \delta, \rho > 0$ small enough such that $\rho < \theta/2$ and $p > 1$ then the following bound hold*

$$\mathbb{E} \left[\left\| (X^\varepsilon)^{\diamond 2} - (X^{\varepsilon'})^{\diamond 2} \right\|_{\mathcal{C}^{\theta/2-\rho}([0,T], \mathcal{C}^{-1-3/(2p)-\delta-2\theta})}^{2p} \right] \lesssim_{\theta,p,\delta} C(\varepsilon, \varepsilon')^p$$

and due to the fact that $(X_0^\varepsilon)^{\diamond 2} = 0$ and $(X^{\diamond 2})_0 = 0$ we see that the sequence $(X^\varepsilon)^{\diamond 2}$ converges in $L^{2p}(\Omega, \mathcal{C}^{\theta/2-\rho}([0,T], \mathcal{C}^{-1-3/(2p)-\delta-3\theta}))$ to random field noted by $X^{\diamond 2}$.

4.4.3 Renormalization for $I(X^3)$

As the computations are quite similar, we only prove the equivalent of the L^2 estimate in proposition (4.4.5). Furthermore we only prove it for a fixed t and not for an increment.

Proof of (4.19). By a simple computation we have that

$$I((X_t^\varepsilon)^{\diamond 3}) = \sum_{k \in \mathbb{Z}^3} \left(\int_0^t \mathcal{F}((X_s^\varepsilon)^{\diamond 3})(k) e^{-|k|^2|t-s|} ds \right) e_k$$

and then

$$\begin{aligned} \mathbb{E} \left[\left| \Delta_q I((X_t^\varepsilon)^{\diamond 3}) \right|^2 \right] &= 6 \sum_{\substack{k \in \mathbb{Z}^3 \\ k_{123}=k}} |\theta(2^{-q}k)|^2 \prod_{i=1,\dots,3} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^s d\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2(|t-s|+|t-\sigma|)} \\ &= \sum_k |\theta(2^{-q}k)|^2 \Xi^{\varepsilon,1}(k), \end{aligned}$$

where

$$\begin{aligned} \Xi^{\varepsilon,1}(k) &= \sum_{k_{123}=k, k_i \neq 0} \prod_{i=1,\dots,3} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^s d\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2(|t-s|+|t-\sigma|)} \\ &\lesssim \sum_{\substack{k_{123}=k \\ \max_{i=1,\dots,3} |k_i|=|k_1|}} \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2} \int_0^t ds \int_0^s d\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2|t-s|} \\ &\lesssim_T \frac{1}{|k|^{2-\rho}} \sum_{\substack{k_{123}=k \\ \max_{i=1,\dots,3} |k_i|=|k_1|}} \frac{1}{|k_1|^{4-\rho} |k_2|^2 |k_3|^2} \lesssim_T \frac{1}{|k|^{4-4\rho}} \left(\sum_{k_2} |k_2|^{-3-\rho} \right)^2. \end{aligned}$$

We have used that

$$\int_0^t ds \int_0^s d\sigma e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|-|k|^2(|t-s|+|t-\sigma|)} \lesssim_T \frac{1}{|k_1|^{2-\rho}|k|^{2-\rho}} \int_0^t ds \int_0^s d\sigma |t-s|^{-1+\rho/2} |s-\sigma|^{-1+\rho/2}$$

for $\rho > 0$ small enough. Using again the Gaussian hypercontractivity we have

$$\mathbb{E} \left[\left\| \Delta_q I((X_t^\varepsilon)^{\diamond 3}) \right\|_{L^{2p}}^{2p} \right] \lesssim 2^{-2pq(1/2-\rho)}$$

and then the Besov embedding gives

$$\sup_{t \in [0, T], \varepsilon} \mathbb{E} \left[\|I((X_t^\varepsilon)^{\diamond 3})\|_{1/2-\rho-3/p} \right] < +\infty.$$

The same computation gives

$$\sup_{t \in [0, T]} \mathbb{E} \left[\|I((X_t^\varepsilon)^{\diamond 3}) - I((X_t^{\varepsilon'})^{\diamond 3})\|_{1/2-\rho-3/p}^{2p} \right] \xrightarrow{|\varepsilon'-\varepsilon| \rightarrow 0} 0$$

and this gives the needed convergence. \square

4.4.4 Renormalization for $\pi_0(I(X^{\diamond 3}), X)$

Here, we only prove the L^2 estimate for the term $I(X^{\diamond 3})X$ instead of $\pi_0(I(X^{\diamond 3}), X)$ since the computations in the two cases are essentially similar. We remark that in that case we do not need a renormalization.

Proof of (4.20). We have the following representation formula

$$\mathbb{E} \left[\left| \Delta_q \left(I((X^\varepsilon)^{\diamond 3}) X^\varepsilon \right) (t) \right|^2 \right] = \sum_k |\theta(2^{-q}k)|^2 (6I_1^\varepsilon(t)(k) + 18I_2^\varepsilon(t)(k) + 18I_3^\varepsilon(t)(k))$$

with

$$\begin{aligned} I_1^\varepsilon(t)(k) &= 2 \sum_{k_{1234}=k} \prod_{i=1, \dots, 4} \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|} \\ &\lesssim \sum_{\substack{k_{1234}=k \\ \max_{i=1,2,3} |k_i|=|k_1|}} \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|} \\ &= I_{11}^\varepsilon(t)(k) + I_{12}^\varepsilon(t)(k) \end{aligned}$$

and

$$\begin{aligned} I_{11}^\varepsilon(t)(k) &\lesssim \sum_{\substack{k_{1234}=k \\ \max_{i=1,2,3} |k_i|=|k_1| \\ |k_{123}| \leq |k_4|}} \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|} \\ &\lesssim \frac{1}{|k|^2} \sum_{k_2, k_3, k_1, \max |k_i|=|k_1|} \frac{1}{|k_1|^{4-\rho} |k_2|^2 |k_3|^2 |k_{123}|^{2-\rho}} \\ &\lesssim \frac{1}{|k|^2} \sum_{k_1, k_2, k_3} \frac{1}{|k_2|^{3+\rho} |k_3|^{3+\rho} |k_{123}|^{3+\rho}} \lesssim |k|^{-2} \end{aligned}$$

for $\rho > 0$ small enough. Hence we obtain the needed result for I_1^ε . We can treat the second term by a similar computation, indeed

$$\begin{aligned} I_{12}^\varepsilon(t)(k) &\lesssim \sum_{\substack{k_{1234}=k \\ \max_{i=1,2,3} |k_i|=|k_1| \\ |k_{123}| \geq |k_4|}} \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} \int_0^t ds \int_0^s d\sigma e^{-|k_{123}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\sigma|} \\ &\lesssim |k|^{-2+\rho} \sum_{k_2, k_3, k_4} |k_2|^{-3-\rho} |k_3|^{-3-\rho} |k_4|^{-3-\rho} \lesssim |k|^{-2+\rho} \end{aligned}$$

with $\rho > 0$ small enough; this gives the bound for I_1^ε . More precisely we have $I_1^\varepsilon(t)(k) \lesssim |k|^{-2+\rho}$ for $\rho > 0$ small enough. Let us focus on the second term $I_2^\varepsilon(t)(k)$ which is given by

$$\begin{aligned} I_2^\varepsilon(t)(k) &= \sum_{\substack{k_{12}=k \\ k_3, k_4}} \prod_{i=1}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^t d\sigma e^{-(|k_1|^2+|k_2|^2)|s-\sigma|-(|k-k_3|^2+|k_3|^2)|t-s|-(|k_4|^2+|k-k_4|^2)|t-\sigma|} \\ &\lesssim \sum_{\substack{k_{12}=k \\ \max_{i=1,2} |k_i|=|k_1| \\ k_3, k_4}} \frac{1}{|k_1|^{1-\rho} |k_2|^{3+\rho} |k_3|^2 |k_4|^2} \int_0^t ds \int_0^t d\sigma e^{-(|k-k_3|^2+|k_3|^2)|t-s|-(|k_4|^2+|k-k_4|^2)|t-\sigma|} \\ &\lesssim_\rho \frac{1}{|k|^{1-\rho}} \left(\sum_{k_3} \frac{1}{|k_3|^2} \int_0^t ds e^{-(|k_3|^2+|k-k_3|^2)|t-s|} \right)^2 \\ &\lesssim_{\rho, T} \frac{1}{|k|^{1-\rho}} \left(\sum_{k_3 \neq 0, k} \frac{1}{|k_3|^2 |k-k_3|^{2-\rho}} \right)^2 \lesssim_{T, \rho} \frac{1}{|k|^{3-3\rho}} \end{aligned}$$

and we obtain the bound for I_2^ε . We notice that

$$\begin{aligned} I_3^\varepsilon(t)(k) &= \sum_{\substack{k_{12}=k \\ k_3, k_4}} \prod_{i=1}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t ds \int_0^t d\sigma e^{-(|k_1|^2+|k_2|^2)|s-\sigma|-(|k+k_3|^2+|k_3|^2)|t-s|-(|k_4|^2+|k+k_4|^2)|t-\sigma|} \\ &= I_2^\varepsilon(t)(k) \end{aligned}$$

We have

$$\sup_{t \in [0, T], \varepsilon} \mathbb{E} \left[\left| \Delta_q \left(I((X^\varepsilon)^{\diamond 3} X^\varepsilon) \right) (t) \right|^2 \right] \lesssim_{\rho, T} 2^{q(1+\rho)}$$

which is the wanted bound. \square

4.4.5 Renormalization for $\pi_0(I(X^{\diamond 2}), X^{\diamond 2})$

We only prove the crucial estimate for a renormalization of $\pi_0(I((X^\varepsilon)^{\diamond 2}, (X^\varepsilon)^{\diamond 2}))$. We recall that since all the other terms of the product $I(X^\varepsilon)^{\diamond 2} \diamond (X^\varepsilon)^{\diamond 2}$ are well-defined and converge to a limit with a good regularity, only this term need to be checked.

Proof of (4.21). Let us begin by giving the computation for the first term. Indeed a chaos decomposition gives

$$\begin{aligned}
 & -\pi_0(I((X^\varepsilon)^{\diamond 2})(t), (X_t^\varepsilon)^{\diamond 2}) = \\
 & \sum_{k \in \mathbb{Z}^3} \sum_{|i-j| \leq 1} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \\
 & \times \int_0^t ds e^{-|k_{12}|^2|t-s|} : \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) X_t^\varepsilon(k_3) X_t^\varepsilon(k_4) : e_k \\
 & + 4 \sum_{k \in \mathbb{Z}^3} \sum_{|i-j| \leq 1} \sum_{k_{13}=k, k_2} \theta(2^{-i}(|k_{12}|) \theta(2^{-j}|k_{2(-3)}|) |f(\varepsilon k_2)|^2 \int_0^t ds e^{-(|k_{12}|^2+|k_2|^2)|t-s|} |k_2|^{-2} : X_s^\varepsilon(k_1) \hat{X}_t^\varepsilon(k_3) : e_k \\
 & + 2 \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{12}|) |f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 \frac{1 - e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_{12}|^2)}
 \end{aligned}$$

where $::$ denotes the usual Gaussian Wick product. Let us focus on the last term

$$A^\varepsilon(t) = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} |\theta(2^{-i}k_{12})| |\theta(2^{-j}k_{12})| |f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 \frac{1 - e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_{12}|^2)} = C_2^\varepsilon + I_3^\varepsilon(t)$$

where I_3^ε is defined below. Moreover is not difficult to see that

$$\lim_{\varepsilon \rightarrow 0} C_2^\varepsilon = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \frac{\theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{12}|)}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_{12}|^2)} = +\infty.$$

To obtain the needed convergence we have to estimate the following term

$$\begin{aligned}
 I_1^\varepsilon(t) &= \sum_{k \in \mathbb{Z}^3} \sum_{|i-j| \leq 1} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \int_0^t ds e^{-|k_{12}|^2|t-s|} : \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) X_t^\varepsilon(k_3) X_t^\varepsilon(k_4) : e_k \\
 I_2^\varepsilon(t) &= \sum_{k \in \mathbb{Z}^3} \sum_{|i-j| \leq 1} \sum_{k_{13}=k, k_2} \theta(2^{-i}(|k_{12}|) \theta(2^{-j}|k_{2(-3)}|) |f(\varepsilon k_2)|^2 \int_0^t ds e^{-(|k_{12}|^2+|k_2|^2)|t-s|} |k_2|^{-2} X_s^\varepsilon(k_1) \hat{X}_t^\varepsilon(k_3) e_k
 \end{aligned}$$

and

$$I_3^\varepsilon(t) = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{12}|) \frac{|f(\varepsilon k_1)|^2 |f(\varepsilon k_2)|^2 e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_{12}|^2)}.$$

We notice that for the deterministic part we have the following bound

$$I_3^\varepsilon(t) \lesssim t^{-\rho} \sum_{k_1, k_2, |k_1| \leq |k_2|} \frac{1}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_{12}|^2)^{1+\rho}} \lesssim t^{-\rho} \sum_{k_1, k_2, |k_1| \leq |k_2|} |k_2|^{-4-2\rho} |k_1|^{-2} \lesssim_\rho t^{-\rho}$$

and then the dominated convergence gives for $\rho > 0$

$$\sup_{t \in [0, T]} t^\rho |I_3^\varepsilon(t) - I_3(t)| \xrightarrow{\varepsilon \rightarrow 0} 0$$

with

$$I_3(t) = \sum_{|i-j| \leq 1} \sum_{k_1, k_2} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{12}|) \frac{e^{-(|k_1|^2+|k_2|^2+|k_{12}|^2)t}}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |k_{12}|^2)}$$

and this gives the bound for the deterministic part. Let us focus on $I_1^\varepsilon(t)$ and $I_2^\varepsilon(t)$. A simple computation gives

$$\begin{aligned}
 \mathbb{E} [\Delta_q | I_1^\varepsilon(t)^2] &= 2 \sum_{k \in \mathbb{Z}^3} \sum_{i \sim j \sim i' \sim j'} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{12}|) \theta(2^{-j'}|k_{34}|) \theta(2^{-q}|k|)^2 \prod_{l=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
 &\quad \times \int_0^t \int_0^t ds d\sigma e^{-|k_{12}|^2(|t-s|+|t-\sigma|)-(|k_1|^2+|k_2|^2)|s-\sigma|} \\
 &+ 2 \sum_{k \in \mathbb{Z}^3} \sum_{i \sim j \sim i' \sim j'} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{12}|) \theta(2^{-j'}|k_{34}|) \theta(2^{-q}|k|)^2 \prod_{l=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
 &\quad \times \int_0^t \int_0^t ds d\sigma e^{-(|k_{12}|^2+|k_1|^2+|k_2|^2)|t-s|-(|k_{34}|^2+|k_3|^2+|k_4|^2)|t-\sigma|} \\
 &+ 2 \sum_{k \in \mathbb{Z}^3} \sum_{i \sim j; i' \sim j'} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{14}|) \theta(2^{-j'}|k_{23}|) \theta(2^{-q}|k|)^2 \prod_{l=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
 &\quad \times \int_0^t \int_0^t ds d\sigma e^{-(|k_{12}|^2+|k_2|^2)|t-s|-(|k_{14}|^2+|k_4|^2)|t-\sigma|-|k_1|^2|s-\sigma|} \\
 &\equiv \sum_{j=1}^3 I_{1,j}^\varepsilon(t).
 \end{aligned}$$

Let us begin by treating the first term. As usual by symmetry we have

$$\begin{aligned}
 I_{1,1}^\varepsilon(t) &\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i} \sum_{\substack{k_{1234}=k \\ |k_1| \leq |k_2|, |k_3| \leq |k_4| \\ \max_{l=1,\dots,4} |k_l| = |k_2|}} \theta(2^{-q}|k|) \theta(2^{-i}|k_{12}|) \prod_{i=l}^4 |k_l|^{-2} \int_0^t \int_0^t ds d\sigma e^{-|k_{12}|^2(|t-s|+|t-\sigma|)-|k_2|^2|s-\sigma|} \\
 &\quad + \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i} \sum_{\substack{k_{1234}=k \\ |k_1| \leq |k_2|, |k_3| \leq |k_4| \\ \max_{l=1,\dots,4} |k_l| = |k_4|}} \theta(2^{-q}|k|) \theta(2^{-i}|k_{12}|) \prod_{i=l}^4 |k_l|^{-2} \int_0^t \int_0^t ds d\sigma e^{-|k_{12}|^2(|t-s|+|t-\sigma|)-|k_2|^2|s-\sigma|} \\
 &\equiv A_1^\varepsilon(t) + A_2^\varepsilon(t).
 \end{aligned}$$

We notice that if $\max_{l=1,\dots,4} |k_l| = |k_1|$ then $|k| \lesssim |k_1|$, then

$$A_1^\varepsilon(t) \lesssim \sum_{k \in \mathbb{Z}^3} |k|^{-1+2\eta} \theta(2^{-q}|k|) \sum_{\substack{k_{1234}=k \\ |k_1| \leq |k_2|, |k_3| \leq |k_4| \\ \max_{l=1,\dots,4} |k_l| = |k_2|}} |k_1|^{-3-\eta/3} |k_3|^{-3-\eta/3} |k_4|^{-3-\eta/3} \sum_{q \lesssim i} 2^{-i(2-\eta)} \lesssim t^\eta 2^{3q\eta}$$

where we have used

$$\int_0^t \int_0^t ds d\sigma e^{-|k_{12}|^2(|t-s|+|s-\sigma|)-|k_2|^2|s-\sigma|} \lesssim t^\eta \frac{1}{|k_2|^{2-\eta} |k_{12}|^{2-\eta}}.$$

By a similar argument we have

$$A_2^\varepsilon(t) \lesssim \sum_{k \in \mathbb{Z}^3} |k|^{-1+4\eta} \theta(2^{-q}|k|) \sum_{\substack{k_{1234}=k \\ |k_1| \leq |k_2|, |k_3| \leq |k_4| \\ \max_{l=1,\dots,4} |k_l| = |k_4|}} |k_1|^{-3-\eta} |k_2|^{-3-\eta} |k_3|^{-3-\eta} \sum_{q \lesssim i} 2^{-i(2-\eta)} \lesssim t^\eta 2^{5q\eta}$$

and then $\sup_\varepsilon I_{1,1}(t) \lesssim t^\eta 2^{5q\eta}$. Let us treat the second term $I_{1,2}^\varepsilon(t)$. we have

$$\begin{aligned}
 I_{1,2}^\varepsilon(t) &\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i \sim j} \sum_{\substack{k_{1234}=k \\ |k_1| \leq |k_2|, |k_3| \leq |k_4| \\ \max_{l=1, \dots, 4} |k_l| = |k_2|}} \theta(2^{-q}|k|) \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \prod_{i=l}^4 |k_l|^{-2} \\
 &\times \int_0^t \int_0^t ds ds' e^{-(|k_{12}|^2 + |k_1|^2 + |k_2|^2)|t-s| - (|k_{34}|^2 + |k_3|^2 + |k_4|^2)|s-\sigma|} \\
 &\lesssim \sum_{k \in \mathbb{Z}^3} |k|^{-1+4\eta} \sum_{q \lesssim i \sim j} \sum_{\substack{k_{1234}=k \\ |k_1| \leq |k_2|, |k_3| \leq |k_4| \\ \max_{l=1, \dots, 4} |k_l| = |k_2|}} \theta(2^{-q}|k|) \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \frac{1}{|k_1|^2 |k_2|^{3+3\eta} |k_3|^2 |k_4|^2 |k_{34}|^{2-\eta}} \\
 &\lesssim t^\eta 2^{q(2+4\eta)} \sum_{q \lesssim j} 2^{-j(2-\eta)} \sum_l |l|^{-3-\eta} \lesssim t^\eta 2^{5q\eta}.
 \end{aligned}$$

We have to treat the last term in the fourth chaos. A similar computation gives

$$\begin{aligned}
 I_{1,3}^\varepsilon(t) &\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i \sim j; q \lesssim i' \sim j'} \sum_{k_{1234}=k} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{14}|) \theta(2^{-j'}|k_{23}|) \theta(2^{-q}|k|)^2 \prod_{l=1}^4 \frac{|f(\varepsilon k_l)|^2}{|k_l|^2} \\
 &\times \int_0^t \int_0^t ds ds' e^{-(|k_{12}|^2 + |k_2|^2)|t-s| - (|k_{14}|^2 + |k_4|^2)|t-\sigma|} \\
 &\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i' \sim j'} \sum_{\substack{k_{1234}=k \\ |k_4| \leq |k_2|, |k_1| \leq |k_3|}} \theta(2^{-i}|k_{12}|) \theta(2^{-j}|k_{34}|) \theta(2^{-i'}|k_{14}|) \theta(2^{-q}|k|)^2 \prod_{l=1}^4 \frac{1}{|k_l|^2} \\
 &\times \int_0^t \int_0^t ds ds' e^{-|k_2|^2|t-s| - |k_{14}|^2|t-\sigma|} \\
 &\lesssim t^\eta 2^{-q(2-\eta)} \sum_{k \in \mathbb{Z}^3} \theta(2^{-q}|k|) \sum_{\substack{k_{1234}=k \\ |k_4| \leq |k_2|, |k_1| \leq |k_3|}} \frac{1}{|k_1|^2 |k_2|^{4-\eta} |k_3|^2 |k_4|^2}.
 \end{aligned}$$

We still need to bound the sum

$$\sum_{\substack{k_{1234}=k \\ |k_4| \leq |k_2|, |k_1| \leq |k_3|}} \frac{1}{|k_1|^2 |k_2|^{4-\eta} |k_3|^2 |k_4|^2}$$

for that we notice that when $|k_3| \leq |k_2|$ we can use the bound

$$\frac{1}{|k_1|^2 |k_2|^{4-\eta} |k_3|^2 |k_4|^2} \lesssim |k|^{-1+4\eta} |k_1|^{-3-\eta} |k_3|^{-3-\eta} |k_4|^{-3-\eta}$$

and in the case $|k_2| \leq |k_3|$ we can use that

$$\frac{1}{|k_1|^2 |k_2|^{4-\eta} |k_3|^2 |k_4|^2} \lesssim |k|^{-1+4\eta} |k_1|^{-2} |k_2|^{-4+\eta} |k_3|^{-1+4\eta} |k_4|^{-2} \lesssim |k|^{-1+4\eta} |k_1|^{-3-\eta} |k_2|^{-3-\eta} |k_4|^{-3-\eta}$$

where we have used that $|k_4| \leq |k_2|$ and then we can conclude that $\sup_\varepsilon I_{1,3}^\varepsilon(t) \lesssim t^\eta 2^{5q\eta}$. This gives the needed bound for the term lying in the chaos of order four; in fact, we have

$$\sup_\varepsilon \mathbb{E} [\Delta_q |I_1^\varepsilon(t)|^2] \lesssim t^\eta 2^{5q\eta}.$$

Let us focus on the term lying in second chaos.

$$\begin{aligned}
 \mathbb{E} [|\Delta_q I_2^\varepsilon(t)|^2] &= \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i \sim j, q \lesssim i' \sim j'} \sum_{k_{13}=k, k_2, k_4} \theta(2^{-i}|k_{12}|) \theta(2^j|k_{2(-3)}|) \theta(2^{-i'}|k_{14}|) \theta(2^{-j'}|k_{4(-3)}|) \prod_{i=1}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \\
 &\quad \times \int_0^t \int_0^t ds d\sigma e^{-(|k_{12}|^2 + |k_2|^2)|t-s| - (|k_{14}|^2 + |k_4|^2)|t-\sigma| - |k_1|^2|s-\sigma|} \\
 &\quad + \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i \sim j, q \lesssim i' \sim j'} \sum_{k_{13}=k, k_2, k_4} \theta(2^{-i}|k_{12}|) \theta(2^j|k_{2(-3)}|) \theta(2^{-i'}|k_{34}|) \theta(2^{-j'}|k_{4(-3)}|) \prod_{i=1}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \\
 &\quad \times \int_0^t \int_0^t ds d\sigma e^{-(|k_{12}|^2 + |k_2|^2 + |k_1|^2)|t-s| - (|k_{34}|^2 + |k_4|^2 + |k_3|^2)|t-\sigma|} \\
 &\equiv I_{2,1}^\varepsilon(t) + I_{2,2}^\varepsilon(t).
 \end{aligned}$$

We treat these two terms separately. In fact, by symmetry, we have

$$\begin{aligned}
 I_{2,1}^\varepsilon(t) &\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i \sim j, q \lesssim i' \sim j'} \sum_{\substack{k_{13}=k \\ k_2, k_4, |k_1| \leq |k_3|}} \theta(2^{-q}|k|)^2 \theta(2^{-i}|k_{12}|) \theta(2^j|k_{2(-3)}|) \theta(2^{-i'}|k_{14}|) \theta(2^{-j'}|k_{4(-3)}|) \\
 &\quad \times \prod_{i=1}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \int_0^t \int_0^t ds d\sigma e^{-(|k_{12}|^2 + |k_2|^2)|t-s| - (|k_{14}|^2 + |k_4|^2)|t-\sigma|} \\
 &\lesssim t^\eta \sum_{|k| \sim q} |k|^{-1+\eta} \sum_{q \lesssim i, i'} \theta(2^{-i}|k_{12}|) \theta(2^{-i'}|k_{14}|) \sum_{k_1, k_2, k_4} |k_1|^{-3-\eta} |k_2|^{-3-\eta} |k_3|^{-3-\eta} |k_{12}|^{-1+2\eta} |k_{14}|^{-1+2\eta} \\
 &\lesssim 2^{q(2+\eta)} \sum_{q \lesssim i, i'} 2^{-(i+i')(1-2\eta)} \lesssim t^\eta 2^{3q\eta}
 \end{aligned}$$

which gives the first bound. The second term has a similar bound, indeed

$$\begin{aligned}
 I_{2,2}^\varepsilon(t) &\lesssim \sum_{k \in \mathbb{Z}^3} \sum_{q \lesssim i, q \lesssim i'} \sum_{k_{13}=k, k_2, k_4, |k_1| \leq |k_3|} \theta(2^{-i}|k_{12}|) \theta(2^{-i'}|k_{34}|) \prod_{i=1}^4 \frac{|f(\varepsilon k_i)|^2}{|k_i|^2} \\
 &\quad \times \int_0^t \int_0^t ds d\sigma e^{-(|k_{12}|^2 + |k_2|^2)|t-s| - (|k_{34}|^2 + |k_4|^2)|t-\sigma|} \lesssim t^\eta 2^{3q\eta}
 \end{aligned}$$

which ends the proof. \square

4.4.6 Renormalization for $\pi_0(I(X^{\diamond 3}), X^{\diamond 2})$

Here again we only give the crucial bound, but for $I(X^{\diamond 3}) \diamond X^{\diamond 2}$ instead of $\pi_{0\diamond}(I(X^{\diamond 3}), X^{\diamond 2})$.

Proposition 4.4.7. *For all $T > 0$, $t \in [0, T]$, $\delta, \delta' > 0$ and all $\gg \nu > 0$ small enough, there exists two constants and $C > 0$ depending on T , δ, δ' and ν such that for all $q \geq -1$,*

$$\mathbb{E}[t^{\delta'+\delta} |\Delta_q(I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} - 3C_2^\varepsilon X_t^\varepsilon)|^2] \leq Ct^\delta 2^{q(1+\nu)}.$$

Proof. Thanks to a straightforward computation we have

$$-I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} = I_t^{(1)} + I_t^{(2)} + I_t^{(3)}$$

where

$$I_t^{(1)} = \sum_{k \neq 0} e_k \sum_{\substack{k_{12345} = k \\ k_i \neq 0}} \int_0^t ds e^{-|k_1+k_2+k_3|^2|t-s|} : \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) \hat{X}_s^\varepsilon(k_3) \hat{X}_t^\varepsilon(k_4) \hat{X}_t^\varepsilon(k_5) :$$

$$I_t^{(2)} = 6 \sum_{k \neq 0} e_k \sum_{\substack{k_3, k_{124} = k \\ k_i \neq 0}} \int_0^t ds e^{-|k_1+k_2+k_3|^2|t-s|} \frac{e^{-|k_3|^2|t-s|}}{|k_3|^2} f(\varepsilon k_3)^2 : \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) \hat{X}_t^\varepsilon(k_4) :$$

and

$$I_t^{(3)} = 6 \sum_{k \neq 0} e_k \int_0^t ds \sum_{k_1 \neq 0, k_2 \neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|} \hat{X}_s^\varepsilon(k)$$

Hence,

$$\begin{aligned} - \left(I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} - 3C_2^\varepsilon X_t^\varepsilon \right) &= I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} I_t^{(3)} \\ &\quad + (I_t^{(3)} - \tilde{I}_t^{(3)}) + (\tilde{I}_t^{(3)} - 3\tilde{C}_2^\varepsilon(t) X_t^\varepsilon) + 3(C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)) X_t^\varepsilon \end{aligned}$$

where we remind that

$$C_2^\varepsilon = \sum_{k_1 \neq 0, k_2 \neq 0} \frac{f(\varepsilon k_1) f(\varepsilon k_2)}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_1 + k_2|^2)}$$

and where we have defined

$$\tilde{I}_t^{(3)} = 6 \sum_{k \neq 0} e_k \int_0^t ds \sum_{k_1 \neq 0, k_2 \neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|} \hat{X}_t^\varepsilon(k)$$

and

$$\tilde{C}_2^\varepsilon = 2 \int_0^t ds \sum_{k_1 \neq 0, k_2 \neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2} e^{-(|k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|}.$$

Hence for $q \geq -1$,

$$\begin{aligned} \mathbb{E}[|\Delta_q(I((X_t^\varepsilon)^{\diamond 3})(X_t^\varepsilon)^{\diamond 2} - 3C_2^\varepsilon X_t^\varepsilon)|^2] &\lesssim \mathbb{E}[|\Delta_q(I_t^{(1)})|^2] + \mathbb{E}[|\Delta_q(I_t^{(2)})|^2] + \mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \\ &\quad + \mathbb{E}[|\Delta_q(\tilde{I}_t^{(3)} - \tilde{C}_2^\varepsilon(t) X_t^\varepsilon)|^2] + |C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)|^2 \mathbb{E}[|\Delta_q X_t^\varepsilon|^2]. \end{aligned}$$

Terms in the first chaos. Let us first deal with the "deterministic" part, here $C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)$. An obvious computation gives for all $\delta' > 0$, $|C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)|^2 \lesssim_{\delta'} 1/t^{\delta'}$. Furthermore, $\mathbb{E}[|\Delta_q X_t^\varepsilon|^2] \lesssim 2^q$, hence for all $\delta' > 0$,

$$|C_2^\varepsilon - \tilde{C}_2^\varepsilon(t)|^2 \mathbb{E}[|\Delta_q X_t^\varepsilon|^2] \lesssim 2^q / t^{\delta'}$$

Let us deal with $\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2]$. For $k \neq 0$ we define

$$a_k(t-s) = \sum_{k_1 \neq 0, k_2 \neq 0} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|}$$

such that

$$\begin{aligned}
 & \mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \\
 &= \mathbb{E} \left[\left| \int_0^t \sum_k \theta(2^{-q}k) e_k a_k(t-s) (\hat{X}_s^\varepsilon(k) - \hat{X}_t^\varepsilon(k)) \right|^2 \right] \\
 &= \int_{[0,t]^2} d\bar{s} ds \sum_{\substack{k \neq 0 \\ \bar{k} \neq 0}} e_k e_{\bar{k}} \theta(2^{-q}k) \theta(2^{-q}\bar{k}) a_k(t-s) a_{\bar{k}}(t-\bar{s}) \mathbb{E}[(\hat{X}_s^\varepsilon(k) - \hat{X}_t^\varepsilon(k))(\hat{X}_{\bar{s}}^\varepsilon(\bar{k}) - \hat{X}_t^\varepsilon(\bar{k}))]
 \end{aligned}$$

But

$$\begin{aligned}
 \mathbb{E}[(\hat{X}_s^\varepsilon(k) - \hat{X}_t^\varepsilon(k))(\hat{X}_{\bar{s}}^\varepsilon(\bar{k}) - \hat{X}_t^\varepsilon(\bar{k}))] &= \delta_{k=-\bar{k}} \frac{f(\varepsilon k)^2}{|k|^2} (e^{-|s-\bar{s}||k|^2} - e^{-|t-\bar{s}||k|^2} - e^{-|t-s||k|^2} + 1) \\
 &\lesssim \delta_{k=-\bar{k}} \frac{f(\varepsilon k)^2}{|k|^2} |k|^{2\eta} |t-s|^{\eta/2} |t-\bar{s}|^{\eta/2}.
 \end{aligned}$$

Hence

$$\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \lesssim \sum_{k \neq 0} \theta(2^{-q}k)^2 \frac{f(\varepsilon k)^2}{|k|^{2(1-\eta)}} \left(\int_0^t ds |t-s|^{\eta/2} a_k(|t-s|) \right)^2$$

and

$$\begin{aligned}
 \int_0^t ds |t-s|^{\eta/2} a_k(|t-s|) &= \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} \int_0^t ds |t-s|^{\eta/2} e^{-(|k+k_1+k_2|^2+|k_1|^2+|k_2|^2)|t-s|} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2} \\
 &\lesssim \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} |k_1|^{-3-\eta'} |k_2|^{-3-\eta'} \int_0^t ds |t-s|^{-1+(\eta/2-\eta')} \lesssim t^{\eta/2-\eta'}
 \end{aligned}$$

for $\eta/2 - \eta' > 0$. Then we have

$$\mathbb{E}[|\Delta_q(I_t^{(3)} - \tilde{I}_t^{(3)})|^2] \lesssim 2^{q(1+2\eta)} t^{\eta-2\eta'}.$$

We have furthermore

$$\mathbb{E}[|\Delta_q(\tilde{I}_t^{(3)} - C_2^\varepsilon X_t^\varepsilon)|^2] = \sum_{k \neq 0} \frac{f(\varepsilon k)^2}{|k|^2} \theta(2^{-q}k)^2 b_k(t)^2$$

with

$$b_k(t) = \int_0^t \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{|k_1|^2 |k_2|^2} e^{-(|k_1|^2+|k_2|^2)|t-s|} \{e^{-|k_1+k_2|^2|t-s|} - e^{-|k_1+k_2+k|^2|t-s|}\}.$$

Using that

$$|e^{-|k_1+k_2+k|^2|t-s|} - e^{-|k_1+k_2|^2|t-s|}| \lesssim |t-s|^\eta |k|^\eta (|k| + \max\{|k_1|, |k_2|\})^\eta$$

we have the following bound

$$b_k(t) \lesssim \int_0^t \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} |k_1|^{-3-\eta'} |k_2|^{-3-\eta''} |k|^\eta (|k| + \max\{|k_1|, |k_2|\})^\eta |t-s|^{-1+(\eta-\eta'/2-\eta''/2)}.$$

We can suppose that $\max\{|k_1|, |k_2|\} = |k_1|$ as the expression is symmetric in k_1, k_2 , then if $|k| > |k_1|$,

$$b_k(t) \lesssim t^{(\eta-\eta'/2-\eta''/2)} |k|^{2\eta}$$

for $\eta - \eta'/2 - \eta''/2 > 0$. Furthermore if $|k_1| > |k|$, and $\eta' > \eta$ then

$$b_k(t) \lesssim t^{(\eta-\eta'/2-\eta''/2)} |k|^\eta \sum_{\substack{k_1 \neq 0 \\ k_2 \neq 0}} |k_1|^{-3-(\eta'-\eta)} |k_2|^{-3-\eta''} \lesssim t^{(\eta-\eta'/2-\eta''/2)} |k|^\eta.$$

Hence, there exists $\delta > 0$ and $\nu > 0$ such that

$$\mathbb{E}[|\Delta_q(\tilde{I}_t^{(3)} - 3C_2^\varepsilon X_t^\varepsilon)|^2] \lesssim t^\delta 2^{(1+\nu)q}.$$

Terms in the third chaos. Let us define $c_{k_1, k_2}(t-s) = \sum_{k_3 \neq 0} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-(|k_1+k_2+k_3|^2+|k_3|^2)|t-s|}$ such that

$$I_t^{(2)} = 6 \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{124} = k}} e_k \int_0^t ds c_{k_1, k_2}(t-s) : \hat{X}_s(k_1) \hat{X}_s(k_2) \hat{X}_t^\varepsilon(k_4) :$$

But for all suitable variables we have

$$\begin{aligned} & \mathbb{E}[: \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) \hat{X}_t^\varepsilon(k_4) :: \hat{X}_{\bar{s}}^\varepsilon(\bar{k}_1) \hat{X}_{\bar{s}}^\varepsilon(\bar{k}_2) \hat{X}_t^\varepsilon(\bar{k}_4) :] \\ &= 2\delta_{k_1 = -\bar{k}_1} \frac{f(\varepsilon k_1)^2}{|k_1|^2} \delta_{k_2 = -\bar{k}_3} \frac{f(\varepsilon k_2)^2}{|k_2|^2} \delta_{k_3 = -\bar{k}_3} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|} \\ &+ 2\delta_{k_1 = -\bar{k}_1} \frac{f(\varepsilon k_1)^2}{|k_1|^2} \delta_{k_2 = -\bar{k}_3} \frac{f(\varepsilon k_2)^2}{|k_2|^2} \delta_{k_3 = -\bar{k}_2} \frac{f(\varepsilon k_3)^2}{|k_3|^2} e^{-|k_1|^2|s-\bar{s}|} e^{-(|k_3|^2)|t-\bar{s}|} e^{-(|k_2|^2)|t-s|} \\ &\quad \times e^{-|k_1|^2|s-\bar{s}|} e^{-(|k_3|^2)|t-\bar{s}|} \end{aligned}$$

and by an easy computation the following holds $\mathbb{E}[|\Delta_q(I_t^{(2)})|^2] = E_t^{2,1} + E_t^{2,2}$ with

$$E_t^{2,1} = 2 \int_0^t ds \int_0^s d\bar{s} \sum_{\substack{k, k_i \neq 0 \\ k_{124} = k}} \theta(2^{-q}k)^2 \prod_i \frac{f(\varepsilon k_i)^2}{|k_i|^2} c_{k_1, k_2}(t-s) c_{k_1, k_2}(t-\bar{s}) e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|}$$

and

$$\begin{aligned} E_t^{2,1} &= 2 \int_0^t ds \int_0^s d\bar{s} \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124} = k}} \theta(2^{-q}k)^2 \prod_i \frac{f(\varepsilon k_i)^2}{|k_i|^2} \\ &\quad \times c_{k_1, k_2}(t-s) c_{k_1, k_4}(t-\bar{s}) e^{-|k_1|^2|s-\bar{s}|} e^{-|k_4|^2|t-\bar{s}|} e^{-|k_2|^2|t-s|}. \end{aligned}$$

In $E_t^{2,1}$, we have a symmetry in k_1, k_2 , hence we can assume that $|k_1| \geq |k_2|$. Furthermore, we have $c_{k_1, k_2}(t-s) \lesssim |t-s|^{-\frac{1+\eta}{2}}$ and $c_{k_1, k_2}(t-\bar{s}) \lesssim |s-\bar{s}|^{-\frac{1+\eta}{2}}$. If we assume that $|k_1| \geq |k_4|$ and that $\eta'/2 - \eta > 0$, then

$$\begin{aligned} E_t^{2,1} &\lesssim \int_0^t ds \int_0^s d\bar{s} |t-s|^{-\frac{1+\eta}{2}} |s-\bar{s}|^{-1+(\eta'/2-\eta)} \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124} = k}} \theta(2^{-q}k)^2 \frac{1}{|k_1|^{3-\eta'} |k_2|^2 |k_4|^2} \\ &\lesssim t^\delta \sum_{k \neq 0} \frac{\theta(2^{-q}k)^2}{|k|^{1-\eta''}} \sum_{k_2, k_3} |k_2|^{-3-\frac{\eta''-\eta'}{2}} |k_4|^{-3-\frac{\eta''-\eta'}{2}} \lesssim t^\delta 2^{q(2+\eta'')} \end{aligned}$$

for $\eta'' > \eta'$. When $|k_4| \geq |k_1|$ it is almost the same computation.

In $E_t^{2,2}$, we can assume that $|k_2| \geq |k_4|$, so

$$\begin{aligned} E_t^{2,2} &\lesssim \int_0^t ds \int_0^t d\bar{s} \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124} = k \\ |k_2| \lesssim |k_4|}} \theta(2^{-q}k)^2 |k_1|^{-3+\eta'} |k_2|^{-3+\eta'} |k_4|^2 |t-s|^{-1+\frac{\eta'-\eta}{2}} |s-\bar{s}|^{-1+\frac{\eta'-\eta}{2}} \\ &\lesssim t^\delta \sum_{\substack{k \neq 0, k_i \neq 0, \\ k_{124} = k}} \theta(2^{-q}k)^2 |k|^{-1+\eta''} |k_1|^{-3+\eta'} |k_2|^{-3+\eta'} |k_4|^2 \max(|k_i|)^{1-\eta''} \lesssim t^\delta 2^{q(1+\eta'')} \end{aligned}$$

that we decompose as in the previous term whether $|k_1| \geq |k_4|$ or $|k_4| \geq |k_1|$.

Terms in the fifth chaos. For all suitable variables, we have

$$\begin{aligned} &\mathbb{E}[: \hat{X}_s^\varepsilon(k_1) \hat{X}_s^\varepsilon(k_2) \hat{X}_s^\varepsilon(k_3) \hat{X}_t^\varepsilon(k_4) \hat{X}_t^\varepsilon(k_5) :: \hat{X}_{\bar{s}}^\varepsilon(\bar{k}_1) \hat{X}_{\bar{s}}^\varepsilon(\bar{k}_2) \hat{X}_{\bar{s}}^\varepsilon(\bar{k}_3) \hat{X}_t^\varepsilon(\bar{k}_4) \hat{X}_t^\varepsilon(\bar{k}_5) :] \\ &= 12 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_i = -\bar{k}_i} e^{-|s-\bar{s}|(|k_1|^2+|k_2|^2+|k_3|^2)} \\ &\quad + 72 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_1 = -\bar{k}_1} \delta_{k_2 = -\bar{k}_2} \delta_{k_3 = -\bar{k}_4} \delta_{k_4 = -\bar{k}_3} \delta_{k_5 = -\bar{k}_5} e^{-|s-\bar{s}|(|k_1|^2+|k_2|^2)-|t-s||k_3|^2-|t-\bar{s}||k_4|^2} + \\ &\quad + 36 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \delta_{k_1 = -\bar{k}_1} \delta_{k_2 = -\bar{k}_4} \delta_{k_3 = -\bar{k}_5} \delta_{k_4 = -\bar{k}_3} \delta_{k_5 = -\bar{k}_2} e^{-|s-\bar{s}||k_1|^2-|t-s|(|k_2|^2+|k_3|^2)-|t-\bar{s}|(|k_4|^2+|k_5|^2)} \end{aligned}$$

Then

$$\mathbb{E}[|\Delta_q I_t^1|^2] = E_t^{1,1} + E_t^{1,2} + E_t^{1,3}$$

with

$$\begin{aligned} E_t^{1,1} &= 12 \int_{[0,t]^2} ds d\bar{s} \theta(2^{-q}k)^2 \sum_k \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} e^{-|k_{123}|^2|t-s|} e^{-(|k_1|^2+|k_2|^2+|k_3|^2)|s-\bar{s}|} \\ E_t^{1,2} &= 72 \int_{[0,t]^2} ds d\bar{s} \sum_{k_{12345} = k} \theta(2^{-q}k)^2 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \\ &\quad \times e^{-(|k_{123}|^2+|k_3|^2)|t-s|} e^{-(|k_{124}|^2+|k_4|^2)|t-\bar{s}|} e^{-|s-\bar{s}|(|k_1|^2+|k_2|^2)} \end{aligned}$$

and

$$E_t^{1,3} = 36 \int_0^t ds \int_0^t d\bar{s} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \\ \times e^{-(|k_{123}|^2 + |k_2|^2 + |k_3|^2)|t-s|} e^{-(|k_{145}|^2 + |k_5|^2 + |k_4|^2)|t-\bar{s}|} e^{-|s-\bar{s}||k_1|^2}$$

id est

$$E_t^{1,3} = e^{-(|k_{123}|^2 + |k_2|^2 + |k_3|^2)|t-s|} e^{-(|k_{145}|^2 + |k_5|^2 + |k_4|^2)|t-\bar{s}|} e^{-|s-\bar{s}||k_1|^2}$$

Estimation of $E_t^{1,1}$. Let us rewrite it in a form better to handle

$$E_t^{1,1} = 12 \int_{[0,t]^2} ds d\bar{s} \sum_{\substack{k, k \neq 0 \\ k_1 + k_2 + l = k \\ l_1 + l_2 + l_3 = l \\ k_i \neq 0, l_i \neq 0}} \theta(2^{-q}k)^2 \prod_{i=1}^2 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \prod_{i=1}^3 \frac{f(\varepsilon l_i)^2}{|l_i|^2} e^{-|l|^2|t-s|} e^{-(|l_1|^2 + |l_2|^2 + |l_3|^2)|s-\bar{s}|}$$

Thanks to the symmetries of this term, we can always assume that $|k_1| = \max(|k_i|)$ and $l_1 = \max(|l_i|)$.

For $l = 0$, we have

$$\int_{[0,t]^2} ds d\bar{s} \sum_{\substack{k, k \neq 0 \\ k_1 + k_2 = k \\ l_1 + l_2 + l_3 = 0 \\ k_i \neq 0, l_i \neq 0}} \theta(2^{-q}k)^2 \prod_{i=1}^2 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \prod_{i=1}^3 \frac{f(\varepsilon l_i)^2}{|l_i|^2} e^{-(|l_1|^2 + |l_2|^2 + |l_3|^2)|s-\bar{s}|} \\ \lesssim \int_{[0,t]^2} ds d\bar{s} \sum_{k \neq 0} \theta(2^{-q}k)^2 |k|^{-1+\eta} \sum_{k_2 \neq 0} |k|^{-3-\eta} \sum_{l_2 \neq 0, l_3 \neq 0} |l_2|^{-4+\eta} |l_3|^{-4+\eta} |s-\bar{s}|^{-1+\eta} \\ \lesssim 2^{q(2+\eta)} t.$$

Let us assume that $|l| = \max(|l|, |k_1|)$; as we have the following estimate $|l_1|^{-1} \lesssim |l|^{-1}$, the following bound holds

$$\int_{[0,t]^2} ds d\bar{s} \sum_{k \neq 0} \theta(2^{-q}k)^2 |k|^{-1+\eta} \sum_{\substack{k_1 k_2 \neq 0 \\ l_2, l_3 \neq 0}} |k_1|^{-4+9\eta/2} |k_2|^{-4+9\eta/2} (|t-s||s-\bar{s}|)^{-1+\eta} |l_2|^{-3-\eta} |l_3|^{-3-\eta} \\ \lesssim t^\eta 2^{q(2+\eta)}$$

The case in which $|k_1| = \max(|l|, |k_1|)$ is quite similar, and the conclusion holds for $E_t^{1,1}$.

Estimation of $E_t^{1,2}$. This term is symmetric in k_1, k_2 and in k_3, k_4 . Hence, we can assume that $|k_1| \geq |k_2|$ and $|k_3| \geq |k_4|$. First let us assume that $|k_5| = \max\{|k_i|\}$. Then

$$\begin{aligned}
 E_t^{1,2} &\lesssim \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2 \int_0^t ds \int_0^s d\bar{s} (|t-s||s-\bar{s}|)^{-1+\eta} \\
 &\quad \times |k_1|^{-4+2\eta} |k_2|^{-2} |k_3|^{-4+2\eta} |k_4|^{-2} |k_5|^{-(1+\eta')} |k|^{-(1-\eta')} \\
 &\lesssim t^\eta \sum_k \theta(2^{-q}k)^2 |k|^{-(1-\eta')} \sum_{k_{12345} = k} |k_1|^{-7/2+2\eta} |k_2|^{-3-\eta'/2} |k_3|^{-7/2+2\eta} |k_4|^{-3-\eta'/2} \\
 &\lesssim t^\eta 2^{(2+\eta')q}
 \end{aligned}$$

for η small enough.

Then assume that $\max\{|k_i|\} = |k_1|$

$$\begin{aligned}
 E_t^{1,2} &\lesssim t^\delta \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k_1|^{-4+2\eta} |k_2|^{-2} |k_3|^{-3+\eta'} |k_4|^{-3+\eta'} |k_5|^{-2} \int_0^t ds \int_0^s d\bar{s} |t-s|^{-1+\eta'} |s-\bar{s}|^{-1+\eta} \\
 &\lesssim t^{\eta'} \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k|^{-1+\eta''} |k_2|^{-3-\eta''} |k_3|^{-7/2+(2\eta+\eta''+\eta')/2} |k_4|^{-7/2+(2\eta+\eta''+\eta')/2} |k_5|^{-3-\eta''} \\
 &\lesssim t^\delta 2^{(2+\eta')q}
 \end{aligned}$$

For $\max\{|k_i|\} = |k_3|$

$$\begin{aligned}
 &\lesssim t^\delta \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k_1|^{-4+\eta} |k_2|^{-2} |k_3|^{-4+\eta'} |k_4|^{-2} |k_5|^{-2} \\
 &\lesssim t^\delta \sum_{\substack{k \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k_1|^{-3+\eta+1/4} |k_2|^{-3+1/4} |k_3|^{-1+\eta'} |k_4|^{-3+1/4} |k_5|^{-3+1/4} \lesssim t^\delta 2^{(2+\eta')q}
 \end{aligned}$$

hence there exists $\delta > 0$ and $\nu > 0$ such that

$$E_t^{1,2} \lesssim t^\delta 2^{(2+\nu)q}.$$

Estimation of $E_t^{1,3}$. Let us deal with this last term. Here the symmetries are in k_2, k_3 and k_4, k_5 . Then we can suppose that $|k_2| \geq |k_3|$ and $|k_4| \geq |k_5|$. Furthermore, the role of k_2, k_3 and k_4, k_5 are symmetrical, then we can assume that $|k_1| \geq |k_4|$

$$\begin{aligned}
 E_t^{1,3} &= \int_{[0,t]^2} ds d\bar{s} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 \prod_{i=1}^5 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \\
 &\quad \times e^{-(|k_{123}|^2 + |k_2|^2 + |k_3|^2)|t-s|} e^{-(|k_{145}|^2 + |k_5|^2 + |k_4|^2)|t-\bar{s}|} e^{-|s-\bar{s}||k_1|^2}
 \end{aligned}$$

If $|k_1| = \max(|k_i|)$ then

$$\begin{aligned} &\lesssim \int_{[0,t]^2} ds d\bar{s} (|t-s||t-\bar{s}|)^{-1+\eta} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k|^{-1+\eta} (|k_2||k_3||k_3||k_4|)^{-7/4+3\eta/4} \\ &\lesssim 2^{q(2+\eta)} t^\eta \end{aligned}$$

If $|k_2| = \max(|k_i|)$ then

$$\begin{aligned} &\lesssim 2 \int_0^t \int_0^s ds d\bar{s} (|t-s||s-\bar{s}|)^{-1+\eta} \sum_{\substack{k \neq 0, k_i \neq 0 \\ k_{12345} = k}} \theta(2^{-q}k)^2 |k|^{-1+\eta} (|k_1||k_3||k_3||k_4|)^{-7/4+3\eta/4} \\ &\lesssim t^\eta 2^{q(2+\eta)}. \end{aligned}$$

□

4.5 Appendix

4.5.1 A commutation lemma

We give the proof of the Lemma 4.2.5. This proof is from Gubinelli, Imkeller and Perkowski, and can be found in the first version of [38] and also in [67] Lemmas 5.3.20 and 5.5.7. In fact we give a stronger result, and apply it with $\varphi(k) = \exp(-|k|^2/2)$.

Lemma 4.5.1. *Let $\alpha < 1$ and $\beta \in \mathbb{R}$. Let $\varphi \in \mathcal{S}$, let $u \in C^\alpha$, and $v \in C^\beta$. Then for every $\varepsilon > 0$ and every $\delta \geq -1$ we have*

$$\|\varphi(\varepsilon \mathcal{D})\pi_{<}(u, v) - \pi_{<}(u, \varphi(\varepsilon \mathcal{D})v)\|_{\alpha+\beta+\delta} \lesssim \varepsilon^{-\delta} \|u\|_\alpha \|v\|_\beta,$$

where

$$\varphi(\mathcal{D})u = \mathcal{F}^{-1}(\varphi \mathcal{F}u).$$

Proof. We define for $j \geq -1$,

$$S_{j-1}u = \sum_{i=-1}^{j-2} \Delta_i u$$

$$\varphi(\varepsilon \mathcal{D})\pi_{<}(u, v) - \pi_{<}(u, \varphi(\varepsilon \mathcal{D})v) = \sum_{j \geq -1} (\varphi(\varepsilon \mathcal{D})(S_{j-1}u \Delta_j v) - S_{j-1}u \Delta_j \varphi(\varepsilon \mathcal{D})v),$$

and every term of this series has a Fourier transform with support in an annulus of the form $2^j \mathcal{A}$. Lemma 2.69 in [6] implies that it suffices to control the L^∞ norm of each term. Let $\psi \in \mathcal{D}$ with support in an annulus be such that $\psi \equiv 1$ on \mathcal{A} . We have

$$\begin{aligned} &\varphi(\varepsilon dD)(S_{j-1}u \Delta_j v) - S_{j-1}u \Delta_j \varphi(\varepsilon \mathcal{D})v \\ &= (\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(dD)(S_{j-1}u \Delta_j v) - S_{j-1}u (\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(dD)\Delta_j v \\ &= [(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(dD), S_{j-1}u] \Delta_j v, \end{aligned}$$

where $[(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]$ denotes the commutator. In the proof of Lemma 2.97 in [6], it is shown that writing the Fourier multiplier as a convolution operator and applying a first order Taylor expansion and then Young's inequality yields

$$\begin{aligned} & \|[(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \\ & \lesssim \sum_{\eta \in \mathbb{N}^d: |\eta|=1} \|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))\|_{L^1} \|\partial^\eta S_{j-1}u\|_{L^\infty} \|\Delta_j v\|_{L^\infty}. \end{aligned} \quad (4.25)$$

Now $\mathcal{F}^{-1}(f(2^{-j}\cdot)g(\varepsilon\cdot)) = 2^{jd} \mathcal{F}^{-1}(fg(\varepsilon 2^j\cdot))(2^j\cdot)$ for every f, g , and thus we have for every multi-index η of order one

$$\begin{aligned} & \|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))\|_{L^1} \\ & \leq 2^{-j} \|\mathcal{F}^{-1}((\partial^\eta \psi)(2^{-j}\cdot)\varphi(\varepsilon\cdot))\|_{L^1} + \varepsilon \|\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\partial^\eta \varphi(\varepsilon\cdot))\|_{L^1} \\ & = 2^{-j} \|\mathcal{F}^{-1}((\partial^\eta \psi)\varphi(\varepsilon 2^j\cdot))\|_{L^1} + \varepsilon \|\mathcal{F}^{-1}(\psi \partial^\eta \varphi(\varepsilon 2^j\cdot))\|_{L^1} \\ & \lesssim 2^{-j} \|(1 + |\cdot|)^{2d} \mathcal{F}^{-1}((\partial^\eta \psi)\varphi(\varepsilon 2^j\cdot))\|_{L^\infty} + \varepsilon \|(1 + |\cdot|)^{2d} \mathcal{F}^{-1}(\psi \partial^\eta \varphi(\varepsilon 2^j\cdot))\|_{L^\infty} \\ & = 2^{-j} \|\mathcal{F}^{-1}((1 - \Delta)^d((\partial^\eta \psi)\varphi(\varepsilon 2^j\cdot)))\|_{L^\infty} + \varepsilon \|\mathcal{F}^{-1}((1 - \Delta)^d(\psi \partial^\eta \varphi(\varepsilon 2^j\cdot)))\|_{L^\infty} \\ & \lesssim 2^{-j} \|(1 - \Delta)^d((\partial^\eta \psi)\varphi(\varepsilon 2^j\cdot))\|_{L^\infty} + \varepsilon \|(1 - \Delta)^d(\psi \partial^\eta \varphi(\varepsilon 2^j\cdot))\|_{L^\infty}, \end{aligned} \quad (4.26)$$

where the last step follows because ψ has compact support. For j satisfying $\varepsilon 2^j \geq 1$ we obtain

$$\|x^\eta \mathcal{F}^{-1}(\varphi(\varepsilon\cdot)\psi(2^{-j}\cdot))\|_{L^1} \lesssim (\varepsilon + 2^{-j})(\varepsilon 2^j)^{2d} \sum_{\eta: |\eta| \leq 2d+1} \|\partial^\eta \varphi(\varepsilon 2^j\cdot)\|_{L^\infty(\text{supp}(\psi))}, \quad (4.27)$$

where we used that ψ and all its partial derivatives are bounded, and where $L^\infty(\text{supp}(\psi))$ means that the supremum is taken over the values of $\partial^\eta \varphi(\varepsilon 2^j\cdot)$ restricted to $\text{supp}(\psi)$. Now φ is a Schwartz function, and therefore it decays faster than any polynomial. Hence, there exists a ball \mathcal{B}_δ such that for all $x \notin \mathcal{B}_\delta$ and all $|\eta| \leq 2d + 1$ we have

$$|\partial^\eta \varphi(x)| \leq |x|^{-2d-1-\delta}. \quad (4.28)$$

Let $j_0 \in \mathbb{N}$ be minimal such that $2^{j_0} \varepsilon \mathcal{A} \cap \mathcal{B}_\delta = \emptyset$ and $\varepsilon 2^{j_0} \geq 1$. Then the combination of (4.25), (4.27), and (4.28) shows for all $j \geq j_0$ that

$$\begin{aligned} & \|[(\psi(2^{-j}\cdot)\varphi(\varepsilon\cdot))(\mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \\ & \lesssim (\varepsilon + 2^{-j})(\varepsilon 2^j)^{2d} \sum_{\eta: |\eta| \leq 2d+1} \|(\partial^\eta \varphi)(\varepsilon 2^j\cdot)\|_{L^\infty(\text{supp}(\psi))} 2^{j(1-\alpha)} \|u\|_\alpha 2^{-j\beta} \|v\|_\beta \\ & \lesssim (\varepsilon + 2^{-j})(\varepsilon 2^j)^{2d} (\varepsilon 2^j)^{-2d-1-\delta} 2^{j(1-\alpha-\beta)} \|u\|_\alpha \|v\|_\beta \\ & \lesssim (1 + (\varepsilon 2^j)^{-1}) \varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)} \|u\|_\alpha \|v\|_\beta. \end{aligned}$$

Here we used that $\alpha < 1$ in order to obtain $\|\partial^\eta S_{j-1}u\|_{L^\infty} \lesssim 2^{j(1-\alpha)} \|u\|_{L^\infty}$. Since $\varepsilon 2^j \geq 1$, we have shown the desired estimate for $j \geq j_0$. On the other side Lemma 2.97 in [6] implies for every $j \geq -1$ that

$$\|[\varphi(\varepsilon \mathcal{D}), S_{j-1}u]\Delta_j v\|_{L^\infty} \lesssim \varepsilon \max_{\eta \in \mathbb{N}^d: |\eta|=1} \|\partial^\eta S_{j-1}u\|_{L^\infty} \|\Delta_j v\|_{L^\infty} \lesssim \varepsilon 2^{j(1-\alpha-\beta)} \|u\|_\alpha \|v\|_\beta.$$

Hence, we obtain for $j < j_0$, i.e. for j satisfying $2^j \varepsilon \lesssim 1$, that

$$\|[\varphi(\varepsilon \mathcal{D}), S_{j-1} u] \Delta_j v\|_{L^\infty} \lesssim (\varepsilon 2^j)^{1+\delta} \varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)} \|u\|_\alpha \|v\|_\beta \lesssim \varepsilon^{-\delta} 2^{-j(\alpha+\beta+\delta)} \|u\|_\alpha \|v\|_\beta,$$

where we used that $\delta \geq -1$. This completes the proof. \square

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