

Time-domain numerical modeling of poroelastic waves: the Biot-JKD model with fractional derivatives

Soutenance de thèse
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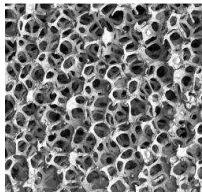
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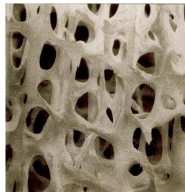
- Porous media

- Solid matrix
- Pores saturated by a fluid
- Wavelengths $\lambda \gg$ radius of the pores r
- Porous \equiv continuous medium

ceramic



cancellous bone



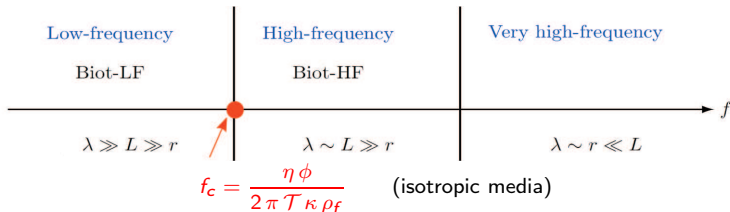
- Biot (1956)

- 2 compressional waves and 1 shear wave
- **Slow compressional wave** very sensitive to the **dissipation** into the pores

- Validation

- Experimental observation of the slow compressional wave: Plona (1980)
- Homogenization method: Burridge and Keller (1981), Auriault, Boutin and Geindreau (2009)

Introduction: JKD



Biot: 2 frequency regimes

- Low-frequency regime (LF) $f < f_c$: Poiseuille flow
- High-frequency regime (HF) $f > f_c$: Prandtl boundary layer

Specificities in the HF regime

- Viscous dissipation depending on the geometry of the pores
 - parallel plane walls, circular duct: Biot, 1956
 - random pores: Johnson, Koplik and Dashen (JKD), 1987
- Frequency-domain: square root of the frequency
- Time-domain: **fractional derivatives** \equiv convolution product, non local-in-time

Methods of resolution in the time-domain

	analytical: Green's functions	numerical
LF	<ul style="list-style-type: none">● Szabo (1994)● Norton and Novarini (2003)● Chen and Holm (2003)	<ul style="list-style-type: none">● finite differences● boundary elements● pseudo-spectral methods● finite elements● spectral-finite elements● discontinuous Galerkin
HF	<ul style="list-style-type: none">● Fella and Depollier (2000)	<ul style="list-style-type: none">● direct approach● diffusive approximation

Numerical methods in the HF regime

Direct approach

- Discretization of the **convolution product**
- **Masson-Pride**, *Geophysics*, 75-5 (2010)
 - ⊖ Storage of a large number of time steps (~ 20 time steps)
 - ⊖ No stability analysis in the general case

Diffusive approximation (DA)

- Based on a **diffusive representation**
- Continuum of diffusive variable: Matignon (1998)
- Gaussian quadratures **Lu-Hanyga**, *J. Comput. Phys.*, 208-2 (2005)
 - ⊖ Very slow convergence (~ 25 diffusive variables)
 - ⊖ $f \ll f_c$: No degenerescence towards the Biot-LF model

- Thesis

- Before: time-domain code for the 2D Biot equations in the LF regime
Chiavassa and Lombard (2011)
- Aim: **time-domain** numerical method in the **HF regime**

- Requirements

- Small number of **additional arrays** (determination *a priori*)
- Degenerescence towards the Biot-LF model at low frequencies
- **Stability** of the numerical scheme ensured in the general case

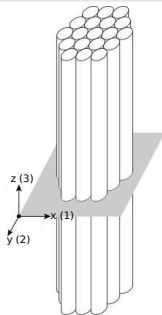
- Strategy: modify the **diffusive approximation**

- **Problem statement**
 - Biot-JKD model
 - Diffusive representation
 - Properties
- **Diffusive approximation**
 - Biot-DA model
 - Properties
- **Numerical modeling**
 - Splitting
 - ADER 4 scheme
 - Immersed interface method
- **Numerical experiments**
 - Validation
 - Complex media
- **Conclusion and perspectives**

Problem statement

Hypotheses

- Elastic matrix
- Continuous single fluid phase
- Wavelengths $\lambda \gg$ radius of the pores r
- Small perturbations (linear)
- Thermo-mechanical effects neglected



2D transversely isotropic media: 17 positive physical parameters

- Fluid: ρ_f, η, K_f
- Solid: ρ_s, K_s

- Porous: $\phi, \mathcal{T}_1, \mathcal{T}_3, \kappa_1, \kappa_3, \Lambda_1, \Lambda_3, \mathbf{C}^u = \begin{pmatrix} c_{11}^u & c_{13}^u & 0 \\ c_{13}^u & c_{33}^u & 0 \\ 0 & 0 & c_{55}^u \end{pmatrix}, c_{12}^u$

8 variables

$$\xi = -\nabla \cdot \phi (\mathbf{u}_f - \mathbf{u}_s), \quad \underline{\underline{\epsilon}} = \frac{1}{2} (\nabla \mathbf{u}_s + \nabla \mathbf{u}_s^T), \quad \underline{\underline{\sigma}}, \quad p$$

2D Biot-JKD model

Constitutive laws and conservation of momentum:

$$\left\{ \begin{array}{l} \boldsymbol{\sigma} = \mathbf{C}^u \boldsymbol{\varepsilon} - m \boldsymbol{\beta} \boldsymbol{\xi} \\ p = -m (\boldsymbol{\beta}^T \boldsymbol{\varepsilon} - \boldsymbol{\xi}) \\ \rho \frac{\partial \mathbf{v}_s}{\partial t} + \rho_f \frac{\partial \mathbf{w}}{\partial t} = \nabla \cdot \boldsymbol{\sigma} \\ \rho_f \frac{\partial \mathbf{v}_s}{\partial t} + \begin{pmatrix} \rho_{w1} & 0 \\ 0 & \rho_{w3} \end{pmatrix} \frac{\partial \mathbf{w}}{\partial t} + \begin{pmatrix} \frac{\eta}{\kappa_1} F_1(t) & 0 \\ 0 & \frac{\eta}{\kappa_3} F_3(t) \end{pmatrix} * \mathbf{w} = -\nabla p \end{array} \right.$$

Low-frequency (Biot-LF)

$$\widehat{F}_i^{LF}(\omega) = 1$$

$$F_i^{LF}(t) * w_i(t) = w_i$$

High-frequency (Biot-JKD)

$$\widehat{F}_i^{JKD}(\omega) = \frac{1}{\sqrt{\Omega_i}} (j\omega + \Omega_i)^{1/2}, \quad \Omega_i = \frac{\Lambda_i^2 \phi^2 \eta}{4 \mathcal{T}_i^2 \kappa_i^2 \rho_f}$$

$$\begin{aligned} F_i^{JKD}(t) * w_i(t) &= \frac{1}{\sqrt{\Omega_i}} (D + \Omega_i)^{1/2} w_i \\ &= \frac{e^{-\Omega_i t}}{\sqrt{\pi \Omega_i t}} * \left(\frac{\partial w_i}{\partial t} + \Omega_i w_i \right) \end{aligned}$$

$(D + \Omega_i)^{1/2}$: shifted fractional derivatives of order 1/2

Diffusive representation

$$\text{Diffusive representation: } \frac{1}{\sqrt{t}} = \int_0^\infty \frac{1}{\sqrt{\pi\theta}} e^{-\theta t} d\theta$$

$$\begin{aligned}(D + \Omega)^{1/2} w &= \frac{e^{-\Omega t}}{\sqrt{\pi t}} * \left(\frac{\partial w}{\partial t} + \Omega w \right) \quad \text{non local-in-time} \\ &= \int_0^\infty \frac{1}{\pi \sqrt{\theta}} \int_0^t e^{-(\theta+\Omega)(t-\tau)} \left(\frac{\partial w}{\partial t} + \Omega w \right) d\tau d\theta \\ &= \int_0^\infty \frac{1}{\pi \sqrt{\theta}} \psi(\theta, t) d\theta \quad \text{diffusive representation}\end{aligned}$$

Ordinary differential equation **local-in-time** satisfied by ψ :

$$\begin{cases} \frac{\partial \psi}{\partial t} = -(\theta + \Omega) \psi + \frac{\partial w}{\partial t} + \Omega w \\ \psi(\theta, 0) = 0 \end{cases}$$

Properties of the Biot-JKD model

Property (Decrease of the energy)

Let $E = E_1 + E_2 + E_3 \geq 0$

$$\begin{cases} E_1 = \frac{1}{2} \int_{\mathbb{R}^2} \left(\rho \mathbf{v}_s^T \mathbf{v}_s + 2 \rho_f \mathbf{v}_s^T \mathbf{w} + \rho_{w1} w_x^2 + \rho_{w3} w_z^2 \right) dx dz, \\ E_2 = \frac{1}{2} \int_{\mathbb{R}^2} \left((\boldsymbol{\sigma} + p \boldsymbol{\beta})^T \mathbf{C}^{-1} (\boldsymbol{\sigma} + p \boldsymbol{\beta}) + \frac{1}{m} p^2 \right) dx dz, \\ E_3 = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\eta}{\pi} \int_0^\infty \sum_{i=1,3} \frac{(w_i - \psi_i)^2}{\kappa_i \sqrt{\Omega_i \theta} (\theta + 2 \Omega_i)} d\theta dx dz \end{cases}$$

$$\frac{dE}{dt} = - \int_{\mathbb{R}^2} \frac{\eta}{\pi} \int_0^\infty \sum_{i=1,3} \frac{1}{\kappa_i \sqrt{\Omega_i \theta} (\theta + 2 \Omega_i)} ((\theta + \Omega_i) \psi_i^2 + \Omega_i w_i^2) d\theta dx dz \leq 0$$

The energy E of the Biot-JKD system decreases: the model is **stable**

Property (Hyperbolicity)

The Biot-JKD system is **hyperbolic**

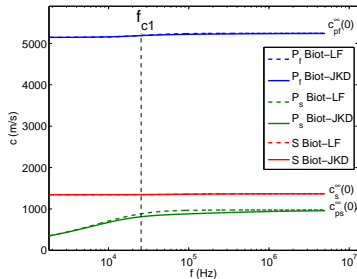
Properties of the Biot-JKD model: dispersion analysis

$$\begin{aligned} \mathbf{V} &= (v_x, v_z, w_x, w_z)^T = \mathbf{V}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}, \\ \mathbf{T} &= (\sigma_{xx}, \sigma_{xz}, \sigma_{zz}, -p)^T = \mathbf{T}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}, \end{aligned}$$

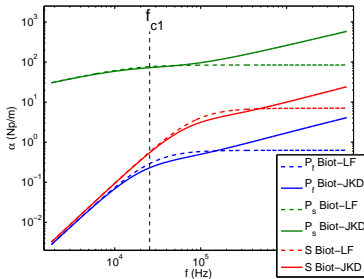
Dispersion relation: $\det \left(\mathbf{\Gamma}^{-1} \mathbf{L} \mathbf{C} - \left(\frac{\omega}{k} \right)^2 \mathbf{I}_4 \right) = 0$, $\mathbf{\Gamma}$: Christoffel tensor

- isotropic media: 2 purely compressional and 1 purely shear waves
- transversely isotropic media: 2 quasi-compressional and 1 quasi-shear waves

phase velocities $\omega / \Re(k)$



attenuations $-\Im(k)$



transversely isotropic medium glass-epoxy

Diffusive approximation of the Biot-JKD model

Biot-DA model

Quadrature formula on N points

$$(D + \Omega)^{1/2} w = \int_0^\infty \frac{1}{\pi \sqrt{\theta}} \psi(\theta, t) d\theta \simeq \sum_{\ell=1}^N a_\ell \psi(\theta_\ell, t) = \sum_{\ell=1}^N a_\ell \psi_\ell(t)$$

Biot-DA system

$$\left\{ \begin{array}{l} \sigma = \mathbf{C}^u \varepsilon - m \beta \xi \\ p = -m (\beta^T \varepsilon - \xi) \\ \rho \frac{\partial \mathbf{v}_s}{\partial t} + \rho_f \frac{\partial \mathbf{w}}{\partial t} = \nabla \cdot \underline{\sigma} \\ \rho_f \frac{\partial \mathbf{v}_s}{\partial t} + \begin{pmatrix} \rho_{w1} & 0 \\ 0 & \rho_{w3} \end{pmatrix} \frac{\partial \mathbf{w}}{\partial t} + \sum_{\ell=1}^N \begin{pmatrix} \frac{\eta}{\kappa_1} \frac{1}{\sqrt{\Omega_1}} a_\ell^x & 0 \\ 0 & \frac{\eta}{\kappa_3} \frac{1}{\sqrt{\Omega_3}} a_\ell^z \end{pmatrix} \psi_\ell = -\nabla p \\ \frac{\partial \psi_\ell}{\partial t} = - \begin{pmatrix} \theta_\ell^x + \Omega_1 & 0 \\ 0 & \theta_\ell^z + \Omega_3 \end{pmatrix} \psi_\ell + \frac{\partial \mathbf{w}}{\partial t} + \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_3 \end{pmatrix} \mathbf{w}, \quad \ell = 1, \dots, N \end{array} \right.$$

Velocity-stress formulation. First-order system of $8 + 2N$ variables

$$\mathbf{U} = (v_{s,x}, v_{s,z}, w_x, w_z, \sigma_{xx}, \sigma_{xz}, \sigma_{zz}, p, \psi_1^x, \psi_1^z, \dots, \psi_N^x, \psi_N^z)^T$$
$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial z} = -\mathbf{S} \mathbf{U}$$

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$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial z} = -\mathbf{S} \mathbf{U}$$

Properties of the Biot-DA model

Property (Hyperbolicity)

*Unconditionally: the Biot-DA system is **hyperbolic***

Property (Dispersion analysis)

*Unconditionally: 2 quasi-compressional waves and 1 quasi-shear wave
High-frequency limits of the phase velocities unchanged, no spurious wave*

Property (Decrease of the energy)

*If positive coefficients: **decreasing energy** \Rightarrow model **stable***

Property (Eigenvalues of \mathbf{S})

*If positive coefficients: **non-zero eigenvalues** $s_\ell^{x,z}$ of \mathbf{S} satisfy*

$$0 < s_1^x < \theta_1^x + \Omega_1 < \cdots < s_N^x < \theta_N^x + \Omega_1 < s_{N+1}^x,$$

$$0 < s_1^z < \theta_1^z + \Omega_1 < \cdots < s_N^z < \theta_N^z + \Omega_3 < s_{N+1}^z$$

Consequences:

- $s_\ell^{x,z} \geq 0 \Rightarrow$ *stability of the diffusive part*
- $R(\mathbf{S}) > \max(\theta_N^x + \Omega_1, \theta_N^z + \Omega_3)$ *stability of the numerical scheme*

Properties of the Biot-DA model

Property (Hyperbolicity)

Unconditionally: the Biot-DA system is hyperbolic

Property (Dispersion analysis)

*Unconditionally: 2 quasi-compressional waves and 1 quasi-shear wave
High-frequency limits of the phase velocities unchanged, no spurious wave*

Property (Decrease of the energy)

If positive coefficients: decreasing energy \Rightarrow model stable

Property (Eigenvalues of \mathbf{S})

If positive coefficients: non-zero eigenvalues $s_\ell^{x,z}$ of \mathbf{S} satisfy

$$0 < s_1^x < \theta_1^x + \Omega_1 < \dots < s_N^x < \theta_N^x + \Omega_1 < s_{N+1}^x,$$

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Consequences:

- $s_\ell^{x,z} \geq 0 \Rightarrow$ stability of the diffusive part
- $R(\mathbf{S}) > \max(\theta_N^x + \Omega_1, \theta_N^z + \Omega_3)$ stability of the numerical scheme

Determination of the quadrature coefficients: error of model

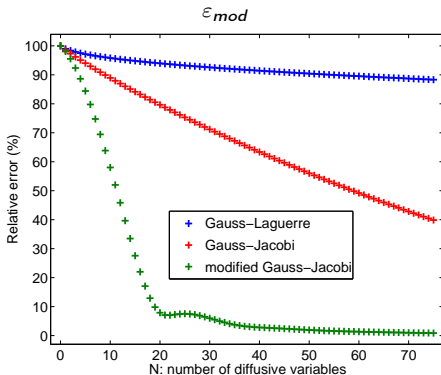
- Key feature: compute the abscissae θ_ℓ and the weights a_ℓ ($\ell = 1, \dots, N$)
- Viscous operators

$$\left\{ \begin{array}{l} \widehat{F}^{JKD}(\omega) = \frac{1}{\sqrt{\Omega}}(j\omega + \Omega)^{1/2} \quad \text{Biot-JKD} \\ \widehat{F}^{DA}(\omega) = \frac{j\omega + \Omega}{\sqrt{\Omega}} \sum_{\ell=1}^N \frac{a_\ell}{\theta_\ell + j\omega + \Omega} \quad \text{Biot-DA} \end{array} \right.$$

- Criterion: error of model $\varepsilon_{mod} = \left\| \frac{\widehat{F}^{DA}(\omega)}{\widehat{F}^{JKD}(\omega)} - 1 \right\|_2$
- **Minimization** of a positive cost function

Determination of the coefficients: Gaussian quadratures

- Yuan and Agrawal (2002): Gauss-Laguerre quadrature
- Diethelm (2008): Gauss-Jacobi quadrature
- Birk (2010): modified Gauss-Jacobi quadrature



Error of model ε_{mod} at $N = 5$: 97.16%, 94.03%, 84.42%

$\varepsilon_{mod} < 10\% \Rightarrow N > 20$

Isotropic medium Cold-Lake sandstone

Determination of the coefficients: optimization

$$\min_{(\theta_\ell, a_\ell)} \chi^2, \quad \chi^2 = \sum_{k=1}^{2N} \left| \frac{\widehat{F}_{DA}(\omega_k)}{\widehat{F}_{JKD}(\omega_k)} - 1 \right|^2 = \sum_{k=1}^{2N} \left| \sum_{\ell=1}^N a_\ell \frac{(j\omega_k + \Omega)^{1/2}}{\theta_\ell + j\omega_k + \Omega} - 1 \right|^2$$

- ω_k distributed linearly on a logarithmic scale: $\omega_k \in [\omega_0/10, 10\omega_0]$
- Optimization on the frequency range of interest

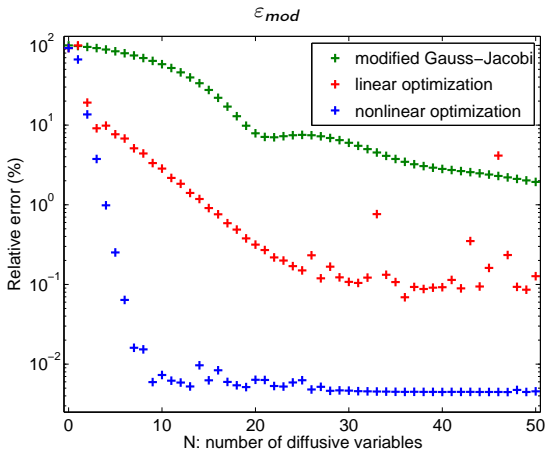
Linear optimization

- Emmerich and Korn (1987): **linear unconstrained least-squares minimization**
- θ_ℓ distributed linearly on a logarithmic scale: $\theta_\ell \in [\omega_0/10, 10\omega_0]$
- optimization of the **weights**
- **negative weights**

Nonlinear optimization

- Reikik and Brenner (2011): **nonlinear constrained minimization problem (Shor's algorithm)**
- optimization of the **abscissae** and the **weights**
- sufficient condition of stability: $\theta_\ell \geq 0, a_\ell \geq 0$
- numerical accuracy of exponential matrix: $\theta_\ell \leq 100\omega_0$
- initial values of the coefficients: modified Gauss-Jacobi quadrature

Determination of the coefficients: discussion



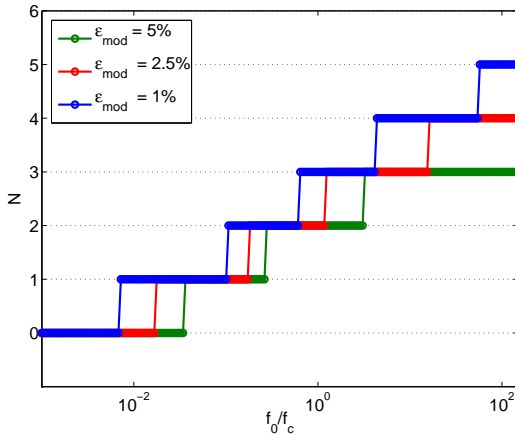
Error of model ε_{mod} at $N = 5$: 84.42%, 6.78%, 0.25%

Best method: nonlinear optimization

Determination *a priori* of N

A priori parametric determination of the number of diffusive variables in terms of

- the error of model ε_{mod}
- the ratio f_0/f_c (f_0 : central frequency of the source, f_c : frequency of transition)



Numerical modeling

- $8 + 2N$ variables. Velocity-stress formulation

$$\mathbf{U} = (v_{s,x}, v_{s,z}, w_x, w_z, \sigma_{xx}, \sigma_{xz}, \sigma_{zz}, \mathbf{p}, \psi_1^x, \psi_1^z, \dots, \psi_N^x, \psi_N^z)^T$$

- First-order hyperbolic system with source term

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial z} = -\mathbf{S}\mathbf{U}$$

- Explicit finite differences, $\Delta_x = \Delta_z$
- Condition of stability of an unsplit discretization

$$\Delta t \leq \min \left(\underbrace{\frac{\Delta x}{\max_{\varphi \in [0, \pi/2]} c_{pf}^\infty(\varphi)}}_{\text{CFL}}, \underbrace{\frac{2}{R(\mathbf{S})}}_{R(\mathbf{S}) \gg 1} \right)$$

Strang splitting

- Strategy: **splitting**. Successive resolutions of
 - the **propagative part**

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial z} = \mathbf{0} \quad (\mathbf{H}_p)$$

- the **diffusive part**

$$\frac{\partial \mathbf{U}}{\partial t} = -\mathbf{S}\mathbf{U} \quad (\mathbf{H}_d)$$

- Strang splitting, of second order

$$\left\{ \begin{array}{l} \mathbf{U}^{(1)} = \mathbf{H}_d(\Delta t/2) \mathbf{U}^n \\ \mathbf{U}^{(2)} = \mathbf{H}_p(\Delta t) \mathbf{U}^{(1)} \\ \mathbf{U}^{n+1} = \mathbf{H}_d(\Delta t/2) \mathbf{U}^{(2)} \end{array} \right.$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial z} = \mathbf{0} \quad (H_p)$$

Explicit finite difference scheme **ADER 4**

$$\mathbf{U}_{IJ}^{n+1} = \mathbf{U}_{IJ}^n + \Delta t \mathcal{H} \left(\mathbf{U}_{I+i, J+j}^n \right), \quad i, j = -2, \dots, 2$$

- Order 4 in time and space
- One time-step scheme, centered in space, stencil: 25 points
- Numerical dispersion of order 4, numerical dissipation of order 6
- Stability condition: $CFL = \max_{\varphi \in [0, \pi/2]} c_{pf}^{\infty}(\varphi) \frac{\Delta t}{\Delta x} \leq 1$

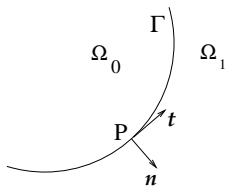
$$\frac{\partial \mathbf{U}}{\partial t} = -\mathbf{S} \mathbf{U} \quad (H_d)$$

- The matrix \mathbf{S} does not depend on time: **exact integration**

$$\mathbf{H}_d(\Delta t) \mathbf{U} = e^{-\Delta t \mathbf{S}} \mathbf{U}$$

- Computation of exponential matrices: **preprocessing**
- "Scaling and squaring method" : Moler and Van Loan (2003). Exponential matrix $e^{-\Delta t \mathbf{S}}$ computed by (6,6) Padé approximation
- Eigenvalues of \mathbf{S} positive \Rightarrow **unconditionally stable**

Piecewise homogeneous media



Interfaces and Cartesian grid

- Geometry of the interfaces
- Jump conditions $[\mathbf{v}_s] = \mathbf{0}$, $[\mathbf{w} \cdot \mathbf{n}] = 0$, $[\boldsymbol{\sigma} \cdot \mathbf{n}] = \mathbf{0}$, $[p] = 0$
- Same order of the numerical scheme than in homogeneous media

Immersed interface method

- Chiavassa and Lombard (2011): isotropic poroelastic waves in the LF regime
- **No change** required in the HF regime
- Contribution: **anisotropic** poroelastic waves

Summary

- Strang splitting
- propagative part: ADER 4 scheme
- diffusive part: exact integration
- immersed interface method

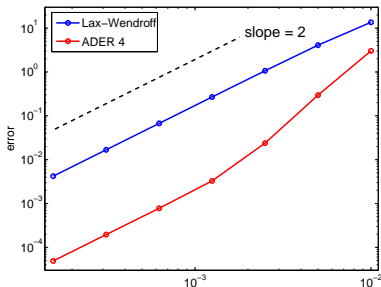
⊕ stability condition $CFL = \max_{\varphi \in [0, \pi/2]} c_{pf}^{\infty}(\varphi) \frac{\Delta t}{\Delta x} \leq 1$

independent of the quadrature coefficients $\theta_{\ell}^{x,z}, a_{\ell}^{x,z}$

- ⊖ the matrices **A**, **B** and **S** do not commute: the theoretical order of the global scheme decreases from 4 to 2

- order 2 measured

Error of the numerical scheme



Numerical experiments

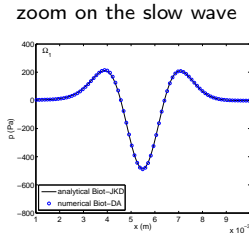
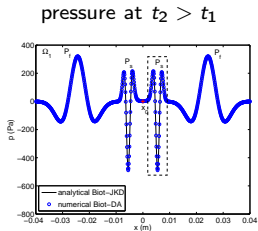
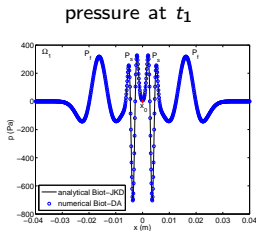
Parameters and media

- $f_0 = 200$ kHz
- CFL = 0.95
- $N = 3$ diffusive variables

	medium	symmetry	ε_{mod}
Ω_0	Cold-Lake sandstone	isotropic	3.75%
Ω_1	Berea sandstone	isotropic	1.21%
Ω_2	cancellous bone	isotropic	1.15%
Ω_3	glass-epoxy	transversely isotropic	1.61%
Ω_4	Berea sandstone	transversely isotropic	1.61%

1D numerical experiments (1/5)

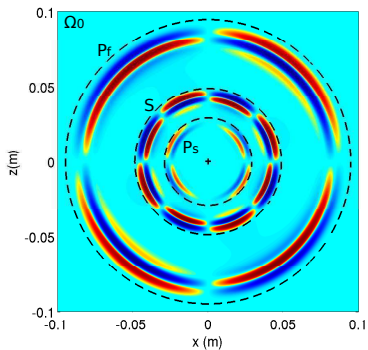
- Objective: **validation** diffusive approximation + numerical modeling 1D
- Numerical **Biot-DA** vs semi-analytical (Fourier synthesis) **Biot-JKD**
- Dispersion** and **attenuation** of the slow compressional wave



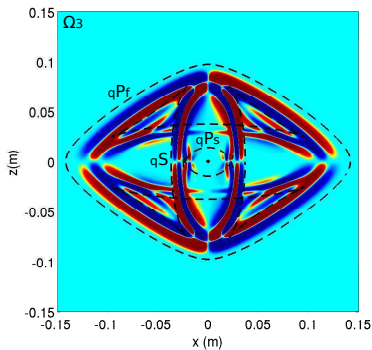
2D numerical experiments: homogeneous media (2/5)

- Objective: **validation** 2D homogeneous + anisotropy
- Black dotted line: **theoretical wavefront**

σ_{zz} (isotropic)

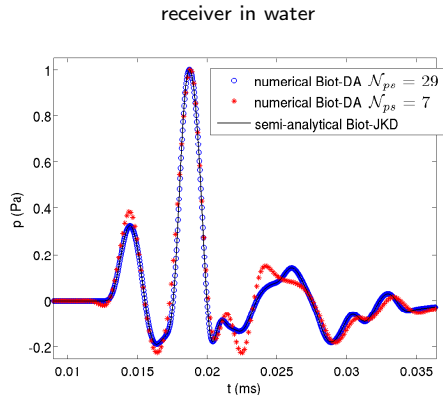
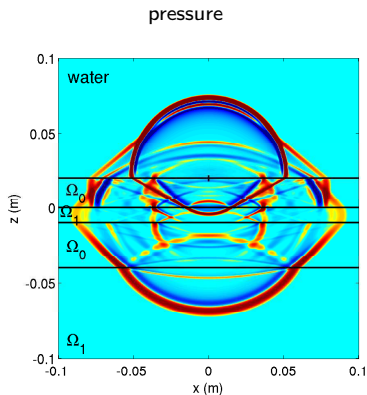


σ_{zz} (transversely isotropic)



2D numerical experiments: multilayered media (3/5)

- Objective: **validation** 2D heterogeneous (plane interfaces)
- Top: water, bottom: layers of isotropic sandstones
- **Seismic** applications
- **Semi-analytical solutions** computed by G. Lefeuvre-Mesgouez and A. Mesgouez (Université d'Avignon)



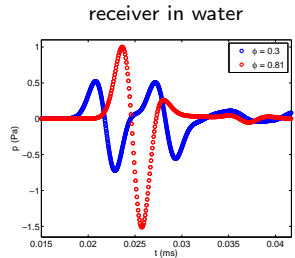
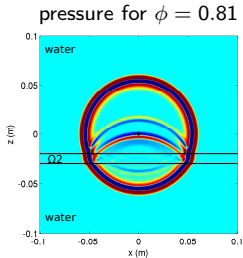
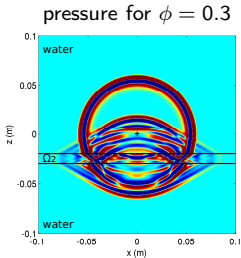
\mathcal{N}_{ps} : number of grid points by slow compressional wavelength

2D numerical experiments: cancellous bone (4/5)

- **Biological** medium
- Berryman (1980), Ashman and Rho (1988)

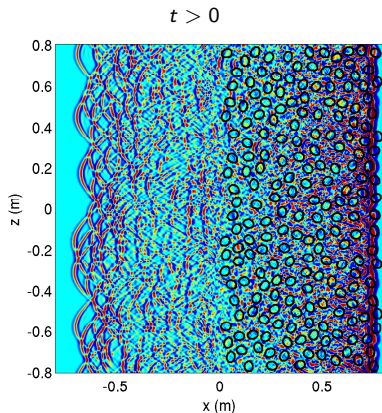
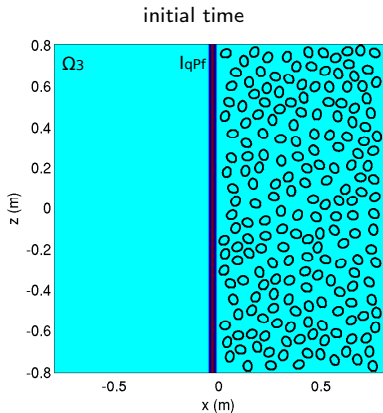
$$\mu = \frac{E_s (1 - \phi)^n}{2(1 + \nu_b)}, \quad K_b = \frac{E_s (1 - \phi)^n}{3(1 - 2\nu_b)}, \quad \mathcal{T} = 1 - r(1 - 1/\phi)$$

- Inverse problems \Rightarrow **porosity**

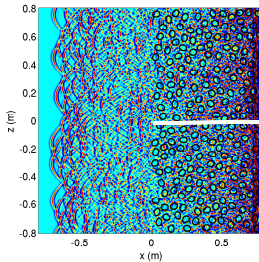


2D numerical experiments: multiple scattering (5/5)

- Wave propagation in **complex media**
- Plane wave in transversely isotropic medium
- Ellipsoidal scatterers **randomly distributed**: concentration 25%
- Properties of **effective medium**

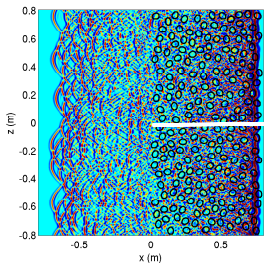


2D numerical experiments: multiple scattering (5/5)

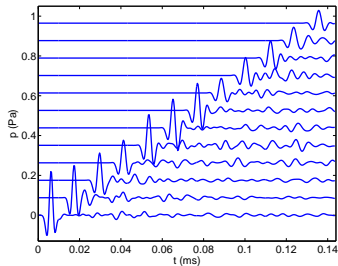


row of receivers
→

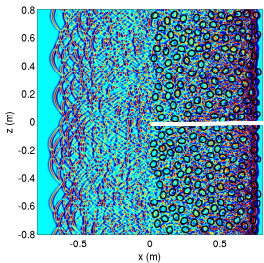
2D numerical experiments: multiple scattering (5/5)



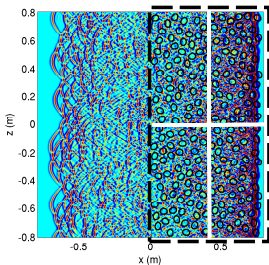
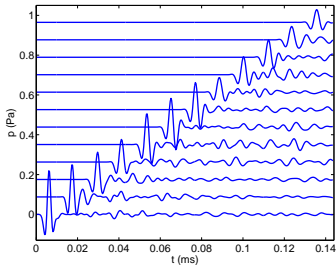
row of receivers
→



2D numerical experiments: multiple scattering (5/5)

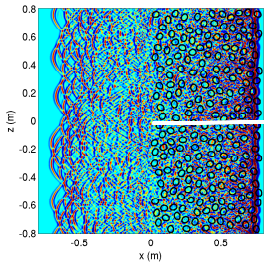


row of receivers
→

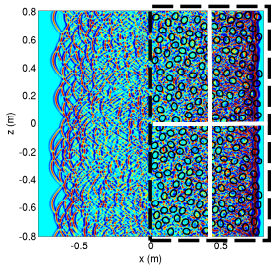
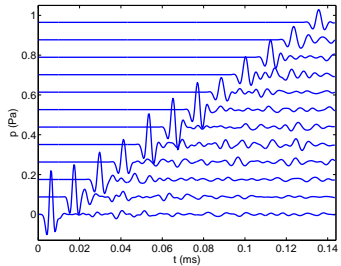


coherent field
→
after summation

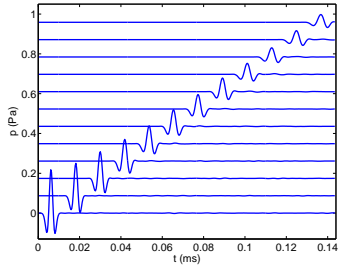
2D numerical experiments: multiple scattering (5/5)



row of receivers
→

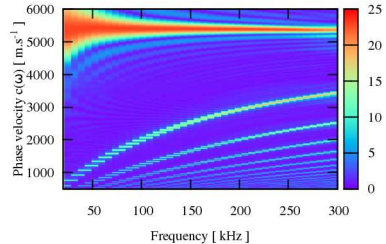


coherent field
→
after summation

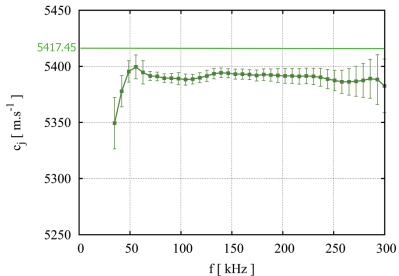


2D numerical experiments: multiple scattering (5/5)

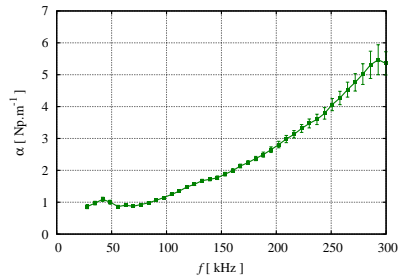
- M. Chekroun (Université du Maine)
- $p - \omega$ transform of the coherent field
- Phase velocity and attenuation of the **effective medium** in terms of the frequency



effective phase velocity

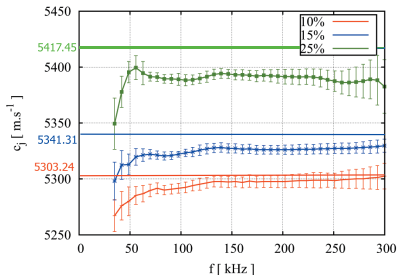


effective attenuation

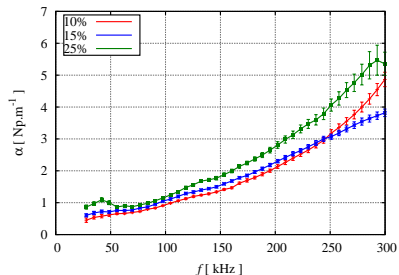


2D numerical experiments: multiple scattering (5/5)

effective phase velocity



effective attenuation



	Models of effective medium	Numerical comparisons
Elastic	<ul style="list-style-type: none"> - ISA, Waterman-Truell (1961) - Conoir-Norris (2010) 	Chekroun and al (2012)
Biot	<ul style="list-style-type: none"> - Tournat-Pagneux-Lafarge-Jaouen¹ (2004) - Luppe-Conoir-Robert² (2008) 	First step

¹ "equivalent fluid" model, isotropic, cylindrical scatterers

² inviscid media ($\eta = 0$), isotropic, cylindrical scatterers

Conclusion

	Masson-Pride	Lu-Hanyga	thesis
1D	20	25	3
2D	40	50	6
3D	60	75	9

Code in C language

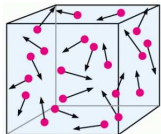
- preprocessing
 - determination of the **number of diffusive variables**
 - computation of the quadrature **coefficients**
- propagative part unchanged
 - physical variables: same algorithm than in the LF regime
 - fluxes of the diffusive variables \equiv fluxes of the filtration velocity
- diffusive part integrated exactly
- immersed interface method
 - same algorithm than in the LF regime
 - extended to **anisotropic** poroelastic media
- **few modifications** compared to the 2D framework

PDE + time-fractional derivatives

- **thermics** of porous media (Johnson, Champoux, Allard, Lafarge)
- fractional **viscoelasticity**: Maxwell, Kelvin-Voigt, Zener, Andrade (Bellis and al)
- **tube with a set of Helmholtz resonator** (Mercier and al)
- **electromagnetism**: Maxwell equations + Cole-Cole, Cole-Davidson, Havriliak-Negami models of complex relative susceptibility

PDE + space-fractional derivatives

- porous media: Lafarge, Nemati, 2012
- model of polydisperse sedimentation: Esipov, 1995
- diffusive representation of **space-fractional derivatives** ?
- postdoc (Uppsala University, G. Kreiss): **subdiffusion** and **superdiffusion** of biological cells



Diffusion

$$\langle r(t)^2 \rangle = K t$$
$$\frac{\partial u}{\partial t} = K \Delta u$$

Superdiffusion

$$\langle r(t)^2 \rangle = K t^\alpha, \quad \alpha > 1$$
$$\frac{\partial u}{\partial t} = K D_x^{1+\alpha} u$$



G.F.A. de L'Hôpital
(1661–1704)

What if the
order will be
 $n = \frac{1}{2}$?

It will lead to a
paradox, from which
one day useful
consequences will be
drawn.

$$\frac{d^n f}{dt^n}$$



G.W. Leibniz
(1646–1716)



G.F.A. de L'Hôpital
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movie

Thank you for your attention!



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