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École Centrale des Arts et Manufactures

THÈSE DE DOCTORAT

Spécialité : Mathématiques

Présentée par

Paul BALANÇA

Pour obtenir le grade de

DOCTEUR de l'ÉCOLE CENTRALE PARIS

Régularité fine de processus stochastiques et analyse 2-microlocale

---

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Soutenue le 6 février 2014 devant le jury composé de :

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“ Voyez-vous dear Achille Talon, l’important en cas d’aléa, c’est de toujours savoir avec exactitude ce qui va se passer ! ”

Duc Arthur de Glouswallow

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Résumé

Les travaux présentés dans cette thèse s'intéressent à la géométrie fractale de processus stochastiques à travers le prisme d'un outil appelé l'analyse 2-microlocale. Ce dernier est issu d'une autre branche des mathématiques, l'analyse fonctionnelle et l'étude des équations aux dérivées partielles, et s'est avéré être pertinent pour décrire la géométrie fine de fonctions déterministes ou de processus aléatoires, généralisant notamment les exposants de Hölder classiques.

Nous envisageons ainsi dans ce manuscrit différentes classes de processus, traitant en premier lieu le cas des martingales continues et de l'intégrale stochastique d'Ito. La régularité 2-microlocale de ces derniers fait notamment apparaître un autre concept, la pseudo frontière 2-microlocale, étroitement lié à son aîné. Nous appliquons également ce formalisme d'étude à une classe de processus gaussiens : le mouvement brownien multifractionnaire. Nous caractérisons ainsi sa régularité 2-microlocale et hölderienne, et déterminons dans un deuxième temps la forme générale de la dimension fractale de ses trajectoires. Dans notre étude portant sur les processus de Lévy, nous combinons le formalisme 2-microlocale à l'analyse multifractale, permettant alors de mettre en évidence des comportements géométriques n'étant pas captés par les outils usuels. Nous obtenons également en corollaire le spectre multifractal des processus fractionnaires de Lévy. Enfin, dans une dernière partie, nous nous intéressons à la définition et aux propriétés de certains processus de Markov multiparamètres, pouvant être plus généralement indicés par des ensembles.

Mots clés : analyse 2-microlocale, géométrie fractale, régularité trajectorielle, spectre multifractal, processus de Lévy, martingales, mouvement Brownien multifractionnaire, processus de Markov multiparamètres.

Abstract

The work presented in this thesis concerns the study of the fractal geometry of stochastic processes using the formalism of 2-microlocal analysis. The latter has been introduced in another branch of mathematics -functional analysis- but has also proved to be relevant to describe the geometry of deterministic functions or random processes, extending in particular the classic Hölder exponents.

Several classes of processes are investigated in this manuscript, beginning with continuous martingales and Ito integrals. In particular, the characterisation of the 2-microlocal regularity of the latter leads to the introduction of a closely related concept: the pseudo 2-microlocal frontier. We also investigate using this formalism a class of Gaussian processes called multifractional Brownian motion and obtain a fine description of its Hölder and 2-microlocal behaviours. In addition, we characterize entirely the Hausdorff and Box dimensions of its graph. In our study of Lévy processes, we combine the 2-microlocal formalism and multifractal analysis to describe their regularity, exhibiting in particular some subtle geometrical behaviours which are not captured by classic tools. Furthermore, as a corollary of this result, we also determine the multifractal spectrum of another family of processes: the fractional Lévy processes. Lastly, we also define a class of multiparameter and set-indexed Markov processes and study its properties.

Keywords : 2-microlocal analysis, fractal geometry, sample path regularity, multifractal spectrum, Lévy processes, martingales, multifractional Brownian motion, multiparameter Markov processes.

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0 Introduction

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La géométrie fractale est une notion de nos jours présente dans de nombreuses branches des mathématiques et utilisée pour de nombreuses applications dans des domaines aussi variés que la physique, la biologie, le traitement du signal et l'architecture. Cependant, son émergence n'est que très récente à l'échelle de l'histoire des mathématiques, et les objets qu'elle étudie ont pendant longtemps été délaissés par la majorité des mathématiciens car ne rentrant pas dans les cadres classiques en analyse et géométrie. Un exemple classique est la *fonction de Weierstrass* définie sous la forme suivante :

$$\forall x \in [0,1]; \quad f_W(x) = \sum_{k=1}^{\infty} b^{-k\alpha} \cos(b^k x), \quad \text{où } b > 1 \text{ et } \alpha \in (0,1). \tag{1}$$

Elle fut longtemps avant tout considérée comme un objet pathologique n'ayant pour seul intérêt que de montrer qu'il existe des fonctions continues nulle part dérivables. Dans la même lignée, on peut également citer la fonction de Cantor [1] montrant qu'il ne suffit pas d'être dérivable presque partout pour que le théorème fondamental de l'analyse s'applique ($f_C'(x) = 0$ pour presque tout $x \in [0,1]$).

Le regard des mathématiciens, et plus largement de la communauté scientifique, sur ces objets a évolué lors de la seconde moitié du XXe siècle, sous l'influence des idées de Benoît

[1] Également connue sous le nom *d'escalier du diable*.

MANDELBROT. Inventeur du néologisme *fractale*, ce dernier montra à travers ses travaux (voir par exemple les ouvrages [123, 124]) que de nombreux phénomènes naturels faisaient en réalité apparaître des comportements très irréguliers, dits fractals, justifiant alors le développement d'un cadre et de nouveaux outils pour leur étude. Bien que n'étant pas défini de manière formelle, on regroupe généralement sous le terme de *fractale* des objets vérifiant l'une des propriétés suivantes (voir notamment la définition donnée par FALCONER [67]) :

(i) sa dimension fractale, définie dans la suite, est supérieure à la dimension topologique (et est généralement non entière) ;

(ii) il est auto-similaire, c'est-à-dire conserve les mêmes propriétés géométriques à n'importe quelle échelle ;

(iii) il est trop irrégulier pour être décrit par les outils de la géométrie classique.

La géométrie fractale s'est avérée être impliquée à de nombreux domaines des mathématiques : probabilités, théorie des nombres, systèmes dynamiques et équations différentielles parmi eux.

Les probabilités modernes ont joué un rôle particulièrement important dans l'émergence de la géométrie fractale car faisant apparaître des objets aléatoires typiquement de nature irrégulière et rugueuse. Ces phénomènes aléatoires sont d'autant plus intéressants qu'il s'avère que leur géométrie est paradoxalement parfois plus facile à étudier que celle de d'objets déterministes comme la fonction de Weierstrass.

Le processus stochastique le plus important et célèbre est certainement le *mouvement brownien*. Il fut tout d'abord décrit par BROWN au XIXe siècle et ensuite défini plus rigoureusement par EINSTEIN puis WIENER au début du XXe. Rappelons qu'il s'agit de l'unique processus Gaussien $B = (B_t)_{t\in\mathbf{R}_+}$ centré et dont la covariance est égale à $\mathbb{E}[B_sB_t] = \min(s,t)$, pour tous $s,t \in \mathbf{R}_+$. Ce processus stochastique vérifie trois propriétés fondamentales :

(i) les accroissements sont indépendants, c.-à-d. pour tous $s \le t \in \mathbf{R}_+$:

$$B_t - B_s \text{ est indépendant de } \mathscr{F}_s := \sigma\big(\{B_u : u \le s\}\big);$$

(ii) les accroissements sont stationnaires, c.-à-d. pour tout $h \ge 0$:

$$\{B_{t+h} - B_h\}_{t\in\mathbf{R}_+} \overset{\mathscr{L}}{=} \{B_t\}_{t\in\mathbf{R}_+};$$

(iii) auto-similarité d'ordre $\frac{1}{2}$, c.-à-d. pour tout $c > 0$:

$$\{B_{ct}\}_{t\in\mathbf{R}_+} \overset{\mathscr{L}}{=} \{c^{1/2}B_t\}_{t\in\mathbf{R}_+}.$$

On retrouve, plus ou moins intuitivement, la traduction géométrique de ces propriétés sur les trajectoires du mouvement brownien (Figure 1). On y observe que ces dernières sont bien continues mais clairement irrégulières, à l'image de la fonction de Weierstrass. Notons également que l'auto-similarité du mouvement brownien est définie au sens probabiliste, et est donc plus faible que son équivalent déterministe vérifié pour la fonction de Weierstrass.

Dans la suite de cette introduction, nous donnons un aperçu des différents outils d'analyse de la régularité qui apparaissent dans les chapitres suivants, et plus généralement dans la littérature portant sur les propriétés trajectorielles de processus stochastiques. Conjointement sont présentés les résultats principaux existant dans cette branche des probabilités, qui concerne au premier lieu la géométrie du mouvement brownien mais également les propriétés trajectorielles de différentes extensions (processus de Lévy et mouvement brownien fractionnaire principalement).

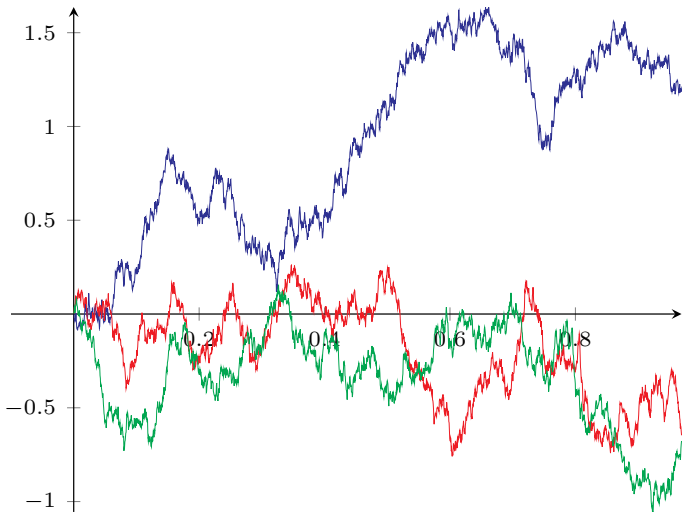


FIGURE 1 – Quelques trajectoires browniennes $t \mapsto B_t(\omega)$

0.1 Régularité hölderienne

Bien que l'introduction des espaces de Hölder soit bien antérieure au développement de la géométrie fractale, ils se sont avérés être un outil intéressant et pertinent pour l'étude et la caractérisation de la régularité.

Précisons que dans la suite, nous considérons des fonctions, ou des processus aléatoires, au minimum càdlàgs, c'est-à-dire qu'en tout $t \in \mathbf{R}$, f est continue à droite de t et il existe une limite à gauche $f(t)_-$. Afin d'analyser les propriétés géométriques fines dans le voisinage d'un point $t \in \mathbf{R}$, nous nous intéressons à la définition locale des espaces de Hölder.

Définition 0.1 (Espaces de Hölder) *Une fonction càdlàg f définie sur $\mathbf{R}$ appartient à* l'espace de Hölder ponctuel C_t^α, *avec $\alpha > 0$ et $t \in \mathbf{R}$, si et seulement si*

$$\forall u \in B(t,\rho); \quad |f(u) - P(u)| \leq C|t-u|^\alpha,$$

où ρ et C sont deux constantes strictement positives et P est un polynôme d'ordre inférieur à α.

De plus, la fonction f appartient à l'espace de Hölder local $\widetilde{C}_t^\alpha$, $\alpha \in (0,1)$, *si*

$$\forall u, v \in B(t,\rho); \quad |f(u) - f(v)| \leq C|u-v|^\alpha.$$

Il faut noter que la définition des *espaces de Hölder locaux* peut être étendue à tout $\alpha > 0$, en analysant les accroissements des dérivées n-ièmes de f. Cependant, cette complexité supplémentaire n'est pas nécessaire dans le cadre d'étude qui nous intéresse, les trajectoires de processus stochastiques s'avérant généralement non dérivables. Grâce à la définition de ces espaces fonctionnels, nous pouvons introduire les coefficients de Hölder ponctuel et local.

Définition 0.2 (Exposants de Hölder) *Soit f une fonction càdlàg sur $\mathbf{R}$ et $t \in \mathbf{R}$.* L'exposant de Hölder ponctuel *de f en t est défini comme :*

$$\alpha_{f,t} = \sup\{\alpha : f \in C_t^\alpha\}.$$

Similairement, l'exposant de Hölder local *de f en t est égal à*

$$\widetilde{\alpha}_{f,t} = \sup\{\alpha : f \in \widetilde{C}_t^\alpha\}.$$

Par convention, on fixe $\sup\{\emptyset\} = -\infty$.

Notons que lorsque les exposants de Hölder appartiennent à l'intervalle $(0,1)$, ils peuvent être directement caractérisés sous la forme suivante :

$$\begin{aligned}\alpha_{f,t} &= \sup\left\{\alpha : \limsup_{\rho\to 0} \frac{|f(t+\rho)-f(t)|}{\rho^\alpha} < \infty\right\} \\ &= \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t,\rho)} \frac{|f(u)-f(v)|}{\rho^\alpha} < \infty\right\}\end{aligned} \tag{2}$$

et

$$\widetilde{\alpha}_{f,t} = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t,\rho)} \frac{|f(u)-f(v)|}{|u-v|^\alpha} < \infty\right\}. \tag{3}$$

Les exposants de Hölder ponctuel et local ont donc pour objectif de décrire le comportement hölderien fin de f, localement autour d'un point $t \in \mathbf{R}$. Ainsi, si l'on considère le simple exemple de la fonction puissance $x \mapsto |x|^\alpha$, $\alpha \in (0,1)$ dans le voisinage de 0, on retrouve bien que l'exposant α régit la régularité hölderienne locale : $\alpha_{x^\alpha,0} = \widetilde{\alpha}_{x^\alpha,0} = \alpha$.

Il existe une subtile différence entre les exposants ponctuel et local qui peut être illustrée par l'exemple classique de la fonction *chirp* $f_c : x \mapsto |x|^\alpha \sin(|x|^{-\beta})$, où $\alpha \in (0,1)$ et $\beta > 0$. En estimant l'amplitude des accroissements de f_c autour de 0, nous obtenons à l'aide des

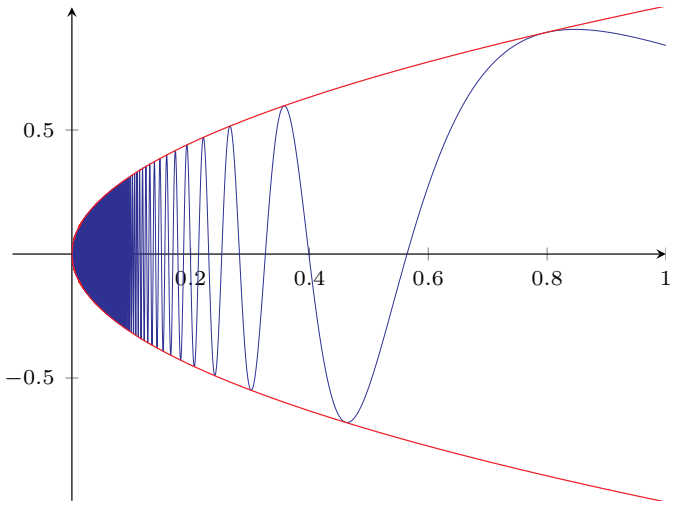


FIGURE 2 – Fonction chirp, $\alpha = 0.5$ et $\beta = 1$ et enveloppe $x \mapsto |x|^\alpha$

caractérisations (2) et (3) la régularité hölderienne de f_c (voir [60] pour le détail du calcul) :

$$\alpha_{f,0} = \alpha \quad \text{et} \quad \widetilde{\alpha}_{f,0} = \frac{\alpha}{1+\beta}. \tag{4}$$

Intuitivement, on s'aperçoit ainsi que l'exposant ponctuel prend uniquement en compte l'enveloppe $|x|^\alpha$ de f_c dans un voisinage de t, alors que le coefficient local considère toutes ses oscillations autour de t, faisant ainsi intervenir les deux paramètres α et β. La Figure 2 rend compte graphiquement de cette différence dans le cas de la fonction chirp.

Dans les deux exemples précédents, nous observons que la régularité hölderienne s'avère non-triviale en un seul point (en dehors de 0, $\alpha_{f,t} = \widetilde{\alpha}_{f,t} = \infty$). A l'inverse, la fonction de Weierstrass introduite précédemment offre un exemple déterministe dont la géométrie est partout irrégulière. Plus précisément, comme présenté par FALCONER [67], nous obtenons que

$$\forall t \in \mathbf{R}; \quad \alpha_{f_W,t} = \alpha \quad \text{et} \quad \widetilde{\alpha}_{f_W,t} = \alpha. \tag{5}$$

On remarque que le résultat sur le coefficient ponctuel induit notamment la non-dérivabilité de f_W en tout point.

Si la régularité hölderienne de la fonction de Weierstrass se trouve être constante sur $\mathbf{R}$, Daoudi, Lévy Véhel et Meyer [49] ont caractérisé la forme que peut prendre l'exposant local $t \mapsto \widetilde{\alpha}_{f,t}$. Plus précisément, pour toute fonction $\alpha(\cdot)$ s'exprimant comme la limite inférieure d'une suite de fonctions continues, il est possible de construire f telle que $\widetilde{\alpha}_{f,t} = \alpha(t)$ pour tout $t \in \mathbf{R}$. Jaffard [92] a étudié le problème similaire pour le coefficient de Hölder ponctuel ainsi qu'autre type d'exposant appelé exposant de chirp.

L'analyse de la géométrie fine des processus stochastiques peut être réalisée de la même manière que sur des objets déterministes, et donc notamment via l'étude des coefficients hölderiens, en essayant de montrer qu'une propriété est vérifiée avec probabilité 1, c.-à-d. sur presque toutes les trajectoires. Ainsi, si l'on considère le mouvement brownien, il a été démontré (voir par exemple l'ouvrage récent de Mörters et Peres [143]) que :

$$\text{presque sûrement} \quad \forall t \in \mathbf{R}_+; \quad \alpha_{B,t} = \frac{1}{2} \quad \text{et} \quad \widetilde{\alpha}_{B,t} = \frac{1}{2}. \tag{6}$$

Le résultat précédent est généralement qualifié *d'uniforme* sur la trajectoire, c.-à-d. pour presque tout $\omega \in \Omega$ fixé, les exposants de Hölder ponctuel et local ont été déterminés en tout point $t \in \mathbf{R}_+$. Une propriété plus faible consiste à fixer t, et ensuite à étudier la régularité presque sûre en ce point t. Si dans le cas du mouvement brownien, les quantificateurs en t et ω peuvent être inversés sans altérer le résultat, nous verrons par la suite que ce n'est pas le cas pour tous les processus stochastiques.

Notons également que lors de l'étude des propriétés trajectorielles d'un processus stochastique, il est plus aisé de trouver la borne inférieure pour les coefficients de Hölder, celle-ci étant généralement une conséquence directe de l'application du théorème d'extension de Kolmogorov. La borne supérieure nécessite quant à elle une étude plus fine du comportement des accroissements du processus, ce qui dans le cas du mouvement brownien est facilité par l'indépendance et la stationnarité de ces derniers.

0.1.1 Mouvement brownien fractionnaire

Malgré son universalité en probabilités, le mouvement brownien s'est avéré être trop restrictif pour modéliser un certain nombre de phénomènes, notamment à cause de l'indépendance des accroissements et la régularité $\frac{1}{2}$-hölderienne.

Une généralisation admettant une gamme de comportements plus large est donc rapidement apparue dans la littérature probabiliste. Appelé *mouvement brownien fractionnaire* (m.b.f.), ce processus fut mis en évidence par Kolmogorov dans les années 1940, mais ne suscita un véritable intérêt dans la communauté probabiliste que quelques années après, notamment suite aux travaux de Mandelbrot et Van Ness [125].

Définition 0.3 **(Mouvement brownien fractionnaire)** *Soit $H \in (0,1)$ un* exposant de Hurst. *On appelle* mouvement brownien fractionnaire *le processus gaussien B^H défini sous la forme intégrale suivante :*

$$\forall t \in \mathbf{R}_+; \quad B_t^H = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_{\mathbf{R}} \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}W_u, \tag{7}$$

où $x_+ := \max(0, x)$ pour tout $x \in \mathbf{R}$.

Rappelons que l'intégrale apparaissant dans la définition précédente est prise au sens de Wiener, c.-à-d. définie comme le processus gaussien centré indexé par $L^2(\mathbf{R})$ et dont la

covariance vérifie :

$$\forall f, g \in L^2(\mathbf{R}); \quad \mathbb{E}\left[\int_{\mathbf{R}} f \,\mathrm{d}W \int_{\mathbf{R}} g \,\mathrm{d}W\right] = \int_{\mathbf{R}} f(u)g(u)\,\mathrm{d}u.$$

Observons que le noyau du m.b.f. appartient bien à $L^2(\mathbf{R})$ pour tout $t \in \mathbf{R}$ étant donné que $\left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right)^2 \sim_{-\infty} c_{H,t}(-u)^{2H-3}$.

Le mouvement brownien fractionnaire admet également une représentation sous forme harmonisable :

$$\forall t \in \mathbf{R}_+; \quad B_t^H = c_H \int_{\mathbf{R}} \frac{\mathrm{e}^{iut} - 1}{|u|^{H+1/2}} \,\mathrm{d}\widehat{W}_u, \tag{8}$$

où c_H est une constante de renormalisation. La mesure gaussienne complexe $\widehat{W}$ est définie de manière à ce que pour tout $A \in \mathscr{B}(\mathbf{R}_+)$, $\widehat{W}(-A) = \overline{\widehat{W}(A)}$ et $\widehat{W}$ est une mesure de Wiener complexe sur $\mathbf{R}_+$. SAMORODNITSKY et TAQQU [167] présente une construction détaillée de $\widehat{W}$ et montre entre autres l'égalité $\int_{\mathbf{R}} f(u)\,\mathrm{d}W_u \overset{\mathscr{L}}{=} \int_{\mathbf{R}} \widehat{f}(u)\,\mathrm{d}\widehat{W}_u$, pour toute fonction $f \in L^2(\mathbf{R})$.

Bien que ces deux représentations soient les plus communes et utilisées, d'autre constructions du m.b.f., incluant notamment la décomposition en ondelettes du m.b.f. (voir par exemple les travaux de DOUKHAN, OPPENHEIM et TAQQU [51], AYACHE et TAQQU [14]), ont été mises en évidence et étudiées dans la littérature.

Le m.b.f. étant un processus gaussien centré, il est entièrement caractérisé par sa structure de covariance. Cette dernière s'obtient aisément à l'aide d'une des représentations précédentes :

$$\forall s, t \in \mathbf{R}_+; \quad \mathbb{E}[B_s^H B_t^H] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{9}$$

On remarque donc que lorsque l'exposant de Hurst H est égal à $\frac{1}{2}$, on retrouve bien la covariance du mouvement brownien. La généralisation de ce dernier ne conserve cependant pas toutes ces propriétés lorsque $H \neq \frac{1}{2}$: les accroissements ne sont pas indépendants et le m.b.f. n'est ni une semimartingale ni un processus de Markov(voir l'ouvrage de NOURDIN [146] pour plus de détails). Néanmoins, pour tout $H \in (0,1)$, le m.b.f. reste

(i) à accroissements stationnaires ;

(ii) auto-similaire d'exposant H.

De plus, la régularité hölderienne du mouvement brownien fractionnaire se trouve être paramétrée par cet exposant de Hurst :

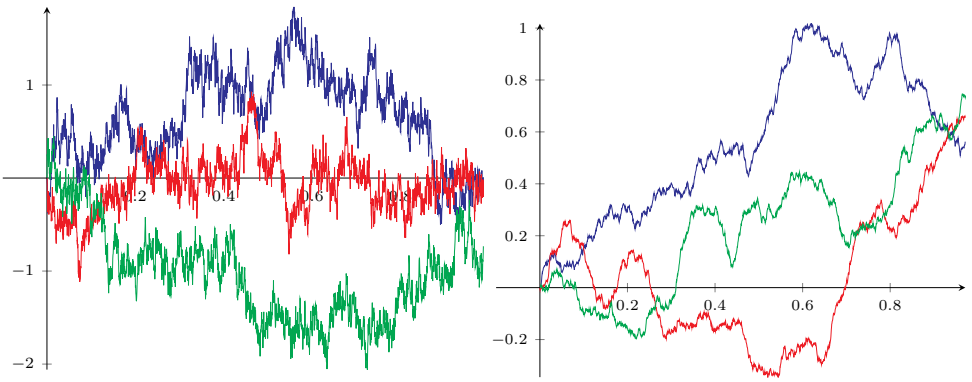
$$\text{presque sûrement} \quad \forall t \in \mathbf{R}_+; \quad \alpha_{B^H,t} = H \quad \text{et} \quad \widetilde{\alpha}_{B^H,t} = H. \tag{10}$$

L'influence du paramètre de Hurst sur la géométrie des trajectoires est perceptible sur les Figures 3a et 3b, où l'on s'aperçoit de la plus large gamme de comportements que peut modéliser le m.b.f..

0.1.2 Mouvement brownien multifractionnaire

Si le mouvement brownien fractionnaire permet de modéliser de manière satisfaisante de nombreux phénomènes, l'hypothèse de stationnarité empêche de l'appliquer à certains cas de figure. Partant de ce constat, PELTIER et LÉVY VÉHEL [154] et BENASSI, JAFFARD et ROUX [26] ont indépendamment introduit une généralisation du m.b.f. appelée *mouvement brownien multifractionnaire* (m.b.m.).

L'idée principale étant de permettre aux propriétés du processus d'évoluer en fonction du temps, le m.b.m. n'est plus paramétré par un simple coefficient mais par une fonction



(a) $H = 0.3$ (b) $H = 0.7$

FIGURE 3 – Trajectoires de mouvement brownien fractionnaire

continue $H : \mathbf{R} \to (0,1)$. La définition du m.b.m. est alors obtenue via une simple extension de l'Équation (7) :

$$\forall t \in \mathbf{R}_+; \quad X_t = \frac{1}{\Gamma\big(H(t)+\frac{1}{2}\big)} \int_{\mathbf{R}} \Big[(t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2}\Big] \mathrm{d}W_u. \tag{11}$$

Une représentation harmonisable peut également être utilisée :

$$\forall t \in \mathbf{R}_+; \quad \widetilde{X}_t = C(H(t)) \int_{\mathbf{R}} \frac{\mathrm{e}^{iut} - 1}{|u|^{H(t)+1/2}} \, \mathrm{d}\widehat{W}_u, \tag{12}$$

où $C(.)$ est une fonction continue positive. Contrairement au mouvement brownien fractionnaire, STOEV et TAQQU [172] ont montré que les processus X et $\widetilde{X}$ n'ont en général pas la même loi. Néanmoins, la différence entre les deux est ténue, et il s'avère en pratique que les démonstrations sur une représentation s'adaptent souvent aisément à l'autre, notamment dans le cas des propriétés géométriques des trajectoires.

La forme générale de la covariance du m.b.m. est bien plus complexe que pour le m.b.f. (voir par exemple [154] pour une expression complète). Cependant, un développement limité de la variance des accroissements permet d'avoir un bon aperçu de la structure fine de corrélation du m.b.m.. Ainsi, comme l'a mis en évidence HERBIN [76], dans tout voisinage de $t > 0$:

$$\mathbb{E}\big[X_u - X_v\big]^2 \asymp |u-v|^{2H(t)} + |H(u) - H(v)|^2. \tag{13}$$

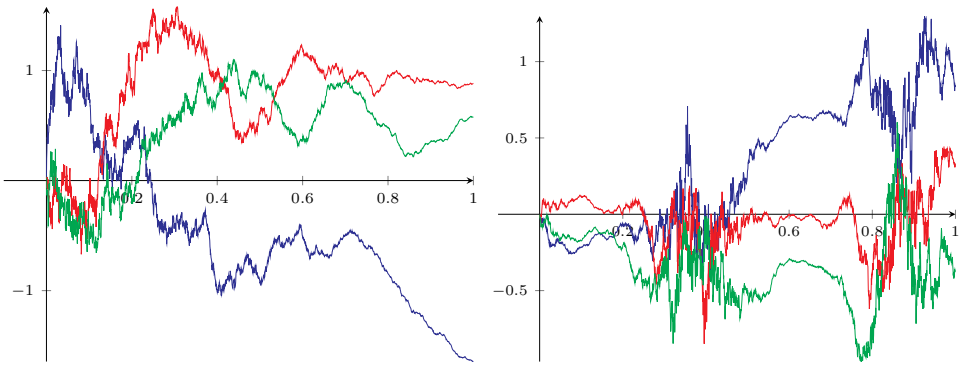
Une hypothèse facilitant l'étude des propriétés du m.b.m. apparaît fréquemment dans la littérature sur le sujet :

$$H \text{ est une fonction } \beta\text{-hölderienne telle que } \forall t \in \mathbf{R}, \;\; H(t) < \beta. \tag{$\mathscr{H}_0$}$$

Elle est vérifié par une grande classe d'applications $H(\cdot)$ comprenant notamment les fonctions continuement dérivables. Sous $\mathscr{H}_0$, la variance des accroissements est alors équivalente au premier terme $|u-v|^{2H(t)}$ apparaissant dans l'Équation (13), laissant penser que le m.b.m. se comporte localement autour de t comme un m.b.f. de paramètre $H(t)$. Cette observation se vérifie sur la régularité hölderienne d'un $\mathscr{H}_0$-m.b.m. qui est égale à :

$$\text{presque sûrement} \quad \forall t \in \mathbf{R}_+; \quad \alpha_{X,t} = H(t) \quad \text{et} \quad \widetilde{\alpha}_{X,t} = H(t). \tag{14}$$

Les Figures 4a et 4b donnent une illustration du comportement du $\mathscr{H}_0$-m.b.m. pour deux fonctions H différentes.



(a) $H : t \mapsto \frac{1}{4} + \frac{t}{2}$ (b) $H : t \mapsto \frac{1}{4}\big(2 + \sin(4\pi t)\big)$

FIGURE 4 – Trajectoires de mouvement brownien multifractionnaire

Si un $\mathscr{H}_0$-mouvement brownien multifractionnaire n'est ni auto-similaire ni à accroissements stationnaires, il vérifie néanmoins une propriété d'*auto-similarité asymptotique locale* (voir [154] pour une démonstration complète) :

$$\forall t \in \mathbf{R}_+; \quad \left\{ \frac{X_{t+\rho u} - X_t}{\rho^{H(t)}} \right\}_{u \in \mathbf{R}_+} \xrightarrow[\rho \to 0]{\mathscr{L}} \{B_u^{H(t)}\}_{u \in \mathbf{R}_+}. \tag{15}$$

Cette convergence vers un m.b.f. a lieu au sens faible dans l'espace des trajectoires continues. Les deux résultats précédents tendent donc à clairement nous indiquer que le comportement d'un $\mathscr{H}_0$-mouvement brownien multifractionnaire est localement celui d'un m.b.f. de paramètre $H(t)$.

Néanmoins, HERBIN [76] a montré que dans le cas général, la situation n'est plus aussi simple. Ainsi, lorsque l'hypothèse $\mathscr{H}_0$ n'est plus vérifiée, il s'avère que la régularité du m.b.m. prend une forme plus complexe :

$$\forall t \in \mathbf{R}_+; \quad \alpha_{X,t} = H(t) \wedge \alpha_{H,t} \quad \text{et} \quad \widetilde{\alpha}_{X,t} = H(t) \wedge \widetilde{\alpha}_{H,t} \quad \text{p.s.}, \tag{16}$$

où $x \wedge y := \min(x, y)$. Notons que ce résultat est valable à t fixé, et non de manière uniforme sur toute la trajectoire. L'autosimilarité asymptotique vers un m.b.f. n'est également plus vérifiée en général (voir [76] pour plus de détails).

On parle de *mouvement brownien multifractionnaire irrégulier* pour décrire le cas où les exposants de Hölder de H en t sont inférieurs à $H(t)$, et où le comportement diffère donc complètement d'un m.b.f. localisé. Les propriétés géométriques fines de ce m.b.m. irrégulier sont étudiées en détails dans le Chapitre 2.

0.1.3 Module de continuité et loi du logarithme itéré

L'exposant de Hölder local nous indique que dans un voisinage de t, les accroissements $X_u - X_v$ se comportent au maximum comme la fonction $|u - v|^{\alpha_{X,t}}$. Il est naturel de se demander si un estimé plus précis peut être obtenu, afin d'affiner l'approximation par une fonction puissance. Cette question correspond à déterminer le *module de continuité* de X, c.-à-d. la fonction optimale $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ telle que

$$\forall u, v \in B(t, \rho); \quad |X_u - X_v| \leq \omega(|u - v|),$$

dans un voisinage de t. Le module de continuité du mouvement brownien a été obtenu par Paul LÉVY [118, 117]. Ce résultat pionnier a ensuite été étendu à toute une classe de processus

gaussiens par MARCUS [126]. Le sujet est abordé en détail dans les ouvrages récents de MARCUS et ROSEN [127] et TALAGRAND [177] qui discutent notamment en profondeur des arguments de chaînage nécessaires à l'obtention de ce type de résultat. En particulier, le mouvement brownien fractionnaire vérifie pour tout $H \in (0,1)$:

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \frac{|B^H_{t+h} - B^H_t|}{h^H\sqrt{2\log(1/h)}} = 1 \quad \text{p.s.} \tag{17}$$

On s'aperçoit donc que le module de continuité du m.b.f. fait apparaître un terme de correction logarithmique $\log(1/h)$, qui implique notamment que le m.b.f. ne peut être H-hölderien.

Il est également possible d'obtenir un raffinement de la régularité hölderienne ponctuelle à l'aide de la *loi du logarithme itéré*. Initialement démontrée par KHINTCHINE pour le mouvement brownien, elle a ensuite été étendue à une famille de processus gaussiens par OREY [150]. Ainsi, pour tout $t \in \mathbf{R}_+$:

$$\limsup_{h\to 0} \frac{|B^H_{t+h} - B^H_t|}{h^H\sqrt{2\log\log(1/h)}} = 1 \quad \text{p.s.} \tag{18}$$

Il est intéressant d'observer que le terme correcteur $\log\log(1/h)$ est différent de celui apparaissant dans le module de continuité (Équation (17)). Cette observation implique donc que l'amplitude des accroissements à t fixé est donc légèrement différente du comportement uniforme sur toute la trajectoire.

Notons qu'une autre forme de loi du logarithme itéré apparaît dans la littérature, sous le nom de *loi de Chung*. Elle s'exprime de la manière suivante, pour tout $t \in \mathbf{R}_+$:

$$\liminf_{h\to 0} \frac{M^t_h}{h^H\sqrt{\log\log(1/h)}} = c_H \quad \text{p.s. où} \quad M^t_h := \sup_{|u-t|\le h} |B^H_u - B^H_t| \tag{19}$$

et c_H est constante positive, égale à $\pi/\sqrt{8}$ lorsque $H = 1/2$. Le résultat originel sur le mouvement brownien est du à CHUNG [43] et sa généralisation au m.b.f. à MONRAD et ROOTZÉN [142].

Ces lois du logarithme itéré (18) et (19) ont récemment été étendues au $\mathscr{H}_0$-mouvement brownien multifractionnaire, respectivement par BENASSI, JAFFARD et ROUX [26] et BOUFOUSSI, DOZZI et GUERBAZ [37].

0.2 Dimensions fractales

Le concept de *dimension fractale* est prédominant en géométrie fractale. L'idée sous-jacente est de trouver une quantité intrinsèque à un ensemble qui caractérise sa structure fine, et en particulier son occupation de l'espace. Intuitivement, il s'agit ainsi d'étendre l'idée de dimension topologique afin de pouvoir différencier plus finement les sous-ensembles de $\mathbf{R}^d$. Considérons par exemple le célèbre ensemble de Cantor défini par récurrence comme il suit :

$$C = \bigcap_{n=0}^{\infty} C_n \quad \text{où } C_0 = [0,1] \text{ et pour tout } n \in \mathbf{N},\ C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3}\right).$$

Il serait donc naturel qu'un outil permette de caractériser la structure d'un tel ensemble, en tenant compte à la fois de sa non-dénombrabilité, mais également de son caractère négligeable dans $[0,1]$.

On remarque rapidement dans la littérature sur le sujet (voir par exemple les ouvrages de MATTILA [130] et FALCONER [67]) qu'il n'existe pas une unique manière d'introduire le concept de dimension fractale. Commençons donc par décrire la *dimension de Minkowski*, également appelée *dimension de boîtes*, qui est la plus simple à définir et calculer (que ce soit théoriquement ou numériquement). Soit E un sous-ensemble de $\mathbf{R}^d$. Rappelons qu'on appelle δ-recouvrement de E une collection finie ou dénombrable d'ensembles $(E_k)_{k\in\mathbf{N}}$ telle que

$$E \subset \bigcup_{k\in\mathbf{N}} E_k \quad \text{et pour tout } k \in \mathbf{N},\ \ |E_k| \leq \delta$$

où $|E_k|$ désigne le diamètre de E_k pour la norme euclidienne. Pour tout $\delta > 0$, définissons alors :

$$M(E,\delta) = \min\{k \geq 1 : E_1, \dots, E_k \text{ est un } \delta\text{-recouvrement de } E \text{ de taille } k\}. \tag{20}$$

Définition 0.4 (Dimension de Minkowski) *Pour tout ensemble $E \subset \mathbf{R}^d$, on appelle respectivement* dimensions inférieure et supérieure de Minkowski *de E :*

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta\to 0} \frac{\log M(E,\delta)}{\log(1/\delta)} \quad \textit{et} \quad \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta\to 0} \frac{\log M(E,\delta)}{\log(1/\delta)}. \tag{21}$$

On parle de la dimension de Minkowski *de E lorsque la limite précédente existe, c.-à-d.*

$$\dim_{\mathrm{B}} E = \lim_{\delta\to 0} \frac{\log M(E,\delta)}{\log(1/\delta)}. \tag{22}$$

Par convention, l'on fixe $\dim_{\mathrm{B}} \emptyset = -\infty$.

La dimension de Minkowski stricto sensu n'est donc pas forcément définie pour tout ensemble, des contre-exemples ad-hoc vérifiant $\underline{\dim}_{\mathrm{B}} E < \overline{\dim}_{\mathrm{B}} E$ pouvant être facilement construits. Nous observons également que dans la définition (20) de $M(E,\delta)$, il est possible de se restreindre à un recouvrement de E par des boules de diamètre δ. FALCONER [67] présente une liste plus exhaustive des différentes définitions équivalentes de la dimension de boîtes.

De manière intuitive, calculer la dimension de boîtes correspond donc à terminer à l'exposant s vérifiant $M(E,\delta) \sim_0 \delta^{-s}$, où $M(E,\delta)$ est le nombre minimum de boules de taille δ nécessaire pour recouvrir E. Ainsi, par l'ensemble de Cantor C défini précédemment, on s'aperçoit qu'il faut pour tout $k \in \mathbf{N}$, 2^k boules de diamètre 3^{-k} pour le recouvrir, impliquant :

$$\dim_{\mathrm{B}} C = \frac{\log 2}{\log 3}.$$

Comme on peut s'en apercevoir, l'avantage principal de la dimension de Minkowski réside dans la simplicité de sa définition, qui la rend généralement facile à calculer. Néanmoins, elle possède également un défaut majeur. En effet, il s'avère que cette dimension fractale n'est pas stable par union dénombrable, c.-à-d. l'égalité

$$\dim_{\mathrm{B}} \bigcup_{k=1}^{\infty} E_k = \sup_{k\in\mathbf{N}}\{\dim_{\mathrm{B}} E_k\}, \tag{23}$$

n'est pas vérifiée en toute généralité. Pour s'en convaincre, il suffit simplement de considérer l'ensemble des rationnels $\mathbf{Q}$ qui vérifie $\dim_{\mathrm{B}} \mathbf{Q} = 1$, alors que la dimension de tout singleton est nulle. On note par ailleurs qu'il peut sembler étrange que $\mathbf{Q}$ ait la même dimension que l'intervalle $[0,1]$.

La *dimension de Hausdorff* constitue une approche qui, bien que plus complexe, permet de pallier le problème précédemment soulevé. Introduite par Felix HAUSDORFF en 1919, elle repose sur la même idée de recouvrement d'ensemble, tout en ajoutant un degré de liberté sur la taille des ensembles utilisés pour ce dernier. Afin de définir cette dimension, commençons par introduire les mesures de Hausdorff $\mathscr{H}^s$, $s > 0$. Pour tout ensemble $E \subset \mathbf{R}^d$, $\mathscr{H}^s(E) := \lim_{\delta\to 0} \mathscr{H}^s_\delta(E)$, où

$$\mathscr{H}^s_\delta(E) = \inf\left\{\sum_{k=0}^{\infty} |E_k|^s : (E_k)_{k\in\mathbf{N}} \text{ est un } \delta\text{-recouvrement de } E\right\}.$$

La limite précédente existe du fait de la croissance de la fonction $\mathscr{H}^s_\delta(E)$ en δ. La démonstration de la sigma-additivité de $\mathscr{H}^s$ s'avère pour le moins technique, et peut être trouvée dans les ouvrages de ROGERS [163] et MATTILA [130].

Définition 0.5 **(Dimension de Hausdorff)** *Pour tout ensemble $E \subset \mathbf{R}^d$, on appelle* dimension de Hausdorff *de E :*

$$\dim_{\mathrm{H}} E = \inf\{s : \mathscr{H}^s(E) = 0\} = \sup\{s : \mathscr{H}^s(E) = \infty\}. \tag{24}$$

Les propriétés de la fonction $s \mapsto \mathscr{H}^s(E)$ implique l'existence de la dimension de Hausdorff, qui correspond donc à la valeur où apparaît la transition entre ∞ et 0 pour $\mathscr{H}^s(E)$. On vérifie aisément que pour tout ensemble E, $\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E \leq \overline{\dim}_{\mathrm{B}} E$. L'égalité est notamment vérifiée pour l'ensemble de Cantor, dont la dimension de Hausdorff est égale à $\log 2/\log 3$, mais pas pour l'ensemble des rationnels $\mathbf{Q}$ qui vérifie $\dim_{\mathrm{H}} \mathbf{Q} = 0$. De plus, il s'avère donc que la dimension de Hausdorff est, elle, bien stable par union dénombrable (Équation (23)).

La calcul de la dimension de Hausdorff est généralement plus compliqué que pour la dimension de Minkowski, à cause de la classe plus large de recouvrement qui est considérée. La borne supérieure est souvent déterminée en construisant un δ-recouvrement ad-hoc tel que

$$\sum_{k=0}^{\infty} |E_k|^s = 0 \quad \text{avec} \quad s > \dim_{\mathrm{H}} E.$$

L'obtention de la borne inférieure est plus compliquée, et repose généralement sur un des résultats suivants :

Mass distribution principle. Soit μ une mesure de probabilité sur E, $\varepsilon > 0$ et $c > 0$ tels que

$$\mu(U) \leq c|U|^s \quad \text{pour tout ensemble } U \text{ vérifiant } |U| \leq \varepsilon. \tag{25}$$

Alors, la dimension de Hausdorff de E est supérieure à s.

Lemme de Frostman. Pour toute mesure μ sur E, définissons sa *s-énergie* de la manière suivante :

$$\mathscr{E}_s(\mu) = \iint_{E^2} \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{\|x-y\|^s}.$$

De plus, introduisons la *s-capacité* de F

$$\mathrm{Cap}_s(F) = \inf\{\mathscr{E}_s(\mu)^{-1} : \mu \text{ est une mesure de probabilité sur } F\}.$$

Alors, le lemme de Frostman affirme que $\dim_{\mathrm{H}} F = \sup\{s \geq 0 : \mathrm{Cap}_s(F) > 0\}$.

L'intérêt de ces deux résultats fondamentaux est qu'ils permettent de ramener l'obtention de la borne inférieure à la construction d'une mesure sur F vérifiant certaines propriétés (notamment $\mathscr{E}_s(\mu) < \infty$ pour le lemme de Frostman), évitant ainsi d'avoir à faire l'étude fine de tous les recouvrements possibles de F. La démonstration de ces deux résultats peut être trouvée dans les ouvrages de FALCONER [67] et MATTILA [130].

Pour terminer ce bref aperçu des outils en géométrie fractale, signalons que d'autres dimensions fractales ont été définies et étudiées dans la littérature. En particulier, on peut citer la *dimension de packing* introduite plus tardivement par TRICOT [183, 182] et qui s'apparente à une dimension de boîtes modifiée afin d'en corriger les principaux défauts (voir [130] et [63] où cette notion est développée plus en détail). Le concept de *dimension de Fourier* apparaît également dans la littérature (voir par exemple [67] et [97]).

0.2.1 Dimensions de graphe et d'images du m.b.f.

Les processus stochastiques faisant apparaître des objets dont la structure est clairement fractale, il est naturel de chercher à caractériser leur géométrie à travers l'analyse de la dimension fractale. Comme premier objet d'étude, on peut simplement considérer le graphe des trajectoires d'un processus X :

$$\mathrm{Gr}(X) = \{(s, X_s) : s \in \mathbf{R}_+\}.$$

Comme remarqué précédemment, la borne supérieure de la dimension de Hausdorff est obtenue en construisant un recouvrement ad-hoc. Dans le cas d'une fonction, ou un processus, α-hölderien, BESICOVITCH a montré que le graphe satisfait forcément $\dim_{\mathrm{H}} \mathrm{Gr}(f) \leq \overline{\dim}_{\mathrm{B}} \mathrm{Gr}(f) \leq 2 - \alpha$ (voir l'article originel retranscrit dans [62] pour le détail de la construction du recouvrement). Dans de nombreux cas, il s'avère que cet estimé correspond à la valeur recherchée. Ainsi, en utilisant cette borne et le lemme de Frostman pour la borne inférieure, on obtient aisément les dimensions de Hausdorff et de boîtes du graphe du m.b.f. :

$$\dim_{\mathrm{H}} \mathrm{Gr}(B^H) = \dim_{\mathrm{B}} \mathrm{Gr}(B^H) = 2 - H \quad \text{p.s.} \tag{26}$$

Il est intéressant d'observer que l'obtention de la dimension de graphe s'avère plus aisée pour l'objet aléatoire qu'est le m.b.f. que pour la fonction de Weierstrass déterministe (Équation (1)). Ainsi, la dimension du graphe de cette dernière, bien que conjecturée, reste inconnue et le meilleur estimé connu à ce jour a été obtenu par MAULDIN et WILLIAMS [131] :

$$2 - \alpha - \frac{c}{\log b} \;\leq\; \dim_{\mathrm{H}} \mathrm{Gr}(f_C) \;\leq\; 2 - \alpha,$$

où c est une constante positive. On remarque que l'encadrement converge bien vers la valeur conjecturée lorsque $b \to \infty$. C'est d'ailleurs en partie ce paradoxe que motiva l'introduction d'une fonction de Weierstrass généralisée par HUNT [86] sous la forme :

$$X_t = \sum_{k=1}^{\infty} b^{-k\alpha} \cos\left(b^k t + \theta_n\right),$$

où $(\theta_n)_{n\in\mathbf{N}}$ est une collection de variables aléatoires indépendantes de loi uniforme sur $[0, 2\pi]$. Les techniques utilisées sur le m.b.f. permettent de la même manière d'obtenir la dimension de graphe de ce processus, qui vaut donc $2 - \alpha$.

La dimension fractale du graphe du $\mathscr{H}_0$-m.b.m. a également été obtenue, dans l'article de PELTIER et LÉVY VÉHEL [154] :

$$\dim_{\mathrm{H}} \mathrm{Gr}(X) = \dim_{\mathrm{B}} \mathrm{Gr}(X) = 2 - \min_{u\in\mathbf{R}_+} H(u) \quad \text{p.s.}$$

Ce dernier résultat reste valable sous une forme similaire lorsqu'on se restreint à n'importe quel sous-intervalle de temps $[a, b]$.

La géométrie fractale du mouvement brownien fractionnaire s'avère d'autant plus riche que l'on s'intéresse à son extension multidimensionnelle et multiparamètre. Pour tout $H \in (0,1)$, on définit ainsi le processus $\{X_t = (X_t^1, \dots, X_t^d) ; t \in \mathbf{R}_+^N\}$ à valeurs dans $\mathbf{R}^d$ comme l'unique processus Gaussien centré de covariance :

$$\forall 1 \leq i, j \leq d, \ \ \forall s, t \in \mathbf{R}_+^N; \quad \mathbb{E}\left[X_s^i X_t^j\right] = \frac{\delta_{ij}}{2}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right].$$

Ce processus est généralement appelé le *mouvement brownien fractionnaire de Lévy*, et coïncide donc avec le m.b.f. lorsque $N = 1$. Notons que d'autre généralisations multiparamètres du m.b.f. ont été définies : le *drap brownien fractionnaire* introduit par KAMONT [98], étendant le drap brownien classique, et le *mouvement brownien fractionnaire multiparamètre* dérivé de la théorie des processus indicés par des ensembles (voir les travaux de HERBIN et MERZBACH [79, 81]).

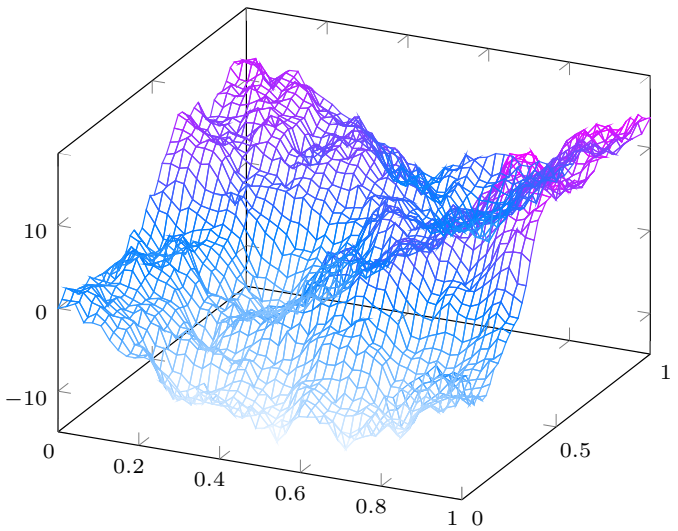


FIGURE 5 – Mouvement brownien fractionnaire multiparamètre

Une large littérature existe sur les propriétés fractales du m.b.f. multiparamètre, en commençant par l'étude du mouvement brownien de Lévy ($H = \frac{1}{2}$) dans [201, 151], pour ensuite s'étendre à des processus gaussiens plus généraux [1, 45, 192, 9]. De manière identique au m.b.f., les coefficients ponctuel et local de X valent H avec probabilité 1 et la dimension fractale des trajectoires du m.b.f. multiparamètre est égale à :

$$\dim_{\mathrm{H}} \mathrm{Gr}(X) = \dim_{\mathrm{B}} \mathrm{Gr}(X) = \min\left\{\frac{N}{H}, N + d(1-H)\right\}. \tag{27}$$

On observe ainsi une transition dans la géométrie fractale des trajectoires, suivant que le processus est récurrent ($N > Hd$) ou transient ($N < Hd$) dans $\mathbf{R}^d$.

L'étude des propriétés géométriques du m.b.f. peut être affinée en s'intéressant à la dimension fractale de l'image d'ensembles par le m.b.f.. Plus précisément, si F est un ensemble borélien de $\mathbf{R}_+^N$, on souhaite caractériser la dimension de $X(F)$ directement en fonction des propriétés géométriques de F. KAHANE [97] a présenté cette analyse pour le m.b.f., montrant

notamment :

$$\dim_{\mathrm{H}} X(F) = \min\left\{ d, \frac{1}{H}\dim_{\mathrm{H}} F \right\} \tag{28}$$

On s'aperçoit donc que la dimension de Hausdorff de $X(F)$ est directement reliée à celle de F. Le résultat ne s'avère pas aussi simple et condensé lorsqu'on considère la dimension de packing [178, 193, 64] ou le drap fractionnaire anisotrope [189, 198, 171, 197].

Le résultat précédent fournit la dimension de Hausdorff pour un ensemble F fixé au préalable. Similairement à la régularité hölderienne, il est naturel de se demander si un résultat uniforme existe, c.-à-d. avec probabilité 1 pour tous les ensembles boréliens de $\mathbf{R}^d$. Il s'avère que la réponse fait apparaître une dichotomie similaire à celle présente pour la dimension du graphe.

(i) Si $N \leq Hd$, alors avec probabilité 1 :

$$\text{pour tout borélien } F \in \mathscr{B}(\mathbf{R}^d), \quad \dim_{\mathrm{H}} X(F) = \frac{1}{H}\dim_{\mathrm{H}} F. \tag{29}$$

Ce résultat uniforme a été présenté dans différents travaux [141, 103, 189, 190].

(ii) Lorsque $N > Hd$, une version plus faible a été mise en évidence par KAUFMAN [99], et ensuite généralisée dans [103, 190]. Ainsi, avec probabilité 1 :

$$\text{pour tout } F \in \mathscr{B}(\mathbf{R}^d) \text{ et presque tout } t \in \mathbf{R}_+^N, \quad \dim_{\mathrm{H}} X(F+t) = \frac{1}{H}\dim_{\mathrm{H}} F. \tag{30}$$

L'existence d'un énoncé affaibli lorsque $N > Hd$ est étroitement lié aux propriétés géométriques des ensembles de niveau $X^{-1}\{x\}$, $x \in \mathbf{R}^d$, du m.b.f. multiparamètre. La littérature sur le sujet est également vaste, en commençant par les travaux sur le mouvement brownien de TAYLOR [180, 179], par la suite généralisés par ADLER [2], KAHANE [97], MONRAD et PITT [141], XIAO [192] notamment. Ainsi, lorsque $N > Hd$, on obtient avec probabilité 1 :

$$\text{pour tout ensemble fermé } E \subset \mathbf{R}^d, \quad \dim_{\mathrm{H}} X^{-1}(E) = N - Hd + H\dim_{\mathrm{H}} E. \tag{31}$$

En considérant simplement l'ensemble de niveau $F = X^{-1}\{0\}$, on s'aperçoit donc que la propriété uniforme (29) ne peut être vérifiée lorsque $N > Hd$. Notons que les propriétés géométriques des images réciproques $X^{-1}(F)$ dérivent de l'analyse fine de la régularité hölderienne du temps local du m.b.f. (voir par exemple les développements [15, 20, 191]).

Quand bien même l'objectif de cette introduction n'est pas de décrire en détail les démonstrations, il est intéressant d'observer que les résultats présentés dans cette partie reposent en grande partie sur une propriété purement probabiliste vérifiée par le m.b.f.. Appelée *local nondeterminism* (LND), elle prend la forme suivante :

$$\forall t \in \mathbf{R}_+^N, \ \forall \rho \in (0, \rho_0); \quad \mathrm{Var}(X_t \,|\, X_s : \|s-t\| \geq \rho\,) \geq c\,\rho^{2H},$$

où c et ρ_0 sont des constantes positives. L'intérêt de cette inégalité est qu'en utilisant le caractère gaussien du processus X, elle permet d'obtenir un estimé suffisamment précis de la loi de X_t connaissant X en dehors du voisinage $B(t,\rho)$, qui est l'élément clé pour ensuite analyser précisément le comportement des accroissements de X. D'une certaine manière, cet argument sur le m.b.f. remplace l'indépendance des accroissements lors de l'étude du mouvement brownien.

Initialement introduite par BERMAN [30] pour l'étude du temps du local de processus gaussiens, elle s'est avérée par la suite être un élément central pour l'analyse des propriétés géométriques des trajectoires. Une présentation plus approfondie de la propriété LND et de son utilisation a été réalisée par GEMAN et HOROWITZ [72], et plus récemment par XIAO [194, 195, 197].

0.2.2 Analyse multifractale et processus de Lévy

L'analyse multifractale est une branche en géométrie fractale qui cherche à combiner harmonieusement le concept de dimension fractale et les idées de régularité hölderienne présentées dans la partie précédente. On parle généralement de comportement *multifractal* pour une fonction f, ou également une mesure, lorsque la fonction $t \mapsto \alpha_{f,t}$ est très irrégulière, c.-à-d. au minimum discontinue. L'analyse multifractale a donc pour but de mettre en évidence une structure sous-jacente, malgré cet aspect à première vue chaotique du comportement hölderien.

Elle fut développée en premier lieu afin de caractériser la géométrie fractale de mesures. L'idée apparaît ainsi dans les travaux de MANDELBROT [124] à travers l'introduction des cascades multiplicatives aléatoires. Le formalisme multifractal à proprement parler a été développé par FRISCH et PARISI [71] et HALSEY et al. [74] pour les mesures dans le cadre de l'étude de la turbulence de fluides, et ensuite étendu aux fonctions par JAFFARD [91, 90]. L'outil principal intervenant en analyse multifractale est le *spectre multifractal*, et plus particulièrement le spectre fin pour ce qui nous intéresse.

Définition 0.6 **(Spectre multifractal)** *Soit f une fonction. On appelle* spectre multifractal *de f l'application $h \mapsto d_f(h)$ définie comme :*

$$\forall h \in \mathbf{R}_+; \quad d_f(h) = \dim_{\mathrm{H}} E_h \quad \text{où} \quad E_h = \{t : \alpha_{f,t} = h\}. \tag{32}$$

Le spectre caractérise donc la géométrie fractale des ensembles de niveau E_h du coefficient ponctuel, permettant ainsi d'avoir un aperçu global de la "fréquence" d'apparition d'un certain comportement hölderien. L'analyse multifractale s'intéresse donc à la quantification des phénomènes rares apparaissant sur les trajectoires, étant donné que E_h s'avère souvent être un ensemble négligeable. Le spectre est dit *dégénéré* lorsque $\alpha_{f,t}$ est constant et égal à un certain α. Dans ce cas, nous obtenons simplement $d_f(\alpha) = 1$ et $d_f(h) = -\infty$ pour tout $h \neq \alpha$.

Afin de faciliter l'étude de cette notion, un formalisme multifractal a été développé, permettant alors sous certaines conditions d'exprimer le spectre comme une transformée de Legendre plus facile à obtenir. N'utilisant dans ce manuscrit que la définition originelle du spectre, nous ne présentons pas plus en détail ces éléments, le lecteur pouvant se référer à l'importante littérature sur le sujet pour plus de précisions (voir par exemple les ouvrages [68, 69, 75]).

Comme indiqué précédemment, le spectre multifractal s'avère un outil intéressant lorsque la fonction $t \mapsto \alpha_{f,t}$ est chaotique. Les processus stochastiques présentés jusqu'à maintenant ne rentrent pas dans cette classe, le mouvement brownien fractionnaire ayant une régularité ponctuelle constante et celle du m.b.m. étant paramétrée par une fonction continue.

A l'inverse, JAFFARD [94] a montré que le spectre multifractal s'avère être le cadre pertinent pour caractériser le comportement hölderien des *processus de Lévy*. Rappelons que X est un processus de Lévy à valeurs dans $\mathbf{R}^d$ si et seulement si

(i) $X_0 = 0$ presque sûrement;

(ii) X est à accroissements stationnaires;

(iii) X est à accroissements indépendants;

(iv) les trajectoires de X sont p.s. càdlàgs.

Le mouvement brownien ainsi les processus de Poisson sont donc des processus de Lévy classiques. La loi des processus de Lévy est entièrement déterminée par la formule de Lévy–Khintchine (voir les références classiques sur le sujet [31, 168, 113]) : pour tout $t \in \mathbf{R}_+$ et tout $\lambda \in \mathbf{R}^d$, $\mathbb{E}[e^{i\langle\lambda, X_t\rangle}] = e^{t\psi(\lambda)}$, où la fonction ψ prend la forme suivante :

$$\forall \lambda \in \mathbf{R}^d; \quad \psi(\lambda) = i\langle a, \lambda\rangle - \frac{1}{2}\langle \lambda, Q\lambda\rangle + \int_{\mathbf{R}^d} \left(e^{i\langle\lambda,x\rangle} - 1 - i\langle\lambda, x\rangle \mathbf{1}_{\{\|x\|\leq 1\}}\right)\pi(\mathrm{d}x).$$

Dans l'expression précédente, Q est une matrice définie positive et π une mesure de Lévy, c.-à-d. une mesure de Radon sur $\mathbf{R}^d \setminus \{0\}$ telle que $\int_{\mathbf{R}^d} (1 \wedge \|x\|^2)\, \pi(\mathrm{d}x) < \infty$. Notons dans la formule de Lévy–Khintchine, le coefficient a correspond à un simple drift, la matrice Q représente la composante brownienne et la mesure π caractérise la partie à sauts dans le processus de Lévy.

Les *processus α-stable* constituent un autre classique exemple de processus de Lévy, vérifiant en dimension $d = 1$: $a = 0, Q = 0$ et

$$\pi(\mathrm{d}x) = \begin{cases} c_1 |x|^{-1-\alpha}\, \mathrm{d}x & \text{sur} \quad (0, \infty) \\ c_2 |x|^{-1-\alpha}\, \mathrm{d}x & \text{sur} \quad (-\infty, 0), \end{cases}$$

où $\alpha \in (0, 2)$ et $c_1 = c_2$ lorsque $\alpha = 1$. Les processus α-stable sont donc purement à sauts et avec une intensité paramétrée par α. La Figure 6 permet de se rendre compte de l'aspect de ses trajectoires, en notant e particulier l'amplitude fluctuante des sauts qui la compose.

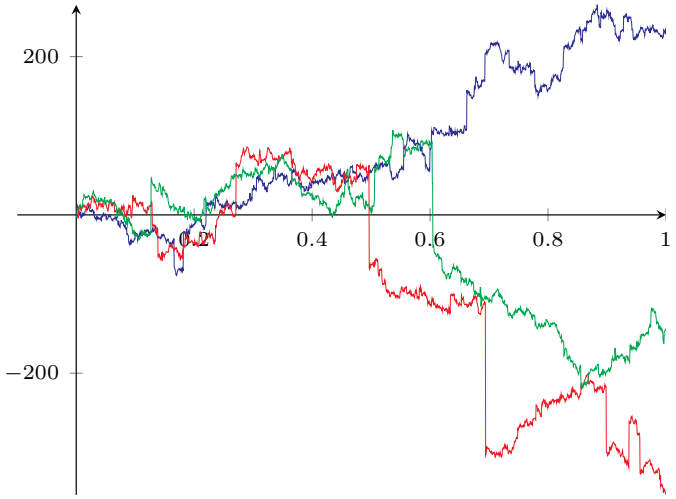


FIGURE 6 – Trajectoires de processus de Lévy alpha-stable ($\alpha = 1.5$)

Afin de simplifier notre présentation, nous supposons dans le reste de cette partie que $Q = 0$, c.-à-d. le processus X n'a pas de composante brownienne. Dans le cadre de l'étude de la régularité hölderienne des processus de Lévy, il s'avère judicieux d'introduire un exposant, appelé *exposant de Blumenthal–Getoor* [34] et défini comme :

$$\beta = \inf\left\{ \delta \geq 0 : \int_{\mathbf{R}^d} \left(1 \wedge \|x\|^\delta\right) \pi(\mathrm{d}x) < \infty \right\} \tag{33}$$

D'après la définition d'une mesure de Lévy, nous savons que $\beta \in [0, 2]$.

Concernant la régularité hölderienne des processus de Lévy, on s'aperçoit facilement que du fait de la densité des sauts, $\widetilde{\alpha}_{X,t} = 0$ pour tout $t \in \mathbf{R}_+$. Pour ce qui du coefficient ponctuel, PRUITT [160] a démontré que pour tout $t \in \mathbf{R}_+$ fixé, $\alpha_{X,t} = 1/\beta$ presque sûrement. Néanmoins, à l'inverse du mouvement brownien, il s'avère ici impossible d'avoir un résultat hölderien uniforme sur toute la trajectoire, justifiant l'emploi par JAFFARD [94] de l'analyse multifractale pour l'étude des processus de Lévy.

Théorème 0.1 *Soit X un processus de Lévy sans composante brownienne et tel que $\beta > 0$. Alors, avec probabilité* 1*, son spectre multifractal est égal à :*

$$\forall h \in \mathbf{R}_+; \quad d_X(h) = \begin{cases} \beta h & \textit{si } h \in [0, 1/\beta] \\ -\infty & \textit{sinon.} \end{cases}$$

La régularité ponctuelle d'un processus de Lévy fluctue donc dans l'intervalle $[0, 1/\beta]$, ce qui est cohérent avec le résultat de PRUITT [160] et la densité des sauts sur $\mathbf{R}$. Le spectre multifractal permet alors de caractériser finement la géométrie des phénomènes rares que sont les points où $\alpha_{X,t} < 1/\beta$.

Notons que d'autres propriétés fractales plus classiques ont également été étudiées sur les processus de Lévy : dimension du graphe [34, 140, 38] et images d'ensemble fractals [107] entre autres. Signalons également que RIEDI [162] a réalisé une étude approfondie et détaillée d'un certain nombre de processus stochastiques affichant une géométrie multifractale.

L'adjectif multifractal s'avère désigner dans la littérature probabiliste moderne une classe plus large de comportements fractals qui, sans directement faire appel au formalisme multifractal, affichent des similarités avec celui-ci. Dans le cas du mouvement brownien fractionnaire, nous avons précédemment remarqué que le spectre multifractal n'est pas un outil pertinent pour son étude. Néanmoins, il demeure possible de s'intéresser de près aux variations de la correction logarithmique apparaissant dans la régularité ponctuelle. Plus précisément, nous remarquons que le terme correctif présent dans la loi du logarithme itéré (18) n'est pas le même que celui se manifestant dans le module de continuité (17). Cette observation laisse donc à penser que la régularité hölderienne fine n'est pas la même tout le long des trajectoires du m.b.f..

Une façon de caractériser cette subtile variation est d'étudier les *points rapides* du m.b.f., définis comme il suit :

$$\forall \lambda > 0; \quad F_H(\lambda) = \left\{ t \in \mathbf{R}_+ : \limsup_{h \to 0} \frac{|B^H_{t+h} - B^H_t|}{h^H \sqrt{2\log(1/h)}} \geq \lambda \right\}.$$

L'ensemble $F_H(\lambda)$ regroupe donc les points ayant un comportement hölderien de l'ordre du module de continuité, donc plus rapide que la loi du logarithme itéré, mais pour une certaine constante λ. La géométrie de ces ensembles a tout d'abord été caractérisée dans le cas le mouvement brownien par OREY et TAYLOR [152], ce résultat ayant été récemment étendu au m.b.f. par KHOSHNEVISAN et SHI [102].

Théorème 0.2 **(Points rapides du m.b.f.)** *Soit $(B^H_t)_{t \in \mathbf{R}_+}$ un mouvement brownien fractionnaire, $H \in (0,1)$. Les points rapides de B^H vérifient alors :*

$$\forall \lambda > 0; \quad \dim_{\mathrm{H}} F_H(\lambda) = 1 - \lambda^2 \quad p.s.$$

Ainsi, quand bien même ce théorème ne détermine pas un spectre multifractal stricto sensu, on observe clairement une similitude dans la forme avec l'étude réalisée sur les processus de Lévy. Notons finalement que de manière analogue, PERKINS [157], DAVIS et PERKINS [50] et BASS et BURDZY [23] ont décrit la géométrie fractale des *points lents* du mouvement brownien, c.-à-d. les temps t où localement $|B_u - B_t| \leq c|u-t|^{1/2}$ pour une certaine constante $c > 0$.

0.3 Analyse 2-microlocale

Une composante importante de cette thèse est l'utilisation de *l'analyse 2-microlocale* pour l'étude et la caractérisation de la régularité de processus stochastiques. Cet outil a été initialement introduit par BONY [35] dans le but d'analyser les singularités de solutions d'équations aux dérivées partielles. Il s'est avéré être par la suite un outil intéressant dans un cadre bien plus large d'étude de la régularité, déterministe comme stochastique.

0.3.1 Espaces 2-microlocaux

L'analyse 2-microlocale repose sur l'introduction des *espaces 2-microlocaux* pour les distributions tempérées $f \in \mathscr{S}'(\mathbf{R})$. La première caractérisation de ces espaces fonctionnels est donc

due à BONY [35] et repose sur la décomposition de Littlewood–Paley. Plus précisément, soit $\phi \in \mathscr{S}(\mathbf{R})$ une fonction à décroissance rapide telle que sa transformée de Fourier $\widehat{\phi}$ vérifie :

$$\widehat{\phi}(\xi) = \begin{cases} 1 & \text{si} \quad |\xi| \leq 1/2; \\ 0 & \text{si} \quad |\xi| \geq 1. \end{cases}$$

Pour tout $j \in \mathbf{N}$, définissons $\phi_j(x) = 2^j \phi(2^j x)$, $\psi_j = \phi_{j+1} - \phi_j$, $S_0 f = \phi * f$ et $\Delta_j f = \psi_j * f$. Il est alors connu que la distribution f peut se décomposer sous la forme suivante, dans l'espace des distributions tempérées :

$$f = S_0 f + \sum_{j=0}^{\infty} \Delta_j f.$$

Notons que les éléments de la décomposition sont des fonctions appartenant à $L^2(\mathbf{R})$. De plus amples détails sur cette décomposition peuvent être trouvés dans l'ouvrage de MEYER [138]. La définition des espaces 2-microlocaux *globaux* est alors étroitement liée aux propriétés des fonctions apparaissant dans la décomposition de Littlewood–Paley.

Définition 0.7 (Espaces 2-microlocaux globaux) *Soit $t \in \mathbf{R}$ et $(s', \sigma) \in \mathbf{R}^2$. Une distribution tempérée f appartient à* l'espace 2-microlocal global $C^{\sigma,s'}_{t,global}$ *si et seulement si il existe une constante $C > 0$ telle que pour tout $u \in \mathbf{R}$*

$$|S_0 f(u)| \leq C\big(1 + |u-t|\big)^{-s'} \quad \textit{et} \quad |\Delta_j f(x)| \leq C\, 2^{-j\sigma}\big(2^{-j} + |u-t|\big)^{-s'},$$

pour tout $j \in \mathbf{N}$.

Notons qu'il a été montré (voir par exemple [138]) que la caractérisation précédente des espaces 2-microlocaux globaux est indépendante de la fonction ϕ choisie.

Comme présenté par ECHELARD [60], la décomposition de Littlewood–Paley peut être interprétée comme l'application de filtres passe-bande ψ_j sur la distribution f, sélectionnant ainsi les fréquences comprises entre 2^{j-1} et 2^{j+1}. L'appartenance à un espace 2-microlocal $C^{\sigma,s'}_{t,global}$ impose donc une certaine vitesse de décroissance aux composantes hautes fréquences de la distribution f. De plus, nous observons que la majoration $2^{-j\sigma}(2^{-j} + |u-t|)^{-s'}$ dépend de la distance $|u-t|$ à la puissance $-s'$, impliquant donc également une forme de localisation temporelle. Ainsi, on s'aperçoit donc que l'appartenance à un espace 2-microlocal décrit un localisation temporelle et fréquentielle de la distribution f.

La définition précédente a l'inconvénient de dépendre du comportement de la distribution f en l'infini, excluant de ce fait un certain nombre de fonctions très régulières mais divergeant trop rapidement (par exemple $t \mapsto \exp(t^2)$). En conséquence, on définit véritablement les *espaces 2-microlocaux* sur les distributions $\mathscr{D}'(\mathbf{R})$ de la manière suivante.

Définition 0.8 (Espaces 2-microlocaux) *Soit $f \in \mathscr{D}'(V)$ une distribution, où V est un ouvert, et $t \in V$. On dit alors que f appartient à l'espace 2-microlocal $C^{\sigma,s'}_t$ s'il existe une distribution tempérée $g \in C^{\sigma,s'}_{t,global}$ et un ouvert U tels que*

$$f = g \ \textit{ sur } U, \textit{ où } \ t \in U \subset V.$$

D'après la définition précédente, les espaces 2-microlocaux sont donc une notion purement localisée autour du point d'étude. Le théorème qui suit, obtenu par MEYER [138], permet d'avoir une caractérisation plus condensée de ces espaces fonctionnels.

Théorème 0.3 *Soit $f \in \mathscr{D}'(V)$ une distribution, où V est un ouvert, et $t \in V$. Alors f appartient à l'espace 2-microlocal $C^{\sigma,s'}_t$ si et seulement si il existe un voisinage $U \subset V$ de t, une fonction $g \in C^{\infty}_0(\mathbf{R})$ et $C > 0$ tels que*

$$\forall j \in \mathbf{N},\ \forall x \in U; \quad |\Delta_j(gf)(x)| \leq C\, 2^{-j\sigma}\big(2^{-j} + |u-t|\big)^{-s'},$$

où g vérifie

$$\forall x \in \mathbf{R}; \quad g(x) = \begin{cases} 1 & si \quad x \in U; \\ 0 & si \quad x \notin V. \end{cases}$$

Cette localisation des espaces 2-microlocaux implique notamment que les fonctions infiniment dérivables appartiennent à n'importe quel espace $C_t^{\sigma,s'}$. En particulier, l'ajout d'une fonction polynomiale n'influence donc pas l'appartenance, ou non, à un espace 2-microlocal.

Notons que dans ce manuscrit, nous utilisons la paramétrisation (s', σ) pour les espaces 2-microlocaux. La convention en la matière peut légèrement différer suivant les publications et ouvrages (voir par exemple les travaux de JAFFARD [93], JAFFARD et MEYER [96], MEYER [138], LÉVY VÉHEL et SEURET [119] où apparaît le couple (s', s), où $s := \sigma - s'$).

Un aspect important et intéressant de l'analyse 2-microlocale réside dans la stabilité des espaces sous l'action d'opérateurs intégro-différentiels. Plus précisément, pour toute distribution $f \in \mathscr{D}'(V)$ et tout $t \in \mathbf{R}$:

$$\forall n \in \mathbf{N}; \quad f \in C_t^{\sigma,s'} \Longleftrightarrow f^{(n)} \in C_t^{\sigma-n,s'}. \tag{34}$$

Ainsi, l'appartenance des dérivées de f aux espaces 2-microlocaux se déduit directement de l'étude des propriétés 2-microlocales de f.

En réalité, cette caractéristique des espaces 2-microlocaux peut s'étendre à des opérateurs pseudo-différentiels telle que l'intégrale fractionnaire. L'intégration fractionnaire est un sujet très vaste, comme peut en témoigner l'ouvrage de référence de SAMKO, KILBAS et MARICHEV [166], et nous n'introduisons donc ici que les concepts et résultats de base. Pour commencer, si h est une fonction localement intégrable, on peut définir *l'intégrale fractionnaire de Riemann–Liouville* d'ordre $\alpha > 0$ de h de la façon suivante :

$$\forall t \in \mathbf{R}; \quad (I_{b\pm}^{\alpha} h)(t) = \frac{1}{\Gamma(\alpha)} \int_b^t (t-u)_{\pm}^{\alpha-1} h(u)\,\mathrm{d}u, \tag{35}$$

où b est un point fixé, $\Gamma(\cdot)$ désigne la fonction gamma, $x_+ := \max(x, 0)$ et $x_- := \max(-x, 0)$. Le réel α correspond à l'ordre d'intégration qui, lorsqu'il est entier, redonne bien une intégrale itérée classique. L'opérateur d'intégration fractionnaire ainsi défini vérifie une propriété de semi-groupe, c.-à-d.

$$\forall \alpha, \beta \geq 0; \quad I_{b\pm}^{\alpha} I_{b\pm}^{\beta} h = I_{b\pm}^{\alpha+\beta} h.$$

Le concept de dérivée fractionnaire peut être introduit de manière similaire. Pour tout $\alpha \in (0, 1)$, l'opérateur $D_{b\pm}^{\alpha}$ est défini comme :

$$\forall t \in \mathbf{R}; \quad (D_{b\pm}^{\alpha} h)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_b^t (t-u)_{\pm}^{-\alpha} h(u)\,\mathrm{d}u, \tag{36}$$

sous réserve d'existence de la dérivée (SAMKO, KILBAS et MARICHEV [166] présentent plusieurs conditions suffisantes portant sur h). Notons que d'après la formule précédente, une dérivée fractionnaire d'ordre α se définit donc comme la dérivée classique de l'intégrale fractionnaire d'ordre $1 - \alpha$. L'opérateur $D_{b\pm}^{\alpha}$ peut ensuite être étendu à tout ordre en utilisant cette même construction et les dérivées successives. Dans la suite, nous utiliserons la notation $I_{b\pm}^{\alpha} h$ avec $\alpha < 0$ pour désigner une dérivée fractionnaire d'ordre $-\alpha$.

Lorsque la fonction h est intégrable sur $\mathbf{R}$, l'intégrale fractionnaire peut être alors définie sur tout l'axe réel :

$$\forall t \in \mathbf{R}; \quad (I_{\pm}^{\alpha} h)(t) = \frac{1}{\Gamma(\alpha)} \int_{\mathbf{R}} (t-u)_{\pm}^{\alpha-1} h(u)\,\mathrm{d}u, \tag{37}$$

La propriété de semi-groupe est toujours satisfaite pour cette définition et la dérivée fractionnaire sur $\mathbf{R}$ peut également être introduite de manière similaire. On s'observe alors que l'intégrale fractionnaire peut être vue comme un produit de convolution, qui s'exprime alors de manière très simple dans l'espace de Fourier :

$$\widehat{I_{\pm}^{\alpha}h}(x) = (\mp ix)^{-\alpha}\widehat{h}(x), \tag{38}$$

pour $\alpha \in (-\infty, 1)$.

La généralisation de l'opérateur $I_{\pm}^{\alpha}$ aux distributions tempérées s'avère légèrement plus complexe, principalement à cause de la non-stabilité de l'espace Schwartz par intégration fractionnaire. Une façon de contourner ce problème consiste donc à introduire un nouvel espace de distribution, appelé *espace de Lizorkin*, qui reste étroitement lié à $\mathscr{S}'(\mathbf{R})$. Commençons par définir l'espace des fonctions test associées :

$$\Phi(\mathbf{R}) = \left\{\varphi \in \mathscr{S}(\mathbf{R}) : \int_{\mathbf{R}} u^k \varphi(u)\,\mathrm{d}u = 0, \ \ \forall k \in \mathbf{N}\right\}.$$

Alternativement, il est possible d'introduire $\Phi(\mathbf{R})$ comme l'espace des fonctions $\varphi \in \mathscr{S}(\mathbf{R})$ dont les dérivées successives de la transformée de Fourier $\widehat{\varphi}$ sont nulles en zéro. Grâce à cette propriété, l'intégrale fractionnaire d'une fonction $\varphi \in \Phi(\mathbf{R})$ est bien définie pour tout $\alpha \in \mathbf{R}$, via la caractérisation par la transformée de Fourier (Équation (38)).

L'espace des distributions de Lizorkin $\Phi'(\mathbf{R})$ est alors simplement défini comme le dual topologique de $\Phi(\mathbf{R})$. Comme observé par SAMKO, KILBAS et MARICHEV [166], il coïncide avec l'espace quotient

$$\Phi'(\mathbf{R}) = \mathscr{S}'(\mathbf{R})/\Phi_0'(\mathbf{R}),$$

où $\Phi_0'(\mathbf{R}) = \{f \in \mathscr{S}'(\mathbf{R}) : \forall \varphi \in \Phi(\mathbf{R}),\ \langle f, \varphi\rangle = 0\}$. D'après la définition de $\Phi(\mathbf{R})$, $\Phi_0'(\mathbf{R})$ correspond simplement à l'ensemble des fonctions polynomiales sur $\mathbf{R}$. Ainsi, en simplifiant notre propos, les distributions de Lizorkin s'apparentent ainsi aux distributions tempérées auxquelles est soustraite la composante polynomiale. L'intégrale fractionnaire sur $\Phi'(\mathbf{R})$ est alors définie de manière classique en utilisant l'espace dual :

$$\forall \alpha \in \mathbf{R}; \quad \left\langle I_{\pm}^{\alpha} f, \varphi\right\rangle := \left\langle f, I_{\mp}^{\alpha}\varphi\right\rangle$$

où $f \in \Phi'(\mathbf{R})$ et $\varphi \in \Phi(\mathbf{R})$. Cette définition coïncide bien avec la formule classique (37), avec $\alpha \in (0, 1)$ dans le cas des fonctions intégrables. SAMKO, KILBAS et MARICHEV [166] ont réalisé un examen bien plus complet et détaillé des différentes approches existantes pour définir le calcul fractionnaire.

Grâce à cette généralisation de l'intégrale fractionnaire aux distributions, la propriété fondamentale de stabilité des espaces 2-microlocaux peut être étendue au cadre fractionnaire.

Théorème 0.4 *Soit f une distribution de Lizorkin. Alors $t \in \mathbf{R}$, f vérifie*

$$\forall \alpha \in \mathbf{R}; \quad f \in C_t^{\sigma,s'} \Longleftrightarrow I_{\pm}^{\alpha} f \in C_t^{\sigma+\alpha,s'}. \tag{39}$$

Notons qu'il n'est pas contradictoire avec la définition des espaces 2-microlocaux de considérer la frontière de la classe d'équivalence dans l'espace de Lizorkin, étant donné que nous avons remarqué précédemment qu'une composante polynomiale ne modifie pas l'appartenance à un espace 2-microlocal. Cette propriété des espaces 2-microlocaux a été étendue par JAFFARD [93] à une classe plus générale d'opérateurs différentiels vérifiant certaines conditions.

Si les espaces 2-microlocaux sont un outil puissant, leur définition originelle (0.8) n'est pas des plus aisées à manipuler. Afin de pallier ce défaut et faciliter leur étude, d'autres caractérisations ont émergé dans la littérature.

JAFFARD [93] a ainsi obtenu une caractérisation des espaces 2-microlocaux à partir de la décomposition d'une distribution dans une base d'ondelettes. Plus précisément, considérons une fonction à décroissance rapide $\psi \in \mathscr{S}(\mathbf{R})$ telle que $\{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)\}$ est une base orthonormale de $L^2(\mathbf{R})$ (des exemples de telles fonctions sont donnés par MEYER [136, 137, 139]). Les coefficients d'ondelettes de $f \in \mathscr{S}'(\mathbf{R})$ sont alors définis comme :

$$\forall j, k \in \mathbf{Z}; \quad d_{j,k} = 2^j \langle f, \psi(2^j x - k)\rangle. \tag{40}$$

La constante 2^j peut être remplacée par $2^{j/2}$ si l'on souhaite que les coefficients soient correctement renormalisés.

Théorème 0.5 *Soit $f \in \mathscr{S}'(\mathbf{R})$ une distribution tempérée, $(\psi_{j,k})_{j,k\in\mathbf{Z}}$ une base d'ondelettes et $(s', \sigma) \in \mathbf{R}^2$. Supposons que les N premiers moments de ψ sont nuls, avec $N > \max(\sigma - s', \sigma)$.*

Alors, pour tout $t \in \mathbf{R}$, f appartient à l'espace 2-microlocal $C_t^{\sigma,s'}$ si et seulement si il existe $C > 0$ tel que

$$|d_{j,k}| \leq C 2^{-j\sigma} \left(2^{-j} + |t - k2^{-j}|\right)^{-s'},$$

pour tous $j, k \in \mathbf{Z}$ satisfaisant $|t - k2^{-j}| \leq 1$.

Une caractérisation similaire utilisant la transformée en ondelette continue a également été présentée par JAFFARD [93]. Notons que l'émergence d'une définition des espaces 2-microlocaux à l'aide des ondelettes est assez naturelle, étant donné que nous avons remarqué précédemment que l'appartenance à ces derniers caractérise une localisation à la fois temporelle et fréquentielle.

Dans les chapitres suivants, nous utiliserons principalement la définition temporelle des espaces 2-microlocaux pour les fonctions classiques. Celle-ci fut d'abord mise en évidence par KOLWANKAR et LÉVY VÉHEL [109] pour les coefficients $(s', s) \in (-1, 0) \times (0, 1)$, avant d'être ensuite étendue à presque tout le domaine 2-microlocal dans les travaux de SEURET et LÉVY VÉHEL [169] et ECHELARD [60]. Pour simplifier notre propos, commençons par énoncer cette caractérisation lorsque $\sigma \in (0, 1)$.

Théorème 0.6 *Soit f une fonction, $\sigma \in (0, 1)$ et $s' \in \mathbf{R}$ tels que $s \notin \mathbf{N}$.*
Alors f appartient à l'espace 2-microlocal $C_t^{\sigma,s'}$, $t \in \mathbf{R}$, si et seulement si il existe $C > 0$, $\rho > 0$ et un polynôme P_t tels que :

$$|f(u) - P_t(u) - f(v) + P_t(v)| \leq C|u - v|^{\sigma} \left(|u - t| + |v - t|\right)^{-s'}, \tag{41}$$

pour tous $u, v \in B(t, \rho)$. Le polynôme P_t est unique s'il est supposé de degré inférieur à $\lfloor s \rfloor$, et correspond alors au développement de Taylor de f d'ordre $\lfloor s \rfloor$ en t.

Le résultat précédent fournit donc une caractérisation temporelle des espaces 2-microlocaux en basant simplement sur les variations de la fonction f. Afin de l'étendre à tout $\sigma \in \mathbf{R} \setminus \mathbf{N}$, nous utilisons le Théorème (0.4) indiquant qu'il suffit d'étudier les accroissements des dérivées et intégrales successives de f. Ainsi, f appartient à $C_t^{\sigma,s'}$, avec $s \notin \mathbf{N}$ et $\sigma \notin \mathbf{Z}$ si et seulement si pour tout $u, v \in B(t, \rho)$:

$$\left|I_{\pm}^m f(u) - P_m(u) - I_{\pm}^m f(v) + P_m(v)\right| \leq C|u - v|^{\sigma+m} \left(|u - t| + |v - t|\right)^{-s'}, \tag{42}$$

où $m \in \mathbf{Z}$ tel que $-m < \sigma < -m + 1$ et P_m est un polynôme. Notons que cette idée de soustraire une composante polynomiale à la fonction apparaît également dans la définition utilisant la projection dans une base d'ondelettes. En effet, dans cette dernière, les N premiers moments de l'ondelette ψ sont supposés nuls, avec $N > \max(\sigma - s', \sigma)$, indiquant donc que toute composante polynomiale est ignorée dans les coefficients $(d_{j,k})_{j,k\in\mathbf{Z}^2}$.

Il pourrait s'avérer tentant d'utiliser la caractérisation précédente (3.4) lorsque $\sigma \in \mathbf{Z}$. En effet, on remarque en particulier que pour $\sigma = 0$, l'Équation (3.4) correspond parfaitement

à la Définition 0.1 des espaces de Hölder ponctuels. Malheureusement, si pour tout $\alpha > 0$, l'inclusion $C_t^\alpha \subset C_t^{0,-\alpha}$ est satisfaite, l'égalité ensembliste n'est pas vérifiée en général. JAFFARD et MEYER [96] et MEYER [138] ont ainsi construit un exemple de distribution appartenant à $C_t^{0,-\alpha}$ mais non C_t^α. Ce dernier montre cependant que toute fonction f vérifie :

$$\text{si} \quad \omega_f(h) = \mathrm{O}(\log h) \quad \text{alors} \quad f \in C_t^\alpha \Longleftrightarrow f \in C_t^{0,-\alpha}, \tag{43}$$

où ω_f désigne le module de continuité de f. Notons que ce déficit de caractérisation lorsque $\sigma \in \mathbf{Z}$ n'est cependant pas gênant dans la perspective d'étudier la frontière 2-microlocale définie dans la suite.

0.3.2 Frontière 2-microlocale et processus stochastiques

De manière similaire à la régularité hölderienne, la définition des espaces 2-microlocaux induit naturellement l'introduction d'un outil, appelé *frontière 2-microlocale*, caractérisant la régularité locale d'une fonction f.

Définition 0.9 **(Frontière 2-microlocale)** *Soit f une distribution et $t \in \mathbf{R}$. La* frontière 2-microlocale *de f en t est définie comme la fonction suivante :*

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,t}(s') = \sup\{\sigma : f \in C_t^{\sigma,s'}\}. \tag{44}$$

Cette courbe correspond à l'enveloppe supérieure du domaine 2-microlocal $E(f,t) = \{(s',\sigma) : f \in C_t^{\sigma,s'}\}$.

La validité de la définition précédente repose sur des propriétés simples d'inclusion que vérifient les espaces 2-microlocaux. Notons que comme précédemment, la convention $s' \mapsto \sigma_{f,t}(s')$ pour la frontière 2-microlocale n'est pas la seule apparaissant dans la littérature ($s' \mapsto s(s')$ est par exemple utilisée dans [169, 119]). Toujours en se reposant sur les propriétés des espaces 2-microlocaux, on peut facilement vérifier que :

(i) $\sigma_{f,t}(\cdot)$ une fonction croissante et concave ;

(ii) $\sigma_{f,t}(\cdot)$ admet des dérivées à gauche et à droite comprises dans l'intervalle $[0,1]$.

La caractérisation de la régularité locale par une fonction permet d'apporter plus d'information que ne le peuvent de simples coefficients, permettant ainsi une analyse plus fine du comportement par rapport aux outils classiques en géométrie fractale.

Il s'avère ainsi que la frontière 2-microlocale englobe et étend les exposants de Hölder décrits dans la Section 0.1. Plus précisément, si f est une fonction continue, alors

- $\widetilde{\alpha}_{f,t} = \sigma_{f,t}(0)$;
- $\alpha_{f,t} = -\inf\{s' : \sigma_{f,t}(s') \geq 0\}$ sous l'hypothèse $\omega_f(h) = \mathrm{O}(\log h)$.

La frontière 2-microlocale peut également être reliée à d'autres exposants de régularité apparaissant dans la littérature.

Definition 0.10 **(Exposant de chirp)** *Soit $f \in L^1_{loc}(\mathbf{R})$ et $t \in \mathbf{R}$.* L'exposant de chirp *de f en t est défini comme*

$$\beta^c_{f,t} = \sup\Big\{\beta : \exists h \in \mathbf{R},\ \forall n \in \mathbf{N};\ I^n_\pm f \in C_t^{h+n(1+\beta)}\Big\}.$$

Definition 0.11 **(Exposant weak scaling)** *Soit $f \in L^1_{loc}(\mathbf{R})$ et $t \in \mathbf{R}$.* L'exposant "weak scaling" *de f en t est défini comme*

$$\beta^w_{f,t} = \sup\Big\{\beta : \exists n \in \mathbf{N};\ I^n_\pm f \in C_t^{n+\beta}\Big\}.$$

Ces deux exposants ont été introduits par MEYER [138] et s'intéressent donc au comportement oscillant de f après un grand nombre d'intégrations.

Definition 0.12 (Exposant d'oscillations) *Soit $f \in L^1_{loc}(\mathbf{R})$ et $t \in \mathbf{R}$.* L'exposant d'oscillations *de f en t est défini comme*

$$\beta^o_{f,t} = \left.\frac{\mathrm{d}\alpha_{I^\varepsilon_\pm f,t}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0_+} - 1,$$

Ce dernier exposant a été présenté par ARNEODO et al. [7] et caractérise donc les variations de f après intégration d'ordre infinitésimal.

On s'aperçoit que les trois exposants précédents s'intéressent au comportement hölderien de f après intégration, induisant naturellement une connexion avec la frontière 2-microlocale. Plus précisément, comme présenté par LÉVY VÉHEL et SEURET [119], les correspondances suivantes peuvent être obtenues :

$$\beta^c_{f,t} = \left\{ \left.\frac{\mathrm{d}\sigma_{f,t}}{\mathrm{d}s'}\right|_{s'=-\infty} \right\}^{-1} - 1;$$

$$\beta^w_{f,t} = \lim_{s'\to-\infty} \sigma_{f,t}(s') - s';$$

$$\beta^o_{f,t} = \left\{ \left.\frac{\mathrm{d}\sigma_{f,t}}{\mathrm{d}s'}\right|_{s'=-\alpha_{f,t-}} \right\}^{-1} - 1.$$

Ces différentes propriétés sont illustrées par la Figure 7 à travers un exemple de frontière 2-microlocale.

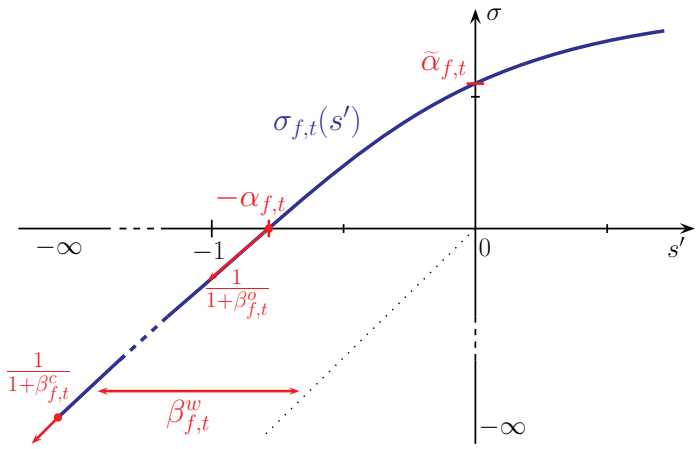


FIGURE 7 – Propriétés de la frontière 2-microlocale

L'intérêt principal de la frontière 2-microlocale par rapport aux exposants classiques réside dans sa prédictibilité sous intégration fractionnaire. Ainsi, grâce au Théorème 0.4 sur les espaces 2-microlocaux, on en déduit que

$$\forall \alpha \in \mathbf{R}; \quad \sigma_{I^\alpha_\pm f,t} = \sigma_{f,t} + \alpha. \tag{45}$$

La Figure 8a offre une illustration de cette propriété fondamentale. Elle permet également de se rendre facilement compte dans quel cas le coefficient ponctuel n'est pas stable après intégration fractionnaire.

Afin de donner une meilleure intuition de cet outil, analysons la frontière 2-microlocale des exemples introduits dans les parties précédentes : $x \mapsto |x|^\alpha$ et $f_c : x \mapsto |x|^\alpha \sin(|x|^{-\beta})$.

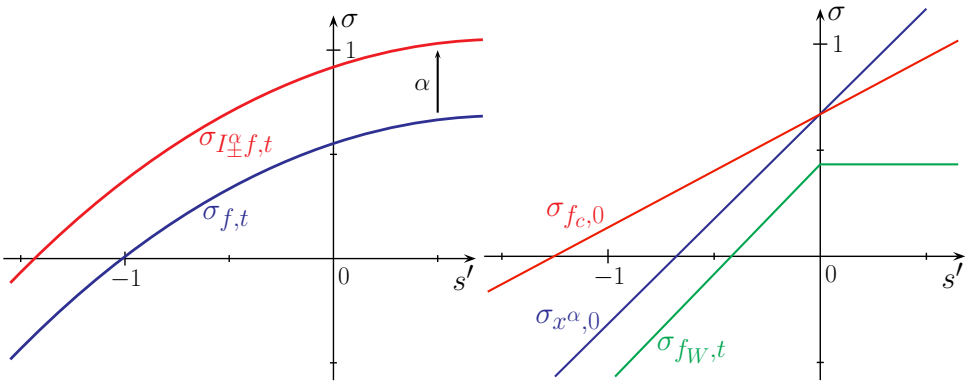
Nous obtenons alors :

$$\forall s' \in \mathbf{R}; \quad \sigma_{|x|^\alpha,0}(s') = \alpha + s' \quad \text{et} \quad \sigma_{f_c,0}(s') = \frac{\alpha + s'}{1+\beta}.$$

Le calcul précis a été présenté par LÉVY VÉHEL et SEURET [119] et ECHELARD [60]. Pour $t \neq 0$, la frontière est simplement égale à $+\infty$ du fait du caractère infiniment dérivable de ces deux fonctions. La frontière de la fonction de Weierstrass peut également être facilement déterminée :

$$\forall t \in \mathbf{R}, \ \forall s' \in \mathbf{R}; \quad \sigma_{f_W,t}(s') = (\alpha + s') \wedge \alpha.$$

Notons que les frontières ainsi obtenues sont cohérentes avec les exposants de Hölder présentés dans la Section 0.1. La Figure 8b donne une illustration de ces différents exemples.



(a) Stabilité de la frontière sous intégration (b) Exemples de frontières 2-microlocales

FIGURE 8 – Illustrations de la frontière 2-microlocale

Signalons enfin que par analogie avec l'analyse multifractale, un formalisme 2-microlocal a été introduit et développé par LÉVY VÉHEL et SEURET [119] et ECHELARD [60], afin de faciliter le calcul de la frontière 2-microlocale à l'aide d'une transformée de Legendre.

L'utilisation de l'analyse 2-microlocale pour l'étude la régularité fine des processus stochastiques a été initiée par HERBIN et LÉVY VÉHEL [78]. Parmi les résultats de ce travail fondateur, une variation du théorème d'extension de Kolmogorov adapté à la frontière 2-microlocale a été obtenue sous la forme suivante.

Théorème 0.7 *Soit $(X_t)_{t\in\mathbf{R}_+}$ un processus aux trajectoires càdlàgs tel que*

$$\forall \rho \leq \rho_0, \ \forall u, v \in B(t_0, \rho); \quad \mathbb{E}\left[\, |X_u - X_v|^\eta \,\right] \leq C|u-v|^{1+\mu}\rho^{-\nu},$$

où $\eta, \mu > 0$, $\nu < 0$ et C, ρ_0 sont deux constantes positives. Alors, la frontière 2-microlocale de X en t vérifie :

$$\sigma_{X,t}\left(\frac{\nu}{\eta}\right) \geq \frac{\mu}{\eta} \quad p.s.$$

Ce résultat permet donc d'obtenir une borne inférieure sur la frontière 2-microlocale en étudiant les moments successifs des accroissements d'un processus. Similairement aux coefficients de Hölder, l'estimé de la borne supérieure nécessite une analyse plus fine des oscillations des trajectoires. HERBIN et LÉVY VÉHEL [78] ont étudié plus en détail cette question sur plusieurs processus gaussiens comprenant le mouvement Brownien multifractionnaire,

l'intégrale de Wiener et la fonction de Weierstrass généralisée. Ainsi, avec probabilité 1, la frontière 2-microlocale d'un $\mathscr{H}_0$-m.b.m. est égale à :

$$\forall t \in \mathbf{R}_+, \ \ \forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \big(H(t) + s'\big) \wedge H(t). \tag{46}$$

Notons que ce résultat est uniforme sur toute la trajectoire, et que nous retrouvons la même forme de frontière 2-microlocale que pour la fonction de Weierstrass déterministe.

0.4 Organisation et résultats principaux

La suite du manuscrit de cette thèse est organisée en quatre chapitres traitant de problèmes distincts. Moyennant quelques redondances avec cette introduction, leur lecture est complètement indépendante. Les trois premiers traitent directement de l'analyse de la régularité fine des certaines classes de processus stochastiques en utilisant le formalisme 2-microlocal, se situant donc clairement dans la lignée du titre de cette thèse. Le quatrième s'éloigne légèrement de cette problématique de départ pour s'intéresser aux processus de Markov multiparamètres, et même plus généralement, les processus de Markov indicés par une classe abstraite d'ensembles. La liste de publications et pré-publications correspondant à ces chapitres est disponible à la fin de cette partie.

Les résultats obtenus dans le Chapitre 1 se place dans la directe lignée du travail réalisé par HERBIN et LÉVY VÉHEL [78] sur l'application du formalisme 2-microlocal pour l'étude de la géométrie des trajectoires de processus stochastiques. Réalisé en collaboration avec E. HERBIN, il s'intéresse à l'étude approfondie de la frontière 2-microlocale de martingales et de l'intégrale stochastique classique. Dans cette perspective, il s'est avéré intéressant et judicieux d'introduire un outil dérivant de l'analyse 2-microlocale : la *pseudo frontière 2-microlocale*. L'idée sous-jacente est de définir une famille d'espaces fonctionnels et une frontière de manière similaire à l'analyse 2-microlocale, mais en introduisant une différence importante : la composante polynomiale d'une fonction n'est plus mise de côté dans l'étude des oscillations. La caractérisation temporelle 1.2 de ces espaces est ainsi très similaire à la Définition 0.8, avec pour unique différence la composante polynomiale qui n'est plus retranchée. La pseudo frontière 2-microlocale, notée $\Sigma_{f,t}$ dans la suite, qui en découle continue à vérifier certaines propriétés héritées de la frontière 2-microlocale. En effet, ces deux concepts s'avèrent être reliés par le résultat suivant.

Théorème (1.1) *Soit f une fonction continue sur $\mathbf{R}$. Alors, pour tout $t \in \mathbf{R}$,*

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t}(s') = \sigma_{f,t}(s') \wedge \left(s' + p_{f,t}\right) \wedge 1,$$

où $p_{f,t}$ désigne la multiplicité de la fonction f en t.

On s'aperçoit bien grâce à cette relation que la pseudo frontière ne peut satisfaire entièrement les propriétés de stabilité de la frontière originelle. Ainsi, si le Théorème 1.3 montre que son évolution peut être prédite après intégration classique, la relation précédente implique que ce n'est plus possible dans le cas d'un opérateur de dérivation.

Grâce à l'introduction du concept de pseudo frontière, il est alors possible de caractériser entièrement les propriétés trajectorielles des martingales continues.

Théorème (1.6) *Soit M une martingale locale continue et $\langle M \rangle$ sa variation quadratique. Alors, avec probabilité 1, pour tout $t \in \mathbf{R}_+$,*

$$\forall s' \geq -\alpha_{M,t}; \quad \sigma_{M,t}(s') = \Sigma_{M,t}(s') = \frac{1}{2}\Sigma_{\langle M \rangle,t}(2s').$$

La régularité 2-microlocale d'une martingale s'avère donc entièrement déterminée par les oscillations de sa variation quadratique, et ce de manière uniforme sur la trajectoire. Notons que cette propriété peut être étendue à une classe plus large de processus que sont les mouvements browniens multifractionnaires composés avec un changement de temps aléatoire (Théorème 1.11).

L'intérêt principal de cette étude approfondie des martingales est de pouvoir ensuite transposer les résultats précédents à l'intégrale stochastique par rapport à des martingales locales continues, étendant alors le résultat de HERBIN et LÉVY VÉHEL [78] sur l'intégrale de Wiener. L'application directe permet donc de connaître le comportement 2-microlocal d'une

intégrale de la forme $\int_0^t H_u \, \mathrm{d}M_u$ en fonction des variations de $\int_0^t H_u^2 \, \mathrm{d}\langle M \rangle_u$. Il reste intéressant d'essayer de caractériser plus finement le comportement de ce dernier terme en fonction de la régularité de M et H en t. Dans le cadre général, il s'avère compliqué d'obtenir un résultat complet (voir Théorème 1.13) du fait du couplage qui peut exister entre M et H. Cependant, l'étude s'avère plus aisée dans le cas particulier de l'intégration par rapport au mouvement Brownien.

Théorème (1.14) *Soit $X_t = \int_0^t H_u \, \mathrm{d}B_u$ l'intégrale d'un processus adapté H par rapport à un mouvement brownien B. Alors, avec probabilité 1, pour tout $t \in \mathbf{R}_+$*

$$\forall s' \geq -\alpha_{X,t}; \quad \sigma_{X,t}(s') = \Sigma_{X,t}(s') = \begin{cases} \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2} & \text{si } H_t(\omega) \neq 0 \\ \frac{1}{2}\left(1 + \Sigma_{H^2,t}(2s')\right) \wedge \frac{1}{2} & \text{si } H_t(\omega) = 0. \end{cases}$$

On s'aperçoit donc que la régularité d'une intégrale stochastique est généralement celle du mouvement Brownien, les propriétés fines de H n'intervenant uniquement qu'en ses points d'annulation. Ce résultat est appliqué à différents exemples dans la fin du chapitre, avec notamment l'analyse de la régularité fine de solutions d'EDS.

Le Chapitre 2 s'intéresse à l'étude des propriétés trajectorielles d'un processus en particulier : le mouvement Brownien multifractionnaire. Il vise en premier à caractériser en toute généralité les comportements hölderiens ponctuel et local d'un m.b.m., étendant ainsi les travaux d'Herbin [76] et Ayache [8]. Cette étude s'appuie sur une représentation déterministe du m.b.m., obtenue dans la Proposition 2.1, qui exprime un mbm sous la forme d'une intégrale fractionnaire d'un mouvement brownien. Par exemple, dans le cas $H(t) > 1/2$, nous observons que

$$X_t = \frac{1}{\Gamma\left(H(t) - \frac{1}{2}\right)} \int_{\mathbf{R}} B_u \left[(t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2} \right] \mathrm{d}u \quad \text{p.s.}$$

Une expression légèrement plus complexe, mais néanmoins similaire, existe également dans le cas $H(t) \leq 1/2$. L'intérêt d'une telle représentation apparaît lors de l'étude de la régularité 2-microlocale du m.b.m.. En effet, grâce aux propriétés de stabilité de la frontière, la question de la régularité du m.b.m. se ramène à la connaissance de la géométrie fine des trajectoires du mouvement brownien, abondamment étudiée dans la littérature.

Proposition (2.5) *Soit $(t,H) \mapsto B(t,H)$ un champ brownien fractionnaire. Pour tous $k \in \mathbf{N}$ et $H \in (0,1)$, désignons par $B^{H,k}$ le processus $B^{H,k} : t \mapsto \partial_H^k B(t,H)$. Alors, avec probabilité 1, pour tous $k \in \mathbf{N}$, $H \in (0,1)$ et $t \in \mathbf{R}$,*

$$\forall s' \in \mathbf{R}; \quad \sigma_{B^{H,k},t}(s') = \left(H + s'\right) \wedge H.$$

Ce résultat uniforme sur le champ fractionnaire et ses dérivées partielles en H permet de déduire une borne sur la frontière 2-microlocale du m.b.m. (Proposition 2.6), mais également d'établir précisément la forme du coefficient ponctuel hölderien du m.b.m..

Proposition (2.7) *Soit X un mouvement brownien multifractionnaire issu d'un champ $(B(t,H)$. Alors, avec probabilité 1, pour tout $t \in \mathbf{R}$,*

$$\alpha_{X,t} = H(t) \wedge m_{t,H(t)} \alpha_{H,t} \quad \text{où } m_{t,H(t)} := \inf\{k \geq 1 : \partial_H^k B(t,H(t)) \neq 0\}.$$

Le coefficient $m_{t,H(t)}$ correspond donc la multiplicité du champ $B(t,H)$ en $(t,H(t))$.

Cette forme du coefficient ponctuel confirme et étend notamment l'observation inattendue réalisée par Ayache [8] : les trajectoires du m.b.m. peuvent avoir une régularité ponctuelle

hölderienne aléatoire. Ce dernier avait d'ailleurs montré qu'à l'inverse, le coefficient local d'un m.b.m. reste purement déterministe.

La deuxième partie du Chapitre 2 s'intéresse à la géométrie des trajectoires du m.b.m. du point de vue de leur dimension fractale. En premier lieu, nous étudions donc la dimension locale du graphe en tout point $t \in \mathbf{R}$. Nous étendons ainsi les résultats obtenus par PELTIER et LÉVY VÉHEL [154] sur le $\mathscr{H}_0$-m.b.m..

Théorème (2.8) *Soit X un mouvement brownien multifractionnaire. Alors, avec probabilité* 1,

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \dim_{\mathrm{B},t} \mathrm{Gr}(X) = \big(2 - H(t)\big) \vee \dim_{\mathrm{B},t} \mathrm{Gr}(H),$$

lorsque la dimension de boîte existe. Dans le cas contraire, l'égalité est satisfaite indépendamment par les dimensions de boîte inférieure et supérieure.

La dimension de Hausdorff du graphe se trouve être plus délicate à caractériser. Similairement au m.b.f. avec drift variable récemment étudié par PERES et SOUSI [156], il s'avère intéressant et judicieux d'introduire une métrique parabolique sur $\mathbf{R}^2$ paramétrée par $H > 0$:

$$\varrho_H\big((s,x);(t,y)\big) = \max\big(|s-t|^H, |x-y|\big).$$

Une dimension de Hausdorff parabolique, notée $\dim_{\mathrm{H}}(A;\varrho_H)$, relative à cette métrique peut alors être définie.

Théorème (2.9) *Soit X un mouvement brownien multifractionnaire. Alors, avec probabilité* 1,

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \dim_{\mathrm{H},t} \mathrm{Gr}(X) = 1 + H(t)\big(\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H);\varrho_{H(t)}\big) - 1\big),$$

où $\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H);\varrho_{H(t)}\big)$ désigne donc la dimension de Hausdorff locale du graphe de H pour la métrique $\varrho_{H(t)}$.

L'exemple 2.3 qui suit montre bien que la dimension de Hausdorff pour la métrique parabolique est la bonne notion pour caractériser la dimension du graphe du m.b.m. et qu'elle ne peut en toute généralité être reliée au la dimension classique $\dim_{\mathrm{H},t} \mathrm{Gr}(H)$. Un résultat similaire peut être obtenu dans le cas de la dimension de Hausdorff de l'image d'un ensemble fractal.

Théorème (2.10) *Soit X un mouvement brownien multifractionnaire et $F \subset \mathbf{R}$ un ensemble. Alors, avec probabilité* 1

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \dim_{\mathrm{H},X(t)} X(F) = \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F);\varrho_{H(t)}\big).$$

Bien que pouvant sembler contre-intuitives, les deux égalités précédentes étendent bien celles connues sur le $\mathscr{H}_0$-m.b.m., c.-à-d. $\dim_{\mathrm{H},t} \mathrm{Gr}(X) = 2 - H(t)$ et $\dim_{\mathrm{H},X(t)} X(F) = H(t)^{-1} \dim_{\mathrm{H},t} F$.

Les résultats de cette deuxième partie sur la géométrie fractale des trajectoires du m.b.m. montre bien que son comportement inattendu pour un processus gaussien ne se limite pas à la régularité hölderienne, les exemples 2.1 et 2.5 apportant alors une illustration de ses propriétés particulières.

Le Chapitre 3 s'intéresse à la généralisation des travaux de JAFFARD [94] sur le spectre multifractal des processus de Lévy. L'idée sous-jacente est de combiner l'analyse 2-microlocale au formalisme multifractal afin de raffiner le spectre de singularités et mettre en avant de nouveaux comportements. Pour cela, nous étendons l'étude des iso-valeurs du coefficient de Hölder ponctuel d'un processus de Lévy X en introduisant les deux collections d'ensembles $(\widetilde{E}_h)_{h\geq 0}$ et $(\widehat{E}_h)_{h\geq 0}$ définies par :

$$\widetilde{E}_h = \big\{t \in E_h : \forall s' \in \mathbf{R};\ \sigma_{X,t}(s') = (h+s') \wedge 0\big\} \quad \text{and} \quad \widehat{E}_h = E_h \setminus \widetilde{E}_h,$$

où $E_h = \{t \in \mathbf{R} : \alpha_{X,t} = h\}$. Intuitivement, il s'agit de faire la différence entre un comportement 2-microlocal classique, c.-à-d. une frontière dont la pente est égale à 1, et une géométrie plus inhabituelle, correspondant donc à une régularité 2-microlocale non triviale. Le principal résultat de ce chapitre porte alors sur la géométrie fractale de ces ensembles.

Théorème (3.1) *Soit X un processus de Lévy. Alors, avec probabilité* 1

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widetilde{E}_h \cap V) = \dim_{\mathrm{H}}(E_h \cap V) = \begin{cases} \beta h & si \quad h \in [0, 1/\beta'); \\ 1 & si \quad h = 1/\beta'; \\ -\infty & si \quad h \in (1/\beta', +\infty], \end{cases}$$

où $\mathscr{O}$ désigne les ouverts non vide de $\mathbf{R}$ et β' est égal au coefficient β de Blumenthal–Getoor (3.7) si X n'a pas de composante brownienne ou 2 dans le cas contraire.

De plus, la collection $(\widehat{E}_h)_{h \in \mathbf{R}_+}$ vérifie presque sûrement

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) \leq \begin{cases} 2\beta h - 1 & si \quad h \in (1/2\beta, 1/\beta'); \\ -\infty & si \quad h \in [0, 1/2\beta] \cup [1/\beta', +\infty]. \end{cases}$$

Étant donné que $\dim_{\mathrm{H}}(\widehat{E}_h \cap V) < \dim_{\mathrm{H}}(\widetilde{E}_h \cap V)$ pour tout $h \in \mathbf{R}_+$, ce résultat implique donc que du point de vue de la dimension de Hausdorff, le comportement naturel de la frontière 2-microlocale prédomine et a une géométrie fractale équivalente au spectre multifractal originellement obtenu par Jaffard [94]. De plus, les Proposition 3.4 et Théorème 3.5 permettent de montrer que l'exposant de Blumenthal–Getoor ne caractérise pas entièrement les propriétés de la collection $(\widehat{E}_h)_{h \in \mathbf{R}_+}$, et donc la géométrie fine des trajectoires des processus de Lévy.

En corollaire de ce résultat principal, il est intéressant d'étudier ce qui s'apparente aux coupes de niveau de la frontière 2-microlocale, c.-à-d. les ensembles

$$E_{\sigma,s'} = \{t \in \mathbf{R}_+ : \forall u' > s';\ X_\cdot \in C_t^{\sigma,u'} \ \text{ and } \ \forall u' < s';\ X_\cdot \notin C_t^{\sigma,u'}\}.$$

Notons que le cas $\sigma = 0$, ils correspondent à la collection $(E_h)_{h \in \mathbf{R}_+}$.

Corollaire (3.3) *Soit X un processus de Lévy. Alors, avec probabilité* 1*, pour tout $\sigma \in [-1, 0]$,*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \beta s & si \quad s \in [0, 1/\beta'); \\ 1 & si \quad s = 1/\beta'; \\ -\infty & sinon, \end{cases}$$

où par définition $s = \sigma - s'$. De plus, pour tout $s' \in \mathbf{R}$, $E_{0,s'} = E_{-s'}$ et l'ensemble $E_{\sigma,s'}$ est vide lorsque $\sigma > 0$.

Le résultat décrit dans ce corollaire constitue une généralisation directe du spectre multifractal, étant donné que ce dernier correspond au cas $\sigma = 0$. On remarque d'ailleurs que le comportement pathologique mis en lumière dans le Théorème 3.1 par la collection $(\widehat{E}_h)_{h \in \mathbf{R}_+}$ disparaît complètement dans cette version plus faible.

Cette étude de la régularité fine des processus de Lévy s'avère avoir une portée plus large, permettant en effet d'apporter de nouveaux résultats sur une autre classe de processus : le *linear fractional stable motion*. Ce dernier est défini comme

$$X_t = \int_{\mathbf{R}} \left[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right] M_{\alpha,\beta}(\mathrm{d}u),$$

où $M_{\alpha,\beta}$ est une mesure aléatoire α-stable et $H \in (0, 1)$. Grâce à une représentation déterministe de ce processus et les propriétés de stabilité des espaces 2-microlocaux, les résultats précédents peuvent se transposer complètement à ce processus.

Théorème (3.6) *Soit X un linear fractional stable motion paramétré par $\alpha \in [1,2)$ et $H \in (0,1)$. Alors, avec probabilité 1, pour tout $\sigma \in \left[H - \frac{1}{\alpha} - 1, H - \frac{1}{\alpha}\right]$,*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \alpha(s-H)+1 & \textit{si} \;\; s \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \textit{sinon.} \end{cases}$$

où comme précédemment $s = \sigma - s'$. De plus, pour tout $s' \in \mathbf{R}$, l'ensemble $E_{\sigma,s'}$ est vide lorsque $\sigma > H - \frac{1}{\alpha}$.

Ce théorème s'avère avoir deux intérêts majeurs. En premier lieu, le cadre 2-microlocal permet complètement d'unifier en terme de régularité le cas $H > 1/\alpha$ où le processus est continu et celui où $H < 1/\alpha$ des trajectoires sont partout non-bornées. Dans cette dernière situation, l'analyse 2-microlocale est encore pertinente car les trajectoires s'avèrent être des distributions. Enfin, le deuxième intérêt du théorème précédent est de pouvoir déduire le spectre multifractal du LFSM lorsque $H > 1/\alpha$. On obtient alors avec probabilité 1

$$\forall V \in \mathscr{O}; \quad d_X(h,V) = \begin{cases} \alpha(h-H)+1 & \text{si} \;\; h \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{sinon.} \end{cases}$$

Les résultats obtenus dans ce chapitre ouvrent des perspectives futures de recherche, notamment en ce qui concerne l'étude plus fine de la géométrie fractale des ensembles $(\widehat{E}_h)_{h \in \mathbf{R}_+}$ en fonction des propriétés de la mesure de Lévy.

Le Chapitre 4 n'est pas dans la directe lignée des parties précédentes, et porte donc sur une thématique légèrement différente. Il vise à définir et étudier une propriété de Markov pour une large classe de processus indicés par des ensembles. Plus précisément, le formalisme *set-indexed* utilisé dans ce chapitre a pour but de fournir un cadre général d'étude de processus stochastiques qui peuvent être indicés par des collections très différentes pouvant être tout aussi bien $\mathbf{R}^N$, une variété riemannienne ou un arbre.

Le formalisme *set-indexed* étant assez long à introduire, nous ne donnons pas les énoncés précis des résultats ici même et renvoyons donc le lecteur directement au Chapitre 4 pour leur obtention. Néanmoins, il est possible donner un aperçu de ces derniers et de décrire la motivation sous-jacente à l'introduction de la propriété $\mathscr{C}$-Markov pour les processus indicés par des ensembles. Sans même considérer la question dans ce cadre très général, il s'avère que la définition d'une propriété de Markov pour les processus multiparamètres est déjà une question non-triviale. Une large littérature s'est intéressée à ce sujet pendant les années 80 et 90, en se concentrant surtout sur les processus à deux paramètres. Parmi les multiples définitions existantes, signalons celle introduite par CAIROLI [39] : un processus $(X_{s,t})_{s,t}$ est dit $*$-*Markov* si

$$\mathbb{P}(X_{s+h,t+k} \in \Gamma \,|\, \mathscr{F}^*_{s,t}) = \mathbb{P}(X_{s+h,t+k} \in \Gamma \,|\, X_{s,t}, X_{s+h,t}, X_{s,t+k}),$$

pour tous $s,t \in \mathbf{R}_+$, $h,k > 0$ et où $\mathscr{F}^*_{s,t} := \sigma\big(\{X_{u,v}\,; u \leq s \text{ ou } v \leq t\}\big)$. Cette définition, qui reste somme toute assez simple, a l'avantage d'être vérifiée par un grande nombre de processus, dont en premier lieu le drap brownien. Malheureusement, on s'aperçoit rapidement qu'elle ne peut directement être généralisée au cadre multiparamètre $\mathbf{R}^N_+$.

Pour les processus multiparamètres, l'approche généralement considérée est celle décrite dans l'ouvrage de KHOSHNEVISAN [100] dont la définition constitue une généralisation directe de la formule classique à un paramètre. Ainsi, un processus $(X_t)_{t \in \mathbf{R}^N_+}$ est dit multiparamètre-Markov si

$$\forall t,h \in \mathbf{R}^N_+; \quad \mathbb{E}[f(X_{t+h}) \,|\, \mathscr{F}_t] = \mathbb{E}[f(X_{t+h}) \,|\, X_t],$$

où la filtration $(\mathscr{F}_t)_{t \in \mathbf{R}^N_+}$ est supposée vérifier $\mathbb{E}\big[\,\mathbb{E}[Y \,|\, \mathscr{F}_t] \;\big|\; \mathscr{F}_s\big] = \mathbb{E}[Y \,|\, \mathscr{F}_{s \wedge t}]$ pour tous $s,t \in \mathbf{R}^N_+$ et toute variable aléatoire bornée Y $\mathscr{F}_t$-mesurable. Cette définition a clairement

l'avantage de la simplicité, mais possède un inconvénient majeur : le drap brownien n'est markovien selon ce sens.

L'approche présentée dans le Chapitre 4 a notamment pour objectif de pallier les défauts des précédentes définitions. L'utilisation du formalisme set-indexed pour introduire la propriété $\mathscr{C}$-Markov oblige certes une plus grande abstraction par rapport au cadre multiparamètre, mais a l'avantage de permettre des notations plus condensées et une étude plus générale. Cette dernière est réalisée dans la première partie du chapitre, où suite à la Définition 4.2, nous montrons que les processus $\mathscr{C}$-Markov satisfont un certain nombre de propriétés naturelles. En particulier, il s'avère que similairement à la théorie classique, un processus $\mathscr{C}$-Markov est entièrement déterminé par sa loi initiale et ses probabilités de transition (Théorème 4.2), et qu'inversement, partant d'une mesure de probabilités et d'un noyau $\mathscr{C}$-markovien, il est possible de construire le processus canonique associé (Théorème 4.3).

Les processus multiparamètres $\mathscr{C}$-markoviens sont étudiés plus précisément dans la deuxième partie de ce chapitre, montrant notamment qu'il constitue bien une généralisation des approches précédentes. Ainsi, il y est montré que le cadre $\mathscr{C}$-Markov s'avère complètement équivalent à la propriété $*$-Markov pour les processus à deux paramètres. De plus, à la fois le drap brownien et les processus usuels multiparamètre-Markov s'avèrent vérifier la propriété $\mathscr{C}$-Markov. Enfin, généralisant un résultat de la théorie classique, le Théorème 4.10 montre que sous une hypothèse du type Feller sur le noyau de transition, le processus $\mathscr{C}$-Markov associé admet une modification dont les trajectoires sont presque sûrement continues à droite.

La troisième partie, réalisée en collaboration avec E. Herbin, est dédiée à la description et l'étude d'un processus d'Ornstein–Uhlenbeck indicé par les ensembles. Ce dernier est défini grâce à la propriété $\mathscr{C}$-Markov, en introduisant des probabilités de transition généralisant le noyau originel. Un des intérêts de ce processus est ainsi de fournir un exemple $\mathscr{C}$-markovien sortant de la famille classique des processus à accroissements indépendants. Signalons que dans le cas multiparamètre, il s'avère prendre la forme intégrale suivante :

$$\forall t \in \mathbf{R}_+^N; \quad X_t = e^{-\langle \alpha, t\rangle}\left[X_0 + \sigma \int_{(-\infty,t]\setminus(-\infty,0]} e^{\langle \alpha, u\rangle} \mathrm{d}W_u\right],$$

où $\sigma > 0$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{R}_+^N$, W est un drap brownien et X_0 est une variable aléatoire indépendante de W.

Quand bien même ce chapitre ne traite pas directement de la régularité des trajectoires de processus stochastiques, nous pensons que le formalisme $\mathscr{C}$-Markov qui y est construit pourrait s'avérer être un cadre intéressant d'étude des propriétés géométriques de processus de Markov, dans la lignée des résultats décrits par Khoshnevisan [100].

(Pré-)publications issues de la thèse

[1] P. Balança et E. Herbin. 2-microlocal analysis of martingales and stochastic integrals. *Stochastic Process. Appl.* **122**.6 (2012), p. 2346–2382. arXiv :1107.6016.

[2] P. Balança. Sample paths properties of irregular multifractional Brownian motion. *En préparation* (2013).

[3] P. Balança. Fine regularity of Lévy processes and linear (multi)fractional stable motion. *Pré-publication* (2013). arXiv :1302.3140.

[4] P. Balança. An increment type set-indexed Markov property. *En révision* (2012). arXiv :1207.6568.

[5] P. Balança et E. Herbin. A set-indexed Ornstein–Uhlenbeck process. *Electron. Commun. Probab.* **17**.39 (2012), p. 1–14. arXiv :1203.5524.

0.5 Organisation and main results

The manuscript of this Ph.D. thesis is organised in four chapters which tackle different questions. Their reading is completely independent in return of a few redundancies in their introduction. The first three chapters are directly linked up with the main topic of this thesis: the study of sample path properties of stochastic processes using the framework of 2-microlocal analysis. These three works are respectively related to continuous martingales and stochastic integrals, the multifractional Brownian motion and Lévy processes. In the fourth chapter, we are interested in a different problematic: the definition and the study of a multiparameter, and more broadly set-indexed, Markov property. A list of the preprints and publications corresponding to these chapters is available at the end of this section.

The results obtained in the Chapter 1 are directly in line with the seminal work of HERBIN and LÉVY VÉHEL [78] on stochastic 2-microlocal analysis. Hence, in collaboration with E. HERBIN, we have studied the 2-microlocal frontier of continuous martingales and classic stochastic integrals. To investigate this question, we have introduced another regularity tool closely related to 2-microlocal analysis: the *pseudo 2-microlocal frontier*. The main idea behind this concept is, on the contrary to the classic 2-microlocal frontier, to keep the local polynomial component of a function in the study of its regularity. This difference becomes obvious when you compare Definitions 0.8 and 1.2. Denoted $\Sigma_{f,t}$, the pseudo frontier is closely connected the 2-microlocal frontier by the following result.

Theorem (1.1) *Suppose f is a continuous function. Then, for all $t \in \mathbf{R}$,*

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t}(s') = \sigma_{f,t}(s') \wedge \left(s' + p_{f,t}\right) \wedge 1,$$

where $p_{f,t}$ denotes the multiplicity of f at t.

Owing to the previous formula, the pseudo 2-microlocal frontier is not completely predictable under the action of fractional operators. More precisely, Theorem 1.3 proves that its behaviour is still known after classic integration, but the same result does not hold under differential operators.

Using this new notion, we have been able to characterise entirely and uniformly the 2-microlocal frontier of continuous martingales.

Theorem (1.6) *Suppose M is a continuous local martingale and let $\langle M \rangle$ denote its quadratic variation. Then, with probability 1 and for all $t \in \mathbf{R}_+$,*

$$\forall s' \geq -\alpha_{M,t}; \quad \sigma_{M,t}(s') = \Sigma_{M,t}(s') = \frac{1}{2}\Sigma_{\langle M \rangle,t}(2s').$$

Hence, the oscillations of the quadratic variation entirely determine the 2-microlocal regularity of a martingale. Interestingly, we prove in Theorem 1.11 that this property can be extended to any multifractional Brownian motion composed with a random time change.

One of the interesting aspects of the previous study is the deduction of the regularity of any stochastic integral with respect to continuous local martingales, therefore extending the result of HERBIN and LÉVY VÉHEL [78] on Wiener's integral. The direct application of the previous theorem implies that the 2-microlocal behaviour of the process $t \mapsto \int_0^t H_u \, \mathrm{d}M_u$ is characterised by the oscillations of $t \mapsto \int_0^t H_u^2 \, \mathrm{d}\langle M \rangle_u$. It would be interesting to determine more precisely the pseudo frontier of the latter in terms of the regularities of $\langle M \rangle$ and H at t. The coupling which may exist between these two processes makes it difficult to obtain a general result (see Theorem 1.13). Nevertheless, the study is easier in the case of an integral with respect to Brownian motion.

Theorem (1.14) *Suppose $X_t = \int_0^t H_u \, \mathrm{d}B_u$ is the integral of an adapted continuous process H with respect to a Brownian motion B. Then, with probability 1, for all $t \in \mathbf{R}_+$*

$$\forall s' \geq -\alpha_{X,t}; \quad \sigma_{X,t}(s') = \Sigma_{X,t}(s') = \begin{cases} \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2} & \text{if } H_t(\omega) \neq 0 \\ \frac{1}{2}\left(1 + \Sigma_{H^2,t}(2s')\right) \wedge \frac{1}{2} & \text{if } H_t(\omega) = 0. \end{cases}$$

Hence, we note that a stochastic integral has most of the time the same regularity as Brownian motion, and the behaviour of H only intervenes at exceptional times corresponding to its zeros. Finally, the end of the chapter is devoted to the application of this result to several examples of SDEs.

Chapter 2 focuses on the sample path properties of a specific process: the multifractional Brownian motion. Its aim is to extend the results of HERBIN [76] et AYACHE [8] to obtain a uniform characterisation of its regularity, for any Hurst function considered. One important ingredient in this study is the deterministic representation of the mBm presented in Proposition 2.1. For instance, when $H(t) > 1/2$,

$$X_t = \frac{1}{\Gamma\left(H(t) - \frac{1}{2}\right)} \int_{\mathbf{R}} B_u \left[(t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2}\right] \mathrm{d}u \quad \text{p.s.}$$

A similar expression, even though slightly more complex, exists when $H(t) \leq 1/2$. The interesting aspect of this representation is that using the 2-microlocal formalism, we can deduce uniform properties on the mBm from the sample path regularity of Brownian motion, knowing that the latter has been extensively studied in the literature.

Proposition (2.5) *Suppose $(t,H) \mapsto B(t,H)$ is a fractional Brownian field. For any $k \in \mathbf{N}$ and any $H \in (0,1)$, let us denote by $B^{H,k}$ the process $B^{H,k} : t \mapsto \partial_H^k B(t,H)$. Then, with probability 1, for every $k \in \mathbf{N}$, every $H \in (0,1)$ and all $t \in \mathbf{R}$,*

$$\forall s' \in \mathbf{R}; \quad \sigma_{B^{H,k},t}(s') = \left(H + s'\right) \wedge H.$$

This uniform result on the fractional Brownian field leads to a lower bound on the 2-microlocal frontier (Proposition 2.6). It also allows to deduce the general form of the pointwise Hölder exponent of the mBm.

Proposition (2.7) *Suppose X is a multifractional Brownian motion defined using the fractional field $B(t,H)$. Then, with probability 1, for all $t \in \mathbf{R}$,*

$$\alpha_{X,t} = H(t) \wedge m_{t,H(t)} \alpha_{H,t} \quad \text{où } m_{t,H(t)} := \inf\{k \geq 1 : \partial_H^k B(t,H(t)) \neq 0\}.$$

The exponent $m_{t,H(t)}$ therefore corresponds to the multiplicity of $B(t,H)$ at $(t,H(t))$.

This result extends to the work AYACHE [8] who has shown that the sample paths of a mBm may have a non-deterministic pointwise regularity, on the contrary to the local Hölder exponent.

The second part of the chapter is devoted to the fractal dimensions of the trajectories. We first study the local dimension of the graph, extending the result of PELTIER and LÉVY VÉHEL [154].

Theorem (2.8) *Suppose X is a multifractional Brownian motion. Then, with probability 1,*

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \dim_{\mathrm{B},t} \mathrm{Gr}(X) = \left(2 - H(t)\right) \vee \dim_{\mathrm{B},t} \mathrm{Gr}(H),$$

when the Box dimension exists. Otherwise, the equality stands independently for the lower and upper Box dimensions.

The Hausdorff dimension is slightly more complex to determine. Similarly to a recent work of PERES and SOUSI [156] on fractional Brownian motion with drift, it is relevant to introduce a parabolic metric on $\mathbf{R}^2$ parametrised by $H > 0$:

$$\varrho_H\big((s,x);(t,y)\big) = \max\big(|s-t|^H, |x-y|\big).$$

The Hausdorff dimension of any set A with respect to the metric ϱ_H is then simply denoted $\dim_{\mathrm{H}}(A;\varrho_H)$.

Theorem (2.9) *Suppose X is a multifractional Brownian motion. Then, with probability* 1,

$$\forall t \in \mathbf{R}\setminus\{0\}; \quad \dim_{\mathrm{H},t}\mathrm{Gr}(X) = 1 + H(t)\big(\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H);\varrho_{H(t)}\big) - 1\big),$$

where $\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H);\varrho_{H(t)}\big)$ denotes the local Hausdorff dimension of the graph of H at t w.r.t. $\varrho_{H(t)}$.

Example 2.3 which follows proves that the parabolic Hausdorff dimension is the relevant notion to use, since the classic Hausdorff dimension of the graph of H can not always be related to $\dim_{\mathrm{H},t}\mathrm{Gr}(X)$. A similar result can also be obtained on the Hausdorff dimension of images of fractal sets by the mBm.

Theorem (2.10) *Suppose X is a multifractional Brownian motion and let $F \subset \mathbf{R}$. Then, with probability* 1,

$$\forall t \in \mathbf{R}\setminus\{0\}; \quad \dim_{\mathrm{H},X(t)}X(F) = \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F);\varrho_{H(t)}\big).$$

Even though the two previous equalities may seem counter-intuitive, they may verify that they extend the well-known formulas satisfied by an $\mathscr{H}_0$-mBm, i.e. $\dim_{\mathrm{H},t}\mathrm{Gr}(X) = 2 - H(t)$ et $\dim_{\mathrm{H},X(t)}X(F) = H(t)^{-1}\dim_{\mathrm{H},t}F$. Interestingly, the results presented in this second part show that, similarly to the pointwise regularity, the Hausdorff dimension of the mBm also presents some unusual and unexpected properties for a Gaussian process.

The aim of Chapter 3 is to extend the work of JAFFARD [94] on the multifractal spectrum of Lévy processes. The main idea is to combine 2-microlocal analysis with the classic multifractal formalism in order to refine the spectrum of singularities determined by the latter. For this purpose, we focus on the fractal geometry of two collections of sets $(\widetilde{E}_h)_{h\geq 0}$ and $(\widehat{E}_h)_{h\geq 0}$ respectively defined by:

$$\widetilde{E}_h = \big\{t \in E_h : \forall s' \in \mathbf{R};\ \sigma_{X,t}(s') = (h+s')\wedge 0\big\} \quad \text{and} \quad \widehat{E}_h = E_h \setminus \widetilde{E}_h,$$

where $E_h = \{t \in \mathbf{R} : \alpha_{X,t} = h\}$. Intuitively, the first collection gathers the times at which the 2-microlocal behaviour is rather classic (i.e. a slope equal to 1), whereas the second one focuses on the unusual frontiers. The main result presented in this chapter is related to the Hausdorff dimension of these sets.

Theorem (3.1) *Suppose X is a Lévy process. Then, with probability* 1

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widetilde{E}_h \cap V) = \dim_{\mathrm{H}}(E_h \cap V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta'); \\ 1 & \text{if } h = 1/\beta'; \\ -\infty & \text{if } h \in (1/\beta', +\infty], \end{cases}$$

where $\mathscr{O}$ denotes the collection of non-empty open sets in $\mathbf{R}$ and β' is equal to the Blumenthal–Getoor exponent (3.7) if X has no Brownian component, or 2 *otherwise.*

Furthermore, the collection $(\widehat{E}_h)_{h\in\mathbf{R}_+}$ satisfies almost surely

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) \leq \begin{cases} 2\beta h - 1 & \text{if } h \in (1/2\beta, 1/\beta'); \\ -\infty & \text{if } h \in [0, 1/2\beta] \cup [1/\beta', +\infty]. \end{cases}$$

The previous result in particular implies that $\dim_{\mathrm{H}}(\widehat{E}_h \cap V) < \dim_{\mathrm{H}}(\widetilde{E}_h \cap V)$ for all $h \in \mathbf{R}_+$, where the last term is equal to the multifractal spectrum obtained by JAFFARD [94]. Therefore, it proves that from a Hausdorff dimension point of view, the 2-microlocal frontier has "most of the time" a classic and typical shape. Furthermore, Proposition 3.4 and Theorem 3.5 show that the geometry of the collection $(\widehat{E}_h)_{h\in\mathbf{R}_+}$ depends on the form of the Lévy measure, and not only the Blumenthal–Getoor exponent.

As a corollary, it is interesting to study the dimension of the level sets of the 2-microlocal frontier, defined by

$$E_{\sigma,s'} = \{t \in \mathbf{R}_+ : \forall u' > s';\ X_\cdot \in C_t^{\sigma,u'} \ \text{ and } \ \forall u' < s';\ X_\cdot \notin C_t^{\sigma,u'}\}.$$

Note that the case $\sigma = 0$ corresponds to the collection $(E_h)_{h\in\mathbf{R}_+}$.

Corollary (3.3) *Suppose X is a Lévy process. Then, with probability 1, for all $\sigma \in [-1,0]$,*

$$\forall V \in \mathscr{O};\quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \beta s & \text{if } \ s \in [0, 1/\beta'); \\ 1 & \text{if } \ s = 1/\beta'; \\ -\infty & \text{otherwise,} \end{cases}$$

where $s := \sigma - s'$. Moreover, for all $s' \in \mathbf{R}$, $E_{0,s'} = E_{-s'}$ and the set $E_{\sigma,s'}$ is empty when $\sigma > 0$.

This corollary is direct generalisation of the multifractal spectrum as the latter corresponds to $\sigma = 0$. Interestingly, we observe that the unusual 2-microlocal behaviour described in Theorem 3.1 with the collection $(\widehat{E}_h)_{h\in\mathbf{R}_+}$ disappears in this weaker result.

The fine geometric properties of the sample paths of Lévy processes happen to be useful for the study of another class of processes: the linear fractional stable motion, defined by

$$X_t = \int_{\mathbf{R}} \left[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right] M_{\alpha,\beta}(\mathrm{d}u),$$

where $M_{\alpha,\beta}$ is an α-stable random measure and $H \in (0,1)$. Using a deterministic representation of this process and the predictability of the 2-microlocal frontier under fractional operators, we extend the previous results to the LFSM.

Théorème (3.6) *Suppose X is a linear fractional stable motion parametrised by $\alpha \in [1,2)$ and $H \in (0,1)$. Then, with probability 1, for all $\sigma \in \left[H - \frac{1}{\alpha} - 1, H - \frac{1}{\alpha}\right]$,*

$$\forall V \in \mathscr{O};\quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \alpha(s-H)+1 & \text{if } \ s \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{otherwise.} \end{cases}$$

where $s := \sigma - s'$. Moreover, for all $s' \in \mathbf{R}$, the set $E_{\sigma,s'}$ is empty when $\sigma > H - \frac{1}{\alpha}$.

The first interesting point of this result is that the 2-microlocal formalism unifies in terms of regularity the continuous $H > 1/\alpha$ and nowhere bounded $H < 1/\alpha$ cases, where in the latter, trajectories happen to be distributions in $\mathscr{D}'(\mathbf{R})$. Furthermore, when sample paths are continuous, we can deduce the multifractal spectrum of the LFSM. Hence, with probability 1,

$$\forall V \in \mathscr{O};\quad d_X(h,V) = \begin{cases} \alpha(h-H)+1 & \text{if } \ h \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{otherwise.} \end{cases}$$

Some of the results obtained in this chapter open some new interesting research perspectives, in particular on the fine characterisation of the geometry of the sets $(\widehat{E}_h)_{h\in\mathbf{R}_+}$.

Chapter 4 is not directly related to the 2-microlocal regularity of stochastic processes and is interested in a slightly different subject. It aims to introduce and investigate a Markov property

using the formalism of set-indexed processes. The latter provides a framework to define and study processes indexed by a general collection of sets, which can represent for instance $\mathbf{R}^N$ or a tree as well.

The *set-indexed* formalism is rather heavy to introduce, and thus, we will not discuss the precise results here, and we refer directly to Chapter 4 for them. Nevertheless, it is possible to give an insight of the motivations behind the introduction of the $\mathscr{C}$-Markov property. The question of defining a well-defined Markov property is already a complicate problem in the multiparameter setting. A large literature has investigated this problem during the 80s and the early 90s, especially for two-parameter processes. Among the different definitions, the work of CAIROLI [39] is closely related to this chapter. In the latter, a process $(X_{s,t})_{s,t}$ is said to be $*$-*Markov* if

$$\mathbb{P}(X_{s+h,t+k} \in \Gamma \,|\, \mathscr{F}^*_{s,t}) = \mathbb{P}(X_{s+h,t+k} \in \Gamma \,|\, X_{s,t}, X_{s+h,t}, X_{s,t+k}),$$

for all $s, t \in \mathbf{R}_+$, all $h, k > 0$ and where $\mathscr{F}^*_{s,t} := \sigma\big(\{X_{u,v}\,; u \leq s \text{ or } v \leq t\}\big)$. This definition has the advantage of being rather simple and is satisfied by a large class of two-parameter processes, including the Brownian sheet. Nevertheless, its generalisation to the multiparameter setting is not straightforward.

Another multiparameter approach is described in the work of KHOSHNEVISAN [100]. It constitutes a direct extension of the one-parameter definition: a process $(X_t)_{t\in\mathbf{R}^N_+}$ is said to be *multiparameter-Markov* if

$$\forall t, h \in \mathbf{R}^N_+; \quad \mathbb{E}[f(X_{t+h}) \,|\, \mathscr{F}_t] = \mathbb{E}[f(X_{t+h}) \,|\, X_t],$$

where the filtration $(\mathscr{F}_t)_{t\in\mathbf{R}^N_+}$ is assumed to satisfy $\mathbb{E}\big[\,\mathbb{E}[Y \,|\, \mathscr{F}_t] \,\big|\, \mathscr{F}_s\,\big] = \mathbb{E}[Y \,|\, \mathscr{F}_{s\wedge t}]$ for all $s, t \in \mathbf{R}^N_+$ and any bounded random variable Y. Even though quite simple, this definition has a major drawback: the Brownian sheet is not a multiparameter-Markov process.

The approach described in Chapter 4 aims to solve these two issues. The set-indexed formalism used is certainly more abstract than the multiparameter setting, but it allows to keep sufficiently simple notations and to define a Markov property for a large class of processes. The $\mathscr{C}$-Markov property is introduced at the beginning of the chapter (Definition 4.2) and its general properties are studied in the first section. In particular, we prove that a $\mathscr{C}$-Markov process is entirely characterised by its initial distribution and its $\mathscr{C}$-transition system (Theorem 4.2). Conversely, Theorem 4.3 shows that a canonical process can be constructed from any probability measure and $\mathscr{C}$-Markov kernel.

Multiparameter processes are considered more deeply in the second part of this chapter. In particular, we prove that the $*$-Markov definition is completely equivalent to the $\mathscr{C}$-Markov formalism for two-parameter processes. Furthermore, the classic multiparameter-Markov processes also happen to satisfy the $\mathscr{C}$-Markov property. Finally, extending a classic result of the theory of Markov processes, we prove that a multiparameter $\mathscr{C}$-Markov process which satisfies a Feller-type assumption almost surely has right-continuous sample paths.

In collaboration with E. HERBIN, we have studied in the third part of set-indexed Ornstein–Uhlenbeck process. The latter is defined using the $\mathscr{C}$-Markov property, introducing for this purpose specific $\mathscr{C}$-Markov transition probabilities which extend the well-known one-parameter kernel. One of the interesting aspects of this process is to provide an example with correlated increments for the $\mathscr{C}$-Markov property. Note that in the multiparameter setting, it has the following integral representation

$$\forall t \in \mathbf{R}^N_+; \quad X_t = e^{-\langle \alpha, t\rangle}\left[X_0 + \sigma \int_{(-\infty,t]\setminus(-\infty,0]} e^{\langle \alpha, u\rangle} \mathrm{d}W_u\right],$$

where $\sigma > 0$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{R}^N_+$, W is a Brownian sheet and X_0 is a random variable independent of W.

Even though this chapter is not directly related to the study of sample path properties of stochastic process, we think the formalism developed in it could be an interesting foundation for the investigation of some geometric properties, in particular to extend the results presented by Khoshnevisan [100].

Ph.D. publications and preprints

[1] P. Balança and E. Herbin. 2-microlocal analysis of martingales and stochastic integrals. *Stochastic Process. Appl.* **122**.6 (2012), pp. 2346–2382. arXiv:1107.6016.

[2] P. Balança. Sample paths properties of irregular multifractional Brownian motion. *In preparation* (2013).

[3] P. Balança. Fine regularity of Lévy processes and linear (multi)fractional stable motion. *Preprint* (2013). arXiv:1302.3140.

[4] P. Balança. An increment type set-indexed Markov property. *In revision* (2012). arXiv:1207.6568.

[5] P. Balança and E. Herbin. A set-indexed Ornstein–Uhlenbeck process. *Electron. Commun. Probab.* **17**.39 (2012), pp. 1–14. arXiv:1203.5524.

1 Martingales and stochastic integrals

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1.1 Introduction

Sample path properties of stochastic processes are widely studied since the 1970s (see e.g. [175, 28, 29, 151] as foundational works). This field of research is still very active and a non-exhaustive list of authors and recent works includes Dalang [46], Khoshnevisan [105, 46], Lawler [115], Mountford [20], Xiao [134, 105, 20], etc. Among the variety of measures of regularity, pointwise and local Hölder exponents are the most recurrent tools used in the literature. Nevertheless, these two exponents (and as well the moduli of continuity) lack of stability under (pseudo-)differential operators or multiplication by a power function, and therefore do not completely characterize the local regularity of a function or a stochastic process at a given point.

Simple examples can illustrate these issues. Consider the deterministic "chirp" function

$$t \longmapsto f(t) = |t - t_0|^{\alpha} \sin\left(|t - t_0|^{-\beta}\right)$$

where α, β, t_0 are positive real numbers. As described in [60], this function has a non-trivial regularity at t_0. Indeed, its pointwise and local Hölder exponents at t_0 (see Section 2 for definitions) are equal to

$$\alpha_{f,t_0} = \alpha \qquad \text{and} \qquad \widetilde{\alpha}_{f,t_0} = \frac{\alpha}{1+\beta}.$$

Using notations from [166], $I_0^\gamma f : t \mapsto \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} f(u) \mathrm{d}u$ denotes f fractional integration of order γ. Then, the pointwise and Hölder exponents of $I_0^\gamma f$ at t_0 are known to be equal to

$$\alpha_{I_0^\gamma f,t_0} = \alpha + \frac{\gamma}{1+\beta} \qquad \text{and} \qquad \widetilde{\alpha}_{I_0^\gamma f,t_0} = \frac{\alpha}{1+\beta} + \gamma.$$

Similarly, the study of the local regularity of the function $t \mapsto |t-t_0|^\gamma f(t)$ leads to

$$\alpha_{|t-t_0|^\gamma f,t_0} = \alpha + \gamma \qquad \text{and} \qquad \widetilde{\alpha}_{|t-t_0|^\gamma f,t_0} = \frac{\alpha+\gamma}{1+\beta}.$$

Hence, in both cases one can observe that the behaviour of local regularity of $I_0^\gamma f$ and $|t-t_0|^\gamma f(t)$ can not be completely deduced from pointwise and local Hölder exponents of f (it does not correspond to a simple translation of coefficient γ).

This deterministic example can be easily transposed into a stochastic context using multifractional Brownian motion, a Gaussian process introduced in [154] and [26] and applied as a probabilistic model in different fields (e.g. [32] and [33]). This process, denoted X^H is parametrized by a deterministic function $H : \mathbf{R} \to (0,1)$ and has interesting regularity properties. In the particular case of a so-called regularity function set to $H(t) = a + b \cdot f(t)$ and under some conditions on coefficients a and b, the regularity of X^H at t_0 almost surely satisfies

$$\alpha_{X^H,t_0} = \alpha_{f,t_0} \qquad \text{and} \qquad \widetilde{\alpha}_{X^H,t_0} = \widetilde{\alpha}_{f,t_0},$$

proving that stochastic processes can also have non-trivial behaviours(see [76]).

These two different examples illustrate the fact that pointwise and local Hölder exponents are not sufficient to describe entirely the local regularity of a deterministic function or the sample paths of a stochastic process. In this context, *2-microlocal analysis* is a tool that provides a finer characterization. In particular, it allows to describe how pointwise and local exponents evolve under the action of (pseudo-)differential operators and under multiplication by power functions. If it has been first introduced in a deterministic frame (PDE precisely, see [35]), a stochastic approach has been recently developed in [78]. This previous work exhibited a Kolmogorov-like criterion which gives an almost surely lower bound for the 2-microlocal frontier of a stochastic process. It also focused on the regularity of Gaussian processes at a fixed point $t_0 \in \mathbf{R}_+$. Thereby, the regularity of a Gaussian process X at any fixed t_0 is almost surely characterized by its incremental variance $\mathbb{E}[X_t - X_s]^2$. In particular, when considering the following Wiener integral $X_t = \int_0^t \eta(u) \mathrm{d}W_u$, it implies that the regularity of its sample paths is described by the behaviour of the deterministic function $t \mapsto \int_0^t \eta^2(u)\,\mathrm{d}u$.

Given this result, a natural goal is to generalize the statement to any stochastic integral

$$\forall t \in \mathbf{R}_+; \quad X_t = \int_0^t H_u \,\mathrm{d}M_u,$$

where H is a progressive continuous process and M is a local continuous martingale. In fact, we first prove in Theorem 1.6 a uniform result on continuous martingales which links up the regularity of the process to the regularity of its quadratic variation. In the specific case of

stochastic integrals, this 2-microlocal analysis result can be used to derive local behaviour of sample paths from the regularity of the integrand and the integrator.

Through these theorems and Examples 1.2, 1.5 and 1.6, we show that local regularity of martingales and stochastic integrals can vary along sample paths and may not be deterministic. Similar behaviours have already been exhibited in the literature, from the slow points of Brownian motion [152, 157] to more recent work on Lévy [94], multifractional [14] and Markov processes [196, 21].

Using the 2-microlocal frontier of stochastic integrals, we finally describe how to obtain regularity results for stochastic differential equations. In particular, if it is already known that Hölder regularity of the coefficients have an impact on the existence and uniqueness of solutions (e.g. [145] in the case of SPDE), we establish for SDE that it also subtly affects the local behaviour of the solution.

The paper is organized as follows: we start by a preliminary section which recalls properties of the classic 2-microlocal frontier in Section 1.2. Section 1.3 introduces another deterministic tool, the pseudo 2-microlocal frontier, closely related to the previous one. Our main result on the 2-microlocal frontier of continuous martingales is proved in Section 1.4. Results concerning stochastic integrals and stochastic differential equations are respectively developed in Sections 1.5 and 1.6. Finally, some technical proofs of deterministic and intermediate results are gathered in Appendix 1.7.

1.2 Preliminaries: Classic 2-microlocal analysis

The starting point of *2-microlocal analysis* is the definition of specific functional spaces, called *2-microlocal spaces* and denoted by $C_{t_0}^{\sigma,s'}$ where $\sigma, s' \in \mathbf{R}$ and $t_0 \in \mathbf{R}$ is a given point. Actually, as noted by HERBIN and LÉVY VÉHEL [78], it is usually sufficient to focus the study on spaces $C_{t_0}^{\sigma,s'}$, where $\sigma \in (0,1)$ and $s' \in \mathbf{R}$.

Definition 1.1 *A continuous function f belongs to $C_{t_0}^{\sigma,s'}$ if there exist $C > 0$, $\rho > 0$ and a polynomial P_t such that for all $u, v \in B(t_0, \rho)$,*

$$\left|\big(f(u) - P_t(u)\big) - \big((f(v) - P_t(v)\big)\right| \leq C|u-v|^{\sigma}\left(|u-t_0| + |v-t_0|\right)^{-s'}. \tag{1.1}$$

As proved by ECHELARD [60], P_t in fact corresponds to the Taylor expansion of f of order $\lfloor \sigma - s' \rfloor$ at t_0 (if $\lfloor \sigma - s' \rfloor > 0$, otherwise P_t is set to 0).

In many situations, the study can even be restricted to spaces $C_{t_0}^{\sigma,s'}$, where $(s',\sigma) \in \sigma_{0,0} := \{(s',\sigma) : \sigma \in [0,1) \text{ and } \sigma - s' \in [0,1)\}$. In this case, $P_t = 0$ and the previous characterization simply becomes

$$\forall u, v \in B(t_0, \rho); \quad |f(u) - f(v)| \leq C|u-v|^{\sigma}\left(|u-t_0| + |v-t_0|\right)^{-s'}.$$

Note that the definition of 2-microlocal spaces when $\sigma \notin (0,1)$ is slightly more complex and is given at the end of this section since it is little-used in this article.

The pointwise Hölder exponent of a function f is characterized as the supremum of the family of α such that f belongs to $C_{t_0}^{\alpha}$. Likewise, the *2-microlocal frontier* of f at t_0 is defined as the map $s' \mapsto \sigma_{f,t_0}(s')$ such that

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,t_0}(s') = \sup\{\sigma \in \mathbf{R} : f \in C_{t_0}^{\sigma,s'}\}. \tag{1.2}$$

We note equation 1.1 implies that $C_{t_0}^{\sigma,s'} \subset C_{t_0}^{\sigma',s'}$ when $\sigma' \leq \sigma$, therefore showing the 2-microlocal frontier exists and is well-defined.

It has been proved (see e.g. [60]) that the 2-microlocal frontier $s' \mapsto \sigma_{f,t_0}(s')$ satisfies several interesting properties:

- $\sigma_{f,t_0}(\cdot)$ is a concave and non-decreasing function;
- $\sigma_{f,t_0}(\cdot)$ has left- and right-derivatives between 0 and 1.

Furthermore, if α_{f,t_0} and $\widetilde{\alpha}_{f,t_0}$ respectively denote the pointwise and local Hölder exponents of f at t_0, then

- $\widetilde{\alpha}_{f,t_0} = \sigma_{f,t_0}(0)$;
- $\alpha_{f,t_0} = -\inf\{s' : \sigma_{f,t_0}(s') \geq 0\}$,

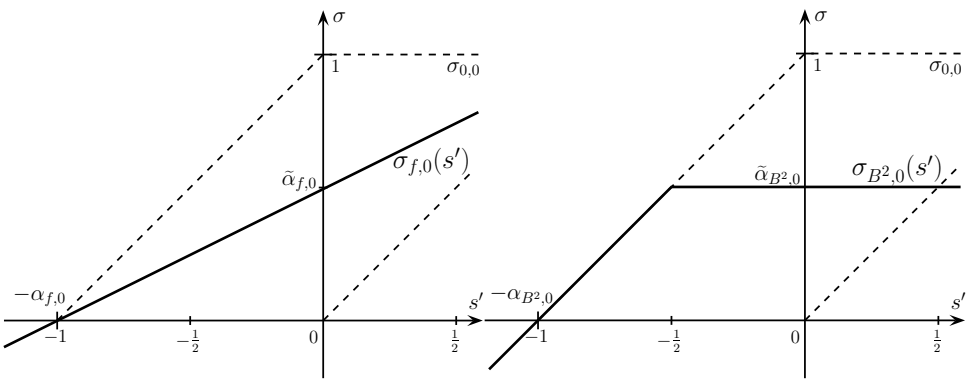
where the last equality holds under the assumption $\omega_f(h) = \mathrm{O}(\log(h))$. We use the convention $\alpha_{f,t_0} = \infty$ in the case σ_{f,t_0} is strictly positive. Let us also recall that several other regularity exponents such as the chirp, weak and oscillation exponents introduced in [7] and [138] that can be retrieved from the 2-microlocal frontier (see [60] for an exhaustive list).

As an example, we consider the Chirp function $f : t \mapsto |t|^\alpha \sin\big(|t|^{-\beta}\big)$ introduced previously. Its 2-microlocal frontier at $t_0 = 0$ (see Figure 1.1a) is

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,0}(s') = \frac{s' + \alpha}{1 + \beta}.$$

Stochastic examples can also be exhibited. Indeed, let us simply consider the random process $X : t \mapsto B_t^2$ where B is a Brownian motion. Based on results from [78], we observe that X almost surely has the following 2-microlocal frontier at $t_0 = 0$

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,0}(s') = \big(1 + s'\big) \wedge \frac{1}{2},$$



(a) Chirp function $f : x \mapsto x\sin(x^{-1})$ (b) Square of Brownian motion $X : t \mapsto B_t^2$

Figure 1.1: Examples of 2-microlocal frontiers at $t_0 = 0$

These two examples clearly illustrate the fact that pointwise and local Hölder exponents are not sufficient to describe entirely the local regularity of a function, whereas the complete characterization of the 2-microlocal frontier gives a wider and finer insight.

In particular, the knowledge of the 2-microlocal frontier allows to predict the evolution of regularity if a (pseudo-)differential operator is applied to f. Let first recall the definition of the Riemann-Liouville fractional integral of order $\alpha \in \mathbf{R}_+$ for a continuous function f:

$$I_{x+}^\alpha f : t \mapsto \frac{1}{\Gamma(\alpha)} \int_x^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u,$$

where $x \in \mathbf{R}$ is a fixed point. For every $\alpha \in \mathbf{R}_+$, the regularity of $I^{\alpha}_{x+}f$ satisfies

$$\forall s' \in \mathbf{R}; \quad \sigma_{I^{\alpha}_{x+}f,t_0}(s') = \sigma_{f,t_0}(s') + \alpha,$$

for all $t_0 > x$. Similarly, the function $g_\alpha : t \mapsto |t - t_0|^\gamma (f(t) - f(t_0))$ has its 2-microlocal frontier at t_0 equal to

$$s' \longmapsto \sigma_{g_\alpha,t_0}(s') = \sigma_{f,t_0}(s' + \alpha).$$

For the sake of completeness, we conclude this section with the definition of 2-microlocal spaces when $\sigma \notin (0,1)$. For all $\sigma < 0$ and $s' \in \mathbf{R}$, a continuous function f is said to belong to $C^{\sigma,s'}_{t_0}$ if there exist $C > 0$, $\rho > 0$ and a polynomial P such that for all $u, v \in B(t_0, \rho)$

$$\left|\left(I^m_{x+}f(u) - P(u)\right) - \left(\left(I^m_{x+}f(v) - P(v)\right)\right|\right. \leq C|u - v|^{\sigma+m}\left(|u - t_0| + |v - t_0|\right)^{-s'}, \tag{1.3}$$

where x is a fixed point such that $x < t_0$ and $m = -\lfloor \sigma \rfloor$.

The characterization of 2-microlocal spaces given in Equations (1.1) and (1.3) has been first introduced in [109] and then developed in [169, 60]. Equivalent definitions based Fourier [35] or wavelet [93, 138] transforms have been studied in the literature. Finally, even though there does not exist a time-domain characterization when $\sigma \in \mathbf{N}$ or $s \in \mathbf{N}$, it does not affect the results developed in this article since the 2-microlocal frontier is not sensitive to this singularity.

1.3 Pseudo 2-microlocal analysis

Owing to definitions (1.1) and (1.3), the simple polynomial function $t \mapsto (t - t_0)^k$, $k \in \mathbf{N}$, has a 2-microlocal frontier at t_0 equal to $+\infty$, on the contrary to the case $k \in \mathbf{R} \setminus \mathbf{N}$. Intuitively, we understand that 2-microlocal analysis does not take into account the polynomial component of a function around t_0. Nevertheless, it might sometimes be necessary and interesting to consider the regularity of this factor. Indeed, if we simply consider a Brownian motion B, we know that its quadratic variation is equal to $\langle B \rangle_t = t$, and therefore, if we want to obtain a result which links up the regularities of B and $\langle B \rangle$, we need a proper tool to characterize the variations of $\langle B \rangle$.

The concept of *pseudo 2-microlocal analysis* introduced in this section has this purpose. The first step consists in the definition of *pseudo 2-microlocal spaces* $\underline{C}^{s,s'}_{t_0}$ and a *pseudo 2-microlocal frontier* Σ_{f,t_0} which are similar to classic spaces $C^{s,s'}_{t_0}$ and frontier σ_{f,t_0}, but which also take into account polynomials in the characterization of regularity. Then, properties of this particular frontier are studied, and in particular Theorem 1.1 is able to link up classic and pseudo 2-microlocal frontiers.

1.3.1 Pseudo 2-microlocal spaces and frontier

Definition 1.2 (Pseudo 2-microlocal spaces)

1. *Let $\sigma \in [0, +\infty)$, $s' \in \mathbf{R}$ and $t_0 \in \mathbf{R}$. A continuous function f is said to belong to $\underline{C}^{\sigma,s'}_{t_0}$ if there exist $C > 0$ and $\rho > 0$ such that for all $u, v \in B(t_0, \rho)$,*

$$|f(u) - f(v)| \leq C|u - v|^{\sigma}\left(|u - t_0| + |v - t_0|\right)^{-s'}.$$

2. *Let $\sigma \in (-\infty, 0)$, $s' \in \mathbf{R}$ and $t_0 \in \mathbf{R}$. A continuous function f is said to belong to $\underline{C}^{\sigma,s'}_{t_0}$ if there exist $C > 0$ and $\rho > 0$ such that for all $u, v \in B(t_0, \rho)$,*

$$\left|I^m_{t_0+}(f - f(t_0))(u) - I^m_{t_0+}(f - f(t_0))(v)\right| \leq C|u - v|^{\sigma+m}\left(|u - t_0| + |v - t_0|\right)^{-s'},$$

where $m = -\lfloor \sigma \rfloor$.

We note this definition differs from the Characterizations (1.1) and (1.3) of 2-microlocal spaces in the polynomial component which is subtracted in the classic case. Similarly to Equation (1.2), the pseudo 2-microlocal frontier Σ_{f,t_0} is defined by

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t_0}(s') = \sup\left\{\sigma \in \mathbf{R} : f \in \underline{\mathbf{C}}_{t_0}^{\sigma,s'}\right\}.$$

The idea of a *pseudo 2-microlocal frontier* has first been introduced by HERBIN and LÉVY VÉHEL [78] in the particular case $(s',\sigma) \in \sigma_{0,0}$. The definition above extends it to the whole 2-microlocal domain.

As a corollary, we also define the *pseudo pointwise and local Hölder exponents*, which might differ from classic ones.

Definition 1.3 *Let f be continuous function and $t_0 \in \mathbf{R}$. Pseudo pointwise and local Hölder exponents of f at t_0 are respectively defined as*

$$\underline{\boldsymbol{\alpha}}_{f,t_0} = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t_0,\rho)} \frac{|f(u)-f(v)|}{\rho^\alpha} < \infty\right\}$$

and

$$\underline{\widetilde{\boldsymbol{\alpha}}}_{f,t_0} = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t_0,\rho)} \frac{|f(u)-f(v)|}{|u-v|^\alpha} < \infty\right\}.$$

These coefficients satisfy $\underline{\boldsymbol{\alpha}}_{f,t_0} = -\inf\{s' : \Sigma_{f,t_0}(s') \geq 0\}$ and $\underline{\widetilde{\boldsymbol{\alpha}}}_{f,t_0} = \Sigma_{f,t_0}(0)$.

We observe that the definitions of the classic and pseudo 2-microlocal spaces coincide inside the domain $\sigma_{0,0}$. Hence, if the graph of one of the frontiers belongs to $\sigma_{0,0}$, both frontiers must coincide. For instance, sample paths of a Brownian motion B satisfy this condition, since it has been proved in [78] that almost surely for all $t \in \mathbf{R}_+$,

$$\forall s' \in \mathbf{R}; \quad \sigma_{B,t}(s') = \left(\frac{1}{2}+s'\right) \wedge \frac{1}{2} \qquad \Longrightarrow \qquad (s',\sigma_{B,t}(s')) \in \sigma_{0,0}.$$

Theorem 1.1 completely characterizes the relationship between the classic and pseudo 2-microlocal frontiers.

Example 1.1 We now illustrate the previous concepts on a simple example: $f : x \mapsto |x|^\alpha$, where $\alpha > 0$. We prove that the pseudo 2-microlocal frontier of f at 0 is equal to

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,0}(s') = (\alpha + s') \wedge 1.$$

1. To obtain the lower bound, we use a simple result from [60]: there exist $C > 0$ and $\rho > 0$ such that for all $u, v \in B(0,\rho)$,

$$|f(u)-f(v)| \leq C|u-v|\big(|u|+|v|\big)^{\alpha-1},$$

which implies for every $\sigma \in [0,1]$,

$$\begin{aligned}|f(u)-f(v)| &= |f(u)-f(v)|^\sigma \cdot |f(u)-f(v)|^{1-\sigma}\\ &\leq C|u-v|^\sigma\big(|u|+|v|\big)^{\sigma(\alpha-1)} \cdot \big(|u|+|v|\big)^{\alpha(1-\sigma)}\\ &= C|u-v|^\sigma\big(|u|+|v|\big)^{\alpha-\sigma}.\end{aligned}$$

Thus, for all $s' \in [-\alpha, 1-\alpha]$, f belongs to $\underline{\mathbf{C}}_0^{\sigma,s'}$ when $\sigma \leq (s'+\alpha) \wedge 1$.

2. On the other side, we note that for all $u \in \mathbf{R}$, $|f(u) - f(0)| = |u|^\alpha = |u|^{\alpha+s'}|u|^{-s'}$, and therefore $\Sigma_{|x|^\alpha,0}(s') \leq \alpha + s'$. Furthermore, as f is differentiable, for all $s' \in \mathbf{R}$, $\rho > 0$ and $\varepsilon > 0$, we know that

$$\sup_{u,v \in B(0,\rho)} \frac{|f(u) - f(v)|}{|u-v|^{1+\varepsilon}\big(|u| + |v|\big)^{-s'}} = +\infty,$$

and therefore $\Sigma_{f,0}(s') \leq 1$.

If we compare classic and pseudo 2-microlocal frontiers of $x \mapsto |x|^\alpha$, we notice that for all $s' \in \mathbf{R}$

$$\Sigma_{|x|^\alpha,0}(s') = (\alpha + s') \wedge 1 \quad \text{and} \quad \sigma_{|x|^\alpha,0}(s') = \begin{cases} \alpha + s' & \text{if } \alpha \in \mathbf{R}_+ \setminus \mathbf{N}; \\ +\infty & \text{if } \alpha \in \mathbf{N}. \end{cases}$$

Therefore, it shows that the pseudo 2-microlocal frontier indeed takes into account polynomials and does not make any distinction between integer and non-integers powers in terms of regularity. Furthermore, this example also illustrates the fact that classic and pseudo 2-microlocal frontiers do not coincide in general.

Remark 1.1 We recalled previously that there exist characterizations of 2-microlocal spaces using wavelet (or Fourier) transform. Hence, we know from [93] that f belongs to $C^{\sigma,s'}_{t_0}$ if and only if

$$\forall j,k \in \mathbf{Z} \text{ s.t. } |t_0 - k2^{-j}| \leq 1; \quad |d_{j,k}| \leq C2^{-j\sigma}\big(2^{-j} + |k2^{-j} - t_0|\big)^{-s'},$$

where $N > \max(\sigma, \sigma - s')$, $d_{j,k} = 2^j\langle f, \psi(2^jx - k)\rangle$, $\psi \in \mathscr{S}(\mathbf{R})$ has N vanishing moments and is such that $\{\psi_{j,k} = 2^{j/2}\psi(2^jx - k)\}_{(j,k)\in\mathbf{Z}^2}$ forms an orthonormal basis of $L^2(\mathbf{R})$.

Therefore, a natural question is to wonder if this characterization can be adapt to pseudo 2-microlocal spaces. In fact, a simple calculation proves that if we replace the vanishing moments hypothesis by

$$\forall k \in \{1, \dots, N\}; \quad \langle x^k, \psi\rangle \neq 0,$$

where as previously $N > \max(\sigma, \sigma - s')$, then this definition becomes a characterization of pseudo 2-microlocal spaces.

1.3.2 Properties

In this second part, we prove a few important results related to the pseudo 2-microlocal frontier and which are useful later in the article. The following main theorem gives a general formula which links up classic and pseudo 2-microlocal frontiers for continuous functions.

Theorem 1.1 *Let f be a continuous function and t_0 be in $\mathbf{R}$. We define p_{f,t_0} as the integer*

$$p_{f,t_0} = \inf\{n \geq 1 : f^{(n)}(t_0) \text{ exists and } f^{(n)}(t_0) \neq 0\},$$

with the usual convention $\inf\{\emptyset\} = +\infty$. *Then, the pseudo 2-microlocal frontier of f at t_0 is equal to*

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t_0}(s') = \sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0}) \wedge 1,$$

unless f is locally constant at t_0, which implies in that specific case:

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t_0}(s') = \sigma_{f,t_0}(s') = +\infty.$$

We note that this result is consistent with our previous calculation of the pseudo frontier of $x \mapsto |x|^\alpha$ in Example 1.1. Furthermore, properties on the map $s' \mapsto \Sigma_{f,t_0}(s')$ can be deduced as a simple corollary.

Corollary 1.2 *Similarly to the classic frontier, the pseudo 2-microlocal frontier of f at t_0 satisfies*

- Σ_{f,t_0} *is a concave and non-decreasing function;*
- Σ_{f,t_0} *has left and right derivatives between* 0 *and* 1.

Using the previous theorem, we can also obtain a useful result which illustrates the behaviour of the pseudo 2-microlocal frontier under an integral operator.

Theorem 1.3 *Let f be a continuous function and F be*

$$\forall t \in \mathbf{R}; \quad F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

Then, for all t_0 in $\mathbf{R}$, the pseudo 2-microlocal frontier of F at t_0 is equal to:

$$\forall s' \in \mathbf{R}; \quad \Sigma_{F,t_0}(s') = \begin{cases} (1+s') \wedge 1 & \text{if } f(t_0) \neq 0; \\ (\Sigma_{f,t_0}(s')+1) \wedge 1 & \text{if } f(t_0) = 0 \text{ and is not locally constant}; \\ +\infty & \text{if } f \text{ is locally equal to } o \text{ at } t_0, \end{cases}$$

For the sake of readability, technical proofs of Theorems 1.1 and 1.3 are given in Appendix 1.7.

To end this section, we establish a result on the pseudo 2-microlocal frontier of composed functions, which is necessary in Section 1.4 to determine the regularity of martingales.

Proposition 1.4 *Let f and g be two continuous functions and let h be the composition $g \circ f$. Then, for every $t \in \mathbf{R}$, the pseudo 2-microlocal frontier $\Sigma_{h,t}$ of h at t satisfies the inequality*

$$\Sigma_{h,t}\left(s'_f \cdot \Sigma_{g,f(t)}(s'_g) + s'_g \cdot \underline{\boldsymbol{\alpha}}_{f,t}\right) \geq \Sigma_{g,f(t)}(s'_g) \cdot \Sigma_{f,t}(s'_f),$$

for all $s'_f \in [-\underline{\boldsymbol{\alpha}}_{f,t}, +\infty)$ and $s'_g \in [-\underline{\boldsymbol{\alpha}}_{g,f(t)}, 0]$.

Proof. Let $t \in \mathbf{R}$, $s'_f \geq -\underline{\boldsymbol{\alpha}}_{f,t}$ and $s'_g \in [-\underline{\boldsymbol{\alpha}}_{g,f(t)}, 0]$. For every $\varepsilon > 0$, there exist $C_f, C_g > 0$ and $\rho > 0$ such for all $u, v \in B(t, \rho)$,

$$|f(u) - f(v)| \leq C_f |u - v|^{\Sigma_{f,t}(s'_f) - \varepsilon} \left(|u - t| + |v - t|\right)^{-s'_f}$$

and for all $x, y \in B(f(t), \rho)$

$$|g(x) - g(y)| \leq C_g |x - y|^{\Sigma_{g,f(t)}(s'_g) - \varepsilon} \left(|x - f(t)| + |y - f(t)|\right)^{-s'_g}.$$

Therefore, we obtain

$$\begin{aligned} &|h(u) - h(v)| \\ &= |(g \circ f)(u) - (g \circ f)(v)| \\ &\leq C_g |f(u) - f(v)|^{\Sigma_{g,f(t)}(s'_g) - \varepsilon} \left(|f(u) - f(t)| + |f(v) - f(t)|\right)^{-s'_g} \\ &\leq C |u - v|^{(\Sigma_{g,f(t)}(s'_g) - \varepsilon)(\Sigma_{f,t}(s'_f) - \varepsilon)} \left(|u - t| + |v - t|\right)^{-s'_f \cdot (\Sigma_{g,f(t)}(s'_g) - \varepsilon) - s'_g \cdot (\underline{\boldsymbol{\alpha}}_{f,t} - \varepsilon)}, \end{aligned}$$

since $|f(u) - f(t)| \leq C_f |u - t|^{\underline{\boldsymbol{\alpha}}_{f,t} - \varepsilon}$ and $-s'_g \geq 0$. Then, using the continuity of the pseudo 2-microlocal frontier, we get the expected inequality. □

Using the previous proposition, we can obtain simpler inequalities, which nevertheless might be less accurate with specific functions f and g.

Corollary 1.5 *Let f and g be two continuous functions and let h be the composition $g \circ f$. Then, for any $x \in \mathbf{R}$, the pseudo 2-microlocal frontier $\Sigma_{h,x}$ of h at x satisfies the inequalities:*

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{h,x}; \quad \Sigma_{h,x}(s') \geq \underline{\widetilde{\boldsymbol{\alpha}}}_{g,f(x)} \cdot \Sigma_{f,x}(s'/\underline{\widetilde{\boldsymbol{\alpha}}}_{g,f(x)}),$$

and

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{h,x}; \quad \Sigma_{h,x}(s') \geq \underline{\widetilde{\boldsymbol{\alpha}}}_{f,x} \cdot \Sigma_{g,f(x)}(s'/\underline{\boldsymbol{\alpha}}_{f,x}).$$

In particular, if we consider the pseudo pointwise and local Hölder exponents, we obtain: and thus, in particular of pseudo pointwise exponents:

$$\underline{\boldsymbol{\alpha}}_{h,x} \geq \underline{\boldsymbol{\alpha}}_{g,f(x)} \cdot \underline{\boldsymbol{\alpha}}_{f,x} \qquad \textit{and} \qquad \underline{\widetilde{\boldsymbol{\alpha}}}_{h,x} \geq \underline{\widetilde{\boldsymbol{\alpha}}}_{g,f(x)} \cdot \underline{\widetilde{\boldsymbol{\alpha}}}_{f,x}$$

Proof. The first two inequalities are obtained using the previous Proposition 1.4 in the particular cases $s'_g = 0$ and $s'_f = 0$. The inequalities on pseudo exponents are deduced from the previous ones, when $s' = \underline{\boldsymbol{\alpha}}_{g,f(x)} \cdot \underline{\boldsymbol{\alpha}}_{f,x}$ and $s' = 0$. □

1.4 Stochastic 2-microlocal analysis of martingales

In the remaining of the article, we study the 2-microlocal frontier of stochastic processes. As noted in [78], when we consider a random process X, the classic $\sigma_{X(\omega),t_0}(s',\omega)$ and pseudo $\Sigma_{X(\omega),t_0}(s',\omega)$ 2-microlocal frontiers at t_0 become random functions. The stochastic 2-microlocal frontier $(s',\omega) \mapsto \sigma_{X(\omega),t_0}(s',\omega)$ is clearly measurable as it is continuous on the variable s' and the process X is progressive.

As previously announced, we first study in Theorem 1.6 the pseudo 2-microlocal frontier of continuous martingales. We deduce from this result uniform lower bounds for semimartingales in Proposition 1.8. Then, we construct an example of martingale to exhibit interesting sample path properties. Finally, we prove how Theorem 1.6 can be easily extended to a time changed multifractional Brownian motion.

1.4.1 2-microlocal frontier of continuous martingales

In this part, we study the regularity of continuous local martingales in order to obtain a relationship between the pseudo 2-microlocal frontiers of the martingale on one hand and its quadratic variation on the other hand. The following theorem states our main result.

Theorem 1.6 *Let M be a continuous local martingale and $\langle M \rangle$ be its quadratic variation. Then, with probability one, for all $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontiers of M at t satisfies the following equality:*

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}; \quad \Sigma_{M,t}(s') = \frac{1}{2}\Sigma_{\langle M \rangle,t}\left(2s'\right). \tag{1.4}$$

Consequently, pseudo pointwise and local Hölder exponents are equal to

$$\underline{\boldsymbol{\alpha}}_{M,t} = \frac{\underline{\boldsymbol{\alpha}}_{\langle M \rangle,t}}{2} \quad \textit{and} \quad \underline{\widetilde{\boldsymbol{\alpha}}}_{M,t} = \frac{\underline{\widetilde{\boldsymbol{\alpha}}}_{\langle M \rangle,t}}{2}.$$

For the sake of readability, we divide the proof of expression (1.4) in two parts (lower and upper bounds).

Proof of the lower bound

Let us first prove that almost surely for all $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of M at t satisfies

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}; \quad \Sigma_{M,t}(s') \geq \frac{1}{2}\Sigma_{\langle M\rangle,t}\left(2s'\right).$$

Without any restriction, we can suppose that $M_0 = 0$. Let set for all $t \in \mathbf{R}_+$ $T_t = \inf\{s : \langle M\rangle_s > t\}$. Then, according to the extended Dubins-Schwarz Theorem (5.1.7 in [161]), we know there exists an enlargement $(\widetilde{\Omega}, \widetilde{\mathscr{F}}_t, \widetilde{\mathbb{P}})$ of the probability space $(\Omega, \mathscr{F}_{T_t}, \mathbb{P})$ and a Brownian motion $\widetilde{\beta}$ on $\widetilde{\Omega}$ such that the process

$$B_t = M_{T_t} + \int_0^t \mathbf{1}_{\{s>\langle M\rangle_\infty\}} \mathrm{d}\widetilde{\beta}_s$$

is a $(\widetilde{\mathscr{F}}_t)_t$ Brownian motion and for every $t \in \mathbf{R}_+$, $M_t = B_{\langle M\rangle_t}$. We note that the enlargement has the following form: $\widetilde{\Omega} = \Omega \times \Omega'$, $\widetilde{\mathscr{F}}_t = \mathscr{F}_{T_t} \otimes \mathscr{F}'_t$ and $\widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$.

Based on results from [78], it is known that $\widetilde{\mathbb{P}}$-almost surely for all $t \in \mathbf{R}_+$, the classic and pseudo local Hölder exponents of the Brownian motion B are equal to: $\underline{\widetilde{\boldsymbol{\alpha}}}_{B,t} = \widetilde{\alpha}_{B,t} = 1/2$. Then, as M corresponds to the composition of B and $\langle M\rangle$, from Corollary 1.5, we obtain that $\widetilde{\mathbb{P}}$-almost surely for all $t \in \mathbf{R}_+$,

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}; \quad \Sigma_{M,t}(s') \geq \underline{\widetilde{\boldsymbol{\alpha}}}_{B,t} \cdot \Sigma_{\langle M\rangle,t}\left(s'/\underline{\widetilde{\boldsymbol{\alpha}}}_{B,t}\right) = \frac{1}{2}\Sigma_{\langle M\rangle,t}\left(2s'\right).$$

More precisely, there exists $\widetilde{\Omega}_0 \in \widetilde{\mathscr{F}}$ such that $\widetilde{\mathbb{P}}(\widetilde{\Omega}_0) = 1$ and for all $\widetilde{\omega} \in \widetilde{\Omega}_0$ and $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of $M_\cdot(\widetilde{\omega})$ at t satisfies

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{M(\widetilde{\omega}),t}(\widetilde{\omega}); \quad \Sigma_{M(\widetilde{\omega}),t}(s', \widetilde{\omega}) \geq \frac{1}{2}\Sigma_{\langle M\rangle(\widetilde{\omega}),t}\left(2s', \widetilde{\omega}\right).$$

Let $\omega \in \Omega$, according to the definition of the enlargement $\widetilde{\Omega}$, we know that for all $\omega' \in \Omega'$,

$$\Sigma_{M,t}(s', (\omega, \omega')) = \Sigma_{M,t}(s', \omega) \quad \text{and} \quad \Sigma_{\langle M\rangle,t}(s', (\omega, \omega')) = \Sigma_{\langle M\rangle,t}(s', \omega).$$

Therefore, the set $\widetilde{\Omega}_0$ has the form $\widetilde{\Omega}_0 = \Omega_0 \times \Omega'$ where $\Omega_0 \in \mathscr{F}$ and $\mathbb{P}(\Omega_0) = 1$. Hence, we have proved that $\mathbb{P}$-almost surely for all $t \in \mathbf{R}_+$, $\Sigma_{M,t}(s') \geq 2^{-1}\Sigma_{\langle M\rangle,t}\left(2s'\right)$ for all $s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}$.

Proof of the upper bound

The second step is to establish that almost surely for all $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of M at t satisfies

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}; \quad \Sigma_{M,t}(s') \leq \frac{1}{2}\Sigma_{\langle M\rangle,t}\left(2s'\right).$$

As previously, using Dubins-Schwarz Theorem, there exists a Brownian motion B such that almost surely for all $t \in \mathbf{R}_+$, we have $M_t = B_{\langle M\rangle_t}$. Then, to prove the upper bound, we use the following technical lemma satisfied by the Brownian motion B.

Lemma 1.1 *Let B be a Brownian motion. Then, there exists an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, $N \in \mathbf{N}$, $\varepsilon > 0$, there exists $h(\omega) > 0$ such that for all $\rho \leq h(\omega)$ and $t \in [0, N]$, we have*

$$\sup_{u,v \in B(t,\rho)} \{|B_u - B_v|\} \geq \rho^{1/2+\varepsilon}.$$

This lemma is a corollary of a more general result proved on the multifractional Brownian motion at the end of this section (see Proposition 1.10).

Therefore, let us set $\omega \in \Omega_0$, $N \in \mathbf{N}$ and $\varepsilon > 0$. We denote by $\langle M\rangle^N$ the increasing process $\langle M\rangle_t^N = \langle M\rangle_t \wedge N$ and M^N the compound process $M_t^N = B_{\langle M\rangle_t^N}$. According to the definition of the pseudo 2-microlocal frontier, for $s' \in \mathbf{R}$ and $t \in \mathbf{R}_+$, there exist sequences $(s_n(\omega))_n, (t_n(\omega))_n$ such that

$$\lim_{n\to+\infty} s_n(\omega) = \lim_{n\to+\infty} t_n(\omega) = t$$

and for all $n \in \mathbf{N}$,

$$\frac{\left|\langle M\rangle_{t_n}^N - \langle M\rangle_{s_n}^N\right|}{|t_n - s_n|^{\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon}\left(|t-t_n|+|t-s_n|\right)^{-s'}}(\omega) \geq 1. \tag{1.5}$$

Without any loss of generality, we can also suppose for all $n \in \mathbf{N}$ that $s_n(\omega) \leq t_n(\omega)$ and $|\langle M\rangle_{t_n}^N(\omega) - \langle M\rangle_{s_n}^N(\omega)| \leq h(\omega)$, where the variable $h(\omega)$ is defined in Lemma 1.1.

Then, since $\langle M\rangle_\cdot^N(\omega) \leq N$, for each $n \in \mathbf{N}$ there exists $u_n(\omega), v_n(\omega) \in [\langle M\rangle_{s_n}^N, \langle M\rangle_{t_n}^N]$ such that

$$|B_{u_n} - B_{v_n}| \geq \left|\langle M\rangle_{t_n}^N - \langle M\rangle_{s_n}^N\right|^{1/2+\varepsilon}. \tag{1.6}$$

As $t \mapsto \langle M\rangle_t(\omega)$ is a continuous non-decreasing function, there exist $x_n(\omega), y_n(\omega)$ such that $\langle M\rangle_{x_n}^N = u_n$ and $\langle M\rangle_{y_n}^N = v_n$ and $[x_n, y_n] \subset [s_n, t_n]$.
Then, using inequalities (1.5) and (1.6), we obtain

$$\begin{aligned}\left|M_{x_n}^N - M_{y_n}^N\right| &= \left|B_{u_n} - B_{v_n}\right| \\ &\geq \left|\langle M\rangle_{t_n}^N - \langle M\rangle_{s_n}^N\right|^{1/2+\varepsilon} \\ &\geq |t_n - s_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon)}\left(|t-t_n|+|t-s_n|\right)^{-s'\cdot(1/2+\varepsilon)}.\end{aligned}$$

Let now distinguish the two different cases.

1. If $s' \leq 0$, we note that $|x_n - y_n| \leq |s_n - t_n|$ and $|t-x_n|+|t-y_n| \leq 2\left(|t-t_n|+|t-s_n|\right)$. Hence, there exists $C > 0$ such that for all $n \in \mathbf{N}$,

$$|M_{x_n}^N - M_{y_n}^N| \geq C|x_n - y_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon)}\left(|t-x_n|+|t-y_n|\right)^{-s'\cdot(1/2+\varepsilon)},$$

   since $-s'\cdot(1/2+\varepsilon) \geq 0$ and $(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon) \geq 0$.

2. If $s' \geq 0$, up to an extraction, we can suppose the following convergence

$$\lim_{n\to\infty} \frac{|t_n - s_n|}{|t-t_n|+|t-s_n|} = m,$$

   where $m \in [0,1]$ since the sequence is positive and bounded by 1.

   - If $m = 0$, as $[x_n, y_n] \subset [s_n, t_n]$, we obtain

$$\left(|t-t_n|+|t-s_n|\right) \sim_{n\to\infty} \left(|t-x_n|+|t-y_n|\right).$$

     Therefore, since we also have $|x_n - y_n| \leq |s_n - t_n|$, for all $n \in \mathbf{N}$ large enough, we get

$$\left|M_{x_n}^N - M_{y_n}^N\right| \geq C|x_n - y_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon)}\left(|t-x_n|+|t-y_n|\right)^{-s'\cdot(1/2+\varepsilon)},$$

     where C is a positive constant.

- If $m > 0$, we know that $|t_n - s_n| \sim_n m \cdot \big(|t - t_n| + |t - s_n|\big)$. Therefore, from (1.5), there exists $C > 0$ such that for all $n \in \mathbf{N}$ large enough

$$\left|\langle M\rangle^N_{t_n} - \langle M\rangle^N_{s_n}\right| \geq C\big(|t - t_n| + |t - s_n|\big)^{\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon-s'},$$

which necessarily implies that $\Sigma_{\langle M\rangle^N,t}(s') = \underline{\boldsymbol{\alpha}}_{\langle M\rangle^N,t} + s'$, according to Definition 1.3 for $\underline{\boldsymbol{\alpha}}_{\langle M\rangle^N,t}$. Then, for all $n \in \mathbf{N}$ large enough, we obtain

$$\begin{aligned}
\left|M^N_{x_n} - M^N_{y_n}\right| &\geq |t_n - s_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon)}\big(|t - t_n| + |t - s_n|\big)^{-s'\cdot(1/2+\varepsilon)} \\
&\geq C|t_n - s_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon-s')\cdot(1/2+\varepsilon)} \\
&= C|t_n - s_n|^{(\underline{\boldsymbol{\alpha}}_{\langle M\rangle^N,t}+\varepsilon)\cdot(1/2+\varepsilon)} \\
&\geq C|x_n - y_n|^{(\underline{\boldsymbol{\alpha}}_{\langle M\rangle^N,t}+\varepsilon)\cdot(1/2+\varepsilon)} \qquad \text{as } \underline{\boldsymbol{\alpha}}_{\langle M\rangle^N,t} + \varepsilon > 0 \\
&= C|x_n - y_n|^{(\underline{\boldsymbol{\alpha}}_{\langle M\rangle^N,t}+\varepsilon-s')\cdot(1/2+\varepsilon)} \cdot |x_n - y_n|^{-s'\cdot(1/2+\varepsilon)} \\
&\geq C|x_n - y_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon)}\big(|t - x_n| + |t - y_n|\big)^{-s'\cdot(1/2+\varepsilon)},
\end{aligned}$$

since $-s' \leq 0$ and $|x_n - y_n| \leq \big(|t - x_n| + |t - y_n|\big)$.

To summarize, in each case, we have proved that for all $s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}$ there exists $C > 0$ such that for all $n \in \mathbf{N}$ large enough,

$$|M^N_{x_n} - M^N_{y_n}| \geq C|x_n - y_n|^{(\Sigma_{\langle M\rangle^N,t}(s')+\varepsilon)\cdot(1/2+\varepsilon)}\big(|t - x_n| + |t - y_n|\big)^{-s'\cdot(1/2+\varepsilon)},$$

which proves that for all $N \in \mathbf{N}$, $\varepsilon > 0$ and $s' \in [-\underline{\boldsymbol{\alpha}}_{M,t}, 0]$, with probability one

$$\forall t \in \mathbf{R}_+; \quad \Sigma_{M^N,t}\big(s' \cdot (1/2 + \varepsilon)\big) \leq (\Sigma_{\langle M\rangle^N,t}(s') + \varepsilon) \cdot (1/2 + \varepsilon).$$

We observe that for all $\omega \in \Omega_0$ and $t \in \mathbf{R}_+$, there exists $N_0(\omega)$ such that for all $N \geq N_0(\omega)$,

$$\forall s' \in \mathbf{R}; \quad \Sigma_{M^N,t}(s') = \Sigma_{M,t}(s') \qquad \text{and} \qquad \Sigma_{\langle M\rangle^N,t}(s') = \Sigma_{\langle M\rangle,t}(s').$$

Therefore, as $N \to +\infty$, we obtain with probability one

$$\forall t \in \mathbf{R}_+; \quad \Sigma_{M,t}\big(s' \cdot (1/2 + \varepsilon)\big) \leq (\Sigma_{\langle M\rangle,t}(s') + \varepsilon) \cdot (1/2 + \varepsilon).$$

Then, using a sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ which converges to zero and $s' \in \mathbf{Q}$ (sufficient as the pseudo 2-microlocal frontier is continuous), we get the expected inequality. With probability one and for all $t \in \mathbf{R}_+$, $\Sigma_{M,t}(s') \leq 2^{-1}\Sigma_{\langle M\rangle,t}\big(2s'\big)$ for all $s' \geq -\underline{\boldsymbol{\alpha}}_{M,t}$.

□

Remark 1.2 In this context of martingales, one could imagine to introduce a more general definition of the 2-microlocal frontier where the Euclidean metric is replaced by other distance on $\mathbf{R}_+$. In particular, we could choose the random metric $d_{\langle M\rangle}(x, y) = |\langle M\rangle_x - \langle M\rangle_y|$ (which is in fact a pseudo-metric) and characterize the pseudo 2-microlocal local frontier $\Sigma^{d_{\langle M\rangle}}_{M,t}$ of the martingale M with respect to this metric. Using Dubins-Schwarz, we can prove show that we would obtain:

$$\forall s' \in [-\tfrac{1}{2}, 0]; \quad \Sigma^{d_{\langle M\rangle}}_{M,t}(s') = \tfrac{1}{2} + s'.$$

However, on the contrary to Theorem 1.6, this expression of the 2-microlocal frontier of M does not capture all the regularity of M with respect the Euclidean metric.

Theorem 1.6 characterizes the regularity of a martingale M in terms of pseudo 2-microlocal frontier. In the following corollary, we extend this property to the 2-microlocal frontier of M.

Corollary 1.7 *Suppose M is a continuous local martingale and $\langle M\rangle$ denotes its quadratic variation. Then, with probability one, for all $t \in \mathbf{R}$, M satisfies*

$$\forall s' \geq -\alpha_{M,t}; \quad \sigma_{M,t}(s') = \Sigma_{M,t}(s') = \frac{1}{2}\Sigma_{\langle M\rangle,t}\left(2s'\right).$$

In particular, the pointwise and local Hölder exponents of M and $\langle M\rangle$ satisfy almost surely

$$\forall t \in \mathbf{R}_+; \quad \alpha_{M,t} = \underline{\boldsymbol{\alpha}}_{M,t} = \frac{\underline{\boldsymbol{\alpha}}_{\langle M\rangle,t}}{2} \quad \textit{and} \quad \widetilde{\alpha}_{M,t} = \underline{\widetilde{\boldsymbol{\alpha}}}_{M,t} = \frac{\underline{\widetilde{\boldsymbol{\alpha}}}_{\langle M\rangle,t}}{2}.$$

Proof. According to Theorem 1.1, we know that almost surely for all $t \in \mathbf{R}_+$ when M is not locally constant at t,

$$\forall s' \in \mathbf{R}; \quad \Sigma_{M,t}(s') = \sigma_{M,t}(s') \wedge (s' + p_{M,t}) \wedge 1,$$

where $p_{M,t} = \inf\{n \geq 1 : M^{(n)}(t) \text{ exists and } M^{(n)}(t) \neq 0\}$, with the usual convention $\inf\{\emptyset\} = +\infty$.

Owing to Theorem 1.6, we know that with probability one, $\Sigma_{M,t} < 1$ for all $t \in \mathbf{R}_+$. Hence, it is sufficient to prove that $p_{M,t} = \infty$ to obtain the expected equality between the classic and pseuo 2-microlocal frontiers of the martingale. Similarly to the proof of Theorem 1.1, up to an enlargement argument, there exists a Brownian motion such that almost surely for all $t \in \mathbf{R}_+$, $M_t = B_{\langle M\rangle_t}$.

Dvoretzky, Erdős, and Kakutani [59] have proved that with probability one, the Brownian motion has no increasing point, i.e. for all $x \in \mathbf{R}$, there does no exist $\delta > 0$ such that

$$\max_{x-\delta \leq u \leq x} B_u = B_x = \min_{x \leq u \leq x+\delta} B_u.$$

As a consequence, for any $x \in \mathbf{R}$, there exists a decreasing (or increasing) sequence $x_n \to x$ such that $B_{x_n} = B_x$.

Let us set $\omega \in \Omega$ and $t \in \mathbf{R}_+$. Owing to the previous remark on the Brownian motion, there exists $t_n \to t$ such that $M_{t_n} = M_t$ for all $n \in \mathbf{N}$. Suppose there exists $k \in \mathbf{N} \setminus \{0\}$ such that $M^{(k)}(t)$ exists. Then, we note that

$$M^{(1)}(t)(\omega) = \lim_{n\to\infty} \frac{M_{t_n} - M_t}{t_n - t}(\omega) = 0.$$

Hence, by induction, if the previous equality holds for any $i \in \{1, \ldots, k-1\}$, we necessarily have

$$M^{(k)}(t)(\omega) = \lim_{n\to\infty} \frac{M_{t_n} - M_t}{(t_n - t)^k}(\omega) = 0.$$

Therefore, with probability and for all $t \in \mathbf{R}_+$, we have proved that $p_{M,t} = \infty$, implying using Theorem 1.1 that $\sigma_{M,t}(s') = \Sigma_{M,t}(s')$ for all $s' \geq -\alpha_{M,t}$. □

To end this section, we establish a lower bound for the pseudo 2-microlocal frontier of semimartingales.

Proposition 1.8 *Suppose $X = M + A$ is a continuous semimartingale, where M is a local continuous martingale and A is a continuous finite variation process. Then, with probability one, for all $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of X at t satisfies*

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}; \quad \Sigma_{X,t}(s') \geq \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s').$$

Furthermore, when $t \in \mathbf{R}_+$ and $s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}$ are such that X satisfies one of these two conditions

1. $\Sigma_{M,t}(s') \neq \Sigma_{A,t}(s')$;
2. *A is locally monotonic at* t,

then the equality $\Sigma_{X,t}(s') = \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$ *holds.*

Proof. Let $t \in \mathbf{R}_+$, $s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}$ and $\sigma < \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$. According to the definition of the pseudo 2-microlocal frontier, we know that M and A belong to $\underline{\boldsymbol{C}}_t^{\sigma,s'}$. Therefore, $X = M + A \in \underline{\boldsymbol{C}}_t^{\sigma,s'}$, which proves that $\Sigma_{X,t}(s') \geq \sigma$, for all $\sigma < \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$.

Let now consider the two cases where the equality holds.

1. If $\Sigma_{M,t}(s') > \Sigma_{A,t}(s')$. Let suppose $\Sigma_{X,t}(s') > \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$, then there exists σ such that $\Sigma_{A,t}(s') < \sigma$, $\Sigma_{M,t}(s') > \sigma$ and $\Sigma_{X,t}(s') > \sigma$. Therefore, M and X belong to $\underline{\boldsymbol{C}}_t^{\sigma,s'}$, and $A = X - M$ as well. But this is in contradiction with the inequality $\Sigma_{A,t}(s') < \sigma$, and therefore we must have
$$\Sigma_{X,t}(s') = \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s').$$
The case $\Sigma_{M,t}(s') < \Sigma_{A,t}(s')$ is treated similarly.

2. Let $\sigma > \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$. To prove the equality, we must find sequences $(s_n)_n$ and $(t_n)_n$ which converges to t and such that
$$\forall n \in \mathbf{N}; \quad |X_{s_n} - X_{t_n}| \geq |s_n - t_n|^{\sigma} \left(|t - s_n| + |t - t_n|\right)^{-s'},$$
Without any loss of generality, let us suppose A is locally increasing. Then, according to the proof of Theorem 1.6 (and Lemma 1.1), we can find such sequences for M and which satisfy for all $n \in \mathbf{N}$, $s_n \leq t_n$ and $M_{s_n} \leq M_{t_n}$. Then, if we consider increments of X, we get
$$\begin{aligned} X_{t_n} - X_{s_n} &= M_{t_n} - M_{s_n} + A_{t_n} - A_{s_n} \\ &\geq M_{t_n} - M_{s_n} \qquad \text{as } A \text{ is locally increasing,} \\ &\geq |s_n - t_n|^{\sigma} \left(|t - s_n| + |t - t_n|\right)^{-s'}. \end{aligned}$$
This inequality proves that $X \notin \underline{\boldsymbol{C}}_t^{\sigma,s'}$ for all $\sigma > \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$, and therefore that $\Sigma_{X,t}(s') = \Sigma_{M,t}(s') \wedge \Sigma_{A,t}(s')$.

□

To end this section, we present a few examples of martingales where the regularity can be entirely determined. To begin with, we note that the Brownian motion is a simple martingale which has a deterministic 2-microlocal frontier
$$\text{a.s. } \forall t \in \mathbf{R}_+, \forall s' \in \mathbf{R}; \quad \sigma_{B,t}(s') = \Sigma_{B,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2},$$
as initially proved by HERBIN and LÉVY VÉHEL [78] (and confirmed by Theorem 1.6).

In the following, we exhibit stochastic processes which have more eccentric regularity. In particular, we construct martingales with a non-deterministic 2-microlocal frontier, which shows that the range of possible behaviours is different than the Gaussian case detailed in [78]. The constructions of the specific quadratic variations are detailed in Appendix 1.7.2.

In this first example, we prove that there exist martingales with a non-trivial 2-microlocal frontier, similar to the "chirp" regularity.

Example 1.2 In Lemma 1.5 is exhibited a continuous non-decreasing function g_α such that at a given t_0,

$$\forall s' \geq -1; \quad \Sigma_{g_\alpha,t_0}(s') = \left(\frac{s'+1}{1-\log_2(\alpha)}\right) \wedge 1,$$

where α is a parameter in $(0,1)$.

Based on this deterministic function, we construct a martingale. Let β be a Brownian motion and U a uniform variable on $[0,1]$ independent of β. We easily verify that $(g_U(t))_{t\in\mathbf{R}_+}$ is time change for the Brownian motion β and therefore, we set the following martingale

$$\forall t \in \mathbf{R}_+ \quad M_t = \beta_{g_U(t)}.$$

Then, using Theorem 1.6 and Corollary 1.7, we obtain the 2-microlocal frontier of M:

$$\forall s' \geq -\tfrac{1}{2}; \quad \sigma_{M,t_0}(s') = \frac{1}{2}\Sigma_{\langle M\rangle,t_0}(2s') = \left(\frac{s'+\frac{1}{2}}{1-\log_2(U)}\right) \wedge \frac{1}{2},$$

since $\langle M\rangle_t = g_U(t)$.

Similarly, we can also consider the time change $\left(g_{\left(2^{-|\beta_{t_0}|}\right)}(t)\right)_{t\in\mathbf{R}_+}$. It can be easily checked that it satisfies the necessary hypotheses, and therefore, in this case the 2-microlocal frontier of the martingale is equal to

$$\forall s' \geq -\tfrac{1}{2}; \quad \sigma_{M,t_0}(s') = \frac{1}{2}\Sigma_{\langle M\rangle,t_0}(2s') = \left(\frac{s'+\frac{1}{2}}{1+|M_{t_0}|}\right) \wedge \frac{1}{2}.$$

Therefore, on the contrary to results presented in [78] on Gaussian processes, at a fixed t_0, there exist martingales with a random and non trivial 2-microlocal frontier which can even depend on the values of the martingale itself. We note that stochastic processes with such regularity properties are called self-regulating processes and have already been exhibited in the literature (e.g. [61] and [21]).

The main drawback of the previous example is that the martingale only has an interesting behaviour at a given t_0 and everywhere else, the 2-microlocal frontier is equal to $\frac{1}{2}+s'$ or $+\infty$. Therefore, we construct in the next example an interesting martingale which is constant almost everywhere and has a particular and self-regulated 2-microlocal frontier elsewhere. This example is based on the construction described in Lemma 1.6.

Example 1.3 There exists a continuous martingale $(M_t(\omega))_{t\in[0,1]}$ and a closed Cantor-like random set $E(\omega)$ such that for all $\omega \in \Omega$ and $t \in [0,1]$, the measure of $E(\omega)$ is null and the regularity of M satisfies

1. if $t \notin E(\omega)$, M is locally constant at t, and therefore

$$\forall s' \in \mathbf{R}; \quad \sigma_{M,t}(s')(\omega) = +\infty.$$

2. if $t \in E(\omega)$, M has the following 2-microlocal frontier

$$\forall s' \in \mathbf{R}; \quad \sigma_{M,t}(s')(\omega) = \left(s' + \frac{1}{2(\alpha+|M_t(\omega)|)}\right) \wedge \frac{1}{2(\alpha+|M_t(\omega)|)},$$

where $\alpha > 1$ is a fixed parameter.

The construction of this martingale is detailed in Lemmas 1.6 and 1.7.

Even if the 2-microlocal frontier obtained in this example is more classic, we notice that M has an interesting regularity on the whole interval $[0,1]$ since at every point t where M is not locally constant ($t \in E(\omega)$), it has a self-regulated behaviour.

Finally, we also observe that the structure of martingales does not allow to extend this kind of particular regularity to all points on the trajectory. Indeed, a simple consequence of Theorem 1.6 and the monotonicity of the quadratic variation is that almost surely, for almost all $t \in \mathbf{R}_+$

$$\widetilde{\alpha}_{M,t} = \begin{cases} \infty & \text{if } M \text{ is locally constant at } t; \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Therefore, a martingale displays almost everywhere the same regularity as Brownian motion.

1.4.2 Time changed multifractional Brownian motion

Theorem 1.6 involves the $\frac{1}{2}$-Hölder regularity of Brownian motion. It is a natural question to investigate the case of more general processes whose local regularities can be prescribed. Among these, fractional and multifractional Brownian motions, which are natural extensions of Brownian motion, are well-known examples.

Both are Gaussian processes, respectively introduced in [125] and [26], [154]. In this work, we will always consider a multifractional Brownian motion (mBm), with Hurst function $H : \mathbf{R} \to (0,1)$, which has the following form:

$$X_t = \frac{1}{\Gamma\big(H(t)+\frac{1}{2}\big)} \int_{\mathbf{R}} \Big[(t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2} \Big] \, \mathrm{d}W_u,$$

even though it has been proved in [172] that the general mBm has a more complex structure. In the case $H = \frac{1}{2}$, we obtain a classic Brownian motion. The Hölder regularity of this process has been widely studied in the literature (see e.g. [154], [10], [14], [76] for sample paths regularity and [134], [37] for local time properties). Therefore, we know that Hölder exponents of X at t only depend on $H(t)$ and the regularity of H at t.

Throughout, we will always assume that the Hurst function H satisfies the assumption $\mathscr{H}_0$:

$$H \text{ is a } \beta\text{-Hölder continuous function such that } \forall t \in \mathbf{R},\ \ H(t) < \beta. \qquad (\mathscr{H}_0)$$

Let us recall the 2-microlocal frontier of the $\mathscr{H}_0$-mBm obtained in [78].

Proposition 1.9 *Let X be a multifractional Brownian motion whose Hurst function H satisfies the hypothesis $\mathscr{H}_0$. Then, with probability one, for all $t \in \mathbf{R}_+$, classic and pseudo 2-microlocal frontier are equal to*

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \Sigma_{X,t}(s') = \big(s' + H(t)\big) \wedge H(t).$$

In particular, we have $\alpha_{X,t} = \widetilde{\alpha}_{X,t} = H(t)$.

Proof. The proof in [78] only concerns the classic 2-microlocal frontier with $s' \in [-H(t), 0]$. The equality can be easily extended to every $s' \leq 0$ since we know the 2-microlocal frontier must be concave and have left- and right-derivatives in the interval $[0,1]$.

Furthermore, using Theorem 1.1, since X is nowhere differentiable, we know that $\Sigma_{X,t} = \sigma_{X,t} \wedge 1$. Finally, if $s' \geq 0$, using Lemma 1.4, we obtain the equality $\Sigma_{X,t}(s') = H(t)$ as the function H is continuous, and also $\sigma_{X,t}(s') = H(t)$ as a consequence. □

We now prove a technical result, related to mBm's increments, which is used to extend our main theorem to a time changed mBm.

Lemma 1.2 *Suppose X is an Hi_0-multifractional Brownian motion. Then, almost surely for all $p \in \mathbf{N}$ and $\varepsilon > 0$, there exists $N(\omega) \in \mathbf{N}$ such that:*

$$\forall n \geq N(\omega),\ \forall i \in \{0, \dots, 2^{n+p}-1\},\ \exists u, v \in [t_i^{(n)},\ t_{i+1}^{(n)}]; \quad |X_u - X_v| \geq \rho_n^{H_i^{(n)}+\varepsilon},$$

where for all $i, n \in \mathbf{N}$, $\rho_n = 2^{-n}$, $t_i^{(n)} = i2^{-n}$ *and* $H_i^{(n)} = \sup_{u \in [t_i^{(n)}, t_{i+1}^{(n)}]} H(u)$.

Proof. As noted previously, we use the following integral representation of the mBm:

$$\begin{aligned} X_t &= \frac{1}{\Gamma\big(H(t)+\frac{1}{2}\big)} \int_{-\infty}^{t} \Big[(t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2}\Big] \, \mathrm{d}W_u \\ &:= C(H(t)) \int_{-\infty}^{t} K\big(u,t,H(t)\big) \, \mathrm{d}W_u. \end{aligned}$$

Let $\varepsilon > 0$ and $p \in \mathbf{N}$. For every $N \in \mathbf{N}$, we consider the following event A_N:

$$A_N = \bigcap_{n \geq N} \bigcap_{0 \leq i < 2^{n+p}} \Big\{ \exists u, v \in [t_i^{(n)},\ t_{i+1}^{(n)}] : |X_u - X_v| \geq \rho_n^{H_i^{(n)}+\varepsilon} \Big\},$$

We fix $n \geq N$ and $i \in \{0, \dots, 2^p - 1\}$. We denote $\Delta t_n = \rho_n^{1+\varepsilon}$ and $m = \lfloor 2^{n\varepsilon} \rfloor$ and we divide the interval $[t_i^{(n)}, t_{i+1}^{(n)}]$ in m smaller intervals $[u_k, u_{k+1}]$ such that $u_{k+1} = u_k + \Delta t_n$ for all $k \in \{0, \dots, m\}$.
Let us then consider the events

$$E_i^{(n)} = \bigcap_{k=1}^{m} \Big\{ |X_{u_k} - X_{u_{k-1}}| < \rho_n^{H_i^{(n)}+\varepsilon} \Big\}.$$

We estimate an upper bound for $\mathbb{P}(E_i^{(n)})$,

$$\begin{aligned} \mathbb{P}(E_i^{(n)}) &= \mathbb{P}\Bigg(\bigcap_{k=1}^{m} \Big\{ |X_{u_k} - X_{u_{k-1}}| < \rho_n^{H_i^{(n)}+\varepsilon} \Big\} \Bigg) \\ &= \mathbb{E}\Bigg[\prod_{k=1}^{m-1} \mathbf{1}_{\left\{ |X_{u_k} - X_{u_{k-1}}| < \rho_n^{H_i^{(n)}+\varepsilon} \right\}} \mathbb{E}\Big[\mathbf{1}_{\left\{ |X_{u_m} - X_{u_{m-1}}| < \rho_n^{H_i^{(n)}+\varepsilon} \right\}} \,\Big|\, W_x, x \leq u_{m-1} \Big] \Bigg] \end{aligned}$$

Let consider the last term

$$\begin{aligned} &\mathbb{P}\big(|X_{u_m} - X_{u_{m-1}}| < \rho_n^{H_i^{(n)}+\varepsilon} \,\big|\, W_x, x \leq u_{m-1} \big) \\ &= \mathbb{P}\Big(\Big| C\big(H(u_m)\big) \int_{u_{m-1}}^{u_m} (u_m - u)^{H(u_m)-1/2} \mathrm{d}W_u + Y_m \Big| < \rho_n^{H_i^{(n)}+\varepsilon} \,\Big|\, W_x, x \leq u_{m-1} \Big), \end{aligned}$$

where $Y_m = \int_{-\infty}^{u_{m-1}} \big[C\big(H(u_m)\big) K\big(u, u_m, H(u_m)\big) - C\big(H(u_{m-1})\big) K\big(u, u_{m-1}, H(u_{m-1})\big) \big] \mathrm{d}W_u$.

We note that Y_m is $\sigma\{W_x, x \leq u_{m-1}\}$-measurable, whereas the stochastic integral on the interval $[u_{m-1}, u_m]$ is independent of $\sigma\{W_x, x \leq u_{m-1}\}$. Furthermore, this last term is a centered Gaussian random variable with the following variance:

$$\sigma_m^2 = C\big(H(u_m)\big)^2 \int_{u_{m-1}}^{u_m} (u_m - u)^{2H(u_m)-1} \mathrm{d}u = \frac{C\big(H(u_m)\big)^2}{2H(u_m)} \big(\Delta t_n\big)^{2H(u_m)}.$$

Therefore, we obtain the inequality

$$\begin{aligned} \mathbb{P}\big(|X_{u_m} - X_{u_{m-1}}| < \rho_n^{H_i^{(n)}+\varepsilon} \,\big|\, W_x, x \leq u_{m-1} \big) &= \frac{1}{\sigma_m \sqrt{2\pi}} \int_{-\rho_n^{H_i^{(n)}+\varepsilon}}^{\rho_n^{H_i^{(n)}+\varepsilon}} \exp\Big(-\frac{(u - Y_m)^2}{2\sigma_m^2} \Big) \mathrm{d}u \\ &\leq \frac{2}{\sqrt{2\pi}} \frac{\rho_n^{H_i^{(n)}+\varepsilon}}{\sigma_m}. \end{aligned}$$

Using previous expressions, we get (where K is positive constant)

$$\begin{aligned}\frac{2}{\sqrt{2\pi}}\frac{\rho_n^{H_i^{(n)}+\varepsilon}}{\sigma_m} &= \frac{2\sqrt{H(u_m)}}{\sqrt{\pi}C\big(H(u_m)\big)}\frac{\rho_n^{H_i^{(n)}+\varepsilon}}{\Delta t_n^{H(u_m)}}\\ &= \frac{2\sqrt{H(u_m)}}{\sqrt{\pi}C\big(H(u_m)\big)}\rho_n^{H_i^{(n)}-H(u_m)+\varepsilon(1-H(u_m))} \le 2^{-n\varepsilon(1-b)+K},\end{aligned}$$

as H and C are continuous, $H(u_m) \le H_i^{(n)}$ and $H(t) \in [a,b]$ for all $t \in \mathbf{R}_+$. Thus, by induction on $k \in \{1,\dots,m\}$, we have

$$\mathbb{P}(E_i^{(n)}) = \mathbb{P}\left(\bigcap_{k=1}^{m}\left\{|X_{u_k} - X_{u_{k-1}}| < \rho_n^{H_i^{(n)}+\varepsilon}\right\}\right) \le 2^{(-n\varepsilon(1-b)+K)m} \le 2^{(-n\varepsilon(1-b)+K)(2^{n\varepsilon}-1)}.$$

Therefore, we obtain:

$$\begin{aligned}&\mathbb{P}\big(\forall u,v \in [t_i^{(n)}, t_i^{(n+1)}] : |X_u - X_v| < \rho_n^{H_i^{(n)}+\varepsilon}\big)\\ &\le \mathbb{P}\left(\bigcap_{k=1}^{m}\left\{|X_{u_k} - X_{u_{k-1}}| < \rho_n^{H_i^{(n)}+\varepsilon}\right\}\right) \le 2^{(-n\varepsilon(1-b)+K)(2^{n\varepsilon}-1)}.\end{aligned}$$

Finally, if we consider the event A_N:

$$\begin{aligned}\mathbb{P}(A_N^c) &= \mathbb{P}\left(\bigcup_{n\ge N}\bigcup_{0\le i<2^{n+p}}\left\{\forall u,v \in [t_i^{(n)}, t_i^{(n+1)}] : |X_u - X_v| < \rho_n^{H_i^{(n)}+\varepsilon}\right\}\right)\\ &\le \sum_{n\ge N}\sum_{0\le i<2^{n+p}}\mathbb{P}\big(\forall u,v \in [t_i^{(n)}, t_i^{(n+1)}] : |X_u - X_v| < \rho_n^{H_i^{(n)}+\varepsilon}\big)\\ &\le \sum_{n\ge N} 2^{(-n\varepsilon(1-b)+K)(2^{n\varepsilon}-1)+n+p} \le \widetilde{K}2^{-N}.\end{aligned}$$

Therefore, using Borel-Cantelli Lemma on $(A_N^c)_{N\in\mathbf{N}}$, we prove the expected result. □

Proposition 1.10 *Suppose X is an $\mathscr{H}_0$-multifractional Brownian motion. Then, with probability one, for any $T \ge 0$ and all $\varepsilon > 0$, there exists $h(\omega) > 0$ such that for all $\rho \le h(\omega)$ and $t \in [0,T]$,*

$$\sup_{u,v\in B(t,\rho)}\left\{|X_u - X_v|\right\} \ge \rho^{H(t)+\varepsilon}.$$

Proof. Let $T > 0$ and $\varepsilon > 0$, there exists $p \in \mathbf{N}$ such that $T \le 2^p$. Using notations from Lemma 1.2, we consider $h(\omega) = 2^{-N(\omega)}$. Then, for every $\rho \le h(\omega)$ and $t \in [0,T]$, there exist $n,k \in \mathbf{N}$ such that $2^{-(n+1)} \le \rho \le 2^{-n}$ and $[k2^{-(n+1)}, (k+1)2^{-(n+1)}] \subseteq [t-\rho, t+\rho]$.

Then, from Lemma 1.2, there exist $u,v \in [k2^{-(n+1)}, (k+1)2^{-(n+1)}]$ such that

$$|X_u - X_v| \ge 2^{-(n+1)(H_{k,n+1}+2\varepsilon)}.$$

As H is uniformly continuous on the interval $[0,T]$, $h(\omega)$ can be chosen small enough such that for all $n,k \in \mathbf{N}$, $n \ge N$, $H_{k,n} - H(t) \le \varepsilon$ for every $t \in [k2^{-n}, (k+1)2^{-n}]$.
Therefore, we have

$$|X_u - X_v| \ge K2^{-n(H_{k,n+1}+3\varepsilon)} \ge K\rho^{H(t)+\varepsilon},$$

which proves the result. □

Remark 1.3 Results similar to Proposition 1.10 have previously been obtained in the literature, using technics based on the study of local times. Thereby, inequality (8.8.26) in [2] is equivalent to our result in the particular case of fractional Brownian motion.

Furthermore, Theorem 3.6 in [13] states that for all $\delta > 0$, $\varepsilon > 0$,

$$\liminf_{\rho\to 0} \inf_{t\in[\delta,1]} \sup_{u\in B(t,\rho)} \frac{|X_t - X_u|}{\rho^{\overline{H}+\varepsilon}} > 0 \quad \text{a.s.},$$

where $\overline{H} = \max_{t\in[\delta,1]} H(t)$. This last property is slightly weaker than our result since Proposition 1.10 is equivalent to

$$\liminf_{\rho\to 0} \inf_{t\in[0,1]} \sup_{u\in B(t,\rho)} \frac{|X_t - X_u|}{\rho^{H(t)+\varepsilon}} > 0 \quad \text{a.s. for all } \varepsilon > 0.$$

We finally present the main result of this section, which gives the classic 2-microlocal frontier of a time changed multifractional Brownian motion. As the proof is not modified compare to the martingale specific case, we only recall the major steps. The proof of the first side inequality does not change at all, we only use the uniform regularity of the mBm recalled previously in Proposition 1.9. The converse inequality is also shown similarly, using Proposition 1.10 proved below.

Theorem 1.11 *Suppose X is an $\mathscr{H}_0$-multifractional Brownian motion and U is a continuous positive process. We denote by Z the compound process:*

$$\forall t \in \mathbf{R}_+; \quad Z_t = X_{U_t}.$$

Then, with probability one, for all $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of Z at t verifies the following equality:

$$\forall s' \geq -\alpha_{Z,t}; \quad \Sigma_{Z,t}(s') = H(U_t) \cdot \Sigma_{U,t}\left(\frac{s'}{H(U_t)}\right).$$

Consequently, with probability one, the pseudo pointwise and local Hölder exponents satisfy:

$$\forall t \in \mathbf{R}_+; \quad \underline{\boldsymbol{\alpha}}_{Z,t} = H(U_t) \cdot \underline{\boldsymbol{\alpha}}_{U,t} \quad \textit{and} \quad \underline{\widetilde{\boldsymbol{\alpha}}}_{Z,t} = H(U_t) \cdot \underline{\widetilde{\boldsymbol{\alpha}}}_{U,t}.$$

Remark 1.4 Properties of time changed fractional Brownian motions have already been studied in the literature. It has first appeared in [122] because of its particular multifractal properties. The multifractal spectra is obtained in [162] using similar technics to the ones used in Theorem 1.11.

1.5 2-microlocal frontier of stochastic integrals

As stated in the introduction, we want to extend the characterization of the Wiener integral's regularity obtained in [78]. The latter result applies to any Wiener integral X which has the following form: $X_t = \int_0^t \eta(u)\mathrm{d}W_u$, where η is an L^2-deterministic function. Then, according to Theorem 4.12 in [78], for all $t_0 \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of X at t_0 is almost surely given by

$$\forall s' \in [-\underline{\boldsymbol{\alpha}}_{X,t}, 0]; \quad \Sigma_{X,t_0}(s') = \frac{1}{2}\Sigma_{\int_0^\cdot \eta^2(u)\mathrm{d}u, t_0}(2s').$$

In this section, we will consider the extension of this equality to stochastic integrals $X_t = \int_0^t H_u \mathrm{d}M_u$, where M is a continuous local martingale and H is a continuous progressive stochastic process. Here, the process H is supposed to be continuous to avoid technical problems on

the definition of its 2-microlocal frontier, but this assumption might be weakened if one wants to consider a larger class of integrands.

We begin by expressing an equality which is a straight forward consequence of our previous analysis of the regularity of martingales.

Theorem 1.12 *Suppose $\{X_t; t \in \mathbf{R}_+\}$ is a stochastic integral defined by*

$$\forall t \in \mathbf{R}_+; \quad X_t = \int_0^t H_u \mathrm{d}M_u,$$

where M is a continuous local martingale and H is a continuous progressive stochastic process. Then, with probability one, for all $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of X at t is equal to

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}; \quad \Sigma_{X,t}(s') = \frac{1}{2}\Sigma_{\int_0^\cdot H_u^2 \mathrm{d}\langle M\rangle_u, t}\left(2s'\right).$$

Proof. This result is a simple application of Theorem 1.6, using the well-known equality

$$\forall t \in \mathbf{R}_+; \quad \langle \int_0^\cdot H_u \mathrm{d}M_u \rangle_t = \int_0^t H_u^2 \mathrm{d}\langle M\rangle_u.$$

□

Theorem 1.12 is obviously an improvement of the result stated on the Wiener integral. Firstly, we obtain an equality for any stochastic integral with respect to a continuous local martingale. Furthermore, a more subtle enhancement lies in the fact that the Theorem 1.12 gives a uniform almost sure result, i.e. "almost surely for all $t \in \mathbf{R}_+$", whereas the original one is not. We present at the end of this section an example where the difference between these two kind of results appears.

The remaining of this section is devoted to the characterization of the pseudo 2-microlocal frontier of stochastic integral using the regularities of both the martingale M and the integrand H. We begin with a technical lemma related to the regularity of the quadratic variation.

Lemma 1.3 *The pseudo 2-microlocal frontier of the increasing process $A = \int_0^\cdot H_s^2 \mathrm{d}\langle M\rangle_s$ satisfies, for all $\omega \in \Omega$ and $t \in \mathbf{R}_+$,*

(i) if $H_t(\omega) \neq 0$,

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{A,t}; \quad \Sigma_{A,t}(s') = \Sigma_{\langle M\rangle,t}(s');$$

(ii) if $H_t(\omega) = 0$,

$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{A,t}; \quad \Sigma_{A,t}(s') \geq \Sigma_{\langle M\rangle,t}(s' + 2\underline{\boldsymbol{\alpha}}_{H,t}),$$

where $\underline{\boldsymbol{\alpha}}_{H,t}$ denotes the pseudo pointwise exponent of H at t.

Proof.

1. Since H is continuous and $H_t \neq 0$, there exist $C_1, C_2, \rho > 0$ such that for all $u \in B(t,\rho)$, we have $C_1 \leq |H_u|^2 \leq C_2$. Therefore, we observe that for all $u \leq v \in B(t,\rho)$,

$$C_1|\langle M\rangle_v - \langle M\rangle_u| \leq \left|\int_0^v H_s^2 \mathrm{d}\langle M\rangle_s - \int_0^u H_s^2 \mathrm{d}\langle M\rangle_s\right| \leq C_2|\langle M\rangle_v - \langle M\rangle_u|,$$

Based on the definition of the pseudo 2-microlocal frontier, these two inequalities prove the first point.

2. When $H_t = 0$, we observe that for all $s' \geq -\underline{\boldsymbol{\alpha}}_{\langle M\rangle,t}$ and $\varepsilon > 0$, there exists $C > 0$ and $\rho > 0$ such that for all $u \leq v \in B(t,\rho)$,

$$\begin{aligned}\left|\int_0^v H_s^2 \mathrm{d}\langle M\rangle_s - \int_0^u H_s^2 \mathrm{d}\langle M\rangle_s\right| &= \left|\int_u^v (H_s - H_t)^2 \mathrm{d}\langle M\rangle_s\right| \\ &\leq C\int_u^v |s-t|^{2\underline{\boldsymbol{\alpha}}_{H,t}-\varepsilon}\mathrm{d}\langle M\rangle_s \\ &\leq C\left(\langle M\rangle_v - \langle M_u\rangle\right)\left(|u-t|+|v-t|\right)^{2\underline{\boldsymbol{\alpha}}_{H,t}-\varepsilon} \\ &\leq C|u-v|^{\Sigma_{\langle M\rangle,t}(s')-\varepsilon}\left(|u-t|+|v-t|\right)^{-s'+2\underline{\boldsymbol{\alpha}}_{H,t}-\varepsilon}\end{aligned}$$

which proves that $\Sigma_{A,t}(s' - 2\underline{\boldsymbol{\alpha}}_{H,t} + \varepsilon) \geq \Sigma_{\langle M\rangle,t}(s') - \varepsilon$. Hence, when $\varepsilon \to 0$, the expected inequality is obtained using the continuity of the pseudo 2-microlocal frontier.

□

We can now derive a general lower bound for the pseudo 2-microlocal frontier of a stochastic integral with respect to a continuous local martingale.

Theorem 1.13 *Suppose $\{X_t; t \in \mathbf{R}_+\}$ is the stochastic integral defined by*

$$\forall t \in \mathbf{R}_+; \quad X_t = \int_0^t H_u \mathrm{d}M_u,$$

where M is a continuous local martingale and H is a continuous progressive stochastic process.

Then, there exists an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ and $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of the stochastic integral X satisfies

1. *if $H_t(\omega) \neq 0$,*
$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}; \quad \Sigma_{X,t}(s') = \Sigma_{M,t}(s');$$
2. *if $H_t(\omega) = 0$,*
$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}; \quad \Sigma_{X,t}(s') \geq \Sigma_{M,t}(s' + \underline{\boldsymbol{\alpha}}_{H,t}),$$
where $\underline{\boldsymbol{\alpha}}_{H,t}$ is the pseudo pointwise exponent of H at t.

Proof.

1. When $H_t(\omega) \neq 0$, Lemma 1.3 leads to
$$\forall s' \geq -\frac{\underline{\boldsymbol{\alpha}}_{\langle M\rangle,t}}{2}; \quad \frac{1}{2}\Sigma_{\int_0^\bullet H_u^2 \mathrm{d}\langle M\rangle_u,t}(2s') = \frac{1}{2}\Sigma_{\langle M\rangle,t}(2s').$$
Hence, the application of Theorem 1.6 allows to obtain the expected equality.
2. In the other case, $H_t(\omega) \neq 0$, the same Lemma 1.3 implies
$$\forall s' \geq -\frac{\underline{\boldsymbol{\alpha}}_{\int_0^\bullet H_u^2 \mathrm{d}\langle M\rangle_u,t}}{2}; \quad \frac{1}{2}\Sigma_{\int_0^\bullet H_u^2 \mathrm{d}\langle M\rangle_u,t}(2s') \geq \frac{1}{2}\Sigma_{\langle M\rangle,t}(2s' + 2\underline{\boldsymbol{\alpha}}_{H,t}),$$
which also induces the expected inequality, using Theorem 1.6.

□

Theorem 1.13 can be improved in the case of a stochastic integral with respect to Brownian motion since in this case, the integral with respect to the quadratic variation is reduced to a classic Lebesgue integral.

Theorem 1.14 *Suppose $\{X_t; t \in \mathbf{R}_+\}$ is a stochastic integral defined by*

$$\forall t \in \mathbf{R}_+; \quad X_t = \int_0^t H_u \mathrm{d}B_u,$$

where B is a Brownian motion and H is a continuous progressive stochastic process.

Then, there exists an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ and $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of the stochastic integral X satisfies

1. *if $H_t(\omega) \neq 0$,*
$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \sigma_{B,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2};$$
2. *if $H_t(\omega) = 0$,*
$$\forall s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}; \quad \sigma_{X,t}(s') = \Sigma_{X,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{H^2,t}(2s')}{2}\right) \wedge \frac{1}{2},$$
unless H is locally equal to zero at t, which induces in that case: $\sigma_{X,t} = \infty$.

Proof.

1. To obtain the first equality, we use the previous Theorem 1.13, since we know that almost surely for all $t \in \mathbf{R}_+$,
$$\forall s' \in \mathbf{R}; \quad \sigma_{B,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2}.$$
The formula is extended to all $s' \leq -\frac{1}{2}$ using properties of the pseudo 2-microlocal frontier (concavity and derivatives in $[0,1]$) proved in Corollary 1.2.
2. When $H_t(\omega) = 0$, we observe that if H is locally equal to zero, we obtain $\sigma_{X,t} = \infty$.
In the opposite case, from Theorem 1.1, the pseudo 2-microlocal frontier of the integral satisfies
$$\forall s' \in \mathbf{R}; \quad \Sigma_{\int_0^\cdot H_s^2 \mathrm{d}s, t}(s') = \left(1 + \Sigma_{H^2,t}(s')\right) \wedge 1.$$
Hence, Theorem 1.6 yields the expected formula.

□

Remark 1.5 There is no contradiction between the second point from Theorems 1.13 and 1.14. Indeed, we know that for all $t \in \mathbf{R}_+$ such that $H_t(\omega) = 0$,

$$\forall s' \in \mathbf{R}; \quad \Sigma_{\langle B \rangle, t}(s' + 2\underline{\boldsymbol{\alpha}}_{H,t}) = \left(1 + s' + 2\underline{\boldsymbol{\alpha}}_{H,t}\right) \wedge 1.$$

Furthermore, for all $s' \leq -2\underline{\boldsymbol{\alpha}}_{H,t}$, still using properties of the pseudo 2-microlocal frontier from Corollary 1.2, we obtain

$$\Sigma_{H^2,t}(s') \geq s' + 2\underline{\boldsymbol{\alpha}}_{H,t},$$

which proves the expected inequality for all $s' \in \mathbf{R}$

$$\Sigma_{X,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{H^2,t}(2s')}{2}\right) \wedge \frac{1}{2} \geq \left(\frac{1}{2} + s' + \underline{\boldsymbol{\alpha}}_{H,t}\right) \wedge \frac{1}{2} = \Sigma_{B,t}(s' + \underline{\boldsymbol{\alpha}}_{H,t}).$$

As an application, we consider two examples of stochastic integral whose 2-microlocal frontier can be completely determined.

Example 1.4 Let us consider an Ornstein–Uhlenbeck process X. It admits the following representation:

$$\forall t \in \mathbf{R}_+; \quad X_t = X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \int_0^t \sigma e^{\theta(s-t)} \mathrm{d}B_s,$$

where $\theta > 0$, μ and $\sigma > 0$ are parameters and B is a Brownian motion.

Then, almost surely for all $t \in \mathbf{R}_+$, classic and pseudo 2-microlocal frontiers of X are equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \Sigma_{X,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2},$$

and in particular $\alpha_{X,t} = \widetilde{\alpha}_{X,t} = \frac{1}{2}$.

This result is a consequence of Theorem 1.14 and the strict positivity of the exponential function. Therefore, the stochastic integral $\int_0^t \sigma e^{\theta s} \mathrm{d}B_s$ has almost surely the following pseudo 2-microlocal frontier, for all $t \in \mathbf{R}_+$:

$$\forall s' \in \mathbf{R}; \quad \Sigma_{\int_0^\cdot \sigma e^{\theta s} \mathrm{d}B_s, t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2}.$$

Furthermore, as $t \mapsto e^{-\theta t}$ is a positive C^∞ function, we can easily see that other terms in the expression do not modify the regularity. Finally, Corollary 1.7 extends the equality to the classic 2-microlocal frontier.

Remark 1.6 As the Ornstein-Uhlenbeck process is a Gaussian process, the 2-microlocal frontier can be directly obtained using techniques and results introduced by Herbin and Lévy Véhel [78]. Another way to get this result is to note that Lamperti transform of Brownian motion does not modify the regularity.

The second example illustrates the impact of integrand's regularity on the 2-microlocal frontier of the integral.

Example 1.5 Let B be a Brownian motion and X an $\mathscr{H}_0$-multifractional Brownian motion adapted to B filtration. We consider the following stochastic integral:

$$\forall t \in \mathbf{R}_+; \quad Z_t = \int_0^t X_s \, \mathrm{d}B_s.$$

Then, with probability one, for all $t \in \mathbf{R}_+$, classic and pseudo 2-microlocal frontier of this integral satisfy:

1. if $X_t(\omega) \neq 0$,

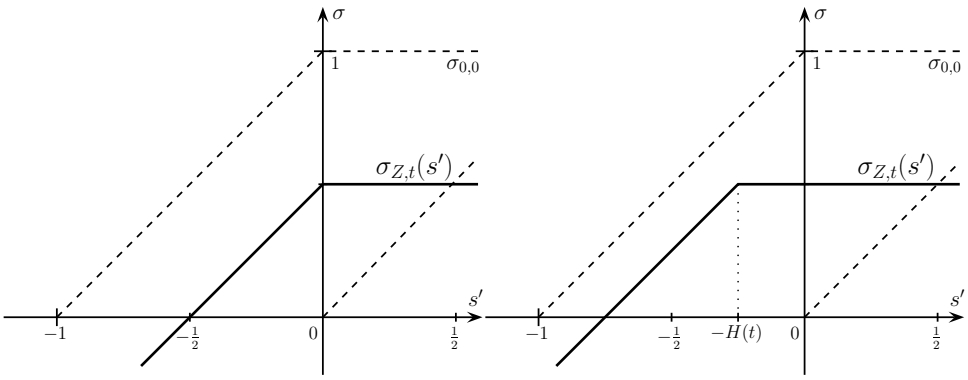
$$\forall s' \in \mathbf{R}; \quad \sigma_{Z,t}(s') = \Sigma_{Z,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2},$$

   and in particular $\alpha_{Z,t} = \widetilde{\alpha}_{Z,t} = \frac{1}{2}$;

2. if $X_t(\omega) = 0$,

$$\forall s' \in \mathbf{R}; \quad \sigma_{Z,t}(s') = \Sigma_{Z,t}(s') = \left(\frac{1}{2} + H(t) + s'\right) \wedge \frac{1}{2},$$

   and in particular $\alpha_{X,t} = \frac{1}{2} + H(t)$ and $\widetilde{\alpha}_{X,t} = \frac{1}{2}$.



(a) if $X_t(\omega) \neq 0$ (b) if $X_t(\omega) = 0$

Figure 1.2: 2-microlocal frontier of $\int_0^\bullet X_s \mathrm{d}B_s$ at t

These two frontiers are depicted in Figures 1.2a and 1.2b.

Before proving these results, we first note that the stochastic integral is well-defined, as for all $t \in \mathbf{R}_+$:

$$\mathbb{E}\left[\int_0^t (X_s)^2 \mathrm{d}s\right] = \int_0^t C(H(s)) s^{2H(s)} \mathrm{d}s < \infty.$$

According to Theorem 1.14, we only have to characterize the pseudo 2-microlocal frontier $\Sigma_{(X)^2,t}$ in the case $X_t(\omega) = 0$.

1. For the lower bound, the frontier of X is known to be equal to

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = \big(H(t) + s'\big) \wedge H(t).$$

Therefore, for all $s' \geq -H(t)$ and $\varepsilon > 0$, there exist $\rho > 0$ and $C > 0$ such that for all $u, v \in B(t, \rho)$,

$$\begin{aligned} |X_u^2 - X_v^2| &= |X_u - X_v| \cdot |X_u + X_v| \\ &\leq C|u-v|^{(H(t)+s')\wedge H(t)-\varepsilon} \big(|t-u| + |t-v|\big)^{-s'} \cdot \big(|t-u| + |t-v|\big)^{H(t)-\varepsilon} \\ &= C|u-v|^{(2H(t)+\tilde{s}')\wedge H(t)-\varepsilon} \big(|t-u| + |t-v|\big)^{-\tilde{s}'-\varepsilon}, \end{aligned}$$

where $\tilde{s}' = s' - H(t) \geq -2H(t)$. Hence, it proves that for all $s' \geq -2H(t)$,

$$\forall s' \geq -2H(t); \quad \Sigma_{(X)^2,t}(s') \geq \big(2H(t) + s'\big) \wedge H(t).$$

2. To obtain the upper bound, we note that Lemma 1.4 implies that $\Sigma_{(X)^2,t} \leq H(t)$.

Then, for all $\varepsilon > 0$, there exists a sequence $(u_n)_n$ such that for all $n \in \mathbf{N}$, $|X_{u_n} - X_t| \geq |u_n - t|^{H(t)+\varepsilon}$. Thus, for all $n \in \mathbf{N}$, we also obtain $|X_{u_n}^2 - X_t^2| = |X_{u_n} - X_t|^2 \geq |u_n - t|^{2H(t)+2\varepsilon}$, which proves that

$$\forall s' \geq -2H(t); \quad \Sigma_{(X)^2,t}(s') \leq 2H(t) + s'.$$

Therefore, we have obtained the equality $\Sigma_{(X)^2,t}(s') = \big(2H(t) + s'\big) \wedge H(t)$ for all $s' \in \mathbf{R}$, where the extension to all $s' \leq -2H(t)$ is once more obtained using properties of the pseudo 2-microlocal frontier (concavity and derivatives between 0 and 1).

Hence, for all $s' \geq -\underline{\boldsymbol{\alpha}}_{X,t}$, we get

$$\Sigma_{Z,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{X^2,t}(2s')}{2}\right) \wedge \frac{1}{2} = \left(\frac{1}{2} + H(t) + s'\right) \wedge \frac{1}{2}.$$

The same properties allow to extend this equality to all $s' \in \mathbf{R}$. Finally, Corollary 1.7 is applied to get the 2-microlocal frontier.

Remark 1.7 In the case X is a fractional Brownian motion, the multifractal spectrum of the integral can also be deduced. Indeed, the pointwise exponent $\alpha_{Z,t}$ of Z can only take two values: $1/2$ and $1/2+H$. Furthermore, it has been proved in the literature that the Hausdorff dimension of the zeros of a fBm is equal to $1-H$. Hence, we are able to determine the multifractal spectrum:

$$d_Z(h) := \dim_{\mathrm{H}}\{t \in \mathbf{R}_+ : \alpha_{Z,t} = h\} = \begin{cases} 1-H & \text{if } h = \frac{1}{2}+H; \\ 1 & \text{if } h = \frac{1}{2}; \\ -\infty & \text{otherwise.} \end{cases}$$

Note the spectrum of singularities is degenerated, and therefore quite different from the spectrum of Lévy processes (see e.g. [94]).

Remark 1.8 Example 1.5 also shows that there is a main difference between uniform results "a.s. $\forall t \in \mathbf{R}_+$" and simpler ones "$\forall t \in \mathbf{R}_+$ a.s.". Indeed, we know that for all $t \in \mathbf{R}_+$, almost surely $X_t \neq 0$. Therefore, for all $t \in \mathbf{R}_+$, the 2-microlocal frontier of the stochastic integral is almost surely equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{\int_0^\bullet X_s \mathrm{d}B_s, t}(s') = \left(\frac{1}{2}+s'\right) \wedge \frac{1}{2},$$

which is obviously a less precise characterization of the regularity.

1.6 Application to stochastic differential equations

Stochastic differential equations (SDE) are often used as a mathematical model of natural phenomenons which present a random behaviour. Local regularity constitutes an interesting indicator to verify how "correctly" a model fits observable data. In this section, we present a method to obtain the 2-microlocal frontier (or at least a lower bound for the local regularity) of SDE solutions.

To simplify our statement and the expressions, we focus on homogeneous diffusions which have the following form:

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t,$$

where a and b are continuous functions with respective pseudo pointwise Hölder exponents $\underline{\boldsymbol{\alpha}}_{a,x}$ and $\underline{\boldsymbol{\alpha}}_{b,x}$ at $x \in \mathbf{R}$.

We assume the existence of a solution X to this SDE. Owing to Theorem 1.14, there exists an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ and $t \in \mathbf{R}_+$, the pseudo 2-microlocal frontier of X and thus the local regularity is completely determined. Similarly to the aforementioned result, we have to distinguish different cases, as illustrated by Equations (1.7), (1.8), (1.9) and (1.10) on the pseudo 2-microlocal frontier $\Sigma_{X,t}$.

1. If $a(X_t(\omega)) \neq 0$, Theorem 1.14 directly implies

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = \left(\frac{1}{2}+s'\right) \wedge \frac{1}{2}. \tag{1.7}$$

2. If $a(X_t(\omega)) = 0$, but is not locally equal to zero, and $b(X_t(\omega)) \neq 0$, the stochastic integral $\int_0^\bullet a(X_s)\mathrm{d}B_s$ satisfies

$$\forall s' \in \mathbf{R}; \quad \Sigma_{\int_0^\bullet a(X_s)\mathrm{d}B_s, t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{a^2(X_\bullet),t}(2s')}{2}\right) \wedge \frac{1}{2}.$$

Furthermore, using Theorem 1.3, as $b(X_t(\omega)) \neq 0$ and $s \mapsto b(X_s(\omega))$ is continuous, we have

$$\forall s' \in \mathbf{R}; \quad \Sigma_{\int_0^\bullet b(X_s)\mathrm{d}s,t}(s') = (1+s') \wedge 1.$$

These two expressions lead to

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{a^2(X_\bullet),t}(2s')}{2}\right) \wedge (1+s') \wedge \frac{1}{2}$$

and, in particular,

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') \leq (1+s') \wedge \frac{1}{2} \quad \text{and} \quad \sigma_{X,t}(s') \leq \frac{1}{2}.$$

We can easily check that $\underline{\boldsymbol{\alpha}}_{a^2(X_\bullet),t} \geq 2\underline{\boldsymbol{\alpha}}_{a,X_t}\underline{\boldsymbol{\alpha}}_{X,t}$. Thus, we get the following inequality for the negative part of the pseudo frontier:

$$\forall s' \leq -\underline{\boldsymbol{\alpha}}_{a^2(X_\bullet),t}; \quad \Sigma_{a^2(X_\bullet),t}(s') \geq 2\underline{\boldsymbol{\alpha}}_{a,X_t}\underline{\boldsymbol{\alpha}}_{X,t} + s'.$$

Using the previous inequalities, we obtain:

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') \geq \left(\frac{1}{2} + \underline{\boldsymbol{\alpha}}_{a,X_t}\underline{\boldsymbol{\alpha}}_{X,t} + s'\right) \wedge (1+s') \wedge \frac{1}{2},$$

and, in particular

$$\underline{\boldsymbol{\alpha}}_{X,t} \geq \left(\frac{1}{2} + \underline{\boldsymbol{\alpha}}_{a,X_t}\underline{\boldsymbol{\alpha}}_{X,t}\right) \wedge 1.$$

Therefore, we have $\underline{\boldsymbol{\alpha}}_{X,t} \geq \left(\frac{1}{2(1-\underline{\boldsymbol{\alpha}}_{a,X_t})_+}\right) \wedge 1$, which leads to the following estimate of the pseudo frontier of X

$$\forall s' \in \mathbf{R}; \quad \left(\frac{1}{2(1-\underline{\boldsymbol{\alpha}}_{a,X_t})_+} + s'\right) \wedge (1+s') \wedge \frac{1}{2} \leq \Sigma_{X,t}(s') \leq (1+s') \wedge \frac{1}{2}. \tag{1.8}$$

In particular, we note that when $\underline{\boldsymbol{\alpha}}_{a,X_t} \geq \frac{1}{2}$, we get

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = (1+s') \wedge \frac{1}{2}.$$

3. If $a(X_t(\omega)) = 0$, but is not locally equal to zero, and $b(X_t(\omega)) = 0$, we similarly obtain

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{a^2(X_\bullet),t}(2s')}{2}\right) \wedge \left(1 + \Sigma_{b(X_\bullet),t}(s')\right) \wedge \frac{1}{2},$$

as $b(X_t(\omega)) = 0$ and using the translation property of the pseudo frontier. This inequality leads to

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = \left(\frac{1}{2} + \underline{\boldsymbol{\alpha}}_{a,X_t}\underline{\boldsymbol{\alpha}}_{X,t} + s'\right) \wedge (1 + \underline{\boldsymbol{\alpha}}_{b,X_t}\underline{\boldsymbol{\alpha}}_{X,t} + s') \wedge \frac{1}{2}$$

and in particular

$$\underline{\boldsymbol{\alpha}}_{X,t} \geq \left(\frac{1}{2} + \underline{\boldsymbol{\alpha}}_{a,X_t}\underline{\boldsymbol{\alpha}}_{X,t}\right) \wedge (1 + \underline{\boldsymbol{\alpha}}_{b,X_t}\underline{\boldsymbol{\alpha}}_{X,t}).$$

Therefore, if $\underline{\boldsymbol{\alpha}}_{a,X_t} < 1$ or $\underline{\boldsymbol{\alpha}}_{b,X_t} < 1$, we obtain:

$$\underline{\boldsymbol{\alpha}}_{X,t} \geq \left(\frac{1}{2(1-\underline{\boldsymbol{\alpha}}_{a,X_t})_+}\right) \wedge \left(\frac{1}{(1-\underline{\boldsymbol{\alpha}}_{b,X_t})_+}\right),$$

which leads to

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') \geq \left(\frac{1}{2(1-\underline{\boldsymbol{\alpha}}_{a,X_t})_+} + s'\right) \wedge \left(\frac{1}{(1-\underline{\boldsymbol{\alpha}}_{b,X_t})_+} + s'\right) \wedge \frac{1}{2}. \tag{1.9}$$

In the case $\underline{\boldsymbol{\alpha}}_{a,X_t} \geq 1$ and $\underline{\boldsymbol{\alpha}}_{b,X_t} \geq 1$, the previous inequalities imply that $\underline{\boldsymbol{\alpha}}_{X,t} = +\infty$ and $\sigma_{X,t} = \Sigma_{X,t} = \frac{1}{2}$.

4. Finally, if $a(X_\bullet(\omega))$ is locally equal to zero at t, we locally obtain a classic differential equation $\mathrm{d}X_t = b(X_t)\mathrm{d}t$. Similarly to previous calculations, we get the following inequality:

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') \geq \left(\frac{1}{(1-\alpha_{b,X_t})_+} + s'\right) \wedge 1. \tag{1.10}$$

We note in the previous results that a self-regulating aspect appears in the regularity of diffusions, since the 2-microlocal frontier at t can depend on the value of $a(X_t(\omega))$ and $b(X_t(\omega))$. The behaviour is also illustrated in the following two Examples 1.6 and 1.7.

For the sake of readability, we have only considered homogeneous diffusions. But, results could be easily extended to non-homogeneous diffusions $\mathrm{d}X_t = a(t, X_t, \omega)\,\mathrm{d}B_t + b(t, X_t, \omega)\,\mathrm{d}t$. In this case, lower bounds obtained would depend on the stochastic Hölder exponents of a and b along the two directions, i.e. $(\underline{\boldsymbol{\alpha}}_{a(\bullet,X_t),t}, \underline{\boldsymbol{\alpha}}_{a(t,\bullet),X_t})(\omega)$ and $(\underline{\boldsymbol{\alpha}}_{b(\bullet,X_t),t}, \underline{\boldsymbol{\alpha}}_{b(t,\bullet),X_t})(\omega)$.

More generally, we can see that the previous methodology can be extended to any kind of stochastic differential equation to get, at least, a lower bound for the local regularity of its solution.

To end this section, we apply theses techniques to obtain the exact regularity in two different examples of SDE. In the first one, we characterize the 2-microlocal frontier of the square of δ-dimensional Bessel processes, which generalizes the result on $t \mapsto B_t^2$ described in the introduction.

Example 1.6 We know that for every $\delta > 0$ and $x \geq 0$, there exists a unique strong continuous solution to the following equation (see [161]): $Z_t = x + 2\int_0^t \sqrt{Z_s}\mathrm{d}\beta_s + \delta t$, which is called the square of δ-dimensional Bessel process started at x ($BESQ^\delta(x)$).

Hence, for every $\delta > 0$ and $x \geq 0$, with probability one, for all $t \in \mathbf{R}_+$, the 2-microlocal frontier of the $BESQ^\delta(x)$ Z satisfies

1. If $Z_t(\omega) \neq 0$,

$$\forall s' \in \mathbf{R}; \quad \sigma_{Z,t}(s') = \Sigma_{Z,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2},$$

and in particular $\alpha_{Z,t} = \widetilde{\alpha}_{Z,t} = \frac{1}{2}$.

2. If $Z_t(\omega) = 0$, the pseudo frontier satisfies the equation

$$\Sigma_{Z,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{Z,t}(2s')}{2}\right) \wedge (1+s') \wedge \frac{1}{2},$$

which has a unique solution,

$$\forall s' \in \mathbf{R}; \quad \sigma_{Z,t}(s') = \Sigma_{Z,t}(s') = (1+s') \wedge \frac{1}{2},$$

and in particular $\alpha_{Z,t} = 1$ and $\widetilde{\alpha}_{Z,t} = \frac{1}{2}$.

These equalities are simple consequences of the previous results on diffusions. Indeed, since $\alpha_{a,x} = \frac{1}{2}$ when $x = 0$, Equations (1.7) and (1.8) prove the equality on the pseudo frontier.

To obtain the classic 2-microlocal frontier, we can apply Corollary 1.7 on the stochastic integral $\int_0^t \sqrt{Z_s}\, \mathrm{d}\beta_s$ and note that according to Definition 1.1, the frontier is not modified when a polynomial component (i.e. δt) is added.

Remark 1.9 As proved in [161], if $\delta \geq 2$, the set $\{0\}$ is transient and therefore, almost surely for any $t \in \mathbf{R}_+$, the 2-microlocal frontier is equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{Z,t}(s') = \Sigma_{Z,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2}.$$

On the contrary, if $\delta < 2$, we know that $\{0\}$ is recurrent, and the distinctions between $Z_t(\omega) \neq 0$ and $Z_t(\omega) = 0$ is necessary.

Example 1.7 In the second example, we obtain the 2-microlocal frontier of the Heston model. This model has been introduced in [84] to describe the price of an asset which has a stochastic volatility and therefore, it consists of two correlated SDE. The interesting point in this model is that the regularity of the asset S depends on the value of the volatility v, as proved below.

In the Heston model, S and v are solutions of the following system of equations:

$$\begin{cases} \mathrm{d}S_t = \mu S_t + \sqrt{v_t} S_t \mathrm{d}W_t^S \\ \mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}W_t^v, \end{cases}$$

where μ, κ, θ and ξ are deterministic positive parameters and (W_t^S, W_t^v) is a Brownian motion where W^S and W^v may be dependent.

Then, there exists an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ and $t \in \mathbf{R}_+$, 2-microlocal frontiers of S and v at t are given by

1. if $v_t(\omega) \neq 0$,
$$\forall s' \in \mathbf{R}; \quad \sigma_{S,t}(s') = \sigma_{v,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2};$$
2. if $v_t(\omega) = 0$,
$$\forall s' \in \mathbf{R}; \quad \sigma_{S,t}(s') = \sigma_{v,t}(s') = (1 + s') \wedge \frac{1}{2}.$$

We obtain the 2-microlocal frontier of v similarly to the previous example. Then, if we consider S, as it is stricly positive, $\sqrt{v_t} S_t = 0$ if and only if $v_t = 0$. Therefore, in this case we obtain the equation

$$\Sigma_{S,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{v,t}(2s')}{2}\right) \wedge (1 + s') \wedge \frac{1}{2},$$

which leads to the expected solution.

Remark 1.10 According to [5], if the parameters satisfy the inequality $2\kappa\theta \geq \xi^2$, then the process v is almost surely positive. Therefore, similarly to square of Bessel processes, the result is simplified in that case as $v_t = 0$ almost surely never happens.

These two examples show that it is possible to get an exact result, more precise than lower bounds obtained at the beginning of the section, when the expressions of functions a and b are explicit and when we are able to characterize the regularity of compositions $a \circ X$ and $b \circ X$.

Another point illustrated by these examples and the previous results on diffusions is that the regularity of the solutions of these SDE are almost everywhere equal to $\frac{1}{2} + s'$ and change

only at exceptional points (e.g. zeros of BESQ$^{\delta}$). Even when these equations are driven by martingales other than the Brownian motion, owing to Theorems 1.6 and 1.13, interesting regularities are still exceptional. Nevertheless, many natural phenomenons have a regularity different from $\frac{1}{2}$ (see e.g. [69]), and thus can not be represented by these classic diffusions. Therefore, an interesting next step would be to extend our work to stochastic differential equations driven by fractional (and multifractional) Brownian motion (see e.g. [148]), in order to determine the possibilities it brings in term of regularity.

1.7 Proofs of technical results

Appendices gather some deterministic and technical results which are used along the article.

1.7.1 Pseudo 2-microlocal analysis

Propositions and theorems related to pseudo 2-microlocal spaces and frontier are proved in this appendix. The first result is an extension of a well-known property on local Hölder exponents: values of the local exponent around a point t_0 have an influence on the pseudo 2-microlocal frontier of f at t_0.

Lemma 1.4 *Let f be a continuous function and t_0 be in $\mathbf{R}$. Then, the pseudo 2-microlocal frontier at t_0 satisfies:*

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t_0}(s') \leq \liminf_{u \to t_0, u \neq t_0} \underline{\widetilde{\boldsymbol{\alpha}}}_{f,u},$$

where $\underline{\widetilde{\boldsymbol{\alpha}}}_{f,u}$ is the pseudo local Hölder exponent of f at u.

Proof. Let t_0 be in $\mathbf{R}$. There exists a sequence $(u_n)_{n\in\mathbf{N}}$ which converges to t_0 and such that

$$\lim_{n\in\mathbf{N}} \underline{\widetilde{\boldsymbol{\alpha}}}_{f,u_n} = \liminf_{u \to t_0, u \neq t_0} \underline{\widetilde{\boldsymbol{\alpha}}}_{f,u} := \alpha_0.$$

According to the definition of the pseudo local Hölder exponent, we know that for all $\varepsilon > 0$, $\rho > 0$ and $n \in \mathbf{N}$,

$$\sup_{u,v\in B(u_n,\rho)} \frac{\left|f(u) - f(v)\right|}{|u-v|^{\underline{\widetilde{\boldsymbol{\alpha}}}_{f,u_n}+\varepsilon}} = +\infty.$$

We use this equality to prove the upper bound on the pseudo 2-microlocal frontier.

Let first note that when $s' \geq 0$, for all $u, v \in B(u_n, \rho)$,

$$\frac{1}{\left(|u-t_0| + |v-t_0|\right)^{-s'}} \geq (2|u_n - t_0| - 2\rho)^{s'} > |u_n - t_0|^{s'},$$

when ρ is small enough. Therefore, according to the first equation, for all $n \in \mathbf{N}$, $s' \geq 0$ and $\rho, \varepsilon > 0$

$$\sup_{u,v\in B(u_n,\rho)} \frac{\left|f(u) - f(v)\right|}{|u-v|^{\underline{\widetilde{\boldsymbol{\alpha}}}_{f,u_n}+\varepsilon}\left(|u-t_0| + |v-t_0|\right)^{-s'}} = +\infty,$$

which leads to

$$\sup_{u,v\in B(t_0,2|u_n-t_0|)} \frac{\left|f(u) - f(v)\right|}{|u-v|^{\underline{\widetilde{\boldsymbol{\alpha}}}_{f,u_n}+\varepsilon}\left(|u-t_0| + |v-t_0|\right)^{-s'}} = +\infty,$$

proving the expected inequality as $\lim_{n\to+\infty} \underline{\widetilde{\boldsymbol{\alpha}}}_{f,u_n} = \underline{\boldsymbol{\alpha}}_0$.

The case $s' \leq 0$ is treated similarly, using the inequality $\left(|u-t_0| + |v-t_0|\right)^{s'} \geq 1$ when $|u_n - t_0|$ and ρ are sufficiently small. □

We know prove Theorem 1.1 which links up the pseudo 2-microlocal frontier to the classic one in the following way:

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,x_0}(s') = \sigma_{f,x_0}(s') \wedge (s' + p_{f,x_0}) \wedge 1,$$

unless f is locally constant at t_0.

Proof of Theorem 1.1. Let first note that in the particular case f is locally constant at t_0, we easily observe that both classic and pseudo 2-microlocal frontiers are equal to $+\infty$. Therefore, we suppose from now that f is not locally constant.

1. Let first prove the following inequality $\Sigma_{f,t_0}(s') \leq 1$ for all $s' \in \mathbf{R}$. We proceed by contradiction: if there exist $\varepsilon > 0$ and $s' \in \mathbf{R}$ such that $\Sigma_{f,t_0}(s') \geq 1 + 2\varepsilon$, then, there exist $\rho > 0$ and $C > 0$ such that

$$\forall u, v \in B(t_0, \rho); \quad |f(u) - f(v)| \leq C|u - v|^{1+\varepsilon} \big(|u - t_0| + |v - t_0|\big)^{-s'}.$$

In particular, for each $k \in \mathbf{N}^*$, there exists $C_k > 0$ such that

$$\forall u, v \in [t_0 + \tfrac{\rho}{k}, t_0 + \rho]; \quad |f(u) - f(v)| \leq C_k|u - v|^{1+\varepsilon},$$

proving that f is differentiable with null derivatives on the interval $[t_0 + \frac{\rho}{k}, t_0 + \rho]$. As it is satisfies for every $k \in \mathbf{N}^*$ and f is continuous at t_0, it must be constant on the interval $[t_0 - \rho, t_0 + \rho]$, which is in contradiction with the assumption of f not locally constant.

2. Let now show the inequality

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t_0}(s') \geq \sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0}) \wedge 1,$$

where $p_{f,t_0} = \inf\{n \geq 1 : f^{(n)}(t_0) \text{ exists and } f^{(n)}(t_0) \neq 0\}$.

Let $s' \in \mathbf{R}$ and let first assume that $\Sigma_{f,t_0}(s') \geq 0$ (the general case is explained at the end of the proof). For all $\sigma < \sigma_{f,t_0}(s')$, there exist $\rho > 0$ and a polynomial P such that for all $u, v \in B(t_0, \rho)$,

$$\Big|(f(u) - P(u)) - (f(v) - P(v))\Big| \leq |u - v|^{\sigma} \big(|u - t_0| + |v - t_0|\big)^{-s'}.$$

It has been observed in [60] that P can be chosen as the Taylor expansion of f at t_0,

$$P(u) = \sum_{k=0}^{N} a_k (u - t_0)^k.$$

Furthermore, it is also proved in [60] that for each $k \in \mathbf{N}^*$ and all $s' \in \mathbf{R}$,

$$\Big|(u - t_0)^k - (v - t_0)^k\Big| \leq C_k|u - v|^{(k+s')\wedge 1} \big(|u - t_0| + |v - t_0|\big)^{-s'},$$

in the neighbourhood of t_0.

Then, according to the definition of p_{f,t_0}, we observe that necessarily $a_k = 0$ for each $k \in \{1, \ldots, p_{f,t_0} - 1\}$. Therefore, there exist $\rho > 0$ and $C > 0$ such that for all $u, v \in B(t_0, \rho)$,

$$|P(u) - P(v)| \leq C|u - v|^{(p_{f,t_0}+s')\wedge 1} \big(|u - t_0| + |v - t_0|\big)^{-s'}.$$

Finally, we obtain that for all $u, v \in B(t_0, \rho)$,

$$\begin{aligned} |f(u) - f(v)| &\leq \Big|(f(u) - P(u)) - (f(v) - P(v))\Big| + |P(u) - P(v)| \\ &\leq C|u - v|^{\sigma\wedge(p_{f,t_0}+s')\wedge 1} \big(|u - t_0| + |v - t_0|\big)^{-s'}, \end{aligned}$$

which proves the expected inequality.

3. Let finally show the converse inequality

$$\forall s' \in \mathbf{R}; \quad \Sigma_{f,t_0}(s') \leq \sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0}) \wedge 1.$$

The boundedness $\Sigma_{f,t_0}(s') \leq 1$ has already been proved. Let $s' \in \mathbf{R}$ with $\Sigma_{f,t_0}(s') \geq 0$.

Let first suppose that $\sigma_{f,t_0}(s') > (s' + p_{f,t_0})$. Then, the function f and the polynomial P are such that

$$\limsup_{\rho \to 0} \sup_{u,v \in B(t_0,\rho)} \frac{(f(u) - P(u)) - (f(v) - P(v))}{|u - v|^{s' + p_{f,t_0}} \left(|u - t_0| + |v - t_0|\right)^{-s'}} = 0.$$

Furthermore, we know from [60] that

$$\limsup_{\rho \to 0} \sup_{u,v \in B(t_0,\rho)} \frac{|(u - t_0)^{p_{f,t_0}} - (v - t_0)^{p_{f,t_0}}|}{|u - v|^{s' + p_{f,t_0}} \left(|u - t_0| + |v - t_0|\right)^{-s'}} > 0.$$

Since $a_{p_{f,t_0}} \neq 0$ and $a_k = 0$ for each $k < p_{f,t_0}$, we have

$$\limsup_{\rho \to 0} \sup_{u,v \in B(t_0,\rho)} \frac{|P(u) - P(v)|}{|u - v|^{s' + p_{f,t_0}} \left(|u - t_0| + |v - t_0|\right)^{-s'}} > 0,$$

which implies

$$\limsup_{\rho \to 0} \sup_{u,v \in B(t_0,\rho)} \frac{|f(u) - f(v)|}{|u - v|^{s' + p_{f,t_0}} \left(|u - t_0| + |v - t_0|\right)^{-s'}} > 0,$$

and thus $\Sigma_{f,t_0}(s') \leq (s' + p_{f,t_0}) = \sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0})$. The case $\sigma_{f,t_0}(s') < (s' + p_{f,t_0})$ is treated similarly.

Finally, let suppose $\sigma_{f,t_0}(s') = s' + p_{f,t_0}$. As P corresponds to the Taylor expansion of f at t_0 and for each $k < p_{f,t_0}$, $a_k = 0$, we have

$$\lim_{u \to t_0} \frac{f(u) - f(t_0)}{(u - t_0)^{p_{f,t_0}}} = \lim_{u \to t_0} \frac{f(u) - f(t_0)}{(u - t_0)^{p_{f,t_0} + s'} (u - t_0)^{-s'}} = a_{p_{f,t_0}} \neq 0,$$

proving that $\Sigma_{f,t_0}(s') \leq (s' + p_{f,t_0})$.

4. We have proved that $\Sigma_{f,t_0}(s') = \sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0}) \wedge 1$ for all $s' \in \mathbf{R}$ such that $\Sigma_{f,t_0}(s') \geq 0$.

Let $k \in \mathbf{N}^*$ and $s' \in \mathbf{R}$ such that $\Sigma_{f,t_0}(s') \in [-k, -k + 1]$. According to Definitions 1.3 and 1.2, we observe that both pseudo and classic 2-microlocal frontiers are defined using increments of the kth integration of f at t_0: $I^k_{t_0}(f)$. Based on this remark, we can adapt the previous reasoning to every $I^k_{t_0}(f)$, $k \in \mathbf{N}$, therefore proving the equality for all $s' \in \mathbf{R}$.

□

We end this section with the proof of Theorem 1.3 related to the behaviour of the pseudo 2-microlocal frontier when a function is integrated.

Proof of Theorem 1.3. Recall that the function F is defined by $F(t) = \int_0^t f(s)\,\mathrm{d}s$. If f is locally equal to 0 at t_0, F is clearly constant in some neighbourhood of t_0, and therefore, the frontiers of f and F are both equal to $+\infty$.

1. Then, let consider the case $f(t_0) \neq 0$. Without any loss of generality, we can suppose $f(t_0) > 0$. Since f is continuous, there exist $\rho, C_1, C_2 > 0$ such that for all $u, v \in B(t_0, \rho)$,

$$C_1|u-v| \leq |F(v) - F(u)| \leq C_2|u-v|.$$

F locally behaves like the function $x \mapsto x$, and thus has the following pseudo frontier: $\Sigma_{F,t_0}(s') = (1+s') \wedge 1$ for all $s' \in \mathbf{R}$.

2. Finally, let suppose $f(t_0) = 0$, but is not locally equal to 0.

We know that the classic 2-microlocal frontier of F satisfies $\sigma_{F,t_0} = \sigma_{f,t_0} + 1$. Furthermore, as f is continuous, $F' = f$ and therefore, as $F'(t_0) = 0$, the exponent p_{F,t_0} defined previously is simply equal to $p_{f,t_0} + 1$.

Theorem 1.1 implies for all $s' \in \mathbf{R}$,

$$\Sigma_{f,t_0}(s') = \sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0} + 1) \wedge 1.$$

Therefore, we obtain

$$\begin{aligned}
\Sigma_{F,t_0}(s') &= \sigma_{F,t_0}(s') \wedge (s' + p_F, t_0 + 1) \wedge 1 \\
&= \big(\sigma_{f,t_0}(s') + 1\big) \wedge (s' + p_{f,t_0} + 2) \wedge 2 \wedge 1 \\
&= \Big(\big(\sigma_{f,t_0}(s') \wedge (s' + p_{f,t_0} + 1) \wedge 1\big) + 1\Big) \wedge 1 \\
&= \big(\Sigma_{f,t_0}(s') + 1\big) \wedge 1.
\end{aligned}$$

□

1.7.2 Examples of martingales

Examples 1.2 and 1.3 describe martingales with a particular local regularity. This section details the construction of the quadratic variation which leads to these interesting 2-microlocal frontiers.

Lemma 1.5 *There exists a deterministic non-decreasing function f_α which has the following pseudo 2-microlocal frontier at* 0

$$\forall s' \geq -1; \quad \Sigma_{f_\alpha,0}(s') = \left(\frac{s'+1}{1-\log_2(\alpha)}\right) \wedge 1,$$

where $\alpha \in [0,1]$ and $\log_2(\alpha) = \frac{\log(\alpha)}{\log(2)}$.

Proof. The construction of this example is similar to Cantor's function. Beginning with the identity $f_0 : x \mapsto x$, we iterate the construction using the following process. Let first choose a parameter $\alpha \in [0,1]$. At each step N, the function f_{N-1} is equal to the identity on the interval $[0, \frac{1}{2^N}]$. Then, f_N is obtained by modifying f_{N-1} on this interval as described in figure 1.3.

The sequence $(f_N)_{N\in\mathbf{N}}$ uniformly converges to a continuous function f since it satisfies a Cauchy criterion

$$\forall n, p \in \mathbf{N}; \quad \|f_{n+p} - f_n\|_\infty \leq 2^{-n}.$$

Let now describe the pseudo 2-microlocal frontier of this function f at 0.

1. We first consider one-side inequality,

$$\forall s' \geq -1; \quad \Sigma_{f_\alpha,0}(s') \leq \left(\frac{s'+1}{1-\log_2(\alpha)}\right) \wedge 1.$$

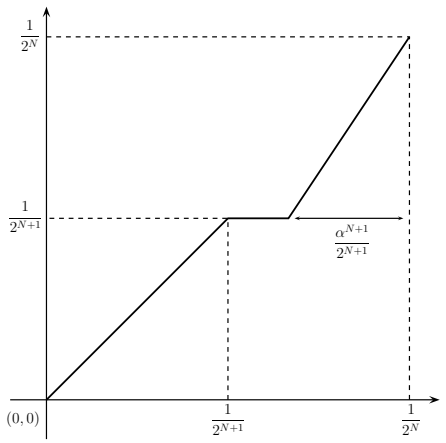


Figure 1.3: Function f_N obtained on the interval $[0, \frac{1}{2^N}]$.

As usually when one wants to obtain an upper bound for a regularity exponent, specific sequences $(s_n)_{n\in\mathbf{N}}$ and $(t_n)_{n\in\mathbf{N}}$ are constructed to capture the irregularity of the function. More precisely, let $t_n = 2^{-n}$ and $s_n = t_n - \left(\frac{\alpha}{2}\right)^{n+1}$. Then, for every $n \in \mathbf{N}$, $\sigma \in [0,1]$ and $s' \geq -1$, we have

$$\begin{aligned}|f(s_n) - f(t_n)| &= 2^{-(n+1)} = 2^{-(n+1)(1+s')} \cdot 2^{s'(n+1)} \\ &= |t_n - s_n|^{\frac{1+s'}{1-\log_2(\alpha)}} \cdot 2^{s'(n+1)} \geq C|t_n - s_n|^{\frac{1+s'}{1-\log_2(\alpha)}} \left(|t_n| + |s_n|\right)^{-s'},\end{aligned}$$

which shows this first inequality.

2. Let now prove the converse inequality. Let $s' \geq -1$ and $\sigma < \left(\frac{s'+1}{1-\log_2(\alpha)}\right) \wedge 1$. Then, we have to show that there exist $\rho > 0$ and $C > 0$ such that

$$\forall u, v \in B(0,\rho); \quad |f(u) - f(v)| \leq C|u - v|^{\sigma} \left(|u| + |v|\right)^{-s'}.$$

Let first observe that for all $u \in \mathbf{R}_+$, we have $f(u) \leq u$.

Let $u \leq v \in [0,1]$ and $m \in \mathbf{N}$ such that $2^{-(m+1)} \leq v \leq 2^{-m}$. We distinguish two different cases.

- If $u \leq 2^{-(m+2)}$, then $2^{-(m+2)} \leq |u - v| \leq 2^{-(m-1)}$. Therefore for all $s' \geq -1$, we obtain

$$\begin{aligned}|f(u) - f(v)| &\leq f(u) + f(v) \\ &\leq 2^{-(m-1)} = 2^{-(m-1)(1+s')} \cdot 2^{s'(m-1)} \\ &\leq C|u - v|^{1+s'} \left(|u| + |v|\right)^{-s'} \leq C|u - v|^{\sigma} \left(|u| + |v|\right)^{-s'},\end{aligned}$$

  since $\sigma < \left(\frac{s'+1}{1-\log_2(\alpha)}\right) \wedge 1 \leq 1 + s'$ and $|u - v| \in [0,1]$.

- If $u \geq 2^{-(m+2)}$. For sake of simplicity, we suppose $u \geq 2^{-(m+1)}$ (the general case is as simple, but longer to argue). Then, according to the construction of the function f_n,

we observe that

$$\begin{aligned}|f(u)-f(v)| &\leq |f(u)-f(v)|^{\sigma}\cdot|f(u)-f(v)|^{1-\sigma}\\ &\leq \left(|u-v|\alpha^{-(n+1)}\right)^{\sigma}\cdot\left(f(u)+f(v)\right)^{1-\sigma}\\ &\leq |u-v|^{\sigma}2^{-(n+1)\sigma\log_2(\alpha)}\left(|u|+|v|\right)^{1-\sigma}\\ &\leq C|u-v|^{\sigma}\left(|u|+|v|\right)^{1-\sigma+\sigma\log_2(\alpha)}\\ &\leq C|u-v|^{\sigma}\left(|u|+|v|\right)^{-s'},\end{aligned}$$

since $-s' \leq 1-\sigma+\sigma\log_2(\alpha)$.

Based on these different cases, we obtain the inequality

$$\forall s' \geq -1; \quad \Sigma_{f_\alpha,0}(s') \geq \left(\frac{s'+1}{1-\log_2(\alpha)}\right)\wedge 1,$$

which concludes the proof.

We note that the pointwise Hölder $\alpha_{f,0}$ exponent does not depend on α and is always equal to 1, whereas the local Hölder exponent $\widetilde{\alpha}_{f,0}$ is equal to $\frac{1}{1-\log_2(\alpha)}$. Therefore in the case of $\alpha = 1$, the function is simply the identity and $\widetilde{\alpha}_{f,0} = 1$ and on the contrary, when $\alpha = 0$, $\widetilde{\alpha}_{f,0} = 0$ and the function f has jumps at every $\frac{1}{2^n}$ (but is still continuous at 0). □

Lemma 1.6 *Let $H : \mathbf{R} \to (0,\gamma]$ be a continuous function such that $\gamma < \frac{1}{2}$. Then, there exists a deterministic increasing function $f : [0,1] \to \mathbf{R}$ and a null-measure closed set E, f is locally constant on E^c and for all $x_0 \in E$, the 2-microlocal frontier of f at x_0 is equal to*

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,x_0}(s') = \Sigma_{f,x_0}(s') = \left(s' - \frac{1}{\log_2\left(H(f(x_0))\right)}\right)\wedge\frac{-1}{\log_2\left(H(f(x_0))\right)}.$$

Proof. We first describe the construction of the function f. Let $H : \mathbf{R} \to (0,\gamma]$ be a continuous function such that $\gamma < \frac{1}{2}$ and let $f_0 : x \mapsto x$ be the identity on $[0,1]$. Then we construct by induction a sequence of functions $(f_n)_{n\in\mathbf{N}}$ which converges. We define $x_0 = 0$ and $y_0 = 0$. Then, by induction, we consider for every $(\varepsilon_0 = 0, \varepsilon_1, \cdots, \varepsilon_n) \in \{0,1\}^{n+1}$ the following construction:

$$\begin{aligned}&\alpha_{\varepsilon_0\cdots\varepsilon_n} = H(y_{\varepsilon_0\cdots\varepsilon_n}) && l_{\varepsilon_0\cdots\varepsilon_n} = \alpha_{\varepsilon_0\cdots\varepsilon_n} l_{\varepsilon_0\cdots\varepsilon_{n-1}},\\ &x_{\varepsilon_0\cdots\varepsilon_n 0} = x_{\varepsilon_0\cdots\varepsilon_n} && x_{\varepsilon_0\cdots\varepsilon_n 1} = x_{\varepsilon_0\cdots\varepsilon_n} + (1-\alpha_{\varepsilon_0\cdots\varepsilon_n})l_{\varepsilon_0\cdots\varepsilon_{n-1}},\\ &y_{\varepsilon_0\cdots\varepsilon_n 0} = y_{\varepsilon_0\cdots\varepsilon_n} && y_{\varepsilon_0\cdots\varepsilon_n 1} = y_{\varepsilon_0\cdots\varepsilon_n} + 2^{-(n+1)}.\end{aligned}$$

The following figure 1.4 illustrates this construction of the function f_n and the associated closed set E_n.

We define the closed set E as the intersection of the collection $(E_n)_{n\in\mathbf{N}}$:

$$E = \bigcap_{n\in\mathbf{N}} E_n,$$

and the function f as the uniform limit of the sequence $(f_n)_{n\in\mathbf{N}}$, which is a Cauchy sequence as for all n, $\|f_{n+1} - f_n\|_\infty \leq 2^{-(n+1)}$.

We may first note that the set E has a null-measure. Indeed, for every $n \in \mathbf{N}$, E_n satisfies the following inequality:

$$\lambda(E) \leq \lambda(E_n) \leq 2^n\gamma^n \xrightarrow[n\to\infty]{} 0.$$

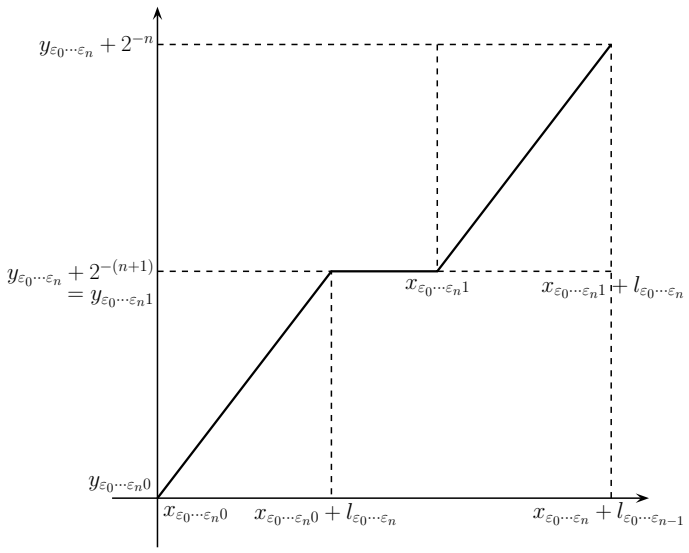


Figure 1.4: Function f_{n+1} obtained on the interval $[x_{\varepsilon_0\cdots\varepsilon_n}, x_{\varepsilon_0\cdots\varepsilon_n} + l_{\varepsilon_0\cdots\varepsilon_{n-1}}]$

Let now establish the 2-microlocal frontier $\sigma_{f,x}$ of f at every $x \in [0,1]$. We first note that for every $x \in E^c$, there exists $\varepsilon > 0$ such that f is constant on the interval $[x-\varepsilon, x+\varepsilon]$. Therefore, in these case, the 2-microlocal frontier corresponds to

$$\forall x \in E^c, \forall s' \in \mathbf{R}; \quad \sigma_{f,x}(s') = +\infty.$$

From now, let consider $x \in E$ and show that the 2-microlocal frontier of f at this point satisfies

$$\forall s' \leq 0; \quad \sigma_{f,x}(s') = s' - \frac{1}{\log_2\big(H(f(x))\big)}.$$

According to the construction of f, for every $x \in E$ there exists a sequence $(\varepsilon_n)_{n\in\mathbf{N}} \in \{0,1\}^{\mathbf{N}}$ such that

$$x = \sum_{i=0}^{\infty} \varepsilon_{i+1} \alpha_{\varepsilon_0} \alpha_{\varepsilon_0\varepsilon_1} \cdots \alpha_{\varepsilon_0\cdots\varepsilon_{i-1}} (1 - \alpha_{\varepsilon_0\cdots\varepsilon_i}).$$

Based on this representation, the corresponding value $f(x)$ is

$$f(x) = \sum_{i=0}^{\infty} \varepsilon_i 2^{-i}.$$

1. Let first obtain a lower bound for the 2-microlocal frontier.

   Let $x_0 \in E$, $\rho > 0$ and $x, y \in B(x_0, \rho) \cap E$ which are represented by the sequences $(\varepsilon_n^x)_n$ and $(\varepsilon_n^y)_n$. We note N be the first $n \in \mathbf{N}$ such that $\varepsilon_n^x \neq \varepsilon_n^y$. Without any restriction and to simplify the calculus, we can suppose that $\forall n \geq N$, $\varepsilon_n^y = 0$. Then, we have

$$\begin{aligned} |f(x) - f(y)| &= \sum_{i=N}^{\infty} \varepsilon_i^x 2^{-i} \\ &\leq K 2^{-N} = K e^{-N\log(2)}. \end{aligned}$$

   We know that $\lim_{N\to+\infty} \alpha_{\varepsilon_0\cdots\varepsilon_N} = H(f(x_0))$ uniformly, therefore

$$\lim_{N\to+\infty} \frac{\log(\alpha_{\varepsilon_0} \ldots \alpha_{\varepsilon_0\cdots\varepsilon_{N-1}})}{N} = \log\big(H(f(x_0))\big)$$

Let $\varepsilon > 0$, then ρ can be chosen small enough such that

$$\left| \frac{\log(\alpha_{\varepsilon_0} \dots \alpha_{\varepsilon_0 \cdots \varepsilon_{N-1}})}{N} - \log\big(H(f(x_0))\big) \right| \leq \varepsilon,$$

which leads to the inequality

$$N \log(2) \geq \frac{\log(\alpha_{\varepsilon_0} \dots \alpha_{\varepsilon_0 \cdots \varepsilon_{N-1}})}{\big(1 - \frac{\varepsilon}{\log(2)}\big) \log_2\big(H(f(x_0))\big)}.$$

Let $\beta = \frac{-1}{\left(1 - \frac{\varepsilon}{\log(2)}\right) \log_2\left(H(f(x_0))\right)}$, then we obtain

$$\begin{aligned}
|f(x) - f(y)| &= \sum_{i=N}^{\infty} \varepsilon_i^x 2^{-i} \\
&\leq K 2^{-N} \\
&\leq K \big(\alpha_{\varepsilon_0} \alpha_{\varepsilon_0 \varepsilon_1} \cdots \alpha_{\varepsilon_0 \cdots \varepsilon_{N-1}}\big)^{\beta} \\
&\leq K \left[\sum_{i=N-1}^{\infty} \varepsilon_{i+1} \alpha_{\varepsilon_0} \alpha_{\varepsilon_0 \varepsilon_1} \cdots \alpha_{\varepsilon_0 \cdots \varepsilon_{i-1}} (1 - \alpha_{\varepsilon_0 \cdots \varepsilon_i}) \right]^{\beta} \\
&= K |x - y|^{\beta}.
\end{aligned}$$

Therefore, for every $s' \leq 0$, we get

$$|f(x) - f(y)| \leq K|x - y|^{\beta} \leq K|x - y|^{\beta + s'} \big(|x - x_0| + |y - x_0|\big)^{-s'}.$$

As this inequality holds for every $\varepsilon > 0$, we obtain the expected lower bound

$$\forall s' \leq 0; \quad \sigma_{f,x_0}(s') \geq s' - \frac{1}{\log_2\big(H(f(x_0))\big)}.$$

2. Conversely, let now consider $x_0 \in E$ and the corresponding representation $(\varepsilon_n)_{n \in \mathbf{N}}$. Then, for every $n \in \mathbf{N}$, let x_n and y_n be $x_n = x_{\varepsilon_0 \dots \varepsilon_n 0}$ and $y_n = x_{\varepsilon_0 \dots \varepsilon_n 0} + l_{\varepsilon_0 \dots \varepsilon_n}$.

   We know that $|f(y_n) - f(x_n)| = 2^{-(n+1)}$ for every $n \in \mathbf{N}$ and $x_0 \in [x_n, y_n]$. Therefore, $|t_n - s_n| = |t_n - x_0| + |s_n - x_0|$, and for all $s' \leq 0$ and $\sigma \in (0, 1)$

$$\begin{aligned}
&\log\left(\frac{|f(y_n) - f(x_n)|}{|x_n - y_n|^{\sigma} \big(|t_n - x_0| + |s_n - x_0|\big)^{-s'}} \right) \\
&= \log\left(\frac{|f(y_n) - f(x_n)|}{|x_n - y_n|^{\sigma - s'}} \right) \\
&= -(n+1)\log(2) - (\sigma - s')\log(l_{\varepsilon_0 \dots \varepsilon_n}) \\
&= -(n+1)\log(2) - (\sigma - s')\big(\log(\alpha_{\varepsilon_0}) + \cdots + \log(\alpha_{\varepsilon_0 \cdots \varepsilon_n})\big) \\
&= -(n+1)\left[\log(2) + (\sigma - s') \frac{1}{n+1} \big(\log(\alpha_{\varepsilon_0}) + \cdots + \log(\alpha_{\varepsilon_0 \cdots \varepsilon_n})\big) \right] \\
&\sim -(n+1)\left[\log(2) + (\sigma - s')\log(f(H(x_0))) \right] \\
&\xrightarrow[n \to \infty]{} +\infty,
\end{aligned}$$

   if $\sigma > s' - \frac{1}{\log_2\left(H(f(x_0))\right)}$.

Therefore, we have proved that for every $x_0 \in E$, classic and pseudo 2-microlocal frontier at x_0 are equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,x_0}(s') = \Sigma_{f,x_0}(s') = \left(s' - \frac{1}{\log_2\big(H(f(x))\big)}\right) \wedge \frac{-1}{\log_2\big(H(f(x))\big)},$$

where the upper bound for $s' \geq 0$ is obtained using Lemma 1.4 and the continuity of functions H and f. □

Based on the previous deterministic lemma, we can detail the construction of the martingale in example 1.3.

Lemma 1.7 *There exists a continuous martingale $(M_t(\omega))_{t\in[0,1]}$ and a Cantor-like closed random set $E(\omega)$ such that for all $\omega \in \Omega$ and $t \in [0,1]$, the measure of $E(\omega)$ is null and the regularity of M satisfies*

1. *if $t \notin E(\omega)$, then M is locally constant at t, and therefore*

$$\forall s' \in \mathbf{R}; \quad \sigma_{M,t}(s')(\omega) = +\infty.$$

2. *if $t \in E(\omega)$, then M has the following 2-microlocal frontier*

$$\forall s' \in \mathbf{R}; \quad \sigma_{M,t}(s')(\omega) = \left(s' + \frac{1}{2(\alpha + |M_t(\omega)|)}\right) \wedge \frac{1}{2(\alpha + |M_t(\omega)|)},$$

where $\alpha > 1$ is a fixed parameter.

Proof. Let β be a Brownian motion and $\alpha > 1$. For every $\omega \in \Omega$, we consider the function $H : x \to 2^{-(\alpha+|\beta_x|)}$. As described in the previous Lemma 1.6, we construct the corresponding function $f(t,\omega)$ and random set $E(\omega)$. We have proved $f(\cdot,\omega)$ has the following 2-microlocal frontier at $t \in E(\omega)$

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,t}(s') = \Sigma_{f,t}(s') = \left(s' + \frac{1}{\alpha + |\beta_{f(t,\omega)}(\omega)|}\right) \wedge \frac{1}{\alpha + |\beta_{f(t,\omega)}(\omega)|},$$

since $\log_2\big(H(f(t,\omega))\big) = \log_2\big(2^{-(\alpha+|\beta_{f(t,\omega)}(\omega)|)}\big) = \alpha + |\beta_{f(t,\omega)}(\omega)|$.

From the construction algorithm used in the previous lemma, we can verify that for all $t \in [0,1]$, $f(t,.)$ is a bounded β-stopping time, and therefore the following process M is a martingale:

$$\forall \omega \in \Omega, \forall t \in [0,1]; \quad M_t(\omega) = \beta_{f(t,\omega)}(\omega).$$

Then, using Theorem 1.6 and the regularity properties of the function f, we obtain the expected 2-microlocal frontier. □

2

Irregular multifractional Brownian motion

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2.1 Introduction

The multifractional Brownian motion (mBm) has been independently introduced by Peltier and Lévy Véhel [154] and Benassi, Jaffard, and Roux [26] as a natural extension of the well-known fractional Brownian motion (fBm). The main idea behind these two works was to drop the stationary assumption on the process, and thus allow the Hurst exponent to change as time passes. In this way, the mBm is parametrized by a function $t \mapsto H(t)$, usually continuous, and happens to be an interesting stochastic model for non-stationary phenomena (e.g. signal processing, traffic on internet, terrain modelling, …).

Since its introduction, several authors have investigated sample path properties of the mBm. For instance, Benassi, Jaffard, and Roux [26] and Peltier and Lévy Véhel [154] have respectively studied the law of iterated logarithm and the Hölder regularity of its trajectories. The latter has also determined the Hausdorff dimension of the graph. Moreover, the fine covariance structure have been analysed and refined by Ayache, Cohen, and Lévy Véhel [10], Herbin [76] and Stoev and Taqqu [172]. Finally, the local time and the Hausdorff dimension of level sets have recently been considered by Boufoussi, Dozzi, and Guerbaz [37]. Several multiparameter extensions, which generalise the classic Lévy fractional Brownian motion and fractional Brownian sheet, have also been introduced and investigated by Herbin [76] Meerschaert, Wu, and Xiao [134] and Ayache, Shieh, and Xiao [13].

Fine geometric properties obtained in the aforementioned works usually rely on a key

assumption on the behaviour of the Hurst function:

$$H \text{ is a } \beta\text{-Hölder continuous function such that } \forall t \in \mathbf{R},\ \ H(t) < \beta. \qquad (\mathscr{H}_0)$$

Due to this hypothesis, the study of the $\mathscr{H}_0$-mBm is usually easier and leads to results closely related to the counterparts on fractional Brownian motion. This behaviour of the mBm is due to its covariance structure which, under $\mathscr{H}_0$, is locally equivalent to one of fBm with Hurst parameter $H(t)$:

$$\forall u, v \in B(t,\rho); \quad \mathbb{E}\big[(X_u - X_v)^2\big] \asymp |u-v|^{2H(t)},$$

for any $t \in \mathbf{R} \setminus \{0\}$.

The study of the general multifractional Brownian motion, i.e. when the assumption $\mathscr{H}_0$ does not hold any more, has only been considered by HERBIN [76] and AYACHE [8]. They have both investigated the Hölder regularity of sample paths, and more precisely the pointwise and local Hölder exponents, which are defined by

$$\forall t \in \mathbf{R}; \quad \alpha_{X,t} = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t,\rho)} \frac{|X_u - X_v|}{\rho^\alpha} < \infty\right\} \tag{2.1}$$

and

$$\forall t \in \mathbf{R}; \quad \widetilde{\alpha}_{X,t} = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{u,v\in B(t,\rho)} \frac{|X_u - X_v|}{|u-v|^\alpha} < \infty\right\}. \tag{2.2}$$

These two exponents aim to characterise the local Hölder regularity of the trajectory around t, by comparing its oscillations to a power of respectively ρ or $|u-v|$. For instance, it is known that the fractional Brownian motion B^H satisfies with probability one and for all $t \in \mathbf{R}$, $\alpha_{B^H,t} = \widetilde{\alpha}_{B^H,t} = H$.

The general form of the Hölder regularity of the multifractional Brownian motion has first been obtained by HERBIN [76], who proved that for all $t > 0$

$$\alpha_{X,t} = H(t) \wedge \alpha_{H,t} \quad \text{and} \quad \widetilde{\alpha}_{X,t} = H(t) \wedge \widetilde{\alpha}_{H,t} \quad \text{a.s.} \tag{2.3}$$

We note that the previous result only holds for a fixed $t > 0$, and thus not uniformly on the sample paths. We also observe that the geometric properties of a common mBm are not simply an extension of those on fBm, but display a much more complicated structure where the fine properties of the Hurst function intervene. AYACHE [8] has recently extended this study, obtaining a uniform characterisation of the local Hölder exponent and proving that its pointwise exponent can behave randomly as time passes. Throughout, we will described as *irregular multifractional Brownian motion* the resulting process whose sample path properties are unusual and unexpected.

The aim of this work is therefore to push further the study of the irregular mBm. In Section 2.2, we first discuss the existence of an alternative deterministic representation of the fractional Brownian field, which will then be used to study more precisely the geometric properties of the mBm. Section 2.3 is devoted to the 2-microlocal and Hölder regularity of the trajectories, thus extends the results obtained by HERBIN [76] and AYACHE [8]. Hausdorff and Box dimensions of the graph and images of fractal sets are investigated in Section 2.4, exhibiting further connections with the geometry of the Hurst function. Finally, Appendix 2.5 gathers a few deterministic results related to the 2-microlocal frontier and the fractal dimensions. Several examples are also given in these different sections to illustrate the main results.

2.2 Deterministic representation

The question of the equivalence of different mBm's representations is known to be non-trivial. On one side, PELTIER and LÉVY VÉHEL [154] have introduced the mBm as the following stochastic integral

$$\forall t \in \mathbf{R}; \quad X_t = \frac{1}{\Gamma\big(H(t)+\frac{1}{2}\big)} \int_{\mathbf{R}} \Big[(t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2}\Big] \mathrm{d}W_u.$$

Whereas on the other hand, BENASSI, JAFFARD, and ROUX [26] have considered a real-harmonizable definition

$$\forall t \in \mathbf{R}; \quad \widetilde{X}_t = \int_{\mathbf{R}} \frac{\mathrm{e}^{iut}-1}{|u|^{H(t)+1/2}} \, \mathrm{d}\widehat{W}_u.$$

Even though it was at first thought that X and $\widetilde{X}$ had the same law, which is the case for the fBm, STOEV and TAQQU [172] have proved that the covariance structure is slightly different between these two definitions. Considering this fact, they have suggested to name multifractional Brownian motion any process X which has the following form

$$\forall t \in \mathbf{R}; \quad X_t = a^+ B^+(t, H(t)) + a^- B^-(t, H(t)),$$

where $(a^+, a^-) \in \mathbf{R}^2 \setminus \{0,0\}$ and the fractional Brownian fields $(t,H) \mapsto B^{\pm}(t,H)$ are defined by

$$B^{\pm}(t,H) = \frac{1}{\Gamma\big(H+\frac{1}{2}\big)} \int_{\mathbf{R}} \Big[(t-u)_{\pm}^{H-1/2} - (-u)_{\pm}^{H-1/2}\Big] \mathrm{d}W_u, \tag{2.4}$$

for all $t \in \mathbf{R}$ and $H \in (0,1)$.

The next proposition presents an alternative deterministic representation for the fractional Brownian field.

Proposition 2.1 *Suppose $(B_t)_{t\in\mathbf{R}}$ is a Brownian motion such that $B_t := W([0,t])$, where W is a Wiener measure. Then, for all $t \in \mathbf{R}$ and $H \in (0,1)$, the fractional Brownian field $B^{\pm}(t,H)$ is almost surely equal to:*

- *if $(t,H) \in \mathbf{R} \times \big(\frac{1}{2}, 1\big)$,*

$$B^{\pm}(t,H) = \frac{\pm 1}{\Gamma\big(H-\frac{1}{2}\big)} \int_{\mathbf{R}} B_u \Big[(t-u)_{\pm}^{H-3/2} - (-u)_{\pm}^{H-3/2}\Big] \mathrm{d}u. \tag{2.5}$$

- *if $(t,H) \in \mathbf{R} \times \big(0, \frac{1}{2}\big]$ and T is a fixed number such that $T \in (\pm\infty, t)$,*

$$\begin{aligned} B^{\pm}(t,H) = {} & \frac{\pm\big(H-\frac{1}{2}\big)}{\Gamma\big(H+\frac{1}{2}\big)} \int_{\mathbf{R}} (B_u - B_t \mathbf{1}_{\{\pm t \geq \pm T\}})(t-u)_{\pm}^{H-3/2} - B_u(-u)_{\pm}^{H-3/2} \mathrm{d}u \\ & \pm B_t (t-T)_{\pm}^{H-1/2}, \end{aligned} \tag{2.6}$$

or equivalently,

$$\begin{aligned} B^{\pm}(t,H) = {} & \frac{\pm 1}{\Gamma\big(H+\frac{1}{2}\big)} \frac{\mathrm{d}}{\mathrm{d}s} \left(\int_{[\pm T, s]} B_u (s-u)_{\pm}^{H-\frac{1}{2}} \mathrm{d}u \right) \{(t)-(0)\} \\ & \pm \frac{H-\frac{1}{2}}{\Gamma\big(H+\frac{1}{2}\big)} \int_{(\pm\infty, \pm T]} B_u \Big[(t-u)_{\pm}^{H-3/2} - (-u)_{\pm}^{H-3/2}\Big] \mathrm{d}u. \end{aligned} \tag{2.7}$$

Proof. Without any loss of generality, we may only prove the equality for the term $B^+(t,H)$. Let first consider the case $t \in \mathbf{R}$ and $H \in (\frac{1}{2}, 1)$. Using Ito's lemma on the product $B_u(t-u)_+^{H-1/2}$, we obtain for any $x \leq t$

$$B_t(t-t)^{H-1/2} = B_x(t-x)^{H-1/2} + \int_x^t (t-u)^{H-1/2} \mathrm{d}B_u - (H-\tfrac{1}{2}) \int_x^t B_u(t-u)^{H-3/2} \mathrm{d}u.$$

Hence, subtracting the case $t = 0$,

$$\begin{aligned} B_x \left[(t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right] = &- \int_x^t \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}B_u \\ &+ (H-\tfrac{1}{2}) \int_x^t B_u \left[(t-u)_+^{H-3/2} - (-u)_+^{H-3/2} \right] \mathrm{d}u. \end{aligned}$$

The Brownian motion is known to satisfy $\lim_{u\to\pm\infty} |B_u|/|u|^{1/2+\varepsilon} = 0$. Moreover, we observe that $\left[(t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right] \sim_{-\infty} (H-\frac{1}{2}) t^{H-1/2} (-x)^{H-3/2}$. Therefore, as $\frac{3}{2} - H > \frac{1}{2}$,

$$B_x \left[(t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right] \xrightarrow[x\to-\infty]{a.s.} 0.$$

Similarly, using the dominated convergence theorem,

$$\int_x^t B_u \left[(t-u)_+^{H-3/2} - (-u)_+^{H-3/2} \right] \mathrm{d}u \xrightarrow[x\to-\infty]{a.s.} \int_{-\infty}^t B_u \left[(t-u)_+^{H-3/2} - (-u)_+^{H-3/2} \right] \mathrm{d}u,$$

Finally, owing to the L^2-continuity of the stochastic integral,

$$\int_x^t \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}B_u \xrightarrow[x\to-\infty]{L^2(\Omega)} \int_{-\infty}^t \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}B_u,$$

which proves the expected equality.

Let now consider the second case $H \in (0, \frac{1}{2})$. The calculus is quite similar, with only a slight modification in the application of Ito's lemma between T and $t - \varepsilon$.

$$\begin{aligned} &(B_{t-\varepsilon} - B_t)\varepsilon^{H-1/2} + B_t(t-T)^{H-1/2} \\ &= B_T(t-T)^{H-1/2} + \int_T^{t-\varepsilon} (t-u)^{H-1/2} \mathrm{d}B_u - (H-\tfrac{1}{2}) \int_T^{t-\varepsilon} (B_u - B_t)(t-u)^{H-3/2} \mathrm{d}u, \end{aligned}$$

for any $\varepsilon > 0$. Owing to the Hölder continuity of the sample paths of B, the previous expression converges when $\varepsilon \to 0$. In addition, the calculus presented in the case $H > \frac{1}{2}$ holds as well on the interval $[x, T]$, therefore proving Equation (2.6). Finally, the second form (2.7) of the fractional Brownian field is obtained using an alternative representation of the Riemann-Liouville fractional derivative of a function on the interval $[T, t]$ (see the book of SAMKO, KILBAS, and MARICHEV [166] for more information on the subject). □

This type of representation has already been exhibited and studied by PICARD [158] for the fractional Brownian motion, even though his proof is based on a different approach (approximation of the Brownian motion with Lipschitz functions). Also note that the representation (2.6) holds when $H > \frac{1}{2}$.

Proposition 2.2 *Let $(B_t)_{t\in\mathbf{R}}$ be a continuous Brownian motion and $B(t,H)$ be the the fractional Brownian field defined in Proposition 2.1.*

Then, for all $\omega \in \Omega$ and for every $(t_0, H_0) \in \mathbf{R} \times (0,1)$, the field $(t,H) \mapsto B(t,H)(\omega)$ is continuous at (t_0, H_0).

Proof. As previously, we only need to consider the component $B^+(t,H)$. The continuity on the domains $\mathbf{R}\times(0,\frac{1}{2})$ and $\mathbf{R}\times(\frac{1}{2},1)$ is a simple consequence of the dominated convergence theorem: the deterministic integral is split in two parts on which the theorem can be applied with different bounds.

Therefore, let $H_0=\frac{1}{2}$, $t_0\in\mathbf{R}$ and $\varepsilon>0$. Since B is continuous, there exists $\alpha>0$ such that for every $t\in B(t_0,\alpha)$, $|B_t-B_{t_0}|<\varepsilon$. Furthermore, still using the dominated convergence theorem, we observe that for any $\alpha>0$ the term

$$\frac{1}{\Gamma(H-\frac{1}{2})}\int_{-\infty}^{t-\alpha}B_u\big[(t-u)_+^{H-3/2}-(-u)_+^{H-3/2}\big]\mathrm{d}u,$$

is continuous in the neighbourhood of (t_0,H_0), and converges to 0 when $H\to\frac{1}{2}$. Hence, there exists $\gamma>0$ such that for every $H\in[\frac{1}{2},\frac{1}{2}+\gamma]$ and $t\in B(t_0,\gamma)$, its norm is dominated by ε. Let now consider the remaining term:

$$\begin{aligned}
&\left|\frac{1}{\Gamma(H-\frac{1}{2})}\int_{t-\alpha}^{t}B_u(t-u)_+^{H-3/2}\mathrm{d}u-B_{t_0}\right|\\
&\le\frac{1}{\Gamma(H-\frac{1}{2})}\int_{t-\alpha}^{t}|B_u-B_{t_0}|(t-u)_+^{H-3/2}\mathrm{d}u+|B_{t_0}|\cdot\left|\frac{\alpha^{H-1/2}}{\Gamma(H+\frac{1}{2})}-1\right|\\
&\le\frac{\varepsilon\alpha^{H-1/2}}{\Gamma(H+\frac{1}{2})}+|B_{t_0}|\cdot\left|\frac{\alpha^{H-1/2}}{\Gamma(H+\frac{1}{2})}-1\right|\\
&\le 2\varepsilon(1+|B_{t_0}|),
\end{aligned}$$

for all $H\in[\frac{1}{2},\frac{1}{2}+\gamma]$ and $t\in B(t_0,\gamma)$, with $\gamma>0$ sufficiently small. Note that we have used the Hölder continuity of the Brownian motion for the second upper bound. This inequality therefore proves the continuity when $H\ge\frac{1}{2}$.

When $H\le\frac{1}{2}$, the continuity is obtained using representation (2.6), as we observe that the first term simply converges to 0 and the second one to B_t. □

Owing to the Hölder continuity of the sample paths of Brownian motion, we obtain a deterministic representation of the fractional Brownian field $(t,H)\mapsto B(t,H)$ which is continuous in both variables. It will be used in the two next sections to study in the fine geometric properties of multifractional Brownian motion, which will be partly deduced from those satisfied by Brownian motion.

Remark 2.1 As an illustration, the self-similarity and stationarity of the increments of fractional Brownian motion can be directly deduced from the deterministic representation of $B(t,H)$. Indeed, for any $c>0$

$$\begin{aligned}
\{B_{ct}^H\}_{t\in\mathbf{R}}&=\left\{\frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbf{R}}B_u\Big[(ct-u)_+^{H-3/2}-(-u)_+^{H-3/2}\Big]\mathrm{d}u\right\}_{t\in\mathbf{R}}\\
&=\left\{\frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbf{R}}B_{cv}\Big[(ct-cv)_+^{H-3/2}-(-cv)_+^{H-3/2}\Big]c\mathrm{d}v\right\}_{t\in\mathbf{R}}\\
&=c^H\left\{\frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbf{R}}\widetilde{B}_v\Big[(t-v)_+^{H-3/2}-(-v)_+^{H-3/2}\Big]\mathrm{d}v\right\}_{t\in\mathbf{R}}\overset{\mathscr{L}}{=}\{c^HB_t^H\}_{t\in\mathbf{R}},
\end{aligned}$$

where $(\widetilde{B}_v)_{v\in\mathbf{R}} = (\sqrt{c}B_{cv})_{v\in\mathbf{R}}$ is a Brownian motion. Similarly, for any $h \in \mathbf{R}_+$,

$$\begin{aligned}\{B^H_{t+h} - B^H_t\}_{t\in\mathbf{R}} &= \left\{\frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbf{R}} B_u\Big[(t+h-u)^{H-3/2}_+ - (h-u)^{H-3/2}_+\Big]\mathrm{d}u\right\}_{t\in\mathbf{R}}\\ &= \left\{\frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbf{R}} (B_{v+h}-B_h)\Big[(t-v)^{H-3/2}_+ - (-v)^{H-3/2}_+\Big]\mathrm{d}v\right\}_{t\in\mathbf{R}}\\ &= \left\{\frac{1}{\Gamma(H-\frac{1}{2})}\int_{\mathbf{R}} \widetilde{B}_v\Big[(t-v)^{H-3/2}_+ - (-v)^{H-3/2}_+\Big]\mathrm{d}v\right\}_{t\in\mathbf{R}} \overset{\mathscr{L}}{=} \{B^H_t\}_{t\in\mathbf{R}},\end{aligned}$$

where $(\widetilde{B}_v)_{v\in\mathbf{R}} = (B_{v+h} - B_h)_{v\in\mathbf{R}}$.

2.2.1 Well-balanced case

As observed by STOEV and TAQQU [172], the introduction of the mBm is more delicate in the well-balanced case $a^+ = a^-$. Indeed, when $H = \frac{1}{2}$, the present definition of the fractional Brownian field $B(t,H)$ implies that $B(\cdot,\frac{1}{2}) = 0$ almost surely. Hence, to obtain a well-defined mBm, one needs to change the normalisation term, and more specifically, add a diverging factor at $H = \frac{1}{2}$: $1/(H-\frac{1}{2})$.

Considering this slightly modified definition of the well-balanced mBm, we observe that it corresponds to a specific compact deterministic expression.

Proposition 2.3 *Suppose $(B_t)_{t\in\mathbf{R}_+}$ is a continuous Brownian motion and $a^+ = a^- = 1$. Then, for all $(t,H) \in \mathbf{R}\times(0,1)$,*

$$B(t,H) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbf{R}} B_u\Big[(t-u)^{<H-3/2>} - (-u)^{<H-3/2>}\Big]\mathrm{d}u \quad a.s. \tag{2.8}$$

where for any $x \in \mathbf{R}$, $x^{<\alpha>} := \mathrm{sign}(x)\,|x|^\alpha$.

Proof. Let $(t,H) \in \mathbf{R}\times(0,1)$, $\delta > 0$ and $T_\pm = t \mp \delta$. Owing to the representation obtained in Proposition 2.1 and using the slightly different re-normalisation,

$$B(t,H) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbf{R}} (B_u - B_t\mathbf{1}_{|t-u|\le\delta})\Big[(t-u)^{<H-3/2>} - (-u)^{<H-3/2>}\Big]\mathrm{d}u.$$

Without any restriction, we may assume that $0 \notin B(t,\delta)$. Then, let us observe the component of the integral corresponding to the interval $[t-\delta, t+\delta]$. It is equal to the limit of

$$\int_{B(t,\delta)\setminus B(t,\varepsilon)} (B_u - B_t)(t-u)^{<H-3/2>}\mathrm{d}u$$

when $\varepsilon \to 0$. For any $\varepsilon > 0$, we note that $\int_{B(t,\delta)\setminus B(t,\varepsilon)} B_t(t-u)^{<H-3/2>}\mathrm{d}u = 0$. Furthermore,

$$\begin{aligned}\int_{B(t,\delta)\setminus B(t,\varepsilon)} B_u(t-u)^{<H-3/2>}\mathrm{d}u &= \int_{[t-\delta,t-\varepsilon]} (B_u - B_{2t-u})(t-u)^{H-3/2}\mathrm{d}u\\ &\longrightarrow_{\varepsilon\to 0} \int_{[t-\delta,t]} (B_u - B_{2t-u})(t-u)^{H-3/2}\mathrm{d}u,\end{aligned}$$

since B is Hölder continuous at t. Hence, the integral converges when $\varepsilon \to 0$, proving Equation (2.8). □

On the contrary to the formulas obtained in Proposition 2.1, we observe that the representation (2.8) is an improper deterministic integral when $H \leq \frac{1}{2}$.

Also note that we may check that when $H = \frac{1}{2}$, the corresponding Gaussian process is, up to a constant, a Brownian motion. STOEV and TAQQU [172] have shown that it has the following stochastic integral representation

$$B\big(t, \tfrac{1}{2}\big) = \frac{1}{\Gamma\big(H + \frac{1}{2}\big)} \int_{\mathbf{R}} \Big[\log|t-u| - \log|u|\Big] \mathrm{d}W_u.$$

As proved in the next proposition, the well-balanced case happens to be a classic mBm defined with respect to the latter Brownian motion.

Proposition 2.4 *Suppose B is a continuous Brownian motion and $B(t,H)$ denotes the well-balanced fractional Brownian field. Define the Brownian motion $\widetilde{B}$ by*

$$\forall t \in \mathbf{R}; \quad \widetilde{B}_t = \frac{1}{2\Gamma\big(H + \frac{1}{2}\big)} \int_{\mathbf{R}} \Big[\log|t-u| - \log|u|\Big] \mathrm{d}B_u.$$

Then, for all $t \in \mathbf{R}$ and $H \in (0,1)$,

$$B(t,H) = \frac{1}{\big(H - \frac{1}{2}\big)\tan\big((H + \frac{1}{2})\frac{\pi}{2}\big)} \big(\widetilde{B}^+(t,H) - \widetilde{B}^-(t,H)\big) \quad \textit{a.s.}$$

where $\widetilde{B}^{\pm}(t,H)$ designates the fractional Brownian field with respect to $\widetilde{B}$. Note that the normalising function is continuous at $H = \frac{1}{2}$.

Proof. There exist several ways to prove this result. One might directly obtain the expression of the fractional Brownian field $\widetilde{B}^{\pm}(t,H)$ using Propositions 2.1 and 2.3, but the calculus is unfortunately a bit tricky.

The proof happens to be simpler if we consider the Fourier representation of $B(t,H)$ obtained by STOEV and TAQQU [172]. For all $t \in \mathbf{R}$ and $H \in (0,1)$,

$$B^{\pm}(t,H) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbf{R}} \frac{\mathrm{e}^{it\xi} - 1}{|\xi|^{H+1/2}} \mathrm{e}^{\mp i\,\mathrm{sign}(\xi)(H+1/2)\pi/2} \mathrm{d}\widehat{W}(\xi),$$

where $\widehat{W}$ is a complex Gaussian measure such that $\widehat{W}$ is a complex Wiener measure on $\mathbf{R}_+$ and $\overline{\widehat{W}(A)} = \widehat{W}(-A)$ for any $A \in \mathscr{B}(R_+)$. The well-balanced fractional Brownian field then has the following representation

$$B(t,H) = \frac{2\cos\big((H + \frac{1}{2})\frac{\pi}{2}\big)}{\sqrt{2\pi}\big(H - \frac{1}{2}\big)} \int_{\mathbf{R}} \frac{\mathrm{e}^{it\xi} - 1}{|\xi|^{H+1/2}} \mathrm{d}\widehat{W}(\xi).$$

Let now introduce a slightly modified complex Gaussian measure $\widetilde{W}$, given by

$$\forall A \in \mathscr{B}(\mathbf{R}_+); \quad \widetilde{W}(A) := i\widehat{W}(A) \quad \text{and} \quad \widetilde{W}(-A) := \overline{\widetilde{W}(A)} = -i\widehat{W}(-A).$$

Then, $\widetilde{B}^{\pm}(t,H)$ denotes the fractional Brownian field defined with respect to $\widetilde{W}$. Since $\widetilde{W}(\mathrm{d}\xi) = i\,\mathrm{sign}(\xi)\widehat{W}(\mathrm{d}\xi)$, we obtain

$$\begin{aligned}\widetilde{B}_+(t,H) - \widetilde{B}_-(t,H) &= \frac{2\sin\big((H + \frac{1}{2})\frac{\pi}{2}\big)}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{\mathrm{e}^{it\xi} - 1}{|\xi|^{H+1/2}} (-i)\,\mathrm{sign}(\xi)\,\mathrm{d}\widetilde{W}(\xi) \\ &= \big(H - \tfrac{1}{2}\big)\tan\big((H + \tfrac{1}{2})\tfrac{\pi}{2}\big)\, B(t,H).\end{aligned}$$

To end the proof, we need to check that the field $\widetilde{B}^{\pm}(t,H)$ corresponds to the process defined in the result, i.e. the integral with respect to the Brownian motion $\widetilde{B}$. For this purpose, let simply consider the previous equality when $H=\frac{1}{2}$. We first note that $\lim_{H\to 1/2}(H-\frac{1}{2})\tan\big((H+\frac{1}{2})\frac{\pi}{2}\big)=1$. Hence,

$$\widetilde{B}_t := \frac{1}{2}B\big(t,\tfrac{1}{2}\big) = \frac{1}{2}\Big(\widetilde{B}_+\big(t,\tfrac{1}{2}\big)-\widetilde{B}_-\big(t,\tfrac{1}{2}\big)\Big).$$

Owing to Equation (2.4), this last equality proves that $\widetilde{B}$ is the Brownian motion appearing in the definition of $B^{\pm}(t,H)$. □

The equality in law between the two processes can be obtained more directly using Theorem 4.1 from [172]. Nevertheless, Proposition 2.4 indicates precisely how these two fractional Brownian fields are related, and proves that the well-balanced one is simply a classic field with a change of Brownian motion. Besides, this statement helps us to simply the study of the sample path properties of the mBm since we do not need any more to consider separately the specific case $a^+=a^-$.

2.3 Hölder and 2-microlocal regularity

Following the convention initiated by [172], we will call *multifractional Brownian motion* a process X which has the following form:

$$\forall t\in\mathbf{R};\quad X_t = a^+B^+\big(t,H(t)\big)+a^-B^-\big(t,H(t)\big),$$

where $(a^-,a^+)\in\mathbf{R}^2\setminus\{(0,0)\}$. As previously said, we may also suppose without any loss of generality that $a^+\neq a^-$.

The Hurst function $t\mapsto H(t)\in(0,1)$ will be assume to be at least a deterministic càdlàg function. Note nevertheless that several of the following results remain valid if $H(\cdot)$ is replaced by a stochastic process with càdlàg sample paths. This last model has already been considered by PAPANICOLAOU and SØLNA [153] and AYACHE and TAQQU [14], where the latter has obtained some general sample path properties. For any càdlàg function $H(\cdot)$, the set $\{t\in\mathbf{R}:\Delta H(t)\neq 0\}$ is countable, and therefore we can easily see from the definition of the mBm that with probability one,

$$\forall t\in\mathbf{R};\quad \Delta H(t)\neq 0 \quad\Longleftrightarrow\quad \Delta X_t\neq 0.$$

Furthermore, ΔX_t is a centered Gaussian variable whose variance is proportional to $\Delta H(t)^2$.

Since we will use the 2-microlocal formalism to study sample path properties of the mBm, let us recall a few elements concerning this framework. Initially introduced by BONY [35] to study the singularities of the solutions of PDEs, HERBIN and LÉVY VÉHEL [78] have recently initiated its use in a stochastic frame, in order to characterise the fine regularity of processes. This formalism is based on the introduction of functional spaces, called *2-microlocal spaces*.

Definition 2.1 *Let $t\in\mathbf{R}$, $s'\leq 0$ and $\sigma\in(0,1)$ such that $\sigma-s'\notin\mathbf{N}$. A function $f:\mathbf{R}\to\mathbf{R}$ belongs to the* 2-microlocal space $C_t^{\sigma,s'}$ *if there exist $C>0$, $\rho>0$ and a polynomial P_t such that*

$$\Big|\big(f(u)-P_t(u)\big)-\big((f(v)-P_t(v)\big)\Big|\leq C|u-v|^{\sigma}\big(|u-t|+|v-t|\big)^{-s'}, \tag{2.9}$$

for all $u,v\in B(t,\rho)$.

This time domain definition is due to KOLWANKAR and LÉVY VÉHEL [109] and SEURET and LÉVY VÉHEL [169]. Note that the original definition is based on the analysis of f in the

Fourier space. To extend Definition 2.9 to any $\sigma \in \mathbf{R} \setminus \mathbf{Z}$, we rely on the following property of 2-microlocal spaces (see e.g. [96]),

$$\forall \alpha > 0; \quad f \in C_t^{\sigma,s'} \quad \Longleftrightarrow \quad I_\pm^\alpha f \in C_t^{\sigma+\alpha,s'}, \tag{2.10}$$

where $I_\pm^\alpha f$ denotes the fractional integral of f of order α: $\big(I_\pm^\alpha f\big)(t) = \int_{\mathbf{R}} (t-u)_\pm^{\alpha-1} f(u)\,\mathrm{d}u$. As a consequence of this property, one only needs to apply Definition 2.1 to iterated integrals or differentials of f to obtain a generalisation of the time-domain characterisation of the 2-microlocal spaces.

Following the introduction of these functional spaces, we can define a related regularity tool, named the *2-microlocal frontier* and given by

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,t}(s') = \sup\{\sigma \in \mathbf{R} : f \in C_t^{\sigma,s'}\}. \tag{2.11}$$

We observe that using the 2-microlocal formalism, the local regularity at t is characterised by a function, and not simply a coefficient. Owing to inclusion properties of the 2-microlocal spaces, the map $s' \mapsto \sigma_{f,t}(s')$ is well-defined and satisfies several properties:

- $\sigma_{f,t}(\cdot)$ is a concave non-decreasing function;
- $\sigma_{f,t}(\cdot)$ has left and right derivatives between 0 and 1.

Furthermore, if the modulus of continuity of f is such that $\omega_f(h) = \mathrm{O}(1/|\log(h)|)$, then the pointwise and local Hölder exponents can be retrieved from the frontier:

$$\alpha_{f,t} = -\inf\{s' : \sigma_{f,t}(s') \geq 0\} \quad \text{and} \quad \widetilde{\alpha}_{f,t} = \sigma_{f,t}(0).$$

For instance, Herbin and Lévy Véhel [78] have determined the 2-microlocal frontier of the fractional Brownian motion B^H, $H \in (0,1)$: with probability one, for all $t \in \mathbf{R}$

$$\forall s' \in \mathbf{R}; \quad \sigma_{B^H,t}(s') = \big(H + s'\big) \wedge H.$$

Before investigating the regularity of the multifractional Brownian motion, we extend the previous equality to the fractional Brownian field and its H-partial derivatives.

Proposition 2.5 *Suppose $B(t,H)$ is a fractional Brownian field. For any $H \in (0,1)$ and any $k \in \mathbf{N}$, let $B^{H,k}$ denotes the process*

$$\forall t \in \mathbf{R}; \quad B_t^{H,k} = \partial_H^k B(t,H).$$

Then, with probability one, for all $H \in (0,1)$, $k \in \mathbf{N}$ and $t \in \mathbf{R}$

$$\forall s' \in \mathbf{R}; \quad \sigma_{B^{H,k},t}(s') = \big(H + s'\big) \wedge H. \tag{2.12}$$

Proof. Since we use the deterministic representation obtained in Proposition 2.1, we can set a fix $\omega \in \Omega$ in the proof. For the sake of readability, let us first consider $H > \frac{1}{2}$ and remove the normalisation constant $1/\Gamma\big(H - \frac{1}{2}\big)$. Then, for any $k \in \mathbf{N}$, the partial derivative with respect to H is equal to

$$\partial_H^k B^\pm(t,H) = \int_{\mathbf{R}} B_u \Big[(t-u)_\pm^{H-3/2} \log^k(t-u)_\pm - (-u)_\pm^{H-3/2} \log^k(-u)_\pm\Big]\,\mathrm{d}u.$$

Suppose (T_+, T_-) is a non-empty open interval. Since the 2-microlocal frontier is a purely local property of the sample paths, we may restrict our study to $t \in (T_+, T_-)$ with any loss of generality. Then, we note that the component

$$t \mapsto \int_{(\mp\infty, T_\pm]} B_u \Big[(t-u)_\pm^{H-3/2} \log^k(t-u)_\pm - (-u)_\pm^{H-3/2} \log^k(-u)_\pm\Big]\,\mathrm{d}u$$

is a smooth function, and therefore does not have any influence on the 2-microlocal regularity. Hence, we focus on the study of the remaining integral

$$t \in (T_+, T_-) \mapsto \int_{[T_+,T_-]} B_u (t-u)_\pm^{H-3/2} \log^k (t-u)_\pm \mathrm{d}u,$$

when $H > \frac{1}{2}$. Similarly, the case $H < \frac{1}{2}$ is reduced to the analysis of

$$t \in (T_+, T_-) \mapsto \frac{\mathrm{d}}{\mathrm{d}s} \left(\int_{[T_\pm,s]} B_u (s-u)_\pm^{H-\frac{1}{2}} \log^k (t-u)_\pm \mathrm{d}u \right)(t).$$

If we define the process $Y_t = B_t \mathbf{1}_{t\in[T_+,T_-]}$, we observe that the two previous formulas correspond respectively to fractional-like integral and derivative of Y on the real axis. Compared to the usual definition, there is only a slight modification in the kernel with a logarithmic term.

Furthermore, when $H < \frac{1}{2}$, the fractional-like derivative is defined as the classic derivative of a fractional-like integral of order $H + \frac{1}{2}$. Since we know the behaviour of the 2-microlocal regularity under derivation, we may restrict our study to the fine analysis of the following process

$$t \in (T_+, T_-) \longmapsto \left(I_\pm^{\alpha,k} Y\right)(t) := \int_{\mathbf{R}} Y_u (t-u)_\pm^{\alpha-1} \log^k (t-u)_\pm \mathrm{d}u,$$

for any $\alpha \in (0,1)$ and $k \in \mathbf{N}$.

We expect the 2-microlocal frontier of this integral to behave similarly to the one of a classic fractional integral of Y, since the only difference lies in a logarithmic term. This easiest way to prove this statement is to compute the Fourier transform of the corresponding pseudo-differential operator and show it has an effect equivalent to the fractional integration in the Fourier space.

The computation of Fourier transform of the tempered distribution $u_\pm^{\alpha-1} \log^k u_\pm$ is similar to the classic calculus on $u_\pm^{\alpha-1}$. We first consider the Laplace transform, with $\Re(z) > 0$. Since this transform is analytic on the previous domain, we can restrict ourselves to the case that $z > 0$. Then,

$$\begin{aligned} \forall z > 0; \quad \int_{\mathbf{R}_+} u^{\alpha-1} \log^k(u) \, \mathrm{e}^{-zu} \, \mathrm{d}u &= z^{-\alpha} \int_{\mathbf{R}_+} v^{\alpha-1} \left(\log v - \log z\right)^k \mathrm{e}^{-v} \mathrm{d}v \\ &= z^{-\alpha} \sum_{j=0}^{k} (-1)^j \log^j(z) \, C_k^j \underbrace{\int_{\mathbf{R}_+} v^{\alpha-1} \log^{k-j}(v) \, \mathrm{e}^{-v}}_{F(\alpha,k-j)}. \end{aligned}$$

As the last term is also analytic, it implies the equality for any $z \in \mathbf{C}$ such that $\Re(z) > 0$.

To obtain the Fourier transform of $u_\pm^{\alpha-1} \log^k u_\pm$ as a tempered distribution, we simply have to consider $z = i\xi + \varepsilon$ and let $\varepsilon \to 0$. The scalar product with a test function converges, and proves that the generalized Fourier transform is equal to

$$\begin{aligned} &\mathscr{F}_g \left(u_+^{\alpha-1} \log^k u_+\right)(\xi) \\ &= e^{-i \operatorname{sign}(\xi) \frac{\pi}{2} \alpha} |\xi|^{-\alpha} \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{k} (-1)^j \left(\log|\xi| + i \operatorname{sign}(\xi) \tfrac{\pi}{2}\right)^j \cdot C_k^j F(\alpha, k-j). \end{aligned}$$

Therefore, the Fourier transform corresponds to a weighted sum of elements $|\xi|^{-\alpha} \log^n |\xi|$, with a phase correction.

Based on the Fourier transform on this pseudo-differential operator, we use the arguments presented by JAFFARD [93] to prove the stability of 2-microlocal spaces. The aforementioned work is based on the original definition of 2-microlocal spaces which makes use of the Littlewood–Paley decomposition of tempered distributions. The latter basically corresponds to the application of band-pass filters, which are then composed with a fractional operator in the Fourier space. Nevertheless, due to the logarithm term which appears in $I_{\pm}^{\alpha,k}$, we obtain a slightly weaker property, i.e.

$$\forall s', \sigma \in \mathbf{R}; \quad Y \in C_t^{\sigma,s'} \Longrightarrow I_{\pm}^{\alpha,k} Y \in C_t^{\sigma-\varepsilon,s'}$$

for any $\varepsilon > 0$. Furthermore and conversely,

$$\forall s', \sigma \in \mathbf{R}; \quad I_{\pm}^{\alpha,k} Y \in C_t^{\sigma,s'} \Longrightarrow Y \in C_t^{\sigma-\varepsilon,s'}.$$

The same property holds for any weighted sum of $I_{-}^{\alpha,k} Y$ and $I_{+}^{\alpha,k} Y$, and therefore, the 2-microlocal frontier of $I_{\pm}^{\alpha,k} Y$ satisfies for any $H \in (0,1)$ and $k \in \mathbf{N}$

$$\forall s' \in \mathbf{R}; \quad \sigma_{I_{\pm}^{\alpha,k} Y,t}(s') = \sigma_{Y,t}(s') + H - \frac{1}{2} = (s' + H) \wedge H.$$

It concludes the proof, since we have shown previously that the 2-microlocal frontier of $B^{H,k}$ is equal to the regularity $I_{\pm}^{\alpha,k} Y$. □

As outlined in the introduction, the Hölder regularity of a general multifractional Brownian motion has already been investigated by HERBIN [76] and AYACHE [8]. In particular, the latter has proved that with probability one, the local regularity is given by

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \widetilde{\alpha}_{X,t} = H(t) \wedge \widetilde{\alpha}_{H,t}.$$

Note that the behaviour at $t = 0$ is slightly different as the form of the variance of increments differs at this point.

In the following, we obtain a first estimate for the general 2-microlocal frontier. The latter happens to depend on the multiplicity of the component $h \mapsto \big(B(t,h) - B(t,H)\big)$ of the fractional field at $(t,H) \in \mathbf{R} \times (0,1)$, i.e.

$$m_{t,H} = \inf\{k \in \mathbf{N} \setminus \{0\} : \partial_H^k B(t,H) \neq 0\}.$$

Proposition 2.6 *Suppose X is a multifractional Brownian motion with Hurst function H. Then, with probability one, the 2-microlocal frontier of X at any $t \in \mathbf{R}$ satisfies*

$$\sigma_{X,t}(s') \geq \big(s' + H(t)\big) \wedge H(t) \wedge \sigma_{H,t}(s' + H(t)) \wedge m_{t,H(t)} \sigma_{H,t}(s'/m_{t,H(t)}).$$

for all $s' \geq -\alpha_{X,t}$. Furthermore, when $m_{t,H(t)} = 1$, the frontier is equal to

$$\forall s' \geq -\alpha_{X,t}; \quad \sigma_{X,t}(s') = \big(s' + H(t)\big) \wedge H(t) \wedge \sigma_{H,t}(s').$$

Proof. Without any loss of generality, we can assume that $\widetilde{\alpha}_{H,t} > 0$, since otherwise the lower bound is equal to 0. Then, let $\varepsilon > 0$, $\rho > 0$, $t \in \mathbf{R}$ and u, v be in the neighbourhood $B(t,\rho)$. We first estimate the size of the increments $X_u - X_v$:

$$B(u,H_u) - B(v,H_v) = \big(B(v,H_u) - B(u,H_u)\big) + \big(B(v,H_v) - B(v,H_u)\big).$$

Using a simple integration by parts, we observe that the first term is equal to

$$B(v,H_u) - B(u,H_u) = B(v,H_t) - B(u,H_t) + \int_{H_t}^{H_u} \big(\partial_H B(v,h) - \partial_H B(u,h)\big) \mathrm{d}h.$$

Owing to Lemma 2.5, for any $\varepsilon > 0$, there exists $c_1 > 0$ such that

$$\forall u, v \in B(t,\rho),\ \forall h \in [H_t, H_u]; \quad \left|\partial_H B(v,h) - \partial_H B(u,h)\right| \leq c_1 |u-v|^{h-\varepsilon}.$$

Therefore, the integral term satisfies

$$\left|\int_{H_t}^{H_u} \left(\partial_H B(v,h) - \partial_H B(u,h)\right) \mathrm{d}h\right| \leq c_1 |H_t - H_u| \cdot |v-u|^{H_t \wedge H_u - \varepsilon},$$

and is thus negligible in front of the first one. Still using Lemma 2.5, we know that for all $u, v \in B(t,\rho)$

$$|B(v,H_t) - B(u,H_t)| \leq c_2 |u-v|^{\sigma} \left(|u-t| + |v-t|\right)^{-s'},$$

where $s' \geq -H_t$ and $\sigma < (H_t + s') \wedge H_t$. This inequality corresponds to the first term in the lower bound.

Using Taylor's formula, we can obtain an estimate of the second component $(B(v,H_v) - B(v,H_u))$.

$$B(v,H_v) - B(v,H_u) = \sum_{k=1}^{m-1} \frac{\partial_H^k B(v,H_u)}{k!} (H_v - H_u)^k + \int_{H_u}^{H_v} \frac{\partial_H^m B(v,h)}{(m-1)!} (H_v - h)^{m-1} \mathrm{d}h,$$

where m denotes the coefficient m_{t,H_t} previously introduced. There exists a neighbourhood $B(t,\rho)$ of t such that for every $k \in \{1,\dots,m-1\}$ and for all $v \in B(t,\rho)$, $|\partial_H^k B(v,H_u)| \leq c_3 |v-t|^{H_t - \varepsilon}$. Hence the term $k=1$ is dominant in the sum and for all $u, v \in B(t,\rho)$

$$|\partial_H B(v,H_u)| \cdot |H_v - H_u| \leq c_4 |u-v|^{\sigma} \left(|u-t| + |v-t|\right)^{-s+H_t-\varepsilon},$$

for any $s' \geq -\alpha_{H,t}$ and $\sigma < \sigma_{H,t}(s')$. This inequality leads to the second term in the lower bound.

Finally, let consider the last integral. $\partial_H^m B(v,h)$ can be simply bound by a constant, implying that the integral is upper bounded by $|H_u - H_v|^m$. The third lower bound is a direct consequence of this estimate.

Let now study the case $m_{t,H(t)} = 1$. The lower bound is clearly equal to $(H_t + s') \wedge H_t \wedge \sigma_{H,t}(s')$. Let set $s' \geq -\alpha_{X,t}$. Suppose first that $(H_t + s') \wedge H_t < \sigma_{H,t}(s')$. Then, the term $B(v,H_u) - B(u,H_u)$ will dominate in the increment $X_u - X_v$, therefore implying $\sigma_{X,t}(s') \leq (H_t + s') \wedge H_t$. Otherwise, we observe that

$$B(v,H_v) - B(v,H_u) = \partial_H B(v,H_v)(H_u - H_v) + \int_{H_v}^{H_u} \partial_H^2 B(v,h)(H_v - h) \mathrm{d}h,$$

There exists $\rho > 0$ and $c_5 > 0$ such that for all $v \in B(t,\rho)$, $|\partial_H B(v,H_v)| > c_5$, implying that the regularity of this component is $\sigma_{H,t}(s')$ Furthermore, as previously noted, the second term is upper-bounded by $(H_u - H_v)^2$, and thus is negligible in front of the first one. Therefore, the increment $X_u - X_v$ behave similarly to $H_u - H_v$, proving the expected equality. □

We observe in the previous proof that the study of the general form of the 2-microlocal frontier is quite technical, in particular because of the interaction between the oscillations of the Hurst function H and the derivatives of the fractional Brownian field $B(t,H)$. Nevertheless, a more precise estimate of the pointwise regularity can be obtained with probability one.

Proposition 2.7 *Suppose X is a multifractional Brownian motion with Hurst function H. Then, with probability one, the pointwise exponent is equal to*

$$\forall t \in \mathbf{R}; \quad \alpha_{X,t} = H(t) \wedge m_{t,H(t)} \alpha_{H,t}.$$

Proof. Let us proceed similarly to the proof of Proposition 2.6 and study the increment $X_u - X_t = \big(B(u,H_u) - B(u,H_t)\big) + \big(B(u,H_t) - B(t,H_t)\big)$. Owing to Lemma 2.5, the pointwise regularity of the second term is exactly $H(t)$.

Then, applying a slightly different Taylor development on the first increment,

$$B(u,H_u) - B(u,H_t) = \sum_{k=1}^{m} \frac{\partial_H^k B(u,H_t)}{k!}(H_u - H_t)^k + \int_{H_t}^{H_u} \frac{\partial_H^{m+1} B(u,h)}{m!}(H_u - h)^m \mathrm{d}h.$$

For every $k \in \{1,\dots,m-1\}$, there exists a simple upper bound : $|\partial_H^k B(u,H_t)||H_u - H_t|^k \leq c_1|u-t|^{H_t + k\alpha_{H,t} - \varepsilon}$. Hence, the pointwise regularity of this term is strictly greater than $H(t)$, and thus negligible. In the case $k = m$, we note that there exists $\rho > 0$ and $c_2 > 0$ such that for all $u \in B(t,\rho)$, $|\partial_H^k B(u,H_t)| \geq c_2$. Hence, the pointwise exponent of this component is exactly $m \cdot \alpha_{X,t}$. Finally, similarly to the previous proof, the last term can be simply upper bounded by $c_3|H_u - H_v|^{m+1} \leq c_3|u-t|^{(m+1)\alpha_{H,t}}$. The last estimate is sufficient to conclude the proof. □

The previous result constitutes an extension of the estimate obtained by Ayache [8]. As pointed out in the latter, it shows that the multifractional Brownian motion can have a non-deterministic pointwise regularity. More precisely, Ayache [8] has proved that for any $\varepsilon > 0$, with positive probability,

$$\dim_{\mathrm{H}}\{t \in [a,b] : m_{t,H(t)} \geq 1\} \geq 1 - \inf_{u \in [a,b]} H(u) - \varepsilon.$$

In the view of the result obtained in Proposition 2.7, it would be interesting to study the Hausdorff dimension of the following sets

$$\forall k \in \mathbf{N}; \quad E_k = \{t \in [a,b] : m_{t,H(t)} = k\}.$$

This question seems to be much more difficult than the previous estimate, since it involves the successive derivatives $\partial_H^k B(t,H)$ of the fractional field and thus the correlation existing between them. Finally, we also note that in terms of regularity, the behaviour at $t = 0$ is particular, since the multiplicity $m_{0,H(0)}$ is equal to infinity.

To end this section, we describe som examples of Hurst function where the 2-microlocal regularity can be entirely determined.

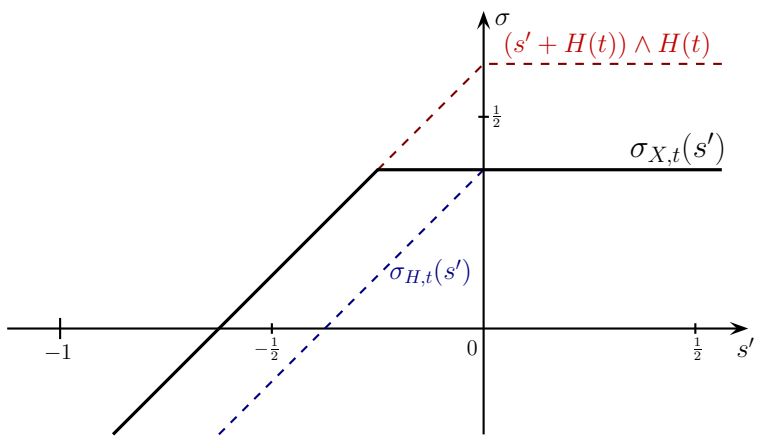

Figure 2.1: 2-microlocal frontier of X at t in the case $m_{t,H(t)} > 1$ (ex. 1).

Example 2.1 Suppose $H : \mathbf{R} \to (\frac{1}{2}, \frac{3}{4})$ is an Hurst function such that for all $t \in \mathbf{R}$, the 2-microlocal local frontier of H at t is equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{H,t}(s') = \left(s' + \frac{3}{8}\right) \wedge \frac{3}{8}.$$

Note that this type of function can be easily constructed from sample paths of a fractional Brownian motion with parameter $H = \frac{3}{8}$.

Then, the 2-microlocal frontier of the corresponding multifractional Brownian motion satisfies with probability one:

- if $m_{t,H_t} = 1$,
$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \sigma_{H,t}(s') = \left(s' + \frac{3}{8}\right) \wedge \frac{3}{8};$$
- if $m_{t,H_t} > 1$,
$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \big(s' + H(t)\big) \wedge \frac{3}{8}.$$

This last case is illustrated in Figure 2.1. To obtain the complete determination, we use Proposition 2.6, noting that it implies the inequality

$$\begin{aligned}\sigma_{X,t}(s') &\geq \big(s' + H(t)\big) \wedge H(t)\left(s' + \frac{3}{8} + H(t)\right) \wedge \frac{3}{8} \wedge \left(s' + \frac{3m_{t,H(t)}}{8}\right) \wedge \frac{3m_{t,H(t)}}{8} \\ &= \big(s' + H(t)\big) \wedge \frac{3}{8},\end{aligned}$$

since $H(t) \in (\frac{1}{2}, \frac{3}{4})$. Then, the upper bound is a consequence of the pointwise exponent obtained in Proposition 2.7 and the following property of the 2-microlocal frontier:

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') \leq \liminf_{s \to t} \widetilde{\alpha}_{X,t},$$

which is proved in the Appendix (Lemma 2.2).

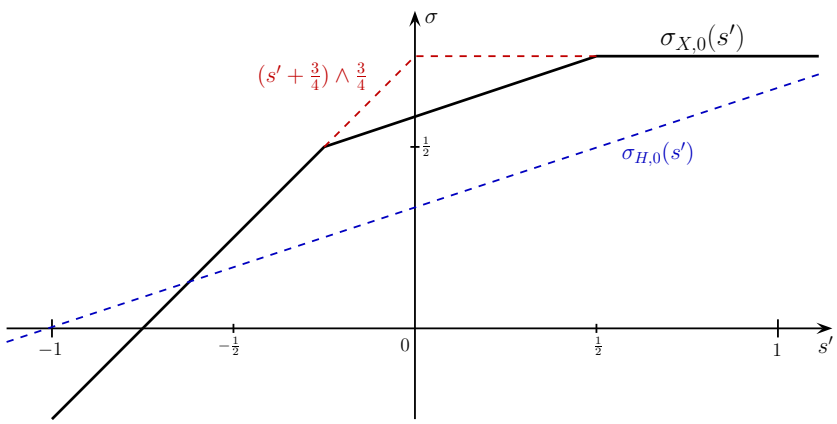

Figure 2.2: 2-microlocal frontier of X at 0 (ex. 2).

In this second example, we illustrate the particular behaviour which may appear at $t = 0$.

Example 2.2 Consider the Hurst function $H : t \mapsto \frac{3}{4} + t\sin\left(\frac{1}{t^2}\right)$, which locally corresponds to the "chirp" function. Then, the 2-microlocal frontier of the multifractional Brownian motion at 0 is equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,0}(s') = \left(s' + \frac{3}{4}\right) \wedge \left(\frac{s'}{3} + \frac{7}{12}\right) \wedge \frac{3}{4}.$$

Figure 2.2 gives an illustration of this particular and hybrid 2-microlocal frontier.

To obtain this estimate, we need to refine the proof of Proposition 2.6. Let first recall that the 2-microlocal frontier of H at 0 is equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{H,0}(s') = \frac{1}{3}(s' + 1).$$

Then, if we consider the regularity of the term $\partial_H B(v, H_0)(H_v - H_v)$, we know there exists for any $\varepsilon > 0$ a sequence $u_n \to 0$ such that $|\partial_H B(u_n, H_0)| \geq u_n^{H_0 - \varepsilon}$. Then, owing to form of the Hurst function, there exists $v_n \to 0$ such that $|H(u_n) - H(v_n)| \geq |u_n - v_n|^{\sigma}(u_n + v_n)^{-s'}$ where $\sigma < \sigma_{H,0}(s')$. Combining the two, we obtain that the 2-microlocal frontier of the component $\partial_H B(v, H_0)(H_v - H_v)$ is

$$\forall s' \geq -\frac{7}{12}; \quad \sigma(s') = \sigma_{H,0}(s' + H(0)) = \left(\frac{s'}{3} + \frac{7}{12}\right).$$

Furthermore, we know that $m_{0,H(0)} = \infty$, inducing that the other terms in Taylor development will be negligible.

We note that the 2-microlocal frontier obtained in this example is consistent with the estimate of the Hölder exponents proved in Proposition 2.7 and in [8].

2.4 Dimensions of graph and images

Following the seminal works of HERBIN [76] and AYACHE [8], we have been able to determine entirely the form of the pointwise regularity of the mBm, and in addition, display the influence of the fine properties of H on the geometry of the sample paths. In this section, we aim to continue this investigation on fractal dimensions of the graph and the images of fractal sets, in particular to show that some unusual and unexpected behaviours can occur as well on these quantities.

We first recall a few notations on fractal dimensions. For any set $A \subset \mathbf{R}^d$, we respectively denote by $\dim_{\mathrm{H}} A$ and $\underline{\dim}_{\mathrm{B}} A$ (or $\overline{\dim}_{\mathrm{B}} A$) the Hausdorff and the lower (or upper) dimensions of A. In the case $\underline{\dim}_{\mathrm{B}} A = \overline{\dim}_{\mathrm{B}} A$, we denote by $\dim_{\mathrm{B}} A$ the common quantity. The reader might refer to the book of FALCONER [67] for the precise definition and the basic properties of these fractal dimensions.

As we investigate the fractal geometry of the multifractional Brownian motion, which is a purely non-stationary process, it is relevant to consider the localised fractal dimension. For all $A \subset \mathbf{R}^d$ and $x \in \mathbf{R}^d$, it is defined by

$$\dim_{*,x} A = \lim_{\rho \to 0} \dim_* \big(A \cap B(x, \rho)\big),$$

where $\dim_*$ denotes any fractal dimension. Note that the previous limit always exists as the right term is decreasing when $\rho \to 0$. We also observe that the map $x \mapsto \dim_{*,x} A$ is upper semi-continuous. Finally, the fractal dimension can be retrieve from the localised values by the formula: $\dim_* A = \sup_{x \in \mathbf{R}^d} \dim_{*,x} A$.

In this first part of this section, we investigate the local fractal dimension of the graph of the mBm. For any function f, its graph on the set V will be denoted $\mathrm{Gr}(f, V)$, i.e.

$$\mathrm{Gr}(f, V) = \big\{(t, f(t)) : t \in V\big\}.$$

In the case $V = \mathbf{R}$, we will simply use the notation $\mathrm{Gr}(f)$. To simplify, we will write $\dim_{*,t} \mathrm{Gr}(f)$ for the local fractal dimension at $(t, f(t))$. The latter is therefore equivalently defined by

$$\dim_{*,t} \mathrm{Gr}(f, V) := \lim_{\rho \to 0} \dim_* \mathrm{Gr}(f, V \cap B(t, \rho)). \tag{2.13}$$

Several deterministic properties satisfied by the local graph dimension are presented in the Appendix 2.5, including some connections with the 2-microlocal frontier.

The graph dimension of an $\mathscr{H}_0$-multifractional Brownian motion has already been determined by PELTIER and LÉVY VÉHEL [154], proving that with probability one

$$\forall t \in \mathbf{R}; \quad \dim_{\mathrm{H},t} \mathrm{Gr}(f) = \dim_{\mathrm{B},t} \mathrm{Gr}(f) = 2 - H(t). \tag{2.14}$$

In the next result, we investigate the extension of this formula to the Box dimension of the graph of a general multifractional Brownian motion.

Theorem 2.8 *Suppose X is a multifractional Brownian motion. Then, with probability one, it satisfies*

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \underline{\dim}_{\mathrm{B},t} \mathrm{Gr}(X) = \big(2 - H(t)\big) \vee \underline{\dim}_{\mathrm{B},t} \mathrm{Gr}(H)$$

and

$$\forall t \in \mathbf{R} \setminus \{0\}; \quad \overline{\dim}_{\mathrm{B},t} \mathrm{Gr}(X) = \big(2 - H(t)\big) \vee \overline{\dim}_{\mathrm{B},t} \mathrm{Gr}(H).$$

As a consequence, the graph has a local Box dimension at t if and only if H does.

Proof. We first obtain the upper bound for both the lower and upper Box dimensions. Let $t \in \mathbf{R}$ and $\rho > 0$. Owing to Lemma 2.5, with probability one, there exist $c_1(\omega)$ and $\varepsilon > 0$ such that for all $u, v \in B(t, \rho)$,

$$|X_u - X_v| \le c_1 |u - v|^{H(t) - \varepsilon} + c_1 |H(u) - H(v)|.$$

Let $N_\delta(H)$ denotes the smallest number of balls of diameter δ necessary to cover the graph of H on the interval $[t - \rho, t + \rho]$. Up to a constant, we can assume that this cover is organised on columns of size δ, as usually presented in the case of a graph covering. Owing to the upper bound of the increment $X_u - X_v$, we know that by adding $\lceil \delta^{H(t)-\varepsilon} \delta^{-1} \rceil$ balls to every column covering, we obtain a covering of the graph of the process X. Hence, there exists a constant c_2 such that

$$\forall \delta > 0; \quad N_\delta(X) \le c_2 \big(\rho \delta^{H(t)-\varepsilon-2} + N_\delta(H)\big)$$

and therefore

$$\begin{aligned} \frac{\log(N_\delta(X))}{-\log(\delta)} &\le \frac{\log(c_2 \rho \delta^{H(t)-\varepsilon-2} + c_2 N_\delta(H))}{-\log(\delta)} \\ &\le \max\left(\frac{\log(c_3 \delta^{H(t)-\varepsilon-2})}{-\log(\delta)}, \frac{\log(c_3 N_\delta(H))}{-\log(\delta)} \right). \end{aligned}$$

Considering the inferior and superior limits of this inequality, we respectively obtain the upper bound of the lower and upper Box dimensions.

To prove the lower bound $2 - H(t)$, we use classic potential methods that have already been presented by PELTIER and LÉVY VÉHEL [154] and HERBIN, ARRAS, and BARRUEL [77] for the multifractional Brownian motion. Briefly, for any interval $[a, b] \subset \mathbf{R}_+$, there exists $C > 0$ such that $\mathbb{E}[X_u - X_v]^2 \ge C|u - v|^{2\underline{H}}$, where $\underline{H} = \min_{u \in [a,b]} H(u)$. This estimate is sufficient, using a potential argument to prove that almost surely,

$$2 - \min_{u \in [a,b]} H(u) \le \dim_{\mathrm{H}} \mathrm{Gr}(X, [a, b]) \le \underline{\dim}_{\mathrm{B}} \mathrm{Gr}(X, [a, b]) \le \overline{\dim}_{\mathrm{B}} \mathrm{Gr}(f, [a, b]).$$

Considering $a, b \in \mathbf{Q}$, we obtain the inequality with probability one for any rational interval. Then, we note that Equation (2.13) holds as well if we use in the limit rational intervals containing t. Hence, owing to the continuity of H, with probability one and for all $t \in \mathbf{R}$, we have $2 - H(t) \leq \underline{\dim}_{\mathrm{B},t} \operatorname{Gr}(X)$ and $2 - H(t) \leq \overline{\dim}_{\mathrm{B},t} \operatorname{Gr}(X)$.

To determine the remaining lower bound, let us first consider the case $m_{t,H(t)} = 1$. ρ can be chosen small enough such that for all (v,h) in neighbourhood of $(t, H(t))$, $|\partial_H B(v,h)| > c_4$, where c_4 is a positive constant. Then, for all $u, v \in B(t,\rho)$

$$X_u - X_v = B(u, H(u)) - B(v, H(v)) + \int_{H(v)}^{H(u)} \partial_H B(v,h)\,\mathrm{d}h.$$

Hence, in the neighbourhood of t,

$$c_4|H(u) - H(v)| \leq \left| \int_{H(v)}^{H(u)} \partial_H B(v,h)\,\mathrm{d}h \right| \leq |X_u - X_v| + c_5|u - v|^{H(t)-\varepsilon}.$$

Using the covering argument described at the beginning of the proof, we obtain that the inequality $\underline{\dim}_{\mathrm{B},t} \operatorname{Gr}(H) \leq \big(2 - H(t)\big) \vee \underline{\dim}_{\mathrm{B},t} \operatorname{Gr}(X)$ and $\overline{\dim}_{\mathrm{B},t} \operatorname{Gr}(H) \leq \big(2 - H(t)\big) \vee \overline{\dim}_{\mathrm{B},t} \operatorname{Gr}(X)$. Since we already know that the lower and upper Box dimensions are greater than $2 - H(t)$, it induces the expected lower bound.

Finally, let us extend the inequality to any $t \in \mathbf{R} \setminus \{0\}$. We have already noted that the function $t \mapsto \dim_{*,t} \operatorname{Gr}(H)$ is upper semi-continuous for every dimension considered. As a consequence, there exists a deterministic countable set $E \subset \mathbf{R} \setminus \{0\}$ such that for all t, there is a sequence $(t_n)_{n\in\mathbf{N}} \in E$ satisfying

$$\dim_{*,t} \operatorname{Gr}(H) = \lim_{n\to\infty} \dim_{*,t_n} \operatorname{Gr}(H),$$

Since for every $s \in E$, $\partial_H B(s, H(s))$ is a centred Gaussian variable with a positive variance, with probability one, for all $s \in E$, $\partial_H B(s, H(s)) \neq 0$. Then, since the local dimension is upper semi-continuous, we obtain

$$\begin{aligned}\underline{\dim}_{\mathrm{B},t} \operatorname{Gr}(X) &\geq \limsup_{n\in\mathbf{N}} \underline{\dim}_{\mathrm{B},t_n} \operatorname{Gr}(X) \\ &= \limsup_{n\in\mathbf{N}} \big(2 - H(t_n)\big) \vee \underline{\dim}_{\mathrm{B},t_n} \operatorname{Gr}(H) = \big(2 - H(t)\big) \vee \underline{\dim}_{\mathrm{B},t} \operatorname{Gr}(H).\end{aligned}$$

The same argument holds as well for the upper Box dimension. Finally, as the upper bound on the local dimension obtained earlier in the proof is uniform on the sample path, we obtain the expected equality. □

Interestingly, we have obtained an expression of the Box dimension of the graph which is similar to the behaviour observed on a fractional Brownian motion with variable drift (see [42, 155]). This common property happens to extend as well to the Hausdorff dimension of the graph, recently obtained by PERES and SOUSI [156] on the fBm with variable drift.

The Hausdorff dimension of the graph happens to be more complicate to analyse. For this purpose, we have to introduce a more general dimension called the *parabolic Hausdorff dimension*. We first define for all $t \in \mathbf{R}$ a *parabolic metric* ϱ_H on $\mathbf{R}^2$, with $H > 0$:

$$\varrho_H\big((u,x);(v,y)\big) := \max\big(|u - v|^H, |x - y|\big).$$

For any set $A \subset \mathbf{R}^2$, we denote by $\dim_{\mathrm{H}}(A; \varrho_H)$ the *parabolic Hausdorff dimension* of A. It is defined similarly to the classic Hausdorff dimension using covering balls relatively to the metric ϱ_H, i.e. it corresponds to the infimum of $s \geq 0$ for which

$$\lim_{\delta\to 0} \inf\left\{ \sum_{i=0}^{\infty} \operatorname{diam}(O_i\,; \varrho_H)^s : (O_i)_{i\in N} \text{ is a } \delta\text{-cover of } A \right\} < \infty$$

Note that given the form of the parabolic metric, a ϱ_H-ball $B((t,x),\delta)$ has the following form: $B((t,x),\delta) = [t-\delta^{1/H}, t+\delta^{1/H}] \times [x-\delta, x+\delta]$.

The parabolic Hausdorff dimension has first been considered by TAYLOR and WATSON [181] for the study of polar sets of the heat equation. Recently, it has been proved to be useful by KHOSHNEVISAN and XIAO [104] to the analysis of the image of Brownian motion. Finally, as previously outlined, PERES and SOUSI [156] have shown it is also a natural idea for the study of fractional Brownian motion with variable drift. Note that the convention used in the latter is slightly different from ours.

Theorem 2.9 *Suppose X is a multifractional Brownian motion. Then, with probability one, it satisfies*

$$\forall t \in \mathbf{R}\setminus\{0\}; \quad \dim_{\mathrm{H},t} \mathrm{Gr}(X) = 1 + H(t)\big(\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) - 1\big).$$

Proof. To begin with, we prove that with probability one,

$$\forall t \in \mathbf{R}\setminus\{0\}; \quad \dim_{\mathrm{H},t}\big(\mathrm{Gr}(X)\,;\varrho_{H(t)}\big) = \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big)$$

The sketch of the proof of this equality is similar to Theorem 2.8, as it uses as well a covering argument. Let us set $\delta > 0$, $\rho > 0$ and $\gamma > \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big)$. Suppose $(O_i)_{i\in\mathbf{N}}$ is a δ-cover of $\mathrm{Gr}(H,[t-\rho,t+\rho])$ made of $\varrho_{H(t)}$-balls and such that

$$\sum_{i=0}^{\infty} \delta_i^{\gamma} < \infty \quad \text{where } \delta_i := \mathrm{diam}(O_i\,;\varrho_{H(t)}) \quad \forall i \in \mathbf{N}.$$

Then, we observe that $|X_u - X_v| \le c_1|u-v|^{H(t)-\varepsilon} + c_1|H(u)-H(v)|$ for all $u,v \in B(t,\rho)$. For any $i \in \mathbf{N}$, we need at most $2\delta_i^{1-\varepsilon/H(t)} \cdot \delta_i^{-1}$ $\varrho_{H(t)}$-balls of diameter δ_i to cover the part of the graph related to the term $|u-v|^{H(t)-\varepsilon}$. Hence, from $(O_i)_{i\in\mathbf{N}}$, we construct a δ-cover $(V_k)_{k\in\mathbf{N}}$ of $\mathrm{Gr}(X,[t-\rho,t+\rho])$ which satisfies

$$\sum_{k=0}^{\infty} \mathrm{diam}(V_k\,;\varrho_{H(t)})^s \le c_2 \sum_{i=0}^{\infty} \delta_i^{-\varepsilon/H(t)} \cdot \delta_i^s.$$

The last series is finite when $s \ge \gamma + \varepsilon/H(t)$. Hence, considering the limits $\varepsilon \to 0$, $\rho \to 0$ and $\gamma \to \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big)$, we obtain the first inequality $\dim_{\mathrm{H},t}\big(\mathrm{Gr}(X)\,;\varrho_{H(t)}\big) \le \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big)$.

To prove the other side inequality, we proceed similarly to the proof of Theorem 2.8. Hence, we first assume that $m_{t,H(t)} = 1$. In this case, the estimate of the increments presented in Theorem 2.8 and the previous reasoning imply that $\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) \le \dim_{\mathrm{H},t}\big(\mathrm{Gr}(X)\,;\varrho_{H(t)}\big)$. Then, we use the upper semi-continuity of the function $t \mapsto \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big)$, obtained in Lemma 2.6, to extend the inequality to any $t \in \mathbf{R}\setminus\{0\}$.

The upper bound of $\dim_{\mathrm{H},t}\mathrm{Gr}(X)$ is then a consequence of the Lemma 2.5, with $H_1 = 1$ and $H_2 = H(t)$. With probability one, for all $t \in \mathbf{R}$, we have $\dim_{\mathrm{H}}\mathrm{Gr}(X,[t-\rho,t+\rho]) \le 1 + H(t)\big(\dim_{\mathrm{H}}\big(\mathrm{Gr}(X,[t-\rho,t+\rho])\,;\varrho_{H(t)}\big) - 1\big)$. Therefore, when $\rho \to 0$, we obtain the first upper bound with probability one: for all $t \in \mathbf{R}\setminus\{0\}$, $\dim_{\mathrm{H},t}\mathrm{Gr}(X) \le 1 + H(t)\big(\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_t\big) - 1\big)$.

To prove the lower bound, we make use of potential arguments on the parabolic and euclidean metrics. We first set $t \in \mathbf{R}\setminus\{0\}$. Then, let $\rho > 0$ and $\gamma < \dim_{\mathrm{H}}\big(\mathrm{Gr}(X,B(t,\rho))\,;\varrho_{H(t)}\big)$. Using Frostman's lemma applied on the parabolic metric ϱ_t, there exists a probability measure μ on $B(t,\rho)$ such that

$$\mathscr{E}_\gamma(\mu_H) = \iint_{B(t,\rho)^2} \frac{\mu(\mathrm{d}u)\,\mu(\mathrm{d}v)}{\big(|u-v|^{H(t)} + |H(u)-H(v)|\big)^{\gamma}} < \infty.$$

Suppose μ_X denotes the image of the measure μ by the map $u \mapsto (u, X_u)$. To prove the lower on the dimension of the graph of X, we need to study s-energy of μ_X, i.e.

$$\mathscr{E}_s(\mu_X) = \iint_{\mathbf{R}^4} \frac{\mu_X(\mathrm{d}x)\,\mu_X(\mathrm{d}y)}{\|x-y\|^s} = \iint_{B(t,\rho)^2} \frac{\mu(\mathrm{d}u)\,\mu(\mathrm{d}v)}{\big(|u-v|^2 + |X_u - X_v|^2\big)^{-s/2}},$$

where $s > 1$. For a fixed $\varepsilon > 0$, let us split the previous term into two components, respectively denoted $\mathscr{E}_s^1$ and $\mathscr{E}_s^2$:

$$\iint_{B(t,\rho)^2} \frac{\mu(\mathrm{d}u)\,\mu(\mathrm{d}v)}{\big(|u-v|^2 + |X_u - X_v|^2\big)^{-s/2}} \Big(\mathbf{1}_{\{|H(u)-H(v)| \le |u-v|^{H(t)(1-\varepsilon)}\}} + \mathbf{1}_{\{|H(u)-H(v)| > |u-v|^{H(t)(1-\varepsilon)}\}} \Big).$$

Let us prove $\mathscr{E}_s^1$ is finite. For this purpose, we estimate its expectation

$$\mathbb{E}[\mathscr{E}_s^1] = \iint_{B(t,\rho)^2} \mathbb{E}\Big[\big(|u-v|^2 + |X_u - X_v|^2\big)^{-s/2}\Big] \mathbf{1}_{\{|H(u)-H(v)| \le |u-v|^{H(t)(1-\varepsilon)}\}} \mu(\mathrm{d}u)\,\mu(\mathrm{d}v).$$

Using the notation $\sigma_{u,v}^2 := \mathbb{E}[X_u - X_v]^2$, the inner term is equal to

$$\begin{aligned}
\mathbb{E}\Big[\big(|u-v|^2 + |X_u - X_v|^2\big)^{-s/2}\Big] &= \frac{2}{\sigma_{u,v}\sqrt{2\pi}} \int_{\mathbf{R}_+} \big(x^2 + |u-v|^2\big)^{-s/2} \exp\Big(-\frac{x^2}{2\sigma_{u,v}^2}\Big)\,\mathrm{d}x \\
&= c_1 \int_{\mathbf{R}_+} \big(r\sigma_{u,v}^2 + |u-v|^2\big)^{-s/2} r^{-1/2} \mathrm{e}^{-r/2}\,\mathrm{d}r.
\end{aligned}$$

Splitting the integral into two terms, and using classic approximations (see e.g. [67]), the previous term is upper bounded by

$$\int_0^{|u-v|^2/\sigma_{u,v}^2} |u-v|^{-s} r^{-1/2}\,\mathrm{d}r + \int_{|u-v|^2/\sigma_{u,v}^2}^{\infty} r^{-s/2} \sigma_{u,v}^{-s} r^{-1/2}\,\mathrm{d}r \le c_2 |u-v|^{1-s} \sigma_{u,v}^{-1}.$$

We recall that HERBIN [76] has proved that

$$\sigma_{u,v}^2 = \mathbb{E}[X_u - X_v]^2 \asymp |u-v|^{2H(t)} + |H(u) - H(v)|^2$$

in the neighbourhood of t. Hence, on the domain $\{|H(u) - H(v)| \le |u-v|^{H(t)(1-\varepsilon)}\}$, we observe that

$$\begin{aligned}
|u-v| &= \big(|u-v|^{2H(t)(1-\varepsilon)}\big)^{1/2H(t)(1-\varepsilon)} \\
&\ge c_3 \big(|u-v|^{2H(t)(1-\varepsilon)} + |H(u)-H(v)|^2\big)^{1/2H(t)(1-\varepsilon)} \\
&\ge c_4\, \sigma_{u,v}^{1/H(t)(1-\varepsilon)}.
\end{aligned}$$

Hence, the expectation of $\mathscr{E}_s^1$ is upper bounded by

$$\begin{aligned}
\mathbb{E}[\mathscr{E}_s^1] &\le c_4 \iint_{B(t,\rho)^2} \sigma_{u,v}^{-1+(1-s)/H(t)(1-\varepsilon)}\,\mu(\mathrm{d}u)\,\mu(\mathrm{d}v) \\
&\le c_5 \iint_{B(t,\rho)^2} \big(|u-v|^{H(t)} + |H(u)-H(v)|\big)^{-1+(1-s)/H(t)(1-\varepsilon)}\,\mu(\mathrm{d}u)\,\mu(\mathrm{d}v).
\end{aligned}$$

The term $\mathscr{E}_s^1$ is therefore almost surely finite when $-1 + (1-s)/H(t)(1-\varepsilon) > -\gamma$, i.e. when

$$s < 1 + (\gamma - 1)\big(H(t)(1-\varepsilon)\big) \quad \text{where } \gamma < \dim_{\mathrm{H}}\big(\mathrm{Gr}(X, B(t,\rho)); \varrho_{H(t)}\big).$$

Hence, considering the limits on γ and ε, the upper bound obtained on s corresponds to the expected estimate.

Let now consider the second term $\mathscr{E}_s^2$. We know that $m_{t,H(t)} = 1$ almost surely. Recall that the increment $X_u - X_v$ has the following form

$$X_u - X_v = B(u, H(u)) - B(v, H(u)) + \int_{H(v)}^{H(u)} \partial_H B(v,h)\,\mathrm{d}h.$$

If ρ is sufficiently small, there exists a positive constant $c_1 > 0$ such that for all $s \in B(t,\rho)$ and $h \in [\min_{B(t,\rho)} H(u), \max_{B(t,\rho)} H(u)]$, we have $|\partial_H B(v,h)| \geq c_1$. Furthermore, there also exists $c_2 > 0$ such that

$$\forall u, v \in B(t,\rho); \quad \Big|B(u,H(u)) - B(v,H(u))\Big| \leq c_2 |u-v|^{H(t)(1-\varepsilon/2)}.$$

Using the previous estimates, we observe that

$$\begin{aligned}|X_u - X_v| &\geq \left|\int_{H(v)}^{H(u)} \partial_H B(v,h)\,\mathrm{d}h\right| - \Big|B(u,H(u)) - B(v,H(u))\Big| \\ &\geq c_1 |H(u) - H(v)| \big(1 - c_2 |H(u)-H(v)|^{-1} \cdot |u-v|^{H(t)(1-\varepsilon/2)}\big)\end{aligned}$$

Hence, on the domain $\{|H(u)-H(v)| > |u-v|^{H(t)(1-\varepsilon)}\}$, we obtain

$$|X_u - X_v| \geq c_1 |H(u)-H(v)| \big(1 - c_2 |u-v|^{H(t)\varepsilon/2}\big) \geq c_3 |H(u)-H(v)|,$$

when ρ is sufficiently small. Therefore, the energy term $\mathscr{E}_s^2$ satisfies

$$\begin{aligned}\mathscr{E}_s^2 &\leq c_4 \iint_{B(t,\rho)} |H(u)-H(v)|^{-s} \mathbf{1}_{\{|H(u)-H(v)| > |u-v|^{H(t)(1-\varepsilon)}\}}\, \mu(\mathrm{d}u)\,\mu(\mathrm{d}v) \\ &\leq c_5 \iint_{B(t,\rho)^2} \big(|u-v|^{H(t)} + |H(u)-H(v)|\big)^{-s}\, \mu(\mathrm{d}u)\,\mu(\mathrm{d}v).\end{aligned}$$

The last integral is finite when $s < \gamma$. But, since $\gamma > 1$, we remark that $\gamma(1-H(t)) > 1 - H(t)$ and thus $\gamma > 1 + H(t)(\gamma - 1)$ showing that $\mathscr{E}_s^2$ is always finite when $s < 1 + (\gamma-1)H(t)$. Therefore, we have proved that the s-energy $\mathscr{E}_s(\mu_X)$ is almost surely finite for all $s < 1 + (\gamma - 1)\big(H(t)(1-\varepsilon)\big)$. Considering the limits $\varepsilon \to 0$ and $\gamma \to \dim_{\mathrm{H},t}\big(\mathrm{Gr}(X)\,;\varrho_{H(t)}\big)$, we obtain

$$\dim_{\mathrm{H},t} \mathrm{Gr}(X) \geq 1 + H(t)\big(\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) - 1\big) \quad \text{a.s.}$$

The lower bound has been obtained for a fixed $t \in \mathbf{R} \setminus \{0\}$. To extend the latter uniformly on $\mathbf{R} \setminus \{0\}$, we proceed similarly to Theorem 2.8. Shortly, as proved in Lemma 2.6, the map $t \mapsto \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big)$ is upper semi-continuous. Hence, there exists a countable set E such that for any t, there is sequence $(t_n)_{n\in\mathbf{N}} \in E$ satisfying

$$\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) = \lim_{n\to\infty} \dim_{\mathrm{H},t_n}\big(\mathrm{Gr}(H)\,;\varrho_{H(t_n)}\big).$$

The equality on the dimension of the graph holds almost surely for all $s \in E$. Therefore, with probability one, for all $t \in \setminus\{0\}$,

$$\begin{aligned}\dim_{\mathrm{H},t} \mathrm{Gr}(X) &\geq \limsup_{n\in\mathbf{N}} \dim_{\mathrm{H},t_n} \mathrm{Gr}(X) \\ &= 1 + \limsup_{n\in\mathbf{N}} H(t_n)\big(\dim_{\mathrm{H},t_n}\big(\mathrm{Gr}(H)\,;\varrho_{H(t_n)}\big) - 1\big) \\ &= 1 + H(t)\big(\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) - 1\big).\end{aligned}$$

This last inequality concludes the proof since the upper bound is already uniform on $\mathbf{R}$. □

Note that Theorem 2.9 does not contradict the result obtained by PELTIER and LÉVY VÉHEL [154] on $\mathscr{H}_0$-mbms. Indeed, due to the form of the parabolic metric $\varrho_{H(t)}$, covering the graph of H in the neighbourhood of t is equivalent to covering a straight line, implying that the local dimension is equal to: $\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) = H(t)^{-1}$. Hence, the application of Theorem 2.9 proves that with probability one

$$\forall t \in \mathbf{R}\setminus\{0\};\quad \dim_{\mathrm{H},t}\mathrm{Gr}X = 2 - H(t),$$

where X is an $\mathscr{H}_0$-multifractional Brownian motion.

It is interesting to observe that the forms of the Hausdorff and Box dimensions of the graphs of mBm and fBm with variable drift are the same, even though these two processes have a quite different Gaussian structure. Indeed, the first one is a centered process whose incremental variance satisfies

$$\mathbb{E}[X_u - X_v]^2 \asymp |u-v|^{2H(t)} + |H(u) - H(v)|^2,$$

whereas the latter still has the covariance of fBm, but a mean equal to the drift $f(t)$. As a consequence of this behaviour, the unusual examples constructed by PERES and SOUSI [156] also apply to the mBm.

Example 2.3 As proved in [156], there exists a continuous function $H : \mathbf{R} \mapsto [1/2, 1)$ such that for all $t \in \mathbf{R}$, $\widetilde{\alpha}_{X,t} = \log(2)/\log(3) := \theta$. Furthermore, at every $t \in \mathbf{R}$,

$$\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{1/2}\big) = \log_2\big(5^{2\theta}+1\big) \quad \text{and} \quad \dim_{\mathrm{B},t}\mathrm{Gr}(H) = 2-\theta.$$

Then, considering the mBm X with Hurst function H, we first observe that almost surely for all $t > 0$, $\widetilde{\alpha}_{X,t} = \theta$. Furthermore, as a consequence of the properties of H, with probability one, at every $t > 0$ such that $H(t) = 1/2$, we obtain

$$\dim_{\mathrm{H},t}\mathrm{Gr}(X) = \frac{1+\log_2\big(5^{2\theta}+1\big)}{2} < \dim_{\mathrm{B},t}\mathrm{Gr}(X) = 2-\theta.$$

Furthermore, as outlined by PERES and SOUSI [156], the local Hausdorff dimension also satisfies $\dim_{\mathrm{H},t}\mathrm{Gr}(X) > \max\big\{\dim_{\mathrm{H},t}\mathrm{Gr}(H), 3/2\big\}$.

This example displays the particular behaviours of the mBm in terms of fractal geometry. Firstly, the last remark proves that the parabolic Hausdorff dimension is the relevant concept for this study, as in this case, $\dim_{\mathrm{H},t}\mathrm{Gr}(H)$ happens to insufficient to characterise entirely the Hausdorff dimension of the graph. Furthermore, this example also shows that the Box and Hausdorff dimensions of the graph are not necessarily equal. Finally, we also note that in contrary to fBm, the fractal dimension of the graph can be completely unrelated to the local Hölder exponent.

As shown in the second example, some more classic behaviours also occur.

Example 2.4 Suppose H is a sample path of a fractional Brownian motion with Hurst exponent α. Up to a rescaling, it can be supposed to have values in the interval $(0,1)$. Then, using some classic estimates (see e.g. [67]), we can obtain that

$$\forall t \in \mathbf{R};\quad \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H)\,;\varrho_{H(t)}\big) = \max\left\{\frac{1}{H(t)}, 1+\frac{1-\alpha}{H(t)}\right\}.$$

Then, owing to Theorem 2.9, a mBm X with Hurst function H satisfies with probability one:

$$\forall t \in \mathbf{R}\setminus\{0\};\quad \dim_{\mathrm{H},t}\mathrm{Gr}(X) = \dim_{\mathrm{B},t}\mathrm{Gr}(X) = \max\{2-\alpha, 2-H(t)\} = 2-\alpha_{X,t}.$$

This example clearly displays a more common fractal geometry, where the Hausdorff and Box dimensions are both equal to $2-\alpha_{X,t}$. Nevertheless, the latter still has an irregular behaviour, since its geometric properties are not any more characterised by $H(t)$ when $\alpha < H(t)$.

In the second part of this section, we are interested in studying the fractal dimension of an image $X(F)$, where F can be any fractal set. A large literature has already investigated the question of the geometry of images of Gaussian processes. It is a well-known result (see e.g. [97]) that for any fixed Borel set $F \in (0,\infty)$,

$$\dim_{\mathrm{H}} B^H(F) = \min\left\{1, \frac{\dim_{\mathrm{H}} F}{H}\right\} \quad \text{a.s.} \tag{2.15}$$

MOUNTFORD [144], KHOSHNEVISAN, WU, and XIAO [103], WU and XIAO [189] have generalized the previous result to (N,d)-(fractional) Brownian sheet, and in particular, obtained uniform results on F under some assumption on N and d.

Adapting classic arguments from KAHANE [97], we can easily extend Equation (2.15) to an $\mathscr{H}_0$-mBm, i.e. for any fixed Borel set $F \in (0,\infty)$ and all $t \in (0,\infty)$

$$\dim_{\mathrm{H},X(t)} X(F) = \min\left\{1, \frac{\dim_{\mathrm{H},t} F}{H(t)}\right\} \quad \text{a.s.} \tag{2.16}$$

In the irregular case, the statement is not as simple and straightforward. Furthermore, the *parabolic Hausdorff dimension* again happens to be the relevant tool to describe the geometry of $X(F)$.

Theorem 2.10 *Suppose X is a multifractional Brownian motion. Then, with probability one, it satisfies*

$$\forall t \in \mathbf{R}\setminus\{0\}; \quad \dim_{\mathrm{H},X(t)} X(F) = \min\Big\{1, \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big)\Big\}.$$

Proof. The proof of this result is similar to the work of PERES and SOUSI [156] on the drifted fBm. Let us first remark that

$$\dim_{\mathrm{H},X(t)} X(F) \le \dim_{\mathrm{H},t} \mathrm{Gr}(X,F).$$

This inequality is a simple consequence of the property of the Hausdorff under the application of Lipschitz function (here, the projection $p:(t,X(t)) \mapsto X(t)$). Then, using Lemma 2.1 from [156], we obtain $\dim_{\mathrm{H},t} \mathrm{Gr}(X,F) \le \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big)$, thus proving the upper bound.

To obtain the lower bound, we use a classic potential argument. Owing to Frostman's lemma, for any $\rho > 0$ and any $\gamma < \dim_{\mathrm{H}}\big(\mathrm{Gr}(H,F\cap B(t,\rho))\,;\varrho_{H(t)}\big)$, there exists a probability measure μ on $B(t,\rho)$ such that

$$\iint_{B(t,\rho)^2} \frac{\mu(\mathrm{d}u)\,\mu(\mathrm{d}v)}{\big(|u-v|^{H(t)} + |H(u)-H(v)|\big)^{\gamma}} < \infty.$$

Recall, as proved by HERBIN [76] the covariance of the mBm has the following form

$$\sigma^2_{u,v} := \mathbb{E}[X_u - X_v]^2 \asymp |u-v|^{2H(t)} + |H(u)-H(v)|^2$$

Suppose μ_X denotes the image of the measure μ under the map $u \mapsto X_u$. Then, for any $s \in (0,1)$, the expectation of its s-energy is equal to

$$\begin{aligned}\mathbb{E}\left[\iint_{\mathbf{R}^2} \frac{\mu_X(\mathrm{d}x)\,\mu_X(\mathrm{d}y)}{\|x-y\|^s}\right] &= \iint_{\mathbf{R}^2} \mathbb{E}\big[\,|X_u - X_v|^{-s}\,\big]\,\mu(\mathrm{d}u)\,\mu(\mathrm{d}v) \\ &= c_2 \iint_{\mathbf{R}^2} \sigma^{-2s}_{u,v}\,\mu(\mathrm{d}u)\,\mu(\mathrm{d}v)\end{aligned}$$

Therefore, owing to the form of the covariance, the integral is finite when $s \leq \gamma$. Hence, considering the limit $\rho \to 0$, we obtain $\dim_{\mathrm{H},X(t)} X(F) \geq \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big)$, which concludes the proof. □

The local parabolic Hausdorff dimension $\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big)$ might not seem to be a very intuitive notion at first. To have a better intuition of the result described in Theorem 2.10, we may first note, like for the dimension of the graph, that it does not contradict Equation 2.16 in the case of an $\mathscr{H}_0$-multifractional Brownian motion. Indeed, when $H(t) < \widetilde{\alpha}_{H,t}$, due to the property of the parabolic metric $\varrho_{H(t)}$, it is completely equivalent to cover F or $\mathrm{Gr}(H,F)$ in the neighbourhood of t. Hence, in this case

$$\begin{aligned}\dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big) &= \lim_{\rho\to 0} \dim_{\mathrm{H}}\big(\mathrm{Gr}(H,F\cap B(t,\rho))\,;\varrho_t\big)\\ &= \lim_{\rho\to 0} H(t)^{-1}\dim_{\mathrm{H}}\big(F\cap B(t,\rho)\big) = H(t)^{-1}\dim_{\mathrm{H},t} F.\end{aligned}$$

The next example shows that different behaviours can occur when the mBm is irregular.

Example 2.5 Let H be a trajectory of fractional Brownian motion defined on the interval $[0,1]$, with Hurst index $\alpha \in (0,1)$. Up to a rescaling and a translation, the sample path satisfies $H([0,1]) \subset (0,1)$.

We first assume that $\alpha < H(t)$. Owing to Equation (2.15), there exists a Borel set F such that $\dim_{\mathrm{H},H(t)} H(F) = \alpha^{-1}\dim_{\mathrm{H},t} F$. Furthermore, since the proof of this statement is based on Frostman's lemma, there exists a probability measure on $F \cap B(t,\rho)$ such that $\iint_{\mathbf{R}^2} |H(u)-H(v)|^{-\gamma}\mu(\mathrm{d}u)\mu(\mathrm{d}v) < \infty$ where $\gamma < \dim_{\mathrm{H},H(t)} H(F)$. Therefore, still using Frostman's lemma, we obtain that $\alpha^{-1}\dim_{\mathrm{H},t} F \leq \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big)$. The other side inequality is simply obtained by observing that $\widetilde{\alpha}_{X,t} = \alpha$, inducing the upper bound using classic covering arguments. Hence, we obtain

$$\dim_{\mathrm{H},X(t)} X(F) = \dim_{\mathrm{H},H(t)} H(F) = \frac{1}{\alpha}\dim_{\mathrm{H},t} F \quad \text{a.s.} \tag{2.17}$$

For the second example, we consider the set $F = H^{-1}\{\frac{1}{2}\}$, which might be assumed to be non-empty. Monrad and Pitt [141] have proved that $\dim_{\mathrm{H}} F = 1-\alpha$. In this case, as the Hurst function H is constant on the set F, we easily prove that the dimension of $\mathrm{Gr}(H,F)$ is characterised by the dimension on F. Therefore,

$$\dim_{\mathrm{H},X(t)} X(F) = \dim_{\mathrm{H},t}\big(\mathrm{Gr}(H,F)\,;\varrho_t\big) = \frac{1}{H(t)}\dim_{\mathrm{H},t} F = \frac{1-\alpha}{H(t)} \quad \text{a.s.} \tag{2.18}$$

The two previous examples show that the estimate obtained in Theorem 2.10 can lead to very different results, depending on the properties of the set F chosen. In particular, we observe that the Hausdorff geometry of images displays a richer structure than the local Hölder regularity. Indeed, in the irregular case, we know that the later only depends on $H(\cdot)$ sample path properties, whereas Equations (2.17) and (2.18) show a more complex geometry of the image $X(F)$. In the first case, we observe that the formula depends only on the local exponent of H and the Hausdorff dimension of F, whereas in the second one, Equation (2.18) presents a case where the Hausdorff dimension follows the classic formula (2.15), even though the mBm is irregular at t.

To conclude this section, we discuss the question of obtaining the Hausdorff dimension of the level sets $X^{-1}(\{x\})$. This problem has already been investigated by Boufoussi, Dozzi, and Guerbaz [36] for the $\mathscr{H}_0$-multifractional Brownian motion, proving in particular that for all $t \in \mathbf{R}\setminus\{0\}$,

$$\dim_{\mathrm{H},t} X^{-1}(\{X_t\}) = 1 - H(t) \quad \text{a.s.}$$

It is quite natural to wonder if the previous result can be extended to the irregular mBm. The upper bound can easily extended, since it is a direct consequence of the Hölder regularity of the process at t. Furthermore, we note as well that the proof from [36] of the lower bound stands as well in the irregular case. As a consequence, we easily obtain that for all $t \in \mathbf{R} \setminus \{0\}$,

$$1 - H(t) \leq \dim_{\mathrm{H},t} X^{-1}(\{X_t\}) \leq 1 - \min\big(H(t), \widetilde{\alpha}_{H,t}\big) \quad \text{a.s.}$$

Hence, lower and upper bounds are not equal in the irregular case, and the remaining question is whether the lower one can be improved. The sketch of the proof of this inequality is rather classic and relies on the study of the Hölder continuity of the local time (see e.g. the seminal works of BERMAN [30] and GEMAN and HOROWITZ [72] on the subject). The key point to prove the Hölder regularity is a property called the *local nondeterminism* (LND). In the case of the mBm, it states that for every $n \in \mathbf{N}$, there exists $c_n > 0$ such that for all $(t_1, \dots, t_n, t) \in (\varepsilon, \infty)^{n+1}$

$$\mathrm{Var}(X_t \,|\, X_{t_1}, \dots, X_{t_n}) \geq c_n \min_{1 \leq k \leq n} |t - t_k|^{2H(t)}.$$

To obtain a better lower bound for the local dimension of the level set, we would need to improve the exponent in the right term. Unfortunately, the next lemma proves that it is optimal under a mild assumption on the Hurst function.

Lemma 2.1 *Let $t \in \mathbf{R}$ and suppose there exists a sequence $s_i \to t$ such that $H(s_i) = H(t)$. Then, for any $n \in \mathbf{N}$, any $\varepsilon > 0$ and any $c_n > 0$, there exists $(t_1, \dots, t_n) \in \mathbf{R}^n$ such that*

$$\mathrm{Var}(X_t \,|\, X_{t_1}, \dots, X_{t_n}) \leq c_n \min_{1 \leq k \leq n} |t - t_k|^{2H(t) - \varepsilon}.$$

Proof. For all $(t_1, \dots, t_n) \in \mathbf{R}^n$ and every $k \in \{1, \dots, n\}$, we observe that $\mathrm{Var}(X_t \,|\, X_{t_1}, \dots, X_{t_n}) \leq \mathrm{Var}(X_t \,|\, X_{t_k}) \leq \mathbb{E}\big[(X_t - X_{t_k})^2\big]$. Using elements $t_k \in \{s_i\}_{i \in \mathbf{N}}$ and the estimate of the variance of the increments, we obtain that

$$\mathrm{Var}(X_t \,|\, X_{t_1}, \dots, X_{t_n}) \leq C \min_{1 \leq k \leq n} |t - t_k|^{2H(t)}.$$

Let set $\rho > 0$. We may assume that $t_k \in B(t, \rho)$ for every $k \in \{1, \dots, n\}$, implying that

$$\mathrm{Var}(X_t \,|\, X_{t_1}, \dots, X_{t_n}) \leq C\rho^{\varepsilon} \min_{1 \leq k \leq n} |t - t_k|^{2H(t) - \varepsilon}.$$

Using *rho* sufficiently small, it proves that the property of local nondeterminism does not stand with the exponent $2H(t) - \varepsilon$, for any $\varepsilon > 0$. □

Hence, the classic exponent $2H(t)$ in the LND property of the mBm is optimal. Intuitively, it can be understood as following: in the neighbourhood of t, the Hurst function can influence the Hölder regularity, but is still a deterministic component and thus does not bring any randomness which could increase the conditional variance. As the consequence, the latter is only a result of the local fBm form of the multifractional Brownian motion.

Therefore, the estimate of the Hausdorff dimension of the level sets can not be improved using the classic methods based on the LND property, and it remains an open question to determine the precise form of the fractal dimension in terms of geometric properties of the Hurst function H.

2.5 Appendix: a few deterministic properties

We gather in this appendix a few deterministic properties which are related to 2-microlocal analysis and the local graph dimension. Let us begin with a simple upper bound on the 2-microlocal which extends the classic property of lower semi-continuity on the local Hölder exponent.

Lemma 2.2 *Suppose f is a continuous function and $t \in \mathbf{R}$ such that* $\liminf_{u\to t} \widetilde{\alpha}_{f,t} < 1$. *Then, the 2-microlocal frontier of f at t satisfies*

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,t}(s') \leq \liminf_{u\to t} \widetilde{\alpha}_{f,u}.$$

Proof. Let us set $\rho > 0$ and $\varepsilon > 0$. There exists $s \in B(t,\rho)$ such that $\widetilde{\alpha}_{f,s} \leq \liminf_{u\to t} \widetilde{\alpha}_{f,u} + \varepsilon$. Then, we can find $u_n, v_n \to s$ such that

$$|f(u_n) - f(v_n)| \geq n\,|u_n - v_n|^{\widetilde{\alpha}_{f,s}+\varepsilon}.$$

Therefore, for any $s' \in \mathbf{R}$, we have

$$\frac{|f(u_n) - f(v_n)|}{|u_n - v_n|^{\widetilde{\alpha}_{f,s}+\varepsilon}\big(|u_n - t| + |v_n - t|\big)^{-s'}} \geq \frac{n}{\big(|u_n - t| + |v_n - t|\big)^{-s'}} \longrightarrow_{n\to+\infty} +\infty,$$

proving that $\sigma_{f,t}(s') \leq \liminf_{u\to t} \widetilde{\alpha}_{f,u} + \varepsilon$, for any $\varepsilon > 0$. □

In the next two lemmas, we present an extension of a well-known property connecting the local dimension of the graph to the local Hölder exponent: $\overline{\dim}_{\mathrm{B},t}\,\mathrm{Gr}(f) \leq 2 - \widetilde{\alpha}_{f,t}$.

Lemma 2.3 *Suppose f is a continuous function and $t \in \mathbf{R}$. Then,*

$$\underline{\dim}_{\mathrm{B},t}\,\mathrm{Gr}(f) \leq \overline{\dim}_{\mathrm{B},t}\,\mathrm{Gr}(f) \leq 2 - \big(\sigma_{f,t}(1) \wedge 1\big).$$

Proof. We need to construct a local cover of the graph which is sufficiently efficient. Let first set $t \in \mathbf{R}$ and $\sigma < \sigma_{f,t}(1) \wedge 1$. Then, there exists $\rho > 0$ and $C > 0$ such that for all $u, v \in B(t,\rho)$,

$$|f(u) - f(v)| \leq C|u - v|^{\sigma}\big(|u - t| + |v - t|\big)^{-1}.$$

The cover is constructed similarly to the classic proof. Hence, if N_δ denotes the number of δ-box necessary to cover $\mathrm{Gr}(f, [t-\rho, t+\rho])$, we have

$$\begin{aligned} N_\delta &\leq C\delta^{-1} + \sum_{k=-\lceil\rho/\delta\rceil}^{\lceil\rho/\delta\rceil} \sup_{u,v\in t+[k\delta,(k+1)\delta]} |f(u) - f(v)| \cdot \delta^{-1} \\ &\leq C\delta^{-1} + \sum_{k=-\lceil\rho/\delta\rceil}^{\lceil\rho/\delta\rceil} C\delta^{\sigma}(k\delta)^{-1} \cdot \delta^{-1} \\ &= C\delta^{-1} + 2C\delta^{\sigma-2} \sum_{k=1}^{\lceil\rho/\delta\rceil} k^{-1} \leq 2C\,\delta^{\sigma-2}\big(1 + \log\lceil\rho/\delta\rceil\big). \end{aligned}$$

since $2 - \sigma > 1$. Then,

$$\overline{\dim}_{\mathrm{B}}\,\mathrm{Gr}(f, [t-\rho, t+\rho]) = \limsup_{\delta\to 0} \frac{N_\delta}{-\log\delta} \leq 2 - \sigma,$$

for all $\sigma < \sigma_{f,t_0}(1) \wedge 1$, which proves to the expected inequality. □

Lemma 2.4 *Suppose f is a continuous function and $t \in \mathbf{R}$. Then,*

$$\dim_{\mathrm{H},t}\,\mathrm{Gr}(f) \leq 2 - \big(\sigma_{f,t}(+\infty) \wedge 1\big).$$

Proof. As the estimate we want to obtain is different than the one in the previous lemma, we need to adopt a constructing procedure which is slightly different. Let $t \in \mathbf{R}$ and $\sigma < \sigma_{f,t}(+\infty) \wedge 1$. Then, there exist $s' \geq 0$, $\rho > 0$ and $C > 0$ such that for all $u, v \in B(t, \rho)$

$$|f(u) - f(v)| \leq C|u - v|^{\sigma} \left(|u - t| + |v - t|\right)^{-s'}.$$

Let us set $\delta > 0$ and $\gamma > 1$. The δ-cover of $\mathrm{Gr}(f, [t - \rho, t + \rho])$ is constructed as following. There exists $n \in \mathbf{N}$ such that for all $k \geq n$, $k^{-\gamma} < \delta$. To cover the set $[t - \rho, t + \rho]$, we first choose successive intervals $(I_k)_{k \geq n}$ of size $k^{-\gamma}$ in the neighbourhood of t. Then, the rest is simply covered by intervals of size δ. The first type of intervals cover a set whose length is equal to:

$$\sum_{k=n}^{+\infty} k^{-\gamma} \geq \int_n^{+\infty} x^{-\gamma} \mathrm{d}x = C_\gamma n^{1-\gamma}.$$

Then, it induces that the number M_δ of intervals of size δ which are necessary is upper bounded by

$$M_\delta \leq \delta^{-1}\left(2\rho - C_\gamma n^{1-\gamma}\right).$$

Let now consider the cover $(O_i)_{i \in \mathbf{N}}$ of $\mathrm{Gr}(f, [t - \rho, t + \rho])$ which can be defined from the previous intervals (still using the same idea of covering by columns). Then, it satisfies for any $s > 0$

$$\sum_{i \in \mathbf{N}} |O_i|^s \leq \sum_{k=n}^{+\infty} (k^{-\gamma})^s \cdot k^{\gamma} \sup_{u,v \in I_k} |f(u) - f(v)| + \sum_{k=1}^{M_\delta} \delta^s \cdot \delta^{-1} \sup_{u,v \in \widetilde{I}_k} |f(u) - f(v)|.$$

Let estimate the first term,

$$\begin{aligned}\sum_{k=n}^{+\infty} (k^{-\gamma})^s \cdot k^{\gamma} \sup_{u,v \in I_k} |f(u) - f(v)| &\leq \sum_{k=n}^{+\infty} (k^{-\gamma})^{\sigma+s-1} |x_k - t|^{-s'} \\ &\leq C_\gamma \sum_{k=n}^{+\infty} (k^{-\gamma})^{\sigma+s-1} \cdot k^{s'(\gamma-1)} = C_\gamma \sum_{k=n}^{+\infty} (k^{-1})^{\gamma(\sigma+s-1)-s'(\gamma-1)}.\end{aligned}$$

where we have used that $d(t, I_k) \sim k^{1-\gamma}$. The previous series converges if s satisfies $\gamma(\sigma + s - 1) - s'(\gamma - 1) > 1$, i.e. if $s > 1 - \sigma + s'(1 - \gamma^{-1}) + \gamma^{-1}$.

Let now consider the second term in the estimate the covering.

$$\begin{aligned}\sum_{k=1}^{M_\delta} \delta^s \cdot \delta^{-1} \sup_{u,v \in \widetilde{I}_k} |f(u) - f(v)| &\leq \sum_{k=1}^{M_\delta} \delta^{\sigma+s-1} |\widetilde{x}_k - t|^{-s'} \\ &\leq \delta^{\sigma+s-1} \sum_{k=1}^{M_\delta} \left(C_\gamma n^{1-\gamma} + k\delta\right)^{-s'}.\end{aligned}$$

The sum can upper bounded by

$$\begin{aligned}\int_0^{M_\delta} \left(C_\gamma n^{1-\gamma} + x\delta\right)^{-s'} \mathrm{d}x &\leq \delta^{-1} \int_0^{\rho} \left(C_\gamma n^{1-\gamma} + v\right)^{-s'} \mathrm{d}v \\ &\leq c_2\, \delta^{-1} \cdot n^{(\gamma-1)(s'-1)} \\ &\leq c_3\, \delta^{-1} \cdot \delta^{-(\gamma-1)(s'-1)/\gamma}.\end{aligned}$$

Combining the previous estimate, we obtain

$$\sum_{k=1}^{M_\delta} \delta^s \cdot \delta^{-1} \sup_{u,v\in\widetilde{I}_k} |f(u)-f(v)| \le c_4\, \delta^{\sigma+s-2-(\gamma-1)(s'-1)/\gamma}.$$

Hence, to converge when $s \to 0$, we must have $s > 2-\sigma+(\gamma-1)(s'-1)/\gamma$. Therefore, combining the two conditions we have obtained, the Hausdorff dimension satisfies

$$\dim_{\mathrm{H}} \mathrm{Gr}(f,[t-\rho,t+\rho]) \le \min\{1-\sigma+s'(1-\gamma^{-1})+\gamma^{-1}, 2-\sigma+(\gamma-1)(s'-1)/\gamma\},$$

for all $\gamma > 1$ and $s' \in \mathbf{R}$. Then, considering the limit $\gamma \to 1$, we obtain $\dim_{\mathrm{H}} \mathrm{Gr}(f,[t-\rho,t+\rho]) \le 2-\sigma$, which implies the expected inequality. □

We observe that the two new upper bounds are consistent with the original one, since $\widetilde{\alpha}_{f,t} = \sigma_{f,t}(0) \le \sigma_{f,t}(1) \le \sigma_{f,t}(+\infty)$. Finally, let us give a few examples to illustrate in which case it brings a better estimate of the Hausdorff or Box dimensions.

Example 2.6 Suppose B^H is a fractional Brownian motion of parameter $H \in (0,1)$. Then, it is known that with probability one, for all $t \in \mathbf{R}$,

$$\dim_{\mathrm{H},t} \mathrm{Gr}(B^H) = \dim_{\mathrm{B},t} \mathrm{Gr}(B^H) = 2-H \quad \text{and} \quad \sigma_{B^H,t}(s') = (H+s')\wedge H,$$

for all $s' \in \mathbf{R}$. Hence, in this case, the estimate obtain with the local Hölder exponent is already optimal and Lemmas 2.3 and 2.4 bring the same upper bound.

On the contrary, let now consider the "chirp" function $f : x \mapsto |x|^\alpha \sin(|x|^{-\beta})$, with $\alpha,\beta > 0$. As proved by ECHELARD [60], its 2-microlocal frontier at 0 is equal to

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,0}(s') = \frac{s'+\alpha}{1+\beta}.$$

Then, if we look at the Box dimension, Lemma 2.3 induces that

$$\underline{\dim}_{\mathrm{B},0} \mathrm{Gr}(f) \le \overline{\dim}_{\mathrm{B},0} \mathrm{Gr}(f) \le 2-\left(\frac{1+\alpha}{1+\beta}\right)\wedge 1.$$

Some simple estimations of the increments near 0 proves that the previous estimate is optimal, mainly because close to 0, the oscillations are too tight to allow a "smarter" cover of the graph.

Furthermore, the estimate given by Lemma 2.4 on the Hausdorff dimension proves that $\dim_{\mathrm{H},0} \mathrm{Gr}(f) = 1$. Hence, for this specific example, Lemmas 2.3 and 2.4 give better estimates, which are proved to be optimal, whereas the classic one only leads to a rough upper bound $2-\alpha/(1+\beta)$.

To end this section, we prove two useful lemmas related to the parabolic Hausdorff dimension. The first one constitutes an extension of an inequality proved by PERES and SOUSI [156] (Lemma 2.1).

Lemma 2.5 *For all $A \subset \mathbf{R}^2$ and all $H_1 > H_2 > 0$, we have*

$$\dim_{\mathrm{H}}(A;\varrho_{H_2}) + \frac{1}{H_2} - \frac{1}{H_1} \;\le\; \dim_{\mathrm{H}}(A;\varrho_{H_1}) \;\le\; 1+\frac{H_2}{H_1}\big(\dim_{\mathrm{H}}(A;\varrho_{H_2})-1\big).$$

As a consequence, if $H(\cdot)$ is a positive continuous function, then the map $t \mapsto \dim_{\mathrm{H}}(A;\varrho_{H(t)})$ is also continuous.

Proof. Suppose $A \subset \mathbf{R}^2$ and $H_1 > H_2$. Let us first prove the upper bound on $\dim_{\mathrm{H}}(A;\varrho_{H_1})$. For any $\gamma > \dim_{\mathrm{H}}(A;\varrho_{H_2})$, there exists a ϱ_{H_2}-cover $(O_i)_{i\in\mathbf{N}}$ of A such that

$$\sum_{i=0}^{\infty} \delta_i^{\gamma} < \infty \quad \text{where} \quad \delta_i := \operatorname{diam}(O_i\,;\varrho_{H_2}) \ \ \forall i \in \mathbf{N}.$$

We denote by ρ_i the quantity $\delta_i^{H_1/H_2}$. Without any loss of generality, we may assume that for every $i \in \mathbf{N}$, O_i is a rectangle of size $\delta_i^{1/H_2} \times \delta_i$ (as a simple consequence of the definition of ϱ_{H_2}). We want to construct a ϱ_{H_1}-cover of A from $(O_i)_{i\in\mathbf{N}}$. Since $\rho_i^{1/H_1} = \delta_i^{1/H_2}$, we need to split O_i along the space axis to obtain ϱ_{H_1}-balls. More precisely, there will be at most $2\delta_i/\rho_i = 2\delta_i^{1-H_1/H_2}$ resulting ϱ_{H_1}-balls of diameter ρ_i. Hence, we obtain a ϱ_{H_1}-cover $(V_k)_{k\in\mathbf{N}}$ of A such that

$$\sum_{k=0}^{\infty} \operatorname{diam}(V_k\,;\varrho_{H_1})^s \le 2\sum_{i=0}^{\infty} \delta_i^{1-H_1/H_2} \cdot \delta_i^{sH_1/H_2}.$$

The last sum is finite when $1+H_1/H_2(s-1) > \gamma$, i.e. when $s > 1+(\gamma-1)H_2/H_1$. Considering $\gamma \to \dim_{\mathrm{H}}(A;\varrho_{H_2})$, we obtain the expected upper bound.

The lower bound is obtained using a similar reasoning. For any $s > \dim_{\mathrm{H}}(A;\varrho_{H_1})$, there exists a ϱ_{H_1}-cover $(V_k)_{k\in\mathbf{N}}$ of A such that

$$\sum_{k=0}^{\infty} \rho_k^{s} < \infty \quad \text{where} \quad \rho_k := \operatorname{diam}(V_k\,;\varrho_{H_1}) \ \ \forall k \in \mathbf{N}.$$

Similarly, we may assume that V_k is a rectangle of size $\rho_k^{1/H_1} \times \rho_k$ for every $k \in \mathbf{N}$. As $H_1 > H_2$, we note that $\rho^{1/H_2} < \rho_k^{1/H_1}$. Hence, to obtain a ϱ_{H_2}-cover of A, we split the previous balls along the time axis, in at most $2\rho_k^{1/H_1-1/H_2}$ sub-balls. Therefore, we obtain a ϱ_{H_2}-cover $(O_i)_{i\in\mathbf{N}}$ of A such that

$$\sum_{i=0}^{\infty} \operatorname{diam}(O_i\,;\varrho_{H_2})^{\gamma} \le 2\sum_{k=0}^{\infty} \rho_k^{1/H_1-1/H_2} \cdot \rho_k^{\gamma}.$$

This sum is finite when $\gamma + 1/H_1 - 1/H_2 > s$, inducing the lower bound on $\dim_{\mathrm{H}}(A;\varrho_{H_1})$.

The continuity of the map $t \mapsto \dim_{\mathrm{H}}(A;\varrho_{H(t)})$ is straightforward as we observe that lower and upper bounds converge to $\dim_{\mathrm{H}}(A;\varrho_{H_1})$ when $H_2 \to H_1$. □

The previous lemma helps us to extend the property of upper semi-continuity to the local parabolic Hausdorff dimension.

Lemma 2.6 *Suppose $A \subset \mathbf{R}^2$ and $H(\cdot)$ is a positive continuous function. Then, the map $t \mapsto \dim_{\mathrm{H},t}(A;\varrho_{H(t)})$ is upper semi-continuous, i.e. for all $t \in \mathbf{R}$*

$$\dim_{\mathrm{H},t}(A;\varrho_{H(t)}) \ge \limsup_{s\to t} \dim_{\mathrm{H},s}(A;\varrho_{H(s)}).$$

Proof. For any $t \in \mathbf{R}$, the local parabolic Hausdorff dimension satisfies

$$\dim_{\mathrm{H},t}(A;\varrho_{H(t)}) \ge \limsup_{s\to t} \dim_{\mathrm{H},s}(A;\varrho_{H(t)})$$

The proof of this inequality is simply the same as the one for the usual Hausdorff dimension. Let $\delta > 0$. Owing to the uniform bounds obtained in Lemma 2.5, there exists $c_\delta > 0$ such that for all $s \in B(t,\delta)$ and all $\rho > 0$,

$$\left|\dim_{\mathrm{H}}(A\cap B(s,\rho)\,;\varrho_{H(t)}) - \dim_{\mathrm{H}}(A\cap B(s,\rho)\,;\varrho_{H(s)})\right| \le c_\delta |H(s)-H(t)|.$$

Hence, considering the limit $\rho \to 0$, we obtain $|\dim_{\mathrm{H},s}(A;\varrho_{H(t)}) - \dim_{\mathrm{H},s}(A;\varrho_{H(s)})| \leq c_\delta |H(s) - H(t)|$, proving, jointly with the first inequality, the upper semi-continuity of the map $t \mapsto \dim_{\mathrm{H},t}(A;\varrho_{H(t)})$. $\square$

3

Lévy processes and linear fractional stable motion

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3.1 Introduction

The study of sample path continuity and Hölder regularity of stochastic processes is a very active field of research in probability theory. The existing literature provides a variety of uniform results on local regularity, especially on the modulus of continuity, for rather general classes of random fields (see e.g. Marcus and Rosen [127], Adler and Taylor [4] on Gaussian processes and Xiao [199] for more recent developments).

On the other hand, the structure of pointwise regularity is generally more complex as the latter often tends to behave erratically as time passes. This type of sample path behaviour was first put into light on Brownian motion by Orey and Taylor [152] and Perkins [157]. They respectively studied *fast* and *slow points* which characterize logarithmic variations of the pointwise modulus of continuity, and proved that the sets of times with a given pointwise regularity have a distinct fractal geometry. Khoshnevisan and Shi [102] have recently extended this study of fast points to fractional Brownian motion.

Lévy processes with a jump compound also present an interesting pointwise behaviour. Indeed, Jaffard [94] has proved that despite the random variations of the pointwise exponent, the level sets of the latter show a specific fractal structure. This seminal work has been enhanced and extended by Durand [57], Durand and Jaffard [58] and Barral et al. [21]. Particularly, the latter have proved that Markov processes have a range of admissible pointwise behaviours wider and richer than Lévy processes. In the aforementioned works, *multifractal*

analysis happens to be the key concept to study and characterise the local fluctuations of the pointwise regularity. In order to be more specific, we recall a few definitions.

Definition 3.1 **(Pointwise exponent)** *A function $f : \mathbf{R} \to \mathbf{R}^d$ belongs to C_t^α, where $t \in \mathbf{R}$ and $\alpha > 0$, if there exist $C > 0$, $\rho > 0$ and a polynomial P_t of degree less than α such that*

$$\forall u \in B(t,\rho); \quad \|f(u) - P_t(u)\| \leq C|t-u|^\alpha.$$

The pointwise Hölder exponent *of f at t is then defined by $\alpha_{f,t} = \sup\{\alpha \geq 0 : f \in C_t^\alpha\}$, where by convention* $\sup\{\emptyset\} = 0$.

Multifractal analysis is interested in the fractal geometry of the level sets of the pointwise exponent, which are also called the *iso-Hölder sets* of f:

$$E_h = \{t \in \mathbf{R} : \alpha_{f,t} = h\} \quad \text{for every } h \in \mathbf{R}_+ \cup \{+\infty\}. \tag{3.1}$$

The geometry of the collection $(E_h)_{h\in\mathbf{R}_+}$ is then studied through its Hausdorff dimension, defining in this way the *local spectrum of singularities* of f. Usually denoted $d_f(h,V)$, it is defined by

$$d_f(h,V) = \dim_{\mathrm{H}}(E_h \cap V) \quad \text{for every } h \in \mathbf{R}_+ \cup \{+\infty\} \text{ and } V \in \mathscr{O}, \tag{3.2}$$

where $\mathscr{O}$ designates the collection of nonempty open sets of $\mathbf{R}$ and $\dim_{\mathrm{H}}$ is the Hausdorff dimension, with by convention $\dim_{\mathrm{H}}(\emptyset) = -\infty$ (we refer to [67] for the complete definition of the latter).

Even though $(E_h)_{h\in\mathbf{R}_+}$ are random sets, stochastic processes such as Lévy processes [94], Lévy processes in multifractal time [22] and fractional Brownian motion have a deterministic multifractal spectrum. Furthermore, these random fields are also said to be *homogeneous* since the quantity $d_X(h,V)$ is independent of the open set V for any $h \in \mathbf{R}_+$. In addition, when the pointwise exponent is constant along sample paths, the spectrum is described as *degenerate*, i.e. its support is reduced to a single point (e.g. the Hurst exponent H in the case of f.B.m.). Nevertheless, note that BARRAL et al. [21] and DURAND [56] have provided examples of respectively Markov jump processes and wavelet random series with a non-homogeneous and random spectrum of singularities.

As outlined in Equations (3.1) and (3.2), multifractal analysis usually focuses on the structure of pointwise regularity. Unfortunately, as presented by MEYER [138], the pointwise Hölder exponent suffers of a couple of drawbacks: it lacks of stability under the action of pseudo-differential operators and it is not always characterised by the wavelets coefficients. In addition, several simple deterministic examples such as the *Chirp function* $t \mapsto |t|^\alpha \sin\left(|t|^{-\beta}\right)$ show that it does not fully capture the local geometry and oscillations of a function.

Several approaches, such as the *oscillating, chirp and weak scaling exponents* introduced by ARNEODO et al. [7] and MEYER [138], have emerged in the literature to address the issues of the pointwise exponent and supplement the latter by characterising other aspects of the local regularity. Interestingly, the aforementioned concepts are embraced by a single framework called *2-microlocal analysis*. It has first been introduced by BONY [35] in the deterministic frame to study singularities of generalised solutions of PDEs. Several authors have then investigated in [93, 96, 138, 119] this framework more deeply, determining in particular the close link between the 2-microlocal formalism and the previous scaling exponents. More recently, HERBIN and LÉVY VÉHEL [78] have developed a stochastic approach of this framework to investigate the fine regularity of stochastic processes such as Gaussian processes, martingales and stochastic integrals.

Similarly to the pointwise Hölder exponent, the introduction of this formalism starts with the definition of appropriate functional spaces, named *2-microlocal spaces*. We begin with a simpler, but narrower, definition to give an intuition of these concepts.

Definition 3.2 *Suppose $t \in \mathbf{R}$, $s' \in \mathbf{R}$ and $\sigma \in (0,1)$ such that $\sigma - s' \notin \mathbf{N}$. A function $f : \mathbf{R} \to \mathbf{R}^d$ belongs to the* 2-microlocal space $C_t^{\sigma,s'}$ *if there exist $C > 0$, $\rho > 0$ and a polynomial P_t such that for all $u, v \in B(t,\rho)$:*

$$\left\| f(u) - P_t(u) - f(v) + P_t(v) \right\| \leq C|u-v|^{\sigma} \left(|u-t| + |v-t| \right)^{-s'}. \tag{3.3}$$

In addition, P_t is unique if we suppose its degree is smaller than $\sigma - s'$. In this case, it corresponds to the Taylor polynomial of order $\lfloor \sigma - s' \rfloor$ of f at t.

The 2-microlocal spaces are therefore parametrised by a pair (s', σ) of real numbers and we clearly observe on Equation (3.3) that they tend to extend the ideas behind classic Hölder spaces. To define these elements for any $\sigma \in \mathbf{R} \setminus \mathbf{Z}$, we need to slightly complexify the form of the increments considered.

Definition 3.3 *Suppose $t \in \mathbf{R}$ and $b < t$ is fixed. In addition, consider $s' \in \mathbf{R}$, $\sigma \in \mathbf{R} \setminus \mathbf{Z}$ and $k \in \mathbf{Z}$ such that $\sigma - s' \notin \mathbf{N}$ and $\sigma + k \in (0,1)$. A function $f : \mathbf{R} \to \mathbf{R}^d$ belongs to the* 2-microlocal space $C_t^{\sigma,s'}$ *if there exist $C > 0$, $\rho > 0$ and a polynomial $P_{t,k}$ such that for all $u, v \in B(t,\rho)$:*

$$\left\| I_{b+}^k f(u) - P_{t,k}(u) - I_{b+}^k f(v) + P_{t,k}(v) \right\| \leq C|u-v|^{\sigma+k} \left(|u-t| + |v-t| \right)^{-s'}, \tag{3.4}$$

where $I_{b+}^k f$ designates the derivative of order $-k$ when $k \leq 0$ and the iterated integral of order k when $k > 0$, i.e. $\left(I_{b+}^k f\right)(u) := 1/\Gamma(k-1) \int_b^u (u-s)^{k-1} f(s)\,\mathrm{d}s$.

The time-domain characterisation (3.3)-(3.4) of 2-microlocal spaces has first been obtained by KOLWANKAR and LÉVY VÉHEL [109] in the case $\sigma \in (0,1)$ and then extended by SEURET and LÉVY VÉHEL [169] and ECHELARD [60] to $\sigma \in \mathbf{R} \setminus \mathbf{Z}$. Note that the previous characterisation does not depend on the value of the constant b, since a modification of the latter simply induces an adjustment of the polynomial.

Even though, we restrict ourselves in Definitions 3.2-3.3 to usual functions, 2-microlocal spaces were originally introduced by BONY [35] for tempered distributions $\mathscr{S}'(\mathbf{R})$. The first definition given by BONY [35] hence relies on the Littlewood–Paley decomposition of distributions, and thereby corresponds to a description in the Fourier space. Another characterisation based on wavelet coefficients has also been presented by JAFFARD [93]. Also note that the characterisation is in fact equivalent the *localised 2-microlocal spaces* which are also defined for distributions in $\mathscr{D}'(\mathbf{R})$ (we refer to [138] for a more precise distinction between global and local definitions of the 2-microlocal spaces).

One major property of the 2-microlocal spaces is their stability under the action of pseudo-differential operators. In particular, as proved by JAFFARD and MEYER [96, Th 1.1], they satisfy

$$\forall \alpha > 0; \quad f \in C_t^{\sigma,s'} \iff I_+^{\alpha} f \in C_t^{\sigma+\alpha,s'}, \tag{3.5}$$

where the fractional integral of f of order $\alpha \geq 0$ is defined by: $\left(I_+^{\alpha} f\right)(u) := 1/\Gamma(\alpha) \int_{\mathbf{R}} (u-s)_+^{\alpha-1} f(s)\,\mathrm{d}s$. Note that the latter definition of the operator I_+^{α} coincides with the fractional integral presented in [96] for tempered distributions (we refer to the book of SAMKO, KILBAS, and MARICHEV [166] for an extensive study of the subject).

Similarly to the pointwise Hölder exponent, the introduction of 2-microlocal spaces leads naturally to the definition of a regularity tool named the *2-microlocal frontier*:

$$\forall s' \in \mathbf{R}; \quad \sigma_{f,t}(s') = \sup\{\sigma \in \mathbf{R} : f \in C_t^{\sigma,s'}\}.$$

Due to several inclusion properties of the 2-microlocal spaces, the map $s' \mapsto \sigma_{f,t}(s')$ is well-defined and satisfies:

- $\sigma_{f,t}(\cdot)$ is a concave non-decreasing function;
- $\sigma_{f,t}(\cdot)$ has left and right derivatives between 0 and 1.

Furthermore, as a consequence of Equation 3.5, $\sigma_{f,t}(\cdot)$ is stable under the action of pseudo-differential operators.

As a function, the 2-microlocal frontier $\sigma_{f,t}(\cdot)$ offers a more complete and richer description of the local regularity and cover in particular the usual Hölder exponents:

$$\widetilde{\alpha}_{f,t} = \sigma_{f,t}(0) \quad \text{and} \quad \alpha_{f,t} = -\inf\{s' : \sigma_{f,t}(s') \geq 0\},$$

where the last equality has been proved by MEYER [138] under the assumption $\omega(h) = \mathrm{O}(1/|\log(h)|)$ on the modulus of continuity of f. Several other scaling exponents previously outlined can also be retrieved from the frontier: the *chirp* and *weak scaling* exponents introduced by MEYER [138] are given by:

$$\beta^c_{f,t} = \left\{ \frac{\mathrm{d}\sigma_{f,t}}{\mathrm{d}s'}\bigg|_{s'\to-\infty} \right\}^{-1} - 1 \quad \text{and} \quad \beta^w_{f,t} = \lim_{s'\to-\infty} \sigma_{f,t}(s') - s';$$

These two elements characterise the asymptotic regularity of a function after a large number of integrations and the latter was been specifically introduced to supplement the pointwise exponent in multifractal analysis. The *oscillating* exponent defined by ARNEODO et al. [7] can also be retrieved from the 2-microlocal frontier:

$$\beta^o_{f,t} = \left\{ \frac{\mathrm{d}\sigma_{f,t}}{\mathrm{d}s'}\bigg|_{s'=-\alpha_{f,t}-} \right\}^{-1} - 1.$$

The latter aims to capture the oscillating behaviour by studying the regularity after infinitesimal integrations. Note that the original definition of these exponents are based on Hölder spaces (see [170] for an extensive review).

In the stochastic framework, Brownian motion provides a simple example of 2-microlocal frontier: with probability one and for all $t \in \mathbf{R}$

$$\forall s' \in \mathbf{R}; \quad \sigma_{B,t}(s') = \left(\frac{1}{2} + s'\right) \wedge \frac{1}{2}. \tag{3.6}$$

Using the terminology of MEYER [138], since $\beta^w_{B,t} = \alpha_{B,t}$ and $\beta^o_{B,t} = 0$, Brownian motion is said to have *cusp singularities*. On the other hand, *oscillating singularities* appear when the slope of the frontier is strictly smaller than 1 at $s' = -\alpha_{f,t}$, or equivalently, when $\beta^w_{B,t} > \alpha_{B,t}$. This oscillating behaviour is well-illustrated by the chirp function whose frontier and scaling exponents at 0 respectively are equal to $\sigma_{f,0}(s') = (\alpha + s')(1+\beta)$, $\alpha_{f,0} = 0$, $\beta^c_{f,0} = \beta^o_{f,0} = \beta$ and $\beta^w_{f,0} = \infty$.

In this paper, we combine the 2-microlocal formalism with the classic use of multifractal analysis to obtain a finer and richer description of the regularity of Lévy processes. Following the path of [94, 57, 58], we extend the multifractal description (Section 3.2) to the aforementioned scaling exponents and the 2-microlocal frontier. We present in particular how this formalism allows to capture and describe the oscillating singularities of Lévy processes. The fractal structure of the latter is determined for a few classes of Lévy processes which include alpha-stable processes.

This finer analysis of the sample path properties of Lévy processes happens to be very useful for the study of another class of processes named linear fractional stable motion (LFSM). The LFSM is a common α-stable self-similar process with stationary increments which can be seen as the extension of the fractional Brownian motion to the non-Gaussian frame. In Section 3.3, we completely characterize the multifractal nature of the LFSM, unifying the geometrical description of the sample paths independently of their boundedness. This analysis is also extended the multifractional generalisation of the LFSM.

3.1.1 Statement of the main results

As it is well known, an $\mathbf{R}^d$-valued Lévy process $(X_t)_{t\in\mathbf{R}_+}$ has stationary and independent increments. Furthermore, its law is determined by the Lévy–Khintchine formula (see e.g. [168]): for all $t \in \mathbf{R}_+$ and $\lambda \in \mathbf{R}^d$, $\mathbb{E}[e^{i\langle\lambda,X_t\rangle}] = e^{t\psi(\lambda)}$ where ψ is given by

$$\forall \lambda \in \mathbf{R}^d; \quad \psi(\lambda) = i\langle a,\lambda\rangle - \frac{1}{2}\langle\lambda, Q\lambda\rangle + \int_{\mathbf{R}^d} \left(e^{i\langle\lambda,x\rangle} - 1 - i\langle\lambda,x\rangle \mathbf{1}_{\{\|x\|\le 1\}}\right)\pi(\mathrm{d}x).$$

In the previous expression, Q is a non-negative symmetric matrix and π is the Lévy measure, i.e. a positive Radon measure on $\mathbf{R}^d \setminus \{0\}$ such that $\int_{\mathbf{R}^d}(1\wedge\|x\|^2)\,\pi(\mathrm{d}x) < \infty$. Throughout this paper, it will always be assumed that $\pi(\mathbf{R}^d) = +\infty$ since otherwise, the Lévy process corresponds to the sum of a simple compound Poisson process with drift and a Brownian motion whose regularity is well-known.

Sample path properties of Lévy processes are known to depend on the growth of the Lévy measure near the origin. More precisely, BLUMENTHAL and GETOOR [34] have defined the following exponents β and β',

$$\beta = \inf\left\{\delta \ge 0 : \int_{\mathbf{R}^d} \left(1\wedge\|x\|^\delta\right)\pi(\mathrm{d}x) < \infty\right\} \quad \text{and} \quad \beta' = \begin{cases} \beta & \text{if } Q = 0; \\ 2 & \text{if } Q \ne 0. \end{cases} \tag{3.7}$$

Owing to π's definition, $\beta, \beta' \in [0,2]$. PRUITT [160] proved that $\alpha_{X,0} \overset{\text{a.s.}}{=} 1/\beta$ when $Q = 0$. Note that several other exponents have been introduced in the literature to study the sample path properties of Lévy processes (see e.g. [106, 101] for some recent developments).

JAFFARD [94] has studied the spectrum of singularities of Lévy processes under the following assumption on the measure π,

$$\sum_{j\in\mathbf{N}} 2^{-j}\sqrt{C_j \log(1+C_j)} < \infty, \quad \text{where } C_j = \int_{2^{-j-1}<\|x\|\le 2^{-j}} \pi(\mathrm{d}x). \tag{3.8}$$

Under the Hypothesis (3.8), Theorem 1 in [94] states that the multifractal spectrum of a Lévy process X is almost surely equal to

$$\forall V \in \mathcal{O}; \quad d_X(h,V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta'); \\ 1 & \text{if } h = 1/\beta'; \\ -\infty & \text{if } h \in (1/\beta', +\infty]. \end{cases} \tag{3.9}$$

DURAND [57] has extended this result to Hausdorff g-measures, where g is a gauge function, and DURAND and JAFFARD [58] have generalized the study to multivariate Lévy fields.

In this work, we first establish in Proposition 3.9 a new proof of the multifractal spectrum (3.9) which does not require Assumption (3.8). Results obtained by DURAND [57] on Hausdorff g-measure are also indirectly extended using this method.

In order to refine and extend the spectrum of singularities (3.9) using the 2-microlocal formalism, we are interested the fractal geometry of the collections of sets $(\widetilde{E}_h)_{h\in\mathbf{R}_+}$ and $(\widehat{E}_h)_{h\in\mathbf{R}_+}$ respectively defined by

$$\widetilde{E}_h = \{t \in E_h : \forall s' \in \mathbf{R};\ \sigma_{X,t}(s') = (h+s')\wedge 0\} \quad \text{and} \quad \widehat{E}_h = E_h \setminus \widetilde{E}_h.$$

The introduction of these two collections corresponds to the natural distinction between two types of singularities: the family $(\widetilde{E}_h)_{h\in\mathbf{R}_+}$ gathers the *cusp singularities* of Lévy processes, i.e. times at which the slope of the 2-microlocal frontier is equal to 1, whereas the collection

$(\widehat{E}_h)_{h\in\mathbf{R}_+}$ regroups the oscillating singularities of the process, i.e. when $\beta^w_{X,t} > \alpha_{X,t}$ and $\beta^o_{X,t} > 0$.

In our first important result, we provide a general description of the fractal geometry of these singularities.

Theorem 3.1 *Suppose X is a Lévy process such that $\beta > 0$. Then, with probability one, the* cusp singularities $(\widetilde{E}_h)_{h\in\mathbf{R}_+}$ *of X satisfy*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widetilde{E}_h \cap V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta'); \\ 1 & \text{if } h = 1/\beta'; \\ -\infty & \text{if } h \in (1/\beta', +\infty]. \end{cases} \tag{3.10}$$

Furthermore, the oscillating singularities $(\widehat{E}_h)_{h\in\mathbf{R}_+}$ *of X are such that*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) \leq \begin{cases} 2\beta h - 1 & \text{if } h \in (1/2\beta, 1/\beta'); \\ -\infty & \text{if } h \in [0, 1/2\beta] \cup [1/\beta', +\infty], \end{cases} \tag{3.11}$$

where the 2-microlocal frontier at $t \in \widehat{E}_h$ verifies $\sigma_{X,t}(s') \leq \left(\frac{h+s'}{2\beta h}\right) \wedge \left(\frac{1}{\beta'} + s'\right) \wedge 0$ for all $s' \in \mathbf{R}$.

Remark 3.1 Theorem 3.1 induces that $\dim_{\mathrm{H}}(\widehat{E}_h) < \dim_{\mathrm{H}}(\widetilde{E}_h)$ for every $h \in [0, 1/\beta']$. Therefore, in terms of Hausdorff dimension, chirp oscillations that might appear on a Lévy process are always singular compared to the common cusp behaviour.

We also note that even though sample paths of Lévy processes do not satisfy the condition $\omega(h) = \mathrm{O}(1/|\log(h)|)$ outlined in the introduction, Theorem 3.1 nevertheless ensures that the pointwise Hölder exponent can be retrieved from the 2-microlocal frontier at any $t \in \mathbf{R}_+$ using the formula $\alpha_{X,t} = -\inf\{s' : \sigma_{X,t}(s') \geq 0\}$. As a consequence, the pointwise regularity of Lévy processes can also be characterised by its wavelet coefficients.

The determination of the 2-microlocal regularity of Lévy processes allows to deduce the behaviour of several scaling exponents. In particular, we are interested in the multifractal spectrum of the *weak scaling exponent*, whose level sets are defined as:

$$E^w_h = \{t \in \mathbf{R} : \beta^w_{X,t} = h\} \quad \text{for every } h \in \mathbf{R}_+ \cup \{+\infty\}.$$

Corollary 3.2 *Suppose X is a Lévy process such that $\beta > 0$. Then, with probability one*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E^w_h \cap V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta'); \\ 1 & \text{if } h = 1/\beta'; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.12}$$

Furthermore, the oscillating exponent *is such that $\beta^o_{X,t} \leq \max(0, 2\beta h - 1)$ and*

$$\dim_{\mathrm{H}}\{t \in E_h : \beta^o_{X,t} > 0\} \leq \begin{cases} 2\beta h - 1 & \text{if } h \in (1/2\beta, 1/\beta'); \\ -\infty & \text{otherwise.} \end{cases} \tag{3.13}$$

Finally, the chirp scaling exponent *satisfies $\beta^c_{X,t} = 1$ for all $t \in \mathbf{R}$.*

According to Corollary 3.2, the multifractal spectrum associated to the weak scaling exponent is the same as the classic one (3.9) despite the oscillating singularities which might exist. We also note that the latter do not influence the chirp scaling exponent, showing that chirp oscillations tend to disappear after multiple integrations.

Following the ideas presented by MEYER [138], it is also natural to investigate geometrical properties of the sets $(E_{\sigma,s'})_{\sigma,s'\in\mathbf{R}}$ defined by

$$E_{\sigma,s'} = \{t \in \mathbf{R}_+ : \forall u' > s';\ X_\cdot \in C_t^{\sigma,u'} \text{ and } \forall u' < s';\ X_\cdot \notin C_t^{\sigma,u'}\}.$$

This collection of sets can be seen as the level sets of the 2-microlocal frontier for a fixed σ.

Corollary 3.3 *Suppose X is a Lévy process such that $\beta > 0$. Then, with probability one and for all $\sigma \in \mathbf{R}_-$,*

$$\forall V \in \mathscr{O};\quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \beta s & \text{if } s \in [0, 1/\beta'); \\ 1 & \text{if } s = 1/\beta'; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.14}$$

where s denotes the common 2-microlocal parameter $s = \sigma - s'$. Furthermore, for all $s' \in \mathbf{R}$, $E_{0,s'} = E_{-s'}$ and $E_{\sigma,s'}$ is empty if $\sigma > 0$.

As for the weak scaling exponent, we obtain in Corollary 3.3 a multifractal spectrum which takes the same form as Equation (3.9) (note that the latter corresponds to the case $\sigma = 0$). In addition, the oscillating singularities are also not captured by these scaling exponents and the spectrum associated.

Theorem 3.1 provides an upper bound of the Hausdorff dimension of the oscillating singularities of a general Lévy process. In Section 3.2.3, we obtain the exact estimates for some specific classes of Lévy processes, proving in particular that the Blumenthal–Getoor exponent does not entirely characterise the structure of these chirp oscillations.

Proposition 3.4 *Suppose π is a Lévy measure on $\mathbf{R}$ such that $\pi(\mathbf{R}_\pm) = 0$ and X is a Lévy process with generating triplet (a, Q, π). Then, with probability one, $\widehat{E}_h = \emptyset$ for all $h \in \mathbf{R}_+$, i.e.*

$$\forall t \in \mathbf{R}_+,\ \forall s' \in \mathbf{R};\quad \sigma_{X,t}(s') = (\alpha_{X,t} + s') \wedge 0.$$

Note in particular that subordinators do not have oscillating singularities, which is quite understandable because of their monotonicity.

Nevertheless, these singularities might appear as well for rather natural classes of processes, including alpha-stable Lévy processes.

Theorem 3.5 *Suppose X is a Lévy process parametrised by $(0, 0, \pi)$, where the Lévy measure π has the following form*

$$\pi(\mathrm{d}x) = a_1 |x|^{-1-\alpha_1} \mathbf{1}_{\mathbf{R}_+} \mathrm{d}x + a_2 |x|^{-1-\alpha_2} \mathbf{1}_{\mathbf{R}_-} \mathrm{d}x, \tag{3.15}$$

and $a_1, a_2 > 0$ and $\alpha_1, \alpha_2 \in (0, 2)$.

Then, the Blumenthal–Getoor exponent of π is equal to $\beta = \max(\alpha_1, \alpha_2)$ and with probability one, the oscillating singularities of X satisfy

$$\forall V \in \mathscr{O};\quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) = \begin{cases} (\alpha_1 + \alpha_2)h - 1 & \text{if } h \in (1/(\alpha_1 + \alpha_2), 1/\beta); \\ -\infty & \text{otherwise.} \end{cases} \tag{3.16}$$

One of the interesting aspects of the previous result is to show that the Hausdorff dimension of the oscillating singularities of Lévy processes is not necessarily governed by the Blumenthal–Getoor exponent, but also takes into account the symmetrical aspect of the Lévy measure. Furthermore, Theorem 3.5 proves that the upper bound obtained in Theorem 3.1 is optimal, since in the case of an alpha-stable process parametrised by (α, β_α), with probability one

$$\forall V \in \mathscr{O};\quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) = \begin{cases} 2\alpha h - 1 & \text{if } h \in (1/2\alpha, 1/\alpha) \text{ and } \beta_\alpha \in (-1, 1); \\ -\infty & \text{otherwise.} \end{cases} \tag{3.17}$$

Note that owing to Proposition 3.4, an alpha-stable process whose skewness parameter β_α is equal to 1 or -1 does not have oscillating singularities.

The fine 2-microlocal structure presented Theorems 3.1 and 3.5 happens to be interesting outside the scope of Lévy processes. More precisely, it allows to characterized the multifractal nature of the linear fractional stable motion (LFSM). The latter is a fractional extension of alpha-stable Lévy processes and is usually defined by the following stochastic integral (see e.g. [167])

$$X_t = \int_{\mathbf{R}} \left\{ (t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right\} M_\alpha(\mathrm{d}u), \tag{3.18}$$

where M_α is an alpha-stable random measure parametrised by $\alpha \in (0,2)$ and $\beta_\alpha \in [-1,1]$, and $H \in (0,1)$ is the Hurst exponent. Several regularity properties have been determined in the literature. In particular, sample paths are known to be nowhere bounded [121] if $H < 1/\alpha$ and Hölder continuous when $H > 1/\alpha$. In this latter case, TAKASHIMA [176], KÔNO and MAEJIMA [110] proved that the pointwise and local Hölder exponents satisfy almost surely $H - 1/\alpha \leq \alpha_{X,t} \leq H$ and $\widetilde{\alpha}_{X,t} = H - 1/\alpha$. Throughout this paper, we will assume that $\alpha \in [1,2)$, which is required to obtain Hölder continuous sample paths ($H > 1/\alpha$).

Using an alternative representation of LFSM presented in Proposition 3.10, we enhance the aforementioned regularity results and obtain a precise description of the multifractal structure of the LFSM.

Theorem 3.6 *Suppose X is a linear fractional stable motion parametrized by $\alpha \in [1,2)$, $\beta_\alpha \in [-1,1]$ and $H \in (0,1)$. Then, with probability one and for all $\sigma \leq H - \frac{1}{\alpha}$*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \alpha(s-H)+1 & \text{if } s \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.19}$$

where $s = \sigma - s'$. When $\sigma > H - \frac{1}{\alpha}$, $E_{\sigma,s'}$ is empty for all $s' \in \mathbf{R}$.
In addition, the weak scaling exponent satisfies with probability one

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_h^w \cap V) = \begin{cases} \alpha(h-H)+1 & \text{if } h \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.20}$$

Finally, the chirp scaling exponent $\beta^c_{X,t}$ is equal to 1 for all $t \in \mathbf{R}$.

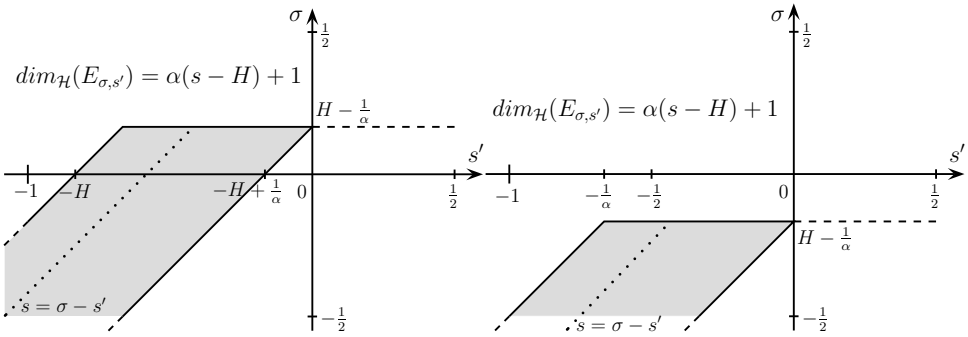
Therefore, we observe that the multifractal structure presented in Theorem 3.6 corresponds to the spectrum of alpha-stable processes translated by a factor $H - \frac{1}{\alpha}$. Interestingly, we also note that on the contrary to usual Hölder exponents, the weak scaling exponent and the 2-microlocal formalism allow to describe the multifractal nature of the LFSM independently of the continuity of its sample paths, unifying the continuous ($H > \frac{1}{\alpha}$) and unbounded ($H < \frac{1}{\alpha}$) cases (see Figure 3.1). In the latter case, the 2-microlocal domain is located strictly below the s'-axis, implying that sample paths are nowhere bounded. Nevertheless, the proof of Theorem 3.6 ensures in this case the existence of a modification of the LFSM such that sample paths are distributions in $\mathscr{D}'(\mathbf{R})$ whose 2-microlocal regularity can be studied as well.

In addition, the classic multifractal spectrum can be explicated when sample paths are Hölder continuous ($H > \frac{1}{\alpha}$).

Corollary 3.7 *Suppose X is a linear fractional stable motion parametrized by $\alpha \in [1,2)$, $\beta_\alpha \in [-1,1]$ and $H \in (0,1)$, with $H > 1/\alpha$. Then, with probability one, the multifractal spectrum of X is given by*

$$\forall V \in \mathscr{O}; \quad d_X(h,V) = \begin{cases} \alpha(h-H)+1 & \text{if } h \in \left[H - \frac{1}{\alpha}, H\right]; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.21}$$

An equivalent multifractal structure is presented in Proposition 3.11 for a similar class of processes called *fractional Lévy processes* (see [25, 128, 44]).



(a) Continuous sample paths: $H = \frac{5}{6}$ and $\alpha = \frac{3}{2}$ (b) Unbounded sample paths: $H = \frac{1}{2}$ and $\alpha = \frac{3}{2}$

Figure 3.1: Domains of admissible 2-microlocal frontiers for the LFSM

The LFSM admits a natural multifractional extension which has been introduced and studied in [174, 173, 54]. The definition of the linear multifractional stable motion (LMSM) is based on Equation (3.18), where the Hurst exponent H is replaced by a function $t \mapsto H(t)$. STOEV and TAQQU [174] and AYACHE and HAMONIER [11] have obtained lower and upper bounds on Hölder exponents which are similar to LFSM results: for all $t \in \mathbf{R}_+, H(t)-1/\alpha \leq \alpha_{X,t} \leq H(t)$ and $\widetilde{\alpha}_{X,t} = H(t) - 1/\alpha$ almost surely. AYACHE and HAMONIER [11] have also investigated the existence of an optimal local modulus of continuity.

Theorem 3.6 can be generalized to the LMSM in the continuous case. More precisely, we assume that the Hurst function satisfies the following assumption,

$$H : \mathbf{R} \to \left(\tfrac{1}{\alpha}, 1\right) \text{ is } \delta\text{-Hölderian, with } \delta > \sup_{u\in\mathbf{R}} H(u). \qquad (\mathscr{H}_0)$$

Since the LMSM is clearly a non-homogeneous process, it is natural to focus on the study of the spectrum of singularities localized at $t \in \mathbf{R}_+$, i.e.

$$\forall t \in \mathbf{R}_+ \quad d_X(h,t) = \lim_{\rho\to 0} d_X(h, B(t,\rho)) = \lim_{\rho\to 0} \dim_{\mathrm{H}}(E_h \cap B(t,\rho)).$$

Theorem 3.8 *Suppose X is a linear multifractional stable motion parametrized by $\alpha \in (1,2)$, $\beta_\alpha \in [-1,1]$ and an $(\mathscr{H}_0)$-Hurst function H.*

Then, with probability one, for all $t \in \mathbf{R}$ and for all $\sigma < H(t) - \frac{1}{\alpha}$,

$$\lim_{\rho\to 0} \dim_{\mathrm{H}}\big(E_{\sigma,s'} \cap B(t,\rho)\big) = \begin{cases} \alpha\big(s - H(t)\big) + 1 & \text{if } s \in \big[H(t) - \frac{1}{\alpha}, H(t)\big]; \\ -\infty & \text{otherwise.} \end{cases} \qquad (3.22)$$

where $s = \sigma - s'$. Furthermore, the set $E_{\sigma,s'} \cap B(t,\rho)$ is empty for any $\sigma > H(t) - \frac{1}{\alpha}$ and $\rho > 0$ sufficiently small.

Theorem 3.8 extends the results presented in [174, 173], and also ensures that the localized multifractal spectrum is equal to

$$\forall t \in \mathbf{R}_+; \quad d_X(h,t) = \begin{cases} \alpha\big(h - H(t)\big) + 1 & \text{if } h \in \big[H(t) - \frac{1}{\alpha}, H(t)\big]; \\ -\infty & \text{otherwise.} \end{cases} \qquad (3.23)$$

Moreover, we observe that Proposition 3.10 and Theorem 3.8 still hold when the Hurst function $H(\cdot)$ is a continuous random process. Thereby, similarly to the works of BARRAL et al. [21] and DURAND [56], it provides a class stochastic processes whose spectrum of singularities, given by Equation (3.23), is non-homogeneous and random.

3.2 Lévy processes

In this section, X will designate a Lévy process parametrized by the generating triplet (a, Q, π). The Lévy-Itō decomposition states that it can represented as the sum of three independent processes B, N and Y, where B is a d-dimensional Brownian motion, N is a compound Poisson process with drift and Y is a Lévy process characterized by $\big(0, 0, \pi(\mathrm{d}x)\mathbf{1}_{\{\|x\|\leq 1\}}\big)$.

Without any loss of generality, we restrict the study to the time interval $[0, 1]$. Furthermore, as outlined in the introduction, we also assume that the Blumenthal–Getoor β is strictly positive. As noted by JAFFARD [94], the component N does not affect the regularity of X since its trajectories are piecewise linear with a finite number of jumps. Sample path properties of Brownian motion are well-known and therefore, we first focus in this section on the study of the jump process Y.

It is well-known that the process Y can be represented as a compensated integral with respect to a Poisson measure $J(\mathrm{d}t, \mathrm{d}x)$ of intensity $\mathscr{L}^1 \otimes \pi$:

$$Y_t = \lim_{\varepsilon\to 0}\left[\int_{[0,t]\times D(\varepsilon,1)} x\, J(\mathrm{d}s, \mathrm{d}x) - t\int_{D(\varepsilon,1)} x\, \pi(\mathrm{d}x)\right], \tag{3.24}$$

where for all $0 \leq a < b$, $D(a, b) := \{x \in \mathbf{R}^d : a < \|x\| \leq b\}$. Moreover, as presented in [168, Th. 19.2], the convergence is almost surely uniform on any bounded interval. In the rest of this section, for any $m \in \mathbf{R}_+$, Y^m will denote the Lévy process:

$$Y_t^m = \lim_{\varepsilon\to 0}\left[\int_{[0,t]\times D(\varepsilon,2^{-m})} x\, J(\mathrm{d}s, \mathrm{d}x) - t\int_{D(\varepsilon,2^{-m})} x\, \pi(\mathrm{d}x)\right]. \tag{3.25}$$

Finally, in the following proofs, c and C will denote positive constants which can change from a line to another. More specific constants will be written $c_1, c_2, \ldots$Furthermore, we will write $u_n \asymp v_n$ when there exists two constants c_1, c_2 independent of n such that $c_1\, v_n \leq u_n \leq c_2\, v_n$ for every $n \in \mathbf{N}$.

3.2.1 Pointwise exponent

We extend in this section the multifractal spectrum (3.9) to any Lévy process. To begin with, we prove two technical lemmas that will be extensively used in the rest of the article.

Lemma 3.1 *For any $\delta > \beta$, there exists a positive constant $c(\delta)$ such that for all $m \in \mathbf{R}_+$*

$$\mathbb{P}\left(\sup_{t\leq 2^{-m}} \left\|Y_t^{m/\delta}\right\|_1 \geq m2^{-m/\delta}\right) \leq c(\delta)e^{-m}.$$

Proof. Let $\delta > \beta$. We observe that for any $m \in \mathbf{R}_+$,

$$\left\{\sup_{t\leq 2^{-m}} \left\|Y_t^{m/\delta}\right\|_1 \geq m2^{-m/\delta}\right\} = \bigcup_{\varepsilon\in\{-1,1\}^d}\left\{\sup_{t\leq 2^{-m}} \langle \varepsilon, Y_t^{m/\delta}\rangle \geq m2^{-m/\delta}\right\}$$

Hence, it is sufficient to prove that there exists $c(\delta) > 0$ such that for any $\varepsilon \in \{-1, 1\}^d$,

$$\mathbb{P}\left(\sup_{t\leq 2^{-m}} \langle \varepsilon, Y_t^{m/\delta}\rangle \geq m2^{-m/\delta}\right) \leq c(\delta)e^{-m}.$$

Let $\lambda = 2^{m/\delta}$ and $M_t = e^{\lambda\langle\varepsilon, Y_t^{m/\delta}\rangle}$ for all $t \in \mathbf{R}_+$. According to Theorem 25.17 in [168], we have $\mathbb{E}[M_t] = \exp\{t\int_{D(0,2^{-m/\delta})}\big(e^{\lambda\langle\varepsilon,x\rangle} - 1 - \lambda\langle\varepsilon, x\rangle\big)\pi(\mathrm{d}x)\}$. Furthermore, we observe that

for all $s \le t \in \mathbf{R}_+$,

$$\mathbb{E}[M_t \mid \mathscr{F}_s] = M_s \exp\left\{(t-s)\int_{D(0,2^{-m/\delta})} \left(e^{\lambda\langle\varepsilon,x\rangle} - 1 - \lambda\langle\varepsilon,x\rangle\right)\pi(\mathrm{d}x)\right\} \ge M_s,$$

since for any $y \in \mathbf{R}$, $e^y - 1 - y \ge 0$. Hence, M is a positive submartingale, and using Doob's inequality (Theorem 1.7 in [161]), we obtain

$$\mathbb{P}\left(\sup_{t\le 2^{-m}} \langle \varepsilon, Y_t^{m/\delta}\rangle \ge m2^{-m/\delta}\right) = \mathbb{P}\left(\sup_{t\le 2^{-m}} M_t \ge e^m\right) \le e^{-m}\mathbb{E}[M_{2^{-m}}].$$

For all $y \in [-1,1]$, we note that $e^y - 1 - y \le y^2$. Thus, for any $m \in \mathbf{R}_+$,

$$\mathbb{E}[M_{2^{-m}}] \le \exp\left\{2^{-m}\int_{D(0,2^{-m/\delta})} \lambda^2\langle\varepsilon,x\rangle^2\pi(\mathrm{d}x)\right\} \le \exp\left\{2^{-m}\int_{D(0,2^{-m/\delta})} \lambda^2\|x\|^2\pi(\mathrm{d}x)\right\}.$$

If $\beta < 2$, let us set $\gamma > 0$ such that $\beta < \gamma < 2$ and $\gamma < \delta$. Then,

$$\begin{aligned}
2^{-m}\int_{D(0,2^{-m/\delta})} \lambda^2\|x\|^2\pi(\mathrm{d}x) &= 2^{-m(1-2/\delta)}\int_{D(0,2^{-m/\delta})} \|x\|^\gamma \cdot \|x\|^{2-\gamma}\pi(\mathrm{d}x) \\
&\le 2^{-m(1-2/\delta)}2^{-m/\delta(2-\gamma)}\int_{D(0,1)} \|x\|^\gamma\pi(\mathrm{d}x) \\
&= 2^{-m(1-\gamma/\delta)}\int_{D(0,1)} \|x\|^\gamma\pi(\mathrm{d}x) \le \int_{D(0,1)} \|x\|^\gamma\pi(\mathrm{d}x),
\end{aligned}$$

since $\gamma < \delta$. If $\beta = 2$, we simply observe that

$$2^{-m}\int_{D(0,2^{-m/\delta})} \lambda^2\|x\|^2\pi(\mathrm{d}x) \le 2^{-m(1-2/\delta)}\int_{D(0,1)} \|x\|^2\pi(\mathrm{d}x) \le \int_{D(0,1)} \|x\|^2\pi(\mathrm{d}x),$$

as $\delta > 2$. Therefore, there exists $c(\delta) > 0$ such that for all $m \in \mathbf{R}_+$, $\mathbb{E}[M_{2^{-m}}] \le c(\delta)$, concluding the proof of this lemma. □

Lemma 3.2 *Suppose $\delta > \beta$. Then, with probability one, there exist $c_1 > 0$ and $M(\omega) > 0$ such that*

$$\forall u, v \in [0,1] : |u-v| \le 2^{-m}; \quad \left\|Y_u^{m/\delta} - Y_v^{m/\delta}\right\| \le c_1\, m2^{-m/\delta} \tag{3.26}$$

for any $m \ge M(\omega)$.

Proof. We first note that for any $m \in \mathbf{R}_+$ and any $\delta > \beta$,

$$\begin{aligned}
&\left\{\sup_{u,v\in[0,1]:|u-v|\le 2^{-m}} \left\|Y_u^{m/\delta} - Y_v^{m/\delta}\right\| \ge 3m2^{-m/\delta}\right\} \\
&\subseteq \bigcup_{k=0}^{2^m-1}\left\{\sup_{t\le 2^{-m}} \left\|Y_{t+k2^{-m}}^{m/\delta} - Y_{k2^{-m}}^{m/\delta}\right\| \ge m2^{-m/\delta}\right\}.
\end{aligned}$$

Therefore, the stationarity of Lévy processes and Lemma 3.1 yield

$$\mathbb{P}\left(\sup_{u,v\in[0,1]:|u-v|\le 2^{-m}} \left\|Y_u^{m/\delta} - Y_v^{m/\delta}\right\| \ge 3m2^{-m/\delta}\right) \le 2^m c(\delta)e^{-m} = c(\delta)e^{-cm}.$$

Using the latter estimate and Borel–Cantelli lemma, we obtain Equation (3.26). □

Let us recall the definition of the collection of random sets $(A_\delta)_{\delta>0}$ introduced by JAFFARD [94]. For every $\omega \in \Omega$, $S(\omega)$ denotes the countable set of jumps of $Y_\cdot(\omega)$. Moreover, for any $\varepsilon > 0$, let A_δ^ε be

$$A_\delta^\varepsilon = \bigcup_{\substack{t\in S(\omega)\\ \|\Delta Y_t\|\le\varepsilon}} \left[t - \|\Delta Y_t\|^\delta, t + \|\Delta Y_t\|^\delta\right].$$

Then, the random set A_δ is defined by $A_\delta = \limsup_{\varepsilon\to 0^+} A_\delta^\varepsilon$. As noted in [94], if $t \in A_\delta$, we necessarily have $\alpha_{Y,t} \le \frac{1}{\delta}$. The other side inequality is obtained in the next statement which extends Proposition 2 from [94].

Proposition 3.9 *Suppose $\delta > \beta$. Then, with probability one, for all $t \in [0,1] \setminus S(\omega)$:*

$$t \notin A_\delta \quad \Longrightarrow \quad \alpha_{Y,t} \ge \tfrac{1}{\delta}.$$

Proof. Suppose $\omega \in \Omega$, $t \notin A_\delta$, $u \in [0,1]$ and $m \in \mathbf{N}$ such that $2^{-(m+1)\delta} \le |t-u| < 2^{-m\delta}$. Since $t \notin A_\delta$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$, $t \notin A_\delta^\varepsilon$. The component

$$\int_{[t,u]\times D(\varepsilon_0,1)} x\, J(\mathrm{d}s,\mathrm{d}x) - (u-t)\int_{D(\varepsilon_0,1)} x\,\pi(\mathrm{d}x)$$

is piecewise linear, and therefore does not influence the pointwise exponent $\alpha_{Y,t}$. Without any loss of generality, we may assume that $2^{-m} \le \varepsilon_0$. Then, for any jump ΔY_s such that $\|\Delta Y_s\| \in [2^{-m}, \varepsilon_0]$, we have $\|\Delta Y_s\|^\delta \ge 2^{-m\delta} \ge |t-u|$, implying that

$$\int_{[t,u]\times D(2^{-m},\varepsilon_0)} x\, J(\mathrm{d}s,\mathrm{d}x) = 0.$$

Furthermore, using Lemma 3.2, we obtain

$$\left\|Y_u^m - Y_t^m\right\| \le c\, m 2^{-m} \le c \log\left(|t-u|^{-1}\right)|t-u|^{1/\delta},$$

assuming that $|t-u|$ is sufficiently small. Therefore, the remaining term to estimate corresponds to $-(u-t)\int_{D(2^{-m},\varepsilon_0)} x\,\pi(\mathrm{d}x)$. To study the latter, we distinguish two different cases, depending on the polynomial component we subtract in Definition 3.1.

1. If $\delta \ge 1$, let us set $P_t \equiv 0$. Then,

$$\begin{aligned}\left\|(u-t)\int_{D(2^{-m},\varepsilon_0)} x\,\pi(\mathrm{d}x)\right\| &\le c\,|t-u| \int_{D(2^{-m},\varepsilon_0)} \|x\|^\delta \cdot \|x\|^{1-\delta}\,\pi(\mathrm{d}x)\\ &\le c\,|t-u|\cdot 2^{-m(1-\delta)} \int_{D(2^{-m},\varepsilon_0)} \|x\|^\delta\,\pi(\mathrm{d}x) \le c\,|t-u|^{1/\delta}.\end{aligned}$$

2. If $\delta < 1$ (and thus $\beta < 1$), we set $P_t(u) \equiv -(u-t)\int_{D(0,\varepsilon_0)} \pi(\mathrm{d}x)$, which corresponds to the linear drift of the Lévy process. We observe that $-(u-t)\int_{D(2^{-m},\varepsilon_0)} x\,\pi(\mathrm{d}x) - P_t(u) = (u-t)\int_{D(0,2^{-m})} x\,\pi(\mathrm{d}x)$. Then, similarly to the previous case, the latter satisfies

$$\begin{aligned}\left\|(u-t)\int_{D(0,2^{-m})} x\,\pi(\mathrm{d}x)\right\| &\le c\,|t-u| \int_{D(0,2^{-m})} \|x\|^\delta \cdot \|x\|^{1-\delta}\,\pi(\mathrm{d}x)\\ &\le c\,|t-u|\cdot 2^{-m(1-\delta)} \int_{D(0,2^{-m})} \|x\|^\delta\,\pi(\mathrm{d}x) \le c\,|t-u|^{1/\delta}.\end{aligned}$$

Therefore, owing to the previous estimates, we have proved that $\|Y_u - Y_t - P_t(u)\| \leq c_0 \log(|t-u|^{-1})|t-u|^{1/\delta}$, where the constant c_0 is independent of u. The latter inequality and Definition 3.1 prove that $\alpha_{Y,t} \geq \frac{1}{\delta}$. □

Proposition 3.9 ensures that almost surely

$$\forall h > 0; \quad E_h = \left(\bigcap_{\delta<1/h} A_\delta\right) \setminus \left(\bigcup_{\delta>1/h} A_\delta\right) \setminus S \quad \text{and} \quad E_0 = \left(\bigcap_{\delta>0} A_\delta\right) \cup S. \tag{3.27}$$

Furthermore, since the estimate of the Hausdorff dimension obtained in [94] does not rely on Assumption (3.8), the Lévy process Y satisfies with probability one

$$\forall V \in \mathcal{O}; \quad \dim_{\mathrm{H}}(E_h \cap V) = \begin{cases} \beta h & \text{if} \quad h \in [0, 1/\beta]; \\ -\infty & \text{otherwise.} \end{cases}$$

3.2.2 2-microlocal frontier of Lévy processes

We now aim to refine the multifractal spectrum of Lévy processes by studying their 2-microlocal structure. Let us begin with a few basics remarks and estimates on their 2-microlocal frontier. Firstly, according to [138, Th. 3.13], with probability one, for all $t \in [0,1]$ and for any $-s' < \alpha_{Y,t}$, the sample path $Y_\cdot(\omega)$ belongs to the 2-microlocal space $C_t^{0,s'}$. Furthermore, owing to the density of the set of jumps $S(\omega)$ in $[0,1]$, necessarily $Y_\cdot(\omega) \notin C_t^{\sigma,s'}$ for any $\sigma > 0$ and all $s' \in \mathbf{R}$. Hence, since the 2-microlocal frontier is a concave function with left- and right-derivatives between 0 and 1, with probability one and for all $t \in [0,1]$:

$$\forall s' \in \mathbf{R}_+; \quad \sigma_{Y,t}(s') \geq (\alpha_{Y,t} + s') \wedge 0 \quad \text{and} \quad \sigma_{Y,t}(s') \leq 0.$$

Therefore, we are interested in obtaining finer estimates of the negative component of the 2-microlocal frontier of Y. As outlined in the introduction and Definitions 3.2-3.3, we need to analyse the following type of increments in the neighbourhood of t:

$$\left\| \int_b^u (u-s)_+^{k-1} Y_s \, \mathrm{d}s - P_{t,k}(u) - \int_b^v (v-s)_+^{k-1} Y_s \, \mathrm{d}s + P_{t,k}(v) \right\| \tag{3.28}$$

where $b < t$ is fixed and $k \geq 1$. The polynomial component to be subtracted can be estimate using our work on pointwise exponent. Indeed, when $k = 0$, the $P_{t,0} \equiv P_t$ where the latter has been presented in the proof of Proposition 3.9,. Then, the consistency of the definition of the 2-microlocal spaces imposes that $P_{t,k-1}$ must correspond to the derivative of $P_{t,k}$. This last property shows us that the form of $P_{t,k}$ can be completely deduce from the known polynomial P_t.

For the sake of readability, we divide the proof of Theorem 3.1 and its corollaries in several different technical lemmas. To begin with, we give simple estimates on the jumps of a Lévy process.

Lemma 3.3 *For any $\varepsilon > 0$, there exists an increasing sequence $(m_n)_{n\in\mathbf{N}}$ such that with probability one, for all $t \in [0,1]$ and for every $n \geq N(\omega)$*

$$\exists u \in B(t, 2^{-m_n\alpha}); \quad \|\Delta Y_u\| \geq 2^{-m_n} \quad \textit{and} \quad J\left(B(u, 2^{-m_n\gamma}) \times D(2^{-m_n(1+\varepsilon)}, 1)\right) = 1,$$

where $\alpha = \beta(1-2\varepsilon)$ and $\gamma = \beta(1+4\varepsilon)$.

Proof. Suppose $m \in \mathbf{N}$, $\varepsilon > 0$, $\alpha = \beta(1-2\varepsilon)$ and $\gamma = \beta(1+4\varepsilon)$. Let I be an interval such that $I = I_1 \cup I_2 \cup I_3$, where I_1, I_2, I_3 are three consecutive and disjoint intervals of size $2^{-m_n\gamma}$. Then, we are interested in the following event:

$$\begin{aligned} A = &\{J\big(I_1, D(2^{-m},1)\big) = 0\} \cap \{J\big(I_3, D(2^{-m},1)\big) = 0\} \cap \\ &\{J\big(I_2, D(2^{-m},1)\big) = 1\} \cap \{J\big(I, D(2^{-m(1+\varepsilon)}, 2^{-m})\big) = 0\}, \end{aligned}$$

Since J is a Poisson measure, A corresponds to the intersection of independent events whose probability is equal to

$$\mathbb{P}(A) = 2^{-m\gamma}\pi(D(2^{-m},1)) \cdot \exp\big(-3 \cdot 2^{-m\gamma}\pi(D(2^{-m},1)) + 2^{-m\gamma}\pi(D(2^{-m(1+\varepsilon)}, 2^{-m}))\big).$$

As described in [34], β can be defined by $\beta = \inf\{\delta \geq 0 : \limsup_{r\to 0} r^{\delta}\pi\big(B(r,1)\big) < \infty\}$. Therefore, there exists $r_0 > 0$ such that for all $r \in (0, r_0]$, $\pi\big(B(r,1)\big) \leq r^{-\beta(1+\varepsilon)}$. Hence, for any $m \in \mathbf{N}$ sufficiently large:

$$\mathbb{P}(A) \geq 2^{-m\gamma}\pi(D(2^{-m},1)) \exp\big(-2^{-m\beta\varepsilon+1}\big) \geq 2^{-m\gamma-1}\,\pi\big(D(2^{-m},1)\big).$$

Furthermore, according to the definition of β, there also exists an increasing sequence $(m_n)_{n\in\mathbf{N}}$ such that for all $n \in \mathbf{N}$, $\pi(D(2^{-m_n},1)) \geq 2^{m_n\beta(1-\varepsilon)}$. Therefore, along this sequence, we obtain $\mathbb{P}(A) \geq 2^{-m_n 5\beta\varepsilon-1}$ for every $n \in \mathbf{N}$.

Let now consider an interval $\mathscr{I}$ of size $2^{-m_n\alpha}$. There exist at most $2^{-m_n\alpha+m_n\gamma}$ disjoint sub-intervals I of size $3\cdot 2^{-m_n\gamma}$. We designate by B the event where A is not satisfied by all these sub-elements I. Owing to the previous estimate of $\mathbb{P}(A)$ and the independence of these different events, we obtain

$$\mathbb{P}(B) = \big(P(A^c)\big)^{2^{m_n(\gamma-\alpha)}} \leq \big(1 - 2^{-m_n 5\beta\varepsilon-1}\big)^{2^{m_n(\gamma-\alpha)}}.$$

Note that $\gamma - \alpha = 6\beta\varepsilon$. Hence, $\log\big(\mathbb{P}(B)\big) \leq -2^{-m_n 5\beta\varepsilon-1} \cdot 2^{m_n 6\beta\varepsilon} = -2^{m_n\beta\varepsilon-1}$ and the probability $\mathbb{P}(B)$ satisfies $\mathbb{P}(B) \leq \exp\big(-2^{m_n\beta\varepsilon-1}\big)$.

Finally, we know there exist at most $2^{m_n\alpha+1}$ disjoint intervals J of size $2^{-m_n\alpha}$ inside $[0,1]$. We denote by B_n the event where B is satisfied for one of the previous interval $\mathscr{I}$. Since B_n is the reunion of events, we obtain

$$\mathbb{P}(B_n) \leq 2^{m_n\alpha+1} \cdot \exp\big(-2^{m_n\beta\varepsilon-1}\big) \leq \exp\big(-2^{m_n\beta\varepsilon-1} + c\, m_n\alpha\big).$$

Therefore, $\sum_{n\in\mathbf{N}} \mathbb{P}(B_n) < \infty$ and owing to Borel–Cantelli lemma, with probability one, there exists $N(\omega)$ such that for every $n \geq N(\omega)$, $\omega \in B_n^c$. The latter inclusion means that for every interval $\mathscr{I}$ previous defined, there exists a sub-element I such that the event A is satisfied on I, which proves this lemma. □

The previous lemma will help us to obtain a uniform upper bound on the 2-microlocal frontier of Y.

Lemma 3.4 *With probability one, for all $t \in [0,1]$, the 2-microlocal frontier of Y at t satisfies*

$$\forall s' \in \mathbf{R}; \quad \sigma_{Y,t}(s') \leq \left(\frac{1}{\beta} + s'\right) \wedge 0. \tag{3.29}$$

Proof. Let us first observe that to obtain an upper bound of the 2-microlocal frontier of the $\mathbf{R}^d$-valued $Y = (Y_1, \ldots, Y_d)$ process, it is sufficient to prove this bound holds for one the component Y_i. Furthermore, we also know that each of these components is a one-dimensional Lévy process and there exists $i \in \{1, \ldots, d\}$ such that the Blumenthal–Getoor exponent of Y_i is equal to β. Hence, considering these two remarks, we may assume without any loss of generality that we study only one component, and thus $d = 1$.

Let us set $t \in [0,1]$. We need to evaluate the size of the increments described in Equation (3.28). Hence, let us first determine the form of the local process $Y(u,k) := (I^k_{b+}Y)(u) - P_{t,k}(u)$ used. We know that when $k=0$, the polynomial component is described in Proposition 3.9, and thus we define the local process $Y(s,0)$ in the neighbourhood of t as following:

$$\forall u \in \mathbf{R}; \quad Y(u,0) = Y_u - Y_t - P_t(u).$$

Then, since the polynomial component must correspond to the Taylor development of the process at t, we define the elements $Y(\cdot,k)$ be induction:

$$\forall u \in \mathbf{R}; \quad Y(u,k) = \int_t^u Y(s,k-1)\,\mathrm{d}s.$$

One can easily verify that the derivative of $Y(\cdot,k)$ is $Y(\cdot,k-1)$ and $Y(t,k)=0$, proving that the Taylor development of $Y(\cdot,k)$ at t is $P \equiv 0$. Therefore, this construction procedure ensures that the difference between $Y(\cdot,k)$ and $I^k_{b+}Y$ corresponds to the polynomial function appearing in the definition of the 2-microlocal spaces.

Hence, we need to show in this proof that for any $k \in \mathbf{N}$, the increments of the process $Y(\cdot,k)$ are sufficiently large in the neighbourhood of t. More precisely, we will show by induction that there exist $t_{n,k} \to_n t$, $\rho_{n,k} > 0$ and $\delta_{n,k} > 0$ such that for every $k \in \mathbf{N}$ and all $n \in \mathbf{N}$:

$$\forall u \in (t_{n,k}, t_{n,k} + \rho_{n,k}); \quad |Y(u,k)| \geq \delta_{n,k}. \tag{3.30}$$

To initialize the induction with $k=0$, we make use of the estimate obtained in Lemma 3.3: there exists an increasing sequence $(m_n)_{n\in\mathbf{N}}$ such that with probability one, for all $t \in [0,1]$ and for every $n \geq N(\omega)$

$$\exists v \in B(t, 2^{-m_n\alpha}); \quad |\Delta Y_v| \geq 2^{-m_n} \quad \text{and} \quad J\big(B(v, 2^{-m_n\gamma}) \times D(2^{-m_n(1+\varepsilon)}, 1)\big) = 1,$$

where $\alpha = \beta(1-2\varepsilon)$ and $\gamma = \beta(1+4\varepsilon)$. Since the reasoning which follows is completely symmetric, we may assume without any loss of generality that $v \geq t$ and $\Delta Y_v \geq 0$. Let us set $n \geq N(\omega)$ and a proper $v \geq t$. We know there is no other jump of size greater than $2^{-m_n(1+\varepsilon)}$ inside the ball $B(v, 2^{-m_n\gamma})$. Therefore, for all $u \in B(v, 2^{-m_n\gamma})$,

$$Y_u - Y_v = -\Delta Y_v \mathbf{1}_{\{u<v\}} - (u-v)\int_{D(2^{-m_n(1+\varepsilon)},1)} x\,\pi(\mathrm{d}x) + Y_u^{m_n(1+\varepsilon)} - Y_v^{m_n(1+\varepsilon)}.$$

Furthermore, according to Lemma 3.2, the norm of the latter increment satisfies:

$$\left|Y_u^{m_n(1+\varepsilon)} - Y_v^{m_n(1+\varepsilon)}\right| \leq c_1\, m_n 2^{-m_n(1+\varepsilon)},$$

as we note that $|u-v| \leq 2^{-m_n\beta(1+4\varepsilon)} = 2^{-m_n(1+\varepsilon)\beta(1+4\varepsilon)/(1+\varepsilon)}$ with $\beta(1+4\varepsilon)/(1+\varepsilon) > \beta$. Then, similarly to the proof of Proposition 3.9, we need to distinguish two different cases.

1. If $\beta \geq 1$, $P_t \equiv 0$ and thus $Y(u,0) = Y_u - Y_t$. Let us first assume that $Y(v,0) \geq 2^{-m_n-1}$ and set $t_{n,0} = v$ and $\rho_{n,0} = 2^{-m_n\gamma}$. Then, for all $u \in (t_{n,0}, t_{n,0} + \rho_{n,0})$:

$$|Y(u,0)| \geq |Y(v,0)| - \left|Y_u^{m_n(1+\varepsilon)} - Y_v^{m_n(1+\varepsilon)}\right| - \left|(u-v)\int_{D(2^{-m_n(1+\varepsilon)},1)} x\,\pi(\mathrm{d}x)\right|.$$

Using the estimates presented in Proposition 3.9, we obtain an upper bound of the last term:

$$\left|(u-v)\int_{D(2^{-m_n(1+\varepsilon)},1)} x\,\pi(\mathrm{d}x)\right| \leq c\,2^{-m_n\gamma}\cdot 2^{-m_n(1+\varepsilon)(1-\beta(1+\varepsilon))} \leq c\,2^{-m_n(1+\varepsilon)},$$

since $-\gamma+\beta(1+\varepsilon)^2=-\beta(2\varepsilon-\varepsilon^2)<0$. Hence, $|Y(u,0)|\geq 2^{-m_n-1}-c\,2^{-m_n(1+\varepsilon)}\geq 2^{-m_n-2}$ for any n sufficiently large.

Let now assume that $Y(v,0)\leq 2^{-m_n-1}$. Since $\Delta Y_v\geq 2^{-m_n}$, we necessarily have $Y(v_-,0)\leq -2^{-m_n-1}$. Then, we set in this case $t_{n,0}=v-2^{-m_n\gamma}$ and $\rho_{n,0}=2^{-m_n\gamma}$, and obtain as well $|Y(u,0)|\geq 2^{-m_n-1}-c\,2^{-m_n(1+\varepsilon)}\geq 2^{-m_n-2}$.

2. If $\beta<1, P_t(u)\equiv-(u-t)\int_{D(0,1)}\pi(\mathrm{d}x)$. Similarly to the previous case, we first assume that $Y(v,0)\geq 2^{-m_n-1}$ and set $t_{n,0}=v$ and $\rho_{n,0}=2^{-m_n\gamma}$. Then, for all $u\in[t_{n,0},t_{n,0}+\rho_{n,0}]$:

$$|Y(u,0)|\geq|Y(v,0)|-\left|Y_u^{m_n(1+\varepsilon)}-Y_v^{m_n(1+\varepsilon)}\right|-\left|(u-v)\int_{D(0,2^{-m_n(1+\varepsilon)})}x\,\pi(\mathrm{d}x)\right|,$$

where the latter element satisfies

$$\left|(u-v)\int_{D(0,2^{-m_n(1+\varepsilon)})}x\,\pi(\mathrm{d}x)\right|\leq c\,2^{-m_n\gamma}\cdot 2^{-m_n(1+\varepsilon)(1-\beta(1+\varepsilon))}\leq c\,2^{-m_n(1+\varepsilon)}.$$

Hence, $|Y(u,0)|\geq 2^{-m_n-1}-c\,2^{-m_n(1+\varepsilon)}\geq 2^{-m_n-2}$ for any n sufficiently large. In the case $Y(v,0)\leq 2^{-m_n-1}$, we observe that $Y(v_-,0)\leq -2^{-m_n-1}$. Therefore, setting $t_{n,0}=v-2^{-m_n\gamma}$ and $\rho_{n,0}=2^{-m_n\gamma}$, we obtain $|Y(u,0)|\geq 2^{-m_n-1}-c\,2^{-m_n(1+\varepsilon)}\geq 2^{-m_n-2}$.

Therefore, in both cases, we have proved that

$$\forall u\in(t_{n,0},t_{n,0}+\rho_{n,0});\quad |Y(u,0)|\geq\delta_{n,0},$$

where $\rho_{n,0}=2^{-m_n\gamma}$, $\delta_{n,0}=2^{-m_n-1}$ and $B(t_{n,0},\rho_{n,0})\subset B(t,2^{-m_n\alpha+1})$.

Let now assume that Equation (3.30) is satisfied for $k\in\mathbf{N}$. Without any loss of generality, we may suppose that $Y(u,k)\geq\delta_{n,k}$ on the interval $(t_{n,k},t_{n,k}+\rho_{n,k})$ (otherwise, simply consider the process $-Y(u,k)$ in the following reasoning). In this case, the function $u\mapsto\int_t^u Y(s,k)\,\mathrm{d}s$ is strictly increasing on the previous interval.

Let us first assume that $\int_t^{t_{n,k}+\rho_{n,k}/2}Y(s,k)\,\mathrm{d}s\geq 0$. In this case, we set $t_{n,k+1}=t_{n,k}+3/4\rho_{n,k}$, $\rho_{n,k+1}=\rho_{n,k}/4$ and $\delta_{n,k+1}=\rho_{n,k}\delta_{n,k}/4$. Then, for all $u\in(t_{n,k+1},t_{n,k+1}+\rho_{n,k+1})$

$$\begin{aligned}Y(u,k+1)\geq\int_{t_{n,k}+\rho_{n,k}/2}^{u}Y(s,k)\,\mathrm{d}s&\geq(u-t_{n,k}-\rho_{n,k}/2)\,\delta_{n,k}\\&\geq\rho_{n,k}\delta_{n,k}/4=\delta_{n,k+1},\end{aligned}$$

In the other case $\int_t^{t_{n,k}+\rho_{n,k}/2}Y(s,k)\,\mathrm{d}s\leq 0$, we consider the set of parameters $t_{n,k+1}=t_{n,k}$, $\rho_{n,k+1}=\rho_{n,k}/4$ and $\delta_{n,k+1}=\rho_{n,k}\delta_{n,k}/4$. Then, for all $u\in(t_{n,k+1},t_{n,k+1}+\rho_{n,k+1})$

$$\begin{aligned}Y(u,k+1)\leq-\int_{u}^{t_{n,k}+\rho_{n,k}/2}Y(s,k)\,\mathrm{d}s&\leq-(t_{n,k}+\rho_{n,k}/2-u)\,\delta_{n,k}\\&\leq-\rho_{n,k}\delta_{n,k}/4=-\delta_{n,k+1},\end{aligned}$$

Therefore, assuming that Equation (3.30) holds for $k\in\mathbf{N}$, we have proved that it does too for $k+1$ with $\rho_{n,k+1}=\rho_{n,k}/4$, $\delta_{n,k+1}=\rho_{n,k}\delta_{n,k}/4$ and $B(t_{n,k+1},\rho_{n,k+1})\subset B(t_{n,k},\rho_{n,k})\subset B(t,2^{-m_n\alpha+1})$.

Finally, the lower bound on $Y_{n,k}$ presented in Equation (3.30) will now help us to obtain the expected bound on the 2-microlocal frontier. Owing to the previous definitions, for every

$k \in \mathbf{N}$, $|t_{n,k} - t| \leq 2^{-m_n \alpha + 1}$ and there exist $c_k > 0$ independent of $n \in \mathbf{N}$ such that $\delta_{n,k} = c_k\, 2^{-m_n k\gamma} \cdot 2^{-m_n}$. Hence, for every $n \in \mathbf{N}$,

$$|Y(t_{n,k}, k) - Y(t,k)| \geq \delta_{n,k} \geq c_k\, 2^{-m_n(1+k\gamma)} \geq c_k\, |t_{n,k} - t|^{(1+k\gamma)/\alpha},$$

where we recall that $\alpha = \beta(1-2\varepsilon)$ and $\gamma = \beta(1+4\varepsilon)$. Therefore, this last inequality proves that the pointwise exponent of $Y(\cdot,k)$ at t satisfies $\alpha_{Y(\cdot,k),t} \leq (1+k\gamma)/\alpha \to_{\varepsilon\to 0} 1/\beta + k$. Owing to the Definition 3.3 of the 2-microlocal spaces, this last inequality induces that with probability one, for any $t \in [0,1]$ and all $s' \in \mathbf{R}$, $\sigma_{Y,t}(s') \leq \left(\frac{1}{\beta} + s'\right) \wedge 0$. □

As we have obtained a uniform upper bound on the 2-microlocal frontier, we now study more precisely the regularity of Y at times where $\alpha_{Y,t} < 1/\beta$. To begin with, we prove a simple lemma related to the number of jumps inside an interval.

Lemma 3.5 *Suppose $\delta > \beta$, $\varepsilon > 0$ and $k \in \mathbf{N}$ are such that $\delta > \beta(1+2\varepsilon)(k+1)/k$. Then, with probability one, there exists $M(\omega)$ such that:*

$$\forall t \in [0,1]; \quad J\big(B(t, 2^{-m\delta}), D(2^{-m(1+\varepsilon)}, 1)\big) \leq k, \tag{3.31}$$

for every $m \geq M(\omega)$.

Proof. Let $m \in \mathbf{N}$ and I be an interval of size $2^{-m\delta+2}$. Since J is a Poisson random measure,

$$\mathbb{P}\big(J\big(I, D(2^{-m(1+\varepsilon)}, 1)\big) > k\big) = \exp(-\lambda_m)\left\{\sum_{\ell=k+1}^{+\infty} \frac{\lambda_m^\ell}{\ell!}\right\},$$

where $\lambda_m = 2^{-m\delta+2}\, \pi\big(D(2^{-m(1+\varepsilon)}, 1)\big) \leq 2^{-m\delta+2+m\beta(1+2\varepsilon)} \to 0$. Hence, we obtain the inequality $\mathbb{P}\big(J\big(I, D(2^{-m(1+\varepsilon)}, 1)\big) > k\big) \leq c\, \lambda_m^{k+1}$.

Considering a covering of the interval $[0,1]$ with $\lceil 2^{m\delta} \rceil$ overlapping sub-elements I of size $2^{-m\delta+2}$, we denote by B_m the event where at least one of these intervals has more than k jumps inside. Then,

$$\mathbb{P}(B_m) \leq c\, 2^{m\delta} \cdot \lambda_m^{k+1} \leq c\, 2^{m\delta - m\delta(k+1) + m\beta(k+1)(1+2\varepsilon)}.$$

Since $\delta k > \beta(1+2\varepsilon)(k+1)$, there exists $\gamma > 0$ such $\mathbb{P}(B_m) \leq c\, 2^{-m\gamma}$. Therefore, $\sum_{m\in\mathbf{N}} \mathbb{P}(B_m) < \infty$ and owing to Borel–Cantelli lemma, there exists $M(\omega)$ such that for every $m \geq M(\omega)$, $J\big(I, D(2^{-m(1+\varepsilon)}, 1)\big) \leq k$. Finally, since we consider intervals I of size $2^{-m\delta+2}$ covering $[0,1]$ and overlapping, we have proved that for all $t \in [0,1]$, $J\big(B(t, 2^{-m\delta}), D(2^{-m(1+\varepsilon)}, 1)\big) \leq k$. □

In the next lemma, we start with the study of the 2-microlocal frontier of Y at points $t \in [0,1]$ where $\alpha_{Y,t} \in [0, 1/2\beta]$.

Lemma 3.6 *With probability one, for all $h \in [0, 1/2\beta]$, the singularities of Y satisfy $\widetilde{E}_h = E_h$ and $\widehat{E}_h = \emptyset$, i.e. for all $t \in E_h$*

$$\forall s' \in \mathbf{R}; \quad \sigma_{Y,t}(s') = (\alpha_{Y,t} + s') \wedge 0.$$

Proof. Suppose $h \in [0, 1/2\beta]$ and $t \in E_h \setminus S(\omega)$ (t is not a jump time). Since we already know that $\sigma_{Y,t}(s') \geq (h + s') \wedge 0$, we need to only prove the other side inequality. For that purpose, we will proceed similarly to the proof of Lemma 3.4.

More precisely, let us set $\varepsilon > 0$ and $\delta > \max\big(2\beta(1+2\varepsilon), 1/(h+\varepsilon)\big)$. Since $t \in E_h$ and owing to Equation (3.27), there exist two sequences $(v_n)_{n\in\mathbf{N}}$ and $(m_n)_{n\in\mathbf{N}}$ such that

$$\forall n \in \mathbf{N}; \quad v_n \in B\big(t, 2^{-m_n/(h+\varepsilon)}\big) \quad \text{and} \quad \big\|\Delta Y_{v_n}\big\| \geq 2^{-m_n}.$$

Without any loss of generality, we may assume that $v_n \geq t$. Furthermore, owing to Lemma 3.5, $J\big(B(v_n, 2^{-m_n\delta}), D(2^{-m_n(1+\varepsilon)}, 1)\big) = 1$, i.e. there is no other jump larger than $2^{-m_n(1+\varepsilon)}$ in the neighbourhood of v_n. Then, similarly to the proof of Lemma 3.4, we need to distinguish two different cases.

1. If $\beta \geq 1$, $P_t \equiv 0$ and thus $Y(u,0) = Y_u - Y_t$. Consider $n \in \mathbf{N}$ and first assume that $\|Y(v_n,0)\| \geq 2^{-m_n-1}$. Let us also set $t_n = v_n$ and $\rho_n = 2^{-m_n\delta}$. Then, for all $u \in (t_n, t_n + \rho_n)$:

$$\|Y(u,0)\| \geq \|Y(v_n,0)\| - \left\|Y_u^{m_n(1+\varepsilon)} - Y_{v_n}^{m_n(1+\varepsilon)}\right\| - \left\|(u-v_n)\int_{D(2^{-m_n(1+\varepsilon)},1)} x\,\pi(\mathrm{d}x)\right\|.$$

   Still using the same estimates as in the proof of Lemma 3.5, we know that the last two terms are upper bounded by $c\,2^{-m_n(1+\varepsilon)}$, proving that $\|Y(u,0)\| \geq 2^{-m_n-2}$ for any n sufficiently large. The case $\|Y(v_n,0)\| \leq 2^{-m_n-1}$ is treated completely similarly, using $t_n = v_n - 2^{-m_n\delta}$ and $\rho_n = 2^{-m_n\delta}$.

2. If $\beta < 1$, $P_t(u) \equiv -(u-t)\int_{D(0,1)} \pi(\mathrm{d}x)$. Assuming first that $\|Y(v_n,0)\| \geq 2^{-m_n-1}$. we still set $t_n = v_n$ and $\rho_n = 2^{-m_n\delta}$. Then, for all $u \in (t_n, t_n + \rho_n)$:

$$\|Y(u,0)\| \geq \|Y(v_n,0)\| - \left\|Y_u^{m_n(1+\varepsilon)} - Y_{v_n}^{m_n(1+\varepsilon)}\right\| - \left\|(u-v_n)\int_{D(0,2^{-m_n(1+\varepsilon)})} x\,\pi(\mathrm{d}x)\right\|,$$

   As previously, the last two terms are upper bounded by $c\,2^{-m_n(1+\varepsilon)}$, proving that $\|Y(u,0)\| \geq 2^{-m_n-2}$ for any n sufficiently large. The case $\|Y(v_n,0)\| \leq 2^{-m_n-1}$ is treated similarly using $t_n = v_n - 2^{-m_n\delta}$ and $\rho_n = 2^{-m_n\delta}$.

Therefore, we have proved in both cases that for all $u \in (t_n, t_n + \rho_n)$, with n sufficiently large, $\|Y(u,0)\| \geq 2^{-m_n-2}$. Reproducing the same reasoning as in the proof of Lemma 3.4, there exists s_n such that for every $n \in \mathbf{N}$, $s_n \in B(t, 2^{-m_n/(h+\varepsilon)})$ and

$$\|Y(s_n,1)\| \geq c\,2^{-m_n} \cdot 2^{-m_n\delta} \geq |t - s_n|^{(h+\varepsilon)(1+\delta)}.$$

Hence, $\alpha_{Y(\cdot,1),t} \leq (h+\varepsilon)(1+\delta)$. Considering the limit $\varepsilon \to 0$ and $\delta \to 1/h$, we obtain $\alpha_{Y(\cdot,1),t} \leq h+1$. The latter inequality is sufficient to prove that $\sigma_{Y,t}(s') = (h+s') \wedge 0$ for all $s' \in \mathbf{R}$.

To conclude this proof, let us consider the case $t \in S(\omega)$. We observe that for all $u \geq t$,

$$\int_t^u Y_s\,\mathrm{d}s = (u-t)Y_t + \int_t^u (Y_s - Y_t)\,\mathrm{d}s \quad \text{with} \quad \left\|\int_t^u (Y_s - Y_t)\,\mathrm{d}s\right\| = \mathrm{o}(|t-u|),$$

as Y is right-continuous. Similarly, for all $u \leq t$, $\int_u^t Y_s\,\mathrm{d}s = (t-u)Y_{t-} + \mathrm{o}(|t-u|)$. Therefore, since $\Delta Y_t = Y_t - Y_{t-} \neq 0$, there does not exist a polynomial P_t which can cancel both terms $(u-t)Y_t$ and $(t-u)Y_{t-}$, proving that $\sigma_{Y,t}(s') = s' \wedge 0$ for all $s' \in \mathbf{R}$. □

In the last technical lemma, we focus on the particular case $\alpha_{Y,t} \in (1/2\beta, 1/\beta)$ and try to distinguish oscillating singularities from the common cusp behaviour.

Lemma 3.7 *With probability one, for all $h \in (1/2\beta, 1/\beta)$, Y satisfies*

$$\forall V \in \mathcal{O}; \quad \dim_{\mathrm{H}}(\widetilde{E}_h \cap V) = \beta h \quad \textit{and} \quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) \leq 2\beta h - 1\ (< \beta h). \tag{3.32}$$

Furthermore, for any $t \in \widehat{E}_h$ and all $s' \in \mathbf{R}$, $\sigma_{Y,t}(s') \leq (h+s')/2\beta h$.

Proof. On the contrary to the previous lemma, we know that some oscillating singularities might appear at a given time t. Hence, the first step in this proof is to isolate these behaviours and estimate the fractal dimension of the corresponding set of times.

For that purpose, let us set $\delta \in (\beta, 2\beta)$ and $\varepsilon > 0$. We are interested in the *double-jump* configurations, i.e. when two jumps greater than $2^{-m(1+\varepsilon)}$ are sufficiently close. More precisely, suppose I is an interval of size $2^{-m\delta+1}$ and p_m designates the probability of obtaining at least two jumps greater than $2^{-m(1+\varepsilon)}$ inside I. Then,

$$c\, 2^{-m2\delta} \cdot 2^{m2\beta} \leq p_m = \mathbb{P}\big(J\big(I, D(2^{-m(1+\varepsilon)}, 1)\big) \geq 2\big) \leq C\, 2^{-m2\delta} \cdot 2^{m2\beta(1+2\varepsilon)}$$

where we may assume that $\delta > \beta(1+2\varepsilon)$. We consider $\lceil 2^{m\delta} \rceil$ consecutive, but disjoint, intervals of size $2^{-m\delta+1}$ which are sufficient to cover $[0,1]$. Then, if we denote by N_m^1 the number of intervals with the previous configuration, it follows a Binomial distribution of parameters $\lceil 2^{m\delta} \rceil$ and p_m. Moreover, Chernoff's inequality induces that

$$\mathbb{P}\big(N_m^1 \geq c_0\, 2^{-m\delta+m2\beta(1+2\varepsilon)}\big) \leq \exp\big(-c\, 2^{-m\delta+m2\beta}\big),$$

where $2\beta > \delta$. Let us consider now the same configurations of intervals translated by $2^{-m\delta}$ and denote by N_m^2 the corresponding Binomial random variable. Owing to the previous estimate and Borel–Cantelli lemma, with probability one, there exists $M(\omega)$ such that for every $m \geq M(\omega)$, $N_m^1 \leq c_0\, 2^{-m\delta+m2\beta(1+2\varepsilon)}$ and $N_m^2 \leq c_0\, 2^{-m\delta+m2\beta(1+2\varepsilon)}$.

Then, let T_m index the previous intervals with a double-jump configuration and $F(\delta, \varepsilon)$ designate the following set:

$$F(\delta, \varepsilon) = \limsup_{m\to\infty} \bigcup_{I \in T_m} \big[c(I) - 2^{-m\delta+2}, c(I) + 2^{-m\delta+2}\big],$$

where $c(I)$ denotes the center of any interval $I \in T_m$. Using a simple covering based on intervals of size $2^{-m\delta}$, we can obtain an upper bound of the Hausdorff dimension of $F(\delta, \varepsilon)$. More precisely, for any $m_0 \in \mathbf{N}$, the series

$$\sum_{m=m_0}^{+\infty} c\,|T_m| \cdot \big(2^{-m\delta}\big)^{\gamma} \leq c \sum_{m=m_0}^{+\infty} 2^{-m(\delta(1+\gamma)-2\beta(1+2\varepsilon))},$$

converges when $\delta(1+\gamma) > 2\beta(1+2\varepsilon)$, i.e. $\gamma > 2\beta(1+2\varepsilon)/\delta - 1$. Therefore, $\dim_{\mathrm{H}} F(\delta, \varepsilon) \leq 2\beta(1+2\varepsilon)/\delta - 1$ almost surely.

Let now set $h \in (1/2\beta, 1/\beta)$. We aim to prove that $\widehat{E}_h \subset F(\delta, \varepsilon)$ for any $\delta < 1/h$ and $\varepsilon > 0$. For that purpose, we need to show that for every $t \in \widetilde{E}_h \setminus F(\delta, \varepsilon)$, the 2-microlocal frontier at t satisfies $\sigma_{Y,t}(s') \leq (h + s')$. As $t \in E_h$, there exist two sequences $(v_n)_{n\in\mathbf{N}}$ and $(m_n)_{n\in\mathbf{N}}$ such that

$$\forall n \in \mathbf{N}; \quad v_n \in B\big(t, 2^{-m_n/(h+\varepsilon)}\big) \quad \text{and} \quad \big\|\Delta Y_{v_n}\big\| \geq 2^{-m_n}.$$

We may assume that ε is sufficiently small to satisfy $2^{-m_n/(h+\varepsilon)} \leq 2^{-m\delta}$, i.e. $\delta < 1/(h+\varepsilon)$. Furthermore, since $t \notin F(\delta, \varepsilon)$, for every m sufficiently large, there is no double-jump configuration in the neighbourhood of t and v_n, meaning that $J\big(B(v_n, 2^{-m_n\delta}), D(2^{-m_n(1+\varepsilon)}, 1)\big) = 1$: there does not exist other jump larger than $2^{-m_n(1+\varepsilon)}$ in the neighbourhood of v_n. Therefore, we obtain the configuration presented in the proof of Lemma 3.6, and as the latter remains valid, we have

$$\forall s' \in \mathbf{R}; \quad \sigma_{Y,t}(s') \leq (h + s').$$

This upper bound shows that $\widehat{E}_h \subset F(\delta, \varepsilon)$, and considering the limits $\delta \to 1/h$ and $\varepsilon \to 0$, it induces the inequality $\dim_{\mathrm{H}} \widehat{E}_h \leq 2\beta h - 1$. Furthermore, since $2\beta h - 1 < \beta h$ and $E_h = \widetilde{E}_h \cup \widehat{E}_h$, we have also proved that $\dim_{\mathrm{H}} \widehat{E}_h = \beta h$.

To conclude this lemma, we obtain an upper bound of the 2-microlocal frontier in the case $t \in \widehat{E}_h$. Since the sketch of the proof is similar to Lemmas 3.4 and 3.6, we only present the main elements. Still using the previous two sequences $(v_n)_{n\in\mathbf{N}}$ and $(m_n)_{n\in\mathbf{N}}$, Lemma 3.5 induces that

$$J\left(B(v_n, 2^{-m_n 2\beta(1+3\varepsilon)}), D(2^{-m_n(1+\varepsilon)}, 1)\right) = 1.$$

Then, using the methodology presented in Lemma 3.6, there exists $(s_n)_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$, $s_n \in B(t, 2^{-m_n/(h+\varepsilon)})$ and

$$\|Y(s_n, 1)\| \geq c\, 2^{-m_n} \cdot 2^{-m_n 2\beta(1+3\varepsilon)} \geq |t - s_n|^{(h+\varepsilon)(1+2\beta(1+3\varepsilon))},$$

Hence, $\alpha_{Y(\cdot,1),t} \leq (h+\varepsilon)(1+2\beta(1+3\varepsilon)) \to_{\varepsilon\to 0} h(1+2\beta)$, and using the reasoning presented in Lemma 3.6, we obtain $\sigma_{Y,t} \leq (h+s')/2\beta h$ for all $s' \in \mathbf{R}$. □

Before finally proving Theorem 3.1 and its corollaries, we recall the following result on the increments of a Brownian motion. The proof can be found in [2] (inequality (8.8.26)).

Lemma 3.8 *Let B be a d-dimensional Brownian motion. There exists an event Ω_0 of probability one such that for all $\omega \in \Omega_0$, $\varepsilon > 0$, there exists $h(\omega) > 0$ such that for all $\rho \leq h(\omega)$ and $t \in [0,1]$, we have*

$$\sup_{u,v\in B(t,\rho)} \{\|B_u - B_v\|\} \geq \rho^{1/2+\varepsilon}.$$

Proof of Theorem 3.1. We use the notations introduced at the beginning of the section. As previously said, the compound Poisson process N can be ignored since it does not influence the final regularity. Furthermore, if $Q = 0$, and therefore $B \equiv 0$ and $\beta' = \beta$, Lemmas 3.4, 3.6 and 3.7 on the component Y yields Theorem 3.1.

Otherwise, the Lévy process X corresponds to the sum of the Brownian motion B and the jump component Y. Still using Lemmas 3.4, 3.6 and 3.7, it is sufficient to prove that with probability one and for all $t \in [0,1]$, $\sigma_{X,t} = \sigma_{B,t} \wedge \sigma_{Y,t}$. Owing to the definition of 2-microlocal frontier, we already know that $\sigma_{X,t} \geq \sigma_{B,t} \wedge \sigma_{Y,t}$. Furthermore, when $\sigma_{B,t}(s') \neq \sigma_{Y,t}(s')$, the upper bound is straightforward, and thus $\sigma_{X,t}(s') = \sigma_{B,t}(s') \wedge \sigma_{Y,t}(s')$.

Therefore, to obtain Theorem 3.1, we have to prove that with probability one, for all $t \in [0,1]$, $\sigma_{X,t} \leq \sigma_{B,t} = s' \mapsto (1/2 + s') \wedge 1/2$. For that purpose, we distinguish two different cases.

1. If $\beta' = \beta = 2$, we only need to slightly modify the proof of Lemma 3.4. More precisely, owing to Lévy's modulus of continuity, the increments of the Brownian motion satisfy:

$$\forall u, v : |u - v| \leq 2^{-m_n\gamma}; \quad \|B_u - B_v\| \leq c\, m_n 2^{-m_n\gamma/2} = c\, m_n 2^{-m_n(1+4\varepsilon)},$$

   since $\gamma = \beta(1+4\varepsilon)$. Therefore, the term due to the increments of the Brownian motion does not influence the rest of the estimates presented in the proof, ensuring that $\sigma_{X,t}(s') \leq (1/2 + s') \wedge 0$, for all $s' \in \mathbf{R}$.

2. If $\beta < 2$, let $\delta = 2$ and $\varepsilon > 0$. According to Lemma 3.5, there exist $k \in \mathbf{N}$ and $M(\omega) \in \mathbf{N}$ such that for all $m \geq M$, there are at most k jumps of size greater than $2^{-m(1+3\varepsilon)}$ in any interval of size $2^{-m\delta}$. Hence, there always exists a sub-interval I of size $c_0\, 2^{-\delta m}$ with no jump greater than $2^{-m(1+3\varepsilon)}$ inside.

   Still using Lemma 3.2, we know that for all $m \geq M(\omega)$

$$\forall u, v \in [0,1] : |u - v| \leq 2^{-\delta m}; \quad \left\|Y_u^{m(1+3\varepsilon)} - Y_v^{m(1+3\varepsilon)}\right\| \leq c_1 m 2^{-m(1+3\varepsilon)}.$$

Let us set $t \in [0,1]$ and I be one of the previous interval of size $c_0\, 2^{-\delta m}$. According to Lemma 3.8, there exist $u, v \in I$ such that $\|B_u - B_v\| \geq c_0\, 2^{-m(1+2\varepsilon)}$. Then,

$$\|X_u - X_v\| \geq \|B_u - B_v\| - \left\|Y_u^{m(1+3\varepsilon)} - Y_v^{m(1+3\varepsilon)}\right\| - |u-v| \cdot \left\|\int_{D(2^{-m(1+3\varepsilon)},1)} x\, \pi(\mathrm{d}x)\right\|,$$

where $|u-v| \cdot \left\|\int_{D(2^{-m(1+3\varepsilon)},1)} x\, \pi(\mathrm{d}x)\right\| \leq c\, 2^{-m(1+3\varepsilon)}$. Hence, we obtain a lower bound of the increments on the interval I, ensuring that the rest of the proof presented in Lemma 3.5 holds similarly.

□

Proof of Corollary 3.2. Recall that $\beta^w_{X,t} = \lim_{s'\to-\infty} \sigma_{X,t}(s') - s'$. Hence, using the global upper bound on the 2-microlocal frontier proved in Theorem 3.1, we know that $\beta^w_{X,t} \leq 1/\beta'$ with probability one. In addition, owing to the geometrical properties of the frontier, we observe that for every $h \in [0, 1/\beta']$

$$\forall h \in [0, 1/\beta'); \quad \widetilde{E}_h \subseteq E^w_h \subseteq \widetilde{E}_h \cup \bigcup_{h'<h} \widehat{E}_{h'}.$$

The first inclusion clearly shows that $\dim_{\mathrm{H}}(E^w_h \cap V) \geq \dim_{\mathrm{H}}(\widetilde{E}_h \cap V) = \beta h$. In addition, we also know that for every $h' < h$, $\dim_{\mathrm{H}} \widehat{E}_{h'} \leq 2\beta h' - 1 < \beta h$, which proves the other side inequality.

To obtain the upper bound on the oscillating exponent, we only need to note that according to its characterisation using the 2-microlocal frontier,

$$\forall h \in [0, 1/\beta'); \quad \{t \in E_h : \beta^o_{X,t} > 0\} = \widehat{E}_h.$$

Finally, the chirp exponent is equal to one because of the upper bound $\sigma_{X,t}(s') \leq (1/\beta' + s')$. □

Proof of Corollary 3.3. Owing to upper bound on the 2-microlocal frontier obtained in Theorem 3.1, the case $\sigma = 0$ corresponds to the classic spectrum of singularity. Hence, let us set $\sigma < 0$. We recall that s denotes the parameter $\sigma - s'$. If $s \geq 1/\beta'$ or $s < 0$, the result is straight forward using Theorem 3.1 and properties of the 2-microlocal frontier.

Therefore, we suppose that $s \in [0, 1/\beta')$ and note that $E_{\sigma,s'} = \{t \in \mathbf{R}_+ : \sigma_{X,t}(s') = \sigma\}$, since the negative component of the 2-microlocal frontier of X can not be constant. Hence, similarly to the previous corollary, $E_{\sigma,s'}$ satisfies

$$\forall s \in [0, 1/\beta'); \quad \widetilde{E}_s \subseteq E_{\sigma,s'} \subseteq \widetilde{E}_s \cup \bigcup_{h<s} \widehat{E}_h.$$

These two inclusions lead to the same estimates, and therefore the expected equality on the Hausdorff dimension. □

3.2.3 Oscillating singularities of some classes of Lévy processes

In this section, we aim to understand more precisely the oscillating singularities of Lévy processes captured by the collection of sets $(\widehat{E}_h)_{h\in\mathbf{R}_+}$. Note that to simply our presentation, we assume that $d = 1$.

Let us begin with the proof of Proposition 3.4 where we present a class Lévy processes with no chirp oscillations. Recall that in this case, we consider Lévy measures such that $\pi(\mathbf{R}_\pm) = 0$.

Proof of Proposition 3.4. In order to prove that $\widehat{E}_h = \emptyset$ for all $h \in \mathbf{R}_+$, we extend Lemma 3.6 to any $h \in [0, 1/\beta)$. We may assume without any loss of generality that $\pi(\mathbf{R}_-) = 0$. We still consider the two sequences $(v_n)_{n\in\mathbf{N}}$ and $(m_n)_{n\in\mathbf{N}}$ such that

$$\forall n \in \mathbf{N}; \quad v_n \in B\big(t, 2^{-m_n/(h+\varepsilon)}\big) \quad \text{and} \quad \big|\Delta Y_{v_n}\big| \geq 2^{-m_n}.$$

where we suppose that $v_n \geq t$. We first assume that $\beta \geq 1$ and set $t_n = v_n$ and $\rho_n = 2^{-m_n\delta}$. Then, since the Lévy process only has positive jumps, we necessarily have for all $u \in [t_n, t_n + \rho_n]$,

$$|Y(u,0)| \geq |Y(v_n,0)| - \big|Y_u^{m_n(1+\varepsilon)} - Y_{v_n}^{m_n(1+\varepsilon)}\big| - \left|(u - v_n)\int_{D(2^{-m_n(1+\varepsilon)},1)} x\,\pi(\mathrm{d}x)\right|,$$

According to the proof presented in Lemma 3.6, this inequality is sufficient to show that $\sigma_{X,t}(s') = (\alpha_{X,t} + s') \wedge 0$. The case $\beta \leq 1$ is then treated completely similarly. □

Proposition 3.4 proves in particular that Lévy subordinators, in which case $\beta \leq 1$, only have cusp singularities.

The second important class of Lévy processes we consider are characterised by the following Lévy measure

$$\pi(\mathrm{d}x) = a_1\,|x|^{-1-\alpha_1}\,\mathbf{1}_{\mathbf{R}_+}\mathrm{d}x + a_2\,|x|^{-1-\alpha_2}\,\mathbf{1}_{\mathbf{R}_-}\mathrm{d}x,$$

where $a_1, a_2 > 0$ and $\alpha_1, \alpha_2 \in (0,2)$.

The proof of Theorem 3.5 is rather technical and will be divided in several parts for the sake of readability. To begin with, we present two technical lemmas related to the Binomial distribution. Recall that Chernoff's inequality states that for any $\varepsilon \in (0,1)$,

$$\mathbb{P}\big(N \leq np(1-\varepsilon)\big) \leq \exp\big(-np\,\varepsilon^2/2\big). \tag{3.33}$$

and

$$\mathbb{P}\big(N \geq np(1+\varepsilon)\big) \leq \exp\big(-np\,\varepsilon^2/2\big). \tag{3.34}$$

where N follows a Binomial distribution of parameters n and p.

Lemma 3.9 *Suppose N follows a Binomial distribution with parameters n and p. Then, there exists $c > 0$ such that when $n > c$ and $p < 1/c$,*

$$\begin{aligned}&\mathbb{P}\Big(\#\{\textit{empty intervals of size} \geq 1/p\,\mathrm{LL}(1/p)\} \geq np\,\mathrm{LL}(1/p)h(p)^4\Big)\\ &\geq 1 - \exp\big(-np/8\,\mathrm{LL}(1/p)^2\big),\end{aligned}$$

using the following notations: $\mathrm{LL}(1/p) := \log(\log(1/p))$ *and* $h(p) := \exp(-1/\mathrm{LL}(p))$.

Proof. To obtain this lower bound, we first estimate the probability of obtaining an empty interval of size of $l_0 = \lfloor 1/p\,\mathrm{LL}(1/p)\rfloor$. Setting $p_0 := (1-p)^{l_0}$, we note that

$$\log(p_0) \geq \frac{1}{p\,\mathrm{LL}(1/p)}\log(1-p) = \frac{-2}{\mathrm{LL}(1/p)}.$$

Therefore, we obtain $p_0 \geq h(p)^2$ when c is sufficiently large.

Let n_0 denote the number of disjoint sub-intervals of size l_0 and N_0 follow a Binomial distribution of parameters n_0 and p_0. Owing to Chernoff's inequality,

$$\mathbb{P}\big(N_0 \geq n_0 p_0\,h(p)\big) \geq 1 - \exp\big(-n_0 p_0\,g(p)^2/2\big),$$

where $g(p) := 1 - h(p)$. Note that $n_0 \geq np\,\mathrm{LL}(1/p)h(p)$ and $g(p) \asymp 1/\mathrm{LL}(1/p)$ when c is large enough. Therefore, using the previous estimates,

$$\mathbb{P}\big(N_0 \geq np\,\mathrm{LL}(1/p)h(p)^4\big) \geq 1 - \exp\big(-np/8\,\mathrm{LL}(1/p)^2\big),$$

proving the lemma. □

Lemma 3.10 *Suppose N follows a Binomial distribution with parameters n and p. Then, there exists $c > 0$ such that when $n > c$ and $p < 1/c$*

$$\mathbb{P}\Big(\#\{\textit{successes spaced by} \geq 1/p\} \geq np/6\Big) \geq 1 - \exp\big(-np/16\big).$$

Proof. The sketch of the proof is similar to Lemma 3.9. The set $\{1, \dots, n\}$ can be divided in n_0 intervals of size $l_0 = \lceil 1/p \rceil$. The probability p_0 of obtaining at least a success in one of these intervals is equal to:

$$p_0 = 1 - (1-p)^{1/p} \longrightarrow_{p\to 0} 1 - \mathrm{e}^{-1} \geq 1/2.$$

Furthermore, only considering one third of the previous intervals, i.e. $n_0/3$, we consider the Binomial distribution $B(n_0/3, p_0)$. Still using Chernoff's inequality, we obtain

$$\mathbb{P}\big(N_0 \geq np/6\big) \geq 1 - \exp\big(-np/16\big),$$

which concludes the proof. □

Proof of Theorem 3.5. As observed in the proof of Theorem 3.1, chirp singularities appear when a compensation phenomena between jumps exists. Hence, the main goal of the proof is to characterise in more details this particular behaviour in the case of the Lévy measure we consider. Firstly, we clearly note the Blumenthal–Getoor exponent β of π is equal to $\max(\alpha_1, \alpha_2)$.

Hausdorff dimension (upper-bound). To obtain a tighter upper bound of the Hausdorff dimension, we need to enhance the estimates presented in Lemma 3.7. We have observed in the proof of Proposition 3.4 that oscillating singularities do not appear when there are jumps of the same size. Hence, we are interested in the *double-jump* configurations with jumps of opposite signs.

Suppose $\delta \in (\beta, \alpha_1+\alpha_2)$, $\varepsilon > 0$ and I is an interval of size $2^{-j\delta+1}$. Hence, we are interested in the following type of configurations: $J\big(I, (2^{-j(1+\varepsilon)}, 1]\big) \geq 1$ and $J\big(I, [-1, -2^{-j(1+\varepsilon)})\big) \geq 1$. The probability p_j of such an event satisfies:

$$p_j \asymp 2^{-j\delta} 2^{j\alpha_1(1+\varepsilon)} \cdot 2^{-j\delta} 2^{j\alpha_2(1+\varepsilon)} = 2^{-j2\delta + j(\alpha_1+\alpha_2)(1+\varepsilon)}.$$

Using this probability, the rest of the proof is rather similar to Lemma 3.7. We consider $\lceil 2^{j\delta} \rceil$ consecutive, but disjoint, intervals of size $2^{-j\delta+1}$ sufficient to cover $[0,1]$ and we denote by N_j^1 the number of intervals with the previous configuration. Owing to Chernoff's inequality,

$$\mathbb{P}\big(N_j^1 \geq c_0\, 2^{-j\delta + j(\alpha_1+\alpha_2)(1+\varepsilon)}\big) \leq \exp\big(-c\, 2^{-j\delta + j(\alpha_1+\alpha_2)}\big),$$

where $(\alpha_1+\alpha_2) > \delta$. Consider now the same configurations of intervals translated by $2^{-j\delta}$ and denote by N_k^2 the corresponding Binomial random variable. Using Borel–Cantelli lemma, with probability one, there exists $M(\omega)$ such that for every $j \geq M(\omega)$, $N_j^1 \leq c_0\, 2^{-j\delta + j(\alpha_1+\alpha_2)(1+\varepsilon)}$ and $N_j^2 \leq c_0\, 2^{-j\delta + j(\alpha_1+\alpha_2)(1+\varepsilon)}$. Using the same notation T_j, we define

$$F(\delta, \varepsilon) = \limsup_{j\to\infty} \bigcup_{I \in T_j} \big[c(I) - 2^{-j\delta+2}, c(I) + 2^{-j\delta+2}\big],$$

where $c(I)$ denotes the center of any interval $I \in T_j$. Then, for any $j_0 \in \mathbf{N}$

$$\sum_{j=j_0}^{+\infty} c\,|T_j| \cdot \left(2^{-j\delta}\right)^{\gamma} \leq c \sum_{j=j_0}^{+\infty} 2^{-j(\delta(1+\gamma)-(\alpha_1+\alpha_2)(1+\varepsilon))},$$

is finite when $\delta(1+\gamma) > (\alpha_1+\alpha_2)(1+\varepsilon)$, i.e. $\gamma > (\alpha_1+\alpha_2)(1+\varepsilon)/\delta - 1$. Therefore, $\dim_{\mathrm{H}} F(\delta,\varepsilon) \leq (\alpha_1+\alpha_2)(1+\varepsilon)/\delta - 1$.

The rest of the proof of Lemma 3.7 does not change, proving that for any $\delta < 1/h$ and $\varepsilon > 0$, $\widehat{E}_h \subset F(\delta,\varepsilon)$. Therefore, with probability one,

$$\forall h \in \left(1/(\alpha_1+\alpha_2), 1/\beta\right); \quad \dim_{\mathrm{H}} \widehat{E}_h \leq (\alpha_1+\alpha_2)h - 1.$$

Finally, when $h \notin \left(1/(\alpha_1+\alpha_2), 1/\beta\right)$, the proof of Lemma 3.6 can also be similarly adapted to prove that $\widehat{E}_h = \emptyset$ with probability one.

Construction (lower-bound). In order to prove the lower bound of the Hausdorff dimension, we need to construct a proper set of times which will show oscillating singularities.

For our construction procedure, we will need a set of parameters $\boldsymbol{p} = (\delta, \delta', \delta'', \gamma, \rho)$ such that $\delta' < \delta < \delta'' \in (\beta, \alpha_1+\alpha_2)$, $\delta < \gamma \in (\beta, \alpha_1+\alpha_2)$, $\delta' < \sqrt{\beta\delta}$ and $\delta > \sqrt{\delta'\delta''}$. In addition, we also define the sequence $j_n = (\delta/\delta')^n \to \infty$.

Our first step is to construct collections of intervals such that for every t inside, there is no jump of size 2^{-j} or greater closer than $2^{-j\delta}$ for all $j \leq j_0$, where j_0 is a given value. More precisely, we define by induction a collection, indexed by the random variables $\mathscr{S}_n$, of disjoint intervals of size $2^{-j_n\delta}$ in the following way. Suppose $\mathscr{S}_n$ is defined such that for every t inside an interval, there is no jump greater than $2^{-j_{n+1}}$ closer than $2^{-j_n\delta}$ of t. In every interval of size $2^{-j_n\delta}$, we consider consecutive sub-intervals of size $2^{-j_{n+1}\delta}$ with no jumps greater than $2^{-j_{n+2}}$ inside. Removing the left and right elements of these collections, we obtain the family $\mathscr{S}_{n+1}$, therefore corresponding to the offspring of $\mathscr{S}_n$. Owing to this construction procedure, we know that the remaining intervals satisfy the expected property, i.e. for any t inside, there is no jump greater than $2^{-j_{n+2}}$ closer than $2^{-j_{n+1}\delta}$.

In order to determine the number of this kind of intervals, we need to estimate the law of $|\mathscr{S}_{n+1}|$ conditionally to $|\mathscr{S}_n|$. For any $n \in \mathbf{N}$, let us denote by p_{n+1} the probability of obtaining at least one jump greater $2^{-j_{n+2}}$ inside an interval of size $2^{-j_{n+1}\delta}$. Note that for every n sufficiently large, an interval of size $2^{-j_n\delta}$ can be divided in at least $2^{-j_n\delta}/2^{-j_{n+1}\delta}h(p_{n+1}) = 2^{j_{n+1}(\delta-\delta')}h(p_{n+1})$ sub-intervals. Furthermore, let M_{n+1} designate the following random variable:

$$M_{n+1} = \#\{\text{family} \geq 1/p_{n+1}\,\mathrm{LL}(1/p_{n+1}) \text{ of consecutive empty intervals of size } 2^{-j_{n+1}}\}.$$

According to Lemma 3.9,

$$\begin{aligned}&\mathbb{P}\left(M_{n+1} \geq s_0\, 2^{j_{n+1}\delta} p_{n+1}\,\mathrm{LL}(1/p_{n+1})h(p_{n+1})^5 \;\middle|\; |\mathscr{S}_n| \geq s_0\, 2^{j_n\delta}\right)\\ &\geq 1 - \exp\left(-s_0\, 2^{j_{n+1}\delta} p_{n+1}/8\,\mathrm{LL}(1/p_{n+1})^2\right),\end{aligned}$$

for any $s_0 \in \mathbf{R}_+$ such that $s_0\, 2^{j_n\delta} \geq 1$. As previously outlined, for every collection of consecutive empty intervals, we remove the extremal elements to constitute the family $\mathscr{S}_{n+1}$. Noting that $1/p_{n+1}\,\mathrm{LL}(1/p_{n+1}) - 2 \geq h(p_{n+1})/p_{n+1}\,\mathrm{LL}(1/p_{n+1})$ for any n sufficiently large, we therefore obtain

$$\begin{aligned}&\mathbb{P}\left(|\mathscr{S}_{n+1}| \geq s_0\, 2^{j_{n+1}\delta} h(p_{n+1})^6 \;\middle|\; |\mathscr{S}_n| \geq s_0\, 2^{j_n\delta}\right)\\ &\geq 1 - \exp\left(-s_0\, 2^{j_{n+1}\delta} p_{n+1}/8\,\mathrm{LL}(1/p_{n+1})^2\right).\end{aligned}$$

Furthermore, the probability of obtaining an empty interval of size $2^{-j_{n+1}\delta}$ is equal to:

$$\begin{aligned} q_{n+1} &:= \mathbb{P}\big(J\big([0, 2^{-j_{n+1}\delta}] \times D\big(2^{-j_{n+2}}, 2^{-j_{n+1}}\big)\big) = 0\big) \\ &\asymp \exp\big(-c\, 2^{j_{n+2}\beta - j_{n+1}\delta}\big) = \exp\big(-c\, 2^{-j_{n+1}\delta(1-\beta/\delta')}\big) \longrightarrow 1. \end{aligned}$$

Hence, $p_{n+1} \asymp 2^{-j_{n+1}\delta(1-\beta/\delta')}$ for any n sufficiently large, and there exists $c_1 > 0$ such that

$$\mathbb{P}\big(|\mathscr{S}_{n+1}| \geq s_0\, 2^{j_{n+1}\delta} h(p_{n+1})^6 \,\big|\, |\mathscr{S}_n| \geq s_0\, 2^{j_n\delta}\big) \geq 1 - \exp\big(-c_1\, s_0\, 2^{j_{n+1}\beta\delta/\delta'}/(n+1)^2\big).$$

Furthermore, note $2^{j_{n+1}\beta\delta/\delta' - j_n\delta} = 2^{j_{n+1}(\beta\delta/\delta' - \delta')}$. Since we have assumed that $\delta' < \sqrt{\beta\delta}$ and $s_0\, 2^{j_n\delta} \geq 1$, there exists $r > 0$ such that

$$\mathbb{P}\big(|\mathscr{S}_{n+1}| \geq s_0\, 2^{j_{n+1}\delta} h(p_{n+1})^6 \,\big|\, |\mathscr{S}_n| \geq s_0\, 2^{j_n\delta}\big) \geq 1 - \exp\big(-2^{j_{n+1}r}\big).$$

Therefore, by induction, the law of $|\mathscr{S}_{n+m}|$ satisfies

$$\mathbb{P}\left(|\mathscr{S}_{n+m}| \geq s_0\, 2^{j_{n+m}\delta} \prod_{k=1}^{m} h(p_{n+k})^6 \,\middle|\, |\mathscr{S}_n| \geq s_0\, 2^{j_n\delta}\right) \geq \prod_{k=1}^{m}\Big(1 - \exp\big(-2^{j_{n+k}r}\big)\Big).$$

Finally, we note that $\prod_{k=1}^{m} h(p_{n+k})^6 = \exp\big(-c\sum_{k=1}^{m} 1/(n+k)\big) \geq \exp\big(-c\log(n+m)\big)$, implying that

$$\mathbb{P}\left(|\mathscr{S}_{n+m}| \geq s_0\, 2^{j_{n+m}\delta - c_4\log(n+m)} \,\middle|\, |\mathscr{S}_n| \geq s_0\, 2^{j_n\delta}\right) \geq \prod_{k=1}^{m}\Big(1 - \exp\big(-2^{j_{n+k}r}\big)\Big), \tag{3.35}$$

where c_4 is a constant independent of n and m.

The previous bound gives us an estimate of the probability of obtaining intervals without any jump in a given neighbourhood. Using this estimate, we will be able to construct our main collection of nested intervals indexed by $\mathscr{T}$ such that a most scales 2^{-j_n}, there is no jump in the neighbourhood, and at specific ones $2^{-j_{n(\ell)}}$, a particular double-jump configuration appears. To construct this collection, let us first define this sequence $\big(n(\ell)\big)_{\ell\in\mathbf{N}}$:

$$n(0) = 1 \quad \text{and} \quad n(\ell+1) = 2^{n(\ell)} \quad \forall \ell \in \mathbf{N}.$$

For every $\ell \in \mathbf{N}$, we are interested in the following type of configuration: in an interval of size $2^{-j_{n(\ell)}\delta}/3$, there exist two jumps ΔX_u and ΔX_v of opposite sign inside the middle third and such that $|u - v| \leq 2^{-j_{n(\ell)}\gamma}$, $|\Delta X_u|, |\Delta X_v| \in [2^{-j_{n(\ell)}-1}, 2^{-j_{n(\ell)}}]$ and $|\Delta X_v - \Delta X_u| \leq 2^{-j_{n(\ell)}\rho}$. Using the independence property on the Poisson measure J, the probability r_ℓ of the previous event can be lower bounded by

$$\begin{aligned} r_\ell &\geq c_2 \exp\big(-3c_0\, 2^{-j_{n(\ell)}\delta + j_{n(\ell)+1}\beta}\big) \cdot 2^{-j_{n(\ell)}(\delta-\alpha_1)} \cdot 2^{-j_{n(\ell)}(\gamma-\alpha_2-1+\rho)} \\ &\geq c_3\, 2^{-j_{n(\ell)}(\delta+\gamma-\alpha_1-\alpha_2-1+\rho)}, \end{aligned}$$

where c_2, c_3 are positive constants.

The collection of intervals T_ℓ is constructed by induction. T_0 is initialised with the singleton corresponding to the interval $[0, 1]$. Then, assuming T_ℓ is defined, for any $I \in T_\ell$, we consider the sub-intervals of size $2^{-j_{n(\ell+1)}\delta}$ with a double-jump configuration and with no jump in the neighbourhood at all intermediate scales 2^{-j_n}, $n(\ell) < n < n(\ell+1)$. Note that if none satisfy the previous conditions, we avoid the extinction of the tree by selecting just one random sub-interval of size $2^{-j_{n(\ell+1)}\delta}$.

We aim to estimate the size of $T_{\ell+1}$ conditionally to T_ℓ. For any $I \in T_\ell$, we denote by the $c(I)$ the middle point between the two jump times inside I. Then, for every integer $k \in$

$[j_{n(\ell)}\delta, j_{n(\ell)}\delta'']$, we want to estimate the number double-jumps configurations inside the interval of size 2^{-k}:

$$I_k := \left[2^{-k} - c(I), 2^{-k-1} - c(I)\right) \cup \left(c(I) + 2^{-k-1}, c(I) + 2^{-k}\right].$$

We designate by $\mathscr{S}_{n(\ell)+1,k}$ the number of sub-intervals of size $2^{-j_{n(\ell)+1}\delta}$ inside I_k which are empty. Using Chernoff's inequality, the latter satisfies

$$\mathbb{P}\left(|\mathscr{S}_{n(\ell)+1,k}| \geq 2^{j_{n(\ell)+1}\delta-k-2}\right) \geq 1 - \exp\left(-2^{j_{n(\ell)+1}\delta-k-4}\right) \geq 1 - \exp\left(-2^{j_{n(\ell)+1}r}\right).$$

The last inequality is due to $j_{n(\ell)+1}\delta - k \geq j_{n(\ell)+1}\delta - j_{n(\ell)}\delta'' > j_{n(\ell)+1}r$, if r is sufficiently small. Therefore, using the estimates obtained above,

$$\begin{aligned}\mathbb{P}\left(|\mathscr{S}_{n(\ell+1)-1,k}| \geq 2^{j_{n(\ell+1)-1}\delta - c_5 n(\ell) - k}\right) &\geq \prod_{k=1}^{n(\ell+1)-1}\left(1 - \exp\left(-2^{j_{n(\ell)+k}r}\right)\right)\\ &\geq \left(1 - \exp\left(-2^{j_{n(\ell)}r}\right)\right)^{n(\ell+1)},\end{aligned}$$

where c_5 is a positive constant and we recall that $n(\ell+1) = 2^{n(\ell)}$. An interval of size $2^{-j_{n(\ell+1)-1}\delta}$ can be divided in at least $2^{j_{n(\ell+1)}(\delta-\delta')-1}$ sub-intervals. Hence, if we denote by $M_{\ell,k}$ the number of double-jump configurations existing among the sub-intervals of $\mathscr{S}_{n(\ell+1)-1,k}$, Lemma 3.10 induces that

$$\mathbb{P}\left(|M_{\ell,k}| \geq 2^{-j_{n(\ell+1)}(\delta+\gamma-\alpha_1-\alpha_2-1+\rho)} \cdot 2^{j_{n(\ell+1)}\delta - c_5 n(\ell) - k - 4}\right) \geq \left(1 - \exp\left(-2^{j_{n(\ell)}r}\right)\right)^{n(\ell+1)+1}.$$

We observe that the intervals I_k are disjoints for different integers k. Hence, the probability of the intersection of the previous event for every $k \in [j_{n(\ell)}\delta, j_{n(\ell)}\delta'']$ satisfies

$$\begin{aligned}&\mathbb{P}\left(\bigcap_{k \geq j_{n(\ell)}\delta}^{j_{n(\ell)}\delta''} |M_{\ell,k}| \geq 2^{j_{n(\ell+1)}(\alpha_1+\alpha_2-\gamma+1-\rho) - c_6 n(\ell) - k}\right)\\ &\geq \left(1 - \exp\left(-2^{j_{n(\ell)}r}\right)\right)^{(n(\ell+1)+1)2\beta j_{n(\ell)}},\end{aligned}$$

since we assume that $\delta'' \leq \alpha_1 + \alpha_2 \leq 2\beta$. The previous construction procedure leads to estimate of size of $T_{\ell+1}$. Therefore, conditionally to the event $\left\{|T_\ell| \geq k_0 2^{j_{n(\ell)}(\alpha_1+\alpha_2-\gamma+1-\rho) - 2j_{n(\ell-1)}\delta}\right\}$, we obtain

$$\begin{aligned}&\mathbb{P}\left(|T_{\ell+1}| \geq k_0 2^{j_{n(\ell+1)}(\alpha_1+\alpha_2-\gamma+1-\rho) - 2j_{n(\ell)}\delta} \,\middle|\, |T_\ell| \geq k_0 2^{j_{n(\ell)}(\alpha_1+\alpha_2-\gamma+1-\rho) - 2j_{n(\ell-1)}\delta}\right)\\ &\geq \left(1 - \exp\left(-2^{j_{n(\ell)}r}\right)\right)^{(n(\ell+1)+1)2\beta j_{n(\ell)} 2^{j_{n(\ell)}c_7}} \geq \left(1 - \exp\left(-2^{j_{n(\ell)}r}\right)\right)^{2^{j_{n(\ell)}c_8}}.\end{aligned}$$

For any $\ell_0 \in \mathbf{N}$, we know that the construction ensures that $|T_{\ell_0}| \geq 1$ almost surely. Hence, choosing $k_0 = 2^{-c_9 j_{n(\ell_0)}}$, with the proper constant c_9, we obtain that the following lower bound

$$\mathbb{P}\left(\bigcap_{\ell > \ell_0} |T_\ell| \geq 2^{j_{n(\ell)}(\alpha_1+\alpha_2-\gamma+1-\rho) - 2j_{n(\ell-1)}\delta - c_9 j_{n(\ell_0)}}\right) \geq \prod_{\ell > \ell_0}\left(1 - \exp\left(-2^{j_{n(\ell-1)}r}\right)\right)^{2^{j_{n(\ell-1)}c_8}}.$$

Considering the logarithm of the right term, we observe

$$\sum_{\ell > \ell_0} \log\left(1 - \exp\left(-2^{j_{n(\ell-1)}r}\right)\right)^{2^{j_{n(\ell-1)}c_8}} \geq -\sum_{\ell > \ell_0} 2^{j_{n(\ell-1)}c_8} \cdot \exp\left(-2^{j_{n(\ell-1)}r}\right) \longrightarrow_{\ell_0 \to \infty} 0.$$

Hence, the previous probability converges to 1 for any set of parameters $\boldsymbol{p} = (\delta, \delta', \delta'', \rho)$. Since the family of events considered is increasing with ℓ_0, it implies that almost surely there

exists $\ell_0(\omega)$ such that $|T_\ell| \geq 2^{j_{n(\ell-1)}(\alpha_1+\alpha_2-\gamma+1-\rho)-2j_{n(\ell)}\delta-c_9 j_{n(\ell_0)}}$ for all $\ell > \ell_0(\omega)$. Furthermore, owing to the construction procedure described previously, we also know that for every $I \in T_\ell$, the interval $I_{\ell,k}$ contains at least $2^{j_{n(\ell+1)}(\alpha_1+\alpha_2-\gamma+1-\rho)-c_6 n(\ell)-k}$ proper double-jump configurations.

Hausdorff dimension (lower-bound). The previous estimates now allow us to study more precisely the oscillating behaviour of the Lévy process. Suppose $h \in (1/(\alpha_1+\alpha_2), 1/\beta)$ and $\boldsymbol{p}$ is a set of parameters such that $\delta < 1/h < \delta''$ and $1/h < \gamma$. Then, let us define the set of interest $G(h,\boldsymbol{p})$ as following:

$$G(h,\boldsymbol{p}) = \bigcap_{\ell>\ell_0} \bigcup_{I\in T_\ell} \Big\{ \big[c(I) - 2^{-j_{n(\ell)}/h+1}, c(I) - 2^{-j_{n(\ell)}/h-1}\big] \\ \cup \big[c(I) + 2^{-j_{n(\ell)}/h-1}, c(I) + 2^{-j_{n(\ell)}/h+1}\big] \Big\},$$

where $c(I)$ still denotes the middle point of any double-jump interval $I \in T_\ell$ and $\ell_0(\omega)$ corresponds to the random index previously defined. Owing to the construction of the tree T, we note that $G(h,\boldsymbol{p})$ corresponds to to the intersection of collections $T_{\ell,h}$ nested intervals of size $3\cdot 2^{-j_{n(\ell)}/h-1}$. Furthermore, according to the estimates obtained in the previous paragraph, we know that every $I \in T_{\ell,h}$ contains at least $2^{j_{n(\ell+1)}(\alpha_1+\alpha_2-\gamma+1-\rho)-c_6 n(\ell)-j_{n(\ell)}/h-1}$ sub-elements separated by $2^{j_{n(\ell+1)}(\gamma-\alpha_1-\alpha_2-1+\rho)}$.

In order the estimate the Hausdorff dimension of the set $G(h,\boldsymbol{p})$, we construct by induction a mass measure μ on it. To begin with, μ_{ℓ_0} attributes an equivalent weight on every interval $I \in T_{\ell_0,h}$. Then, similarly to the procedure on Cantor's set, $\mu_{\ell+1}$ is defined on the intervals $I \in T_{\ell+1,h}$ such that the weight $\mu_\ell(J)$, $J \in T_{\ell,h}$, is equally distributed on its offspring. The measure μ is then defined as the limit of the sequence $(\mu_\ell)_{\ell>\ell_0}$, which clearly exists since the cumulative distribution functions uniformly converge on $[0,1]$.

Since every $I \in T_{\ell,h}$ contains at least $2^{j_{n(\ell+1)}(\alpha_1+\alpha_2-\gamma+1-\rho)-c_6 n(\ell)-j_{n(\ell)}/h}$ elements, we first note that

$$\forall I \in T_{\ell,h}; \quad \mu(I) \leq \prod_{k=\ell_0+1}^{\ell} 2^{-j_{n(k)}(\alpha_1+\alpha_2-\gamma+1-\rho)+c_{10} j_{n(k-1)}} \\ \leq 2^{-j_{n(\ell)}(\alpha_1+\alpha_2-\gamma+1-\rho)+c_{11} n(\ell)}.$$

As we aim to use the usual *mass distribution principle* to determine a gauge function g such that the Hausdorff measure $\mathscr{H}_g$ of $G(h,\boldsymbol{p})$ is positive, we need to obtain an upper bound of $\mu\big(B(t,r)\big)$ for any $t \in [0,1]$ and $r > 0$ sufficiently small. There exists $\ell \in \mathbf{N}$ such that $2^{-j_{n(\ell)}/h} \leq r < 2^{-j_{n(\ell-1)}/h}$ and without any loss of generality, we may assume that $\ell > \ell_0$. Furthermore, as $r < 2^{-j_{n(\ell-1)}/h}$, we may also suppose that $B(t,r) \subset J$ where $J \in T_{\ell-1,h}$ (otherwise, consider the intersection $B(t',r')$ between $B(t,r)$ and the closest element J). Since the subintervals $I \subset J \in T_{\ell,h}$ are separated by at least $2^{j_{n(\ell+1)}(\gamma-\alpha_1-\alpha_2-1+\rho)}$, we know that the ball $B(t,r)$ intersects with at most $r\,2^{j_{n(\ell)}(\gamma-\alpha_1-\alpha_2-1+\rho)+1}$ of them. Hence, as we note that $\mu(I)$ is the same for every $I \subset J \in T_{\ell,h}$, we obtain

$$\mu\big(B(t,r)\big) \leq r\,2^{j_{n(\ell)}(\gamma-\alpha_1-\alpha_2-1+\rho)+1}\mu(I) \\ \leq r\,2^{j_{n(\ell-1)}/h+c_6 n(\ell-1)+1}\mu(J),$$

since $\mu(I) \leq \mu(J)/2^{j_{n(\ell)}(\alpha_1+\alpha_2-\gamma+1-\rho)-c_6 n(\ell-1)-j_{n(\ell-1)}/h}$. Furthermore, we also know that $\mu(J) \leq 2^{-j_{n(\ell-1)}(\alpha_1+\alpha_2-\gamma+1-\rho)+c_{11} n(\ell-1)}$, showing that

$$\mu\big(B(t,r)\big) \leq r\,2^{-j_{n(\ell-1)}(\alpha_1+\alpha_2-\gamma+1-\rho)+j_{n(\ell-1)}/h+c_{12} n(\ell-1)}.$$

Furthermore, $\rho > 1$ may be chosen such that $\alpha_1 + \alpha_2 - \gamma - 1/h + 1 - \rho < 0$, implying that $1-(\alpha_1+\alpha_2-\gamma+1-\rho)h > 0$ and $r^{1-(\alpha_1+\alpha_2-\gamma+1-\rho)h} \leq 2^{-j_{n(\ell-1)}/h+j_{n(\ell-1)}(\alpha_1+\alpha_2-\gamma+1-\rho))}$. Finally, since $j_{n(\ell-1)} = \left(\delta/\delta'\right)^{n(\ell-1)} \leq c\log\left(1/r\right)$, there exist $c_{13}, c_{14} > 0$ such that

$$\mu\big(B(t,r)\big) \leq c_{13}\log(1/r)^{c_{14}}\, r^{(\alpha_1+\alpha_2-\gamma+1-\rho)h}.$$

Using the mass distribution principle (see [68] for instance), this inequality proves that $G(h,\boldsymbol{p})$ has a positive g-Hausdorff measure, where the gauge function g is defined by $g(r) = \log(1/r)^{c_{14}} r^{(\alpha_1+\alpha_2-\gamma+1-\rho)h}$.

Therefore, if we restrict ourselves to rational parameters $\boldsymbol{p}$, we have proved that with probability one, for all $h \in (1/(\alpha_1+\alpha_2), 1/\beta)$, $\dim_{\mathrm{H}} G(h,\boldsymbol{p}) \geq (\alpha_1+\alpha_2-\gamma+1-\rho)h$.

2-microlocal frontier (lower-bound). In this last step of the proof, we aim to show that the 2-microlocal frontier of every $t \in G(h,\boldsymbol{p})$ has a chirp oscillation form.

Let us set $\omega \in \Omega$ and $t \in G(h,\boldsymbol{p})$. As previously outlined in this work, we know that we may ignore the component of Lévy process which corresponds to the jumps of size greater than $2^{-j_{n(\ell_0)}}$. Furthermore, owing to the construction of the set $G(h,\boldsymbol{p})$, we know that for every $\ell \in \mathbf{N}$, the distance between t and the closest jump time s such that $|\Delta X_s| \geq 2^{-j_{n(\ell)}}$, satisfies

$$2^{-j_{n(\ell)}-2} \leq |s-t| \leq 2^{-j_{n(\ell)}+1}.$$

Therefore, owing to the characterisation of the set E_h, we know that $G(h,\boldsymbol{p}) \subset E_h$, i.e. $\alpha_{X,t} = h$.

We aim to prove that the 2-microlocal frontier of X at t shows a chirp oscillation behaviour: $\sigma_{X,t}(s') > h+s'$ for all $s' < -h$. Similarly to the proof of Theorem 3.1, we therefore investigate the pointwise regularity of the integral of X. Lastly, we first assume that $\beta > 1$.

Let us set $u \in \mathbf{R}$, $\varepsilon > 0$ and $h' := h(1+\varepsilon) > h$. There exist $m > 0$ such that $2^{-(m+1)/h'} < |t-u| \leq 2^{-m/h'}$. Furthermore, let $\ell \in \mathbf{N}$ be the greatest integer such that $j_{n(\ell)} \leq m$. We have to distinguish two different cases depending on the value of m.

Let us first suppose that $j_{n(\ell)}(1+\varepsilon) \leq m$. Since $1/h' < \beta$, Lemma 3.2 implies that

$$\forall v \in B(t,|u-t|); \quad |X_v^m - X_t^m| \leq c\,\log\big(|u-t|^{-1}\big)\,|u-t|^{h'}.$$

Furthermore, we note that $|t-u| \leq 2^{-j_{n(\ell)}(1+\varepsilon)/h'} < 2^{-j_{n(\ell)}h}$ as $h' < h(1+\varepsilon)$. It implies that there is no jump time s whose size is greater than 2^{-m} and such that $s \in B(t,|u-t|)$. Using in addition the estimates of the drift obtained in Proposition 3.9, we obtain

$$\forall v \in B(t,|u-t|); \quad |X_v - X_t| \leq c\,\log\big(|u-t|^{-1}\big)\,|u-t|^{h'}.$$

Therefore,

$$\left|\int_t^u (X_v - X_t)\,\mathrm{d}v\right| \leq c\,\log\big(|u-t|^{-1}\big)\,|u-t|^{1+h'},$$

where we recall that $h' > h$.

We now consider the second case $j_{n(\ell)} \leq m \leq j_{n(\ell)}(1+\varepsilon)$. As previously, we know that for all $v \in B(t,|u-t|)$, $|X_v^m - X_t^m| \leq c\,\log\big(|u-t|^{-1}\big)\,|u-t|^{h'}$. Nevertheless, in this case, there might be jump of size $2^{-j_{n(\ell)}}$ inside the interval $B(t,|u-t|)$. Owing to the construction of the set $G(h,\boldsymbol{p})$, the contribution to $|\int_t^u (X_v - X_t)\,\mathrm{d}v|$ is upper-bounded by

$$\begin{aligned} 2^{-j_{n(\ell)}} \cdot 2^{-j_{n(\ell)}\gamma} + 2^{-j_{n(\ell)}\rho}\,|u-t| &\leq c\,|u-t|^{h'(1+\gamma)/(1+\varepsilon)} + c\,|u-t|^{\rho\,h'/(1+\varepsilon)+1} \\ &= c\,|u-t|^{h(1+\gamma)} + c\,|u-t|^{\rho\,h+1}. \end{aligned}$$

In the previous exponents, we note that $\rho > 1$ implies $\rho\,h+1 > h+1$ and we can choose $\gamma > 1/h$, i.e. $h(1+\gamma) > h+1$. Hence, we have shown there exists $\varepsilon_0 > 0$ such that

$$\left|\int_t^u (X_v - X_t)\,\mathrm{d}v\right| \leq c|u-t|^{1+h+\varepsilon_0},$$

This last inequality proves that the regularity at t is singular, as the 2-microlocal frontier satisfies

$$\forall s' \leq -h; \quad \sigma_{X,t}(s') \geq \frac{s'+h}{1+\varepsilon_0}.$$

Therefore, with probability one, for all $h \in (1/(\alpha_1+\alpha_2), 1/\beta)$, $G(h,\boldsymbol{p}) \subset \widehat{E}_h$, and $\dim_{\mathrm{H}} \widehat{E}_h \geq (\alpha_1+\alpha_2-\gamma+1-\rho)h$. Considering rational parameters such that $\gamma \to 1/h$ and $\rho \to 1$, we obtain $\dim_{\mathrm{H}} \widehat{E}_h \geq (\alpha_1+\alpha_2)h-1$. Finally, since the previous reasoning holds on any interval $[a,b]$ where $a,b \in \mathbf{Q}$, with probability one, we have the expected lower bound of the Hausdorff dimension:

$$\forall V \in \mathscr{O},\ \forall h \in (1/(\alpha_1+\alpha_2)), 1/\beta); \quad \dim_{\mathrm{H}}(\widehat{E}_h \cap V) \geq (\alpha_1+\alpha_2)h-1.$$

□

3.3 Linear (multi)fractional stable motion

The linear fractional stable motion (LFSM) is a stochastic process that has been considered by several authors: MAEJIMA [121], TAKASHIMA [176], KÔNO and MAEJIMA [110], SAMORODNITSKY and TAQQU [167], AYACHE, ROUEFF, and XIAO [12], AYACHE and HAMONIER [11]. Its general integral form is defined by

$$\begin{aligned} X_t = \int_{\mathbf{R}} \Big\{ & a^+ \big[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}\big] \\ & + a^- \big[(t-u)_-^{H-1/\alpha} - (-u)_-^{H-1/\alpha}\big] \Big\} M_\alpha(\mathrm{d}u), \end{aligned} \tag{3.36}$$

where $H \in (0,1)$, $(a^+,a^-) \in \mathbf{R}^2 \setminus (0,0)$ and M_α is an α-stable random measure on $\mathbf{R}$ with Lebesgue control measure λ and skewness intensity $\beta_\alpha(\cdot) \in [-1,1]$. Throughout this paper, it is assumed that β_α is constant, and equal to zero when $\alpha = 1$. In this context, for any Borel set $A \subset \mathbf{R}$, the characteristic function of $M_\alpha(A)$ is given by

$$\mathbb{E}\big[e^{i\theta M_\alpha(A)}\big] = \begin{cases} \exp\{-\lambda(A)|\theta|^\alpha\big(1-i\beta_\alpha \operatorname{sign}(\theta)\tan(\alpha\pi/2)\big)\} & \text{if } \alpha \in (0,1)\cup(1,2); \\ \exp\{-\lambda(A)|\theta|\} & \text{if } \alpha = 1. \end{cases}$$

For the sake of readability, we consider in the rest of the section the particular case $(a^+,a^-) = (1,0)$ (even though as stated [167], the law of the process depends on values (a^+,a^-) chosen).

To begin with, we present in the next statement an alternative representation for the two-parameter field $(t,H) \mapsto X(t,H) = \int_{\mathbf{R}} \{(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}\} M_\alpha(\mathrm{d}u)$. In the case $H \geq H/\alpha$, the formula has been previously obtained by TAKASHIMA [176].

Proposition 3.10 *For all $t \in \mathbf{R}$ and $H \in (0,1)$, the random variable $X(t,H)$ satisfies*

$$X(t,H) \overset{\text{a.s.}}{=} \begin{cases} C_H \displaystyle\int_{\mathbf{R}} L_u \Big\{(t-u)_+^{H-1/\alpha-1} - (-u)_+^{H-1/\alpha-1}\Big\} \mathrm{d}u & \text{if } H \in [\tfrac{1}{\alpha},1); \\ L_t & \text{if } H = \tfrac{1}{\alpha} \\ C_H \displaystyle\int_{\mathbf{R}} \Big\{(L_u - L_t)(t-u)_+^{H-1/\alpha-1} - L_u(-u)_+^{H-1/\alpha-1}\Big\} \mathrm{d}u & \text{if } H \in (0,\tfrac{1}{\alpha}], \end{cases} \tag{3.37}$$

where $C_H = H - 1/\alpha$ and L is an α-stable Lévy process defined by

$$\forall t \in \mathbf{R}_+ \quad L_t = M_\alpha([0,t]) \quad \textit{and} \quad \forall t \in \mathbf{R}_- \quad L_t = -M_\alpha([t,0]).$$

Proof. For the sake of readiness, we present in the proof for any $H \in (0,1)$, even though the first case can be found in [176]. Suppose $t \in \mathbf{R}$ and $H \in (0,1)$. Since $(L_t)_{t\in\mathbf{R}}$ is an α-stable Lévy process, it has càdlàg sample paths. According to [6] (chap. 4.3.4), the theory of the stochastic integration based α-stable Lévy processes coincide integrals with respect to α-stable random measure. Therefore, the r.v. $X(t,H)$ is almost surely equal to $\int_{\mathbf{R}}\{(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}\}\,\mathrm{d}L_u$. Let $\varepsilon > 0$ and $b < t$. Using a classic integration by parts, we obtain

$$\begin{aligned} L_{t-\varepsilon}\varepsilon^{H-1/\alpha} - L_s(t-b)^{H-1/\alpha} &= \int_b^{t-\varepsilon} (t-u)^{H-1/\alpha}\,\mathrm{d}L_u \\ &\quad - \left(H - \frac{1}{\alpha}\right)\int_b^{t-\varepsilon} L_u(t-u)^{H-1/\alpha-1}\,\mathrm{d}u. \end{aligned} \tag{3.38}$$

1. If $H \in \left(\frac{1}{\alpha}, 1\right)$, $H - 1/\alpha > 0$. Hence, $\int_b^{t-\varepsilon} L_u(t-u)^{H-1/\alpha-1}\,\mathrm{d}u$ almost surely converges to $\int_b^t L_{u-}(t-u)^{H-1/\alpha-1}\,\mathrm{d}u$ when $\varepsilon \to 0$. Similarly, $\int_b^{t-\varepsilon}(t-u)^{H-1/\alpha}\,\mathrm{d}L_u$ converges in $L^\alpha(\Omega)$. Therefore, using Equation (3.38) with $t = 0$ and $b < 0$, we obtain almost surely

$$\begin{aligned} \int_b^t \left\{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\right\}\mathrm{d}L_u &= C_H \int_b^t L_u\left\{(t-u)^{H-1/\alpha-1} - (-u)^{H-1/\alpha-1}\right\}\mathrm{d}u \\ &\quad - L_b\left\{(t-b)^{H-1/\alpha} - (-b)^{H-1/\alpha}\right\}. \end{aligned}$$

   When $b \to -\infty$, the left-term clearly converges to $X(t,H)$ in $L^\alpha(\Omega)$. According to [160], we know that almost surely for any $\varepsilon > 0$, $\limsup_{u\to-\infty} u^{1/\alpha+\varepsilon}|L_u| = 0$. Furthermore, we also have $(t-u)^{H-1/\alpha-1} - (-u)^{H-1/\alpha-1} \sim_{-\infty} (-u)^{H-1/\alpha-2}$ and $(t-b)^{H-1/\alpha} - (-b)^{H-1/\alpha} \sim_{-\infty} (-b)^{H-1/\alpha-1}$. Therefore, as $H < 1$ and using the dominated convergence theorem, the right-term almost surely converges to the expected integral.

2. If $H \in \left(0, \frac{1}{\alpha}\right)$, we observe that Equation (3.38) can be slightly transformed into

$$\begin{aligned} &(L_{t-\varepsilon} - L_t)\varepsilon^{H-1/\alpha} - (L_b - L_t)(t-b)^{H-1/\alpha} \\ &= \int_b^{t-\varepsilon} (t-u)^{H-1/\alpha}\,\mathrm{d}L_u - \left(H - \frac{1}{\alpha}\right)\int_b^{t-\varepsilon} (L_u - L_t)(t-u)^{H-1/\alpha-1}\,\mathrm{d}u. \end{aligned}$$

   According to [160], $\alpha_{Y,t} \overset{\text{a.s.}}{=} 1/\alpha$. Therefore, up to an extracted sequence, the previous expression almost surely converges when $\varepsilon \to 0$ and using a similar formula for $t = 0$, we obtain

$$\begin{aligned} &\int_b^t \left\{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\right\}\mathrm{d}L_u \\ &= C_H \int_b^t \left\{(L_u - L_t)(t-u)^{H-1/\alpha-1} - L_u(-u)^{H-1/\alpha-1}\right\}\mathrm{d}u \\ &- L_b\left\{(t-b)^{H-1/\alpha} - (-b)^{H-1/\alpha}\right\} + L_t(t-b)^{H-1/\alpha}. \end{aligned}$$

   The property $\limsup_{u\to-\infty} u^{1/\alpha+\varepsilon}|L_u| = 0$ and the previous equivalents finally prove Equation (3.37).

To end this proof, let us consider the integral representation in the particular case $H = 1/\alpha$. In fact, Equation (3.37) is a slightly misuse since the expression does not exist. Nevertheless, we prove that it converges almost surely to $X(t, 1/\alpha) = L_t$ when $H \to 1/\alpha$.

Suppose first that $H \nearrow 1/\alpha$ and rewrite $X(t,H)$ as

$$X(t,H) = C_H \int_{\mathbf{R}} \left\{ (L_u - L_t \mathbf{1}_{u \geq b})(t-u)_+^{H-1/\alpha-1} - L_u(-u)_+^{H-1/\alpha-1} \right\} \mathrm{d}u + L_t(t-b)^{H-1/\alpha},$$

The first component of the expression converges to zero since $C_H \to_{H \to 1/\alpha} 0$ and $\alpha_{Y,t} \overset{\text{a.s.}}{=} 1/\alpha$. As the second part simply converges to L_t, we get the expected limit. The case $H \searrow 1/\alpha$ is treated similarly. □

Note that PICARD [158] has determined a similar representation for fractional Brownian motion.

Proof of Theorem 3.6. Let us set $H \in (0,1)$ and $\alpha \in [1,2)$. In order to obtain the multifractal structure of the LFSM, we first relate the 2-microlocal frontier of X at t to the frontier of the alpha-stable process L.

1. If $H > 1/\alpha$, we note that the representation obtained in Proposition 3.10 is defined almost surely for all $t \in \mathbf{R}$. Therefore, let us set $\omega \in \Omega$ and $t \in \mathbf{R}$. As previously, we can assume that $t \in [0,1]$. Then,

$$\begin{aligned} X_t = C_H \int_b^t L_u(t-u)_+^{H-1/\alpha-1} \mathrm{d}u + C_H \int_b^0 L_u(-u)_+^{H-1/\alpha-1} \mathrm{d}u \\ + C_H \int_{-\infty}^b L_u \left\{ (t-u)_+^{H-1/\alpha-1} - (-u)_+^{H-1/\alpha-1} \right\} \mathrm{d}u, \end{aligned}$$

   where $b < 0$ is fixed. The second term is simply a constant that does not influence the regularity. Similarly, using the dominated convergence theorem, we note that the third one is a smooth function on the interval $[0,1]$, and therefore has no impact on the 2-microlocal frontier.

   Therefore, we only need to focus on the first term. Let us define the process $Y_u = L_u \mathbf{1}_{\{u \geq t\}}$. Since the 2-microlocal spaces and frontier presented in Definitions 3.2 and 3.3 are localised at a point t, we necessarily have $\sigma_{L,t} = \sigma_{Y,t}$. Furthermore, we note that

$$Z_t := C_H \int_b^t L_u(t-u)_+^{H-1/\alpha-1} \mathrm{d}u = C_H \int_{\mathbf{R}} Y_u(t-u)_+^{H-1/\alpha-1} \mathrm{d}u = (I_+^{H-1/\alpha} Y)(t)$$

   Owing to the property of stability of 2-microlocal spaces under fractional integration (see see Theorem 1.1 in [95]), we obtain $\sigma_{Z,t} = \sigma_{Y,t} + H - 1/\alpha$, and therefore $\sigma_{X,t} = \sigma_{L,t} + H - 1/\alpha$.

2. If $H < 1/\alpha$, we first observe that according to the multifractal spectrum of alpha-stable processes and $H > 0$, $\dim_{\mathrm{H}}(\{t \in \mathbf{R} : \alpha_{L,t} \leq 1/\alpha - H\}) < 1$. Hence, for almost every $\omega \in \Omega$, Formula (3.37) is well-defined almost everywhere on $\mathbf{R}$. Anywhere else, we may simply assume that $X(t,H)$ is set to zero. We will explain later why the value 0 at these particular times does not modify the 2-microlocal frontier.

   Similarly to the previous case $H > 1/\alpha$, the regularity of X only depends on the behaviour of the component

$$Z : t \longmapsto C_H \int_b^t (L_u - L_t)(t-u)_+^{H-1/\alpha-1} \mathrm{d}u.$$

One might recognize a Marchaud fractional derivative (see e.g. [166]). Let us modify this expression to exhibit a more classic form of fractional derivative. For almost all $s \in [0,1]$ and $\varepsilon > 0$, we have

$$\begin{aligned}\int_b^{s-\varepsilon} L_u(s-u)^{H-1/\alpha}\,\mathrm{d}u &= C_H \int_b^{s-\varepsilon} L_u\,\mathrm{d}u \int_{u+\varepsilon}^{s} (v-u)^{H-1/\alpha-1}\,\mathrm{d}v + \varepsilon^{H-1/\alpha}\int_b^{s-\varepsilon} L_u\,\mathrm{d}u \\ &= C_H \int_b^{s-\varepsilon} \mathrm{d}u \int_{u+\varepsilon}^{s} (L_u - L_v)(v-u)^{H-1/\alpha-1}\,\mathrm{d}v \\ &\quad + \varepsilon^{H-1/\alpha}\int_b^{s-\varepsilon} L_u\,\mathrm{d}u + C_H \int_b^{s-\varepsilon} \mathrm{d}u \int_{u+\varepsilon}^{s} L_v(v-u)^{H-1/\alpha-1}\,\mathrm{d}v\end{aligned}$$

The last two terms are equal to

$$\varepsilon^{H-1/\alpha}\int_b^{s-\varepsilon} L_u\,\mathrm{d}u - \varepsilon^{H-1/\alpha}\int_{b+\varepsilon}^{s} L_v\,\mathrm{d}v + \int_{b+\varepsilon}^{s} L_v(v-b)^{H-1/\alpha}\,\mathrm{d}v,$$

which converges to $\int_b^s L_v(v-b)^{H-1/\alpha}\mathrm{d}v$ as $\varepsilon \to 0$ since $H - 1/\alpha > -1$. Similarly, as we consider times at which $\alpha_{L,s} > 1/\alpha - H$, the dominated convergence theorem implies that the first term converges to $C_H \int_b^s \mathrm{d}v \int_b^v (L_u - L_v)(v-u)^{H-1/\alpha-1}\,\mathrm{d}u$. Therefore,

$$\int_b^s L_u(s-u)^{H-1/\alpha}\,\mathrm{d}u = C_H \int_b^s \mathrm{d}v \int_b^v (L_u - L_v)(v-u)^{H-1/\alpha-1}\,\mathrm{d}u + \int_b^s L_v(v-b)^{H-1/\alpha}\,\mathrm{d}v.$$

According to classic real analysis results, the previous expression is differentiable almost everywhere on the interval $[0,1]$, and therefore

$$Z_t \overset{\text{a.e.}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_b^t L_u(t-u)^{H-1/\alpha}\,\mathrm{d}u - L_t(t-b)^{H-1/\alpha},$$

for almost all $t \in [0,1]$. Note that the last two formulas ensure that $Z_\bullet \in L^1_{\mathrm{loc}}(\mathbf{R})$, and thus $X_\bullet \in L^1_{\mathrm{loc}}(\mathbf{R})$ with probability one.

Let us now explain in which sense we investigate the 2-microlocal regularity of X. As previously outlined in the introduction, in the case $H < 1/\alpha$, sample paths of LFSM are nowhere bounded. As a consequence, it is meaningless to consider the usual Hölder regularity. On the other hand, the 2-microlocal formalism has been introduced in a more general frame which are distributions $\mathscr{D}'(\mathbf{R})$. Since we have previously proved that $X_\bullet \in L^1_{\mathrm{loc}}(\mathbf{R})$, with probability one, $X_\bullet$ is a distribution whose 2-microlocal frontier is well-defined. We refer to [138] for a complete presentation of the 2-microlocal spaces for distributions. Also note that in this context, we can modify the values of X_t on the negligible set $\{t \in \mathbf{R} : \alpha_{L,t} \le 1/\alpha - H\}$ without modifying $X_\bullet$ in the sense of distributions.

Then, let first consider the term $Y : t \mapsto \int_b^t L_u(t-u)^{H-1/\alpha}\,\mathrm{d}u$. Since $H - 1/\alpha > -1$, it is a Riemann–Liouville fractional integral of order $H - 1/\alpha + 1 > 0$. Hence, using the techniques previously presented, we obtain that $\sigma_{Y,t} = \sigma_{L,t} + H - 1/\alpha + 1$. Furthermore, the almost everywhere derivative $\frac{\mathrm{d}}{\mathrm{d}t}\int_b^t L_u(t-u)^{H-1/\alpha}\,\mathrm{d}u$ coincide with the derivative in the sense distribution. Still using the stability of 2-microlocal spaces, the 2-microlocal frontier of the latter is therefore equal to $\sigma_{L,t} + H - 1/\alpha$. In addition, the 2-microlocal frontier of $t \mapsto L_t(t-b)^{H-1/\alpha}$ is equal to $\sigma_{L,t}$ (the multiplication with a locally smooth function having no effect). Hence, as $\sigma_{L,t} > \sigma_{L,t} + H - 1/\alpha$, we have proved that $\sigma_{Z,t} = \sigma_{L,t} + H - 1/\alpha$, and thus $\sigma_{X,t} = \sigma_{L,t} + H - 1/\alpha$ with probability one.

Therefore, in both cases, we have proved that with probability one and for all $t \in [0,1]$,

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \sigma_{L,t}(s') + H - 1/\alpha.$$

Then, using the same reasoning as in the proof of Corollary 3.3, we observe that

$$\forall s \in [0, 1/\alpha]; \quad \widetilde{E}_{s-H+1/\alpha}(L) \subseteq E_{\sigma,s'}(X) \subseteq \widetilde{E}_{s-H+1/\alpha}(L) \cup \bigcup_{h<s-H+1/\alpha} \widehat{E}_h(L).$$

for any $\sigma < H - 1/\alpha$. Furthermore, since : is an alpha-stable process, Theorem 3.1 induces that

$$\dim_{\mathrm{H}} \widetilde{E}_{s-H+1/\alpha}(L) = \alpha\big(s - H + 1/\alpha\big) = \alpha(s-H) + 1$$

and $\dim_{\mathrm{H}} \widehat{E}_h(L) < \alpha(s+H) - 1$ for every $h < s - H + 1/\alpha$. These two estimates clearly prove the spectrum presented in Equation (3.19). Finally, the spectrum of singularity for the weak scaling exponent is obtained similarly. □

Another class of processes similar to the LFSM has been introduced and studied in [25, 128, 44]. Named *fractional Lévy processes*, it is defined by

$$X_t = \frac{1}{\Gamma(d+1)} \int_{\mathbf{R}} \Big\{ (t-u)_+^d - (-u)_+^d \Big\} L(\mathrm{d}u),$$

where $d \in (0, 1/2)$ and L is a Lévy process enjoying $Q = 0$ (no Brownian component), $\mathbb{E}[L(1)] = 0$ and $\mathbb{E}[L(1)^2] < +\infty$. Owing to this last assumption on L, LFSMs are not fractional Lévy processes. Nevertheless, their multifractal regularity can be determined as well.

Proposition 3.11 *Suppose X is a fractional Lévy process parametrized by $d \in (0, 1/2)$. Then, with probability one and for all $\sigma \leq d$,*

$$\forall V \in \mathscr{O}; \quad \dim_{\mathrm{H}}(E_{\sigma,s'} \cap V) = \begin{cases} \beta(s-d) & \text{if } s \in \big[d, d + \frac{1}{\beta}\big]; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.39}$$

where β designates the Blumenthal–Getoor exponent of the Lévy process L. Furthermore, for all $s' \in \mathbf{R}$, $E_{\sigma,s'}$ is empty if $\sigma > d$.

Proof. MARQUARDT [128] has established (Theorem 3.4) a representation of fractional Lévy processes equivalent to Proposition 3.10:

$$X_t = \frac{1}{\Gamma(d)} \int_{\mathbf{R}} L_u \Big\{ (t-u)_+^{d-1} - (-u)_+^{d-1} \Big\} \, \mathrm{d}u.$$

Based on this result, a straightforward adaptation of the proof of Theorem 3.6 yields Equation (3.39). □

Similarly to the LFSM, this statement refines regularity results established in [25, 128] and proves that the multifractal spectrum of a fractional Lévy process is equal to

$$\forall V \in \mathscr{O}; \quad d_X(h, V) = \begin{cases} \beta(h-d) & \text{if } h \in \big[d, d + \frac{1}{\beta}\big]; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.40}$$

Let us finally conclude this section with the proof of Theorem 3.8.

Proof of Theorem 3.8. Suppose $(X_t)_{t\in\mathbf{R}}$ is a linear multifractional stable motion with $\alpha \in (1,2)$ and Hurst function $H(\cdot) \in (1/\alpha, 1)$. According to the representation obtained in Proposition 3.10, X_t is almost surely equal to $X(t, H(t))$.

To begin with, we first use the uniform estimate of the local Hölder exponent obtained by AYACHE and HAMONIER [11, Th. 8.1] to obtain an upper bound on the 2-microlocal frontier. The latter have proved that with probability one and for all $t \in \mathbf{R}$, $\widetilde{\alpha}_{X,t} = H(t) - 1/\alpha$. In addition, the 2-microlocal frontier is known to satisfy the inequality $\sigma_{X,t} \leq \liminf_{u\to t} \widetilde{\alpha}_{X,u}$ for any $t \in \mathbf{R}$, which proves that $\sigma_{X,t} \leq H(t) - 1/\alpha$ with probability one.

Let us now set $\omega \in \Omega$ and $t \in \mathbf{R}$ and decompose X into two parts:

$$X_u = X(u, H(t)) + \big(X(u, H(u)) - X(u, H(t))\big).$$

According to the proof of Theorem 3.6, we already that the 2-microlocal frontier of the first component $X(\cdot, H(t))$ is equal to $\sigma_{L,t} + H(t) - 1/\alpha$. As a consequence, we have to prove that the second term $Y_u := X(u, H(u)) - X(u, H(t))$ is negligible in terms of 2-microlocal regularity. For that purpose, we observe that for any $u, v \in B(t, \rho)$

$$\begin{aligned} Y_u - Y_v &= X(u, H(u)) - X(u, H(t)) - X(v, H(v)) + X(v, H(t)) \\ &= \int_{H(t)}^{H(u)} \partial_H X(u, h)\, \mathrm{d}h - \int_{H(t)}^{H(v)} \partial_H X(v, h)\, \mathrm{d}h. \end{aligned}$$

Therefore, since H is δ-Hölderian,

$$\begin{aligned} |Y_u - Y_v| &\leq \int_{H(t)}^{H(u)} \left|\partial_H X(u, h) - \partial_H X(v, h)\right| \mathrm{d}h + \int_{H(u)}^{H(v)} \left|\partial_H X(v, h)\right| \mathrm{d}h \\ &\leq c\, |u - t|^{\delta} \cdot |u - v|^{\gamma} + c\, |u - v|^{\delta} \end{aligned}$$

where $\gamma < \inf_{u\in B(t,\rho)} H(u) - 1/\alpha$ and $\delta > \sup_{u\in\mathbf{R}} H(u)$. Using Definition 3.2 of the 2-microlocal spaces, this inequality proves that $\sigma_{Y,t} \geq (\delta + s') \wedge (H - 1/\alpha)$ for all $s' \in \mathbf{R}$. Since $\delta > H(t)$, $\sigma_{X,t} \leq H(t) - 1/\alpha$ and $\sigma_{L,t}(s') \leq (1/\alpha + s') \wedge 0$, we therefore obtain with probability one and for all $t \in \mathbf{R}$

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \sigma_{L,t}(s') + H(t) - 1/\alpha.$$

Then, the multifractal structure described in Equation (3.22) is obtained using the same arguments as in the proof of Theorem 3.6. □

Remark 3.2 In the case $H(\cdot)$ does not satisfy the assumption $\delta > \sup_{t\in\mathbf{R}} H(t)$, the proof of Theorem 3.8 can be modified to extend the statement and generalize results obtained in [173]. This complete study is made in [17] for the multifractional Brownian motion. For the sake of clarity, we prefer to focus in this work on $(\mathscr{H}_0)$-Hurst functions and the multifractal structure of the LMSM presented in Theorem 3.8.

Remark 3.3 Even though it is assumed all along this section that $H(\cdot)$ is deterministic, owing to the deterministic representation presented in Proposition 3.10, Theorems 3.6 and 3.8 still hold if $H(\cdot)$ is a continuous random process. Hence, based on these results, a class of random processes with random and non-homogeneous multifractal spectrum can be easily constructed. A similar extension of the multifractional Brownian motion has been introduced and studied by AYACHE and TAQQU [14].

4

An increment type set-indexed Markov property

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4.1 Introduction

The Markov property is a central concept of the classic theory of real-parameter stochastic processes. Its extension to processes indexed by a partially ordered collection is a non-trivial problem and multiple attempts exist in the literature to obtain a satisfactory definition. This question has been first investigated by Lévy [117] who introduced the *sharp Markov* property for two-parameter processes: a process $(X_t)_{t\in[0,1]^2}$ is said to be *sharp Markov* with respect to a set $A \subset [0,1]^2$ if the σ-fields $\mathscr{F}_A$ and $\mathscr{F}_{A^c}$ are conditionally independent given $\mathscr{F}_{\partial A}$, where for any $V \subset [0,1]^2$, $\mathscr{F}_V := \sigma(\{X_t; t \in V\})$. Russo [165] proved that processes with independent increments are *sharp Markov* with respect to any finite unions of rectangles. Later, Dalang and Walsh [48, 47] characterized entirely the collection of sets with respect to which processes with independent increments are *sharp Markov*.

Since the Brownian sheet was known not to satisfy this property with respect to simple sets (e.g. triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$), McKean [133] has introduced a weaker Markov property called *germ-Markov*. Similarly to the sharp Markov property, a process is said to be *germ-Markov* with respect to a set $A \subset [0,1]^2$ if $\mathscr{F}_A$ and $\mathscr{F}_{A^c}$ are conditionally independent given $\mathscr{G}_{\partial A} := \cap_V \mathscr{F}_V$, where the intersection is taken over all open sets V containing ∂A. As shown by Russo [165], the Brownian sheet turns out to be *germ-Markov* with respect to any open set.

A third Markov property, named $*$-*Markov*, arises in the two-parameter literature. It has been first presented by Cairoli [39], and then widely studied by Nualart and Sanz [149], Korezlioglu, Lefort, and Mazziotto [111], Mazziotto [132]. In particular, the latter established that $*$-*Markov* processes satisfy both *sharp* and *germ-Markov* properties with respect to respectively finite unions of rectangles and convex domains. Moreover, as stated in [147, 120, 203], the concept of $*$-*transition function* can be naturally introduced and leads to a complete characterization of the finite-dimensional distributions of $\mathbf{R}^2$-indexed processes. For the sake of readability, the precise definition of $*$-Markov processes is given in the core of the article.

This overview of the two-parameter Markov literature is clearly not exhaustive and a more complete picture is given later, including in particular the modern multiparameter approach presented by Khoshnevisan [100].

Recently, Ivanoff and Merzbach [89] have introduced a set-indexed formalism which allows to define and study a wider class of processes indexed by a collection of sets. Based on this framework, several extensions of classic real-parameter processes have been investigated, including Lévy processes [82], martingales [89] and fractional Brownian motion [79, 80]. The study of the Markovian aspects of set-indexed processes led to the definition of *sharp Markov* and *Markov* properties in [88], and of the *set-Markov* property in [18]. The first paper mainly focused on the adaptation of Paul Lévy's ideas to the set-indexed formalism. The latter approach, also called $\mathscr{Q}$*-Markov*, introduced a stronger property, leading to the definition of a transition system which characterizes the law of a *set-Markov* process. More precisely, Theorem 1 in [18] states that, given an initial distribution, if a transition system satisfies a Chapman–Kolmogorov Equation (4.14) and a supplementary invariance Assumption (4.15), a corresponding *set-Markov* process can be constructed.

In the present work, we suggest a different approach, named $\mathscr{C}$*-Markov*, for the introduction of a set-indexed Markov property. Our main goal is to obtain a natural definition of the transition probabilities which leads to satisfactory characterization and construction theorems. As later presented in Section 4.3, this new set-indexed Markov property also appears to be a natural extension of some existing multiparameter Markov properties.

To introduce the definition of $\mathscr{C}$*-Markov* processes, let us first recall the notations used in the set-indexed formalism. Throughout, we consider set-indexed processes $X = \{X_A; A \in \mathscr{A}\}$, where the indexing collection $\mathscr{A}$ is made up of compact subsets of a locally compact metric space $\mathscr{T}$. Moreover, $\mathscr{A}$ is assumed to satisfy the following conditions:

(i) it is closed under arbitrary intersections and $A^\circ \neq A$ for all $A \in \mathscr{A}, A \neq \mathscr{T}$;

(ii) $\emptyset' := \bigcap_{A \in \mathscr{A}} A$ is a nonempty set (it plays a role equivalent to 0 in $\mathbf{R}_+^N$);

(iii) there is an increasing sequence $(B_n)_{n \in \mathbf{N}}$ of sets in $\mathscr{A}$ such that $\mathscr{T} = \cup_{n \in \mathbf{N}} B_n$;

(iv) *Shape hypothesis*: for any $A, A_1, \ldots, A_k \in \mathscr{A}$ with $A \subseteq \cup_{i=1}^k A_i$, there exists $i \in \{1, \ldots, k\}$ such that $A \subseteq A_i$.

For the sake of readability, we restrict the assumptions on $\mathscr{A}$ to those required for the development of the $\mathscr{C}$*-Markov* approach. Supplementary properties such as *separability from above* will be added later when necessary. The complete definition of an indexing collection can be found in the work of Ivanoff and Merzbach [89].

Several examples of indexing collection have been presented in the literature (see e.g. [89, 18, 79]). Among them, let us mention:

- rectangles of $\mathbf{R}_+^N$: $\mathscr{A} = \{[0,t] : t \in \mathbf{R}_+^N\}$. This collection is equivalent to the usual multiparameter setting on $\mathbf{R}_+^N$;
- subsets of the N-dimensional unit sphere $\mathscr{S}_N$: $\mathscr{A} = \{A_\varphi : \varphi \in [0,\pi]^{N-1} \times [0,2\pi)\}$, where $A_\varphi := \{u \in \mathscr{S}_N : \phi_i(u) \leq \varphi_i,\ 1 \leq i \leq N\}$, $\phi_i(u)$ denoting the ith angular coordinates of u, $1 \leq i \leq N$;
- branches of a tree T parametrized by $G_T \subset \cup_{n\in\mathbf{N}}(\mathbf{N}^*)^n$: $\mathscr{A} = G_T$, endowed with the usual intersection on tree structures. Note that $\mathscr{A}$ is a discrete indexing collection.

More complex indexing collections can be obtained by considering the Cartesian product of simpler ones. This construction procedure is described more thoroughly in Section 4.2.5.

The notations $\mathscr{A}(u)$ and $\mathscr{C}$ respectively refer to the class of finite unions of sets from $\mathscr{A}$ and to the collection of increments $C = A \setminus B$, where $A \in \mathscr{A}$ and $B \in \mathscr{A}(u)$. As mentioned in [89], the assumption *Shape* on $\mathscr{A}$ implies the existence of a unique *extremal representation* $\{A_i\}_{1\leq i\leq k}$ of every $B \in \mathscr{A}(u)$, i.e. such that $B = \cup_{i=1}^k A_i$ and for every $i \neq j$, $A_i \nsubseteq A_j$. Hence, from now on, to any increment $C \in \mathscr{C}$ is associated the unique $A \in \mathscr{A}$ and $B = \cup_{i=1}^k A_i$ such that $C = A \setminus B$, $B \subseteq A$ and $\{A_i\}_{i\leq k}$ is the extremal representation of B.

Then, for any $C = A \setminus B$, $B = \cup_{i=1}^k A_i$, $\mathscr{A}_C$ denotes the following subset of $\mathscr{A}$,

$$\mathscr{A}_C = \{U \in \mathscr{A}_\ell;\, U \nsubseteq B^\circ\} := \{U_C^1, \cdots, U_C^p\}, \qquad \text{where } p = |\mathscr{A}_C| \tag{4.1}$$

and $\mathscr{A}_\ell$ denotes the semilattice $\{A_1 \cap \cdots \cap A_k, \ldots, A_1 \cap A_2, A_1 \ldots, A_k\} \subset \mathscr{A}$. For any set-indexed stochastic process X and any $C \in \mathscr{C}$, the notation $\mathbf{X}_C$ refers to the random vector $\mathbf{X}_C = \left(X_{U_C^1}, \ldots, X_{U_C^p}\right)$. Similarly, $\mathbf{x}_C$ is used to denote a vector of variables $(x_{U_C^1}, \ldots, x_{U_C^p})$.

Lastly, for any set-indexed filtration $\mathscr{F} = \{\mathscr{F}_A; A \in \mathscr{A}\}$, collections $(\mathscr{F}_B)_{B\in\mathscr{A}(u)}$ and $(\mathscr{G}_C^*)_{C\in\mathscr{C}}$ are respectively defined as

$$\mathscr{F}_B := \bigvee_{A\in\mathscr{A}, A\subseteq B} \mathscr{F}_A \qquad \text{and} \qquad \mathscr{G}_C^* := \bigvee_{B\in\mathscr{A}(u), B\cap C=\emptyset} \mathscr{F}_B. \tag{4.2}$$

The family $(\mathscr{G}_C^*)_{C\in\mathscr{C}}$ is usually called the *strong history*. Note that these filtrations do not necessarily satisfy any outer-continuity property, on the contrary to the augmented filtrations defined later.

We can now present the set-indexed increment Markov property, abbreviated $\mathscr{C}$*-Markov* property ($\mathscr{C}$ denoting the class of increment sets) which is studied in the present work.

Definition 4.1 (**$\mathscr{C}$-Markov property**) *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and E be a metric space. An E-valued set-indexed process X is said to be $\mathscr{C}$-*Markov *with respect to (w.r.t.) the filtration $(\mathscr{F}_A)_{A\in\mathscr{A}}$ if it is adapted to $(\mathscr{F}_A)_{A\in\mathscr{A}}$ and if it satisfies*

$$\mathbb{E}[f(X_A) \,|\, \mathscr{G}_C^*] = \mathbb{E}[f(X_A) \,|\, \mathbf{X}_C] \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

for all $C = A \setminus B \in \mathscr{C}$ and any bounded measurable function $f : E \to \mathbf{R}$.

$\mathscr{C}$-Markov processes are first studied in the general set-indexed framework (Section 4.2). We introduce the concept of $\mathscr{C}$*-transition system*, usually designated $\mathscr{P} = \{P_C(\mathbf{x}_C; \mathrm{d}x_A); C \in \mathscr{C}\}$, associated to the $\mathscr{C}$-Markov property. It extends in a natural way the classic definition of transition operators for one-parameter processes, and in particular, the Chapman–Kolmogorov equation which becomes:

$$\forall C \in \mathscr{C}, A' \in \mathscr{A}; \qquad P_C f = P_{C'} P_{C''} f \quad \text{where } C' = C \cap A', C'' = C \setminus A' \tag{4.4}$$

and f is a bounded measurable function. This notion happens to properly suit the $\mathscr{C}$-Markov property. Indeed, our main result (Theorems 4.2 and 4.3 in Section 4.2.1) states that the initial law of $X_{\emptyset'}$ and the transition probabilities naturally deduced from a $\mathscr{C}$-Markov process form a $\mathscr{C}$-transition system which characterizes entirely the finite-dimensional distributions. Moreover, given an initial measure ν and $\mathscr{C}$-transition system $\mathscr{P}$, we state that a corresponding $\mathscr{C}$-Markov process can be constructed on the canonical space.

As presented in Section 4.2.2, the $\mathscr{C}$-Markov property has connections with the set-indexed Markovian literature, especially the work of IVANOFF and MERZBACH [88]. However, we note there that it clearly differs from the $\mathscr{Q}$-*Markov* approach of BALAN and IVANOFF [18].

$\mathscr{C}$-Markov processes appear to satisfy several interesting properties (Sections 4.2.3 and 4.2.4). Theorem 4.6 proves that the natural filtration of $\mathscr{C}$-Markov process displays a *conditional independence* property. Moreover, classic *simple* and *strong Markov* properties can be extended to the $\mathscr{C}$-Markov formalism (Theorems 4.7 and 4.9) under an homogeneity assumption on the $\mathscr{C}$-transition system, and for the latter, with the help of the notion of $\mathscr{C}$-Feller processes. To illustrate these different concepts, several examples are given in Section 4.2.5, including an overview of set-indexed Lévy processes and the construction of a set-indexed $\mathscr{S}\alpha\mathscr{S}$ Ornstein–Uhlenbeck process.

Section 4.3 is devoted to the specific properties satisfied by multiparameter $\mathscr{C}$-Markov processes. Specifically, Theorem 4.10 states that multiparameter $\mathscr{C}$-Feller processes have a right-continuous modification which is $\mathscr{C}$-Markov with respect to the augmented filtration. Connections with the multiparameter Markovian literature are also investigated, especially with the two-parameter $*$-Markov property and the recent multiparameter developments presented by KHOSHNEVISAN [100]. Several examples of common multiparameter $\mathscr{C}$-Markov processes are given at the end of the section.

Finally, the Gaussian set-indexed Ornstein–Uhlenbeck process is studied in more details in Section 4.4, focusing in particular on the construction procedure and the characterisation in the stationary case.

4.2 Set-indexed $\mathscr{C}$-Markov property

We begin this section with a few observations and remarks.

In this work are considered E-valued set-indexed processes $X = \{X_A; A \in \mathscr{A}\}$, where (E, d_E) is a locally compact separable metric space endowed with the Borel sigma-algebra $\mathscr{E}$. If $(E, \mathscr{E}) = (\mathbf{R}^d, \mathscr{B}(\mathbf{R}^d))$, then the assumption *shape* on the indexing collection allows to define the extension ΔX of X on the classes $\mathscr{A}(u)$ and $\mathscr{C}$. It is given by the following inclusion-exclusion formulas:

$$\Delta X_B := \sum_{i=1}^{k} X_{A_i} - \sum_{i<j} X_{A_i \cap A_j} + \cdots + (-1)^{k+1} X_{A_1 \cap \cdots \cap A_k} \quad \text{and} \quad \Delta X_C := X_A - \Delta X_B.$$

Due to Lemma 3.4 by IVANOFF and MERZBACH [88] and using the notations previously introduced, we observe that the previous formulas can be equivalently written as follows:

$$\Delta X_B = \sum_{i=1}^{|\mathscr{A}_C|} (-1)^{\varepsilon_i} X_{U_C^i} \qquad \text{and} \qquad \Delta X_C = X_A - \left[\sum_{i=1}^{|\mathscr{A}_C|} (-1)^{\varepsilon_i} X_{U_C^i} \right], \tag{4.5}$$

where $(-1)^{\varepsilon_i}$ corresponds the sign in front of $X_{U_C^i}$ in the inclusion-exclusion formula. In other words, Equation (4.5) states that every term X_V such that $V \notin \mathscr{A}_C$ is cancelled by another element in the inclusion-exclusion formula.

In consequence, we note that Definition 4.1 can be equivalently written as follows: an $\mathbf{R}^d$-valued set-indexed process X is $\mathscr{C}$-Markov with respect to $(\mathscr{F}_A)_{A \in \mathscr{A}}$ if for all $C \in \mathscr{C}$ and any

measurable function $f : E \to \mathbf{R}_+$,

$$\mathbb{E}[f(\Delta X_C) \,|\, \mathscr{G}_C^*] = \mathbb{E}[f(\Delta X_C) \,|\, \mathbf{X}_C] \quad \mathbb{P}\text{-a.s.}$$

Indeed, owing to Equation (4.5), ΔX_B is measurable with respect to $\mathscr{G}_C^*$ and $\sigma(\mathbf{X}_C)$, and therefore, the equality $X_A = \Delta X_C + \Delta X_B$ and classic properties of the conditional expectation imply the equivalence of the two definitions.

Finally, let us recall that the natural filtration of a set-indexed process X is defined by $\mathscr{F}_A = \sigma(\{X_V; V \subseteq A, V \in \mathscr{A}\})$ for all $A \in \mathscr{A}$. Filtrations are always supposed to be complete. Based on Definition 4.1, we observe that any $\mathscr{C}$-Markov process is always $\mathscr{C}$-Markov with respect to its natural filtration.

4.2.1 $\mathscr{C}$-transition system: characterization and construction results

In the light of Definition 4.1 of $\mathscr{C}$-Markov processes, there is a natural way to introduced the concept of $\mathscr{C}$*-transition system*.

Definition 4.2 (**$\mathscr{C}$-transition system**) *A collection $\mathscr{P} = \{P_C(\mathbf{x}_C; \mathrm{d}x_A);\, C \in \mathscr{C}\}$ is called a* $\mathscr{C}$-transition system *if it satisfies the following properties:*

1. *for all $C \in \mathscr{C}$, $P_C(\mathbf{x}_C; \mathrm{d}x_A)$ is a transition probability, i.e. for all $\mathbf{x}_C \in E^{|\mathscr{A}_C|}$, $P_C(\mathbf{x}_C; \cdot)$ is a probability measure on $(E, \mathscr{E})$ and for all $\Gamma \in \mathscr{E}$, $P_C(\cdot\,; \Gamma)$ is a measurable function;*

2. *for all $x \in E$ and $\Gamma \in \mathscr{E}$, $P_\emptyset(x; \Gamma) = \delta_x(\Gamma)$;*

3. *for all $C \in \mathscr{C}$ and any $A' \in \mathscr{A}$, let $C' = C \cap A'$ and $C'' = C \setminus A'$. Then, $\mathscr{P}$ satisfies a Chapman–Kolmogorov like equation:*

$$P_C(\mathbf{x}_C; \Gamma) = \int_E P_{C'}(\mathbf{x}_{C'}; \mathrm{d}x_{A'})\, P_{C''}(\mathbf{x}_{C''}; \Gamma), \tag{4.6}$$

for all $\mathbf{x}_C \in E^{|\mathscr{A}_C|}$, $\mathbf{x}_{C'} \in E^{|\mathscr{A}_{C'}|}$, $\mathbf{x}_{C''} \in E^{|\mathscr{A}_{C''}|}$ and $\Gamma \in \mathscr{E}$.

Remark 4.1 Let $C = A \setminus B \in \mathscr{C}$ and $A' \in \mathscr{A}$. We observe that if $A' \subseteq B$ or $A \subseteq A'$, then $C' = \emptyset$ or $C'' = \emptyset$ respectively, and therefore, Equation (4.6) is straight forward since one of the term is a Dirac distribution.

On the other hand, if $A' \nsubseteq B$ and $A' \subset A$, Equation (4.6) is still consistent. Indeed, as $C'' = A \setminus (A' \cup B)$, and $A' \nsubseteq B$, we have $A' \in \mathscr{A}_{C''}$, and thus, the variable $x_{A'}$ is one of the component of the vector $\mathbf{x}_{C''}$.

We also note that Equation (4.6) implicitly induces that the integral does not depend on any variable x_V with $V \notin \mathscr{A}_C$, since these terms do not appear in the left-term.

Remark 4.2 In the particular case of $\mathscr{T} = \mathbf{R}_+$ and $\mathscr{A} = \{[0, t];\, t \in \mathbf{R}_+\}$, Definition 4.2 corresponds to the usual definition of a transition system. Indeed, as $\mathscr{A}(u) = \mathscr{A}$ and $\mathscr{C} = \{(s, t];\, s, t \in \mathbf{R}_+^2\}$, $\mathscr{P}$ is indexed by $\mathbf{R}_+^2$: $\mathscr{P} = \{P_{s,t}(x; \mathrm{d}y);\, (s, t) \in \mathbf{R}_+^2\}$. Then, Definition 4.2 states that:

1. for all $s, t \in \mathbf{R}_+^2$, $P_{s,t}(x; \mathrm{d}y)$ is a transition probability;

2. for all $s \in \mathbf{R}_+$, $P_{s,s}(x, \mathrm{d}y) = \delta_x(\mathrm{d}y)$;

3. for all $s < s' < t \in \mathbf{R}_+$, $x \in E$ and $\Gamma \in \mathscr{E}$, $\mathscr{P}$ satisfies $P_{s,t}(x; \Gamma) = \int_E P_{s,s'}(x; \mathrm{d}y)\, P_{s',t}(y; \Gamma)$ (using previous notations, $C = (s, t]$, $C' = (s', t]$ and $C'' = (s, s']$).

Proposition 4.1 proves that Definition 4.2 of a $\mathscr{C}$-transition systems is coherent with the $\mathscr{C}$-Markov property.

Proposition 4.1 *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and X be a $\mathscr{C}$-Markov process w.r.t. $(\mathscr{F}_A)_{A\in\mathscr{A}}$. For all $C = A\setminus B \in \mathscr{C}$, define $P_C(\mathbf{x_C}; \mathrm{d}y)$ as follows:*

$$\forall \mathbf{x_C} \in E^{|\mathscr{A}_C|}, \Gamma \in \mathscr{E}; \quad P_C(\mathbf{x_C}; \Gamma) := \mathbb{P}(X_A \in \Gamma \,|\, \mathbf{X_C} = \mathbf{x_C}).$$

Then, the collection $\mathscr{P} = \{P_C(\mathbf{x_C}; \mathrm{d}y);\ C \in \mathscr{C}\}$ is a $\mathscr{C}$-transition system.

Proof. Let us verify the different points of Definition 4.2.

1. Clearly, for any $C = A\setminus B \in \mathscr{C}$, $P_C(\mathbf{x_C}; \mathrm{d}x_A)$ is a transition probability.

2. The equality $P_\emptyset(x; \mathrm{d}y) = \delta_x(\mathrm{d}y)$ holds since, $\mathscr{G}^*_\emptyset = \vee_{V\in\mathscr{A}}\mathscr{F}_V$ and therefore, for any f bounded measurable function and any $A \in \mathscr{A}$, $\mathbb{E}[\,f(X_A) \,|\, \mathscr{G}^*_\emptyset\,] = f(X_A)$.

3. For all $C = A\setminus B \in \mathscr{C}$ and all $A' \in \mathscr{A}$ such that $A' \subseteq A$ and $A' \nsubseteq B$, let $C' = C \cap A' = A' \setminus (A' \cap B)$ and $C'' = C \setminus A' = A \setminus (A' \cup B)$. Since $C'' \subset C$, we observe that $\mathscr{G}^*_C \subseteq \mathscr{G}^*_{C''}$ and thus, for any $\Gamma \in \mathscr{E}$,

$$P_C(\mathbf{X_C}; \Gamma) = \mathbb{P}(X_A \in \Gamma \,|\, \mathscr{G}^*_C) = \mathbb{P}\big(\mathbb{P}(X_A \in \Gamma \,|\, \mathscr{G}^*_{C''}) \,\big|\, \mathscr{G}^*_C\big) = \mathbb{E}[\,P_{C''}(\mathbf{X_{C''}}; \Gamma) \,|\, \mathscr{G}^*_C\,]. \tag{4.7}$$

Furthermore, $\mathscr{G}^*_C \subseteq \mathscr{G}^*_{C'}$ and the vector $\mathbf{X_{C'}}$ is $\mathscr{G}^*_C$-measurable ($A' \cap B \subset B$). Hence, for any positive measurable function h,

$$\mathbb{E}[\,h(X_{A'}) \,|\, \mathscr{G}^*_C\,] = \mathbb{E}\big[\,\mathbb{E}[\,h(X_{A'}) \,|\, \mathscr{G}^*_{C'}\,] \,\big|\, \mathscr{G}^*_C\,\big] = \int_{\mathbf{R}} P_{C'}(\mathbf{X_{C'}}; \mathrm{d}x_{A'})\, h(x_{A'}). \tag{4.8}$$

As $C'' = A\setminus(A' \cup B)$, we observe that $X_{A'}$ is the only term in the vector $\mathbf{X_{C''}}$ which is not $\mathscr{G}^*_C$-measurable. Therefore, using a monotone class argument, Equations (4.7) and (4.8) lead to the expected equality.

□

The next two theorems gather the main result of this section: for any $\mathscr{C}$-Markov process, its initial distribution and its $\mathscr{C}$-transition system characterize entirely the law of the process. Conversely, from any probability measure and any $\mathscr{C}$-transition system, a corresponding canonical $\mathscr{C}$-Markov process can be constructed. To our knowledge, such a result does not exist for other set-indexed Markov properties, or at least require some tricky technical assumption in the case of $\mathscr{Q}$-Markov.

Theorem 4.2 *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and X be a $\mathscr{C}$-Markov process w.r.t. $(\mathscr{F}_A)_{A\in\mathscr{A}}$.*

Then, the initial distribution ν (i.e. the law of $X_{\emptyset'}$) and the $\mathscr{C}$-transition system $\mathscr{P} = \{P_C(\mathbf{x_C}; \mathrm{d}x_A);\ C \in \mathscr{C}\}$ of the process X characterize entirely its law.

Proof. We have to express the finite-dimensional distributions of X in terms of ν and $\mathscr{P}$.

Let $A_1, \dots, A_k \in \mathscr{A}$ and $f : E^k \to \mathbf{R}_+$ be a measurable function. Without any loss of generality, we can assume that $\mathscr{A}_\ell = \{A_0 = \emptyset', A_1, \dots, A_k\}$ is a finite semilattice with consistent numbering, i.e. stable under intersection and such that $A_j \subseteq A_i$ implies $j \le i$.

Let $C_1, \dots, C_k \in \mathscr{C}$ be the left-neighbourhoods $C_i = A_i \setminus (\cup_{j=0}^{j-1} A_j)$, $1 \le i \le k$. Since $\mathscr{A}_\ell$ is semilattice, $\mathscr{A}_{C_i} \subset \mathscr{A}_\ell$ for any $i \in \{1, \dots, k\}$.

Then, if we consider the law of $(X_{A_1}, \dots, X_{A_k})$, we observe that

$$\begin{aligned}\mathbb{E}[f(X_{A_1}, \dots, X_{A_k})] &= \mathbb{E}\big[\,\mathbb{E}[\,f(X_{A_1}, \dots, X_{A_k}) \,|\, \mathscr{G}^*_{C_k}\,]\,\big] \\ &= \mathbb{E}\Big[\int_E P_{C_k}(\mathbf{X_{C_k}}; \mathrm{d}x_{A_k}) f(X_{A_1}, \dots, X_{A_{k-1}}, x_{A_k})\Big],\end{aligned}$$

using a monotone class argument and the $\mathscr{G}^*_{C_k}$-measurability of the vector $(X_{A_1},\dots,X_{A_{k-1}})$. Therefore, by induction,

$$\begin{aligned}
&\mathbb{E}[f(X_{A_1},\dots,X_{A_k})]\\
&=\mathbb{E}\Big[\int_{E^k} P_{C_1}(X_{\emptyset'};\mathrm{d}x_{A_1})P_{C_2}(\mathbf{X}_{C_2};\mathrm{d}x_{A_2})\cdots P_{C_k}(\mathbf{X}_{C_k};\mathrm{d}x_{A_k})f(x_{A_1},\dots,x_{A_k})\Big]\\
&=\int_{E^{k+1}} \nu(\mathrm{d}x_0)P_{C_1}(x_0;\mathrm{d}x_{A_1})P_{C_2}(\mathbf{X}_{C_2};\mathrm{d}x_{A_2})\cdots P_{C_k}(\mathbf{X}_{C_k};\mathrm{d}x_{A_k})f(x_{A_1},\dots,x_{A_k}),
\end{aligned}$$

since $\mathscr{A}_{C_1}=\{\emptyset'\}$. □

To obtain the construction theorem, we need the following technical lemma.

Lemma 4.1 *Consider a $\mathscr{C}$-transition system $\mathscr{P}=\{P_C(\mathbf{x}_C;\mathrm{d}x_A);\ C\in\mathscr{C}\}$ and two semilattices $\mathscr{A}^1=\{A^1_0=\emptyset',A^1_1,\dots,A^1_{n_1}\}$ and $\mathscr{A}^2=\{A^2_0=\emptyset',A^2_1,\dots,A^2_{n_2}\}$ consistently numbered and such that $\mathscr{A}^1\subseteq\mathscr{A}^2\subset\mathscr{A}$.*

Let $C^1_1,\dots,C^1_{n_1}\in\mathscr{C}$ and $C^2_1,\dots,C^2_{n_2}\in\mathscr{C}$ be the left-neighbourhoods of $\mathscr{A}^1$ and $\mathscr{A}^2$, respectively, i.e. $C^1_i=A^1_i\setminus(\cup_{j=0}^{i-1}A^1_j)$ and $C^2_i=A^2_i\setminus(\cup_{j=0}^{i-1}A^2_j)$. Then, for all $x\in E$ and any positive measurable function $f:E^{n_1}\mapsto\mathbf{R}_+$, $\mathscr{P}$ satisfies

$$\begin{aligned}
&\int_{E^{n_1}} P_{C^1_1}(x;\mathrm{d}x_{A^1_1})P_{C^1_2}(\mathbf{x}_{C^1_2};\mathrm{d}x_{A^1_2})\cdots P_{C^1_{n_1}}(\mathbf{x}_{C^1_{n_1}};\mathrm{d}x_{A^1_{n_1}})f(x_{A^1_1},\dots,x_{A^1_{n_1}})\\
&=\int_{E^{n_2}} P_{C^2_1}(x;\mathrm{d}x_{A^2_1})P_{C^2_2}(\mathbf{x}_{C^2_2};\mathrm{d}x_{A^2_2})\cdots P_{C^2_{n_2}}(\mathbf{x}_{C^2_{n_2}};\mathrm{d}x_{A^2_{n_2}})f(x_{A^1_1},\dots,x_{A^1_{n_1}}).
\end{aligned}\tag{4.9}$$

We note that since $\mathscr{A}^1\subseteq\mathscr{A}^2$, every variable $x_{A^1_i}$ has a corresponding term $x_{A^2_{j_i}}$ where $j_i\in\{1,\dots,n_2\}$, ensuring the consistency of the second term in Equation (4.9).

Proof. We proceed by induction on n_2. Let $k\in\{1,\dots,n_2\}$ be the highest index such that $A^2_k\in\mathscr{A}^2$ and $A^2_k\notin\mathscr{A}^1$. Let $\mathscr{A}^{2'}$ be $\mathscr{A}^2\setminus\{A^2_k\}$. We observe that $\mathscr{A}^{2'}$ is semilattice, since otherwise, we would have $A^2_k\in\mathscr{A}^1$. We consider two different cases.

1. Suppose first that for every $i\in\{k+1,\dots,n_2\}$, $A^2_k\notin\mathscr{A}_{C^2_i}$. Then, we know that for any $i\in\{1,\dots,n_2\}\setminus\{k\}$, the term $x_{A^2_k}$ does not appear in the component $P_{C_i}(\mathbf{x}_{C^2_i};\mathrm{d}x_{A^2_i})$. Hence, we can integrate over the variable $x_{A^2_k}$, and since $\int_E P_{C^2_k}(\mathbf{x}_{C^2_k};\mathrm{d}x_{A^2_k})=1$, we obtain a formula which corresponds to the case of the semilattice $\mathscr{A}^{2'}$.

2. Assume now that there exist $i_m>\cdots>i_1>k$ such that $A^2_k\in\mathscr{A}_{C^2_{i_j}}$ for every $j\in\{1,\dots,m\}$. Let us first suppose that $m\geq 2$.

   Then, there exists $i_0\in\{1,\dots,i_1\}$ such that $A^2_{i_0}=A^2_{i_1}\cap A^2_{i_2}$. As $A^2_k\subseteq A^2_{i_1}\cap A^2_{i_2}$, we have $k\leq i_0$. Furthermore, $A^2_{i_0}=A^2_k$ is not possible since $A^2_{i_1},A^2_{i_2}$, and thus $A^2_{i_0}$, belong to $\mathscr{A}^1$. Thereby, we necessarily have $A^2_k\in\mathscr{A}_{C^2_{i_0}}$, which implies $i_0=i_1$ because of i_1's definition.

   Hence, $A^2_{i_1}\subset A^2_{i_2}$. But, since $A^2_k\subset A^2_{i_1}$, we have

$$C^2_{i_2}:=A^2_{i_2}\setminus(\cup_{j=0}^{i_2-1}A^2_j)=A^2_{i_2}\setminus(\cup_{j=0,j\neq k}^{i_2-1}A^2_j).$$

   Furthermore, we note that $\{A^2_1,\dots,A^2_{i_2-1}\}\setminus\{A^2_k\}$ is a semilattice (otherwise we would have $A^2_k\in\mathscr{A}^1$), therefore stable under intersections. Hence, due to the previous observations,

$\mathscr{A}_{C^2_{i_2}} \subseteq \{A^2_j\,;\, 0 \leq j < i_2 \text{ and } j \neq k\}$, which is in contradiction with the assumption $A^2_k \in \mathscr{A}_{C^2_{i_2}}$.

Therefore, $m = 1$ and without any loss of generality, we can assume that $i_1 = k+1$. The variable $x_{A^2_k}$ is only present in the vector $\mathbf{x}_{C^2_{k+1}}$ and thus, our problem is reduced to the computation of the sub-integral $\int_E P_{C^2_k}(\mathbf{x}_{C^2_k}; \mathrm{d}x_{A^2_k})\, P_{C^2_{k+1}}(\mathbf{x}_{C^2_{k+1}}; \Gamma)$, where $\Gamma \in \mathscr{E}$.

Let $C \in \mathscr{C}$ be $C = A^2_{k+1} \setminus (\cup_{j=0}^{k-1} A^2_j)$. We observe that $C^2_k = C \cap A^2_k$ and $C^2_{k+1} = C \setminus A^2_k$. Therefore, owing to the Chapman–Kolmogorov Equation (4.6),

$$P_C(\mathbf{x}_C; \Gamma) = \int_E P_{C^2_k}(\mathbf{x}_{C^2_k}; \mathrm{d}x_{A^2_k})\, P_{C^2_{k+1}}(\mathbf{x}_{C^2_{k+1}}; \Gamma).$$

Finally, if we insert the previous formula in Equation (4.9), we obtain an integral which also corresponds to the case of the semilattice $\mathscr{A}^{2'}$.

In both cases, the right-integral is simplified into a formula corresponding to the case of the semilattice $\mathscr{A}^{2'} = \mathscr{A}^2 \setminus \{A^2_k\}$. Hence, by induction on the size of $\mathscr{A}^2$, we obtain Equation (4.9). □

We are now able to prove that a canonical $\mathscr{C}$-Markov process can be constructed from any initial probability measure and any $\mathscr{C}$-transition system.

In the following proposition, Ω designates the canonical space $E^{\mathscr{A}}$ endowed with its usual σ-field $\mathscr{F}$ generated by coordinate applications. X denotes the canonical process, i.e. for all $A \in \mathscr{A}$ and all $\omega \in \Omega$, $X_A(\omega) = \omega(A)$ and $(\mathscr{F}_A)_{A \in \mathscr{A}}$ is its natural filtration (as defined at the beginning of this section).

Theorem 4.3 *Let $\mathscr{P} = \{P_C(\mathbf{x}_C; \mathrm{d}x_A);\, C \in \mathscr{C}\}$ be a $\mathscr{C}$-transition system and ν be a probability measure ν on $(E, \mathscr{E})$.*

Then, there exists a unique probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathscr{F})$ such that X is a $\mathscr{C}$-Markov process w.r.t. $(\mathscr{F}_A)_{A \in \mathscr{A}}$ and whose initial measure and $\mathscr{C}$-transition system are ν and $\mathscr{P}$, respectively:

$$X_{\emptyset'} \sim \nu \qquad \text{and} \qquad \mathbb{P}_\nu(X_A \in \Gamma \mid \mathscr{G}^*_C) = P_C(\mathbf{X}_C; \Gamma) \quad \mathbb{P}_\nu\text{-a.s.}$$

for all $C = A \setminus B \in \mathscr{C}$ and all $\Gamma \in \mathscr{E}$.

Proof. To construct the measure $\mathbb{P}_\nu$, we need to define for any $A_1, \ldots, A_k \in \mathscr{A}$ a probability measure $\mu_{A_1 \ldots A_k}$ on $(E^k, \mathscr{E}^{\otimes k})$ such that this family of probabilities satisfies the usual consistency conditions of Kolmogorov's extension theorem.

1. Let $A_1, \ldots, A_k$ be k distinct sets in $\mathscr{A}$ and π be a permutation on $\{1, \ldots, k\}$. Then,

$$\mu_{A_1 \ldots A_k}(\Gamma_1 \times \cdots \times \Gamma_k) = \mu_{A_{\pi(1)} \ldots A_{\pi(k)}}(\Gamma_{\pi(1)} \times \cdots \times \Gamma_{\pi(k)}) \tag{4.10}$$

for all $\Gamma_1, \ldots, \Gamma_k \in \mathscr{E}$.

2. Let $A_1, \ldots, A_k, A_{k+1}$ be $k+1$ distinct sets in $\mathscr{A}$. Then,

$$\mu_{A_1 \ldots A_k}(\Gamma_1 \times \cdots \times \Gamma_k) = \mu_{A_1 \ldots A_k A_{k+1}}(\Gamma_1 \times \cdots \times \Gamma_k \times E) \tag{4.11}$$

for all $\Gamma_1, \ldots, \Gamma_k \in \mathscr{E}$.

For any $A_1, \ldots, A_k$ distinct sets in $\mathscr{A}$, we consider $\mathscr{A}' = \{A'_0 = \emptyset', A'_1, \ldots, A'_m\}$ the smallest semilattice generated by $\{\emptyset', A_1, \ldots, A_k\}$. Let $C_1, \ldots, C_m \in \mathscr{C}$ be the left-neighbourhoods on

$\mathscr{A}'$: $C_i = A'_i \setminus (\cup_{j=0}^{i-1} A'_j)$. Then, we define the probability measure $\mu_{A_1 \dots A_k}$ on $(E^k, \mathscr{E}^{\otimes k})$ as follows:

$$\begin{aligned}
&\mu_{A_1 \dots A_k}(\Gamma_1 \times \cdots \times \Gamma_k) \\
&= \int_{E^{m+1}} \nu(\mathrm{d}x_0) P_{C_1}(x_0; \mathrm{d}x_{A'_1}) P_{C_2}(\mathbf{x}_{C_2}; \mathrm{d}x_{A'_2}) \cdots P_{C_m}(\mathbf{x}_{C_m}; \mathrm{d}x_{A'_m}) \mathbf{1}_{\Gamma_1}(x_{A_1}) \cdots \mathbf{1}_{\Gamma_k}(x_{A_k}).
\end{aligned}$$

where $\Gamma_1, \dots, \Gamma_k \in \mathscr{E}$. Note that for every variable x_{A_i}, $i \in \{1, \dots, k\}$, there exists a corresponding $x_{A'_{j_i}}$, $j_i \in \{1, \dots, m\}$. Moreover, the formula above does not depend on the numbering of $\mathscr{A}'$, since neither C_i nor $\mathbf{x}_{C_i}$, $i \in \{1, \dots, m\}$, are changed with a different consistent ordering. These two remarks ensure that the previous definition is consistent.

Consider now the first consistency condition that must satisfy $\mu_{A_1 \dots A_k}$. Let $A_1, \dots, A_k$ be k distinct sets in $\mathscr{A}$, $\Gamma_1, \dots, \Gamma_k$ be in $\mathscr{E}$ and π be a permutation on $\{1, \dots, k\}$. Then, we know that families $\{A_1, \dots, A_k\}$ and $\{A_{\pi(1)}, \dots, A_{\pi(k)}\}$ lead to the same semilattice $\mathscr{A}' = \{A'_0 = \emptyset', A'_1, \dots, A'_m\}$. Hence, since the definition of $\mu_{A_1 \dots A_k}$ does not depend on the ordering of the semilattice $\mathscr{A}'$, Equation (4.10) is verified.

In order to check the second consistency condition, let $A_1, \dots, A_{k+1}$ be $k+1$ distinct sets in $\mathscr{A}$ and $\Gamma_1, \dots, \Gamma_k$ be in $\mathscr{E}$. Let $\mathscr{A}^1 = \{A^1_0 = \emptyset', A^1_1, \dots, A^1_{n_1}\}$ and $\mathscr{A}^2 = \{A^2_0 = \emptyset', A^2_1, \dots, A^2_{n_2}\}$ be the two semilattices generated by $\{A_1, \dots, A_k\}$ and $\{A_1, \dots, A_k, A_{k+1}\}$, respectively. We clearly have $\mathscr{A}^1 \subseteq \mathscr{A}^2$. Let finally $f : E^{n_1} \mapsto \mathbf{R}_+$ be a positive measurable function defined as follows:

$$\forall (x_{A^1_1}, \dots, x_{A^1_{n_1}}) \in E^{n_1}; \quad f(x_{A^1_1}, \dots, x_{A^1_{n_1}}) = \mathbf{1}_{\Gamma_1}(x_{A_1}) \cdots \mathbf{1}_{\Gamma_k}(x_{A_k}).$$

The definition of f is consistent since $\{A_1, \dots, A_k\} \subseteq \mathscr{A}^1$. Furthermore, if we apply Lemma 4.1 to $\mathscr{A}^1$ and $\mathscr{A}^2$, we obtain

$$\begin{aligned}
&\int_{E^{n_1+1}} \nu(\mathrm{d}x_0) P_{C^1_1}(\mathbf{x}_{C^1_0}; \mathrm{d}x_{A^1_1}) \cdots P_{C^1_{n_1}}(\mathbf{x}_{C^1_{n_1}}; \mathrm{d}x_{A^1_{n_1}}) \mathbf{1}_{\Gamma_1}(x_{A_1}) \cdots \mathbf{1}_{\Gamma_k}(x_{A_k}) \\
&= \int_{E^{n_2+1}} \nu(\mathrm{d}x_0) P_{C^2_1}(\mathbf{x}_{C^2_0}; \mathrm{d}x_{A^2_1}) \cdots P_{C^2_{n_2}}(\mathbf{x}_{C^2_{n_2}}; \mathrm{d}x_{A^2_{n_2}}) \mathbf{1}_{\Gamma_1}(x_{A_1}) \cdots \mathbf{1}_{\Gamma_k}(x_{A_k}) \mathbf{1}_{\mathbf{R}}(x_{A_{k+1}}),
\end{aligned}$$

which exactly corresponds to the second consistency Equation (4.11).

Therefore, using Kolmogorov's extension theorem (see e.g. Appendix A in [100]), there exists a probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathscr{F})$ such that for all $A_1, \dots, A_k$ in $\mathscr{A}$ and $\Gamma_1, \dots, \Gamma_k$ in $\mathscr{E}$,

$$\mathbb{P}_\nu(X_{A_1} \in \Gamma_1, \dots, X_{A_k} \in \Gamma_k) = \mu_{A_1 \dots A_k}(\Gamma_1 \times \cdots \times \Gamma_k),$$

where X designates the canonical process. A usual monotone class argument ensures the uniqueness of this probability measure.

Our last point in this proof is to ensure that under $\mathbb{P}_\nu$, X is a $\mathscr{C}$-Markov process w.r.t. $(\mathscr{F}_A)_{A \in \mathscr{A}}$ whose initial measure and $\mathscr{C}$-transition system are ν and $\mathscr{P}$. The first point is clear, since for any $\Gamma_0 \in \mathscr{E}$, $\mathbb{P}_\nu(X_{\emptyset'} \in \Gamma_0) = \nu(\Gamma_0)$.

Consider now $C = A \backslash B$ in $\mathscr{C}$, where $B = \cup_{i=1}^k A_i$. Let $A'_1, \dots, A'_m$ be in $\mathscr{A}$ such that $A'_i \cap C = \emptyset$ for every $i \in \{1, \dots, m\}$. Without any loss of generality, we may assume that $\mathscr{A}' = \{A'_0 = \emptyset', A'_1, \dots, A'_m, A\}$ is a semilattice with consistent numbering such that $\{A_i\}_{i \le k} \subseteq \mathscr{A}'$. Let $C_1, \dots, C_m \in \mathscr{C}$ be the left-neighbourhoods $C_i = A'_i \setminus (\cup_{j=0}^{i-1} A'_j)$. We note that $C = A \setminus (\cup_{j=0}^m A'_j)$ and $X_{A'_i}$ is $\mathscr{G}^*_C$-measurable for every $i \in \{1, \dots, m\}$. Therefore, for any $\Gamma, \Gamma_1, \dots, \Gamma_m \in \mathscr{E}$, we

have

$$\begin{aligned}
&\mathbb{E}_\nu\left[\mathbf{1}_{\Gamma_1}(X_{A'_1})\cdots\mathbf{1}_{\Gamma_m}(X_{A'_m})\,\mathbb{P}_\nu(X_A\in\Gamma\mid\mathscr{G}_C^*)\right]\\
&=\mathbb{P}_\nu\left(X_{A'_1}\in\Gamma_1,\dots,X_{A'_m}\in\Gamma_m,X_A\in\Gamma\right)\\
&=\int_E\nu(\mathrm{d}x_0)\Big\{\int_{E^{m+1}}P_{C_1}(x_0;\mathrm{d}x_{A'_1})\cdots P_{C_m}(\mathbf{x}_{\boldsymbol{C_m}};\mathrm{d}x_{A'_m})\,P_C(\mathbf{x}_{\boldsymbol{C}};\mathrm{d}x_A)\\
&\qquad\qquad\mathbf{1}_{\Gamma_1}(x_{A'_1})\cdots\mathbf{1}_{\Gamma_m}(x_{A'_m})\,\mathbf{1}_\Gamma(x_A)\Big\}
\end{aligned}$$

The integral over x_A does not depend on the other terms, and is equal to $\int_E P_C(\mathbf{x}_{\boldsymbol{C}};\mathrm{d}x_A)\,\mathbf{1}_\Gamma(x_A)=P_C(\mathbf{x}_{\boldsymbol{C}};\Gamma)$. Hence,

$$\begin{aligned}
&\mathbb{E}_\nu\left[\mathbf{1}_{\Gamma_1}(X_{A'_1})\cdots\mathbf{1}_{\Gamma_m}(X_{A'_m})\,\mathbb{P}_\nu(X_A\in\Gamma\mid\mathscr{G}_C^*)\right]\\
&=\int_{E^{m+1}}\nu(\mathrm{d}x_0)P_{C_1}(x_0;\mathrm{d}x_{A'_1})\cdots P_{C_m}(\mathbf{x}_{\boldsymbol{C_m}};\mathrm{d}x_{A'_m})\,\mathbf{1}_{\Gamma_1}(x_{A'_1})\cdots\mathbf{1}_{\Gamma_m}(x_{A'_m})P_C(\mathbf{x}_{\boldsymbol{C}};\Gamma)\\
&=\mathbb{E}_\nu\left[\mathbf{1}_{\Gamma_1}(X_{A'_1})\cdots\mathbf{1}_{\Gamma_m}(X_{A'_m})\,P_C(\mathbf{X}_{\boldsymbol{C}};\Gamma)\right].
\end{aligned}$$

A monotone class argument allows to conclude the proof of the $\mathscr{C}$-Markov property, i.e. $\mathbb{P}_\nu(X_A\in\Gamma\mid\mathscr{G}_C^*)=P_C(\mathbf{X}_{\boldsymbol{C}};\Gamma)$ $\mathbb{P}_\nu$-almost surely for all $C\in\mathscr{C}$ and $\Gamma\in\mathscr{E}$. □

Common notations $\mathbb{P}_x(\,.\,)$ and $\mathbb{E}_x[\,.\,]$ are used later in the case of Dirac initial distributions $\nu=\delta_x$, $x\in E$. Similarly to the classic Markov theory, if Z is a bounded random variable on the probability space $(\Omega,\mathscr{F},\mathbb{P})$, a monotone class argument shows that the map $x\mapsto\mathbb{E}_x[Z]$ is measurable and for any probability measure ν on $(E,\mathscr{E})$,

$$\mathbb{E}_\nu[Z]=\int_E\nu(\mathrm{d}x)\,\mathbb{E}_x[Z].$$

4.2.2 Set-indexed Markov properties

As outlined in the introduction, several set-indexed Markov properties have already been investigated in the literature. Hence, it seems natural to wonder if the $\mathscr{C}$-Markov property is related to some of them.

To begin with, let us recall the definitions of the *Markov* and *sharp Markov* properties presented by IVANOFF and MERZBACH [88].

Definition 4.3 (**Markov SI processes [88]**) *A set-indexed process X is said to be* Markov *if for all $B\in\mathscr{A}(u)$ and all $A_1,\dots,A_k\in\mathscr{A}$ such that $A_i\nsubseteq B$, $i\in\{1,\dots,k\}$,*

$$\mathbb{E}\left[f(X_{A_1},\dots,X_{A_k})\,\middle|\,\mathscr{F}_B\right]=\mathbb{E}\left[f(X_{A_1},\dots,X_{A_k})\,\middle|\,\mathscr{F}_{\partial B}\cap\mathscr{F}_{\cup_{i=1}^k A_i}\right]\qquad\mathbb{P}\text{-a.s.}$$

where $\mathscr{F}_{\partial B}$ denotes the σ-field $\sigma\left(\{X_A;A\in\mathscr{A},A\subseteq B,A\nsubseteq B^\circ\}\right)$ and f is a measurable function $f:\mathbf{R}^k\to\mathbf{R}_+$.

Definition 4.4 (**sharp Markov processes [88]**) *A set-indexed process X is said to be* sharp Markov *if for all $B\in\mathscr{A}(u)$,*

$$\mathscr{F}_B\perp\mathscr{F}_{B^c}\mid\mathscr{F}_{\partial B},\tag{4.12}$$

where $\mathscr{F}_{B^c}$ is defined as the σ-field $\sigma\left(\{X_A;A\in\mathscr{A},A\nsubseteq B\}\right)$. We recall that for any σ-fields $\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3$, the notation $\mathscr{F}_1\perp\mathscr{F}_2\mid\mathscr{F}_3$ means that $\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent given $\mathscr{F}_3$.

We may also compare our definition to the *set-Markov* property introduced by BALAN and IVANOFF [18] and deeply studied in both [18] and [19].

Definition 4.5 (set-Markov processes [18]) *A set-indexed process X is said to be* set-Markov *if for all $B, B' \in \mathscr{A}(u)$, $B \subseteq B'$ and for any measurable function $f : \mathbf{R} \to \mathbf{R}_+$,*

$$\mathbb{E}\big[\, f(\Delta X_{B'}) \,\big|\, \mathscr{F}_B \,\big] = \mathbb{E}\big[\, f(\Delta X_{B'}) \,\big|\, \Delta X_B \,\big] \qquad \mathbb{P}\text{-a.s.}$$

It has been proved in [88, 18] that the aforementioned Markov properties satisfy the following implications:

$$\textit{set-Markov} \Rightarrow \textit{sharp Markov} \quad \text{and} \quad \textit{Markov} \Rightarrow \textit{sharp Markov}.$$

The latter is an equivalence when the filtration verifies an assumption of *conditional orthogonality* (see Definition 2.3 in [88] for more details).

Proposition 4.4 *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and X be a $\mathscr{C}$-Markov process with respect to $(\mathscr{F}_A)_{A\in\mathscr{A}}$.*

Then, X is also a Markov *and a* Sharp Markov *process.*

Proof. Let X be *$\mathscr{C}$-Markov* on $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$. Since the *Markov* property implies *Sharp Markov*, we only have to prove that X is *Markov*.

Let $B = \cup_{i=1}^{l} A_i^1 \in \mathscr{A}(u)$ and $A_1, \dots, A_k \in \mathscr{A}$ such that $A_i \nsubseteq B$ for every $i \in \{1, \dots, k\}$. The collection of sets $\{A_1, \dots, A_k, A_1^1, \dots, A_l^1\}$ generates a semilattice $\mathscr{A}' = \{A'_0 = \emptyset', A'_1, \dots, A'_m\}$. Without any loss of generality, we may assume that the consistent numbering of $\mathscr{A}'$ is such that

$$\exists p \le m : \forall i \in \{1, \dots, m\}; \quad A'_i \nsubseteq B \Longleftrightarrow i \ge p.$$

$C_1, \dots, C_m \in \mathscr{C}$ designate the usual left-neighbourhoods $C_i = A'_i \setminus \cup_{j=0}^{i-1} A'_j$. Note that for all $i \in \{p, \dots, m\}$, $\mathscr{F}_B \subseteq G^*_{C_i}$ as $C_i \cap B = \emptyset$. Then, for any measurable function $f : \mathbf{R}^k \to \mathbf{R}_+$,

$$\begin{aligned}
\mathbb{E}[\, f(X_{A_1}, \dots, X_{A_k}) \,|\, \mathscr{F}_B \,] &= \mathbb{E}\big[\, \mathbb{E}[\, f(X_{A_1}, \dots, X_{A_k}) \,|\, \mathscr{G}^*_{C_m} \,] \,\big|\, \mathscr{F}_B \,\big] \\
&= \mathbb{E}\Big[\int_{\mathbf{R}} P_{C_m}(\mathbf{X}_{\boldsymbol{C_m}}; \mathrm{d}x_{A'_m}) f(X_{A_1}, \dots, X_{A_{k-1}}, x_{A'_m}) \,\Big|\, \mathscr{F}_B \Big],
\end{aligned}$$

where we assume that $A'_m = A_k$. Hence, we obtain by induction

$$\begin{aligned}
&\mathbb{E}[\, f(X_{A_1}, \dots, X_{A_k}) \,|\, \mathscr{F}_B \,] \\
&= \mathbb{E}\Big[\int_{\mathbf{R}^{m-p+1}} P_{C_p}(\mathbf{X}_{\boldsymbol{C_p}}; \mathrm{d}x_{A'_p}) \cdots P_{C_m}(\mathbf{X}_{\boldsymbol{C_m}}; \mathrm{d}x_{A'_m}) f(x_{A_1}, \dots, x_{A_k}) \,\Big|\, \mathscr{F}_B \Big] \\
&= \int_{\mathbf{R}^{m-p+1}} P_{C_p}(\mathbf{X}_{\boldsymbol{C_p}}; \mathrm{d}x_{A'_p}) \cdots P_{C_m}(\mathbf{X}_{\boldsymbol{C_m}}; \mathrm{d}x_{A'_m}) f(x_{A_1}, \dots, x_{A_k}), \qquad (4.13)
\end{aligned}$$

since for all $i \in \{1, \dots, p-1\}$, $X_{A'_i}$ is $\mathscr{F}_B$-measurable ($A'_i \subseteq B$).

Consider $i \in \{p, \dots, m\}$. Since $A'_i \subseteq \cup_{j=1}^{k} A_j$, we know that $\mathbf{X}_{\boldsymbol{C_i}}$ is $\mathscr{F}_{\cup_{j=1}^{k} A_j}$-measurable. Furthermore, let $U \in \mathscr{A}_{\boldsymbol{C_i}}$ such that $U \neq A'_p, \dots, A'_m$. Due to the definition of p and as $C_i \cap B = \emptyset$, $U \subseteq B$ and $U \nsubseteq B^\circ$. Hence, X_U is $\mathscr{F}_{\partial B}$-measurable.

As we integrate over the variables $x_{A'_p}, \dots, x_{A'_m}$ in Equation (4.13), the only random variables X_U left are such that $U \in \mathscr{A}_{\boldsymbol{C_i}}$ and $U \neq A'_p, \dots, A'_m$. Therefore, the integral (4.13) is $\mathscr{F}_{\cup_{j=1}^{k} A_j}$- and $\mathscr{F}_{\partial B}$-measurable, proving that

$$\mathbb{E}[\, f(X_{A_1}, \dots, X_{A_k}) \,|\, \mathscr{F}_B \,] = \mathbb{E}\big[\, f(X_{A_1}, \dots, X_{A_k}) \,\big|\, \mathscr{F}_{\partial B} \cap \mathscr{F}_{\cup_{i=1}^{k} A_i} \,\big].$$

□

An interesting discussion lies in the comparison between the *set-Markov* and $\mathscr{C}$*-Markov* properties. We first observe that they both satisfy a few basic features which may be expected from any set-indexed Markov definition: they imply the *sharp-Markov* property and processes with independent increments (see Example 4.1) are both $\mathscr{C}$*-Markov* and *set-Markov*. Nevertheless, one might quickly notice that they are not equivalent. Indeed, if we simply consider the empirical process $X_A = \sum_{i=1}^n \mathbf{1}_{\{Z_j \in A\}}$, where $(Z_i)_{i \leq n}$ are i.i.d. random variables, we know from [18] that X is *set-Markov* whereas a simple calculus shows that it is not $\mathscr{C}$*-Markov*. Conversely, the *set-indexed* $\mathscr{S}\alpha\mathscr{S}$ *Ornstein–Uhlenbeck process* presented later is a $\mathscr{C}$*-Markov* process which is not *set-Markov* (see Section 4.2.5).

The $\mathscr{C}$*-Markov* and *set-Markov* properties can be seen as two consistent ways to present a set-indexed Markov property leading to the definition of a transition system, $\mathscr{P}$ and $\mathscr{Q}$ respectively, which characterizes entirely the finite-dimensional distributions. If both $\mathscr{P}$ and $\mathscr{Q}$ satisfy a Chapman–Kolmogorov like equation,

$$P_C f = P_{C'} P_{C''} f \quad \text{and} \quad Q_{BB''} f = Q_{BB'} Q_{B'B''} f, \tag{4.14}$$

an important difference lies in the indexing family used. On the one hand, $\mathscr{P} = \{P_C(\mathbf{x_C}; \mathrm{d}x_A); C \in \mathscr{C}\}$ considers variables indexed by $\mathscr{A}$, whereas on the other hand, $\mathscr{Q} = \{Q_{BB'}(x_B; \mathrm{d}x_{B'}); B, B' \in \mathscr{A}(u), B \subseteq B'\}$ used $\mathscr{A}(u)$ as indexing collection.

In fact, as stated in [18], the $\mathscr{Q}$-Markov property is defined on the global extension ΔX on the collection $\mathscr{A}(u)$. This strong feature explains why a supplementary assumption is needed on the transition system (Theorem 1 in [18]), to ensure the existence of ΔX. More precisely, the construction of $\mathscr{Q}$-Markov process requires that the integral

$$\int_{\mathbf{R}^{m+1}} \mu(\mathrm{d}x_0)\, \mathbf{1}_{\Gamma_0}(x_0) \prod_{i=1}^m \mathbf{1}_{\Gamma_i}(x_i - x_{i-1})\, Q_{\cup_{j=0}^{i-1} A_j, \cup_{j=0}^{i} A_j}(x_{i-1}; \mathrm{d}x_i), \tag{4.15}$$

is independent of the numbering of $\mathscr{A}'$, for any semilattice $\mathscr{A}' = \{A_0, \ldots, A_m\}$. Note that this last assumption is not necessary when $\mathscr{Q}$ is homogeneous in a certain sense (see the work of HERBIN and MERZBACH [82]).

On the other hand, the $\mathscr{C}$-Markov property adopts a different approach. The indexing collection is assumed to satisfy the *Shape* assumption so that any set-indexed process is well-defined on $\mathscr{A}(u)$ and the definition of a $\mathscr{C}$-Markov process makes sense (in particular the random vector $\mathbf{X_C}$, $C \in \mathscr{C}$, exists). This hypothesis is not necessary if one only considers processes with independent increments, such as the set-indexed Lévy processes, since these are independently random scattered measures (see [82] for more details on IRSM). Nevertheless, one can easily check that this Shape assumption is required to define some other classes of $\mathscr{C}$-Markov processes, such as the *set-indexed Ornstein–Uhlenbeck processes* presented in Section 4.2.5.

One might also wonder why, on the contrary to the *set-Markov* property and some other set-indexed references, we assume that $\emptyset \notin \mathscr{A}$. The explanation lies in the characterization and construction Theorems 4.2 and 4.3. If we assume that $\emptyset \in \mathscr{A}$, we observe that $\mathscr{A} \subset \mathscr{C}$, and in particular $\emptyset' \in \mathscr{C}$. Thereby, the transition system $\mathscr{P}$ incorporates the transition probability $\mathscr{P}_{\emptyset'}(x_\emptyset, \mathrm{d}x_{\emptyset'})$. But since $X_\emptyset = 0$ is always assumed (simply for consistency), we observe that the measure $\mathscr{P}_{\emptyset'}(x_\emptyset, \mathrm{d}x_{\emptyset'})$ in fact corresponds to the law of $X_{\emptyset'}$. This statement clearly contradicts our will to separate in Theorems 4.2 and 4.3 the initial distribution ν of a process from its transition system $\mathscr{P}$.

To end this section, let us finally note that when branches of a tree form the indexing collection (as presented in the introduction), $\mathscr{C}$-Markov processes are in fact Markov chains indexed by trees, as defined in the seminal article of BENJAMINI and PERES [27] (see also the recent work of DURAND [55]).

4.2.3 Features of $\mathscr{C}$-Markov processes

Following the definition of a set-indexed Markov property, several simple and natural questions arise. Among them, we study in this section the projections of $\mathscr{C}$-Markov processes on flows, conditional independence of filtrations and homogeneity of transition probabilities.

To begin with, let us recall that an *elementary flow* is a continuous increasing function $f : [a,b] \subset \mathbf{R}_+ \to \mathscr{A}$, i.e.

(i) Increasing: $\forall s,t \in [a,b]; \quad s < t \Rightarrow f(s) \subseteq f(t)$;

(ii) Outer-continuous: $\forall s \in [a,b); \quad f(s) = \bigcap_{v>s} f(v)$;

(iii) Inner-continuous: $\forall s \in (a,b); \quad f(s) = \overline{\bigcup_{u<s} f(u)}$.

As we might expect, the elementary projections of $\mathscr{C}$-Markov processes happen to be one-parameter Markov processes.

Proposition 4.5 *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and X be a $\mathscr{C}$-Markov process w.r.t. $(\mathscr{F}_A)_{A\in\mathscr{A}}$.*

Then, for any elementary flow $f : [a,b] \to \mathscr{A}$, the projection X^f of X along f is a Markov process w.r.t. the filtration $\mathscr{F}^f = (\mathscr{F}_{f(s)})_{s\in[a,b]}$. Furthermore, X^f has the following transition probabilities,

$$\forall s \le t \in [a,b]; \quad P^f_{s,t}(x_s; \mathrm{d}x_t) = P_{f(t)\setminus f(s)}(x_{f(s)}; \mathrm{d}x_{f(t)}),$$

where $\mathscr{P}$ is the $\mathscr{C}$-transition system of X.

Proof. We simply check that X^f is a Markov process with the expected transition probabilities. Let f be an elementary flow and $s \le t$ be in $[a,b]$. Note that $\mathscr{F}_{f(s)} \subseteq \mathscr{G}^*_{f(t)\setminus f(s)}$ and $\mathscr{A}_{f(t)\setminus f(s)} = \{f(s)\}$. Then, for any $\Gamma \in \mathscr{E}$,

$$\begin{aligned}\mathbb{P}\big(X^f_t \in \Gamma \,\big|\, \mathscr{F}^f_s\big) &= \mathbb{P}\big(\mathbb{P}(X_{f(t)} \in \Gamma \mid \mathscr{G}^*_{f(t)\setminus f(s)}) \,\big|\, \mathscr{F}_{f(s)}\big) \\ &= \mathbb{E}[\,P_{f(t)\setminus f(s)}(X_{f(s)}; \Gamma) \mid \mathscr{F}_{f(s)}\,] = P^f_{s,t}(X^f_s; \Gamma).\end{aligned}$$

□

Note that the converse result is not true: it is not sufficient to have Markov elementary projections to obtain a $\mathscr{C}$-Markov process. Furthermore, Proposition 4.5 can not be extended to *simple flows*, i.e. continuous increasing function $f : [a,b] \to \mathscr{A}(u)$, as this property constitutes a characterization of *set-Markov* processes (Proposition 2 in [18]).

The *conditional independence* of filtrations is an important property in the theory of multiparameter processes. Also named *commuting property* or (F4), it has first been introduced by CAIROLI and WALSH [40] for two-parameter processes and recently extended to the set-indexed formalism by IVANOFF and MERZBACH [89].

Theorem 4.6 (**Conditional independence**) *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and X be a $\mathscr{C}$-Markov w.r.t. $(\mathscr{F}_A)_{A\in\mathscr{A}}$. Suppose that $(\mathscr{F}_A)_{A\in\mathscr{A}}$ is included in the σ-field $\mathscr{F}^0_\infty := \sigma(\{X_A; A \in \mathscr{A}\})$.*

Then, the filtration $(\mathscr{F}_A)_{A\in\mathscr{A}}$ satisfies the conditional independence *property (CI), i.e. for all $U, V \in \mathscr{A}$ and any bounded random variable Y on the space $(\Omega, \mathscr{F}^0_\infty, \mathbb{P})$,*

$$\mathbb{E}\big[\,\mathbb{E}[\,Y \mid \mathscr{F}_U\,] \,\big|\, \mathscr{F}_V\,\big] = \mathbb{E}\big[\,\mathbb{E}[\,Y \mid \mathscr{F}_V\,] \,\big|\, \mathscr{F}_U\,\big] = \mathbb{E}[\,Y \mid \mathscr{F}_{U\cap V}\,] \quad \mathbb{P}\text{-a.s.}$$

Proof. Let $U, V \in \mathscr{A}$. If $U \subseteq V$ or $V \subseteq U$, the equality is straight-forward. Hence, we suppose throughout that $U \nsubseteq V$ and $V \nsubseteq U$. Using a monotone class argument, we only need to prove that

$$\mathbb{E}\big[\,\mathbb{E}[f_1(X_{A_1})\cdots f_k(X_{A_k})\,|\,\mathscr{F}_U\,]\,\big|\,\mathscr{F}_V\,\big] = \mathbb{E}[f_1(X_{A_1})\cdots f_k(A_k)\,|\,\mathscr{F}_{U\cap V}\,],$$

for every $k \in \mathbf{N}$, any $A_1,\dots,A_k \in \mathscr{A}$ and any $f_1,\dots,f_k : E \to \mathbf{R}_+$ measurable functions. Without any loss of generality, we assume that $\mathscr{A}' = \{A_0 = \emptyset', A_1,\dots,A_k\}$ is a semilattice with a consistent numbering and which contains the sets U, V and $U \cap V$. We denote as usually $(C_i)_{i\le k}$ the left-neighbourhoods in $\mathscr{A}'$, i.e. $C_i = A_i \setminus (\cup_{j=0}^{i-1} A_j)$. Finally, up to a re-ordering of $\mathscr{A}'$, we can suppose there exist $k_{U\cap V}, k_V \in \mathbf{N}$ such that $k_{U\cap V} < k_V$,

$$\forall i \in \{1,\dots,k_{U\cap V}\};\quad A_i \subseteq A_{k_{U\cap V}} := U\cap V \quad\text{and}\quad \forall i \in \{1,\dots,k_V\};\quad A_i \subseteq A_{k_V} := V.$$

Then, as $\mathscr{F}_V \subseteq \mathscr{G}^*_{C_k}$, the $\mathscr{C}$-Markov property induces

$$\begin{aligned}
&\mathbb{E}\big[\,\mathbb{E}[f_1(X_{A_1})\cdots f_k(X_{A_k})\,|\,\mathscr{F}_V\,]\,\big|\,\mathscr{F}_U\,\big]\\
&= \mathbb{E}\big[\,\mathbb{E}[\,\mathbb{E}[f_1(X_{A_1})\cdots f_k(X_{A_k})\,|\,\mathscr{G}^*_{C_k}\,]\,|\,\mathscr{F}_V\,]\,\big|\,\mathscr{F}_U\,\big]\\
&= \mathbb{E}\Big[\,\mathbb{E}\Big[\int_E P_{C_k}(\mathbf{X}_{\boldsymbol{C_k}}; \mathrm{d}x_{A_k}) f_1(X_{A_1})\cdots f_k(x_{A_k})\,\Big|\,\mathscr{F}_V\Big]\,\Big|\,\mathscr{F}_U\Big],
\end{aligned}$$

Therefore, by induction,

$$\begin{aligned}
&\mathbb{E}\big[\,\mathbb{E}[f_1(X_{A_1})\cdots f_k(X_{A_k})\,|\,\mathscr{F}_V\,]\,\big|\,\mathscr{F}_U\,\big]\\
&= \mathbb{E}\Big[\,\mathbb{E}\Big[\int_{E^{k-k_V}} P_{C_{k_V+1}}(\mathbf{X}_{\boldsymbol{C_{k_V+1}}}; \mathrm{d}x_{A_{k_V+1}})\cdots P_{C_k}(\mathbf{X}_{\boldsymbol{C_k}}; \mathrm{d}x_{A_k})\\
&\qquad\qquad f_1(X_{A_1})\cdots f_{k_V}(X_{A_{k_V}}) f_{k_V+1}(x_{A_{k_V+1}})\cdots f_k(x_{A_k})\,\Big|\,\mathscr{F}_V\Big]\,\Big|\,\mathscr{F}_U\Big].
\end{aligned}$$

Every random variable X_{A_i} left in the previous integral is such that $A_i \subseteq V$, and thus is $\mathscr{F}_V$-measurable. Furthermore, for every $i \in \{k_{U\cap V}+1,\dots,k_V\}$,

$$U \cap C_i \subseteq U \cap (A_i \setminus U\cap V) = (U\cap A_i)\setminus(U\cap V) = \emptyset \quad \text{since } A_i \subseteq V.$$

Therefore, $\mathscr{F}_U \subseteq \mathscr{G}^*_{C_i}$ and we obtain

$$\begin{aligned}
&\mathbb{E}\big[\,\mathbb{E}[f_1(X_{A_1})\cdots f_k(X_{A_k})\,|\,\mathscr{F}_V\,]\,\big|\,\mathscr{F}_U\,\big]\\
&= \mathbb{E}\Big[\int_{E^{k-k_V}} P_{C_{k_V+1}}(\mathbf{X}_{\boldsymbol{C_{k_V+1}}}; \mathrm{d}x_{A_{k_V+1}})\cdots P_{C_k}(\mathbf{X}_{\boldsymbol{C_k}}; \mathrm{d}x_{A_k})\\
&\qquad\qquad f_1(X_{A_1})\cdots f_{k_V+1}(x_{A_{k_V+1}})\cdots f_k(x_{A_k})\,\Big|\,\mathscr{F}_U\Big]\\
&= \mathbb{E}\Big[\int_{E^{k-k_{U\cap V}}} P_{C_{k_{U\cap V}+1}}(\mathbf{X}_{\boldsymbol{C_{k_{U\cap V}+1}}}; \mathrm{d}x_{A_{k_{U\cap V}+1}})\cdots P_{C_k}(\mathbf{X}_{\boldsymbol{C_k}}; \mathrm{d}x_{A_k})\\
&\qquad\qquad f_1(X_{A_1})\cdots f_{k_{U\cap V}}(X_{A_{k_{U\cap V}}}) f_{k_{U\cap V}+1}(x_{A_{k_{U\cap V}+1}})\cdots f_k(x_{A_k})\,\Big|\,\mathscr{F}_U\Big]
\end{aligned}$$

Finally, every random variable X_{A_i} left in the integral is $\mathscr{F}_{U\cap V}$-measurable (as $A_i \subseteq U\cap V$ for every $i \in \{1,\dots,k_{U\cap V}\}$), proving that

$$\mathbb{E}\big[\,\mathbb{E}[f_1(X_{A_1})\cdots f_k(X_{A_k})\,|\,\mathscr{F}_V\,]\,\big|\,\mathscr{F}_U\,\big] = \mathbb{E}[f_1(X_{A_1})\cdots f_k(A_k)\,|\,\mathscr{F}_{U\cap V}\,].$$

□

In the next theorem, we investigate the extension of the *simple Markov property* to the $\mathscr{C}$-Markov formalism. In order to introduce the concept of *homogeneous $\mathscr{C}$-transition system*, we suppose that there exists a collection of shift operators $(\theta_U)_{U\in\mathscr{A}}$ on $\mathscr{A}$ which satisfies:

(i) for any $U \in \mathscr{A}$ such that $U^\circ \neq \emptyset$, we have $A \subset (\theta_U(A))^\circ$ for all $A \in \mathscr{A}$;

(ii) for any $A \in \mathscr{A}$, $U \mapsto \theta_U(A)$ is an increasing monotone outer-continuous function with $\theta_{\emptyset'}(A) = A$;

(iii) for any $U \in \mathscr{A}$, θ_U is stable under intersections and $\theta_U(\emptyset') = U$.

For all $U \in \mathscr{A}$, the operator θ_U is extended on $\mathscr{A}(u)$ and $\mathscr{C}$ in the following way:

$$\theta_U(B) = \cup_{i=1}^k \theta_U(A_i) \quad \text{and} \quad \theta_U(C) = \theta_U(A) \setminus \theta_U(B),$$

where $B = \cup_{i=1}^k A_i \in \mathscr{A}(u)$ and $C = A \setminus B \in \mathscr{C}$.

To illustrate this notion, let simply consider the multiparameter setting $\mathscr{A} = \{[0,t] : t \in \mathbf{R}_+^N\}$. In this framework, the natural family $(\theta_u)_{u\in\mathscr{T}}$ of shift operators corresponds to the usual translation on $\mathbf{R}_+^N$, i.e.

$$\forall u, t \in \mathbf{R}_+^N; \quad \theta_u(t) = \theta_{[0,u]}\big([0,t]\big) := [0, t+u].$$

One can easily verify that $(\theta_u)_{u\in\mathbf{R}_+^N}$ satisfy the previous conditions.

Definition 4.6 (**Homogeneous $\mathscr{C}$-transition system**) *A $\mathscr{C}$-transition system $\mathscr{P}$ is said to be* homogeneous *with respect to $(\theta_U)_{U\in\mathscr{A}}$ if it satisfies*

$$\forall U \in \mathscr{A}, \forall C = A \setminus B \in \mathscr{C}; \quad P_C(\mathbf{x}_C; \mathrm{d}x_A) = P_{\theta_U(C)}(\mathbf{x}_{\theta_U(C)}; \mathrm{d}x_{\theta_U(A)}).$$

A $\mathscr{C}$-Markov process is said to be homogeneous *w.r.t $(\theta_U)_{U\in\mathscr{A}}$ if its $\mathscr{C}$-transition system is itself homogeneous.*

We can now establish a set-indexed *simple $\mathscr{C}$-Markov property*. In the following theorem, Ω and X refer to the canonical space $E^{\mathscr{A}}$ and the canonical process $X_A(\omega) = \omega(A)$, respectively.

Theorem 4.7 (**Simple $\mathscr{C}$-Markov property**) *Let $\mathscr{P}$ be an homogeneous $\mathscr{C}$-transition system w.r.t $(\theta_U)_{U\in\mathscr{A}}$. Then, the canonical $\mathscr{C}$-Markov process X satisfies*

$$\mathbb{E}_\nu\left[f(X\circ\theta_U) \,\middle|\, \mathscr{F}_U\right] = \mathbb{E}_{X_U}\left[f(X)\right] \quad \mathbb{P}_\nu\text{-a.s.},$$

for any $U \in \mathscr{A}$, any measurable function $f : E^{\mathscr{A}} \to \mathbf{R}_+$ and any initial measure ν.

Proof. Let $A_1, \ldots, A_k$ be in $\mathscr{A}$. Without any loss of generality, we can suppose that $\mathscr{A}' = \{A_0 = \emptyset', A_1, \ldots, A_k\}$ is a semilattice with consistent numbering. As usually, $C_1, \ldots, C_k \in \mathscr{C}$ denote the left-neighbourhoods $C_i = A_i \setminus (\cup_{j=0}^{i-1} A_j)$. Then, for any measurable function $h : E^k \to \mathbf{R}_+$ and any $U \in \mathscr{A}$,

$$\begin{aligned}
&\mathbb{E}_\nu\left[h(X_{\theta_U(A_1)}, \ldots, X_{\theta_U(A_k)}) \,\middle|\, \mathscr{F}_U\right] \\
&= \mathbb{E}_\nu\left[\mathbb{E}_\nu\left[h(X_{\theta_U(A_1)}, \ldots, X_{\theta_U(A_k)}) \,|\, \mathscr{G}^*_{\theta_U(C_k)}\right] \,\middle|\, \mathscr{F}_U\right] \\
&= \mathbb{E}_\nu\left[\int_{\mathbf{R}} P_{\theta_U(C_k)}(\mathbf{X}_{\theta_U(C_k)}; \mathrm{d}x_{\theta_U(A_k)})\, h(X_{\theta_U(A_1)}, \ldots, X_{\theta_U(A_{k-1})}, x_{\theta_U(A_k)}) \,\middle|\, \mathscr{F}_U\right],
\end{aligned}$$

since for every $i \in \{1,\dots,k\}$, $\theta_U(C_i) \cap U = \emptyset$, and thus $\mathscr{F}_U \subseteq \mathscr{G}^*_{\theta_U(C_k)}$. By induction,

$$\begin{aligned}
&\mathbb{E}_\nu\left[h(X_{\theta_U(A_1)},\dots,X_{\theta_U(A_k)}) \,\middle|\, \mathscr{F}_U\right] \\
&= \int_{\mathbf{R}^k} P_{\theta_U(C_1)}(X_U;\mathrm{d}x_{\theta_U(A_1)})\dots P_{\theta_U(C_k)}(\mathbf{x}_{\boldsymbol{\theta_U(C_k)}};\mathrm{d}x_{\theta_U(A_k)})\,h(x_{\theta_U(A_1)},\dots,x_{\theta_U(A_k)}) \\
&= \int_{\mathbf{R}^k} P_{C_1}(X_U;\mathrm{d}x_{A_1})\dots P_{C_k}(\mathbf{x}_{\boldsymbol{C_k}};\mathrm{d}x_{A_k})\,h(x_{A_1},\dots,x_{A_k}) \\
&= \mathbb{E}_{X_U}[h(X_{A_1},\dots,X_{A_k})],
\end{aligned}$$

as $\mathscr{P}$ is homogeneous and $\theta_U(C_1) = \theta_U(A_1) \setminus \theta_U(\emptyset') = \theta_U(A_1) \setminus U$. A monotone class argument allows to conclude the proof. □

4.2.4 $\mathscr{C}$-Feller processes

The Feller property plays a major role in the classic theory of Markov processes, motivating its introduction into the $\mathscr{C}$-Markov formalism. For this purpose, we need to reinforce the assumptions on the indexing collection $\mathscr{A}$, so that a pseudo-metric can be defined on the latter and on the class of increments $\mathscr{C}$.

Definition 4.7 (**Separability from above, see [89]**) *There exist an increasing sequence of finite subclasses $\mathscr{A}_n = \{A_1^n,\dots,A_{k_n}^n\}$ of $\mathscr{A}$ closed under intersections and such that $B_n \in \mathscr{A}_n$; a sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ decreasing to 0 and a sequence of functions $g_n : \mathscr{A} \to \mathscr{A}_n \cup \{\mathscr{T}\}$ such that*

(i) for any $A \in \mathscr{A}$, $A = \cap_n g_n(A)$;

(ii) g_n preserves arbitrary intersections and finite unions;

(iii) for any $A \in \mathscr{A}$, $A \subseteq (g_n(A))^\circ$;

(iv) $g_n(A) \subseteq g_m(A)$ if $n \geq m$;

(v) for any $A \in \mathscr{A}$, $d_H(A, g_n(A)) \leq \varepsilon_n$ for all $n \in \mathbf{N}$ such that $g_n(A) \subseteq B_n$,

where d_H denotes the Hausdorff distance between two sets.

The family $(g_n)_{n\in\mathbf{N}}$ plays a role similar to the dyadic rational numbers in $\mathbf{R}^N$. Note that the previous definition is slightly stronger than the usual one presented by IVANOFF and MERZBACH [89], since every g_n is supposed to be $\mathscr{A}_n$-valued, and not $\mathscr{A}_n(u)$. For any filtration $(\mathscr{F}_A)_{A\in\mathscr{A}}$, the collection $(g_n)_{n\in\mathbf{N}}$ induces the definition of the augmented filtration $(\widetilde{\mathscr{F}}_A)_{A\in\mathscr{A}}$:

$$\forall A \in \mathscr{A}; \quad \widetilde{\mathscr{F}}_A := \bigcap_{n\in\mathbf{N}} \mathscr{F}_{g_n(A)}.$$

The filtrations $(\widetilde{\mathscr{F}}_B)_{B\in\mathscr{A}(u)}$ and $(\widetilde{\mathscr{G}}^*_C)_{C\in\mathscr{C}}$, given by Equation (4.2), denote the extensions of $\widetilde{\mathscr{F}}$ on $\mathscr{A}(u)$ and $\mathscr{C}$, respectively. As proved by IVANOFF and MERZBACH [89], the augmented filtration is monotone outer-continuous, i.e. $\widetilde{\mathscr{F}}_B = \cap_{n\in\mathbf{N}}\widetilde{\mathscr{F}}_{B_n}$ for any decreasing sequence $(B_n)_{n\in\mathbf{N}}$ in $\mathscr{A}(u)$ such that $B = \cap_{n\in\mathbf{N}} B_n$.

In the classic theory of Markov processes, a Feller semigroup P must satisfy the condition $P_t f \to_{t\to 0} f$. Therefore, to adapt such a property to the $\mathscr{C}$-Markov formalism, one must define a pseudo-norm $\|\cdot\|_{\mathscr{C}}$ on the class of increments $\mathscr{C}$. For this purpose, let us introduce a few more notations related to $\mathscr{C}$. For any $C = A \setminus B \in \mathscr{C}$, we assume that the numbering $B = \cup_{i=1}^k A_i$ is such that $d_H(A,A_1) \leq \dots \leq d_H(A,A_k)$. Then, we define a map $\psi_C : \{U_C^0, U_C^1,\dots,U_C^p\} \to \{U_C^1,\dots,U_C^p\}$ constructed as follows (we recall that the definition of $(U_C^i)_{i\leq p}$ is given in the introduction):

1. $\psi_C(U_C^0) = A_1$, where by convention $U_C^0 = A$;
2. $\psi_C(U_C^1) = A_1$, where it is assumed that $U_C^1 = A_1$;
3. let $i \in \{2, \dots, p\}$ and $V = A_1 \cap U_C^i$. Owing to Equation (4.5), we observe that $X_{A_1} = \sum_{i=1}^p (-1)^{\varepsilon_i} X_{A_1 \cap U_C^i}$, which implies the existence of $j > i$ such that $V = A_1 \cap U_C^j$. Consequently, we set
$$\psi_C(U_C^i) = U_C^j \quad \text{and} \quad \psi_C(U_C^j) = U_C^j. \tag{4.16}$$
By induction, we obtain a complete definition of ψ_C.

We note that the construction of ψ_C implies that ΔX satisfies
$$\Delta X_C = \left(X_A - X_{\psi_C(A)}\right) - \left[\sum_{i=1}^p (-1)^{\varepsilon_i} \left(X_{U_C^i} - X_{\psi_C(U_C^i)}\right)\right]. \tag{4.17}$$
The map ψ_C leads to the definition of the pseudo-norms $\|C\|_{\mathscr{C}}$ and $\|\mathbf{x}_C\|_{\mathscr{C}}$:
$$\|C\|_{\mathscr{C}} := \max_{0 \le i \le p} d_H\left(U_C^i, \psi_C(U_C^i)\right) \quad \text{and} \quad \|\mathbf{x}_C\|_{\mathscr{C}} := \max_{0 \le i \le p} d_E\left(x_{U_C^i}, x_{\psi_C(U_C^i)}\right). \tag{4.18}$$
Finally, for any $m \in \mathbf{N}$, let $\mathscr{C}^m$ denotes the sub-class $\{C \in \mathscr{C} : |\mathscr{A}_C| \le m \text{ and } C \subseteq B_m\}$.

Definition 4.8 (**$\mathscr{C}$-Feller transition system**) *A $\mathscr{C}$-transition system $\mathscr{P}$ is said to be* Feller *if it satisfies the following conditions:*

1. *for any $f \in C_0(E)$ and for all $C \in \mathscr{C}$,*
$$P_C f \in C_0\left(E^{|\mathscr{A}_C|}\right),$$
where $C_0(E^k)$, $k \in \mathbf{N}$, denotes the usual space of continuous functions that vanish at infinity;
2. *for any $f \in C_0(E)$ and for every $m \in \mathbf{N}$,*
$$\lim_{\rho \to 0} \sup_{\substack{C = C' \cup C'' \in \mathscr{C}^m \\ \|C'\|_{\mathscr{C}} \le \rho}} \sup_{\substack{\mathbf{x}_{C''}, \mathbf{x}_{C'} \\ \|\mathbf{x}_{C'}\|_{\mathscr{C}} \le \rho}} \left|P_C f(\mathbf{x}_C) - P_{C''} f(\mathbf{x}_{C''})\right| = 0. \tag{4.19}$$
where, similarly to Definition 4.2, the sets C, C' and C'' are such that $C' = C \cap A'$ and $C'' = C \setminus A'$, with $A' \in \mathscr{A}$.

*A $\mathscr{C}$-Markov process X is said to be $\mathscr{C}$-*Feller *if its $\mathscr{C}$-transition system $\mathscr{P}$ is Feller.*

The definition of a $\mathscr{C}$-Feller transition system, and particularly Equation (4.19), may seem a bit cumbersome, but in the one-dimensional case and if $\mathscr{P}$ is homogeneous, it is equivalent to the usual Feller property $\lim_{t \to 0} \sup_{s \ge 0} \|P_{s+t} f - P_s f\|_\infty = 0$.

Moreover, in the case of simple increments $C' = U \setminus V$ and $C'' = \emptyset$, it corresponds to a more common formula:
$$\lim_{\rho \to 0} \sup_{\substack{U, V \in \mathscr{A}, U \subseteq V \subset B_m \\ d_H(U, U \cap V) \le \rho}} \left\|P_{U \setminus V} f - f\right\|_\infty = 0, \tag{4.20}$$
since $\|U \setminus V\|_{\mathscr{C}} = d_H(U, U \cap V)$ and $\|\mathbf{x}_{U \setminus V}\|_{\mathscr{C}} = d_E(x_U, x_{U \cap V})$.

The existence of interesting $\mathscr{C}$-Feller processes which are not homogeneous (e.g. the well-known multiparameter Brownian sheet) motivates the separation between the homogeneous and Feller hypotheses, at the cost of a more complex definition of the latter (Equation (4.19)).

Finally, in this section are considered $\mathscr{C}$-Markov processes X which have outer-continuous sample paths, meaning that:
$$\forall A \in \mathscr{A},\ \forall \varepsilon > 0,\ \exists \alpha > 0; \quad A \subseteq U \in \mathscr{A} \text{ and } d_H(A, U) \le \alpha \implies d_E(X_A, X_U) \le \varepsilon. \tag{4.21}$$

We can now extend a classic result of the theory of Markov processes.

Theorem 4.8 (**$\mathscr{C}$-Markov w.r.t the augmented filtration**) *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space and X be a $\mathscr{C}$-Feller process w.r.t $(\mathscr{F}_A)_{A\in\mathscr{A}}$. In addition, suppose X has outer-continuous sample paths.*

Then, X is a $\mathscr{C}$-Markov process with respect to the augmented filtration $(\widetilde{\mathscr{F}}_A)_{A\in\mathscr{A}}$, i.e.

$$\mathbb{E}[\, f(X_A) \,|\, \widetilde{\mathscr{G}}_C^* \,] = \mathbb{E}[\, f(X_A) \,|\, \mathbf{X}_C \,] \quad \mathbb{P}\text{-a.s.}$$

for all $C = A \setminus B \in \mathscr{C}$ and any measurable function $f : E \mapsto \mathbf{R}_+$.

Proof. Let $C = A \setminus B$ be in $\mathscr{C}$, with $B = \cup_{j=1}^k A_j$. We assume that the numbering of $\mathscr{A}_C = \{U_C^1, \dots, U_C^p\}$ is consistent. Let $\rho_0 = \max_{1\le i<j\le p} d_H(U_C^i, U_C^j) > 0$. Then, for all $n \in \mathbf{N}$ and for every $i \in \{1, \dots, p+1\}$, let C_i^n and D_i^n denote the following increments,

$$C_i^n = A \setminus \left(B \cup \bigcup_{j=1}^{i-1} g_n(U_C^j) \right) \quad \text{and} \quad D_i^n = g_n(U_C^i) \setminus \left(B \cup \bigcup_{j=1}^{i-1} g_n(U_C^j) \right),$$

where by convention $g_n(U_C^{p+1}) = A$. For every $i \in \{1, \dots, p\}$, let $D_i^n = g_n(U_C^i) \setminus \left(\cup_{j=1}^{k_i} A_j^i\right)$ be the extremal representation of D_i^n. As previously said, it is assumed that the numbering is such that $d_H(g_n(U_C^i), A_1^i) \le \cdots \le d_H(g_n(U_C^i), A_{k_i}^i)$. For every j, due to the definition of D_i^n, there exist l_j such that

$$A_j^i = g_n(U_C^i) \cap U_C^{l_j} \quad \text{or} \quad A_j^i = g_n(U_C^i) \cap g_n(U_C^{l_j}) = g_n(U_C^i \cap U_C^{l_j}).$$

Since $U_C^i \subseteq B$, there exists $j \le k_i$ such that $U_C^i \subseteq A_j^i \subset g_n(U_C^i)$. Therefore, as the family $(g_n)_n$ satisfies assumption *(v)*, we have $d_H(g_n(U_C^i), A_j^i) \to 0$ and A_j^i converges to U_C^i.

If we consider the particular case of A_1^i, we obtain that $d_H(g_n(U_C^i), A_1^i) \to 0$ and A_1^i converges to an element $V := U_C^i \cap U_C^{l_1} \in \mathscr{A}_C$. Let suppose $V \ne U_C^i$, implying that $d_H(V, U_C^i) \ge \rho_0$. We know that $d_H(A_1^i, V) \to 0$, $d_H(g_n(U_C^i), A_1^i) \to 0$ and $d_H(g_n(U_C^i), U_C^i) \to 0$, which is in contradiction with the previous assumption. Therefore, $U_C^i \cap U_C^{l_1} = U_C^i$.

Then, let $V_n \in \mathscr{A}_{D_i^n}$. According to the definition of D_i^n, V_n must have the following general form

$$V_n = U_C^{j_V} \cap g_n(U_C^{l_V}) \cap g_n(U_C^i) \quad \text{where } j_V, l_V \in \{1, \dots, p\}.$$

Therefore, $(V_n)_{n\in\mathbf{N}}$ is a decreasing sequence which converges to $U_C^{j_V} \cap U_C^{l_V} \cap U_C^i := U_C^{k_V}$. Since A_1^i converges to U_C^i, $V_n \cap A_1^i$ is a decreasing sequence whose limit is also $U_C^{k_V}$. Hence, according to definition (4.16) of $\psi_{D_i^n}$, we know that both V_n and $\psi_{D_i^n}(V_n)$ converge to $U_C^{k_V}$, and as the family $(g_n)_n$ satisfies the assumption *(v)*, for any $\rho > 0$, there exists N_ρ such that

$$\forall n \ge N_\rho; \quad d_H\left(V_n, \psi_{D_i^n}(V_n)\right) \le \rho.$$

Furthermore, since the sample paths of X are outer-continuous, N_ρ can be suppose large enough to satisfy

$$\forall n \ge N_\rho; \quad d_E\left(X_{V_n}, X_{U_C^{k_V}}\right) \le \rho \quad \text{and} \quad d_E\left(X_{\psi_{D_i^n}(V_n)}, X_{U_C^{k_V}}\right) \le \rho$$

Finally, as the previous two properties are satisfied uniformly for any $V_n \in \mathscr{A}_{D_i^n}$ and $i \in \{1, \dots, p\}$,

$$\forall n \ge N_\rho; \quad \left\| D_i^n \right\|_{\mathscr{C}} \le \rho \quad \text{and} \quad \left\| \mathbf{X}_{D_i^n} \right\|_{\mathscr{C}} \le \rho. \tag{4.22}$$

Let $\varepsilon > 0$ and $f \in C_0(E)$. According to Definition 4.8, there exists $\rho > 0$ such that

$$\sup_{\substack{C=C'\cup C''\in\mathscr{C}^m\\ \|C'\|_{\mathscr{C}}\leq\rho}} \sup_{\substack{\mathbf{x}_{C''},\mathbf{x}_{C'}\\ \|\mathbf{x}_{C'}\|_{\mathscr{C}}\leq\rho}} \left|P_C f(\mathbf{x}_C) - P_{C''} f(\mathbf{x}_{C''})\right| \leq \varepsilon.$$

Note that for every $i \in \{1,\dots,p\}$, $C_{i+1}^n = C_i^n \setminus g_n(U_C^i)$ and $D_i^n = C_i^n \cap g_n(U_C^i)$, and there exists $m \in \mathbf{N}$ such that C_i^n and D_i^n belong to $\mathscr{C}^m$.

Therefore, owing to the Feller property and Inequalities (4.22),

$$\forall i \in \{1,\dots,p\}; \quad \left|P_{C_i^n} f(\mathbf{X}_{C_i^n}) - P_{C_{i+1}^n} f(\mathbf{X}_{C_{i+1}^n})\right| \leq \varepsilon.$$

Thereby, by induction and since $C_1^n = C$, we have $\left|P_C f(\mathbf{X}_C) - P_{C_{p+1}^n} f(\mathbf{X}_{C_{p+1}^n})\right| \leq p\varepsilon$, for all $n \geq N_\rho$, which proves that $P_{C_{p+1}^n} f(\mathbf{X}_{C_{p+1}^n}) \xrightarrow[n\to\infty]{} P_C f(\mathbf{X}_C)$ almost surely.

As we observe that for all $n \in \mathbf{N}$, $\widetilde{\mathscr{G}}_C^* \subset \mathscr{G}_{C_{p+1}^n}^*$, we can eventually prove the $\mathscr{C}$-Markov property with respect to the augmented filtration:

$$\mathbb{E}[f(X_A) \,|\, \widetilde{\mathscr{G}}_C^*] = \mathbb{E}\big[\,\mathbb{E}[f(X_A) \,|\, \mathscr{G}_{C_{k+1}^n}^*]\,\big|\,\widetilde{\mathscr{G}}_C^*\,\big] = \mathbb{E}\big[\,P_{C_{p+1}^n} f(\mathbf{X}_{C_{p+1}^n})\,\big|\,\widetilde{\mathscr{G}}_C^*\,\big].$$

By the dominated convergence theorem,

$$\mathbb{E}[f(X_A) \,|\, \widetilde{\mathscr{G}}_C^*] = \lim_{n\to\infty} \mathbb{E}\big[\,P_{C_{p+1}^n} f(\mathbf{X}_{C_{p+1}^n})\,\big|\,\widetilde{\mathscr{G}}_C^*\,\big] = \mathbb{E}[P_C f(\mathbf{X}_C) \,|\, \widetilde{\mathscr{G}}_C^*] = P_C f(\mathbf{X}_C).$$

since $\mathbf{X}_C$ is $\widetilde{\mathscr{G}}_C^*$-measurable. □

Before establishing a *strong $\mathscr{C}$-Markov property*, we recall the definition of a *simple stopping set*, as presented by IVANOFF and MERZBACH [89]. A *simple stopping set* is an random variable $\xi : \Omega \to \mathscr{A}$ such that for all $A \in \mathscr{A}$, $\{\omega : A \subseteq \xi(\omega)\} \in \mathscr{F}_A$. ξ is said to be bounded if there exists $V \in \mathscr{A}$ such that $\xi \subseteq V$ almost surely. As the assumption *Shape* holds on the collection $\mathscr{A}$, we can define the σ-algebra of the events prior to ξ:

$$\widetilde{\mathscr{F}}_\xi = \{F \in \mathscr{F} : F \cap \{\xi \subseteq B\} \in \widetilde{\mathscr{F}}_B,\ \forall B \in \mathscr{A}(u)\}.$$

According to [89], if X has outer-continuous sample paths, then X_ξ is $\widetilde{\mathscr{F}}_\xi$-measurable.

Theorem 4.9 (**Strong $\mathscr{C}$-Markov property**) *Let $(\Omega, \mathscr{F}, (\mathscr{F}_A)_{A\in\mathscr{A}}, \mathbb{P})$ be a complete probability space, X be a $\mathscr{C}$-Feller process w.r.t. $(\mathscr{F}_A)_{A\in\mathscr{A}}$ and ξ be a bounded simple stopping set. Moreover, suppose X has outer-continuous sample paths and is homogeneous w.r.t. $(\theta_U)_{U\in\mathscr{A}}$.*

Then, X satisfies a strong $\mathscr{C}$-Markov property*:*

$$\mathbb{E}_\nu\big[\,f(X\circ\theta_\xi)\,\big|\,\widetilde{\mathscr{F}}_\xi\,\big] = \mathbb{E}_{X_\xi}\big[\,f(X)\,\big] \quad \mathbb{P}_\nu\text{-a.s.}$$

for any initial measure ν and any measurable function $f : E^{\mathscr{A}} \to \mathbf{R}_+$.

Proof. Let F be in $\widetilde{\mathscr{F}}_\xi$, $\mathscr{A}' = \{A_0 = \emptyset', A_1, \dots, A_k\}$ be a finite semilattice, $C_1, \dots, C_k \in \mathscr{C}$ be the left-neighbourhoods $C_i = A_i \setminus (\cup_{j=0}^{i-1} A_j)$, $i \in \{1,\dots,k\}$.

According to Lemma 1.5.4 in [89], for every $n \in \mathbf{N}$, $g_n(\xi)$ is a discrete simple stopping set such that $\xi \subset g_n(\xi)$, $\xi = \cap_n g_n(\xi)$ and $\widetilde{\mathscr{F}}_\xi \subseteq \widetilde{\mathscr{F}}_{g_n(\xi)}$. Hence, $F \in \widetilde{\mathscr{F}}_{g_n(\xi)}$ and for any $h \in C_0(E)$,

$$\mathbb{E}_\nu[\mathbf{1}_F\, h(X_{\theta_\xi(A_k)})] = \lim_{n\to\infty} \mathbb{E}_\nu[\mathbf{1}_F\, h(X_{\theta_{g_n(\xi)}(A_k)})],$$

since X and $U \mapsto \theta_U$ are outer-continuous. Furthermore, $g_n(\xi)$ is a discrete stopping set and $F \cap \{g_n(\xi) = U\} \in \widetilde{\mathscr{F}}_U \subset \widetilde{\mathscr{G}}^*_{\theta_U(C_k)}$. Hence,

$$\begin{aligned}\mathbb{E}_\nu[\mathbf{1}_F\, h(X_{\theta_\xi(A_k)})] &= \lim_{n\to\infty} \sum_{U\in\mathscr{A}_n} \mathbb{E}_\nu\left[\,\mathbf{1}_{F\cap\{g_n(\xi)=U\}}\, h(X_{\theta_U(A_k)})\,\right] \\ &= \lim_{n\to\infty} \sum_{U\in\mathscr{A}_n} \mathbb{E}_\nu\left[\,\mathbf{1}_{F\cap\{g_n(\xi)=U\}}\, \mathbb{E}_\nu[\,h(X_{\theta_U(A_k)}) \,|\, \widetilde{\mathscr{G}}^*_{\theta_U(C_k)}]\,\right].\end{aligned}$$

Owing to the $\mathscr{C}$-Markov property and the homogeneity of $\mathscr{P}$,

$$\begin{aligned}\mathbb{E}_\nu[\mathbf{1}_F\, h(X_{\theta_\xi(A_k)})] &= \lim_{n\to\infty} \sum_{U\in\mathscr{A}_n} \mathbb{E}_\nu\left[\,\mathbf{1}_{F\cap\{g_n(\xi)=U\}}\, P_{\theta_U(C_k)} h(\mathbf{X}_{\boldsymbol{\theta}_U(\boldsymbol{C}_k)})\,\right] \\ &= \lim_{n\to\infty} \sum_{U\in\mathscr{A}_n} \mathbb{E}_\nu\left[\,\mathbf{1}_{F\cap\{g_n(\xi)=U\}}\, P_{C_k} h(\mathbf{X}_{\boldsymbol{\theta}_U(\boldsymbol{C}_k)})\,\right] \\ &= \lim_{n\to\infty} \mathbb{E}_\nu[\mathbf{1}_F\, P_{C_k} h(\mathbf{X}_{\boldsymbol{\theta}_{g_n(\xi)}(\boldsymbol{C}_k)})].\end{aligned}$$

Finally, since $P_{C_k} h \in C_0(E^{|\mathscr{A}_{C_k}|})$ and X is outer-continuous, the dominated convergence theorem leads to

$$\mathbb{E}_\nu[\mathbf{1}_F\, h(X_{\theta_\xi(A_k)})] = \mathbb{E}_\nu[\mathbf{1}_F\, P_{C_k} h(\mathbf{X}_{\boldsymbol{\theta}_\xi(\boldsymbol{C}_k)})].$$

A simple induction argument extends the equality to any collection $h_1, \dots, h_k$ in $C_0(E)$,

$$\begin{aligned}&\mathbb{E}_\nu[\mathbf{1}_F\, h_1(X_{\theta_\xi(A_1)}) \cdots h_k(X_{\theta_\xi(A_k)})] \\ &= \mathbb{E}_\nu[\mathbf{1}_F\, h_1(X_{\theta_\xi(A_1)}) \cdots h_{k-1}(X_{\theta_\xi(A_{k-1})})\, P_{C_k} h_k(\mathbf{X}_{\boldsymbol{\theta}_\xi(\boldsymbol{C}_k)})] \\ &= \mathbb{E}_\nu\left[\mathbf{1}_F \int_{\mathbf{R}^k} P_{C_1}(X_\xi; \mathrm{d}x_{A_1}) \dots P_{C_k}(\mathbf{x}_{\boldsymbol{C}_k}; \mathrm{d}x_{A_k})\, h_1(x_{A_1}) \dots h_k(x_{A_k})\right],\end{aligned}$$

using $\theta_\xi(\emptyset') = \xi$. This last formula proves that

$$\mathbb{E}_\nu\left[\,\mathbf{1}_F\, h_1(X_{\theta_\xi(A_1)}) \cdots h_k(X_{\theta_\xi(A_k)})\,\right] = \mathbb{E}_\nu\left[\,\mathbf{1}_F\, \mathbb{E}_{X_\xi}[h_1(X_{A_1}) \cdots h_k(X_{A_k})]\,\right],$$

which leads to the expected result, by a monotone class argument. □

The outer-continuity of sample paths is an important assumption in Theorems 4.8 and 4.9. In the general set-indexed framework, the question of the regularity of processes is known to be non-trivial, and far more complex than in the multiparameter setting. For instance, ADLER [2] has presented indexing collections on which the set-indexed Brownian motion is not continuous (and even not bounded). There exist several approach in the literature to study this question, including the metric entropy theory [2, 3] or more recently, the set-indexed formalism [83].

The càdlàguity of $\mathscr{C}$-Markov processes is studied in Section 4.3 within the multiparameter setting.

4.2.5 Applications and examples

To conclude this general study of the $\mathscr{C}$-Markov property, we present several examples of $\mathscr{C}$-Markov processes which are natural extensions of classic one- and multi-parameter processes.

Example 4.1 (Processes with independent increments) A set-indexed process X has independent increments if for any disjoints sets $C_1, \dots, C_k \in \mathscr{C}$, $\Delta X_{C_1}, \dots, \Delta X_{C_k}$ are independent random variables. Or equivalently, if for any $C \in \mathscr{C}$, the increment ΔX_C is independent of the σ-field $\mathscr{G}^*_C$.

Let P_{X_C} denote the law of the increment ΔX_C. Then, for all $C = A \setminus B \in \mathscr{C}$ and any $\Gamma \in \mathscr{E}$,

$$\begin{aligned}\mathbb{P}(X_A \in \Gamma \mid \mathscr{G}_C^*) &= \mathbb{P}(\Delta X_C \in \Gamma - \Delta X_B \mid \mathscr{G}_C^*) \\ &= P_{X_C}(\Gamma - \Delta X_B) := P_C(\mathbf{X_C}; \Gamma) \quad \text{owing to Equation (4.5).}\end{aligned}$$

Therefore, as we might have expected, set-indexed processes with independent increments satisfy the $\mathscr{C}$-Markov property. As proved by BALAN and IVANOFF [18], they are also *set-Markov*.

Example 4.2 (Set-indexed Lévy processes) Set-indexed Lévy processes have been defined and studied by ADLER and FEIGIN [3], BASS and PYKE [24] in the particular case of subsets of $\mathbf{R}^N$, and by HERBIN and MERZBACH [82] in the general set-indexed formalism. In the latter, a set-indexed Lévy process X on a measure space $(\mathscr{T}, m)$ is characterized as follows:

1) $X_{\emptyset'} = 0$ almost surely;

2) X has independent increments;

3) X has m-stationary $\mathscr{C}_0$-increments, i.e. for all $V \in \mathscr{A}$ and for all increasing sequences $(U_i)_{i \le n}$ and $(A_i)_{i \le n}$ in $\mathscr{A}$ such that $m(U_i \setminus V) = m(A_i)$,

$$(\Delta X_{U_1 \setminus V}, \dots, \Delta X_{U_n \setminus V}) \stackrel{\mathrm{d}}{=} (\Delta X_{A_1}, \dots, \Delta X_{A_n});$$

4) X is continuous in probability.

According to [82], for any set-indexed Lévy process X, there exists an infinitely divisible measure μ on $(E, \mathscr{E})$ such that for all $C \in \mathscr{C}$, $P_{X_C} = \mu^{m(C)}$. Since X has independent increments, it is a $\mathscr{C}$-Markov process with the following $\mathscr{C}$-transition system $\mathscr{P}$,

$$\forall C = A \setminus B \in \mathscr{C}, \quad \forall \Gamma \in \mathscr{E}; \quad P_C(\mathbf{x_C}; \Gamma) = \mu^{m(C)}(\Gamma - \Delta x_B). \tag{4.23}$$

Conversely, one can easily prove, using the $\mathscr{C}$-Markov property, that a Lévy process can be constructed from any infinitely divisible measure μ. For this purpose, one simply needs to show that $\mathscr{P}$ satisfies the three conditions of Definition 4.2.

1) For any $C \in \mathscr{C}$, $P_C(\mathbf{x_C}; \mathrm{d}x_A)$ is clearly a transition probability;

2) As $m(\emptyset) = 0$, we have $P_\emptyset(x; \mathrm{d}y) = \delta_x(\mathrm{d}y)$;

3) For any $C = A \setminus B \in \mathscr{C}$ and $A' \in \mathscr{A}$, let $C' = C \cap A'$ and $C'' = C \setminus A'$ in $\mathscr{C}$. Then, we observe that

$$\mu^{m(C)} = \mu^{m(C')} * \mu^{m(C'')} \quad \text{and} \quad x_{A'} + \Delta x_B = \Delta x_{A' \cup B} + \Delta x_{A' \cap B},$$

using the equality $m(C) = m(C') + m(C'')$ and the inclusion-exclusion principle. These two equations induce the Chapman–Kolmogorov formula (4.6).

Note that the construction procedure presented by HERBIN and MERZBACH [82] is completely equivalent to the proof of Theorem 4.3 (in fact more general in the case of Lévy processes since the *Shape* assumption is not required on the collection $\mathscr{A}$).

One can also check that the $\mathscr{C}$-transition system is homogeneous w.r.t $(\theta_U)_{U \in \mathscr{A}}$ if and only if the measure m is compatible with the family operators, i.e. $\forall U \in \mathscr{A}$, $\forall C \in \mathscr{C}$; $m(C) = m(\theta_U(C))$.

Furthermore, if the measure m and the Hausdorff metric d_H satisfy the following mild condition,

$$\forall m \in \mathbf{N} \quad \exists K_m > 0 \quad \forall C \in \mathscr{C}^m; \quad m(C) \le K_m \|C\|_{\mathscr{C}}, \tag{4.24}$$

then the $\mathscr{C}$-transition system $\mathscr{P}$ of a Lévy process is Feller. Indeed, suppose $f \in C_0(E)$ and $C = C' \cup C'' \in \mathscr{C}$. Then,

$$P_C f(\mathbf{x}_C) = \int_E \mu^{m(C'')}(\mathrm{d}x_A)\left\{\int_E f(x_A + x_{A'} + \Delta x_B)\,\mu^{m(C')}(\mathrm{d}x_{A'})\right\}. \tag{4.25}$$

Due to the Feller property of one-parameter Lévy processes,

$$\lim_{\rho\to 0} \sup_{\substack{C'\in\mathscr{C}\\ m(C')\le\rho}} \sup_{y\in E} \left|\int_E f(x_{A'} + y)\,\mu^{m(C')}(\mathrm{d}x_{A'}) - f(y)\right| = 0.$$

Furthermore, if $\|\mathbf{x}_{C'}\| \le \rho$, Equation (4.17) implies that $|\Delta x_{C'}| \le m\rho$. Hence, since $\Delta x_{B''} = \Delta x_{C'} + \Delta x_B$ and $f \in C_0(E)$,

$$\lim_{\rho\to 0} \sup_{\substack{C'\in\mathscr{C}\\ m(C')\le\rho}} \sup_{\substack{\mathbf{x}_{C''},\mathbf{x}_{C'}\\ \|\mathbf{x}_{C'}\|_{\mathscr{C}}\le\rho}} \left|\int_E f(x_A + x_{A'} + \Delta x_B)\,\mu^{m(C')}(\mathrm{d}x_{A'}) - f(x_A + \Delta x_{B''})\right| = 0.$$

Assumption (4.24), Equation (4.25) and the last equality induce the Feller property:

$$\lim_{\rho\to 0} \sup_{\substack{C=C'\cup C''\in\mathscr{C}^m\\ \|C'\|_{\mathscr{C}}\le\rho}} \sup_{\substack{\mathbf{x}_{C''},\mathbf{x}_{C'}\\ \|\mathbf{x}_{C'}\|_{\mathscr{C}}\le\rho}} \left|P_C f(\mathbf{x}_C) - P_{C''} f(\mathbf{x}_{C''})\right| = 0.$$

Note that in the particular case of the set-indexed Brownian motion, $\mathscr{P}$ is characterized by the following transition densities

$$\forall C = A\setminus B \in \mathscr{C};\quad p_C(\mathbf{x}_C; y) = \frac{1}{\left(2\pi\, m(C)\right)^{d/2}} e^{-\frac{\|y-\Delta x_B\|_2^2}{2m(C)}}.$$

Finally, we observe that Theorem 4.6 extends the Cairoli-Walsh Commutation Theorem (see e.g. in [185]) to the set-indexed formalism, showing that the Brownian motion history is a commuting filtration.

Example 4.3 (Set-indexed α-stable Ornstein–Uhlenbeck process) The classic Ornstein–Uhlenbeck (OU) process is a well-known stochastic process which has the following integral representation (see e.g. [167]):

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-u)} M(\mathrm{d}u), \tag{4.26}$$

where $\lambda > 0$ and M is symmetric α-stable random measure ($\alpha \in (0, 2]$) with Lebesgue control measure.

A set-indexed extension of this process can not be directly deduced from Equation (4.26), Nevertheless, as X is also a Markov process, a $\mathscr{C}$-transition system which generalizes the OU Markov kernel can be introduced.

More precisely, on the space $(E, \mathscr{E}) = (\mathbf{R}, \mathscr{B}(\mathbf{R}))$, a *set-indexed α-stable Ornstein–Uhlenbeck* process is defined as the $\mathscr{C}$-Markov process with any initial distribution ν and the $\mathscr{C}$-transition system $\mathscr{P}$ characterized by

$$\forall \mathbf{x}_C \in E^{|\mathscr{A}_C|};\quad P_C f(\mathbf{x}_C) = \mathbb{E}\left[f\left(\sigma_C X + e^{-\lambda m(A)}\left[\sum_{i=1}^{|\mathscr{A}_C|}(-1)^{\varepsilon_i}\, x_{U_C^i}\, e^{\lambda m(U_C^i)}\right]\right)\right] \tag{4.27}$$

where $\alpha \in (0,2]$, X is a symmetric α-stable variable $S_\alpha(1,0,0)$, $f : E \to \mathbf{R}_+$ is a measurable function and σ_C is defined by

$$\sigma_C^\alpha = \frac{\sigma^\alpha}{\alpha\lambda}\left(1 - e^{-\alpha\lambda m(A)}\left[\sum_{i=1}^{|\mathscr{A}_C|}(-1)^{\varepsilon_i} e^{\alpha\lambda m(U_C^i)}\right]\right). \tag{4.28}$$

Let us prove that $\mathscr{P}$ is a well-defined $\mathscr{C}$-transition system.

1) For any $C \in \mathscr{C}$, $P_C(\mathbf{x}_C; \mathrm{d}x_A)$ is a transition probability;

2) $P_\emptyset(x; \mathrm{d}y) = \delta_x(\mathrm{d}y)$ since $\sigma_\emptyset = 0$;

3) for any $C = A \setminus B \in \mathscr{C}$ and $A' \in \mathscr{A}$, let $C' = C \cap A'$ and $C'' = C \setminus A'$ in $\mathscr{C}$. Let us show that for any measurable function f, $P_C f(\mathbf{x}_C) = \int_{E^2} P_{C'}(\mathbf{x}_{C'}; dx_{A'}) P_{C''}(\mathbf{x}_{C''}; \mathrm{d}x_A) f(x_A)$. Owing to Equation (4.27),

$$P_{C''} f(\mathbf{x}_{C''}) = \mathbb{E}\left[f\left(\sigma_{C''}X + e^{-\lambda m(A)}\left[\sum_{i=2}^{|\mathscr{A}_{C''}|}(-1)^{\varepsilon_i} x_{U_{C''}^i} e^{\lambda m(U_{C''}^i)}\right] + x_{A'} e^{-\lambda m(A\setminus A')}\right)\right],$$

where the numbering of $\mathscr{A}_{C''}$ is assumed to satisfy $U_{C''}^1 = A'$. Suppose $\widetilde{X}$ is random variable $S_\alpha(1,0,0)$ independent of X. Then,

$$\begin{aligned} P_{C'}P_{C''} f = \mathbb{E}\Bigg[f\Bigg(&\sigma_{C''}X + e^{-\lambda m(A)}\left[\sum_{i=2}^{|\mathscr{A}_{C''}|}(-1)^{\varepsilon_i''} x_{U_{C''}^i} e^{\lambda m(U_{C''}^i)}\right] \\ &+ \left\{\sigma_{C'}\widetilde{X} + e^{-\lambda m(A')}\left[\sum_{i=1}^{|\mathscr{A}_{C'}|}(-1)^{\varepsilon_i'} x_{U_{C'}^i} e^{\lambda m(U_{C'}^i)}\right]\right\} e^{-\lambda m(A\setminus A')}\Bigg)\Bigg]. \end{aligned}$$

Let $h : \mathscr{A} \to \mathbf{R}$ be the deterministic map $h(A) = x_A e^{\lambda m(A)}$. Owing to Equation (4.5), it has an additive extension Δh on $\mathscr{A}(u)$ which satisfies:

$$\Delta h(B \cup A') = \sum_{i=1}^{|\mathscr{A}_{C''}|}(-1)^{\varepsilon_i''} x_{U_{C''}^i} e^{\lambda m(U_{C''}^i)} \quad \text{and} \quad \Delta h(B \cap A') = \sum_{i=1}^{|\mathscr{A}_{C'}|}(-1)^{\varepsilon_i'} x_{U_{C'}^i} e^{\lambda m(U_{C'}^i)}$$

Furthermore, using the inclusion-exclusion principle,

$$\Delta h(B \cup A') - h(A') + \Delta h(B \cap A') = \Delta h(B) = \sum_{i=1}^{|\mathscr{A}_C|}(-1)^{\varepsilon_i} x_{U_C^i} e^{\lambda m(U_C^i)}.$$

Therefore,

$$P_{C'}P_{C''} f(\mathbf{x}_C) = \mathbb{E}\left[f\left(\sigma_{C''}X + \sigma_{C'}e^{-\lambda m(A\setminus A')}\widetilde{X} + e^{-\lambda m(A)}\Delta h(B)\right)\right].$$

Since X and $\widetilde{X}$ are two independent $\mathscr{S}\alpha\mathscr{S}$ variables, we know that $\sigma_{C''}X + \sigma_{C'}e^{-\lambda m(A\setminus A')} \sim S_\alpha(\sigma, 0, 0)$, where

$$\sigma^\alpha = \sigma_{C'}^\alpha + \sigma_{C''}^\alpha e^{-\alpha\lambda m(A\setminus A')}.$$

According to the Definition (4.28) of σ_C and the inclusion-exclusion principle (4.5), we obtain $\sigma = \sigma_C$, therefore proving the Chapman–Kolmogorov equation $P_C f = P_{C'}P_{C''} f$.

Hence, we have shown that such a $\mathscr{C}$-Markov process exists. One can easily verify that it corresponds to the Markov kernel of the usual Ornstein–Uhlenbeck process in the case of the one-parameter indexing collection $\mathscr{A} = \{[0,t]; t \in \mathbf{R}_+\}$.

Similarly to the set-indexed Lévy processes, we observe that $\mathscr{P}$ is homogeneous when the measure m is compatible with the family of operators $(\theta_U)_{U\in\mathscr{A}}$ and Feller under the mild condition (4.24).

The Gaussian particular case $\alpha = 2$ is studied in more details in Section 4.4. The latter describes a more constructive approach of the set-indexed Ornstein–Uhlenbeck process which helps to better understand the form of the transition probabilities exhibited in Equations (4.27) and (4.28). In particular, it is proved that if ν is the Dirac distribution δ_x, $x \in \mathbf{R}$, then X is Gaussian process such that for all $U, V \in \mathscr{A}$

$$\mathbb{E}_x[X_U] = xe^{-\lambda m(U)} \quad \text{and} \quad \mathrm{Cov}_x(X_U, X_V) = \frac{\sigma^2}{2\lambda}\left(e^{-\lambda m(U\Delta V)} - e^{-\lambda(m(U)+m(V))}\right).$$

Example 4.4 (**Set-indexed additive Lévy and product processes**) From the previous examples of $\mathscr{C}$-Markov processes, more complex objects can be constructed. Named *product* and *additive Lévy* set-indexed processes, they still satisfy the $\mathscr{C}$-Markov property and constitute a generalization of the multiparameter examples presented by KHOSHNEVISAN [100].

For this purpose, we first need to introduce the idea of *product indexing collections*, previously mentioned in the Introduction. Given m indexing collections $\mathscr{A}_1, \dots, \mathscr{A}_m$ on respectively $\mathscr{T}_1, \dots, \mathscr{T}_m$, we define the product indexing collection $\mathscr{A}$ as follows:

$$\mathscr{A} = \{A_1 \times \cdots \times A_m : A_i \in \mathscr{A}_i,\ 1 \le i \le m\}, \tag{4.29}$$

on the space $\mathscr{T} = \mathscr{T}_1 \times \cdots \times \mathscr{T}_m$. Conditions $(i) - (iii)$ of the definition of an indexing collection can be easily checked. The *Shape assumption* is slightly more difficult to verify. Up to an induction reasoning on m, we can assume that $m = 2$. Then, let $B = \cup_{j=1}^k (A_{1,j} \times A_{2,j})$ be an element of $\mathscr{A}(u)$ and $A = A_1 \times A_2 \subseteq B$ belong to $\mathscr{A}$. Let also $I_1 = \{j : A_1 \subseteq A_{1,j}\}$ and $I_2 = \{j : A_2 \subseteq A_{2,j}\}$. We have to prove that $I_1 \cap I_2 \neq \emptyset$. Suppose on the contrary that $I_1 \cap I_2 = \emptyset$ and set

$$t_1 \in A_1 \setminus \left(\cup_{j\notin I_1} A_{1,j}\right) \quad \text{and} \quad t_2 \in A_2 \setminus \left(\cup_{j\notin I_2} A_{2,j}\right).$$

The existence of (t_1, t_2) is due to the Shape assumption on $\mathscr{A}_1$ and $\mathscr{A}_2$. Then, let $j \in \{1, \dots, k\}$. If $j \notin I_1$ and $j \notin I_2$, then $t_1 \notin A_{1,j}$ and $t_2 \notin A_{2,j}$. Moreover, if $j \in I_1$, then $j \notin I_2$ and thus $t_2 \notin A_{2,j}$. Similarly, if $j \in I_2$, then $t_1 \notin A_{1,j}$. Therefore, $(t_1, t_2) \in A_1 \times A_2$ but $(t_1, t_2) \notin B$, which contradicts the hypothesis $A \subseteq B$. Hence, $I_1 \cap I_2 \neq \emptyset$ and $\mathscr{A}$ satisfies the Shape assumption.

A class of approximating functions $(g_n)_{n\in\mathbf{N}}$ on $\mathscr{A}$ can be easily deduced from existing ones on $\mathscr{A}_1, \dots, \mathscr{A}_m$. Moreover, any family of filtrations indexed by $\mathscr{A}_1, \dots, \mathscr{A}_m$ leads to an $\mathscr{A}$-indexed extension defined as follows:

$$\forall A = A_1 \times \cdots \times \mathscr{A}_m \in \mathscr{A}; \quad \mathscr{F}_A = \bigvee_{i=1}^m \mathscr{F}^i_{A_i}.$$

Throughout, for any $C = A \setminus B \in \mathscr{C}$, $C_1, \dots, C_m$ will denote the following increments:

$$C_i = A_i \setminus \left(\cup_{j=1}^k A_{i,j}\right) \in \mathscr{C}_i, \quad 1 \le i \le m. \tag{4.30}$$

Note that C does not correspond to the direct product of $C_1, \dots, C_m$.

Based on this construction procedure, we are able to introduce larger and richer classes of $\mathscr{C}$-Markov processes by combining previous examples.

Let us first describe the family of *product $\mathscr{C}$-Markov processes*. Given m independent $\mathscr{C}$-Markov processes $X^1, \dots, X^m$, we define the product process X as the direct product of the latter:

$$\forall A = A_1 \times \cdots \times A_m \in \mathscr{A}; \quad X_A = X^1_{A_1} \otimes \cdots \otimes X^m_{A_m}. \tag{4.31}$$

Owing to the independence of $X^1, \dots, X^m$, a monotone class argument shows that X is a $\mathscr{C}$-Markov process. Moreover, its transition probabilities are given by

$$\forall C \in \mathscr{C}; \quad P_C(\mathbf{x}_C; \mathrm{d}x_A) = \prod_{i=1}^{m} P^i_{C_i}(\mathbf{x}_{C_i}; \mathrm{d}x_{A_i}), \tag{4.32}$$

where C_i is the increment defined in Equation (4.30) and $P^i_{C_i}(\mathbf{x}_{C_i}; \mathrm{d}x_{A_i})$ denotes the transition probabilities of X^i, $1 \le i \le m$. A slightly technical calculus also shows that if $X^1, \dots, X^m$ are $\mathscr{C}$-Feller processes, then X is also $\mathscr{C}$-Feller.

Note that, still using the notation introduced in Equation (4.30), any increment of X can be written as

$$\forall C \in \mathscr{C}; \quad \Delta X_C = \Delta X^1_{C_1} \otimes \cdots \otimes \Delta X^m_{C_m}.$$

Particularly, if $X^1, \dots, X^m$ are processes with independent increments, then so does X.

A second class of processes we are interested in is the family of *set-indexed additive Lévy processes*. Suppose that $X^1, \dots, X^m$ are $\mathbf{R}^d$-valued set-indexed Lévy processes, characterized by $(\nu_1, m_1), \dots, (\nu_m, m_m)$, respectively. Then, X is defined as

$$\forall A = A_1 \times \cdots \times A_m \in \mathscr{A}; \quad X_A = X^1_{A_1} + \cdots + X^m_{A_m}. \tag{4.33}$$

Similarly to product processes, increments of X have the following form,

$$\forall C \in \mathscr{C}; \quad \Delta X_C = \Delta X^1_{C_1} + \cdots + \Delta X^m_{C_m},$$

showing in particular that X has independent increments. Its transition probabilities can be obtained as well:

$$\forall C \in \mathscr{C}; \quad P_C(\mathbf{x}_C; \Gamma) = \left(\nu_1^{m_1(C_1)} * \cdots * \nu_m^{m_m(C_m)}\right)(\Gamma - \Delta x_B), \tag{4.34}$$

where $\Gamma \in \mathscr{B}(\mathbf{R}^d)$. The transition system $\mathscr{P}$ is Feller if the mild assumption (4.24) holds on $m_1, \dots, m_m$. It is homogeneous as well if $X^1, \dots, X^m$ are.

Note that X is not necessarily a *set-indexed Lévy process*, as defined previously in Example 4.2. Nevertheless, a sufficient condition is the existence of a measure ν and $\alpha_1, \dots, \alpha_N$ in $\mathbf{R}_+^*$ such that $\nu = \nu_1^{\alpha_1} = \cdots = \nu_N^{\alpha_N}$. In this case, we observe that X is the Lévy process characterised by (ν, m_α), where the measure m_α is defined as follows:

$$\forall A = A_1 \times \cdots \times A_m; \quad m_\alpha(A) = \sum_{i=1}^{m} \alpha_i \, m_i(A_i). \tag{4.35}$$

Conversely, set-indexed Lévy processes are usually not additive processes (e.g. the well-known multiparameter Brownian sheet), since in general, a measure m does not have the form displayed in Equation (4.35).

4.3 Multiparameter $\mathscr{C}$-Markov processes

The extension of the one-parameter Markov property to the multiparameter setting has been intensively investigated and a large literature already exists on the subject. Since the set-indexed

formalism covers multiparameter processes, it is therefore natural to study more precisely the $\mathscr{C}$-Markov property in this setting.

Consequently, in this section, $\mathscr{T}$ and $\mathscr{A}$ designate $\mathbf{R}_+^N$ and the indexing collection $\{[0,t]; t \in \mathbf{R}_+^N\}$, respectively. As previously noted, the natural translation on $\mathbf{R}_+^N$ leads to the definition of the shift operators $\theta_u(t) = \theta_{[0,u]}([0,t]) := [0, t+u]$, for all $u, t \in \mathbf{R}_+^N$.

4.3.1 Right-continuous modification of $\mathscr{C}$-Feller processes

We begin by extending a classic result of the theory of Markov processes: multiparameter $\mathscr{C}$-Feller processes have a right-continuous modification which is $\mathscr{C}$-Markov with respect to the augmented filtration.

In the rest of the section, E_Δ denotes the usual one-point compactification of E, i.e. the set $E \cup \{\Delta\}$ endowed with the following topology: $A \subset E_\Delta$ is open if either $A \subset E$ is open in the topology of E or there exists a compact $K \subset E$ such that $A = E_\Delta \setminus K$.

Theorem 4.10 (**Càd modification**) *Let $(X_t)_{t\in\mathbf{R}_+^N}$ be a multiparameter process, $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$ be a filtration and $(\mathbb{P}_x)_{x\in E}$ be a collection of probability measures such that for every $x \in E$, $\mathbb{P}_x(X_0 = x) = 1$ and X is a $\mathscr{C}$-Feller process on $(\Omega, \mathscr{F}, \mathbb{P}_x)$ w.r.t. $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$.*
Then, there exists an E_Δ-valued process $(\widetilde{X}_t)_{t\in\mathbf{R}_+^N}$ such that for every $x \in E$:

(i) $t \mapsto \widetilde{X}_t$ is right-continuous $\mathbb{P}_x$-a.s.;

(ii) for all $t \in \mathbf{R}_+$, $\widetilde{X}_t = X_t$ $\mathbb{P}_x$-a.s.;

(iii) $\widetilde{X}$ is a $\mathscr{C}$-Feller process on $(\Omega, \mathscr{F}, \mathbb{P}_x)$ w.r.t. the augmented filtration $(\widetilde{\mathscr{F}}_t)_{t\in\mathbf{R}_+^N}$.

Proof. Let $\mathscr{H} = \{\varphi_n; n \in \mathbf{N}\}$ be a countable collection of functions in $C_0^+(E)$ which separate points in E_Δ, i.e., for any $x, y \in E_\Delta$, there exists $\varphi \in \mathscr{H}$ such that $\varphi(x) \neq \varphi(y)$. For all $r \in \mathbf{Q}_+^N$ and any $\varphi \in \mathscr{H}$, the process M^r is defined as follows:

$$\forall t \preceq r \in \mathbf{R}_+^N; \quad M_t^r = P_{[0,r]\setminus[0,t]}\varphi(X_t).$$

For every $x \in E$, M^r is multiparameter martingale with respect to the natural filtration $(\mathscr{F}_t^0)_{t\in\mathbf{R}_+^N}$ of X (i.e. $\mathscr{F}_t^0 := \sigma(\{X_s; s \preceq t, s \in \mathbf{R}_+^N\})$):

$$\begin{aligned}\forall t \prec t' \preceq r \in \mathbf{R}_+^N; \quad \mathbb{E}_x[M_{t'}^r \mid \mathscr{F}_t^0] &= \mathbb{E}_x[P_{[0,r]\setminus[0,t']}\varphi(X_{t'}) \mid \mathscr{F}_t^0] \\ &= P_{[0,t']\setminus[0,t]}\big(P_{[0,r]\setminus[0,t']}\varphi\big)(X_t) \\ &= P_{[0,r]\setminus[0,t]}\varphi(X_t) = M_t^r,\end{aligned}$$

using the Chapman–Kolmogorov Equation (4.6).

Owing to Theorem 4.6, the filtration $(\mathscr{F}_t^0)_{t\in\mathbf{R}_+^N}$ is commuting. Then, as stated by [100] (and originally proved in [16]), since M^r is a bounded multiparameter martingale with respect to a commuting filtration, there exists an event $\Lambda_{\varphi,r} \in \mathscr{F}$ such that for every $x \in E$, $\mathbb{P}_x(\Lambda_{\varphi,r}^c) = 0$ and for all $\omega \in \Lambda_{\varphi,r}$,

$$\lim_{s\downarrow t, s\in\mathbf{Q}_+^N} M_s^r(\omega) \text{ exists for all } t \prec r \in \mathbf{R}_+^N. \tag{4.36}$$

Let Λ denote the event defined by

$$\Lambda = \bigcap_{\varphi\in\mathscr{H}, r\in\mathbf{Q}_+^N} \Lambda_{\varphi,r}.$$

Since the union is countable, $\mathbb{P}_x(\Lambda^c) = 0$ for every $x \in E$.

Let us prove $\lim_{s\downarrow t, s\in\mathbf{Q}_+^N} X_s(\omega)$ exists for all $\omega \in \Lambda$. We proceed by contradiction. Suppose there exist $\omega \in \Lambda$, $t \in \mathbf{R}_+^N$, and two decreasing sequences $(s_n^1)_{n\in\mathbf{N}}, (s_n^2)_{n\in\mathbf{N}}$ in $\mathbf{Q}_+^N$ such that $\lim_{n\in\mathbf{N}} s_n^1 = t$, $\lim_{n\in\mathbf{N}} s_n^2 = t$ and

$$X_t^1(\omega) := \lim_{s_n^1\downarrow t} X_{s_n^1}(\omega) \neq \lim_{s_n^2\downarrow t} X_{s_n^2}(\omega) := X_t^2(\omega),$$

where the two limits stand in E_Δ.

Let φ be a separating function in $\mathscr{H}$ such that $\varepsilon = |\varphi(X_t^1(\omega)) - \varphi(X_t^2(\omega))| > 0$ and $(u_n)_{n\in\mathbf{N}}$ be the decreasing sequence $u_n = (s_n^1 \curlyvee s_n^2) + \frac{1}{n} \in \mathbf{Q}_+^N$.

Since the transition system $\mathscr{P}$ of X is Feller, Equation (4.20) implies the existence of $\alpha > 0$ such that

$$\forall u \preceq v \preceq 2t \in \mathbf{R}_+^N; \quad \|u - v\|_\infty \leq \alpha \quad \Longrightarrow \quad \left\|P_{[0,v]\setminus[0,u]}\varphi - \varphi\right\|_\infty \leq \frac{\varepsilon}{8}.$$

Furthermore, since $u_n \to_n t$, there exists $k \in \mathbf{N}$ such that for all $n \geq k$, $\|u_n - t\| \leq \alpha$, $\|u_k - s_n^1\| \leq \alpha$ and $\|u_k - s_n^2\| \leq \alpha$. Therefore, for all $n \geq k$,

$$\left\|P_{[0,u_k]\setminus[0,s_n^1]}\varphi - \varphi\right\|_\infty \leq \frac{\varepsilon}{8} \quad \text{and} \quad \left\|P_{[0,u_k]\setminus[0,s_n^2]}\varphi - \varphi\right\|_\infty \leq \frac{\varepsilon}{8}.$$

Owing to Equation (4.36) and as $\omega \in \Lambda_{\varphi,u_k}$, k can be chosen large enough such that for all $n \geq k$, we have $|M_{s_n^1}^{u_k}(\omega) - M_{s_n^2}^{u_k}(\omega)| \leq \frac{\varepsilon}{8}$, i.e.

$$\forall n \geq k; \quad \left|P_{[0,u_k]\setminus[0,s_n^1]}\varphi(X_{s_n^1}(\omega)) - P_{[0,u_k]\setminus[0,s_n^2]}\varphi(X_{s_n^2}(\omega))\right| \leq \frac{\varepsilon}{8}.$$

Finally, as the function φ is continuous, k can be supposed to satisfy

$$\forall n \geq k; \quad \left|\varphi(X_t^1(\omega)) - \varphi(X_{s_n^1}(\omega))\right| \leq \frac{\varepsilon}{8} \quad \text{and} \quad \left|\varphi(X_t^2(\omega)) - \varphi(X_{s_n^2}(\omega))\right| \leq \frac{\varepsilon}{8}.$$

Due to previous inequalities, for all $n \geq k$,

$$\begin{aligned}
\left|\varphi(X_t^1(\omega)) - \varphi(X_t^2(\omega))\right| &\leq \left|\varphi(X_t^1(\omega)) - \varphi(X_{s_n^1}(\omega))\right| \\
&\quad + \left|\varphi(X_{s_n^1}(\omega)) - P_{[0,u_k]\setminus[0,s_n^1]}\varphi(X_{s_n^1}(\omega))\right| \\
&\quad + \left|P_{[0,u_k]\setminus[0,s_n^1]}\varphi(X_{s_n^1}(\omega)) - P_{[0,u_k]\setminus[0,s_n^2]}\varphi(X_{s_n^2}(\omega))\right| \\
&\quad + \left|\varphi(X_{s_n^2}(\omega)) - P_{[0,u_k]\setminus[0,s_n^2]}\varphi(X_{s_n^2}(\omega))\right| \\
&\quad + \left|\varphi(X_t^2(\omega)) - \varphi(X_{s_n^2}(\omega))\right| \\
&\leq \frac{5}{8}\varepsilon,
\end{aligned}$$

which clearly contradicts the definition of ε.

Hence, for all $\omega \in \Lambda$ and all $t \in \mathbf{R}_+^N$, $\lim_{s\downarrow t, s\in\mathbf{Q}_+^N} X_s(\omega)$ exists in E_Δ and the process $\widetilde{X}$ can be defined as follows:

$$\forall t \in \mathbf{R}_+^N; \quad \widetilde{X}_t = \begin{cases} \lim_{s\downarrow t, s\in\mathbf{Q}_+^N} X_s(\omega) & \text{if } \omega \in \Lambda \\ x_0 & \text{if } \omega \notin \Lambda, \end{cases}$$

where x_0 is an arbitrary point in E. This new process $\widetilde{X}$ is clearly an E_Δ-valued process with right-continuous sample paths.

Let us now prove $\widetilde{X}$ is a modification of X. Let φ_1 and φ_2 be in $C_0(E)$. Then, for every $x \in E$ and all $t \in \mathbf{R}_+^N$, the dominated convergence theorem and the Feller property induce

$$\begin{aligned}\mathbb{E}_x\left[\varphi_1(\widetilde{X}_t)\varphi_2(X_t)\right] &= \lim_{s\downarrow t, s\in\mathbf{Q}_+^N} \mathbb{E}_x\left[\varphi_1(X_s)\varphi_2(X_t)\right] \\ &= \lim_{s\downarrow t, s\in\mathbf{Q}_+^N} \mathbb{E}_x\left[\mathbb{E}_x[\varphi_1(X_s)\,|\,\mathscr{F}_t]\,\varphi_2(X_t)\right] \\ &= \lim_{s\downarrow t, s\in\mathbf{Q}_+^N} \mathbb{E}_x\left[P_{[0,s]\setminus[0,t]}\varphi_1(X_t)\,\varphi_2(X_t)\right] = \mathbb{E}_x\left[\varphi_1(X_t)\varphi_2(X_t)\right].\end{aligned}$$

A classic monotone class argument extends this equality to any measurable function $f : E^2 \to \mathbf{R}_+$, and in particular shows that

$$\forall x \in E,\ \forall t \in \mathbf{R}_+^N; \quad \mathbb{P}_x(X_t = \widetilde{X}_t) = 1.$$

The process $\widetilde{X}$ is clearly adapted to the filtration $(\widetilde{\mathscr{F}}_t)_{t\in\mathbf{R}_+^N}$ and using Theorem 4.8, we obtain the $\mathscr{C}$-Markov property with respect to the augmented filtration. □

Note that the martingale argument used in the previous proof can not be directly transpose to the set-indexed formalism since there does not exist any result on set-indexed martingales stating the existence of a right-continuous modification (see [89] for more details on the theory of set-indexed martingales).

4.3.2 $\mathscr{C}$-Markov and multiparameter Markov properties

A large literature exists on multiparameter Markov properties, especially in the case of two-parameter processes. It includes the works [116, 133, 135, 48, 47] on the previously mentioned *sharp-Markov* and *germ-Markov* properties, but also the literature on Gaussian Markov random fields [159, 112, 41] and strong Markov properties [65, 66, 135, 108]. Different surveys and books [164, 52, 53, 87] have been written on these distinct topics. In the context of this article, we focus on two existing multiparameter properties which can be directly linked up to our work.

To begin with, let us mentioned the two-parameter $*$-*Markov* property introduced by CAIROLI [39]. A process $(X_{s,t})_{s,t}$ is said to be $*$-*Markov* if for all $s, t \in \mathbf{R}_+$, $h, k > 0$ and any $\Gamma \in \mathscr{E}$,

$$\mathbb{P}(X_{s+h,t+k} \in \Gamma \,|\, \mathscr{G}^*_{s,t}) = \mathbb{P}(X_{s+h,t+k} \in \Gamma \,|\, X_{s,t}, X_{s+h,t}, X_{s,t+k}), \tag{4.37}$$

where $\mathscr{G}^*_{s,t}$ correspond to the strong history previously introduced (Equation (4.2)). The $*$-Markov property has been widely studied and considered in the literature, leading to several interesting results, including a martingale characterization in the Gaussian case [149], the equivalence of several different definitions [111], some properties on transition probabilities [120] and the càdlàguity of sample paths [203]. We note that the $*$-Markov property also appears in the study of a two-parameter Ornstein–Uhlenbeck process [187, 202].

As stated in the following proposition, it turns out that $\mathscr{C}$-Markov and $*$-Markov properties are equivalent in the two-parameter formalism.

Proposition 4.11 *Let $\mathscr{T} = \mathbf{R}_+^2$, $\mathscr{A}$ be the indexing collection $\{[0,t]; t \in \mathbf{R}_+^2\}$ and $(X_{s,t})_{s,t}$ be a two-parameter process.*

*Then, X is a $*$-*Markov *process if and only if it satisfies the $\mathscr{C}$-*Markov *property.*

Proof. Let X be a two-parameter $\mathscr{C}$-Markov process, $s, t \in \mathbf{R}_+$, $h, k > 0$ and $\Gamma \in \mathscr{E}$. Define the increment $C = A \setminus B$ as $[0,(s+h,t+k)] \setminus \big([0,(s,t)] \cup [0,(s+h,t)] \cup [0,(s,t+k)]\big)$. We observe that $\mathbf{X}_C = (X_{s,t}, X_{s+h,t}, X_{s,t+k})$ and $\mathscr{G}^*_{s,t} = \mathscr{G}^*_C$. Hence, the $\mathscr{C}$-Markov property $\mathbb{P}(X_A \in \Gamma \,|\, \mathscr{G}^*_C) = \mathbb{P}(X_A \in \Gamma \,|\, \mathbf{X}_C)$ corresponds to $*$-Markov's definition (Equation (4.37)).

Conversely, let X be a $*$-Markov process and $C = A \setminus B$ be an increment where $A = [0,(s,t)]$ and $B = \cup_{i=1}^{n}[0,(s_i,t_i)]$. Without any loss of generality, we can assume that $s_1 < s_2 < \ldots < s_n$. Since we consider an extremal representation of B, it must also satisfy $t_1 > t_2 > \ldots > t_n$. Therefore, $\mathscr{A}_C$ has the following form

$$\mathscr{A}_C = \{[0,(s_1,t_1)],[0,(s_1,t_2)],[0,(s_2,t_2)],\ldots,[0,(s_{n-1},t_n)],[0,(s_n,t_n)]\}.$$

The conditional expectation w.r.t. to such a collection of random variables $\mathbf{X}_C$ has already been considered in the work of KOREZLIOGLU, LEFORT, and MAZZIOTTO [111], and the $\mathscr{C}$-Markov property is therefore a direct consequence of Theorem 3.7 obtained in the latter. □

Proposition 4.11 shows that the $\mathscr{C}$-Markov property offers an elegant way to extend the two-parameter $*$-Markov property to multiparameter processes, and more largely, to set-indexed processes.

Moreover, the set-indexed formalism allows to simplify notations and concepts compared to the $*$-Markov framework. Indeed, as stated in [120, 203], the concept of $*$-transition function (P^1,P^2,P) has been introduced to characterize the law of $*$-Markov process. The corresponding notion of $\mathscr{C}$-transition system $\mathscr{P}$ replaces the triplet (P^1,P^2,P), and reduces the different consistency hypotheses on (P^1,P^2,P) to the single Chapman–Kolmogorov Equation (4.6). We also note that Theorem 4.10 extends the regularity result obtained by ZHOU and ZHOU [203] on two-parameter $*$-Markov processes.

A second interesting Markov property is the *multiparameter Markov* property presented by KHOSHNEVISAN [100] and widely studied in the literature (see e.g. [85, 132, 188, 100]).

An E_Δ valued process $X = \{X_t;\, t \in \mathbf{R}_+^N\}$ is said to be a *multiparameter Markov* process if there exists a filtration $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$ and a family of operators $\mathcal{T} = \{\mathcal{T}_t;\, t \in \mathbf{R}_+^N\}$ such that for every $x \in E$, there exists a probability measure $\mathbb{P}_x$ which satisfies

(i) X is adapted to $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$;

(ii) $t \mapsto X_t$ is right-continuous $\mathbb{P}_x$-a.s.;

(iii) for all $t \in \mathbf{R}_+^N$, $\mathscr{F}_t$ is $\mathbb{P}_x$-complete; moreover, $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$ is a commuting filtration, i.e. for all $s,t \in \mathbf{R}_+^N$ and for any bounded $\mathscr{F}_t$-measurable random variable Y,

$$\mathbb{E}_x[\,Y \mid \mathscr{F}_s\,] = \mathbb{E}_x[\,Y \mid \mathscr{F}_{s\wedge t}\,] \quad \mathbb{P}_x\text{-a.s.}$$

(iv) for all $s,t \in \mathbf{R}_+^N$ and for any $f \in C_0(E)$,

$$\mathbb{E}_x[\,f(X_{t+s}) \mid \mathscr{F}_s\,] = \mathcal{T}_t f(X_s) \quad \mathbb{P}_x\text{-a.s.}$$

(v) $\mathbb{P}_x(X_0 = x) = 1$.

Furthermore, X is said to be a *Feller* process if

(i) for all $t \in \mathbf{R}_+^N$, $\mathcal{T}_t : C_0(E) \to C_0(E)$;

(ii) for any $f \in C_0(E)$, $\lim_{t\to 0}\|\mathcal{T}_t f - f\|_\infty = 0$.

Proposition 4.12 *Let $(\mathbb{P}_x)_{x\in E}$ be a family of probabilities and X be a multiparameter homogeneous $\mathscr{C}$-Feller process w.r.t $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$. Suppose X has right-continuous sample paths $\mathbb{P}_x$-a.s., $\mathbb{P}_x(X_0 = x) = 1$ for every $x \in E$ and $\mathscr{F}_t \subseteq \sigma(\{X_t;\, t \in \mathbf{R}_+^N\})$ for all $t \in \mathbf{R}_+^N$.*

Then, X is a multiparameter Markov and Feller process *w.r.t. the filtration $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$ and with the following transition operators $\mathcal{T} = \{\mathcal{T}_t;\, t \in \mathbf{R}_+^N\}$:*

$$\forall t \in \mathbf{R}_+^N,\ \forall f : \mathbf{R} \to \mathbf{R}_+ \text{ measurable};\quad \mathcal{T}_t f = P_{[0,t]\setminus\{0\}} f,$$

where $\mathscr{P}$ denotes the Markov kernel of X.

Proof. We have to verify the different points of the previous definition of a *multiparameter Markov process*.

(i) X is clearly adapted to $(\mathscr{F}_t)_{t\in\mathbf{R}_+^N}$;

(ii) X is right-continuous $\mathbb{P}_x$-a.s.;

(iii) $(\mathscr{F}_t)_{t\in\mathbf{R}_+^n}$ is a commuting filtration according to Theorem 4.6;

(iv) for all $s,t\in\mathbf{R}_+^N$ and for any $f\in C_0(E)$,

$$\begin{aligned}\mathbb{E}_x[f(X_{t+s})\,|\,\mathscr{F}_s] &= P_{[0,t+s]\setminus[0,s]}f(X_s) && \text{since } X \text{ is } \mathscr{C}\text{-Markov}\\ &= P_{[0,t]\setminus\{0\}}f(X_s) && \text{since } \mathscr{P} \text{ is homogeneous}\\ &:= \mathcal{T}_t f(X_s);\end{aligned}$$

(v) for all $x\in\mathbf{R}$, $\mathbb{P}_x(X_0=x)=1$.

Finally, the Feller conditions on $\mathcal{T}$ are easily verified using Equation (4.20) and Definition 4.8 of a Feller $\mathscr{C}$-transition system. □

As stated by KHOSHNEVISAN [100], the family of operators $\mathcal{T}$ characterizes the two-dimensional marginals of X, but it is still unknown whether it determines entirely the law of X. Proposition 4.12 partially answers this question since when X is also $\mathscr{C}$-Markov, we know that the larger family $\mathscr{P}$ of transition probabilities completely characterizes the finite-dimensional distributions of X.

Since it is not known whether the multiparameter Markov property gives a complete picture of the law of a process X, it is not possible to obtain a general result which would state that multiparameter Markov processes are also $\mathscr{C}$-Markov. Nevertheless, we show in the next examples that usual multiparameter Markov processes investigated in the literature are also $\mathscr{C}$-Markov.

Example 4.5 (Additive Lévy and product processes) As observed by KHOSHNEVISAN [100], *multiparameter additive Lévy* and *product processes* constitute interesting and rich collections of multiparameter Markov processes. Their generic set-indexed form has already been described in Example 4.4, Section 4.2.5. Hence, we only make a few remarks specific to the multiparameter setting.

The general $\mathscr{C}$-Markov characterization of product processes (Equation (4.32)) stands as well for *multiparameter product processes*. Hence, let us simply illustrate this class by a two-parameter process named the *bi-Brownian motion* and defined as

$$\forall t\in\mathbf{R}_+^2;\quad Y_t = B^1_{t_1}\otimes B^2_{t_2}$$

where B^1 and B^2 are two independent $\mathbf{R}^d$-valued Brownian motion. Owing to Equation (4.32), the Markov kernel of Y is characterized by the following transition densities, for all $C\in\mathscr{C}$,

$$p_C(\mathbf{x}_C;(u\otimes v)) = \frac{1}{(2\pi)^d(\lambda_1(C_1)\lambda_1(C_2))^{d/2}}\exp\left(-\frac{\|u-\Delta x^1_{B_1}\|^2}{2\lambda_1(C_1)}-\frac{\|v-\Delta x^2_{B_2}\|^2}{2\lambda_1(C_2)}\right),$$

where λ_1 denotes the Lebesgue measure on $\mathbf{R}$ and C_1, C_2 are the increments defined in Equation (4.30). The family of transition operators $\mathcal{T}$ can be easily retrieve from this last equality.

The common definition of an *additive Lévy process* is usually slightly more restrictive than Equation (4.33), as it usually refers to a process $X=\{X_t;\ t\in\mathbf{R}_+^N\}$ defined as follows:

$$\forall t\in\mathbf{R}_+^N;\quad X_t = X^1_{t_1}+\cdots+X^N_{t_N}.$$

where $t = (t_1, \dots, t_N)$ and $X^1, \dots, X^N$ are N independent one-parameter Lévy processes. The form of its transition probabilities still corresponds to Equation (4.34), where every measure m_i, $1 \le i \le N$, is nevertheless replaced by the Lebesgue measure λ_1 on $\mathbf{R}$. As previously noted, this class is far from containing all multiparameter Lévy processes, since for instance, the Brownian sheet is not an additive process.

Example 4.6 (Multiparameter Ornstein–Uhlenbeck processes) The *set-indexed Gaussian Ornstein–Uhlenbeck* process presented in the next Section 4.4 has an integral representation in the multiparameter setting. More precisely,

$$\forall t \in \mathbf{R}_+^N; \quad X_t = e^{-\langle \alpha, t\rangle}\left[X_0 + \sigma \int_{(-\infty,t]\setminus(-\infty,0]} e^{\langle \alpha, u\rangle} \mathrm{d}W_u\right],$$

where $\sigma > 0$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{R}^N$ with $\alpha_i > 0$, W is the Brownian sheet and X_0 is a random variable independent of W. As previously outlined in Example 4.3, X satisfies the $\mathscr{C}$-Markov property.

A different, but also natural, multiparameter extension of the Ornstein–Uhlenbeck process has been suggested in the literature (e.g. see [187, 186] and [73]). It is defined by

$$\forall t \in \mathbf{R}_+^N; \quad Y_t = e^{-\langle \alpha, t\rangle}\left[Y_0 + \sigma \int_{[0,t]} e^{\langle \alpha, u\rangle} \mathrm{d}W_u\right].$$

As proved by WANG [186], the two-parameter process Y is $*$-Markov. As proved in Section 4.4, it is also a multiparameter $\mathscr{C}$-Markov process. Moreover, suppose $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ denote the $\mathscr{C}$-transition systems of X and Y, respectively. Then, theses two are related by the following equality:

$$\widetilde{P}_C(\mathbf{x}_C; \mathrm{d}x_A) = \begin{cases} \delta_{\Delta x_B}(\mathrm{d}x_A) & \text{for all } C \in \mathscr{C} \text{ s.t. } C \subseteq S; \\ P_C(\mathbf{x}_C; \mathrm{d}x_A) & \text{for all } C \in \mathscr{C} \text{ s.t. } C \cap S = \emptyset, \end{cases}$$

where S denotes the set $\{t \in \mathbf{R}_+^N : \prod_{i=1}^N t_i = 0\}$.

4.4 A set-indexed Ornstein–Uhlenbeck process

In this section, we study in more details the Gaussian set-indexed Ornstein–Uhlenbeck which has been introduced previously. Let recall that the one-dimensional OU process has been originaly introduced by UHLENBECK and ORNSTEIN [184] and then widely investigated and used in the literature to represent phenomena in physics, biology and finance (e.g. see [70, 114, 129]). A well-known integral representation of the real-parameter OU process $X = \{X_t;\ t \in \mathbf{R}_+\}$ is given by

$$\forall t \in \mathbf{R}_+; \quad X_t = X_0\, e^{-\lambda t} + \int_0^t \sigma\, e^{\lambda(s-t)}\, \mathrm{d}W_s, \tag{4.38}$$

where λ and σ are positive parameters and the initial distribution $\nu = \mathscr{L}(X_0)$ is independent of the Brownian motion W. Furthermore, X is a Markov process characterized by the following transition densities, for all $t \in \mathbf{R}_+$ and $x, y \in \mathbf{R}$;

$$p_t(x; y) = \frac{1}{\sigma_t\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_t^2}\left(y - xe^{-\lambda t}\right)^2\right] \qquad \text{where } \sigma_t^2 = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right). \tag{4.39}$$

Two particular cases of initial distribution will be of specific interest in the sequel:

(i) If $\nu = \delta_x$, $x \in \mathbf{R}$, X is a Gaussian process with the following mean and covariance, for all $s, t \in \mathbf{R}_+$,

$$\mathbb{E}_x[X_t] = xe^{-\lambda t} \qquad \text{and} \qquad \mathrm{Cov}_x(X_s, X_t) = \frac{\sigma^2}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right). \tag{4.40}$$

(ii) If $\nu \sim \mathcal{N}(0, \frac{\sigma^2}{2\lambda})$, X is a *stationary Ornstein–Uhlenbeck process*, i.e. a zero-mean Gaussian process such that

$$\forall s, t \in \mathbf{R}_+; \quad \mathbb{E}_\nu[X_s X_t] = \frac{\sigma^2}{2\lambda} e^{-\lambda|t-s|}. \tag{4.41}$$

As previously outlined, a set-indexed extension of the OU process cannot be directly derived from the integral representation (4.38). Hence, we first focus on the stationary process described in Equation (4.41). A natural way to extend this covariance to the set-indexed framework is to substitute the absolute value $|t-s|$ with $d(U,V)$ where d is a distance defined on the elements of $\mathscr{A}$. Similarly to the case of the set-indexed fractional Brownian motion described in [79], we consider the choice $d(U,V) = m(U \Delta V)$, where Δ denotes the symmetric difference.

In Section 4.4.1, we first define a *stationary set-indexed Ornstein–Uhlenbeck* process (ssiOU) as a zero-mean Gaussian process $X = \{X_U; U \in \mathscr{A}\}$ such that

$$\forall U, V \in \mathscr{A}; \quad \mathbb{E}[X_U X_V] = \frac{\sigma^2}{2\lambda} e^{-\lambda m(U \Delta V)}, \tag{4.42}$$

where λ and σ are positive parameters. Stationarity and Markov properties of this set-indexed process are studied, and lead to the complete characterization proved in Theorem 4.17. Then, using the Markov kernel obtained, we are able to introduce in Definition 4.10 a general *set-indexed Ornstein–Uhlenbeck process*, whose law is consistent with the covariance structure (4.40) in particular case of initial Dirac distributions.

Finally, in Section 4.4.3, we prove that in the multiparameter case, the set-indexed Ornstein–Uhlenbeck has a natural integral representation which generalizes expression (4.38).

4.4.1 A stationary set-indexed Ornstein–Uhlenbeck process

In this section, we define a set-indexed extension of the stationary Ornstein–Uhlenbeck process, defined by the Gaussian covariance structure (4.41).

Definition and first properties

As a preliminary to the definition, we need to prove that the expected covariance function of the process is positive definite in the same way as Lemma 2.9 of [79].

Lemma 4.2 *If $\mathscr{A}$ is an indexing collection, m a Radon measure on the σ-field generated by $\mathscr{A}$ and λ, σ positive constants, the function $\Gamma : \mathscr{A} \times \mathscr{A} \to \mathbf{R}$ defined by*

$$\forall U, V \in \mathscr{A}; \quad \Gamma(U,V) = \frac{\sigma^2}{2\lambda} e^{-\lambda m(U \Delta V)},$$

is positive definite.

Proof. Let $f_1, f_2, \ldots, f_k$ be in $L^2(m)$ and $u_1, u_2, \ldots, u_k$ be in $\mathbf{R}$. Let $\mathscr{V}$ be the vector space $\mathscr{V} = \mathrm{span}(f_1, \ldots, f_k)$. Since $f \mapsto e^{-\frac{1}{2}\|f\|^2_{L^2(m)}}$ is positive definite, there exists a Gaussian vector X on the finite-dimensional space $\mathscr{V}$ such that

$$\forall \lambda > 0, \ \forall f \in \mathscr{V}; \quad \mathbb{E}\left[e^{i\sqrt{2\lambda}\langle X, f\rangle}\right] = e^{-\lambda\|f\|^2_{L^2(m)}}.$$

The non-negative definition of $f \mapsto e^{-\lambda\|f\|^2_{L^2(m)}}$ can be written

$$\sum_{i=1}^{k}\sum_{j=1}^{k} u_i u_j e^{-\lambda\|f_i-f_j\|^2_{L^2}} = \sum_{i=1}^{k}\sum_{j=1}^{k} u_i u_j \mathbb{E}\left[\, e^{i\sqrt{2\lambda}\langle X, f_i - f_j\rangle}\,\right]$$
$$= \left\| \sum_{i=1}^{k} u_i e^{i\sqrt{2\lambda}\langle X, f_i\rangle} \right\|^2_{L^2(\Omega)} \geq 0.$$

For any $U_1, \ldots, U_k \in \mathscr{A}$, the previous result is applied to $f_1 = \mathbf{1}_{U_1}, \ldots, f_k = \mathbf{1}_{U_k} \in L^2(m)$. As in the proof of Lemma 2.9 in [79], we remark that

$$\forall i, j \in \{1, \ldots, k\}; \quad m(U_i \Delta U_j) = m\big(|\mathbf{1}_{U_i} - \mathbf{1}_{U_j}|\big) = \left\| f_i - f_j \right\|^2_{L^2(m)},$$

and we deduce

$$\sum_{i=1}^{k}\sum_{j=1}^{k} u_i u_j e^{-\lambda m(U_i \Delta U_j)} \geq 0$$

which proves the result. □

According to Lemma 4.2, we can define

Definition 4.9 *Given the indexing collection $\mathscr{A}$ and positive real numbers λ and σ, any Gaussian process $\{X_U;\ U \in \mathscr{A}\}$ such that for all $U, V \in \mathscr{A}$,*

$$\mathbb{E}[X_U] = 0 \quad \textit{and} \quad \mathbb{E}[X_U X_V] = \frac{\sigma^2}{2\lambda} e^{-\lambda m(U \Delta V)},$$

is called a stationary set-indexed Ornstein–Uhlenbeck (ssiOU) process.

The covariance structure of the Gaussian process coming from Definition 4.9 directly implies the L^2-continuity and stationarity properties.

Proposition 4.13 *The stationary set-indexed Ornstein–Uhlenbeck process X of Definition 4.9 is L^2-monotone inner- and outer-continuous, i.e. for any increasing sequence $(U_n)_{n\in\mathbf{N}}$ in $\mathscr{A}$, such that $\overline{\cup_{k\in\mathbf{N}} U_k} \in \mathscr{A}$ and for any decreasing sequence $(V_n)_{n\in\mathbf{N}}$ in $\mathscr{A}$,*

$$\lim_{n\to\infty} \mathbb{E}\left[\, |X_{U_n} - X_{\overline{\cup_{k\in\mathbf{N}} U_k}}|^2 \,\right] = 0 \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{E}\left[\, |X_{V_n} - X_{\cap_{k\in\mathbf{N}} V_k}|^2 \,\right] = 0.$$

Proof. Let $(U_n)_{n\in\mathbf{N}}$ be an increasing sequence in $\mathscr{A}$ such that $\overline{\cup_{k\in\mathbf{N}} U_k} \in \mathscr{A}$. Then using equation (4.42), we have

$$\forall n \in \mathbf{N}; \quad \mathbb{E}\left[\, |X_{U_n} - X_{\overline{\cup_{k\in\mathbf{N}} U_k}}|^2 \,\right] = \frac{\sigma^2}{2\lambda}\big(2 - 2e^{-\lambda m(\overline{\cup_{k\in\mathbf{N}} U_k} \setminus U_n)}\big).$$

According to Assumption *(iii)* on $\mathscr{A}$ and m, $\lim_{n\to\infty} m(\overline{\cup_{k\in\mathbf{N}} U_k} \setminus U_n) = 0$. Therefore, the L^2-monotone inner-continuity follows, and similarly, the outer-continuity of X. □

The stationarity increments property for set-indexed processes has been introduced in [80] in the context of fractional Brownian motion, and it has constitued the key property to derive deep understanding of the set-indexed Lévy processes in [82]. The stationarity property defined below is closely related to these two previous works.

Proposition 4.14 *The stationary set-indexed Ornstein–Uhlenbeck process X of Definition 4.9 is m-stationary, i.e. for any $k \in \mathbf{N}$, $V \in \mathscr{A}$ and increasing sequences $(U_i)_{1\leq i\leq k}$ and $(A_i)_{1\leq i\leq k}$ in $\mathscr{A}$ such that $m(U_i \setminus V) = m(A_i)$ for all $i \in \{1, \ldots, k\}$, X satisfies*

$$(X_{U_1}, \ldots, X_{U_k}) \overset{(d)}{=} (X_{A_1}, \ldots, X_{A_k}).$$

Proof. Let V, $(U_i)_{1\le i\le k}$ and $(A_i)_{1\le i\le k}$ be as in the statement. Without any loss of generality, we suppose that $V \subseteq U_i$. Then, for all $j \ge i$, as $U_i \subseteq U_j$ and $A_i \subseteq A_j$,

$$\begin{aligned} m(U_i \Delta U_j) = m(U_j) - m(U_i) &= m(U_j \setminus V) - m(U_i \setminus V) \\ &= m(A_j) - m(A_i) = m(A_j \Delta A_i). \end{aligned}$$

Therefore, we deduce the expected equality, since X is a centered Gaussian process and for all $i, j \in \{1, \dots, k\}$, $\mathbb{E}[X_{U_i} X_{U_j}] = \dfrac{\sigma^2}{2\lambda} e^{-\lambda m(U_i \Delta U_j)} = \dfrac{\sigma^2}{2\lambda} e^{-\lambda m(A_j \Delta A_i)} = \mathbb{E}[X_{A_i} X_{A_j}]$. □

We observe that the definition of stationarity is given in a strict sense, since it concerns the invariance of finite-dimensional distributions under a form of *measure-invariant translation*. In the classic theory of stationary random fields, a weaker property relying on the correlation function is usually defined (see [200]): $C(s,t) = \mathbb{E}[X_s X_t]$ only depends on the difference $t - s$. The weak definition of stationarity for one-parameter processes can be naturally extended to the multiparameter case, but it appears that this straightforward extension is not the most relevant. Indeed, the stationarity of increments defined using Lebesgue measure or their invariance under translation appeared to be more interesting to study multiparameter processes (see e.g. Lévy and fractional Brownian sheets), and this fact explains the form of the set-indexed extension for the stationarity property.

Markov property and characterisation

As we have constructed a set-indexed Ornstein–Uhlenbeck process, we may now check that it is a $\mathscr{C}$-Markov process whose transition probabilites are consistent with Example 4.3.

Proposition 4.15 *The stationary set-indexed Ornstein–Uhlenbeck process X of Definition 4.9 is a $\mathscr{C}$-Markov process with respect to its natural filtration $(\mathscr{F}_A)_{A\in\mathscr{A}}$, i.e. for all $C = A \setminus B$ with $A \in \mathscr{A}$, $B \in \mathscr{A}(u)$ and all Borel function $f : \mathbf{R} \to \mathbf{R}_+$, X satisfies*

$$\mathbb{E}[f(X_A) \,|\, \mathscr{G}_C^*] = \mathbb{E}[f(X_A) \,|\, \mathbf{X_C}] := P_C f(\mathbf{X_C}) \quad \mathbb{P}\text{-a.s.} \tag{4.43}$$

Proof. Let $C = A \setminus B$ be in $\mathscr{C}$, $\{A_i\}_{i\le k}$ be the extremal representation of B and U be in $\mathscr{A}$ such that $U \cap C = \emptyset$. We first note that $U \cap A = (U \cap C) \cup (U \cap B) = U \cap B \in \mathscr{A}$. Thus, since $\mathscr{A}$ satisfies the *Shape* hypothesis, there exists $l \in \{1, \dots, k\}$ such that $U \cap B = U \cap A_l$. Consider the following quantity I_U,

$$\begin{aligned} I_U &= \mathbb{E}\Big[X_U\Big(X_A - \sum_{i=1}^{k} X_{A_i} e^{-\lambda m(A\setminus A_i)} + \sum_{1\le i<j\le k} X_{A_i\cap A_j} e^{-\lambda m(A\setminus A_i\cap A_j)} + \\ &\qquad \cdots + (-1)^k X_{A_1\cap\cdots\cap A_k} e^{-\lambda m(A\setminus A_1\cap\cdots\cap A_k)}\Big)\Big] \\ &= \frac{\sigma^2}{2\lambda}\Big(e^{-\lambda m(A\Delta U)} - \sum_{i=1}^{k} e^{-\lambda(m(A_i\Delta U)+m(A\setminus A_i))} + \sum_{1\le i<j\le k} e^{-\lambda(m(A_i\cap A_j\Delta U)+m(A\setminus A_i\cap A_j))} + \\ &\qquad \cdots + (-1)^k e^{-\lambda(m(A_1\cap\cdots\cap A_k\Delta U)+m(A\setminus A_1\cap\cdots\cap A_k))}\Big) \\ &= \frac{\sigma^2}{2\lambda} e^{-\lambda(m(A)+m(U))}\Big(e^{-2\lambda m(A\cap U)} - \sum_{i=1}^{k} e^{-2\lambda m(A_i\cap U)} + \sum_{1\le i<j\le k} e^{-2\lambda m(A_i\cap A_j\cap U)} + \\ &\qquad \cdots + (-1)^k e^{-2\lambda m(A_1\cap\cdots\cap A_k\cap U)}\Big). \end{aligned}$$

Let us introduce the set-indexed function $h : A \mapsto e^{-2\lambda m(A\cap U)}$. Since the assumption *Shape* holds, h admits an extension Δh on $\mathscr{A}(u)$ based on an inclusion-exclusion formula. Thus, we

have $I_U = \frac{\sigma^2}{2\lambda} e^{-\lambda(m(A)+m(U))}\big(h(A\cap U) - \Delta h(B\cap U)\big)$. But since $A\cap U = B\cap U = A_l \cap U \in \mathscr{A}$ and h coincides with Δh on $\mathscr{A}$, we obtain $I_U = 0$.

Therefore, $I_U = 0$ for all $U \in \mathscr{A}$ such that $U \cap C = \emptyset$ and as X is a Gaussian process, we can claim that the random variable

$$X_A - \sum_{i=1}^{k} X_{A_i} e^{-\lambda m(A\setminus A_i)} + \cdots + (-1)^k X_{A_1\cap\cdots\cap A_k} e^{-\lambda m(A\setminus A_1\cap\cdots\cap A_k)}$$

and $\mathscr{G}_C^*$ are independent. Since the previous random variable is expressed as an inclusion-exclusion formula, equation (4.5) shows that it can also be expressed as

$$X_A - Z_C \quad \text{with} \quad Z_C = e^{-\lambda m(A)}\left[\sum_{i=1}^{n}(-1)^{\varepsilon_i} X_{U_C^i}\, e^{\lambda m(U_C^i)}\right].$$

Notice that Z_C is $\mathbf{X_C}$-measurable (and then $\mathscr{G}_C^*$-measurable) and $X_A - Z_C$ is independent of $\mathscr{G}_C^*$ (and then independent of $\mathbf{X_C}$). Hence, using a classic property of the conditional expectation, we have

$$\mathbb{E}[\,f(X_A)\,|\,\mathscr{G}_C^*\,] = \mathbb{E}[\,f(X_A - Z_C + Z_C)\,|\,\mathscr{G}_C^*] = \mathbb{E}[\,f(X_A)\,|\,Z_C\,].$$

We similarly obtain the equality $\mathbb{E}[\,f(X_A)\,|\,\mathbf{X_C}\,] = \mathbb{E}[\,f(X_A)\,|\,Z_C\,]$ which ends the proof. □

Intuitively, the $\mathscr{C}$-Markov property can be understood as following: For any increment $C = A\setminus B$, the σ-field $\mathscr{G}_C^*$ represents the past, described as *strong* as it contains all the information inside the regions B satisfying $C = A\setminus B$. The vector $\mathbf{X_C}$ itself gathers the minimum information related to the "border" points of C, and finally, X_A represents the future value of the process. Then, Equation (4.43) simply states that conditioning the future with respect the full history $\mathscr{G}_C^*$ or the vector $\mathbf{X_C}$ are equivalent.

According to Proposition 4.4, we can deduce that the set-indexed Ornstein–Uhlenbeck process also satisfies set-indexed *sharp-Markov* and *Markov* properties whose definitions can be found in [88]. In the multiparameter case, it implies that this process is *sharp-Markov* and *germ-Markov* with respect to finite unions of rectangles (see [88]). The question whether this implication remains true for more complex sets has not been investigated yet (see [48, 47] for answers in the particular case of Brownian and Lévy sheets).

As a consequence of the previous proposition, we can derive the $\mathscr{C}$-transition system $\mathscr{P}$ and the initial law that characterize entirely a ssiOU process.

Corollary 4.16 *The $\mathscr{C}$-transition system $\mathscr{P} = \{P_C(\mathbf{x_C};\Gamma);\, C\in\mathscr{C}, \Gamma\in\mathscr{B}(\mathbf{R})\}$ of the stationary set-indexed Ornstein Uhlenbeck process of Definition 4.9 is characterized by the following transition densities, for all $C = A\setminus B \in \mathscr{C}$:*

$$p_C(\mathbf{x_C}; y) = \frac{1}{\sigma_C\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_C^2}\left(y - e^{-\lambda m(A)}\left[\sum_{i=1}^{n}(-1)^{\varepsilon_i}\, x_{U_C^i}\, e^{\lambda m(U_C^i)}\right]\right)^2\right], \tag{4.44}$$

where

$$\sigma_C^2 = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda m(A)}\left[\sum_{i=1}^{n}(-1)^{\varepsilon_i} e^{2\lambda m(U_C^i)}\right]\right).$$

Furthermore, the initial law is given by $X_\emptyset \sim \mathcal{N}(0, \frac{\sigma^2}{2\lambda})$.

Proof. Let $C = A\setminus B$ be in $\mathscr{C}$ and let Z_C and Y_C be the following Gaussian variables

$$Z_C = e^{-\lambda m(A)}\left[\sum_{i=1}^{n}(-1)^{\varepsilon_i} X_{U_C^i}\, e^{\lambda m(U_C^i)}\right] \quad \text{and} \quad Y_C = X_A - Z_C.$$

Since the process X is centered, $\mathbb{E}[Z_C] = \mathbb{E}[Y_C] = 0$. We note σ_C^2 the variance of Y_C. Using the independence of Y_C and $\mathscr{G}_C^*$, shown in the proof of Proposition 4.15, and the fact that Z_C is $\mathscr{G}_C^*$-measurable, we have for any measurable function $f : \mathbf{R} \to \mathbf{R}_+$,

$$\begin{aligned}\mathbb{E}[f(X_A) \,|\, \mathscr{G}_C^*] =& \mathbb{E}[f(Z_C + Y_C) \,|\, \mathscr{G}_C^*] \\ =& \frac{1}{\sigma_C\sqrt{2\pi}} \int_{\mathbf{R}} f(u + Z_C) \exp\left(-\frac{u^2}{2\sigma_C^2}\right) \mathrm{d}u \\ =& \frac{1}{\sigma_C\sqrt{2\pi}} \int_{\mathbf{R}} f(v) \exp\left(-\frac{(v - Z_C)^2}{2\sigma_C^2}\right) \mathrm{d}v := \int_{\mathbf{R}} f(v)\, p_C(\mathbf{X_C}; v)\, \mathrm{d}v.\end{aligned}$$

Equation (4.44) follows from this last equality. It remains to prove the expression of the variance σ_C^2. We first note that, as $X_{U_C^i}$ is $\mathscr{G}_C^*$-measurable and Y_C is independent of $\mathscr{G}_C^*$, $\mathbb{E}[X_{U_C^i} Y_C] = 0$ for any $i \in \{1, \dots, n\}$. Therefore, we have

$$\begin{aligned}\sigma_C^2 = \mathbb{E}[X_A Y_C] = \mathbb{E}[X_A^2] - e^{-\lambda m(A)} \left[\sum_{i=1}^n (-1)^{\varepsilon_i}\, \mathbb{E}\left[X_A X_{U_C^i}\right] e^{\lambda m(U_C^i)}\right] \\ = \frac{\sigma^2}{2\lambda}\left(1 - e^{-\lambda m(A)} \left[\sum_{i=1}^n (-1)^{\varepsilon_i}\, e^{-\lambda m(A \Delta U_C^i)}\, e^{\lambda m(U_C^i)}\right]\right) \\ = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda m(A)} \left[\sum_{i=1}^n (-1)^{\varepsilon_i} e^{2\lambda m(U_C^i)}\right]\right).\end{aligned}$$

□

The following result shows that properties exhibited in Propositions 4.13, 4.14 and 4.15 lead to a complete characterization of the stationary set-indexed Ornstein–Uhlenbeck process.

Theorem 4.17 *A set-indexed mean-zero Gaussian process $X = \{X_U;\, U \in \mathscr{A}\}$ is a stationary set-indexed Ornstein–Uhlenbeck process if and only if the three following properties hold:*

(i) L^2-monotone inner- and outer-continuity;

(ii) m-stationarity;

(iii) $\mathscr{C}$-Markov property.

Proof. We already know that the stationary set-indexed Ornstein–Uhlenbeck process of Definition 4.9 satisfies these three properties. Conversely, let X be a zero-mean Gaussian set-indexed process which is L^2-monotone inner- and outer-continuous, m-stationary and $\mathscr{C}$-Markov. Without any loss of generality, we suppose $\mathbb{E}[X_{\emptyset'}^2] = 1$.

We first consider an increasing and continuous function $f : \mathbf{R}_+ \to \mathscr{A}$, i.e. an *elementary flow* in the terminology of [89, 80], such that $f(0) = \emptyset'$. Since m is monotonically continuous on $\mathscr{A}$ (Condition *(iii)* of the indexing collection), the function $\theta : t \mapsto m[f(t)]$ is continuous, $\theta(0) = 0$ and the pseudo-inverse $\theta^{-1}(t) = \inf\{u : \theta(u) > t\}$ satisfies $\theta \circ \theta^{-1}(t) = t$. Then, the projected one-parameter process $X^{m,f} = \{X_{f \circ \theta^{-1}(t)};\ t \in \mathbf{R}_+\}$ is a centered one-parameter Gaussian process which is L^2-continuous, stationary (see [80]) and Markov. Therefore, $X^{m,f}$ is a one-dimensional Ornstein–Uhlenbeck process (see e.g. [167]). Since $\mathbb{E}[(X_0^f)^2] = 1$, there exists $\lambda_f > 0$ such that for all $s, t \in \mathbf{R}_+$,

$$\mathbb{E}[X_s^{m,f} X_t^{m,f}] = e^{-\lambda_f |t-s|} = e^{-\lambda_f |m[f \circ \theta^{-1}(t)] - m[f \circ \theta^{-1}(s)]|} = e^{-\lambda_f m[f \circ \theta^{-1}(s) \Delta f \circ \theta^{-1}(t)]}.$$

Let us prove the constant λ_f does not depend on the function f. Let f_1 and f_2 be two different elementary flows which satisfy the previous conditions. Then, as $m(\emptyset') = 0$, for any

$t > 0$, we know that $m(f_1 \circ \theta_1^{-1}(t) \setminus \emptyset') = m(f_2 \circ \theta_2^{-1}(t)) = t$, and therefore, according to the m-stationarity of X, $(X_t^{m,f_1}, X_0^{m,f_1}) \stackrel{\mathrm{d}}{=} (X_t^{m,f_2}, X_0^{m,f_2})$, which implies $e^{-\lambda_{f_1} t} = \mathbb{E}[X_t^{m,f_1} X_0^{m,f_1}] = \mathbb{E}[X_t^{m,f_2} X_0^{m,f_2}] = e^{-\lambda_{f_2} t}$, i.e. $\lambda_{f_1} = \lambda_{f_2} := \lambda$.

For all $U, V \in \mathscr{A}$ such that $U \subseteq V$, there exists f which goes through U and V. We obtain

$$\mathbb{E}[X_U X_V] = e^{-\lambda |m(V) - m(U)|} = e^{-\lambda m(U \Delta V)}.$$

Finally let $U, V \in \mathscr{A}$. From the previous equation, we observe that

$$\mathbb{E}[(X_V - e^{-\lambda m(V \setminus U)} X_{U \cap V}) X_{U \cap V}] = e^{-\lambda m(V \Delta (U \cap V))} - e^{-\lambda m(V \setminus U)} = 0.$$

Therefore, since X is a Gaussian process, $\mathbb{E}[X_V \,|\, X_{U \cap V}] = e^{-\lambda m(V \setminus U)} X_{U \cap V}$, and using the $\mathscr{C}$-Markov property applied to $C = U \setminus V$ with the fact $\mathbf{X}_{U \setminus V} = X_{U \cap V}$, we obtain the expected covariance,

$$\begin{aligned}
\mathbb{E}[X_U X_V] = \mathbb{E}\big[X_U \, \mathbb{E}[X_V \,|\, \mathscr{G}^*_{U \setminus V}]\big] &= \mathbb{E}\big[X_U \, \mathbb{E}[X_V \,|\, X_{U \cap V}]\big] \\
&= e^{-\lambda m(V \setminus U)} \, \mathbb{E}\big[X_U X_{U \cap V}\big] \\
&= e^{-\lambda m(V \setminus U)} \cdot e^{-\lambda m(U \setminus V)} = e^{-\lambda m(U \Delta V)}.
\end{aligned}$$

□

4.4.2 Definition of a general *set-indexed Ornstein–Uhlenbeck process*

Using the $\mathscr{C}$-Markov property obtained in Proposition 4.15 and the $\mathscr{C}$-transition system $\mathscr{P}$ from Corollary 4.16, we can finally define a general *set-indexed Ornstein–Uhlenbeck process*.

Definition 4.10 *A process X is called a* set-indexed Ornstein–Uhlenbeck process *if*

(i) $X_{\emptyset'} \sim \nu$, where ν is a given initial probability distribution;

(ii) X is $\mathscr{C}$-Markov with a $\mathscr{C}$-transition system given by (4.44).

Theorem 4.3 proves the existence of such processes in the canonical probability space $(\mathbf{R}^{\mathscr{A}}, \mathbb{P}_\nu)$ for any initial probability distribution ν. Then, $\mathbb{P}_\nu$ is the probability measure on $\mathbf{R}^{\mathscr{A}}$ under which the canonical process defined by $X_U(\omega) = \omega(U)$ for all $\omega \in \mathbf{R}^{\mathscr{A}}$ is a set-indexed Ornstein–Uhlenbeck process. In the particular case of Dirac initial distribution, the complete determination of the laws of X is given by the following result.

Proposition 4.18 *For any $x \in \mathbf{R}$, under the probability $\mathbb{P}_x$, the canonical set-indexed Ornstein–Uhlenbeck process X is the Gaussian process defined by the covariance structure*

$$\forall U \in \mathscr{A}; \quad \mathbb{E}_x[X_U] = x \, e^{-\lambda m(U)}, \tag{4.45}$$

$$\forall U, V \in \mathscr{A}; \quad \mathrm{Cov}_x(X_U, X_V) = \frac{\sigma^2}{2\lambda}\big(e^{-\lambda m(U \Delta V)} - e^{-\lambda(m(U) + m(V))}\big). \tag{4.46}$$

Proof. We first check that X is a Gaussian process under the probability $\mathbb{P}_x$.
Let $A_1, \ldots, A_k \in \mathscr{A}$ and $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$. Without any loss of generality, we can suppose that $\mathscr{A}_\ell = \{A_0 = \emptyset', A_1, \ldots, A_k\}$ is a semilattice and we denote $C_i = A_i \setminus (\cup_{j=0}^{i-1} A_j)$ for all

$i \in \{1, \dots, k\}$. Then, using notations from Corollary 4.16, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left(i\sum_{j=1}^{k}\lambda_j X_{A_j}\right)\right] \\
&= \mathbb{E}_x\left[\exp\left(i\sum_{j=1}^{k-1}\lambda_j X_{A_j}\right)\mathbb{E}_x\left[\exp(i\lambda_k X_{A_k}) \,\middle|\, \mathscr{G}^*_{C_k}\right]\right] \\
&= \mathbb{E}_x\left[\exp\left(i\sum_{j=1}^{k-1}\lambda_j X_{A_j}\right)\exp(i\lambda_k Z_{C_k})\,\mathbb{E}_x\left[\exp(i\lambda_k Y_{C_k})\right]\right] \\
&= \exp\left(-\frac{\lambda_k^2\sigma^2_{C_k}}{2}\right)\mathbb{E}_x\left[\exp\left(i\sum_{j=1}^{k-1}\lambda'_j X_{A_j}\right)\right],
\end{aligned}$$

since Z_{C_k} is weighted sum of X_V, $V \in \{A_0, \dots, A_{k-1}\}$. Therefore, by induction on k, we get the characteristic function of a Gaussian variable.

In order to obtain the mean and the covariance functions, we consider the case $k = 3$, with the semi-lattice $\{\emptyset', A_1 = A_2 \cap A_3, A_2, A_3\}$. We compute

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\Big(i(\lambda_2 X_{A_2} + \lambda_3 X_{A_3})\Big)\right] \\
&= \exp\Big(-\frac{1}{2}\lambda_3^2\sigma^2_{C_3}\Big)\,\mathbb{E}_x\left[\exp\Big(i(\lambda_2 X_{A_2} + \lambda_3 Z_{C_3})\Big)\right] \\
&= \exp\Big(-\frac{1}{2}\lambda_3^2\sigma^2_{C_3}\Big)\,\mathbb{E}_x\left[\exp\Big(i(\lambda_2 X_{A_2} + \lambda_3 e^{-\lambda m(A_3\setminus A_1)} X_{A_1})\Big)\right].
\end{aligned}$$

Using the $\mathscr{C}$-Markov property applied to $C_2 = A_2 \setminus A_1$, we get

$$\begin{aligned}
\mathbb{E}_x\left[\exp\Big(i(\lambda_2 X_{A_2} + \lambda_3 X_{A_3})\Big)\right] &= \exp\Big(-\frac{1}{2}\lambda_3^2\sigma^2_{C_3} - \frac{1}{2}\lambda_3^2\sigma^2_{C_2}\Big) \\
&\quad \cdot \mathbb{E}_x\left[\exp\Big(i\big(\lambda_2 e^{-\lambda m(A_2\setminus A_1)} + \lambda_3 e^{-\lambda m(A_3\setminus A_1)}\big)X_{A_1}\Big)\right].
\end{aligned}$$

Then, the $\mathscr{C}$-Markov property applied to $C_1 = A_1 \setminus \emptyset'$ leads to

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\Big(i(\lambda_2 X_{A_2} + \lambda_3 X_{A_3})\Big)\right] \\
&\quad = \exp\Big(-\frac{1}{2}\lambda_3^2\sigma^2_{C_3} - \frac{1}{2}\lambda_3^2\sigma^2_{C_2} - \frac{1}{2}\big(\lambda_2 e^{-\lambda m(A_2\setminus A_1)} + \lambda_3 e^{-\lambda m(A_3\setminus A_1)}\big)^2\sigma^2_{C_1}\Big) \\
&\qquad \times \mathbb{E}_x\left[\exp\Big(i\big(\lambda_2 e^{-\lambda m(A_2\setminus A_1)} + \lambda_3 e^{-\lambda m(A_3\setminus A_1)}\big)e^{-\lambda m(A_1)}X_{\emptyset'}\Big)\right] \\
&\quad = \exp\Big(-\frac{1}{2}\lambda_3^2\sigma^2_{C_3} - \frac{1}{2}\lambda_3^2\sigma^2_{C_2} - \frac{1}{2}\big(\lambda_2 e^{-\lambda m(A_2\setminus A_1)} + \lambda_3 e^{-\lambda m(A_3\setminus A_1)}\big)^2\sigma^2_{C_1}\Big) \\
&\qquad \times \exp\Big(i\big(\lambda_2 e^{-\lambda m(A_2)} + \lambda_3 e^{-\lambda m(A_3)}\big)x\Big).
\end{aligned}$$

The mean of X comes from the last line. The covariance is obtained from the cross term in front of $\lambda_2\lambda_3$:

$$\begin{aligned}
\sigma^2_{C_1} e^{-\lambda m(A_2\setminus A_1)} e^{-\lambda m(A_3\setminus A_1)} &= \frac{\sigma^2}{2\lambda}\big(1 - e^{-2\lambda m(A_1)}\big)e^{-\lambda m(A_2\Delta A_3)} \\
&= \frac{\sigma^2}{2\lambda}\big(e^{-\lambda m(A_2\Delta A_3)} - e^{-\lambda(m(A_2)+m(A_3))}\big),
\end{aligned}$$

since $A_1 = A_2 \cap A_3$ and $\sigma^2_{C_1} = \frac{\sigma^2}{2\lambda}\big(1 - e^{-2\lambda m(A_1)}\big)$. □

4.4.3 Multiparameter Ornstein–Uhlenbeck process

In the particular case of the indexing collection $\mathscr{A} = \{[0,t]; t \in \mathbf{R}_+\}$ endowed with the Lebesgue measure m, the set-indexed Ornstein–Uhlenbeck processes studied in Sections 4.4.1 and 4.4.2 reduce to the classical one-dimensional Ornstein–Uhlenbeck process.

In the multiparameter setting, a natural extension of the stationary Ornstein–Uhlenbeck process can be defined by

$$\forall t \in \mathbf{R}_+^N; \quad Y_t = \int_{-\infty}^{t} \sigma\, e^{\langle \alpha, u-t\rangle} \mathrm{d}W_u. \tag{4.47}$$

where $\sigma > 0$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{R}^N$ with $\alpha_i > 0$ and W is the Brownian sheet. The covariance of this process is given by

$$\mathbb{E}[Y_s Y_t] = \prod_{i=1}^{N} \int_{-\infty}^{s_i \wedge t_i} \sigma^2 e^{\alpha_i (u_i - s_i - t_i)}\, \mathrm{d}u_i = \frac{\sigma^2}{\prod_{i=1}^N \alpha_i} \exp\Big\{ -\sum_{i=1}^{N} \alpha_i (s_i + t_i - s_i \wedge t_i) \Big\}.$$

Hence, Y is a stationary set-indexed Ornstein–Uhlenbeck process on the space $\mathbf{R}_+^N$ endowed with the indexing collection $\mathscr{A} = \{[0,t]; t \in \mathbf{R}_+^N\}$ and the measure m_α defined on the Borel σ-field by

$$\forall A \in \mathscr{B}(\mathbf{R}^N); \quad m_\alpha(A) = \sum_{i=1}^{N} \alpha_i \lambda_1(A \cap e_i), \tag{4.48}$$

where λ_1 is the Lebesgue measure on $\mathbf{R}$ and $e_1, \dots, e_N$ are the axes of $\mathbf{R}^N$: $e_1 = \mathbf{R} \times \{0\}^{N-1}$, $e_2 = \{0\} \times \mathbf{R} \times \{0\}^{N-2}, \dots$

The following proposition extends this result to the general set-indexed Ornstein–Uhlenbeck process defined in Section 4.4.2, proving that it also has a natural integral representation in the particular multiparameter case.

Proposition 4.19 *Let $Y = \{Y_t;\ t \in \mathbf{R}_+^N\}$ be the multiparameter process defined by*

$$\forall t \in \mathbf{R}_+^N; \quad Y_t = e^{-\langle \alpha, t\rangle} \Big[Y_0 + \sigma \int_{(-\infty,t]\setminus(-\infty,0]} e^{\langle \alpha, u\rangle} \mathrm{d}W_u \Big], \tag{4.49}$$

where $\sigma > 0$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{R}^N$ with $\alpha_i > 0$ for each $i \in \{1, \dots, N\}$, W is the Brownian sheet and Y_0 is a random variable independent of W.
Then, Y is a set-indexed Ornstein–Uhlenbeck process of Definition 4.10 on the space $(\mathscr{T}, \mathscr{A}, m_\alpha)$, with $\mathscr{A} = \{[0,t];\ t \in \mathbf{R}_+^N\}$ and m_α defined in (4.48).

Proof. First we observe that the measure m_α satisfies, for all $s, t \in \mathbf{R}_+^N$,

$$m_\alpha([0,s] \cap [0,t]) = \sum_{i=1}^{N} \alpha_i (s_i \wedge t_i) = \langle \alpha, s \curlywedge t \rangle,$$

where $s \curlywedge t := (s_1 \wedge t_1, \dots, s_N \wedge t_N)$. Let $t_1, \dots, t_k$ be in $\mathbf{R}_+^N$ and $\lambda_1, \dots, \lambda_k$ in $\mathbf{R}$. For any fixed $x_0 \in \mathbf{R}$, Y^{x_0} denotes the Gaussian process defined by

$$\forall t \in \mathbf{R}_+^N; \quad Y_t^{x_0} = e^{-\langle \alpha, t\rangle} \Big[x_0 + \sigma \int_{A_t} e^{\langle \alpha, u\rangle} \mathrm{d}W_u \Big],$$

where $A_t = (-\infty, t] \setminus (-\infty, 0]$.

Let Y be the $\mathbf{R}_+^N$-indexed process defined by (4.49) and denote by ν the law of Y_0. Since Y_0 and W are independent, we have

$$\mathbb{E}\left[e^{i\sum_{j=1}^k \lambda_j Y_{t_j}}\right] = \int_{\mathbf{R}} \mathbb{E}\left[e^{i\sum_{j=1}^k \lambda_j Y_{t_j}^{x_0}}\right] \nu(\mathrm{d}x_0),$$

Let us determine the mean and covariance of the process Y^{x_0}, for any $x_0 \in \mathbf{R}$:

$$\forall t \in \mathbf{R}_+^N; \quad \mathbb{E}\left[Y_t^{x_0}\right] = x_0\, e^{-\langle \alpha, t\rangle} = x_0\, e^{-m_\alpha([0,t])}$$

and for all $s, t \in \mathbf{R}_+^N$,

$$\begin{aligned}
\mathrm{Cov}(Y_s^{x_0}, Y_t^{x_0}) &= \sigma^2 e^{-\langle \alpha, s+t\rangle} \mathbb{E}\left[\int_{A_s} e^{\langle \alpha, u\rangle} \mathrm{d}W_u \int_{A_t} e^{\langle \alpha, u\rangle} \mathrm{d}W_u\right] \\
&= \sigma^2 e^{-\langle \alpha, s+t\rangle} \int_{A_{s \curlywedge t}} e^{2\langle \alpha, u\rangle} \mathrm{d}u \\
&= \frac{\sigma^2}{2^n \prod_{j=1}^n \alpha_j} e^{-\langle \alpha, s+t\rangle} \left(e^{2\langle \alpha, s \curlywedge t\rangle} - 1\right) \\
&= \frac{\sigma^2}{2^n \prod_{j=1}^n \alpha_j} e^{-m_\alpha([0,s]) - m_\alpha([0,t])} \left(e^{2m_\alpha([0,s]\cap[0,t])} - 1\right) \\
&= \frac{\widetilde{\sigma}^2}{2} \left(e^{-m_\alpha([0,s]\Delta[0,t])} - e^{-(m_\alpha([0,s]) + m_\alpha([0,t]))}\right) \\
&= \mathrm{Cov}_{x_0}(X_{[0,s]}, X_{[0,t]}),
\end{aligned}$$

where X is the canonical set-indexed Ornstein–Uhlenbeck process with parameters $(\widetilde{\sigma}, \lambda = 1)$ with notations of Proposition 4.18. Therefore, the process Y^{x_0} has the same law as $X_{[0,\bullet]}$ starting from x_0 and

$$\mathbb{E}\left[e^{i\sum_{j=1}^k \lambda_j Y_{t_j}^{x_0}}\right] = \mathbb{E}_{x_0}\left[e^{i\sum_{j=1}^k \lambda_j X_{[0,t_j]}}\right].$$

Consequently

$$\mathbb{E}\left[e^{i\sum_{j=1}^k \lambda_i Y_{t_j}}\right] = \int_{\mathbf{R}} \mathbb{E}_{x_0}\left[e^{i\sum_{j=1}^k \lambda_j X_{[0,t_j]}}\right] \nu(\mathrm{d}x_0) = \mathbb{E}_\nu\left[e^{i\sum_{j=1}^k \lambda_j X_{[0,t_j]}}\right],$$

which states that Y and $X_{[0,\bullet]}$ have the same law and concludes the proof. □

Remark 4.3 We have exhibited an unusual measure m_α on $\mathbf{R}^N$, which only charges the axes $(e_i)_{i\leq N}$. This measure is also interesting when the set-indexed Brownian motion (siBM) is considered on the space $(\mathscr{T}, \mathscr{A}, m_\alpha)$ with $\alpha = (1, \ldots, 1)$, as it corresponds to a classic multiparameter process called the additive Brownian motion (see e.g. [100]). Conversely, since we know that the Brownian sheet is a siBM on the space $(\mathscr{T}, \mathscr{A}, \lambda)$, where λ is the Lebesgue measure, we could also define a different multiparameter Ornstein–Uhlenbeck process using the Lebesgue measure instead of m_α.

Remark 4.4 A different multiparameter extension of the Ornstein–Uhlenbeck process has already been introduced in the literature (e.g. see [187, 186] and [73]). It admits an integral representation given by,

$$\forall t \in \mathbf{R}_+^N; \quad Y_t = e^{-\langle \alpha, t\rangle}\left[Y_0 + \sigma \int_0^t e^{\langle \alpha, u\rangle} \mathrm{d}W_u\right]. \tag{4.50}$$

If we consider a Markov point of view, the definition given in Proposition 4.19 seems more natural. Indeed, as described previously, the transition probabilities of the process described in Equation (4.50) do not strictly correspond to those of the set-indexed Ornstein–Uhlenbeck, and can not be extended to the set-indexed formalism. Furthermore, we observe that the model (4.50) does not embrace the natural stationary case described in equation (4.47).

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