



Etude de l'existence et de la stabilité de dynamiques explosives pour des problèmes paraboliques critiques.

Rémi Schweyer

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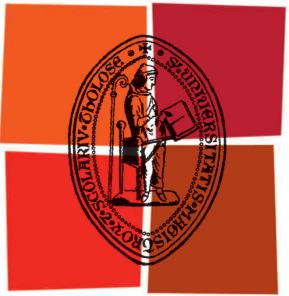
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Université
de Toulouse

THÈSE

En vue de l'obtention du
DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par :

Université Toulouse III Paul Sabatier (UT3 Paul Sabatier)

Discipline ou spécialité :

Mathématiques appliquées

Présentée et soutenue par

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le : 17 mai 2013

Titre :

Etude de l'existence et de la stabilité de dynamiques explosives pour des problèmes paraboliques critiques.

École doctorale :

Mathématiques Informatique Télécommunications (MITT)

Unité de recherche :

UMR 5219

Directeur de thèse :

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Hatem Zaag

Pierre Degond
Jean Dolbeault
Eric Lombardi
Yvan Martel

A mes grands-parents,

Remerciements

En premier lieu, je souhaiterais remercier chaleureusement Pierre Raphaël pour ces trois années de thèse. Tout d'abord pour la qualité, l'intérêt, et l'exposition des trois problèmes avec lesquels j'ai passé mes journées et accessoirement mes nuits. Les perspectives de recherche pour chacun d'entre eux sont nombreuses et particulièrement motivantes. Pouvoir disposer en fin de thèse d'un spectre aussi large de recherche est une chance inestimable. Je voudrais le remercier aussi pour la pédagogie qu'il a utilisée à mon égard, à la fois au service de ma découverte de cette partie du monde de l'analyse non linéaire, mais aussi au service de l'apprentissage du métier de chercheur, en me présentant en toute franchise les difficultés mais aussi les joies et les plaisirs de cette belle activité. Enfin, malgré le nombre restreint de mois passés à travailler au même endroit, sa très grande réactivité aux emails a permis de garder une grande qualité d'échanges tout du long de ces trois dernières années.

Je voudrais dire un grand merci à Hatem Zaag et à Nader Masmoudi. Bien évidemment pour avoir pris le temps de lire et de juger ma thèse et pour la qualité de leurs rapports. Mais aussi pour les échanges que l'on a pu avoir durant celle-ci.

J'ai un souvenir ému de ma visite à Villetaneuse en février 2012. D'une part parce qu'il s'agissait de ma première présentation en séminaire. D'autre part pour la longue discussion que j'ai pu avoir avec Hatem Zaag portant sur l'environnement du résultat obtenu pour l'équation de la chaleur. Cette mise en perspective, révélant tout l'intérêt du travail effectué, a été par la suite une importance source de motivation.

Le séjour à New-York en décembre 2012 et la rencontre de Nader Masmoudi restera comme un des moments forts de ces trois dernières années. Cette semaine passée à réfléchir du matin au soir au cas complexe de la dynamique explosives en temps infini, pour le problème de Patlak-Keller-Segel parabolique elliptique en masse critique, a été fantastique, et m'a confirmé définitivement mon attrait pour la recherche.

Tout naturellement, mes remerciements s'adressent aussi aux autres membres de mon jury : Pierre Degond, Jean Dolbeault, Yvan Martel et Eric Lombardi. J'ai été très touché par l'intérêt qu'ils ont pu porter à mon travail.

Je désirerais remercier vivement Denis Matignon. Son soutien et son aide ont été déterminants dans la réussite de ma réorientation du monde de l'ingénieur vers celui de la recherche académique. Son cours d'optimisation lors de la seconde année de Supaéro a été pour moi comme une révélation, une redécouverte des mathématiques. Il occupe ainsi un rôle majeur dans la réussite de cette thèse, rôle que je n'ai pas toujours su mesurer à sa juste valeur.

Ces remerciements s'adressent encore au laboratoire, à l'IMT, et à l'ensemble des membres qui en font partie. A la fois les chercheurs, mais aussi l'ensemble du personnel administratif qui par leur soutien m'a permis d'optimiser le temps consacré à la recherche.

Comment ne pas remercier l'ensemble des doctorants du labo, notamment mes "cobureaux", Laetitia, Mathieu, Fabien, Minh, mais aussi Elissar, et mes compagnons de TDs, Pierre-Henri et Cyrille. Sans oublier Tej-eddine, rencontré à New-York. O combien l'entraide a été précieuse ! Le passage, opéré lors de la thèse, de simple étudiant à chercheur est parsemé de nombreux obstacles, nombreux doutes, nombreuses questions. Pouvoir échanger sur nos cheminement personnels est quelque chose d'absolument crucial pour ne pas s'égarer en chemin, pour ne pas rester seul à tourner en boucle sur des questions existentielles.

Les longues discussions avec Adrien Blanchet et Phillippe Laurençot lors de ma dernière année de thèse m'ont beaucoup apportées. Incontestablement sur le plan scientifique, et tout particulièrement sur l'équation de Patlak-Keller-Segel. Aussi incontestablement sur ces questions existentielles que sont les questions de perspectives de carrière, et toutes ces questions quelque peu angoissantes déviant trop souvent notre esprit de la recherche.

Le séjour à Vancouver, entre la rédaction de ce manuscrit de thèse, et sa soutenance, m'a permis de rencontrer Tai-Peng Tsai, Stephen Gustafson, Nabile Boussaid et Slim Ibrahim notamment, que je voudrais chaleureusement remercier. Leur accueil a été formidable, et les nombreuses pistes de recherches entrevues là-bas sont particulièrement excitantes, et déboucheront je l'espère sur de futurs travaux.

Enfin, il me paraît important de mentionner ici toutes les personnes évoluant hors de la sphère professionnelle, mais sans qui cette thèse n'aurait sûrement jamais pu se faire. Toute ma famille bien entendu, qui m'a toujours apporté un soutien sans faille, et qui m'a laissé la liberté de faire les études que je voulais, aussi longues soient-elles. Mais aussi l'ensemble de mes amis toulousains, avec qui j'ai partagé des moments inoubliables. Que c'est important de se sentir entouré et soutenu, tout particulièrement lors des passages à vide, inhérent à la vie d'un thésard. Pour conclure, je voudrais remercier tous les coureurs du groupe Run à la fac, ainsi que Maria, pour l'ensemble des entraînements de ces trois années. Cette activité physique intense a été un complément idéal à la thèse.

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Introduction générale

Dans ce manuscrit, nous présenterons trois résultats d'existence de dynamique explosive pour des problèmes paraboliques critiques : le flot de la chaleur harmonique, l'équation de la chaleur semi-linéaire, et le problème de Patlak-Keller-Segel parabolique-elliptique.

La stratégie mise en place dans la preuve de ces trois résultats est basée sur la théorie de la modulation, mise en place pour la première fois par Martel et Merle dans [61] pour l'équation de Korteweg-De-Vries généralisée critique. Utilisée tout d'abord dans le contexte des problèmes hamiltoniens, et notamment l'équation de Schrödinger non-linéaire [69], [67], [68], [71] et [70], son adaptation pour les problèmes géométriques (flot de la chaleur harmonique [91], "wave maps" [88] et "Schrödinger maps" [72]) a permis de mieux comprendre la dynamique explosive de ces problèmes, pour des solutions possédant une certaine symétrie. La première partie de cette thèse en fait la présentation.

Les deux parties suivantes seront consacrées à l'équation de la chaleur non-linéaire. Dans un premier temps, nous étudierons l'équation dans un contexte général, et nous verrons notamment l'influence sur la dynamique des solutions explosives du nombre choisi en exposant dans le terme non-linéaire. Le théorème principal de [94] sera ensuite énoncé, puis commenté, avant de proposer des axes de recherches en découlant. Dans un second temps, nous donnerons une preuve simplifiée du résultat, avec en particulier pour commencer une partie formelle en toute dimension dans le cadre de l'énergie critique. Nous profitons de la relative simplicité de ce problème pour donner les éléments importants de la stratégie de notre démonstration.

Dans la quatrième partie sera exposée le modèle de Patlak-Keller-Segel décrivant une accumulation d'amibes, soumis à une chimiotaxie. Le fait que l'opérateur linéarisé de notre problème n'est plus un opérateur auto-adjoint, et surtout le fait que la non-linéarité est ici non locale, a entraîné des modifications importantes dans la stratégie de la preuve.

Dans les trois dernières parties seront insérés les trois articles [91], [94] et [89], dans leur version soumise à publication.

Chapitre 1

Flot de la chaleur harmonique

Dans ce chapitre, nous allons présenter le théorème de [91], où nous décrivons de façon précise une dynamique explosive stable pour le flot de la chaleur harmonique. Nous allons commencer par rappeler ce qu'est une application harmonique, et le problème parabolique d'évolution lui étant associé. Ensuite, nous étudierons ce problème dans le cadre restreint des solutions à symétrie corotationnelle. Pour conclure, nous énoncerons le théorème de [91], avant de le commenter et de proposer quelques pistes de prolongement de cette étude.

1.1 Introduction

1.1.1 Application harmonique

Dans les prochains paragraphes, nous utiliserons la convention de sommation d'Einstein. De plus, pour une matrice $a = (a_{ij})_{1 \leq i,j \leq n}$, nous noterons $a = (a^{ij})$ la matrice inverse.

Soit (\mathcal{M}, γ) une surface de Riemann, avec le tenseur métrique $\gamma = (\gamma_{\alpha\beta})_{1 \leq \alpha, \beta \leq 2}$. Soit (\mathcal{N}, g) une variété de dimension n , avec pour tenseur métrique $g = (g_{ij})_{1 \leq i,j \leq n}$, inclus isométriquement dans \mathbb{R}^k . Pour toute application différentiable $u : \mathcal{M} \rightarrow \mathcal{N}$, on lui associe une densité d'énergie que l'on notera $e(u)$, définie par

$$e(u) = \frac{1}{2} \gamma_{\alpha\beta}(x) g_{i,j}(u) \frac{\partial}{\partial x^\alpha} u^i \frac{\partial}{\partial x^\beta} u^j \quad (1.1.1)$$

en un point $u(x)$, où nous avons utilisé les coordonnées locales $x = (x_1, x_2) \in \mathcal{M}$ et $u = (u_1, \dots, u_n) \in \mathcal{N}$. Ainsi, nous définissons l'énergie de l'application u , que l'on notera $E(u)$ par

$$E(u) = \int_{\mathcal{M}} e(u) d\mathcal{M}. \quad (1.1.2)$$

Définition 1.1.1 (Application harmonique). *Une application harmonique u entre \mathcal{M} et \mathcal{N} est un point d'extremum de l'énergie E .*

Une telle application u vérifie l'équation d'Euler-Lagrange suivante :

$$\Delta_{\mathcal{M}} u - \gamma^{\alpha\beta} A_u \left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right) = 0 \quad (1.1.3)$$

avec $\Delta_{\mathcal{M}}$ l'opérateur de Laplace-Beltrami sur la surface \mathcal{M} , et A la seconde forme fondamentale de la variété $\mathcal{N} \rightarrow \mathbb{R}^k$. De façon équivalente, (1.1.3) peut s'écrire

$$\mathbb{P}_{T_u N}(\Delta_{\mathcal{M}} u) = 0. \quad (1.1.4)$$

où $\mathbb{P}_{T_u N}$ est la projection sur l'espace tangent à \mathcal{N} de u . Dans la suite, nous nous restreindrons au cas où $\mathcal{M} = \mathbb{R}^2$. De plus, on peut voir dans [56] par exemple qu'il existe des variétés \mathcal{N} , sur lesquelles il n'existe pas d'application harmonique. Nous introduirons un peu plus loin le problème (1.1.6) d'évolution associé à (1.1.4). Struwe montre dans [96] que si une solution de ce problème explose en temps fini, alors elle converge en un certain sens vers une application harmonique. La non-existence d'application harmonique pour la variété \mathcal{N} entraîne ainsi l'existence globale de la solution. C'est pourquoi, nous restreindrons aussi notre étude à des variétés \mathcal{N} particulières décrites ci-dessous :

Définition 1.1.2 (Variété \mathcal{N} admissible). *Soit l'arc paramétré \mathcal{A} du plan, régulier défini par*

$$u \in [-\pi, \pi] \mapsto \begin{vmatrix} g(u) \\ z(u) \end{vmatrix}, \quad (g')^2 + (z')^2 = 1,$$

où

$$(H) \quad \begin{cases} g \in \mathcal{C}^\infty(\mathbb{R}) \text{ est impair et } 2\pi \text{ périodique,} \\ g(0) = g(\pi) = 0, \quad g(u) > 0 \text{ for } 0 < u < \pi, \\ g'(0) = 1, \quad g'(\pi) = -1, \end{cases} \quad (1.1.5)$$

Alors, on dit que \mathcal{N} est une variété admissible si \mathcal{N} peut être paramétrée par :

$$(\theta, u) \in [0, 2\pi] \times [0, \pi] \mapsto \begin{vmatrix} g(u) \cos \theta \\ g(u) \sin \theta \\ z(u) \end{vmatrix}$$

On peut voir que les variétés admissibles sont des surfaces de révolution de \mathbb{R}^3 , compactes et de classe \mathcal{C}^∞ , avec la métrique $(du)^2 + g^2(u)(d\theta)^2$. On trouvera une preuve de cela dans [29] par exemple.

1.1.2 Problème d'évolution

A l'équation (1.1.4) décrivant les applications harmoniques, on lui associe le problème d'évolution parabolique suivant :

$$(\text{HHF}) \quad \begin{cases} \partial_t v = \mathbb{P}_{T_v \mathcal{N}}(\Delta v) \\ v|_{t=0} = v_0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad v(t, x) \in \mathcal{N} \quad (1.1.6)$$

avec \mathcal{N} une variété admissible selon la définition (1.1.2). Ce problème a été pour la première fois posé par Eells et Sampson dans [25].

En plus de cette équation d'évolution, on peut introduire deux autres types d'équation d'évolution : Les "Schrödinger maps" et les "wave maps", que nous allons écrire dans le cas où $\mathcal{N} = \mathbb{S}^2$:

$$(\text{SM}) \quad \begin{cases} u \wedge \partial_t u = \Delta u + |\nabla u|^2 u \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(t, x) \in \mathbb{S}^2, \quad (1.1.7)$$

$$(WM) \quad \begin{cases} \partial_{tt}u - \Delta u = (|\partial_t u|^2 - |\nabla u|^2)u & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(t, x) \in \mathbb{S}^2, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases} \quad (1.1.8)$$

Nous ne rentrerons pas dans les détails concernant les résultats d'existence, d'unicité ou de globalité pour ces deux problèmes. Nous les mentionnons car ils forment avec le problème du flot de la chaleur harmonique, un ensemble cohérent. De plus, des résultats analogues à [91] ont été établis, par Raphaël et Rodnianski pour les "wave maps" dans [88], et par Merle, Raphaël et Rodnianski dans [72] pour les "Schrödinger maps".

On peut trouver dans [96] une preuve de l'existence et de l'unicité d'une solution du problème du flot de la chaleur harmonique. En fait, Struwe démontre que si la solution de (1.1.6) explose en temps fini, alors cela ne peut se produire qu'en un nombre fini de points, et dans un sens précisé par le théorème 1.1.5, elle converge vers une application harmonique régulière.

Par souci de complétude, nous allons donner les quatre théorèmes principaux de [96]. Le premier décrit l'existence et l'unicité des solutions, le second l'existence globale au sens des distributions, le troisième la convergence, aux points d'explosion, vers une application harmonique régulière, et enfin, le dernier nous présente une décomposition en bulles de concentration des solutions. Pour cela, on introduit les notations suivantes :

$$\begin{aligned} B_R(x) : & \quad \{y \in \mathbb{R}^2, |x - y| < R\} \\ E_R(u; x) : & \quad \int_{B_R(x)} e(u) \, dx \\ \epsilon_R(t) : & \quad \sup_{x \in \mathbb{R}^2} E_R(u(t); x) \\ \Omega^T : & \quad \Omega \times [0, T] \\ V(\Omega^T, \mathcal{N}) : & \quad \{u : \Omega^T \rightarrow \mathcal{N}, u \text{ mesurable tel que} \end{aligned}$$

$$\operatorname{esssup}_{\tau \leq t \leq T} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\tau}^T \int_{\mathbb{R}^2} |\nabla^2 u|^2 + |\partial_t u|^2 \, dxdt < \infty$$

Enfin, il existe un nombre $\epsilon_1(\mathcal{N})$, tel que les théorèmes suivants soient vrais. En général, ce nombre n'est pas explicite.

Théorème 1.1.3 (Existence et unicité). $\forall u_0 \in H^{1,2}(\mathbb{R}^2, \mathcal{N})$, $\exists u \in \bigcap_{T' < T} V(\mathbb{R}^{2T'}, \mathcal{N})$ solution de (1.1.6), avec $u(., 0) = u_0$. T est caractérisé par la condition suivante :

$$\limsup_{T' \rightarrow T} \epsilon_R(T') > \epsilon_1(\mathcal{N}) \quad \forall R > 0$$

u est unique et régulière sur $\mathbb{R}^2 \times [0, \infty[$ à l'exception d'un nombre L fini de points $(x^l, T)_{1 \leq l \leq L}$ tel que :

$$\limsup_{T' \rightarrow T} E_R(u(., T'); x^l) > \epsilon_1(\mathcal{N}) \quad \forall R > 0$$

De plus, si u_0 est régulière (sur $\Omega \subset \mathbb{R}^2$), u est régulière en $t = 0$ (sur $\Omega \subset \mathbb{R}^2$). Enfin, l'énergie est finie et décroissante sur $[0, T[$. Enfin, si la caractérisation de T n'est pas réalisée, pour aucun $T > 0$, alors la solution existe sur $[0, \infty[$.

Théorème 1.1.4 (Existence globale au sens des distributions). *$\forall u_0 \in H^{1,2}(\mathbb{R}^2, \mathcal{N})$, il existe une unique solution u de (1.1.6) sur $\mathbb{R}^2 \times [0, \infty[$, excepté un nombre fini de points $(x^l, T^l)_{1 \leq l \leq L}$ caractérisés par la condition suivante :*

$$\limsup_{\substack{T \rightarrow T^l \\ T < T^l}} E_R(u(, T); x^l) > \epsilon_1(\mathcal{N}), \quad \forall R > 0$$

Théorème 1.1.5 (Convergence, aux points d'explosion, vers une application harmonique régulière). *Soit u la solution de (1.1.6) construite au théorème précédent et (x_0, T) un point d'explosion. Alors, $\exists x_m \rightarrow x_0$, $\exists t_m \rightarrow T$, $t_m < T$, $\exists R_m \rightarrow 0$, et $\exists \bar{u}_0 : \mathbb{R}^2 \rightarrow \mathcal{N}$ une application harmonique régulière, tel que*

$$\lim_{m \rightarrow \infty} u(R_m \cdot x_m + x_0, t_m) \rightarrow \bar{u}_0 \text{ dans } H_{loc}^{2,2}(\mathbb{R}^2, \mathcal{N})$$

De plus, \bar{u}_0 a une énergie finie.

Remarque 1.1.6. Si la condition d'exception du premier théorème ne se vérifie jamais, alors u est régulière sur \mathbb{R}^{2^∞} . De plus, comme $\partial_t u \in L^2(\mathbb{R}^{2^\infty})$, en utilisant le même principe que dans la démonstration du dernier théorème, on montre que $u(., t_m)$ converge fortement vers \bar{u}_∞ dans $H_{loc}^{2,2}(\mathbb{R}^2, \mathcal{N})$ et dans $H^{1,2}(\Omega, \mathcal{N})$, quelque soit $\Omega \subset\subset \mathbb{R}^2$. Par passage à la limite, \bar{u}_∞ est une application harmonique, avec $E(\bar{u}_\infty) \leq E(u_0)$.

Théorème 1.1.7 (Décomposition en bulles de concentration). *Soit u la solution de (1.1.6) construite au théorème (1.1.4) et \bar{u}_0^l , $1 \leq l \leq L$ les L applications harmoniques, construites au théorème (1.1.5), pour les L points de singularité. Soit \bar{u}_∞ , l'application harmonique limite quand $t \rightarrow +\infty$, de la remarque (1.1.6). Alors,*

$$E(\bar{u}_\infty) + \sum_{1 \leq l \leq L} E(\bar{u}_0^l) \leq E(u_0) \tag{1.1.9}$$

Il est important de noter que ces théorèmes n'impliquent pas l'existence de solutions explosives en temps fini. Les premiers résultats de telles solutions sont dûs à Coron et Ghidaglia [17], puis Chang, Ding, Ye [14], et Qing, Tian [86]. Pour un historique plus détaillé, le lecteur intéressé pourra consulter l'article de Topping [98]. Notons cependant les travaux de Freire [27] et [28] sur l'unicité des solutions de (1.1.6).

1.2 Restriction aux solutions corotationnelles

Le problème (1.1.6), même dans le cas où la variété \mathcal{N} est admissible selon la définition (1.1.2), est encore assez mal compris dans le cas général. En imposant aux solutions une certaine symétrie, on récupère assez d'informations, permettant par la suite de connaître de façon relativement fine la dynamique explosive pour des solutions 1-corotationnelles de (1.1.6).

Commençons par cette définition du degré d'homotopie, nombrant jouant un rôle majeur dans la dynamique des solutions de (1.1.6).

Définition 1.2.1 (Degré d'homotopie). *Soit u une application régulière $u : \mathbb{R}^2 \rightarrow \mathcal{N}$. On appelle degré d'homotopie de l'application u , que l'on note $\deg(u)$, le nombre entier non nul :*

$$k = \deg(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} (\partial_1 u \wedge \partial_2 u) \cdot u. \quad (1.2.1)$$

Il est important de noter que le degré d'homotopie d'une solution de notre problème sera toujours celui de la donnée initiale associée. On peut à présent introduire une première classe spécifique de solutions, ayant une symétrie additionnelle.

Définition 1.2.2 (Application k -équivariante). *On appelle application k -équivariante une application $u : \mathbb{R}^2 \rightarrow \mathcal{N}$ telle que :*

- $\deg(u) = k$,
- Il existe des applications $(u_1, u_2, u_3) \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ telles que :

$$u(t, x) = \exp^{k\theta R} \begin{vmatrix} u_1(t, r) \\ u_2(t, r) \\ u_3(t, r), \end{vmatrix} \quad \text{avec } R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1.2.2)$$

Cette symétrie sera préservée au cours du temps, pour des données initiales k -équivariantes, par le problème (1.1.6), mais aussi pour les problèmes (1.1.7) et (1.1.8). En fait, on va définir une seconde classe spécifique de solutions, solutions k -équivariantes, avec une symétrie supplémentaire. Dans cette définition, nous utiliserons à nouveau les notations de la définition (1.1.2) décrivant le paramétrage des variétés \mathcal{N} que nous considérons.

Définition 1.2.3 (Application k -corotationnelle). *On appelle application k -corotationnelle une application $u : \mathbb{R}^2 \rightarrow \mathcal{N}$ de la forme*

$$v(t, x) = \begin{vmatrix} g(u(t, r)) \cos(k\theta) \\ g(u(t, r)) \sin(k\theta) \\ z(u(t, r)) \end{vmatrix} \quad (1.2.3)$$

Remarquons que cela correspond à prendre dans la définition (1.2.2) $u_2 = 0$. Cette nouvelle classe de solutions est conservée par le flot (1.2.4), mais aussi par le flot (1.1.8), associé aux "wave maps". Par contre, ce n'est pas le cas pour les "Schrödinger maps" (1.1.7). Nous verrons dans la partie (1.3), concernant les commentaires du théorème 1.3.1, que cette distinction jouera un rôle déterminant dans la différence de stabilité, pour les trois équations d'évolution que nous étudions, des régimes explosifs décrits dans [72], [88] et [91].

Sauf mention contraire, nous ne considérerons dans la suite de cette partie que des applications k -rotationnelles. L'équation (1.1.6) peut se réécrire alors, pour cette classe de fonctions :

$$\begin{cases} \partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2 \frac{f(u)}{r^2} = 0, & f = gg' \\ u_{t=0} = u_0 \end{cases} \quad (1.2.4)$$

L'énergie de Dirichlet (1.1.2) devient pour ces applications k -corotationnelles :

$$E(u) = \int_0^{+\infty} \left[|\partial_r u|^2 + k^2 \frac{(g(u))^2}{r^2} \right] r dr. \quad (1.2.5)$$

qui est invariante par le changement d'échelle :

$$u_\lambda(t, r) = u\left(\frac{t}{\lambda^2}, \frac{r}{\lambda}\right). \quad (1.2.6)$$

Le problème est donc énergie critique. On peut remarquer aussi que l'énergie de Dirichlet est dissipée par le flot :

$$\frac{d}{dt} \left\{ \int_{\mathbb{R}^2} |\nabla v|^2 \right\} = -2 \int_{\mathbb{R}^2} |\partial_t v|^2 \leq 0. \quad (1.2.7)$$

Si on fixe comme conditions au bord :

$$u(0) = 0, \quad u(r) = \pi, \quad r \rightarrow \infty \quad (1.2.8)$$

qui correspond à une application reliant de façon continue les deux points d'intersection entre la surface de révolution \mathcal{N} , et son axe, alors l'énergie (1.2.5) est minimisée sur une classe homotopique k donnée par les solutions de l'ODE suivante :

$$r \partial_r Q_k = k g(Q_k). \quad (1.2.9)$$

A invariance d'échelle près, l'application harmonique solution de (1.2.9) est unique. Pour plus d'informations, on pourra consulter [18]. Dans le cas où $\mathcal{N} = \mathbb{S}^2$, ie $g(u) = \sin(u)$ alors Q_k est donné explicitement par

$$Q_k(r) = 2 \arctan(r^k), \quad (1.2.10)$$

qui correspond à une solution

$$Q_k = \begin{cases} \frac{2r^k}{1+r^{2k}} \cos(k\theta) \\ \frac{2r^k}{1+r^{2k}} \sin(k\theta). \\ \frac{1-r^{2k}}{1+r^{2k}} \end{cases} \quad (1.2.11)$$

Dans la suite, nous noterons Q à la place de Q_1 , et \mathcal{Q} la solution de (1.2.4) associée.

Pour conclure cette partie, avant de présenter le résultat de [91], donnons quelques résultats déjà connus. Van Den Berg, Hulshof et King ont établi dans [99] une étude formelle, dans le cas où la surface $\mathcal{M} = \mathbb{D}^2$. Ils prédisent formellement dans le cas où $k = 1$ l'existence d'un régime explosif stable avec

$$u(t, r) \sim Q\left(\frac{r}{\lambda(t)}\right), \quad \lambda(t) \sim \frac{(T-t)^{l+1}}{|\log(T-t)|^{\frac{2l+2}{2l+1}}}, \quad l \in \mathbb{N} \quad (1.2.12)$$

Guan, Gustaffson et Tsai [36] et Gustaffson, Nakanishi et Tsai [37] prouvent que les solutions de (1.2.4) avec des données initiales proches du soliton Q_k dans un certain sens, sont globales en temps pour $k \geq 2$. De plus, si $k \geq 3$, la solution reste dans un voisinage de Q_k . Un phénomène d'explosion ne peut donc avoir lieu. Par contre, pour $k = 2$, la solution peut osciller indéfiniment, ou même exploser en temps infini. Pour les solutions 1-corotationnelles, la situation est beaucoup plus délicate. Comme mentionné précédemment, Chang, Ding et Ye [14], Coron et Ghildaglia [11] et Qing et Tian [86] ont prouvé l'existence de solutions de ce type explosant en temps fini. Angenent, Hulshof et Matano ont établi récemment dans [1] une première estimation de la vitesse d'explosion en temps fini $T < \infty$ pour ce type de solutions :

$$\lambda(t) = o(T-t) \quad \text{quand } t \rightarrow T. \quad (1.2.13)$$

1.3 Présentation et commentaires de [91]

Donnons donc l'énoncé du résultat principal de cet article, décrivant une dynamique stable d'explosion, pour des données initiales corotationnelles, autour du soliton \mathcal{Q} , au sens de la topologie induite par l'énergie.

Théorème 1.3.1 (Dynamique d'explosion stable pour des solutions 1-corotationnelles, [91]). *Soit $k = 1$ et g satisfaisant (1.1.5). Soit \mathcal{Q} l'application harmonique de plus basse énergie donnée par :*

$$\mathcal{Q}(x) = \begin{cases} g(Q(t, r)) \cos(\theta) \\ g(Q(t, r)) \sin(\theta) \\ z(Q(t, r)) \end{cases} \quad (1.3.1)$$

Alors, il existe un espace ouvert \mathcal{O} de données initiales 1-corotationnelles, de la forme

$$v_0 = \mathcal{Q} + \varepsilon_0, \quad \varepsilon_0 \in \mathcal{O} \subset \dot{H}^1 \cap \dot{H}^4$$

tel que la solution $v \in C([0, T), \dot{H}^1 \cap \dot{H}^4)$ de (1.1.6) explose en temps fini $0 < T = T(v_0) < +\infty$ selon la dynamique universelle décrite ci-dessous :

(i) Universalité de la bulle de concentration : *Il existe un profil asymptotique $v^* \in \dot{H}^1$ et $\lambda \in \mathcal{C}^1([0, T), \mathbb{R}_+^*)$ tel que*

$$\lim_{t \rightarrow T} \left\| v(t, x) - \mathcal{Q} \left(\frac{x}{\lambda(t)} \right) - v^* \right\|_{\dot{H}^1} = 0. \quad (1.3.2)$$

(ii) Dynamique connue finement : *La vitesse de concentration est donnée par :*

$$\lambda(t) = c(v_0)(1 + o(1)) \frac{T - t}{|\log(T - t)|^2} \quad \text{as } t \rightarrow T \quad (1.3.3)$$

pour un $c(v_0) > 0$.

(iii) Régularité du profil asymptotique : *On a de plus la régularité suivante :*

$$v^* \in \dot{H}^2. \quad (1.3.4)$$

Commençons par quelques commentaires :

- On peut tout d'abord noter que l'on retrouve le résultat prédit (1.2.12) dans [99], pour la première vitesse. Le cas des autres vitesses est traité dans [90].
- Comme annoncé un peu plus haut, il est intéressant de mettre en parallèle ce résultat, avec ceux établis pour les "Schrödinger maps" [72], et les "wave maps" [88]. Dans ce dernier sont prouvés l'existence et la stabilité d'une dynamique explosive en temps fini, pour chaque classe k de données initiales corotationnelles. On retrouve donc, pour $k = 1$, une dynamique finalement assez similaire, avec malgré tout une vitesse de concentration différente. Pour les "Schrödinger maps", les données initiales sont considérées 1-équivariantes. Nous rappelons que la classe des solutions corotationnelles n'est pas préservée par cette équation d'évolution. Dans [72] est démontrée l'existence d'une variété de codimension 1 de solutions initiales lisses 1-équivariantes ayant la dynamique suivante :

$$\lim_{t \rightarrow T} \left\| u(t, r) - e^{\Theta^* R} Q \left(\frac{r}{\lambda(t)} \right) - u^* \right\|_{\mathcal{H}} = 0 \quad (1.3.5)$$

pour $\Theta^* \in \mathbb{R}$, $u^* \in \dot{H}^1$ à la vitesse donnée par (1.3.3) :

$$\lambda(t) = c(u_0)(1 + o(1)) \frac{T - t}{|\log(T - t)|^2}.$$

Le fait de devoir maîtriser la rotation de la solution au cours du temps entraîne la perte de stabilité dans la formation de singularités.

- Par rapport aux travaux antérieurs de Struwe [96] notamment, on obtient ici une meilleure régularité du profil asymptotique, grâce à une connaissance précise de la vitesse (1.3.3) d'explosion. Cela indique un lien étroit entre ce taux de concentration, relativement éloigné de la vitesse auto-similaire ($\lambda(t) \sim \sqrt{T - t}$), et cette régularité. On pourra lire par exemple [92] pour une discussion similaire.

Voici quelques perspectives à cette étude :

- Il serait intéressant de voir ce qu'il en est pour des solutions initiales 1-équivariantes. La seconde remarque que l'on a faite ci-dessus nous amène à penser que cette perte de symétrie se traduira par la perte de stabilité du régime.
- Cette dernière perspective sera une étape supplémentaire dans la compréhension du modèle complet. On peut remarquer que notre preuve ne repose en aucun cas sur le principe du maximum. Ainsi, on peut envisager de l'étendre au cas le plus général, qui se pose alors comme un système non-linéaire, fortement couplé.
- Enfin, le problème particulièrement intéressant de la description complète autour du soliton du flot pourra être entrevu, en se basant notamment sur les travaux de Merle, Martel et Raphaël récents [62], [63] et [64] sur une telle description pour l'équation de Korteweg-De-Vries généralisée en dimension 1.

Pour conclure cette présentation de [91], il faut mentionner le fait que la preuve est très proche de celle de [94], pour la chaleur non linéaire. Ici, l'opérateur linéarisé H ne possède pas de valeurs propres strictement négatives. C'est une conséquence directe de la factorisation $H = A^*A$, où A est un opérateur linéaire d'ordre 1. Cela simplifie un peu les choses. Par contre, le fait que la non-linéarité ne soit plus une simple puissance rend la gestion de la non-linéarité un peu plus complexe. Enfin, il faut faire attention au transfert de régularité notamment, entre le problème simplifié (1.2.4) et le problème initial (1.1.6). C'est pour cette raison que nous avons choisi de ne pas donner des éléments de preuve pour ce résultat, les différences étant trop mineures. Nous renvoyons donc au chapitre 5 le lecteur intéressé par la preuve complète du théorème 1.3.1, et à la partie 3 pour une idée fine de la stratégie mise en place.

Chapitre 2

Equation de la chaleur non-linéaire

Dans cette partie, nous allons nous pencher sur l'équation de la chaleur semi-linéaire suivante :

$$\begin{cases} \partial_t u - \Delta u - u^p = 0, & p > 1. \\ u(., 0) = u_0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N. \end{cases} \quad (2.0.1)$$

Il est bien connu - voir par exemple [3] pour des données initiales $u_0 \in H^\alpha$, avec $\alpha \in [0, 2]$ - que pour toute donnée initiale $u_0 \in L^\infty(\mathbb{R}^N)$, le problème (2.0.1) possède une solution unique $u \in C([0, T), L^\infty(\mathbb{R}^N))$, où le temps maximal de vie de la solution T est tel que soit $T = +\infty$, soit :

$$T < \infty \text{ et } \lim_{t \rightarrow T} \|u(., t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

Dans la suite de cette partie, nous différencierons deux types d'explosion en temps fini, qui correspondent au dernier cas, en reprenant la nomenclature introduite par Matano et Merle dans [65].

Définition 2.0.2 (Explosion de type I et type II). *Soit u une solution de (2.0.1) explosant en temps fini T . Alors nous dirons que l'explosion associée à u est de type I si la quantité $\|u(., t)(T-t)^{\frac{1}{p-1}}\|_{L^\infty(\mathbb{R}^N)}$ reste bornée au cours du temps. Dans le cas contraire, nous dirons qu'il s'agit d'une explosion de type II.*

Il est intéressant de remarquer que la borne

$$\|u(., t)\|_{L^\infty(\mathbb{R}^N)} \leq C(T-t)^{-\frac{1}{p-1}} \quad (2.0.2)$$

correspondant à une solution u explosant de type I est la même que celle de l'EDO suivante :

$$\partial_t u = u^p. \quad (2.0.3)$$

Plus précisément, la solution stationnaire en espace $u(t) = \kappa(T-t)^{\frac{-1}{p-1}}$, explosant en temps fini au temps T , est solution de (2.0.1), mais aussi de (2.0.3). La borne (2.0.2) est donc optimale. On verra même par la suite que, dans le régime sous critique, toutes les solutions explosives ont exactement cette vitesse de concentration.

De plus, l'équation (2.0.1) admet une énergie

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} \quad (2.0.4)$$

qui est décroissante au cours du temps. $\left(\frac{dE(u)}{dt} = - \int_{\mathbb{R}^N} |\partial_t u|^2\right)$. Dans la formule de l'énergie (2.0.4), le premier terme correspond à la partie cinétique, et le second à la partie potentielle de l'énergie.

2.1 Introduction au cas énergie critique :

Introduisons pour commencer l'invariance par changement d'échelle. Soit $\lambda > 0$. Soit u une solution de (2.0.1). Alors, la fonction

$$u_\lambda(x, t) = \lambda^{\frac{-2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N \quad (2.1.1)$$

est aussi solution de (2.0.1) en prenant $(u_0)_\lambda$ comme donnée initiale.

Définition 2.1.1 (Espace critique). *On appelle espace critique l'espace de Sobolev homogène \dot{H}^{s_c} , où $s_c = s_c(p, N)$, qui vérifie :*

$$\|u_\lambda(., t)\|_{\dot{H}^{s_c}} = \left\| u\left(., \frac{t}{\lambda^2}\right) \right\|_{\dot{H}^{s_c}}, \quad \forall t \in \mathbb{R}^+ \quad (2.1.2)$$

On a alors $s_c = \frac{N}{2} - \frac{2}{p-1}$

Si on note

$$2^* = \begin{cases} +\infty, & \text{pour } N \in \{1, 2\}, \\ \frac{2N}{N-2}, & \text{pour } N \geq 3, \end{cases} \quad (2.1.3)$$

le coefficient de Sobolev critique, comme on a l'injection de \dot{H}^1 dans L^{2^*} , on peut remarquer que le cas $p = 2^* - 1$ est le cas critique où le terme cinétique et le terme de potentiel dans l'énergie ont un poids équivalent. En reprenant la définition (2.1.1), on conclut que pour $N \geq 3$ et $p = 2^* - 1 = \frac{N+2}{N-2}$, alors on a $s_c = 1$. En fait, le problème est plus que \dot{H}^1 critique, il est énergie critique, son énergie étant alors invariante par changement d'échelle.

On peut ainsi définir trois zones distinctes :

$$\begin{cases} p < 2^* - 1 & \text{Problème énergie sous-critique} \\ p = p_c = 2^* - 1 & \text{Problème énergie critique} \\ p > 2^* - 1 & \text{Problème énergie sur-critique} \end{cases} \quad (2.1.4)$$

2.1.1 Problème énergie sous-critique

Gidas et Spruck montre dans [30] que l'équation elliptique

$$\Delta u + u^p = 0, \quad p < 2^* - 1 \quad (2.1.5)$$

n'admet pas d'autres solutions positives que la solution triviale nulle. Ainsi, il n'existe aucune solution stationnaire positive, autre que la fonction nulle à l'équation (2.0.1) dans la zone sous-critique.

Dans [33], [31] et [32], Giga et Kohn établissent que si $N \leq 2$ ou si $1 < p < \frac{3N+8}{3N-4} < p_c$ alors il n'existe pas de solution de (2.0.1) explosant de type II. Ce résultat a été complété par Giga, Matsui et Sasayama dans [34] et [35] à toute la zone sous-critique. Enfin, Merle et Zaag ont montré dans [73] et [74] que si u explosait de type I, en sous-critique, alors

$$\|u(., t)\|_{L^\infty(\mathbb{R}^N)} \underset{t \rightarrow T}{\sim} \kappa(T - t)^{-\frac{1}{p-1}}.$$

2.1.2 Problème énergie sur-critique

La situation dans le domaine énergie sur-critique est plus délicate. Les principaux résultats sur ce sujet sont relativement récents -moins de 10 ans- et concernent à ma connaissance uniquement le cas de fonctions radiales.

Contrairement au cas sous critique, le système suivant :

$$\begin{cases} \Phi'' + \frac{N-1}{r}\Phi' + \Phi^p = 0, \\ \Phi(0) = \alpha, \quad \Phi'(0) = 0, \quad \alpha > 0 \end{cases} \quad (2.1.6)$$

correspondant à des solutions stationnaires et radiales de (2.0.1) admet une solution positive, que l'on notera Φ_α , d'après l'identité de Pohožaev [84]. Par souci de cohérence, nous noterons dans la suite Φ_∞ la solution stationnaire, positive et singulière suivante :

$$\Phi_\infty(r) = c_\infty r^{-\frac{2}{p-1}} \quad (2.1.7)$$

avec

$$c_\infty = \left\{ \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right\}^{\frac{1}{p-1}}$$

Cette solution singulière de (2.0.1), couplée avec la fonction introduite dans la définition suivante, constituent des outils importants dans l'établissement des preuves d'existence, et de non existence d'explosion de type II en énergie sur-critique.

Définition 2.1.2. Soit f une fonction positive, non identiquement nulle. On associe à cette fonction le nombre $\mathcal{Z}(f) \in [0, +\infty]$ défini comme la borne supérieure des j tels qu'il existe $0 < r_1 < r_2 < \dots < r_{j+1}$ avec $f(r_i) \cdot f(r_{i+1}) < 0$ pour $i \in [1, j]$.

A l'ensemble de ces solutions stationnaires, Joseph et Lundgren ont déterminé dans [46] un coefficient portant leur nom

$$p_{JL} = \begin{cases} +\infty & \text{for } N \leq 10 \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{for } N \geq 11 \end{cases}, \quad (2.1.8)$$

intimement lié, comme le montre le lemme suivant, aux propriétés des solutions stationnaires de (2.0.1) :

Lemme 2.1.3 (Caractérisation de p_{JL}). Soit le coefficient p_{JL} défini par (2.1.8). Alors, pour tout $\alpha > 0$, on a :

- Si $p_c < p < p_{JL}$, alors $\mathcal{Z}(\Phi_\alpha - \Phi_\infty) = \infty$.
- Si $p > p_{JL}$, alors $\mathcal{Z}(\Phi_\alpha - \Phi_\infty) < \infty$.

Cette différence fondamentale de comportement des solutions stationnaires entraîne une distinction de deux régions du domaine sur-critique, dans lesquelles on a approximativement les propriétés suivantes :

- Si $p_c < p < p_{JL}$, alors l'explosion de type II est impossible.
- Si $p > p_{JL}$, alors il existe des solutions qui explosent de type II.

Plus précisément, Matano et Merle ont montré dans [65] le théorème suivant :

Théorème 2.1.4 (Matano, Merle, [65], surcritique). *Soit $p_c < p < p_{JL}$. Soit u une solution de (2.0.1), associée à une donnée initiale $u_0 \in L^\infty(\mathbb{R}^N)$ à symétrie radiale. Supposons qu'elle explose en norme L^∞ en temps fini $T < +\infty$. De plus, supposons qu'il existe $t_0 \in [0, T)$ tel que*

$$(H1) \quad \mathcal{Z}(u_t(t_0)) < \infty,$$

$$(H2) \quad \mathcal{Z}(u(t_0) - \Phi_\infty) < \infty \text{ et } \mathcal{Z}(u(t_0) + \Phi_\infty) < \infty.$$

Alors u explose de type I, ie il existe C_0 tel que :

$$\|u(., t)\|_{L^\infty(\mathbb{R}^N)} \leq C_0(T-t)^{-\frac{1}{p-1}}$$

Dans [66], ils améliorent ce théorème en supprimant l'hypothèse (H1). Récemment Mizoguchi a remplacé l'hypothèse (H2) dans [77] par l'hypothèse suivante :

- Il existe des constantes $C, K > 0$ telles que $|u(x, t)| \leq C$ pour $|x| \geq K$ et $t \in [0, T]$.

Ce raffinement permet de conclure à la non-existence d'explosion de type II pour des données initiales radiales, positives, et décroissantes, cas non exclu avec les hypothèses (H1) et (H2).

Pour $p > p_{JL}$, Herrero et Velázquez ont montré formellement dans [38] et [39] l'existence de solutions explosives de type II. Ce résultat a été démontré rigoureusement par Mizoguchi dans [75]. En outre, introduisons le coefficient de Lepin :

$$p_L = \begin{cases} +\infty & \text{for } N \leq 10 \\ 1 + \frac{6}{N-1} (> p_{JL}) & \text{for } N \geq 11 \end{cases}, \quad (2.1.9)$$

jouant un rôle crucial sur le nombre de valeurs propres strictement négatives de l'opérateur linéarisé de (2.0.1), après être passé en variables autosimilaires. Pour $p > p_L$, Mizoguchi obtient dans [77] la vitesse précise d'explosion de type II pour des données initiales vérifiant un certain nombres d'hypothèses, *a priori* techniques pour la plupart.

On peut naturellement se poser deux questions, qui sont encore des problèmes ouverts :

- Que se passe-t-il pour $p = p_{JL}$?
- Quel est la dynamique des solutions explosives de type II pour $p_{JL} < p < p_L$?

2.1.3 Problème énergie critique

On se place dans cette sous partie dans le cas où le problème est énergie critique, ie où l'énergie associée au problème (2.0.1) définie par (2.0.4) est laissée invariante par le changement d'échelle (2.1.1). On se place donc en dimension supérieure ou égale à 3, pour un coefficient $p = p_c = \frac{N+2}{N-2} = 2^* - 1$.

Contrairement au cas énergie surcritique, les solutions radiales, stationnaires de (2.0.1) Φ_α sont connues ici explicitement :

$$\Phi_\alpha(r) = \frac{\alpha}{\left(1 + \frac{\alpha^{\frac{4}{N-2}} r^2}{N(N-2)}\right)^{\frac{N-2}{2}}}, \quad \text{pour } r \geq 0. \quad (2.1.10)$$

En fait, notons

$$Q_N(r) = \left(\frac{1}{1 + \frac{r^2}{N(N-2)}} \right)^{\frac{N-2}{2}}. \quad (2.1.11)$$

En utilisant la notation (2.1.1) pour le changement d'échelle, nous obtenons :

$$\{(Q_N)_\lambda, \lambda > 0\} = \{\Phi_\alpha, \alpha > 0\}.$$

Matano et Merle ont prouvé dans [65] le théorème suivant :

Théorème 2.1.5 (Matano, Merle, [65], critique). *Soit u_0 une donnée initiale radiale, positive, et bornée. Soit u la solution de (2.0.1) associée à cette donnée initiale. Alors si u explose en temps fini, l'explosion est de type I.*

Maintenant, Herrero et Velázquez suggèrent dans [38] la possibilité d'une explosion de type II en énergie critique pour des solutions changeant de signe. Avec Filippas, ils construisent formellement dans [26] des solutions explosives de type II, avec la dynamique suivante :

Théorème 2.1.6 (Filippas, Herrero, Velázquez, [26]). *Il existe des données initiales $u_0 = Q_\lambda(0) + \varepsilon(0)$, explosant de type II en temps fini $T < \infty$, où la fonction $\lambda(t)$ vérifie selon la dimension N :*

- $N = 3 \quad \lambda(t) \sim (T-t)^{2(l+1)}$ avec $l \in \mathbb{N}$,
- $N = 4 \quad \lambda(t) \sim \frac{(T-t)^{l+1}}{|\log(T-t)|^{\frac{2l+2}{2l+1}}} \text{ avec } l \in \mathbb{N}$,
- $N = 5 \quad \lambda(t) \sim (T-t)^{\frac{2}{3}(l+1)}$ avec $l \in \mathbb{N}$,
- $N = 6 \quad \lambda(t) \sim (T-t)^{\frac{1}{4}} |\log(T-t)|^{\frac{15}{16}}$.

On peut ajouter deux remarques à ce théorème :

1. Les auteurs précisent que dans le cas de la dimension supérieure ou égale à 7, ils ne peuvent pas décider de l'existence formelle de solution explosant de type II. Nous verrons dans la partie suivante les raisons de cette indécision.
2. Il ne s'agit que d'une approche formelle du problème. Ainsi, la gestion de la fonction d'erreur ε n'est pas évoquée. La méthode que l'on utilise apporte justement une réponse à cette question.

2.2 Présentation du résultat [94]

Dans cette sous-partie, nous allons donner le résultat d'existence de solution explosive de type II, en dimension 4, et le commenter. Afin d'améliorer la clarté de notre exposé, nous allons rappeler, en dimension 4, l'équation que l'on étudie, ainsi qu'un certain nombre de quantités lui étant associées. Le problème suivant :

$$\begin{cases} \partial_t u - \Delta u - u^3 = 0, \\ u(., 0) = u_0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^4. \end{cases} \quad (2.2.1)$$

admet comme énergie :

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^4} u^4. \quad (2.2.2)$$

De plus, la solution stationnaire radiale :

$$Q(r) = \frac{1}{1 + \frac{r^2}{8}}. \quad (2.2.3)$$

est l'unique, à invariance par changement d'échelle près, solution radiale du problème stationnaire :

$$\Delta Q + Q^3 = 0. \quad (2.2.4)$$

2.2.1 Sur le spectre de l'opérateur linéarisé H

En calculant pour toute direction u suffisamment lisse et localisée la quantité

$$\frac{dE(Q + \varepsilon u)}{d\varepsilon} \Big|_{\varepsilon=0} = - \int (\Delta Q + Q^3) u,$$

on montre que le soliton Q est un point extremum de l'énergie. En fait, on peut même raffiner ce résultat, en utilisant le fait que :

$$\frac{d^2E(Q + \varepsilon u)}{d\varepsilon^2} \Big|_{\varepsilon=0} = \frac{1}{2}(Hu, u),$$

où l'opérateur H est le linéarisé de la partie spatiale de (2.2.1), autour de l'objet stationnaire Q , et est donné par :

$$Hu = -\Delta u - 3Q^2u. \quad (2.2.5)$$

En remarquant que $(HQ, Q) < 0$, on en déduit que Q n'est pas un minimiseur de l'énergie. En fait, on peut trouver dans [47] par exemple la démonstration de l'unicité de la valeur propre strictement négative de H . L'espace propre associé à cette dernière, que l'on notera $-\zeta$, avec $\zeta > 0$, est généré par une fonction Ψ , positive, radiale, et très bien localisée. Ces propriétés spectrales de l'opérateur linéarisé restent valables en toute dimension, pour l'exposant critique $p = p_c$.

2.2.2 Résultat et commentaires

Théorème 2.2.1 (Existence de solutions explosives de type II pour $N = 4$, [94]). *Soit Q la solution stationnaire donnée par (2.2.4). Alors $\forall \alpha^* > 0$, il existe une donnée initiale radiale $u_0 \in H^1(\mathbb{R}^4)$ avec*

$$E(Q) < E(u_0) < E(Q) + \alpha^* \quad (2.2.6)$$

tel que la solution u de l'équation (2.2.1) associée à cette donnée initiale explose en temps fini $T = T(u_0) < \infty$ dans un régime de type II dans la dynamique suivante :

Il existe $u^ \in \dot{H}^1$ tel que :*

$$\nabla \left[u(t, x) - \frac{1}{\lambda(t)} Q \left(\frac{x}{\lambda(t)} \right) \right] \rightarrow \nabla u^* \quad \text{in } L^2 \quad \text{as } t \rightarrow T \quad (2.2.7)$$

à la vitesse

$$\lambda(t) = c(u_0) (1 + o(1)) \frac{T-t}{|\log(T-t)|^2} \quad \text{as } t \rightarrow T \quad (2.2.8)$$

pour $c(u_0) > 0$. En outre, nous avons la régularité suivante pour le profil asymptotique :

$$\Delta u^* \in L^2 \quad (2.2.9)$$

Voici quelques remarques que l'on peut faire sur ce résultat, pour lequel nous donnerons une stratégie de preuve approfondie dans le chapitre 3 :

1. Dans cet article, seul le cas radial a été traité. Comme pour les autres preuves, le cas non radial ne pose conceptuellement aucune difficulté. Il faudrait notamment rajouter dans la construction un paramètre pour contrôler la possibilité nouvelle pour la solution approchée de translater, en se servant notamment du fait que $H(\nabla Q) = 0$. On peut lire par exemple [69], où Merle et Raphaël traitent le cas non radial pour l'équation de Schrödinger non linéaire, pour se faire une idée des changements que cela occasionne. De plus, cela rendrait la preuve plus technique, notamment dans la partie maîtrise de l'énergie.
2. Ce résultat est le premier résultat rigoureux dans le domaine énergie critique, décrivant précisément une dynamique de type II. Remarquons que l'on retrouve la vitesse prédictive par Filippas, Herrero et Velázquez dans [26], pour $N = 4$ et $l = 0$.
3. Contrairement au résultat pour le flot de la chaleur harmonique [91], nous n'avons pas ici d'information sur la stabilité du régime explosif. Plus précisément, la présence dans ce problème d'une valeur propre strictement négative pour l'opérateur linéarisé H , engendre une direction d'instabilité. Ainsi, nous arrivons à gérer cette difficulté, en reprenant la stratégie mise en place dans [41] par Hillaret et Raphaël dans le cadre de l'équation des ondes $\partial_{tt}u = \Delta u + u^3$, en dimension 4, basée sur un argument développé dans le lemme 6 de [20]. Malgré la puissance de cet argument, cela ne nous permet pas d'affirmer quoi que ce soit sur l'ensemble des données initiales, concernées par le théorème 2.2.1. Cependant, nous pouvons conjecturer sans trop nous avancer que la dynamique est ici instable, et que cet ensemble est une variété de codimension 1.

Voici plusieurs perspectives de prolongement de ce résultat :

1. Il serait intéressant de traiter le cas non radial. Cela constituerait un pas supplémentaire vers le problème particulièrement excitant de la description complète du flot autour du soliton, à l'instar du travail récent de Martel, Merle et Raphaël pour gKdV unidimensionnel [62], [63] et [64].
2. En utilisant la stratégie présentée dans le cadre de la partie suivante pour obtenir formellement les vitesses d'explosion possibles en dimension N , il est envisageable d'obtenir des dynamiques explosives de type II, pour toutes les vitesses annoncées dans [26]. La question liée, et encore loin d'être claire, est de prouver que seules ces vitesses-là sont possibles pour une explosion de type II.
3. Toujours dans le cadre où l'exposant p est tel que l'on se place dans le cadre énergie critique, la méthode mise en place dans [94] donne l'espoir d'aborder le problème aux autres dimensions, notamment la dimension 3, qui est profondément reliée à des problématiques physiques, notamment dans des modèles de combustion, voir par exemple [9]. Alors que la question en dimension 6 n'apparaît pas évidente à trancher, et laisse entrevoir, pourquoi pas, une explosion en temps infini, il semble naturel de conjecturer la non-existence d'explosion de type II en dimension $N \geq 7$, d'après l'analyse formelle que l'on va présenter à présent.
4. Une autre question ouverte est d'établir l'existence, ou la non-existence de dynamique explosive de type II en prenant comme exposant celui de Joseph et Lundgren $p = p_{JL}$, défini en (2.1.8). D'après [66], il semblerait que le profil de la bulle de concentration ne soit pas ici, à invariance par changement d'échelle près, un profil lisse comme la solution

stationnaire Q, définie en (2.2.4), mais la solution (2.1.7) stationnaire singulière en l'origine Φ_∞ , ce qui inévitablement amènera à revoir en profondeur la stratégie de preuve.

Chapitre 3

Présentation de la méthode de preuve pour l'équation de la chaleur

Dans cette partie, nous allons présenter de façon relativement précise la preuve du théorème 2.2.1. Celle-ci se déroule en quatre temps :

- Construction d'une solution approchée.
- Equation de modulation des paramètres et description de l'espace des données initiales.
- Méthode d'énergie pour contrôler le terme d'erreur.
- Conclusion du théorème.

Lors de la première étape, nous allons exposer le cas de la dimension N , afin de retrouver le résultat de Filippas, Herrero et Velázquez [26]. Nous en profiterons pour voir comment adapter la construction de la solution approchée dans l'optique d'obtenir l'ensemble des vitesses prédictes dans [26]. Cette stratégie a été rigoureusement mise en place dans [90] pour le problème du flot de la chaleur harmonique. Cela ne pose aucun problème d'adapter cette preuve au cas de la chaleur en dimension 4 et d'obtenir le théorème suivant :

Théorème 3.0.2 (Existence de solutions à explosion lente de type II pour $N = 4$). *Soit Q la solution stationnaire donnée par (2.2.4). Alors $\forall \alpha^* > 0$, il existe une donnée initiale radiale $u_0 \in H^1(\mathbb{R}^{2L+2})$ avec*

$$E(Q) < E(u_0) < E(Q) + \alpha^* \quad (3.0.1)$$

tel que la solution u de l'équation (2.2.1) associée à cette donnée initiale explose en temps fini $T = T(u_0) < \infty$ dans un régime de type II dans la dynamique suivante :

Soit $L \in \mathbb{N}$. Alors il existe $u^ \in \dot{H}^1$ tel que :*

$$\nabla \left[u(t, x) - \frac{1}{\lambda(t)} Q \left(\frac{x}{\lambda(t)} \right) \right] \rightarrow \nabla u^* \quad \text{in } L^2 \quad \text{as } t \rightarrow T \quad (3.0.2)$$

à la vitesse

$$\lambda(t) = c(u_0) (1 + o(1)) \frac{(T-t)^{L+1}}{|\log(T-t)|^{\frac{2L+2}{2L+1}}} \quad \text{as } t \rightarrow T \quad (3.0.3)$$

pour $c(u_0) > 0$.

Pour conclure la présentation de ce théorème, ajoutons que l'ensemble des données initiales forment *a priori* une variété de codimension $L+1$. Le régime de concentration est donc d'autant plus instable que la vitesse de formation de la singularité est lente.

3.1 Etude formelle du cas critique en dimension N :

3.1.1 Changement de variables

Nous étudions donc l'équation suivante, pour $N \geq 3$:

$$\begin{cases} \partial_t u - \Delta u - u^{\frac{N+2}{N-2}} = 0, \\ u(., 0) = u_0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N. \end{cases} \quad (3.1.1)$$

Comme vu précédemment, il existe une famille de solutions radiales et stationnaires :

$$\{(Q_N)_\lambda, \lambda > 0\}$$

avec

$$Q_N(r) = \left(\frac{1}{1 + \frac{r^2}{N(N-2)}} \right)^{\frac{N-2}{2}}.$$

Le problème présente l'invariance par changement d'échelle suivante :

$$u_\lambda(t, r) = \frac{1}{\lambda^{\frac{N-2}{2}}} u\left(\frac{t}{\lambda^2}, \frac{r}{\lambda}\right).$$

Dans la suite de cet exposé, nous ne considérerons plus que des fonctions radiales. C'est pourquoi, nous noterons toujours r la variable d'espace dans les variables d'origine.

On va ici construire une solution approchée de (3.1.1). L'idée cruciale est de faire un changement de variables adapté à la dynamique recherchée, qui n'est pas le changement de variable auto-similaire standard. En effet, le défaut majeur de ce dernier est de fixer par avance une vitesse de "zoom", qui ne correspond pas à la vitesse de concentration. Dans notre stratégie, c'est la construction qui va dicter la vitesse d'explosion, et non l'inverse. On cherche donc une application $\lambda(t)$ (de classe \mathcal{C}^1), strictement positive, touchant zéro au temps $T < \infty$, qui sera le temps où la singularité se forme, telle que la solution u de (3.1.1) se décompose comme :

$$u(t, r) = (Q_N)_{\lambda(t)}(r) + \omega(t, r). \quad (3.1.2)$$

ω est donc un terme d'erreur, dû au fait que la fonction λ dépend du temps. On va le décomposer en deux parties. La première sera celle que l'on va construire par la suite. C'est elle qui implique la dynamique du système. La seconde partie sera une erreur que l'on pourrait qualifier d'"erreur pure", au sens où il ne s'agit que d'un terme à contrôler pour s'assurer qu'il ne perturbe pas autre mesure la dynamique explosive provenant du premier terme de ω .

Soit u une solution de (3.1.1). Si on injecte $u_{\lambda(t)}$ dans cette équation, on obtient :

$$\lambda^2(t) \partial_t u\left(\frac{r}{\lambda(t)}\right) - \lambda(t) \lambda'(t) \Lambda u\left(\frac{r}{\lambda(t)}\right) = \Delta u\left(\frac{r}{\lambda(t)}\right) + u^{\frac{N+2}{N-2}}\left(\frac{r}{\lambda(t)}\right) \quad (3.1.3)$$

où l'opérateur Λ est défini par :

$$\Lambda u(y) = \frac{N-2}{2} u(y) + y \cdot \nabla u(y) = \frac{\partial u_\lambda}{\partial \lambda} \Big|_{\lambda=1} \quad (3.1.4)$$

La dernière définition nous permet de voir le lien étroit entre cet opérateur et le changement d'échelle.

Posons donc comme changement de variables :

$$s = \int_0^t \frac{d\tau}{\lambda^2(\tau)} \text{ et } y = \frac{r}{\lambda}. \quad (3.1.5)$$

qui nous donne :

$$\partial_s u - \frac{\lambda_s}{\lambda} \Lambda u = \Delta u + u^{\frac{N+2}{N-2}}. \quad (3.1.6)$$

avec $\lambda_s = \lambda'(s)$. Remarquons que dans la dynamique recherchée, on a $0 < \lambda(t) = o(\sqrt{T-t})$. La fonction $s(t)$ réalise ainsi une bijection de $[0, T[$ dans \mathbb{R}^+ . Cette condition n'est pas vérifiée en dimension 6 pour la vitesse annoncée par Filippas, Herrero et Velázquez dans [26], ce qui nous permet de douter de cette dernière. La situation n'est pas vraiment claire pour l'instant. Est-il possible d'exploser en temps fini ? Ou a-t-on un phénomène au niveau de la dimension analogue à celui de l'indice d'homotopie des solutions du problème (1.1.6) du flot de la chaleur harmonique, qui nous ferait pencher pour une explosion en temps infini, comme pour les solutions 2-corotationnelles de (1.1.6) ?

Nous voyons apparaître dans l'équation (3.1.6) la quantité centrale dans notre preuve $-\frac{\lambda_s}{\lambda}$. On peut voir que cette quantité est positive, et tend vers zéro au moment de la formation de la singularité. Introduisons un nouveau paramètre dépendant uniquement du temps :

$$b_1(s) = -\frac{\lambda_s}{\lambda}(1 + o(1)). \quad (3.1.7)$$

Contrairement au cas de l'équation des ondes étudiée dans [41] par Hillaret et Raphaël, ou de celle des "wave maps" [88], on doit relâcher la contrainte $b_1(s) = -\frac{\lambda_s}{\lambda}$. Comme nous le verrons par la suite, cet assouplissement nous permettra de prendre deux conditions d'orthogonalités pour le terme d'"erreur pure", indispensable dans son contrôle.

En réalité, nous prendrons comme paramètre $b(s) = (b_1(s), \dots, b_L(s))$, où $|b_k| \lesssim b_1^k$. Lorsque l'on reviendra à la preuve du cas de la première vitesse en dimension 4, nous noterons b à la place de b_1 , afin d'alléger les notations.

(3.1.6) devient donc

$$\partial_s u + b_1(s) \Lambda u = \Delta u + u^{\frac{N+2}{N-2}} + \left(b_1(s) + \frac{\lambda_s}{\lambda} \right) \Lambda u. \quad (3.1.8)$$

3.1.2 A propos de l'opérateur H_N^k :

Linéarisons la partie spatiale de (3.1.8) autour du soliton Q_N .

$$\begin{aligned} \Delta(Q_N + \varepsilon) + (Q_N + \varepsilon)^{\frac{N+2}{N-2}} &= \Delta Q_N + Q_N^{\frac{N+2}{N-2}} + \Delta \varepsilon + \frac{N+2}{N-2} Q_N^{\frac{4}{N-2}} \varepsilon + O(\varepsilon^2). \\ &= -H_N(\varepsilon) + O(\varepsilon^2). \end{aligned}$$

où

$$H_N \varepsilon = -\Delta \varepsilon - \mathcal{V}_N \varepsilon = -\Delta \varepsilon - \frac{N+2}{N-2} Q_N^{\frac{4}{N-2}} \varepsilon, \quad (3.1.9)$$

\mathcal{V}_N étant un terme de potentiel. Rappelons qu'une autre façon d'interpréter cet opérateur provient de :

$$\frac{d^2 E(Q_N + \mu\varepsilon)}{d\mu^2} \Big|_{\mu=0} = \frac{1}{2}(H_N\varepsilon, \varepsilon).$$

Maintenant, en se servant du fait que $(Q_N)_\lambda$ est solution de l'EDO elliptique, pour tout $\lambda > 0$:

$$\Delta u + u^{\frac{N+2}{N-2}} = 0$$

et de la définition de l'opérateur Λ (3.1.4), on montre facilement que la fonction radiale :

$$\Lambda Q_N(y) = \frac{N-2}{2 \left(1 + \frac{y^2}{N(N-2)}\right)^{\frac{N}{2}}} \left(1 - \frac{y^2}{N(N-2)}\right) \quad (3.1.10)$$

est dans le noyau de H_N . Grâce à ce zéro explicite, on peut résoudre l'équation

$$H_N u = f, \quad (3.1.11)$$

où f est une fonction donnée, que l'on supposera suffisamment régulière. (Dans la suite, nous prendrons toujours f de classe \mathcal{C}^∞ .) Pour les fonctions radiales, cette équation est une EDO du second ordre, à coefficients non constants. Si on note Γ_N le second zéro de H_N , alors le Wronskien nous donne la relation suivante :

$$\Gamma'_N \Lambda Q_N - \Gamma_N (\Lambda Q_N)' = \frac{-1}{y^{N-1}}$$

et donc :

$$\Gamma_N(y) = -\Lambda Q_N(y) \int_1^y \frac{ds}{s^{N-1}(\Lambda Q_N)^2(s)} \quad (3.1.12)$$

On constate facilement que cette fonction est singulière en l'origine, et satisfait les asymptotiques suivantes :

$$\frac{d^k \Gamma_N}{dy^k}(y) = \begin{cases} O(y^{-(N-2+k)}) & \text{lorsque } y \rightarrow 0 \\ O(y^{-k}) & \text{lorsque } y \rightarrow \infty. \end{cases}$$

De là, une solution explicite de (3.1.11) est donnée par :

$$u(y) = \Gamma_N(y) \int_0^y f(s) \Lambda Q_N(s) s^{N-1} ds - \Lambda Q_N(y) \int_0^y f(s) \Gamma_N(s) s^{N-1} ds. \quad (3.1.13)$$

Malgré le fait que la fonction Γ_N soit singulière en l'origine, u est bien définie, notamment en l'origine, pour des fonctions f régulières. On notera dans la suite de façon abusive $u = H_N^{-1}f$, et en répétant i fois le processus H_N^{-i} .

Définition 3.1.1 (Noyau de l'opérateur H_N^k). *On définit par récurrence les profils $(T_{i,N})_{i \in \mathbb{N}}$ par :*

$$T_{0,N} = \Lambda Q_N \quad \text{et} \quad T_{i+1,N} = -H^{-1}T_{i,N} \quad \forall i \in \mathbb{N}$$

ou de manière équivalente

$$T_{i,N} = H^{-i} \Lambda Q_N.$$

Il est trivial de remarquer que, pour tout $j \in \mathbb{N}$:

$$Vect(T_{i,N})_{0 \leq i \leq j} \subset Ker(H_N^{j+1})$$

En fait, si on se restreint aux fonctions régulières, il y a même égalité. Dans la suite, on aura besoin des asymptotiques suivantes en $+\infty$:

$$T_{1,N} = \begin{cases} c_N y^{4-N} + O(y^{3-N}) & \text{pour } N \neq 4 \\ -4\log y + 2 + O\left(\frac{\log^2 y}{y^2}\right) & \text{pour } N = 4. \end{cases} \quad (3.1.14)$$

où $(c_N)_{N \neq 4}$ sont des constantes. Il faut faire attention au fait que les notations ici ne correspondent pas à celle de l'article [94].

Avant de construire la solution approchée, rappelons que le spectre de l'opérateur H_N possède une unique valeur propre strictement négative, et que le supplémentaire orthogonal au sous-espace propre associé à cette valeur propre est stable par H_N . Ainsi, la stratégie de gestion de la direction négative, que nous expliquerons par la suite, reste entièrement valable pour établir la généralisation du théorème 2.2.1 à toutes les vitesses. (Voir le théorème 3.0.2).

3.1.3 Construction d'une solution approchée

En omettant dans un premier temps le dernier terme de (3.1.8), on cherche une fonction proche de Q_N , que l'on notera $Q_{N,b}$ minimisant dans un certain sens l'erreur Ψ_b définie par :

$$\Psi_{b,N} = \partial_s Q_{N,b} + b_1(s)\Lambda Q_{N,b} - \Delta Q_{N,b} - Q_{N,b}^{\frac{N+2}{N-2}} \quad (3.1.15)$$

Pour commencer, afin de comprendre la difficulté soulevée par le problème en dimension supérieure ou égale à 7, nous prendrons $L = 1$, et donc nous allégerons la notation b_1 en b . Nous allons chercher une solution $Q_{N,b}$ sous la forme d'une solution polynomiale en b :

$$Q_{N,b}(s, y) = Q_N(y) + b(s)T_{1,N}(y) + b^2(s)S_{2,N}(y) + \dots \quad (3.1.16)$$

Volontairement, nous n'écrivons pas tout les termes, mais seulement ceux utiles à la compréhension de la méthode. Il est crucial de voir que $S_{2,N}$ ne dépend pas de la variable s . En outre, rappelons que b est une fonction positive, tendant vers zéro au moment de l'explosion. Dans le type de régime explosif dans lequel on se place, on a même $b_s = O(b^2)$. Si on injecte le développement (3.1.16) dans (3.1.15), on peut écrire l'erreur $\Psi_{b,N}$, en classant par puissance de b comme :

$$\begin{aligned} \Psi_{b,N} &= \Delta Q_N + Q_N^{\frac{N+2}{N-2}} \\ &+ b(H_N T_{1,N} + \Lambda Q_N) \\ &+ b^2 (H_N S_{1,N} + \Lambda T_{1,N} + \mathcal{N}(T_{1,N})) + b_s T_{1,N} \\ &+ b^3 \Lambda S_{2,N} + \dots \end{aligned}$$

avec \mathcal{N} un opérateur non linéaire, tel que $\mathcal{N}(T_{1,N}) = o(\Lambda T_{1,N})$. Du fait que Q_N est une solution stationnaire de (2.0.1), et du choix du profil $T_{1,N}$ donné dans la définition 3.1.1, les deux premières lignes du membre de droite sont nuls.

Il nous reste à déterminer $S_{2,N}$. Nous avons vu auparavant que nous voulions une loi de modulation $b_s = O(b^2)$. Nous allons la chercher de la forme

$$b_s = \beta_N b^2 + o(b^2),$$

et déterminer la constante β_N optimale, afin de minimiser la croissance asymptotique de $S_{2,N}$, tout en minimisant, pour certaines normes que nous verrons par la suite, la taille de $\Psi_{b,N}$ dans la région parabolique $y \leq \frac{1}{\sqrt{b}}$. Le terme $b^2 S_{2,N}$ deviendra alors négligeable devant $b T_{1,N}$, dans cette zone de l'espace, dépendant du temps via le paramètre b . $S_{2,N}$ est déterminé par :

$$H_N S_{2,N} = -\Lambda T_{1,N} - \mathcal{N}(T_{1,N}) - \beta_N T_{1,N}. \quad (3.1.17)$$

Rappelons que $T_{1,N}$ vérifie les asymptotiques suivantes :

$$T_{1,N} = \begin{cases} c_N y^{4-N} + O(y^{3-N}) & \text{pour } N \neq 4 \\ -4\log y + 2 + O\left(\frac{\log^2 y}{y^2}\right) & \text{pour } N = 4. \end{cases} \quad (3.1.18)$$

On en déduit celle de $\Lambda T_{1,N}$:

$$\Lambda T_{1,N} = \begin{cases} c_N \left(3 - \frac{N}{2}\right) y^{4-N} + O(y^{3-N}) & \text{pour } N \neq 4 \\ -4\log y - 2 + O\left(\frac{\log^2 y}{y^2}\right) & \text{pour } N = 4. \end{cases} \quad (3.1.19)$$

Comme $\mathcal{N}(T_{1,N}) = o(\Lambda T_{1,N})$, le choix optimal pour β_N est celui qui vérifie :

$$\Lambda T_{1,N} + \beta_N T_{1,N} = o(T_{1,N}),$$

soit

$$\beta_N = \left(\frac{N}{2} - 3\right). \quad (3.1.20)$$

ce qui nous donne comme loi pour le paramètre b :

$$b_s = \left(\frac{N}{2} - 3\right) b^2 (1 + o(1)). \quad (3.1.21)$$

Le paramètre de modulation $b(s)$ étant recherché sous la forme d'une fonction positive, tendant vers 0 au moment de l'explosion, le problème de la dimension $N \geq 7$ apparaît ici distinctement, de même que l'indécision pour la dimension $N = 6$.

Pour la dimension 4, on introduira par la suite un terme de radiation, afin de supprimer la constante, dans l'asymptotique du terme $S_{1,4}$. Cela ajoutera un terme logarithmique à la loi de b , que l'on retrouve dans la loi de $\lambda(t)$ prédite dans [26]. Nous verrons cela plus en détail dans la preuve consacrée à cette dimension.

3.1.4 Construction toutes vitesses pour $N \leq 5$.

On suppose maintenant que $N \leq 5$. Nous allons seulement donner les grandes lignes de la construction dans le but d'obtenir de façon formelle toutes les vitesses, en particulier le système dynamique satisfait par $b(s) = (b_1(s), \dots, b_L(s))$.

Pour obtenir la première vitesse, nous avons décomposé la partie principale de la solution approchée $Q_{N,b}$ sur le noyau de H_N^2 . Sur le même principe, nous allons décomposer la solution approchée $Q_{N,b}$ sur le noyau de H_N^{L+1} pour obtenir la $L^{\text{ième}}$ vitesse :

$$Q_{N,b}(y) = Q_N(y) + \sum_{i=1}^L b_i(s) T_{i,N}(y) + \sum_{i=2}^{L+2} S_{i,N}(s, y) \quad (3.1.22)$$

Dans le régime que l'on va étudier, les paramètres $b_i(s)$ vérifient $b_1^i \lesssim b_i \lesssim b_1^i$. Cela entraînera qu'au bord de la zone parabolique ($y \sim B_0 = \frac{1}{\sqrt{b}}$), les profils $(b_i T_{N,i})_{1 \leq i \leq L}$ seront de même taille que Q_N , à une correction logarithmique près pour la dimension 4. Nous chercherons les profils $(S_{i,N}(s, y))_{2 \leq i \leq L+2}$ sous la forme $P_{i,N}(s) \mathcal{S}_{i,N}(y)$, où la partie temporelle $P_{i,N}(s)$, est un polynôme en $(b_j)_{1 \leq j \leq L}$, tel qu'elle soit de taille équivalente à b_1^i . A l'instar de ce que nous avions vu pour la première vitesse, les profils $(S_{i,N})_{2 \leq i \leq L+2}$ seront négligeables devant Q_N dans la zone parabolique.

Pour obtenir cette dernière propriété pour les profils $S_{i,N}$, nous devons injecter la décomposition (3.1.22) dans l'expression de l'erreur (3.1.15), et classer les termes par puissance de b_1^i . Rappelons que nous nous plaçons sous l'hypothèse $b_1^i \lesssim b_i \lesssim b_1^i$.

Ainsi, nous obtenons pour les termes en $O(b_1)$:

$$b_1(H_N T_{1,N} + \Lambda Q_N) = 0,$$

l'annulation provenant de notre choix du profil stationnaire $T_{1,N}$. En $O(b_1^2, b_2, (b_1)_s)$, nous avons :

$$(b_1)_s T_{1,N} + b_1^2 \Lambda T_{1,N} + b_2 H_N T_{2,N} + H_N S_{2,N} = b_1^2 \mathcal{N}(T_{1,N}).$$

Or $H_N T_{2,N} = -T_{1,N}$. De plus, le terme non linéaire $\mathcal{N}(T_{1,N})$ est un terme d'ordre inférieur devant $T_{1,N}$. En se servant des asymptotiques des profils $T_{1,N}$ (3.1.18), et $\Lambda T_{1,N}$ (3.1.19), nous avons l'équivalence suivante :

$$(b_1)_s T_{1,N} + b_1^2 \Lambda T_{1,N} + b_2 H_N T_{2,N} \sim \left\{ (b_1)_s + \left(3 - \frac{N}{2} \right) b_1 - b_2 \right\} T_{1,N}$$

En prenant comme contrainte sur les lois de modulations :

$$(b_1)_s + \left(3 - \frac{N}{2} \right) b_1^2 - b_2 = 0, \quad (3.1.23)$$

le profil $S_{2,N}$ devient alors d'ordre négligeable.

Plus généralement, à la $k^{\text{ième}}$ étape, nous obtenons :

$$(b_k)_s T_{k,N} + b_1 b_k \Lambda T_{k,N} + b_{k+1} H_N T_{k+1,N} + H_N S_{k+1,N} = b_1^{k+1} \mathcal{N}(T_{1,N}, \dots, T_{k,N}).$$

avec la convention $b_{L+1} = 0$. L'équivalence

$$\Lambda T_{k,N} \sim \left(2k + 1 - \frac{N}{2} \right) T_{k,N} \quad (3.1.24)$$

nous amène à prendre comme contrainte sur la loi $b_k(s)$:

$$(b_k)_s + \left(2k + 1 - \frac{N}{2} \right) b_1 b_k - b_{k+1} = 0. \quad (3.1.25)$$

A nouveau, en dimension 4, une correction logarithmique sera nécessaire. Le paramètre $b(s)$ vérifie ainsi le système dynamique suivant :

$$\begin{cases} (b_k)_s + \left(2k+1 - \frac{N}{2} + \frac{2\delta_{4,N}}{|\log b_1|}\right) b_1 b_k - b_{k+1} = 0, & 1 \leq k \leq L, \quad b_{L+1} \equiv 0, \\ -\frac{\lambda_s}{\lambda} = b_1, \\ \frac{ds}{dt} = \frac{1}{\lambda^2}. \end{cases} \quad (3.1.26)$$

où $\delta_{i,j}$ est le symbole de Kronecker. En recherchant une solution explicite de ce système sous la forme

$$b_k(s) = \frac{c_{k,N}}{s^k} + \frac{d_{k,N}}{s^k \log s}$$

où bien entendu le dernier terme est nul en dimension 3 et 5, on obtient les relations suivantes :

$$\begin{cases} c_{1,N} = \frac{L}{2L+1-\frac{N}{2}}, \\ c_{k+1,N} = -\frac{(L-k)(1-\frac{N}{2})}{2L+1-\frac{N}{2}} c_{k,N}, & 1 \leq k \leq L-1, \end{cases} \quad (3.1.27)$$

$$\begin{cases} d_{1,4} = -\frac{2L}{(2L-1)^2}, \\ d_{k+1,4} = -\frac{L-k}{2L-1} d_{k+1} + \frac{4L(L-k)}{(2L-1)^2} c_k, & 1 \leq k \leq L-1. \end{cases} \quad (3.1.28)$$

Plaçons nous dans le cadre de la dimension 3 ou 5, pour éviter la correction logarithmique de la dimension 4. Le lecteur intéressé par la dimension 4 pourra consulter le lemme 2.8 de l'article [90]. On doit donc résoudre le système suivant, pour obtenir la loi de λ :

$$\begin{cases} -\frac{\lambda_s}{\lambda} = \frac{c_{1,N}}{s}, \\ \frac{ds}{dt} = \frac{1}{\lambda^2}. \end{cases}$$

On en déduit de la première ligne de ce système que $(s^{c_{1,N}} \lambda(s))_s = 0$. Couplé à la seconde ligne, ce résultat implique que :

$$\lambda(t) = (T-t)^{\frac{c_{1,N}}{2c_{1,N}-1}} = (T-t)^{\frac{L}{\frac{N}{2}-1}}$$

On retrouve les différentes vitesses d'explosion $\lambda(t)$ prédites par Filippas, Herrero et Velázquez dans [26].

3.2 Construction en dimension 4

Dorénavant, nous n'étudierons plus que le problème en dimension 4, pour la première vitesse, cadre du théorème 2.2.1. Dans cette partie, bien que nous venons de voir la méthode de construction de la solution approchée, nous allons la reprendre, notamment pour introduire la radiation corrigeant logarithmiquement les lois λ et b . De plus, les profils introduits seront localisés, via une fonction cut-off, pour tenir compte de leur croissance problématique, en dehors de la zone parabolique.

Proposition 3.2.1 (Construction de la solution approchée). *Soit $M > 0$ suffisamment grand. Alors, il existe une constante universelle $b^*(M)$ suffisamment petite, tel que la suite soit vraie. Soit $b \in]0, b^*(M)[$. Posons $B_1 = \frac{|\log b|}{\sqrt{b}}$. Alors il existe des profils T_1, T_2 et T_3 , tel que*

$$Q_b(y) = Q(y) + bT_1(y) + b^2T_2(y) + b^3T_3(y) = Q(y) + \alpha(y) \quad (3.2.1)$$

générant une erreur

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b - (Q_b)^3 + b\Lambda Q_b \quad (3.2.2)$$

qui satisfait les contrôles suivants :

$$\int_{y \leq 2B_1} |H\Psi_b|^2 \lesssim b^4 |\log b|^2, \quad (3.2.3)$$

$$\int_{y \leq 2B_1} \frac{1}{1+y^8} |\Psi_b|^2 \lesssim b^6, \quad (3.2.4)$$

$$\int_{y \leq 2B_1} |H^2 \Psi_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (3.2.5)$$

ainsi que le flux :

$$\frac{(\Psi_b, H\Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \quad (3.2.6)$$

où Φ_M est donnée par (3.3.1).

Avant de donner les éléments importants de la démonstration de cette proposition, voici quelques remarques que nous pouvons faire :

- Les notations adoptées ici sont celles de l'article [94]. En particulier, les profils T_2 et T_3 correspondent aux profils $S_{2,4}$, et $S_{3,4}$ de la précédente partie, et non aux éléments du noyau de H^4 . Le profil T_1 restant bien celui défini précédemment.
- L'erreur (3.2.2) est légèrement différente de celle donnée par (3.1.15). Nous tenons déjà compte ici du fait que $(b_1)_s = -b_1^2(1 + o(1))$.
- Les bornes (3.2.3), (3.2.4) et (3.2.5) seront utilisées par la suite dans la gestion de la taille, en un certain sens, du terme d'erreur, via une méthode d'énergie. En particulier, on peut interpréter la troisième borne comme une version raffinée de la seconde, où la structure de l'opérateur H^2 a été prise en compte. Cela a permis d'obtenir sur la borne (3.2.4) un gain de l'ordre de $\frac{1}{|\log b|^2}$, absolument crucial pour assurer que l'erreur commise ne perturbe pas la dynamique explosive de la solution approchée.
- Nous reviendrons plus tard sur la signification du terme de flux (3.2.6), terme jouant un rôle primordial dans les lois de modulations.

Donnons à présent une preuve allégée de la proposition 3.2.1, déclinée en plusieurs étapes .

Etape 1 Calcul de l'erreur.

En injectant la décomposition de Q_b (3.2.1) dans l'expression du terme d'erreur (3.2.2), nous obtenons, en classant par puissance de b :

$$\begin{aligned}\Psi_b &= b(HT_1 + \Lambda Q) \\ &+ b^2(HT_2 - T_1 + \Lambda T_1 - 3QT_1^2) \\ &+ b^3(HT_3 - 2T_2 + \Lambda T_2 - 6QT_1T_2 - T_1^3) \\ &+ b^4\Lambda T_3 + R_1(T_1, T_2, T_3),\end{aligned}\tag{3.2.7}$$

le terme R_1 étant un polynôme en T_i , contenant des termes d'ordre $(b^j)_{j \geq 4}$. Nous devons maintenant déterminer les profils T_i pour obtenir une erreur Ψ_b satisfaisant les contrôles de la proposition 3.2.1.

Etape 2 Détermination de T_1 .

Nous avons déjà vu cette étape dans les parties antérieures. Rappelons que T_1 satisfait l'équation $HT_1 + \Lambda Q = 0$, et a le comportement asymptotique suivant en $+\infty$:

$$\Lambda^i T_1(y) = -4 \left(\log y - \frac{1}{2} + i \right) + O \left(\frac{(\log y)^2}{y^2} \right), \quad \text{for } 0 \leq i \leq 2.\tag{3.2.8}$$

Dans ce problème, le comportement des profils à l'origine ne posant aucun problème, nous omettrons systématiquement leur gestion.

Etape 3 Introduction de la radiation Σ_b .

La seconde ligne du membre de droite de l'erreur (3.2.7) amène à première vue à l'équation :

$$HT_2 = T_1 - \Lambda T_1 - 3QT_1^2\tag{3.2.9}$$

Le choix $b_s = -b^2$ permettant dès à présent d'exécuter la croissance logarithmique du membre de droite de (3.2.9). Ainsi,

$$T_1 - \Lambda T_1 - 3QT_1^2 = -4 + O \left(\frac{|\log y|^2}{y^2} \right) \quad \text{quand } y \rightarrow +\infty.$$

Pour autant, si T_2 est solution de (3.2.9), alors les profils Q et $b^2 T_2$ seront de même ordre de grandeur au bord de la zone parabolique, ce qui n'est pas le but recherché. Pour arriver à nos fins, *ie* avoir un profil T_2 d'ordre inférieur, il va nous falloir introduire un terme de radiation Σ_b . C'est l'introduction de ce dernier qui modifiera la loi de modulation de $b(s)$, en rajoutant un terme logarithmique. Le profil T_2 que nous déterminerons dans l'étape suivante, sera alors solution de :

$$HT_2 = T_1 - \Lambda T_1 - 3QT_1^2 + \Sigma_b.\tag{3.2.10}$$

Pour bien comprendre le choix effectué pour la radiation, point relativement délicat de la preuve, redonnons quelques propriétés de l'opérateur H .

En ne considérant que les fonctions radiales, le noyau de cet opérateur ne possède qu'une unique direction régulière ΛQ . A l'aide des fonctions de Green, on peut calculer explicitement la seconde solution de

$$H\Gamma = \Gamma_{rr} + \frac{3\Gamma_r}{r} + 3Q^2\Gamma = 0. \quad (3.2.11)$$

Cette solution est singulière en l'origine, comme le montrent ses asymptotes :

$$\Gamma(y) = \begin{cases} O\left(\frac{1}{y^2}\right) & \text{quand } y \rightarrow 0, \\ \frac{1}{16} + O\left(\frac{\log y}{y^2}\right) & \text{quand } y \rightarrow +\infty. \end{cases} \quad (3.2.12)$$

On est ainsi en mesure de connaître l'ensemble des solutions régulières en l'origine, pour une fonction f régulière donnée, de $Hu = f$, qui sont données par

$$u = \Gamma(y) \int_0^y f(\tau) \Lambda Q(\tau) \tau^3 d\tau - \Lambda Q(y) \int_0^y f(\tau) \Gamma(\tau) \tau^3 d\tau + c \Lambda Q(y), \quad c \in \mathbb{R}. \quad (3.2.13)$$

Revenons au choix de Σ_b . D'après ce que nous avons vu, nous cherchons Σ_b vérifiant les contraintes suivantes :

– Asymptotique en $+\infty$:

$$\Sigma_b(y) = 4 + O\left(\frac{|\log y|^i}{y^2}\right) \quad (3.2.14)$$

– Respect des bornes (3.2.3), (3.2.4) et (3.2.5) de la proposition 3.2.1. L'ajout de la radiation ammène dans l'erreur Ψ_b le terme $b^2\Sigma_b$. La radiation Σ_b doit donc satisfaire les bornes suivantes :

$$\int |H\Sigma_b|^2 \lesssim |\log b|, \quad \int \frac{1}{1+y^8} |\Sigma_b|^2 \lesssim b^2, \quad \int |H^2\Sigma_b|^2 \lesssim \frac{b^2}{|\log b|^2}. \quad (3.2.15)$$

Posons

$$c_b = \frac{64}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} = \frac{2}{|\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right)\right) \quad (3.2.16)$$

et

$$d_b = c_b \int_0^{B_0} \chi_{\frac{B_0}{4}} \Lambda Q \Gamma = O\left(\frac{1}{b|\log b|}\right). \quad (3.2.17)$$

Alors la fonction Σ_b solution de

$$H\Sigma_b = c_b \chi_{\frac{B_0}{4}} \Lambda Q + d_b H[(1 - \chi_{3B_0}) \Lambda Q] \quad (3.2.18)$$

donnée par

$$\Sigma_b(y) = \Gamma(y) \int_0^y c_b \chi_{\frac{B_0}{4}} (\Lambda Q)^2 - \Lambda Q(y) \int_0^y c_b \chi_{\frac{B_0}{4}} \Gamma \Lambda Q + d_b (1 - \chi_{3B_0}) \Lambda Q.$$

est la bonne candidate. Σ_b est bien connue loin de la frontière de la zone parabolique :

$$\Sigma_b = \begin{cases} c_b T_1 & \text{for } y \leq \frac{B_0}{4} \\ 64\Gamma & \text{for } y \geq 6B_0 \end{cases} \quad (3.2.19)$$

Etape 4 Détermination des profils T_2 et T_3 .

Nous prenons donc comme profil T_2 la solution donnée par (3.2.13) de

$$HT_2 = T_1 - \Lambda T_1 - 3QT_1^2 + \Sigma_b.$$

avec Σ_b la radiation introduite à l'étape 3 (3.2.18). Si on arrête la décomposition de Q_b à T_2 , le terme dominant de l'erreur devient $b^3\Lambda T_2$. Or on ne pourra contrôler assez finement Ψ_b , à cause de la borne ci-dessous, elle-même optimale.

$$\int_{y \leq 2B_1} |H^2 b^3 \Lambda T_2|^2 \lesssim \frac{b^6}{|\log b|}.$$

C'est pourquoi, l'introduction du profil T_3 , annulant la troisième ligne du membre de droite de l'erreur (3.2.7), s'avère nécessaire, et nous donnera les bornes souhaitées.

Etape 5 Calcul du flux.

Il est préférable de garder cette étape pour le moment où nous introduirons la direction Φ_M . Pour autant, mentionnons que l'erreur est de la forme :

$$\Psi_b = b^2 \Sigma_b + b^4 \Lambda T_3 (1 + o(1)) \quad (3.2.20)$$

Ceci constitue la fin de la preuve de la proposition 3.2.1.

Nous avons pour l'instant décomposé notre solution approchée sur des profils stationnaires, $Q_b = Q(y) + b(s)T_1(y) + b^2(s)T_2(y) + b^3(s)T_3(y)$. Or, nous pouvons remarquer qu'à s fixé, les profils $b^i T_i$ ont une croissance problématique hors de la zone parabolique $y \sim B_0$. En effet, le profil Q devient alors négligeable devant les profils $b^i T_i$, ce qui n'a pas de sens dans l'optique de décrire une solution proche en un certain sens du soliton. C'est pourquoi, on va localiser les profils, via une fonction cut-off dépendant du temps. Pour autant, il ne faut pas que l'erreur engendrée par la localisation soit supérieure à celle acquise à la proposition 3.2.1 précédente.

Soit χ la fonction cut-off régulière tel que :

$$\chi(y) = \begin{cases} 1 & \text{pour } y \leq 1, \\ 0 & \text{pour } y \geq 2. \end{cases}$$

et sa version paramétrée par $B > 0$:

$$\chi_B(y) = \chi\left(\frac{y}{B}\right).$$

Alors, nous avons la proposition suivante :

Proposition 3.2.2 (Localisation des profils au voisinage de B_1). *Soit une application de classe C^1 , $s \mapsto b(s)$ définie sur $[0, s_0]$ avec la borne à priori $\forall s \in [0, s_0]$,*

$$0 < b(s) < b^*(M), \quad |b_s| \leq 10b^2. \quad (3.2.21)$$

Soit la solution approchée localisée

$$\tilde{Q}_b(s, y) = Q + b\tilde{T}_1 + b^2\tilde{T}_2 + b^3\tilde{T}_3 = Q + \tilde{\alpha}$$

où

$$\tilde{T}_i = \chi_{B_1} T_i, \quad 1 \leq i \leq 3.$$

Alors

$$\partial_s \tilde{Q}_b - \Delta \tilde{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \tilde{Q}_b - \tilde{Q}_b^3 = Mod(t) + \tilde{\Psi}_b \quad (3.2.22)$$

avec

$$Mod(t) = - \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2) \quad (3.2.23)$$

et où $\tilde{\Psi}_b$ satisfait les bornes sur $[0, s_0]$:

(i) Contrôle de l'erreur :

$$\int |H\tilde{\Psi}_b|^2 \lesssim b^4 |\log b|^2, \quad (3.2.24)$$

$$\int \frac{1}{1+y^8} |\tilde{\Psi}_b|^2 \lesssim b^6, \quad (3.2.25)$$

$$\int |H^2 \tilde{\Psi}_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (3.2.26)$$

(ii) Calcul du flux : Soit Φ_M donnée par (3.3.1), alors :

$$\frac{(H\tilde{\Psi}_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \quad (3.2.27)$$

La preuve de cette proposition n'apporte rien à la compréhension de la méthode. Notons néanmoins plusieurs choses :

- Nous avons introduit le terme Mod , terme de modulation, contenant les erreurs que l'on commet en prenant $b = -\frac{\lambda_s}{\lambda}$ et $b_s = -b^2$. Cela ne change pas la définition de l'erreur par rapport à celle (3.2.2) de la proposition précédente, mais permet de mettre en valeur le fait que ces erreurs sont à première approximation, supportée respectivement par ΛQ et T_1 , éléments du noyau de H^2 . De plus, l'erreur tient compte aussi de la nouvelle dépendance temporelle des profils \tilde{T}_i .
- Dans la preuve de cette proposition, on va séparer l'erreur $\tilde{\Psi}_b$, en deux parties, l'une contenant les termes liés à la localisation $\tilde{\Psi}_b^{(1)}$, et l'autre, les termes provenant de la dérivation temporelle des profils \tilde{T}_i , que l'on note \tilde{R} , en cohérence avec les notations de l'article [94]. Le meilleur contrôle que l'on arrive à effectuer sur $\tilde{\Psi}_b^{(1)}$ est :

$$\int |H^2 \tilde{\Psi}_b^{(1)}|^2 \lesssim \frac{b^6}{|\log b|^2} + \frac{b^2 |\log b|^2}{B_1^8} + \frac{b^4 |\log b|^2}{B_1^4} \lesssim \frac{b^6}{|\log b|^2},$$

ce qui nous dicte le choix de $B_1 = \frac{|\log b|}{\sqrt{b}}$.

Indiquons, par souci de complétude, qu'une seconde localisation sera aussi nécessaire, autour de B_0 , pour gérer l'énergie \mathcal{E}_2 , que nous verrons par la suite. Cet artifice étant sûrement plus technique qu'autre chose, nous ne nous épancherons pas sur le sujet.

3.3 Solution exacte et théorie de la modulation

Nous allons ici décomposer une solution de notre problème (3.1.1) en deux parties. Une sera la solution approchée $\tilde{Q}_{b,\lambda}$, que l'on vient de construire, et l'autre sera un terme d'erreur que l'on va devoir maîtriser. La notation $\tilde{Q}_{b,\lambda}$ permet d'exhiber la double dépendance de la solution approchée avec les paramètres temporels b et λ . Grâce à cette flexibilité, nous allons pouvoir fixer deux directions d'orthogonalité pour le terme d'erreur, assurant ainsi l'unicité de la décomposition. Après avoir écrit l'équation satisfaite par l'erreur, nous la projeterons sur les deux directions d'orthogonalités, afin d'obtenir les équations des paramètres de modulation, appelées plus succinctement équations de modulation.

3.3.1 Directions d'orthogonalité

Comme nous l'avons remarqué lors de la présentation de la proposition sur la localisation des profils T_i , l'inexactitude des équations de modulation est supportée principalement par les directions ΛQ et T_1 , *i.e.* par les solutions régulières du noyau de H^2 . Le principal inconvénient de ces fonctions est de décroître trop doucement, et en particulier de ne pas être pas $L^2_{rad}(\mathbb{R}^4)$.

Soit $M > 0$, une constante suffisamment grande. Introduisons donc la direction :

$$\Phi_M = \chi_M \Lambda Q - c_M H(\chi_M \Lambda Q) \quad (3.3.1)$$

avec

$$c_M = \frac{(\chi_M \Lambda Q, T_1)}{(H(\chi_M \Lambda Q), T_1)}.$$

Le second terme du membre de droite de (3.3.1) est un terme correctif, permettant d'assurer l'orthogonalité entre Φ_M et T_1 , ainsi qu'entre ΛQ et $H\Phi_M$. De plus par construction nous avons :

$$\int |\Phi_M|^2 \lesssim |\log M|, \quad (3.3.2)$$

et les produits scalaires suivants :

$$(\Lambda Q, \Phi_M) = (-HT_1, \Phi_M) = (\chi_M \Lambda Q, \Lambda Q) = 64 \log M (1 + o_{M \rightarrow +\infty}(1)). \quad (3.3.3)$$

Les directions Φ_M et $H\Phi_M$ sont une bonne approximation du noyau de H^2 au sens où elles assurent la coercitivité de cet opérateur, comme le montre le lemme suivant :

Lemme 3.3.1 (Coercitivité de H^2). *Soit $M \geq 1$ une constante suffisamment grande. Soit Φ_M donné par (3.3.1). Alors il existe une constante universelle $C(M) > 0$ telle que pour tout $u \in H^4_{rad}(\mathbb{R}^4)$ satisfaisant les conditions d'orthogonalités suivantes :*

$$(u, \Phi_M) = 0, \quad (Hu, \Phi_M) = 0$$

alors :

$$\begin{aligned} & \int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2} \\ & + \int \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} \int \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ & + \int \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} \leq C(M) \int |H^2(u)|^2. \end{aligned} \quad (3.3.4)$$

3.3.2 Unicité de la décomposition

Soit u une solution du problème (3.1.1). Alors on va décomposer cette solution comme suit :

$$u(t, r) = \frac{1}{\lambda(t)} (\tilde{Q}_b + \varepsilon) \left(t, \frac{r}{\lambda(t)} \right) = \frac{1}{\lambda(t)} \tilde{Q}_{b(t)} \left(t, \frac{r}{\lambda(t)} \right) + w(t, r). \quad (3.3.5)$$

La première décomposition sera privilégiée avec les nouvelles variables (s, y) alors que la seconde sera utilisée avec les variables d'origine.

En imposant les deux conditions d'orthogonalité suivantes :

$$(\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0, \quad (3.3.6)$$

on assure l'unicité de la décomposition (3.3.5), avec la condition initiale de petitesse sur b :

$$0 < b(0) < b^*(M) \ll 1,$$

$b^*(M)$ étant une constante universelle choisie suffisamment petite. En effet, c'est une simple conséquence du théorème des fonctions implicites, l'application

$$(b, \lambda) \rightarrow (u, \Phi_M) = \left(\left(\tilde{Q}_{b(t)} \right)_{\lambda(t)}, \Phi_M \right)$$

étant de classe \mathcal{C}^1 et d'après le choix de Φ_M et la construction :

$$\begin{vmatrix} \left(\frac{\partial}{\partial \lambda} (\tilde{Q}_b)_\lambda, \Phi_M \right) & \left(\frac{\partial}{\partial b} (\tilde{Q}_b)_\lambda, \Phi_M \right) \\ \left(\frac{\partial}{\partial \lambda} (\tilde{Q}_b)_\lambda, H\Phi_M \right) & \left(\frac{\partial}{\partial b} (\tilde{Q}_b)_\lambda, H\Phi_M \right) \end{vmatrix}_{\lambda=1, b=0} = \begin{vmatrix} (-\Lambda Q, \Phi_M) & 0 \\ 0 & (T_1, H\Phi_M) \end{vmatrix} = -(\Lambda Q, \Phi_M)^2 \neq 0. \quad (3.3.7)$$

3.3.3 Équation vérifiée par l'erreur ε

Rappelons les formules de passage des variables d'origine aux nouvelles variables suivantes :

$$u(t, r) = \frac{1}{\lambda} v(s, y), \quad \partial_t u = \frac{1}{\lambda^2} \left(\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v \right)_\lambda.$$

En injectant la décomposition (3.3.5) dans l'équation (3.1.1), on obtient l'équation suivante pour ε :

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon + H \varepsilon = F - Mod = \mathcal{F}, \quad (3.3.8)$$

où nous redonnons l'expression du terme de modulation introduit à la proposition 3.2.2 :

$$Mod(t) = - \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2)$$

et où nous avons noté

$$F = -\tilde{\Psi}_b + L(\varepsilon) + N(\varepsilon), \quad (3.3.9)$$

avec L l'opérateur linéaire provenant de la différence entre la linéarisation autour du soliton Q , et de la solution approchée \tilde{Q}_b :

$$L(\varepsilon) = 3(\tilde{Q}_b^2 - Q^2)\varepsilon, \quad (3.3.10)$$

et N étant un terme purement non linéaire :

$$N(\varepsilon) = 3\tilde{Q}_b\varepsilon^2 + \varepsilon^3. \quad (3.3.11)$$

3.3.4 Equations de modulation

Nous sommes à présent en mesure d'établir les équations de modulation, ou plus précisément, de borner l'imprécision commise sur ces équations, notamment en fonction de l'erreur ε . Voici la proposition que l'on obtient :

Proposition 3.3.2 (Equations de modulation). *Soit $\mathcal{E}_4 = \int |H^2(\varepsilon)|^2$. Alors :*

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}, \quad (3.3.12)$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log M}} \left(\sqrt{\mathcal{E}_4} + \frac{b^2}{|\log b|} \right). \quad (3.3.13)$$

Pour obtenir ces bornes nous allons successivement projeter l'équation (3.3.8) sur les deux directions d'orthogonalité. Au vu de l'importance de cette proposition dans notre analyse, et de la taille de la preuve, nous la donnons ici en très grande partie.

Commençons par poser

$$V(t) = |b_s + b^2| + \left| \frac{\lambda_s}{\lambda} + b \right|,$$

mesurant ainsi l'ordre de grandeur de l'erreur commise dans les équations de modulation. Procédons en deux étapes :

Etape 1 Loi de b :

Nous projetons l'équation (3.3.8) sur la direction $H\Phi_M$, puis nous estimons chaque terme.

$$\begin{aligned} & (\partial_s \varepsilon, H\Phi_M) + (H\varepsilon, H\Phi_M) - \left(\frac{\lambda_s}{\lambda} \Lambda \varepsilon, H\Phi_M \right) \\ &= - \left(\tilde{\Psi}_b, H\Phi_M \right) - (Mod(t), H\Phi_M) + (L(\varepsilon), H\Phi_M) + (N(\varepsilon), H\Phi_M). \end{aligned}$$

D'abord, d'après le choix d'orthogonalité (3.3.6), nous avons l'annulation suivante :

$$(\partial_s \varepsilon, H\Phi_M) = \partial_s (H\varepsilon, \Phi_M) = 0. \quad (3.3.14)$$

De plus, en utilisant l'inégalité de Cauchy-Schwartz, on obtient :

$$(H\varepsilon, H\Phi_M) = \left(H^2 \varepsilon, \Phi_M \right) \lesssim \|H^2 \varepsilon\|_{L^2} \|\Phi_M\|_{L^2} \lesssim \sqrt{\mathcal{E}_4 \log M}. \quad (3.3.15)$$

Le terme de modulation (3.2.23) est estimé en utilisation la localisation $\text{Supp}(\Phi_M) \subset [0, 2M]$ provenant de (3.3.1), ainsi que les choix de construction :

$$\begin{aligned} (H(Mod(t)), \Phi_M) &= - \left(b + \frac{\lambda_s}{\lambda} \right) (H\Lambda \tilde{Q}_b, \Phi_M) + (b_s + b^2) (H(\tilde{T}_1 + 2b\tilde{T}_2), \Phi_M) \\ &= -(\Lambda Q, \Phi_M)(b_s + b^2) + O(c(M)b|V(t)|). \end{aligned}$$

On montre sans difficulté le rôle mineur joué par les autres termes :

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H\Phi_M \right) \right| \lesssim C(M)b(\mathcal{E}_4 + |V(t)|).$$

D'après les produits scalaires (3.3.3), nous pouvons conclure en supposant le flux (3.2.27) :

$$\begin{aligned} b_s + b^2 &= \frac{(\tilde{\Psi}_b, H\Phi_M)}{(\Lambda Q, \Phi_M)} + O\left(\frac{\sqrt{\log M \mathcal{E}_4}}{\log M}\right) + O(C(M)b|V(t)|) \\ &= -\frac{2b^2}{|\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right)\right) + O\left(\sqrt{\frac{\mathcal{E}_4}{\log M}} + C(M)b|V(t)|\right) \end{aligned} \quad (3.3.16)$$

Comme $\Psi_b = b^2 \Sigma_b + b^4 \Lambda T_3(1 + o(1))$, (3.2.27) se montre facilement, chose que nous n'avions pas faite dans le cadre de la preuve de la proposition 3.2.1 :

$$\begin{aligned} \frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} &= \frac{1}{(\Lambda Q, \Phi_M)} \left[\left(-b^2 c_b \chi_{\frac{B_0}{4}} \Lambda Q, \Phi_M \right) + O(C(M)b^3) \right] \\ &= -c_b b^2 + O(C(M)b^3) = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \end{aligned}$$

Etape 2 Loi de λ .

Maintenant, projetons (3.3.8) sur la direction Φ_M afin d'obtenir :

$$(\text{Mod}(t), \Phi_M) = -(\tilde{\Psi}_b, \Phi_M) - (\partial_s \varepsilon + H\varepsilon, \Phi_M) - \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M\right).$$

Le choix des directions d'orthogonalité implique (3.3.6) :

$$(\partial_s \varepsilon + H\varepsilon, \Phi_M) = 0.$$

Comme ci-dessus, nous avons la borne suivante pour les termes d'ordre mineur :

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M \right) \right| \lesssim C(M)b\sqrt{\mathcal{E}_4}.$$

Le terme de modulation devient :

$$\begin{aligned} (\text{Mod}(t), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda}\right) (\Lambda \tilde{Q}_b, \Phi_M) + (b_s + b^2) (\tilde{T}_1 + 2b\tilde{T}_2, \Phi_M) \\ &= -4\log M (1 + o_{M \rightarrow +\infty}(1)) \left(\frac{\lambda_s}{\lambda} + b\right) + O(C(M)b|V(t)|), \end{aligned}$$

et connaissant précisément la radiation dans la zone parabolique (3.2.19), le terme d'erreur $\tilde{\Psi}_b$ donne :

$$\left| (\tilde{\Psi}_b, \Phi_M) \right| \lesssim b^2 |(\Sigma_b, \Phi_M)| + O(C(M)b^3) = c_b b^2 |(T_1, \Phi_M)| + O(C(M)b^3) = O(C(M)b^3).$$

D'où :

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim b^3 C(M) + b C(M) O\left(\sqrt{\mathcal{E}_4} + |V(t)|\right). \quad (3.3.17)$$

Avec (3.3.16), nous obtenons la borne

$$|V(t)| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}.$$

L'injection de cette borne dans (3.3.17) implique (3.3.12) et (3.3.13). Ceci conclue la preuve de la proposition 3.3.2.

Par rapport aux bornes de cette proposition, on constate qu'elles dépendent de b , mais aussi de M , et de \mathcal{E}_4 . Au moment de conclure la preuve du théorème 2.2.1, nous prendrons $M = \frac{1}{b^\delta}$, avec δ suffisamment petit. L'erreur qu'engendrera le fait de ne plus prendre M constante sera contrôlée sans difficulté. Par contre, tout l'enjeu de la suite de la démonstration est de trouver des solutions initiales, satisfaisant la borne : $\mathcal{E}_4 \lesssim \frac{b^4}{|\log b|^2}$, nous permettant de conclure. D'après les conditions d'orthogonalité vérifiées par ε , et le lemme 3.3.1, contrôler \mathcal{E}_4 au cours du temps permet aussi de contrôler des normes à poids sur ε , et ainsi de maîtriser cette erreur dans un espace fonctionnel adéquat.

3.4 Contrôle de l'erreur ε via une méthode d'énergie

Pour obtenir toutes les informations contenues dans le théorème 2.2.1, notamment celles concernant le profil asymptotique, nous avons besoin de contrôler l'erreur avec trois types de normes adaptées à notre étude :

- Norme d'énergie :

$$\mathcal{E}_1 = \int |\nabla \varepsilon|^2 \quad (3.4.1)$$

- Normes de Sobolev d'ordre supérieur :

$$\mathcal{E}_2 = \int |H\varepsilon|^2 = \int |\varepsilon_2|^2, \quad \mathcal{E}_4 = \int |H^2\varepsilon|^2 = \int |\varepsilon_4|^2 \quad (3.4.2)$$

avec $\varepsilon_i = H^i \varepsilon$ for $i \in \{2; 4\}$.

Les conditions d'orthogonalité (3.3.6) assurent que les objets définis ci-dessus sont bien des normes pour ε .

Pour contrôler ces quantités, nous allons procéder à une méthode d'énergie, basée sur une récurrence. On va supposer des bornes sur ces normes, qui seront vérifiées initialement, ainsi que sur un intervalle de temps $[0, t_1[$, avec $t_1 > 0$. Via des fonctionnelles de Lyapounov, nous montrerons que ces bornes peuvent être en fait améliorées, et donc en particulier, elles restent vraies jusqu'au temps de vie maximal de la solution. Dans ce problème, la gestion de la direction d'instabilité, correspondant à la direction ψ liée à l'unique valeur propre $-\zeta$ strictement négative de l'opérateur H , modifie très légèrement la conclusion de cette méthode, mais n'en change pas la philosophie générale.

3.4.1 Description des solutions initiales

On a déjà vu que la petitesse de $b(0)$, *ie*

$$0 < b(0) < b^*(M) \ll 1, \quad (3.4.3)$$

assurait l'unicité de la décomposition (3.3.5). En outre, on impose à la solution initiale les contraintes suivantes, qui sont très larges par rapport à celle de la récurrence :

$$|\mathcal{E}_1(0)| \leq b(0)^2, \quad (3.4.4)$$

et

$$|\mathcal{E}_2(0)| + |\mathcal{E}_4(0)| \leq b(0)^{10}. \quad (3.4.5)$$

De plus, nous devons absolument contrôler la valeur de la projection de la solution initiale sur la direction d'instabilité ψ , qui non contrôlée peut anéantir la dynamique explosive recherchée. Pour cela posons :

$$\kappa(t) = (\varepsilon(t), \psi), \quad (3.4.6)$$

et

$$a^+ = \kappa(0) = (\varepsilon(0), \psi). \quad (3.4.7)$$

Nous allons imposer :

$$|a^+| \leq \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}. \quad (3.4.8)$$

Pour autant, les contraintes que nous venons d'énoncer ne constituent en rien une description parfaite de l'ensemble des données initiales évoluant selon la dynamique du théorème 2.2.1. Nous verrons notamment apparaître une contrainte supplémentaire sur a^+ .

3.4.2 Bornes de la récurrence

Soit K une constante suffisamment grande, indépendante de M . On suppose les bornes suivantes :

– Contrôle de $b(t)$:

$$0 < b(t) < 10b(0). \quad (3.4.9)$$

– Contrôle de l'erreur :

$$\int |\nabla \varepsilon(t)|^2 \leq 10\sqrt{b(0)}, \quad (3.4.10)$$

$$|\mathcal{E}_2(t)| \leq Kb^2(t)|\log b(t)|^5, \quad (3.4.11)$$

$$|\mathcal{E}_4(t)| \leq K \frac{b^4(t)}{|\log b(t)|^2}. \quad (3.4.12)$$

– Borne *a priori* du mode instable :

$$|\kappa(t)| \leq 2 \frac{b(t)^{\frac{5}{2}}}{|\log b(t)|}. \quad (3.4.13)$$

La régularité des solutions de (3.1.1), couplée aux conditions initiales requises sur $\varepsilon(0)$, prouve l'existence d'un temps $t_1 > 0$, tel que les bornes ci-dessus soient vraies sur l'intervalle $[0, t_1]$. De plus on a la régularité des paramètres : $(\lambda, b) \in C^1([0, t_1], \mathbb{R}_+^* \times \mathbb{R})$.

Définition 3.4.1 (Temps de sortie). Soit $a^+ \in \left[-2 \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}; 2 \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|} \right]$. Soit $T(a^+)$ le temps de vie maximal de la solution de (2.2.1), vérifiant initialement les bornes (3.4.3), (3.4.4), (3.4.5) et (3.4.8). On appelle temps de sortie le temps $T_1(a^+) > 0$, définie comme la borne supérieure de $T \in (0, T(a^+))$, tel que pour tout $t \in [0; T]$, les estimées (3.4.9), (3.4.10), (3.4.11), (3.4.12) et (3.4.13) soient vraies.

3.4.3 Résultat principal :

L'objectif de la suite de notre étude sera de démontrer la proposition suivante.

Proposition 3.4.2. *Il existe $a^+ \in \left[-2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}; 2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|} \right]$ tel que*

$$T_1(a^+) = T(a^+) \quad (3.4.14)$$

et la solution correspondante de (2.2.1) explose en temps fini selon la dynamique décrite par le théorème 2.2.1.

Un des points clés de la preuve de cette proposition est l'utilisation des variables d'origine, et non des nouvelles variables, associées au changement d'échelle. Rappelons que dans les variables d'origine, la solution u du problème (3.1.1) que nous recherchons se décompose de façon unique comme :

$$u(t, r) = \frac{1}{\lambda(t)} \tilde{Q}_{b(t)} \left(t, \frac{r}{\lambda(t)} \right) + w(t, r).$$

Nous noterons par la suite :

$$f_\lambda(r) = \frac{1}{\lambda} f \left(\frac{r}{\lambda} \right). \quad (3.4.15)$$

De plus, nous adapterons la notation au terme de potentiel, en raison de sa nature quadratique :

$$\tilde{V}(y) = \frac{1}{\lambda^2} V \left(\frac{r}{\lambda} \right).$$

L'équation (3.3.8), vérifiée par l'erreur ε devient :

$$\partial_t w + H_\lambda w = \frac{1}{\lambda^2} \mathcal{F}_\lambda, \quad (3.4.16)$$

où \mathcal{F} a été définie par :

$$\mathcal{F} = -\tilde{\Psi}_b + L(\varepsilon) + N(\varepsilon) - Mod.$$

Maintenant, de la même façon que précédemment, définissons $w_i = H_\lambda^i w$ pour $i \in \{2; 4\}$, qui vérifient respectivement :

$$\partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right), \quad (3.4.17)$$

$$\partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda \left(\partial_t \tilde{V} w \right) + H_\lambda^2 \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right). \quad (3.4.18)$$

De plus, nous avons les égalités suivantes :

$$\frac{1}{\lambda^2} \mathcal{E}_2 = \int |Hw|^2 = \int |w_2|^2, \quad \frac{1}{\lambda^6} \mathcal{E}_4 = \int |H^2 w|^2 = \int |w_4|^2. \quad (3.4.19)$$

Alors qu'on ne sait pas maîtriser les variations de \mathcal{E}_4 et de \mathcal{E}_2 , il en est autrement pour $\int |w_4|^2$ et $\int |w_2|^2$, comme le montrent les deux propositions suivantes :

Proposition 3.4.3 (Contrôle au niveau \dot{H}^4). *Nous avons :*

$$\frac{d}{dt} \left\{ \frac{1}{\lambda^6} \left[\mathcal{E}_4 + O \left(\sqrt{b} \frac{b^4}{|\log b|^2} \right) \right] \right\} \leq C \frac{b}{\lambda^8} \left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|} \sqrt{\mathcal{E}_4} \right] \quad (3.4.20)$$

pour une constante universelle $C > 0$ indépendante de M et de la constante de la récurrence K de (3.4.9), (3.4.10), (3.4.11), (3.4.12), pour un $b^*(M)$ définie par (3.4.3) choisie suffisamment petit.

Proposition 3.4.4 (Contrôle au niveau \dot{H}^2). *Nous avons :*

$$\frac{d}{dt} \hat{\mathcal{E}}_2 \lesssim \frac{b^3 |\log b|^2}{\lambda^4} \quad (3.4.21)$$

Ici, $\hat{\mathcal{E}}_2$ est une version modifiée de \mathcal{E}_2 , venant d'une différence de localisation de la solution approchée autour de B_0 , au lieu de B_1 . Ce souci technique ne présentant pas un intérêt majeur, nous le passerons sous silence. Le lecteur intéressé pourra consulter [94].

D'après les équations de modulation décrites dans la proposition 3.3.2, nous montrerons que l'intégrale $\int_0^t \frac{b^5}{\lambda^8 |\log b|^2}$ est divergente au voisinage du point d'explosion, et plus précisément :

$$\int_0^t \frac{b^5}{\lambda^8 |\log b|^2} \lesssim \frac{b^4}{\lambda^6 |\log b|^2}, \quad (3.4.22)$$

ce qui nous permettra de raffiner le contrôle de l'erreur au niveau \dot{H}^4 (3.4.12). C'est en partie l'objet de la proposition ci-dessous :

Proposition 3.4.5 (Bornes améliorées). *Soit K , la constante de récurrence de (3.4.9), (3.4.10), (3.4.11), et (3.4.12) choisie suffisamment grande. Alors, $\forall t \in [0, t_1]$:*

$$0 \leq b(t) \leq 2b(0), \quad (3.4.23)$$

$$\int |\nabla \varepsilon|^2 \leq \sqrt{b(0)}, \quad (3.4.24)$$

$$|\mathcal{E}_2(t)| \leq \frac{K}{2} b^2(t) |\log b(t)|^5, \quad (3.4.25)$$

$$|\mathcal{E}_4(t)| \leq \frac{K}{2} \frac{b^4(t)}{|\log b(t)|^2}. \quad (3.4.26)$$

La borne (3.4.23) découlera naturellement des équations de modulation, la borne (3.4.24) de la décroissance de l'énergie. Cela permet d'assurer que les contraintes (3.4.9), (3.4.10), (3.4.11), et (3.4.12) ne seront pas enfreintes. Seule la direction d'instabilité (3.4.13) peut être à l'origine d'une sortie précoce du régime recherché, avant le temps de vie maximal de la solution.

Le lemme suivant nous apporte des informations sur l'évolution du mode instable :

Lemme 3.4.6 (Contrôle du mode instable). *Pour tout $t \in [0, T_1(a^+)]$, nous avons*

$$\left| \frac{d\kappa}{ds} - \zeta \kappa \right| \leq \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (3.4.27)$$

Un argument classique de continuité, provenant du lemme 6 de [20], nous permettra d'assurer l'existence d'un a^+ tel que la borne (3.4.13) soit respectée pour tout temps, ce qui conclura la proposition 3.4.2.

Donnons maintenant quelques éléments de preuve, notamment de la proposition 3.4.3 décrivant le contrôle de l'erreur au niveau \dot{H}^4 . Nous montrerons ensuite la borne (3.4.26) de la proposition 3.4.5, avant d'étudier le mode instable.

3.4.4 Preuve de la proposition 3.4.3

Tout d'abord, en se servant des équations vérifiées par w_2 (3.4.17), et par w_4 (3.4.18), on va prouver le lemme suivant, décrivant une identité d'énergie modifiée au niveau \dot{H}^4 :

Lemme 3.4.7 (Identité d'énergie au niveau \dot{H}^4).

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left\{ w_4^2 - 2\partial_t \tilde{\mathcal{V}} w w_4 \right\} \\ &= - \int w_4 H_\lambda w_4 + \int (\partial_t \tilde{\mathcal{V}})^2 w_2 w - \int \partial_{tt} \tilde{\mathcal{V}} w w_4 + \int H_\lambda (\partial_t \tilde{\mathcal{V}} w) \partial_t \tilde{\mathcal{V}} w \\ &+ \int w_4 H_\lambda^2 \frac{1}{\lambda^2} \mathcal{F}_\lambda - \int H_\lambda (\partial_t \tilde{\mathcal{V}} w) H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) - \int \partial_t \tilde{\mathcal{V}} w_4 \frac{1}{\lambda^2} \mathcal{F}_\lambda. \end{aligned} \quad (3.4.28)$$

Preuve du lemme 3.4.7 : Tout d'abord, en se servant de l'équation (3.4.18) :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int w_4^2 &= \int w_4 \partial_t w_4 \\ &= \int w_4 \left(-H_\lambda w_4 - \partial_t \tilde{\mathcal{V}} w_2 - H_\lambda (\partial_t \tilde{\mathcal{V}} w) + H_\lambda^2 \frac{1}{\lambda^2} \mathcal{F}_\lambda \right). \end{aligned}$$

Nous allons maintenant traiter séparément les second et troisième termes.

$$\begin{aligned} & - \int w_4 \partial_t \tilde{\mathcal{V}} w_2 \\ &= \int \partial_t \tilde{\mathcal{V}} w_2 \left(\partial_t w_2 + \partial_t \tilde{\mathcal{V}} w - H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &= \int (\partial_t \tilde{\mathcal{V}})^2 w_2 w - \int \partial_t \tilde{\mathcal{V}} w_2 H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) + \int \partial_t \tilde{\mathcal{V}} \partial_t \left(\frac{w_2^2}{2} \right) \\ &= \int (\partial_t \tilde{\mathcal{V}})^2 w_2 w - \int \partial_t \tilde{\mathcal{V}} w_2 H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) - \frac{1}{2} \int \partial_{tt} \tilde{\mathcal{V}} w_2^2 + \frac{1}{2} \frac{d}{dt} \left(\int \partial_t \tilde{\mathcal{V}} w_2^2 \right). \end{aligned}$$

Le troisième terme du membre de droite devient :

$$\begin{aligned} - \int w_4 H_\lambda (\partial_t \tilde{\mathcal{V}} w) &= \int H_\lambda (\partial_t \tilde{\mathcal{V}} w) \left(\partial_t w_2 + \partial_t \tilde{\mathcal{V}} w - H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &= \int H_\lambda (\partial_t \tilde{\mathcal{V}} w) \left(\partial_t \tilde{\mathcal{V}} w - H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &+ \frac{d}{dt} \int H_\lambda (\partial_t \tilde{\mathcal{V}} w) w_2 - \int \partial_t \left[H_\lambda (\partial_t \tilde{\mathcal{V}} w) \right] w_2. \end{aligned}$$

Enfin, on modifie le dernier terme par :

$$-\int \partial_t H_\lambda (\partial_t \tilde{\mathcal{V}} w) w_2 = \int (\partial_t \tilde{\mathcal{V}})^2 w_2 w - \int \partial_{tt} \tilde{\mathcal{V}} w w_4 - \int \partial_t \tilde{\mathcal{V}} \partial_t w w_4,$$

et

$$\begin{aligned} -\int \partial_t \tilde{\mathcal{V}} \partial_t w w_4 &= \int \partial_t \tilde{\mathcal{V}} \left(w_2 - \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) w_4 \\ &= -\int \partial_t \tilde{\mathcal{V}} \frac{1}{\lambda^2} \mathcal{F}_\lambda w_4 + \int \partial_t \tilde{\mathcal{V}} w_2 \left(-\partial_t w_2 - \partial_t \tilde{\mathcal{V}} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &= -\int \partial_t \tilde{\mathcal{V}} \frac{1}{\lambda^2} \mathcal{F}_\lambda w_4 + \int \partial_t \tilde{\mathcal{V}} w_2 \left(-\partial_t \tilde{\mathcal{V}} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &\quad - \frac{1}{2} \frac{d}{dt} \left(\int \partial_t \tilde{\mathcal{V}} w_2^2 \right) + \frac{1}{2} \int \partial_{tt} \tilde{\mathcal{V}} w_2^2. \end{aligned}$$

Ceci conclut la preuve du lemme 3.4.7.

Il faut estimer désormais chaque terme présent dans l'identité d'énergie (3.4.28). Il faut séparer les termes de la seconde ligne, termes quadratiques, et les termes de la dernière ligne liés à la fonction \mathcal{F}_λ .

Comme nous l'avons déjà vu, les conditions d'orthogonalités (3.3.6) imposées à l'erreur assure la coercitivité des opérateurs H , et H^2 . En utilisant des inégalités de Hardy, et en se plaçant dans le cadre des bornes (3.4.9), (3.4.10), (3.4.11), (3.4.12) et (3.4.13), on montre le lemme suivant, nous décrivant des estimées d'interpolation :

Lemme 3.4.8 (Estimées d'interpolation). *Nous avons, avec des constantes dépendant a priori de M :*

$$\begin{aligned} &\int \frac{|H(\varepsilon)|^2}{y^4(1+|\log y|^2)} + \int \frac{|\partial_y H(\varepsilon)|^2}{y^2(1+|\log y|^2)} + \int \frac{|\partial_{yy} H(\varepsilon)|^2}{(1+|\log y|^2)} \\ &+ \int \frac{|\varepsilon|^2}{y^4(1+y^4)(1+|\log y|^2)} + \int \frac{|\partial_y^i \varepsilon|^2}{y^{8-2i}(1+|\log y|^2)} \lesssim \mathcal{E}_4, \quad 1 \leq i \leq 4, \end{aligned} \tag{3.4.29}$$

$$\int \frac{|\varepsilon|^2}{y^4(1+|\log y|^2)} + \int \frac{|\partial_y^i \varepsilon|^2}{y^{4-2i}} \lesssim \mathcal{E}_2, \quad 1 \leq i \leq 2, \tag{3.4.30}$$

$$\int_{y \geq 1} \frac{1+|\log y|^C}{y^{8-2i}(1+|\log y|^2)} |\partial_y^i \varepsilon|^2 \lesssim b^4 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \tag{3.4.31}$$

$$\int_{y \geq 1} \frac{1+|\log y|^C}{y^{6-2i}(1+|\log y|^2)} |\partial_y^i \varepsilon|^2 \lesssim b^3 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \tag{3.4.32}$$

$$\int_{y \geq 1} \frac{1+|\log y|^C}{y^{4-2i}(1+|\log y|^2)} |\partial_y^i \varepsilon|^2 \lesssim b^2 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 1, \tag{3.4.33}$$

$$\left\| \varepsilon (1+|\log y|^C) \right\|_{L_{y \geq 1}^\infty}^2 \lesssim b^2 |\log b|^{C_1(C)}, \tag{3.4.34}$$

$$\|\varepsilon\|_{L_{y \leq 1}^\infty}^2 + \|\partial_y \varepsilon\|_{L_{y \leq 1}^\infty}^2 + \|y \partial_{yy} \varepsilon\|_{L_{y \leq 1}^\infty}^2 \lesssim \frac{b^4}{|\log b|^2}, \quad (3.4.35)$$

$$\|y \partial_y \varepsilon\|_{L^\infty}^2 \lesssim b^2 |\log b|^5, \quad (3.4.36)$$

$$\left\| \frac{\varepsilon}{1+y} \right\|_{L^\infty}^2 + \|\partial_y \varepsilon\|_{L^\infty}^2 \lesssim b^3 |\log b|^C, \quad (3.4.37)$$

$$\left\| \frac{\varepsilon}{1+y^2} \right\|_{L^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{1+y} \right\|_{L^\infty}^2 + \|\partial_{yy} \varepsilon\|_{L_{y \geq 1}^\infty}^2 \lesssim b^4 |\log b|^C. \quad (3.4.38)$$

Avec l'ensemble de ces estimées, nous traitons facilement le cas des termes quadratiques, avec les bornes suivantes sur le potentiel :

$$|\partial_t \tilde{\mathcal{V}}| \lesssim \frac{b}{\lambda^4} \frac{1}{1+y^4} \quad |\partial_{tt} \tilde{\mathcal{V}}| \lesssim \frac{b}{\lambda^6} \frac{1}{1+y^4}. \quad (3.4.39)$$

Maintenant, supposons :

$$\int \frac{1}{1+y^8} \mathcal{F}^2 \lesssim \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right] \quad (3.4.40)$$

$$\int |H^2 \mathcal{F}|^2 \lesssim b^2 \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right]. \quad (3.4.41)$$

Alors, nous pouvons traiter les termes de la dernière ligne comme suit :

$$\begin{aligned} & \left| \int w_4 H_\lambda^2 \frac{1}{\lambda^2} \mathcal{F}_\lambda - \int H_\lambda (\partial_t \tilde{V} w) H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) - \int \partial_t \tilde{V} w_4 \frac{1}{\lambda^2} \mathcal{F}_\lambda \right| \\ & \lesssim \frac{1}{\lambda^8} \left\{ \int \left(\varepsilon_4^2 + b^2 \frac{\varepsilon^2}{1+y^8} \right)^{\frac{1}{2}} \left(\int |H^2 \mathcal{F}|^2 \right)^{\frac{1}{2}} + b \left(\int \varepsilon_4^2 \right)^{\frac{1}{2}} \left(\int \frac{1}{1+y^8} \mathcal{F}^2 \right)^{\frac{1}{2}} \right\} \\ & \lesssim \frac{b}{\lambda^8} \left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|} \sqrt{\mathcal{E}_4} \right], \end{aligned}$$

ce qui conclut la preuve de la proposition 3.4.3, à condition de démontrer (3.4.40) et (3.4.41). Rappelons que :

$$\mathcal{F} = -\tilde{\Psi}_b - \text{Mod}(t) + L(\varepsilon) + N(\varepsilon).$$

Nous prouvons les bornes (3.4.40) et (3.4.41) en étudiant successivement le cas des 4 termes ci-dessus. Nous laissons ces preuves de côté. Elles sont basées essentiellement sur les estimées issues de la construction de la solution approchée \tilde{Q}_b pour le terme de modulation, et sur les estimées d'interpolations pour les deux derniers. Le cas de $\tilde{\Psi}_b$ a été exactement traité dans la proposition 3.2.2.

Nous voyons ici toute l'importance du gain $\frac{1}{|\log b|^2}$ obtenu en se servant de la structure de l'opérateur H^2 . C'est une des différences fondamentales avec le cas de l'équation des ondes traité dans [41]. Le gain $\frac{1}{|\log b|^2}$ peut être obtenu dès le niveau \dot{H}^2 dans le cadre de l'équation des ondes, ce que nous n'avons pas réussi à obtenir pour l'équation de la chaleur, comme le montre le contrôle \dot{H}^2 (3.2.3) de l'erreur $\tilde{\Psi}_b$.

3.4.5 Amélioration de la borne (3.4.12) :

Montrons comment la proposition 3.4.3, couplée aux équations de modulation (3.3.12) et (3.3.13), permettent d'améliorer la borne (3.4.12) en la borne (3.4.26) de la proposition 3.4.5. En intégrant en temps l'inégalité de la proposition 3.4.3, on obtient $\forall t \in [0, t_1]$,

$$\begin{aligned} \mathcal{E}_4(t) &\leq 2 \left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \left[\mathcal{E}_4(0) + C \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} \right] + \frac{b^4(t)}{|\log b(t)|^2} \\ &+ C \left[1 + \frac{K}{\log M} + \sqrt{K} \right] \lambda^6(t) \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\tau \end{aligned} \quad (3.4.42)$$

pour une constante $C > 0$ indépendante de M .

Soient les constantes

$$\alpha_1 = 2 - \frac{C_1}{\sqrt{\log M}}, \quad \alpha_2 = 2 + \frac{C_2}{\sqrt{\log M}} \quad (3.4.43)$$

pour des constantes C_1, C_2 suffisamment grandes. Couplées aux équations de modulation (3.3.12), (3.3.13) et de la borne (3.4.12), on trouve :

$$\begin{aligned} \frac{d}{ds} \left\{ \frac{|\log b|^{\alpha_i} b}{\lambda} \right\} &= \frac{|\log b|^{\alpha_i}}{\lambda} \left[\left(1 - \frac{\alpha_i}{|\log b|} \right) b_s - \frac{\lambda_s}{\lambda} b \right] \\ &= \frac{|\log b|^{\alpha_i}}{\lambda} \left[\left(1 - \frac{\alpha_i}{|\log b|} \right) b_s + b^2 + O \left(\frac{b^3}{|\log b|} \right) \right] \\ &= \left(1 - \frac{\alpha_i}{|\log b|} \right) \frac{|\log b|^{\alpha_i}}{\lambda} \left[b_s + b^2 \left(1 + \frac{\alpha_i}{|\log b|} + O \left(\frac{1}{|\log b|^2} \right) \right) \right] \\ &\begin{cases} \leq 0 & \text{for } i = 1 \\ \geq 0 & \text{for } i = 2. \end{cases} \end{aligned}$$

En intégrant entre 0 et t , on obtient :

$$\frac{b(0)}{\lambda(0)} \left(\frac{|\log b(0)|}{|\log b(t)|} \right)^{\alpha_2} \leq \frac{b(t)}{\lambda(t)} \leq \frac{b(0)}{\lambda(0)} \left(\frac{|\log b(0)|}{|\log b(t)|} \right)^{\alpha_1}. \quad (3.4.44)$$

Cela prouve en particulier, en utilisant la borne initiale (3.4.5) et la borne (3.4.9) :

$$\left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \mathcal{E}_4(0) \leq (b(t)|\log b(t)|^{\alpha_2})^6 \frac{\mathcal{E}_0}{(b(0)|\log b(0)|^{\alpha_2})^6} \leq \frac{b^4(t)}{|\log b(t)|^2}, \quad (3.4.45)$$

$$\begin{aligned} C \left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} &\lesssim \left(\frac{b(t)|\log b(t)|^{\alpha_2}}{b(0)|\log b(0)|^{\alpha_2}} \right)^6 \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} \\ &\lesssim C(b(t))^{4+\frac{1}{4}} \leq \frac{b^4(t)}{|\log b(t)|^2}. \end{aligned} \quad (3.4.46)$$

Comme $b = -\lambda \lambda_t + O \left(\frac{b^2}{|\log b|} \right)$, on trouve exactement :

$$\begin{aligned} \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\sigma &= \frac{1}{6} \left[\frac{b^4}{\lambda^6 |\log b|^2} \right]_0^t - \frac{1}{6} \int_0^t \frac{b_t b^3}{\lambda^6 |\log b|^2} \left(4 + \frac{2}{|\log b|} \right) d\tau \\ &+ O \left(\int_0^t \frac{b}{\lambda^8} \frac{b^5}{|\log b|^2} d\tau \right) \end{aligned}$$

qui implique en utilisant $|b_s + b^2| \lesssim \frac{b^2}{|\log b|}$ et la borne (3.4.12) :

$$\lambda^6(t) \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\sigma \lesssim \left[1 + O\left(\frac{1}{|\log b_0|}\right) \right] \frac{b^4(t)}{|\log b(t)|^2}.$$

En injectant cette dernière inégalité, ainsi que celles (3.4.45) et (3.4.46) dans (3.4.42), on trouve

$$\mathcal{E}_4(t) \leq C \frac{b^4(t)}{|\log b(t)|^2} \left[1 + \frac{K}{\log M} + \sqrt{K} \right]$$

pour une constante $C > 0$ indépendante de K et M , ce qui prouve (3.4.26) pour K suffisamment grand, indépendant M .

Nous ne montrerons pas ici les autres bornes de la proposition 3.4.5. Reste la gestion du mode instable à examiner pour conclure la preuve de la proposition 3.4.2.

3.4.6 Gestion du mode instable

Pour commencer, redonnons l'inégalité du lemme 3.4.6 décrivant la dynamique du mode instable, où nous rappelons que $\kappa(s) = (\varepsilon(s), \psi)$, ψ étant une direction propre associée à la valeur propre négative $-\zeta$.

$$\left| \frac{d\kappa}{ds} - \zeta \kappa \right| \leq \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (3.4.47)$$

Cette inégalité s'obtient en projetant l'équation de l'erreur ε (3.3.8) sur la direction ψ .

Nous allons prouver la proposition 3.4.2 par contradiction. Supposons que pour tout $a^+ \in \left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}\right]$:

$$T_1(a^+) < T(a^+). \quad (3.4.48)$$

D'après la définition 3.4.1 du temps de sortie, cela entraîne que :

$$\left| \kappa(T_1(a^+)) \right| = 2 \frac{b(T_1(a^+))^{\frac{5}{2}}}{|\log b(T_1(a^+))|}. \quad (3.4.49)$$

Un argument classique de continuité développé dans [20] au lemme 3 assure la continuité de l'application suivante :

$$\begin{aligned} \left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right] &\rightarrow \mathbb{R}^+ \\ a^+ &\rightarrow T_1(a^+). \end{aligned}$$

et donc de :

$$\Phi = \begin{cases} \left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right] &\rightarrow \mathbb{R} \\ a^+ &\rightarrow \kappa(T_1(a^+)) \frac{|\log b(T_1(a^+))|}{2b(T_1(a^+))^{\frac{5}{2}}}. \end{cases}$$

Comme la direction d'instabilité est responsable de la sortie du régime, la contrainte (3.4.49) nous donne :

$$\Phi \left(\left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right] \right) \subset \{-1; 1\}. \quad (3.4.50)$$

Or, la dynamique du mode instable d'écrite au lemme 3.4.6 (3.4.47) entraîne que :

$$\Phi \left(-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right) = -1; \quad \Phi \left(2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right) = 1. \quad (3.4.51)$$

Une contradiction apparaît alors avec la continuité de l'application Φ . Ceci conclue la preuve de la proposition 3.4.2.

3.5 Conclusion de la preuve du théorème 2.2.1 :

On est désormais en mesure de conclure la preuve du théorème 2.2.1. Pour cela, on a déjà démontré les bornes suivantes pour une solution de (3.1.1), pour tout $[0, T_1(a^+))$, $0 < T_1(a^+) \leq +\infty$:

- Estimations d'énergie

$$\forall t \in [0, T_1(a^+)] \quad \mathcal{E}_1(t) \leq \delta(b^*), \quad \mathcal{E}_2(t) \lesssim b(t)^2 |\log b(t)|^5, \quad \mathcal{E}_4(t) \lesssim \frac{b(t)^4}{|\log b(t)|^2} \quad (3.5.1)$$

- Lien entre les lois $b(t)$ $\lambda(t)$: il existe deux constantes $\alpha_1, \alpha_2 > 0$ telles que :

$$C(u_0)b(t)|\log b(t)|^{\alpha_1} \leq \lambda(t) \leq C'(u_0)b(t)|\log b(t)|^{\alpha_2} \quad (3.5.2)$$

Etape 1 Temps fini d'explosion.

Afin d'alléger les notations, notons $T \leq +\infty$ le temps de vie de la solution u . De (3.5.1), (3.5.2),

$$-\frac{d}{dt}\sqrt{\lambda} = -\frac{1}{2\lambda\sqrt{\lambda}}\frac{\lambda_s}{\lambda} \gtrsim \frac{b}{\lambda\sqrt{\lambda}} \gtrsim C(u_0) > 0$$

et ainsi λ s'annule en temps fini

$$T < +\infty.$$

Les bornes (3.5.1), couplées à un argument standard de caractère localement bien posé dans H^4 de la solution, assure que l'explosion correspond à

$$\lambda(t) \rightarrow 0 \quad \text{quand} \quad t \rightarrow T \quad (3.5.3)$$

et donc avec (3.5.2) :

$$\lambda(T) = b(T) = 0. \quad (3.5.4)$$

Etape 2 Obtention de la loi de vitesse de formation de la singularité.
Tout d'abord, les bornes (3.5.1) permettent de raffiner les loi de modulations :

$$\left| b + \frac{\lambda_s}{\lambda} \right| \lesssim \frac{b^2}{|\log b|} \quad (3.5.5)$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log M}} \frac{b^2}{|\log b|}. \quad (3.5.6)$$

Posons

$$B_\delta = \frac{1}{b^\delta}. \quad (3.5.7)$$

Comme M est une constante suffisamment grande, et que b évolue lentement, on va prendre $M = B_\delta$. Bien évidemment, cela va créer de nouvelles erreurs, que l'on gère sans difficulté. Comme il s'agit d'un point technique, qui n'apporte rien de très intéressant dans cette preuve allégée, nous renvoyons le lecteur intéressé à [94]. (3.5.6) devient ainsi :

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log B_\delta}} \frac{b^2}{|\log b|} \lesssim \frac{b^2}{|\log b|^{\frac{3}{2}}}. \quad (3.5.8)$$

En utilisant que $\lim_{s \rightarrow +\infty} b(s) = 0$, et en intégrant en temps, on trouve :

$$b(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right). \quad (3.5.9)$$

L'estimée (3.5.5) nous donnant l'information sur la mesure de l'incertitude entre la loi b et la quantité $\frac{-\lambda_s}{\lambda}$, nous concluons :

$$-\frac{\lambda_s}{\lambda} = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right).$$

On peut réécrire cette égalité par

$$\left| \frac{d}{ds} \log \left(\frac{s \lambda(s)}{(\log s)^2} \right) \right| \lesssim \frac{1}{s |\log s|^{\frac{3}{2}}}$$

et en intégrant en temps, on montre l'existence de $\kappa(u_0) > 0$ tel que :

$$\frac{s \lambda(s)}{(\log s)^2} = \frac{1}{\kappa(u_0)} \left[1 + O\left(\frac{1}{|\log s|^{\frac{3}{2}}}\right) \right].$$

En prenant le logarithme de l'expression ci-dessus, on obtient la borne :

$$|\log \lambda| = |\log s| \left[1 + O\left(\frac{|\log \log s|}{\log s}\right) \right]$$

et donc

$$\frac{1}{s} = \kappa(u_0) \frac{\lambda}{|\log \lambda|^2} (1 + o(1)).$$

En injectant cela dans (3.5.9), on obtient :

$$-\lambda\lambda_t = -\frac{\lambda_s}{\lambda} = \frac{1}{s}(1+o(1)) = \kappa(u_0)\frac{\lambda}{|\log\lambda|^2}(1+o(1)). \quad (3.5.10)$$

D'où

$$-|\log\lambda|^2\lambda_t = \kappa(u_0)(1+o(1)).$$

En intégrant en temps entre t et T avec $\lambda(T) = 0$, on trouve la loi

$$\lambda(t) = \kappa(u_0)\frac{T-t}{|\log(T-t)|^2}[1+o(1)],$$

et (3.0.3) est prouvé.

Etape 3 Mesure de l'énergie concentrée.

Reste à prouver (3.0.2), (2.2.9). Pour cela, nous allons utiliser la borne suivante, qui est une simple conséquence de (3.5.1) et de notre construction :

$$\forall t \in [0, T], \quad \|\Delta\tilde{u}(t, x)\|_{L^2} \leq C(v_0). \quad (3.5.11)$$

où

$$\tilde{u}(t, x) = u(t, x) - \frac{1}{\lambda(t)}Q\left(\frac{x}{\lambda(t)}\right) \quad (3.5.12)$$

La régularité de $\tilde{u}(t, x)$ hors de l'origine est une conséquence classique de l'effet régularisant des équations paraboliques. Ainsi, il existe $u^* \in \dot{H}^1$ tel que

$$\forall R > 0, \quad \nabla u(t) \rightarrow \nabla u^* \text{ in } L^2(|x| \geq R) \text{ as } t \rightarrow T.$$

En outre, la dissipation de l'énergie montre que $\tilde{u} \in \dot{H}^1$, et donc d'après la décomposition (3.5.12) et la borne uniforme (3.5.11) :

$$\nabla\tilde{u}(t) \rightarrow \nabla u^* \text{ in } L^2 \text{ and } \Delta u^* \in L^2$$

ce qui conclut la preuve de (3.0.2), (2.2.9) et donc du théorème 2.2.1.

Chapitre 4

Modèle de Patlak-Keller-Segel d'une accumulation de cellules liée à une chimiotaxie.

Dans cette partie, nous allons étudier le problème parabolique-elliptique suivant :

$$(PKSpe) \quad \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla \phi_u), \\ \phi_u = \frac{1}{2\pi} \log|x| \star u \\ u|_{t=0} = u_0 > 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad (4.0.1)$$

qui est une version simplifiée du modèle parabolique-parabolique :

$$(PKSpp) \quad \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v), \\ c \partial_t v = \Delta v - u \\ u|_{t=0} = u_0 > 0, \\ v|_{t=0} = v_0 > 0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, c > 0. \quad (4.0.2)$$

Après avoir présenté les origines biologiques de ce modèle, nous exposerons quelques résultats importants, notamment liés à l'existence d'une masse critique, jouant un effet seuil sur la dynamique des solutions. Enfin, après avoir énoncé le théorème principal de [91], nous le commenterons, et donnerons quelques perspectives de prolongement de cette recherche.

4.1 Origine biologique du modèle de Patlak-Keller-Segel :

Jusqu'à la découverte en 1935 par K.B. Raper du myxomycète *Dictyostelium discoideum*, la classification des êtres vivants distinguait deux grands groupes parfaitement distincts : les êtres unicellulaires et les êtres pluricellulaires. Le myxomycète *Dictyostelium discoideum* possède dans son cycle de vie quelque chose de remarquable, depuis très largement étudié. Il passe de l'état d'amibe, unicellulaire, à une organisation pluricellulaire comme on peut le voir sur la figure 4.1.

Lors de la formation du "fruiting body", sorte de champignon, un cinquième de la population d'amibes se sacrifie pour en former la structure, afin que les autres puissent repasser à l'état unicellulaire, sous forme de spores, et être disséminés pour atteindre si possible un milieu plus

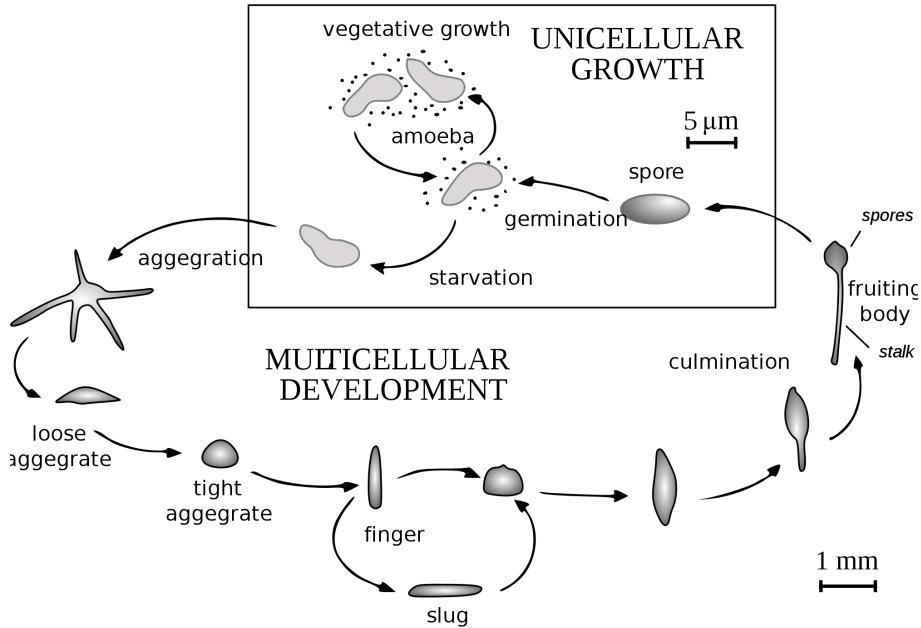


FIGURE 4.1 – Cycle de vie de *Dictyostelium discoideum*, Source : Wikipédia

favorable et ainsi reformer une nouvelle colonie d'amibes. C'est ce qui lui confère le qualificatif d'amibes sociales.

Le modèle que l'on considère ici, s'intéresse uniquement au passage de l'amibe à l'être pluricellulaire. Cette accumulation de cellules est lié à une chemiotaxie. Le milieu devenant hostile, bien souvent par pénurie de nutriment, l'amibe *Dictyostelium discoideum* sécrète un produit chimique, le cyclique Adenosine Monophosphate (cAMP), attirant les autres amibes.

Expérimentalement, il a été observé que le nombre de myxomycètes, impliqués dans le processus de concentration, jouait un rôle déterminant. En effet, lorsque que ceux-ci ne sont pas assez nombreux, l'accumulation ne se produit pas. Cohen et Roberson donne dans [16] la concentration seuil pour *Dictyostelium discoideum* : $5 \cdot 10^4$ myxomycètes par cm^2 . Pour les lecteurs intéressés par ces expérimentations, on peut renvoyer à [85]. Modéliser une première fois par Patlak en 1953 [80], Keller et Segel ont proposé, en 1971, dans une série d'articles [49], [50] et [51], un modèle comprenant quatre équations paraboliques couplées, décrivant chacune l'évolution temporelle et spatiale d'une concentration : celle de myxomycètes, celle de chemoattractant, mais aussi celle d'une enzyme produite par les amibes, dégradant le cAMP, et enfin celle du complexe chimique, formé par l'enzyme et le chemoattractant lors de cette dégradation. Keller et Segel ont mentionné dès leur premier article sur le sujet qu'au risque de perdre un peu de précision dans la description du phénomène, il était "plus raisonnable" d'étudier le système parabolique-parabolique :

$$\begin{cases} u_t = \nabla \cdot (k_1(u, v) \nabla u - k_2(u, v) \nabla v), & x \in \Omega, t > 0 \\ v_t = k_c \Delta v - k_3(v)v + uf(v), & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

où $u(x, t)$ représente la densité du myxomycète *Dictystelium discoideum*, $v(x, t)$ celle du chemoattractant, k_c la constante de diffusivité du chemoattractant, k_i et f des fonctions à détermi-

miner, traduisant le couplage du système, Ω étant le domaine d'étude inclus dans \mathbb{R}^N .

Le système (4.0.2), écrit pour la première fois par Nanjundiah dans [79], est de la forme du système précédent, pour des fonctions k_i et f bien choisies, et $c = \frac{1}{k_c}$. Sans entrer dans les détails, plus la diffusivité k_c du chemoattractant est grande, et donc plus la constante c est proche de 0, plus cet élément chimique se propage rapidement dans le système. A la limite, en prenant $c = 0$, ie en considérant le système parabolique elliptique (4.0.1), la propagation du cAMP dans le milieu se fait instantanément. Au vu de la différence de vitesse de déplacement d'une molécule par rapport à celle d'une amibe, cette approximation n'est pas totalement décorrélée de la réalité. Sauf mention explicite de notre part, nous nous placerons désormais toujours dans ce cadre là.

Nous avons vu deux choses très importantes sur le myxomycète *Dictyostelium discoideum*, que nous aimions retrouver dans le modèle mathématique : la notion de masse critique, et l'existence de solutions explosives. On appelle masse du système la quantité suivante, qui est conservée au cours du temps :

$$M = \int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u(0, x) dx \quad (4.1.1)$$

En 1981, Childress et Percus ont montré dans [15] l'impossibilité de formation de singularité en dimension 1, mais aussi la possibilité de solutions explosives en dimension supérieure à 3, et ont conjecturé l'existence d'une masse seuil, ayant les propriétés désirées, en dimension 2.

Pour un historique plus complet de ce problème, nous invitons le lecteur à lire les articles [43] et [44] de Horstmann, dans lesquels est faite une revue détaillée de l'ensemble des résultats jusqu'en 2004. On pourra aussi consulter [42] et [82] pour des références plus récentes.

4.2 Notion de masse critique

Rappelons le système parabolique-elliptique que l'on considère :

$$(PKSpe) \quad \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla \phi_u), \\ \phi_u = \frac{1}{2\pi} \log|x| \star u \\ u|_{t=0} = u_0 > 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2,$$

où ϕ_u est une solution de l'équation de Poisson $\Delta \phi_u = u$. Dans [78], Nagai propose une preuve de l'existence locale de solution lisse de ce problème, basée sur des arguments liés à des théorèmes de point fixe.

A ce problème, on peut associer l'invariance par changement d'échelle suivante

$$u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x) \quad \text{pour } \lambda > 0, \quad (4.2.1)$$

qui laisse la masse de la solution invariante :

$$M = \int_{\mathbb{R}^2} u_\lambda(t, x) dx = \int_{\mathbb{R}^2} u(\lambda^2 t, x) dx.$$

Le problème est dit alors L^1 critique, ce qui explique notamment sa complexité en dimension 2. On définit en outre une énergie associée à une solution u de (4.0.1), appelée énergie libre, définie par :

$$\mathcal{F}(u) = \int u \log u + \frac{4\pi}{M} \int u \phi_u \leq \mathcal{F}(u_0). \quad (4.2.2)$$

qui est dissipée au cours du temps. Le premier terme de cette énergie est un terme d'entropie. [2] regroupe un certain nombre de considérations à ce sujet. Le second est lié au potentiel chimique du chemoattractant.

Jäger et Luckhaus établissent dans [45] en 1992 le premier résultat rigoureux, portant sur la masse critique. Ils obtiennent qu'en deçà d'une certaine masse, constante indéterminée dans leur étude, il y a existence globale des solutions, et qu'au-delà d'une autre masse, possiblement la même, la formation de singularité devient possible. Ce dernier résultat a été montré pour des solutions radiales dans une boule. L'idée de la preuve de l'existence globale, en prenant l'espace \mathbb{R}^2 comme domaine d'étude, consiste à montrer la décroissance de l'entropie :

$$\frac{d}{dt} \int u \log u = -4 \int |\nabla \sqrt{u}|^2 + \int u^2 \quad (4.2.3)$$

pour une masse $M < \frac{4}{C_{GNS}^2} < 8\pi$, où C_{GNS} est la constante optimale de l'inégalité de Gagliardo-Nirenberg-Sobolev :

$$\|v\|_{L^4(\mathbb{R}^2)}^2 \leq C_{GNS} \|\nabla v\|_{L^2(\mathbb{R}^2)}^2 \|v\|_{L^2(\mathbb{R}^2)}^2 \quad (4.2.4)$$

dont la valeur numérique est calculée dans [102]. Cette approche s'est donc vue bloquée par cette constante.

Beckner dans [4], ainsi que Carlen et Loss dans [12] établissent l'inégalité logarithmique de Hardy-Littlewood-Sobolev suivante :

$$\int u \log u + \frac{4\pi}{M} \int u \phi_u \geq M[1 + \log \pi - \log M]. \quad (4.2.5)$$

pour une solution u de (4.0.1) de masse M , suffisamment décroissante. Cette inégalité est la base de la preuve du résultat suivant de [8], annoncé dans [24] :

$$\frac{d}{dt} \int |x|^2 u(t, x) dx = 4 \left(1 - \frac{M}{8\pi}\right) M \quad (4.2.6)$$

pour une donnée initiale u_0 de second moment fini. Cette identité du viriel montre que pour $M > 8\pi$, la solution de (4.0.1) pour de telles données initiales ne peut être globale en temps. Cet argument non constructif est à mettre en relation avec celui utilisé pour l'équation de Schrödinger non linéaire, amenant à la même conclusion pour des solutions d'énergie strictement négative. En fait, la contrainte second moment fini est juste technique, et il est possible d'établir le même résultat pour toutes solutions initiales, suffisamment décroissantes en un certain sens. Dans [8], Blanchet, Dolbeault et Perthame prouvent l'existence globale des solutions de masse inférieure à 8π .

Pour cette valeur seuil 8π de la masse, une solution radiale stationnaire du problème (4.0.1) est explicitement connue :

$$Q(r) = \frac{8}{(1+r^2)^2}. \quad (4.2.7)$$

L'invariance par changement d'échelle nous donne ainsi une famille de solutions stationnaires pour cette masse. En outre, cette solution est un minimiseur de l'inégalité logarithmique de Hardy-Littlewood-Sobolev (4.2.5), confirmant ainsi l'optimalité de cette borne.

Avant d'énoncer le théorème principal de [89], donnons quelques résultats de dynamique des solutions, en fonction de leur masse :

- Masse sous-critique : le résultat le plus complet à ma connaissance est celui de Campos et Dolbeault [11]. Il est bien connu que dans ce cadre là, toutes les solutions convergent localement vers la fonction nulle, excluant ainsi la possibilité de solution stationnaire. Par contre, en se plaçant en variables auto-similaires, il existe un unique objet stationnaire $u_{\infty,M}$, dépendant de la masse M . Ils montrent la vitesse de convergence suivante :

$$\int_{\mathbb{R}^2} \frac{|u(t,x) - u_{\infty,M}(x)|^2}{u_{\infty,M}} dx \leq Ce^{-2t}, \quad \forall t > 0,$$

$u(t,x)$ étant la solution du problème (4.0.1), après le passage en variables auto-similaires.

- Masse-critique : ce cas est un peu plus compliqué, venant notamment du fait qu'il faut dissocier le cas où le second moment de la solution initiale, conservé par le flot en masse critique comme le montre (4.2.6), est fini ou non. Dans [7], Blanchet, Carillo et Masmoudi montrent qu'un second moment fini entraîne une convergence en temps infini vers une masse de Dirac égale à 8π . Ce résultat a été complété par Senba dans [95], où il montre l'existence de solutions strictement positives radiales explosant en temps infini satisfaisant la borne :

$$\lim_{t \rightarrow \infty} \frac{\|u(.,t)\|_{\infty}}{(\log t)^2} = C, \quad C > 0. \quad (4.2.8)$$

Pour des solutions de second moment infini, Blanchet, Carlen et Carillo définissent dans [6], une certaine distance $(\mathcal{H}_{\mu})_{\mu>0}$ ayant la propriété suivante :

$$\mathcal{H}_{\mu}(Q_{\lambda}) = \begin{cases} 0 & \text{si } \mu = \lambda \\ +\infty & \text{sinon} \end{cases}$$

Pour chaque solution stationnaire Q_{λ} est associée un rayon r_{λ} , décrivant une zone d'attraction, dans laquelle toute solution initiale converge au sens de la distance \mathcal{H}_{λ} vers la solution stationnaire Q_{λ} . En dehors de ces bassins d'attraction, la situation n'est pas encore clairement établie, même s'il paraît raisonnable de conjecturer qu'il n'y a pas de formation de singularité pour les solutions initiales de second moment infini.

- Masse sur-critique : on a déjà évoqué l'argument du viriel, permettant de manière non constructive de prouver l'explosion en temps fini de toute solution initiale de masse sur-critique. Comme en masse sous-critique, cela donne la non-existence de solutions stationnaires dans ce cas-là. Herrero et Velázquez dans une série d'articles [40], [100] et [101] ont obtenu un premier résultat formel de dynamique explosive pour des solutions radiales, proche en un certain sens du soliton Q . Ils obtiennent notamment le profil et la vitesse de la concentration, mais ne peuvent rien dire quant à la stabilité de ce régime.

4.3 Résultat, commentaires et perspectives.

Avant d'énoncer le résultat principal de [89], nous devons au préalable introduire les espaces fonctionnels adaptés à notre étude. Posons l'espace L^2 à poids :

$$\|\varepsilon\|_{L_Q^2} = \left(\int \frac{\varepsilon^2}{Q} \right)^{\frac{1}{2}} \quad (4.3.1)$$

ainsi que l'espace H^2 à poids :

$$\|\varepsilon\|_{H_Q^2} = \|\Delta\varepsilon\|_{L_Q^2} + \left\| \frac{\nabla\varepsilon}{1+|x|} \right\|_{L_Q^2} + \|\varepsilon\|_{L^2}. \quad (4.3.2)$$

On peut maintenant introduire l'espace d'énergie \mathcal{E} , associée à la norme suivante :

$$\|\varepsilon\|_{\mathcal{E}} = \|\varepsilon\|_{H_Q^2} + \|\varepsilon\|_{L^1}. \quad (4.3.3)$$

Avec ces notations, on peut énoncer le théorème principal de [89], décrivant un ensemble ouvert de données initiales radiales, proches au sens de la norme d'énergie du soliton, explosant en temps fini, avec une dynamique connue précisément, et notamment, avec une bulle et une vitesse de concentration universelles.

Théorème 4.3.1 (Dynamique stable d'explosion pour le modèle de Patlak-Keller-Segel). *Il existe un espace de données initiales de la forme*

$$u_0 = Q + \varepsilon_0 \in \mathcal{E}, \quad u_0 > 0, \quad \|\varepsilon_0\|_{\mathcal{E}} \ll 1$$

tel que la solution correspondante $u \in \mathcal{C}([0, T), \mathcal{E})$ de (4.0.1) satisfasse les points suivants :

(i) Masse très légèrement sur-critique :

$$8\pi < \int u_0 < 8\pi + \alpha^*$$

pour $0 < \alpha^ \ll 1$ pouvant être choisi arbitrairement petit ;*

(ii) Explosion : *La solution explose en temps fini $0 < T < +\infty$;*

(iii) Bulle de concentration universelle : *La solution admet pour tout temps $t \in [0, T)$ une décomposition*

$$u(t, x) = \frac{1}{\lambda^2(t)}(Q + \varepsilon)\left(t, \frac{x}{\lambda(t)}\right)$$

avec

$$\|\varepsilon(t)\|_{H_Q^2} \rightarrow 0 \quad \text{quand } t \rightarrow T \quad (4.3.4)$$

et la vitesse universelle de concentration :

$$\lambda(t) = \sqrt{T-t} e^{-\sqrt{\frac{|\log(T-t)|}{2}} + O(1)} \quad \text{quand } t \rightarrow T. \quad (4.3.5)$$

(iv) Stabilité : *La dynamique d'explosion décrite ci-dessus est stable pour des petites perturbations de la donnée initiale dans \mathcal{E} :*

$$v_0 > 0, \quad \|v_0 - u_0\|_{\mathcal{E}} < \epsilon(u_0).$$

Faisons à présent quelques remarques sur ce résultat tout en présentant des perspectives de cette étude.

Tout d'abord, la dynamique explosive exposée dans cet article est la même que celle obtenue par Herrero et Velázquez dans [40], [100] et [101]. L'ajout majeur de cet article, outre le fait de montrer la robustesse de la méthode pour obtenir rigoureusement la dynamique de la formation

de la masse de Dirac, est de répondre à la question de la stabilité posée dans [101]. Pour autant, celle-ci n'exclut en aucun cas d'autres dynamiques, comme présenté dans [90] pour le flot de la chaleur harmonique. On peut s'attendre ici à retrouver un ensemble infini de vitesses de concentration, rendant le régime d'autant plus instable que la vitesse est lente. En outre, nous nous restreignons à des solutions de masses très légèrement sur-critiques, et nous ne décrivons qu'une petite partie du voisinage du soliton. Il est possible que loin du soliton, des régimes très différents apparaissent. On voit ici poindre trois axes de recherche :

- Etude des régimes excités, en revisitant la méthode utilisée dans [90].
- Description complète des dynamiques au voisinage du soliton. Une avancée majeure a été faite en ce sens très récemment par Merle, Martel et Raphaël dans [62], [63] et [64], où est présentée cette description pour l'équation de Korteweg-De-Vries généralisée unidimensionnelle. Cela impliquerait notamment de comprendre en profondeur le rôle du second moment dans le cadre masse critique.
- Explosion loin du soliton. Dans [19], Côte, Kenig, Lawrie et Schlag donne pour l'équation des ondes (1.1.8) pour des solutions initiales équivariantes, le nombre de points d'explosion maximal possible, ainsi que l'énergie concentrée en chacun de ces points, pour des solutions initiales d'énergie inférieure à trois fois celle de l'énergie minimale d'une application harmonique. La méthodologie mise en place laisse l'espoir de mieux comprendre le cas des solutions initiales de masse M telle que : $8\pi < M < 16\pi$.

Pour revenir aux remarques sur le théorème, un autre point important est la non-localité présente dans le terme non-linéaire, dû au produit de convolution. C'est en partie cette spécificité de ce problème qui le rend actuellement si attractif. Pour autant, en ne considérant que les solutions radiales, Herrero et Velázquez ont montré dans [40] que l'équation vérifiée par la masse partielle de la solution donnée par $m(r) = \int_0^r u(y)ydy$ résout ce problème, comme le montre l'expression ci-dessous :

$$\partial_t m = \partial_{rr} m + \frac{\partial_r m}{r} (m - 1).$$

Il est important de souligner que nous n'avons pas utilisé cette localisation du problème. Ainsi, il semble envisageable de reprendre la trame de démonstration pour aborder le cas non-radial. Comme pour le problème de la chaleur, cela impliquerait dans la construction un contrôle de la translation de la solution approchée. De plus, la preuve deviendrait alors beaucoup plus technique, sans qu'*a priori* il n'y ait d'obstacle conceptuel.

Pour conclure, donnons une dernière piste de recherche intéressante, qui est actuellement travaillée avec Adrien Blanchet. Qu'en est-il pour le problème parabolique-parabolique (4.0.2) ? On peut tout d'abord remarquer que la solution stationnaire pour le problème (4.0.1) Q est aussi une solution stationnaire de ce problème. Quel est l'impact d'une vitesse finie de déplacement du chemoattractant sur la dynamique explosive considérée ?

Chapitre 5

Article [91] le flot de la chaleur harmonique

5.1 Introduction

5.1.1 Setting of the problem

The harmonic heat flow between two embedded Riemannian manifolds $(N, g_N), (M, g_M)$ is the gradient flow associated to the Dirichlet energy of maps from $N \rightarrow M$:

$$\begin{cases} \partial_t v = \mathbb{P}_{T_v M}(\Delta_{g_N} v) & (t, x) \in \mathbb{R} \times N, \quad v(t, x) \in M \\ v|_{t=0} = v_0 \end{cases} \quad (5.1.1)$$

where $\mathbb{P}_{T_v M}$ is the projection onto the tangent space to M at v . The special case $N = \mathbb{R}^2$, $M = \mathbb{S}^2$ corresponds to the harmonic heat flow to the 2-sphere

$$\partial_t v = \Delta v + |\nabla v|^2 v, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad v(t, x) \in \mathbb{S}^2 \quad (5.1.2)$$

which appears in crystal physics and is related to the Landau Lifschitz equation of ferromagnetism, we refer to [99], [1], [36], [37] and references therein for a complete introduction to this class of problems. We shall from now on restrict our discussion to the case :

$$N = \mathbb{R}^2.$$

Local existence of solutions emanating from smooth data is well known. Note that the Dirichlet energy is dissipated by the flow

$$\frac{d}{dt} \left\{ \int_{\mathbb{R}^2} |\nabla v|^2 \right\} = -2 \int_{\mathbb{R}^2} |\partial_t v|^2$$

and left invariant by the scaling symmetry

$$u_\lambda(t, x) = u(\lambda^2 t, \lambda x).$$

Hence the problem is *energy critical* and a singularity formation by energy concentration is possible. By the works of Struwe [96], Ding and Tian [23], Qing and Tian [86] (see Topping [98]

for a complete history of the problem), it is known that concentration implies the bubbling off of a non trivial harmonic map at a finite number of blow up points

$$v(t_i, a_i + \lambda(t_i)x) \rightarrow Q_i, \quad \lambda(t_i) \rightarrow 0 \quad (5.1.3)$$

locally in space. In particular, this shows the existence of a global in time flow on negatively curved targets where no nontrivial harmonic map exists.

5.1.2 Corotational flows

The existence of blow up solutions has been proved in various different geometrical settings, see in particular Chang, Ding, Ye [14], Coron and Ghidaglia [17], Qing and Tian [86], Topping [98]. We shall restrict in this paper onto flows with symmetries which are better understood. Let a smooth closed curve in the plane parametrized by arclength

$$u \in [-\pi, \pi] \mapsto \begin{vmatrix} g(u) \\ z(u) \end{vmatrix}, \quad (g')^2 + (z')^2 = 1,$$

where

$$(H) \quad \begin{cases} g \in C^\infty(\mathbb{R}) \text{ is odd and } 2\pi \text{ periodic,} \\ g(0) = g(\pi) = 0, \quad g(u) > 0 \text{ for } 0 < u < \pi, \\ g'(0) = 1, \quad g'(\pi) = -1, \end{cases} \quad (5.1.4)$$

then the revolution surface M with parametrization

$$(\theta, u) \in [0, 2\pi] \times [0, \pi] \mapsto \begin{vmatrix} g(u) \cos \theta \\ g(u) \sin \theta \\ z(u) \end{vmatrix},$$

is a smooth¹ compact revolution surface of \mathbb{R}^3 with metric $(du)^2 + (g(u))^2(d\theta)^2$. Given a homotopy degree $k \in \mathbb{Z}^*$, the k-corotational reduction to (5.1.1) corresponds to solutions of the form

$$v(t, r) = \begin{vmatrix} g(u(t, r)) \cos(k\theta) \\ g(u(t, r)) \sin(k\theta) \\ z(u(t, r)) \end{vmatrix} \quad (5.1.5)$$

which leads to the semilinear parabolic equation² :

$$\begin{cases} \partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2 \frac{f(u)}{r^2} = 0, \\ u_{t=0} = u_0 \end{cases} \quad f = gg'. \quad (5.1.6)$$

The k-corotational Dirichlet energy becomes

$$E(u) = \int_0^{+\infty} \left[|\partial_r u|^2 + k^2 \frac{(g(u))^2}{r^2} \right] r dr \quad (5.1.7)$$

and is minimized among maps with boundary conditions

$$u(0) = 0, \quad \lim_{r \rightarrow +\infty} u(r) = \pi \quad (5.1.8)$$

1. see eg [29]

2. see (5.4.27)

onto the least energy harmonic map Q_k which is the unique -up to scaling- solution to

$$r\partial_r Q_k = kg(Q_k) \quad (5.1.9)$$

satisfying (5.1.8), see for example [18].

In the case of \mathbb{S}^2 target $g(u) = \sin u$, the harmonic map is explicitly given by

$$Q_k(r) = 2 \tan^{-1}(r^k). \quad (5.1.10)$$

In the series of works by Guan, Gustaffson, Tsai [36], Gustaffson, Nakanishi, Tsai [37], Q_k is proved to be *stable* by the flow (5.1.6) for $k \geq 3$, and in particular no blow up will occur near Q_k . Moreover, eternally oscillating solutions and infinite time blow up solutions are exhibited for $k = 2$.

We shall from now on and for the rest of the paper restrict attention to the degree $k = 1$ which generates the least energy harmonic map $Q \equiv Q_1$. For \mathbb{D}^2 initial manifold and \mathbb{S}^2 target, the formal analysis by Van den Berg, Hulshof and King [99] suggests through matching asymptotics the existence of a *stable generic* blow up regime with

$$u(t, r) \sim Q\left(\frac{r}{\lambda(t)}\right), \quad \lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}.$$

In this direction, Angenent, Hulshof and Matano exhibit in [1] a class of corotational solutions which blow up in finite time with an estimate :

$$\lambda(t) = o(T-t) \text{ as } t \rightarrow T.$$

The maximum principle plays an important role in this analysis. The sharp description of the singularity formation and in particular the understanding of the generic regime thus remain open.

More generally, let us recall that the derivation of the blow up speed for energy critical parabolic problems is poorly understood, and for example the derivation of sharp asymptotics of type II blow up for the energy critical semilinear problem

$$\partial_t u = \Delta u + u^{\frac{N+2}{N-2}}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad N \geq 3$$

is open.

5.1.3 Statement of the result

Let \mathcal{Q} the least energy harmonic map with degree 1 generated by the $Q \equiv Q_1$ solution to (5.1.9), explicitly :

$$\mathcal{Q}(x) = \begin{cases} g(Q(t, r)) \cos \theta \\ g(Q(t, r)) \sin \theta \\ z(Q(t, r)) \end{cases} \quad (5.1.11)$$

For an integer $i \geq 1$, we let \dot{H}^i be the completion of $\mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ for the norm

$$\|v\|_{\dot{H}^i} = \|\Delta^{\frac{i}{2}} v\|_{L^2}.$$

We claim the existence and stability of a universal blow up regime emerging from 1-equivariant smooth data arbitrarily close to \mathcal{Q} in the energy critical topology, together with sharp asymptotics on the singularity formation.

Théorème 5.1.1 (Stable blow up dynamics for the 1-corotational heat flow). *Let $k = 1$ and g satisfy (5.1.4). Let \mathcal{Q} be the least energy harmonic map given by (5.1.11). Then there exists an open set \mathcal{O} of 1-corotational initial data of the form*

$$v_0 = \mathcal{Q} + \varepsilon_0, \quad \varepsilon_0 \in \mathcal{O} \subset \dot{H}^1 \cap \dot{H}^4$$

such that the corresponding solution $v \in \mathcal{C}([0, T), \dot{H}^1 \cap \dot{H}^4)$ to (5.1.1) blows up in finite time $0 < T = T(v_0) < +\infty$ according to the following universal scenario :

- (i) *Universality of the concentrating bubble : there exists an asymptotic profile $v^* \in \dot{H}^1$ and $\lambda \in \mathcal{C}^1([0, T), \mathbb{R}_+^*)$ such that*

$$\lim_{t \rightarrow T} \left\| v(t, x) - \mathcal{Q} \left(\frac{x}{\lambda(t)} \right) - v^* \right\|_{\dot{H}^1} = 0. \quad (5.1.12)$$

- (ii) *Sharp asymptotics : the blow up speed is given by*

$$\lambda(t) = c(v_0)(1 + o(1)) \frac{T-t}{|\log(T-t)|^2} \quad \text{as } t \rightarrow T \quad (5.1.13)$$

for some $c(v_0) > 0$.

- (iii) *Regularity of the asymptotic profile : there holds the additional regularity*

$$\Delta v^* \in L^2. \quad (5.1.14)$$

In other words, there exists a generic blow up regime with the law (5.1.13) as predicted in [99] for $g(u) = \sin u$, and blow up in this regime occurs by the concentration of a universal and quantized bubble of energy.

Comments on the result :

1. *Energy method* : Following the strategy developed in [69], [71], [87], [88], [72], our strategy of proof proceeds first through the construction of suitable approximate solutions, and then the control of the remaining radiation through a robust *energy method*. In particular, we make no use of the maximum principle, and hence we expect our strategy to be applicable to more complicated parabolic *systems* among which the full problem (5.1.1). Note also that parabolic problems cannot be solved backwards in time and involve smooth data. In this sense the construction of blow up solutions requires to follow the flow of smooth solutions forward in time and cannot be achieved by solving from blow up time for rough data as in [61], [54]. The set of initial data we construct in the proof of Theorem 5.1.1 contains compactly supported C^∞ 1-corotational functions.

2. *Regularity of the asymptotic profile* : The regularity of the asymptotic profile (5.1.14) is a completely new feature with respect to the regularity obtained in [70], [88] where the profile is just in the critical space. This would also allow one to quantify the convergence rate (5.1.12) and bound the error polynomially in time, which is a substantial improvement on the general convergence (5.1.3). This shows also the close relation between the blow up rate which is far above selfsimilarity³ and the regularity of u^* , see [92] for related discussions, and explains formally why the problem under consideration should be thought of as "one derivative" above the

3. corresponding to the law $\lambda(t) \sim \sqrt{T-t}$.

wave map problem considered in [88].

3. Comparison with wave and Schrödinger maps : This result lies in the continuation of the works [88], [72] on the derivation of stable or codimension one blow up dynamics for the wave map :

$$(WM) \quad \begin{cases} \partial_{tt}u - \Delta u = (|\partial_t u|^2 - |\nabla u|^2)u & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(t, x) \in \mathbb{S}^2, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases} \quad (5.1.15)$$

and the Schrödinger map :

$$(SM) \quad \begin{cases} u \wedge \partial_t u = \Delta u + |\nabla u|^2 u & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(t, x) \in \mathbb{S}^2, \\ u|_{t=0} = u_0, \end{cases} \quad (5.1.16)$$

in both cases from $\mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{S}^2$. For the wave map, a stable blow up dynamics within the k-corotational symmetry class (5.1.5) is exhibited in [88] for all homotopy number $k \geq 1$ with an almost self similar blow up speed, see also [91]. In [72], the Schrödinger map problem is considered within the class of functions with 1-equivariant symmetry ie of the form

$$v(t, x) = e^{\theta R} w(t, r), \quad w(t, r) = \begin{vmatrix} w_1(t, r) \\ w_2(t, r) \\ w_3(t, r) \end{vmatrix} \quad (5.1.17)$$

with

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{\theta R} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.1.18)$$

A codimension one set of smooth initial data is exhibited for which concentration occurs

$$\lim_{t \rightarrow T} \left\| v(t, x) - e^{\Theta^* R} \mathcal{Q} \left(\frac{x}{\lambda(t)} \right) - v^* \right\|_{\dot{H}^1} = 0 \quad (5.1.19)$$

for some $\Theta^* \in \mathbb{R}$, $v^* \in \dot{H}^1$ at the speed given by (5.1.13) :

$$\lambda(t) = c(v_0)(1 + o(1)) \frac{T-t}{|\log(T-t)|^2}.$$

Note that the 1-equivariant symmetry is also preserved by the wave map and the harmonic heat flow (5.1.2), and the 1-corotational symmetry⁴ (5.1.5) corresponds to the 1-equivariant symmetry (5.1.17) with $w_2 \equiv 0$, and hence such maps are *not allowed to rotate around the e_z axis*. This extra degree of freedom in 1-equivariant symmetry is shown in [72] to stabilize the system and leads to a codimension one blow up phenomenon for the Schrödinger map. We expect the same phenomenon to occur here, and we conjecture that the blow up solutions constructed in Theorem 5.1.1 correspond to a codimension one phenomenon for the full problem (5.1.2).

Acknowledgments : The authors would like to thank Michael Struwe for pointing out to them the relevance of this problem in the continuation of the work [88]. The authors would

4. which is not preserved by the Schrödinger map.

also like to thank the anonymous referee for his careful reading of the paper which helped improved the presentation. Part of this work was done while P.R was visiting the ETH, Zurich, which he would like to thank for its kind hospitality. Both authors are supported by the French ERC/ANR project SWAP.

Notations : We introduce the differential operator

$$\Lambda f = y \cdot \nabla f \quad (\text{energy critical scaling}).$$

Given a positive number $b > 0$, we let

$$B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}. \quad (5.1.20)$$

Given a parameter $\lambda > 0$, we let

$$u_\lambda(r) = u(y) \quad \text{with} \quad y = \frac{r}{\lambda}.$$

We let χ be a positive nonincreasing smooth cut off function with

$$\chi(y) = \begin{cases} 1 & \text{for } y \leq 1, \\ 0 & \text{for } y \geq 2. \end{cases}$$

Given a parameter $B > 0$, we will denote :

$$\chi_B(y) = \chi\left(\frac{y}{B}\right). \quad (5.1.21)$$

We shall systematically omit the measure in all radial two dimensional integrals and note :

$$\int f = \int_0^{+\infty} f(r) r dr.$$

5.1.4 Strategy of the proof

Let us give a brief insight into the strategy of the proof of Theorem 5.1.1 which adapts the strategy developed in [88], [72].

(i). *Formal derivation of the law*

Let us look for a modulated solution $u(t, r)$ of (5.1.6) in renormalized form

$$u(t, r) = z(s, y), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}$$

which leads to the self similar equation :

$$\partial_s z - \Delta z + b \Lambda z + \frac{f(z)}{y^2} = 0, \quad b = -\frac{\lambda_s}{\lambda}. \quad (5.1.22)$$

The speed of concentration $\lambda(t)$ is an unknown of the problem, but we know a priori from energy constraint that the renormalized flow $z(s, y)$ must evolve close to the harmonic map $Q(y)$. This

forces in particular the drift $b = -\frac{\lambda_s}{\lambda}$ to be *uniformly small in time*. We therefore proceed to a slow variable expansion⁵ and look for a solution to (5.1.22) of the form :

$$z(s, y) = Q(y) + b(s)T_1(y) + b^2(s)T_2(y) + \dots \quad (5.1.23)$$

and where the time dependance appears only in the modulation function b through the ODE

$$b_s(s) = -c_2b^2(s) + c_3b^3(s) + \dots \quad (5.1.24)$$

Here the profiles $T_1(y), T_2(y) \dots$ and constants c_2, c_3, \dots are unkwnowns to be determined. Plugging in the expansion (5.1.23), (5.1.24) into (5.1.22) and matching the powers of b yields at order b :

$$HT_1 + \Lambda Q = 0 \quad (5.1.25)$$

and at order b^2 :

$$HT_2 + \Lambda T_1 - c_2T_1 + \frac{f''(Q)}{y^2}T_1^2 = 0. \quad (5.1.26)$$

Here H is the linearized Hamiltonian close to Q ie the Schrödinger operator

$$H = -\Delta + \frac{f'(Q)}{y^2}.$$

It displays a remarkable algebraic structure⁶, and is in particular nonnegative with a resonance⁷ generated by the energy critical scaling symmetry :

$$H(\Lambda Q) = 0$$

where from explicit computation⁸

$$\Lambda Q \sim \frac{c}{y} \text{ as } y \rightarrow \infty \text{ and thus } \Lambda Q \notin L^2. \quad (5.1.27)$$

The Green's functions of H are explicit, and a smooth solution to (5.1.25) can be computed which displays the *growth* at infinity :

$$T_1(y) \sim y \log y + cy + O\left(\frac{|\log y|^2}{y}\right) \text{ as } y \rightarrow +\infty.$$

We now aim at solving (5.1.26) with the *least possible growth* for T_2 . The choice $c_2 = 1$ cancels the worst $y \log y$ growth :

$$\Lambda T_1 - T_1 = y + O\left(\frac{|\log y|^2}{y}\right) \quad (5.1.28)$$

which after inversion of (5.1.26) leads to a growth :

$$T_2(y) \sim y^3 \text{ as } y \rightarrow +\infty. \quad (5.1.29)$$

5. as standard in various settings, for example geometric optics, see for example [97].

6. see section 5.2.2.

7. ie a non L^2 eigenstate for the zero eigenvalue.

8. Note that this slow decay is intrinsic to the $k = 1$ degree case, as higher degree yield faster decay, see (5.1.10).

An essential feature here is that an additional *logarithmic gain* can be obtained by manufacturing an expansion

$$b_s = -b^2 \left(1 + \frac{2}{|\log b|} \right) \quad (5.1.30)$$

which leads to an improved control after the parabolic zone :

$$T_2(y) = O \left(\frac{y}{b|\log b|} \right) \ll y^3 \quad \text{for } y \geq \frac{1}{\sqrt{b}}. \quad (5.1.31)$$

Similar lower order corrections in the law for b like (5.1.30) occurred in [81], [71], [88], [72] and can be derived from a *flux type computation*, or equivalently a boundary term in energy critical Pohozaev type integration by parts generated by the explicit growth (5.1.28). Here the gain is very explicit and relies on the direct introduction of the *radiative correction*, see step 3 of the proof of Proposition 5.2.4 and Remark 5.2.7.

The above heuristical argument thus leads to the system of ODE's :

$$-\frac{\lambda_s}{\lambda} = b, \quad b_s = -b^2 \left(1 + \frac{2}{|\log b|} \right), \quad \frac{ds}{dt} = \frac{1}{\lambda^2}. \quad (5.1.32)$$

We integrate this with initial data

$$0 < b(0) \ll 1, \quad \lambda(0) > 0$$

and easily show that the corresponding scaling parameter $\lambda(t)$ touches zero in finite time $0 < T = T(b(0), \lambda(0)) < +\infty$, this is blow up, at a rate :

$$\lambda(t) = c_{b_0, \lambda_0} (1 + o(1)) \frac{T-t}{|\log(T-t)|^2} \quad \text{as } t \rightarrow T,$$

this is (5.1.13).

(ii). Decomposition of the flow and modulation equations

Let then

$$Q_{b(s)}(y) = Q(y) + b(s)T_2(y) + b^2(s)T_2(y) \dots$$

a sufficiently high order correction to (5.1.22) in the regime (5.1.24) measured by the error :

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2}$$

which is formally cubic in b , and consider an initial data of the form

$$u_0(y) = Q_{b_0}(y) + \varepsilon_0(y), \quad |\varepsilon_0(y)| \ll b_0 \ll 1$$

in some suitable sense. We introduce a modulated decomposition of the flow

$$u(t, r) = (Q_{b(t)} + \varepsilon) \left(t, \frac{r}{\lambda(t)} \right) \quad (5.1.33)$$

where the modulation parameters $(b(t), \lambda(t))$ are chosen in order to manufacture the orthogonality conditions :

$$(\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0. \quad (5.1.34)$$

Here $\Phi_M(y)$ is some fixed direction depending on some large constant M which generates an approximation of the kernel⁹ of H^2 , see (5.3.5). This orthogonal decomposition, which for each fixed time t directly follows from the implicit function theorem, now allows us to compute the modulation equations governing the parameters $(b(t), \lambda(t))$. The Q_b construction is precisely manufactured to produce the expected ODE's¹⁰ :

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \ll \|\varepsilon\|_{loc} + b^2 \left(1 + \frac{2}{|\log b|} \right) \quad (5.1.35)$$

where $\|\varepsilon\|_{loc}$ measures a *local in space* interaction with the harmonic map :

$$\|\varepsilon\|_{loc}^2 \sim \int \frac{|\varepsilon|^2}{1+y^8}.$$

(iii). *Control of radiation and monotonicity formula.*

We now need to show that *in the regime we are focusing on*, the radiation ε does not perturb the expected dynamical system (5.1.32), and from (5.1.35) this will follow from a bound :

$$\int \frac{|\varepsilon|^2}{1+y^8} \lesssim \frac{b^4}{|\log b|^2}. \quad (5.1.36)$$

In other words, the infinite dissipative dynamics of the radiation ε is fully controlled by the one dimensional projection b , and does not perturb the leading order dynamics. Controlling radiation is the heart of the analysis. Related bounds were derived in [37] for $k \geq 2$ using parabolic propagator estimates which become more delicate in the presence of resonances and with slow decay of the harmonic map, and we use here the approach developed in [88], [92]. The idea is to rewrite the decomposition (5.1.33) in original non rescaled variables as

$$u(t, r) = Q_{b(t)} \left(t, \frac{r}{\lambda(t)} \right) + w(t, r), \quad w(t, r) = \varepsilon \left(t, \frac{r}{\lambda(t)} \right). \quad (5.1.37)$$

The equation for $w(t, r)$ is roughly of the form¹¹

$$\partial_t w + H_{\lambda(t)} w = \frac{1}{\lambda^2} \Psi_{b(t)} \left(\frac{r}{\lambda} \right) + (\text{lower order terms}) \quad (5.1.38)$$

where H_λ denotes the linearized operator close to the focusing harmonic map :

$$H_\lambda = -\Delta + \frac{f'(Q_\lambda)}{r^2}, \quad Q_\lambda(r) = Q \left(\frac{r}{\lambda} \right).$$

We now perform an energy type identity on (5.1.38) and aim at controlling Sobolev type norms on w build on iterates of the Hamiltonian H_λ . The difficulty is twofold : first we need to treat the terms induced by the time dependance of the Hamiltonian H_λ , and here we use mostly the algebraic structure of H and *dissipation*; second we need to estimate suitable weighted Sobolev norms of Ψ_b . Here let us insist onto the fact that Ψ_b contains a priori high order terms in b

9. Smooth solutions to $H^2 u = 0$ lie in $\text{Span}(\Lambda Q, T_1)$ and therefore decay badly near $+\infty$, hence the need for a localized approximation provided by Φ_M .

10. see Lemma 5.3.2.

11. see (5.3.27).

against functions which grow in y , and hence it will be small in weighted norms only as expressed by Proposition 5.2.4. The energy bound we derive relies on a monotonicity formula which takes the form of the Sobolev \dot{H}^4 bound -see Proposition 5.3.4 for a precise statement- :

$$\frac{d}{dt} \|H_\lambda^2 w\|_{L^2}^2 \lesssim \frac{b}{\lambda^8} \left[\frac{b^4}{|\log b|^2} + (\text{lower order terms}) \right]. \quad (5.1.39)$$

(iv). *Closing the bootstrap.*

We now integrate in time the monotonicity bound (5.1.39) assuming that the parameters follow to leading order the regime (5.1.32). This first yields using the rescaling (5.1.37) :

$$\|H^2 \varepsilon(t)\|_{L^2}^2 \lesssim \left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \|H^2 \varepsilon(0)\|_{L^2}^2 + \lambda^6(t) \int_0^t \frac{b^5(\tau)}{\lambda^8(\tau) |\log b(\tau)|^2} d\tau.$$

We next integrate the ODE (5.1.32) to derive the bounds :

$$\lambda^6(t) \lesssim b^5(t), \quad \lambda^6(t) \int_0^t \frac{b^5(\tau)}{\lambda^8(\tau) |\log b(\tau)|^2} d\tau \lesssim \frac{b^4(t)}{|\log b(t)|^2}$$

and finally obtain the pointwise bound :

$$\|H^2 \varepsilon(t)\|_{L^2}^2 \lesssim \frac{b^4(t)}{|\log b(t)|^2}.$$

We now observe that *provided we removed a suitable localization of the kernel of H^2* , this is the choice of orthogonality conditions (5.1.34), H^2 is a coercive operator which satisfies weighted Hardy type bounds, see Lemma 5.5.3 :

$$\int \frac{|\varepsilon|^2}{1+y^8} \lesssim \|H^2 \varepsilon\|_{L^2}^2 \lesssim \frac{b^4}{|\log b|^2} \quad (5.1.40)$$

and (5.1.36) is proved.

The above *linear* argument paves the way to close bootstrap bounds for $(b, \lambda, \varepsilon)$ in the decomposition (5.1.33). The remaining standard difficulty is to control the *nonlinear terms* in particular in the monotonicity estimate (5.1.39). This involves some technicalities including interpolation bounds build on the Hardy bounds (5.1.40), see Lemma 5.5.4, and a careful treatment of both the singularity of the nonlinearity at the origin induced by the corotational change of variables, and the slow decay of functions at infinity. The analysis requires a careful track of polynomial weights both at the origin and infinity in the Hardy type estimates we use.

Finally, let us observe that the key in the above proof is to provide a sufficiently small error $\frac{b^4}{|\log b|^2}$ in the right hand side of (5.1.39), and this error is from (5.1.38) directly related to the size of Ψ_b in suitable norms, and hence to the order of expansion in terms of power of b in the construction of Q_b . An explicit computation reveals that we need to construct at least the first three terms in the Q_b expansion according to (5.2.18), see Remark 5.2.6. Moreover, from direct computations again, the dominant terms in the size of Ψ_b leave in the parabolic zone $y \sim \frac{1}{\sqrt{b}}$ where improved decay bounds from (5.1.29) to (5.1.31) induced by the introduction of suitable radiative corrections will turn out to be essential¹².

12. see in particular the proof of (5.2.23).

This paper is organized as follows. In section 5.2, we construct the approximate self similar solutions Q_b and obtain sharp estimates on the error term Ψ_b . In section 5.3, we set up the bootstrap argument, Proposition 5.3.1, and derive the fundamental Sobolev \dot{H}^4 control, Proposition 5.3.4, which is the heart of the analysis. In section 5.4, we close the bootstrap bounds which easily imply the blow up statement of Theorem 5.1.1. Some further slight improvement of the obtained bounds yields the regularity (5.1.14) of the asymptotic profile.

5.2 Construction of the approximate profile

We follow the scheme of proof in [88], [72] and proceed in this section with the construction of suitable approximate solutions to the renormalized flow (5.1.22).

5.2.1 Asymptotics of the 1-corotational harmonic map

Let us start with recalling the structure of the harmonic map Q in the context of (5.1.6) which is the unique -up to scaling- solution to

$$\Lambda Q = g(Q), \quad Q(0) = 0, \quad \lim_{r \rightarrow +\infty} Q(r) = \pi. \quad (5.2.1)$$

This equation can be integrated explicitly and leads to the following asymptotics :

Lemme 5.2.1 (Asymptotics of the harmonic map). *There holds $Q \in C^\infty([0, +\infty), [0, \pi])$ with the Taylor expansions¹³ :*

$$Q(y) = \sum_{i=0}^p c_i y^{2i+1} + O(y^{2p+3}) \quad \text{as } y \rightarrow 0, \quad c_0 \neq 0, \quad (5.2.2)$$

$$Q(y) = \pi - \frac{2}{y} - \sum_{i=1}^p \frac{d_i}{y^{2i+1}} + O\left(\frac{1}{y^{2p+3}}\right) \quad \text{as } y \rightarrow +\infty. \quad (5.2.3)$$

Remarque 5.2.2. Recall that (5.2.1) is subject to the scaling symmetry $Q(y) \rightarrow Q(\lambda y)$, $\lambda > 0$. The scaling constant $\lambda > 0$ affects the decay of the map near $+\infty$, and we define Q to be the unique solution with from (5.2.3) :

$$\Lambda Q(y) \sim \frac{2}{y} \quad \text{for } y \rightarrow +\infty.$$

This arbitrary choice has been made to match the more familiar case of the round sphere :

$$Q(y) = 2 \tan^{-1}(y), \quad \Lambda Q(y) = \frac{2y}{1+y^2} \sim \frac{2}{y} \quad \text{as } y \rightarrow +\infty.$$

Proof of Lemma 5.2.1 From the Taylor expansion of g at 0 and π given by (5.1.4), we have :

$$\int_a^\pi \frac{\pi - \tau - g(\tau)}{(\pi - \tau)g(\tau)} d\tau + \log(\pi - a) \rightarrow \begin{cases} +\infty & \text{as } a \rightarrow 0 \\ -\infty & \text{as } a \rightarrow \pi \end{cases}$$

and we may thus find $a \in (0, \pi)$ such that

$$\int_a^\pi \frac{\pi - \tau - g(\tau)}{(\pi - \tau)g(\tau)} d\tau + \log(\pi - a) = \log 2. \quad (5.2.4)$$

13. up to scaling

We then let

$$G(u) = \int_a^u \frac{d\tau}{g(\tau)} \quad (5.2.5)$$

which from (5.1.4) is a diffeomorphism from $(0, \pi)$ onto $(-\infty, +\infty)$. Let now Q be the normalized solution to (5.2.1) given by

$$Q(y) = G^{-1}(\log y), \quad y \in [0, +\infty). \quad (5.2.6)$$

We compute near π from the normalization (5.2.4) :

$$\begin{aligned} G(u) &= \int_a^u \frac{d\tau}{g(\tau)} = \int_a^\pi \frac{\pi - \tau - g(\tau)}{(\pi - \tau)g(\tau)} d\tau + \int_a^u \frac{d\tau}{\pi - \tau} \\ &= \int_a^\pi \frac{\pi - \tau - g(\tau)}{(\pi - \tau)g(\tau)} d\tau + \log(\pi - a) - \log(\pi - u) - \int_u^\pi \frac{\pi - \tau - g(\tau)}{(\pi - \tau)g(\tau)} d\tau \\ &= -\log\left(\frac{\pi - u}{2}\right) + \sum_{i=1}^p \tilde{d}_i (\pi - u)^{2i} + O((\pi - u)^{2p+2}) \quad \text{as } u \rightarrow \pi \end{aligned} \quad (5.2.7)$$

and near the origin :

$$\begin{aligned} G(u) &= \log u - \log a + \int_0^u \frac{\tau - g(\tau)}{\tau g(\tau)} d\tau - \int_0^a \frac{\tau - g(\tau)}{\tau g(\tau)} d\tau \\ &= \log u + \tilde{c}_0 + \sum_{i=1}^p \tilde{c}_i u^{2i} + O(u^{2p+2}) \quad \text{as } u \rightarrow 0, \end{aligned}$$

and these developments together with (5.2.6) now yield (5.2.2), (5.2.3). This concludes the proof of Lemma 5.2.1.

5.2.2 The linearized Hamiltonian

We recall in this section the structure of the linearized operator close to Q . Let the potentials

$$Z = g'(Q), \quad V = Z^2 + \Lambda Z = f'(Q), \quad \tilde{V} = (1 + Z)^2 - \Lambda Z, \quad (5.2.8)$$

then the linearized operator close to Q is the Schrödinger operator :

$$H = -\Delta + \frac{V}{y^2}. \quad (5.2.9)$$

As observed in [91], an important consequence of the Bogomolny's factorization of the Dirichlet energy (5.1.7) is the decomposition

$$H = A^* A$$

where A^* is the adjoint of A with respect to the $L^2(\mathbb{R}^2)$ scalar product, explicitly :

$$A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1+Z}{y}.$$

The kernels of A and A^* on \mathbb{R}_+^* are explicit :

$$Au = 0 \quad \text{iff } u \in \text{Span}(\Lambda Q), \quad A^*u = 0 \quad \text{iff } u \in \text{Span}\left(\frac{1}{y\Lambda Q}\right), \quad (5.2.10)$$

and thus the kernel of H on \mathbb{R}_+^* is :

$$Hu = 0 \text{ iff } u \in \text{Span}(\Lambda Q, \Gamma) \quad (5.2.11)$$

with

$$\Gamma(y) = \Lambda Q \int_1^y \frac{dx}{x(\Lambda Q(x))^2} = \begin{cases} O\left(\frac{1}{y}\right) \text{ as } y \rightarrow 0, \\ \frac{y}{4} + O\left(\frac{\log y}{y}\right) \text{ as } y \rightarrow +\infty. \end{cases} \quad (5.2.12)$$

In particular, H is a positive operator on \dot{H}_{rad}^1 with a resonance ΛQ at the origin induced by the energy critical scaling invariance. We also introduce the conjugate Hamiltonian

$$\tilde{H} = AA^* = -\Delta + \frac{\tilde{V}}{y^2} \quad (5.2.13)$$

which is definite positive by construction and (5.2.10), see Lemma 5.5.2. Finally, let us compute using (5.1.4), (5.2.2), (5.2.3) the behavior of Z, V, \tilde{V} at 0 and $+\infty$ which will be fundamental in our analysis :

$$Z(y) = \begin{cases} 1 + \sum_{i=1}^p c_i y^{2i} + O(y^{2p+2}) & \text{as } y \rightarrow 0, \\ -1 + \sum_{i=1}^p \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) & \text{as } y \rightarrow +\infty, \end{cases} \quad (5.2.14)$$

$$V(y) = \begin{cases} 1 + \sum_{i=1}^p c_i y^{2i} + O(y^{2p+2}) & \text{as } y \rightarrow 0, \\ 1 + \sum_{i=1}^p \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) & \text{as } y \rightarrow +\infty, \end{cases} \quad (5.2.15)$$

$$\tilde{V}(y) = \begin{cases} 4 + \sum_{i=1}^p c_i y^{2i} + O(y^{2p+2}) & \text{as } y \rightarrow 0, \\ \sum_{i=1}^p \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) & \text{as } y \rightarrow +\infty, \end{cases} \quad (5.2.16)$$

where $(c_i)_{i \geq 1}$ stands for some generic sequence of constants which depend on the Taylor expansion of g at 0 and π .

Remarque 5.2.3. The exact values for the \mathbb{S}^2 target $g(u) = \sin u$ are given by :

$$Z(y) = \frac{1 - y^2}{1 + y^2}, \quad V(y) = \frac{y^4 - 6y^2 + 1}{(1 + y^2)^2}, \quad \tilde{V}(y) = 2(1 + Z) = \frac{4}{1 + y^2}.$$

5.2.3 Slowly modulated approximate profiles

We now aim at constructing a suitable approximate solution to (5.1.22) near the harmonic map Q with *moderate growth* as $y \rightarrow +\infty$ by adapting the slowly modulated ansatz approach developed in [67], [52], [88], [72]. We will see that this naturally leads to the leading order modulation equation¹⁴ :

$$b_s = -b^2(1 + o(1)). \quad (5.2.17)$$

Proposition 5.2.4 (Construction of the approximate profile). *Let $M > 0$ be a large enough universal constant. Then there exists $0 < b^*(M) \ll 1$ such for all $0 < b < b^*(M)$, there exist smooth profiles $(T_i(y, b))_{1 \leq i \leq 3}$ such that the following holds true. Let the approximate profile*

$$Q_b(y) = Q(y) + bT_1(b, y) + b^2T_2(b, y) + b^3T_3(b, y) = Q(y) + \alpha(b, y) \quad (5.2.18)$$

14. see Remark 5.2.7

and define the error :

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2}, \quad (5.2.19)$$

then there holds the bounds :

(i) Weighted bounds :

$$\int_{y \leq 2B_1} |H\Psi_b|^2 \lesssim b^4 |\log b|^2, \quad (5.2.20)$$

$$\int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} |H\Psi_b|^2 \lesssim \frac{b^4}{|\log b|^2}, \quad (5.2.21)$$

$$\int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^2} |AH\Psi_b|^2 \lesssim \frac{b^5}{|\log b|^2}, \quad (5.2.22)$$

$$\int_{y \leq 2B_1} |H^2\Psi_b|^2 \lesssim \frac{b^6}{|\log b|^2} \quad (5.2.23)$$

(ii) Flux computation : Let Φ_M be given by (5.3.5), then :

$$\frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \quad (5.2.24)$$

Remarque 5.2.5. From the proof, the profiles T_i are smooth and display some explicit algebraic growth near $y \rightarrow +\infty$, see the precise bounds (5.2.30), (5.2.31), (5.2.45), (5.2.50). Also the dependence on b of the profiles T_i is lower order, and we shall in the sequel omit this dependence and note

$$T_i(y), \alpha(y)$$

to ease notations. Observe that the decomposition (5.2.18) in the regime (5.2.17) leads to leading order to :

$$\partial_s Q_b - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2} = \Psi_b + (\text{lower order terms})$$

and hence Q_b is a high order approximate solution to the renormalized flow (5.1.22) in the regime (5.2.17). The flux computation (5.2.24) is the heart of the logarithmic correction to the ODE $b_s = -b^2$ and will lead to the correct modulation equation (5.1.30), see (5.3.32).

Remarque 5.2.6. The key here is the estimate (5.2.23) which is the leading order term driving the energy bound (5.3.4). This high order error requires an order three expansion of the profile in (5.2.18).

Proof of Proposition 5.2.4

We recall that given a small parameter $0 < b \ll 1$, we define the large scales B_0, B_1 given by (5.1.20). B_0 is the boundary of the parabolic zone $y \lesssim \frac{1}{\sqrt{b}}$ where most estimates will take place, and B_1 is a suitable size after which the profiles T_i cease to make sense in the dynamical description of the solution, and hence we will proceed to localization of the profiles for $y \geq 2B_1$ in Proposition 5.2.8. We therefore need a control of T_i, Ψ_b in the zone $y \leq 2B_1$ only.

step 1 Expansion in powers of b .

Let us compute the error (5.2.19) for a general decomposition (5.2.18) using a Taylor expansion :

$$\begin{aligned} f(Q_b) &= f(Q) + b [T_1 f'(Q)] + b^2 \left[T_2 f'(Q) + \frac{1}{2} T_1^2 f''(Q) \right] \\ &\quad + b^3 \left[T_3 f'(Q) + T_1 T_2 f''(Q) + \frac{1}{6} f^{(3)}(Q) T_1^3 \right] + R_1 + R_2 \end{aligned}$$

with

$$R_1 = \frac{1}{2} f''(Q) \left[\alpha^2 - b^2 T_1^2 - 2b^3 T_1 T_2 \right] + \frac{1}{6} f^{(3)}(Q) \left[\alpha^3 - b^3 T_1^3 \right], \quad (5.2.25)$$

$$R_2 = \frac{\alpha^4}{6} \int_0^1 (1-\tau)^3 f^{(4)}(Q + \tau\alpha) d\tau. \quad (5.2.26)$$

Hence from (5.2.19) :

$$\begin{aligned} \Psi_b &= b (HT_1 + \Lambda Q) \\ &\quad + b^2 \left(HT_2 - T_1 + \Lambda T_1 + \frac{f''(Q)}{2y^2} T_1^2 \right) \\ &\quad + b^3 \left(HT_3 - 2T_2 + \Lambda T_2 + \frac{f''(Q)}{y^2} T_1 T_2 + \frac{1}{6} \frac{f^{(3)}(Q)}{y^2} T_1^3 \right) \\ &\quad + b^4 \Lambda T_3 + \frac{1}{y^2} [R_1 + R_2]. \end{aligned} \quad (5.2.27)$$

Step 2 Construction of T_1 .

We may invert H explicitly from (5.2.11) and a smooth solution at the origin to $Hu = -f$ is given by :

$$u = \Gamma(y) \int_0^y f \Lambda Q x dx - \Lambda Q(y) \int_0^y f \Gamma x dx. \quad (5.2.28)$$

Observe that if f admits the Taylor expansion at the origin

$$f(y) = c_1 y + c_3 y^3 + O(y^5),$$

then

$$u(y) = d_3 y^3 + O(y^5) \text{ as } y \rightarrow 0. \quad (5.2.29)$$

Indeed, the Wronskian relation $\Gamma'(\Lambda Q) - (\Lambda Q)' \Gamma = \frac{1}{y}$ implies

$$A\Gamma = -\Gamma' + \frac{Z}{y}\Gamma = -\Gamma' + \frac{(\Lambda Q)'}{\Lambda Q}\Gamma = -\frac{1}{y\Lambda Q}$$

from which using $A\Lambda Q = 0$:

$$Au = A\Gamma \int_0^y f \Lambda Q x dx = -\frac{1}{y\Lambda Q} \int_0^y f \Lambda Q x dx = c_2 y^2 + c_4 y^4 + O(y^6)$$

near the origin for some constants (c_2, c_4) . We now integrate using $u = O(y^3)$ at the origin from (5.2.28) and thus :

$$u = -\Lambda Q \int_0^y \frac{Au}{\Lambda Q} dx = d_3 y^3 + O(y^5).$$

We now let T_1 be the solution to $HT_1 + \Lambda Q = 0$ given by

$$T_1(y) = \Gamma(y) \int_0^y (\Lambda Q)^2 x dx - \Lambda Q(y) \int_0^y \Lambda Q \Gamma x dx.$$

We compute from Lemma 5.2.1 and (5.2.28), (5.2.29) the behavior :

$$T_1(y) = \begin{cases} y \log y + e_0 y + O\left(\frac{(\log y)^2}{y}\right) & \text{as } y \rightarrow +\infty \\ d_3 y^3 + O(y^5) & \text{as } y \rightarrow 0 \end{cases} \quad (5.2.30)$$

for some universal constant e_0 , and similarly :

$$\Lambda^i T_1(y) = \begin{cases} y \log y + (e_0 + i)y + O\left(\frac{(\log y)^2}{y}\right) & \text{as } y \rightarrow +\infty \\ d_{3,i} y^3 + O(y^5) & \text{as } y \rightarrow 0 \end{cases} \quad \text{for } 1 \leq i \leq 3. \quad (5.2.31)$$

Note that T_1 does not depend on b :

$$\frac{\partial T_1}{\partial b} = 0.$$

Step 3 Construction of the radiation Σ_b .

We now construct the radiation term which will allow us to capture further cancellation near the parabolic zone $y \sim \frac{1}{\sqrt{b}}$, see Remark 5.2.7. Recall the definition (5.1.20) and let :

$$c_b = \frac{4}{\int \chi_{\frac{B_0}{4}} (\Lambda Q)^2} = \frac{2}{|\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right) \right) \quad (5.2.32)$$

and

$$d_b = c_b \int_0^{B_0} \chi_{\frac{B_0}{4}} \Lambda Q \Gamma y dy = \frac{C}{b |\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right) \right) \quad (5.2.33)$$

Let Σ_b be the solution to

$$H\Sigma_b = -c_b \chi_{\frac{B_0}{4}} \Lambda Q + d_b H[(1 - \chi_{3B_0}) \Lambda Q] \quad (5.2.34)$$

given by

$$\Sigma_b(y) = \Gamma(y) \int_0^y c_b \chi_{\frac{B_0}{4}} (\Lambda Q)^2 x dx - \Lambda Q(y) \int_0^y c_b \chi_{\frac{B_0}{4}} \Gamma \Lambda Q x dx + d_b (1 - \chi_{3B_0}) \Lambda Q(y) \quad (5.2.35)$$

Observe that by definition :

$$\Sigma_b = \begin{cases} c_b T_1 & \text{for } y \leq \frac{B_0}{4} \\ 4\Gamma & \text{for } y \geq 6B_0. \end{cases} \quad (5.2.36)$$

We now estimate for $6B_0 \leq y \leq 2B_1$:

$$\Sigma_b(y) = y + O\left(\frac{\log y}{y}\right) \quad \Lambda\Sigma_b(y) = y + O\left(\frac{\log y}{y}\right) \quad (5.2.37)$$

and for $y \leq 6B_0$:

$$\begin{aligned} \Sigma_b(y) &= c_b \left(\frac{y}{4} + O\left(\frac{\log y}{y}\right) \right) \left[\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 x dx \right] - c_b \Lambda Q(y) \int_1^y O(1) x dx \\ &= y \frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b|}\right) \end{aligned} \quad (5.2.38)$$

and similarly for $y \leq 6B_0$:

$$\Lambda^i \Sigma_b(y) = y \frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b|}\right) \quad \text{for } 0 \leq i \leq 2.$$

The equation (5.2.34) and the cancellation $A\Lambda Q = H\Lambda Q = 0$ yield the bounds :

$$\int |H\Sigma_b|^2 \lesssim \frac{1}{|\log b|}, \quad \int \frac{1+|\log y|^2}{1+y^4} |H\Sigma_b|^2 \lesssim \frac{1}{|\log b|^2}, \quad (5.2.39)$$

$$\int \frac{1+|\log y|^2}{1+y^2} |AH\Sigma_b|^2 \lesssim b^2, \quad \int |H^2\Sigma_b|^2 \lesssim \frac{b^2}{|\log b|^2}. \quad (5.2.40)$$

Step 4 Construction of T_2 .

Let

$$\Sigma_2 = \Sigma_b + T_1 - \Lambda T_1 - \frac{f''(Q)}{2y^2} T_1^2. \quad (5.2.41)$$

First observe from (5.1.4) that $f = gg'$ is odd and 2π periodic and thus :

$$\forall k \geq 0, \quad |f^{(2k)}(u)| + |f^{(2k)}(\pi - u)| \lesssim C_k |u|, \quad |f^{(2k+1)}(u)| \lesssim 1$$

which implies :

$$\forall k \geq 0, \quad |f^{(2k)}(Q)| \lesssim \frac{y}{1+y^2}. \quad (5.2.42)$$

We estimate from (5.2.12), (5.2.30), (5.2.31), (5.2.37) : for $6B_0 \leq y \leq 2B_1$,

$$\Sigma_2(y) = O\left(\frac{|\log y|^2}{y}\right),$$

and for $y \leq 6B_0$, there holds the behavior (5.2.29) at the origin and the desired cancellation :

$$\begin{aligned} \Sigma_2(y) &= y \frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} - y + O\left(\frac{1+y}{|\log b|}\right) + O\left(\frac{|\log y|^2}{1+y}\right) \\ &= O\left(\frac{1+y}{|\log b|}(1+|\log(y\sqrt{b})|)\right). \end{aligned}$$

Remarque 5.2.7. The above cancellation is due both to the presence of the T_1 term in the RHS of (5.2.41) which follows from the choice of modulation equation $b_s = -b^2$ to leading order and cancels the $y \log y$ growth of T_1 , and the radiation term Σ_b which is designed to cancel the remaining y growth in $T_1 - \Lambda T_1$.

We similarly estimate : for $0 \leq i \leq 2$,

$$|\Lambda^i \Sigma_2| \lesssim \frac{y^3}{1+y^2} \left(\mathbf{1}_{y \leq 1} + \frac{1 + \log(y\sqrt{b})}{|\log b|} \mathbf{1}_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y} \mathbf{1}_{y \geq 6B_0}. \quad (5.2.43)$$

We now let T_2 be the solution to

$$HT_2 = \Sigma_2 \quad (5.2.44)$$

given by

$$T_2(y) = -\Gamma(y) \int_0^y \Sigma_2 \Lambda Q x dx + \Lambda Q(y) \int_0^y \Sigma_2 \Gamma x dx$$

which satisfies (5.2.29) and the estimate from (5.2.43) :

$$\forall y \leq 2B_1, \quad |\Lambda^i T_2(y)| \lesssim \frac{y^5}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right), \quad \text{for } 0 \leq i \leq 3 \quad (5.2.45)$$

We also have the rougher bound :

$$\forall y \leq 2B_1, \quad |T_2(y)| \lesssim y^3. \quad (5.2.46)$$

T_2 also displays a small dependence on b through Σ_2 which will be evaluated in the proof of Proposition 5.2.8.

Step 5 Construction of T_3 .

Let

$$\Sigma_3 = 2T_2 - \Lambda T_2 - \frac{f''(Q)}{y^2} T_1 T_2 - \frac{1}{6} \frac{f^{(3)}(Q)}{y^2} T_1^3, \quad (5.2.47)$$

then Σ_3 satisfies (5.2.29) and we estimate from (5.2.31), (5.2.42), (5.2.45) :

$$\forall y \leq 2B_1, \quad |\Lambda^i \Sigma_3(y)| \lesssim \frac{y^5}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right), \quad \text{for } 0 \leq i \leq 2 \quad (5.2.48)$$

We then let T_3 be the solution to

$$HT_3 = \Sigma_3 \quad (5.2.49)$$

given by :

$$T_3(y) = -\Gamma(y) \int_0^y \Sigma_3 \Lambda Q + \Lambda Q(y) \int_0^y \Sigma_3 \Gamma$$

which satisfies (5.2.29) and the estimates from (5.2.48) :

$$\forall y \leq 2B_1, \quad |\Lambda^i T_3(y)| \lesssim \frac{y^7}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right), \quad 0 \leq i \leq 1, \quad (5.2.50)$$

$$|T_3(y)| \lesssim y^3(1+y^2). \quad (5.2.51)$$

T_3 also displays a small dependence on b which will be evaluated in the proof of Proposition 5.2.8. We claim the bounds for $i = 0, 1$:

$$\int_{y \leq 2B_1} |H\Lambda^i T_3|^2 \lesssim \frac{|\log b|^2}{b^4}, \quad \int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} |H\Lambda^i T_3|^2 \lesssim \frac{1}{b^4 |\log b|^2} \quad (5.2.52)$$

$$\int \frac{1 + |\log y|^2}{1 + y^2} |A H \Lambda^i T_3|^2 \lesssim \frac{|\log b|^6}{b^2}, \quad \int_{y \leq 2B_1} |H^2 \Lambda^i T_3|^2 \lesssim \frac{1}{b^2 |\log b|^2}. \quad (5.2.53)$$

Proof of (5.2.52), (5.2.53) : Observe from a simple rescaling argument that for any function f :

$$H\Lambda f = 2Hf + \Lambda Hf - \frac{\Lambda V}{y^2} f. \quad (5.2.54)$$

Hence from (5.2.49), (5.2.15) :

$$\begin{aligned} H(\Lambda T_3) &= 2\Sigma_3 + \Lambda\Sigma_3 - \frac{\Lambda V}{y^2} T_3 = 2\Sigma_3 + \Lambda\Sigma_3 + O(y) \\ H^2(\Lambda T_3) &= H(2\Sigma_3 + \Lambda\Sigma_3) + O\left(\frac{1}{1+y}\right). \end{aligned} \quad (5.2.55)$$

We thus estimate from (5.2.48), (5.1.20) for $i = 0, 1$:

$$\begin{aligned} \int_{y \leq 2B_1} |H\Lambda^i T_3|^2 &\lesssim \int_{y \leq 2B_1} \left| \frac{1+y}{b|\log b|} \right|^2 \lesssim \frac{B_1^4}{b^2 |\log b|^2} \lesssim \frac{|\log b|^2}{b^4}, \\ \int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} |H\Lambda^i T_3|^2 &\lesssim \int_{y \leq 2B_1} \frac{1}{b^2 |\log b|^2} \frac{(1+y^2)(1+|\log y|^2)}{1+y^4} \lesssim \frac{1}{b^4 |\log b|^2}, \end{aligned} \quad (5.2.56)$$

and using the rough bound (5.2.51) :

$$\int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^2} |A H \Lambda^i T_3|^2 \lesssim \int_{y \leq 2B_1} \frac{(1+y^4)(1+|\log y|^2)}{1+y^2} \lesssim B_1^4 |\log b|^2 \lesssim \frac{|\log b|^6}{b^2}.$$

The second bound in (5.2.53) is more subtle and requires further cancellations with respect to (5.2.50). Indeed, from (5.2.54), (5.2.47), (5.2.44) :

$$\begin{aligned} H\Sigma_3 &= 2HT_2 - H\Lambda T_2 + O\left(\frac{|\log y|^3}{y}\right) = \Lambda\Sigma_2 + O\left(\frac{|\log y|^5}{y}\right), \\ H\Lambda\Sigma_3 &= -2\Lambda\Sigma_2 - \Lambda^2\Sigma_2 + O\left(\frac{|\log y|^5}{y}\right) \end{aligned}$$

and injecting this into (5.2.55) with (5.2.43) yields¹⁵ :

$$\begin{aligned} \int_{y \leq 2B_1} |H^2(\Lambda T_3)|^2 &\lesssim \int_{y \leq 2B_1} \left| \frac{y^3}{1+y^2} \left(\mathbf{1}_{y \leq 1} + \frac{1 + \log(y\sqrt{b})}{|\log b|} \mathbf{1}_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y} \mathbf{1}_{y \geq 6B_0} \right|^2 \\ &\lesssim \frac{1}{b^2 |\log b|^2} \end{aligned}$$

15. the key here is that Σ_2 decays at infinity from the cancellation $H\Gamma = 0$, and hence the control becomes independent of B_1

and (5.2.53) is proved.

step 6 Estimate on the error.

By construction, we have from (5.2.27) :

$$\Psi_b = b^2 \Sigma_b + b^4 \Lambda T_3 + \frac{1}{y^2} (R_1 + R_2). \quad (5.2.57)$$

We inject into the formulas (5.2.25), (5.2.26) the rough bounds (5.2.31), (5.2.46), (5.2.51) and the definition of B_1 (5.1.20), and obtain the rough bound :

$$\forall y \leq 2B_1, \quad \left| \frac{d^i}{dy^i} \frac{R_1(y)}{y^2} \right| + \left| \frac{d^i}{dy^i} \frac{R_2(y)}{y^2} \right| \lesssim b^4 y^{5-i} \mathbf{1}_{y \leq 1} + \frac{b^4 y^5 |\log b|^C}{1 + y^{2+i}}, \quad 0 \leq i \leq 4.$$

This yields :

$$\begin{aligned} \int_{y \leq 2B_1} \left| H \left(\frac{R_1}{y^2} \right) \right|^2 + \left| H \left(\frac{R_2}{y^2} \right) \right|^2 &\lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{y^{10}}{y^8} \lesssim b^6 |\log b|^C, \\ \int_{y \leq 2B_1} \left| AH \left(\frac{R_1}{y^2} \right) \right|^2 + \left| AH \left(\frac{R_2}{y^2} \right) \right|^2 &\lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{y^{10}}{1 + y^{10}} \lesssim b^7 |\log b|^C, \\ \int_{y \leq 2B_1} \left| H^2 \left(\frac{R_1}{y^2} \right) \right|^2 + \left| H^2 \left(\frac{R_2}{y^2} \right) \right|^2 &\lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{y^{10}}{1 + y^{12}} \lesssim b^8 |\log b|^C. \end{aligned}$$

Injecting these bounds together with (5.2.39), (5.2.40), (5.2.52), (5.2.53) into (5.2.57) yields (5.2.20), (5.2.21), (5.2.22), (5.2.23).

We now prove the flux computation (5.2.24). From (5.2.57), (5.2.34) :

$$\begin{aligned} \frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} &= \frac{1}{(\Lambda Q, \Phi_M)} \left[\left(-b^2 c_b \chi_{\frac{B_0}{4}} \Lambda Q, \Phi_M \right) + O(C(M)b^3) \right] \\ &= -c_b b^2 + O(C(M)b^3) = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right), \end{aligned}$$

for $|b| < b^*(M)$ small enough. This concludes the proof of Proposition 5.2.4

5.2.4 Localization of the profile

The Q_b profile constructed above displays a very specific growth in the parabolic zone $y \sim B_0 \sim \frac{1}{\sqrt{b}}$. We claim that further away, this growth becomes irrelevant in the description of the full solution, and we may therefore proceed to a simple localization procedure for $y \geq 2B_1 \sim \frac{|\log b|}{\sqrt{b}} \gg B_0$.

Proposition 5.2.8 (Localization of the profile). *Let $(T_i)_{1 \leq i \leq 3}$ be given by Proposition 5.2.4. Let a C^1 map $s \mapsto b(s)$ defined on $[0, s_0]$ with a priori bound : $\forall s \in [0, s_0]$,*

$$0 < b(s) < b^*(M), \quad |b_s| \leq 10b^2. \quad (5.2.58)$$

Let the localized profile

$$\tilde{Q}_b(s, y) = Q + b\tilde{T}_1 + b^2\tilde{T}_2 + b^3\tilde{T}_3 = Q + \tilde{\alpha}$$

where

$$\tilde{T}_i = \chi_{B_1} T_i, \quad 1 \leq i \leq 3.$$

Then

$$\partial_s \tilde{Q}_b - \Delta \tilde{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \tilde{Q}_b + \frac{f(\tilde{Q}_b)}{y^2} = Mod(t) + \tilde{\Psi}_b \quad (5.2.59)$$

with

$$Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b\right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2) \quad (5.2.60)$$

and where $\tilde{\Psi}_b$ satisfies the bounds on $[0, s_0]$:

(i) Weighted bounds :

$$\int |H\tilde{\Psi}_b|^2 \lesssim b^4 |\log b|^2, \quad (5.2.61)$$

$$\int \frac{1 + |\log y|^2}{1 + y^4} |H\tilde{\Psi}_b|^2 \lesssim \frac{b^4}{|\log b|^2}, \quad (5.2.62)$$

$$\int \frac{1 + |\log y|^2}{1 + y^2} |AH\tilde{\Psi}_b|^2 \lesssim \frac{b^5}{|\log b|^2}, \quad (5.2.63)$$

$$\int |H^2\tilde{\Psi}_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (5.2.64)$$

(ii) Flux computation : Let Φ_M be given by (5.3.5), then :

$$\frac{(H\tilde{\Phi}_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right), \quad (5.2.65)$$

Proof of Proposition 5.2.8

step 1 Localization.

Let

$$\Psi_b^{(1)} = -b^2(\tilde{T}_1 + 2b\tilde{T}_2) - \Delta \tilde{Q}_b + b\Lambda \tilde{Q}_b + \frac{f(\tilde{Q}_b)}{y^2}.$$

We compute the action of localization which produces an error localized in $[B_1, 2B_1]$ up to the term $(1 - \chi_{B_1})\Lambda Q$:

$$\begin{aligned} \Psi_b^{(1)} &= \chi_{B_1} \Psi_b + b(1 - \chi_{B_1})\Lambda Q + b\Lambda \chi_{B_1} \alpha - \alpha \Delta \chi_{B_1} - 2\partial_y \chi_{B_1} \partial_y \alpha \\ &+ \frac{1}{y^2} [f(Q + \chi_{B_1} \alpha) - f(Q) - \chi_{B_1} (f(Q + \alpha) - f(Q))]. \end{aligned} \quad (5.2.66)$$

We estimate in brute force from (5.2.31), (5.2.46), (5.2.51) and the choice of B_1 :

$$\forall y \leq 2B_1, \quad |\alpha(y)| \lesssim by \left(|\log y| + \frac{by^2}{|\log b|} \right) \lesssim by |\log y|$$

and thus :

$$|b(1 - \chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha| \lesssim \frac{b}{y}\mathbf{1}_{y \geq B_1} + b^2y\log y\mathbf{1}_{B_1 \leq y \leq 2B_1},$$

$$\begin{aligned} \left| \frac{1}{y^2} [f(Q + \chi_{B_1}\alpha) - f(Q) - \chi_{B_1}(f(Q + \alpha) - f(Q))] \right| &\lesssim \frac{|\alpha(y)|}{y^2} \mathbf{1}_{B_1 \leq y \leq 2B_1} \\ &\lesssim \frac{b\log y}{y} \mathbf{1}_{B_1 \leq y \leq 2B_1} \end{aligned}$$

from which using (5.2.20), (5.2.21), (5.2.22), (5.2.23) :

$$\begin{aligned} \int |H\Psi_b^{(1)}|^2 &\lesssim b^4|\log b|^2 + \int_{B_1 \leq y \leq 2B_1} \left[\frac{b^2|\log y|^2}{y^6} + \frac{b^4|\log y|^2}{y^2} \right] \lesssim b^4|\log b|^2, \\ \int \frac{1 + |\log y|^2}{1 + y^4} |H\Psi_b^{(1)}|^2 &\lesssim \frac{b^4}{|\log b|^2} + \int_{B_1 \leq y \leq 2B_1} \left[\frac{b^2|\log y|^2}{y^8} + \frac{b^4|\log y|^2}{y^4} \right] \lesssim \frac{b^4}{|\log b|^2}, \\ \int \frac{1 + |\log y|^2}{1 + y^2} |AH\Psi_b^{(1)}|^2 &\lesssim \frac{b^5}{|\log b|^2} + \int_{B_1 \leq y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^2} \left[\frac{b^2}{y^8} + \frac{b^4|\log y|^2}{y^4} \right] \lesssim \frac{b^5}{|\log b|^2}, \\ \int |H^2\Psi_b^{(1)}|^2 &\lesssim \frac{b^6}{|\log b|^2} + \int_{B_1 \leq y \leq 2B_1} \left[\frac{b^2|\log y|^2}{y^{10}} + \frac{b^4|\log y|^2}{y^6} \right] \\ &\lesssim \frac{b^6}{|\log b|^2} + \frac{b^2|\log b|^2}{B_1^8} + \frac{b^4|\log b|^2}{B_1^4} \lesssim \frac{b^6}{|\log b|^2}. \end{aligned}$$

Remarque 5.2.9. This last estimate and (5.2.56) govern the choice $B_1 = \frac{|\log b|}{\sqrt{b}}$.

step 2 Control of time derivatives.

We now compute from (5.2.59) :

$$\tilde{\Phi}_b = \Psi_b^{(1)} + \tilde{R}, \quad \tilde{R} = b_s(3b^2\tilde{T}_3 + b\frac{\partial\tilde{T}_1}{\partial b} + b^2\frac{\partial\tilde{T}_2}{\partial b} + b^3\frac{\partial\tilde{T}_3}{\partial b}) \quad (5.2.67)$$

and estimate all terms. From (5.2.34),

$$\frac{\partial c_b}{\partial b} = O\left(\frac{1}{b|\log b|^2}\right), \quad \frac{\partial d_b}{\partial b} = O\left(\frac{1}{b^2|\log b|}\right) \quad (5.2.68)$$

and thus from (5.2.35) :

$$\begin{aligned} \frac{\partial\Sigma_b}{\partial b}(y) &= \frac{\partial c_b}{\partial b}T_1\mathbf{1}_{y \leq \frac{B_0}{2}} + O\left(\frac{1}{b^2y|\log b|}\mathbf{1}_{\frac{B_0}{2} \leq y \leq 6B_0}\right) \\ &= O\left(\frac{y}{b|\log b|}\mathbf{1}_{y \leq B_0} + \frac{1}{b^2y|\log b|}\mathbf{1}_{\frac{B_0}{2} \leq y \leq 6B_0}\right). \end{aligned}$$

We inject this estimate into the explicit formulas for T_2, T_3 and conclude :

$$\frac{\partial \tilde{T}_1}{\partial b} = O\left(\frac{y \log y}{b} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1}\right), \quad (5.2.69)$$

$$\frac{\partial \tilde{T}_2}{\partial b} = O\left(\frac{1+y^3}{b|\log b|} \mathbf{1}_{y \leq B_0} + \frac{y}{b^2|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 2B_1}\right), \quad (5.2.70)$$

$$\frac{\partial \tilde{T}_3}{\partial b} = O\left(\frac{1+y^5}{b|\log b|} \mathbf{1}_{y \leq B_0} + \frac{y^3}{b^2|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 2B_1}\right). \quad (5.2.71)$$

This yields together with the a priori bound (5.2.58) the pointwise control :

$$|\tilde{R}| \lesssim b^2 \left[\frac{b(1+y^3)}{|\log b|} \mathbf{1}_{y \leq B_0} + \frac{by^3}{|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 6B_1} + y \log y \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right]$$

and hence the bounds :

$$\begin{aligned} \int |H\tilde{R}|^2 &\lesssim b^4 \int \left| \frac{b(1+y)}{|\log b|} \mathbf{1}_{y \leq B_0} + \frac{by}{|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 6B_1} + \frac{\log y}{y} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right|^2 \\ &\lesssim b^4 |\log b|^2, \\ & \int \frac{1+|\log y|^2}{1+y^4} |H\tilde{R}|^2 + \int \frac{1+|\log y|^2}{1+y^2} |AH\tilde{R}|^2 \\ &\lesssim b^4 \int \frac{1+|\log y|^2}{1+y^4} \left| \frac{b(1+y)}{|\log b|} \mathbf{1}_{y \leq B_0} + \frac{by}{|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 6B_1} + \frac{\log y}{y} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right|^2 \\ &\lesssim \frac{b^5}{|\log b|^2} \end{aligned}$$

We now track for more cancellation when applying H^2 . Indeed, from (5.2.52), (5.2.50) :

$$\int |H^2 \tilde{T}_3|^2 \lesssim \frac{1}{b^2 |\log b|^2} + \int \left| \frac{y^3}{by^4 |\log b|} \mathbf{1}_{B_1 \leq y \leq 2B_1} \right|^2 \lesssim \frac{1}{b^2 |\log b|^2}.$$

From direct inspection :

$$\int \left| H^2 \left(\frac{\partial T_1}{\partial b} \right) \right|^2 \lesssim \int_{B_1 \leq y \leq 2B_1} \left| \frac{\log y}{by^3} \right|^2 \lesssim \frac{|\log b|^2}{b^2 B_1^4} \lesssim \frac{1}{|\log b|^2}.$$

Next :

$$H^2 \left(\frac{\partial \tilde{T}_2}{\partial b} \right) = \chi_{B_1} H^2 \left(\frac{\partial T_2}{\partial b} \right) + O\left(\frac{1}{y^3 b^2 |\log b|} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1}\right)$$

and from (5.2.44), (5.2.41) :

$$\begin{aligned} H^2 \left(\frac{\partial T_2}{\partial b} \right) &= H \left(\frac{\partial \Sigma_2}{\partial b} \right) = H \left(\frac{\partial \Sigma}{\partial b} \right) \\ &= O\left(\frac{1}{b(1+y)|\log b|^2} \mathbf{1}_{y \leq 2B_0} + \frac{1}{b(1+y)|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 2B_0}\right) \end{aligned}$$

which yields the bound :

$$\begin{aligned} \int \left| H^2 \left(\frac{\partial \tilde{T}_2}{\partial b} \right) \right|^2 &\lesssim \int \left| \frac{1}{b(1+y)|\log b|^2} \mathbf{1}_{y \leq 2B_0} + \frac{1}{b(1+y)|\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 2B_0} \right. \\ &\quad \left. + \frac{1}{y^3 b^2 |\log b|} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right|^2 \lesssim \frac{1}{b^2 |\log b|^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} H^2 \left(\frac{\partial \tilde{T}_3}{\partial b} \right) &= \chi_{B_1} H^2 \left(\frac{\partial T_3}{\partial b} \right) + O \left(\frac{1}{yb^2 |\log b|} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right) \\ &= O \left(\frac{1+y}{b|\log b|} \mathbf{1}_{y \leq B_0} + \frac{1}{yb^2 |\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 2B_0} + \frac{1}{yb^2 |\log b|} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right) \end{aligned}$$

from which :

$$\begin{aligned} \int \left| H^2 \left(\frac{\partial \tilde{T}_3}{\partial b} \right) \right|^2 &\lesssim \int \left| \frac{1+y}{b|\log b|} \mathbf{1}_{y \leq B_0} + \frac{1}{yb^2 |\log b|} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 2B_0} + \frac{1}{yb^2 |\log b|} \mathbf{1}_{\frac{B_1}{2} \leq y \leq 2B_1} \right|^2 \\ &\lesssim \frac{1}{b^4 |\log b|^2}. \end{aligned}$$

We inject these estimates into (5.2.67) and obtain :

$$\int |H^2 \tilde{R}|^2 \lesssim \frac{b^6}{|\log b|^2}.$$

Injecting the collection of estimates of step 1 and step 2 into (5.2.67) now yields the expected bounds (5.2.61), (5.2.62), (5.2.63), (5.2.64).

step 3 Flux computation.

By definition, $\Psi_b^{(1)} = \Psi_b$ on $\text{Supp}(\Phi_M) \subset [0, 2M]$, and from (5.2.69), (5.2.70), (5.2.71) and (5.2.50) :

$$|\tilde{R}| \lesssim C(M)|b_s|b \lesssim C(M)b^3 \text{ on } \text{Supp}(\Phi_M).$$

This estimate together with (5.2.24) now yields (5.2.65).

This concludes the proof of Proposition 5.2.8.

We introduce a second localization of the profile near B_0 which will be used only to capture some further cancellation in the proof of the regularity claim (5.1.14).

Lemme 5.2.10 (Second localization). *Let a C^1 map $s \mapsto b(s)$ defined on $[0, s_0]$ with a priori bound (5.1.14). Let the localized profile*

$$\hat{Q}_b(s, y) = Q + b\hat{T}_1 + b^2\hat{T}_2 + b^3\hat{T}_3 = Q + \hat{\alpha} \tag{5.2.72}$$

where

$$\hat{T}_i = \chi_{B_0} T_i, \quad 1 \leq i \leq 3.$$

Let the radiation :

$$\zeta_b = \tilde{\alpha} - \hat{\alpha} \quad (5.2.73)$$

and the error

$$\partial_s \hat{Q}_b - \Delta \hat{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \hat{Q}_b + \frac{f(\hat{Q}_b)}{y^2} = \widehat{Mod}(t) + \hat{\Psi}_b$$

with

$$\widehat{Mod}(t) = - \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \hat{Q}_b + (b_s + b^2)(\hat{T}_1 + 2b\hat{T}_2). \quad (5.2.74)$$

Then there holds the bounds :

$$\int |H\zeta_b|^2 \lesssim b^2 |\log b|^2, \quad \Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1+y^{8-2i}} \lesssim b^4 |\log b|^C, \quad (5.2.75)$$

$$\Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1+y^{2(3-i)}} \lesssim b^3 |\log b|^C, \quad (5.2.76)$$

$$Supp(H\hat{\Psi}_b) \subset [0, 2B_0] \quad \text{and} \quad \int |H\hat{\Psi}_b|^2 \lesssim b^4 |\log b|^2. \quad (5.2.77)$$

Remarque 5.2.11. Note that this localization near B_0 displays the same properties like the one near B_0 at the \mathcal{H}^2 level and (5.2.61), (5.2.77) are comparable. The estimate corresponding to (5.2.64) would however be worse for $\hat{\Psi}_b$ due to the terms induced by localization, see Remark 5.2.9. Hence we will use the B_1 localization to control high derivatives norms, see Proposition 5.3.4 and the control of \mathcal{E}_4 below, and B_0 localization for lower order control, see section 5.4.1 and the control of \mathcal{E}_2 .

Proof of Lemma 5.2.10 By construction,

$$\zeta_b = (\chi_{B_1} - \chi_{B_0})(bT_1 + b^2T_2 + b^3T_3)$$

and thus from (5.2.45), (5.2.50) :

$$\begin{aligned} \int |H\zeta_b|^2 &\lesssim \int_{y \leq 2B_1} \left| \frac{by \log y}{y^2} + \frac{b^2 y}{by^2 |\log b|} + \frac{b^3 y^3}{by^2 |\log b|} \right|^2 \lesssim b^2 |\log b|^2, \\ \Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1+y^{8-2i}} &\lesssim \int_{B_0 \leq y \leq 2B_1} \frac{1}{1+y^8} \left[b^2 y^2 |\log y|^2 + \frac{b^6 y^6}{b^2 |\log b|^2} \right] \lesssim b^4 |\log b|^C, \\ \Sigma_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1+y^{2(3-i)}} &\lesssim \int_{B_0 \leq y \leq 2B_1} \frac{1}{1+y^6} \left[b^2 y^2 |\log y|^2 + \frac{b^6 y^6}{b^2 |\log b|^2} \right] \lesssim b^3 |\log b|^C. \end{aligned}$$

The localization property (5.2.77) directly follows from the analogue of the formula (5.2.66), (5.2.67) for $\hat{\Psi}_b$ and the cancellation $H\Lambda Q = 0$, while the proof of the estimate (5.2.77) is very similar to the one of (5.2.61) and left to the reader.

This concludes the proof of Lemma 5.2.10.

5.3 The trapped regime

This section is devoted to the description of the set of initial data and the corresponding trapped regime in which the singularity formation described by Theorem 5.1.1 will occur.

5.3.1 Setting the bootstrap

We describe in this section the set of initial data leading to the blow up scenario of Theorem 5.1.1. Let a 1-corotational map

$$v_0 \in \dot{H}^1 \cap \dot{H}^4 \text{ with } \|\nabla(v_0 - Q)\|_{L^2} \ll 1$$

and $v(t) \in \mathcal{C}([0, T), \dot{H}^1 \cap \dot{H}^4)$, $0 < T \leq +\infty$ be the corresponding solution to (5.1.1). First recall from standard argument the blow up criterion :

$$T < +\infty \text{ implies } \|\Delta v(t)\|_{L^2} \rightarrow +\infty \text{ as } t \rightarrow T. \quad (5.3.1)$$

From Lemma 5.5.1, v admits on a small time interval $[0, t_1]$ a decomposition

$$v(t, x) = \begin{cases} g(u(t, r)) \cos \theta \\ g(u(t, r)) \sin \theta \\ z(u(t, r)) \end{cases} \quad (5.3.2)$$

where $\tilde{\varepsilon}(t, r) = u(t, r) - Q(r)$ satisfies the boundary condition (5.5.3) and the regularity (5.5.4), (5.5.5) displayed in Lemma 5.5.1. Moreover, from the initial smallness

$$\|\nabla \tilde{\varepsilon}(0)\|_{L^2} + \left\| \frac{\tilde{\varepsilon}(0)}{y} \right\|_{L^2} \ll 1,$$

we may from standard modulation argument¹⁶ introduce for each fixed time $t \in [0, t_1]$ the unique decomposition

$$u(t, r) = (\tilde{Q}_{b(t)} + \varepsilon) \left(t, \frac{y}{\lambda(t)} \right), \quad \|\nabla \varepsilon(t)\|_{L^2} + \left\| \frac{\varepsilon(t)}{y} \right\|_{L^2} + |b(t)| \ll 1, \quad \lambda(t) > 0 \quad (5.3.3)$$

where $\varepsilon(t)$ satisfies the orthogonality conditions :

$$\forall t \in [0, t_1], \quad (\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0. \quad (5.3.4)$$

Here given $M > 0$ large enough, Φ_M corresponds to the fixed direction

$$\Phi_M = \chi_M \Lambda Q - c_M H(\chi_M \Lambda Q) \quad (5.3.5)$$

with

$$c_M = \frac{(\chi_M \Lambda Q, T_1)}{(H(\chi_M \Lambda Q), T_1)} = c_\chi \frac{M^2}{4} (1 + o_{M \rightarrow +\infty}(1)).$$

Observe by construction that

$$\int |\Phi_M|^2 \lesssim |\log M|, \quad (\Phi_M, T_1) = 0 \quad (5.3.6)$$

and the scalar products

$$(\Lambda Q, \Phi_M) = (-HT_1, \Phi_M) = (\chi_M \Lambda Q, \Lambda Q) = 4 \log M (1 + o_{M \rightarrow +\infty}(1)) \quad (5.3.7)$$

^{16.} see for example [58], [69], [88] for a further introduction to modulation.

are non degenerate. The existence of the decomposition (5.3.3) is then a standard consequence of the implicit function theorem and the explicit relations

$$\left(\frac{\partial}{\partial \lambda} (\tilde{Q}_b)_\lambda, \frac{\partial}{\partial b} (\tilde{Q}_b)_\lambda \right) |_{\lambda=1, b=0} = (\Lambda Q, T_1)$$

which ensure the nondegeneracy of the Jacobian :

$$\begin{vmatrix} \left(\frac{\partial}{\partial \lambda} (\tilde{Q}_b)_\lambda, \Phi_M \right) & \left(\frac{\partial}{\partial b} (\tilde{Q}_b)_\lambda, \Phi_M \right) \\ \left(\frac{\partial}{\partial \lambda} (\tilde{Q}_b)_\lambda, H\Phi_M \right) & \left(\frac{\partial}{\partial b} (\tilde{Q}_b)_\lambda, H\Phi_M \right) \end{vmatrix} \Big|_{\lambda=1, b=0} = \begin{vmatrix} (\Lambda Q, \Phi_M) & 0 \\ 0 & (T_1, H\Phi_M) \end{vmatrix} = -(\Lambda Q, \Phi_M)^2 \neq 0.$$

From Lemma 5.5.1, we may measure the regularity of the map through the following norms of ε : the energy norm

$$\|\varepsilon\|_{\mathcal{H}}^2 = \int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2}, \quad (5.3.8)$$

and higher order Sobolev norms adapted to the linearized operator

$$\mathcal{E}_{2k} = \int |H^k \varepsilon|^2, \quad 1 \leq k \leq 2. \quad (5.3.9)$$

We now assume the following bounds on initial data which describe an open $\dot{H}^1 \cap H^4$ affine space of 1-corotational initial data around \mathcal{Q} :

- Smallness and positivity of $b(0)$:

$$0 < b(0) < b^*(M) \ll 1. \quad (5.3.10)$$

- Smallness of the excess of energy :

$$\|\varepsilon(0)\|_{\mathcal{H}}^2 \leq b^2(0), \quad (5.3.11)$$

$$|\mathcal{E}_2(0)| + |\mathcal{E}_4(0)| \leq [b(0)]^{10}. \quad (5.3.12)$$

The regularity of the full map $v \in \mathcal{C}^0([0, T], \dot{H}^1 \cap \dot{H}^4)$ by the parabolic heat flow and Lemma 5.5.1 ensure that these estimates hold on some small enough time interval $[0, t_1]$, and from standard argument¹⁷, there also holds the regularity $(\lambda, b) \in \mathcal{C}^1([0, t_1], \mathbb{R}_+^* \times \mathbb{R})$. Given a large enough universal constant $K > 0$ -independent of M - , we assume on $[0, t_1]$ the following bootstrap bounds :

- Control of $b(t)$:

$$0 < b(t) < 10b(0). \quad (5.3.13)$$

- Control of the radiation :

$$\|\varepsilon(t)\|_{\mathcal{H}}^2 \leq 10\sqrt{b(0)}, \quad (5.3.14)$$

$$|\mathcal{E}_2(t)| \leq Kb^2(t)|\log b(t)|^5, \quad (5.3.15)$$

$$|\mathcal{E}_4(t)| \leq K \frac{b^4(t)}{|\log b(t)|^2}. \quad (5.3.16)$$

¹⁷. see [58] for more details in a completely similar setting.

The following proposition describes the contraction of the bootstrap regime and is the core of the proof.

Proposition 5.3.1 (Bootstrap control of b and ε). *Assume that K in (5.3.13), (5.3.14), (5.3.15) and (5.3.16) has been chosen large enough. Then, $\forall t \in [0, t_1]$:*

$$0 < b(t) < 2b(0), \quad (5.3.17)$$

$$\|\varepsilon(t)\|_{\mathcal{H}}^2 \leq \sqrt{b(0)}, \quad (5.3.18)$$

$$|\mathcal{E}_2(t)| \leq \frac{K}{2} b^2(t) |\log b(t)|^5, \quad (5.3.19)$$

$$|\mathcal{E}_4(t)| \leq \frac{K}{2} \frac{b^4(t)}{|\log b(t)|^2}. \quad (5.3.20)$$

In other words, the solution is trapped in the regime (5.3.17), (5.3.18), (5.3.19), (5.3.20) and thus $t_1 = T$, the life time of the solution. We will in section 5.4.2 that this easily implies finite time blow up $T < +\infty$ in the regime described by Theorem 5.1.1. The rest of this section is devoted to developing the tools needed for the proof of Proposition 5.3.1 which will be completed in section 5.4.1.

5.3.2 Equation for the radiation

Let us now pass to renormalized variables¹⁸ :

$$u(t, r) = z(s, y), \quad y = \frac{r}{\lambda(t)}, \quad s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} \quad (5.3.21)$$

and recall from (5.3.3) the decomposition of the flow :

$$z(s, y) = (Q + \tilde{\alpha} + \varepsilon)(s, y).$$

We will use the rescaling formula :

$$\partial_t u = \frac{1}{\lambda^2(t)} (\partial_s z - \frac{\lambda_s}{\lambda} \Lambda z)_\lambda$$

to derive the equation for ε in renormalized variables :

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon + H \varepsilon = F - Mod = \mathcal{F}. \quad (5.3.22)$$

Here H is the linearized operator given by (5.2.9), $Mod(t)$ is given by (5.2.60),

$$F = -\tilde{\Phi}_b + L(\varepsilon) - N(\varepsilon) \quad (5.3.23)$$

where L is the linear operator corresponding to the error in the linearized operator from Q to \tilde{Q}_b :

$$L(\varepsilon) = \frac{f'(Q) - f'(\tilde{Q}_b)}{y^2} \varepsilon \quad (5.3.24)$$

18. we will show that the renormalized time is global : $\lim_{t \rightarrow T} s(t) \rightarrow +\infty$, see (5.4.16).

and the remainder term is the purely nonlinear term :

$$N(\varepsilon) = \frac{f(\tilde{Q}_b + \varepsilon) - f(\tilde{Q}_b) - \varepsilon f'(\tilde{Q}_b)}{y^2}. \quad (5.3.25)$$

We also need to write the flow (5.3.22) in original variables (t, r) . Let

$$w(t, r) = \varepsilon(s, y)$$

and the rescaled operators be :

$$\begin{aligned} A_\lambda &= -\partial_r + \frac{Z_\lambda}{r}, \quad A_\lambda^* = \partial_r + \frac{1 + Z_\lambda}{r}, \\ H_\lambda &= A_\lambda^* A_\lambda = -\Delta + \frac{V_\lambda}{r^2}, \quad \tilde{H}_\lambda = A_\lambda A_\lambda^* = -\Delta + \frac{\tilde{V}_\lambda}{r^2}, \end{aligned} \quad (5.3.26)$$

then (5.3.22) becomes :

$$\partial_t w + H_\lambda w = \frac{1}{\lambda^2} \mathcal{F}_\lambda \quad (5.3.27)$$

5.3.3 Modulation equations

Let us now compute the modulation equations for (b, λ) as a consequence of the choice of orthogonality conditions (5.3.4).

Lemme 5.3.2 (Modulation equations). *There holds the bound on the modulation parameters :*

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}, \quad (5.3.28)$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log M}} \left(\sqrt{\mathcal{E}_4} + \frac{b^2}{|\log b|} \right). \quad (5.3.29)$$

Remarque 5.3.3. Note that this implies in the bootstrap the rough bounds :

$$|b_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \leq 2b^2. \quad (5.3.30)$$

and in particular (5.2.58) holds.

Proof of Lemma 5.3.2

step 1 Law for b.

Let

$$\mathcal{V}(t) = |b_s + b^2| + \left| \frac{\lambda_s}{\lambda} + b \right|$$

We take the inner product of (5.3.22) with $H\Phi_M$ and compute using (5.3.4) :

$$\begin{aligned} (\text{Mod}(t), H\Phi_M) &= -(\tilde{\Phi}_b, H\Phi_M) - (H\varepsilon, H\Phi_M) \\ &\quad - \left(-\frac{\lambda_s}{\lambda} \Lambda\varepsilon - L(\varepsilon) + N(\varepsilon), H\Phi_M \right). \end{aligned} \quad (5.3.31)$$

We first compute from the construction of the profile, (5.2.60), the relations $H\Lambda Q = 0$, $HT_1 = -\Lambda Q$, and the localization $\text{Supp}(\Phi_M) \subset [0, 2M]$ from (5.3.5) :

$$\begin{aligned}(H(\text{Mod}(t)), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda}\right)(H\Lambda\tilde{Q}_b, \Phi_M) + (b_s + b^2)(H(\tilde{T}_1 + 2b\tilde{T}_2), \Phi_M) \\ &= -(\Lambda Q, \Phi_M)(b_s + b^2) + O(c(M)b|\mathcal{V}(t)|).\end{aligned}$$

The linear term in (5.3.31) is estimated¹⁹ from (5.3.6) :

$$|(H\varepsilon, H\Phi_M)| \lesssim \|H^2\varepsilon\|_{L^2} \sqrt{\log M} = \sqrt{\log M \mathcal{E}_4}$$

and the remaining nonlinear term is estimated using the Hardy bounds of Appendix (5.5.2) :

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda\varepsilon + L(\varepsilon) + N(\varepsilon), H\Phi_M \right) \right| \lesssim C(M)b(\sqrt{\mathcal{E}_4} + |\mathcal{V}(t)|).$$

We inject these estimates into (5.3.31) and conclude from (5.3.7) and the fundamental flux computation (5.2.65) :

$$b_s + b^2 = \frac{(\tilde{\Phi}_b, H\Phi_M)}{(\Lambda Q, \Phi_M)} + O\left(\frac{\sqrt{\log M \mathcal{E}_4}}{\log M}\right) + O(C(M)b|\mathcal{V}(t)|)$$

and hence the first modulation equation :

$$b_s + b^2 = -\frac{2b^2}{|\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right)\right) + O\left(\sqrt{\frac{\mathcal{E}_4}{\log M}} + C(M)b|\mathcal{V}(t)|\right). \quad (5.3.32)$$

step 2 Degeneracy of the law for λ .

We now take the inner product of (5.3.22) with Φ_M and obtain :

$$(\text{Mod}(t), \Phi_M) = -(\tilde{\Phi}_b, \Phi_M) - (H\varepsilon, \Phi_M) - \left(-\frac{\lambda_s}{\lambda} \Lambda\varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M\right).$$

Note first that the choice of orthogonality conditions (5.3.4) gets rid of the linear term in ε :

$$(H\varepsilon, \Phi_M) = 0.$$

Next, we compute from (5.3.7) and the orthogonality (5.3.6) :

$$\begin{aligned}(\text{Mod}(t), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda}\right)(\Lambda\tilde{Q}_b, \Phi_M) + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2, \Phi_M) \\ &= -4\log M(1 + o_{M \rightarrow +\infty}(1)) \left(\frac{\lambda_s}{\lambda} + b\right) + O(C(M)b|\mathcal{V}(t)|).\end{aligned}$$

and observe the cancellation from (5.2.36), (5.3.6) :

$$|(\tilde{\Phi}_b, \Phi_M)| \lesssim b^2|(\Sigma_b, \Phi_M)| + O(C(M)b^3) = c_b b^2|(T_1, \Phi_M)| + O(C(M)b^3) = O(C(M)b^3).$$

19. Observe that we do not use the interpolated bounds of Lemma 5.5.4 but directly the definition (5.3.9) of \mathcal{E}_4 , and hence the dependence of the constant in M is explicit what is crucial for the analysis.

Nonlinear terms are easily estimated using the Hardy bounds of Appendix (5.5.2) :

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M \right) \right| \lesssim C(M)b \left(\sqrt{\mathcal{E}_4} + |\mathcal{V}(t)| \right).$$

We thus obtain the modulation equation for scaling :

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim b^3 C(M) + b C(M) O \left(\sqrt{\mathcal{E}_4} + |\mathcal{V}(t)| \right).$$

Combining this with (5.3.32) yields the bound

$$|\mathcal{V}(t)| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}$$

which together with (5.3.32) again now implies the refined bound (5.3.29). This concludes the proof of Lemma 5.3.2.

5.3.4 The Lyapounov monotonicity

We now turn to the core of the argument which is the derivation of a suitable Lyapounov functional at the Sobolev \dot{H}^4 level. The parabolic structure will yield further dissipation with respect to the analysis of dispersive problems in [88], [72], what will allow us to treat a general metric g .

Proposition 5.3.4 (Lyapounov monotonicity). *There holds :*

$$\frac{d}{dt} \left\{ \frac{1}{\lambda^6} \left[\mathcal{E}_4 + O \left(\sqrt{b} \frac{b^4}{|\log b|^2} \right) \right] \right\} \leq C \frac{b}{\lambda^8} \left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|} \sqrt{\mathcal{E}_4} \right] \quad (5.3.33)$$

for some universal constant $C > 0$ independent of M and of the bootstrap constant K in (5.3.13), (5.3.14), (5.3.15), (5.3.16), provided $b^*(M)$ in (5.3.10) has been chosen small enough.

Proof of Proposition 5.3.4

The proof relies on the derivation of the energy identity for suitable derivatives of ε seen in original variables ie w , and repulsivity properties of the corresponding time dependent Hamiltonian \tilde{H}_λ . The control of the solution is then ensured thanks to coercivity properties of the iterated Hamiltonian H, H^2 under the orthogonality conditions (5.3.4), see Lemma 5.5.3. Nonlinear terms will be estimated using the interpolated bounds of Lemma 5.5.4 which will be implicitly used all along the proof.

step 1 Suitable derivatives.

We define the derivatives of w associated with the linearized Hamiltonian H :

$$w_1 = A_\lambda w, \quad w_2 = A_\lambda^* w_1, \quad w_3 = A_\lambda w_2$$

which satisfy from (5.3.27) :

$$\partial_t w_1 + \tilde{H}_\lambda w_1 = \frac{\partial_t Z_\lambda}{r} w + A_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right)$$

$$\partial_t w_2 + H_\lambda w_2 = \frac{\partial_t V_\lambda}{r^2} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \quad (5.3.34)$$

$$\partial_t w_3 + \tilde{H}_\lambda w_3 = \frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) + (AH)_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \quad (5.3.35)$$

We similarly use in the following steps the notation :

$$\varepsilon_1 = A\varepsilon, \quad \varepsilon_2 = A^*\varepsilon_1, \quad \varepsilon_3 = A\varepsilon_2.$$

Observe from (5.3.9) that

$$\mathcal{E}_4 = \int |A^*\varepsilon_3|^2. \quad (5.3.36)$$

We recall the action of time derivatives on rescaling :

$$\partial_t v_\lambda = \frac{1}{\lambda^2} \left(\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v \right)_\lambda.$$

step 2 Modified energy identity.

We compute the energy identity on (5.3.35) using (5.3.26) :

$$\begin{aligned} \frac{1}{2} \frac{d\mathcal{E}_4}{dt} &= \frac{1}{2} \frac{d}{dt} \left\{ \int \tilde{H}_\lambda w_3 w_3 \right\} = \int \tilde{H}_\lambda w_3 \partial_t w_3 + \int \frac{\partial_t \tilde{V}_\lambda}{2r^2} w_3^2 \\ &= - \int (\tilde{H}_\lambda w_3)^2 + b \int \frac{(\Lambda \tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2 - \left(\frac{\lambda_s}{\lambda} + b \right) \int \frac{(\Lambda \tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2 \\ &\quad + \int \tilde{H}_\lambda w_3 \left[\frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) + (AH)_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right]. \end{aligned} \quad (5.3.37)$$

We now aim at using the dissipative term $\int (\tilde{H} w_3)^2$ to treat the quadratic terms in the RHS of (5.3.37). Observe however that this quantity is delicate to use because it is positive but not coercive a priori²⁰. Also one can explicitly compute for the sphere target

$$b \frac{(\Lambda V)_\lambda}{\lambda^2 r^2} = - \frac{b}{\lambda^8} \frac{4}{(1+y^2)^2} < 0,$$

and thus the critical in size quadratic term has the right sign in this case :

$$b \int \frac{(\Lambda \tilde{V})_\lambda}{\lambda^2 r^2} w_3^2 < 0$$

which would allow some simplification of our analysis. However, this sign property does not seem to hold a priori for the general metric g we consider. We nevertheless claim using a similar algebra as in [88], [72] that this term can be treated thanks to a further integration by parts in

20. as can be seen by considering the zero of \tilde{H} given $w_3 = \frac{1}{y\Lambda Q} \int_0^y \tau(\Lambda Q)^2 d\tau$.

time which in the dispersive cases would correspond to a Morawetz type computation. Indeed, we compute from (5.3.34), (5.3.35) :

$$\begin{aligned} \frac{d}{dt} \left\{ \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 w_2 \right\} &= \int \frac{d}{dt} \left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} \right) w_3 w_2 \\ &+ \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_2 \left[-\tilde{H}_\lambda w_3 + \frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) + (AH)_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right] \\ &+ \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 \left[-A_\lambda^* w_3 + \frac{\partial_t V_\lambda}{r^2} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right] \end{aligned}$$

We now integrate by parts to compute using (5.2.8) :

$$\begin{aligned} \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 A_\lambda^* w_3 &= \frac{b}{\lambda^8} \int \frac{\Lambda Z}{y} \varepsilon_3 A^* \varepsilon_3 = \frac{b}{\lambda^8} \int \frac{2(1+Z)\Lambda Z - \Lambda^2 Z}{2y^2} \varepsilon_3^2 \\ &= \frac{b}{\lambda^8} \int \frac{\Lambda \tilde{V}}{2y^2} \varepsilon_3^2 = b \int \frac{(\Lambda \tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2. \end{aligned}$$

Injecting this into the energy identity (5.3.37) yields the modified energy identity :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ \mathcal{E}_4 + 2 \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 w_2 \right\} = - \int (\tilde{H}_\lambda w_3)^2 \\ &- \left(\frac{\lambda_s}{\lambda} + b \right) \int \frac{(\Lambda \tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2 + \int \frac{d}{dt} \left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} \right) w_3 w_2 \\ &+ \int \tilde{H}_\lambda w_3 \left[\frac{\partial_t Z_\lambda}{r} w_2 + \int A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) + (AH)_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right] \\ &+ \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_2 \left[-\tilde{H}_\lambda w_3 + \frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) + (AH)_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right] \\ &+ \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 \left[\frac{\partial_t V_\lambda}{r^2} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right] \end{aligned} \tag{5.3.38}$$

We now aim at estimating all terms in the RHS of (5.3.38). All along the proof, we shall make an implicit use of the coercivity estimates of Lemma 5.5.2 and Lemma 5.5.4.

step 3 Lower order quadratic terms.

We start with treating the lower order quadratic terms in (5.3.38) *using dissipation*. Indeed, we have from (5.2.14), (5.2.15), (5.3.30) the bounds :

$$|\partial_t Z_\lambda| + |\partial_t V_\lambda| \lesssim \frac{b}{\lambda^2} (|\Lambda Z| + |\Lambda V|)_\lambda \lesssim \frac{b}{\lambda^2} \frac{y^2}{1+y^4} \tag{5.3.39}$$

and thus from Cauchy Schwartz, the rough bound (5.3.30) and Lemma 5.5.4 :

$$\begin{aligned} &\int \left| \tilde{H}_\lambda w_3 \left[\frac{\partial_t Z_\lambda}{r} w_2 + \int A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) \right] \right| + \int |\tilde{H}_\lambda w_3| \left| \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_2 \right| \\ &\leq \frac{1}{2} \int |\tilde{H}_\lambda w_3|^2 + \frac{b^2}{\lambda^8} \left[\int \frac{\varepsilon_2^2}{1+y^6} + \int \frac{\varepsilon_1^2}{1+y^8} + \int \frac{\varepsilon^2}{y^2(1+y^8)} \right] \\ &\leq \frac{1}{2} \int |\tilde{H}_\lambda w_3|^2 + \frac{b}{\lambda^8} b |\log b|^C b^4. \end{aligned}$$

All other quadratic terms are lower order by a factor b using again (5.3.30) and Lemma 5.5.4 :

$$\begin{aligned}
& \left| \frac{\lambda_s}{\lambda} + b \right| \int \left| \frac{(\Lambda \tilde{V})_\lambda}{2\lambda^2 r^2} w_3^2 \right| + \int \left| \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_2 \left[\frac{\partial_t Z_\lambda}{r} w_2 + A_\lambda \left(\frac{\partial_t V_\lambda}{r^2} w \right) \right] \right| \\
& + \int \left| \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 \frac{\partial_t V_\lambda}{r^2} w \right| + \left| \int \frac{d}{dt} \left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} \right) w_3 w_2 \right| \\
& \lesssim \frac{b^2}{\lambda^8} \left[\int \frac{\varepsilon_3^2}{1+y^2} + \int \frac{\varepsilon_2^2}{1+y^4} + \int \frac{\varepsilon_1^2}{1+y^8} + \int \frac{\varepsilon^2}{y^2(1+y^8)} \right] \\
& \lesssim \frac{b}{\lambda^8} b |\log b|^C b^4.
\end{aligned}$$

We similarly estimate the boundary term in time :

$$\left| \int \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 w_2 \right| \lesssim \frac{b}{\lambda^6} \left[\int \frac{\varepsilon_3^2}{1+y^2} + \int \frac{\varepsilon_2^2}{1+y^4} \right] \lesssim \frac{b}{\lambda^6} |\log b|^C b^4.$$

We inject these estimates into (5.3.38) to derive the preliminary bound :

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\lambda^6} \left[\mathcal{E}_4 + O \left(\sqrt{b} \frac{b^4}{|\log b|^2} \right) \right] \right\} \leq -\frac{1}{2} \int (\tilde{H}_\lambda w_3)^2 + \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} \\
& + \int \tilde{H}_\lambda w_3 A_\lambda H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) + \int H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \left[\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 + A_\lambda^* \left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_2 \right) \right]
\end{aligned} \tag{5.3.40}$$

with constants independent of M for $|b| < b^*(M)$ small enough. We now aim at estimating all terms in the RHS of (5.3.40).

step 4 Further use of dissipation.

Let us introduce the decomposition from (5.3.22), (5.3.23) :

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1, \quad \mathcal{F}_0 = -\tilde{\Phi}_b - \text{Mod}(t), \quad \mathcal{F}_1 = L(\varepsilon) - N(\varepsilon).$$

The first term in the RHS of (5.3.40) is estimated after an integration by parts :

$$\begin{aligned}
& \left| \int \tilde{H}_\lambda w_3 A_\lambda H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right| \leq \frac{C}{\lambda^8} \|A^* \varepsilon_3\|_{L^2} \|H^2 \mathcal{F}_0\|_{L^2} + \frac{1}{4} \int (\tilde{H}_\lambda w_3)^2 + \frac{C}{\lambda^8} \int |AH\mathcal{F}_1|^2 \\
& \leq \frac{C}{\lambda^8} \left[\|H^2 \mathcal{F}_0\|_{L^2} \sqrt{\mathcal{E}_4} + \|AH\mathcal{F}_1\|_{L^2}^2 \right] + \frac{1}{4} \int (\tilde{H}_\lambda w_3)^2
\end{aligned} \tag{5.3.41}$$

for some universal constant $C > 0$ independent of M . The last two terms in (5.3.40) can be estimated in brute force from Cauchy Schwarz :

$$\begin{aligned}
\left| \int H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_3 \right| & \lesssim \frac{b}{\lambda^8} \left(\int \frac{1+|\log y|^2}{1+y^4} |H\mathcal{F}|^2 \right)^{\frac{1}{2}} \left(\int \frac{\varepsilon_3^2}{y^2(1+|\log y|^2)} \right)^{\frac{1}{2}} \\
& \lesssim \frac{b}{\lambda^8} \sqrt{\mathcal{E}_4} \left(\int \frac{1+|\log y|^2}{1+y^4} |H\mathcal{F}|^2 \right)^{\frac{1}{2}}
\end{aligned} \tag{5.3.42}$$

where constants are independent of M thanks to the estimate (5.5.13) for ε_3 . Similarly :

$$\begin{aligned} & \left| \int H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) A_\lambda^* \left(\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_2 \right) \right| \\ & \lesssim \frac{b}{\lambda^8} \left(\int \frac{1 + |\log y|^2}{1 + y^2} |AH\mathcal{F}|^2 \right)^{\frac{1}{2}} \left(\int \frac{\varepsilon_2^2}{1 + y^4(1 + |\log y|^2)} \right)^{\frac{1}{2}} \\ & \lesssim \frac{b}{\lambda^8} C(M) \sqrt{\mathcal{E}_4} \left(\int \frac{1 + |\log y|^2}{1 + y^2} |AH\mathcal{F}_0|^2 + \int |AH\mathcal{F}_1|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (5.3.43)$$

We now claim the bounds :

$$\int \frac{1 + |\log y|^2}{1 + y^4} |H\mathcal{F}|^2 \lesssim \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M}, \quad (5.3.44)$$

$$\int \frac{1 + |\log y|^2}{1 + y^2} |AH\mathcal{F}_0|^2 \lesssim \delta^* \left(\frac{b^4}{|\log b|^2} + \mathcal{E}_4 \right), \quad (5.3.45)$$

$$\int |H^2 \mathcal{F}_0|^2 \lesssim b^2 \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right], \quad (5.3.46)$$

$$\int |AH\mathcal{F}_1|^2 \lesssim b \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right], \quad (5.3.47)$$

with all \lesssim constants independent of M for $|b| < b^*(M)$ small enough, and where

$$\delta^* = \delta^*(b^*(M)) \rightarrow 0 \text{ as } b^*(M) \rightarrow 0.$$

Injecting these bounds together with (5.3.41), (5.3.42), (5.3.43) into (5.3.40) concludes the proof of (5.3.33). We now turn to the proof of (5.3.44), (5.3.45), (5.3.46), (5.3.47).

step 5 $\tilde{\Phi}_b$ terms.

The contribution of $\tilde{\Phi}_b$ terms to (5.3.44), (5.3.45), (5.3.46) is estimated from (5.2.62), (5.2.63), (5.2.64) which are at the heart of the construction of \tilde{Q}_b and yield the desired bounds.

step 6 $Mod(t)$ terms.

Recall the definition (5.2.60) of $Mod(t)$:

$$Mod(t) = - \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2).$$

For (5.3.44), we estimate using the rough bounds (5.2.46), (5.2.51) and the control of the modulation parameters (5.3.28), (5.3.29) to estimate :

$$\begin{aligned} & \int \frac{1 + |\log y|^2}{1 + y^4} |HMod|^2 \\ & \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int \frac{1 + |\log y|^2}{1 + y^4} |H\Lambda \tilde{Q}_b|^2 + |b_s + b^2|^2 \int \frac{1 + |\log y|^2}{1 + y^4} |H(\tilde{T}_1 + 2b\tilde{T}_2)|^2 \\ & \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 + |b_s + b^2|^2 \lesssim \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M}. \end{aligned}$$

For (5.3.45), we use the cancellations $H\Lambda Q = 0$, $AHT_1 = 0$ and the rough bounds (5.2.46), (5.2.51) to derive the degenerate bounds :

$$\begin{aligned} \int \frac{1 + |\log y|^2}{1 + y^2} |AH\Lambda \tilde{Q}_b|^2 &\lesssim \int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^2} \left| \frac{by|\log y| + b^2(1 + y^3) + b^3(1 + y^5)}{1 + y^3} \right|^2 \\ &\lesssim b^2 \end{aligned}$$

$$\begin{aligned} \int \frac{1 + |\log y|^2}{1 + y^2} |AH(\tilde{T}_1 + b\tilde{T}_2)|^2 &\lesssim \int \frac{1 + |\log y|^2}{1 + y^2} \left| \frac{y\log y}{y^3} \mathbf{1}_{B_1 \leq y \leq 2B_1} + b \frac{1 + y^3}{1 + y^3} \mathbf{1}_{y \leq 2B_1} \right|^2 \\ &\lesssim b \end{aligned}$$

and thus from (5.3.28), (5.3.29) :

$$\int \frac{1 + |\log y|^2}{1 + y^2} |AH\mathcal{M}od|^2 \lesssim b \left[\left(\frac{\lambda_s}{\lambda} + b \right)^2 + (b_s + b^2)^2 \right] \lesssim b \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right].$$

For (5.3.46), we estimate from the rough bounds (5.2.46), (5.2.51) :

$$\begin{aligned} \int |H^2 \Lambda \tilde{Q}_b|^2 &\lesssim \int \left| \frac{by\log y + b^2(1 + y^3) + b^3(1 + y^5)}{1 + y^4} \right|^2 \lesssim b^2, \\ \int |H^2 \tilde{T}_1|^2 &\lesssim \int_{B_1 \leq y \leq 2B_1} \left| \frac{y\log y}{y^4} \right|^2 \lesssim \frac{|\log b|^2}{B_1^4} \lesssim b^2. \end{aligned}$$

The last term is more subtle and we claim :

$$\int |H^2 \tilde{T}_2|^2 \lesssim 1 \tag{5.3.48}$$

which yields

$$\int |H^2 Mod|^2 \lesssim b^2 \left[\left(\frac{\lambda_s}{\lambda} + b \right)^2 + (b_s + b^2)^2 \right] \lesssim b^2 \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right]$$

and concludes the proof of (5.3.46).

Proof of (5.3.48) : First by definition of $\tilde{T}_2 = \chi_{B_1} T_2$, the rough bound (5.2.46) and (5.2.44) :

$$\int |H^2 \tilde{T}_2|^2 \lesssim \left[\int_{B_1 \leq y \leq 2B_1} \left| \frac{y^3}{y^4} \right|^2 + \int_{y \leq 2B_1} |H\Sigma_2|^2 \right] \lesssim 1 + \int_{y \leq 2B_1} |H\Sigma_2|^2. \tag{5.3.49}$$

We now compute from (5.2.41), (5.2.34) :

$$\begin{aligned} H\Sigma_2 &= H\Sigma_b + H(T_1 - \Lambda T_1) + O\left(\frac{y^2 |\log y|^2}{1 + y^5}\right) \\ &= -c_b \chi_{\frac{B_0}{4}} \Lambda Q + d_b H[(1 - \chi_{3B_0}) \Lambda Q] + H(T_1 - \Lambda T_1) + O\left(\frac{y^2 |\log y|^2}{1 + y^5}\right) \\ &= \frac{1}{|\log b|} O\left(\frac{1}{1 + y} \mathbf{1}_{y \leq 3B_0}\right) + H(T_1 - \Lambda T_1) + O\left(\frac{y^2 |\log y|^2}{1 + y^5}\right) \end{aligned}$$

and observe using (5.2.54), the asymptotics (5.2.3) of ΛQ and the fundamental cancellation $(\Lambda + \Lambda^2) \left(\frac{1}{y} \right) = 0$ that :

$$\begin{aligned} HT_1 - H\Lambda T_1 &= HT_1 - \left(2HT_1 + \Lambda HT_1 - \frac{\Lambda V}{y^2} T_1 \right) = \Lambda Q + \Lambda^2 Q + O \left(\frac{y \log y}{1+y^4} \right) \\ &= O \left(\frac{\log y}{1+y^3} \right). \end{aligned}$$

We thus conclude :

$$\int |H\Sigma_2|^2 \lesssim \frac{1}{|\log b|^2} \int_{y \leq 2B_0} \frac{1}{1+y^2} + \int_{y \leq 2B_1} \frac{|\log y|^4}{1+y^6} \lesssim 1$$

which together with (5.3.49) concludes the proof of (5.3.48).

step 7 Small linear term $L(\varepsilon)$.

Let us rewrite from a Taylor expansion :

$$L(\varepsilon) = -\frac{N_2(\tilde{\alpha})}{y^2} \varepsilon \quad \text{with} \quad N_2(\tilde{\alpha}) = f'(Q + \tilde{\alpha}) - f'(Q) = \tilde{\alpha} \int_0^1 f''(Q + \tau \tilde{\alpha}) d\tau. \quad (5.3.50)$$

Near the origin $y \leq 1$, we use $f''(0) = 0$ and the estimate $|\tilde{\alpha}| \lesssim by^3$ near the origin by construction to obtain the high order cancellation

$$N_2(\tilde{\alpha}) \lesssim |\tilde{\alpha}| y \lesssim by^4 \quad (5.3.51)$$

which together with the bounds (5.5.22), (5.5.23), (5.5.24) easily yields :

$$\int_{y \leq 1} \frac{1 + |\log y|^2}{1+y^4} |HL(\varepsilon)|^2 + \int_{y \leq 1} |AHL(\varepsilon)|^2 \lesssim b^6.$$

For $y \geq 1$, we use $f''(\pi) = 0$ and the bounds (5.2.45), (5.2.50) to derive the bound :

$$\begin{aligned} |N_2(\tilde{\alpha})| &\lesssim |\tilde{\alpha}| \left[\frac{1}{1+y} + |\tilde{\alpha}| \right] \lesssim [b|\log y| + b^2 y^2 |\log y|^2] \mathbf{1}_{y \leq 2B_1} \\ &\lesssim b|\log b|^C \mathbf{1}_{y \leq 2B_1}. \end{aligned} \quad (5.3.52)$$

A brute force computation taking further derivatives and using (5.5.17) now yields the control :

$$\begin{aligned} \int_{y \geq 1} |AHL(\varepsilon)|^2 + \int_{y \geq 1} \frac{1 + |\log y|^2}{1+y^4} |HL(\varepsilon)|^2 &\lesssim b^2 |\log b|^C \int_{y \leq 2B_1} \Sigma_{i=0}^3 \frac{|\partial_y^i \varepsilon|^2}{y^{2(4-i)}} \\ &\lesssim b^2 |\log b|^C \mathcal{E}_4 \lesssim b \delta^* \mathcal{E}_4. \end{aligned}$$

This concludes the proof of (5.3.44), (5.3.47) for $L(\varepsilon)$ terms.

step 8 Nonlinear term $N(\varepsilon)$.

Let us now treat the nonlinear term (5.3.25). We split the contribution at the origin and far out and claim :

$$\forall y \leq 1, \quad |AHN(\varepsilon)(y)| \lesssim C(M)b^4|\log y|^4, \quad (5.3.53)$$

$$\int_{y \geq 1} (AHN(\varepsilon))^2 \lesssim C(M) \frac{b^6}{|\log b|^2}. \quad (5.3.54)$$

which implies (5.3.47) for $N(\varepsilon)$. The estimate (5.3.44) follows along similar lines and is in fact simpler and left to the reader.

Proof of (5.3.53) : We need to treat the possible singularity at the origin. For this, let us rewrite by Taylor expansion :

$$N(\varepsilon) = z^2 N_0(\varepsilon) \quad \text{with} \quad z = \frac{\varepsilon}{y}, \quad N_0(\varepsilon) = \int_0^1 (1 - \tau) f''(\tilde{Q}_b + \tau\varepsilon) d\tau,$$

and thus :

$$H(N(\varepsilon)) = -N_0(\varepsilon)\Delta(z^2) - 2z\partial_y z\partial_y N_0(\varepsilon) + z^2 H(N_0(\varepsilon)),$$

$$\begin{aligned} AHN(\varepsilon) &= -A(N_0(\varepsilon))\Delta(z^2) + N_0(\varepsilon)\partial_y\Delta(z^2) - 2A(z\partial_y z)\partial_y(N_0(\varepsilon)) \\ &\quad + 2z\partial_y z\partial_{yy}(N_0(\varepsilon)) + z^2 AH(N_0(\varepsilon)) - 2z\partial_y zH(N_0(\varepsilon)). \end{aligned} \quad (5.3.55)$$

We now use the Hardy bounds (5.5.22), (5.5.23), (5.5.24) and the degeneracy $|Z-1|+|V-1| \lesssim y^2$ for $y \leq 1$ to estimate for $0 < y \leq 1$:

$$|\partial_{yy}\varepsilon| = \left| -H\varepsilon + \frac{A\varepsilon}{y} + \frac{1-Z+V-1}{y^2}\varepsilon \right| \lesssim b^2 y |\log y|,$$

$$|z| = \left| \frac{\varepsilon}{y} \right| \lesssim b^2,$$

$$|\partial_y z| = \left| -\frac{A\varepsilon}{y} + \frac{Z-1}{y^2}\varepsilon \right| \lesssim b^2 y |\log y|,$$

$$|\partial_{yy} z| = \left| \frac{2+Z}{y^2} A\varepsilon - \frac{H\varepsilon}{y} + \partial_y \left(\frac{Z-1}{y^2}\varepsilon \right) \right| \lesssim b^2 |\log y|,$$

$$\begin{aligned} |\partial_{yyy} z| &= \left| \partial_y \left(\frac{2+Z}{y^2} A\varepsilon \right) + \frac{2+Z}{y^2} \left(H\varepsilon - \frac{1+Z}{y} A\varepsilon \right) + \frac{H\varepsilon}{y^2} \right. \\ &\quad \left. - \frac{1}{y} \left(-AH\varepsilon + \frac{Z}{y} H\varepsilon \right) + \partial_{yy} \left(\frac{Z-1}{y^2}\varepsilon \right) \right| \\ &\lesssim \frac{b^2 |\log y|}{y} \end{aligned}$$

with constants depending on M . We now estimate using $f^{(2k)}(0) = 0$: for $0 < y \leq 1$,

$$|N_0(\varepsilon)| \lesssim y,$$

$$|\partial_y N_0(\varepsilon)| = \left| \int_0^1 (1 - \tau) \partial_y(\tilde{Q}_b + \tau\varepsilon) f^{(3)}(\tilde{Q}_b + \tau\varepsilon) d\tau \right| \lesssim 1,$$

$$\begin{aligned} |\partial_{yy}N_0(\varepsilon)| &= \left| \int_0^1 (1-\tau) \left[\partial_{yy}(\tilde{Q}_b + \tau\varepsilon) f^{(3)}(\tilde{Q}_b + \tau\varepsilon) + (\partial_y(\tilde{Q}_b + \tau\varepsilon))^2 f^{(4)}(\tilde{Q}_b + \tau\varepsilon) \right] d\tau \right| \\ &\lesssim y|\log y|^2 \end{aligned}$$

We need to exploit further cancellations for $AN_0(\varepsilon)$ and this requires pushing the Taylor expansion :

$$N_0(\varepsilon) = \frac{1}{2}f''(\tilde{Q}_b) + \varepsilon N_1(\varepsilon), \quad N_1(\varepsilon) = \int_0^1 \int_0^1 (1-\sigma)\tau(1-\tau)f^{(3)}(\tilde{Q}_b + \sigma\tau\varepsilon)d\sigma d\tau.$$

By construction, \tilde{Q}_b is a smooth function at the origin and admits a Taylor expansion

$$\tilde{Q}_b = c_1(b)y + c_2(b)y^3 + O(y^5) \quad \text{with } |c_1(b), c_2(b)| \lesssim 1$$

and hence :

$$|Af''(\tilde{Q}_b)| + |Hf''(\tilde{Q}_b)| + |AH(f''(\tilde{Q}_b))| \lesssim 1.$$

We therefore estimate arguing like for $N_0(\varepsilon)$:

$$|A(\varepsilon N_1(\varepsilon))| = |A\varepsilon N_1(\varepsilon) - \varepsilon\partial_y(N_1(\varepsilon))| \lesssim y^2|\log y|,$$

$$|H(\varepsilon N_1(\varepsilon))| = |N_1(\varepsilon)H\varepsilon - 2\partial_y\varepsilon\partial_y N_1(\varepsilon) - \varepsilon\partial_{yy}N_1(\varepsilon)| \lesssim y|\log y|^2,$$

$$\begin{aligned} |AH(\varepsilon N_1(\varepsilon))| &= |N_1(\varepsilon)AH\varepsilon - (H\varepsilon)\partial_y N_1(\varepsilon) - 2(A\partial_y\varepsilon)\partial_y N_1(\varepsilon) + 2\partial_y\varepsilon\partial_{yy}N_1(\varepsilon) \\ &\quad - A\varepsilon\partial_{yy}N_1(\varepsilon) + \varepsilon\partial_{yyy}N_1(\varepsilon)| \\ &\lesssim |\log y|^4 \end{aligned}$$

Injecting the collection of above estimates into (5.3.55) now yields (5.3.53).

Proof of (5.3.54) : For $y \geq 1$, we estimate from (5.5.25), (5.5.26) :

$$\|z\|_{L^\infty(y \geq 1)} \lesssim b|\log b|^C, \quad \|\partial_y z\|_{L^\infty(y \geq 1)} + \|\frac{z}{y}\|_{L^\infty(y \geq 1)} \lesssim b^{\frac{3}{2}}|\log b|^C.$$

The construction of \tilde{Q}_b yields the bounds for $y \geq 1$:

$$\begin{aligned} |\partial_y \tilde{Q}_b| &\lesssim |\log b|^C \left(\frac{1}{y^2} + b\mathbf{1}_{y \leq 2B_1} \right), \\ |\partial_{yy} \tilde{Q}_b| &\lesssim |\log b|^C \left(\frac{1}{y^3} + \frac{b}{y} \mathbf{1}_{y \leq 2B_1} \right) \lesssim |\log b|^C \left(\frac{1}{y^3} + b^{\frac{3}{2}} \right), \\ |\partial_{yyy} \tilde{Q}_b| &\lesssim |\log b|^C \left(\frac{1}{y^4} + \frac{b}{y^2} \mathbf{1}_{y \leq 2B_1} \right) \lesssim |\log b|^C \left(\frac{1}{y^4} + b^2 \right) \end{aligned}$$

which together with (5.5.25), (5.5.26), (5.5.27) yields the pointwise bounds :

$$|N_0(\varepsilon)| \lesssim 1, \tag{5.3.56}$$

$$|\partial_y N_0(\varepsilon)| \lesssim |\log b|^C \left(\frac{1}{y^2} + b + \|\partial_y\varepsilon\|_{L^\infty(y \geq 1)} \right) \lesssim |\log b|^C \left(\frac{1}{y^2} + b \right), \tag{5.3.57}$$

$$|\partial_{yy}N_0(\varepsilon)| \lesssim |\log b|^C \left[\frac{1}{y^3} + b^{\frac{3}{2}} + \|\partial_{yy}\varepsilon\|_{L^\infty(y \geq 1)} \right] \lesssim |\log b|^C \left[\frac{1}{y^3} + b^{\frac{3}{2}} \right], \quad (5.3.58)$$

$$|\partial_{yyy}N_0(\varepsilon)| \lesssim |\log b|^C \left[\frac{1}{y^4} + b^2 + \|\partial_{yyy}\varepsilon\|_{L^\infty(y \geq 1)} \right] \lesssim |\log b|^C \left[\frac{1}{y^4} + b^2 \right]. \quad (5.3.59)$$

We now compute :

$$H(N(\varepsilon)) = N_0(\varepsilon)H(z^2) - 2z\partial_y z\partial_y N_0(\varepsilon) - z^2\Delta(N_0(\varepsilon)), \quad (5.3.60)$$

$$\begin{aligned} AHN(\varepsilon) &= N_0(\varepsilon)AH(z^2) - \partial_y N_0(\varepsilon)H(z^2) - 2A(z\partial_y z)\partial_y(N_0(\varepsilon)) \\ &\quad + 2z\partial_y z\partial_{yy}(N_0(\varepsilon)) - A(z^2)\Delta(N_0(\varepsilon)) + z^2\partial_y\Delta(N_0(\varepsilon)), \end{aligned}$$

and hence using the L^2 weighted bounds (5.5.17), (5.5.18), (5.5.19) and the L^∞ bounds (5.5.25), (5.5.26) :

$$\begin{aligned} \int_{y \geq 1} |N_0(\varepsilon)AH(z^2)|^2 &\lesssim \int_{y \geq 1} \left[|\partial_y z\Delta z|^2 + |z\partial_y\Delta z|^2 + |\partial_y z\partial_{yy}z|^2 + \frac{|z\partial_y z|^2}{y^4} \right. \\ &\quad \left. + \frac{1}{y^2} \left(|z\Delta z|^2 + |\partial_y z|^4 + \frac{|z|^4}{y^4} \right) \right] \\ &\lesssim b^6 |\log b|^C, \end{aligned}$$

$$\int_{y \geq 1} |\partial_y N_0(\varepsilon)H(z^2)|^2 \lesssim \int_{y \geq 1} \left(\frac{1}{y^2} + b^2 |\log b|^C \right) \left(|z|^2 |\Delta z|^2 + |\partial_y z|^4 + \frac{|z|^4}{y^4} \right) \lesssim b^6 |\log b|^C,$$

$$\begin{aligned} \int_{y \geq 1} |A(z\partial_y z)\partial_y N_0(\varepsilon)|^2 &\lesssim \int_{y \geq 1} \left(\frac{1}{y^2} + b^2 |\log b|^C \right) \left(|z|^2 |\partial_{yy}z|^2 + |\partial_y z|^4 + \frac{|z|^2 |\partial_y z|^2}{y^2} \right) \\ &\lesssim b^6 |\log b|^C, \end{aligned}$$

$$\int_{y \geq 1} |z\partial_y z\partial_{yy}N_0(\varepsilon)|^2 \lesssim |\log b|^C \int_{y \geq 1} |z|^2 |\partial_y z|^2 \left[\frac{1}{y^6} + b^3 \right] \lesssim b^6 |\log b|^C,$$

$$\int_{y \geq 1} |A(z^2)\Delta N_0(\varepsilon)|^2 \lesssim |\log b|^C \int_{y \geq 1} \left[|z|^2 |\partial_y z|^2 + \frac{|z|^4}{y^2} \right] \left[\frac{1}{y^6} + b^3 \right] \lesssim b^6 |\log b|^C,$$

$$\int_{y \geq 1} |z^2\partial_y\Delta(N_0(\varepsilon))|^2 \lesssim |\log b|^C \int_{y \geq 1} |z|^4 \left[\frac{1}{y^8} + b^4 \right] \lesssim b^6 |\log b|^C.$$

This concludes the proof of (5.3.54).

This concludes the proof of (5.3.44), (5.3.45), (5.3.46), (5.3.47) and thus of Proposition 5.3.4.

5.4 Sharp description of the singularity formation

In this section, we start with completing the proof of the bootstrap Proposition 5.3.1. Theorem 5.1.1 will then easily follow.

5.4.1 Closing the bootstrap

We are now in position to close the bootstrap bounds of Proposition 5.3.1.

Proof of Proposition 5.3.1

step 1 Energy bound.

First observe that (5.3.16) and the modulation equation (5.3.29) ensure for M large enough²¹ :

$$b_s \leq 0$$

and the upper bound in (5.3.17) follows. We now claim :

$$b(t) > 0 \text{ on } [0, T_1].$$

Indeed, from (5.3.30), if $b(t_0) = 0$ for some $t_0 \in [0, T_1]$, then $b(t) \equiv 0$ on some $[t_0 - \delta, t_0]$ and thus from (5.3.30), (5.3.16), $\lambda(t) \equiv \lambda(t_0)$ and $u(t) \equiv Q_{\lambda(t_0)}$ on $[t_0 - \delta, t_0]$. We can thus iterate on $\delta > 0$ and conclude that $u(0)$ is initially a harmonic map, a contradiction. This concludes the proof of (5.3.17).

We now prove (5.3.19) which follows from the conservation of energy. Indeed, let

$$\tilde{\varepsilon} = \varepsilon + \tilde{\alpha},$$

then

$$\begin{aligned} E_0 &= \int |\partial_y(Q + \tilde{\varepsilon})|^2 + \int \frac{g^2(Q + \tilde{\varepsilon})}{y^2} \\ &= E(Q) + (H\varepsilon, \varepsilon) + \int \frac{1}{y^2} [g^2(Q + \tilde{\varepsilon}) - 2f(Q)\varepsilon - f'(Q)\tilde{\varepsilon}^2]. \end{aligned} \quad (5.4.1)$$

We now recall from Lemma 5.5.2 and Lemma 5.5.3 the coercivity properties :

$$\begin{aligned} (H\varepsilon, \varepsilon) &\geq c(M) \left[\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} \right], \\ (H\tilde{\varepsilon}, \tilde{\varepsilon}) &\geq c(M) \left[\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} \right] + O(C(M)b^2). \end{aligned}$$

The nonlinear term is estimated from a Taylor expansion :

$$\left| \int \frac{1}{y^2} [g^2(Q + \tilde{\varepsilon}) - 2f(Q)\tilde{\varepsilon} - f'(Q)\tilde{\varepsilon}^2] \right| \lesssim \int \frac{|\tilde{\varepsilon}|^3}{y^2} \lesssim \left(\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} \right)^{\frac{3}{2}}.$$

where we used the Sobolev bound

$$\|\tilde{\varepsilon}\|_{L^\infty}^2 \lesssim \|\partial_y \tilde{\varepsilon}\|_{L^2} \left\| \frac{\tilde{\varepsilon}}{y} \right\|_{L^2}.$$

21. recall that K in the bootstrap bounds is large but independent of M .

We inject these bounds into the conservation of energy (5.4.1) and use the bound on the profile

$$\int |\partial_y \tilde{a}|^2 + \int \frac{|\tilde{a}|^2}{y^2} \lesssim b |\log b|^C$$

and (5.3.12), (5.3.17) to estimate :

$$\begin{aligned} \int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} &\lesssim \int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} + b |\log b|^C \lesssim C(M) |E_0 - E(Q)| + O(C(M)b) \\ &\lesssim C(M)b(0) \leq \sqrt{b(0)} \end{aligned}$$

for $|b(0)| \leq b^*(M)$ small enough, and (5.3.19) is proved.

step 2 Control of \mathcal{E}_4 .

We now close the bootstrap bound (5.3.20) which follows by reintegrating the Lyapounov monotonicity (5.3.33) in the regime governed by the modulation equations (5.3.28), (5.3.29). Indeed, inject the bootstrap bound (5.3.16) into the monotonicity formula (5.3.33) and integrate in time ; this yields : $\forall t \in [0, T_1]$,

$$\begin{aligned} \mathcal{E}_4(t) &\leq 2 \left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \left[\mathcal{E}_4(0) + C \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} \right] + \frac{b^4(t)}{|\log b(t)|^2} \\ &+ C \left[1 + \frac{K}{\log M} + \sqrt{K} \right] \lambda^6(t) \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\tau \end{aligned} \quad (5.4.2)$$

for some universal constant $C > 0$ independent of M .

Let us now consider two constants

$$\alpha_1 = 2 - \frac{C_1}{\sqrt{\log M}}, \quad \alpha_2 = 2 + \frac{C_2}{\sqrt{\log M}} \quad (5.4.3)$$

for some large enough universal constants C_1, C_2 . We compute using the modulation equations (5.3.28), (5.3.29) and the bootstrap bound (5.3.16) :

$$\begin{aligned} \frac{d}{ds} \left\{ \frac{|\log b|^{\alpha_i} b}{\lambda} \right\} &= \frac{|\log b|^{\alpha_i}}{\lambda} \left[\left(1 - \frac{\alpha_i}{|\log b|} \right) b_s - \frac{\lambda_s}{\lambda} b \right] \\ &= \frac{|\log b|^{\alpha_i}}{\lambda} \left[\left(1 - \frac{\alpha_i}{|\log b|} \right) b_s + b^2 + O \left(\frac{b^3}{|\log b|} \right) \right] \\ &= \left(1 - \frac{\alpha_i}{|\log b|} \right) \frac{|\log b|^{\alpha_i}}{\lambda} \left[b_s + b^2 \left(1 + \frac{\alpha_i}{|\log b|} + O \left(\frac{1}{|\log b|^2} \right) \right) \right] \\ &\begin{cases} \leq 0 & \text{for } i = 1 \\ \geq 0 & \text{for } i = 2. \end{cases} \end{aligned}$$

Integrating this from 0 to t yields :

$$\frac{b(0)}{\lambda(0)} \left(\frac{|\log b(0)|}{|\log b(t)|} \right)^{\alpha_2} \leq \frac{b(t)}{\lambda(t)} \leq \frac{b(0)}{\lambda(0)} \left(\frac{|\log b(0)|}{|\log b(t)|} \right)^{\alpha_1}. \quad (5.4.4)$$

This yields in particular using the initial bound (5.3.12) and the bound (5.3.17) :

$$\left(\frac{\lambda(t)}{\lambda(0)}\right)^6 \mathcal{E}_4(0) \leq (b(t)|\log b(t)|^{\alpha_2})^6 \frac{\mathcal{E}_0}{(b(0)|\log b(0)|^{\alpha_2})^6} \leq \frac{b^4(t)}{|\log b(t)|^2}, \quad (5.4.5)$$

$$\begin{aligned} C \left(\frac{\lambda(t)}{\lambda(0)}\right)^6 \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} &\lesssim \left(\frac{b(t)|\log b(t)|^{\alpha_2}}{b(0)|\log b(0)|^{\alpha_2}}\right)^6 \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} \\ &\lesssim C(b(t))^{4+\frac{1}{4}} \leq \frac{b^4(t)}{|\log b(t)|^2}. \end{aligned} \quad (5.4.6)$$

We now compute explicitly using $b = -\lambda\lambda_t + O\left(\frac{b^2}{|\log b|}\right)$ from (5.3.28) :

$$\begin{aligned} \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\sigma &= \frac{1}{6} \left[\frac{b^4}{\lambda^6 |\log b|^2} \right]_0^t - \frac{1}{6} \int_0^t \frac{b_t b^3}{\lambda^6 |\log b|^2} \left(4 + \frac{2}{|\log b|}\right) d\tau \\ &+ O\left(\int_0^t \frac{b}{\lambda^8} \frac{b^5}{|\log b|^2} d\tau\right) \end{aligned}$$

which implies using now $|b_s + b^2| \lesssim \frac{b^2}{|\log b|^2}$ from (5.3.29) and (5.3.16) :

$$\lambda^6(t) \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\sigma \lesssim \left[1 + O\left(\frac{1}{|\log b_0|}\right)\right] \frac{b^4(t)}{|\log b(t)|^2}.$$

Injecting this together with (5.4.5), (5.4.6) into (5.4.2) yields

$$\mathcal{E}_4(t) \leq C \frac{b^4(t)}{|\log b(t)|^2} \left[1 + \frac{K}{\log M} + \sqrt{K}\right]$$

for some universal constant $C > 0$ independent of K and M , and thus (5.3.20) follows for K large enough independent of M .

step 3 Control of \mathcal{E}_2 .

We now close the H^2 bound (5.3.19). This bound is used mostly in the proof of the interpolation estimates of Lemma 5.5.4, and there the power the log is (5.3.19) is irrelevant. It becomes on the contrary critical in the proof of the regularity (5.1.14) and this requires being careful with logarithmic growth. For this reason, the profile \hat{Q}_b localized near B_0 given by (5.2.8) is better adapted to the \mathcal{E}_2 control.

Let then the radiation ζ_b given by (5.2.73) and the new decomposition of the flow :

$$u = (\tilde{Q}_b + \varepsilon)_\lambda = (\hat{Q}_b + \hat{\varepsilon})_\lambda \text{ ie } \hat{\varepsilon} = \varepsilon + \zeta_b \quad (5.4.7)$$

and the renormalization

$$\hat{w}(t, r) = \hat{\varepsilon}(s, y).$$

The equation for \hat{w} is similarly like (5.3.27) :

$$\partial_t \hat{w} + H_\lambda \hat{w} = \frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda, \quad \hat{\mathcal{F}} = -\hat{\Psi}_b - \widehat{\text{Mod}} + \hat{L}(\hat{\varepsilon}) - \hat{N}(\hat{\varepsilon}), \quad (5.4.8)$$

$$\hat{L}(\hat{\varepsilon}) = \frac{f'(Q) - f'(\hat{Q}_b)}{y^2} \hat{\varepsilon}, \quad \hat{N}(\hat{\varepsilon}) = \frac{f(\hat{Q}_b + \hat{\varepsilon}) - f(\hat{Q}_b) - \hat{\varepsilon} f'(\hat{Q}_b)}{y^2}.$$

We then let

$$\hat{\varepsilon}_2 = H\hat{\varepsilon}, \quad \hat{w}_2 = H_\lambda \hat{w},$$

which satisfies from (5.4.8) :

$$\partial_t \hat{w}_2 + H_\lambda \hat{w}_2 = \frac{\partial_t V_\lambda}{r^2} \hat{w} + H_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right).$$

We therefore compute the energy identity :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \int |\hat{w}_2|^2 \right\} &= \int \hat{w}_2 \left[-H_\lambda \hat{w}_2 + \frac{\partial_t V_\lambda}{r^2} \hat{w} + H_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right) \right] \\ &\lesssim - \int |A_\lambda \hat{w}_2|^2 + \frac{1}{\lambda^4} \left[b \|\hat{\varepsilon}_2\|_{L^2} \left\| \frac{\Lambda V}{y^2} \hat{\varepsilon} \right\|_{L^2} + |(\hat{\varepsilon}_2, H\hat{\mathcal{F}})| \right] \end{aligned} \quad (5.4.9)$$

and aim at estimating all terms in the above RHS. The local term is estimated using the decomposition (5.4.7), the estimate (5.2.75) and Lemma 5.5.4 :

$$\int \left| \frac{\Lambda V}{y^2} \hat{\varepsilon} \right|^2 \lesssim b^4 |\log b|^C + \int \frac{|\zeta_b|^2}{1+y^8} \lesssim b^4 |\log b|^C. \quad (5.4.10)$$

We now claim the bound :

$$|(\hat{\varepsilon}_2, H\hat{\mathcal{F}})| \lesssim b^3 |\log b|^2. \quad (5.4.11)$$

Asssume (5.4.11), we then obtain from (5.4.9), (5.4.10) the pointwise bound :

$$\frac{d}{dt} \left\{ \int |\hat{w}_2|^2 \right\} \lesssim \frac{b^3 |\log b|^2}{\lambda^4},$$

which we integrate using also (5.2.75) :

$$\begin{aligned} \mathcal{E}_2(t) &= \lambda^2(t) \|w_2(t)\|_{L^2}^2 \lesssim \|H\zeta_b(t)\|_{L^2}^2 + \lambda^2(t) \|\hat{w}_2(t)\|_{L^2}^2 \\ &\lesssim b^4(t) |\log b(t)|^2 + \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 [\mathcal{E}_2(0) + b^2(0) |\log b(0)|^2] + \lambda^2(t) \int_0^t \frac{b^3 |\log b|^2}{\lambda^4(\tau)} d\tau. \end{aligned} \quad (5.4.12)$$

From (5.3.12), (5.4.4) :

$$\begin{aligned} \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 [\mathcal{E}_2(0) + b^2(0) |\log b(0)|^2] &\lesssim \frac{(b(0))^{10} + b^2(0) |\log b(0)|^2}{(b(0) |\log b(0)|^{\alpha_2})^2} b^2(t) |\log b(t)|^{2\alpha_2} \\ &\leq b^2(t) |\log b(t)|^{4+\frac{1}{4}}, \end{aligned}$$

We now use the bound $b_s \lesssim -b^2$ and (5.4.4) to estimate :

$$\begin{aligned} \lambda^2(t) \int_0^t \frac{b^3 |\log b|^2}{\lambda^4(\tau)} d\tau &\lesssim \lambda^2(t) \int_0^t \frac{-b_t b |\log b|^2}{\lambda^2(\tau)} d\tau \\ &\lesssim \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 b^2(0) |\log b(0)|^{2\alpha_1} \int_0^t \frac{-b_t}{b |\log b|^{2\alpha_1-2}} d\tau \\ &\lesssim \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 b^2(0) |\log b(0)|^{2\alpha_1} \frac{1}{|\log b(0)|^{2\alpha_1-3}} \\ &\lesssim b^2(t) |\log b(t)|^{2\alpha_2} \frac{|\log b(0)|^3}{|\log b(0)|^{2\alpha_2}} \lesssim b^2(t) |\log b(t)|^{4+\frac{1}{4}}. \end{aligned}$$

Injecting these bounds into (5.4.12) yields :

$$\mathcal{E}_2(t) \lesssim b^2(t) |\log b(t)|^{4+\frac{1}{4}}$$

and concludes the proof of (5.3.19).

Proof of (5.4.11) : We estimate the contribution of each term in (5.4.11) coming from the decomposition (5.4.8). First observe from the interpolation bound (5.5.17) and (5.2.75) :

$$\begin{aligned} \int_{y \leq 2B_0} |\hat{\varepsilon}_2|^2 &\lesssim B_0^4 |\log b|^2 \int \frac{|\varepsilon|^2}{(1+y^4)|\log y|^2} + \int |H\zeta_b|^2 \\ &\lesssim C(M)b^2 + b^2 |\log b|^2 \lesssim b^2 |\log b|^2. \end{aligned} \quad (5.4.13)$$

The $\hat{\Psi}_b$ term is now estimated using (5.2.77) and (5.4.13) :

$$|(\hat{\varepsilon}_2, H\hat{\Psi}_b)| \lesssim \|H\hat{\Psi}_b\|_{L^2} \|\hat{\varepsilon}_2\|_{L^2(y \leq 2B_0)} \lesssim (b^4 |\log b|^2 b^2 |\log b|^2)^{\frac{1}{2}} \lesssim b^3 |\log b|^2.$$

We next estimate from (5.2.31), (5.2.45) :

$$\begin{aligned} \int |H\hat{T}_1|^2 &\lesssim \int_{y \leq 2B_0} |\Lambda Q|^2 + \int_{B_0 \leq y \leq 2B_0} \left| \frac{\log y}{y} \right|^2 \lesssim |\log b|^2, \\ \int |H\hat{T}_2|^2 &\lesssim \int_{y \leq 2B_0} \left| \frac{y}{y^2 b |\log b|} \right|^2 \lesssim \frac{1}{b^2 |\log b|}, \end{aligned}$$

and thus from (5.2.74), (5.3.28), (5.3.29) :

$$\begin{aligned} \int |H\widehat{Mod}(t)|^2 &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int |H\Lambda\hat{Q}_b|^2 + |b_s + b^2|^2 \int |H(\hat{T}_1 + b\hat{T}_2)|^2 \\ &\lesssim \frac{b^4}{|\log b|^2} |\log b|^2 \lesssim b^4 |\log b|^2. \end{aligned}$$

Moreover, $\text{Supp}(H\widehat{Mod}) \subset [0, 2B_0]$ and thus with (5.4.13) :

$$|(\hat{\varepsilon}_2, H\widehat{Mod})| \lesssim (b^4 |\log b|^2 b^2 |\log b|^2)^{\frac{1}{2}} \lesssim b^3 |\log b|^2.$$

We now turn to the control of the small linear term $\hat{L}(\hat{\varepsilon})$ which we rewrite as for (5.3.50) :

$$\hat{L}(\hat{\varepsilon}) = -\frac{N_2(\hat{\alpha})}{y^2} \hat{\varepsilon} \quad \text{with} \quad N_2(\hat{\alpha}) = f'(Q + \hat{\alpha}) - f'(Q) = \hat{\alpha} \int_0^1 f''(Q + \tau\hat{\alpha}) d\tau.$$

Near the origin $y \leq 1$, $\hat{\varepsilon} = \varepsilon$ and the high order vanishing (5.3.51) and the bounds (5.5.22), (5.5.23), (5.5.24) easily yield :

$$\int_{y \leq 1} |H\hat{L}(\hat{\varepsilon})|^2 \lesssim b^6.$$

For $y \geq 1$, we estimate like for (5.3.52) :

$$|N_2(\hat{\alpha})| \lesssim b |\log b|^C \mathbf{1}_{y \leq 2B_0},$$

and then a brute force computation and (5.2.75) yield the control :

$$\int_{y \geq 1} |H\hat{L}(\hat{\varepsilon})|^2 \lesssim b^2 |\log b|^C \int_{y \leq 2B_0} \Sigma_{i=0}^2 \frac{|\partial_y^i \hat{\varepsilon}|^2}{y^{2(4-i)}} \lesssim b^2 |\log b|^C b^4 |\log b|^C \lesssim b^5.$$

We also estimate from (5.2.75) and the bootstrap bound (5.3.15) :

$$\|\hat{\varepsilon}_2\|_{L^2}^2 \lesssim b^2 |\log b|^C \quad (5.4.14)$$

and thus

$$|(\hat{\varepsilon}_2, H\hat{L}(\hat{\varepsilon}))| \lesssim (b^2 |\log b|^C b^5)^{\frac{1}{2}} \leq b^3 |\log b|^2.$$

It remains to estimate the nonlinear term. Near the origin, we argue like for the proof of (5.3.53) to derive :

$$\forall y \leq 1, \quad |H(\hat{N}(\hat{\varepsilon}))| = |H(N(\varepsilon))| \lesssim b^4 |\log b|^C,$$

this is left to the reader. For $y \geq 1$, we introduce the decomposition :

$$\hat{N}(\hat{\varepsilon}) = \hat{z}^2 \hat{N}_0(\hat{\varepsilon}) \quad \text{with} \quad \hat{z} = \frac{\hat{\varepsilon}}{y}, \quad \hat{N}_0(\hat{\varepsilon}) = \int_0^1 (1-\tau) f''(\hat{Q}_b + \tau \hat{\varepsilon}) d\tau,$$

and recall the formula (5.3.60) :

$$H(\hat{N}(\varepsilon)) = \hat{N}_0(\hat{\varepsilon}) H(\hat{z}^2) - 2\hat{z} \partial_y \hat{z} \partial_y \hat{N}_0(\hat{\varepsilon}) - \hat{z}^2 \Delta(\hat{N}_0(\hat{\varepsilon})).$$

The estimates (5.3.56), (5.3.57), (5.3.58), (5.3.59) still hold from direct check for $\hat{N}_0(\hat{\varepsilon})$, and we estimate with Lemma 5.5.4 and (5.2.76) :

$$\begin{aligned} \int_{y \geq 1} |\hat{N}_0(\hat{\varepsilon}) H(\hat{z}^2)|^2 &\lesssim \int_{y \geq 1} \left[|\hat{z} \Delta \hat{z}|^2 + \frac{\hat{z}^4}{y^4} + |\partial_y \hat{z}|^4 \right] \\ &\lesssim \left[\left\| \frac{\hat{\varepsilon}}{y} \right\|_{L^\infty(y \geq 1)}^2 + \|\partial_y \hat{\varepsilon}\|_{L^\infty(y \geq 1)}^2 \right] \Sigma_{i=0}^2 \int \frac{|\partial_y^i \hat{\varepsilon}|^2}{1+y^{2(3-i)}} \\ &\lesssim b^2 |\log b|^C b^3 |\log b|^C \lesssim b^5 |\log b|^C. \end{aligned}$$

$$\begin{aligned} \int_{y \geq 1} |\hat{z} \partial_y \hat{z} \partial_y \hat{N}_0(\hat{\varepsilon})|^2 &\lesssim |\log b|^C \left\| \frac{\hat{\varepsilon}}{y} \right\|_{L^\infty(y \geq 1)}^2 \int_{y \geq 1} \left[\frac{1}{y^4} + b^2 \right] \left[\frac{|\partial_y \hat{\varepsilon}|^2}{y^2} + \frac{|\hat{\varepsilon}|^2}{y^4} \right] \\ &\lesssim b^2 |\log b|^C b^3 |\log b|^C \lesssim b^5 |\log b|^C, \end{aligned}$$

$$\int |\hat{z}^2 \Delta(\hat{N}_0(\hat{\varepsilon}))|^2 \lesssim |\log b|^C \left\| \frac{\hat{\varepsilon}}{y} \right\|_{L^\infty(y \geq 1)}^2 \int_{y \geq 1} \left[\frac{1}{y^6} + b^3 \right] \frac{|\hat{\varepsilon}|^2}{y^2} \lesssim b^5 |\log b|^C.$$

We thus conclude using (5.4.14) :

$$|(\hat{\varepsilon}_2, H\hat{N}(\hat{\varepsilon}))| \lesssim (b^2 |\log b|^C b^5)^{\frac{1}{2}} \leq b^3 |\log b|^2.$$

This concludes the proof of (5.4.11).

This concludes the proof of the Proposition 5.3.1.

5.4.2 Proof of Theorem 5.1.1

We are now in position to conclude the proof of Theorem 5.1.1. The proof relies on the reintegration of the modulation equations as in [69], [71], [72], we sketch the argument for the sake of completeness.

step 1 Finite time blow up.

Let $T \leq +\infty$ be the life time of the full map v given by (5.3.2), then the estimates of Proposition 5.3.1 hold on $[0, T)$. From (5.3.30), (5.4.4),

$$-\frac{d}{dt}\sqrt{\lambda} = -\frac{1}{2\lambda\sqrt{\lambda}}\frac{\lambda_s}{\lambda} \gtrsim \frac{b}{\lambda\sqrt{\lambda}} \gtrsim C(u_0) > 0$$

and thus λ touches zero at some finite time $T_0 < +\infty$. Using (5.4.27), (5.4.28), it is easily seen that the estimates of Proposition 5.3.1 and the bootstrap bounds of Proposition (5.5.4) imply :

$$\forall t \in [0, T_0), \quad \|\Delta v(t)\|_{L^2} \lesssim C(t) < +\infty$$

and thus from the blow up criterion (5.3.1) :

$$T_0 = T < +\infty.$$

Observe then from (5.4.4) that this implies

$$\lambda(T) = b(T) = 0. \tag{5.4.15}$$

Moreover, we obtain from (5.3.28) and the estimates of Proposition 5.3.1 the rough bound :

$$|\lambda\lambda_t| = \left| \frac{\lambda_s}{\lambda} \right| \lesssim 1 \quad \text{and thus } \lambda(t) \lesssim \sqrt{T-t},$$

and hence from (5.3.21) :

$$s(t) \rightarrow +\infty \quad \text{as } t \rightarrow T. \tag{5.4.16}$$

step 2 Derivation of the sharp blow up speed.

We now slightly refine our control of b through a logarithmic gain in the modulation equation (5.3.29). We commute (5.3.22) with H and take the inner product with $\chi_{B_\delta}\Lambda Q$ to derive :

$$\begin{aligned} & \frac{d}{ds} \{(H\varepsilon, \chi_{B_\delta}\Lambda Q)\} - (H\varepsilon, \partial_s \chi_{B_\delta}\Lambda Q) + \frac{\lambda_s}{\lambda} (\chi_{B_\delta}\Lambda Q, H\Lambda\varepsilon) + (H^2\varepsilon, \chi_{B_\delta}\Lambda Q) \\ &= \left(H \left[-\tilde{\Phi}_b + L(\varepsilon) - N(\varepsilon) - \text{Mod} \right], \chi_{B_\delta}\Lambda Q \right). \end{aligned} \tag{5.4.17}$$

We now estimate all terms in the above identity. First, for δ small enough, we estimate in brute force :

$$|(H\varepsilon, \partial_s \chi_{B_\delta}\Lambda Q)| + \left| \frac{\lambda_s}{\lambda} (\chi_{B_\delta}\Lambda Q, H\Lambda\varepsilon) \right| + |(H[L(\varepsilon) - N(\varepsilon)], \chi_{B_\delta}\Lambda Q)| \lesssim \frac{b}{b^{C\delta}} \sqrt{\mathcal{E}_4} \lesssim \frac{b^2}{|\log b|^2}.$$

We then estimate the linear term :

$$|(H^2 \varepsilon, \chi_{B_\delta} \Lambda Q)| \lesssim \sqrt{\mathcal{E}_4} \sqrt{|\log b|} \lesssim \frac{b^2}{\sqrt{|\log b|}}.$$

The leading order $\tilde{\Phi}_b$ term is computed from (5.2.57), (5.2.67) :

$$(-H\tilde{\Phi}_b, \chi_{B_\delta} \Psi_b) = -b^2 (H\Sigma_b, \chi_{B_\delta} \Psi_b) + O\left(\frac{b^3}{b^{C\delta}}\right) = b^2 c_b (\Lambda Q, \chi_{B_\delta} \Lambda Q) + O\left(\frac{b^2}{|\log b|^2}\right).$$

Finally, we compute the modulation term from (5.2.60) :

$$\begin{aligned} (-HMod, \chi_{B_\delta} \Lambda Q) &= \left(\frac{\lambda_s}{\lambda} + b\right) (H\Lambda \tilde{Q}_b, \chi_{B_\delta} \Lambda Q) - (b_s + b^2) (H(\tilde{T}_1 + 2b\tilde{T}_2), \chi_{B_\delta} \Lambda Q) \\ &= (b_s + b^2) (\Lambda Q, \chi_{B_\delta} \Lambda Q) + O\left(\frac{b}{b^{C\delta}} \frac{b^2}{|\log b|}\right). \end{aligned}$$

We thus inject the collection of above estimates into (5.4.17) and derive the modulation equation :

$$(b_s + b^2) (\Lambda Q, \chi_{B_\delta} \Lambda Q) = \frac{d}{ds} \{(H\varepsilon, \chi_{B_\delta} \Lambda Q)\} - c_b b^2 (\Lambda Q, \chi_{B_\delta} \Lambda Q) + O\left(\frac{b^2}{\sqrt{|\log b|}}\right)$$

which we rewrite using (5.2.32) and an integration by parts in time :

$$\begin{aligned} &\frac{d}{ds} \left\{ b - \frac{(H\varepsilon, \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)} \right\} + b^2 \left(1 + \frac{2}{|\log b|} \right) \\ &= O\left(\frac{b^2}{|\log b|^{\frac{3}{2}}}\right) + (H\varepsilon, \chi_{B_\delta} \Lambda Q) \frac{(\Lambda Q, \partial_s \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)^2}. \end{aligned} \tag{5.4.18}$$

We now estimate :

$$\begin{aligned} \left| (H\varepsilon, \chi_{B_\delta} \Lambda Q) \frac{(\Lambda Q, \partial_s \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)^2} \right| &\lesssim \frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}} \frac{|b_s|}{b} \lesssim \frac{b^3}{b^{C\delta}}, \\ \left| \frac{(H\varepsilon, \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)} \right| &\lesssim \frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}} \lesssim \frac{b^2}{b^{C\delta}}. \end{aligned}$$

We inject these bounds into (5.4.18) and conclude that the quantity

$$\tilde{b} = b - \frac{(H\varepsilon, \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)} = b + O\left(\frac{b^2}{|\log b|^2}\right) \tag{5.4.19}$$

satisfies the pointwise differential control :

$$\left| \tilde{b}_s + \tilde{b}^2 \left(1 + \frac{2}{|\log \tilde{b}|} \right) \right| \lesssim \frac{\tilde{b}^2}{|\log \tilde{b}|^{\frac{3}{2}}}.$$

Equivalently,

$$\frac{\tilde{b}_s}{\tilde{b}^2 \left(1 + \frac{2}{|\log \tilde{b}|}\right)} + 1 = O\left(\frac{1}{|\log \tilde{b}|^{\frac{3}{2}}}\right).$$

We now integrate this in time using $\lim_{s \rightarrow +\infty} \tilde{b}(s) = 0$ from (5.4.15), (5.4.16), (5.4.19) and get :

$$\tilde{b}(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right)$$

and thus from (5.4.19) :

$$b(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right). \quad (5.4.20)$$

We now inject the modulation equation (5.3.2) and conclude :

$$-\frac{\lambda_s}{\lambda} = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right).$$

We rewrite this as

$$\left| \frac{d}{ds} \log \left(\frac{s \lambda(s)}{(\log s)^2} \right) \right| \lesssim \frac{1}{s |\log s|^{\frac{3}{2}}}$$

and thus integrating in time yields the existence of $\kappa(u) > 0$ such that :

$$\frac{s \lambda(s)}{(\log s)^2} = \frac{1}{\kappa(u)} \left[1 + O\left(\frac{1}{|\log s|^{\frac{3}{2}}}\right) \right].$$

Taking the log yields the bound

$$|\log \lambda| = |\log s| \left[1 + O\left(\frac{|\log \log s|}{\log s}\right) \right]$$

and thus

$$\frac{1}{s} = \kappa(u) \frac{\lambda}{|\log \lambda|^2} (1 + o(1)).$$

Injecting this into (5.4.20) yields :

$$-\lambda \lambda_t = -\frac{\lambda_s}{\lambda} = \frac{1}{s} (1 + o(1)) = \kappa(u) \frac{\lambda}{|\log \lambda|^2} (1 + o(1)) \quad (5.4.21)$$

and thus

$$-|\log \lambda|^2 \lambda_t = \kappa(u) (1 + o(1)).$$

Integrating from t to T with $\lambda(T) = 0$ yields

$$\lambda(t) = \kappa(u) \frac{T-t}{|\log(T-t)|^2} [1 + o(1)],$$

and (5.1.13) is proved. This also implies using (5.3.30) :

$$b(t) = \kappa^2(u) \frac{T-t}{|\log(T-t)|^4} [1 + o(1)]. \quad (5.4.22)$$

In particular :

$$\frac{b}{\lambda} = \frac{k(u)}{|\log b|^2}(1 + o(1)). \quad (5.4.23)$$

step 3 \dot{H}^2 bound.

We now turn to the proof of (5.1.12), (5.1.14). Let $v(t, x)$ be the map associated to $u(t, r)$, explicitly :

$$v(t, x) = \begin{cases} g(u(t, r)) \cos \theta \\ g(u(t, r)) \sin \theta \\ z(u(t, r)) \end{cases}$$

and \mathcal{Q} be given by (5.1.11). Let then

$$\tilde{v}(t, x) = v(t, x) - \mathcal{Q}\left(\frac{x}{\lambda(t)}\right), \quad (5.4.24)$$

and correspondingly

$$\tilde{u}(t, r) = u(t, r) - Q\left(\frac{r}{\lambda(t)}\right) = (\tilde{\alpha} + \varepsilon)_{\lambda(t)}. \quad (5.4.25)$$

We claim the bound :

$$\forall t \in [0, T), \quad \|\Delta \tilde{v}(t, x)\|_{L^2} \leq C(v_0). \quad (5.4.26)$$

Indeed, let the normal vector to the revolution surface M at v be given by

$$\mathbf{n} = \begin{cases} -z'(u) \cos \theta \\ -z'(u) \sin \theta \\ g'(u) \end{cases}$$

and compute the Laplace operator :

$$\Delta v = [\Delta v - (\Delta v \cdot \mathbf{n})\mathbf{n}] + (\Delta v \cdot \mathbf{n})\mathbf{n}$$

with explicitly :

$$\Delta v - (\Delta v \cdot \mathbf{n})\mathbf{n} = \left(\Delta u - \frac{f(u)}{r^2}\right) \begin{cases} g'(u) \cos \theta \\ g'(u) \sin \theta \\ z'(u) \end{cases}, \quad (5.4.27)$$

$$\Delta v \cdot \mathbf{n}\mathbf{n} = \begin{cases} \left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2}\right] \cos \theta \\ \left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2}\right] \sin \theta \\ (\partial_r u)^2 z''(u) - \frac{z(u)(z'(u))^2}{r^2} \end{cases}, \quad (5.4.28)$$

we now claim :

$$\int |\Delta \tilde{v} - (\Delta \tilde{v} \cdot \mathbf{n})\mathbf{n}|^2 \lesssim C(v_0), \quad (5.4.29)$$

$$\int |\Delta \tilde{v} \cdot \mathbf{n}|^2 \lesssim C(v_0), \quad (5.4.30)$$

which implies (5.4.26).

Proof of (5.4.29) : We inject the decomposition (5.4.25) and estimate :

$$\begin{aligned} \int |\Delta \tilde{v} - (\Delta \tilde{v} \cdot \mathbf{n}) \mathbf{n}|^2 &= \int \left| \Delta u - \frac{f(u)}{r^2} \right|^2 = \frac{1}{\lambda^2} \int \left| \Delta(\tilde{\alpha} + \varepsilon) - \frac{f(Q + \tilde{\alpha} + \varepsilon) - f(Q)}{y^2} \right|^2 \\ &\lesssim \frac{1}{\lambda^2} \left[\int |H\varepsilon|^2 + \int |H\tilde{\alpha}|^2 + \int |f''(Q)|^2 \frac{|\tilde{\alpha}|^4 + |\varepsilon|^4}{y^4} + \int \frac{|\tilde{\alpha}|^6 + |\varepsilon|^6}{y^4} \right]. \end{aligned} \quad (5.4.31)$$

We compute from the definition of $\tilde{\alpha}$ and (5.4.23) :

$$\begin{aligned} \int |H\tilde{\alpha}|^2 &\lesssim \int_{y \leq 2B_1} \left[by|\log y| + b^3 \frac{y^3}{b|\log b|} \right]^2 \lesssim b^2 |\log b|^2 \lesssim \lambda^2, \\ \int \frac{|\tilde{\alpha}|^4 + |\tilde{\alpha}|^6}{y^4} &\lesssim \int_{y \leq 2B_1} \frac{1}{y^4} \left[by|\log y| + b^3 \frac{y^3}{b|\log b|} \right]^4 \lesssim b^3 |\log b|^C \lesssim \lambda^2. \end{aligned}$$

Observe now from (5.4.12), (5.1.13), (5.4.22) the bound :

$$\int |H\varepsilon(t)|^2 = \mathcal{E}_2(t) \lesssim C(u_0) \lambda^2(t) \left[1 + \int_0^T \frac{b^3(\tau) |\log b(\tau)|^2}{\lambda^4(\tau)} d\tau \right] \lesssim C(u_0) \lambda^2(t)$$

where we used from (5.1.13), (5.4.22) again :

$$\int_0^T \frac{b^3(\tau) |\log b(\tau)|^2}{\lambda^4(\tau)} d\tau \lesssim \int_0^T \frac{1}{(T-\tau) |\log(T-\tau)|^2} d\tau < +\infty.$$

We now estimate using (5.5.24) :

$$\int_{y \leq 1} |f''(Q)|^2 \frac{|\varepsilon|^4}{y^4} \lesssim b^6 \lesssim \lambda^2,$$

and using $|f''(Q)| \lesssim \frac{y}{1+y^2}$, (5.5.26) and the energy bound :

$$\int_{y \geq 1} |f''(Q)|^2 \frac{|\varepsilon|^4}{y^4} \lesssim \left\| \frac{\varepsilon}{y^2} \right\|_{L^\infty(y \geq 1)}^2 \int \frac{|\varepsilon|^2}{y^2} \lesssim b^3 |\log b|^C \lesssim \lambda^2.$$

Injecting these bounds into (5.4.31) yields (5.4.29).

Proof of (5.4.30) : We first claim :

$$\int \left| (\partial_r u)^2 g''(u) - (\partial_r Q_\lambda)^2 g''(Q_\lambda) \right|^2 \lesssim 1. \quad (5.4.32)$$

Indeed, we estimate :

$$\begin{aligned} \left| (\partial_r u)^2 g''(u) - (\partial_r Q_\lambda)^2 g''(Q_\lambda) \right| &\lesssim |(\partial_r u)^2(g''(u) - g''(Q_\lambda))| + |g''(Q_\lambda)| |\partial_r u|^2 - |\partial_r Q_\lambda|^2 \\ &\lesssim |\tilde{u}| \left[|\partial_r Q_\lambda|^2 + |\partial_r \tilde{u}|^2 \right] + |g''(Q_\lambda)| \left[|\partial_r Q_\lambda| |\partial_r \tilde{u}| + |\partial_r \tilde{u}|^2 \right] \end{aligned}$$

and thus after rescaling using Lemma 5.5.4 :

$$\begin{aligned} & \lambda^2 \int \left| (\partial_r u)^2 g''(u) - (\partial_r Q_\lambda)^2 g''(Q_\lambda) \right|^2 \lesssim \int \frac{|\tilde{\alpha}|^2 + |\varepsilon|^2}{1+y^8} + \int |\partial_y(\tilde{\alpha} + \varepsilon)|^4 |\tilde{\alpha} + \varepsilon|^2 \\ & + \int \frac{1}{1+y^6} |\partial_y(\tilde{\alpha} + \varepsilon)|^2 + \|\partial_y(\tilde{\alpha} + \varepsilon)\|_{L^\infty}^2 \int \frac{1}{1+y^2} |\partial_y(\tilde{\alpha} + \varepsilon)|^2 \\ & \lesssim b^2 + \int (|\partial_y \tilde{\alpha}|^4 + |\partial_y \varepsilon|^4) (|\tilde{\alpha}|^2 + |\varepsilon|^2). \end{aligned}$$

We estimate :

$$\begin{aligned} \int |\partial_y \alpha|^4 |\tilde{\alpha}|^2 & \lesssim b^4 |\log b|^C, \quad \int |\partial_y \varepsilon|^4 |\tilde{\alpha}|^2 \lesssim \|\partial_y \varepsilon\|_{L^\infty}^2 \|\tilde{\alpha}\|_{L^\infty}^2 \lesssim b^4 |\log b|^C, \\ \int |\partial_y \tilde{\alpha}|^4 |\varepsilon|^2 & \lesssim b^4 |\log b|^C \int_{y \leq 2B_1} |\varepsilon|^2 \lesssim b^4 |\log b|^C B_1^8 \int \frac{|\varepsilon|^2}{(1+y^8)|\log b|^C} \lesssim b^4 |\log b|^C, \end{aligned}$$

and using a two dimensional Gagliardo Nirenberg inequality for the radial function $\partial_y \varepsilon$:

$$\int |\partial_y \varepsilon|^4 |\varepsilon|^2 \lesssim b^4 + \|\varepsilon\|_{L^\infty}^2 \|\partial_{yy} \varepsilon\|_{L^2}^2 \|\partial_y \varepsilon\|_{L^2}^2 \lesssim b^4 + \mathcal{E}_2 \lesssim \lambda^2, \quad (5.4.33)$$

which concludes the proof of (5.4.32).

We now claim :

$$\int \left| \frac{g(u)(z'(u))^2 - g(Q_\lambda)(z'(Q_\lambda))^2}{r^2} \right|^2 \lesssim 1. \quad (5.4.34)$$

Indeed,

$$\left| \frac{g(u)(z'(u))^2 - g(Q_\lambda)(z'(Q_\lambda))^2}{r^2} \right| \lesssim \frac{|z'(Q_\lambda)|^2 + |\tilde{u}| + g(Q_\lambda)}{r^2} |\tilde{u}|$$

and then from $z'(0) = z'(\pi) = 0$:

$$\int \frac{|z'(Q)|^4 + |g(Q)|^2}{y^4} |\tilde{\alpha} + \varepsilon|^2 \lesssim \int \frac{|\tilde{\alpha}|^2 + |\tilde{\varepsilon}|^2}{y^2(1+y^4)} \lesssim b^3 |\log b|^C \lesssim \lambda^2,$$

$$\begin{aligned} \int \frac{|\tilde{u}|^2}{y^4} |\tilde{\alpha} + \varepsilon|^4 & \lesssim b^3 |\log b|^C + \int_{y \geq 1} \frac{|\varepsilon|^4}{y^4} \\ & \lesssim b^3 |\log b|^C + \int [|A\varepsilon|^4 + |\partial_y \varepsilon|^4] \lesssim b^3 |\log b|^C + \lambda^2 + \int |\nabla(A\varepsilon)|^2 \int |A\varepsilon|^2 \\ & \lesssim \lambda^2 + \int |A^* A\varepsilon|^2 \lesssim \lambda^2 \end{aligned}$$

where we used (5.4.33), and (5.4.34) follows by rescaling. We estimate along the same lines :

$$\int \left| (\partial_r u)^2 z''(u) - (\partial_r Q_\lambda)^2 z''(Q_\lambda) - \frac{z(u)(z'(u))^2 - z(Q_\lambda)(z'(Q_\lambda))^2}{r^2} \right|^2 \lesssim 1,$$

this is left to the reader. This concludes the proof of (5.4.30)

step 3 Quantization of the focused energy.

We now turn to the proof of (5.1.12), (5.1.14) and adapt the strategy in [70]. The regularity of $v(t, x)$ outside the origin is a standard consequence of parabolic regularity and the fact that in corotational symmetry the nonlinearity is singular at the origin only. Hence there exists $v^* \in \dot{H}^1$ such that

$$\forall R > 0, \quad \nabla u(t) \rightarrow \nabla v^* \text{ in } L^2(|x| \geq R) \text{ as } t \rightarrow T.$$

Moreover, v is \dot{H}^1 bounded by conservation of energy, and thus recalling the decomposition (5.4.24) and the uniform bound (5.4.26) :

$$\nabla \tilde{v}(t) \rightarrow \nabla v^* \text{ in } L^2 \text{ and } \Delta v^* \in L^2$$

which concludes the proof of (5.1.12), (5.1.14). This concludes the proof of Theorem 5.1.1.

5.5 Appendix

5.5.1 Regularity in corotational symmetry

We detail in this appendix the regularity of $\dot{H}^1 \cap \dot{H}^4$ maps with 1-corotational symmetry.

Lemme 5.5.1 (Regularity in corotational symmetry). *Let v be a 1-corotational map with*

$$\|\nabla v - \nabla \mathcal{Q}\|_{\dot{H}^1} \ll 1 \text{ and } \|v\|_{\dot{H}^i} < +\infty, \quad 2 \leq i \leq 4. \quad (5.5.1)$$

Then v admits a representation

$$v(x) = \begin{cases} g(u(y)) \cos \theta \\ g(u(y)) \sin \theta \\ z(u(y)) \end{cases} \quad \text{where } u(y) = Q(y) + \varepsilon(y) \quad (5.5.2)$$

satisfies the boundary conditions

$$\varepsilon(0) = 0, \quad \lim_{y \rightarrow +\infty} \varepsilon(y) = 0, \quad (5.5.3)$$

the Sobolev bounds

$$\sum_{i=1}^4 \int |\partial_y^{(i)} \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} < +\infty \quad (5.5.4)$$

and the regularity at the origin :

$$\begin{aligned} & \int |H^2 \varepsilon|^2 + \int \frac{|H \varepsilon|^2}{y^4(1 + |\log y|)^2} + \int \frac{|AH \varepsilon|^2}{y^2} + \int |\partial_y(AH \varepsilon)|^2 + \int \frac{|\partial_y^3 \varepsilon|^2}{y^2(1 + |\log y|)^2} \\ & + \int \frac{|\partial_y^2 \varepsilon|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y \varepsilon|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\varepsilon|^2}{y^4(1 + |\log y|)^2} \\ & < +\infty. \end{aligned} \quad (5.5.5)$$

Proof of Lemma 5.5.1

step 1 \dot{H}^1, \dot{H}^2 bound.

From (5.5.2),

$$\int |\nabla v|^2 = 2\pi \int \left[|\partial_y u|^2 + \frac{g^2(u)}{y^2} \right]$$

and thus the structure of g , the Sobolev embedding in radial symmetry

$$\|u\|_{L^\infty}^2 \lesssim \|\partial_y u\|_{L^2} \left\| \frac{u}{y} \right\|_{L^2} \quad (5.5.6)$$

and the smallness (5.5.1) ensure :

$$\lim_{y \rightarrow 0} \varepsilon(y) = \lim_{y \rightarrow +\infty} \varepsilon(y) = 0.$$

The energy bound

$$\int |\partial_y \varepsilon|^2 + \frac{|\varepsilon|^2}{y^2} < +\infty \quad (5.5.7)$$

easily follows. Moreover, the energy density

$$e(y) = |\partial_1 v|^2 + |\partial_2 v|^2 = 2 \left[|\partial_y u|^2 + \frac{g^2(u)}{y^2} \right]$$

is bounded near the origin from the regularity $v \in \dot{H}^1 \cap \dot{H}^4$ which implies

$$|\partial_y u| + \frac{|u(y)|}{y} \lesssim 1 \quad \text{for } y \leq 1. \quad (5.5.8)$$

We now recall (5.4.27) which implies using the \dot{H}^2 boundedness of v :

$$\int \left| \Delta u - \frac{f(u)}{y^2} \right|^2 < +\infty.$$

Using (5.5.7), (5.5.8) and the structure of f , we conclude :

$$\int |Hu|^2 < +\infty.$$

step 2 \dot{H}^4 bound.

We now recall (5.4.27), (5.4.28) which we rewrite :

$$\Delta v = H\varepsilon \begin{vmatrix} \cos \theta \\ \sin \theta \\ 0 \end{vmatrix} + F(\varepsilon) \quad (5.5.9)$$

with :

$$\begin{aligned} F(\varepsilon) &= \left(\Delta u - \frac{f(u)}{y^2} \right) \begin{vmatrix} (g'(u) - 1) \cos \theta \\ (g'(u) - 1) \sin \theta \\ z'(u) \end{vmatrix} + \frac{f(Q + \varepsilon) - f(Q) - \varepsilon f'(Q) + (f'(Q) - 1)\varepsilon}{y^2} \begin{vmatrix} \cos \theta \\ \sin \theta \\ 0 \end{vmatrix} \\ &+ \begin{vmatrix} \left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2} \right] \cos \theta \\ \left[(\partial_r u)^2 g''(u) - \frac{g(u)(z'(u))^2}{r^2} \right] \sin \theta \\ (\partial_r u)^2 z''(u) - \frac{z(u)(z'(u))^2}{r^2} \end{vmatrix}, \end{aligned}$$

We then compute $\nabla \Delta v$, $\Delta^2 v$ and use the odd parity of f, g and the cancellation at the origin (5.5.8) to conclude after a brute force computation :

$$\int |\partial_y H\varepsilon|^2 + \int \frac{|H\varepsilon|^2}{y^2} + \int |H^2\varepsilon|^2 < +\infty. \quad (5.5.10)$$

The Sobolev bound (5.5.4) away from the origin now easily follows :

$$\sum_{i=1}^4 \int_{y \geq 1} |\partial_y^{(i)} \varepsilon|^2 < +\infty.$$

step 3 Regularity at the origin.

We claim that (5.5.10) also implies the regularity (5.5.5) at the origin. Indeed, we estimate from (5.5.6) and the two dimensional Hardy inequality with logarithmic loss :

$$\begin{aligned} &\int_{y \leq 1} \left[\frac{|\partial_y H\varepsilon|^2}{y^2(1 + |\log y|^2)} + \frac{|H\varepsilon|^2}{y^4(1 + |\log y|^2)} \right] \\ &\lesssim 1 + \int \frac{|\nabla \Delta v|^2}{y^2(1 + |\log y|^2)} \lesssim 1 + \int |\Delta^2 v|^2 + \int_{1 \leq y \leq 2} |\nabla \Delta v|^2 \\ &\lesssim 1 \end{aligned} \quad (5.5.11)$$

We similarly compute from (5.4.27) after an explicit computation :

$$\int_{y \leq 1} \frac{(AH\varepsilon)^2}{y^2} \lesssim 1 + \int \left| \frac{1}{r} \partial_\theta \partial_1 (\Delta v - (\Delta v \cdot \mathbf{n}) \mathbf{n}) \right|^2 < +\infty.$$

We now observe from

$$\partial_y (\log(\Lambda\phi)) = \frac{Z}{y}$$

that for any function h :

$$A^*h = \partial_y h + \frac{1+Z}{y}h = \frac{1}{y\Lambda\phi} \partial_y (y\Lambda\phi h),$$

and thus using the a priori bound (5.5.8) :

$$A\varepsilon(y) = \frac{1}{y\Lambda\phi(y)} \int_0^y \tau \Lambda\phi(\tau) H\varepsilon(\tau) d\tau.$$

We then estimate from Cauchy Schwarz and Fubbini :

$$\begin{aligned} \int_{y \leq 1} \frac{|A\varepsilon|^2}{y^5(1 + |\log y|^2)} dy &\lesssim \int_{0 \leq y \leq 1} \int_{0 \leq \tau \leq y} \frac{y^5}{y^9(1 + |\log y|^2)} |H\varepsilon(\tau)|^2 dy d\tau \\ &\lesssim \int_{0 \leq \tau \leq 1} |H\varepsilon(\tau)|^2 \left[\int_{\tau \leq y \leq 1} \frac{dy}{y^4(1 + |\log y|^2)} \right] d\tau \lesssim \int_{\tau \leq 1} \frac{|H\varepsilon(\tau)|^2}{\tau^3(1 + |\log \tau|^2)} d\tau \\ &\lesssim 1. \end{aligned} \quad (5.5.12)$$

We now rewrite near the origin :

$$H\varepsilon = -\partial_y^2\varepsilon + \frac{1}{y} \left(-\partial_y\varepsilon + \frac{\varepsilon}{y} \right) + \frac{V-1}{y^2}\varepsilon = -\partial_y^2\varepsilon + \frac{A\varepsilon}{y} + \frac{(V-1)+(1-Z)}{y^2}\varepsilon$$

which implies using (5.5.11), (5.5.12), (5.5.8) :

$$\begin{aligned} \int_{y \leq 1} \frac{|\partial_y^2\varepsilon|^2}{y^4(1 + |\log y|^2)} &\lesssim \int_{y \leq 1} \frac{|H\varepsilon|^2}{y^4(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|A\varepsilon|^2}{y^6(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\varepsilon|^2}{y^4(1 + |\log y|^2)} \\ &\lesssim 1, \\ \int_{y \leq 1} \frac{|\partial_y^3\varepsilon|^2}{y^2(1 + |\log y|^2)} &\lesssim \int_{y \leq 1} \frac{|\partial_y H\varepsilon|^2}{y^2(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|H\varepsilon|^2}{y^4(1 + |\log y|^2)} \\ &+ \int_{y \leq 1} \frac{|\partial_y\varepsilon|^2}{y^2(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\varepsilon|^2}{y^4(1 + |\log y|^2)} \\ &\lesssim 1 \end{aligned}$$

This concludes the proof of Lemma 5.5.1.

5.5.2 Coercivity bounds and interpolation estimates

We recall in this section the coercivity bounds we use involving the operators H, \tilde{H} and their iterate. Let us start with the coercivity of \tilde{H} :

Lemme 5.5.2 (Coercivity of \tilde{H}). *Let ε_3 with*

$$\int \frac{|\varepsilon_3|^2}{y^2} < +\infty,$$

then

$$\mathcal{E}_4 = (\tilde{H}\varepsilon_3, \varepsilon_3) = \|A^*\varepsilon_3\|_{L^2}^2 \geq c_0 \left[\int |\partial_y\varepsilon_3|^2 + \int \frac{|\varepsilon_3|^2}{y^2(1 + |\log y|^2)} \right] \quad (5.5.13)$$

for some universal constant $c_0 > 0$.

Proof of Lemma 5.5.2. The proof is immediate in the case of \mathbb{S}^2 target thanks to the sign

$$\tilde{V} = \frac{4}{1 + y^2} \geq 0$$

which yields (5.5.13) from standard two dimensional weighted Hardy inequality -see [72] for more details-. For a general g however, this sign property is lost and the proof relies on a

standard compactness argument and the explicit knowledge of the kernel of A^* .

We first claim the subcoercivity property :

$$\begin{aligned} (\tilde{H}\varepsilon_3, \varepsilon_3) &\geq c_0 \left(\int |\partial_y \varepsilon_3|^2 + \int_{y \leq 1} \frac{|\varepsilon_3|^2}{y^2} + \int_{y \geq 1} \frac{\varepsilon_3^2}{y^2(1+y^2)} + \int \frac{|\varepsilon_3|^2}{y^2(1+|\log y|^2)} \right) \\ &- \frac{1}{c_0} \int \frac{|\varepsilon_3|^2}{1+y^4} \end{aligned} \quad (5.5.14)$$

for some universal constant $c_0 > 0$. Indeed, we estimate near the origin using the expansion (5.2.16) :

$$\int_{y \leq 1} \frac{\tilde{V}}{y^2} |\varepsilon_3|^2 \geq c_0 \int_{y \leq 1} \frac{|\varepsilon_3|^2}{y^2} - \frac{1}{c_0} \int_{y \leq 1} |\varepsilon_3|^2.$$

For $y \geq 1$, the logarithmic Hardy bound

$$\int \frac{|\varepsilon_3|^2}{y^2(1+|\log y|^2)} \lesssim \int |\nabla \varepsilon_3|^2 + \int_{1 \leq y \leq 2} |\varepsilon_3|^2$$

and the degeneracy from (5.2.16) $|\tilde{V}(y)| \lesssim \frac{1}{y^2}$ yield (5.5.14).

We now prove (5.5.13) and argue by contradiction. Let a sequence $\varepsilon_3^{(n)}$ with

$$\int |\partial_y \varepsilon_3^{(n)}|^2 + \int_{y \leq 1} \frac{|\varepsilon_3^{(n)}|^2}{y^2} + \int_{y \geq 1} \frac{|\varepsilon_3^{(n)}|^2}{y^2(1+|\log y|^2)} = 1, \quad (\tilde{H}\varepsilon_3^{(n)}, \varepsilon_3^{(n)}) \leq \frac{1}{n}, \quad (5.5.15)$$

then the sequence $\varepsilon_3^{(n)}$ is bounded in H_{loc}^1 and thus weakly converges to ε_3^* up to a subsequence. Moreover, (5.5.14),

$$\int \frac{|\varepsilon_3^{(n)}|^2}{1+y^4} \gtrsim c_0 > 0$$

and thus by lower semi continuity of norms and the compactness of the Sobolev embedding $H^1 \hookrightarrow L_{loc}^2$:

$$\int |\partial_y \varepsilon_3^*|^2 + \int_{y \leq 1} \frac{|\varepsilon_3^*|^2}{y^2} + \int_{y \geq 1} \frac{|\varepsilon_3^*|^2}{y^2(1+|\log y|^2)} = 1, \quad \int \frac{|\varepsilon_3^*|^2}{1+y^4} \gtrsim c_0 > 0 \quad (5.5.16)$$

and

$$(\tilde{H}\varepsilon_3^*, \varepsilon_3^*) = \|A^* \varepsilon_3^*\|_{L^2}^2 \leq 0.$$

Hence

$$A^* \varepsilon_3 = 0 \quad \text{ie} \quad \varepsilon_3 = \frac{c}{y \Lambda Q}$$

which contradicts the boundary condition at the origin imposed by (5.5.16). This concludes the proof of (5.5.13) and Lemma 5.5.2.

We now claim the following coercivity properties for H, H^2 which rely on a similar compactness argument and the explicit knowledge of the kernel of H, H^2 . The proof is given in [88], [72] in the case $g(u) = \sin u$, but the same argument applies under our assumptions on g , the key being the behaviour at the origin and infinity (5.2.14), (5.2.15), (5.2.16). The proof is therefore left to the reader :

Lemme 5.5.3 (Coercivity of H, H^2). *Let $M > 0$ and ε radially symmetric satisfying (5.5.3), (5.5.4), (5.5.5) and the orthogonality conditions*

$$(\varepsilon, \Phi_M) = (\varepsilon, H\Phi_M) = 0.$$

Then there exists $C(M) > 0$ such that the following bounds hold :

$$\begin{aligned} (H\varepsilon, \varepsilon) &= \int |A\varepsilon|^2 \geq C(M) \int \left[(\partial_y \varepsilon)^2 + \frac{\varepsilon^2}{y^2} \right], \\ \int (H\varepsilon)^2 &\geq C(M) \int \left[\frac{(\partial_y \varepsilon)^2}{y^2(1 + |\log y|^2)} + \frac{\varepsilon^2}{y^4(1 + |\log y|^2)} \right], \\ \int (H^2 \varepsilon)^2 &\geq C(M) \int \left[\frac{|H\varepsilon|^2}{y^4(1 + |\log y|)^2} + \frac{|\partial_y H\varepsilon|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial_y^3 \varepsilon|^2}{y^2(1 + |\log y|)^2} \right. \\ &\quad \left. + \frac{|\partial_y^2 \varepsilon|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y \varepsilon|^2}{y^2(1 + y^4)(1 + |\log y|)^2} + \int \frac{|\varepsilon|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \right] \end{aligned}$$

We now recall the interpolation bounds on ε needed all along the proof of Proposition 5.3.1 which are a direct consequence of two dimensional weighted Hardy estimates and the coercivity bounds of Lemma 5.5.3. These bounds which hold in the setting of the bootstrap bounds (5.3.13), (5.3.14), (5.3.15), (5.3.16) were explicitly derived in [72] to which we refer for a proof.

Lemme 5.5.4 (Interpolation estimates). *There holds -with constants a priori depending on M - :*

$$\begin{aligned} &\int \frac{|\varepsilon|^2}{y^4(1 + y^4)(1 + |\log y|^2)} + \int \frac{|\partial_y^i \varepsilon|^2}{y^2(1 + y^{6-2i})(1 + |\log y|^2)} \\ &\lesssim \mathcal{E}_4, \quad 1 \leq i \leq 3, \end{aligned} \tag{5.5.17}$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^2(1 + |\log y|^2)(1 + y^{6-2i})} |\partial_y^i \varepsilon|^2 \lesssim b^4 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 3, \tag{5.5.18}$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^2(1 + |\log y|^2)(1 + y^{4-2i})} |\partial_y^i \varepsilon|^2 \lesssim b^3 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 3, \tag{5.5.19}$$

$$\|\varepsilon\|_{L^\infty} \lesssim \delta(\alpha^*), \tag{5.5.20}$$

$$\|A\varepsilon\|_{L^\infty}^2 \lesssim b^2 |\log b|^2, \tag{5.5.21}$$

$$\left\| \frac{A\varepsilon}{y^2(1 + |\log y|)} \right\|_{L^\infty(y \leq 1)}^2 + \left\| \frac{\Delta A\varepsilon}{1 + |\log y|} \right\|_{L^\infty(y \leq 1)}^2 + \left\| \frac{H\varepsilon}{y(1 + |\log y|)} \right\|_{L^\infty(y \leq 1)}^2 \lesssim b^4, \tag{5.5.22}$$

$$\left\| \frac{|H\varepsilon|}{y(1 + |\log y|)} \right\|_{L^\infty(y \leq 1)}^2 \lesssim b^4, \tag{5.5.23}$$

$$\left\| \frac{\varepsilon}{y} \right\|_{L^\infty(y \leq 1)}^2 + \left\| \frac{\partial_y \varepsilon}{\sqrt{1 + |\log y|}} \right\|_{L^\infty(y \leq 1)}^2 \lesssim b^4, \tag{5.5.24}$$

$$\left\| \frac{\varepsilon}{y} \right\|_{L^\infty(y \geq 1)}^2 + \|\partial_y \varepsilon\|_{L^\infty(y \geq 1)}^2 \lesssim b^2 |\log b|^C, \quad (5.5.25)$$

$$\left\| \frac{\varepsilon}{1+y^2} \right\|_{L^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{1+y} \right\|_{L^\infty}^2 + \|\partial_{yy} \varepsilon\|_{L^\infty(y \geq 1)}^2 \lesssim b^3 |\log b|^C, \quad (5.5.26)$$

$$\|\partial_{yyy} \varepsilon\|_{L^\infty(y \geq 1)}^2 \lesssim b^4 |\log b|^C. \quad (5.5.27)$$

Chapitre 6

Article [94] sur l'équation de la chaleur

6.1 Introduction

6.1.1 Setting of the problem

We consider in this paper the energy critical semi linear heat equation

$$\partial_t u - \Delta u - u^3 = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^4 \quad (6.1.1)$$

which is the energy critical four dimensional version of the more general problem

$$\partial_t u - \Delta u - u^p = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad p \geq 2^* - 1 \quad (6.1.2)$$

where

$$2^* = \frac{2N}{N-2}$$

is the Sobolev exponent. There is an important literature devoted to the qualitative description of solutions to (6.1.2), and we refer to [65], [66] for a complete introduction to the history of the problem. For radial data, two type of blow-up regimes are typically expected : type I blow-up which corresponds to a self-similar blow-up, and type II blow-up which displays excited blow-up speeds. Such kind of type II blow-up solutions were formally predicted by Herrero and Velázquez [38] using matching asymptotic procedures for a large value of p , and the corresponding regime displays a polynomial type blow-up speed. A major breakthrough is achieved by Matano and Merle in [65], [66], where the *non-existence of type II* blow-up is shown for

$$2^* - 1 < p < p_c$$

where

$$p_c = \begin{cases} +\infty & \text{for } N \leq 10 \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{for } N \geq 11 \end{cases},$$

and the *existence* of type II blow-up for $p > p_c$ is proved. More precisely, such solutions are obtained as *threshold* dynamics between well known type I blow-up solutions and global dissipative dynamics. A complete classification of these type II regimes is then completed in [76]

where quantized blow-up speeds are exhibited with polynomial rates.

These results leave completely open the question of existence of type II blow-up in the energy critical setting. In fact, in the energy critical setting and even for the parabolic problem, the maximum principle does not seem to yield enough information to control a type II blow-up. The criticality of the problem is reflected by the fact that the total dissipated energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t, x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1}(t, x) dx \quad (6.1.3)$$

is left invariant by the scaling symmetry of the problem

$$u_\lambda(t, x) = \lambda^{\frac{N-2}{2}} u(\lambda^2 t, \lambda x), \quad E(u_\lambda(t)) = E(u(\lambda^2 t)).$$

The study of *critical problems* has attracted a considerable attention for the past ten years in the dispersive community, in particular the study of the mass critical nonlinear Schrödinger equation [81], [69], [67], [68], [71], [70] and geometric problems like wave maps, Schrödinger maps and the harmonic heat flow [54], [93], [88], [72], [91]. In particular, a robust *energy approach* is developed in [88], [72] to construct type II blow-up solutions in the energy critical setting. This strategy is implemented in the parabolic setting in [91] and led to the construction of *stable* blow-up dynamics with sharp asymptotics on the singularity formation for the harmonic heat flow. Note that more type II regimes for dispersive problems were obtained in [54], [53] but rely on the construction of *non-smooth* solutions and a procedure of backwards in time integration of the flow from the singularity which are both non suitable for parabolic problems.

6.1.2 Statement of the result

We carry out in this paper the program which was implemented in [41] to adapt the study of the geometric wave equation in [88] to the semi linear cubic four dimensional wave equation. The main difficulty is the fact that the energy (6.1.3) is non-definite positive, and this induces a non-positive eigenvalue in the spectrum of the linearized operator close to the Talenti-Aubin stationary solution

$$Q(r) = \frac{1}{1 + \frac{r^2}{8}}, \quad (6.1.4)$$

which is the unique up to scaling radially symmetric solution to the stationary problem

$$\Delta Q + Q^3 = 0, \quad Q(r) = \frac{1}{1 + \frac{r^2}{8}}. \quad (6.1.5)$$

This requires building our set of initial data on a suitable codimension one set, and we similarly claim in the continuation of [91] the existence of a type II blow-up dynamics for the energy critical four dimensional problem :

Théorème 6.1.1 (Existence of type II blow-up in dimension $N = 4$). *Let Q be the Talenti-Aubin soliton (6.1.5). Then $\forall \alpha^* > 0$, there exists a radially symmetric initial data $u_0 \in H^1(\mathbb{R}^4)$ with*

$$E(Q) < E(u_0) < E(Q) + \alpha^* \quad (6.1.6)$$

such that the corresponding solution to the energy critical focusing parabolic equation (6.1.1) blows up in finite time $T = T(u_0) < \infty$ in a type II regime according to the following dynamics : there exists $u^* \in \dot{H}^1$ such that :

$$\nabla \left[u(t, x) - \frac{1}{\lambda(t)} Q \left(\frac{x}{\lambda(t)} \right) \right] \rightarrow \nabla u^* \text{ in } L^2 \text{ as } t \rightarrow T \quad (6.1.7)$$

at the speed

$$\lambda(t) = c(u_0) (1 + o(1)) \frac{T-t}{|\log(T-t)|^2} \text{ as } t \rightarrow T \quad (6.1.8)$$

for some $c(u_0) > 0$. Moreover, there hold the regularity of the asymptotic profile :

$$\Delta u^* \in L^2 \quad (6.1.9)$$

Comments on the result

1. In dimension four, the choice $p = 2^* - 1 = 3$ is therefore the only one for which a type II blow-up occurs for radial data. We have decided to focus onto the four dimensional case for the case of simplicity but the construction we propose could be addressed in a much more general setting. Let us insist also that it does not rely on the maximum principle and may therefore be addressed in the non-radial setting and for more complicated systems.

2. The blow-up speed (6.1.8) is also the one obtained for the energy critical harmonic heat flow in [91]. Following the heuristics developed in [99], we conjecture the existence of a sequence of *quantized* blow-up speeds with polynomial rates corrected by suitable logarithmic factors, and (6.1.8) is the *fundamental* which corresponds from the proof to a codimension one in some weak sense manifold of initial data.

The main open problem after this work is to obtain a complete classification of type II blow-up for the energy critical problem, both in the radially symmetric case and the non-symmetric case.

Acknowledgments This work is part of the author's PhD thesis which is supported by the ERC/ANR project SWAP. The author would like to thank his advisor Pierre Raphaël for his guidance and advice.

Notations We introduce the differential operator

$$\Lambda f = f + y \cdot \nabla f \text{ (energy critical scaling).}$$

Given a positive number $b > 0$, we let

$$B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}. \quad (6.1.10)$$

Given a parameter $\lambda > 0$, we let

$$u_\lambda(r) = \frac{1}{\lambda} u(y) \text{ with } y = \frac{r}{\lambda}.$$

We let χ as a smooth cut-off function with

$$\chi(y) = \begin{cases} 1 & \text{for } y \leq 1, \\ 0 & \text{for } y \geq 2. \end{cases}$$

We shall systematically omit the measure in all radial two dimensional integrals and note :

$$\int f = \int_0^{+\infty} f(r) r^3 dr.$$

6.2 Construction of an explicit approximate solution

The aim of this section is to construct an approximate blow-up solution of (6.1.1), which is close to the ground state Q . This approximate solution will be the dominant part of the blow-up profile inside the parabolic zone. We adapt the strategy developed in [72], [88], [91].

Let u be a stationary solution of (6.1.1). Let $\lambda > 0$. Then, $\frac{1}{\lambda}u(\frac{r}{\lambda})$ is also a solution. Consider now that λ is no more a constant, but depends of time. Thus, we obtain the following equation :

$$\lambda(t)^2 \partial_t u \left(\frac{r}{\lambda(t)} \right) - \lambda(t) \lambda'(t) \Lambda u \left(\frac{r}{\lambda(t)} \right) - \Delta u \left(\frac{r}{\lambda(t)} \right) - u^3 \left(\frac{r}{\lambda(t)} \right) = 0. \quad (6.2.1)$$

We then define a rescaled time

$$s = \int_0^t \frac{d\tau}{\lambda^2(\tau)}. \quad (6.2.2)$$

Remark that if $\lambda(t)$ verifies the law (6.1.8) defined in the Theorem 6.1.1, then $s(t)$ is a bijection between $[0, T[$ and \mathbb{R}^+ . We also let the rescaled variable $y(t) = \frac{r}{\lambda(t)}$. Eq. (6.2.1) becomes, using the new variables :

$$\partial_s u - \frac{\lambda_s}{\lambda} \Lambda u - \Delta u - u^3 = 0. \quad (6.2.3)$$

As well as the parameter $\lambda(s)$, we define a new parameter $b(s)$ such that :

$$b = -\frac{\lambda_s}{\lambda} (1 + o(1)), \quad (6.2.4)$$

$$b_s = -b^2 (1 + o(1)). \quad (6.2.5)$$

The modulation laws (6.2.4) and (6.2.5) will be justified thereafter. First, in the following subsection, we consider that

$$b_s = -b^2 \quad \text{and} \quad b + \frac{\lambda_s}{\lambda} = 0, \quad (6.2.6)$$

b being positive.

6.2.1 Construction of explicit approximate blow-up profiles

Proposition 6.2.1 (Construction of the approximate profile). *Let $M > 0$ be large enough. Then, there exists a small enough universal constant $b^*(M)$, such that the following holds. Let $b \in]0, b^*(M)[$. Then there exists profiles T_1 , T_2 and T_3 , such that*

$$Q_b(y) = Q(y) + bT_1(y) + b^2T_2(y) + b^3T_3(y) = Q(y) + \alpha(y)$$

generates an error

$$\Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b - (Q_b)^3 + b\Lambda Q_b \quad (6.2.7)$$

which satisfies :

(i) Weighted bounds :

$$\int_{y \leq 2B_1} |H\Psi_b|^2 \lesssim b^4 |\log b|^2, \quad (6.2.8)$$

$$\int_{y \leq 2B_1} \frac{1}{1+y^8} |\Psi_b|^2 \lesssim b^6, \quad (6.2.9)$$

$$\int_{y \leq 2B_1} |H^2 \Psi_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (6.2.10)$$

(ii) Flux computation : Let Φ_M be given by (6.3.2), then :

$$\frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \quad (6.2.11)$$

Remarque 6.2.2. From the proof, the profiles $(T_i)_{1 \leq i \leq 3}$ display a lower order dependance in b .

Proof of Proposition 6.2.1

Step1. Computation of the error.

We expand Q_b^3 and formulate the error Ψ_b as a polynomial expression in b :

$$\begin{aligned} Q_b^3 &= Q^3 + 3bQ^2 T_1 + b^2 (3Q^2 T_2 + 3QT_1^2) + b^3 (3Q^2 T_3 + 6QT_1 T_2 + T_1^3) \\ &\quad + R_1(T_1, T_2, T_3), \end{aligned}$$

where $R_1(T_1, T_2, T_3)$ is polynomial in $(T_i)_{1 \leq i \leq 3}$ and contains the terms of power $(b^j)_{j \geq 4}$. Hence,

$$\begin{aligned} \Psi_b &= b(HT_1 + \Lambda Q) \\ &\quad + b^2(HT_2 - T_1 + \Lambda T_1 - 3QT_1^2) \\ &\quad + b^3(HT_3 - 2T_2 + \Lambda T_2 - 6QT_1 T_2 - T_1^3) \\ &\quad + b^4 \Lambda T_3 + R_1(T_1, T_2, T_3) \end{aligned} \quad (6.2.12)$$

with

$$H = -\Delta - 3Q^2 = -\Delta - V. \quad (6.2.13)$$

Moreover,

$$V(y) = \frac{3}{\left(1 + \frac{y^2}{8}\right)^2}, \quad (6.2.14)$$

which yields

$$V(y) = \begin{cases} 3 + O(y^2) & \text{as } y \rightarrow 0, \\ \frac{192}{y^4} + O\left(\frac{1}{y^6}\right) & \text{as } y \rightarrow +\infty, \end{cases} \quad (6.2.15)$$

and

$$\Lambda V = \frac{-192(3y^2 - 8)}{(y^2 + 8)^3}. \quad (6.2.16)$$

Step 2. Construction of T_1 .

The spectral structure of the Schrödinger operator H is well known : it has a well localized non positive eigenvalue

$$H\psi = -\zeta\psi, \quad \zeta > 0,$$

and a resonance at the origin induced by the energy critical scaling symmetry :

$$H\Lambda Q = 0, \quad \Lambda Q \notin L^2(\mathbb{R}^4).$$

Hence the Green's functions of H are explicit and the other solution to $H\Gamma = 0$ for $y > 0$ is given by :

$$\Gamma(y) = -\Lambda Q \int_1^y \frac{dx}{x^3(\Lambda Q(x))^2} = \frac{y^2 - 8}{(y^2 + 8)^2} \left(\frac{y^2}{16} + 6\log y - \frac{583}{112} - \frac{4}{y^2} \right) - \frac{64}{(y^2 + 8)^2},$$

which yields

$$\Gamma(y) = \begin{cases} O\left(\frac{1}{y^2}\right) & \text{as } y \rightarrow 0, \\ \frac{1}{16} + O\left(\frac{\log y}{y^2}\right) & \text{as } y \rightarrow +\infty. \end{cases} \quad (6.2.17)$$

We may thus invert H explicitly and the smooth solutions at the origin of

$$Hu = f$$

are given by

$$u = \Gamma(y) \int_0^y f \Lambda Q - \Lambda Q(y) \int_0^y f \Gamma + c \Lambda Q(y), \quad c \in \mathbb{R}. \quad (6.2.18)$$

We let T_1 be the solution of

$$HT_1 + \Lambda Q = 0, \quad (6.2.19)$$

canceled in zero, which means that we choose the constant $c = 0$ in (6.2.18). There hold the behaviors at $r \rightarrow +\infty$

$$\Lambda^i T_1(y) = -4 \left(\log y - \frac{1}{2} + i \right) + O\left(\frac{(\log y)^2}{y^2}\right), \quad \text{for } 0 \leq i \leq 2. \quad (6.2.20)$$

There hold the behaviors at $r \rightarrow 0$

$$\Lambda^i T_1 = O\left(y^2\right), \quad \text{for } 0 \leq i \leq 2. \quad (6.2.21)$$

Hence, for $0 \leq i \leq 2$

$$\begin{aligned} \|\Lambda^i T_1\|_{L^\infty_{y \leq 2B_1}} &\lesssim \log b, \\ H\Lambda^i T_1 &\sim \frac{8}{y^2}, \quad \text{when } y \rightarrow \infty. \end{aligned}$$

Step 3. Construction of the radiation Σ_b .

First, we can notice that the choice $b_s = -b^2$ allows to cancel the $\log y$ growth of the expression $-T_1 + \Lambda T_1 - 3Q T_1^2$. We now construct a radiation term according two specifications. First, it must compensate the 1-growth of the last expression. Moreover the error induced by this term inside the parabolic zone $y \lesssim B_0$ must be sufficiently small in order not to perturb the dynamics of the blow-up.

Let

$$c_b = \frac{64}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} = \frac{2}{|\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right) \right) \quad (6.2.22)$$

and

$$d_b = c_b \int_0^{B_0} \chi_{\frac{B_0}{4}} \Lambda Q \Gamma = O\left(\frac{1}{b |\log b|}\right). \quad (6.2.23)$$

Let Σ_b be the solution to

$$H\Sigma_b = c_b \chi_{\frac{B_0}{4}} \Lambda Q + d_b H[(1 - \chi_{3B_0}) \Lambda Q] \quad (6.2.24)$$

given by

$$\Sigma_b(y) = \Gamma(y) \int_0^y c_b \chi_{\frac{B_0}{4}}(\Lambda Q)^2 - \Lambda Q(y) \int_0^y c_b \chi_{\frac{B_0}{4}} \Gamma \Lambda Q + d_b (1 - \chi_{3B_0}) \Lambda Q.$$

The choice of the constants c_b and d_b yields :

$$\Sigma_b = \begin{cases} c_b T_1 & \text{for } y \leq \frac{B_0}{4} \\ 64\Gamma & \text{for } y \geq 6B_0 \end{cases} \quad \Lambda \Sigma_b = \begin{cases} c_b \Lambda T_1 & \text{for } y \leq \frac{B_0}{4} \\ 64\Lambda\Gamma & \text{for } y \geq 6B_0. \end{cases} \quad (6.2.25)$$

Then the estimates for Σ_b and $\Lambda \Sigma_b$ for $6B_0 \leq y \leq 2B_1$,

$$\Sigma_b(y) = 4 + O\left(\frac{\log y}{y^2}\right) \quad \Lambda \Sigma_b(y) = 4 + O\left(\frac{\log y}{y^2}\right), \quad (6.2.26)$$

which fits the first criterion, that we fixed to construct the radiation. For $\frac{B_0}{4} \leq y \leq 6B_0$, we have :

$$\begin{aligned} \Sigma_b(y) &= c_b \left(\frac{1}{16} + O\left(\frac{\log y}{y^2}\right) \right) \left[\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 \right] - c_b \Lambda Q(y) \int_1^y O(x) dx \\ &= 4 \frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1}{|\log b|}\right). \end{aligned} \quad (6.2.27)$$

$$\begin{aligned} \Lambda \Sigma_b(y) &= \Lambda \Gamma(y) \int_0^y c_b \chi_{\frac{B_0}{4}}(\Lambda Q)^2 - \Lambda^2 Q(y) \int_0^y c_b \chi_{\frac{B_0}{4}} \Gamma \Lambda Q \\ &= c_b \left(\frac{1}{16} + O\left(\frac{\log y}{y^2}\right) \right) \left[\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 \right] - c_b \Lambda^2 Q(y) \int_1^y O(x) dx \\ &= 4 \frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1}{|\log b|}\right). \end{aligned} \quad (6.2.28)$$

Similarly ,

$$\Lambda^2 \Sigma_b(y) = 4 \frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} + O\left(\frac{1}{|\log b|}\right).$$

Eq. (6.2.24) and the cancellation $H\Lambda Q = 0$ yield the bounds :

$$\int |H\Sigma_b|^2 \lesssim \frac{1}{|\log b|}, \quad \int \frac{1}{1+y^8} |\Sigma_b|^2 \lesssim b^2, \quad \int |H^2 \Sigma_b|^2 \lesssim \frac{b^2}{|\log b|^2}. \quad (6.2.29)$$

We will see that this bounds with respect to the second criterion for the construction of the radiation. Furthermore we will see the importance of the term c_b , which modifies the modulation equation of b . It becomes :

$$b_s = -b^2 \left(1 + \frac{2}{|\log b|}\right). \quad (6.2.30)$$

After reintegration, this equation gives the expected blow-up speed (6.1.8).

Step 4. Construction of T_2 .

Define

$$\Sigma_2 = \Sigma_b + T_1 - \Lambda T_1 - 3Q T_1^2. \quad (6.2.31)$$

The profile T_2 will be defined later as the suitable output of H for the argument Σ_2 . Estimate Σ_2 before choosing T_2 . For $y \leq 1$,

$$\Sigma_2 \lesssim y^2. \quad (6.2.32)$$

For $1 \leq y \leq 6B_0$

$$\begin{aligned} \Sigma_2 &= 4 \left(\frac{\int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} - 1 \right) + O\left(\frac{(\log y)^2}{y^2}\right) + O\left(\frac{1}{|\log b|}\right) \\ &\lesssim \frac{1 + \log(y\sqrt{b})}{|\log b|}. \end{aligned} \quad (6.2.33)$$

According to the choice of the modulation law for b and the construction of the radiation, we have, for $y \geq 6B_0$

$$\Sigma_2 \lesssim \frac{(\log y)^2}{y^2}. \quad (6.2.34)$$

Hence, we obtain :

$$|\Sigma_2| \lesssim \frac{y^2}{1+y^2} \left(\mathbf{1}_{y \leq 1} + \frac{1 + \log(y\sqrt{b})}{|\log b|} \mathbf{1}_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y^2} \mathbf{1}_{y \geq 6B_0}. \quad (6.2.35)$$

We have the same bound for $\Lambda \Sigma_2$ and for $\Lambda^2 \Sigma_2$. We now let T_2 be the solution to

$$HT_2 = \Sigma_2 \quad (6.2.36)$$

given by

$$T_2(y) = \Gamma(y) \int_0^y \Sigma_2 \Lambda Q - \Lambda Q(y) \int_0^y \Sigma_2 \Gamma. \quad (6.2.37)$$

We derive from (6.2.35) the bounds :

$$\forall y \leq 2B_1, |\Lambda^i T_2(y)| \lesssim \frac{y^4}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right), \quad 0 \leq i \leq 1 \quad (6.2.38)$$

$$|T_2(y)| \lesssim y^2. \quad (6.2.39)$$

With an explicit calculus, we prove that for any function f :

$$H\Lambda f = 2Hf + \Lambda Hf - \Lambda Vf. \quad (6.2.40)$$

Hence,

$$H(\Lambda T_2) = 2\Sigma_2 + \Lambda\Sigma_2 - \Lambda VT_2 \quad (6.2.41)$$

and

$$|H(\Lambda T_2)| \lesssim \frac{y^2}{1+y^2} \left(\mathbf{1}_{y \leq 1} + \frac{1 + \log(y\sqrt{b})}{|\log b|} \mathbf{1}_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y^2} \mathbf{1}_{y \geq 6B_0}. \quad (6.2.42)$$

Step 5. Construction of T_3 .

In the same way as before, we define

$$\Sigma_3 = -2T_2 + \Lambda T_2 - 6QT_1T_2 - T_1^3. \quad (6.2.43)$$

Notice that we do not need to construct a second radiation term. We estimate from (6.2.20) and (6.2.38)

$$\forall y \leq 2B_1, |\Sigma_3(y)| \lesssim \frac{y^4}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right) \quad (6.2.44)$$

and

$$\forall y \leq 2B_1, |\Lambda\Sigma_3(y)| \lesssim \frac{y^4}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right). \quad (6.2.45)$$

We then let T_3 be the solution to

$$HT_3 = \Sigma_3 \quad (6.2.46)$$

given by :

$$T_3(y) = \Gamma(y) \int_0^y \Sigma_3 \Lambda Q - \Lambda Q(y) \int_0^y \Sigma_3 \Gamma. \quad (6.2.47)$$

Hence,

$$\Lambda T_3(y) = \Lambda \Gamma(y) \int_0^y \Sigma_3 \Lambda Q - \Lambda^2 Q(y) \int_0^y \Sigma_3 \Gamma. \quad (6.2.48)$$

We estimate from (6.2.44)

$$\forall y \leq 2B_1, |\Lambda^i T_3(y)| \lesssim \frac{y^6}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right), \quad 0 \leq i \leq 1 \quad (6.2.49)$$

$$|T_3(y)| \lesssim y^2(1+y^2). \quad (6.2.50)$$

Finally with (6.2.40),

$$H(\Lambda T_3) = 2\Sigma_3 + \Lambda\Sigma_3 - \Lambda VT_3 \quad (6.2.51)$$

and

$$\forall y \leq 2B_1, \quad |H\Lambda T_3(y)| \lesssim \frac{y^4}{1+y^4} \left(\mathbf{1}_{y \leq 1} + \frac{1}{b|\log b|} \mathbf{1}_{y \geq 1} \right). \quad (6.2.52)$$

We have thus the bounds for $i = 0, 1$, using (6.2.52), (6.2.49), (6.2.44) and (6.2.45) :

$$\int_{y \leq 2B_1} |H\Lambda^i T_3|^2 \lesssim \int_{y \leq 2B_1} \frac{1}{b^2 |\log b|^2} \lesssim \frac{B_1^4}{b^2 |\log b|^2} \lesssim \frac{|\log b|^2}{b^4}, \quad (6.2.53)$$

$$\int_{y \leq 2B_1} \frac{1}{1+y^8} |\Lambda^i T_3|^2 \lesssim \frac{1}{b^2 |\log b|^2} \int_{y \leq 2B_1} \frac{1}{1+y^4} \lesssim \frac{1}{b^2}. \quad (6.2.54)$$

The crucial bound to control the error at H^4 level is :

$$\int_{y \leq 2B_1} |H^2 \Lambda^i T_3|^2 \lesssim \frac{1}{b^2 |\log b|^2} \quad \text{for } 0 \leq i \leq 1. \quad (6.2.55)$$

We now prove this bound. A rough estimate loses the huge gain $\frac{1}{|\log b|^2}$, and we need to be more precise. From (6.2.51),

$$H^2(\Lambda T_3) = H(2\Sigma_3 + \Lambda\Sigma_3) + O\left(\frac{1}{1+y^2}\right). \quad (6.2.56)$$

We use again (6.2.40), together with (6.2.43), (6.2.36), to obtain :

$$\begin{aligned} H\Sigma_3 &= -2HT_2 + H\Lambda T_2 + O\left(\frac{|\log y|^3}{y^2}\right) = \Lambda\Sigma_2 + O\left(\frac{|\log y|^5}{y^2}\right), \\ H\Lambda\Sigma_3 &= 2\Lambda\Sigma_2 + \Lambda^2\Sigma_2 + O\left(\frac{|\log y|^5}{y^2}\right) \end{aligned}$$

and injecting this into (6.2.56) with (6.2.35) yields :

$$\begin{aligned} \int_{y \leq 2B_1} |H^2(\Lambda T_3)|^2 &\lesssim \int_{y \leq 2B_1} \left| \frac{y^2}{1+y^2} \left(\mathbf{1}_{y \leq 1} + \frac{1+\log(y\sqrt{b})}{|\log b|} \mathbf{1}_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y^2} \mathbf{1}_{y \geq 6B_0} \right|^2 \\ &\lesssim \frac{1}{b^2 |\log b|^2} \end{aligned}$$

and (6.2.55) is proved.

Step 6. Estimation of the error.

We are in position to estimate the error Ψ_b . According to our construction, we have from (6.2.12)

$$\Psi_b = b^2 \Sigma_b + b^4 \Lambda T_3 + R_1(T_1, T_2, T_3) \quad (6.2.57)$$

We then study the last term, the others being already estimated. The bounds (6.2.21), (6.2.20), (6.2.39) and (6.2.50) yield the bound for $y \leq 2B_1$, $0 \leq i \leq 4$, and $0 \leq j \leq 5$:

$$\begin{aligned} \left| \frac{d^i R_1(y)}{dy^i} \right| &\lesssim b^4 \left(y^{4-i} \mathbf{1}_{y \leq 1} + b^j y^{2(j+1)-i} (1 + |\log y|^2) \mathbf{1}_{y \geq 1} \right) \\ &\lesssim b^4 \left(y^{4-i} \mathbf{1}_{y \leq 1} + y^{2-i} |\log b|^C \mathbf{1}_{y \geq 1} \right). \end{aligned}$$

Hence :

$$\begin{aligned} \int_{y \leq 2B_1} |HR_1|^2 &\lesssim b^8 |\log b|^C \int_{y \leq 2B_1} 1 \lesssim b^6 |\log b|^C, \\ \int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} |HR_1|^2 &\lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} \lesssim b^8 |\log b|^C, \\ \int_{y \leq 2B_1} |H^2 R_1|^2 + \frac{|R_1|^2}{1 + y^8} &\lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{1}{1 + y^4} \lesssim b^8 |\log b|^C. \end{aligned}$$

Injecting these bounds together with (6.2.29), (6.2.53), (6.2.54), (6.2.55) into (6.2.57) yield (6.2.8), (6.2.9), (6.2.10).

We now prove the flux computation (6.2.11), which will be helpful for the improved modulation equations.

$$\begin{aligned} \frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} &= \frac{1}{(\Lambda Q, \Phi_M)} \left[\left(-b^2 c_b \chi_{\frac{B_0}{4}} \Lambda Q, \Phi_M \right) + O(C(M)b^3) \right] \\ &= -c_b b^2 + O(C(M)b^3) = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \end{aligned}$$

Here we recall that M being chosen large enough, we assume $|b| < b^*(M)$ so that the above claim make sense. This concludes the proof of Proposition 6.2.1.

6.2.2 Localization of the profile

Taking a careful look at the profiles $(T_i)_{1 \leq i \leq 3}$, we can notice that for $y \gg B_1$, Q is negligible compared to $bT_1 + b^2T_2 + b^3T_3$. Obviously, this doesn't make sense, because we look for a solution close to Q . So, we must localize the profiles, with cut-off smooth functions. For technical reasons, we use two localizations : one at B_1 , another one at B_0 .

Proposition 6.2.3 (Localization of the profile near B_1). *Let a \mathcal{C}^1 map $s \mapsto b(s)$ be defined on $[0, s_0]$ with a priori bound $\forall s \in [0, s_0]$,*

$$0 < b(s) < b^*(M), \quad |b_s| \leq 10b^2. \quad (6.2.58)$$

Let the localized profile

$$\tilde{Q}_b(s, y) = Q + b\tilde{T}_1 + b^2\tilde{T}_2 + b^3\tilde{T}_3 = Q + \tilde{\alpha}$$

where

$$\tilde{T}_i = \chi_{B_1} T_i, \quad 1 \leq i \leq 3.$$

Then

$$\partial_s \tilde{Q}_b - \Delta \tilde{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \tilde{Q}_b - \tilde{Q}_b^3 = Mod(t) + \tilde{\Psi}_b \quad (6.2.59)$$

with

$$Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b\right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2) \quad (6.2.60)$$

and where $\tilde{\Psi}_b$ satisfies the bounds on $[0, s_0]$:

(i) Weighted bounds :

$$\int |H\tilde{\Psi}_b|^2 \lesssim b^4 |\log b|^2, \quad (6.2.61)$$

$$\int \frac{1}{1+y^8} |\tilde{\Psi}_b|^2 \lesssim b^6, \quad (6.2.62)$$

$$\int |H^2 \tilde{\Psi}_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (6.2.63)$$

(ii) Flux computation : Let Φ_M be given by (6.3.2), then :

$$\frac{(H\tilde{\Psi}_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = -\frac{2b^2}{|\log b|} + O\left(\frac{b^2}{|\log b|^2}\right). \quad (6.2.64)$$

We introduce a second localization at B_0 which will be relevant for H^2 control, see the proof of Proposition 6.5.3.

Proposition 6.2.4 (Second localization). *Let a \mathcal{C}^1 map $s \mapsto b(s)$ be defined on $[0, s_0]$ with a priori bound (6.2.58). Let the localized profile*

$$\hat{Q}_b(s, y) = Q + b\hat{T}_1 + b^2\hat{T}_2 + b^3\hat{T}_3 = Q + \hat{\alpha} \quad (6.2.65)$$

where

$$\hat{T}_i = \chi_{B_0} T_i, \quad 1 \leq i \leq 3.$$

Let the radiation :

$$\zeta_b = \tilde{\alpha} - \hat{\alpha} \quad (6.2.66)$$

and the error

$$\partial_s \hat{Q}_b - \Delta \hat{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \hat{Q}_b - \hat{Q}_b^3 = \widehat{Mod}(t) + \hat{\Psi}_b$$

with

$$\widehat{Mod}(t) = -\left(\frac{\lambda_s}{\lambda} + b\right) \Lambda \hat{Q}_b + (b_s + b^2)(\hat{T}_1 + 2b\hat{T}_2). \quad (6.2.67)$$

Then there hold the bounds :

$$\left\| \partial_y^i \zeta_b \right\|_{L^\infty}^2 \lesssim b^{2+i} |\log b|^C, \quad (6.2.68)$$

$$\int |H\zeta_b|^2 \lesssim b^2 |\log b|^2, \quad \sum_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1+y^{2(3-i)}} \lesssim b^3 |\log b|^C, \quad (6.2.69)$$

$$\int |H^2 \zeta_b|^2 + \sum_{i=0}^2 \int \frac{|\partial_y^i \zeta_b|^2}{1+y^{8-2i}} \lesssim b^4 |\log b|^C, \quad (6.2.70)$$

$$Supp(H\hat{\Psi}_b) \subset [0, 2B_0] \quad \text{and} \quad \int |H\hat{\Psi}_b|^2 \lesssim b^4 |\log b|^2. \quad (6.2.71)$$

The proof follows similar lines as in [91] and is displayed for the reader's convenience in Appendix (6.7.3).

6.3 Presentation of possible solution of Theorem 6.1.1

6.3.1 Uniqueness of the decomposition

We now look for a solution u of (6.1.1) u , which we will decompose in the form of :

$$u = \left(\tilde{Q}_{b(t)} + \varepsilon(t) \right)_{\lambda(t)}. \quad (6.3.1)$$

Naturally, we must fix constraints to obtain the uniqueness of this decomposition. Moreover it's crucial that the radiation term ε doesn't perturb the modulation equation (6.2.30) found during the construction. We will see in the subsection devoted to the modulation equations that it's the case if we have the following inequality :

$$\int |H^2(\varepsilon(t))|^2 \lesssim \frac{b^4(t)}{|\log b(t)|^2}.$$

To control sharply the radiation term ε , we need to ensure suitable orthogonality conditions with respect to the kernel of H^2 . The smooth solutions of $H^2 f = 0$ are situated in $\text{Span}(\Lambda Q, T_1)$. But neither ΛQ nor T_1 is in $L^2(\mathbb{R}^4)$. Therefore, we use an approximation of the kernel, localizing both directions, with the smooth cut-off function χ_M , where $M > 0$ is a large enough constant. More precisely, we let the direction :

$$\Phi_M = \chi_M \Lambda Q - c_M H(\chi_M \Lambda Q) \quad (6.3.2)$$

with

$$c_M = \frac{(\chi_M \Lambda Q, T_1)}{(H(\chi_M \Lambda Q), T_1)} = c_\chi \frac{M^2}{4} (1 + o_{M \rightarrow +\infty}(1)).$$

The second term, which is a corrective term, ensures the orthogonality between Φ_M and T_1 , and the orthogonality between ΛQ and $H\Phi_M$, because H is a self-adjoint operator. Furthermore, we have by construction :

$$\int |\Phi_M|^2 \lesssim |\log M|, \quad (6.3.3)$$

and the scalar products

$$(\Lambda Q, \Phi_M) = (-HT_1, \Phi_M) = (\chi_M \Lambda Q, \Lambda Q) = 64 \log M (1 + o_{M \rightarrow +\infty}(1)). \quad (6.3.4)$$

In Appendix 6.7.1-6.7.3, we argue that we have coercive estimates for the operators H and H^2 under additional orthogonality conditions. As a consequence, we fix for the radiation term ε the orthogonality conditions :

$$(\varepsilon(t), \Phi_M) = (\varepsilon(t), H\Phi_M) = 0. \quad (6.3.5)$$

From a standard argument based on the implicit function theorem, these constraints give us the existence and the uniqueness of the decomposition (6.3.1). First, we have :

$$(b, \lambda) \rightarrow (u, \Phi_M) = \left(\left(\tilde{Q}_{b(t)} \right)_{\lambda(t)}, \Phi_M \right)$$

is a \mathcal{C}^1 map and thus :

$$\begin{vmatrix} \left(\frac{\partial}{\partial \lambda}(\tilde{Q}_b)_\lambda, \Phi_M\right) & \left(\frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda, \Phi_M\right) \\ \left(\frac{\partial}{\partial \lambda}(\tilde{Q}_b)_\lambda, H\Phi_M\right) & \left(\frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda, H\Phi_M\right) \end{vmatrix}_{\lambda=1,b=0} = \begin{vmatrix} (-\Lambda Q, \Phi_M) & 0 \\ 0 & (T_1, H\Phi_M) \end{vmatrix} = -(\Lambda Q, \Phi_M)^2 \neq 0. \quad (6.3.6)$$

We used the orthogonality conditions mentioned at the moment of the conception of Φ_M and the following equality :

$$\left(\frac{\partial}{\partial \lambda}(\tilde{Q}_b)_\lambda, \frac{\partial}{\partial b}(\tilde{Q}_b)_\lambda \right) |_{\lambda=1,b=0} = -(\Lambda Q, T_1).$$

As long as the solution remains in a fixed small neighborhood of Q for the norm \dot{H}^1 what will be ensured for a suitable set of initial data, the implicit function theorem ensures the existence and the uniqueness of the decomposition (6.3.1).

6.3.2 Partial differential equation verified by the radiation and suitable energies

From now on, we always use the last decomposition, and also its reformulation in original variables :

$$u = \frac{1}{\lambda(t)} (\tilde{Q}_b + \varepsilon) \left(t, \frac{r}{\lambda(t)} \right) = \frac{1}{\lambda(t)} \tilde{Q}_{b(t)} \left(t, \frac{r}{\lambda(t)} \right) + w(t, r). \quad (6.3.7)$$

We recall the correspondence between both systems of variables

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} \quad \text{and} \quad y = \frac{r}{\lambda(t)}.$$

We give also the rescaling formulas

$$u(t, r) = \frac{1}{\lambda} v(s, y), \quad \partial_t u = \frac{1}{\lambda^2} \left(\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v \right)_\lambda.$$

We then can inject the decomposition (6.3.7) with the rescaled variables in Eq. (6.1.1) using the one of \tilde{Q}_b (6.2.59) and obtain the following :

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon + H \varepsilon = F - Mod = \mathcal{F}, \quad (6.3.8)$$

where we remind that

$$H = -\Delta - V, \quad Mod(t) = - \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2)$$

and where we denoted :

$$F = -\tilde{\Psi}_b + L(\varepsilon) + N(\varepsilon), \quad (6.3.9)$$

where L is a linear operator coming from the difference between H and H_{B_1} :

$$L(\varepsilon) = H\varepsilon - H_{B_1}\varepsilon = 3(\tilde{Q}_b^2 - Q^2)\varepsilon \quad (6.3.10)$$

with

$$H_{B_1} = -\Delta - 3\tilde{Q}_b^2$$

and the nonlinear term is given by :

$$N(\varepsilon) = 3\tilde{Q}_b\varepsilon^2 + \varepsilon^3. \quad (6.3.11)$$

It's important to remark that we used here the localization of the profiles near B_1 . At the end of this subsection, we will introduce in the same way some new operators with the second localization near B_0 . Before rewriting (6.3.8) with the original variables, we introduce the suitable norms for our study :

- Energy norm

$$\mathcal{E}_1 = \int |\nabla \varepsilon|^2 \quad (6.3.12)$$

- Higher order Sobolev norms

$$\mathcal{E}_2 = \int |H\varepsilon|^2 = \int |\varepsilon_2|^2, \quad \mathcal{E}_4 = \int |H^2\varepsilon|^2 = \int |\varepsilon_4|^2 \quad (6.3.13)$$

with $\varepsilon_i = H^i\varepsilon$ for $i \in \{2; 4\}$.

To work with the original variables, we recall the notation :

$$f_\lambda(y) = \frac{1}{\lambda} f\left(\frac{r}{\lambda}\right).$$

Furthermore we must adapt this notation for the potential term namely because of its quadratic nature :

$$\tilde{V}(y) = \frac{1}{\lambda^2} V\left(\frac{r}{\lambda}\right).$$

Then (6.3.8) becomes :

$$\partial_t w + H_\lambda w = \frac{1}{\lambda^2} \mathcal{F}_\lambda. \quad (6.3.14)$$

We define $w_i = H_\lambda^i w$ for $i \in \{2; 4\}$ which verify respectively :

$$\partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right), \quad (6.3.15)$$

$$\partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda \left(\partial_t \tilde{V} w \right) + H_\lambda^2 \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right). \quad (6.3.16)$$

We have also by substitution :

$$\lambda^2 \mathcal{E}_2 = \int |Hw|^2 = \int |w_2|^2, \quad \lambda^6 \mathcal{E}_4 = \int |H^2 w|^2 = \int |w_4|^2. \quad (6.3.17)$$

We now focus onto the localization near B_0 . Using the definition of the radiation ζ_b (6.2.66), we obtain the new unique decomposition :

$$u = (\hat{Q}_b + \hat{\varepsilon})_\lambda \quad \text{ie} \quad \hat{\varepsilon} = \varepsilon + \zeta_b. \quad (6.3.18)$$

Thus, with this localization, we have :

$$\partial_s \hat{\varepsilon} - \frac{\lambda_s}{\lambda} \Lambda \hat{\varepsilon} + \hat{H} \hat{\varepsilon} = \hat{F} - \widehat{\text{Mod}} = \hat{\mathcal{F}}, \quad (6.3.19)$$

where we remind that

$$\hat{H} = -\Delta - \hat{V}, \quad \widehat{\text{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b\right) \Lambda \hat{Q}_b + (b_s + b^2)(\hat{T}_1 + 2b\hat{T}_2)$$

and where we have noticed

$$\hat{F} = -\hat{\Psi}_b + \hat{L}(\hat{\varepsilon}) + \hat{N}(\hat{\varepsilon}), \quad (6.3.20)$$

where \hat{L} is a linear operator coming from the difference between \hat{H} and \hat{H}_{B_1} :

$$\hat{L}(\varepsilon) = \hat{H}\hat{\varepsilon} - \hat{H}_{B_1}\hat{\varepsilon} = 3(\hat{Q}_b^2 - \hat{Q}^2)\hat{\varepsilon} \quad (6.3.21)$$

with

$$\hat{H}_{B_1} = -\Delta - 3\hat{Q}_b^2$$

and a last purely nonlinear term :

$$\hat{N}(\hat{\varepsilon}) = 3\hat{Q}_b\hat{\varepsilon}^2 + \hat{\varepsilon}^3. \quad (6.3.22)$$

We define likewise the operators $\hat{\varepsilon}_2$, \hat{w} and \hat{w}_2 , which come of course respectively from ε_2 , w , and w_2 . The energy at H^2 level becomes :

$$\hat{\mathcal{E}}_2 = \int |\hat{\varepsilon}_2|^2. \quad (6.3.23)$$

Moreover, with the bounds of the radiation (6.2.69), we can measure the difference between both energies at H^2 level.

$$\hat{\mathcal{E}}_2 \lesssim \mathcal{E}_2 + \int |H\zeta_b|^2 \lesssim \mathcal{E}_2 + b^2 |\log b|^2. \quad (6.3.24)$$

Finally, \hat{w}_2 verifies the following partial differential equation :

$$\partial_t \hat{w}_2 + \hat{H}_\lambda \hat{w}_2 = -\partial_t \tilde{V} \hat{w} + \hat{H}_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right). \quad (6.3.25)$$

6.3.3 Modulation equations

With the choice of orthogonality conditions (6.3.5), we can now measure the error made by taking $b = \frac{-\lambda_s}{\lambda}$ and $b_s = -b^2 \left(1 + \frac{2}{|\log b|}\right)$. This estimations are in the core of our proof. The proof is the same as that of [91], with the exception of a very small difference with the linear and nonlinear operators $L(\varepsilon)$ and $N(\varepsilon)$, which truly brings any difficulty, because of the same interpolation bounds. In view of the importance to this lemma in our proof of Theorem 6.1.1, we considered useful to give an extensive proof.

Lemme 6.3.1 (Modulation equations). *There holds the bound on the modulation parameters :*

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}, \quad (6.3.26)$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log M}} \left(\sqrt{\mathcal{E}_4} + \frac{b^2}{|\log b|} \right). \quad (6.3.27)$$

Remarque 6.3.2. Note that this implies in the bootstrap the rough bounds :

$$|b_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \leq 2b^2. \quad (6.3.28)$$

and in particular (6.2.58) holds.

Proof of Lemma 6.3.1

Step 1. Law for b.

Let

$$V(t) = |b_s + b^2| + \left| \frac{\lambda_s}{\lambda} + b \right|.$$

We take the inner product of (6.3.8) with $H\Phi_M$ and estimate each terms.

$$\begin{aligned} & (\partial_s \varepsilon, H\Phi_M) + (H\varepsilon, H\Phi_M) - \left(\frac{\lambda_s}{\lambda} \Lambda \varepsilon, H\Phi_M \right) \\ &= -(\tilde{\Psi}_b, H\Phi_M) - (Mod(t), H\Phi_M) + (L(\varepsilon), H\Phi_M) + (N(\varepsilon), H\Phi_M). \end{aligned}$$

First according to our choice of orthogonality (6.3.5)

$$(\partial_s \varepsilon, H\Phi_M) = \partial_s (H\varepsilon, \Phi_M) = 0. \quad (6.3.29)$$

Then

$$(H\varepsilon, H\Phi_M) = (H^2 \varepsilon, \Phi_M) \lesssim \|H^2 \varepsilon\|_{L^2} \|\Phi_M\|_{L^2} \lesssim \sqrt{\mathcal{E}_4 \log M}. \quad (6.3.30)$$

From the construction of the profile, (6.2.60) and the localization $\text{Supp}(\Phi_M) \subset [0, 2M]$ from (6.3.2) :

$$\begin{aligned} (H(Mod(t)), \Phi_M) &= -\left(b + \frac{\lambda_s}{\lambda} \right) (H\Lambda \tilde{Q}_b, \Phi_M) + (b_s + b^2) (H(\tilde{T}_1 + 2b\tilde{T}_2), \Phi_M) \\ &= -(\Lambda Q, \Phi_M)(b_s + b^2) + O(c(M)b|V(t)|). \end{aligned}$$

Using the Hardy bounds of Appendix 6.7.2 :

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H\Phi_M \right) \right| \lesssim C(M)b(\mathcal{E}_4 + |V(t)|).$$

We conclude from (6.3.4) and the fundamental flux computation (6.2.64) :

$$\begin{aligned} b_s + b^2 &= \frac{(\tilde{\Psi}_b, H\Phi_M)}{(\Lambda Q, \Phi_M)} + O\left(\frac{\sqrt{\log M \mathcal{E}_4}}{\log M}\right) + O(C(M)b|V(t)|) \\ &= -\frac{2b^2}{|\log b|} \left(1 + O\left(\frac{1}{|\log b|}\right) \right) + O\left(\sqrt{\frac{\mathcal{E}_4}{\log M}} + C(M)b|V(t)|\right) \end{aligned}$$

and (6.3.27) is proved.

Step 2. Degeneracy of the law for λ .

Now we take the inner product of (6.3.8) with Φ_M and obtain :

$$(\text{Mod}(t), \Phi_M) = -(\tilde{\Psi}_b, \Phi_M) - (\partial_s \varepsilon + H\varepsilon, \Phi_M) - \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M \right).$$

From our choice of orthogonality conditions (6.3.5) :

$$(\partial_s \varepsilon + H\varepsilon, \Phi_M) = 0.$$

In the same way as in the last step, using the Hardy bounds of Appendix 6.7.2 :

$$\left| \left(-\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M \right) \right| \lesssim C(M)b\sqrt{\mathcal{E}_4}.$$

Next, we compute from (6.3.4) and the orthogonality (6.3.3) :

$$\begin{aligned} (\text{Mod}(t), \Phi_M) &= - \left(b + \frac{\lambda_s}{\lambda} \right) (\Lambda \tilde{Q}_b, \Phi_M) + (b_s + b^2) (\tilde{T}_1 + 2b\tilde{T}_2, \Phi_M) \\ &= -4\log M (1 + o_{M \rightarrow +\infty}(1)) \left(\frac{\lambda_s}{\lambda} + b \right) + O(C(M)b|V(t)|). \end{aligned}$$

and observe the cancellation from (6.2.25), (6.3.3) :

$$\left| (\tilde{\Psi}_b, \Phi_M) \right| \lesssim b^2 |(\Sigma_b, \Phi_M)| + O(C(M)b^3) = c_b b^2 |(T_1, \Phi_M)| + O(C(M)b^3) = O(C(M)b^3).$$

We thus obtain the modulation equation for scaling :

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim b^3 C(M) + b C(M) O \left(\sqrt{\mathcal{E}_4} + |V(t)| \right). \quad (6.3.31)$$

With (6.3.27), we obtain the bound

$$|V(t)| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4}.$$

Injecting this bound in (6.3.31) implies the refined bound (6.3.27). This concludes the proof of Lemma 6.3.1.

6.3.4 Proof of Theorem 6.1.1

In this section, we conclude the proof of Theorem 6.1.1 assuming the following a priori bounds on the solution on its maximum time interval of existence $[0, T)$, $0 < T \leq +\infty$:

- Energy estimates

$$\forall t \in [0, T[\quad \mathcal{E}_1(t) \leq \delta(b^*), \quad \mathcal{E}_2(t) \lesssim b(t)^2 |\log b(t)|^5, \quad \mathcal{E}_4(t) \lesssim \frac{b(t)^4}{|\log b(t)|^2} \quad (6.3.32)$$

- Link between both laws $b(t)$ and $\lambda(t)$: there exist $\alpha_1, \alpha_2 > 0$ such that :

$$C(u_0)b(t)|\log b(t)|^{\alpha_1} \leq \lambda(t) \leq C'(u_0)b(t)|\log b(t)|^{\alpha_2} \quad (6.3.33)$$

The heart of our analysis in Section 6.4 will be to produce such kind of solutions. We now assume (6.3.32) and (6.3.33) and prove Theorem 6.1.1. The proof adapts the argument in [91] which we sketch for the convenience of the reader.

Step 1. Finite time blow-up.

Let $T \leq +\infty$ be the life time of u . From (6.3.32), (6.3.33),

$$-\frac{d}{dt}\sqrt{\lambda} = -\frac{1}{2\lambda\sqrt{\lambda}}\frac{\lambda_s}{\lambda} \gtrsim \frac{b}{\lambda\sqrt{\lambda}} \gtrsim C(u_0) > 0$$

and thus λ touches zero in finite time which implies

$$T < +\infty.$$

The bounds (6.3.32) and standard H^4 local well posedness theory ensure that blow-up corresponds to

$$\lambda(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow T \quad (6.3.34)$$

and thus, with (6.3.33)

$$\lambda(T) = b(T) = 0. \quad (6.3.35)$$

Step 2. Derivation of the sharp blow-up speed.

To begin, the modulation laws become with the bound (6.3.32) :

$$\left| b + \frac{\lambda_s}{\lambda} \right| \lesssim \frac{b^2}{|\log b|} \quad (6.3.36)$$

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log M}} \frac{b^2}{|\log b|}. \quad (6.3.37)$$

We have defined M as a large enough constant. Let

$$B_\delta = \frac{1}{b^\delta}. \quad (6.3.38)$$

Since the variation of $b(s)$ is very small, we can consider the first time such that $M = B_\delta$ in (6.3.37). Assuming this, prove (6.1.8), and demonstrate afterwards that the made error is negligible for δ enough small. We have thus that (6.3.37) becomes :

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log B_\delta}} \frac{b^2}{|\log b|} \lesssim \frac{b^2}{|\log b|^{\frac{3}{2}}}. \quad (6.3.39)$$

We now integrate this in time using $\lim_{s \rightarrow +\infty} b(s) = 0$:

$$b(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right). \quad (6.3.40)$$

Using this decomposition of $b(s)$ in the modulation equation (6.3.36), we conclude :

$$-\frac{\lambda_s}{\lambda} = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s |\log s|^{\frac{3}{2}}}\right).$$

We rewrite this as

$$\left| \frac{d}{ds} \log \left(\frac{s \lambda(s)}{(\log s)^2} \right) \right| \lesssim \frac{1}{s |\log s|^{\frac{3}{2}}}$$

and thus integrating in time yields the existence of $\kappa(u_0) > 0$ such that :

$$\frac{s \lambda(s)}{(\log s)^2} = \frac{1}{\kappa(u_0)} \left[1 + O\left(\frac{1}{|\log s|^{\frac{3}{2}}}\right) \right].$$

Taking the log yields the bound

$$|\log \lambda| = |\log s| \left[1 + O\left(\frac{|\log \log s|}{\log s}\right) \right]$$

and thus

$$\frac{1}{s} = \kappa(u_0) \frac{\lambda}{|\log \lambda|^2} (1 + o(1)).$$

Injecting this into (6.3.45) yields :

$$-\lambda \lambda_t = -\frac{\lambda_s}{\lambda} = \frac{1}{s} (1 + o(1)) = \kappa(u_0) \frac{\lambda}{|\log \lambda|^2} (1 + o(1)) \quad (6.3.41)$$

and thus

$$-|\log \lambda|^2 \lambda_t = \kappa(u_0) (1 + o(1)).$$

Integrating from t to T with $\lambda(T) = 0$ yields

$$\lambda(t) = \kappa(u_0) \frac{T-t}{|\log(T-t)|^2} [1 + o(1)],$$

and (6.1.8) is proved.

We now prove that the error we made is negligible. Indeed, we take the inner product of (6.3.8) with $H \chi_{B_\delta} \Lambda Q$ and obtain :

$$\begin{aligned} & \frac{d}{ds} \{(H\varepsilon, \chi_{B_\delta} \Lambda Q)\} - (H\varepsilon, \partial_s \chi_{B_\delta} \Lambda Q) + \frac{\lambda_s}{\lambda} (\chi_{B_\delta} \Lambda Q, H \Lambda \varepsilon) + (H^2 \varepsilon, \chi_{B_\delta} \Lambda Q) \\ &= \left(H \left[-\tilde{\Psi}_b + L(\varepsilon) - N(\varepsilon) - Mod \right], \chi_{B_\delta} \Lambda Q \right). \end{aligned} \quad (6.3.42)$$

We must estimate all terms in this identity. First, for δ small enough, we have the rough bound :

$$|(H\varepsilon, \partial_s \chi_{B_\delta} \Lambda Q)| + |\frac{\lambda_s}{\lambda} (\chi_{B_\delta} \Lambda Q, H \Lambda \varepsilon)| + |(H[L(\varepsilon) - N(\varepsilon)], \chi_{B_\delta} \Lambda Q)| \lesssim \frac{b}{b^{C\delta}} \sqrt{\mathcal{E}_4} \lesssim \frac{b^2}{|\log b|^2}.$$

For the linear term, we have immediately :

$$|(H^2 \varepsilon, \chi_{B_\delta} \Lambda Q)| \lesssim \sqrt{\mathcal{E}_4} \sqrt{|\log b|} \lesssim \frac{b^2}{\sqrt{|\log b|}}.$$

The $\tilde{\Psi}_b$ term is computed from (6.2.57) :

$$(-H\tilde{\Psi}_b, \chi_{B_\delta} \Psi_b) = -b^2 (H\Sigma_b, \chi_{B_\delta} \Psi_b) + O\left(\frac{b^3}{b^{C\delta}}\right) = b^2 c_b(\Lambda Q, \chi_{B_\delta} \Lambda Q) + O\left(\frac{b^2}{|\log b|^2}\right).$$

From (6.2.60), we have the following estimate for the modulation term :

$$\begin{aligned} (-HMod, \chi_{B_\delta} \Lambda Q) &= \left(\frac{\lambda_s}{\lambda} + b\right) (H\Lambda\tilde{Q}_b, \chi_{B_\delta} \Lambda Q) - (b_s + b^2) (H(\tilde{T}_1 + 2b\tilde{T}_2), \chi_{B_\delta} \Lambda Q) \\ &= (b_s + b^2) (\Lambda Q, \chi_{B_\delta} \Lambda Q) + O\left(\frac{b}{b^{C\delta}} \frac{b^2}{|\log b|}\right). \end{aligned}$$

We now inject the estimates into (6.3.42) and obtain :

$$(b_s + b^2) (\Lambda Q, \chi_{B_\delta} \Lambda Q) = \frac{d}{ds} \{(H\varepsilon, \chi_{B_\delta} \Lambda Q)\} - c_b b^2 (\Lambda Q, \chi_{B_\delta} \Lambda Q) + O\left(\frac{b^2}{\sqrt{|\log b|}}\right)$$

which we rewrite using (6.2.22) and an integration by parts in time :

$$\begin{aligned} &\frac{d}{ds} \left\{ b - \frac{(H\varepsilon, \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)} \right\} + b^2 \left(1 + \frac{2}{|\log b|} \right) \\ &= O\left(\frac{b^2}{|\log b|^{\frac{3}{2}}}\right) + (H\varepsilon, \chi_{B_\delta} \Lambda Q) \frac{(\Lambda Q, \partial_s \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)^2}. \end{aligned} \tag{6.3.43}$$

We now estimate :

$$\left| (H\varepsilon, \chi_{B_\delta} \Lambda Q) \frac{(\Lambda Q, \partial_s \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)^2} \right| \lesssim \frac{\sqrt{\mathcal{E}_4} |b_s|}{b^{C\delta}} \frac{b}{b} \lesssim \frac{b^3}{b^{C\delta}},$$

$$\left| \frac{(H\varepsilon, \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)} \right| \lesssim \frac{\sqrt{\mathcal{E}_4}}{b^{C\delta}} \lesssim \frac{b^2}{b^{C\delta}}.$$

We inject these bounds into (6.3.43) and conclude that the difference between b and \tilde{b} is given by

$$\tilde{b} = b - \frac{(H\varepsilon, \chi_{B_\delta} \Lambda Q)}{(\Lambda Q, \chi_{B_\delta} \Lambda Q)} = b + O\left(\frac{b^2}{|\log b|^2}\right) \tag{6.3.44}$$

and satisfies the pointwise differential control :

$$\left| \tilde{b}_s + \tilde{b}^2 \left(1 + \frac{2}{|\log \tilde{b}|} \right) \right| \lesssim \frac{\tilde{b}^2}{|\log \tilde{b}|^{\frac{3}{2}}},$$

which we rewrite :

$$\frac{\tilde{b}_s}{\tilde{b}^2 \left(1 + \frac{2}{|\log \tilde{b}|} \right)} + 1 = O \left(\frac{1}{|\log \tilde{b}|^{\frac{3}{2}}} \right).$$

We now integrate this in time using $\lim_{s \rightarrow +\infty} \tilde{b}(s) = 0$ from (6.3.35), (6.3.44) and get :

$$\tilde{b}(s) = \frac{1}{s} - \frac{2}{s \log s} + O \left(\frac{1}{s |\log s|^{\frac{3}{2}}} \right)$$

and thus from (6.3.44) :

$$b(s) = \frac{1}{s} - \frac{2}{s \log s} + O \left(\frac{1}{s |\log s|^{\frac{3}{2}}} \right). \quad (6.3.45)$$

This concludes the proof.

Step 3. Quantization of the focused energy.

We now turn to the proof of (6.1.7), (6.1.9) and adapt the strategy in [70]. We shall need the following bound, which is a direct consequence of our construction and (6.3.32) :

$$\forall t \in [0, T), \quad \|\Delta \tilde{u}(t, x)\|_{L^2} \leq C(v_0). \quad (6.3.46)$$

where

$$\tilde{u}(t, x) = u(t, x) - \frac{1}{\lambda(t)} Q \left(\frac{x}{\lambda(t)} \right) \quad (6.3.47)$$

The regularity of $v(t, x)$ outside the origin is a standard consequence of parabolic regularity. Hence there exists $u^* \in \dot{H}^1$ such that

$$\forall R > 0, \quad \nabla u(t) \rightarrow \nabla u^* \text{ in } L^2(|x| \geq R) \text{ as } t \rightarrow T.$$

Moreover, v is \dot{H}^1 thanks to the dissipation of the total energy, and thus recalling the decomposition (6.3.47) and the uniform bound (6.3.46) :

$$\nabla \tilde{u}(t) \rightarrow \nabla u^* \text{ in } L^2 \text{ and } \Delta u^* \in L^2$$

which concludes the proof of (6.1.7), (6.1.9). This concludes the proof of Theorem 6.1.1.

6.4 Description of the initial data and bootstrap

The proof of Theorem 6.1.1 consists now in the demonstration of the existence of initial data, close to Q , which will satisfy the assumed bounds (6.3.32) and (6.3.33). We have already seen the condition of smallness of b to assure the uniqueness of the decomposition, through the implicit function theorem :

$$0 < b(0) < b^*(M) \ll 1. \quad (6.4.1)$$

In order to have some room with respect to (6.3.32), we fix the initial generous bounds namely :

$$|\mathcal{E}_1(0)| \leq b(0)^2, \quad (6.4.2)$$

and

$$|\mathcal{E}_2(0)| + |\mathcal{E}_4(0)| \leq b(0)^{10}. \quad (6.4.3)$$

Moreover, there is a crucial difference compared to [91]. The linear operator H possesses a negative direction ψ , source of instability, which can be harmful to the blow-up dynamics. We manage this as in [41]. We note

$$\kappa(t) = (\varepsilon(t), \psi), \quad (6.4.4)$$

and

$$a^+ = \kappa(0) = (\varepsilon(0), \psi). \quad (6.4.5)$$

We impose that :

$$|a^+| \leq \frac{2b(0)^{\frac{5}{2}}}{|\log b(0)|}. \quad (6.4.6)$$

The propagation of regularity by the parabolic heat flow ensures that these estimates hold on some time interval $[0, t_1]$ together with the regularity $(\lambda, b) \in C^1([0, t_1], \mathbb{R}_+^* \times \mathbb{R})$. Given a large enough universal constant $K > 0$ -independent of M - , we assume on $[0, t_1]$:

– Control of $b(t)$:

$$0 < b(t) < 10b(0). \quad (6.4.7)$$

– Control of the radiation :

$$\int |\nabla \varepsilon(t)|^2 \leq 10\sqrt{b(0)}, \quad (6.4.8)$$

$$|\mathcal{E}_2(t)| \leq Kb^2(t)|\log b(t)|^5, \quad (6.4.9)$$

$$|\mathcal{E}_4(t)| \leq K \frac{b^4(t)}{|\log b(t)|^2}. \quad (6.4.10)$$

– *A priori* bound on the unstable mode

$$|\kappa(t)| \leq 2 \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.4.11)$$

We may describe the bootstrap regime as follows :

Définition 6.4.1 (Exit time). *Given $a^+ \in \left[-2 \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}, 2 \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}\right]$, we let $T(a^+)$ be the life time of the solution to (6.1.1) with initial data (6.4.1), (6.4.2), (6.4.3) and (6.4.6), and let $T_1(a^+) > 0$ be the supremum of $T \in (0, T(a^+))$, such that for all $t \in [0; T]$, the estimates (6.4.7), (6.4.8), (6.4.9), (6.4.10) and (6.4.11) hold.*

The existence of blow-up solutions in the regime described by Theorem 6.1.1 now follows from the following :

Proposition 6.4.2. *There exists $a^+ \in \left[-2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}, 2\frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}\right]$ such that*

$$T_1(a^+) = T(a^+) \quad (6.4.12)$$

and then corresponding solution of (6.1.1) blows up in finite time in the regime described by Theorem 6.1.1

We shall use the same strategy as in [88], [72], [41], and [91]. We will process in three times :

- First, we shall derive of suitable Lyapunov functionals at Sobolev respectively \dot{H}^4 and \dot{H}^2 levels. That is the most difficult part of the proof, particularly because of the estimates of nonlinear terms, for whose we must make a sharp study. Moreover, we shall see that it was crucial that $|a^+| \lesssim \frac{b(0)^{\frac{5}{2}}}{|\log b(0)|}$.
- Secondly, we will reintegrate this functionals to obtain improved bounds for \mathcal{E}_2 and \mathcal{E}_4 . The bounds (6.4.7) is a direct consequence of the dissipation of energy. Thus, only the last bound (6.4.11) for the unstable direction can be the cause of an exit time less than the life time of the solution.
- Finally, we will study the dynamics of the unstable mode, and see that we can choose an a^+ to obtain Proposition 6.4.2. To ensure the existence of this solution, it is important that $|a^+| \gtrsim g(b(0))$ with $\frac{b(x)^3}{|\log b(x)|} = o(g(x))$ as $x \rightarrow 0$, hence the choice for the bounds of a^+ is in (6.4.6).

6.5 Lyapunov monotonicities

6.5.1 At \dot{H}^4 level

Proposition 6.5.1 (Lyapunov monotonicity \dot{H}^4). *There holds :*

$$\frac{d}{dt} \left\{ \frac{1}{\lambda^6} \left[\mathcal{E}_4 + O\left(\sqrt{b} \frac{b^4}{|\log b|^2}\right) \right] \right\} \leq C \frac{b}{\lambda^8} \left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|} \sqrt{\mathcal{E}_4} \right] \quad (6.5.1)$$

for some universal constant $C > 0$ independent of M and of the bootstrap constant K in (6.4.7), (6.4.8), (6.4.9), (6.4.10), provided $b^*(M)$ in (6.4.1) has been chosen small enough.

Proof.

We recall the partial differential equations satisfied by w_2 and w_4 :

$$\partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right). \quad (6.5.2)$$

$$\partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda \left(\partial_t \tilde{V} w \right) + H_\lambda^2 \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right). \quad (6.5.3)$$

Moreover, we recall the action of time derivate on rescaling :

$$\partial_t v_\lambda(r) = \frac{1}{\lambda^2} \left(\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v \right)_\lambda.$$

Step 1. Energy identity.

Lemme 6.5.2 (Energy identity \dot{H}^4).

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left\{ w_4^2 - 2\partial_t \tilde{V} w w_4 \right\} \\ = & - \int w_4 H_\lambda w_4 + \int (\partial_t \tilde{V})^2 w_2 w - \int \partial_{tt} \tilde{V} w w_4 + \int H_\lambda (\partial_t \tilde{V} w) \partial_t \tilde{V} w \\ + & \int w_4 H_\lambda^2 \frac{1}{\lambda^2} \mathcal{F}_\lambda - \int H_\lambda (\partial_t \tilde{V} w) H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) - \int \partial_t \tilde{V} w_4 \frac{1}{\lambda^2} \mathcal{F}_\lambda. \end{aligned}$$

Proof. We propose here a simplification with respect to the algebra in [91]. Dissipation also allows us to sign some terms and avoid the study of suitable quadratic forms as in [41]. We compute the energy identity :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int w_4^2 &= \int w_4 \partial_t w_4 \\ &= \int w_4 \left(-H_\lambda w_4 - \partial_t \tilde{V} w_2 - H_\lambda (\partial_t \tilde{V} w) + H_\lambda^2 \frac{1}{\lambda^2} \mathcal{F}_\lambda \right). \end{aligned}$$

We now treat separately the second and the third terms.

$$\begin{aligned} & - \int w_4 \partial_t \tilde{V} w_2 \\ = & \int \partial_t \tilde{V} w_2 \left(\partial_t w_2 + \partial_t \tilde{V} w - H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ = & \int (\partial_t \tilde{V})^2 w_2 w - \int \partial_t \tilde{V} w_2 H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) + \int \partial_t \tilde{V} \partial_t \left(\frac{w_2^2}{2} \right) \\ = & \int (\partial_t \tilde{V})^2 w_2 w - \int \partial_t \tilde{V} w_2 H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) - \frac{1}{2} \int \partial_{tt} \tilde{V} w_2^2 + \frac{1}{2} \frac{d}{dt} \left(\int \partial_t \tilde{V} w_2^2 \right). \end{aligned}$$

Now

$$\begin{aligned} - \int w_4 H_\lambda (\partial_t \tilde{V} w) &= \int H_\lambda (\partial_t \tilde{V} w) \left(\partial_t w_2 + \partial_t \tilde{V} w - H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &= \int H_\lambda (\partial_t \tilde{V} w) \left(\partial_t \tilde{V} w - H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\ &+ \frac{d}{dt} \int H_\lambda (\partial_t \tilde{V} w) w_2 - \int \partial_t [H_\lambda (\partial_t \tilde{V} w)] w_2. \end{aligned}$$

The last term becomes

$$- \int \partial_t H_\lambda (\partial_t \tilde{V} w) w_2 = \int (\partial_t \tilde{V})^2 w_2 w - \int \partial_{tt} \tilde{V} w w_4 - \int \partial_t \tilde{V} \partial_t w w_4,$$

and

$$\begin{aligned}
-\int \partial_t \tilde{V} \partial_t w w_4 &= \int \partial_t \tilde{V} \left(w_2 - \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) w_4 \\
&= -\int \partial_t \tilde{V} \frac{1}{\lambda^2} \mathcal{F}_\lambda w_4 + \int \partial_t \tilde{V} w_2 \left(-\partial_t w_2 - \partial_t \tilde{V} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\
&= -\int \partial_t \tilde{V} \frac{1}{\lambda^2} \mathcal{F}_\lambda w_4 + \int \partial_t \tilde{V} w_2 \left(-\partial_t \tilde{V} w + H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \\
&\quad - \frac{1}{2} \frac{d}{dt} \left(\int \partial_t \tilde{V} w_2^2 \right) + \frac{1}{2} \int \partial_{tt} \tilde{V} w_2^2.
\end{aligned}$$

In the following steps, we estimate each term of Lemma 6.5.2 in order to prove Proposition 6.5.1.

Step 2. Lower order quadratic terms.

We have from (6.2.16) and (6.5.4), and the modulation equations, the bounds :

$$|\partial_t \tilde{V}| \lesssim \frac{b}{\lambda^4} \frac{1}{1+y^4} \quad |\partial_{tt} \tilde{V}| \lesssim \frac{b}{\lambda^6} \frac{1}{1+y^4}. \quad (6.5.4)$$

Using (6.7.3), we obtain

$$\begin{aligned}
-\int H_\lambda w_4 w_4 &= -\frac{1}{\lambda^8} \int H \varepsilon_4 \varepsilon_4 \lesssim \frac{1}{\lambda^8} \left(\int \varepsilon_4 \psi \right)^2 \lesssim \frac{\zeta^2}{\lambda^8} \left(\int \varepsilon \psi \right)^2 \\
&\lesssim \frac{1}{\lambda^8} \kappa^2 \lesssim \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2}.
\end{aligned}$$

Remark that :

$$\int H_\lambda (\partial_t \tilde{V} w) \partial_t \tilde{V} w = \int (\partial_t \tilde{V})^2 w_2 w - \int \Delta (\partial_t \tilde{V}) \partial_t \tilde{V} w^2 - 2 \int \partial_{rt} \tilde{V} \partial_t \tilde{V} \partial_r w w.$$

We treat now the two following terms :

$$\begin{aligned}
&\left| 2 \int (\partial_t \tilde{V})^2 w_2 w - \int \partial_{tt} \tilde{V} w w_4 - \int \Delta (\partial_t \tilde{V}) \partial_t \tilde{V} w^2 - 2 \int \partial_{rt} \tilde{V} \partial_t \tilde{V} \partial_r w w \right| \\
&\lesssim \frac{b^2}{\lambda^8} \int \left(\frac{\varepsilon \varepsilon_2}{1+y^8} + \frac{\varepsilon \varepsilon_4}{1+y^4} + \frac{\varepsilon^2}{1+y^{10}} + \frac{\varepsilon \partial_y \varepsilon}{1+y^9} \right) \lesssim \frac{b^2}{\lambda^8} b^4 |\log b|^C.
\end{aligned}$$

The last inequality comes from Cauchy-Schwarz and the bounds (6.7.27) and (6.7.29). Finally, we estimate the boundary term in time

$$\left| \int \partial_t \tilde{V} w w_4 \right| \lesssim \frac{b}{\lambda^6} \left(\int \frac{\varepsilon^2}{1+y^8} \right)^{\frac{1}{2}} \left(\int \varepsilon_4^2 \right)^{\frac{1}{2}} \lesssim \frac{b}{\lambda^6} b^4 |\log b|^C \lesssim \frac{\sqrt{b}}{\lambda^6} \frac{b^4}{|\log b|^2}.$$

Step 3. Further use of dissipation.

First, we claim the following bounds :

$$\int \frac{1}{1+y^8} \mathcal{F}^2 \lesssim \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right] \quad (6.5.5)$$

$$\int |H^2 \mathcal{F}|^2 \lesssim b^2 \left[\frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right]. \quad (6.5.6)$$

Thus,

$$\begin{aligned} & \left| \int w_4 H_\lambda^2 \frac{1}{\lambda^2} \mathcal{F}_\lambda - \int H_\lambda (\partial_t \tilde{V} w) H_\lambda \left(\frac{1}{\lambda^2} \mathcal{F}_\lambda \right) - \int \partial_t \tilde{V} w_4 \frac{1}{\lambda^2} \mathcal{F}_\lambda \right| \\ & \lesssim \frac{1}{\lambda^8} \left\{ \int \left(\varepsilon_4^2 + b^2 \frac{\varepsilon^2}{1+y^8} \right)^{\frac{1}{2}} \left(\int |H^2 \mathcal{F}|^2 \right)^{\frac{1}{2}} + b \left(\int \varepsilon_4^2 \right)^{\frac{1}{2}} \left(\int \frac{1}{1+y^8} \mathcal{F}^2 \right)^{\frac{1}{2}} \right\} \\ & \lesssim \frac{b}{\lambda^8} \left[\frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|} \sqrt{\mathcal{E}_4} \right]. \end{aligned}$$

This concludes the proof of the Proposition 6.5.1. We now turn of the proof of (6.5.5) and (6.5.6). We recall that

$$\mathcal{F} = -\tilde{\Psi}_b - Mod(t) + L(\varepsilon) + N(\varepsilon).$$

Step 4. $\tilde{\Psi}_b$ terms.

The contribution of the $\tilde{\Psi}_b$ terms in (6.5.5) and (6.5.6) has already been proved in Lemma 6.2.3. For that matter, the construction of the approximated solution by the profiles $(T_i)_{1 \leq i \leq 3}$ has been made to obtain these good estimates.

Step 5. $Mod(t)$ terms.

Recall the definition (6.2.60) of $Mod(t)$.

$$Mod(t) = -\left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2).$$

With the modulation equation (6.3.26) and (6.3.27), we have :

$$\left| \frac{\lambda_s}{\lambda} + b \right|^2 + |b_s + b^2|^2 \lesssim \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M}.$$

But

$$\int \frac{1}{1+y^8} |\Lambda \tilde{Q}_b|^2 + \int \frac{1}{1+y^8} |\tilde{T}_1 + 2b\tilde{T}_2|^2 \lesssim 1.$$

Now,

$$\int |H^2 \Lambda \tilde{Q}_b|^2 + \int |H^2 (\tilde{T}_1 + 2b\tilde{T}_2)|^2 \lesssim b^2. \quad (6.5.7)$$

Indeed :

$$\int |H^2 \tilde{T}_1|^2 \lesssim \int_{B_1 \leq y \leq 2B_1} \left| \frac{\log y}{y^4} \right| \lesssim \frac{|\log b|^2}{B_1^4} \lesssim b^2 \quad (6.5.8)$$

There is a whole proof of the estimate for \tilde{T}_2 in [91]. Here is a summary of this demonstration. With the definition of \tilde{T}_2 , and the bound (6.2.38) of T_2 , we have :

$$\int |H^2 \tilde{T}_2|^2 \lesssim \left[\int_{B_1 \leq y \leq 2B_1} \left| \frac{y^2}{y^4} \right|^2 + \int_{y \leq 2B_1} |H \Sigma_2|^2 \right] \lesssim 1 + \int_{y \leq 2B_1} |H \Sigma_2|^2. \quad (6.5.9)$$

From the construction of the radiation Σ_b and the definition of Σ_2 , we compute :

$$\begin{aligned} H\Sigma_2 &= H\Sigma_b + H(T_1 - \Lambda T_1) + O\left(\frac{y|\log y|^2}{1+y^5}\right) \\ &= \frac{1}{|\log b|} O\left(\frac{1}{1+y^2} \mathbf{1}_{y \leq 3B_0}\right) + H(T_1 - \Lambda T_1) + O\left(\frac{y|\log y|^2}{1+y^5}\right). \end{aligned}$$

But

$$\begin{aligned} HT_1 - H\Lambda T_1 &= HT_1 - \left(2HT_1 + \Lambda HT_1 - \frac{\Lambda V}{y^2} T_1\right) \\ &= O\left(\frac{\log y}{1+y^3}\right). \end{aligned}$$

We thus conclude :

$$\int |H\Sigma_2|^2 \lesssim \frac{1}{|\log b|^2} \int_{y \leq 2B_0} \frac{1}{1+y^4} + \int_{y \leq 2B_1} \frac{|\log y|^4}{1+y^8} \lesssim 1,$$

and the contribution of $Mod(t)$ terms to (6.5.5) and (6.5.6) are small enough.

Step 6. Small linear term $L(\varepsilon)$.

We recall the expression of $L(\varepsilon)$:

$$L(\varepsilon) = 3(\tilde{Q}_b^2 - Q^2)\varepsilon.$$

We have, with the rough bounds (6.2.20), (6.2.39) and (6.2.50) :

$$|3(\tilde{Q}_b^2 - Q^2)| \lesssim b$$

and, with (6.7.29)

$$\int \frac{1}{1+y^8} |L(\varepsilon)|^2 \lesssim b^2 \int \frac{1}{1+y^8} \varepsilon^2 \lesssim \frac{b^4}{|\log b|^2}.$$

Let us study the second estimate. In order to do that, let

$$g = 3(\tilde{Q}_b^2 - Q^2). \quad (6.5.10)$$

We have the following bound :

$$|H^j (\partial_y^i g)| \lesssim b \frac{y^k (1+|\log y|)}{1+y^{2+i+2j+k}}, \quad 0 \leq i \leq 4, \quad 0 \leq j \leq 1 \quad (6.5.11)$$

where

$$k = \max\{0; 2-i-2j\}.$$

So, with the bounds (6.5.11) and those of Lemma 6.7.10, we obtain :

$$\begin{aligned}
\int |H^2(L(\varepsilon))|^2 &= \int |H^2(g\varepsilon)|^2 = \int |H(gH\varepsilon - \varepsilon\Delta g - \partial_y g \partial_y \varepsilon)|^2 \\
&\lesssim \int |gH^2(\varepsilon)|^2 + \int |\Delta g H\varepsilon|^2 + \int |\partial_y g \partial_y(H\varepsilon)|^2 \\
&+ \int |\Delta g H\varepsilon|^2 + \int |\varepsilon \Delta^2 g|^2 + \int |\partial_y(\Delta g) \partial_y \varepsilon|^2 \\
&+ \int |H(\partial_y g) \partial_y \varepsilon|^2 + \int |\partial_y g \Delta(\partial_y \varepsilon)|^2 + \int |\partial_{yy} g \partial_{yy} \varepsilon|^2 \\
&\lesssim \frac{b^6}{|\log b|^2}.
\end{aligned}$$

The proof is similar to that of the forthcoming estimates of the nonlinear terms and therefore left to the reader.

Step 7 Nonlinear term $N(\varepsilon)$.

We recall the expression of $N(\varepsilon)$:

$$N(\varepsilon) = 3\tilde{Q}_b \varepsilon^2 + \varepsilon^3.$$

We have, with the rough bounds (6.2.20), (6.2.39) and (6.2.50) :

$$\left| 3(\tilde{Q}_b(y) - Q(y)) \right| \lesssim |b|(1 + |\log y|)$$

and, with (6.7.32)

$$\begin{aligned}
\int \frac{1}{1+y^8} \left| 3(\tilde{Q}_b - Q + Q)\varepsilon^2 \right|^2 &\lesssim b^2 \int \frac{1+|\log y|^2}{1+y^8} \varepsilon^4 + \int \frac{\varepsilon^4}{1+y^{12}} \\
&\lesssim b^2 \|\varepsilon\|_{L^\infty}^4 \int \frac{1+|\log y|^2}{1+y^8} + \left\| \frac{\varepsilon}{1+y^2} \right\|_{L^\infty}^2 \int \frac{\varepsilon^2}{1+y^8} \lesssim \frac{b^4}{|\log b|^2}
\end{aligned}$$

and to conclude :

$$\int \frac{1}{1+y^8} |\varepsilon^3|^2 \lesssim \|\varepsilon\|_{L^\infty}^6 \int \frac{1}{1+y^8} \lesssim \frac{b^4}{|\log b|^2}.$$

For the second bound, let us compute $H(\varepsilon^3)$

$$\begin{aligned}
H(\varepsilon^3) &= \varepsilon^2 H(\varepsilon) - 2\partial_y(\varepsilon^2) \partial_y \varepsilon - \Delta(\varepsilon^2) \varepsilon \\
&= \varepsilon^2 H(\varepsilon) - 2(\partial_y \varepsilon)^2 \varepsilon - \left(\varepsilon \partial_{yy} \varepsilon + (\partial_y \varepsilon)^2 + 3\frac{\varepsilon \partial_y \varepsilon}{y} \right) \varepsilon \\
&= \varepsilon^2 H(\varepsilon) - 3(\partial_y \varepsilon)^2 \varepsilon - \varepsilon^2 \Delta \varepsilon \\
&= 2\varepsilon^2 H(\varepsilon) + V \varepsilon^3 - 3\varepsilon(\partial_y \varepsilon)^2.
\end{aligned}$$

Now, we treat each term separately. First :

$$\begin{aligned}
\int |H(\varepsilon^2 H(\varepsilon))|^2 &\lesssim \int \varepsilon^4 |H^2(\varepsilon)|^2 + \int \Delta(\varepsilon^2)^2 H(\varepsilon)^2 + \int |\partial_y(\varepsilon^2)|^2 |\partial_y H(\varepsilon)|^2 \\
&\lesssim \|\varepsilon\|_{L^\infty}^4 \mathcal{E}_4 + \left(\|\varepsilon\|_{L_{y \geq 1}^\infty}^2 \|\partial_{yy}\varepsilon\|_{L_{y \geq 1}^\infty}^2 + \|\partial_y\varepsilon\|_{L^\infty}^4 + \|\partial_y\varepsilon\|_{L_{y \geq 1}^\infty}^2 \left\| \frac{\varepsilon}{y} \right\|_{L_{y \geq 1}^\infty}^2 \right) \mathcal{E}_2 \\
&+ \left(\|\varepsilon\|_{L_{y \leq 1}^\infty}^2 \|y\partial_{yy}\varepsilon\|_{L_{y \leq 1}^\infty}^2 + \|\partial_y\varepsilon\|_{L_{y \leq 1}^\infty}^2 \|\varepsilon\|_{L_{y \leq 1}^\infty}^2 + \left\| \varepsilon(1 + |\log y|^2) \right\|_{L_{y \geq 1}^\infty}^2 \|y\partial_y\varepsilon\|_{L_{y \geq 1}^\infty}^2 \right) \mathcal{E}_4 \\
&\lesssim \frac{b^6}{|\log b|^2}.
\end{aligned}$$

Secondly, using that

$$|\partial_y^i V| \lesssim \frac{1}{1+y^{4+i}} \quad 0 \leq i \leq 2, \quad (6.5.12)$$

we have :

$$\begin{aligned}
\int |H(V\varepsilon^3)|^2 &\lesssim \int |H(V)|^2 \varepsilon^6 + \int \Delta(\varepsilon^3)^2 V^2 + \int |\partial_y(\varepsilon^3)|^2 |\partial_y V|^2 \\
&\lesssim \|\varepsilon\|_{L^\infty}^2 \left(\left\| \frac{\varepsilon}{1+y} \right\|_{L^\infty}^2 \left\| \frac{\varepsilon}{1+y^2} \right\|_{L^\infty}^2 + \|\partial_y\varepsilon\|_{L^\infty}^4 + \left\| \frac{\varepsilon}{1+y^2} \right\|_{L^\infty}^2 \|\partial_y\varepsilon\|_{L^\infty}^2 \right) \int \frac{1}{1+y^6} \\
&+ \|\varepsilon\|_{L^\infty}^4 \int \frac{|\partial_{yy}\varepsilon|^2}{1+y^8} \lesssim \frac{b^6}{|\log b|^2}.
\end{aligned}$$

Lastly

$$\begin{aligned}
\int |H(\varepsilon(\partial_y\varepsilon)^2)|^2 &\lesssim \int |H(\varepsilon)|^2 |(\partial_y\varepsilon)^2|^2 + \int \varepsilon^2 |\Delta((\partial_y\varepsilon)^2)|^2 + \int \partial_y\varepsilon^2 |\partial_y((\partial_y\varepsilon)^2)|^2 \\
&\lesssim \|\partial_y\varepsilon\|_{L^\infty}^4 \mathcal{E}_2 + \int \varepsilon^2 |\Delta((\partial_y\varepsilon)^2)|^2 \lesssim \frac{b^6}{|\log b|^2} + \int \varepsilon^2 |\Delta((\partial_y\varepsilon)^2)|^2.
\end{aligned}$$

But

$$\begin{aligned}
&\int \varepsilon^2 |\Delta((\partial_y\varepsilon)^2)|^2 \\
&= \int \varepsilon^2 \left(|\partial_{yy}\varepsilon|^2 + \partial_y\varepsilon \partial_y^3 \varepsilon + 3 \frac{\partial_y\varepsilon \partial_{yy}\varepsilon}{y} \right)^2 \\
&= \int \varepsilon^2 \left(\partial_{yy}\varepsilon (H(\varepsilon) + V\varepsilon) + \partial_y\varepsilon \partial_y^3 \varepsilon \right)^2 \\
&\lesssim \|\varepsilon\|_{L^\infty}^2 \left(\|y\partial_{yy}\varepsilon\|_{L_{y \leq 1}^\infty}^2 \mathcal{E}_4 + \|\partial_{yy}\varepsilon\|_{L_{y \geq 1}^\infty}^2 \mathcal{E}_2 + \|\varepsilon\|_{L^\infty}^2 \int \frac{|\partial_{yy}\varepsilon|^2}{1+y^8} \right) \\
&+ \left\| \varepsilon(1 + |\log y|^2) \right\|_{L_{y \geq 1}^\infty}^2 \|y\partial_y\varepsilon\|_{L_{y \geq 1}^\infty}^2 \mathcal{E}_4 + \|\varepsilon\|_{L_{y \leq 1}^\infty}^2 \|\partial_y\varepsilon\|_{L_{y \leq 1}^\infty}^2 \int_{y \leq 1} |\partial_y^3 \varepsilon|^2 \\
&\lesssim \frac{b^6}{|\log b|^2}.
\end{aligned}$$

Thus,

$$\int |H^2(\varepsilon^3)|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (6.5.13)$$

Treat now the other contribution of $N(\varepsilon)$ in the bound (6.5.6). Let

$$f = 3(\tilde{Q}_b - Q).$$

We have the following bounds :

$$|\partial_y^i f| \lesssim b \frac{y^{2-i}(1+|\log y|)}{1+y^2} + \frac{1}{1+y^{2+i}}, \quad 0 \leq i \leq 2 \quad (6.5.14)$$

$$|H(\partial_y^j f)| \lesssim b \frac{y^k}{1+y^{2+j+k}} + \frac{1}{1+y^{4+j}}, \quad 0 \leq j \leq 2 \quad (6.5.15)$$

where

$$k = \begin{cases} 1 & \text{for } j=1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute $H(f\varepsilon^2)$

$$\begin{aligned} H(f\varepsilon^2) &= H(\varepsilon^2)f - \varepsilon^2\Delta f - 2\partial_y f \partial_y(\varepsilon^2) \\ &= H(\varepsilon^2)f + \varepsilon^2 Hf - 2\partial_y f \partial_y(\varepsilon^2) + Vf\varepsilon^2 \end{aligned} \quad (6.5.16)$$

In the same way as in the last proof, we treat each term separately. First :

$$H(H(\varepsilon^2)f) = H^2(\varepsilon^2)f - \Delta f H(\varepsilon^2) - \partial_y f \partial_y H(\varepsilon^2). \quad (6.5.17)$$

Let us estimate the three components using that :

$$H(\varepsilon^2) = 3\varepsilon H(\varepsilon) + 2V\varepsilon^2 - 2(\partial_y \varepsilon)^2 \quad (6.5.18)$$

and,

$$\begin{aligned} H^2(\varepsilon^2) &= 3(\varepsilon H^2(\varepsilon) + |H(\varepsilon)|^2 + V\varepsilon H(\varepsilon) - 2\partial_y \varepsilon \partial_y H(\varepsilon)) \\ &\quad + 2(HV\varepsilon^2 - V\Delta(\varepsilon^2) - 2\partial_y V \partial_y(\varepsilon^2)) \\ &\quad - 2(3\partial_y \varepsilon H(\partial_y \varepsilon) + 2V(\partial_y \varepsilon)^2 - 2(\partial_{yy} \varepsilon)^2) \end{aligned} \quad (6.5.19)$$

and moreover,

$$H(\partial_y \varepsilon) = \partial_y H(\varepsilon) - \frac{\partial_y \varepsilon}{y^2} - \partial_y V \varepsilon \quad (6.5.20)$$

(6.5.17), (6.5.18), (6.5.19) and (6.5.20) together with the bounds (6.5.14), (6.5.15), (6.5.12) and those of Lemma 6.7.10 imply :

$$\int |H(H(\varepsilon^2)f)|^2 \lesssim \frac{b^6}{|\log b|^2} \quad (6.5.21)$$

Let us study the second term of (6.5.16) .

$$H(\varepsilon^2 Hf) = \varepsilon^2 H^2(f) - Hf \Delta(\varepsilon^2) - 2\partial_y(Hf) \partial_y(\varepsilon^2). \quad (6.5.22)$$

The bounds (6.5.14), (6.5.15) and Lemma 6.7.10 yield

$$\int |H(\varepsilon^2 Hf)|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (6.5.23)$$

We estimate the two last terms in (6.5.16) in the same way. This concludes the proof of (6.5.6) and thus of Proposition 6.5.1.

6.5.2 At \dot{H}^2 level

For the H^2 level, we use the profile \hat{Q}_b localized near B_0 . The description of this localization and the estimates of the new error generated by these are given by Lemma 6.2.4 and Section 6.3.2. We recall the equation verified by \hat{w}_2 :

$$\partial_t \hat{w}_2 + \hat{H}_\lambda \hat{w}_2 = -\partial_t \tilde{V} \hat{w} + \hat{H}_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right). \quad (6.5.24)$$

Proposition 6.5.3 (Lyapunov monotonicity \dot{H}^2). *There hold :*

$$\frac{d}{dt} \frac{\hat{\mathcal{E}}_2}{\lambda^2} \lesssim \frac{b^3 |\log b|^2}{\lambda^4} \quad (6.5.25)$$

Proof.

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \frac{\hat{\mathcal{E}}_2}{\lambda^2} &= \int \hat{w}_2 \left[-H_\lambda \hat{w}_2 - \partial_t \tilde{V} \hat{w} + H_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right) \right] \\ &= - \int \hat{w}_2 H_\lambda \hat{w}_2 - \int \partial_t \tilde{V} \hat{w} \hat{w}_2 + \int \hat{w}_2 H_\lambda \left(\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right). \end{aligned} \quad (6.5.26)$$

We study each term separately :

To begin, in agreement with (6.7.3), the *a priori* bound (6.4.11) on the unstable direction, and the bounds (6.2.70) we have

$$\begin{aligned} - \int \hat{w}_2 H_\lambda \hat{w}_2 &\lesssim \frac{1}{\lambda^4} (\hat{\varepsilon}_2, \psi)^2 \\ &\lesssim \frac{1}{\lambda^4} [(\varepsilon_2, \psi)^2 + (H\zeta_b, \psi)^2] \\ &\lesssim \frac{1}{\lambda^4} [\kappa^2 + \|H^2 \zeta_b\|_{L^2}^2] \lesssim \frac{b^3 |\log b|^2}{\lambda^4}. \end{aligned} \quad (6.5.27)$$

The second term is a lower order quadratic term. Recall that :

$$|\partial_t \tilde{V}| \lesssim \frac{b}{\lambda^4} \frac{1}{1+y^4}.$$

Hence, using Cauchy-Schwarz, with the *a priori* bound (6.4.9), (6.3.24) measuring the difference between the two energies at \dot{H}^2 level, and the bounds (6.2.70) and (6.7.27)

$$\left| \int \partial_t \tilde{V} \hat{w} \hat{w}_2 \right| \lesssim \frac{b}{\lambda^4} \hat{\mathcal{E}}_2^{\frac{1}{2}} \left(\int \frac{|\varepsilon|^2 + |\zeta_b|^2}{1+y^8} \right)^{\frac{1}{2}} \lesssim \frac{b}{\lambda^4} b^3 |\log b|^C. \quad (6.5.28)$$

For the last, we don't use exactly the same strategy as for the control \dot{H}^4 . Indeed, for the term of error, the term of modulation, the global L^2 bounds for $\hat{\varepsilon}_2$ we have are too rough. We must then improve them, as both terms are localized for $y \leq 2B_0$. So :

$$\begin{aligned} \int_{y \leq 2B_0} |\hat{\varepsilon}_2|^2 &\lesssim B_0^4 |\log b|^2 \int \frac{|\varepsilon|^2}{(1+y^4)|\log y|^2} + \int |H\zeta_b|^2 \\ &\lesssim C(M)b^2 + b^2 |\log b|^2 \lesssim b^2 |\log b|^2. \end{aligned} \quad (6.5.29)$$

The term of error $\hat{\Psi}_b$ is now estimated using (6.2.71) and the improved bound (6.5.29) :

$$|(\hat{\varepsilon}_2, H\hat{\Psi}_b)| \lesssim \|H\hat{\Psi}_b\|_{L^2} \|\hat{\varepsilon}_2\|_{L^2(y \leq 2B_0)} \lesssim (b^4 |\log b|^2 b^2 |\log b|^2)^{\frac{1}{2}} \lesssim b^3 |\log b|^2. \quad (6.5.30)$$

We next estimate from (6.2.20), (6.2.38) :

$$\begin{aligned} \int |H\hat{T}_1|^2 &\lesssim \int_{y \leq 2B_0} |\Lambda Q|^2 + \int_{B_0 \leq y \leq 2B_0} \left| \frac{\log y}{y} \right|^2 \lesssim |\log b|^2, \\ \int |H\hat{T}_2|^2 &\lesssim \int_{y \leq 2B_0} \left| \frac{y}{y^2 b |\log b|} \right|^2 \lesssim \frac{1}{b^2 |\log b|}, \end{aligned}$$

and thus from (6.2.67), (6.3.26), (6.3.27) :

$$\begin{aligned} \int |H\widehat{Mod}(t)|^2 &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int |H\Lambda\hat{Q}_b|^2 + |b_s + b^2|^2 \int |H(\hat{T}_1 + b\hat{T}_2)|^2 \\ &\lesssim \frac{b^4}{|\log b|^2} |\log b|^2 \lesssim b^4 |\log b|^2. \end{aligned}$$

Moreover, $\text{Supp}(\widehat{HMod}) \subset [0, 2B_0]$ and thus with (6.5.29) :

$$|(\hat{\varepsilon}_2, H\widehat{Mod})| \lesssim (b^4 |\log b|^2 b^2 |\log b|^2)^{\frac{1}{2}} \lesssim b^3 |\log b|^2. \quad (6.5.31)$$

We now claim the following bound for the small linear term, and the nonlinear term :

$$\int |H\hat{L}(\hat{\varepsilon})|^2 + |H\hat{N}(\hat{\varepsilon})|^2 \lesssim b^5. \quad (6.5.32)$$

Assume (6.5.32). Thus,

$$|(\hat{\varepsilon}_2, HL(\hat{\varepsilon}))| \lesssim b |\log b|^C b^{\frac{5}{2}} \lesssim b^3 |\log b|^2. \quad (6.5.33)$$

(6.5.33), together with (6.5.27), (6.5.28), (6.5.30) and (6.5.31) concludes the proof of Proposition 6.5.3.

Proof of (6.5.32) :

We recall that :

$$\hat{L}(\hat{\varepsilon}) = 3(\hat{Q}_b^2 - Q^2)\hat{\varepsilon}.$$

In the same way as previously, we let :

$$\hat{g} = 3(\hat{Q}_b^2 - Q^2),$$

for which we have the bounds :

$$\begin{aligned} |\hat{g}| + |\partial_y \hat{g}| + |H\hat{g}| &\lesssim b \quad \text{for } y \leq 1 \\ \left| \frac{\hat{g}}{1 + |\log y|} \right| + |y\partial_y \hat{g}| + |y^2 H\hat{g}| &\lesssim \frac{b}{1 + y^2} \quad \text{for } y \geq 1. \end{aligned}$$

These bounds together with (6.2.69), (6.2.70), Lemma 6.7.10 and the following decomposition :

$$H\hat{L} = \hat{\varepsilon}H\hat{g} - \hat{g}\Delta\hat{\varepsilon} - 2\partial_y\hat{g}\partial_y\hat{\varepsilon}$$

yield the expected bound for the linear term. We estimate afterwards the nonlinear term. We know that :

$$\hat{N}(\hat{\varepsilon}) = 3Q\hat{\varepsilon}^2 + 3\hat{f}\hat{\varepsilon}^2 + \hat{\varepsilon}^3.$$

where we note

$$\hat{f} = 3(\hat{Q}_b - Q)$$

and we are within the bounds of the profiles $(T_i)_{1 \leq i \leq 3}$:

$$\begin{aligned} |\hat{f} + \partial_y\hat{f} + H\hat{f}| &\lesssim b \quad \text{for } y \leq 1 \\ \left| \frac{\hat{f}}{1 + |\log y|} + y\partial_y\hat{f} + y^2H\hat{f} \right| &\lesssim b \quad \text{for } y \geq 1. \end{aligned}$$

With the estimates L^∞ of Lemma 6.7.10, of the second localization (6.2.69) and (6.2.70), we have :

$$\begin{aligned} \int |H\hat{N}(\hat{\varepsilon})|^2 &\lesssim \frac{\Delta(\hat{\varepsilon}^2)^2}{1 + y^2} + \frac{(\partial_y\hat{\varepsilon}^2)^2}{1 + y^6} + \int \frac{\hat{\varepsilon}^4}{1 + y^8} \\ &+ \|\hat{\varepsilon}\|_{L^\infty}^2 \int |H\hat{f}\hat{\varepsilon}|^2 + |\Delta\hat{\varepsilon}\hat{f}|^2 + |\partial_y\hat{f}\partial_y\hat{\varepsilon}|^2 \\ &+ \|\hat{\varepsilon}\|_{L^\infty}^4 \hat{\mathcal{E}}_2 + \|\hat{\varepsilon}\|_{L^\infty}^2 \int |\Delta\hat{\varepsilon}^2|^2 + \|\hat{\varepsilon}\|_{L^\infty}^2 \|\partial_y\hat{\varepsilon}\|_{L^\infty}^2 \int |\partial_y\hat{\varepsilon}|^2 \\ &\lesssim b^5. \end{aligned}$$

Proposition 6.5.3 is proved.

6.6 Proof of Proposition 6.4.2

6.6.1 Improved bound

The Lyapunov monotonicity property allow us to improve the a priori bounds under the a priori control (6.4.11) on the unstable direction.

Lemme 6.6.1 (Improved bounds under the a priori control (6.4.11)). *Assume that K in (6.4.7), (6.4.8), (6.4.9), and (6.4.10) has been chosen large enough. Then, $\forall t \in [0, t_1]$:*

$$0 \leq b(t) \leq 2b(0), \tag{6.6.1}$$

$$\int |\nabla\hat{\varepsilon}|^2 \leq \sqrt{b(0)}, \tag{6.6.2}$$

$$|\mathcal{E}_2(t)| \leq \frac{K}{2} b^2(t) |\log b(t)|^5, \tag{6.6.3}$$

$$|\mathcal{E}_4(t)| \leq \frac{K}{2} \frac{b^4(t)}{|\log b(t)|^2}. \tag{6.6.4}$$

Proof .

Step 1. Positivity and smallness of $b(t)$.

The proof of (6.6.1) is a direct consequence of the modulation equations. Indeed, this last equation (6.3.27) yields that

$$b_s \leq 0 \quad (6.6.5)$$

We must now prove that $b(t)$ can't be negative. We argue by contradiction. As $b(0) \geq 0$ and $b(t)$ is a continuous function, we suppose that there exists t_0 such that $b(t_0) = 0$. With the modulation equations, we have that :

$$|b_s| \leq 2b^2 \quad (6.6.6)$$

Hence, there exists δ such that $b(t) = 0$ on $[t_0 - \delta, t_0]$, and thus from (6.4.10), $\lambda(t) = \lambda(t_0)$ and $u(t) = Q_{\lambda(t_0)}$ on $[t_0 - \delta, t_0]$. Iterating on $\delta > 0$, we conclude that u_0 is initially a rescaling of Q , a contradiction.

Step 2. Energy bound.

(6.6.2) is a consequence of the decrease of energy. Indeed, let

$$\tilde{\varepsilon} = \tilde{\alpha} + \varepsilon. \quad (6.6.7)$$

Then

$$\begin{aligned} E(u) &= \frac{1}{2} \int |\nabla u|^2 - \frac{1}{4} \int |u|^4 \\ &= \frac{1}{2} \left\{ \int |\nabla Q|^2 + \int |\nabla \tilde{\varepsilon}|^2 \right\} + \int \partial_y Q \partial_y \tilde{\varepsilon} - \frac{1}{4} \int [Q^4 + 4Q^3 \tilde{\varepsilon} + 6Q^2 \tilde{\varepsilon}^2 + 4Q \tilde{\varepsilon}^3 + \tilde{\varepsilon}^4] \\ &= E(Q) + (H\tilde{\varepsilon}, \tilde{\varepsilon}) - \frac{1}{4} \int [4Q \tilde{\varepsilon}^3 + \tilde{\varepsilon}^4]. \end{aligned} \quad (6.6.8)$$

Now,

$$\begin{aligned} (H\tilde{\varepsilon}, \tilde{\varepsilon}) &= (H\varepsilon, \varepsilon) + (H\tilde{\alpha}, \tilde{\alpha}) + 2(\tilde{\alpha}, H\varepsilon) \\ &= (H\varepsilon, \varepsilon) + O(b|\log b|^C). \end{aligned}$$

The last equality comes from Cauchy-Schwarz, the bound (6.4.9) for ε_2 and the inequalities :

$$\|\tilde{\alpha}\|_{L^2}^2 \lesssim |\log b|^C, \quad \|H\tilde{\alpha}\|_{L^2}^2 \lesssim b^4 |\log b|^C.$$

Moreover, using (6.7.10), the orthogonality conditions (6.3.5), and the *a priori* bound on the unstable direction (6.4.11)

$$(H\varepsilon, \varepsilon) \geq c \int |\nabla \varepsilon|^2 - \frac{1}{c} \left[(\varepsilon, \Phi_M)^2 + (\varepsilon, \psi)^2 \right] \gtrsim \int |\nabla \varepsilon|^2 + O\left(\frac{b^5}{|\log b|^2}\right).$$

Thus,

$$(H\tilde{\varepsilon}, \tilde{\varepsilon}) \gtrsim \int |\nabla \varepsilon|^2 + O(b|\log b|^C). \quad (6.6.9)$$

Let us see the nonlinear terms. We recall that :

$$\|\tilde{\varepsilon}\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty} + \|\tilde{\alpha}\|_{L^\infty} \lesssim b|\log b| \quad \left\| \frac{\tilde{\alpha}}{y} \right\|_{L^2}^2 + \|\nabla \tilde{\alpha}\|_{L^2}^2 \lesssim b^2 |\log b|^C.$$

Like this, with the bound (6.4.8) and the fact that :

$$\forall u \in H_{rad}^1(\mathbb{R}^4), \quad \int \frac{|u|^2}{y^2} \lesssim \int |\nabla u|^2, \quad \left(\int |u|^4 \right)^{\frac{1}{4}} \lesssim \left(\int |\nabla u|^2 \right)^{\frac{1}{2}}, \quad (6.6.10)$$

thus

$$\begin{aligned} \int [4Q\tilde{\varepsilon}^3 + \tilde{\varepsilon}^4] &\lesssim \|\tilde{\varepsilon}\|_{L^\infty} \left(\int \frac{|\varepsilon|^2}{1+y^2} + \left\| \frac{\tilde{\alpha}}{y} \right\|_{L^2}^2 \right) + \left(\int |\nabla \varepsilon|^2 \right)^2 + \left(\int |\nabla \tilde{\alpha}|^2 \right)^2 \\ &\lesssim \sqrt{b(0)} \int |\nabla \varepsilon|^2 + O(b|\log b|^C). \end{aligned} \quad (6.6.11)$$

The first inequality of (6.6.10) comes from Lemma (6.7.1). The second is a classical result of Sobolev, Gagliardo and Nirenberg in dimension $N = 4$. This results in general case and its proof is available in [10] with Theorem IX.9. By construction,

$$0 \leq E(u) - E(Q) \lesssim b^2(0)|\log b(0)|^C. \quad (6.6.12)$$

Injecting (6.6.9), (6.6.11) and (6.6.12) into (6.6.8) concludes the proof of (6.6.2).

Step 3. Control of \mathcal{E}_4 .

We argue similarly as in [91]. $\forall t \in [0, t_1]$,

$$\begin{aligned} \mathcal{E}_4(t) &\leq 2 \left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \left[\mathcal{E}_4(0) + C\sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} \right] + \frac{b^4(t)}{|\log b(t)|^2} \\ &+ C \left[1 + \frac{K}{\log M} + \sqrt{K} \right] \lambda^6(t) \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\tau \end{aligned} \quad (6.6.13)$$

for some universal constant $C > 0$ independent of M .

Let us now consider two constants

$$\alpha_1 = 2 - \frac{C_1}{\sqrt{\log M}}, \quad \alpha_2 = 2 + \frac{C_2}{\sqrt{\log M}} \quad (6.6.14)$$

for some large enough universal constants C_1, C_2 . We compute using the modulation equations (6.3.26), (6.3.27) and the bootstrap bound (6.4.10) :

$$\begin{aligned} \frac{d}{ds} \left\{ \frac{|\log b|^{\alpha_i} b}{\lambda} \right\} &= \frac{|\log b|^{\alpha_i}}{\lambda} \left[\left(1 - \frac{\alpha_i}{|\log b|} \right) b_s - \frac{\lambda_s}{\lambda} b \right] \\ &= \frac{|\log b|^{\alpha_i}}{\lambda} \left[\left(1 - \frac{\alpha_i}{|\log b|} \right) b_s + b^2 + O\left(\frac{b^3}{|\log b|}\right) \right] \\ &= \left(1 - \frac{\alpha_i}{|\log b|} \right) \frac{|\log b|^{\alpha_i}}{\lambda} \left[b_s + b^2 \left(1 + \frac{\alpha_i}{|\log b|} + O\left(\frac{1}{|\log b|^2}\right) \right) \right] \\ &\begin{cases} \leq 0 & \text{for } i = 1 \\ \geq 0 & \text{for } i = 2. \end{cases} \end{aligned}$$

Integrating this from 0 to t yields :

$$\frac{b(0)}{\lambda(0)} \left(\frac{|\log b(0)|}{|\log b(t)|} \right)^{\alpha_2} \leq \frac{b(t)}{\lambda(t)} \leq \frac{b(0)}{\lambda(0)} \left(\frac{|\log b(0)|}{|\log b(t)|} \right)^{\alpha_1}. \quad (6.6.15)$$

This yields in particular using the initial bound (6.4.3) and the bound (6.4.7) :

$$\left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \mathcal{E}_4(0) \leq (b(t)|\log b(t)|^{\alpha_2})^6 \frac{\mathcal{E}_0}{(b(0)|\log b(0)|^{\alpha_2})^6} \leq \frac{b^4(t)}{|\log b(t)|^2}, \quad (6.6.16)$$

$$\begin{aligned} C \left(\frac{\lambda(t)}{\lambda(0)} \right)^6 \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} &\lesssim \left(\frac{b(t)|\log b(t)|^{\alpha_2}}{b(0)|\log b(0)|^{\alpha_2}} \right)^6 \sqrt{b(0)} \frac{b^4(0)}{|\log b(0)|^2} \\ &\lesssim C(b(t))^{4+\frac{1}{4}} \leq \frac{b^4(t)}{|\log b(t)|^2}. \end{aligned} \quad (6.6.17)$$

We now compute explicitly using $b = -\lambda \lambda_t + O\left(\frac{b^2}{|\log b|}\right)$ from (6.3.26) :

$$\begin{aligned} \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\sigma &= \frac{1}{6} \left[\frac{b^4}{\lambda^6 |\log b|^2} \right]_0^t - \frac{1}{6} \int_0^t \frac{b_t b^3}{\lambda^6 |\log b|^2} \left(4 + \frac{2}{|\log b|} \right) d\tau \\ &+ O\left(\int_0^t \frac{b}{\lambda^8} \frac{b^5}{|\log b|^2} d\tau \right) \end{aligned}$$

which implies using now $|b_s + b^2| \lesssim \frac{b^2}{|\log b|}$ from (6.3.27) and (6.4.10) :

$$\lambda^6(t) \int_0^t \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2} d\sigma \lesssim \left[1 + O\left(\frac{1}{|\log b_0|}\right) \right] \frac{b^4(t)}{|\log b(t)|^2}.$$

Injecting this together with (6.6.16), (6.6.17) into (6.6.13) yields

$$\mathcal{E}_4(t) \leq C \frac{b^4(t)}{|\log b(t)|^2} \left[1 + \frac{K}{\log M} + \sqrt{K} \right]$$

for some universal constant $C > 0$ independent of K and M , and thus (6.6.4) follows for K large enough independent of M .

Step 4. Control of \mathcal{E}_2 .

Similarly to the control of \mathcal{E}_4 , we give the same proof as well as in [91]. We integrate the monotonicity formula (6.5.25) after recalling the estimate of the difference between $\hat{\mathcal{E}}_2$ and \mathcal{E}_2 :

$$\begin{aligned} \mathcal{E}_2(t) &= \lambda^2(t) \|w_2(t)\|_{L^2}^2 \lesssim \|H\zeta_b(t)\|_{L^2}^2 + \lambda^2(t) \|\hat{w}_2(t)\|_{L^2}^2 \\ &\lesssim b^4(t) |\log b(t)|^2 + \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 [\mathcal{E}_2(0) + b^2(0) |\log b(0)|^2] + \lambda^2(t) \int_0^t \frac{b^3 |\log b|^2}{\lambda^4(\tau)} d\tau. \end{aligned} \quad (6.6.18)$$

From (6.4.3), (6.6.15) :

$$\begin{aligned} \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 [\mathcal{E}_2(0) + b^2(0) |\log b(0)|^2] &\lesssim \frac{(b(0))^{10} + b^2(0) |\log b(0)|^2}{(b(0) |\log b(0)|^{\alpha_2})^2} b^2(t) |\log b(t)|^{2\alpha_2} \\ &\leq b^2(t) |\log b(t)|^{4+\frac{1}{4}}, \end{aligned}$$

We now use the bound $b_s \lesssim -b^2$ and (6.6.15) to estimate :

$$\begin{aligned} & \lambda^2(t) \int_0^t \frac{b^3 |\log b|^2}{\lambda^4(\tau)} d\tau \lesssim \lambda^2(t) \int_0^t \frac{-b_t b |\log b|^2}{\lambda^2(\tau)} d\tau \\ & \lesssim \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 b^2(0) |\log b(0)|^{2\alpha_1} \int_0^t \frac{-b_t}{b |\log b|^{2\alpha_1-2}} d\tau \\ & \lesssim \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 b^2(0) |\log b(0)|^{2\alpha_1} \frac{1}{|\log b(0)|^{2\alpha_1-3}} \\ & \lesssim b^2(t) |\log b(t)|^{2\alpha_2} \frac{|\log b(0)|^3}{|\log b(0)|^{2\alpha_2}} \lesssim b^2(t) |\log b(t)|^{4+\frac{1}{4}}. \end{aligned}$$

Injecting these bounds into (6.6.18) yields :

$$\mathcal{E}_2(t) \lesssim b^2(t) |\log b(t)|^{4+\frac{1}{4}}$$

and concludes the proof of (6.6.3).

6.6.2 Dynamic of the unstable mode

To conclude Proposition 6.4.2, we must study the dynamic of the unstable mode. We recall that $\kappa(t) = (\varepsilon(t), \psi)$.

Lemme 6.6.2 (Control of the unstable mode). *There holds : for all $t \in [0, T_1(a^+)]$,*

$$\left| \frac{d\kappa}{ds} - \zeta \kappa \right| \leq \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.6.19)$$

Proof. We compute the equation satisfied by κ by taking the inner product of (6.3.8) with the well localized direction ψ to get :

$$\frac{d\kappa}{ds} - \zeta \kappa = E(\varepsilon) \quad (6.6.20)$$

with

$$E(\varepsilon) = (-\tilde{\Psi}_b, \psi) + (L(\varepsilon), \psi) + (N(\varepsilon), \psi) - (Mod, \psi) + \frac{\lambda_s}{\lambda} (\Lambda \varepsilon, \psi). \quad (6.6.21)$$

We now estimate all terms of RHS. We recall the exponential localization of ψ as well as the orthogonality $(\psi, \Lambda Q) = 0$. To begin, using (6.2.63)

$$\left| (\tilde{\Psi}_b, \psi) \right| = \left| \frac{1}{\zeta^2} (\tilde{\Psi}_b, H^2 \psi) \right| = \left| \frac{1}{\zeta^2} (H^2 \tilde{\Psi}_b, \psi) \right| \lesssim \left(\int |H^2(\tilde{\Psi}_b)|^2 \right)^{\frac{1}{2}} \lesssim \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.6.22)$$

From the definition (6.3.21) of $L(\varepsilon)$, we have the following bound :

$$|L(\varepsilon)| \lesssim b y^{10} |\varepsilon|.$$

Thus,

$$|(L(\varepsilon), \psi)| \lesssim b \left| \left(\frac{|\varepsilon|}{y^2(1+y^2)(1+|\log y|)}, (1+y^{14})(1+|\log y|)\psi \right) \right| \lesssim \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.6.23)$$

In the same way, with (6.3.11),

$$|N(\varepsilon)| \lesssim \left(by^{10} |\varepsilon| + \left\| \frac{\varepsilon}{1+y} \right\|_{L^\infty}^2 y^2 \right) |\varepsilon|.$$

(6.7.35) and the fact that $\forall i, \|y^i \psi\|_{L^\infty} \lesssim 1$ yield :

$$|(N(\varepsilon), \psi)| \lesssim \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.6.24)$$

With the notation of Lemma 6.2.3, we have

$$\begin{aligned} |(Mod, \psi)| &= \left| \left(-\left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b\tilde{T}_2), \psi \right) \right| \\ &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right| |(\Lambda Q + \Lambda \tilde{\alpha}, \psi)| + |b_s + b^2| |(\tilde{T}_1 + 2b\tilde{T}_2, \psi)|. \end{aligned}$$

But

$$|\Lambda \tilde{\alpha}| + |2b\tilde{T}_2| \lesssim by^{10}$$

and

$$|(\tilde{T}_1, \psi)| = \left| \frac{-1}{\zeta} (\tilde{T}_1, H\psi) \right| = \left| \frac{-1}{\zeta} (H\tilde{T}_1, \psi) \right| \lesssim \left| \int_{y \geq B_1} (H\tilde{T}_1 - \Lambda Q) \psi \right| \lesssim b.$$

Hence, with the modulation equations

$$|(Mod, \psi)| \lesssim \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.6.25)$$

For the last term, we use (6.7.27).

$$\left| \frac{\lambda_s}{\lambda} (\Lambda \varepsilon, \psi) \right| = \left| \frac{\lambda_s}{\lambda} \right| \left| \left(\frac{\Lambda \varepsilon}{y^2(1+y^2)(1+|\log y|)}, y^2(1+y^2)(1+|\log y|)\psi \right) \right| \lesssim \sqrt{b} \frac{b^{\frac{5}{2}}}{|\log b|}. \quad (6.6.26)$$

This concludes the proof of Lemma 6.6.2.

6.6.3 Conclusion

We have at our disposal all the elements to finish the proof using a rough argument which does not give a sharp information on the link between the initial data and the choice of a^+ , and uniqueness for example is not covered at this stage. The keystone of the proof is the fact that the map :

$$\begin{aligned} \left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right] &\rightarrow \mathbb{R}^+ \\ a^+ &\rightarrow T_1(a^+) \end{aligned}$$

is continuous as a consequence of the strictly outgoing behavior on exit (6.6.20) defined by (6.4.11). This classical argument is displayed in detail in [20], Lemma 6, in a more complicated setting and therefore left to the reader. Hence, we also have the continuity of the map

$$\begin{aligned} \left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|} \right] &\rightarrow \mathbb{R}^+ \\ a^+ &\rightarrow \kappa(T_1(a^+)). \end{aligned}$$

In agreement with the dynamics of the unstable mode found in the last subsection, we know that, for $a^+ = 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}$:

$$\frac{d}{ds} \kappa(0) = 2\zeta \frac{b_0^{\frac{5}{2}}}{|\log b_0|} + O\left(\frac{b_0^3}{|\log b_0|}\right) > 0$$

and then

$$\kappa\left(T_1\left(2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}\right)\right) = 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}. \quad (6.6.27)$$

Likewise,

$$\kappa\left(T_1\left(-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}\right)\right) = -2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}. \quad (6.6.28)$$

By continuity, there exists $a^+ \in \left[-2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}; 2 \frac{b_0^{\frac{5}{2}}}{|\log b_0|}\right]$ such that

$$\kappa\left(T_1\left(a^+\right)\right) = 0. \quad (6.6.29)$$

In addition, according to the definition of an exit time and Lemma 6.6.1, we have two choices. Either $|\kappa(T_1(a^+))| = 2 \frac{b(T_1(a^+))^{\frac{5}{2}}}{|\log b(T_1(a^+))|}$ or $T_1(a^+)$ is the life time of the solution. If the first possibility is the good one, the condition (6.6.29) gives

$$2 \frac{b(T_1(a^+))^{\frac{5}{2}}}{|\log b(T_1(a^+))|} = 0.$$

As we have proved that $b(t) > 0$ for $t < T$, where T is the life time of the solution, we have thus the second possibility. Notice that in this case, the two choices tally. This is exactly Proposition 6.4.2.

6.7 Appendix

6.7.1 L^2 coercivity estimates

In this appendix, we at first prove the Hardy inequalities for functions $u \in H_{rad}^2(\mathbb{R}^4)$. We will use afterwards this results to establish properties of weighted sub-coercivity for H and H^2 , which allows us to obtain coercive estimates for these operators under additional orthogonality conditions. These coercivity estimates are crucial in our study. The proof lies in the continuation of the analysis in [41].

Hardy inequalities

Lemme 6.7.1. *There exists a constant C for which there holds, for any $v \in H_{rad}^1(\mathbb{R}^4)$*

$$\left[\int_{\mathbb{R}^4} \frac{v(y)^2}{y^2} \right]^{\frac{1}{2}} + \sup_{y \in \mathbb{R}^4} (|yv(y)|) \leq C \left[\int_{\mathbb{R}^4} |\nabla v(y)|^2 \right]^{\frac{1}{2}}, \quad (6.7.1)$$

$$\|v\|_{L_{y \geq 1}^\infty}^2 \lesssim \int_{y \geq 1} \left(\frac{|\nabla v|^2}{y^2} + \frac{|v|^2}{y^4} \right) + \int_{\frac{1}{2} \leq y \leq 1} \frac{|v|^2}{y^2} \quad (6.7.2)$$

Proof. By integration-by-parts :

$$\begin{aligned} \int \frac{v(y)^2}{y^2} &= \left[\frac{v(y)^2 y^2}{2} \right]_0^\infty - \int \frac{v(y) \partial_y v(y)}{y} \\ &\lesssim \left[\int \frac{v(y)^2}{y^2} \right]^{\frac{1}{2}} \left[\int |\nabla v(y)|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Next :

$$|v(y)| \lesssim \int_y^{+\infty} \frac{|\partial_y v|}{y^3} \lesssim \frac{1}{y} \left(\int |\partial_y v|^2 \right)^{\frac{1}{2}},$$

$$|v^2(y)| \lesssim \int_{1 \leq y \leq 2} |v|^2 + \int_{y \geq 1} \frac{|v||\partial_y v|}{y^3} \lesssim \left(\int_{y \geq 1} \frac{|\partial_y v|^2}{y^2} \right)^{\frac{1}{2}} \left(\int \frac{|v|^2}{y^4} \right)^{\frac{1}{2}}.$$

This concludes the proof of Lemma 6.7.1.

Lemme 6.7.2 (Hardy inequalities). $\forall R > 2, \forall v \in H_{rad}^2(\mathbb{R}^4), \forall \gamma > 0$ there holds the following controls :

$$\int |\partial_{yy} v|^2 + \int \frac{|\partial_y v|^2}{y^2} \lesssim \int (\Delta v)^2, \quad (6.7.3)$$

$$\int_{y \leq R} \frac{|v|^2}{y^4(1 + |\log y|)^2} \lesssim \int_{y \leq R} \frac{|\partial_y v|^2}{y^2} + \int_{1 \leq y \leq 2} |v|^2, \quad (6.7.4)$$

$$\int_{1 \leq y \leq R} \frac{|v|^2}{y^{4+\gamma}(1 + |\log y|)^2} \lesssim \int_{1 \leq y \leq R} \frac{|\nabla v|^2}{y^{2+\gamma}(1 + |\log y|)^2} + C_\gamma \int_{1 \leq y \leq 2} |v|^2. \quad (6.7.5)$$

Proof. Let v be smooth and radially symmetric. (6.7.3) follows from the explicit formula after integration by parts

$$\int (\Delta v)^2 = \int \left(\partial_{yy} v + \frac{3}{y} \partial_y v \right)^2 = \int |\partial_{yy} v|^2 + 3 \int \frac{|\partial_y v|^2}{y^2}.$$

To prove (6.7.4) and (6.7.5), from the one dimensional Sobolev embedding $H^1(1 \leq y \leq 2)$ in $L^\infty(1 \leq y \leq 2)$, we obtain

$$|v(1)|^2 \lesssim \int_{1 \leq y \leq 2} (|v|^2 + |\partial_y v|^2). \quad (6.7.6)$$

Let $f(y) = -\frac{\mathbf{e}_y}{y^3(1+\log y)}$ so that $\nabla \cdot f = \frac{1}{y^4(1+\log y)^2}$, and integrate by parts to get :

$$\begin{aligned} & \int_{1 \leq y \leq R} \frac{|v|^2}{y^4(1 + |\log y|)^2} = \int_{1 \leq y \leq R} |v|^2 \nabla \cdot f \\ &= - \left[\frac{|v|^2}{1 + \log y} \right]_1^R + 2 \int_{y \leq R} \frac{v \partial_y v}{y^3(1 + \log y)} \\ &\lesssim |v(1)|^2 + \left(\int_{y \leq R} \frac{|v|^2}{y^4(1 + |\log y|)^2} \right)^{\frac{1}{2}} \left(\int_{y \leq R} \frac{|\partial_y v|^2}{y^2} \right)^{\frac{1}{2}}. \end{aligned} \quad (6.7.7)$$

Similarly, using $\tilde{f}(y) = -\frac{\mathbf{e}_y}{y^3(1-\log y)}$, we get :

$$\begin{aligned} & \int_{\varepsilon \leq y \leq 1} \frac{|v|^2}{y^4(1 - \log y)^2} = \int_{\varepsilon \leq y \leq 1} |v|^2 \nabla \cdot f \\ &= \left[\frac{|v|^2}{1 - \log y} \right]_\varepsilon^1 + 2 \int_{y \leq 1} \frac{v \partial_y v}{y^3(1 - \log y)} \\ &\lesssim |v(1)|^2 + \left(\int_{y \leq R} \frac{|v|^2}{y^4(1 + |\log y|)^2} \right)^{\frac{1}{2}} \left(\int_{y \leq R} \frac{|\partial_y v|^2}{y^2} \right)^{\frac{1}{2}}. \end{aligned} \quad (6.7.8)$$

(6.7.6), (6.7.7) and (6.7.8) now yield (6.7.4). To prove (6.7.5), let $\gamma > 0$, and

$$f(y) = -\frac{\mathbf{e}_y}{y^{\gamma+3}(1 + \log y)^2}$$

so that for $y \geq 1$

$$\nabla \cdot f(y) = \frac{1}{y^{\gamma+4}(1 + \log y)^2} \left[\gamma + \frac{2}{1 + \log y} \right] \geq \frac{\gamma}{y^{\gamma+4}(1 + \log y)^2}.$$

We then integrate by parts to get :

$$\begin{aligned} & \gamma \int_{1 \leq y \leq R} \frac{|v|^2}{y^{4+\gamma}(1 + |\log y|)^2} \leq \int_{1 \leq y \leq R} |v|^2 \nabla \cdot f \\ &\leq - \left[\frac{|v|^2}{y^{3+\gamma}(1 + |\log y|)^2} \right]_1^R + 2 \int_{1 \leq y \leq R} \frac{|v \partial_y v|}{y^{\gamma+3}(1 + \log y)^2} \\ &\leq C \int_{1 \leq y \leq 2} |v|^2 + 2 \left(\int_{y \leq R} \frac{|v|^2}{y^{4+\gamma}(1 + |\log y|)^2} \right)^{\frac{1}{2}} \left(\int_{y \leq R} \frac{|\nabla v|^2}{y^{2+\gamma}(1 + |\log y|)^2} \right)^{\frac{1}{2}}. \end{aligned}$$

and (6.7.5) follows.

Sub-positive estimates for \mathbf{H}

The following lemma highlights the negative part of the operator H . We recall that this operator possesses a unique non-positive direction ψ .

Lemme 6.7.3. Let $u \in H_{rad}^2(\mathbb{R}^4)$, then there exists a constant $C > 0$ such that :

$$(Hu, u) \geq -C(u, \psi)^2. \quad (6.7.9)$$

Proof. Let $u \in H_{rad}^2(\mathbb{R}^4)$. There exists an unique decomposition of u :

$$u = \kappa\psi + v$$

with the orthogonality condition

$$(\psi, v) = 0.$$

By definition, we have

$$\kappa = \frac{(u, \psi)}{(\psi, \psi)}.$$

Moreover, the uniqueness of the negative direction of H gives

$$(Hv, v) \geq 0.$$

Thus,

$$\begin{aligned} (Hu, u) &= \kappa^2(H\psi, \psi) + \kappa[(Hv, \psi) + (v, H\psi)] + (Hv, v) \\ &= \kappa^2(H\psi, \psi) - 2\zeta\kappa(v, \psi) + (Hv, v) \\ &\geq \frac{(H\psi, \psi)}{(\psi, \psi)^2} (u, \psi)^2 \\ &\geq \frac{-\zeta}{(\psi, \psi)} (u, \psi)^2. \end{aligned}$$

Sub-coercivity estimates

In this subsection, we prove sub-coercivity estimates for H and H^2 which are the key to the proof of coercive estimates for these operators under additional orthogonality conditions.

Lemme 6.7.4 (Sub-coercivity estimates for H). Let $u \in H_{rad}^2(\mathbb{R}^4)$, then there exist constants $\delta > 0, C > 0$ such that :

$$(H\varepsilon, \varepsilon) \geq c \int |\nabla \varepsilon|^2 - \frac{1}{c} [(\varepsilon, \Phi_M)^2 + (\varepsilon, \psi)^2], \quad (6.7.10)$$

$$\int |\partial_{yy} u|^2 + \int \frac{|\partial_y u|^2}{y^2} + \int \frac{u^2}{y^4(1+|\log y|)^2} - C \left[\int \frac{|\partial_y u|^2}{1+y^4} + \int \frac{u^2}{1+y^8} \right] \lesssim \int (Hu)^2. \quad (6.7.11)$$

Proof. (6.7.11) is a direct consequence of the inequalities (6.7.3), (6.7.4) and the following decomposition :

$$\begin{aligned} \int (Hu)^2 &= \int (\Delta u + Vu)^2 = \int (\Delta u)^2 - 2 \int V(\partial_y u)^2 + \int (\Delta V + V^2)u^2 \\ &\gtrsim \left[\int (\Delta u)^2 + \int \frac{u^2}{1+y^6} \right] - C \left[\int \frac{|\partial_y u|^2}{1+y^4} + \int \frac{u^2}{1+y^8} \right] \end{aligned}$$

where we used that

$$V(y) = \frac{192}{1+y^4} \left[1 + O\left(\frac{1}{1+y^2}\right) \right] \quad \text{as } y \rightarrow +\infty.$$

Lemme 6.7.5 (Weighted sub-coercivity for H). *Let $u \in H_{rad}^4(\mathbb{R}^4)$, then there exists a constant C such that :*

$$\begin{aligned} & \int \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ & + \int \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} \\ & - C \left[\int \frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2} \right] \\ & \lesssim \int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2}. \end{aligned} \quad (6.7.12)$$

Remarque 6.7.6. Using (6.7.1), (6.7.4), and (6.7.5), $u \in H_{rad}^4(\mathbb{R}^4)$ yields

$$\begin{aligned} & \int \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ & + \int \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} \\ & + \int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2} < \infty. \end{aligned}$$

Proof of Lemma 6.7.5 : Let $\chi(y)$ be a smooth cut-off function with support in $y \geq 1$ and equal to 1 for $y \geq 2$.

$$\begin{aligned} & \int \chi \frac{|Hu|^2}{y^4(1+|\log y|)^2} = \int \chi \frac{[-\partial_y(y^3 \partial_y u) - y^3 V u]^2}{y^{10}(1+|\log y|)^2} \\ & = \int \chi \frac{|\partial_y(y^3 \partial_y u)|^2}{y^{10}(1+|\log y|)^2} + 2 \int \chi \frac{\partial_y(y^3 \partial_y u) V u}{y^7(1+|\log y|)^2} + \int \chi \frac{V^2 u^2}{y^4(1+|\log y|)^2} \\ & = \int \chi \frac{|\partial_y(y^3 \partial_y u)|^2}{y^{10}(1+|\log y|)^2} - 2 \int \chi \frac{V(\partial_y u)^2}{y^4(1+|\log y|)^2} + \int \chi \frac{V^2 u^2}{y^4(1+|\log y|)^2} \\ & + \int |u|^2 \Delta \left(\chi \frac{V}{y^4(1+|\log y|)^2} \right). \end{aligned}$$

We now observe that for $k \geq 0$

$$|\partial_y^k V(y)| \lesssim \frac{1}{1+y^{4+k}}$$

and thus,

$$\begin{aligned} & \left| -2 \int \chi \frac{V(\partial_y u)^2}{y^4(1+|\log y|)^2} + \int \chi \frac{V^2 u^2}{y^4(1+|\log y|)^2} + \int \chi u^2 \Delta \left(\frac{V}{y^4(1+|\log y|)^2} \right) \right| \\ & \lesssim \int \frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \int \chi \frac{|Hu|^2}{y^4(1+|\log y|)^2} &\gtrsim \int \chi \frac{\partial_y (y^3 \partial_y u)^2}{y^{10}(1+|\log y|)^2} \\ &- C \left[\int \frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2} \right]. \end{aligned}$$

We may apply twice the Hardy inequality (6.7.5) with $\gamma = 8$ and $\gamma = 4$ and get for a sufficiently large universal constant R :

$$\begin{aligned} \int \chi \frac{\partial_y (y^3 \partial_y u)^2}{y^{10}(1+|\log y|)^2} &\gtrsim \int_{y \geq R} \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} - C \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2} \\ &\gtrsim \int_{y \geq R} \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} \\ &- C \left[\int \frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2} \right]. \end{aligned}$$

and finally,

$$\begin{aligned} &\int_{y \geq 2} \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int_{y \geq 2} \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int_{y \geq 2} \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ &- C \left[\int \frac{|u|^2}{y^2(1+y^6)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2} \right] \\ &\lesssim \int \chi \frac{|Hu|^2}{y^4(1+|\log y|)^2}. \end{aligned} \tag{6.7.13}$$

Now, the control of the third derivative for $y \leq 1$ follows from :

$$\begin{aligned} \int \chi \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} &\geq \int \chi \frac{|\partial_y(\Delta u + Vu)|^2}{y^2(1+|\log y|)^2} \gtrsim \int \chi \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} \\ &- C \left[\int \chi \frac{|u|^2}{(1+y^{12})(1+|\log y|)^2} + \int \chi \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \chi \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \right] \end{aligned} \tag{6.7.14}$$

and the fourth derivative from :

$$\begin{aligned} \int \chi \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2} &\geq \int \chi \frac{|\partial_{yy}(\Delta u + Vu)|^2}{(1+|\log y|)^2} \\ &\gtrsim \int \chi \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} - C \left[\int \chi \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \chi \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \right] \end{aligned} \tag{6.7.15}$$

$$- C \left[\int \chi \frac{|u|^2}{(1+y^{12})(1+|\log y|)^2} + \int \chi \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} \right]. \tag{6.7.16}$$

(6.7.13),(6.7.14) and (6.7.16) yield (6.7.12), away from the origin. Let us study this control near the origin. Let $\zeta = (1-\chi)^{\frac{1}{2}}$. With Lemma 6.7.4, we have that :

$$\begin{aligned} &\int \zeta^2 \frac{|Hu|^2}{y^4(1+|\log y|^2)} \gtrsim \int \zeta^2 |Hu|^2 \gtrsim |H\zeta u|^2 - C \int_{1 \leq y \leq 2} (|\partial_y u|^2 + |u|^2) \\ &\gtrsim \int \frac{|\zeta u|^2}{y^4(1+|\log y|^2)} - C \int \frac{|\zeta u|^2}{y^2} \gtrsim \int_{y \leq 1} \frac{|u|^2}{y^4(1+|\log y|^2)} - C \int_{y \leq 2} \frac{|u|^2}{y^2}. \end{aligned}$$

Now, by definition, we have :

$$Hu = -\frac{1}{y^3} \frac{\partial}{\partial_y} (y^3 \partial_y u) - Vu.$$

Hence

$$\partial_y u = -\frac{1}{y^3} \int_{\tau \leq y} (Vu + Hu). \quad (6.7.17)$$

We then estimate from Cauchy-Schwarz and Fubini :

$$\begin{aligned} \int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} &= \int_{y \leq 1} \frac{|\partial_y u|^2}{y^{12}(1 + |\log y|^2)} \left(\int_0^y Vu + Hu \right)^2 \\ &\lesssim \int_{y \leq 1} \frac{|\partial_y u|^2}{y^2(1 + |\log y|^2)} \left(\int_0^y \frac{|Vu(\tau)|^2 + |Hu(\tau)|^2}{\tau^3} \right) \\ &\lesssim \int_{\tau \leq 1} \frac{|Vu(\tau)|^2 + |Hu(\tau)|^2}{\tau^3} \left(\int_{\tau \leq y \leq 1} \frac{1}{y^2(1 + |\log y|^2)} \right) \\ &\lesssim \int_{y \leq 1} \frac{|Hu|^2}{y^4(1 + |\log y|^2)^2}. \end{aligned} \quad (6.7.18)$$

Finally, for the control of the other derivatives near the origin :

$$\partial_{yy} u = -Hu - 3 \frac{\partial_y u}{y} - Vu.$$

Thus,

$$\int_{y \leq 1} \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|^2)} \lesssim \int_{y \leq 1} \frac{|Hu|^2}{y^4(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)}, \quad (6.7.19)$$

$$\begin{aligned} \int_{y \leq 1} \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|^2)} &\lesssim \int_{y \leq 1} \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|^2)} \\ &+ \int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)}, \end{aligned} \quad (6.7.20)$$

$$\begin{aligned} \int_{y \leq 1} \frac{|\partial_y^4 u|^2}{(1 + |\log y|^2)} &\lesssim \int_{y \leq 1} \frac{|\partial_{yy} Hu|^2}{(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_y^3 u|^2}{y^4(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|^2)} \\ &+ \int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)}. \end{aligned} \quad (6.7.21)$$

This concludes the proof.

We now combine the results of Lemma 6.7.4 and Lemma 6.7.5

Lemme 6.7.7 (Sub-coercivity for H^2). *Let $u \in H_{rad}^4(\mathbb{R}^4)$. Then,*

$$\begin{aligned} \int |H^2 u|^2 &\gtrsim \int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \left| \frac{\partial_{yy} Hu}{(1+|\log y|)^2} \right|^2 \\ &+ \int \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ &+ \int \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} \int \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} - C \int \frac{|u|^2}{y^2(1+y^8)(1+|\log y|)^2} \\ &- C \left[\int \frac{|\partial_y u|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{1+y^4} + \int \frac{Hu^2}{1+y^8} \right]. \end{aligned} \quad (6.7.22)$$

Coercivity of H^2

We are now in position to derive the fundamental coercivity property of H^2 at the heart of our analysis.

Lemme 6.7.8 (Coercivity of H^2). *Let $M \geq 1$ be a large enough universal constant. Let Φ_M be given by (6.3.2). Then there exists a universal constant $C(M) > 0$ such that for all $u \in H_{rad}^4(\mathbb{R}^4)$ satisfying the orthogonality conditions :*

$$(u, \Phi_M) = 0, \quad (Hu, \Phi_M) = 0$$

there holds :

$$\begin{aligned} &\int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu|^2}{(1+|\log y|)^2} \\ &+ \int \frac{|u|^2}{y^4(1+y^4)(1+|\log y|)^2} \int \frac{|\partial_y u|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1+|\log y|)^2} \\ &+ \int \frac{|\partial_y^3 u|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u|^2}{(1+|\log y|)^2} \leq C(M) \int |H^2(u)|^2. \end{aligned} \quad (6.7.23)$$

Proof. We argue by contradiction. Let $M > 0$ be fixed and consider a normalized sequence u_n

$$\begin{aligned} &\int \frac{|Hu_n|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu_n|^2}{(1+|\log y|)^2} \\ &+ \int \frac{|u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{y^4(1+|\log y|)^2} \\ &+ \int \frac{|\partial_y^3 u_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u_n|^2}{(1+|\log y|)^2} = 1 \end{aligned} \quad (6.7.24)$$

satisfying the orthogonality conditions

$$(u_n, \Phi_M) = 0, \quad (Hu_n, \Phi_M) = 0$$

and

$$\int |H^2(u_n)|^2 \leq \frac{1}{n}. \quad (6.7.25)$$

The normalization condition implies that the sequence u_n is uniformly bounded in H_{loc}^4 . As a consequence, we can assume that u_n weakly converges in H_{loc}^4 to u_∞ . Moreover, u_∞ satisfies the equation

$$H^2 u_\infty = 0 \quad \text{for } r > 0.$$

Integrating this ODE leads to

$$Hu_\infty = \alpha \Lambda Q + \beta \Gamma \quad \text{for } r > 0.$$

Using the condition $u_\infty \in H_{loc}^4$, we can determine that $\beta = 0$. Hence, the function u_∞ can be written in the form

$$u_\infty = -\alpha T_1 + \gamma \Lambda Q + \delta \Gamma.$$

The condition $u_\infty \in H_{loc}^4$ yields that $\delta = 0$. Passing through the limit in the orthogonality conditions, using that u_n converges to u_∞ weakly in H_{loc}^4 , we conclude that u_∞ satisfies

$$(u_\infty, \Phi_M) = 0, \quad (Hu_\infty, \Phi_M) = 0.$$

We may therefore determine the constants α and γ using (6.3.2), (6.3.5) which yield $\alpha = \gamma = 0$ and thus $u_\infty = 0$.

The sub-coercivity bound (6.7.22) together with (6.7.25) ensures :

$$\begin{aligned} \frac{1}{n} &\gtrsim \int |H^2(u_n)|^2 \gtrsim \int \frac{|Hu_n|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_{yy} Hu_n|^2}{(1+|\log y|)^2} \\ &+ \int \frac{|u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^6(1+|\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{y^4(1+|\log y|)^2} \\ &+ \int \frac{|\partial_y^3 u_n|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^4 u_n|^2}{(1+|\log y|)^2} - C \left[\int \frac{|\partial_y Hu_n|^2}{1+y^4} + \int \frac{Hu_n^2}{1+y^8} \right] \\ &- C \left[\int \frac{|u_n|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} \right]. \end{aligned}$$

Coupling this with the normalization condition we obtain that

$$\int \frac{|\partial_y Hu_n|^2}{1+y^4} + \int \frac{Hu_n^2}{1+y^8} + \int \frac{|u_n|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^4(1+y^4)(1+|\log y|)^2} \geq c$$

for some positive constant $c > 0$. Since u_n weakly converges to u_∞ in H_{loc}^4 on any compact subinterval of $y \in (0, \infty)$, we can pass to the limit to conclude

$$\int \frac{|\partial_y Hu_\infty|^2}{1+y^4} + \int \frac{Hu_\infty^2}{1+y^8} + \int \frac{|u_\infty|^2}{y^2(1+y^8)(1+|\log y|)^2} + \int \frac{|\partial_y u_\infty|^2}{y^4(1+y^4)(1+|\log y|)^2} \geq c.$$

This contradicts the established identity $u_\infty = 0$ and concludes the proof of Lemma 6.7.7.

Coercivity of H

We complement the coercivity property of the operator H^2 , established in the previous section, by the corresponding statement for the operator H , which follows from standard compactness argument. A complete proof is given in [41] with a slightly different orthogonality condition but the proof is the same and therefore left to the reader.

Lemme 6.7.9 (Coercivity of H). *Let $M \geq 1$ be fixed. Then there exists $c(M) > 0$ such that the following holds true. Let $u \in H_{rad}^2$ with*

$$(u, \Phi_M) = 0$$

then

$$\int |\partial_{yy} u|^2 + \int \frac{|\partial_y u|^2}{y^2} + \int \frac{u^2}{y^4(1 + |\log y|)^2} \leq c(M) \int |Hu|^2. \quad (6.7.26)$$

6.7.2 Interpolation estimates

In this appendix, we prove interpolation estimates for ε in the bootstrap regimes which are used all along the proof of Proposition 6.4.2. We recall the norm \mathcal{E}_1 , \mathcal{E}_2 and \mathcal{E}_4 , introduced in (6.3.13), together with their bootstrap bounds :

$$\begin{aligned} \mathcal{E}_1 &= \int |\nabla \varepsilon|^2 \leq K\delta(b^*), \\ \mathcal{E}_2 &= \int |H\varepsilon|^2 \leq Kb^2(t)|\log b(t)|^5, \\ \mathcal{E}_4 &= \int |H^2\varepsilon|^2 \leq K \frac{b^4(t)}{|\log b(t)|^2}. \end{aligned}$$

Lemme 6.7.10 (Interpolation estimates). *There hold -with constants a priori depending on M - :*

$$\begin{aligned} &\int \frac{|H(\varepsilon)|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y H(\varepsilon)|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_{yy} H(\varepsilon)|^2}{(1 + |\log y|^2)} \\ &+ \int \frac{|\varepsilon|^2}{y^4(1 + y^4)(1 + |\log y|^2)} + \int \frac{|\partial_y^i \varepsilon|^2}{y^{8-2i}(1 + |\log y|^2)} \lesssim \mathcal{E}_4, \quad 1 \leq i \leq 4, \end{aligned} \quad (6.7.27)$$

$$\int \frac{|\varepsilon|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y^i \varepsilon|^2}{y^{4-2i}} \lesssim \mathcal{E}_2, \quad 1 \leq i \leq 2, \quad (6.7.28)$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{8-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \lesssim b^4 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \quad (6.7.29)$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{6-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \lesssim b^3 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \quad (6.7.30)$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{4-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \lesssim b^2 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 1, \quad (6.7.31)$$

$$\left\| \varepsilon(1 + |\log y|^C) \right\|_{L_{y \geq 1}^\infty}^2 \lesssim b^2 |\log b|^{C_1(C)}, \quad (6.7.32)$$

$$\|\varepsilon\|_{L_{y \leq 1}^\infty}^2 + \|\partial_y \varepsilon\|_{L_{y \leq 1}^\infty}^2 + \|y \partial_{yy} \varepsilon\|_{L_{y \leq 1}^\infty}^2 \lesssim \frac{b^4}{|\log b|^2}, \quad (6.7.33)$$

$$\|y \partial_y \varepsilon\|_{L^\infty}^2 \lesssim b^2 |\log b|^5, \quad (6.7.34)$$

$$\left\| \frac{\varepsilon}{1+y} \right\|_{L^\infty}^2 + \|\partial_y \varepsilon\|_{L^\infty}^2 \lesssim b^3 |\log b|^C, \quad (6.7.35)$$

$$\left\| \frac{\varepsilon}{1+y^2} \right\|_{L^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{1+y} \right\|_{L^\infty}^2 + \|\partial_{yy} \varepsilon\|_{L_{y \geq 1}^\infty}^2 \lesssim b^4 |\log b|^C. \quad (6.7.36)$$

Proof. (6.7.27) and (6.7.28) are respectively direct consequences of the Lemma 6.7.8 and Lemma 6.7.9 and definition of the norms \mathcal{E}_2 and \mathcal{E}_4 .

To prove (6.7.29), we split the integral at $y = B_0^{20}$.

$$\begin{aligned} & \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{8-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \\ &= \int_{1 \leq y \leq B_0^{20}} \frac{1 + |\log y|^C}{y^{8-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 + \int_{y \geq B_0^{20}} \frac{1 + |\log y|^C}{y^{8-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \\ &\lesssim |\log b|^{C+2} \mathcal{E}_4 \\ &+ \left\| \frac{1 + |\log y|^C}{y^2} \right\|_{L_{y \geq B_0^{20}}^\infty} \left(\int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2}{y^{8-2i}(1 + |\log y|^2)} \right)^{\frac{1}{2}} \left(\int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2}{y^{4-2i}(1 + |\log y|^2)} \right)^{\frac{1}{2}}. \end{aligned}$$

The bounds (6.7.27) and (6.7.28) conclude the proof. The bound (6.7.30) is a direct consequence of the last bound and the bootstrap bound for \mathcal{E}_2 . Indeed :

$$\begin{aligned} & \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{6-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \\ &\lesssim \left(\int_{y \geq 1} \frac{1 + |\log y|^{2C}}{y^{8-2i}(1 + |\log y|^2)} |\partial_y^i \varepsilon|^2 \right)^{\frac{1}{2}} \left(\int \frac{|\partial_y^i \varepsilon|^2}{y^{4-2i}(1 + |\log y|^2)} \right)^{\frac{1}{2}} \end{aligned}$$

and (6.7.30) follows.

The proof of (6.7.31) is the same as that of (6.7.29) using the energy \mathcal{E}_1 and \mathcal{E}_2 . (6.7.32) comes from (6.7.2) and (6.7.31). Indeed :

$$\begin{aligned} & \left\| \varepsilon(1 + |\log y|^C) \right\|_{L_{y \geq 1}^\infty}^2 \\ &\lesssim \int_{y \geq 1} \frac{|\partial_y \varepsilon|^2(1 + |\log y|^C)}{y^2} + \int_{y \geq 1} \frac{\varepsilon^2(1 + |\log y|^C)}{y^4} + \int_{\frac{1}{2} \leq y \leq 1} \frac{\varepsilon^2(1 + |\log y|^C)}{y^2} \\ &\lesssim b^2 |\log b|^C. \end{aligned}$$

Let us prove (6.7.33). Let $a \in [1; 2]$ such that :

$$|\varepsilon(a)| \lesssim \int_1^2 \frac{|\varepsilon(y)|}{y^3} \lesssim \sqrt{\mathcal{E}_4}. \quad (6.7.37)$$

Then using Cauchy-Schwarz

$$\forall y \in [0; 1], |\varepsilon(y)| \lesssim |\varepsilon(a)| + \left| \int_a^y \frac{|\partial_y \varepsilon(y)|}{y^3} \right| \lesssim \sqrt{\mathcal{E}_4}.$$

In the same way, let $a \in [1; 2]$ such that :

$$|\partial_y \varepsilon(a)| \lesssim \int_1^2 \frac{|\partial_y \varepsilon(y)|}{y^3} \lesssim \sqrt{\mathcal{E}_4} \quad (6.7.38)$$

and

$$\forall y \in [0; 1], |\partial_y \varepsilon(y)| \lesssim |\partial_y \varepsilon(a)| + \left| \int_a^y \frac{|\partial_{yy} \varepsilon(y)|}{y^3} \right| \lesssim \sqrt{\mathcal{E}_4}.$$

Finally, let $a \in [1; 2]$ such that :

$$|\partial_{yy} \varepsilon(a)| \lesssim \int_1^2 \frac{|\partial_{yy} \varepsilon(y)|}{y^2} \lesssim \sqrt{\mathcal{E}_4} \quad (6.7.39)$$

and

$$\forall y \in [0; 1], |y \partial_{yy} \varepsilon(y)| \lesssim a |\partial_{yy} \varepsilon(a)| + \left| \int_a^y \left(\frac{|\partial_{yy} \varepsilon(y)|}{y^3} + \frac{|\partial_y^3 \varepsilon(y)|}{y^2} \right) \right| \lesssim \sqrt{\mathcal{E}_4}.$$

The bound (6.7.34) is a direct consequence of Lemma 6.7.1 and (6.7.28).

We now prove (6.7.35) using (6.7.2), (6.7.28) and (6.7.29) :

$$\begin{aligned} & \left\| \frac{\varepsilon}{y} \right\|_{L_{y \geq 1}^\infty}^2 + \|\partial_y \varepsilon\|_{L_{y \geq 1}^\infty}^2 \lesssim \int_{y \geq 1} \left(\frac{\varepsilon^2}{y^6} + \frac{|\partial_y \varepsilon|^2}{y^4} + \frac{|\partial_{yy} \varepsilon|^2}{y^2} \right) + \int_{\frac{1}{2} \leq y \leq 1} \left(\frac{|\varepsilon|^2}{y^4} + \frac{|\partial_y \varepsilon|^2}{y^2} \right) \\ & \lesssim \sum_{i=0}^2 \left(\int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2}{y^{4-2i}(1+|\log y|^2)} \right)^{\frac{1}{2}} \left(\int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2(1+|\log y|^2)}{y^{8-2i}} \right)^{\frac{1}{2}} + \mathcal{E}_4 \lesssim b^3 |\log b|^C. \end{aligned}$$

Similarly, we prove (6.7.36) :

$$\begin{aligned} & \left\| \frac{\varepsilon}{y^2} \right\|_{L_{y \geq 1}^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{y} \right\|_{L_{y \geq 1}^\infty}^2 + \|\partial_{yy} \varepsilon\|_{L_{y \geq 1}^\infty}^2 \\ & \lesssim \int_{y \geq 1} \left(\frac{\varepsilon^2}{y^8} + \frac{|\partial_y \varepsilon|^2}{y^6} + \frac{|\partial_{yy} \varepsilon|^2}{y^4} + \frac{|\partial_y^3 \varepsilon|^2}{y^2} \right) + \int_{\frac{1}{2} \leq y \leq 1} \left(\frac{|\varepsilon|^2}{y^6} + \frac{|\partial_y \varepsilon|^2}{y^4} + \frac{|\partial_{yy} \varepsilon|^2}{y^2} \right) \\ & \lesssim b^4 |\log b|^C. \end{aligned}$$

6.7.3 Localization of the profile

In this appendix, we are going to give the important steps of the proofs of Propositions 6.2.3 and 6.2.4.

To begin, remark that the definition (6.2.59) of $\tilde{\Psi}_b$ in the localization near B_1 gives two types of error. One is the result of the only localization. The other is the effect of the time derivative. Indeed, we can rewrite $\tilde{\Psi}_b$ as follows :

$$\tilde{\Psi}_b = \Psi_b^{(1)} + \tilde{R} \quad (6.7.40)$$

where

$$\Psi_b^{(1)} = -b^2(\tilde{T}_1 + b\tilde{T}_2) - \Delta\tilde{Q}_b + b\Lambda\tilde{Q}_b - (\tilde{Q}_b)^3 \quad (6.7.41)$$

and

$$\tilde{R} = b_s \left(3b^2\tilde{T}_3 + b \frac{\partial\tilde{T}_1}{\partial b} + b^2 \frac{\partial\tilde{T}_2}{\partial b} + b^3 \frac{\partial\tilde{T}_3}{\partial b} \right). \quad (6.7.42)$$

We compute the action of localization which produces an error localized in $[B_1, 2B_1]$ up to the term $(1 - \chi_{B_1})\Lambda Q$:

$$\begin{aligned} \Psi_b^{(1)} &= \chi_{B_1}\Psi_b + b(1 - \chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha \\ &+ (Q + \chi_{B_1}\alpha)^3 - Q^3 - \chi_{B_1}((Q + \alpha)^3 - Q^3). \end{aligned} \quad (6.7.43)$$

We estimate from the rough bounds of $(T_i)_{1 \leq i \leq 3}$ and the choice of B_1 :

$$\forall y \leq 2B_1, \quad |\alpha(y)| \lesssim by \left(\frac{|\log y|}{y} + \frac{by}{|\log b|} \right) \lesssim b|\log y|$$

and thus :

$$\begin{aligned} |b(1 - \chi_{B_1})\Lambda Q + b\Lambda\chi_{B_1}\alpha - \alpha\Delta\chi_{B_1} - 2\partial_y\chi_{B_1}\partial_y\alpha| &\lesssim \frac{b}{y^2} \mathbf{1}_{y \geq B_1} + b^2 \log y \mathbf{1}_{B_1 \leq y \leq 2B_1}, \\ |(Q + \chi_{B_1}\alpha)^3 - Q^3 - \chi_{B_1}((Q + \alpha)^3 - Q^3)| &\lesssim \frac{|\alpha(y)|}{y^3} |\mathbf{1}_{B_1 \leq y \leq 2B_1}| \\ &\lesssim \frac{b \log y}{y^2} \mathbf{1}_{B_1 \leq y \leq 2B_1} \end{aligned}$$

Hence, $\Psi_b^{(1)}$ verifies the bounds (6.2.61), (6.2.62), (6.2.63). For the control of time derivatives, we have to use that :

$$\frac{\partial c_b}{\partial b} = O\left(\frac{1}{b|\log b|^2}\right), \quad \frac{\partial d_b}{\partial b} = O\left(\frac{1}{b^2|\log b|^2}\right), \quad (6.7.44)$$

which are consequences of their definition, and that :

$$\frac{\partial \Sigma_b}{\partial b} = O\left(\frac{1}{b|\log b|} \mathbf{1}_{y \leq \frac{B_0}{2}} + \frac{1}{y^2 b^2 |\log b|^2} \mathbf{1}_{\frac{B_0}{2} \leq y \leq 6B_0}\right). \quad (6.7.45)$$

Using (6.7.44), (6.7.45) together the explicit formula of $(T_i)_{1 \leq i \leq 3}$ yields (6.2.61), (6.2.62), (6.2.63) without difficulty. Only an estimate is more delicate, and requests more cancellation. Indeed, we must use that

$$H^2 \left(\frac{\partial \tilde{T}_2}{\partial b} \right) = H \left(\frac{\partial \Sigma_2}{\partial b} \right) = H \left(\frac{\partial \Sigma_b}{\partial b} \right),$$

and similarly

$$H^2 \left(\frac{\partial \tilde{T}_3}{\partial b} \right) = H \left(\frac{\partial \Sigma_3}{\partial b} \right) = \Lambda \left(\frac{\partial \Sigma_b}{\partial b} \right).$$

The proof of (6.2.64) is left to the reader. For the proof of Proposition 6.2.4, we just remark that, by definition :

$$\zeta_b = (\chi_{B_1} - \chi_{B_0})(bT_1 + b^2T_2 + b^3T_3)$$

This proof is afterwards the same as the previous one.

Chapitre 7

Article [89] sur le modèle de Patlak-Keller-Segel

7.1 Introduction

7.1.1 Setting of the problem

We consider the two dimensional Patlak-Keller-Segel problem :

$$(PKS) \quad \begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla \phi_u), \\ \phi_u = \frac{1}{2\pi} \log|x| * u \\ u|_{t=0} = u_0 > 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad (7.1.1)$$

This degenerate nonlocal diffusion equation is a canonical limit of kinetic models of particles evolving through a nonlocal attractive force, and is one of the model which arises in the description of colonies of bacteria, [49], [80]. This system has attracted a considerable attention for the past twenty years in the mathematical community, in particular in connection with the local regularity of weak solutions, the qualitative behavior of solutions and the possibility of finite time blow up for large enough data corresponding to the aggregation of bacteria. We refer to [6] for an extensive introduction to the literature on these subjects. Our point of view in this paper is to replace the problem within the sets of *critical blow up problems* which for both parabolic and dispersive problems have also attracted a considerable attention for the past ten years.

The existence of unique local smooth solutions in some suitable Sobolev sense can be obtained from standard fixed point arguments as in [78]. The corresponding non negative strong solution satisfies the conservation of mass

$$\int_{\mathbb{R}^2} u(t, x) dx = \int_{\mathbb{R}^2} u(0, x) dx \quad (7.1.2)$$

and the flow dissipates the logarithmically degenerate entropy :

$$\mathcal{F}(u) = \int u \log u + \frac{1}{2} \int u \phi_u \leq \mathcal{F}(u_0). \quad (7.1.3)$$

The scaling symmetry

$$u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x)$$

leaves the L^1 norm unchanged

$$\int_{\mathbb{R}^2} u_\lambda(t, x) dx = \int_{\mathbb{R}^2} u(\lambda^2 t, x) dx$$

and hence the problem is L^1 critical. Note from (7.1.3) that the problem is also *almost energy critical* and therefore particularly degenerate.

7.1.2 Previous results

A fundamental role is played for the analysis by the explicit *ground state* stationary solution

$$Q(x) = \frac{8}{(1 + |x|^2)^2}. \quad (7.1.4)$$

From [4], [12], Q is up to symmetry the unique minimizer of the logarithmic Hardy-Littlewood-Sobolev inequality : $\forall u \geq 0$ with $\int u = M$,

$$\int u \log u + \frac{4\pi}{M} \int \phi_u u \geq M [1 + \log \pi - \log M]. \quad (7.1.5)$$

For smooth well localized data with small mass $\int u_0 < \int Q$, the flow is global and zero is the universal local attractor, see [22], [8], [83], and in this sense Q is the first nonlinear object. For data above the ground state $\int u_0 > \int Q$, the virial type identity

$$\frac{d}{dt} \int |x|^2 u(t, x) dx = 4 \left(1 - \frac{\int u}{\int Q} \right) \int u$$

implies that all smooth well localized data *blow up* in finite time. Note however that as usual, this argument is very unstable by perturbation of the equation, and provides almost no insight into the structure of the singularity formation.

Substantial progress have been made in the critical case $\int u = \int Q$. In [7], finite variance initial data $\int |x|^2 u_0 < +\infty$ with minimal mass are shown to grow up in *infinite time*. The argument is again by contradiction and does not give the associated blow up rate. More dynamical results for domains are obtained in [48]. In [6], it is shown that infinite variance initial data $\int |x|^2 u_0 = +\infty$ exhibit a completely different behavior and generate a global flow asymptotically attracted by the soliton (7.1.4), see also [13] for quantitative convergence rates. The proof involves the use of an additional Lyapounov functional at the minimal mass level, and the importation of tools from optimal transportation thanks to the gradient flow structure of the problem.

All the above subcritical mass results heavily rely on the fact that for $\int u_0 \leq \int Q$, the dissipated entropy (7.1.3) coupled with the variational characterization of Q through the logarithmic HLS (7.1.5) imply a priori uniform bounds on the solution. This structure is completely lost for $\int u_0 > \int Q$, and the description of the flow for mass super critical data is very poorly understood. In the pioneering work [38], Herrero and Velázquez use formal matching asymptotics, the radial reduction of the problem and ODE techniques to produce the first example of blow up solution.

7.1.3 Connection with critical problems

The L^1 critical structure of the (PKS) system is canonical from the point of view of critical problems, and a number of examples extracted both from the parabolic and dispersive worlds

have recently attracted a considerable attention and led to a new approach for the construction of blow up solutions. Examples of such critical flows are the mass critical Non Linear Schrödinger equations of nonlinear optics

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u + u|u|^2 = 0, \\ u|_{t=0} = u_0, \end{cases}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u \in \mathbb{C}, \quad (7.1.6)$$

the geometric energy critical parabolic heat flow to the 2-sphere of crystal physics

$$(HF) \quad \begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u, \\ u|_{t=0} = u_0, \end{cases}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u \in \mathbb{S}^2$$

and its dispersive Schrödinger map analogue in ferromagnetism :

$$(SM) \quad \begin{cases} u \wedge \partial_t u = \Delta u + |\nabla u|^2 u, \\ u|_{t=0} = u_0, \end{cases}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u \in \mathbb{S}^2.$$

For all these problems, a robust approach for the construction of blow up solutions and the study of their possible stability has been developed in the past ten years, see [69], [67], [68], [71], [70] for the mass critical NLS, [88], [72] for the wave and Schrödinger maps, and [91] for the harmonic heat flow. The strategy proceeds in two steps : the construction of suitable approximate solutions through the derivation of the leading order ODE's driving the trajectory of the solution on the modulated manifold of ground states; the control of the exact flow near these approximate profiles through the derivation of suitable Lyapounov functionals and a robust energy method. The first step avoids the common use of matching asymptotics which is most of the time delicate both to implement and to make rigorous, see for example [55] for (NLS), [99] for the heat flow, [21] for (PKS). The second step is a pure energy method which therefore applies both to dispersive problems and parabolic systems, and makes in this last case no use of the maximum principle commonly used for scalar parabolic problems. The sharp knowledge of the spectral structure of the linearized operator, which is typically a delicate problem for (PKS), see [21], is replaced by canonical spectral gap estimates as initiated in [93], [88].

7.1.4 Statement of the result

Our main claim in this paper is that despite the nonlocal structure of the problem and in particular of the linearized operator close to Q , and the almost energy critical degeneracy of the problem¹, the above route map can be implemented for (PKS). We address the radial case only for the sake of simplicity, but the full non radial problem can be analyzed in principle along similar lines.

In order to make our statement precise, let us consider the following function spaces. Let the weighted L^2 space

$$\|\varepsilon\|_{L_Q^2} = \left(\int \frac{\varepsilon^2}{Q} \right)^{\frac{1}{2}} \quad (7.1.7)$$

1. which is reflected by the weakness of the a priori information (7.1.3) for mass super critical initial data.

and the weighted H^2 space :

$$\|\varepsilon\|_{H_Q^2} = \|\Delta\varepsilon\|_{L_Q^2} + \left\| \frac{\nabla\varepsilon}{1+|x|} \right\|_{L_Q^2} + \|\varepsilon\|_{L^2}. \quad (7.1.8)$$

We introduce the energy norm

$$\|\varepsilon\|_{\mathcal{E}} = \|\varepsilon\|_{H_Q^2} + \|\varepsilon\|_{L^1}. \quad (7.1.9)$$

The main result of this paper is the complete description of a *stable* chemotactic blow up with radial data arbitrarily close to the ground state in the L^1 critical topology.

Théorème 7.1.1 (Stable chemotactic blow up). *There exists a set of initial data of the form*

$$u_0 = Q + \varepsilon_0 \in \mathcal{E}, \quad u_0 > 0, \quad \|\varepsilon_0\|_{\mathcal{E}} \ll 1$$

such that the corresponding solution $u \in C([0, T), \mathcal{E})$ to (7.1.1) satisfies the following :

(i) Small super critical mass :

$$8\pi < \int u_0 < 8\pi + \alpha^*$$

for some $0 < \alpha^ \ll 1$ which can be chosen arbitrarily small;*

(ii) Blow up : *the solution blows up in finite time $0 < T < +\infty$;*

(iii) Universality of the blow up bubble : *the solution admits for all times $t \in [0, T)$ a decomposition*

$$u(t, x) = \frac{1}{\lambda^2(t)}(Q + \varepsilon)\left(t, \frac{x}{\lambda(t)}\right)$$

with

$$\|\varepsilon(t)\|_{H_Q^2} \rightarrow 0 \quad \text{as } t \rightarrow T \quad (7.1.10)$$

and the universal blow up speed :

$$\lambda(t) = \sqrt{T-t} e^{-\sqrt{\frac{|\log(T-t)|}{2}}+O(1)} \quad \text{as } t \rightarrow T. \quad (7.1.11)$$

(iv) Stability : *the above blow up dynamics is stable by small perturbation of the data in \mathcal{E} :*

$$v_0 > 0, \quad \|v_0 - u_0\|_{\mathcal{E}} < \epsilon(u_0).$$

Comments on the result

1. *On the stability statement* : Formal arguments to prove the stability of chemotactic blow up are presented in [100], and in fact Theorem 7.1.1 answers one of the problems mentioned in [101]. It is likely that there are many other blow regimes, possibly stable or unstable depending on the tail of the initial data, as is expected for the heat flow [99], see also [37], or known for the super critical heat equation [76], see also [5] for a further illustration of the role of the topology for the long time dynamics. Note that a complete description of the flow near the ground state is obtained for the first time for a critical problem in [62], [63], [64] for the critical (KdV) problem, and this includes pathological blow up rates depending on the structure of the data. In this sense, Theorem 7.1.1 is a first step toward the description of the flow near Q which even in the

radial setting is still a challenging problem.

2. On the non locality : It is known that in the radial setting, the nonlocal (PKS) can be turned to a nonlinear *local* problem for the partial mass, see for example [38]. We shall not use this structure at all, and therefore our approach can in principle be extended to the non radial setting. The non locality of the chemotactic problem induces a nonlinear linearized operator \mathcal{L} close to Q , which due to the slow decay of the soliton makes the use of spectral techniques delicate, see [21] for an attempt in this direction. Moreover, and this is for example a *major* difference with the mass critical (NLS) (7.1.6), the entropy control (7.1.3) is very weak and yields almost no information on the flow near Q . However, following the approach in [88], simple spectral gap estimates can be obtained on \mathcal{L} and its iterates as a consequence of the variational characterization of Q which lead to the control of derivatives of u in suitable weighted L^2 norms. In other words, for the linearized flow close to Q , we control the solution through the control of the natural linearized entropy involving the derivatives of u , and the associated dissipated quantities. The control of derivatives yields through Hardy inequalities the local control of the solution on the soliton core which is the heart of the control of the speed of concentration. We also obtain sharp convergence rates of the solution towards the solitary wave after renormalization.

Our proof provides a robust approach to the construction of blow up regimes in the presence of a nonlocal nonlinearity, and is a first step towards the understanding of the flow near the soliton. It also provides a new insight to attack more complicated chemotactic coupling, in particular the parabolic-parabolic Patlak-Keller-Segel problem for which blow up in the critical setting is open.

Acknowledgements. The authors would like to thank A. Blanchet, J. Dolbeault and P. Laurençot for their interest and support during the preparation of this work. Part of this work was done while P.R was visiting the MIT Mathematics department which he would like to thank for its kind hospitality. This work is supported by the ERC/ANR grant SWAP and the advanced ERC grant BLOWDISOL

Notations. For a given function u , we note its Poisson field

$$\phi_u = \frac{1}{2\pi} \log|x| \star u.$$

We note the real L^2 scalar product

$$(f, g) = \int_{\mathbb{R}^2} f(x)g(x)dx.$$

Given a vector $y \in \mathbb{R}^2$, we will systematically note $r = |y| = \sqrt{y_1^2 + y_2^2}$. We will generically note

$$\delta(z^*) = o(1) \text{ as } z^* \rightarrow 0.$$

We let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a radially symmetric cut off function with

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 2 \end{cases}, \quad \chi(x) \geq 0.$$

We introduce the differential operator

$$\Lambda f = 2f + y \cdot \nabla f = \nabla \cdot (yf).$$

Given $f \in L^2$, we introduce its decomposition into radial and non radial part :

$$f^{(0)}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta, \quad f^{(1)}(x) = f(x) - f^{(0)}(r). \quad (7.1.12)$$

Observe that

$$(\Lambda f)^{(0)} = (r\partial_r f)^{(0)} = r\partial_r f^{(0)} = \Lambda f^{(0)}.$$

Given $b > 0$, we let

$$B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}. \quad (7.1.13)$$

Give two linear operators (A, B) , we note their commutator :

$$[A, B] = AB - BA.$$

7.1.5 Strategy of the proof

We briefly explain the main steps of the proof of Theorem 7.1.1 which follows the approach developed in [88], [72], [91].

Step 1 : Construction of approximate blow up profiles. Let us renormalize the (PKS) flow by considering

$$u(t, x) = \frac{1}{\lambda^2(t)} v(s, y), \quad \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad y = \frac{x}{\lambda}$$

which leads to the renormalized flow

$$\partial_s v + b\Lambda v = \nabla \cdot (\nabla v + v\nabla\phi_v), \quad b = -\frac{\lambda_s}{\lambda}.$$

We look for a slowly modulated approximate solution of the form

$$v(s, y) = Q_{b(s)}(y), \quad Q_b(y) = Q(y) + bT_1(y) + b^2T_2(y)$$

and where the law $b_s = F(b) = O(b^2)$ is an unknown of the problem. The construction of T_1 , T_2 amounts solving stationary in s elliptic problems of the form

$$\mathcal{L}T_j = F_j(T_{j-1}) \quad (7.1.14)$$

where we introduced the linearized operator close the soliton :

$$\mathcal{L}\varepsilon = \nabla \cdot (Q\nabla\mathcal{M}\varepsilon) = \Delta\varepsilon + 2Q\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon + \nabla Q \cdot \nabla\phi_\varepsilon. \quad (7.1.15)$$

This operator displays explicit resonances² induced by the symmetry group of (PKS). In the radial setting, the solution to (7.1.14) is almost explicit, and the slow decay of Q at $+\infty$ induces slowly decaying function T_j with explicit tails at $+\infty$. In particular,

$$T_1(r) \sim \frac{c}{r^2} \quad \text{as } r \rightarrow +\infty$$

2. i.e. slowly decaying zeroes.

and to leading order in terms of tails at $+\infty$, the T_2 equation looks like

$$\mathcal{L}T_2 = -b_s T_1 - b^2 \Lambda T_1 + \text{lot.}$$

We now observe the cancellation

$$\Lambda T_1 \sim \frac{1}{r^4} \quad \text{as } r \rightarrow +\infty$$

and hence the choice $b_s = 0$ yields the most decreasing right hand side. In fact, a more careful analysis leads to the logarithmically degenerate law :

$$b_s = -\frac{2b^2}{|\log b|},$$

see the proof of Proposition 7.3.1. We refer to [88], [72] for a further discussion on the derivation of these logarithmic gains. The outcome is the derivation of the leading order dynamical system

$$\begin{cases} \frac{ds}{dt} = \frac{1}{\lambda^2} \\ b = -\frac{\lambda_s}{\lambda}, \quad b_s = -\frac{2b^2}{|\log b|} \end{cases} \quad (7.1.16)$$

which after reintegration in time implies that λ touches zero in finite time $T < +\infty$, this is blow up, with the asymptotics (7.1.11).

Step 2 : Energy bounds. We now consider an initial data of the form

$$u_0 = Q_{b_0} + \varepsilon_0 \quad \text{with } \|\varepsilon_0\| \lesssim b_0^{10}$$

in some suitable topology. We claim that we can bootstrap the existence of a decomposition

$$u(t, x) = \frac{1}{\lambda^2(t)} (Q_{b(t)} + \varepsilon)(t, \frac{x}{\lambda(t)}) \quad \text{with } \|\varepsilon(t)\| \lesssim (b(t))^{10}$$

with $(\lambda(t), b(t))$ satisfying to leading order the ODE's (7.1.16). Roughly speaking, the equation driving ε is of the form

$$\partial_s \varepsilon = \mathcal{L}\varepsilon - b\Lambda\varepsilon + \Psi_b + \text{Mod} + N(\varepsilon)$$

where $\Psi_b = O(b^3)$ is the error in the construction of Q_b , Mod is the forcing term induced by the modulation parameters

$$\text{Mod} = \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b - b_s \frac{\partial Q_b}{\partial b},$$

and $N(\varepsilon)$ denotes the lower order nonlinear term. In order to treat the $b\Lambda$ correction to the linearized term *which cannot be treated perturbatively in blow up regimes*, we reformulate the problem in terms of the original variables

$$\tilde{u}(t, x) = \frac{1}{\lambda^2} \varepsilon(t, \frac{x}{\lambda})$$

which satisfies

$$\partial_t \tilde{u} = \mathcal{L}_\lambda \tilde{u} + \frac{1}{\lambda^4} [\Psi_b + \text{Mod} + N(\varepsilon)] (t, \frac{x}{\lambda}). \quad (7.1.17)$$

We now control \tilde{u} using an energy method on its derivatives as in [88]. Let \mathcal{M} be the linearized entropy close to Q , see (7.2.1). One can show using the explicit knowledge of the resonances of \mathcal{L} , which itself essentially follows from the variational characterization of Q , that modulo suitable orthogonality conditions which remove the zero modes and correspond to the dynamical adjustment of $(b(t), \lambda(t))$, the generalized linearized HLS energy is coercive on suitable derivatives :

$$(\mathcal{M}\mathcal{L}\varepsilon, \mathcal{L}\varepsilon) \gtrsim \|\mathcal{L}\varepsilon\|_{L_Q^2}^2 \gtrsim \int \frac{|\Delta\varepsilon|^2}{Q} + \int |\varepsilon|^2 \sim \|\varepsilon\|_{H_Q^2}^2.$$

Applying this to \tilde{u} , we are led to consider the natural higher order entropy for (7.1.17) :

$$\frac{d}{dt} (\mathcal{M}_\lambda \mathcal{L}_\lambda \tilde{u}, \mathcal{L}_\lambda \tilde{u})$$

where $\mathcal{M}_\lambda, \mathcal{L}_\lambda$ correspond to the linearization close to the singular bubble $\frac{1}{\lambda^2} Q(\frac{x}{\lambda})$. Two difficulties occur. First one needs to treat the local terms on the soliton core induced by the time dependance $\lambda(t)$ of the renormalized operator $\mathcal{M}_\lambda, \mathcal{L}_\lambda$, and here the dissipated terms together with the sharp control of tails at infinity play a crucial role. An additional problem occurs due to the pointwise control of Mod terms which is not good enough in our regime³, and requires some further integration by parts in time, see Lemma 7.5.5.

The energy method leads roughly to a pointwise bound

$$\frac{d}{dt} (\mathcal{M}_\lambda \mathcal{L}_\lambda \tilde{u}, \mathcal{L}_\lambda \tilde{u}) \lesssim \frac{1}{\lambda^6} \frac{b^4}{|\log b|^2}$$

which after rescaling leads *in the regime* (7.1.16) to the pointwise bound :

$$\|\varepsilon\|_{H_Q^2}^2 \lesssim \frac{b^3}{|\log b|^2}.$$

It turns out that this estimate is sharply good enough to close the ODE for b and show that the ε radiation does not perturb the leading order system (7.1.16), and this concludes the proof of Theorem 7.1.1.

This work is organized as follows. In section 7.2, we prove spectral gap estimates for the nonlocal linearized operator close to Q in the continuation of [88]. In section 7.3, we build the family of approximate self similar profiles which contains the main qualitative informations on the singularity formation, and adapts in the non local setting the strategy developed in [88], [72], [91]. In section 7.4, we start the analysis of the full solution and explicitly display the set of initial data leading to the conclusions of Theorem 7.1.1 and the bootstrap argument to control the radiation. In section 7.5, we exhibit the H_Q^2 monotonicity formula which is the heart of our analysis. In section 7.6, we collect the estimates of sections 7.4, 7.5 to close the bootstrap and complete the description of the singularity formation.

3. This is a technical unpleasant problem which directly relates to the slow decay of Q at infinity. Similar issues already occurred in related settings, see for example [60].

7.2 Spectral gap estimates

This section is devoted to the derivation of spectral gap estimates for the linearized operator close to Q given by (7.1.15). The radial assumption is useless here and we address directly the general case. These energy bounds are at the heart of the control of the radiation correction for flows evolving close to the approximate \tilde{Q}_b blow up profiles. We derive two type of estimates : energy estimates which are mostly a consequence of the linearization close to Q of the sharp log Sobolev estimate⁴ and higher order iterated estimates which rely on the explicit knowledge of the kernel of \mathcal{L} and suitable repulsivity properties at infinity.

7.2.1 Energy bounds

We start with the study of the linearized log Sobolev energy on the weighted L^2 space (7.1.7).

Lemme 7.2.1 (Structure of the linearized energy). *The operator*

$$\mathcal{M}u = \frac{u}{Q} + \phi_u \quad (7.2.1)$$

is a continuous self adjoint operator $\mathcal{M} : L_Q^2 \rightarrow (L_Q^2)^*$ ie :

$$\int Q|\mathcal{M}u|^2 \lesssim \|u\|_{L_Q^2}^2, \quad (7.2.2)$$

$$\forall (u, v) \in L_Q^2 \times L_Q^2, \quad (\mathcal{M}u, v) = (u, \mathcal{M}v). \quad (7.2.3)$$

Moreover, there holds :

(i) Algebraic identities :

$$\mathcal{M}(\Lambda Q) = -2, \quad \mathcal{M}(\partial_i Q) = 0, \quad i = 1, 2. \quad (7.2.4)$$

(ii) Generalized kernel : Let $(u, \phi) \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfy

$$\begin{cases} \nabla \left(\frac{u}{Q} + \phi \right) = 0 \\ \Delta \phi = u \end{cases}, \quad \int \frac{\phi^2}{(1+r^4)(1+|\log r|^2)} < +\infty, \quad (7.2.5)$$

then

$$u \in \text{Span}(\Lambda Q, \partial_1 Q, \partial_2 Q), \quad \phi = \phi_u. \quad (7.2.6)$$

Remarque 7.2.2. This structure is reminiscent from kinetic transport models like the gravitational Vlasov Poisson equation, see [57].

Proof of Lemma 7.2.1. **step 1** Continuity and self adjointness. The continuity of \mathcal{M} as an operator from $L_Q^2 \rightarrow (L_Q^2)^*$ follows from (7.7.2) :

$$\int Q|\mathcal{M}u|^2 \lesssim \int \frac{u^2}{Q} + \left\| \frac{\phi_u}{1+|\log r|} \right\|_{L^\infty}^2 \|(1+|\log r|)Q\|_{L^1} \lesssim \|u\|_{L_Q^2}^2.$$

4. following the celebrated proof by Weinstein [103] for nonlinear Schrödinger equations.

The self adjointness (7.2.3) is equivalent to :

$$\forall (u, v) \in L_Q^2 \times L_Q^2, \quad (\phi_u, v) = (u, \phi_v). \quad (7.2.7)$$

Indeed, the integrals are absolutely convergent arguing like for (7.7.2) :

$$\int | \log|x - y| |u(y)| |v(x)| dx dy \lesssim \|u\|_{L_Q^2} \|v\|_{L_Q^2}$$

and the claim follows from Fubini.

step 2. Proof of (i). By definition of Q :

$$\phi'_Q(r) = \frac{1}{r} \int_0^r \frac{8\tau}{(1+\tau^2)^2} d\tau = \frac{4r}{1+r^2}.$$

Moreover, using the convolution representation

$$\phi_Q(0) = \int_0^{+\infty} \frac{8r \log r}{(1+r^2)^2} dr = 4 \lim_{(\varepsilon, R) \rightarrow (0, +\infty)} \left[\frac{r^2 \log r}{1+r^2} - \frac{1}{2} \log(1+r^2) \right]_\varepsilon^R = 0,$$

and thus :

$$\phi_Q(r) = \int_0^r \frac{4\tau}{1+\tau^2} d\tau = 2\log(1+r^2).$$

We therefore obtain the ground state equation :

$$\log Q + \phi_Q - \log 8 = 0. \quad (7.2.8)$$

We translate with $x_0 \in \mathbb{R}^2$:

$$\log Q(x - x_0) + \phi_Q(x - x_0) - \log 8 = 0$$

and differentiate at $x_0 \in \mathbb{R}^2$ to get :

$$\frac{\nabla Q}{Q} + \nabla \phi_Q = 0, \quad (7.2.9)$$

and hence

$$\mathcal{M}(\nabla Q) = \frac{\nabla Q}{Q} + \phi_{\nabla Q} = \frac{\nabla Q}{Q} + \nabla \phi_Q = 0. \quad (7.2.10)$$

Similarly, given $\lambda > 0$, let $Q_\lambda(y) = \lambda^2 Q(\lambda y)$. We compute using $\int Q = 8\pi$:

$$\phi_{Q_\lambda}(y) = \frac{1}{2\pi} \int \log|x - y| Q_\lambda(y) dy = \frac{1}{2\pi} \int \log\left(\frac{1}{\lambda}|\lambda x - z|\right) Q(z) dz = -4\log\lambda + \phi_Q(\lambda y)$$

and hence from (7.2.8) :

$$-\log 8 + \log Q_\lambda + \phi_{Q_\lambda} + 2\log\lambda = 0.$$

We differentiate this expression at $\lambda = 1$ and obtain :

$$\frac{\Lambda Q}{Q} + \phi_{\Lambda Q} + 2 = 0, \quad (7.2.11)$$

this is (7.2.4).

step 3 Generalized kernel. Let now (u, ϕ) smooth solve (7.2.5), then :

$$\Delta\phi + Q\phi = cQ$$

for some $c \in \mathbb{R}$. Q being radially symmetric, we develop ϕ in real spherical harmonics :

$$\phi(x) = \sum_{k \geq 0} [\phi_{1,k}(r) \cos(k\theta) + \phi_{2,k}(r) \sin(k\theta)]$$

and obtain

$$\forall k \in \mathbb{N}, \quad H_k \phi_{i,k} = -\phi''_{i,k} - \frac{\phi'_{i,k}}{r} + \frac{k^2}{r^2} \phi_{i,k} - Q\phi_{i,k} = -cQ\delta_{k=0}.$$

We omit in the sequel the i -dependance to simplify notations. For $k \geq 2$, we can from standard argument using the decay $Q = O(\frac{1}{r^4})$ at infinity construct a basis of solutions to $H_k \psi_{i,k} = 0$ with at infinity

$$\psi_{1,k} \sim \frac{1}{r^k}, \quad \psi_{2,k} \sim r^k.$$

The a priori bound (7.2.5) implies $\phi_k \in \text{Span}\{\psi_{1,k}\}$ and then the regularity of ϕ and $H_k \phi_k = 0$ easily yield

$$\int |\partial_r \phi_k|^2 + \int \frac{(\phi_k)^2}{r^2} < +\infty, \tag{7.2.12}$$

and hence ϕ_k belongs to the natural energy space associated to H_k . Now

$$\forall k \geq 2, \quad H_k \geq H_1 + \frac{1}{r^2}$$

in the sense of quadratic forms, while from (7.2.10),

$$H_1(\phi'_Q) = 0, \quad \phi'_Q(r) > 0 \quad \text{for } r > 0$$

implies from standard Sturm Liouville argument that ϕ'_Q is the bound state of H_1 . Hence $H_k > 0$ for $k \geq 2$ and then from (7.2.12) :

$$\phi_k = 0 \quad \text{for } k \geq 2. \tag{7.2.13}$$

For $k = 1$, the zeroes of H_1 are explicit :

$$\psi_{1,1}(r) = \phi'_Q(r) = \frac{4r}{1+r^2}, \quad \psi_{2,1}(r) = -\psi_{1,1}(r) \int_1^r \frac{d\tau}{\tau \psi_{1,1}^2(\tau)}.$$

The second zero is singular at the origin $\psi_{2,1}(r) \sim \frac{c}{r}$ and thus the assumed regularity of ϕ implies :

$$\phi_1 \in \text{Span}(\phi'_Q). \tag{7.2.14}$$

For $k = 0$, we first observe from (7.2.4) that

$$\tilde{u}_0 = u_0 + \frac{c}{2} \Lambda Q, \quad \tilde{\phi}_0 = \phi_0 + \frac{c}{2} \phi_{\Lambda Q}$$

satisfies

$$\begin{cases} \frac{\tilde{u}_0}{Q} + \tilde{\phi}_0 = 0 \\ \Delta \tilde{\phi}_0 = \tilde{u}_0 \end{cases} \quad \text{and thus } H_0 \tilde{\phi}_0 = 0 \quad (7.2.15)$$

Let the partial mass be

$$m_0(r) = \int_0^r \tau \tilde{u}_0(\tau) d\tau \quad \text{so that } m'_0 = r \tilde{u}_0, \quad \tilde{\phi}'_0 = \frac{m_0}{r},$$

then :

$$0 = rQ \left(\frac{\tilde{u}_0}{Q} + \tilde{\phi}_0 \right)' = r \left[\tilde{u}'_0 - \frac{Q'}{Q} \tilde{u}_0 + Q \tilde{\phi}'_0 \right] = r \left[\left(\frac{m'_0}{r} \right)' - \frac{Q'}{yQ} m'_0 + \frac{Q}{r} m_0 \right].$$

Equivalently,

$$L_0 m_0 = 0$$

where

$$\begin{aligned} L_0 m_0 &= -m''_0 + \left(\frac{1}{r} + \frac{Q'}{Q} \right) m'_0 - Q m_0 \\ &= -m''_0 - \frac{3r^2 - 1}{r(1+r^2)^2} m'_0 - \frac{8}{(1+r^2)^2}. \end{aligned} \quad (7.2.16)$$

The basis of solutions to this homogeneous equation is explicit⁵ and given by :

$$\psi_0(r) = \frac{r^2}{(1+r^2)^2}, \quad \psi_1(r) = \frac{1}{(1+r^2)^2} [r^4 + 4r^2 \log r - 1]. \quad (7.2.17)$$

The regularity of \tilde{u}_0 implies $m_0(r) \rightarrow 0$ as $r \rightarrow 0$ and thus

$$m_0 \in \text{Span}(\psi_0), \quad \tilde{u}_0 = \frac{m'_0}{r} \in \text{Span}\left(\frac{\psi'_0}{y}\right) = \text{Span}(\Lambda Q)$$

from direct check. Hence $\tilde{u}_0 = c\Lambda Q$ and the regularity of $\tilde{\phi}_0$ at the origin⁶ implies $\tilde{\phi}_0(r) = c\phi_{\Lambda Q}$, but then from (7.2.4), (7.2.15) :

$$0 = \frac{\tilde{u}_0}{Q} + \tilde{\phi}_0 = c\mathcal{M}\Lambda Q = -2c \quad \text{and thus } \tilde{u}_0 = \tilde{\phi}_0 = 0.$$

Together with (7.2.13), (7.2.14), this implies :

$$\phi \in \text{Span}(\phi_{\Lambda Q}, \partial_1 \phi_Q, \partial_2 \phi_Q), \quad u = \Delta \phi \in \text{Span}(\Lambda Q, \partial_1 Q, \partial_2 Q),$$

and (7.2.6) follows. This concludes the proof of Lemma 7.2.1. \square

The explicit knowledge of the kernel of \mathcal{M} given by Lemma 7.2.1 and the variational characterization of Q imply from standard argument the modulated coercivity of the linearized energy.

5. this is due to the fact that the non linear equation satisfied by the partial mass of Q_b is invariant by $m(r) \mapsto m(\lambda r)$, see (7.3.8), and indeed $\psi_0 \in \text{Span}\{rm'_Q\}$, $m_Q(r) = \int_0^r Q(\tau) \tau d\tau$.

6. recall that the only radially symmetric harmonic function in R^2 is $\log r$ which is singular at the origin.

Proposition 7.2.3 (Coercivity of the linearized energy). *There exists a universal constant $\delta_0 > 0$ such that for all $u \in L_Q^2$,*

$$(\mathcal{M}u, u) \geq \delta_0 \int \frac{u^2}{Q} - \frac{1}{\delta_0} \left[(u, \Lambda Q)^2 + (u, 1)^2 + \sum_{i=1}^2 (u, \partial_i Q)^2 \right]. \quad (7.2.18)$$

Proof of Proposition 7.2.3 A slightly weaker version is proved in [100] using explicit and somewhat miraculous computations. We give a simpler and stronger argument by adapting the robust proof by [103], and which here relies on the classical sharp Hardy Littlewood Sobolev, see also [57] for a very similar argument.

step 1 Positivity. We claim :

$$\forall u \in L_Q^2 \text{ with } \int u = 0, \quad (\mathcal{M}u, u) \geq 0. \quad (7.2.19)$$

From standard density argument, it suffices to prove (7.2.19) for $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\int u = 0$. We apply the sharp logarithmic HLS inequality (7.1.5) at minimal mass $M = 8\pi$ to $v = Q + \lambda u > 0$ for $\lambda \in \mathbb{R}$ small enough : $\int v = \int Q = 8\pi$ and thus

$$F(\lambda) = \int v \log v + \frac{1}{2} \int v \phi_v \geq F(0).$$

We compute

$$\begin{aligned} F'(\lambda) &= \int \left[(\log(Q + \lambda u) + 1)u + \frac{1}{2}(u\phi_{Q+\lambda u} + (Q + \lambda u)\phi_u) \right] \\ &= \int [(\log(Q + \lambda u) - \log 8 + \phi_Q)u + \lambda u\phi_u] \end{aligned}$$

where we used (7.2.7) and the vanishing $\int u = 0$, and then Q is a critical point from (7.2.8) :

$$F'(0) = 0,$$

and the Hessian is positive :

$$F''(\lambda)|_{\lambda=0} = \int \left[\frac{u}{Q} + \phi_u \right] u = (\mathcal{M}u, u) \geq 0.$$

step 2 Coercivity. We now claim the spectral gap :

$$I = \inf \left\{ (\mathcal{M}u, u), \quad \|u\|_{L_Q^2} = 1, \quad (u, 1) = (u, \Lambda Q) = (u, \partial_i Q) = 0 \right\} > 0. \quad (7.2.20)$$

We argue by contradiction and consider from (7.2.19) a sequence $u_n \in L_Q^2$ with

$$0 \leq (\mathcal{M}u_n, u_n) \leq \frac{1}{n}, \quad \|u_n\|_{L_Q^2} = 1, \quad (u_n, 1) = (u_n, \Lambda Q) = (u_n, \partial_i Q) = 0.$$

Hence up to a subsequence,

$$u_n \rightharpoonup u \text{ in } L_Q^2 \quad (7.2.21)$$

which implies using the decay $|Q(y)| \lesssim \frac{1}{1+r^4}$:

$$\int \frac{u^2}{Q} \leq \liminf_{n \rightarrow +\infty} \int \frac{u_n^2}{Q} = 1, \quad (u, 1) = (u, \Lambda Q) = (u, \partial_i Q) = 0. \quad (7.2.22)$$

Observe now from (7.7.5) that

$$\int u_n \phi_{u_n} = - \int |\nabla \phi_{u_n}|^2$$

and we claim from standard argument :

$$\nabla \phi_{u_n} \rightarrow \nabla \phi_u \text{ in } L^2. \quad (7.2.23)$$

Assume (7.2.23), then from (7.2.22) :

$$(\mathcal{M}u, u) \leq 0, \quad \int \frac{u^2}{Q} \leq 1$$

and thus from (7.2.19),

$$(\mathcal{M}u, u) = 0, \quad \int \frac{u^2}{Q} = \int |\nabla \phi_u|^2 \leq 1.$$

Moreover, from the normalization of the sequence :

$$(\mathcal{M}u_n, u_n) = \int \frac{u_n^2}{Q} - \int |\nabla \phi_{u_n}|^2 = 1 - \int |\nabla \phi_{u_n}|^2 \leq \frac{1}{n}$$

and thus using (7.2.23) :

$$\int |\nabla \phi_u|^2 = \lim_{n \rightarrow +\infty} \int |\nabla \phi_{u_n}|^2 \geq 1,$$

from which

$$(\mathcal{M}u, u) = 0, \quad \int \frac{u^2}{Q} = 1, \quad (u, 1) = (u, \Lambda Q) = (u, \partial_i Q) = 0. \quad (7.2.24)$$

Hence u is non trivial and attains the infimum. We conclude from standard Lagrange multiplier argument using the selfadjointness of \mathcal{M} that :

$$\mathcal{M}u = a + b\Lambda Q + c \cdot \nabla Q + d \frac{u}{Q}, \quad u \in L_Q^2.$$

Taking the scalar product with u yields $d = 0$ and thus $\mathcal{M}u \in L_Q^2$. We then take the scalar product with $1, \Lambda Q, \partial_i Q$ and obtain :

$$\mathcal{M}u = 0. \quad (7.2.25)$$

Moreover $(u, \phi_u) \in \mathcal{C}^\infty(\mathbb{R}^2)$ from standard bootstrapped regularity argument and $\phi_u \in L^\infty$ from (7.7.4). Hence (u, ϕ_u) satisfy the generalized kernel equation (7.2.5) and $u \in \text{Span}(\Lambda Q, \partial_1 Q, \partial_2 Q)$ from Lemma 7.2.1. The orthogonality conditions on u now imply $u \equiv 0$ which contradicts (7.2.24) and concludes the proof of (7.2.20).

Proof of (7.2.23) : From (7.7.5) :

$$\int |\nabla \phi_u - \nabla \phi_{u_n}|^2 = - \int (u_n - u)(\phi_{u_n} - \phi_u).$$

From (7.7.4), (7.7.5), ϕ_{u_n} is bounded in $H^1(|x| \leq R)$ for any $R > 0$, and thus from the local compactness of Sobolev embeddings and a standard diagonal extraction argument, we may find $\psi \in H^1_{\text{loc}}(\mathbb{R}^2)$ such that up to a subsequence⁷ :

$$\phi_{u_n} \rightarrow \phi \text{ in } L^2_{\text{loc}}(\mathbb{R}^2). \quad (7.2.26)$$

Moreover

$$\phi \in L^\infty(\mathbb{R}^2) \quad (7.2.27)$$

from (7.7.4). We then split the integral in two. For the outer part, we use Cauchy Schwarz and (7.7.4) to estimate :

$$-\int_{|x| \geq R} (u_n - u)(\phi_{u_n} - \phi_u) \lesssim \frac{1}{R} (\|\phi_u\|_{L^\infty} + \|\phi_{u_n}\|_{L^\infty}) (\|u\|_{L_Q^2} + \|u_n\|_{L_Q^2}) \lesssim \frac{1}{R}.$$

For the inner part, we use the strong local convergence (7.2.26), the a priori bound (7.2.27) and the weak convergence (7.2.21) to conclude :

$$\begin{aligned} -\int_{|x| \leq R} (u_n - u)(\phi_{u_n} - \phi_u) &= -\int_{|x| \leq R} (u_n - u)(\phi_{u_n} - \phi) - \int_{|x| \leq R} (u_n - u)(\phi - \phi_u) \\ &\lesssim (\|u_n\|_{L_Q^2} + \|u\|_{L_Q^2}) \|\phi_{u_n} - \phi\|_{L^2(|x| \leq R)} - \int_{|x| \leq R} (u_n - u)(\phi - \phi_u) \\ &\rightarrow 0 \text{ as } n \rightarrow +\infty, \end{aligned}$$

and (7.2.23) is proved.

step 3 Conclusion. Let now $u \in L_Q^2$ and

$$v = u - aQ - b\Lambda Q - c_1\partial_1 Q - c_2\partial_2 Q$$

with

$$a = \frac{(u, 1)}{\|Q\|_{L^1}}, \quad b = \frac{1}{\|\Lambda Q\|_{L^2}^2} [(u, \Lambda Q) + a(Q, \Lambda Q)], \quad c_i = \frac{(u, \partial_i Q)}{\|\partial_i Q\|_{L^2}^2},$$

then

$$(v, 1) = (v, \Lambda Q) = (v, \partial_i Q) = 0$$

and thus from (7.2.20) :

$$(\mathcal{M}v, v) \geq \delta_0 \int \frac{v^2}{Q} \geq \delta_0 \int \frac{u^2}{Q} - \frac{1}{\delta_0} (a^2 + b^2 + c_1^2 + c_2^2)$$

and the claim follows by expanding $(\mathcal{M}v, v)$. This concludes the proof of Proposition 7.2.3.

7. we need to prove that $\phi = \phi_u$

7.2.2 Structure of \mathcal{L}

We now study the linearized operator close to Q of the (PKS) flow given by (7.1.15) for perturbations in \mathcal{E} given by (7.1.9). We define its formal adjoint for the L^2 scalar product by

$$\mathcal{L}^* \varepsilon = \mathcal{M} \nabla \cdot (Q \nabla \varepsilon). \quad (7.2.28)$$

Lemme 7.2.4 (Continuity of \mathcal{L} on \mathcal{E}). (i) *Continuity :*

$$\|\mathcal{L}\varepsilon\|_{L_Q^2} \lesssim \|\varepsilon\|_{\mathcal{E}}. \quad (7.2.29)$$

(ii) *Adjunction* : $\forall (\varepsilon, \tilde{\varepsilon}) \in \mathcal{E}^2$,

$$(\mathcal{L}\varepsilon, \tilde{\varepsilon}) = (\varepsilon, \mathcal{L}^*\tilde{\varepsilon}). \quad (7.2.30)$$

(iii) *Algebraic identities* :

$$\mathcal{L}(\Lambda Q) = \mathcal{L}(\partial_1 Q) = \mathcal{L}(\partial_2 Q) = 0, \quad (7.2.31)$$

$$\mathcal{L}^*(y_1) = \mathcal{L}^*(y_2) = \mathcal{L}^*(1) = 0, \quad \mathcal{L}^*(|y|^2) = -4. \quad (7.2.32)$$

(iv) *Vanishing average* : $\forall \varepsilon \in \mathcal{E}$,

$$(\mathcal{L}\varepsilon, 1) = 0. \quad (7.2.33)$$

Proof of Lemma 7.2.4.

Proof of (i) : We have from the explicit formula (7.1.15) and the energy bound (7.7.6) on the Poisson field :

$$\begin{aligned} \int \frac{|\mathcal{L}\varepsilon|^2}{Q} &\lesssim \int \frac{|\Delta\varepsilon|^2}{Q} + \int Q|\varepsilon|^2 + \int \frac{|\nabla\phi_Q|^2}{Q} |\nabla\varepsilon|^2 + \int \frac{|\nabla Q|^2}{Q} |\nabla\phi_\varepsilon|^2 \\ &\lesssim \|u\varepsilon\|_{H_Q^2}^2 + \int \frac{|\nabla\phi_\varepsilon|^2}{1+r^4} \lesssim \|\varepsilon\|_{\mathcal{E}}^2. \end{aligned}$$

Proof of (ii) : The representation formula

$$\mathcal{L}\varepsilon = \nabla \cdot (Q \nabla \mathcal{M}\varepsilon) \quad (7.2.34)$$

ensures that the formal adjoint of \mathcal{L} for the L^2 scalar product is given by (7.2.28). To justify the integration by parts (7.2.30), we first remark that both integrals are absolutely convergent. Indeed, from (7.2.29) :

$$\int |\mathcal{L}\varepsilon||\tilde{\varepsilon}| \lesssim \|\mathcal{L}\varepsilon\|_{L_Q^2} \|\tilde{\varepsilon}\|_{L^2} \lesssim \|\varepsilon\|_{\mathcal{E}} \|\tilde{\varepsilon}\|_{\mathcal{E}}.$$

Moreover,

$$\mathcal{L}^*\tilde{\varepsilon} = \frac{1}{Q} \nabla \cdot (Q \nabla \tilde{\varepsilon}) + \phi_{\nabla \cdot (Q \nabla \tilde{\varepsilon})}$$

and from $\int \nabla \cdot (Q \nabla \tilde{\varepsilon}) = 0$ and (7.7.4) :

$$\|\phi_{\nabla \cdot (Q \nabla \tilde{\varepsilon})}\|_{L^\infty}^2 \lesssim \|\nabla \cdot (Q \nabla \tilde{\varepsilon})\|_{L_Q^2}^2 \lesssim \int \frac{|Q\Delta\tilde{\varepsilon}|^2 + |\nabla Q \cdot \nabla\tilde{\varepsilon}|^2}{Q} \lesssim \|\tilde{\varepsilon}\|_{H_Q^2}^2,$$

from which :

$$\int |\varepsilon| |\mathcal{L}^*\tilde{\varepsilon}| \lesssim \|\varepsilon\|_{L^1} \|\phi_{\nabla \cdot (Q \nabla \tilde{\varepsilon})}\|_{L^\infty} + \int |\varepsilon| \left[|\Delta\tilde{\varepsilon}| + \frac{|\nabla Q|}{Q} |\nabla\tilde{\varepsilon}| \right] \lesssim \|\varepsilon\|_{\mathcal{E}} \|\tilde{\varepsilon}\|_{\mathcal{E}}.$$

Hence :

$$(\mathcal{L}\varepsilon, \tilde{\varepsilon}) = \lim_{R \rightarrow +\infty} \int_{|x| \leq R} \tilde{\varepsilon} \mathcal{L}\varepsilon.$$

Now

$$\begin{aligned} \int_{|x| \leq R} \tilde{\varepsilon} \mathcal{L}\varepsilon &= \int_{|x| \leq R} \nabla \cdot (Q \nabla \mathcal{M}\varepsilon) \tilde{\varepsilon} = \int_{S_R} Q \tilde{\varepsilon} \partial_r (\mathcal{M}\varepsilon) d\sigma_R - \int_{|x| \leq R} Q \nabla \mathcal{M}\varepsilon \cdot \nabla \tilde{\varepsilon} \\ &= \int_{S_R} [Q \tilde{\varepsilon} \partial_r (\mathcal{M}\varepsilon) - Q \partial_r \tilde{\varepsilon} \mathcal{M}\varepsilon] d\sigma_R + \int_{|x| \leq R} \mathcal{M}\varepsilon \nabla \cdot (Q \nabla \tilde{\varepsilon}). \end{aligned}$$

Moreover,

$$\|\nabla \cdot (Q \nabla \tilde{\varepsilon})\|_{L_Q^2}^2 \lesssim \int \frac{|\nabla \tilde{\varepsilon} \cdot \nabla Q|^2 + Q^2 |\Delta \tilde{\varepsilon}|^2}{Q} \lesssim \|\tilde{\varepsilon}\|_{H_Q^2}^2,$$

and thus from (7.2.3) :

$$\lim_{R \rightarrow +\infty} \int_{|y| \leq R} \mathcal{M}\varepsilon \nabla \cdot (Q \nabla \tilde{\varepsilon}) = (\mathcal{M}\varepsilon, \nabla \cdot (Q \nabla \tilde{\varepsilon})) = (\varepsilon, \mathcal{M}(\nabla \cdot (Q \nabla \tilde{\varepsilon}))) = (\varepsilon, \mathcal{L}^* \tilde{\varepsilon}).$$

Moreover, we estimate from Cauchy Schwarz and (7.7.6) :

$$\left| \int Q \tilde{\varepsilon} \partial_r (\mathcal{M}\varepsilon) \right| \lesssim \int Q |\tilde{\varepsilon}| \left[\frac{|\partial_r \varepsilon|}{Q} + \frac{|\partial_r Q||\varepsilon|}{Q^2} + |\partial_r \phi_\varepsilon| \right] < +\infty$$

and from (7.2.2) :

$$\left| \int Q \partial_r \tilde{\varepsilon} \mathcal{M}\varepsilon \right| \lesssim \left(\int Q |\mathcal{M}\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int Q |\nabla \tilde{\varepsilon}|^2 \right)^{\frac{1}{2}} \lesssim \|\tilde{\varepsilon}\|_{H_Q^2} \|\varepsilon\|_{H_Q^2}$$

and thus we can find a sequence $R_n \rightarrow +\infty$ such that

$$\int_{S_{R_n}} [Q v \partial_r (\mathcal{M}u) - Q \partial_r v \mathcal{M}u] d\sigma_{R_n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

and (7.2.30) follows.

Proof of (iii) The algebraic identities (7.2.31) follow directly from (7.2.4), (7.2.28) :

$$\mathcal{L}^*(y_i) = \mathcal{M}(\nabla \cdot (Q \nabla y_i)) = \mathcal{M}(\partial_i Q) = 0,$$

$$\mathcal{L}^*(|y|^2) = \mathcal{M}(\nabla \cdot (Q \nabla |y|^2)) = \mathcal{M}(2 \Lambda Q) = -4.$$

Proof of (iv) : From (7.2.29), $\mathcal{L}\varepsilon \in L_Q^2 \subset L^1$, and thus from (7.2.34) : $\forall r_n \rightarrow +\infty$,

$$\int \mathcal{L}\varepsilon = \lim_{r_n \rightarrow +\infty} \int_{|x| \leq r_n} \mathcal{L}\varepsilon = \lim_{r_n \rightarrow +\infty} r_n \int_0^{2\pi} Q \partial_r (\mathcal{M}\varepsilon) d\theta.$$

But

$$\int |Q \nabla \mathcal{M}\varepsilon|^2 \lesssim \int \left[|\nabla \varepsilon|^2 + \frac{|\varepsilon|^2}{1+|y|^2} + \frac{|\nabla \phi_\varepsilon|^2}{1+r^4} \right] < +\infty$$

where we used (7.7.6) , and thus we can find a sequence $r_n \rightarrow +\infty$ such that

$$\int_0^{2\pi} |Q \partial_r \mathcal{M}\varepsilon|^2 d\theta = o\left(\frac{1}{r_n^2}\right).$$

Hence

$$r_n \int_0^{2\pi} Q \partial_r (\mathcal{M}\varepsilon) d\theta \lesssim \left(r_n^2 \int_0^{2\pi} |Q \partial_r \mathcal{M}\varepsilon|^2 d\theta \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } r_n \rightarrow +\infty,$$

and (7.2.33) follows. This concludes the proof of Lemma 7.2.4.

We now introduce the directions $\Phi_M, \Phi_M^{(1,2)}$ which are localizations of the generalized kernel of \mathcal{L}^* , and on which we will construct our set of orthogonality conditions. We anticipate on section 7.3.1 and let T_1 be the explicit radially symmetric function given by (7.3.16) and which satisfies from (7.3.46), (7.3.2) :

$$\mathcal{L}T_1 = \Lambda Q, \quad |r^i \partial_r^i T_1| \lesssim \frac{1}{1+r^2}. \quad (7.2.35)$$

We claim :

Lemme 7.2.5 (Direction $\Phi_M, \Phi_M^{(1,2)}$). *Given $M \geq M_0 > 1$ large enough, we define the directions :*

$$\Phi_{0,M}(y) = \chi_M r^2, \quad (7.2.36)$$

$$\Phi_M(y) = \chi_M r^2 + c_M \mathcal{L}^*(\chi_M r^2), \quad c_M = -\frac{(\chi_M |y|^2, T_1)}{(\chi_M |y|^2, \Lambda Q)}, \quad (7.2.37)$$

$$\Phi_{i,M} = \chi_M y_i, \quad i = 1, 2. \quad (7.2.38)$$

Then :

(i) Estimate on Φ_M :

$$\Phi_M(r) = r^2 \chi_M - 4c_M \chi_M + \frac{M^2}{\log M} O\left(\frac{1}{1+r^2} \mathbf{1}_{r \leq 2M} + \mathbf{1}_{M \leq r \leq 2M}\right), \quad (7.2.39)$$

$$(\Phi_M, T_1) = 0, \quad (\Phi_M, \Lambda Q) = -(32\pi) \log M + O_{M \rightarrow +\infty}(1), \quad (7.2.40)$$

$$(\Phi_{i,M}, \partial_i Q) = -\|Q\|_{L^1} + O\left(\frac{1}{M}\right). \quad (7.2.41)$$

(ii) Estimate on scalar products : $\forall \varepsilon \in L^1$,

$$|(\varepsilon, \Phi_M)| \lesssim \int_{r \leq 2M} (1+r^2) |\varepsilon| + \frac{M^2}{\log M} \left[|(\varepsilon, 1)| + \int_{r \geq M} |\varepsilon| \right], \quad (7.2.42)$$

$$|(\varepsilon, \mathcal{L}^* \Phi_{0,M})| \lesssim \int_{r \leq 2M} |\varepsilon|, \quad (7.2.43)$$

$$|(\varepsilon, \mathcal{L}^* \Phi_M)| \lesssim \int_{r \leq 2M} |\varepsilon| + \frac{M^2}{\log M} \int_{r \geq M} \frac{|\varepsilon|}{1+r^2}, \quad (7.2.44)$$

$$|(\varepsilon, \mathcal{L}^* \Phi_M) + c_M^{(1)}(\varepsilon, 1)| + |(\varepsilon, \mathcal{L}^* \Phi_{0,M}) + c_M^{(2)}(\varepsilon, 1)| \lesssim \int_{r \geq M} |\varepsilon| \quad (7.2.45)$$

with

$$c_M^{(i)} = 4 + O\left(\frac{1}{M^2}\right), \quad i = 1, 2,$$

$$|(\varepsilon, \mathcal{L}^* \Phi_{i,M})| \lesssim \int_{M \leq r \leq 2M} \frac{|\varepsilon|}{1+r} + \frac{1}{M^2} \int \frac{|\varepsilon|}{1+r} \quad i = 1, 2. \quad (7.2.46)$$

Remarque 7.2.6. It is important for the analysis to keep a sharp track of the M dependence of constants in Lemma 7.2.5. The estimate (7.2.45) will play a distinguished role in the analysis for the derivation of the blow up speed. The estimates on scalar products (7.2.42), (7.2.44), (7.2.45), (7.2.46) are a sharp measurement of the effect of localization on the exact formulae (7.2.32).

Remarque 7.2.7. For $\varepsilon \in L_Q^2$, (7.2.42) implies by Cauchy Schwarz :

$$|(\varepsilon, \Phi_M)| + |(\varepsilon, \Phi_{0,M})| \lesssim M\|\varepsilon\|_{L_Q^2} + \frac{M^2}{\log M}|(\varepsilon, 1)|. \quad (7.2.47)$$

Also (7.2.44) implies the rough bound

$$|(\varepsilon, \mathcal{L}^* \Phi_M)| \lesssim M\|\varepsilon\|_{L^2}. \quad (7.2.48)$$

Proof of Lemma 7.2.5

Step 1 Proof of (i). We integrate by parts and use the cancellation $\mathcal{L}(\Lambda Q) = 0$ to compute :

$$(\Phi_M, \Lambda Q) = (\chi_M |y|^2, \Lambda Q) = -2 \int |x|^2 \chi_M Q - \int |x|^2 Q x \cdot \nabla \chi_M = -(32\pi) \log M + O(1).$$

This yields using (7.2.35) the upper bound :

$$|c_M| \lesssim \frac{1}{\log M} \int_{|y| \leq 2M} \frac{|y|^2}{1+|y|^2} \lesssim \frac{M^2}{\log M}. \quad (7.2.49)$$

We estimate by definition :

$$\mathcal{L}^*(\chi_M |y|^2) = \mathcal{M} r_M, \quad r_M = \nabla \cdot [Q \nabla (\chi_M |x|^2)] \quad (7.2.50)$$

and thus using (7.5.41) :

$$\begin{aligned} \frac{r_M}{Q} &= \frac{1}{Q} [2\chi_M \Lambda Q + |x|^2 (Q \Delta \chi_M + \nabla Q \cdot \nabla \chi_M) + 4Qx \cdot \nabla \chi_M] \\ &= -4\chi_M + O\left(\frac{1}{1+r^2} \mathbf{1}_{r \leq 2M} + \mathbf{1}_{M \leq r \leq 2M}\right) \end{aligned} \quad (7.2.51)$$

Moreover, solving the Poisson equation for radial fields :

$$|\phi_{r_M}(r)| = \left| \int_r^{+\infty} Q \partial_r (\chi_M r^2) dr \right| \lesssim \frac{1}{1+r^2} \mathbf{1}_{r \leq 2M}$$

and thus

$$\mathcal{L}^*(\chi_M |y|^2) = \mathcal{M} r_m = -4\chi_M + O\left(\frac{1}{1+r^2} \mathbf{1}_{r \leq 2M} + \mathbf{1}_{M \leq r \leq 2M}\right) \quad (7.2.52)$$

which together with (7.2.49) yields (7.2.39).

We now compute the scalar product :

$$(\Phi_M, T_1) = (\chi_M |y|^2 + c_M \mathcal{L}^*(\chi_M |y|^2), T_1) = (\chi_M |y|^2, T_1) + c_M (\mathcal{L} T_1, \chi_M |y|^2) = 0$$

where we used (7.2.35) and the definition (7.2.37) of c_M , and integration by parts which is easily justified using the compact support of $\chi_M|y|^2$ and the decay (7.2.35). We finally compute after an integration by parts :

$$(\partial_i Q, \chi_M x_i) = -\|Q\|_{L^1} + O\left(\frac{1}{M}\right).$$

step 2 Proof of (7.2.42), (7.2.43), (7.2.44), (7.2.45). We claim the pointwise bound :

$$(\mathcal{L}^*)^2(\chi_M r^2) = d_M \chi_M + \frac{\mathbf{1}_{r \geq M}}{1+r^2} \quad \text{with } d_M = O\left(\frac{1}{M^4}\right). \quad (7.2.53)$$

Assume (7.2.53), then from (7.2.49), (7.2.52), (7.2.53) :

$$|(\varepsilon, \mathcal{L}^* \Phi_M)| \lesssim \int_{r \leq 2M} |\varepsilon| + \frac{M^2}{\log M} \left[\frac{1}{M^4} \int_{r \leq M} |\varepsilon| + \int_{r \geq M} \frac{|\varepsilon|}{1+r^2} \right],$$

which implies (7.2.44). The estimate (7.2.43) follows from (7.2.52). Recall now (7.2.50), then :

$$\tilde{r}_M = r_M - 2\Lambda Q = \nabla \cdot [Q \nabla ((\chi_M - 1)|x|^2)] = O\left(\frac{\mathbf{1}_{|y| \geq M}}{1+|y|^4}\right), \quad (7.2.54)$$

and computing the poisson field in radial coordinates :

$$\begin{aligned} \phi_{\tilde{r}_M} &= - \int_r^{+\infty} Q \partial_\tau [(1 - \chi_M) \tau^2] d\tau \\ &= -\mathbf{1}_{r \leq M} \int_M^{+\infty} Q \partial_\tau [(1 - \chi_M) \tau^2] d\tau - \mathbf{1}_{r \geq M} \int_r^{+\infty} Q \partial_\tau [(1 - \chi_M) \tau^2] d\tau \\ &= e_M \chi_M + O\left(\frac{\mathbf{1}_{r \geq M}}{1+r^2}\right) \end{aligned}$$

with

$$e_M = - \int_M^{+\infty} Q \partial_\tau [(1 - \chi_M) \tau^2] d\tau = O\left(\frac{1}{M^2}\right).$$

We then compute from $\mathcal{M}\Lambda Q = -2$:

$$|(\varepsilon, \mathcal{L}^*(\chi_M |y|^2)) + (4 - e_M)(\varepsilon, 1)| = |(\varepsilon, \mathcal{M}\tilde{r}_M - e_M)| \lesssim \int_{r \geq M} |\varepsilon|. \quad (7.2.55)$$

Hence :

$$\begin{aligned} |(\varepsilon, \Phi_M)| &\lesssim |(\varepsilon, |y|^2 \chi_M)| + |c_M| |(\varepsilon, \mathcal{L}^*(\chi_M |y|^2))| \\ &\lesssim \int_{r \leq 2M} r^2 |\varepsilon| + \frac{M^2}{\log M} \left[|(\varepsilon, 1)| + \int_{r \geq M} |\varepsilon| \right] \end{aligned}$$

and (7.2.42) is proved. Moreover, from (7.2.49), (7.2.53), (7.2.55) :

$$\begin{aligned} |(\varepsilon, \mathcal{L}^* \Phi_M) + (4 - e_M - c_M d_M)(\varepsilon, 1)| &\lesssim \int_{r \geq M} |\varepsilon| + \frac{M^2}{\log M} \left[\int_{r \geq M} \frac{|\varepsilon|}{1+r^2} \right] \\ &\lesssim \int_{r \geq M} |\varepsilon| \end{aligned}$$

and (7.2.45) is proved.

Proof of (7.2.53) : First observe from $(\mathcal{L}^*)^2(|y|^2) = \mathcal{L}^*(-4) = 0$ that

$$(\mathcal{L}^*)^2(\chi_M|y|^2) = \mathcal{L}^*f_M \text{ with } f_M = \mathcal{M}\tilde{r}_M.$$

We compute explicitly using the radial representation of the Poisson field

$$\begin{aligned} \mathcal{L}^*f_M &= \mathcal{M}\nabla \cdot (Q\nabla f_M) = \frac{1}{rQ}\partial_r(rQ\partial_r f_M) + \phi_{\frac{1}{r}\partial_r(rQ\partial_r f_M)} \\ &= \partial_r^2 f_M + \left(\frac{1}{r} + \frac{\partial_r Q}{Q}\right)\partial_r f_M - \int_r^{+\infty} Q\partial_\tau f_M d\tau. \end{aligned}$$

We then estimate from (7.2.54) :

$$|\partial_r^i \tilde{r}_M| \lesssim \frac{\mathbf{1}_{r \geq M}}{1+r^{4+i}}, \quad \left|\partial_r^i \left(\frac{\tilde{r}_M}{Q}\right)\right| \lesssim \frac{\mathbf{1}_{r \geq M}}{1+r^i}, \quad i \geq 0.$$

To estimate the Poisson field, we use :

$$\partial_r \phi_{\tilde{r}_M}(r) = \frac{1}{r} \int_0^r \tau \tilde{r}_M d\tau$$

and thus :

$$|\partial_r \phi_{\tilde{r}_M}(r)| \lesssim \mathbf{1}_{r \geq M} \frac{1}{r} \int_M^{+\infty} \frac{\tau d\tau}{1+\tau^4} \lesssim \frac{\mathbf{1}_{r \geq M}}{M^2 r},$$

and similarly :

$$|\partial_r^i \phi_{\tilde{r}_M}(y)| \lesssim \frac{\mathbf{1}_{r \geq M}}{M^2 r^i}, \quad i \geq 1.$$

This yields the bounds

$$|\partial_r^i f_M| \lesssim \frac{\mathbf{1}_{r \geq M}}{1+r^i},$$

$$\begin{aligned} \mathcal{L}^*f_M &= -\mathbf{1}_{r \leq M} \int_M^{+\infty} Q\partial_\tau f_M d\tau + \mathbf{1}_{r \geq M} \int_r^{+\infty} Q\partial_\tau f_M d\tau + O\left(\frac{1}{1+r^2}\right) \\ &= O\left(\frac{\mathbf{1}_{r \geq M}}{1+r^2}\right) + d_M \chi_M \end{aligned}$$

with

$$d_M = - \int_M^{+\infty} Q\partial_\tau f_M d\tau = O\left(\frac{1}{M^4}\right)$$

and (7.2.53) is proved.

step 3 Proof of (7.2.46). We estimate using (7.2.32) :

$$\mathcal{L}^*(\chi_M y_i) = \frac{1}{Q} \nabla \cdot (Q\nabla f_M^{(i)}) + \phi_{\nabla \cdot (Q\nabla f_M^{(i)})}, \quad f_M^{(i)} = (\chi_M - 1)y_i.$$

The local term is estimated in brute force :

$$\left| \frac{1}{Q} \nabla \cdot (Q\nabla f_M^{(i)}) \right| = \left| \Delta f_M^{(i)} + \frac{\nabla Q}{Q} \cdot \nabla f_M^{(i)} \right| \lesssim \frac{\mathbf{1}_{M \leq |y| \leq 2M}}{1+|y|}.$$

We compute the Poisson field by writing :

$$f_M^{(i)} = \zeta_M^{(i)} \frac{y_i}{r}, \quad \zeta_M^{(i)}(r) = (\chi_M - 1)r, \quad \phi_{\nabla \cdot (Q \nabla f_M^{(i)})} = \Phi_{i,M}(r) \frac{y_i}{r}$$

with

$$\begin{aligned} \partial_r^2 \Phi_{i,M} + \frac{\partial_r \Phi_{i,M}}{r} - \frac{\Phi_{i,M}}{r^2} &= g_M^{(i)}, \\ g_M^{(i)} &= \frac{1}{r} \partial_r(r Q \partial_r \zeta_M^{(i)}) - \frac{Q}{r^2} \zeta_M^{(i)} = O\left(\frac{\mathbf{1}_{r \geq M}}{1+r^5}\right). \end{aligned}$$

Letting $v_M^{(i)} = \frac{\Phi_{i,M}}{r}$, we obtain :

$$\begin{aligned} \frac{1}{r^3} \partial_r(r^3 \partial_r v_M^{(i)}) &= \frac{g_M^{(i)}}{r}, \quad \partial_r v_M^{(i)} = \frac{1}{r^3} \int_0^r \tau^2 g_M^{(i)} d\tau = O\left(\frac{\mathbf{1}_{r \geq M}}{M^2 r^3}\right), \\ \Phi_{i,M}(r) &= -r \int_r^{+\infty} \partial_\tau v_M^{(i)} d\tau = O\left(\frac{1}{M^2(1+r)}\right). \end{aligned}$$

We therefore obtain the estimate :

$$|\mathcal{L}^*(\chi_M y_i)| \lesssim \frac{1}{1+r} \left[\mathbf{1}_{M \leq |y| \leq 2M} + \frac{1}{M^2} \right] \quad (7.2.56)$$

and (7.2.46) is proved.

This concludes the proof of Lemma 7.2.5.

7.2.3 H_Q^2 bounds

We now claim the following coercivity property which is the keystone of our analysis. We recall the radial/non radial decomposition (7.1.12). The radial symmetry of Q and the structure of the Poisson field ensure :

$$(\mathcal{L}\varepsilon)^{(0)} = \mathcal{L}\varepsilon^{(0)}.$$

Proposition 7.2.8 (Coercivity of \mathcal{L}). *There exist universal constants $\delta_0, M_0 > 0$ such that $\forall M \geq M_0$, there exists $\delta(M) > 0$ such that the following holds. Let $\varepsilon \in \mathcal{E}$ with*

$$(\varepsilon, \Phi_M) = (\varepsilon, \mathcal{L}^*\Phi_M) = (\varepsilon, \Phi_{i,M}) = (\varepsilon, \mathcal{L}^*\Phi_{i,M}) = 0, \quad i = 1, 2, \quad (7.2.57)$$

then there hold the bounds :

(i) Control of $\mathcal{L}\varepsilon$:

$$(\mathcal{M}\mathcal{L}\varepsilon^{(0)}, \mathcal{L}\varepsilon^{(0)}) \geq \frac{\delta_0(\log M)^2}{M^2} \int \frac{(\mathcal{L}\varepsilon^{(0)})^2}{Q}, \quad (7.2.58)$$

$$(\mathcal{M}\mathcal{L}\varepsilon^{(1)}, \mathcal{L}\varepsilon^{(1)}) \geq \frac{\delta_0}{\log M} \int \frac{(\mathcal{L}\varepsilon^{(1)})^2}{Q} \quad (7.2.59)$$

(ii) Coercivity of \mathcal{L} :

$$\begin{aligned} \frac{1}{\delta(M)} \int \frac{(\mathcal{L}\varepsilon)^2}{Q} &\geq \int (1+r^4)|\Delta\varepsilon|^2 + \int (1+r^2)|\nabla\varepsilon|^2 + \int \varepsilon^2 \\ &+ \int \frac{|\nabla\phi_\varepsilon|^2}{r^2(1+|\log r|)^2}. \end{aligned} \quad (7.2.60)$$

Proof of Proposition 7.2.8. **step 1** Control of $\mathcal{L}\varepsilon$. Let

$$\varepsilon_2 = \mathcal{L}\varepsilon \in L_Q^2.$$

From (7.2.33),

$$(\varepsilon_2, 1) = 0$$

and from the choice of orthogonality conditions (7.2.57) and (7.2.30) :

$$(\varepsilon_2, \Phi_M) = (\mathcal{L}\varepsilon, \Phi_M) = (\varepsilon, \mathcal{L}^*\Phi_M) = 0, \quad (\varepsilon_2, \Phi_{i,M}) = (\mathcal{L}\varepsilon, \Phi_{i,M}) = (\varepsilon, \mathcal{L}^*\Phi_{i,M}) = 0$$

Let :

$$\tilde{\varepsilon}_2 = \varepsilon_2 - a_1 \Lambda Q - c_1 \partial_1 Q - c_2 \partial_2 Q$$

with

$$a_1 = \frac{(\varepsilon_2, \Lambda Q)}{\|\Lambda Q\|_{L^2}^2}, \quad c_i = \frac{(\varepsilon_2, \partial_i Q)}{\|\partial_i Q\|_{L^2}^2}.$$

Then :

$$(\tilde{\varepsilon}_2, \Lambda Q) = 0, \quad (\tilde{\varepsilon}_2, \partial_i Q) = 0, \quad (\tilde{\varepsilon}_2, 1) = (\varepsilon_2, 1) = 0$$

where we used the fundamental L^1 critical degeneracy :

$$(\Lambda Q, 1) = 0.$$

We split into radial and non radial parts and conclude from (7.2.18) :

$$(\mathcal{M}\tilde{\varepsilon}_2^{(0)}, \tilde{\varepsilon}_2^{(0)}) \geq \delta_0 \|\tilde{\varepsilon}_2^{(0)}\|_{L_Q^2}^2, \quad (\mathcal{M}\tilde{\varepsilon}_2^{(1)}, \tilde{\varepsilon}_2^{(1)}) \geq \delta_0 \|\tilde{\varepsilon}_2^{(1)}\|_{L_Q^2}^2. \quad (7.2.61)$$

We now use the orthogonality conditions on ε_2 , the radially of Q and the bounds (7.2.47), (7.2.40) to estimate :

$$\begin{aligned} |a_1| &= \left| \frac{(\tilde{\varepsilon}_2, \Phi_M)}{(\Phi_M, \Lambda Q)} \right| = \left| \frac{(\tilde{\varepsilon}_2^{(0)}, \Phi_M)}{(\Phi_M, \Lambda Q)} \right| \lesssim \frac{1}{|\log M|} \left[M \|\tilde{\varepsilon}_2^{(0)}\|_{L_Q^2} + \frac{M^2}{\log M} |(\tilde{\varepsilon}_2^{(0)}, 1)| \right] \\ &\lesssim \frac{M}{\log M} \|\tilde{\varepsilon}_2^{(0)}\|_{L_Q^2}, \end{aligned}$$

and thus

$$\|\varepsilon_2^{(0)}\|_{L_Q^2} \lesssim \frac{M}{\log M} \|\tilde{\varepsilon}_2^{(0)}\|_{L_Q^2} \quad (7.2.62)$$

On the other hand, thanks to the orthogonality $(\varepsilon_2, 1) = (\varepsilon_2^{(0)}, 1) = 0$ and (7.2.4) :

$$(\mathcal{M}\tilde{\varepsilon}_2^{(0)}, \tilde{\varepsilon}_2^{(0)}) = (\mathcal{M}\varepsilon_2^{(0)} + 2a_1, \varepsilon_2 - a_1 \Lambda Q) = (\mathcal{M}\varepsilon_2^{(0)}, \varepsilon_2^{(0)})$$

and thus from (7.2.61), (7.2.62) :

$$(\mathcal{M}\varepsilon_2^{(0)}, \varepsilon_2^{(0)}) = (\mathcal{M}\tilde{\varepsilon}_2^{(0)}, \tilde{\varepsilon}_2^{(0)}) \geq \delta_0 \|\tilde{\varepsilon}_2^{(0)}\|_{L_Q^2}^2 \gtrsim \frac{\delta_0 (\log M)^2}{M^2} \|\varepsilon_2^{(0)}\|_{L_Q^2}^2,$$

this is (7.2.58).

Similarly, we estimate using (7.2.41), (7.2.46) :

$$|c_i| = \left| \frac{(\tilde{\varepsilon}_2, \Phi_{i,M})}{(\Phi_{i,M}, \partial_i Q)} \right| = \left| \frac{(\tilde{\varepsilon}_2^{(1)}, \Phi_{i,M})}{(\Phi_{i,M}, \partial_i Q)} \right| \lesssim \int_{r \leq 2M} r |\tilde{\varepsilon}_2^{(1)}| \lesssim \sqrt{\log M} \|\tilde{\varepsilon}_2\|_{L_Q^2}$$

and thus

$$\|\varepsilon_2^{(1)}\|_{L_Q^2} \lesssim \sqrt{\log M} \|\tilde{\varepsilon}_2^{(1)}\|_{L_Q^2}.$$

We conclude from $\mathcal{M}\partial_i Q = 0$:

$$(\mathcal{M}\varepsilon_2^{(1)}, \varepsilon_2^{(1)}) = (\mathcal{M}\tilde{\varepsilon}_2^{(1)}, \tilde{\varepsilon}_2^{(1)}) \geq \delta_0 \|\tilde{\varepsilon}_2^{(1)}\|_{L_Q^2}^2 \geq \frac{\delta_0}{\log M} \|\varepsilon_2^{(1)}\|_{L_Q^2}^2,$$

this is (7.2.59).

step 2 Subcoercivity of \mathcal{L} . The spectral gap (7.2.60) follows as in [88], [72], [91] from a compactness argument and the explicit knowledge of the kernel of \mathcal{L} . We first claim as a consequence of two dimensional Hardy inequalities and the explicit repulsive structure of the operator far out the subcoercivity estimate : $\forall \varepsilon \in H_Q^2$,

$$\begin{aligned} \int \frac{(\mathcal{L}\varepsilon)^2}{Q} &\gtrsim \int (1+r^4)(\Delta\varepsilon)^2 + \int \left[\frac{1}{r^2(1+|\log r|)^2} + r^2 \right] |\nabla\varepsilon|^2 + \int \varepsilon^2 + \int \frac{|\nabla\phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \\ &- \int \frac{|\nabla\phi_\varepsilon|^2}{1+r^4} - \int \frac{\varepsilon^2}{1+r^2} - \int |\nabla\varepsilon|^2. \end{aligned} \quad (7.2.63)$$

Indeed, from (7.1.15) :

$$\begin{aligned} \int \frac{(\mathcal{L}\varepsilon)^2}{Q} &\gtrsim \int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} - \int \frac{|\nabla Q|^2}{Q} |\nabla\phi_\varepsilon|^2 - \int Q\varepsilon^2 \\ &\gtrsim \int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} - \int \frac{\varepsilon^2}{1+r^4} - \int \frac{|\nabla\phi_\varepsilon|^2}{1+r^4}. \end{aligned} \quad (7.2.64)$$

We develop :

$$\int \frac{(\Delta\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon)^2}{Q} = \int \frac{(\Delta\varepsilon)^2}{Q} + \frac{(\partial_r\phi_Q \partial_r\varepsilon)^2}{Q} + \int \frac{2}{Q} \Delta\varepsilon \nabla\phi_Q \cdot \nabla\varepsilon.$$

We observe from (7.2.9) :

$$\frac{2\nabla\phi_Q}{Q} = -2 \frac{\nabla Q}{Q^2} = 2\nabla\psi, \quad \psi = \frac{1}{Q}.$$

We now recall the classical Pohozaev integration by parts formula :

$$\begin{aligned} 2 \int \Delta\varepsilon \partial_r\psi \partial_r\varepsilon &= 2 \int_0^{+\infty} \int_0^{2\pi} \left[\partial_r(r\partial_r\varepsilon) + \frac{1}{r} \partial_\theta^2\varepsilon \right] \partial_r\psi \partial_r\varepsilon dr d\theta \\ &= - \int_0^{+\infty} \int_0^{2\pi} (r\partial_r\varepsilon)^2 \partial_r \left(\frac{\partial_r\psi}{r} \right) dr d\theta + \int_0^{+\infty} \int_0^{2\pi} (\partial_\theta\varepsilon)^2 \partial_r \left(\frac{\partial_r\psi}{r} \right) dr d\theta \\ &= - \int \left[(\partial_r\varepsilon)^2 - \left(\frac{1}{r} \partial_\theta\varepsilon \right)^2 \right] \left[\partial_r^2\psi - \frac{\partial_r\psi}{r} \right]. \end{aligned}$$

We compute for $r \gg 1$:

$$\begin{aligned}\psi(r) &= \frac{1}{Q} = \frac{r^4}{8} + O(r^2), \quad \partial_r^2 \psi - \frac{\partial_r \psi}{r} = r^2 + O(1), \\ \phi'_Q(r) &= \frac{1}{r} \int_0^r Q(\tau) \tau d\tau = \frac{4}{r} + O\left(\frac{1}{r^3}\right), \quad \frac{(\partial_r \phi_Q)^2}{Q} = 2r^2 + O(1)\end{aligned}$$

from which :

$$\begin{aligned}&\int \frac{(\Delta \varepsilon + \nabla \phi_Q \cdot \nabla \varepsilon)^2}{Q} \\ &\gtrsim \int (1+r^4)(\Delta \varepsilon)^2 + \int \left[(2r^2 - r^2)(\partial_r \varepsilon)^2 + r^2 \left(\frac{1}{r} \partial_\theta \varepsilon\right)^2 \right] - \int |\nabla \varepsilon|^2.\end{aligned}$$

Injecting this into (7.2.64) yields the lower bound :

$$\int \frac{(\mathcal{L}\varepsilon)^2}{Q} \gtrsim \int (1+r^4)(\Delta \varepsilon)^2 + \int r^2 |\nabla \varepsilon|^2 - \int \frac{\varepsilon^2}{1+r^4} - \int |\nabla \varepsilon|^2 - \int \frac{|\nabla \phi_\varepsilon|^2}{1+r^4}$$

and (7.2.63) follows from the Hardy bounds (7.7.12), (7.7.13) and the relation $\varepsilon = \Delta \phi_\varepsilon$.

step 3 Coercivity of \mathcal{L} . We are now in position to prove the coercivity (7.2.60). We claim that it is enough to prove it for $\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Indeed, if so, let $\varepsilon \in \mathcal{L}$ and

$$\varepsilon_n \rightarrow \varepsilon \text{ in } \mathcal{E}, \quad \varepsilon_n \in \mathcal{C}_c^\infty(\mathbb{R}^2),$$

then

$$\|\mathcal{L}\varepsilon_n\|_{L_Q^2} \rightarrow \|\mathcal{L}\varepsilon\|_{L_Q^2}$$

from (7.2.29). Moreover, from (7.7.6) and Sobolev :

$$\int \frac{|\nabla \phi_{\varepsilon_n} - \nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \lesssim \|\nabla \phi_{\varepsilon_n} - \nabla \phi_\varepsilon\|_{L^\infty}^2 \lesssim \|\varepsilon_n - \varepsilon\|_{\mathcal{E}}^2 \rightarrow 0,$$

and we may thus pass to the limit in (7.2.60) for ε_n and conclude that (7.2.60) holds for ε . In order to prove (7.2.60) for $\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, assume by contradiction that there exists a sequence $\varepsilon_n \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that⁸

$$\begin{aligned}&\int (1+r^4)(\Delta \varepsilon_n)^2 + \int \left[\frac{1}{r^2(1+|\log r|)^2} + r^2 \right] |\nabla \varepsilon_n|^2 + \int \varepsilon_n^2 \\ &+ \int \frac{|\nabla \phi_{\varepsilon_n}|^2}{r^2(1+|\log r|)^2} + \int \frac{\phi_{\varepsilon_n}^2}{(1+r^4)(1+|\log r|)^2} = 1,\end{aligned}\tag{7.2.65}$$

$$(\varepsilon_n, \Phi_M) = (\varepsilon_n, \mathcal{L}^* \Phi_M) = (\varepsilon_n, \partial_i Q) = 0, \quad i = 1, 2$$

and

$$\int \frac{(\mathcal{L}\varepsilon_n)^2}{Q} \leq \frac{1}{n}. \tag{7.2.66}$$

8. Working with $\varepsilon_n \in \mathcal{C}^\infty(\mathbb{R}^2)$ ensures that ϕ_{ε_n} makes sense while only $\nabla \phi_\varepsilon$ is well defined for $\varepsilon \in \mathcal{E}$, and we may thus recover a Hardy type control on ϕ_{ε_n} from (7.7.14).

From (7.2.65) and the local compactness of Sobolev embeddings⁹, we have up to a subsequence :

$$\phi_{\varepsilon_n} \rightarrow \phi \text{ in } H_{\text{loc}}^1 \quad (7.2.67)$$

Similarly, ε_n is bounded in $H^2(\mathbb{R}^2)$ and we may extract up to a subsequence

$$\varepsilon_n \rightharpoonup \varepsilon \text{ in } H^2, \quad \varepsilon_n \rightarrow \varepsilon \text{ in } H_{\text{loc}}^1 \quad (7.2.68)$$

and there holds from standard lower semi continuity estimates the a priori bounds :

$$\begin{aligned} & \int (1+r^4)(\Delta\varepsilon)^2 + \int \left[\frac{1}{r^2(1+|\log r|)^2} + r^2 \right] |\nabla\varepsilon|^2 + \int \varepsilon^2 \\ & + \int \frac{|\nabla\phi|^2}{r^2(1+|\log r|)^2} + \int \frac{\phi^2}{(1+r^4)(1+|\log r|)^2} \lesssim 1. \end{aligned} \quad (7.2.69)$$

The subcoercivity estimate (7.2.63), the initialization (7.2.65), the assumption (7.2.66) and the strong convergences (7.2.67), (7.2.68) yield the non degeneracy :

$$\begin{aligned} & \int \frac{\varepsilon^2}{1+r^2} + \int |\nabla\varepsilon|^2 + \int \frac{|\nabla\phi|^2}{1+r^4} + \int \frac{\phi^2}{1+r^6} \\ & = \lim_{n \rightarrow +\infty} \left[\int \frac{\varepsilon_n^2}{1+r^2} + \int |\nabla\varepsilon_n|^2 + \int \frac{|\nabla\phi_{\varepsilon_n}|^2}{1+r^4} + \int \frac{\phi_{\varepsilon_n}^2}{1+r^6} \right] \\ & \gtrsim 1. \end{aligned} \quad (7.2.70)$$

Passing to the limit in the distribution sense, we also conclude

$$\Delta\phi = \varepsilon \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

Note however that we do not know the relation $\phi = \phi_\varepsilon$.

We now pass to the limit in

$$\mathcal{L}\varepsilon_n = \Delta\varepsilon_n + 2Q\varepsilon_n + \nabla\phi_Q \cdot \nabla\varepsilon_n + \nabla Q \cdot \nabla\phi_{\varepsilon_n}$$

and conclude from (7.2.67), (7.2.68) that

$$\mathcal{L}\varepsilon_n \rightharpoonup \Delta\varepsilon + 2Q\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon + \nabla Q \cdot \nabla\phi \text{ in } \mathcal{D}'(\mathbb{R}^2)$$

while $\mathcal{L}\varepsilon_n \rightarrow 0$ in $L^2(\mathbb{R}^2)$ from (7.2.66) and thus

$$\begin{cases} \Delta\varepsilon + 2Q\varepsilon + \nabla\phi_Q \cdot \nabla\varepsilon + \nabla Q \cdot \nabla\phi = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2). \\ \Delta\phi = \varepsilon \end{cases} \quad (7.2.71)$$

The a priori bound (7.2.69) and standard elliptic regularity estimates ensure the bootstrapped regularity $(\varepsilon, \phi) \in C^\infty(\mathbb{R}^2)$ and thus (7.2.71) holds in strong sense.

Let

$$f = \frac{\varepsilon}{Q} + \phi,$$

9. recall that $\varepsilon_n = \Delta\phi_{\varepsilon_n}$

we rewrite (7.2.71) as the divergence equation :

$$\nabla \cdot (Q \nabla f) = 0. \quad (7.2.72)$$

Moreover, we have the a priori bound from (7.2.69) :

$$\int \frac{Q}{1+r^2} |\nabla f|^2 \lesssim \int \frac{1}{1+r^2} \left[\frac{|\nabla \varepsilon|^2}{Q} + \frac{\varepsilon^2}{(1+y^2)Q} + Q |\nabla \phi|^2 \right] < +\infty \quad (7.2.73)$$

and thus in particular there exists a sequence $r_n \rightarrow +\infty$ such that :

$$\lim_{r_n \rightarrow +\infty} \int_0^{2\pi} Q |\nabla f_n|^2(r_n, \theta) d\theta = 0. \quad (7.2.74)$$

We claim from standard argument in Liouville classification of diffusion equations that this implies

$$\nabla f = 0. \quad (7.2.75)$$

Assume (7.2.75), then from (7.2.69), (7.2.75), (ε, ϕ) satisfies (7.2.5) and hence from Lemma 7.2.1 :

$$\varepsilon \in \text{Span}(\Lambda Q, \partial_1 Q, \partial_2 Q), \quad \phi = \phi_\varepsilon.$$

The orthogonality conditions $(\varepsilon, \Phi_M) = (\varepsilon, \partial_1 Q) = (\varepsilon, \partial_2 Q) = 0$ and (7.2.40) imply $\varepsilon \equiv 0$ and thus $\phi = \phi_\varepsilon \equiv 0$ which contradicts the non degeneracy (7.2.70) and concludes the proof of (7.2.60) for $\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^2)$.

Proof of (7.2.75) : We integrate (7.2.72) on B_r and use the regularity of f at the origin and the spherical symmetry of Q to compute :

$$0 = r \int_0^{2\pi} Q(r) \partial_r f(r, \theta) d\theta = r Q(r) \partial_r \left(\int_0^{2\pi} f(r, \theta) d\theta \right).$$

and hence

$$\partial_r f^{(0)}(r) = 0, \quad f^{(0)}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta. \quad (7.2.76)$$

We then multiply (7.2.72) by f , integrate over B_r and estimate using (7.2.76), the spherical symmetry of Q and the sharp Poincaré inequality on $[0, 2\pi]$:

$$\begin{aligned} \int_{B_r} Q(y) |\nabla f(y)|^2 dy &= r \int_0^{2\pi} Q(r) f(r, \theta) \partial_r f(r, \theta) d\theta = r Q(r) \int_0^{2\pi} [f - f^{(0)}] \partial_r f d\theta \\ &\leq r Q(r) \left(\int_0^{2\pi} (f - f^{(0)})^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} (\partial_r f)^2 d\theta \right)^{\frac{1}{2}} \\ &\leq r Q(r) \left(\int_0^{2\pi} (\partial_\theta f)^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} (\partial_r f)^2 d\theta \right)^{\frac{1}{2}} \\ &\leq \frac{r^2 Q(r)}{2} \int_0^{2\pi} \left[(\partial_r f)^2 + \left(\frac{1}{r} \partial_\theta f \right)^2 \right] d\theta = \frac{r^2}{2} \int_0^{2\pi} Q(r) |\nabla f|^2(r, \theta) d\theta. \end{aligned} \quad (7.2.77)$$

Hence

$$g(r) = \int_{B_r} Q(y) |\nabla f(y)|^2 dy$$

satisfies the differential inequation :

$$g'(r) = r \int_0^{2\pi} Q(r) |\nabla f|^2(r, \theta) d\theta \geq \frac{2g}{r}$$

ie

$$\frac{d}{dr} \left[\frac{g}{r^2} \right] \geq 0. \quad (7.2.78)$$

On the other hand, the boundary condition (7.2.74) and the control (7.2.77) imply

$$\lim_{r_n \rightarrow +\infty} \frac{g(r_n)}{r_n^2} = 0$$

which together with the positivity of g and the monotonicity (7.2.78) implies $g \equiv 0$, and (7.2.75) is proved.

This concludes the proof of Proposition 7.2.8. \square

7.3 Construction of approximate blow up profiles

This section is devoted to the construction of suitable approximate blow up profiles which contain the main qualitative informations on the solution. These profiles are one mass super critical continuation of the exact mass subcritical self similar solutions exhibited in [8].

7.3.1 Approximate blow up profiles

We build the family of approximate self similar solutions. The key in the construction is to track in a sharp way the size of tails at infinity. We start with building radial profiles which contain the leading order terms.

Proposition 7.3.1 (Radial blow up profiles). *Let $M > 0$ enough large, then there exists a small enough universal constant $b^*(M) > 0$ such that the following holds. Let $b \in]0, b^*(M)[$ and B_0, B_1 be given by (7.1.13), then there exist radially symmetric profiles $T_1(r), T_2(b, r)$ such that*

$$Q_b(r) = Q(r) + bT_1(r) + b^2T_2(b, r)$$

is an approximate self similar solution in the following sense. Let the error :

$$\Psi_b = \nabla \cdot (\nabla Q_b + Q_b \nabla \phi_{Q_b}) - b\Lambda Q_b + c_b b^2 \chi_{\frac{B_0}{4}} T_1, \quad (7.3.1)$$

with c_b given by (7.3.20), then there holds :

(i) Control of the tails : $\forall r \geq 0, \forall i \geq 0 :$

$$|r^i \partial_r^i T_1| \lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1}{r^2} \mathbf{1}_{r \geq 1}, \quad (7.3.2)$$

and $\forall r \leq 2B_1, \forall i \geq 0 :$

$$|r^i \partial_r^i T_2| \lesssim r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{r \geq 6B_0}, \quad (7.3.3)$$

$$|b\partial_b r^i \partial_r^i T_2| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{r \geq 6B_0} \right]. \quad (7.3.4)$$

(ii) Control of the error in weighted norms : for $i \geq 0$,

$$\int_{r \leq 2B_1} |r^i \partial_r^i \Psi_b|^2 + \int_{r \leq 2B_1} \frac{|\phi'_{\Psi_b}|^2}{1 + \tau^2} + \int_{r \leq 2B_1} \frac{|\mathcal{L}\Psi_b|^2}{Q} \lesssim \frac{b^5}{|\log b|^2}, \quad (7.3.5)$$

$$\int_{r \leq 2B_1} Q |\nabla \mathcal{M}\Psi_b|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (7.3.6)$$

Remarque 7.3.2. The error Ψ_b displays slowly decaying tail as $r \rightarrow +\infty$, and hence the norm in which is measured the error is essential. This explains why polynomials in b in the error vary from (7.3.5) to (7.3.6). These errors are the leading order terms in the control of the radiation for the full solution, see step 4 of the proof of Proposition 7.5.7.

Proof of Proposition 7.3.1. **step 1** Setting the computation on the mass.

We look for radial profiles and it is therefore simpler -but not necessary- to work with the partial mass :

$$m_b(r) = \int_0^r Q_b(\tau) \tau d\tau. \quad (7.3.7)$$

The Poisson field for radial solutions is given by

$$\phi'_{Q_b} = \frac{m_b}{r}.$$

Let using (7.3.1) :

$$\Phi_b = m_b'' - \frac{m_b'}{r} + \frac{m_b m_b'}{r} - br m_b', \quad \Psi_b = \frac{1}{r} \Phi_b' + c_b b^2 \chi_{\frac{B_0}{4}} T_1. \quad (7.3.8)$$

We proceed to an expansion

$$m_b = m_0 + b m_1 + b^2 m_2$$

where

$$m_0(r) = \int_0^r Q(\tau) \tau d\tau = \frac{4r^2}{1 + r^2} = -\frac{r Q'}{Q} = r \phi'_Q.$$

Correspondingly,

$$Q_b = Q + b T_1 + b^2 T_2 = \frac{m_b'}{r}.$$

We let the linearized operator close to m_0 be given by (7.2.16) :

$$L_0 m = -m'' + \left(\frac{1}{r} + \frac{Q'}{Q} \right) m' - Q m = -m'' - \frac{3r^2 - 1}{r(1 + r^2)} m' - \frac{8}{(1 + r^2)^2} m$$

and obtain the expansion :

$$\begin{aligned} \Phi_b &= b [-L_0 m_1 - r m_0'] + b^2 \left[-L_0 m_2 + \frac{m_1 m_1'}{r} - r m_1' \right] \\ &\quad + b^3 \left[\frac{(m_1 m_2)'}{r} - r m_2' \right] + b^4 \left[\frac{m_2 m_2'}{r} \right]. \end{aligned}$$

step 2 Inversion of L_0 . The Green's functions of L_0 are explicit¹⁰ and the set of radial solutions to the homogeneous problem

$$L_0 m = 0$$

is spanned according to (7.2.17) by :

$$\psi_0(r) = \frac{r^2}{(1+r^2)^2}, \quad \psi_1(r) = \frac{r^4 + 4r^2 \log r - 1}{(1+r^2)^2} \quad (7.3.9)$$

with Wronskian

$$W = \psi'_1 \psi_0 - \psi_1 \psi'_0 = \frac{rQ}{4} = \frac{2r}{(1+r^2)^2}. \quad (7.3.10)$$

Hence a solution to

$$L_0 m = -f$$

can be found by the method of variation of constants :

$$m = A\psi_0 + B\psi_1 \quad \text{with} \quad \begin{cases} A'\psi_0 + B'\psi_1 = 0, \\ A'\psi'_0 + B'\psi'_1 = f, \end{cases} \quad (7.3.11)$$

This leads to

$$B' = \frac{f\psi_0}{W} = \frac{r}{2}f, \quad A' = -\frac{f\psi_1}{W} = -\frac{r^4 + 4r^2 \log r - 1}{2r}f$$

and a solution is given by :

$$m(r) = -\frac{1}{2}\psi_0(r) \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} f(\tau) d\tau + \frac{1}{2}\psi_1(r) \int_0^r \tau f(\tau) d\tau. \quad (7.3.12)$$

We compute

$$\frac{\psi'_0}{r} = \frac{2(1-r^2)}{(1+r^2)^3}, \quad \frac{\psi'_1}{r} = \frac{8(1+r^2 - (r^2-1)\log r)}{(1+r^2)^3}. \quad (7.3.13)$$

and then (7.3.11) yields

$$\begin{aligned} \frac{m'}{r} &= A \frac{\psi'_0}{r} + B \frac{\psi'_1}{r} \\ &= -\frac{1-r^2}{(1+r^2)^3} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} f(\tau) d\tau + \frac{4(1+r^2 - (r^2-1)\log r)}{(1+r^2)^3} \int_0^r \tau f(\tau) d\tau. \end{aligned} \quad (7.3.14)$$

step 3 Construction of m_1, T_1 . We let m_1 be the solution to

$$L_0 m_1 = -rm'_0 = -r^2 Q = -\frac{8r^2}{(1+r^2)^2} = -8\psi_0 \quad (7.3.15)$$

given by (7.3.12), explicitly :

$$m_1(r) = -4\psi_0(r) \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1+\tau^2)^2} d\tau + 4\psi_1(r) \int_0^r \frac{\tau^3}{(1+\tau^2)^2} d\tau.$$

10. This structure is reminiscent from the parabolic heat flow problem and one could show that this operator can be factorized $L_0 = A_0^* A_0$ where the adjoint is taken against $\frac{(1+r^2)^2}{r} dr$ and A_0 is first order, and this explains why all formulas are explicit.

Then from (7.3.14) :

$$\begin{aligned} T_1 &= \frac{m'_1}{r} \\ &= -\frac{8(1-r^2)}{(1+r^2)^3} \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1+\tau^2)^2} d\tau + \frac{32(1+r^2 - (r^2-1)\log r)}{(1+r^2)^3} \int_0^r \frac{\tau^3}{(1+\tau^2)^2} d\tau. \end{aligned} \quad (7.3.16)$$

There holds the behavior at the origin

$$m_1(r) = O(r^4), \quad T_1(r) = O(r^2).$$

For r large, we use the explicit formula :

$$\int_0^r \frac{\tau^3}{(1+\tau^2)^2} d\tau = \frac{\log(1+r^2)}{2} + \frac{1}{2} \left(\frac{1}{1+r^2} - 1 \right)$$

to compute :

$$m_1(r) = 4(\log r - 1) + O\left(\frac{|\log r|^2}{r^2}\right), \quad (7.3.17)$$

$$m'_1(r) = \frac{4}{r} + O\left(\frac{|\log r|^2}{r^3}\right), \quad (7.3.18)$$

$$T_1(r) = \frac{4}{r^2} + O\left(\frac{|\log r|^2}{r^4}\right). \quad (7.3.19)$$

This yields in particular the bound for $i \geq 0$:

$$|r^i \partial_r^i T_1| \lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1}{r^2} \mathbf{1}_{r \geq 1}.$$

step 4 Construction of the radiation. We now introduce the radiation term which will allow us to adjust in a sharp way the tail of T_2 outside the parabolic zone $r \geq B_0$. Let

$$c_b = \frac{1}{\int_0^{+\infty} \chi_{\frac{B_0}{4}} \psi_0(\tau) \tau d\tau} = \frac{2}{|\log b|} \left[1 + O\left(\frac{1}{|\log b|}\right) \right] \quad (7.3.20)$$

$$d_b = 4c_b \int_0^{+\infty} \chi_{\frac{B_0}{4}} \psi_1(\tau) \tau d\tau = O\left(\frac{1}{b|\log b|}\right) \quad (7.3.21)$$

We let the radiation Σ_b be the solution to

$$L_0 \Sigma_b = -8c_b \chi_{\frac{B_0}{4}} \psi_0 + d_b L_0 [(1 - \chi_{3B_0}) \psi_0] \quad (7.3.22)$$

given from (7.3.12) by

$$\begin{aligned} \Sigma_b(r) &= -4c_b \psi_0(r) \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1+\tau^2)^2} \chi_{\frac{B_0}{4}} d\tau + 4c_b \psi_1(r) \int_0^r \frac{\tau^3}{(1+\tau^2)^2} \chi_{\frac{B_0}{4}} d\tau \\ &\quad + d_b (1 - \chi_{3B_0}) \psi_0(r). \end{aligned} \quad (7.3.23)$$

Observe that by definition :

$$\Sigma_b = \begin{cases} c_b m_1 & \text{for } r \leq \frac{B_0}{4} \\ 4\psi_1 & \text{for } r \geq 6B_0 \end{cases}. \quad (7.3.24)$$

and we estimate for $\frac{B_0}{4} \leq r \leq 6B_0$:

$$\Sigma_b(r) = 4 + O\left(\frac{1}{|\log b|}\right).$$

For $r \geq 6B_0$, we observe from (7.3.9), (7.3.24) the degeneracy :

$$\Sigma_b = 4 + O\left(\frac{\log r}{r^2}\right), \quad r\partial_r \Sigma_b = O\left(\frac{\log r}{r^2}\right).$$

This yields in particular the bounds using (7.3.17) :

$$\begin{aligned} r^i \partial_r^i \Sigma_b(r) &= O\left(\frac{r^4}{|\log b|}\right) \quad \text{for } r \leq 1, \quad i \geq 0, \\ \forall 1 \leq r \leq 6B_0, \quad \Sigma_b(r) &= \frac{8\log r}{|\log b|} + O\left(\frac{1}{|\log b|}\right), \\ \forall r \geq 6B_0, \quad \Sigma_b(r) &= 4 + O\left(\frac{\log r}{r^2}\right). \end{aligned} \quad (7.3.25)$$

We now observe an improved bound for derivatives far out. Indeed, from (7.3.14) : for $r \leq 2B_1$,

$$\begin{aligned} \left| \frac{\Sigma'_b}{r} \right| &= - \left| -4c_b \frac{\psi'_0(r)}{r} \int_0^r \frac{\tau(\tau^4 + 4\tau^2 \log \tau - 1)}{(1 + \tau^2)^2} \chi_{\frac{B_0}{4}} d\tau + 4c_b \frac{\psi'_1(r)}{r} \int_0^r \frac{\tau^3}{(1 + \tau^2)^2} \chi_{\frac{B_0}{4}} d\tau \right. \\ &\quad \left. + d_b (1 - \chi_{3B_0}) \frac{\psi'_0(r)}{r} - d_b \chi'_{3B_0} \frac{\psi_0}{r} \right| \\ &\lesssim \frac{1}{|\log b|(1 + r^4)} \left[r^2 \mathbf{1}_{r \leq 1} + r^2 \mathbf{1}_{r \leq 6B_0} + \frac{1}{b} \mathbf{1}_{r \geq 6B_0} \right] \\ &\quad + \frac{1 + |\log r|}{|\log b|(1 + r^4)} [(1 + |\log r|) \mathbf{1}_{1 \leq r \leq 6B_0} + |\log b| \mathbf{1}_{r \geq B_0}] + \frac{1}{b |\log b|(1 + r^4)} \mathbf{1}_{r \geq 3B_0} \\ &\lesssim \frac{1}{|\log b|} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1}{r^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{br^4} \mathbf{1}_{r \geq 6B_0} \right] \end{aligned}$$

and thus : for $i \geq 1$, $r \leq 2B_1$,

$$|r^i \partial_r^i \Sigma_b| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \mathbf{1}_{r \leq 6B_0} + \frac{1}{br^2} \mathbf{1}_{r \geq 6B_0} \right]. \quad (7.3.27)$$

We now estimate the b dependance of Σ_b . From (7.3.20), (7.3.21),

$$\frac{\partial c_b}{\partial b} = O\left(\frac{1}{b |\log b|^2}\right), \quad \frac{\partial d_b}{\partial b} = O\left(\frac{1}{b^2 |\log b|^2}\right),$$

and from (7.3.24),

$$\frac{\partial \Sigma_b}{\partial b} = \begin{cases} \frac{\partial c_b}{\partial b} m_1 & \text{for } r \leq \frac{B_0}{4} \\ 0 & \text{for } r \geq 6B_0 \end{cases}.$$

This leads to the bound :

$$r^i \partial_r^i \frac{\partial \Sigma_b}{\partial b}(r) = O\left(\frac{r^4}{b|\log b|^2}\right) \quad \text{for } r \leq 1,$$

$$r^i \partial_r^i \frac{\partial \Sigma_b}{\partial b}(r) = O\left(\frac{1 + |\log r|}{b|\log b|^2}\right) \quad \text{for } 1 \leq r \leq \frac{B_0}{4}, \quad i \geq 0.$$

In the transition zone $\frac{B_0}{4} \leq r \leq 6B_0$, we estimate from (7.3.23) :

$$\begin{aligned} \left| \frac{\partial \Sigma_b}{\partial b} \right| &\lesssim \frac{1}{b|\log b|} + \frac{|c_b|}{1+r^2} \int \tau \left| \partial_b \chi_{\frac{B_0}{4}} \right| d\tau + |c_b| \int \frac{d\tau}{\tau} \left| \partial_b \chi_{\frac{B_0}{4}} \right| + \frac{|d_b|}{1+r^2} |\partial_b \chi_{3B_0}| \\ &\lesssim \frac{1}{b|\log b|} \end{aligned}$$

and similarly for higher derivatives. This yields the bound : for $i \geq 0$, $r \leq 2B_1$,

$$\left| br^i \partial_r^i \partial_b \Sigma_b \right| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log r|}{|\log b|} \mathbf{1}_{r \leq 6B_0} \right]. \quad (7.3.28)$$

step 5 Construction of m_2, T_2 . We define :

$$\Sigma_2 = \frac{m_1 m'_1}{r} - r m'_1 + \Sigma_b. \quad (7.3.29)$$

We estimate

$$\Sigma_2 = O(r^4) \quad \text{for } r \leq 1.$$

For $1 \leq r \leq 6B_0$, there holds the bound using (7.3.18), (7.3.25) :

$$\begin{aligned} \Sigma_2 &= -4 + \frac{8\log r}{|\log b|} + O\left(\frac{1}{|\log b|}\right) + O\left(\frac{|\log r|^2}{1+r^2}\right) \\ &= O\left(\frac{|\log(\sqrt{b}r)|}{|\log b|}\right) + O\left(\frac{1}{|\log b|}\right) \end{aligned} \quad (7.3.30)$$

and for $r \geq 6B_0$, we have from (7.3.9), (7.3.18), (7.3.24) :

$$\Sigma_2 = -4 + 4\psi_1 + O\left(\frac{|\log r|^2}{1+r^2}\right) = O\left(\frac{|\log r|^2}{1+r^2}\right). \quad (7.3.31)$$

Hence, we obtain : $\forall i \geq 0$,

$$|r^i \partial_r^i \Sigma_2| \lesssim r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{|\log r|^2}{1+r^2} \mathbf{1}_{r \geq 6B_0}. \quad (7.3.32)$$

Remarque 7.3.3. The essential feature of (7.3.32) is the $\frac{1}{|\log b|}$ smallness of the Σ_2 tail in the parabolic zone $r \sim B_0$ which justifies the introduction of the radiation Σ_b in (7.3.29), and this will turn out to modify the modulation equation for b according to (7.1.16).

The b dependance of Σ_2 is estimated from (7.3.28), (7.3.29) :

$$\left| br^i \partial_r^i \partial_b \Sigma_2 \right| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log r|}{|\log b|} \mathbf{1}_{r \leq 6B_0} \right]. \quad (7.3.33)$$

We now let m_2 be the solution to

$$L_0 m_2 = \Sigma_2$$

given by

$$m_2(r) = \frac{1}{2} \psi_0(r) \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} \Sigma_2(\tau) d\tau - \frac{1}{2} \psi_1(r) \int_0^r \tau \Sigma_2(\tau) d\tau. \quad (7.3.34)$$

Near the origin,

$$m_2 = O(r^6).$$

For $1 \leq r \leq 6B_0$,

$$\begin{aligned} |m_2(r)| &\lesssim \frac{1}{1+r^2} \int_0^r \tau^3 \frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} d\tau + \int_0^r \tau \frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} d\tau \\ &\lesssim r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|}. \end{aligned}$$

For $r \geq 6B_0$,

$$\begin{aligned} |m_2(r)| &\lesssim \frac{1}{b|\log b|} + \frac{1}{r^2} \int_{6B_0}^r \tau^3 \frac{|\log \tau|^2}{1+\tau^2} d\tau + \int_{6B_0}^r \tau \frac{|\log \tau|^2}{1+\tau^2} d\tau \\ &\lesssim \frac{1}{b|\log b|} + (\log r)^3. \end{aligned}$$

We now observe the following cancellation when exiting the parabolic zone $r \geq 6B_0$ which will be important to treat the error term in the sequel¹¹ : for $6B_0 \leq r \leq 2B_1$,

$$\begin{aligned} \left| \frac{m'_2}{r} \right| &= \left| \frac{1}{2} \frac{\psi'_0(r)}{r} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} \Sigma_2(\tau) d\tau - \frac{1}{2} \frac{\psi'_1(r)}{r} \int_0^r \tau \Sigma_2(\tau) d\tau \right| \\ &\lesssim \frac{1}{r^4} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} \left[\frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{|\log \tau|^2}{1+\tau^2} \mathbf{1}_{\tau \geq 6B_0} \right] d\tau \\ &\quad + \frac{|\log r|}{r^4} \int_0^r \tau \left[\frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{|\log \tau|^2}{1+\tau^2} \mathbf{1}_{\tau \geq 6B_0} \right] d\tau \\ &\lesssim \frac{1}{r^4} \left[\frac{1}{b^2 |\log b|} + r^2 (\log r)^2 \right] + \frac{\log r}{r^4} \left[\frac{1}{b |\log b|} + (\log r)^3 \right] \\ &\lesssim \frac{1}{r^4 b^2 |\log b|}. \end{aligned} \quad (7.3.35)$$

The collection of above bounds yields the control : $\forall r \leq 2B_1$

$$|m_2| \lesssim r^6 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b |\log b|} \mathbf{1}_{r \geq 6B_0}, \quad (7.3.36)$$

11. Equivalently, one should observe that the T_2 equation is forced by ΛT_1 which enjoys the improved decay at infinity $\Lambda T_1 = O(\frac{|\log r|^2}{r^4})$, see [88] for related phenomenons.

$$|r^i \partial_r^i m_2| \lesssim r^6 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2 |\log b|} \mathbf{1}_{r \geq 6B_0}, \quad i \geq 1. \quad (7.3.37)$$

The b dependance is estimated using (7.3.33) :

$$\begin{aligned} |b \partial_b m_2| &\lesssim \frac{r^2}{1+r^4} \int_0^r \frac{1+\tau^4}{\tau} \frac{1}{|\log b|} \left[\tau^4 \mathbf{1}_{\tau \leq 1} + \frac{1+|\log \tau|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} \right] d\tau \\ &+ \int_0^r \frac{\tau}{|\log b|} \left[\tau^4 \mathbf{1}_{\tau \leq 1} + \frac{1+|\log \tau|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} \right] d\tau \\ &\lesssim \frac{1}{|\log b|} \left[r^6 \mathbf{1}_{r \leq 1} + r^2 \frac{1+|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b} \mathbf{1}_{r \geq 6B_0} \right] \end{aligned} \quad (7.3.38)$$

and for higher derivatives :

$$\begin{aligned} \left| \frac{b \partial_b \partial_r m_2}{r} \right| &\lesssim \frac{1}{(1+r^4)|\log b|} \int_0^r \frac{\tau^4 + 4\tau^2 \log \tau - 1}{\tau} \left[\tau^4 \mathbf{1}_{\tau \leq 1} + \frac{|\log \tau|}{|\log b|} \mathbf{1}_{\tau \leq 6B_0} \right] d\tau \\ &+ \frac{|\log r|}{(1+r^4)|\log b|} \int_0^r \tau \left[\tau^4 \mathbf{1}_{\tau \leq 1} + \frac{|\log \tau|}{|\log b|} \mathbf{1}_{\tau \leq 6B_0} \right] d\tau \\ &\lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^4 b^2} \mathbf{1}_{r \geq 6B_0} \right], \end{aligned}$$

and hence the bounds for $r \leq 2B_1$:

$$|b \partial_b m_2| \lesssim \frac{1}{|\log b|} \left[r^6 \mathbf{1}_{r \leq 1} + r^2 \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b} \mathbf{1}_{r \geq 6B_0} \right], \quad (7.3.39)$$

and for $i \geq 1$:

$$|b \partial_b r^i \partial_r^i m_2| \lesssim \frac{1}{|\log b|} \left[r^6 \mathbf{1}_{r \leq 1} + r^2 \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2} \mathbf{1}_{r \geq 6B_0} \right]. \quad (7.3.40)$$

This yields the estimate on $T_2 = \frac{m'_2}{r}$: for $i \geq 0$, $r \leq 2B_1$:

$$|r^i \partial_r^i T_2| \lesssim r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{r \geq 6B_0},$$

$$|b \partial_b r^i \partial_r^i T_2| \lesssim \frac{1}{|\log b|} \left[r^4 \mathbf{1}_{r \leq 1} + \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{r \geq 6B_0} \right].$$

step 6 Estimate on the error.

By construction, Φ_b given by (7.3.8) satisfies :

$$\Phi_b = -b^2 \Sigma_b + R(r) \quad (7.3.41)$$

with

$$R(r) = b^3 \left[\frac{(m_1 m_2)'}{r} - r m'_2 \right] + b^4 \left[\frac{m_2 m'_2}{r} \right]$$

and the error to self similarity given by (7.3.1) satisfies :

$$\Psi_b = \frac{1}{r} \Phi'_b + b^2 c_b T_1 \chi_{\frac{B_0}{4}} = \frac{R'}{r} - b^2 \left[\frac{\Sigma'_b}{r} - c_b T_1 \chi_{\frac{B_0}{4}} \right]. \quad (7.3.42)$$

We have from direct check the bound¹² : for $i \geq 0$,

$$|r^i \partial_r^i R(r)| \lesssim b^3 \left[r^6 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2 |\log b|} \mathbf{1}_{r \geq 6B_0} \right] \quad (7.3.43)$$

This yields using (7.3.24), (7.3.27), the bound for $r \leq 2B_1$:

$$\begin{aligned} |\Psi_b| &\lesssim b^3 \left[r^4 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^4 b^2 |\log b|} \mathbf{1}_{r \geq 6B_0} \right] \\ &+ \frac{b^2}{|\log b|} \left[\frac{1}{r^2} \mathbf{1}_{\frac{B_0}{4} \leq r \leq 6B_0} + \frac{1}{br^4} \mathbf{1}_{r \geq 6B_0} \right] \end{aligned} \quad (7.3.44)$$

$$(7.3.45)$$

We therefore estimate :

$$\begin{aligned} \int_{r \leq 2B_1} |\Psi_b|^2 &\lesssim b^6 \int_{r \leq 2B_1} \left[\mathbf{1}_{r \leq 1} + \left(\frac{|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} \right)^2 + \frac{1}{r^8 b^4 |\log b|^2} \mathbf{1}_{r \geq 6B_0} \right] \\ &+ \frac{b^4}{|\log b|^2} \int_{\tau \leq 2B_1} \left[\frac{1}{\tau^4} \mathbf{1}_{\frac{B_0}{4} \leq \tau \leq 6B_0} + \frac{1}{b^2 \tau^8} \mathbf{1}_{\tau \geq 6B_0} \right] \\ &\lesssim \frac{b^5}{|\log b|^2} \end{aligned}$$

and similarly for higher derivatives :

$$\int_{r \leq 2B_1} |r^i \partial_r^i \Psi_b|^2 \lesssim \frac{b^5}{|\log b|^2}.$$

Moreover, the radial representation of the Poisson field and (7.3.42) ensure :

$$\phi'_{\Psi_b} = \frac{\Phi_b}{r} + \frac{b^2 c_b}{r} \int_0^r m'_1(\tau) \chi_{\frac{B_0}{4}} d\tau = \frac{R}{r} - \frac{b^2}{r} \int_0^r \left[\frac{\Sigma'_b}{\tau} - c_b T_1 \chi_{\frac{B_0}{4}} \right] \tau d\tau.$$

We estimate from (7.3.24), (7.3.25), (7.3.27) :

$$\left| \frac{b^2}{r} \int_0^r \left[\frac{\Sigma'_b}{\tau} - c_b T_1 \chi_{\frac{B_0}{4}} \right] \tau d\tau \right| \lesssim \frac{b^2}{|\log b|} \left[\frac{\mathbf{1}_{\frac{B_0}{4} \leq r \leq 6B_0}}{r} + \frac{1}{br^3} \mathbf{1}_{r \geq 6B_0} \right]$$

which together with (7.3.43) yields the bound :

$$\begin{aligned} \int_{\tau \leq 2B_1} \frac{|\phi'_{\Psi_b}|^2}{1 + \tau^2} &\lesssim \frac{b^4}{|\log b|^2} \int_{\tau \leq 2B_1} \frac{1}{1 + \tau^2} \left[\frac{\mathbf{1}_{\frac{B_0}{4} \leq \tau \leq 6B_0}}{\tau^2} + \frac{1}{b^2 \tau^6} \mathbf{1}_{\tau \geq 6B_0} \right] \\ &+ b^6 \int_{\tau \leq 2B_1} \frac{1}{1 + \tau^2} \left[\tau^3 \mathbf{1}_{\tau \leq 1} + \tau \frac{1 + |\log(\tau\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq \tau \leq 6B_0} + \frac{1}{\tau^3 b^2 |\log b|} \mathbf{1}_{\tau \geq 6B_0} \right]^2 \\ &\lesssim \frac{b^5}{|\log b|^2}. \end{aligned}$$

12. the worst term is $b^3 r m'_2$

We further estimate from the definition (7.1.15) :

$$\int_{r \leq 2B_1} \frac{|\mathcal{L}\Psi_b|^2}{Q} \lesssim \sum_{i=0}^2 \int_{r \leq 2B_1} (1+r^{2i}) |\partial_r^i \Psi_b|^2 + \int_{r \leq 2B_1} \frac{|\partial_r \phi_{\Psi_b}|^2}{1+r^6} \lesssim \frac{b^5}{|\log b|^2}$$

which concludes the proof of (7.3.5). Finally, we obtain the control of the stronger norm :

$$\begin{aligned} \int_{r \leq 2B_1} Q |\nabla \mathcal{M}\Psi_b|^2 &\lesssim \int_{r \leq 2B_1} (1+r^2) \left[|r \partial_r \Psi_b|^2 + |\Psi_b|^2 \right] + \frac{|\phi'_{\Psi_b}|^2}{1+r^4} \\ &\lesssim b^6 \int_{r \leq 2B_1} \left[1 + \frac{(1+|\log(r\sqrt{b})|^2)(1+r^2)}{|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^6 b^4 |\log b|} \mathbf{1}_{r \geq 6B_0} \right] \\ &+ \frac{b^4}{|\log b|^2} \int_{r \leq 2B_1} \left[\frac{1}{r^2} \mathbf{1}_{\frac{B_0}{4} \leq r \leq 6B_0} + \frac{1}{b^2 r^6} \mathbf{1}_{r \geq 6B_0} \right] + \frac{b^5}{|\log b|^2} \\ &\lesssim \frac{b^4}{|\log b|^2}, \end{aligned}$$

and (7.3.6) is proved. This concludes the proof of Proposition 7.3.1. \square

Remarque 7.3.4. Observe from (7.3.16) and the readily verified formula

$$\mathcal{L} \left(\frac{m'}{r} \right) = -\frac{1}{r} (L_0 m)'$$

that

$$\mathcal{L} T_1 = \mathcal{L} \left(\frac{m'_1}{r} \right) = -\frac{1}{r} (L_0 m_1)' = \frac{8\psi'_0}{r} = \Lambda Q. \quad (7.3.46)$$

7.3.2 Localization

We now proceed to a brute force localization of the tails of T_1, T_2 which become irrelevant strictly outside the parabolic zone $r \geq 2B_1$. We need to be careful with localization due to the non local structure of the problem, and we shall rely again on the specific structure of the Poisson equation for radial fields.

Proposition 7.3.5 (Localization). *Given a small parameter*

$$0 < b \ll 1,$$

let the localized profile :

$$\tilde{Q}_b = Q + b\tilde{T}_1 + b^2\tilde{T}_2 \quad \tilde{T}_i = \chi_{B_1} T_i \quad \text{for } i = 1, 2. \quad (7.3.47)$$

Let the error :

$$\tilde{\Psi}_b = \nabla \cdot (\nabla \tilde{Q}_b + \tilde{Q}_b \nabla \phi_{\tilde{Q}_b}) - b\Lambda \tilde{Q}_b + c_b b^2 \chi_{\frac{B_0}{4}} T_1 \quad (7.3.48)$$

then there holds :

(i) Control of the tails : $\forall i \geq 0$:

$$|r^i \partial_r^i \tilde{T}_1| \lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1}{r^2} \mathbf{1}_{1 \leq r \leq 2B_1}, \quad (7.3.49)$$

$$|b\partial_b r^i \partial_r^i T_1| \lesssim \frac{1}{r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1}, \quad (7.3.50)$$

$$|r^i \partial_r^i \tilde{T}_2| \lesssim r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1}, \quad (7.3.51)$$

$$|b\partial_b r^i \partial_r^i \tilde{T}_2| \lesssim \frac{1}{|\log b|} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{|\log r|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right]. \quad (7.3.52)$$

(ii) Control of the error in weighted norms :

$$\int |r^i \nabla^i \tilde{\Psi}_b|^2 + \int \frac{|\nabla \phi_{\tilde{\Psi}_b}|^2}{1+r^2} + \int \frac{|\mathcal{L}\tilde{\Psi}_b|^2}{Q} \lesssim \frac{b^5}{|\log b|^2} \quad (7.3.53)$$

$$\int Q |\nabla \mathcal{M}\tilde{\Psi}_b|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (7.3.54)$$

(iii) Degenerate flux : Let $\frac{B_0}{20} \leq B \leq 20B_0$ and $\Phi_{0,B} = \chi_B r^2$, then

$$|(\mathcal{L}\tilde{\Psi}_b, \Phi_{0,B})| \lesssim \frac{b^2}{|\log b|}. \quad (7.3.55)$$

Proof of Proposition 7.3.5. **step 1** Terms induced by localization. We compute from (7.3.47) :

$$|b\partial_b \tilde{T}_i| = \left| b\chi_{B_1} \partial_b T_i - b \frac{\partial_b B_1}{B_1} (y\chi') \left(\frac{y}{B_1} \right) \right| \lesssim |b\partial_b T_i| \mathbf{1}_{r \leq 2B_1} + |T_i| \mathbf{1}_{B_1 \leq r \leq 2B_1}.$$

and the bounds (7.3.2), (7.3.3), (7.3.4) now yield (7.3.49), (7.3.50), (7.3.51), (7.3.52).
Let now the partial mass

$$m_b(r) = \int_0^r Q_b \tau d\tau, \quad \tilde{m}_b(r) = \int_0^r \tilde{Q}_b \tau d\tau,$$

and consider the decomposition

$$m_b = m_0 + n_b, \quad \tilde{m}_b = m_0 + \tilde{n}_b, \quad m_0(r) = \int_0^r Q \tau d\tau,$$

then by definition

$$\tilde{n}'_b = \chi_{B_1} n'_b.$$

We compute the error induced by localization at the level of the masses. Let like for (7.3.8) :

$$\tilde{\Psi}_b = \frac{\tilde{\Phi}'_b}{r} + b^2 c_b T_1 \chi_{\frac{B_0}{4}}, \quad \tilde{\Phi}_b = \tilde{m}''_b - \frac{\tilde{m}'_b}{r} + \frac{\tilde{m}_b \tilde{m}'_b}{r} - br \tilde{m}'_b,$$

then :

$$\begin{aligned} \tilde{\Phi}_b &= \tilde{n}''_b - \frac{\tilde{n}'_b}{r} + \frac{1}{r} (\tilde{n}_b m'_0 + m_0 \tilde{n}'_b + \tilde{n}_b \tilde{n}'_b) - br(m'_0 + \tilde{n}'_b) \\ &= \chi_{B_1} \Phi_b + Z_b \end{aligned} \quad (7.3.56)$$

with

$$\begin{aligned} Z_b &= \chi'_{B_1} n'_b + \frac{m'_0}{r} (\tilde{n}_b - \chi_{B_1} n_b) + \frac{1}{r} \chi_{B_1} n'_b (\tilde{n}_b - n_b) \\ &\quad - b(1 - \chi_{B_1}) r m'_0. \end{aligned} \quad (7.3.57)$$

We estimate from (7.3.17), (7.3.36) :

$$\begin{aligned} |n_b| &\lesssim b \left[r^2 \mathbf{1}_{r \leq 1} + |\log r| \mathbf{1}_{r \geq 1} \right] + b^2 \left[r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b|\log b|} \mathbf{1}_{r \geq 6B_0} \right] \\ &\lesssim b \left[r^2 \mathbf{1}_{r \leq 1} + |\log r| \mathbf{1}_{r \geq 1} \right], \end{aligned} \quad (7.3.58)$$

and for $i \geq 1$ from (7.3.37) :

$$\begin{aligned} |r^i \partial_r^i n'_b| &\lesssim b \left[r^2 \mathbf{1}_{r \leq 1} + \mathbf{1}_{r \geq 1} \right] + b^2 \left[r^4 \mathbf{1}_{r \leq 1} + r^2 \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^2 |\log b|} \mathbf{1}_{r \geq 6B_0} \right] \\ &\lesssim b \left[r^2 \mathbf{1}_{r \leq 1} + \mathbf{1}_{r \geq 1} \right]. \end{aligned}$$

This yields the bound :

$$\begin{aligned} |r^i \partial_r^i Z_b| &\lesssim \frac{b}{r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1} + \frac{b|\log b|}{1+r^4} \mathbf{1}_{r \geq B_1} + \frac{1}{r} \mathbf{1}_{B_1 \leq r \leq 2B_1} \frac{b^2 |\log b|}{r} + \frac{b}{r^2} \mathbf{1}_{r \geq B_1} \\ &\lesssim \frac{b}{r^2} \mathbf{1}_{r \geq B_1}. \end{aligned} \quad (7.3.59)$$

We compute :

$$\tilde{\Psi}_b = \chi_{B_1} \frac{\tilde{\Phi}'_b}{r} + \frac{\chi'_{B_1}}{r} \Phi_b + \frac{Z'_b}{r} + b^2 c_b T_1 \chi_{\frac{B_0}{4}} = \chi_{B_1} \Psi_b + \frac{\chi'_{B_1}}{r} \Phi_b + \frac{Z'_b}{r}. \quad (7.3.60)$$

We now estimate from (7.3.41), (7.3.26), (7.3.59) :

$$|\tilde{\Psi}_b - \chi_{B_1} \Psi_b| \lesssim \frac{b^2}{r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1} + \frac{b}{r^4} \mathbf{1}_{r \geq B_1}$$

and hence using (7.3.5) :

$$\int |r^i \partial_r^i \tilde{\Psi}_{b,0}|^2 \lesssim \frac{b^5}{|\log b|^2} + \int_{r \geq B_1} \left[\frac{b^4}{r^4} + \frac{b^2}{r^8} \right] \lesssim \frac{b^5}{|\log b|^2}. \quad (7.3.61)$$

We now estimate using the radial representation of the Poisson field and (7.3.60), (7.3.59) :

$$\phi'_{\tilde{\Psi}_b} = \phi'_{\Psi_b} \quad \text{for } r \leq B_1$$

and for $r \geq B_1$:

$$|\phi'_{\tilde{\Psi}_b}| \lesssim \frac{1}{r} \int_{B_1}^r \left[\frac{b^2}{\tau^2} \mathbf{1}_{B_1 \leq \tau \leq 2B_1} + \frac{b}{\tau^4} \mathbf{1}_{\tau \geq B_1} \right] \tau d\tau \lesssim \frac{b^2}{r} \mathbf{1}_{r \geq B_1}$$

and thus from (7.3.5) :

$$\int \frac{|\phi'_{\tilde{\Psi}_b}|^2}{1+r^2} \lesssim \frac{b^5}{|\log b|^2} + \int_{r \geq B_1} \frac{b^4}{r^4} \lesssim \frac{b^5}{|\log b|^2}.$$

This last estimate together with (7.3.61) implies from (7.1.15) :

$$\int \frac{|\mathcal{L}\tilde{\Psi}_b|^2}{Q} \lesssim \frac{b^5}{|\log b|^2}. \quad (7.3.62)$$

step 2 Degenerate flux.

We now turn to the proof of (7.3.55). We estimate using (7.2.47), (7.3.62) with the cancellation (7.2.33) :

$$|(\mathcal{L}\tilde{\Psi}_b, \Phi_{0,B})| \lesssim B \|\mathcal{L}\tilde{\Psi}_b\|_{L_Q^2} \lesssim \frac{b^{\frac{5}{2}}}{\sqrt{b} |\log b|} \lesssim \frac{b^2}{|\log b|}$$

and (7.3.55) is proved. \square

7.4 The bootstrap argument

In this section, we set up the bootstrap arguments at the heart of the proof of Theorem 7.1.1. We will in particular give an explicit description of the set of initial data and detail the preliminary information on the dynamical flow, in particular modulation equations. The dynamical bounds on radiation in higher weighted Sobolev norms at the heart of the analysis are then derived in section 7.5. The bootstrap is finally closed in section 7.6, and this will easily yield the statements of Theorem 7.1.1.

7.4.1 Geometrical description of the set of initial data

Let us describe explicitly the set of initial data \mathcal{O} for which the conclusions of Theorem 7.1.1 hold.

We first recall the standard modulation argument.

Lemme 7.4.1 (L^1 modulation). *Let $M > M^*$ large enough, then there exists a universal constant $\delta^*(M) > 0$ such that $\forall v \in L^1$ with*

$$\|v - Q\|_{L^1} < \delta^*(M),$$

there exists a unique decomposition

$$v = \frac{1}{\lambda_1^2} (\tilde{Q}_b + \varepsilon_1) \left(\frac{x}{\lambda_1} \right)$$

such that

$$(\varepsilon_1, \Phi_M) = (\varepsilon_1, \mathcal{L}^* \Phi_M) = 0.$$

Moreover,

$$\|\varepsilon_1\|_{L^1} + |\lambda_1 - 1| + |b| \lesssim C(M) \delta^*.$$

Proof of Lemma 7.4.1. This is a standard consequence of the implicit function theorem. Let us denote

$$\text{Mod} = (\lambda_1, b), \quad v_{\text{Mod}} = \lambda_1^2 v(\lambda_1 x) - \tilde{Q}_b$$

and consider the \mathcal{C}^1 functional

$$F(v, \lambda_1, b) = \left[(v_{\text{Mod}}, \Phi_M), (v_{\text{Mod}}, \mathcal{L}^* \Phi_M) \right].$$

Then $F(Q, 1, 0) = 0$, and we compute the Jacobian at $(v, \text{Mod}) = (Q, 1, 0)$:

$$\begin{aligned} \frac{\partial}{\partial \lambda_1} (v_{\text{Mod}}, \Phi_M) |_{(v, \text{Mod})=(Q, 1, 0)} &= \frac{\partial}{\partial \lambda_1} \left(v, \Phi_M \left(\frac{x}{\lambda_1} \right) \right) |_{(v, \text{Mod})=(Q, 1, 0)} \\ &= -(Q, y \cdot \nabla \Phi_M) = (\Lambda Q, \Phi_M) \\ &= -24 \log M + O(1), \end{aligned}$$

$$\frac{\partial}{\partial \lambda_1} (v_{\text{Mod}}, \mathcal{L}^* \Phi_M) |_{(v, \text{Mod})=(Q, 1, 0)} = -(Q, y \cdot \nabla \mathcal{L}^* \Phi_M) = (\mathcal{L} \Lambda Q, \Phi_M) = 0,$$

$$\frac{\partial}{\partial b} (v_{\text{Mod}}, \Phi_M) |_{(v, \text{Mod})=(Q, 1, 0)} = -(T_1, \Phi_M) = 0,$$

$$\begin{aligned} \frac{\partial}{\partial b} (v_{\text{Mod}}, \mathcal{L}^* \Phi_M) |_{(v, \text{Mod})=(Q, 1, 0)} &= -(T_1, \mathcal{L}^* \Phi_M) = -(\Lambda Q, \Phi_M) \\ &= -24 \log M + O(1) \end{aligned}$$

The collection of above computations yields the Jacobian :

$$\left| \frac{\partial F}{\partial \text{Mod}} \right|_{(v, \text{Mod})=(Q, 1, 0)} = (24 \log M + O(1))^2 > 0$$

for M large enough, and the claim follows from the implicit function theorem. \square

We are now in position to describe explicitly the set of initial data leading to the conclusions of Theorem 7.1.1.

Définition 7.4.2 (Description of the open set of stable chemotactic blow up). *Fix $M > M^*$ large enough. For $0 < \alpha^* < \alpha^*(M)$ small enough, we let \mathcal{O} be the set of initial data of the form*

$$u_0 = \frac{1}{\lambda_0^2} (\tilde{Q}_{b_0} + \varepsilon_0) \left(\frac{x}{\lambda_0} \right), \quad \varepsilon_0 \in \mathcal{E}_{\text{rad}},$$

where ε_0 satisfies the orthogonality conditions

$$(\varepsilon_0, \Phi_M) = (\varepsilon_0, \mathcal{L}^* \Phi_M), \tag{7.4.1}$$

and with the following a priori bounds :

- (i) Positivity : $u_0 > 0$.
- (ii) Small super critical mass :

$$\int Q < \int u_0 < \int Q + \alpha^*. \tag{7.4.2}$$

(iii) Positivity and smallness of b_0 :

$$0 < b_0 < \delta(\alpha^*). \quad (7.4.3)$$

(iv) Initial smallness :

$$\|\varepsilon_0\|_{H_Q^2} < b_0^{10}. \quad (7.4.4)$$

Observe using Lemma 7.4.1 that \mathcal{O} is open in $\mathcal{E}_{\text{rad}} \cap \{u > 0\}$. It is also non empty and for example $u_0 = \tilde{Q}_{b_0} \in \mathcal{O}$ for $b_0 > 0$ small enough.¹³ The positivity (7.4.3) of b_0 coupled with the smallness (7.4.4) means that we are imposing a deformation of Q which pushes towards concentration in the regime formally predicted by the construction of \tilde{Q}_b . The challenge is to show dynamically that the solution keeps a similar shape in time meaning that $\varepsilon(t)$ remains negligible in suitable norms with respect to $b(t)$.

7.4.2 Setting up the bootstrap

Let $u_0 \in \mathcal{O}$ and let $u \in \mathcal{C}([0, T), \mathcal{E})$ be the corresponding strong solution to (7.1.1), then using the implicit function theorem, the solution admits on some small time interval $[0, T^*)$ a unique decomposition

$$u(t, x) = \frac{1}{\lambda^2(t)} (\tilde{Q}_{b(t)} + \varepsilon) \left(t, \frac{x}{\lambda(t)} \right) \quad (7.4.5)$$

where $\varepsilon(t)$ satisfies the orthogonality conditions

$$(\varepsilon(t), \Phi_M) = (\varepsilon(t), \mathcal{L}^* \Phi_M) = 0, \quad (7.4.6)$$

and the geometrical parameters satisfy from standard argument¹⁴ $(\lambda(t)) \in \mathcal{C}([0, T^*), \mathbb{R}_+^* \times \mathbb{R}_+^*)$. Using the initial smallness assumption, we may assume on $[0, T^*)$ the bootstrap bounds :

– Positivity and smallness of b :

$$0 < b(t) < 10b_0. \quad (7.4.7)$$

– L^1 bound :

$$\int |\varepsilon(t)| < (\delta^*)^{\frac{1}{4}}, \quad (7.4.8)$$

– Control of ε in smoother norms : let

$$\varepsilon_2 = \mathcal{L}\varepsilon,$$

then :

$$\|\varepsilon_2(t)\|_{L_Q^2}^2 \leq K^* \frac{b^3(t)}{|\log b(t)|^2}, \quad (7.4.9)$$

Here (δ^*, K^*) denote respectively small and large enough universal constants with

$$\delta^* = \delta(\alpha^*) \rightarrow 0 \text{ as } \alpha^* \rightarrow 0,$$

and

$$K^* = K^*(M) \gg 1.$$

We claim that this regime is trapped :

13. Observe that the positivity $\tilde{Q}_{b_0} > 0$ for $b_0 > 0$ directly follows from (7.3.19), (7.3.3).

14. see for example [59] for a complete proof in a related setting.

Proposition 7.4.3 (Bootstrap). *Assume that K^* in (7.4.9) has been chosen large enough, then $\forall t \in [0, T^*)$:*

$$0 < b(t) < 2b_0, \quad (7.4.10)$$

$$\int |\varepsilon(t)| < \frac{1}{2}(\delta^*)^{\frac{1}{4}}, \quad (7.4.11)$$

$$\|\varepsilon_2(t)\|_{L_Q^2}^2 \leq \frac{K^*}{2} \frac{b^3(t)}{|\log b(t)|^2}. \quad (7.4.12)$$

In other words, the regime is trapped and

$$T^* = T.$$

The rest of this section together with section 7.5 are devoted to the exhibition of the main dynamical arguments at the heart of the proof of Proposition 7.4.3 which is completed in section 7.6, and easily implies Theorem 7.1.1. All along the proof, we shall make an extensive use of the interpolation bounds of Appendix 7.7.3 which hold in the bootstrap regime.

Our aim for the rest of this section is to derive the modulation equations driving the parameters (b, λ) .

7.4.3 Setting up the equations

Let the space time renormalization¹⁵

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \quad y = \frac{x}{\lambda(t)}. \quad (7.4.13)$$

We will use the notation :

$$f_\lambda(t, x) = \frac{1}{\lambda^2} f(s, y), \quad r = |y| \quad (7.4.14)$$

which leads to :

$$\partial_t f_\lambda = \frac{1}{\lambda^2} \left[\partial_s f - \frac{\lambda_s}{\lambda} \Lambda f \right]_\lambda. \quad (7.4.15)$$

Let the renormalized density :

$$u(t, x) = v_\lambda(s, y)$$

then

$$\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v = \nabla \cdot (\nabla v + v \nabla \phi_v).$$

We expand using the geometrical decomposition (7.4.5) :

$$v(s, y) = \tilde{Q}_{b(s)}(y) + \varepsilon(s, y)$$

and obtain the renormalized linearized equations :

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon = \mathcal{L}\varepsilon + \mathcal{F}, \quad \mathcal{F} = F + \text{Mod} + E \quad (7.4.16)$$

15. we will show that the rescaled time is global $s(t) \rightarrow +\infty$ as $t \rightarrow T$.

with

$$F = \tilde{\Psi}_b + \Theta_b(\varepsilon) + N(\varepsilon), \quad (7.4.17)$$

$$\tilde{\Psi}_b = \nabla \cdot (\nabla \tilde{Q}_b + \tilde{Q}_b \nabla \phi_{\tilde{Q}_b}) - b \Lambda \tilde{Q}_b - c_b b^2 \chi_{\frac{B_0}{4}} T_1, \quad (7.4.18)$$

$$E = -b_s \frac{\partial \tilde{Q}_b}{\partial b} - c_b b^2 \chi_{\frac{B_0}{4}} T_1, \quad (7.4.19)$$

$$\text{Mod} = \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b, \quad (7.4.20)$$

$$\Theta_b(\varepsilon) = \nabla \cdot [\varepsilon (\nabla \phi_{\tilde{Q}_b} - \nabla \phi_Q) + (\tilde{Q}_b - Q) \nabla \phi_\varepsilon], \quad (7.4.21)$$

$$N(\varepsilon) = \nabla \cdot (\varepsilon \nabla \phi_\varepsilon). \quad (7.4.22)$$

Equivalently, let us consider the fluctuation of density in original variables

$$w(t, x) = \varepsilon_\lambda(s, y),$$

then

$$\partial_t w = \mathcal{L}_\lambda w + \frac{1}{\lambda^2} \mathcal{F}_\lambda \quad (7.4.23)$$

with

$$\mathcal{L}_\lambda w = \nabla \cdot (Q_\lambda \nabla \mathcal{M}_\lambda w), \quad \mathcal{M}_\lambda w = \frac{w}{Q_\lambda} + \phi_w.$$

7.4.4 L^1 bound

We start with closing the L^1 bound on the radiation (7.4.11) which is a consequence of the conservation of mass. This bound is very weak but gives a control of the solution far out which allows us to derive bounds with logarithmic losses, see Appendix 7.7.3.

Lemme 7.4.4 (L^1 bound). *There holds :*

$$\int |\varepsilon| < \frac{1}{2} (\delta^*)^{\frac{1}{4}}. \quad (7.4.24)$$

Proof of Lemma 7.4.4. We introduce the decomposition

$$u = \frac{1}{\lambda^2} (Q + \tilde{\varepsilon}) \left(t, \frac{x}{\lambda(t)} \right) \quad \text{i.e. } \tilde{\varepsilon} = \varepsilon + \tilde{Q}_b - Q$$

and define :

$$\tilde{\varepsilon} = \tilde{\varepsilon}_< + \tilde{\varepsilon}_>, \quad \tilde{\varepsilon}_< = \tilde{\varepsilon} \mathbf{1}_{|\tilde{\varepsilon}| < Q}, \quad \tilde{\varepsilon}_> = \tilde{\varepsilon} \mathbf{1}_{\tilde{\varepsilon} > Q}. \quad (7.4.25)$$

The bootstrap bounds (7.7.18), (7.4.7) with (7.3.49), (7.3.51) imply :

$$\|\tilde{\varepsilon}\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty} + |b| \lesssim \delta(\alpha^*). \quad (7.4.26)$$

Pick a small constant $\eta^* \ll 1$, then $|\tilde{\varepsilon}_<| > \eta^* Q$ implies $\delta(\alpha^*) \gtrsim \eta^* Q$ and thus $|x| \geq r(\alpha^*)$ with $r(\alpha^*) \rightarrow +\infty$ as $\alpha^* \rightarrow 0$ from which :

$$\begin{aligned} \int |\tilde{\varepsilon}_<| &\leq \int |\tilde{\varepsilon}_<| \mathbf{1}_{\eta^* Q < |\tilde{\varepsilon}| < Q} + \int |\tilde{\varepsilon}_<| \mathbf{1}_{|\tilde{\varepsilon}_<| \leq \eta^* Q} \lesssim \int_{|x| \geq r(\alpha^*)} Q + \eta^* \int Q \\ &\lesssim \delta(\alpha^*) + \eta^* \lesssim \delta(\alpha^*). \end{aligned}$$

We now write down the conservation of mass which from (7.4.2) implies

$$\int \tilde{\varepsilon} < \alpha^*$$

from which using by definition $\tilde{\varepsilon}_> > 0$:

$$\int |\tilde{\varepsilon}| \lesssim \delta(\alpha^*). \quad (7.4.27)$$

Together with the bootstrap bound (7.4.7) and (7.3.49), (7.3.51), this yields :

$$\int |\varepsilon| \lesssim \int |\tilde{\varepsilon}| + \sqrt{b} \lesssim \delta(\alpha^*) + \sqrt{\delta^*} < \frac{1}{2}(\delta^*)^{\frac{1}{4}}$$

and concludes the proof of Lemma 7.4.4. \square

7.4.5 Rough modulation equations

We compute the modulation equations for (b, λ) which follow from our choice of orthogonality conditions (7.4.6).

Remarque 7.4.5. Here and in the sequel, we note $C(M)$ a generic large constant which depends on M *independently* of the bootstrap constant $K^*(M)$ in (7.4.9), provided $\alpha^* < \alpha^*(M)$ has been chosen small enough in Definition 7.4.2.

Lemme 7.4.6 (Rough control of the modulation parameters). *There holds the bounds :*

$$|\frac{\lambda_s}{\lambda} + b| \lesssim C(M) \frac{b^2}{|\log b|}, \quad (7.4.28)$$

$$|b_s| \lesssim b^{\frac{3}{2}}. \quad (7.4.29)$$

Remarque 7.4.7. Note that (7.4.28), (7.4.29) imply the bootstrap bound :

$$\left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| \lesssim b^{\frac{3}{2}}. \quad (7.4.30)$$

Proof of Lemma 7.4.6. We make an implicit use of the bootstrap bounds of Proposition 7.7.4. Let

$$V = \left| \frac{\lambda_s}{\lambda} + b \right| + |b_s|,$$

we claim the bounds :

$$\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim b^{\frac{3}{4}} |V| + C(M) \frac{b^2}{|\log b|}, \quad (7.4.31)$$

$$|b_s| \lesssim b^{\frac{3}{2}} + C(M)b|V|. \quad (7.4.32)$$

Summing these bounds yields $V \lesssim b^{\frac{3}{2}}$ which reinjected into (7.4.31), (7.4.32) yields (7.4.28), (7.4.29).

Proof of (7.4.31) : We project (7.4.16) onto Φ_M which is compactly supported in $r \leq 2M$ and compute from (7.4.6), (7.2.39), (7.3.53), (7.4.9) :

$$\begin{aligned} |(\text{Mod} + E, \Phi_M)| &= \left| -\frac{\lambda_s}{\lambda} (\Lambda \varepsilon, \Phi_M) - (F, \Phi_M) \right| \\ &\lesssim C(M) |b| (|V| + \|\varepsilon_2\|_{L_Q^2}) + C(M) \frac{b^2}{|\log b|} \\ &\lesssim b^{\frac{3}{4}} |V| + C(M) \frac{b^2}{|\log b|}. \end{aligned} \quad (7.4.33)$$

We estimate in brute force using (7.2.39), (7.2.40), (7.3.20) :

$$|(E, \Phi_M)| \lesssim C(M) \left[b|V| + |b_s| |(T_1, \Phi_M)| + \frac{b^2}{|\log b|} \right] \lesssim b^{\frac{3}{4}} |V| + C(M) \frac{b^2}{|\log b|}$$

and we further compute using (7.2.40) :

$$\begin{aligned} (\text{Mod}, \Phi_M) &= \left(\frac{\lambda_s}{\lambda} + b \right) (\Lambda Q, \Phi_M) + O(C(M) |b| |V|) \\ &= -(32\pi) \log M + O(1) \left(\frac{\lambda_s}{\lambda} + b \right) + O(C(M) |b| |V|). \end{aligned}$$

Injecting these estimates into (7.4.33) yields (7.4.31).

Proof of (7.4.32) : We let \mathcal{L} go through (7.4.16) and project onto Φ_M . We compute from (7.4.6), (7.2.48), (7.3.53), (7.2.45) with the degeneracy (7.2.33) :

$$\begin{aligned} |(\mathcal{L}\text{Mod} + \mathcal{L}E, \Phi_M)| &= \left| -(\mathcal{L}\varepsilon_2, \Phi_M) - \frac{\lambda_s}{\lambda} (\mathcal{L}\Lambda\varepsilon, \Phi_M) - (\mathcal{L}F, \Phi_M) \right| \\ &= \left| -(\varepsilon_2, \mathcal{L}^*\Phi_M) - \frac{\lambda_s}{\lambda} (\Lambda\varepsilon, \mathcal{L}^*\Phi_M) - (F, \mathcal{L}^*\Phi_M) \right| \\ &\lesssim \frac{\|\varepsilon_2\|_{L_Q^2}}{M} + C(M) \left[(b + V) \int \frac{|\Lambda\varepsilon| + |\nabla\varepsilon|}{1+r^2} + \int \frac{|N(\varepsilon)| + |\Theta_b(\varepsilon)|}{1+r^2} + \frac{b^2}{|\log b|} \right] \\ &\lesssim b^{\frac{3}{2}} + C(M) b |V| \end{aligned} \quad (7.4.34)$$

where the $\tilde{\Psi}_b$ term is estimated from (7.2.48), (7.3.53), and the estimates of $(N(\varepsilon), \Theta_b(\varepsilon))$ terms easily follow from the bootstrap bounds of Appendix 7.7.3. We then estimate :

$$\begin{aligned} (\mathcal{L}E, \Phi_M) &= b_s (\mathcal{L}\tilde{T}_1, \Phi_M) + O \left(C(M) |b_s| b + C(M) \frac{b^2}{|\log b|} \right) \\ &= b_s [-(32\pi) \log M + O_{M \rightarrow +\infty}(1)] + O \left(C(M) b |V| + C(M) \frac{b^2}{|\log b|} \right). \end{aligned}$$

Moreover, using $\mathcal{L}\Lambda Q = 0$:

$$\begin{aligned} |(\mathcal{L}\text{Mod}, \Phi_M)| &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right| |\Lambda(\tilde{Q}_b - Q), \mathcal{L}^*\Phi_M| \lesssim C(M) |V| \int \frac{1}{1+r^2} \left[b \frac{\mathbf{1}_{r \leq 2B_1}}{1+r^2} \right] \\ &\lesssim C(M) b |V|. \end{aligned}$$

Injecting the collection of above bounds into (7.4.34) yields (7.4.32). \square

7.5 H_Q^2 bound

After the construction of the approximate blow up profile and the derivation of the modulation equations of Lemma 7.5.5, we now turn to the second main input of our analysis which is the control of the H_Q^2 norm through the derivation of a suitable Lyapounov functional. Our strategy is to implement an energy method for $\varepsilon_2 = \mathcal{L}\varepsilon$ which breaks the L^1 scaling invariance of the problem.

Here we are facing a technical problem which is that the equation (7.4.16) is forced on the RHS by the b_s term in (7.4.19) which satisfies pointwise the weak estimate (7.4.29) only, and this is not good enough to close (7.4.12). This is a consequence of the fact that the elements of the kernel of \mathcal{L}^* grow too fast in space, and therefore b_s remains of order 1 in ε . We however claim that better estimates hold *up to an oscillation in time*, but this requires the introduction of a second decomposition of the flow, see for example [59] for a related difficulty.

7.5.1 The radiation term

We use the bootstrap bound (7.4.9) to introduce a second decomposition of the flow which will *lift* the parameter b . We recall the notation (7.2.36).

Lemme 7.5.1 (Bound on the lifted parameter). *There exists a unique decomposition*

$$\tilde{Q}_b + \varepsilon = \tilde{Q}_{\hat{b}} + \hat{\varepsilon} \quad (7.5.1)$$

with $\hat{b} \lesssim b \lesssim \hat{b}$ and $\hat{\varepsilon}$ satisfying the orthogonality condition

$$(\hat{\varepsilon}, \mathcal{L}^* \Phi_{0, \hat{B}_0}) = 0, \quad \hat{B}_0 = \frac{1}{\sqrt{\hat{b}}}. \quad (7.5.2)$$

Moreover, there holds the bound :

$$|b - \hat{b}| \lesssim \frac{b}{|\log b|}. \quad (7.5.3)$$

Remarque 7.5.2. It is essential for the rest of the analysis that the constant in (7.5.3) is independent of $K^*(M)$.

Proof of Lemma 7.5.1. The claim follows from the implicit function theorem again. For $|b - \hat{b}| \lesssim b$, let the map $F(v, \hat{b}) = (v - \tilde{Q}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0})$, then $F(\tilde{Q}_b, b) = 0$ and

$$\frac{\partial F}{\partial \hat{b}}|_{(\hat{b}=b, v=\tilde{Q}_b)} = -\left(\frac{\partial \tilde{Q}_b}{\partial b}, \mathcal{L}^* \Phi_{0, B_0}\right).$$

We compute using the bounds of Proposition 7.3.5 :

$$\frac{\partial \tilde{Q}_b}{\partial b} = \chi_{B_1} T_1 + O\left(\frac{1}{1+r^2} \mathbf{1}_{B_1 \leq r \leq 2B_1} + b \left[\frac{1+|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{6B_0 \leq r \leq 2B_1}\right]\right),$$

and thus using (7.2.40), (7.2.43) :

$$\begin{aligned} & \left(\frac{\partial \tilde{Q}_b}{\partial b}, \mathcal{L}^* \Phi_{0, B_0}\right) \\ &= (\chi_{B_1} T_1, \mathcal{L}^* \Phi_{0, B_0}) + b O\left(\int \left[\frac{1+|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1}\right]\right) \\ &= -32\pi \log B_0 + O(1) < 0, \end{aligned}$$

and the existence of the decomposition (7.5.1) with $b \sim \hat{b}$ now follows from the implicit function theorem.

The size of the deformation can be measured thanks to (7.4.9). Indeed, we first claim that for all $\frac{1}{\sqrt{b}} \lesssim B \lesssim \frac{1}{\sqrt{b}}$:

$$(\tilde{Q}_b - Q, \mathcal{L}^* \Phi_{0,B}) = -32\pi b \log B + O(b). \quad (7.5.4)$$

Indeed, we estimate from (7.2.43) :

$$\begin{aligned} & (\tilde{Q}_b - Q, \mathcal{L}^* \Phi_{0,B}) \\ &= b(\mathcal{L}T_1, \Phi_{0,B}) + O\left[\int_{r \leq 2B} |b(\tilde{T}_1 - T_1) + b^2 \tilde{T}_2|\right]. \end{aligned}$$

By assumption,

$$\left| \frac{\log B}{|\log b|} - \frac{1}{2} \right| = \left| \frac{\log(B\sqrt{b})}{|\log b|} \right| \lesssim \frac{1}{|\log b|}, \quad (7.5.5)$$

and we therefore estimate from (7.2.40) :

$$b(\mathcal{L}T_1, \Phi_{0,B}) = b(\Lambda Q, \Phi_B) = b[-(32\pi)\log B + O(1)].$$

We estimate from (7.3.49), (7.3.51) :

$$\int_{r \leq 2B} b|\tilde{T}_1 - T_1| \lesssim \int_{B_1 \leq r \leq 2B_1} \frac{b}{r^2} \lesssim b,$$

$$\begin{aligned} \int_{r \leq 2B} b^2 |\tilde{T}_2| &\lesssim b^2 \int_{r \leq 2B_1} \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\ &\lesssim b \end{aligned}$$

and (7.5.4) follows.

We now claim the bound

$$|b - \hat{b}| \lesssim \frac{|b|}{|\log b|} + \frac{\|\varepsilon_2\|_{L_Q^2}}{\sqrt{b} |\log b|} \quad (7.5.6)$$

which together with the bootstrap bound (7.4.9) implies (7.5.3). Indeed, we take the scalar product of (7.5.7) with $\mathcal{L}^* \Phi_{0,\hat{B}_0}$ and conclude from the choice of orthogonality condition (7.5.2) and (7.5.4), (7.5.5) :

$$|\log b| |b - \hat{b}| \lesssim |b| + |\hat{b}| + |(\varepsilon_2, \Phi_{0,\hat{B}_0})| \lesssim |b| + \frac{\|\varepsilon_2\|_{L_Q^2}}{\sqrt{b}}$$

and (7.5.6) is proved. This concludes the proof of Lemma 7.5.1. \square

We now introduce the radiation term

$$\zeta = \hat{\varepsilon} - \varepsilon = \tilde{Q}_b - \tilde{Q}_{\hat{b}} \quad (7.5.7)$$

which we split in two parts :

$$\zeta = \zeta_{\text{big}} + \zeta_{\text{sm}}, \quad (7.5.8)$$

$$\begin{aligned}\zeta_{\text{big}} &= (b - \hat{b})\chi_{B_1}T_1, \\ \zeta_{\text{sm}} &= \hat{b}(\chi_{B_1} - \chi_{\hat{B}_1})T_1 + \int_{\hat{b}}^b [2b\tilde{T}_2 + b^2\partial_b\tilde{T}_2] db\end{aligned}$$

We let

$$\zeta_2 = \mathcal{L}\zeta.$$

Lemme 7.5.3 (Bounds on the radiation). *There holds the pointwise bounds :*

$$\int |r^i \nabla^i \zeta_{\text{big}}|^2 + \int \frac{|\nabla \phi_{\zeta_{\text{big}}}|^2}{1+r^2} \lesssim \frac{b^2}{|\log b|^2}, \quad (7.5.9)$$

$$\int |r^i \nabla^i \zeta_{\text{sm}}|^2 + \int \frac{|\nabla \phi_{\zeta_{\text{sm}}}|^2}{1+r^2} + \int \frac{|\mathcal{L}\zeta_{\text{sm}}|^2}{Q} + \int \frac{|\mathcal{L}\Lambda\zeta_{\text{sm}}|^2}{Q} \lesssim \frac{b^3}{|\log b|^4}, \quad (7.5.10)$$

$$\int |\mathcal{L}\zeta_{\text{sm}}|^2 \lesssim \frac{b^4}{|\log b|^2}, \quad (7.5.11)$$

$$\int \frac{|\mathcal{L}\zeta_{\text{big}}|^2}{Q} + \int \frac{|\mathcal{L}\Lambda\zeta_{\text{big}}|^2}{Q} \lesssim \frac{b^2}{|\log b|^2}, \quad (7.5.12)$$

$$\forall v \in L_Q^2 \quad \text{with} \quad \int v = 0, \quad |(\mathcal{M}\zeta_2, v)| \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|v\|_{L_Q^2}, \quad (7.5.13)$$

$$|(\mathcal{M}\zeta_2, \zeta_2)| \lesssim \frac{b^3}{|\log b|^2}, \quad (7.5.14)$$

$$\int Q |\nabla \mathcal{M}\zeta_2|^2 \lesssim \frac{b^4}{|\log b|^2}. \quad (7.5.15)$$

Remarque 7.5.4. This lemma quantifies the fact that the radiation is a priori large, in particular larger than ε when comparing (7.5.9) with (7.4.9), but because the leading order term in ζ is supported along T_1 , some degenerate norms like (7.5.13), (7.5.15) are better behaved, and this will be essential to close the H_Q^2 bound for ε , see step 6 of the proof of Proposition 7.5.7.

Proof of Lemma 7.5.3. Proof of (7.5.9) : We have the pointwise bound using (7.3.49) :

$$|r^i \nabla^i \zeta_{\text{big}}| \lesssim |\hat{b} - b| \frac{\mathbf{1}_{r \leq 2B_1}}{1+r^2},$$

and hence from (7.5.3) :

$$\int |r^i \nabla^i \zeta_{\text{big}}|^2 \lesssim |\hat{b} - b|^2 \lesssim \frac{b^2}{|\log b|^2}.$$

We estimate the Poisson field using the radial representation :

$$|\nabla \phi_{\zeta_{\text{big}}} \lesssim |\hat{b} - b| \frac{1 + |\log r|}{1+r} \quad (7.5.16)$$

from which :

$$\int \frac{|\nabla \phi_{\zeta_{\text{big}}}|^2}{1+r^2} \lesssim |\hat{b} - b|^2 \lesssim \frac{b^2}{|\log b|^2}.$$

Proof of (7.5.10), (7.5.11) : We have using (7.5.3), (7.3.3) the pointwise bound :

$$\begin{aligned} |r^i \nabla^i \zeta_{\text{sm}}| &\lesssim |b - \hat{b}| \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^2} \\ &+ b|b - \hat{b}| \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \end{aligned} \quad (7.5.17)$$

from which using (7.5.3) :

$$\begin{aligned} \int |r^i \nabla^i \zeta_{\text{sm}}|^2 &\lesssim \frac{b^2}{|\log b|^2} \int \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^4} \\ &+ \frac{b^4}{|\log b|^2} \int \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|^2}{|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^4 r^8 |\log b|^2} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\ &\lesssim \frac{b^3}{|\log b|^4}. \end{aligned}$$

We recall from (7.3.36), (7.3.51) the bounds :

$$|\phi'_{\tilde{T}_2}| = \left| \frac{m_2}{r} \right| \lesssim r^5 \mathbf{1}_{r \leq 1} + \frac{1 + r|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{rb|\log b|} \mathbf{1}_{r \geq 2B_0} \quad \text{for } r \leq B_1,$$

$$|\phi'_{\tilde{T}_2}| \lesssim \frac{1}{r} \int_0^{2B_1} r |\tilde{T}_2| dr \lesssim \frac{1}{rb|\log b|} \quad \text{for } r \geq 2B_1.$$

We then estimate the Poisson field using the radial representation :

$$|\nabla \phi_{\zeta_{\text{sm}}}| \lesssim \frac{|b - \hat{b}|}{r} \mathbf{1}_{r \geq \frac{B_1}{2}} + b|b - \hat{b}| \left[r^5 \mathbf{1}_{r \leq 1} + \frac{1 + r|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{rb|\log b|} \mathbf{1}_{r \geq 2B_0} \right] \quad (7.5.18)$$

and hence :

$$\begin{aligned} \int \frac{|\nabla \phi_{\zeta_{\text{sm}}}|^2}{1 + r^2} &\lesssim \frac{b^2}{|\log b|^2} \int_{r \geq \frac{B_1}{2}} \frac{1}{r^4} \\ &+ \frac{b^4}{|\log b|^2} \int \frac{1}{1 + r^2} \left[r^5 \mathbf{1}_{r \leq 1} + \frac{1 + r^2|\log(r\sqrt{b})|^2}{|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{r^2 b^2 |\log b|^2} \mathbf{1}_{r \geq 2B_0} \right] \\ &\lesssim \frac{b^3}{|\log b|^4}. \end{aligned}$$

We estimate from (7.1.15), (7.5.17), (7.5.18) :

$$\begin{aligned}
|r^i \nabla^i \mathcal{L} \zeta_{\text{sm}}| &\lesssim |b - \hat{b}| \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^4} \\
&+ b|b - \hat{b}| \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{r^2 |\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^6 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\
&+ \frac{b - \hat{b}}{r^6} \mathbf{1}_{r \geq \frac{B_1}{2}} \\
&+ \frac{b|b - \hat{b}|}{1 + r^5} \left[r^5 \mathbf{1}_{r \leq 1} + \frac{1 + r|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{rb|\log b|} \mathbf{1}_{r \geq 2B_0} \right] \\
&\lesssim |b - \hat{b}| \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^4} \\
&+ b|b - \hat{b}| \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{r^2 |\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^6 |\log b|} \mathbf{1}_{r \geq 2B_0} \right]
\end{aligned} \tag{7.5.19}$$

and thus from (7.5.3) :

$$\begin{aligned}
\int |\mathcal{L} \zeta_{\text{sm}}|^2 &\lesssim \int \left[|b - \hat{b}|^2 \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^8} \right] \\
&+ \frac{b^4}{|\log b|^2} \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|^2}{(1 + r^4)|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^4 r^{12} |\log b|^2} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\
&\lesssim \frac{b^4}{|\log b|^2}, \\
\int \frac{|\mathcal{L} \zeta_{\text{sm}}|^2}{Q} &\lesssim \int \left[|b - \hat{b}|^2 \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^4} \right] \\
&+ \frac{b^4}{|\log b|^2} \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|^2}{|\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^4 r^8 |\log b|^2} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\
&\lesssim \frac{b^3}{|\log b|^4}.
\end{aligned}$$

Finally, pick a well localized function f and let $f_\lambda(x) = \lambda^2 f(\lambda x)$, then from direct check :

$$\mathcal{L} f_\lambda = \lambda^2 \left[\Delta f + 2Q_{\frac{1}{\lambda}} f + \nabla \phi_{Q_{\frac{1}{\lambda}}} \cdot \nabla f + \nabla Q_{\frac{1}{\lambda}} \cdot \nabla \phi_f \right]_\lambda$$

and differentiating this relation at $\lambda = 1$ yields :

$$\mathcal{L} \Lambda f = 2\mathcal{L} f + \Lambda(\mathcal{L} f) - 2(\Lambda Q)f - \nabla \phi_{\Lambda Q} \cdot \nabla f - \nabla(\Lambda Q) \cdot \nabla \phi_f. \tag{7.5.20}$$

Applying this to ζ_{sm} and using (7.5.19), (7.5.17), (7.5.18) yields the pointwise bound :

$$\begin{aligned}
|\mathcal{L}(\Lambda \zeta_{\text{sm}})| &\lesssim |b - \hat{b}| \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^4} \\
&+ b|b - \hat{b}| \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{r^2 |\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^6 |\log b|} \mathbf{1}_{r \geq 2B_0} \right]
\end{aligned}$$

and hence

$$\int \frac{|\mathcal{L}\Lambda\zeta_{\text{sm}}|^2}{Q} \lesssim \frac{b^3}{|\log b|^2}.$$

Proof of (7.5.12), (7.5.13) : We estimate from (7.5.10), (7.2.2) :

$$|(\mathcal{M}\mathcal{L}\zeta_{\text{sm}}, v)| \lesssim \|\mathcal{L}\zeta_{\text{sm}}\|_{L_Q^2} \|v\|_{L_Q^2} \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|v\|_{L_Q^2}.$$

We now compute in brute force using (7.1.15), (7.3.46) :

$$\begin{aligned} \mathcal{L}(\chi_{B_1} T_1) &= \chi_{B_1} [\Delta T_1 + 2QT_1 + \nabla\phi_Q \cdot \nabla T_1] + \nabla Q \cdot \nabla\phi_{\chi_{B_1} T_1} + O\left(\frac{\mathbf{1}_{B_1 \leq r \leq 2B_1}}{1+r^4}\right) \\ &= \chi_{B_1} \Lambda Q + \nabla Q \cdot [\chi_{B_1} \nabla\phi_{T_1} - \nabla\phi_{\chi_{B_1} T_1}] = \Lambda Q + O\left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4}\right) \end{aligned} \quad (7.5.21)$$

and thus :

$$\mathcal{L}\zeta_{\text{big}} = (b - \hat{b})\Lambda Q + \xi, \quad (7.5.22)$$

$$\xi = |b - \hat{b}| O\left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4}\right). \quad (7.5.23)$$

This yields the rough bound using (7.5.3) :

$$\int \frac{|\mathcal{L}\zeta_{\text{big}}|^2}{Q} \lesssim |b - \hat{b}|^2 \lesssim \frac{b^2}{|\log b|^2}.$$

The second estimate in (7.5.12) follows similarly using (7.5.20). We further estimate from (7.5.22), (7.5.23), (7.2.4), (7.2.3), (7.2.33), (7.5.3) :

$$|(\mathcal{M}\mathcal{L}\zeta_{\text{big}}, v)| \lesssim |(-2, v)| + \|v\|_{L_Q^2} \left[|b - \tilde{b}|^2 \int \frac{\mathbf{1}_{r \geq B_1}}{Q(1+r^8)} \right]^{\frac{1}{2}} \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|v\|_{L_Q^2}$$

and (7.5.13) is proved.

Proof of (7.5.14) : From (7.5.22), (7.5.13), (7.5.10) :

$$\begin{aligned} |(\mathcal{M}\zeta_2, \zeta_2)| &= |(\mathcal{M}\zeta_2, \xi + \mathcal{L}\zeta_{\text{sm}})| \lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \left[\|\xi\|_{L_Q^2} + \|\mathcal{L}\zeta_{\text{sm}}\|_{L_Q^2} \right] \\ &\lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \left[\left(\frac{b^2}{|\log b|^2} \int_{r \geq B_1} \frac{1}{1+r^4} \right)^{\frac{1}{2}} + \frac{b^{\frac{3}{2}}}{|\log b|} \right] \lesssim \frac{b^3}{|\log b|^2}. \end{aligned}$$

Proof of (7.5.15) : We estimate from (7.5.19) and Hardy Littlewood Sobolev :

$$\begin{aligned} \int Q |\nabla \mathcal{M}\mathcal{L}\zeta_{\text{sm}}|^2 &\lesssim \int \frac{1}{Q} \left\{ |b - \hat{b}|^2 \frac{\mathbf{1}_{\frac{B_1}{2} \leq r \leq 3B_1}}{r^{10}} \right. \\ &\quad + \left. \frac{b^4}{|\log b|^2} \left[\mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|^2}{r^6 |\log b|^2} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^4 r^{14} |\log b|^2} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \right\} \\ &\quad + \|\nabla\phi_{\mathcal{L}\zeta_{\text{sm}}}\|_{L^4}^2 \\ &\lesssim \frac{b^4}{|\log b|^2} + \|\mathcal{L}\zeta_{\text{sm}}\|_{L^{\frac{4}{3}}}^2 \lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

We estimate from (7.5.22), (7.5.23), (7.2.4), (7.5.3) and Hardy Littlewood Sobolev :

$$\begin{aligned} \int Q |\nabla \mathcal{ML}\zeta_{\text{big}}|^2 &\lesssim \int \frac{1}{Q} \left| |b - \hat{b}| \frac{\mathbf{1}_{r \geq B_1}}{1 + r^5} \right|^2 + \int Q |\nabla \phi_\xi|^2 \\ &\lesssim \frac{|b - \hat{b}|^2}{B_1^4} + \|\xi\|_{L^{\frac{4}{3}}}^2 \lesssim \frac{|b - \hat{b}|^2}{B_1^4} \\ &\lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

This concludes the proof of Lemma 7.5.3. \square

7.5.2 Sharp modulation equation for b

We now compute the sharp modulation equation for b using the lifted parameters \hat{b} .

Lemme 7.5.5 (Sharp modulation equations for b). *There holds the differential inequations :*

$$\left| \hat{b}_s + \frac{2b^2}{|\log b|} \right| \lesssim C(M) \frac{\sqrt{b}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^2}{|\log b|^2}. \quad (7.5.24)$$

Remarque 7.5.6. Note that (7.5.24) coupled with the bootstrap bound (7.4.9) ensures the pointwise control :

$$|\hat{b}_s| \lesssim \frac{b^2}{|\log b|}. \quad (7.5.25)$$

Proof of Lemma 7.5.5. Proof of (7.5.24) : We take the scalar product of (7.4.16) with $\mathcal{L}^* \Phi_{0, \hat{B}_0}$:

$$\begin{aligned} \frac{d}{ds} \left\{ (\tilde{Q}_b - Q + \varepsilon, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \right\} &= (\tilde{Q}_b - Q + \varepsilon, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0}) + (\varepsilon_2, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \\ &\quad - c_b b^2 (\chi_{\frac{B_0}{4}} T_1, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \\ &\quad + \frac{\lambda_s}{\lambda} (\Lambda \varepsilon, \mathcal{L}^* \Phi_{0, B}) + (F, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \\ &\quad + \left(\frac{\lambda_s}{\lambda} + b \right) (\Lambda \tilde{Q}_b, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \end{aligned} \quad (7.5.26)$$

and estimate all terms in (7.5.26). For the boundary term in time, we have from (7.5.7), (7.5.2) :

$$(\tilde{Q}_b - Q + \varepsilon, \mathcal{L}^* \Phi_{0, \hat{B}_0}) = (\tilde{Q}_{\hat{b}} - Q + \hat{\varepsilon}, \mathcal{L}^* \Phi_{0, \hat{B}_0}) = (\tilde{Q}_{\hat{b}} - Q, \mathcal{L}^* \Phi_{0, \hat{B}_0}).$$

We then compute :

$$\frac{d}{ds} (\tilde{Q}_{\hat{b}} - Q, \mathcal{L}^* \Phi_{0, \hat{B}_0}) = (\partial_s \tilde{Q}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0}) + (\tilde{Q}_{\hat{b}} - Q, \mathcal{L}^* \partial_s \Phi_{0, \hat{B}_0}).$$

The leading order term is estimated using (7.2.43) :

$$\begin{aligned} (\partial_s \tilde{Q}_{\hat{b}}, \mathcal{L}^* \Phi_{0, \hat{B}_0}) &= \hat{b}_s (\tilde{T}_1 + \hat{b} \partial_b \tilde{T}_1 + 2\hat{b} \tilde{T}_2 + \hat{b}^2 \partial_b \tilde{T}_2, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \\ &= \hat{b}_s (T_1, \mathcal{L}^* \Phi_{0, \hat{B}_0}) + |\hat{b}_s| O \left(\int |\tilde{T}_1 - T_1| + b |\partial_b \tilde{T}_1| + b |\tilde{T}_2| + b^2 |\partial_b \tilde{T}_2| \right). \end{aligned}$$

We have from (7.2.40) :

$$\hat{b}_s(T_1, \mathcal{L}^* \Phi_{0, \hat{B}_0}) = \hat{b}_s(\Lambda Q, \Phi_{0, \hat{B}_0}) = \hat{b}_s \left[-(32\pi) \log \hat{B}_0 + O(1) \right],$$

and we estimate the error terms un brute force using the estimates of Proposition 7.3.5 :

$$\begin{aligned} & \int |\tilde{T}_1 - T_1| + \hat{b} |\partial_b \tilde{T}_1| + \hat{b} |\tilde{T}_2| + \hat{b}^2 |\partial_b \tilde{T}_2| \lesssim \int_{B_1 \leq r \leq 2B_1} \frac{1}{r^2} \\ & + b \int \left[\frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4 |\log b|} \mathbf{1}_{6B_0 \leq r \leq 2B_1} \right] \\ & \lesssim 1. \end{aligned}$$

We also estimate using the definition (7.2.36) and Proposition 7.3.5 :

$$|(\tilde{Q}_b - Q, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0})| \lesssim |\hat{b}_s| \int_{\hat{B}_0 \leq r \leq 2\hat{B}_0} r^2 |\mathcal{L}(\tilde{Q}_b - Q)| \lesssim |\hat{b}_s| \int_{\frac{B_0}{4} \leq r \leq 4B_0} \frac{r^2}{r^4} \lesssim |\hat{b}_s|. \quad (7.5.27)$$

We therefore have obtained the preliminary bound using also (7.5.3), (7.5.5) :

$$\frac{d}{ds} \left\{ (\tilde{Q}_b - Q + \varepsilon, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \right\} = \hat{b}_s \left[-(32\pi) \log B_0 + O(1) \right].$$

We now estimate the RHS of (7.5.26). The main linear term is treated using (7.2.33), (7.2.45) which yield a sharp bound :

$$|(\varepsilon_2, \mathcal{L}^* \Phi_{0, \hat{B}_0})| \lesssim \int_{r \geq \hat{B}_0} |\varepsilon_2| \lesssim \sqrt{b} \|\varepsilon_2\|_{L_Q^2}.$$

We estimate like for the proof of (7.5.27) :

$$|(\tilde{Q}_b - Q, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0})| \lesssim |\hat{b}_s|.$$

The second linear term is estimated in brute force :

$$\begin{aligned} |(\varepsilon, \partial_s \mathcal{L}^* \Phi_{0, \hat{B}_0})| &= \left| \left(\varepsilon_2, \frac{(\hat{B}_0)_s}{\hat{B}_0} (r\chi') \left(\frac{r}{\hat{B}_0} \right) r^2 \right) \right| \lesssim \frac{|\hat{b}_s|}{b} \|\varepsilon_2\|_{L_Q^2} \left(\int_{B_0 \leq r \leq 2B_0} \frac{r^4}{r^4} \right)^{\frac{1}{2}} \\ &\lesssim \frac{|\hat{b}_s|}{b\sqrt{b}} \|\varepsilon_2\|_{L_Q^2} \lesssim |\hat{b}_s| \end{aligned}$$

where we used the bootstrap bound (7.4.9) in the last step. The leading order *flux* term is computed from (7.2.40), (7.2.43), (7.3.20), (7.5.5) :

$$\begin{aligned} -c_b b^2 (\chi_{\frac{B_0}{4}} T_1, \mathcal{L}^* \Phi_{0, \hat{B}_0}) &= -c_b b^2 \left[(\mathcal{L} T_1, \Phi_{0, \hat{B}_0}) + ((\chi_{\frac{B_0}{4}} - 1) T_1, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \right] \\ &= \frac{2b^2}{|\log b|} \left[1 + O\left(\frac{1}{|\log b|}\right) \right] \left[32\pi \log B_0 + O(1) + \int_{r \leq 2\hat{B}_0} \frac{\mathbf{1}_{r \geq \frac{B_0}{4}}}{1+r^2} \right] \\ &= 32\pi b^2 \left[1 + O\left(\frac{1}{|\log b|}\right) \right]. \end{aligned}$$

Next, from (7.2.43), (7.4.9) :

$$\begin{aligned} \left| \frac{\lambda_s}{\lambda} (\Lambda \varepsilon, \mathcal{L}^* \Phi_{0, \hat{B}_0}) \right| &\lesssim b \int_{r \leq 2B_0} (|\varepsilon| + |y \cdot \nabla \varepsilon|) \lesssim \sqrt{b} (\|\varepsilon\|_{L^2} + \|y \cdot \nabla \varepsilon\|_{L^2}) \\ &\lesssim C(M) \sqrt{b} \|\varepsilon_2\|_{L_Q^2}. \end{aligned}$$

We now treat the F terms. The $\tilde{\Psi}_b$ term is estimated from the degenerate flux estimate (7.3.55) :

$$|(\mathcal{L} \tilde{\Psi}_b, \Phi_{0, \hat{B}_0})| \lesssim \frac{b^2}{|\log b|}.$$

To treat the small linear term $\Theta_b(\varepsilon)$, we first extract from Proposition 7.3.1 the pointwise bounds :

$$|r^i \partial_r^i (\tilde{Q}_b - Q)| \lesssim b \frac{\mathbf{1}_{r \leq 2B_1}}{1+r^2}, \quad |r^i \partial_r^i \nabla \phi_{\tilde{Q}_b - Q}| \lesssim b \frac{1 + |\log r|}{1+r} \quad (7.5.28)$$

which imply the pointwise bounds :

$$|\Theta_b(\varepsilon)| \lesssim b \left[\frac{1 + |\log r|}{1+r} |\nabla \varepsilon| + \frac{|\varepsilon|}{1+r^2} + \frac{|\nabla \phi_\varepsilon|}{1+r^3} \right], \quad (7.5.29)$$

$$|\nabla \Theta_b(\varepsilon)| \lesssim b \left[\frac{1 + |\log r|}{1+r} |\nabla^2 \varepsilon| + \frac{1 + |\log r|}{1+r^2} |\nabla \varepsilon| + \frac{|\nabla^2 \phi_\varepsilon| + |\varepsilon|}{1+r^3} + \frac{|\nabla \phi_\varepsilon|}{1+r^4} \right]. \quad (7.5.30)$$

We also observe by definition the cancellation $(\Theta_b(\varepsilon), 1) = 0$ and therefore estimate from (7.2.45) and Proposition 7.7.4 :

$$\begin{aligned} |(\Theta_b(\varepsilon), \mathcal{L}^* \Phi_{0, \hat{B}_0})| &\lesssim \int_{r \geq \hat{B}_0} |\Theta_b(\varepsilon)| \lesssim b \int \left[\frac{(1 + |\log r|) |\nabla \varepsilon|}{1+r} + \frac{|\varepsilon|}{1+r^2} + \frac{|\nabla \phi_\varepsilon|}{1+r^3} \right] \\ &\lesssim b C(M) \|\varepsilon_2\|_{L_Q^2} \lesssim \frac{b^2}{|\log b|} \end{aligned}$$

and similarly for the nonlinear term using (7.7.19) :

$$|(N(\varepsilon), \mathcal{L}^* \Phi_{0, \hat{B}_0})| \lesssim \int |N(\varepsilon)| \lesssim \int \varepsilon^2 + |\nabla \phi_\varepsilon| |\nabla \varepsilon| \lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 \lesssim \frac{b^2}{|\log b|}.$$

We now treat the terms induced by modulation. We use the L^1 critical relation $\int \Lambda Q = \int \Lambda \tilde{Q}_b = 0$, (7.2.45), (7.4.30) and (7.3.49), (7.3.51) to estimate :

$$\begin{aligned} \left| \frac{\lambda_s}{\lambda} + b \right| |(\mathcal{L} \Lambda \tilde{Q}_b, \Phi_{0, \hat{B}_0})| &\lesssim b^{\frac{3}{2}} \int_{r \geq \hat{B}_0} |\Lambda(\tilde{Q}_b - Q)| \\ &\lesssim b^{\frac{3}{2}} \int_{\frac{B_0}{2} \leq r \leq 2B_1} \frac{b}{1+r^2} \lesssim \frac{b^2}{|\log b|}. \end{aligned}$$

Injecting the collection of above bounds into (7.5.26) yields :

$$\hat{b}_s [-(32\pi) \log B_0 + O(1)] = 32\pi b^2 + O \left(\frac{b^2}{|\log b|} + C(M) \sqrt{b} \|\varepsilon_2\|_{L_Q^2} + |\hat{b}_s| \right).$$

Hence using (7.5.5) :

$$\begin{aligned} \left| \tilde{b}_s + \frac{b^2}{|\log B_0|} \right| &\lesssim \frac{1}{|\log B_0|} \left[\frac{b^2}{|\log b|} + C(M)\sqrt{b}\|\varepsilon_2\|_{L_Q^2} + |\hat{b}_s| \right] \\ &\lesssim \frac{b^2}{|\log b|^2} + C(M)\frac{\sqrt{b}}{|\log b|}\|\varepsilon_2\|_{L_Q^2} + \frac{|\hat{b}_s|}{|\log b|} \end{aligned}$$

and (7.5.24) now follows using the bootstrap bound (7.4.9). \square

7.5.3 The monotonicity formula

We now turn to the heart of the energy method and claim the following monotonicity formula at the H_Q^2 level :

Proposition 7.5.7 (H_Q^2 monotonicity). *There holds :*

$$\begin{aligned} &\frac{d}{dt} \left\{ \frac{(\mathcal{M}\varepsilon_2, \varepsilon_2)}{\lambda^2} + O \left(\frac{C(M)b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{C(M)b^3}{\lambda^2 |\log b|^2} \right) \right\} \\ &\leq \frac{bC(M)}{\lambda^4} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned} \quad (7.5.31)$$

Proof of Proposition 7.5.7

step 1 Introduction of the radiation.

The ε equation (7.4.16) is forced on the RHS by the b_s term which satisfies pointwise the weak estimate (7.4.29) only which is not good enough to close (7.5.7). This is a consequence of the fact that the elements of the kernel of \mathcal{L}^* grow too fast in space. We therefore aim at relying on the sharp bounds of Lemma 7.5.5 and on the lifted parameter \hat{b} . Recall (7.5.7) which yields

$$\partial_s(\tilde{Q}_b + \varepsilon) = \partial_s(\tilde{Q}_{\hat{b}} + \hat{\varepsilon}), \quad \varepsilon = \hat{\varepsilon} - \zeta,$$

and rewrite the ε equation (7.4.16) as :

$$\partial_s \hat{\varepsilon} - \frac{\lambda_s}{\lambda} \Lambda \hat{\varepsilon} = \mathcal{L} \hat{\varepsilon} + \hat{\mathcal{F}}, \quad \hat{\mathcal{F}} = F + \widehat{\text{Mod}} + E_1 + E_2, \quad (7.5.32)$$

with F given by (7.4.17) and :

$$E_1 = -\mathcal{L}\zeta - \frac{\lambda_s}{\lambda} \Lambda \zeta, \quad (7.5.33)$$

$$E_2 = -c_b b^2 \chi_{\frac{B_0}{4}} T_1, \quad (7.5.34)$$

$$\widehat{\text{Mod}} = -\partial_s \tilde{Q}_{\hat{b}} + \left(\frac{\lambda_s}{\lambda} + b \right) \tilde{Q}_b. \quad (7.5.35)$$

step 2 Renormalized variables and energy identity. Let

$$w = \varepsilon_\lambda, \quad \hat{w} = \hat{\varepsilon}_\lambda.$$

We introduce the differential operator of order one :

$$A_\lambda w = Q_\lambda \nabla \mathcal{M}_\lambda w = \nabla w + \nabla \phi_{Q_\lambda} w + Q_\lambda \nabla \phi_w \quad (7.5.36)$$

and the suitable derivatives¹⁶

$$\begin{cases} w_1 = A_\lambda w, & \hat{w}_1 = A_\lambda \hat{w} \\ \varepsilon_1 = A\varepsilon, & \hat{\varepsilon}_1 = A\hat{\varepsilon} \end{cases}, \quad \begin{cases} w_2 = \nabla \cdot w_1 = \mathcal{L}_\lambda w, & \hat{w}_2 = \nabla \cdot \hat{w}_1 = \mathcal{L}_\lambda \hat{w} \\ \varepsilon_2 = \nabla \cdot \varepsilon = \mathcal{L}\varepsilon, & \hat{\varepsilon}_2 = \nabla \cdot \hat{\varepsilon} = \mathcal{L}\hat{\varepsilon} \end{cases} \quad (7.5.37)$$

and similarly for the radiation :

$$\zeta_1 = A\zeta, \quad \zeta_2 = \mathcal{L}\zeta.$$

We compute from (7.5.32) the equation satisfied by $\hat{w}, \hat{w}_1, \hat{w}_2$:

$$\partial_t \hat{w} = \hat{w}_2 + \frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda, \quad (7.5.38)$$

$$\partial_t \hat{w}_1 = A_\lambda \hat{w}_2 + \frac{1}{\lambda^2} A_\lambda \hat{\mathcal{F}}_\lambda + [\partial_t, A_\lambda] \hat{w}, \quad (7.5.39)$$

$$\partial_t \hat{w}_2 = \mathcal{L}_\lambda \hat{w}_2 + \frac{1}{\lambda^2} \mathcal{L}_\lambda \hat{\mathcal{F}}_\lambda + \nabla \cdot ([\partial_t, A_\lambda] \hat{w}). \quad (7.5.40)$$

We compute the energy identity for \hat{w}_2 , and will all along the proof repeatedly use (7.4.15) :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) &= (\partial_t \hat{w}_2, \mathcal{M}_\lambda \hat{w}_2) - \int \frac{\partial_t Q_\lambda}{2Q_\lambda^2} \hat{w}_2^2 \\ &= (\mathcal{L}_\lambda \hat{w}_2 + \frac{\mathcal{L}_\lambda \hat{\mathcal{F}}_\lambda}{\lambda^2} + \nabla \cdot ([\partial_t, A_\lambda] \hat{w}), \mathcal{M}_\lambda \hat{w}_2) - \int \frac{\partial_t Q_\lambda}{2Q_\lambda^2} \hat{w}_2^2 \\ &= - \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + (\hat{w}_2, \frac{1}{\lambda^2} \mathcal{M}_\lambda \mathcal{L}_\lambda F_\lambda) - \int [\partial_t, A_\lambda] \hat{w} \cdot \nabla (\mathcal{M}_\lambda \hat{w}_2) \\ &\quad + \int \frac{\hat{w}_2^2}{\lambda^2 Q_\lambda^2} \left[\left(\frac{\lambda_s}{\lambda} + \hat{b} \right) (\Lambda Q)_\lambda \right] - \int \frac{\hat{b}(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w}_2^2. \end{aligned}$$

step 3 Repulsivity and sharp control of tails at infinity. We need to treat the last quadratic term in (7.5.41) which has no definite sign and critical size for the analysis, and corresponds to an interaction of radiation with the soliton profile. For this, we will use dissipation and repulsivity properties of \mathcal{L} measured in suitable weighted norms. We will in particular use the fundamental degeneracy (7.2.11)¹⁷ :

$$\frac{\Lambda Q}{Q} + 2 = -\phi_{\Lambda Q} = O\left(\frac{1}{1+r^2}\right), \quad (7.5.41)$$

which will allow us to measure in a sharp way the size of tails at infinity. Our approach is deeply related to the construction of suitable Morawetz type multiplier in the dispersive setting of [88],

16. which correspond to the factorization of the linearized operator \mathcal{L} in two operators of order one : $\mathcal{L} = \nabla \cdot (A)$.

17. which is a consequence of the L^1 critical identity $\int \Lambda Q = 0$.

[72].

We compute from (7.5.38), (7.5.40) :

$$\begin{aligned} \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} \right\} &= \int \hat{w}_2 \hat{w} \frac{d}{dt} \left\{ \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \right\} \\ &+ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \left[\mathcal{L}_\lambda \hat{w}_2 + \frac{1}{\lambda^2} \mathcal{L}_\lambda \hat{\mathcal{F}}_\lambda + \nabla \cdot ([\partial_t, A_\lambda] \hat{w}) \right] + \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2 \left[\hat{w}_2 + \frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right]. \end{aligned}$$

Let us compute the leading order terms in this identity. First the potential term :

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \right\} &= \frac{d}{dt} \left\{ \hat{b} \left(\frac{\Lambda Q}{Q^2} \right) \left(\frac{x}{\lambda(t)} \right) \right\} \\ &= \frac{1}{\lambda^2} \left[\hat{b}_s \frac{\Lambda Q}{Q^2} - \hat{b} \frac{\lambda_s}{\lambda} y \cdot \nabla \left(\frac{\Lambda Q}{Q^2} \right) \right] (y) \\ &= \frac{1}{\lambda^2 Q} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s + O \left(\frac{|\hat{b}_s| + \hat{b} |\frac{\lambda_s}{\lambda}|}{1+r^2} \right) \right]. \end{aligned}$$

Next, we observe the integration by parts :

$$\begin{aligned} \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \mathcal{L}_\lambda \hat{w}_2 &= -2\hat{b} \int \frac{\hat{w} \mathcal{L}_\lambda \hat{w}_2}{\lambda^2 Q_\lambda} + \int \frac{\hat{b}(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \mathcal{L}_\lambda \hat{w}_2 \\ &= \frac{2\hat{b}}{\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[\frac{\hat{w}}{Q_\lambda} + \phi_{\hat{w}} \right] - \frac{\hat{b}}{\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\ &= -\frac{2\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) - \frac{\hat{b}}{\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right]. \end{aligned}$$

We therefore obtain :

$$\begin{aligned} \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} \right\} &= \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2^2 - \frac{2\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) \\ &+ \int \frac{\hat{\varepsilon} \hat{\varepsilon}_2}{\lambda^6 Q} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s \right] \\ &- \frac{\hat{b}}{\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\ &+ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \left[\frac{1}{\lambda^2} \mathcal{L}_\lambda \hat{\mathcal{F}}_\lambda + \nabla \cdot ([\partial_t, A_\lambda] \hat{w}) \right] + \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2 \left[\frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right] \\ &+ O \left(\int \left[|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right| \right] \frac{\hat{\varepsilon}_2 \hat{\varepsilon}}{\lambda^6 (1+r^2) Q} \right) \end{aligned} \tag{7.5.42}$$

We now compute the energy identity for \hat{w}_1 using (7.5.39) :

$$\frac{1}{2} \frac{d}{dt} \left\{ \int \frac{\hat{b} \hat{w}_1^2}{\lambda^2 Q_\lambda} \right\} = \frac{1}{2} \int \hat{w}_1^2 \frac{d}{dt} \left\{ \frac{\hat{b}}{\lambda^2 Q_\lambda} \right\} + \hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot \left[A_\lambda \hat{w}_2 + \frac{1}{\lambda^2} A_\lambda \hat{\mathcal{F}}_\lambda + [\partial_t, A_\lambda] \hat{w} \right].$$

We compute the dominant terms :

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\hat{b}}{\lambda^2 Q_\lambda} \right\} &= \frac{d}{dt} \left\{ \frac{\hat{b}}{Q \left(\frac{x}{\lambda(t)} \right)} \right\} = \frac{1}{\lambda^2} \left[\frac{\hat{b}_s}{Q} + \hat{b} \frac{\frac{\lambda_s}{\lambda} y \cdot \nabla Q}{Q^2} \right] \\ &= \frac{1}{\lambda^2 Q} \left[\hat{b}_s - 4\hat{b} \frac{\lambda_s}{\lambda} + O \left(\frac{|\hat{b}_s| + |\hat{b}| \frac{|\lambda_s|}{\lambda}}{1+r^2} \right) \right], \end{aligned}$$

and integrate by parts using (7.5.36) and the definition of \hat{w}_i :

$$\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot A_\lambda \hat{w}_2 = -\frac{\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \nabla \cdot \hat{w}_1) = -\frac{\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2).$$

This leads to the second identity¹⁸ :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \int \frac{\hat{b} \hat{w}_1^2}{\lambda^2 Q_\lambda} \right\} &= -\frac{\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) + \int \frac{\hat{\varepsilon}_1^2}{2\lambda^6 Q} \left[\hat{b}_s - 4\hat{b} \frac{\lambda_s}{\lambda} \right] \\ &+ \hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \left[\frac{1}{\lambda^2} A_\lambda \hat{\mathcal{F}}_\lambda + [\partial_t, A_\lambda] \hat{w} \right] + O \left(\int \frac{\hat{\varepsilon}_1^2}{\lambda^6 Q} \left[\frac{|\hat{b}_s| + |\hat{b}| \frac{|\lambda_s|}{\lambda}}{1+r^2} \right] \right). \end{aligned}$$

We combine this with (7.5.42) to derive :

$$\begin{aligned} \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2 \hat{w} - 2 \int \frac{\hat{b} \hat{w}_1^2}{\lambda^2 Q_\lambda} \right\} &= \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2^2 + \frac{2\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) \\ &+ \int \frac{\hat{\varepsilon}_2 \hat{\varepsilon}_1 + \hat{\varepsilon}_1^2}{\lambda^6 Q} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s \right] \\ &- \frac{\hat{b}}{\lambda^2} \int Q_\lambda \nabla (\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\ &+ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}_\lambda \hat{\mathcal{F}}_\lambda + \hat{w}_2 \frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right] - 4\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot \left[\frac{1}{\lambda^2} A_\lambda \hat{\mathcal{F}}_\lambda \right] \\ &+ \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot ([\partial_t, A_\lambda] \hat{w}) - 4\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot [\partial_t, A_\lambda] \hat{w} \\ &+ O \left(\int \frac{|\hat{\varepsilon}_2 \hat{\varepsilon}_1| + |\hat{\varepsilon}_1|^2}{\lambda^6 Q (1+r^2)} (|\hat{b}_s| + |\hat{b}| \frac{|\lambda_s|}{\lambda}) \right). \end{aligned}$$

We now observe by integration by parts :

$$\int \frac{\hat{\varepsilon}_2 \hat{\varepsilon}_1}{Q} = - \int Q \nabla \mathcal{M} \hat{\varepsilon} \cdot \nabla \left(\frac{\hat{\varepsilon}_1}{Q} \right) = - \int \frac{\hat{\varepsilon}_1^2}{Q} + \int Q \nabla \mathcal{M} \hat{\varepsilon} \cdot \nabla \phi_{\hat{\varepsilon}}$$

and hence the cancellation¹⁹ :

$$\int \frac{\hat{\varepsilon}_2 \hat{\varepsilon}_1 + \hat{\varepsilon}_1^2}{Q} = \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}. \quad (7.5.43)$$

18. This identity reflects the fact that the dual Hamiltonian $\nabla \cdot A_\lambda$ driving the w_1 equation (7.5.39) is repulsive, and this is reminiscent from geometric equations, see [93], [88].

19. which corresponds to an improved decay at infinity, each term in the left hand side of (7.5.43) being too slowly decaying at infinity to be treated separately.

Similarly :

$$\begin{aligned}
& - \int \frac{2}{Q} (\hat{\varepsilon} \mathcal{L} \hat{\mathcal{F}} + \hat{\varepsilon}_2 \hat{\mathcal{F}}) - 4 \int \frac{1}{Q} \hat{\varepsilon}_1 \cdot A \hat{\mathcal{F}} \\
= & 2 \int Q \nabla \mathcal{M} \hat{\mathcal{F}} \cdot \nabla (\mathcal{M} \hat{\varepsilon} - \phi_{\hat{\varepsilon}}) + 2 \int Q \nabla \mathcal{M} \hat{\varepsilon} \cdot \nabla (\mathcal{M} \hat{\mathcal{F}} - \phi_{\hat{\mathcal{F}}}) - 4 \int Q \nabla \mathcal{M} \hat{\varepsilon} \cdot \nabla \mathcal{M} \hat{\mathcal{F}} \\
= & -2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\mathcal{F}}} - 2 \int Q \nabla \mathcal{M} \hat{\mathcal{F}} \cdot \nabla \phi_{\hat{\varepsilon}}
\end{aligned}$$

and thus :

$$\begin{aligned}
& \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \left[\frac{\hat{w}}{\lambda^2} \mathcal{L}_\lambda \hat{\mathcal{F}}_\lambda + \hat{w}_2 \frac{1}{\lambda^2} \hat{\mathcal{F}}_\lambda \right] - 4\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot \left[\frac{1}{\lambda^2} A_\lambda \hat{\mathcal{F}}_\lambda \right] \\
= & \frac{\hat{b}}{\lambda^6} \left\{ \int \frac{\Lambda Q + 2Q}{Q^2} (\hat{\varepsilon} \mathcal{L} \hat{\mathcal{F}} + \varepsilon_2 \hat{\mathcal{F}}) - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\mathcal{F}}} - 2 \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} \hat{\mathcal{F}} \right\}.
\end{aligned}$$

We finally obtain the identity :

$$\begin{aligned}
& \frac{d}{dt} \left\{ \int \frac{\hat{b}(\Lambda Q + 2Q)}{\lambda^4 Q^2} \hat{\varepsilon}_2 \hat{\varepsilon} - 2 \int \frac{\hat{b} \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}}{\lambda^4} \right\} = \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w}_2^2 + \frac{2\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) \\
& + \int \frac{\hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}}{\lambda^6} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s \right] \\
& - \frac{\hat{b}}{\lambda^2} \int Q_\lambda \nabla (\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\
& + \int \frac{\hat{b}(\Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot ([\partial_t, A_\lambda] \hat{w}) - 4\hat{b} \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot [\partial_t, A_\lambda] \hat{w} \\
& + \frac{\hat{b}}{\lambda^6} \left\{ \int \frac{\Lambda Q + 2Q}{Q^2} (\hat{\varepsilon} \mathcal{L} \hat{\mathcal{F}} + \varepsilon_2 \hat{\mathcal{F}}) - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\mathcal{F}}} - 2 \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} \hat{\mathcal{F}} \right\} \\
& + O \left(\int \frac{|\hat{\varepsilon}_2 \hat{\varepsilon}| + |\hat{\varepsilon}_1|^2}{\lambda^6 Q(1+r^2)} (|\hat{b}_s| + \hat{b} |\frac{\lambda_s}{\lambda}|) \right).
\end{aligned}$$

We combine this with the energy identity (7.5.41) and obtain the modified and manageable

energy identity :

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) + \int \frac{\hat{b}(\Lambda Q + 2Q)}{\lambda^4 Q^2} \hat{\varepsilon}_2 \hat{\varepsilon} - 2 \int \frac{\hat{b} \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}}{\lambda^4} \right\} \\
&= - \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + \frac{\hat{b}}{\lambda^2} (\mathcal{M}_\lambda \hat{w}_2, \hat{w}_2) + \frac{1}{\lambda^6} (\hat{\varepsilon}_2, \mathcal{M} \mathcal{L} \hat{\mathcal{F}}) \\
&+ \int \frac{\hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}}{2\lambda^6} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s \right] \\
&+ \frac{\hat{b}}{2\lambda^6} \left\{ \int \frac{\Lambda Q + 2Q}{Q^2} (\hat{\varepsilon} \mathcal{L} \hat{\mathcal{F}} + \hat{\varepsilon}_2 \hat{\mathcal{F}}) - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\mathcal{F}}} - 2 \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} \hat{\mathcal{F}} \right\} \\
&- \frac{\hat{b}}{2\lambda^2} \int Q_\lambda \nabla (\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \\
&- \int [\partial_t, A_\lambda] \hat{w} \cdot \nabla (\mathcal{M}_\lambda \hat{w}_2) + \int \frac{b(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot ([\partial_t, A_\lambda] \hat{w}) - 2b \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot [\partial_t, A_\lambda] \hat{w} \\
&+ \int \frac{\hat{w}_2^2}{\lambda^2 Q_\lambda^2} \left[\left(\frac{\lambda_s}{\lambda} + \hat{b} \right) (\Lambda Q)_\lambda \right] \\
&+ O \left(\int \frac{|\hat{\varepsilon}_2 \varepsilon| + |\hat{\varepsilon}_1|^2}{\lambda^6 Q(1+r^2)} (|\hat{b}_s| + \hat{b} \left| \frac{\lambda_s}{\lambda} \right|) \right).
\end{aligned} \tag{7.5.44}$$

We now aim at estimating all terms in (7.5.44). We will implicitly use the bounds of Lemma 7.7.4. We will make an essential use of the improved decay (7.5.41).

step 4. Boundary term in time. We estimate using Lemma 7.5.3 :

$$\begin{aligned}
\left| \int \frac{\hat{b}(\Lambda Q + 2Q)}{\lambda^4 Q^2} \hat{\varepsilon}_2 \hat{\varepsilon} \right| &\lesssim \frac{b}{\lambda^4} \int \frac{|\hat{\varepsilon}_2 \hat{\varepsilon}|}{(1+r^2)Q} \lesssim \frac{b}{\lambda^4} \left(\int \frac{\hat{\varepsilon}_2^2}{Q} \right)^{\frac{1}{2}} \left(\int \hat{\varepsilon}^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{C(M)b}{\lambda^4} \left[\|\varepsilon_2\|_{L_Q^2}^2 + \|\zeta_2\|_{L_Q^2}^2 + \|\zeta\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 \right] \\
&\lesssim \frac{C(M)b^3}{|\log b|^2}
\end{aligned}$$

and using (7.7.16), (7.7.19) :

$$\begin{aligned}
&\left| \frac{1}{\lambda^4} \int b \hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}} \right| \lesssim \frac{b}{\lambda^4} \left(\int (1+r^2) |\hat{\varepsilon}_1|^2 \right)^{\frac{1}{2}} \left(\int \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^2} \right)^{\frac{1}{2}} \\
&\lesssim \frac{b}{\lambda^4} \left[\int (1+r^2) |\varepsilon_1|^2 + \int \frac{|\nabla \phi_{\varepsilon}|^2}{1+r^2} + \int (1+r^2) |\nabla \zeta|^2 + \int |\zeta|^2 + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right] \\
&\lesssim \frac{b}{\lambda^4} \left[|\log b|^C \left(\|\varepsilon_2\|_{L_Q^2}^2 + b^5 \right) + \frac{b^2}{|\log b|^2} \right] \lesssim \frac{C(M)b^3}{|\log b|^2}.
\end{aligned}$$

step 5. Lower order quadratic terms. We now estimate the lower order quadratic terms in the RHS of (7.5.44). First the terms which are estimated using dissipation, (7.7.15) and Lemma

7.5.3 :

$$\begin{aligned}
& \left| \frac{b}{2\lambda^2} \int Q_\lambda \nabla(\mathcal{M}_\lambda \hat{w}_2) \cdot \nabla \left[2\phi_{\hat{w}} + \frac{(2Q + \Lambda Q)_\lambda}{\lambda^2 Q_\lambda^2} \hat{w} \right] \right| \\
& \leq \frac{1}{100} \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + \frac{Cb^2}{\lambda^6} \int Q \left[|\nabla \phi_{\hat{\varepsilon}}|^2 + \frac{|\nabla \hat{\varepsilon}|^2}{(1+r^4)Q^2} + \frac{\hat{\varepsilon}^2}{(1+r^6)Q^2} \right] \\
& \leq \frac{1}{100} \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + C(M) \frac{b^2}{\lambda^6} \left[\|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \right].
\end{aligned}$$

Next, we observe from (7.5.36) and (7.5.41) the decay estimate :

$$|[\partial_t, A_\lambda] \hat{w}| \lesssim \frac{b}{\lambda^2} \left[\frac{|\hat{\varepsilon}|}{1+r^3} + \frac{|\nabla \phi_{\hat{\varepsilon}}|}{1+r^4} \right]$$

which leads using (7.7.15), (7.7.16), (7.7.19) to the bounds :

$$\begin{aligned}
& \left| \int [\partial_t, A_\lambda] \hat{w} \cdot \nabla(\mathcal{M}_\lambda w_2) \right| \leq \frac{1}{100} \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + C(M) \frac{b^2}{\lambda^6} \int \frac{1}{Q} \left[\frac{|\hat{\varepsilon}|^2}{1+r^6} + \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^8} \right] \\
& \leq \frac{1}{100} \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + C(M) \frac{b^2}{\lambda^6} \left[\|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \right], \\
& \left| \int \frac{b(\Lambda Q)_\lambda}{2\lambda^2 Q_\lambda^2} \hat{w} \nabla \cdot ([\partial_t, A_\lambda] \hat{w}) \right| \lesssim \frac{b^2}{\lambda^6} \int \left[\frac{|\hat{\varepsilon}|}{1+r^3} + \frac{|\nabla \phi_{\hat{\varepsilon}}|}{1+r^4} \right] \left[\frac{|\nabla \hat{\varepsilon}|}{Q} + \frac{|\hat{\varepsilon}|}{(1+r)Q} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + \int |\zeta|^2 + \int (1+r^2) |\nabla \zeta|^2 + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \right], \\
& \left| b \int \frac{\hat{w}_1}{\lambda^2 Q_\lambda} \cdot [\partial_t, A_\lambda] \hat{w} \right| \lesssim \frac{b^2}{\lambda^6} \int \frac{|\hat{\varepsilon}_1|}{Q} \left[\frac{|\hat{\varepsilon}|}{1+r^3} + \frac{|\nabla \phi_{\hat{\varepsilon}}|}{1+r^4} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[\int (1+r^2) |\varepsilon_1|^2 + \int |\varepsilon|^2 + \int \frac{|\nabla \phi_\varepsilon|^2}{1+r^2} + \int |\zeta|^2 + \int (1+r^2) |\nabla \zeta|^2 + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \right].
\end{aligned}$$

We now estimate using from (7.4.28), (7.5.3), (7.5.25) the rough bounds :

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim b, \quad |\hat{b}_s| \lesssim b^2,$$

and therefore :

$$\begin{aligned}
& \left| \int \frac{\hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}}{2\lambda^6} \left[8\hat{b} \frac{\lambda_s}{\lambda} - 2\hat{b}_s \right] \right| \\
& \lesssim \frac{b^2}{\lambda^6} \int |\hat{\varepsilon}_1 \cdot \nabla \phi_{\hat{\varepsilon}}| \lesssim \frac{b^2}{\lambda^6} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + b^4 + \int (1+r^2) |\nabla \zeta|^2 + \int |\zeta|^2 + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right] \\
& \lesssim \frac{b^2}{\lambda^6} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \right],
\end{aligned}$$

and for the error term :

$$\begin{aligned} & \int \frac{|\hat{\varepsilon}_2 \hat{\varepsilon}| + |\hat{\varepsilon}_1|^2}{\lambda^6 Q(1+r^2)} (|b_s| + b \left| \frac{\lambda_s}{\lambda} + b \right| + b^2) \\ & \lesssim \frac{b^2}{\lambda^6} \int [(1+r^4)\hat{\varepsilon}_2^2 + (1+r^2)|\hat{\varepsilon}_1|^2 + \hat{\varepsilon}^2] \\ & \lesssim C(M) \frac{b^2}{\lambda^6} \left[\|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \right]. \end{aligned}$$

We now estimate using the bound (7.4.28), (7.5.3) :

$$\left| \frac{\lambda_s}{\lambda} + \hat{b} \right| \lesssim \frac{b}{|\log b|}.$$

Note that this bound is very bad to treat the radiation term ζ and we will need some additional cancellation. We recall the decomposition (7.5.22), (7.5.23) and first estimate using Lemma 7.5.3 :

$$\begin{aligned} & \left| \int \frac{\hat{w}_2^2}{\lambda^2 Q_\lambda^2} \left[\left(\frac{\lambda_s}{\lambda} + \hat{b} \right) (\Lambda Q)_\lambda \right] \right| \lesssim \frac{b}{|\log b| \lambda^6} \left| \int \frac{(\varepsilon_2 + \mathcal{L}\zeta_{\text{sm}} + \xi + (b - \hat{b})\Lambda Q)^2}{Q^2} \Lambda Q \right| \\ & \lesssim \frac{b}{|\log b| \lambda^6} \left[\|\varepsilon_2\|_{L_Q^2}^2 + \|\mathcal{L}\zeta_{\text{sm}}\|_{L_Q^2}^2 + \|\xi\|_{L_Q^2}^2 + \frac{b^2}{|\log b|^2} \left| \int \left(\frac{\Lambda Q}{Q} \right)^2 \Lambda Q \right| \right. \\ & + \frac{b}{|\log b|} \left| \int (\varepsilon_2 + \mathcal{L}\zeta_{\text{sm}}) \frac{(\Lambda Q)^2}{Q^2} \right| + \frac{b^2}{|\log b|^2} \int_{r \geq B_1} \frac{1}{1+r^4} \left. \right] \\ & \lesssim \frac{b}{\lambda^6} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} + \frac{b^2}{|\log b|^2} \left| \int \left(\frac{\Lambda Q}{Q} \right)^2 \Lambda Q \right| + \frac{b}{|\log b|^2} \left| \int (\varepsilon_2 + \mathcal{L}\zeta_{\text{sm}}) \frac{(\Lambda Q)^2}{Q^2} \right| \right]. \end{aligned}$$

We now use the cancellation (7.5.41) to estimate

$$\left(\frac{\Lambda Q}{Q} \right)^2 = 4 + O \left(\frac{1}{1+r^2} \right)$$

and hence using (7.2.33), (7.5.11) :

$$\left| \int \mathcal{L}\zeta_{\text{sm}} \frac{(\Lambda Q)^2}{Q^2} \right| \lesssim \int \frac{|\mathcal{L}\zeta_{\text{sm}}|}{1+r^2} \lesssim \|\mathcal{L}\zeta_{\text{sm}}\|_{L^2} \lesssim \frac{b^2}{|\log b|}.$$

We now claim the algebraic identity

$$\mathcal{M}(\Lambda^2 Q) = \left(\frac{\Lambda Q}{Q} \right)^2 \tag{7.5.45}$$

which is proved below, and conclude using (7.2.4) :

$$\int \left(\frac{\Lambda Q}{Q} \right)^2 \Lambda Q = (\mathcal{M}(\Lambda^2 Q), \Lambda Q) = -2 \int \Lambda^2 Q = 0.$$

Finally, using (7.5.45), (7.2.4), (7.2.33), (7.5.15) :

$$\begin{aligned} \left| \left(\varepsilon_2, \frac{(\Lambda Q)^2}{Q^2} \right) \right| &\lesssim |(\varepsilon_2, \mathcal{M}(\Lambda^2 Q + 2\Lambda Q))| = \left| \int \mathcal{M}\varepsilon_2 \nabla \cdot (r(\Lambda Q + 2Q)) \right| \\ &\lesssim \left(\int Q |\nabla(\mathcal{M}\varepsilon_2)|^2 \right)^{\frac{1}{2}} \left(\int \frac{r^2(\Lambda Q + 2Q)^2}{Q} \right)^{\frac{1}{2}} \\ &\lesssim \left(\int Q |\nabla(\mathcal{M}\hat{\varepsilon}_2)|^2 + \frac{b^4}{|\log b|^2} \right)^{\frac{1}{2}}. \end{aligned}$$

The collection above bounds yields the admissible control :

$$\left| \int \frac{\hat{w}_2^2}{\lambda^2 Q_\lambda^2} \left[\left(\frac{\lambda_s}{\lambda} + \hat{b} \right) (\Lambda Q)_\lambda \right] \right| \lesssim \frac{b}{\lambda^6} \left[b^{\frac{3}{2}} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right] + \frac{1}{100\lambda^6} \int Q |\nabla(\mathcal{M}\hat{\varepsilon}_2)|^2.$$

Proof of (7.5.45) : For any well localized function, we have the commutator formula :

$$\mathcal{M}\Lambda f = \Lambda\mathcal{M}f + \frac{x \cdot \nabla Q}{Q^2} f - 2\phi_f - \frac{\int f}{2\pi}$$

from which using (7.2.4), (7.2.11) and the critical relation $\int \Lambda Q = 0$:

$$\begin{aligned} \mathcal{M}(\Lambda^2 Q) &= \Lambda(\mathcal{M}\Lambda Q) + \frac{x \cdot \nabla Q}{Q^2} \Lambda Q - 2\phi_{\Lambda Q} = -4 + \left(\frac{\Lambda Q}{Q} \right)^2 - 2 \left[\phi_{\Lambda Q} + \frac{\Lambda Q}{Q} \right] \\ &= \left(\frac{\Lambda Q}{Q} \right)^2. \end{aligned}$$

step 6 Leading order E_2 terms. We claim the bounds :

$$|(\hat{\varepsilon}_2, \mathcal{M}\mathcal{L}E_2)| \lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right], \quad (7.5.46)$$

$$\begin{aligned} &\left| \int \frac{\Lambda Q + 2Q}{Q^2} (\hat{\varepsilon}\mathcal{L}E_2 + \hat{\varepsilon}_2 E_2) - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{E_2} - 2 \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M}E_2 \right| \\ &\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned} \quad (7.5.47)$$

Proof of (7.5.10) : We compute using (7.2.33) and $\mathcal{L}T_1 = \Lambda Q$:

$$\begin{aligned} \left| (\varepsilon_2, \mathcal{M}\mathcal{L}(\chi_{\frac{B_0}{4}} T_1)) \right| &= \left| (\varepsilon_2, \mathcal{M}\mathcal{L}[(1 - \chi_{\frac{B_0}{4}})T_1]) \right| \\ &\lesssim \|\varepsilon_2\|_{L_Q^2} \left(\int \frac{|\mathcal{L}[(1 - \chi_{\frac{B_0}{4}})T_1]|^2}{Q} \right)^{\frac{1}{2}}. \end{aligned}$$

We now estimate using the radial representation of the Poisson field and (7.3.2) :

$$\phi'_{(1 - \chi_{\frac{B_0}{4}})T_1} = \frac{1}{r} \int_0^r (1 - \chi_{\frac{B_0}{4}}) T_1 \tau d\tau = O\left(\frac{\log r}{r} \mathbf{1}_{r \geq \frac{B_0}{4}}\right) \quad (7.5.48)$$

and hence the pointwise bound :

$$|\mathcal{L}[(1 - \chi_{\frac{B_0}{4}})T_1]| \lesssim \frac{1}{1 + r^4} \mathbf{1}_{r \geq \frac{B_0}{4}} \quad (7.5.49)$$

which leads to the bound :

$$\left| (\varepsilon_2, c_b b^2 \mathcal{ML}(\chi_{\frac{B_0}{4}} T_1)) \right| \lesssim \frac{b^2}{|\log b|} \|\varepsilon_2\|_{L_Q^2} \left(\int_{r \geq \frac{B_0}{4}} \frac{1}{1 + r^4} \right)^{\frac{1}{2}} \lesssim b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2}.$$

We treat the term involving the radiation as follows. Arguing like for (7.5.21), we estimate :

$$\mathcal{L}(\chi_{\frac{B_0}{4}} T_1) = \Lambda Q + \tilde{\xi}, \quad \tilde{\xi} = O\left(\frac{\mathbf{1}_{r \geq \frac{B_0}{4}}}{1 + r^4}\right) \quad (7.5.50)$$

from which using (7.5.21), (7.5.23), (7.2.2) :

$$\begin{aligned} |(\mathcal{L}\zeta_{\text{big}}, \mathcal{ML}E_2)| &= c_b b^2 \left| ((b - \hat{b})\Lambda Q + \xi, \mathcal{M}(\Lambda Q + \tilde{\xi})) \right| \\ &\lesssim \frac{b^2}{|\log b|} \left[|b - \hat{b}| \int |\tilde{\xi}| + \int |\xi| + \|\xi\|_{L_Q^2} \|\tilde{\xi}\|_{L_Q^2} \right] \\ &\lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

Similarly, using (7.2.33), (7.5.10), (7.5.50) :

$$\begin{aligned} |(\mathcal{L}\zeta_{\text{sm}}, \mathcal{ML}E_2)| &= c_b b^2 \left| (\mathcal{L}\zeta_{\text{sm}}, \mathcal{M}(\Lambda Q + \tilde{\xi})) \right| \\ &= c_b b^2 \left| (\mathcal{L}\zeta_{\text{sm}}, \mathcal{M}\tilde{\xi}) \right| \lesssim \frac{b^2}{|\log b|} \|\mathcal{L}\zeta_{\text{sm}}\|_{L_Q^2} \|\tilde{\xi}\|_{L_Q^2} \\ &\lesssim \frac{b^2}{|\log b|^2} \left(\frac{b^3}{|\log b|^2} \right)^{\frac{1}{2}} \left(\frac{1}{B_0^2} \right)^{\frac{1}{2}} \lesssim \frac{b^4}{|\log b|^2}, \end{aligned}$$

and (7.5.46) is proved.

Proof of (7.5.47) : We estimate in brute force using (7.5.50), (7.7.15) and Lemma 7.5.3 :

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} (\hat{\varepsilon} \mathcal{L}E_2 + \hat{\varepsilon}_2 E_2) \right| &\lesssim b \frac{b^2}{|\log b|} \int \frac{1}{(1 + r^2)Q} \left[\frac{|\hat{\varepsilon}|}{1 + r^4} + \frac{|\hat{\varepsilon}_2|}{1 + r^2} \right] \\ &\lesssim b \frac{b^2}{|\log b|} \left[\|\varepsilon\|_{L^2} + \|\varepsilon_2\|_{L_Q^2} + \|\zeta_2\|_{L_Q^2} + \|\zeta\|_{L^2} \right] \\ &\lesssim b \left[C(M) \frac{b^2}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

Similarly, using (7.7.16) and Lemma 7.5.3 :

$$\begin{aligned} b \left| \int \hat{\varepsilon}_1 \cdot \nabla \phi_{E_2} \right| &\lesssim b \frac{b^2}{|\log b|} \int \frac{1 + |\log r|}{1 + r} \left[|\varepsilon_1| + |\nabla \zeta| + Q |\nabla \phi_\zeta| + \frac{|\zeta|}{1 + r} \right] \\ &\lesssim b \frac{b^2}{|\log b|} \left(\int (1 + r^2) |\varepsilon_1|^2 + |\zeta|^2 + (1 + r^2) |\nabla \zeta|^2 + \frac{|\nabla \phi_\zeta|^2}{1 + r^2} \right)^{\frac{1}{2}} \\ &\lesssim b \frac{b^2}{|\log b|} \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \lesssim b \left[\frac{b^2}{|\log b|} C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right], \end{aligned}$$

and using an integration by parts :

$$\begin{aligned}
b \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} E_2 \right| &= b |(Q \hat{\varepsilon} + \nabla Q \cdot \nabla \phi_{\hat{\varepsilon}}, \mathcal{M} E_2)| \\
&\lesssim b \frac{b^2}{|\log b|} \int \left[\frac{|\hat{\varepsilon}|}{1+r^4} + \frac{|\nabla \phi_{\hat{\varepsilon}}|}{1+r^5} \right] (1+r^2) \\
&\lesssim b \frac{b^2}{|\log b|} \left(\int |\hat{\varepsilon}|^2 + \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{(1+r^2)(1+|\log r|^2)} \right)^{\frac{1}{2}} \\
&\lesssim b \left[\frac{b^2}{|\log b|} C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right],
\end{aligned}$$

and (7.5.47) is proved.

step 7 $\tilde{\Psi}_b$ terms. We now estimate the $\tilde{\Psi}_b$ type of terms. From (7.3.54),

$$\begin{aligned}
|(\hat{\varepsilon}_2, \mathcal{M} \mathcal{L} \tilde{\Psi}_b)| &\lesssim \left(\int Q |\nabla \mathcal{M} \hat{\varepsilon}_2|^2 \right)^{\frac{1}{2}} \left(\int Q |\nabla \mathcal{M} \tilde{\Psi}_b|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{100} \int Q |\nabla \mathcal{M} \hat{\varepsilon}_2|^2 + C \frac{b^4}{|\log b|^2}.
\end{aligned}$$

Next, from (7.3.53) :

$$\begin{aligned}
\left| \frac{b}{\lambda^6} \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \mathcal{L} \tilde{\Psi}_b \right| &\lesssim \frac{b}{\lambda^6} \int \frac{|\hat{\varepsilon} \mathcal{L} \tilde{\Psi}_b|}{Q(1+r^2)} \lesssim \frac{b}{\lambda^6} \|\hat{\varepsilon}\|_{L^2} \|\mathcal{L} \tilde{\Psi}_b\|_{L_Q^2} \\
&\lesssim C(M) \frac{b}{\lambda^6} \frac{b^{\frac{5}{2}}}{|\log b|} (\|\varepsilon_2\|_{L_Q^2} + \|\zeta\|_{L^2}) \lesssim \frac{b}{\lambda^6} \left[b^2 \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

From (7.3.53), (7.7.15) :

$$\begin{aligned}
\left| \frac{b}{\lambda^6} \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \tilde{\Psi}_b \right| &\lesssim \frac{b}{\lambda^6} \int \frac{|\hat{\varepsilon}_2 \tilde{\Psi}_b|}{Q(1+r^2)} \lesssim \frac{b}{\lambda^6} \|\hat{\varepsilon}_2\|_{L_Q^2} \|\tilde{\Psi}_b\|_{L^2} \\
&\lesssim \frac{b}{\lambda^6} \frac{b^2}{|\log b|} (\|\varepsilon_2\|_{L_Q^2} + \|\zeta\|_{L_Q^2}) \lesssim \frac{b}{\lambda^6} \left[b^2 \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

From (7.3.53), (7.7.16) :

$$\begin{aligned}
\left| \frac{b}{\lambda^6} \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\tilde{\Psi}_b} \right| &\lesssim \frac{b}{\lambda^6} \left(\int (1+r^2) |\hat{\varepsilon}_1|^2 \right)^{\frac{1}{2}} \left(\int \frac{|\nabla \phi_{\tilde{\Psi}_b}|^2}{1+r^2} \right)^{\frac{1}{2}} \\
&\lesssim \frac{b}{\lambda^6} \frac{b^{\frac{5}{2}}}{|\log b|} \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \left(\int |\nabla \zeta|^2 + \int \frac{|\zeta|^2}{1+r^2} + \int \frac{|\nabla \phi_\zeta|^2}{1+r^4} \right)^{\frac{1}{2}} \right] \\
&\lesssim \frac{b}{\lambda^6} \left[b^2 \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

Finally, from (7.3.53), (7.7.19) :

$$\begin{aligned} & \left| \frac{b}{\lambda^6} \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} \tilde{\Psi}_b \right| \lesssim \frac{b}{\lambda^6} \int |\nabla \phi_{\hat{\varepsilon}}| \left[|\nabla \tilde{\Psi}_b| + \frac{|\tilde{\Psi}_b|}{1+r} + Q |\nabla \phi_{\tilde{\Psi}_b}| \right] \\ & \lesssim \frac{b}{\lambda^6} \left(\int \frac{|\nabla \phi_{\varepsilon}|^2}{1+r^2} + \frac{|\nabla \phi_{\zeta}|^2}{1+r^2} \right)^{\frac{1}{2}} \left(\int (1+r^2) \left[|\nabla \tilde{\Psi}_b|^2 + \frac{|\tilde{\Psi}_b|^2}{1+r^2} + Q^2 |\nabla \phi_{\tilde{\Psi}_b}|^2 \right] \right)^{\frac{1}{2}} \\ & \lesssim \frac{b}{\lambda^6} \frac{b^{\frac{5}{2}}}{|\log b|} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \lesssim \frac{b}{\lambda^6} \left[b^2 \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

step 8 Modulation terms. We now treat the modulation parameters given by (7.5.35) which require the introduction of the lifted parameter \hat{b} and the radiation term ζ_b . We decompose :

$$\widehat{\text{Mod}} = \text{Mod}_0 + \text{Mod}_1,$$

$$\text{Mod}_0 = -\partial_s \tilde{Q}_{\hat{b}}, \quad (7.5.51)$$

$$\text{Mod}_1 = \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b = \left(\frac{\lambda_s}{\lambda} + b \right) (\Lambda Q + b \Lambda \tilde{T}_1 + b^2 \Lambda \tilde{T}_2), \quad (7.5.52)$$

and further split :

$$\text{Mod}_0 = \text{Mod}_{0,1} + \text{Mod}_{0,2},$$

$$\text{Mod}_{0,1} = -\hat{b}_s T_1, \quad \text{Mod}_{0,2} = -\hat{b}_s \left(\tilde{T}_1 - T_1 + \hat{b} \frac{\partial \tilde{T}_1}{\partial b} + 2\hat{b}\tilde{T}_2 + \hat{b}^2 \frac{\partial \tilde{T}_2}{\partial b} \right).$$

We claim the bounds :

$$\int |r^i \partial_r^i \text{Mod}_{0,1}|^2 + \int \frac{|\mathcal{L} \text{Mod}_{0,1}|^2}{Q} + \int \frac{|\nabla \phi_{\text{Mod}_{0,1}}|^2}{1+r^2} \lesssim |\hat{b}_s|^2, \quad (7.5.53)$$

$$\int |r^i \partial_r^i \text{Mod}_{0,2}|^2 + \int \frac{|\mathcal{L} \text{Mod}_{0,2}|^2}{Q} + \int \frac{|\nabla \phi_{\text{Mod}_{0,2}}|^2}{1+r^2} \lesssim b |\hat{b}_s|^2, \quad (7.5.54)$$

$$\int |r^i \partial_r^i \text{Mod}_1|^2 + \int \frac{|\nabla \phi_{\text{Mod}_1}|^2}{1+r^2} \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2, \quad (7.5.55)$$

$$\int \frac{|\mathcal{L} \text{Mod}_1|^2}{Q} \lesssim b^2 \left| \frac{\lambda_s}{\lambda} + b \right|^2. \quad (7.5.56)$$

Proof of (7.5.53) : This is a direct consequence of (7.3.18), (7.3.19).

Proof of (7.5.54) : We estimate from (7.3.49), (7.3.50), (7.3.51), (7.3.52) :

$$\begin{aligned} & \left| r^i \partial_r^i \left[\tilde{T}_1 - T_1 + b \frac{\partial \tilde{T}_1}{\partial b} + 2b\tilde{T}_2 + b^2 \frac{\partial \tilde{T}_2}{\partial b} \right] \right| \\ & \lesssim \frac{\mathbf{1}_{B_1 \leq r \leq 2B_1}}{r^2} + b \left[r^2 \mathbf{1}_{r \leq 1} + \frac{1 + |\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} + \frac{1}{b^2 r^4} \mathbf{1}_{B_0 \leq r \leq 2B_1} \right], \end{aligned} \quad (7.5.57)$$

and using the radial representation of the Poisson field :

$$\left| \nabla \phi_{r^i \partial_r^i [\tilde{T}_1 - T_1 + b \frac{\partial \tilde{T}_1}{\partial b} + 2b\tilde{T}_2 + b^2 \frac{\partial \tilde{T}_2}{\partial b}]} \right| \lesssim b(1+r) \mathbf{1}_{r \leq B_0} + \frac{\mathbf{1}_{r \geq B_0}}{r}. \quad (7.5.58)$$

This yields :

$$\begin{aligned} \int |r^i \partial_r^i \text{Mod}_{0,2}|^2 &\lesssim |\hat{b}_s|^2 \int \left[\frac{\mathbf{1}_{B_1 \leq r \leq 2B_1}}{1+r^4} + b^2 \left(\frac{1+|\log(r\sqrt{b})|}{|\log b|} \mathbf{1}_{1 \leq r \leq 6B_0} \right)^2 + \frac{1}{b^2 r^8} \mathbf{1}_{B_0 \leq r \leq 2B_1} \right] \\ &\lesssim b |\hat{b}_s|^2, \\ \int \frac{|\nabla \phi_{\text{Mod}_{0,2}}|^2}{1+r^2} &\lesssim |\hat{b}_s|^2 \int \frac{1}{1+r^2} \left[b^2(1+r^2) \mathbf{1}_{r \leq B_0} + \frac{\mathbf{1}_{r \geq B_0}}{r^2} \right] \lesssim b |\hat{b}_s|^2, \end{aligned}$$

and similarly using the explicit representation (7.1.15) :

$$\int \frac{|\mathcal{L}\text{Mod}_{0,2}|^2}{Q} \lesssim b |\hat{b}_s|^2.$$

Proof of (7.5.55) : We extract from (7.3.49), (7.3.51) the rough bound :

$$|r^i \partial_r^i (\tilde{T}_1 + b\tilde{T}_2)| \lesssim \frac{1}{1+r^2} \mathbf{1}_{r \leq 2B_1}, \quad (7.5.59)$$

$$|\nabla \phi_{r^i \partial_r^i (\tilde{T}_1 + b\tilde{T}_2)}| \lesssim \frac{1+|\log r|}{1+r}. \quad (7.5.60)$$

This yields :

$$\begin{aligned} \int |r^i \partial_r^i \text{Mod}_{1,1}|^2 &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int \frac{1}{1+r^4} \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2, \\ \int \frac{|\nabla \phi_{\text{Mod}_{1,1}}|^2}{1+r^2} &\lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int \frac{1+|\log r|^2}{1+r^4} \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2. \end{aligned}$$

Proof of (7.5.56) : We use the cancellation $\mathcal{L}\Lambda Q = 0$, the bound (7.5.59), (7.5.60) and the explicit formula (7.1.15) to estimate :

$$\int \frac{|\mathcal{L}\text{Mod}_{1,1}|^2}{Q} \lesssim b^2 \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int \left[\frac{1}{1+r^4} \mathbf{1}_{1 \leq r \leq 2B_1} + \frac{1}{1+r^6} \frac{1+|\log r|^2}{1+r^2} \right] \lesssim b^2 \left| \frac{\lambda_s}{\lambda} + b \right|^2.$$

We are now in position to estimate all terms in (7.5.44). From (7.5.13), (7.5.54), (7.5.56) :

$$\begin{aligned} |(\hat{\varepsilon}_2, \mathcal{M}\widehat{\text{Mod}})| &\lesssim |(\mathcal{M}\zeta_2, \mathcal{L}\widehat{\text{Mod}})| + \|\varepsilon_2\|_{L_Q^2} [\|\mathcal{L}\text{Mod}_{0,2}\|_{L_Q^2} + \|\mathcal{L}\text{Mod}_1\|_{L_Q^2}] \\ &\lesssim \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b^{\frac{3}{2}}}{|\log b|} \right) [\|\mathcal{L}\text{Mod}_{0,2}\|_{L_Q^2} + \|\mathcal{L}\text{Mod}_1\|_{L_Q^2}] \\ &\lesssim \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b^{\frac{3}{2}}}{|\log b|} \right) [\sqrt{b} |\hat{b}_s| + b \left| \frac{\lambda_s}{\lambda} + b \right|]. \end{aligned}$$

We now use the pointwise bound (7.4.28) and in a fundamental way the improved bound (7.5.25) which motivates the introduction of the lifted parameter \hat{b}_s to conclude :

$$\begin{aligned} |(\hat{\varepsilon}_2, \mathcal{M}\widehat{\text{Mod}})| &\lesssim \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b^{\frac{3}{2}}}{|\log b|} \right) \left[\sqrt{b} \frac{b^2}{|\log b|} + bC(M) \frac{b^2}{|\log b|} \right] \\ &\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

Next, from (7.5.3), (7.4.28), (7.5.25), (7.7.15) :

$$\begin{aligned}
b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \widehat{\mathcal{L}\text{Mod}} \right| &\lesssim b \|\hat{\varepsilon}\|_{L^2} \|\widehat{\mathcal{L}\text{Mod}}\|_{L_Q^2} \lesssim b \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \|\zeta\|_{L^2} \right] \left[|\hat{b}_s| + b \left| \frac{\lambda_s}{\lambda} + b \right| \right] \\
&\lesssim b \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \frac{b^2}{|\log b|} \\
&\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right], \\
b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \widehat{\text{Mod}} \right| &\lesssim b \left[\|\varepsilon_2\|_{L_Q^2} + \|\zeta\|_{L_Q^2} \right] \|\widehat{\text{Mod}}\|_{L^2} \\
&\lesssim b \left[\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \left[|\hat{b}_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \right] \\
&\lesssim b C(M) \left[\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \frac{b^2}{|\log b|} \\
&\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + C(M) \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
b \left| \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\widehat{\text{Mod}}} \right| &\lesssim b \left[\int (1+r^2) |\varepsilon_1|^2 + \int |\zeta|^2 + (1+r^2) |\nabla \zeta|^2 + \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right]^{\frac{1}{2}} \left[\int \frac{|\nabla \phi_{\widehat{\text{Mod}}}|^2}{1+r^2} \right]^{\frac{1}{2}} \\
&\lesssim b \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \left[|\hat{b}_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \right] \\
&\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + C(M) \frac{b^3}{|\log b|^2} \right],
\end{aligned}$$

and using (7.7.19) :

$$\begin{aligned}
b \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} \widehat{\text{Mod}} \right| &\lesssim b \int |\nabla \phi_{\hat{\varepsilon}}| \left| \widehat{\mathcal{M}\text{Mod}} + \frac{|\widehat{\text{Mod}}|}{1+r} + \frac{|\nabla \phi_{\widehat{\text{Mod}}}|}{1+r^4} \right| \\
&\lesssim b \left(\int \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^2} + \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right)^{\frac{1}{2}} \left(\int (1+r^2) |\nabla \widehat{\text{Mod}}|^2 + |\widehat{\text{Mod}}|^2 + \frac{|\nabla \phi_{\widehat{\text{Mod}}}|^2}{1+r^2} \right)^{\frac{1}{2}} \\
&\lesssim b \left(|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + b^{10} + \frac{b^2}{|\log b|^2} \right)^{\frac{1}{2}} \left[|\hat{b}_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \right] \\
&\lesssim b C(M) \frac{b^2}{|\log b|} \left(|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + b^{10} + \frac{b^2}{|\log b|^2} \right)^{\frac{1}{2}} \\
&\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + C(M) \frac{b^3}{|\log b|^2} \right].
\end{aligned}$$

This concludes the control of the forcing induced by the modulation parameters.

step 9 E_1 terms. We split

$$E_1 = -\zeta_2 + E_{1,1}, \quad E_{1,1} = -\frac{\lambda_s}{\lambda} \Lambda \zeta.$$

ζ_2 terms : From (7.5.15) :

$$\begin{aligned} |(\hat{\varepsilon}_2, \mathcal{M}\mathcal{L}\zeta_2)| &= \left(\int Q |\nabla \mathcal{M}\hat{\varepsilon}_2|^2 \right)^{\frac{1}{2}} \left(\int Q |\nabla \mathcal{M}\zeta_2|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{100} \int Q |\nabla \mathcal{M}\hat{\varepsilon}_2|^2 + C \frac{b^4}{|\log b|^2}. \end{aligned}$$

Next, after an integration by parts :

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \mathcal{L} \zeta_2 \right| &\lesssim b \left(\int Q |\nabla \mathcal{M}\zeta_2|^2 \right)^{\frac{1}{2}} \left(\int |\nabla \hat{\varepsilon}|^2 + \frac{|\hat{\varepsilon}|^2}{1+r^2} \right)^{\frac{1}{2}} \\ &\lesssim \frac{b^3}{|\log b|^2} \left[\frac{b}{|\log b|} + C(M) \|\varepsilon_2\|_{L_Q^2} \right] \\ &\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + C(M) \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

Similarly, using Lemma 7.5.3 :

$$\begin{aligned} b \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M}\zeta_2 \right| &\lesssim b \left(\int \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^4} \right)^{\frac{1}{2}} \left(\int Q |\nabla \mathcal{M}\zeta_2|^2 \right)^{\frac{1}{2}} \\ &\lesssim b \frac{b^2}{|\log b|} \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \\ &\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

To control the last terms, we introduce the decomposition from (7.5.22) :

$$\zeta_2 = (b - \hat{b}) \Lambda Q + \tilde{\zeta}_2, \quad \tilde{\zeta}_2 = \xi + \mathcal{L}\zeta_{\text{sm}}, \quad \xi = |b - \hat{b}| O \left(\frac{\mathbf{1}_{r \geq B_1}}{1+r^4} \right).$$

Observe using Lemma 7.5.3 that :

$$\|\tilde{\zeta}_2\|_{L^2}^2 \lesssim \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|^2} \int_{r \geq B_1} \frac{1}{1+r^8} \lesssim \frac{b^4}{|\log b|^2}, \quad (7.5.61)$$

and using by construction $\int \tilde{\zeta}_2 = 0$ and (7.7.5) :

$$\int |\nabla \phi_{\tilde{\zeta}_2}|^2 \lesssim \|\tilde{\zeta}_2\|_{L_Q^2}^2 \lesssim \frac{b^3}{|\log b|^2}. \quad (7.5.62)$$

We therefore estimate in brute force :

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \tilde{\zeta}_2 \right| &\lesssim b \left[\|\varepsilon_2\|_{L_Q^2} + \|\zeta_2\|_{L_Q^2} \right] \|\tilde{\zeta}_2\|_{L^2} \\ &\lesssim b \frac{b^2}{|\log b|} \left[\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \\ &\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right]. \end{aligned}$$

Similarly,

$$b \int |\varepsilon_1 \cdot \nabla \phi_{\tilde{\zeta}_2}| \lesssim b \left(\int (1+r^2) |\varepsilon_1|^2 \right)^{\frac{1}{2}} \left(\int \frac{|\nabla \phi_{\tilde{\zeta}_2}|^2}{1+r^2} \right)^{\frac{1}{2}} \lesssim b C(M) \|\varepsilon_2\|_{L_Q^2} \frac{b^{\frac{3}{2}}}{|\log b|}.$$

and for the term induced by the radiation using (7.5.62), (7.5.61) and the degeneracy (7.5.41) :

$$\begin{aligned} b \left| \int Q \nabla(\mathcal{M}\zeta) \cdot \nabla \phi_{\tilde{\zeta}_2} \right| &= b \left| \int \zeta_2 \phi_{\tilde{\zeta}_2} \right| \lesssim b |b - \hat{b}| \int |\tilde{\zeta}_2 \phi_{\Lambda Q}| + b \int |\nabla \phi_{\tilde{\zeta}_2}|^2 \\ &\lesssim \frac{b^2}{|\log b|} \|\tilde{\zeta}_2\|_{L^2} + b \frac{b^3}{|\log b|^2} \lesssim \frac{b^4}{|\log b|^2}. \end{aligned}$$

It remains to estimate the term :

$$b(b - \hat{b}) \left[\int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \Lambda Q - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\Lambda Q} \right] = b(b - \hat{b}) \int \hat{\varepsilon}_2 \left[\frac{\Lambda Q(\Lambda Q + 2Q)}{Q^2} + 2\phi_{\Lambda Q} \right].$$

Here we need an additional algebra in order to be able to use dissipation²⁰. We compute from (7.2.11) :

$$\frac{\Lambda Q(\Lambda Q + 2Q)}{Q^2} + 2\phi_{\Lambda Q} = \left(\frac{\Lambda Q}{Q} \right)^2 - 4.$$

We now use (7.5.45), integrate by parts and use the degeneracy (7.2.33) and (7.5.41) to estimate :

$$\begin{aligned} &\left| \int \hat{\varepsilon}_2 \left[\frac{\Lambda Q(\Lambda Q + 2Q)}{Q^2} + 2\phi_{\Lambda Q} \right] \right| = \left| \int \hat{\varepsilon}_2 \mathcal{M}(\Lambda^2 Q) \right| \\ &= \left| \int \hat{\varepsilon}_2 \mathcal{M}(\Lambda^2 Q + 2\Lambda Q) \right| = \left| \int \mathcal{M} \hat{\varepsilon}_2 \nabla \cdot (r(\Lambda Q + 2Q)) \right| \\ &= \left(\int Q |\nabla(\mathcal{M} \hat{\varepsilon}_2)|^2 \right)^{\frac{1}{2}} \left(\int \frac{r^2(\Lambda Q + 2Q)^2}{Q} \right)^{\frac{1}{2}} \lesssim \left(\int Q |\nabla(\mathcal{M} \hat{\varepsilon}_2)|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and hence the control of the last term :

$$\begin{aligned} &\left| b(b - \hat{b}) \left[\int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \Lambda Q - 2 \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\Lambda Q} \right] \right| \\ &\lesssim \frac{b^2}{|\log b|} \left(\int Q |\nabla(\mathcal{M} \hat{\varepsilon}_2)|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{100} \int Q |\nabla(\mathcal{M} \hat{\varepsilon}_2)|^2 + C \frac{b^4}{|\log b|^2}. \end{aligned}$$

20. for which a better bound than for $\|\varepsilon_2\|_{L_Q^2}$ holds in time averaged sense.

$E_{1,1}$ terms : We systematically use the rough pointwise bound from (7.4.28) :

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim b.$$

For the first term, we estimate first using the decomposition (7.5.22), (7.5.23) and (7.5.10) :

$$\begin{aligned} b|(\hat{\varepsilon}_2, \mathcal{ML}(\Lambda\zeta_{\text{sm}}))| &\lesssim b|(\xi + \mathcal{L}\zeta_{\text{sm}}, \mathcal{ML}(\Lambda\zeta_{\text{sm}}))| + b\|\varepsilon_2\|_{L_Q^2}\|\mathcal{L}(\Lambda\zeta_{\text{sm}})\|_{L_Q^2} \\ &\lesssim b \left(\int_{r \geq B_1} \frac{b^2}{|\log b|^2} \frac{1}{(1+r^8)Q} + \int \frac{|\mathcal{L}\zeta_{\text{sm}}|^2}{Q} \right)^{\frac{1}{2}} \|\mathcal{L}(\Lambda\zeta_{\text{sm}})\|_{L_Q^2} + b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} \\ &\lesssim b \left(\frac{b^3}{|\log b|^2} \right)^{\frac{1}{2}} \left(\frac{b^3}{|\log b|^2} \right)^{\frac{1}{2}} + b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} \lesssim \frac{b^4}{|\log b|^2} + b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2}. \end{aligned}$$

We now observe using (7.3.19) the cancellation

$$r^i \partial_r^i (\Lambda T_1) = O \left(\frac{1 + |\log r|}{1 + r^4} \right)$$

which implies

$$\begin{aligned} \int Q|\nabla \mathcal{M}\Lambda(\chi_{B_1}T_1)|^2 &\lesssim \int \frac{|\nabla \phi_{\chi_{B_1}T_1}|^2}{1+r^4} + \int (1+r^4)|\nabla \Lambda(\chi_{B_1}T_1)|^2 + (1+r^2)|\Lambda(\chi_{B_1}T_1)|^2 \\ &\lesssim 1 + \int_{B_1 \leq r \leq 2B_1} \frac{1}{1+r^2} \lesssim 1 \end{aligned}$$

from which :

$$\begin{aligned} b|(\hat{\varepsilon}_2, \mathcal{ML}(\Lambda\zeta_{\text{big}}))| &= b|b - \hat{b}| \left| \int Q \nabla \mathcal{M}\hat{\varepsilon}_2 \cdot \nabla \mathcal{M}\Lambda(\chi_{B_1}T_1) \right| \\ &\leq \frac{1}{100} \int Q|\nabla \mathcal{M}\hat{\varepsilon}_2|^2 + C \frac{b^4}{|\log b|^2} \int Q|\nabla \mathcal{M}\Lambda(\chi_{B_1}T_1)|^2 \\ &\leq \frac{1}{100} \int Q|\nabla \mathcal{M}\hat{\varepsilon}_2|^2 + C \frac{b^4}{|\log b|^2}. \end{aligned}$$

We further estimate using Lemma 7.5.3 :

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \mathcal{L}(b\Lambda\zeta) \right| &\lesssim b^2 \|\hat{\varepsilon}\|_{L^2} \|\mathcal{L}\Lambda\zeta\|_{L_Q^2} \lesssim b \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \frac{b^2}{|\log b|} \\ &\lesssim b \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right], \end{aligned}$$

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 (b\Lambda\zeta) \right| &\lesssim b^2 \|\hat{\varepsilon}_2\|_{L_Q^2} \|\Lambda\zeta\|_{L^2} \lesssim b \left[\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \frac{b^2}{|\log b|} \\ &\lesssim \frac{b^4}{|\log b|^2} + b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2}, \end{aligned}$$

$$\begin{aligned}
& b \left| \int \hat{\varepsilon}_1 \cdot \nabla \phi_{b\Lambda\zeta} \right| \\
& \lesssim b^2 \left(\int (1+r^2) |\varepsilon_1|^2 + \int (1+r^2) |\nabla \zeta|^2 + \int |\zeta|^2 + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right)^{\frac{1}{2}} \left(\int \frac{|\nabla(y \cdot \nabla \phi_\zeta)|^2}{1+r^2} \right)^{\frac{1}{2}} \\
& \lesssim b^2 \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \frac{b}{|\log b|} \lesssim \frac{b^4}{|\log b|^2} + b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2},
\end{aligned}$$

and using Proposition 7.7.4 :

$$\begin{aligned}
& b \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M}(b\Lambda\zeta) \right| \lesssim b^2 \left| (Q\hat{\varepsilon} + \nabla Q \cdot \nabla \phi_{\hat{\varepsilon}}, \frac{\Lambda\zeta}{Q}) \right| + b^2 \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \phi_{\Lambda\zeta} \right| \\
& \lesssim b^2 \|\Lambda\zeta\|_{L^2} \left(\int |\hat{\varepsilon}|^2 + \int \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^2} \right)^{\frac{1}{2}} + b^2 \left(\int \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^4} \right)^{\frac{1}{2}} \left(\int \frac{|\nabla \phi_{\Lambda\zeta}|^2}{1+r^4} \right)^{\frac{1}{2}} \\
& \lesssim b^2 \frac{b}{|\log b|} \left(|\log b|^C \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right) \lesssim \frac{b^4}{|\log b|^2} + b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2}.
\end{aligned}$$

step 10 Non linear terms. We estimate the contribution of the non linear term

$$N(\varepsilon) = \nabla \cdot (\varepsilon \nabla \phi_\varepsilon) = \varepsilon^2 + \nabla \varepsilon \cdot \nabla \phi_\varepsilon.$$

We decompose using (7.2.9) :

$$Q \nabla \mathcal{M} N(\varepsilon) = \nabla N(\varepsilon) + Q \nabla \phi_{N(\varepsilon)} + N(\varepsilon) \nabla \phi_Q = S_1 + S_2 \quad (7.5.63)$$

with

$$S_1 = 2\varepsilon \nabla \varepsilon + \nabla^2 \phi_\varepsilon \cdot \nabla \varepsilon + Q \nabla \phi_{N(\varepsilon)} + N(\varepsilon) \nabla \phi_Q, \quad S_2 = \nabla^2 \varepsilon \cdot \nabla \phi_\varepsilon.$$

We claim the bounds :

$$\forall 2 < p < +\infty, \quad \|\nabla \phi_{N(\varepsilon)}\|_{L^p} \lesssim C(p) |\log b|^{C(p)} \left[\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right], \quad (7.5.64)$$

$$\int |N(\varepsilon)|^2 \lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}, \quad (7.5.65)$$

$$\int \frac{|S_1|^2}{(1+r^2)Q} \lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}, \quad (7.5.66)$$

$$\int \frac{|\nabla \cdot S_1|^2}{Q} \lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}, \quad (7.5.67)$$

$$\int \frac{|S_2|^2}{Q} \lesssim \|\varepsilon_2\|_{L_Q^2}^{4-\frac{1}{100}}, \quad (7.5.68)$$

$$\int (1+r^2) |S_2|^2 \lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}. \quad (7.5.69)$$

Proof of (7.5.64) : We estimate from Sobolev and Plancherel :

$$\|\nabla^2 \phi_\varepsilon\|_{L^\infty} \lesssim \|\nabla^2 \phi_\varepsilon\|_{H^2} \lesssim \|\varepsilon\|_{H^2} \lesssim C(M) \|\varepsilon_2\|_{L_Q^2}. \quad (7.5.70)$$

We then estimate from Hardy Littlewood Sobolev :

$$\|\nabla \phi_{N(\varepsilon)}\|_{L^p} \lesssim \|N(\varepsilon)\|_{L^r} \quad \text{with } r = \frac{2p}{p+2} \in (1, 2).$$

Then from Sobolev and (7.7.20), (7.5.70) :

$$\begin{aligned} \|N(\varepsilon)\|_{L^r} &\lesssim \|\varepsilon\|_{L^{2r}}^2 + \|\nabla \varepsilon \cdot \nabla \phi_\varepsilon\|_{L^r} \lesssim \|\varepsilon\|_{H^2}^2 + \|\nabla \varepsilon \cdot \nabla \phi_\varepsilon\|_{H^1} \\ &\lesssim \|\varepsilon\|_{H^2}^2 + \left\| \frac{\nabla \phi_\varepsilon}{1+r} \right\|_{L^\infty} \left[\|(1+r)\nabla \varepsilon\|_{L^2} + \|(1+r)\nabla^2 \varepsilon\|_{L^2} \right] + \|\nabla^2 \phi_\varepsilon\|_{L^\infty} \|\nabla \varepsilon\|_{L^2} \\ &\lesssim |\log b|^C (\|\varepsilon_2\|_{L_Q^2} + b^{10}) \|\varepsilon_2\|_{L_Q^2} \end{aligned} \quad (7.5.71)$$

and (7.5.64) follows.

Proof of (7.5.65) : It follows from the chain of estimates (7.5.71) with $r = 2$.

Proof of (7.5.66) : We estimate in brute force using (7.5.70), (7.5.64), (7.5.65) :

$$\begin{aligned} \int \frac{|S_1|^2}{(1+r^2)Q} &\lesssim (\|\varepsilon\|_{L^\infty}^2 + \|\nabla^2 \phi_\varepsilon\|_{L^\infty}^2) \int (1+r^2) |\nabla \varepsilon|^2 + \int \frac{|\nabla \phi_{N(\varepsilon)}|^2}{1+r^6} + \int (N(\varepsilon))^2 \\ &\lesssim C(M) \|\varepsilon_2\|_{L_Q^2}^4 + \|\nabla \phi_{N(\varepsilon)}\|_{L^4}^2 + |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20} \\ &\lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}. \end{aligned}$$

Proof of (7.5.67) : We compute :

$$\nabla \cdot S_1 = 2\varepsilon \Delta \varepsilon + 2|\nabla \varepsilon|^2 + \nabla \cdot (\nabla^2 \phi_\varepsilon \cdot \nabla \varepsilon) + \nabla Q \cdot \nabla \phi_{N(\varepsilon)} + 2QN(\varepsilon) + \nabla \phi_Q \cdot \nabla N(\varepsilon)$$

and hence the bound :

$$\begin{aligned} |\nabla \cdot S_1| &\lesssim |\varepsilon| |\Delta \varepsilon| + |\nabla \varepsilon|^2 + |\nabla^3 \phi_\varepsilon| |\nabla \varepsilon| + |\nabla^2 \phi_\varepsilon| |\nabla^2 \varepsilon| + \frac{|\nabla \phi_{N(\varepsilon)}|}{1+r^5} \\ &+ \frac{|\varepsilon|^2 + |\nabla \varepsilon| |\nabla \phi_\varepsilon|}{1+r^4} + \frac{|\varepsilon| |\nabla \varepsilon| + |\nabla^2 \phi_\varepsilon| |\nabla \varepsilon| + |\nabla^2 \varepsilon| |\nabla \phi_\varepsilon|}{1+r}. \end{aligned} \quad (7.5.72)$$

We have the nonlinear estimate using Sobolev :

$$\begin{aligned} \int \frac{|\nabla \varepsilon|^4}{Q} &\lesssim \int |\nabla \varepsilon|^4 + \sum_{i=1}^2 \int |\nabla[(1+x_i)\varepsilon]|^4 \\ &\lesssim \|\varepsilon\|_{H^2}^4 + \sum_{i=1}^2 \left(\int |\nabla[(1+x_i)\varepsilon]|^2 \right)^2 \left(\int |\nabla^2[(1+x_i)\varepsilon]|^2 \right) \\ &\lesssim c(M) \|\varepsilon_2\|_{L_Q^2}^4. \end{aligned}$$

Similarly, using Lemma 7.7.1 with $p = 1$:

$$\begin{aligned}
\int \frac{|\nabla^3 \phi_\varepsilon|^4}{Q} &\lesssim \int |\nabla^2 \phi_\varepsilon|^4 + \sum_{i=1}^2 \int |\nabla[(1+x_i)\nabla^2 \phi_\varepsilon]|^4 \\
&\lesssim \|\nabla^2 \phi_\varepsilon\|_{H^2}^4 + \sum_{i=1}^2 \left(\int |\nabla[(1+x_i)\nabla^2 \phi_\varepsilon]|^2 \right) \left(\int |\nabla^2[(1+x_i)\nabla^2 \phi_\varepsilon]|^2 \right) \\
&\lesssim \|\varepsilon\|_{H^2}^4 + \sum_{i=1}^2 \left(\|\varepsilon\|_{H^2}^2 + \int |x_i|^2 |\nabla^2(\nabla \phi_\varepsilon)|^2 \right) \left(\|\varepsilon\|_{H^2}^2 + \int |x_i|^2 |\nabla^2(\nabla^2 \phi_\varepsilon)|^2 \right) \\
&\lesssim \|\varepsilon\|_{H^2}^4 + \sum_{i=1}^2 \left(\|\varepsilon\|_{H^2}^2 + \int |x_i|^2 |\Delta(\nabla \phi_\varepsilon)|^2 \right) \left(\|\varepsilon\|_{H^2}^2 + \int |x_i|^2 |\Delta(\nabla^2 \phi_\varepsilon)|^2 \right) \\
&= \|\varepsilon\|_{H^2}^4 + \left(\|\varepsilon\|_{H^2}^2 + \int |x_i|^2 |\nabla(\Delta \phi_\varepsilon)|^2 \right) \left(\|\varepsilon\|_{H^2}^2 + \int |x_i|^2 |\nabla^2(\Delta \phi_\varepsilon)|^2 \right) \\
&\lesssim C(M) \|\varepsilon_2\|_{L_Q^2}^4.
\end{aligned}$$

We then estimate using (7.5.64), (7.5.70), (7.5.72), (7.7.15), (7.7.20) :

$$\begin{aligned}
\int \frac{|\nabla \cdot S_1|^2}{Q} &\lesssim \|\varepsilon\|_{L^\infty}^2 \int \frac{|\Delta \varepsilon|^2}{Q} + \int \frac{|\nabla \varepsilon|^4 + |\nabla^3 \phi_\varepsilon|^4}{Q} + \|\nabla^2 \phi_\varepsilon\|_{L^\infty}^2 \int \frac{|\nabla^2 \varepsilon|^2}{Q} \\
&+ \int \frac{|\nabla \phi_{N(\varepsilon)}|^2}{1+r^6} + \|\varepsilon\|_{L^\infty}^4 + \|\frac{\nabla \phi_\varepsilon}{1+r}\|_{L^\infty}^2 \int \frac{|\nabla \varepsilon|^2}{1+r^2} \\
&+ (\|\varepsilon\|_{L^\infty}^2 + \|\nabla^2 \phi_\varepsilon\|_{L^\infty}^2) \int (1+r^2) |\nabla \varepsilon|^2 + \|\frac{\nabla \phi_\varepsilon}{1+r}\|_{L^\infty}^2 \int (1+r^4) |\nabla^2 \varepsilon|^2 \\
&\lesssim C(M) \|\varepsilon_2\|_{L_Q^2}^4 + \|\nabla \phi_{N(\varepsilon)}\|_{L^4}^2 + \|\varepsilon_2\|_{L_Q^2}^2 |\log b|^C \left(\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right) \\
&\lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}.
\end{aligned}$$

Proof of (7.5.68), (7.5.69) : From (7.7.17) :

$$\int \frac{|S_2|^2}{Q} \lesssim \|\nabla \phi_\varepsilon\|_{L^\infty}^2 \int \frac{|\nabla^2 \varepsilon|^2}{Q} \lesssim \|\varepsilon_2\|_{L_Q^2}^{4-\frac{1}{100}}.$$

From (7.7.20) :

$$\int (1+r^2) |S_2|^2 \lesssim \int (1+r^2) |\nabla^2 \varepsilon|^2 |\nabla \phi_\varepsilon|^2 \lesssim \left\| \frac{\nabla \phi_\varepsilon}{1+r} \right\|_{L^\infty}^2 \|\varepsilon\|_{H_Q^2}^2 \lesssim |\log b|^C \|\varepsilon_2\|_{L_Q^2}^4 + b^{20}.$$

We are now in position to obtain admissible bounds for the nonlinear terms in (7.5.44). From (7.5.63), (7.2.2) and the nonlinear estimates (7.5.67), (7.5.69) :

$$\begin{aligned}
|(\varepsilon_2, \mathcal{MLN}(\varepsilon))| &\lesssim |(\mathcal{M}\varepsilon_2, \nabla \cdot S_1)| + |(\nabla \mathcal{M}\varepsilon_2 \cdot S_2)| \\
&\lesssim \|\varepsilon_2\|_{L_Q^2} \|\nabla \cdot S_1\|_{L_Q^2} + \left(\int Q |\nabla \mathcal{M}\varepsilon_2|^2 \right)^{\frac{1}{2}} \left(\int \frac{|S_2|^2}{Q} \right)^{\frac{1}{2}} \\
&\leq \frac{1}{100} \int Q |\nabla \mathcal{M}\varepsilon_2|^2 + C(M) |\log b|^C \|\varepsilon_2\|_{L_Q^2} \left[\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right] + C(M) \|\varepsilon_2\|_{L_Q^2}^{4-\frac{1}{100}} \\
&\leq \frac{1}{100} \int Q |\nabla \mathcal{M}\hat{\varepsilon}_2|^2 + C(M) \left[b b^{\frac{1}{4}} \|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^4}{|\log b|^2} \right].
\end{aligned}$$

We estimate the radiation term using Lemma 7.5.3 :

$$\begin{aligned}
|(\zeta_2, \mathcal{MLN}(\varepsilon))| &\lesssim |(\mathcal{M}\zeta_2, \nabla \cdot S_1)| + |(\nabla \mathcal{M}\zeta_2, S_2)| \\
&\lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \|\nabla \cdot S_1\|_{L_Q^2} + \left(\int Q |\nabla \mathcal{M}\zeta_2|^2 \right)^{\frac{1}{2}} \left(\int \frac{|S_2|^2}{Q} \right)^{\frac{1}{2}} \\
&\lesssim \frac{b^{\frac{3}{2}}}{|\log b|} \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right] + \frac{b^2}{|\log b|} \|\varepsilon_2\|_{L_Q^2}^{2-\frac{1}{50}} \\
&\lesssim \frac{b^4}{|\log b|^2}.
\end{aligned}$$

Next, using the degeneracy (7.5.41) and integrating by parts :

$$\begin{aligned}
b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \mathcal{LN}(\varepsilon) \right| &\lesssim b \int \frac{1}{(1+r^2)Q} \left[|\hat{\varepsilon} \nabla \cdot S_1| + \frac{|\hat{\varepsilon}| |S_2|}{1+r} + |\nabla \varepsilon| |S_2| \right] \\
&\lesssim b (\|\varepsilon\|_{L^2} + \|\zeta\|_{L^2}) \left[\|\nabla \cdot S_1\|_{L_Q^2} + \|(1+r)S_2\|_{L^2} \right] \\
&+ b (\|(1+r)\nabla \varepsilon\|_{L^2} + \|(1+r)\nabla \zeta\|_{L^2}) \|(1+r)S_2\|_{L^2} \\
&\lesssim b \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right) \left[|\log b|^C \|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right] \lesssim \frac{b}{|\log b|} \left[\|\varepsilon_2\|_{L_Q^2}^2 + b^4 \right].
\end{aligned}$$

Using (7.5.64), (7.7.16) :

$$\begin{aligned}
b \left| \int \hat{\varepsilon}_1 \cdot \nabla \phi_{N(\varepsilon)} \right| &\lesssim b \left(\int (1+r^2) |\varepsilon_1|^2 + \int |\zeta|^2 + \int (1+r^2) |\nabla \zeta|^2 + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right)^{\frac{1}{2}} \left(\int \frac{|\nabla \phi_{N(\varepsilon)}|^2}{1+r^2} \right)^{\frac{1}{2}} \\
&\lesssim b \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \|\nabla \phi_{N(\varepsilon)}\|_{L^4} \lesssim b |\log b|^C \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] \left[\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right] \\
&\lesssim \frac{b}{|\log b|} \left[\|\varepsilon_2\|_{L_Q^2}^2 + b^4 \right]
\end{aligned}$$

Using (7.5.64), (7.5.66), (7.5.69) and (7.7.19) :

$$\begin{aligned}
b \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{MLN}(\varepsilon) \right| &\lesssim b \left(\int \frac{|\nabla \phi_\varepsilon|^2}{1+r^2} + \int \frac{|\nabla \phi_\zeta|^2}{1+r^2} \right)^{\frac{1}{2}} \left(\int (1+r^2) (|S_1|^2 + |S_2|^2) \right)^{\frac{1}{2}} \\
&\lesssim b \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right) |\log b|^C \left[[\|\varepsilon_2\|_{L_Q^2}^2 + b^{10}] \right] \\
&\lesssim \frac{b}{|\log b|} \left[\|\varepsilon_2\|_{L_Q^2}^2 + b^4 \right].
\end{aligned}$$

step 11 Small linear terms. We estimate the contribution of the small linear error $\Theta_b(\varepsilon)$ given by (7.4.21). We expand :

$$\Theta_b(\varepsilon) = \nabla \varepsilon \cdot \nabla \phi_{\tilde{Q}_b - Q} + 2(\tilde{Q}_b - Q)\varepsilon + \nabla(\tilde{Q}_b - Q) \cdot \nabla \phi_\varepsilon.$$

We claim the bounds :

$$\int |\Theta_b(\varepsilon)|^2 \lesssim b^2 C(M) \|\varepsilon_2\|_{L_Q^2}^2, \quad (7.5.73)$$

$$\int \frac{|\nabla \phi_{\Theta_b(\varepsilon)}|^2}{1+r^2} \lesssim b^2 C(M) \|\varepsilon_2\|_{L_Q^2}^2, \quad (7.5.74)$$

$$\int Q |\nabla \mathcal{M} \Theta_b(\varepsilon)|^2 \lesssim b^2 C(M) \|\varepsilon_2\|_{L_Q^2}^2. \quad (7.5.75)$$

Proof of (7.5.73), (7.5.74), (7.5.75) : We recall the pointwise bounds (7.5.29), (7.5.30) :

$$\begin{aligned} |\Theta_b(\varepsilon)| &\lesssim b \left[\frac{1+|\log r|}{1+r} |\nabla \varepsilon| + \frac{|\varepsilon|}{1+r^2} + \frac{|\nabla \phi_\varepsilon|}{1+r^3} \right], \\ |\nabla \Theta_b(\varepsilon)| &\lesssim b \left[\frac{1+|\log r|}{1+r} |\nabla^2 \varepsilon| + \frac{1+|\log r|}{1+r^2} |\nabla \varepsilon| + \frac{|\nabla^2 \phi_\varepsilon| + |\varepsilon|}{1+r^3} + \frac{|\nabla \phi_\varepsilon|}{1+r^4} \right]. \end{aligned}$$

The bound (7.5.73) now follows from Proposition 7.7.4. We estimate using (7.7.4), HLS and (7.5.29) :

$$\begin{aligned} \int \frac{|\nabla \phi_{\Theta_b(\varepsilon)}|^2}{1+r^2} &\lesssim \|\nabla \phi_{\Theta_b(\varepsilon)}\|_{L^4}^2 \lesssim \|\Theta_b(\varepsilon)\|_{L^{\frac{4}{3}}}^2 \\ &\lesssim C(M) b^2 \|\varepsilon_2\|_{L_Q^2}^2 \end{aligned} \quad (7.5.76)$$

and (7.5.74) is proved. We then compute like for (7.5.63) :

$$Q \nabla \mathcal{M} \Theta_b(\varepsilon) = \nabla \Theta_b(\varepsilon) + Q \nabla \phi_{\Theta_b(\varepsilon)} + \Theta_b(\varepsilon) \nabla \phi_Q$$

and estimate using (7.5.29), (7.5.30), (7.5.76) and Proposition 7.7.4 :

$$\begin{aligned} \int Q |\nabla \mathcal{M} \Theta_b(\varepsilon)|^2 &\lesssim b^2 \int (1+r^4) \left[\frac{1+|\log r|^2}{1+r^2} |\nabla^2 \varepsilon|^2 + \frac{1+|\log r|^2}{1+r^4} |\nabla \varepsilon|^2 \right. \\ &\quad \left. + \frac{|\nabla^2 \phi_\varepsilon|^2 + |\varepsilon|^2}{1+r^6} + \frac{|\nabla \phi_\varepsilon|^2}{1+r^8} + \frac{|\nabla \phi_{\Theta_b(\varepsilon)}|^2}{1+r^8} \right] \\ &\lesssim b^2 C(M) \|\varepsilon_2\|_{L_Q^2}^2, \end{aligned}$$

this is (7.5.75).

We now estimate all corresponding terms in (7.5.44). From (7.5.75), (7.4.12) :

$$\begin{aligned} |(\hat{\varepsilon}_2, \mathcal{M} \mathcal{L} \Theta_b(\varepsilon))| &= \left| \int Q \nabla \cdot \mathcal{M} \hat{\varepsilon}_2 \cdot \nabla \mathcal{M} \Theta_b(\varepsilon) \right| \lesssim \frac{1}{100} \int Q |\nabla \mathcal{M} \hat{\varepsilon}_2|^2 + C \int Q |\nabla \mathcal{M} \Theta_b(\varepsilon)|^2 \\ &\leq \frac{1}{100} \int Q |\nabla \mathcal{M} \hat{\varepsilon}_2|^2 + C(M) \frac{b^4}{|\log b|^2}. \end{aligned}$$

Next, integrating by parts :

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon} \mathcal{L} \Theta_b(\varepsilon) \right| &\lesssim b \int (1+r^2) \left(|\nabla \hat{\varepsilon}| + \frac{|\hat{\varepsilon}|}{1+r} \right) Q |\nabla \mathcal{M} \Theta_b(\varepsilon)| \\ &\lesssim b \left(\int |\nabla \hat{\varepsilon}|^2 + \frac{|\hat{\varepsilon}|^2}{1+r^2} \right)^{\frac{1}{2}} \left(\int Q |\nabla \mathcal{M} \Theta_b(\varepsilon)|^2 \right)^{\frac{1}{2}} \\ &\lesssim b \left[C(M) \|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right] b \|\varepsilon_2\|_{L_Q^2} \lesssim b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^4}{|\log b|^2}. \end{aligned}$$

Similarly, using (7.5.73) :

$$\begin{aligned} b \left| \int \frac{\Lambda Q + 2Q}{Q^2} \hat{\varepsilon}_2 \Theta_b(\varepsilon) \right| &\lesssim b \|\hat{\varepsilon}_2\|_{L_Q^2} \|\Theta_b(\varepsilon)\|_{L^2} \lesssim b \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right) b C(M) \|\varepsilon_2\|_{L_Q^2} \\ &\lesssim b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^4}{|\log b|^2}. \end{aligned}$$

Using (7.5.74), (7.7.16) :

$$\begin{aligned} b \left| \int \hat{\varepsilon}_1 \cdot \nabla \phi_{\Theta_b(\varepsilon)} \right| &\lesssim b \left(\int (1+r^2) |\hat{\varepsilon}_1|^2 \right)^{\frac{1}{2}} \left(\int \frac{|\nabla \phi_{\Theta_b(\varepsilon)}|^2}{1+r^2} \right)^{\frac{1}{2}} \\ &\lesssim b C(M) \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right) b \|\varepsilon_2\|_{L_Q^2} \lesssim b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^4}{|\log b|^2}, \\ b \left| \int \nabla \phi_{\hat{\varepsilon}} \cdot Q \nabla \mathcal{M} \Theta_b(\varepsilon) \right| &\lesssim b \left(\int \frac{|\nabla \phi_{\hat{\varepsilon}}|^2}{1+r^4} \right)^{\frac{1}{2}} \left(\int Q |\nabla \mathcal{M} \Theta_b(\varepsilon)|^2 \right)^{\frac{1}{2}} \\ &\lesssim b C(M) \left(\|\varepsilon_2\|_{L_Q^2} + \frac{b}{|\log b|} \right) b \|\varepsilon_2\|_{L_Q^2} \\ &\lesssim b \frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^4}{|\log b|^2}. \end{aligned}$$

step 12 Conclusion. Injecting the collection of bounds obtained in steps 4 through 8 into (7.5.44) yields the following preliminary estimate :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ \frac{(\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2)}{\lambda^4} + O\left(\frac{C(M)b^3}{\lambda^4 |\log b|^2}\right) \right\} \\ &\leq -\frac{1}{2} \int Q_\lambda |\nabla \mathcal{M}_\lambda \hat{w}_2|^2 + \frac{b C(M)}{\lambda^6} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right] + \frac{\hat{b}}{\lambda^6} (\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2). \end{aligned}$$

We now make an essential of the precise numerology to treat the remaining $(\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2)$ term which has the wrong sign. We multiply this identity by λ^2 and obtain using the rough bound from (7.4.30), (7.5.3) :

$$\left| \frac{\lambda_s}{\lambda} + \hat{b} \right| \lesssim \frac{b}{|\log b|} \quad (7.5.77)$$

the control :

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ \frac{(\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2)}{\lambda^2} + O\left(\frac{C(M)b^3}{\lambda^2 |\log b|^2}\right) \right\} \\ &\leq \frac{b C(M)}{\lambda^4} \left[\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2} \right] \\ &+ \frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} + \hat{b} \right| |(\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2)| + \frac{1}{\lambda^4} \left| \frac{\lambda_s}{\lambda} \right| \frac{C(M)b^3}{|\log b|^2}. \end{aligned} \quad (7.5.78)$$

We now develop the quadratic term and estimate using (7.5.13), (7.5.14) :

$$\begin{aligned} (\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2) &= (\mathcal{M}\varepsilon_2, \varepsilon_2) + 2(\mathcal{M}\zeta_2, \varepsilon_2) + (\mathcal{M}\zeta_2, \zeta_2) \\ &= (\mathcal{M}\varepsilon_2, \varepsilon_2) + O\left(\frac{b^{\frac{3}{2}}}{|\log b|} \|\varepsilon_2\|_{L_Q^2} + \frac{b^3}{|\log b|^2}\right) \end{aligned}$$

which implies the upper bound

$$|(\mathcal{M}\hat{\varepsilon}_2, \hat{\varepsilon}_2)| \lesssim \|\varepsilon_2\|_{L_Q^2}^2 + \frac{b^3}{|\log b|^2}.$$

Injecting these bounds together with (7.5.77), (7.4.9) into (7.5.78) yields (7.5.31) and concludes the proof of Proposition 7.5.7.

7.6 Sharp description of the singularity formation

We are now in position to conclude the proof of the bootstrap Proposition 7.4.3 from which Theorem 7.1.1 easily follows.

7.6.1 Closing the bootstrap

We now conclude the proof of Proposition 7.4.3.

Proof of Proposition 7.4.3. The L^1 bound (7.4.11) follows from (7.4.24). The pointwise upper bound on b (7.4.10) follows from (7.5.24) which implies $\hat{b}_s < 0$ and the conclusion follows from (7.5.3). The lower bound $b > 0$ follows from the bootstrap bound (7.4.9). Indeed, if $b(s^*) = 0$, then $\varepsilon(s^*) = 0$ from (7.4.9) and then $\int u(s^*) = \int Q = \int u_0$ by conservation mass, and a contradiction follows from (7.4.2).

It remains to close the bound (7.4.12) which is the heart of the analysis, and this follows from the monotonicity formula (7.5.7). Indeed, we rewrite (7.5.31) using (7.4.9) :

$$\frac{d}{dt} \left\{ \frac{(\mathcal{M}\varepsilon_2, \varepsilon_2)}{\lambda^2} + O\left(C(M)\sqrt{K^*} \frac{b^3}{\lambda^2 |\log b|^2}\right) \right\} \lesssim \sqrt{K^*} \frac{b^4}{\lambda^4 |\log b|^2}. \quad (7.6.1)$$

In order to integrate this between $t = 0$ and t^* , we first observe from (7.5.3), (7.5.25), (7.4.28) the bounds

$$|\hat{b}_s| \lesssim \frac{b^2}{|\log b|}, \quad |\lambda\lambda_t + b| \lesssim C(M) \frac{b^2}{|\log b|} \quad (7.6.2)$$

from which :

$$\begin{aligned} \int_0^{t^*} \frac{b^4}{\lambda^4 |\log b|^2} dt &= \int -\lambda_t \frac{\hat{b}^3}{\lambda^3 |\log \hat{b}|^2} dt + O\left(\int_0^{t^*} \frac{b^5}{\lambda^4 |\log b|^5} dt\right) \\ &= \left[\frac{\hat{b}^3}{2\lambda^2 |\log \hat{b}|^2} \right]_0^{t^*} - \int_0^{t^*} \frac{1}{2\lambda^4} \frac{d}{ds} \left[\frac{\hat{b}^3}{|\log \hat{b}|^2} \right] ds + O\left(\int_0^{t^*} \frac{b^5}{\lambda^4 |\log b|^5} dt\right) \\ &= \left[\frac{\hat{b}^3}{2\lambda^2 |\log \hat{b}|^2} \right]_0^{t^*} + O\left(\int_0^{t^*} \frac{b^4}{\lambda^4 |\log b|^3} dt\right) \end{aligned}$$

and hence using the a priori smallness of b :

$$\lambda^2(t) \int_0^{t^*} \frac{b^4}{\lambda^4 |\log b|^2} dt \lesssim \frac{b^3(t)}{|\log b(t)|^2} + \left(\frac{\lambda(t)}{\lambda(0)} \right)^2 \frac{b^3(0)}{|\log b(0)|^2}$$

Integrating (7.6.1) in time, we conclude using (7.2.58) :

$$\begin{aligned} \delta(M) \|\varepsilon_2(t)\|_{L_Q^2}^2 &\lesssim (\mathcal{M}\varepsilon_2(t), \varepsilon_2(t)) \\ &\lesssim C(M) \left\{ \frac{\lambda^2(t)}{\lambda^2(0)} \left[\|\varepsilon_2(0)\|_{L_Q^2}^2 + \sqrt{K^*} \frac{b^3(0)}{|\log b(0)|^2} \right] + \sqrt{K^*} \frac{b^3(t)}{|\log b(t)|^2} \right\} \end{aligned} \quad (7.6.3)$$

for some small enough universal constants $\delta(M), C(M) > 0$ independent of K^* . Moreover, (7.6.2) implies

$$\frac{d}{ds} \left\{ \frac{\hat{b}^3}{\lambda^2 |\log \hat{b}|^2} \right\} > 0 \quad (7.6.4)$$

and thus (7.6.3) and the initialization (7.4.4) ensure :

$$\begin{aligned} \delta(M) \|\varepsilon_2(t)\|_{L_Q^2}^2 &\lesssim C(M) \sqrt{K^*} \left[\lambda^2(t) \frac{b^3(0)}{\lambda^2(0) |\log b(0)|^2} + \frac{b^3(t)}{|\log b(t)|^2} \right] \\ &\lesssim C(M) \sqrt{K^*} \left[\lambda^2(t) \frac{\hat{b}^3(0)}{\lambda^2(0) |\log \hat{b}(0)|^2} + \frac{b^3(t)}{|\log b(t)|^2} \right] \\ &\lesssim C(M) \sqrt{K^*} \frac{\hat{b}^3(t)}{|\log \hat{b}(t)|^2} \lesssim C(M) \sqrt{K^*} \frac{b^3(t)}{|\log b(t)|^2} \end{aligned}$$

and (7.4.12) follows for $K^* = K^*(M)$ large enough. This concludes the proof of Proposition 7.4.3. \square

7.6.2 Sharp control of the singularity formation

We are now in position to conclude the proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. step 1 Let $u_0 \in \mathcal{O}$ and $u \in \mathcal{C}([0, T], \mathcal{E})$ be the corresponding solution to (7.1.1) with lifetime $0 < T \leq +\infty$, then the estimates of Proposition 7.4.3 hold on $[0, T]$. Observe from (7.6.4) the bound

$$\lambda^2 \lesssim b^3$$

from which using (7.6.2)

$$-\lambda \lambda_t \gtrsim b \gtrsim C(u_0) \lambda^{\frac{2}{3}} \text{ and thus } -(\lambda^{\frac{4}{3}})_t \gtrsim C(u_0) > 0$$

implies that $\lambda(t)$ touches zero in some finite time $0 < T_0 < +\infty$. Note that the bounds of Proposition 7.4.3 injected into the decomposition (7.4.5) ensure

$$\|\varepsilon(t)\|_{H^2} \ll 1 \text{ for } 0 \leq t < T_0 \text{ and } \lim_{t \uparrow T_0} \|u(t)\|_{H^2} = +\infty$$

and thus from standard Cauchy theory, the solution blows up at $T = T_0 < +\infty$. Moreover, we obtain from (7.6.2) the rough bound :

$$|\lambda \lambda_t| = \left| \frac{\lambda_s}{\lambda} \right| \lesssim 1 \text{ and thus } \lambda(t) \lesssim \sqrt{T-t},$$

and hence from (7.4.13) :

$$s(t) \rightarrow +\infty \text{ as } t \rightarrow T. \quad (7.6.5)$$

step 2 Blow up speed. We now derive the sharp asymptotics at blow up time by reintegrating the modulation equations for (b, λ) in the vicinity of blow up time or equivalently as $s \rightarrow \infty$. We estimate from (7.5.24), (7.4.12) :

$$\left| \hat{b}_s + \frac{2\hat{b}^2}{|\log \hat{b}|} \right| \lesssim \frac{\hat{b}^2}{|\log \hat{b}|^2}. \quad (7.6.6)$$

We then argue as in [88]. We multiply (7.6.6) by $\frac{|\log \hat{b}|}{\hat{b}^2}$ and obtain :

$$\frac{\hat{b}_s \log \hat{b}}{\hat{b}^2} = -2 + O\left(\frac{1}{|\log \hat{b}|}\right).$$

We use

$$\left(\frac{\log t}{t} + \frac{1}{t} \right)' = -\frac{\log t}{t^2}$$

to conclude after integration :

$$-\frac{\log \hat{b} + 1}{\hat{b}} = 2s + O\left(\int_0^s \frac{d\sigma}{|\log \hat{b}|}\right)$$

and thus

$$\hat{b}(s) = \frac{\log s}{2s} (1 + o(1)), \quad \log \hat{b} = \log \log s - \log s + O(1).$$

This finally yields the asymptotic development near blow up time :

$$\hat{b}(s) = \frac{1}{2s} (\log s - \log \log s) + O\left(\frac{1}{s}\right).$$

We now rewrite the modulation equations for λ using (7.6.2), (7.5.3) :

$$-\frac{\lambda_s}{\lambda} = \hat{b} + O\left(\frac{b}{|\log b|}\right) = \frac{1}{2s} (\log s - \log \log s) + O\left(\frac{1}{s}\right) \quad (7.6.7)$$

which time integration yields :

$$-\log \lambda = \frac{1}{4} \left[(\log s)^2 - 2\log s \log \log s \right] + O(\log s) = \frac{(\log s)^2}{4} \left[1 - \frac{2\log \log s}{\log s} + O\left(\frac{1}{\log s}\right) \right].$$

In particular,

$$\sqrt{|\log \lambda|} = \frac{\log s}{2} \left[1 - \frac{\log \log s}{\log s} + O\left(\frac{1}{\log s}\right) \right]$$

which also implies :

$$e^{2\sqrt{|\log \lambda|}+O(1)} = \frac{s}{\log s}, \quad s = \sqrt{|\log \lambda|} e^{2\sqrt{|\log \lambda|}+O(1)}.$$

We use these relations to rewrite the modulation equation (7.6.7) :

$$-s \frac{\lambda_s}{\lambda} = -\sqrt{|\log \lambda|} e^{2\sqrt{|\log \lambda|}+O(1)} \lambda \lambda_t = \sqrt{|\log \lambda|} + O(1)$$

and thus

$$-\lambda \lambda_t e^{2\sqrt{|\log \lambda|}} = e^{O(1)}.$$

The time integration with boundary condition $\lambda(T) = 0$ yields

$$\lambda(t) = \sqrt{T-t} e^{-\sqrt{\frac{|\log(T-t)|}{2}}+O(1)} \quad \text{as } t \rightarrow T,$$

this is (7.1.11).

Observe now that the chain of above estimates ensures in particular

$$b(t) \rightarrow 0 \quad \text{as } t \rightarrow T$$

which easily implies using Proposition 7.3.5 :

$$\|Q - \tilde{Q}_b\|_{H_Q^2} \rightarrow 0 \quad \text{as } t \rightarrow T,$$

and the strong convergence (7.1.10) now follows from (7.4.12).

This concludes the proof of Theorem 7.1.1. □

7.7 Appendix

7.7.1 Estimates for the Poisson field

This appendix is devoted to the derivation of linear estimates, in particular for the Poisson field in H_Q^2 and \mathcal{E} . We start with weighted H^2 type bounds :

Lemme 7.7.1 (H^2 bound). *Let $p = 1, 2$, then for all $u \in \mathcal{D}(\mathbb{R}^2)$,*

$$\int |x|^{2p} |\nabla^2 u|^2 \lesssim \int |x|^{2p} |\Delta u|^2 + \int |x|^{2p-2} |\nabla u|^2. \quad (7.7.1)$$

Proof of Lemma 7.7.1. We integrate by parts to compute :

$$\begin{aligned} & \int x_1^{2p} (\partial_{11} u + \partial_{22} u)^2 = \int x_1^{2p} [(\partial_{11} u)^2 + (\partial_{22} u)^2] + 2 \int x_1^{2p} \partial_{11} u \partial_{22} u \\ &= \int x_1^{2p} [(\partial_{11} u)^2 + (\partial_{22} u)^2] - 2 \int x_1^{2p} \partial_2 u \partial_{211} u \\ &= \int x_1^{2p} [(\partial_{11} u)^2 + (\partial_{22} u)^2] + 2 \int \partial_{12} u [x_1^{2p} \partial_{12} u + 2px_1^{2p-1} \partial_2 u] \\ &= \int x_1^{2p} [(\partial_{11} u)^2 + (\partial_{22} u)^2 + 2(\partial_{12} u)^2] - 2p(2p-1) \int x_1^{2p-2} (\partial_2 u)^2 \end{aligned}$$

and similarly with x_2 , and (7.7.1) follows. □

We now turn to the linear control of the Poisson field :

Lemme 7.7.2 (Estimates for the Poisson field). *There holds the bounds on the Poisson field :*

(i) *General L_Q^2 bounds :*

$$\|(1 + |\log r|)u\|_{L^1} + \|\phi_u\|_{L^\infty(r \leq 1)} + \left\| \frac{|\phi_u|}{1 + |\log r|} \right\|_{L^\infty(r \geq 1)} \lesssim \|u\|_{L_Q^2}, \quad (7.7.2)$$

$$\|\nabla \phi_u\|_{L^4} \lesssim \|u\|_{L_Q^2}. \quad (7.7.3)$$

(ii) *Improved L_Q^2 bound : if moreover $\int u = 0$, then :*

$$\|\phi_u\|_{L^\infty} \lesssim \|u\|_{L_Q^2}, \quad (7.7.4)$$

$$\int |\nabla \phi_u|^2 = - \int u \phi_u \lesssim \|u\|_{L_Q^2}^2. \quad (7.7.5)$$

(iii) *Energy bound :*

$$\forall 1 \leq p < 2, \quad \|\nabla \phi_u\|_{L^\infty} \leq C_p \|u\|_{L^\infty}^{1-\frac{p}{2}} \|u\|_{L^2}^{p-1} \|u\|_{L^1}^{1-\frac{p}{2}}. \quad (7.7.6)$$

(iv) *Decay in the energy space :* $\forall 0 \leq \alpha < 1$,

$$|u(x)| \leq C_\alpha \frac{\|u\|_\varepsilon}{1 + |x|^\alpha}, \quad (7.7.7)$$

$$|\nabla \phi_u(x)| \leq C_\alpha \frac{\|u\|_\varepsilon}{1 + |x|^{\frac{\alpha}{2}}}. \quad (7.7.8)$$

Proof of Lemma 7.7.2. : Proof of (i)-(ii) : By Cauchy-Schwarz :

$$\int (1 + |\log r|)|u(y)| \lesssim \|u\|_{L_Q^2} \left(\int (1 + |\log r|)^2 Q \right)^{\frac{1}{2}} \lesssim \|u\|_{L_Q^2}.$$

For $|x| \leq 1$, we estimate using Cauchy-Schwarz :

$$\begin{aligned} |\phi_u(x)| &\lesssim \int_{|x-y| \leq 1} |\log(|x-y|)|u(y)| dy + \int_{|x-y| \geq 1} |\log|x-y|||u(y)| dy \\ &\lesssim \|u\|_{L_Q^2} + \int (1 + |\log|y||)|u(y)| \lesssim \|u\|_{L_Q^2}. \end{aligned}$$

For $|x| \geq 1$, we rewrite

$$\begin{aligned} &\left| \phi_u(x) - \frac{\log|x|}{2\pi} \int u \right| \lesssim \int \left| \log\left(\frac{|x-y|}{|x|}\right) \right| |u(y)| dy \\ &\lesssim \int_{|x-y| \geq \frac{|x|}{2}} \left| \log\left(\frac{|x-y|}{|x|}\right) \right| |u(y)| dy + \int_{|x-y| \leq \frac{|x|}{2}} \left| \log\left(\frac{|x-y|}{|x|}\right) \right| |u(y)| dy. \end{aligned}$$

For the outer term, we estimate :

$$\begin{aligned} \int_{|x-y| \geq \frac{|x|}{2}} |\log(\frac{|x-y|}{|x|})| |u(y)| dy &\lesssim \int_{|x-y| \geq \frac{|x|}{2}} \left(\frac{|x-y|}{|x|} \right)^{\frac{3}{4}} |u(y)| dy \lesssim \int \left(1 + \frac{|y|^{\frac{3}{4}}}{|x|^{\frac{3}{4}}} \right) |u(y)| dy \\ &\lesssim \|u\|_{L^1} + \frac{\|u\|_{L_Q^2}}{|x|^{\frac{3}{4}}} \lesssim \|u\|_{L_Q^2}. \end{aligned}$$

On the singularity, we estimate using a simple change of variables :

$$\begin{aligned} \int_{|x-y| \leq \frac{|x|}{2}} |\log(\frac{|x-y|}{|x|})| |u(y)| dy &\lesssim \left(\int_{|x-y| \leq \frac{|x|}{2}} |\log(\frac{|x-y|}{|x|})|^2 dy \right)^{\frac{1}{2}} \left(\int_{|x-y| \leq \frac{|x|}{2}} |u(y)|^2 dy \right)^{\frac{1}{2}} \\ &\lesssim \left(\|x\|^2 \int_{|z| \leq \frac{1}{2}} (\log|z|)^2 dz \int_{|x-y| \leq \frac{|x|}{2}} |u(y)|^2 dy \right)^{\frac{1}{2}} \lesssim \left(\int |y|^2 |u(y)|^2 dy \right)^{\frac{1}{2}} \\ &\lesssim \|u\|_{L_Q^2} \end{aligned}$$

and this concludes the proof of (7.7.2) and (7.7.4).

We then estimate from the 2 dimensional Hardy-Littlewood-Sobolev and Hölder :

$$\|\nabla \phi_u\|_{L^4} \lesssim \frac{1}{|x|} \star u \|_{L^4} \lesssim \|u\|_{L^{\frac{4}{3}}} \lesssim \|u\|_{L_Q^2} \|Q\|_{L^2}^{\frac{1}{2}}$$

and (7.7.3) is proved. Let a smooth cut off function $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$, and $\chi_R(x) = \chi(\frac{x}{R})$, then from (7.7.4) :

$$\begin{aligned} \int \chi_R |\nabla \phi_u|^2 &= \frac{1}{2} \int \Delta \chi_R \phi_u^2 - \int \chi_R u \phi_u \\ &\lesssim \|\phi_u\|_{L^\infty}^2 + \|\phi_u\|_{L^\infty} \|u\|_{L^1} \lesssim \|u\|_{L_Q^2}^2 \end{aligned} \tag{7.7.9}$$

with constants independent of $R > 0$, and hence

$$\int |\nabla \phi_u|^2 < +\infty. \tag{7.7.10}$$

Moreover, integrating by parts :

$$\int_{|x| \leq R} |\nabla \phi_u|^2 = R \int_0^{2\pi} \phi_u \partial_r \phi_u d\theta - \int_{|x| \leq R} u \phi_u.$$

From (7.7.10), we can find a sequence $R_n \rightarrow \infty$ such that

$$R_n^2 \int_0^{2\pi} |\partial_r \phi_u|^2 \rightarrow 0$$

and then from (7.7.4) :

$$R_n \left| \int_0^{2\pi} \phi_u \partial_r \phi_u \right| \lesssim \|\phi_u\|_{L^\infty} \left(R_n^2 \int_0^{2\pi} |\partial_r \phi_u|^2 \right)^{\frac{1}{2}} \rightarrow 0$$

and thus

$$\int |\nabla \phi_u|^2 = \lim_{R_n \rightarrow +\infty} \int_{|x| \leq R_n} |\nabla \phi_u|^2 = - \int u \phi_u.$$

The estimate (7.7.5) now follows from (7.7.4).

Proof of (iii) : Let $1 \leq p < 2$, we estimate in brute force :

$$\begin{aligned} |\nabla \phi_u| &\lesssim \frac{1}{|x|} \star |u| \lesssim \int_{|x-y| \leq R} \frac{|u(y)|}{|x-y|} dy + \int_{|x-y| \geq R} \frac{|u(y)|}{|x-y|} dy \\ &\lesssim \|u\|_{L^\infty} \int_{|z| \leq R} \frac{1}{|z|} + \|u\|_{L^p} \left(\int_{|z| \geq R} \frac{1}{|z|^{p'}} \right)^{\frac{1}{p'}} \lesssim R \|u\|_{L^\infty} + \frac{\|u\|_{L^p}}{R^{1-\frac{2}{p'}}}. \end{aligned}$$

We optimize in R and interpolate :

$$\|\nabla \phi_u\|_{L^\infty} \leq C_p \|u\|_{L^p}^{\frac{p}{2}} \|u\|_{L^\infty}^{1-\frac{p}{2}} \lesssim C_p \|u\|_{L^\infty}^{1-\frac{p}{2}} \|u\|_{L^2}^{p-1} \|u\|_{L^1}^{1-\frac{p}{2}},$$

this is (7.7.6).

Proof of (iv) : By density, it suffices to prove (7.7.7) for $u \in \mathcal{D}(\mathbb{R}^2)$. Let $(v_{i,j} = x_i \partial_j u)_{1 \leq i,j \leq 2}$, we estimate from (7.7.1) with $p = 2$:

$$\int |v_{i,j}|^2 + \int |\nabla v_{i,j}|^2 \lesssim \int (1 + |x|^2) |\nabla u|^2 + \int (1 + |x|^4) [(\partial_{11} u)^2 + (\partial_{22} u)^2 + (\partial_{12} u)^2] \lesssim \|u\|_{H_Q^2}^2$$

and thus from Sobolev :

$$\forall p > 2, \quad \|v_{i,j}\|_{L^p} \lesssim \|v_{i,j}\|_{H^1} \lesssim \|u\|_{H_Q^2}. \quad (7.7.11)$$

We now recall the standard Sobolev bound, see for example [10] :

$$\forall p > 2, \quad \forall f \in \mathcal{D}(\mathbb{R}^2), \quad |f(x) - f(y)| \lesssim |x - y|^{1-\frac{2}{p}} \|\nabla f\|_{L^p}.$$

We may find $|a| \leq 1$ such that

$$|f(a)| \lesssim \left(\int_{|y| \leq 1} |f(y)|^2 dy \right)^{\frac{1}{2}}$$

and hence the growth estimate :

$$|f(x)| \lesssim \left(\int_{|y| \leq 1} |f(y)|^2 dy \right)^{\frac{1}{2}} + |x|^{1-\frac{2}{p}} \|\nabla f\|_{L^p}.$$

We apply this to $f_i = x_i u$ and conclude from (7.7.11), (7.7.18) : $\forall p > 2$ and $i = 1, 2$:

$$\begin{aligned} |x_i u| &\lesssim \left(\int_{|y| \leq 1} |x_i u|^2 dy \right)^{\frac{1}{2}} + |x|^{1-\frac{2}{p}} \|\nabla(x_i u)\|_{L^p} \\ &\lesssim \|u\|_{L^\infty} + |x|^{1-\frac{2}{p}} [\|v_{i,j}\|_{L^p} + \|u\|_{L^p}] \lesssim (1 + |x|^{1-\frac{2}{p}}) \|u\|_{\mathcal{E}} \end{aligned}$$

and hence the decay :

$$|u(x)| \lesssim \frac{\|u\|_{\mathcal{E}}}{1 + |x|^{\frac{2}{p}}}$$

which yields (7.7.7).

Let $0 \leq \alpha < 1$ and $|x| \gg 1$, we estimate the Poisson field in brute force using (7.7.7) :

$$\begin{aligned} |\nabla \phi_u(x)| &\lesssim \int \frac{|u(y)|}{|x-y|} = \int_{|x-y|>|x|^{\frac{\alpha}{2}}} \frac{|u(y)|}{|x-y|} + \int_{|x-y|<|x|^{\frac{\alpha}{2}}} \frac{|u(y)|}{|x-y|} \\ &\lesssim \frac{\|u\|_{L^1}}{|x|^{\frac{\alpha}{2}}} + \int_{|y|\geq|x|^{\frac{1}{2}}, |x-y|<|x|^{\frac{\alpha}{2}}} \frac{|u(y)|}{|x-y|} \lesssim \frac{\|u\|_{L^1}}{|x|^{\frac{\alpha}{2}}} + \frac{\|u\|_{\mathcal{E}}}{1+|x|^{\alpha}} \int_{|z|\leq|x|^{\frac{\alpha}{2}}} \frac{dz}{|z|} \\ &\lesssim \frac{\|u\|_{\mathcal{E}}}{1+|x|^{\frac{\alpha}{2}}} \end{aligned}$$

and (7.7.8) is proved. \square

7.7.2 Hardy bounds

We recall some standard weighted Hardy inequalities :

Lemme 7.7.3 (Weighted Hardy inequality). *There holds the Hardy bounds :*

$$\forall \alpha > -2, \quad \int r^{\alpha+2} |\partial_r u|^2 \geq \frac{(2+\alpha)^2}{4} \int r^\alpha u^2, \quad (7.7.12)$$

$$\int |\Delta \phi|^2 \gtrsim \int \frac{|\nabla \phi|^2}{r^2(1+|\log r|)^2} - \int \frac{|\nabla \phi|^2}{1+r^4}, \quad (7.7.13)$$

$$\int \frac{|\nabla \phi|^2}{r^2(1+|\log r|)^2} \gtrsim \int \frac{\phi^2}{(1+r^4)(1+|\log r|)^2} - \int \frac{\phi^2}{1+r^6}. \quad (7.7.14)$$

Proof of Lemma 7.7.3 Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. We integrate by parts to estimate :

$$\frac{\alpha+2}{2} \int r^\alpha u^2 = - \int r^{\alpha+1} u \partial_r u \leq \left(\int r^\alpha u^2 \right)^{\frac{1}{2}} \left(\int r^{\alpha+2} (\partial_r u)^2 \right)^{\frac{1}{2}}$$

and (7.7.12) follows.

Let now $\phi \in \mathcal{D}(\mathbb{R}^2)$ and consider the radial continuous and piecewise \mathcal{C}^1 function

$$f(r) = \begin{cases} \frac{1}{r(1-\log r)} & \text{for } 0 < r \leq 1, \\ \frac{1}{r(1+\log r)} & \text{for } r \geq 1 \end{cases}, \quad F(x) = f(r) \frac{x}{r}$$

then

$$\nabla \cdot F(x) = \begin{cases} \frac{1}{r^2(1-\log r)^2} & \text{for } 0 < r < 1, \\ -\frac{1}{r^2(1+\log r)^2} & \text{for } r > 1 \end{cases},$$

and thus :

$$\begin{aligned} \int \frac{1}{r^2(1+|\log r|)^2} |\nabla \phi|^2 &\lesssim \int_{|x|\leq 1} \nabla \cdot F |\nabla \phi|^2 - \int_{|x|\geq 1} \nabla \cdot F |\nabla \phi|^2 \\ &\lesssim \int_{r=1} f |\nabla \phi|^2 d\sigma + \int |f| |\nabla(|\nabla \phi|^2)| \\ &\lesssim \int_{r=1} f |\nabla \phi|^2 d\sigma + \left(\int |f|^2 |\nabla \phi|^2 \right)^{\frac{1}{2}} \left(\int |\nabla^2 \phi|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Now by Sobolev :

$$\int_{r=1} f |\nabla \phi|^2 d\sigma \lesssim \int_{\frac{1}{2} \leq r \leq 2} (|\nabla \phi|^2 + |\nabla^2 \phi|^2)$$

and thus

$$\begin{aligned} & \int \frac{1}{r^2(1+|\log r|)^2} |\nabla \phi|^2 \\ & \lesssim \int_{\frac{1}{2} \leq r \leq 2} |\nabla \phi|^2 + \int |\Delta \phi|^2 + \left(\int \frac{1}{r^2(1+|\log r|)^2} |\nabla \phi|^2 \right)^{\frac{1}{2}} \left(\int |\Delta \phi|^2 \right)^{\frac{1}{2}} \end{aligned}$$

which implies (7.7.13).

Similarly, for $r \geq r_0$ large enough :

$$\begin{aligned} \int_{r \geq r_0} \frac{\phi^2}{(1+r^4)(1+|\log r|)^2} & \lesssim - \int_{r \geq r_0} \nabla \cdot \left[\frac{1}{(1+r^3)(1+|\log r|)^2} \frac{y}{|y|} \right] \phi^2 \\ & \lesssim \int_{r=r_0} |\phi|^2 d\sigma + \int_{r \geq r_0} \frac{|\phi||\nabla \phi|}{(1+r^3)(1+|\log r|)^2} \end{aligned}$$

and (7.7.14) follows again from Cauchy Schwarz and Sobolev.

This concludes the proof of Lemma 7.7.3.

7.7.3 Interpolation bounds

We collect in this appendix the bootstrap bounds on ε which are a consequence of the spectral estimates of Proposition 7.2.8 and further interpolation estimates.

Proposition 7.7.4 (Interpolation bounds). *For $i = 0, 1$:*

(i) H_Q^2 bound :

$$\begin{aligned} & \int (1+r^4) |\nabla^2 \varepsilon|^2 + \int (1+r^2) |\nabla \varepsilon|^2 + \int \varepsilon^2 \\ & + \int |\Delta \phi_\varepsilon|^2 + \int \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \lesssim C(M) \|\varepsilon_2\|_{L_Q^2}^2, \end{aligned} \tag{7.7.15}$$

$$\int (1+r^2) |\varepsilon_1|^2 \lesssim C(M) \|\varepsilon_2\|_{L_Q^2}^2. \tag{7.7.16}$$

(ii) L^∞ bounds :

$$\forall 0 < \eta \leq \frac{1}{2}, \quad \|\nabla \phi_\varepsilon\|_{L^\infty} \leq C_\eta \|\varepsilon_2\|_{L_Q^2}^{1-\eta}, \tag{7.7.17}$$

and $\forall 0 \leq \alpha < \frac{1}{2}$,

$$\|(1+|x|^\alpha)\varepsilon\|_{L^\infty} + \left\| (1+|x|^{\frac{\alpha}{2}}) \nabla \phi_\varepsilon(x) \right\|_{L^\infty} \lesssim \delta(\alpha^*). \tag{7.7.18}$$

(iii) H_Q^2 bound with logarithmic loss :

$$\int (1+\log(1+r))^C \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \lesssim |\log b|^C \left(\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right), \tag{7.7.19}$$

(iv) L^∞ bound with loss :

$$\left\| \frac{\nabla \phi_\varepsilon}{1+|x|} \right\|_{L^\infty}^2 \lesssim |\log b|^C \left(\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right). \quad (7.7.20)$$

(v) Weighted bound with loss :

$$\int \frac{|\varepsilon|}{1+r} \lesssim C(M) \sqrt{|\log b|} \|\varepsilon_2\|_{L_Q^2} + b^{10}. \quad (7.7.21)$$

Proof of Proposition 7.7.4. Proof of (i) : The estimate (7.7.15) follows directly from (7.2.58), (7.2.60), our choice of orthogonality conditions (7.4.6) and (7.7.1) with $p = 2$. We then estimate from the definitions (7.5.36), (7.5.37) and (7.7.15) :

$$\begin{aligned} \int (1+r^2)|\varepsilon_1|^2 &\lesssim \int (1+r^2) \left[|\nabla \varepsilon|^2 + |\nabla \phi_Q|^2 |\varepsilon|^2 + Q^2 |\nabla \phi_\varepsilon|^2 \right] \\ &\lesssim \|\varepsilon\|_{H_Q^2}^2 + \int \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \lesssim C(M) \|\varepsilon_2\|_{L_Q^2}^2. \end{aligned}$$

Proof of (ii) : Let $p = 2(1-\eta) \in [1, 2]$, then from (7.7.6), Sobolev, (7.7.15) and the bootstrap bound (7.4.8) :

$$\|\nabla \phi_\varepsilon\|_{L^\infty} \lesssim C_p(M) \|\varepsilon_2\|_{L_Q^2}^{\frac{p}{2}} \|\varepsilon\|_{L^1}^{1-\frac{p}{2}} \lesssim C_\eta \|\varepsilon_2\|_{L_Q^2}^{1-\eta}.$$

The decay bound (7.7.18) follows from the interpolation bounds (7.7.7), (7.7.8), the H_Q^2 bound (7.7.15) and the bootstrap bounds (7.4.8), (7.4.9).

Proof of (iii) : The lossy bound (7.7.19) follows from (7.7.15), (7.7.18) with $\alpha = \frac{1}{2}$. Indeed, let $B = b^{-100}$, then :

$$\begin{aligned} &\int (1+\log(1+r))^C \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \\ &\lesssim \int_{r \leq B} (1+\log(1+r))^C \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} + \int_{r \geq B} (1+\log(1+r))^C \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} \\ &\lesssim |\log b|^{C_1(C)} \int \frac{|\nabla \phi_\varepsilon|^2}{r^2(1+|\log r|)^2} + \int_{r \geq B} \frac{(1+|\log r|)^C}{r^{2+\frac{1}{2}}} \\ &\lesssim |\log b|^{C_1(C)} \left[\|\varepsilon_2\|_{L_Q^2}^2 + \frac{1}{\sqrt{B}} \right] \lesssim |\log b|^{C_1(C)} \left(\|\varepsilon_2\|_{L_Q^2}^2 + b^{10} \right). \end{aligned}$$

Proof of (iv) : From Sobolev :

$$\left\| \frac{\nabla \phi_\varepsilon}{1+|x|} \right\|_{L^\infty}^2 \lesssim \left\| \frac{\nabla \phi_\varepsilon}{1+|x|} \right\|_{H^2}^2 \lesssim \int \frac{|\nabla \phi_\varepsilon|^2}{1+r^2} + \|\varepsilon\|_{H^2}^2$$

and (7.7.20) follows from (7.7.19).

Proof of (7.7.21) : We use the global L^1 bound (7.4.8), (7.7.15) and Cauchy Schwarz to estimate :

$$\begin{aligned} \int \frac{|\varepsilon|}{1+r} &\lesssim \|\varepsilon\|_{L^2} \left(\int_{r \leq b^{-20}} \frac{1}{r^2} \right)^{\frac{1}{2}} + \int_{r \geq b^{-20}} \frac{|\varepsilon|}{1+r} \\ &\lesssim C(M) \sqrt{|\log b|} \|\varepsilon_2\|_{L_Q^2} + b^{20} \|\varepsilon\|_{L^1} \end{aligned}$$

and (7.7.21) is proved. \square

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