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David Godinho Pereira

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et soutenue publiquement par

David GODINHO PEREIRA

le 25 Novembre 2013

**Contribution à l'étude des équations
de Boltzmann, Kac et Keller-Segel à
l'aide d'équations différentielles
stochastiques non linéaires**

Thèse dirigée par Nicolas FOURNIER

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Résumé

L'objet de cette thèse est l'étude de l'asymptotique des collisions rasantes pour les équations de Kac et de Boltzmann ainsi que l'étude de la propagation du chaos pour l'équation de Keller-Segel dans un cadre sous-critique à l'aide d'équations différentielles stochastiques non linéaires.

Le premier chapitre est consacré à l'équation de Kac avec un potentiel Maxwellien. Nous commençons par donner une vitesse de convergence explicite (que l'on pense être optimale) dans le cadre de l'asymptotique des collisions rasantes. Puis nous approchons la solution de l'équation de Kac dans le cadre général, ce qui nous permet de montrer la propagation du chaos pour un système de particules vers cette dernière de manière quantitative.

Dans le deuxième chapitre, nous étudions l'asymptotique des collisions rasantes pour l'équation de Boltzmann avec des potentiels mous et de Coulomb. Nous donnons là encore des vitesses de convergence explicites (mais non optimales).

Enfin dans le troisième et dernier chapitre, nous montrons la propagation du chaos pour l'équation de Keller-Segel dans un cadre sous-critique. Pour cela, nous utilisons des arguments de compacité (tension du système de particules).

Mots clefs. Kac, Boltzmann, Keller-Segel, distance de Wasserstein, collisions rasantes, potentiel Maxwellien, potentiels mous, potentiel de Coulomb, propagation du chaos, système de particules.

Contribution to the study of Boltzmann's, Kac's and Keller-Segel's equations with non-linear Stochastic Differential Equations

This thesis is devoted to the study of the asymptotic of grazing collisions for Kac's and Boltzmann's equations and to the study of the chaos propagation for some sub-critical Keller-Segel equation with non-linear Stochastic Differential Equations.

The first chapter is devoted to the Kac equation with a Maxwellian potential. We start by giving an explicit rate of convergence (than we believe to be optimal) for the asymptotic of grazing collisions. Then, we approximate the solution of Kac's equation in the general case, which allows us to show the chaos propagation for some particle system to this last one in a quantitative way.

In the second chapter, we study the asymptotic of grazing collisions for the Boltzmann equation with soft and Coulomb potentials. We also give explicit rates of convergence (which are not optimal).

Finally in the third and last chapter, we show the chaos propagation for some sub-critical Keller-Segel equation. To this aim, we use compactness arguments (tightness of the particle system).

Keywords. Kac, Boltzmann, Keller-Segel, Wasserstein distance, grazing collisions, Maxwellian potential, soft potentials, Coulomb potential, chaos

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propagation, particle system.

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Introduction

0.1 Présentation de la thèse

Dans cette thèse, nous nous intéresserons successivement aux équations de Kac, Boltzmann et Keller-Segel. Nous étudierons ces équations à l'aide d'outils probabilistes.

L'équation de Boltzmann s'inscrit dans la théorie cinétique des gaz. Elle décrit l'évolution de la vitesse de particules qui se déplacent dans un gaz. On suppose que les particules interagissent à travers des forces répulsives : deux particules se trouvant à une distance r exercent l'une sur l'autre une force proportionnelle à $1/r^s$. Le noyau de collision sera alors de la forme

$$B(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta), \quad \gamma = \frac{s - 5}{s - 1},$$

où θ est l'angle de déviation et $\beta : (0, \pi] \rightarrow [0, \infty)$ est une fonction appelée *la section efficace*. On parle de potentiels *durs* lorsque $s > 5$ (i.e. $\gamma > 0$), d'un potentiel *Maxwellien* lorsque $s = 5$ (i.e. $\gamma = 0$), de potentiels *mous* lorsque $2 < s < 5$ (i.e. $-3 < \gamma < 0$) et d'un potentiel de *Coulomb* lorsque $s = 2$ (i.e. $\gamma = -3$). On peut observer que dans le cas Maxwellien, le noyau de collision ne dépend pas de la vitesse des particules. Voir [3, 15, 23, 71] pour plus de détails sur l'équation de Boltzmann. Nous traitons ici les potentiels *mous* et de Coulomb dans le cas dit *sans cutoff* où l'on suppose que

$$\int_0^\pi \beta(\theta) d\theta = \infty, \quad \int_0^\pi \theta^2 \beta(\theta) d\theta < \infty.$$

Cette hypothèse traduit le fait qu'il y ait une infinité de collisions engendrant de *petites* modifications de vitesse : on appelle celles-ci les *collisions rasantes*. Il est connu que dans l'asymptotique des collisions rasantes (où les autres collisions sont négligeables), la solution de l'équation de Boltzmann tend vers

la solution de l'équation de Landau. Dans le chapitre 2, nous donnons une vitesse explicite pour cette convergence en utilisant la distance de Wasserstein avec coût quadratique. Plus précisément, nous obtenons une vitesse de l'ordre de $\epsilon^{1/2-}$ pour les potentiels mous (on pense que la vitesse optimale est ϵ) et une vitesse de l'ordre de $\left(\frac{1}{\log \frac{1}{\epsilon}}\right)^a$ avec $a > 0$ très petit pour un potentiel de Coulomb (pour une vitesse optimale supposée de $\sqrt{\frac{1}{\log \frac{1}{\epsilon}}}$). En donnant une vitesse de convergence, nous justifions le fait que l'équation de Landau soit une bonne approximation de l'équation de Boltzmann dans le régime des collisions rasantes. À notre connaissance, nous sommes les premiers avec He [41] à donner ce type de résultat. He obtient une meilleure vitesse de convergence (ϵ pour les potentiels mous et de Coulomb) mais pour des solutions beaucoup plus régulières et avec une section efficace ne correspondant pas vraiment à la situation physique dans le cas du potentiel de Coulomb. Nous travaillons dans un premier temps sur l'équation de Kac.

En 1956, Marc Kac définit dans [47] la notion de propagation du chaos. Il considère un système de particules et montre qu'il y a propagation du chaos vers la solution d'une équation intégral-différentielle non linéaire qui est une *caricature* unidimensionnelle de l'équation de Boltzmann. Cette dernière est aujourd'hui connue sous le nom d'équation de Kac. Ce papier est un des premiers à utiliser des probabilités pour l'étude de la théorie cinétique des gaz. En utilisant la terminologie introduite pour l'équation de Boltzmann, nous considérons un potentiel Maxwellien. De même que pour Boltzmann, il est connu que la solution de l'équation de Kac converge vers la solution d'une équation de type Fokker-Planck. Dans le chapitre 1, nous commençons par donner une vitesse de convergence explicite que l'on pense être optimale. En s'appuyant sur ce résultat, nous approchons la solution de l'équation de Kac en remplaçant les collisions rasantes (qui sont en nombre infini) par un terme de type Fokker-Planck (ce qui correspond à une diffusion en terme probabiliste). Ceci nous permet d'obtenir un résultat de propagation du chaos, pour un système de particules que l'on peut simuler, avec une preuve assez simple qui nous donne des estimées quantitatives.

Afin d'étudier les équations de Kac et de Boltzmann, qui sont des équations intégral-différentielles non linéaires sur \mathbb{R} et \mathbb{R}^3 respectivement, nous utilisons des équations différentielles stochastiques non linéaires dirigées par une mesure de Poisson. Le premier à introduire ce type d'E.D.S. pour l'étude de ces équations fut Tanaka dans [64] où il démontra que la distance de

Wasserstein avec coût quadratique entre deux solutions de l'équation de Kac est décroissante en temps. Il étendit ce résultat à l'équation de Boltzmann dans [64]. Cette idée a été reprise plus tard par Graham-Méléard dans [37] afin d'obtenir des propriétés de régularisation pour l'équation de Kac et par Fournier-Mouhot [35], Fournier-Guérin [32] avec des résultats d'unicité pour Boltzmann. Nos résultats s'appuient principalement sur des méthodes de couplage. Un des points clefs de nos preuves est d'approcher des phénomènes discontinus par des phénomènes continus. Plus précisément, comme dans [30], nous estimons la distance de Wasserstein avec coût quadratique entre la loi de l'intégrale d'une fonction déterministe par rapport à une mesure de Poisson compensée et une loi Gaussienne centrée de même variance en utilisant des résultats donnant une vitesse de convergence dans le cadre du théorème central limite ([57] pour Kac et [74] pour Boltzmann). On obtient de meilleurs résultats pour l'équation de Kac. Ceci n'est pas surprenant car de nombreuses difficultés supplémentaires apparaissent lors de l'étude de l'équation de Boltzmann : on travaille en dimension supérieure, on considère des potentiels mous ou de Coulomb au lieu d'un potentiel Maxwellien pour Kac (pour lequel on rappelle que le taux de collisions entre particules ne dépend pas de leur vitesse), et l'équation limite est non linéaire.

L'équation de Keller-Segel décrit le mouvement de cellules (typiquement des bactéries) qui sont attirées par une substance chimique qu'elles produisent : c'est un modèle de chimiotaxie. Nous traitons ici un cas sous-critique de l'équation de Keller-Segel standard. En effet le noyau considéré est $K(x) = \frac{x}{|x|^{\alpha+1}}$ avec $\alpha \in (0, 1)$ au lieu de $\alpha = 1$. Malgré cela, on peut observer que l'on considère un noyau ayant une singularité en zéro. Dans le chapitre 3, nous donnons un résultat de propagation du chaos pour ce modèle (sans introduire de paramètre de *cutoff*). Pour cela, nous montrons la tension pour notre système de particules et l'unicité du processus limite. En particulier, nos preuves s'appuyant sur des méthodes de compacité, nous n'obtenons pas cette fois-ci de vitesse de convergence. Le noyau que nous considérons ayant une singularité, l'existence et l'unicité du système de particules ne s'obtiennent pas en utilisant les arguments classiques pour les E.D.S. à coefficients Lipschitz. En s'inspirant du cas des E.D.S. avec coefficients localement Lipschitz (cf [28]), on montre ici l'existence et l'unicité jusqu'à un certain temps aléatoire d'explosion et le but est ensuite de montrer que ce temps d'explosion est infini p.s.

Les noyaux intervenant dans les équations de Keller-Segel et de Boltz-

mann avec des potentiels mous ont le même type de singularité. Pour montrer l'unicité dans Keller-Segel, nous utilisons donc le même type d'arguments que dans [32] : on considère deux solutions $(f_t)_{t \geq 0}$ et $(g_t)_{t \geq 0}$, on prend un couplage optimal entre ces deux solutions, puis on construit à l'aide de ce couplage optimal deux processus dont les marginales en temps ont pour lois respectives f_t et g_t . On obtient ainsi une estimation de type Grönwall sur la distance de Wasserstein entre f_t et g_t . L'existence d'une solution découle alors de la tension du système de particules. Grâce à des propriétés de renormalisation de la solution, nous obtenons la dissipation de l'entropie pour la solution de l'équation de Keller-Segel. Ceci nous permet de montrer une notion plus forte de propagation du chaos, appelée chaos entropique.

0.2 Distance de Wasserstein

Un dénominateur commun aux trois chapitres de cette thèse est l'utilisation de distances de Wasserstein. Dans les deux premiers chapitres, nos résultats de convergence sont tous basés sur ces distances. Dans le troisième, cette distance nous est utile pour nos résultats d'unicité.

On désigne par $\mathcal{P}(\mathbb{R}^n)$ l'ensemble des mesures de probabilité sur \mathbb{R}^n et, pour $p \geq 1$ par $\mathcal{P}_p(\mathbb{R}^n)$ l'ensemble des $\mu \in \mathcal{P}(\mathbb{R}^n)$ telle que

$$m_p(\mu) := \int_{\mathbb{R}^n} |x|^p \mu(dx) < \infty.$$

Pour $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, on considère $\Pi(\mu, \nu)$ l'ensemble des mesures sur $\mathbb{R}^n \times \mathbb{R}^n$ ayant μ et ν pour première et seconde marginale respectivement.

Définition 0.2.1. *Soit $p \geq 1$ et $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^n)$. La distance de Wasserstein d'ordre p entre μ et ν est définie par*

$$\mathcal{W}_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^p \pi(dx, dy) \right)^{1/p}.$$

On peut définir la distance de Wasserstein de manière équivalente par

$$\mathcal{W}_p(\mu, \nu) = \inf \left\{ \mathbb{E}[|U - V|^p]^{1/p}, U \sim \mu, V \sim \nu \right\},$$

l'infimum étant pris sur l'ensemble des variables aléatoires U de loi μ et V de loi ν . Il est utile de remarquer que l'infimum est en fait un minimum. Plus

précisément, si on fixe U de loi μ , on peut trouver une variable aléatoire V de loi ν telle que

$$\mathbb{E}[|U - V|^p] = \mathcal{W}_p^p(\mu, \nu).$$

Pour $n = 1$, nous pouvons donner une autre définition équivalente (cf Villani [72, Remark 2.19 (iii)]).

Proposition 0.2.2. *Soit $p \geq 1$ et $\mu, \nu \in \mathcal{P}_p(\mathbb{R})$. On a*

$$\mathcal{W}_p^p(\mu, \nu) = \int_0^1 (F_\mu^{-1}(\alpha) - F_\nu^{-1}(\alpha))^p d\alpha,$$

où $F_\mu(x) := \mu((-\infty, x])$ et $F_\nu(x) := \nu((-\infty, x])$ désignent les fonctions de répartition de μ et ν respectivement.

Le lecteur intéressé pourra se rapporter à [72] pour plus de détails sur le sujet.

Un des principaux outils que nous utilisons pour obtenir des vitesses de convergence dans le cadre de l'asymptotique des collisions rasantes pour les équations de Kac et de Boltzmann est l'estimation de la distance de Wasserstein entre la loi de l'intégrale d'une fonction déterministe par rapport à une mesure de Poisson compensée et une Gaussienne centrée de même variance. Pour obtenir ce type d'estimation, on s'appuie sur des résultats donnant une vitesse de convergence dans le cadre du Théorème central limite. Pour des variables aléatoires réelles, on utilise le résultat de Rio suivant.

Théorème 0.2.1. ([57, Theorem 4.1]) *Il existe une constante C_0 telle que pour tout $n \in \mathbb{N}^*$ et pour toute suite $(X_i)_{i \geq 0}$ de variables aléatoires i.i.d. centrées de loi dans $\mathcal{P}_4(\mathbb{R})$,*

$$\mathcal{W}_2^2(\eta_n, \mathcal{N}(0, 1)) \leq C_0 v_n^{-2} \sum_{i=1}^n \mathbb{E}(|X_i|^4),$$

où $\eta_n = \mathcal{L}(v_n^{-1/2} S_n)$, $S_n = \sum_{i=1}^n X_i$ et $v_n = \text{Var}(S_n)$.

On obtient le corollaire suivant que nous utiliserons lors de l'étude de l'équation de Kac.

Corollaire 0.2.3. *Soient E un espace polonais muni d'une mesure positive σ -finie ν , N une mesure de Poisson sur $[0, T] \times E$ de mesure intensité $dt\nu(dz)$ et $H : [0, T] \times E \mapsto \mathbb{R}$ une fonction déterministe telle que $\int_0^t \int_E (H^2(s, z) + H^4(s, z))\nu(dz)ds < \infty$. Alors en posant*

$$X_t = \int_0^t \int_E H(s, z)\tilde{N}(ds, dz), \quad q_t = \int_0^t \int_E H^2(s, z)\nu(dz)ds,$$

on a

$$\mathcal{W}_2^2(\mathcal{L}(X_t), \mathcal{N}(0, q_t)) \leq C_0 \frac{\int_0^t \int_E H^4(s, z)\nu(dz)ds}{q_t},$$

où C_0 est la constante intervenant dans le théorème précédent.

En dimension supérieure, on s'appuie sur un résultat de Zaitsev. Pour $\tau \geq 0$ et $d \in \mathbb{N}$, soit $\mathcal{A}_d(\tau)$ l'ensemble des $F \in \mathcal{P}(\mathbb{R}^d)$ telles que la fonction $\varphi(z) = \log \int_{\mathbb{R}^d} e^{z \cdot x} F(dx)$ soit analytique sur $\{z \in \mathbb{C}^d, |z|\tau < 1\}$ et $|d_u d_v^2 \varphi(z)| \leq |u|\tau \mathbb{D}v \cdot v$ pour tout $u, v \in \mathbb{R}^d$ et $|z|\tau < 1$, où \mathbb{D} est la matrice de covariance de F , et $d_u \varphi$ est la dérivée de φ dans la direction u .

Theorem 0.2.4. *(Zaitsev [74, Theorem 2]) Supposons que $\tau \geq 1$ et que ξ_1, \dots, ξ_n soient des vecteurs aléatoires indépendants tels que $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E}(\xi_k) = 0$, $\text{Cov}(\xi_k) = I_d$, $k = 1, \dots, n$. Alors on peut construire sur un espace de probabilités une famille de vecteurs aléatoires indépendants X_1, \dots, X_n telle que $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$ pour tout $k = 1, \dots, n$ et une famille de vecteurs aléatoires indépendants $Y_1, \dots, Y_n \sim \mathcal{N}(0, I_d)$ telle que*

$$\mathbb{E} \left[\exp \left(\frac{a \Delta_n(X, Y)}{\tau} \right) \right] \leq \exp \left(b \max(1, \log n / \tau^2) \right),$$

où

$$\Delta_n(X, Y) = \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i \right|,$$

et où a, b sont des constantes positives ne dépendants que de d .

En s'appuyant sur ce dernier résultat, on obtient la proposition suivante qui nous est utile pour l'étude de l'équation de Boltzmann.

Proposition 0.2.5. *Soient A un espace mesurable muni d'une mesure positive σ -finie ν , N une mesure de Poisson sur $[0, \infty) \times A$ de mesure intensité $dt\nu(dz)$. On considère une fonction déterministe $h : A \rightarrow \mathbb{R}^d$ et on pose $Z_t = \int_0^t \int_A h(z) \tilde{N}(ds, dz)$, $\mu_t = \mathcal{L}(Z_t)$ et $\Gamma = \int_A h(z)h^*(z)\nu(dz)$. Si $\kappa := \max_{z \in A} |\Gamma^{-1/2}h(z)| \in (0, \infty)$, alors*

$$\mathcal{W}_2^2(\mu_t, \mathcal{N}(0, t\Gamma)) \leq C\kappa^2|\Gamma| \left[\max \left(1, \log \frac{t}{\kappa^2} \right) \right]^2,$$

où C dépend uniquement de d et où $\mathcal{N}(0, t\Gamma)$ est la distribution Gaussienne sur \mathbb{R}^d d'espérance 0 et de matrice de covariance $t\Gamma$.

On peut noter que l'obtention de ce résultat est plus compliquée en dimension supérieure. De plus, il est également plus difficile à appliquer du fait d'hypothèses plus restrictives. On remarque notamment que la fonction h apparaissant dans l'intégrale par rapport à la mesure de Poisson ne peut pas dépendre du temps alors qu'elle le pouvait en dimension 1.

0.3 Asymptotique des collisions rasantes et système de particules pour l'équation de Kac sans cutoff

Le premier chapitre de cette thèse porte sur l'équation de Kac sans cutoff dans le cas Maxwellien. Cette équation est une caricature unidimensionnelle de l'équation de Boltzmann. Elle s'écrit de la façon suivante :

$$(0.3.1) \quad \frac{\partial f_t}{\partial t}(v) = \int_{v_* \in \mathbb{R}} \int_{\theta = -\pi}^{\pi} \left(f_t(v')f_t(v'_*) - f_t(v)f_t(v_*) \right) \beta(\theta) d\theta dv_*,$$

où $t \geq 0$, $v \in \mathbb{R}$ et où les vitesses avant collision sont données par

$$v' = v \cos \theta - v_* \sin \theta, \quad v'_* = v \sin \theta + v_* \cos \theta.$$

La fonction $\beta : [-\pi, \pi] - \{0\} \rightarrow \mathbb{R}_+$ est une fonction paire appelée *section efficace*. On peut interpréter cette équation de la façon suivante : des particules de vitesses v vont apparaître du fait de collisions entre des particules ayant pour vitesse v' et v'_* et dans le même temps, d'autres particules de vitesse v vont disparaître en se choquant avec des particules de vitesse v_* ,

le tout se faisant à un taux proportionnel à $\beta(\theta)$ (on rappelle qu'on traite ici le cas Maxwellien et que le taux de collisions ne dépend donc pas de la vitesse des particules). On peut voir Desvillettes [19] ou Kac [47, 48] pour plus de précisions. Si la section efficace vérifie $\int_0^\pi \beta(\theta)d\theta = \infty$, alors il y a une infinité de collisions sur n'importe quel intervalle de temps. Le cas où on suppose $\int_0^\pi \beta(\theta)d\theta < \infty$ (dit avec cutoff) a été très étudié. Nous nous concentrons dans le Chapitre 1 à la vraie situation physique où l'on suppose uniquement $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$ (cas sans cutoff).

Si $f_0 \in \mathcal{P}_2(\mathbb{R})$ et $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$, on peut voir Toscani-Villani [67] pour l'existence et l'unicité d'une solution (dans un sens faible) à (0.3.1). De plus, il y a conservation de l'énergie pour cette dernière : la solution $(f_t)_{t \geq 0}$ de (0.3.1) vérifie

$$\int_{\mathbb{R}} v^2 f_t(dv) = \int_{\mathbb{R}} v^2 f_0(dv).$$

Jusqu'à la fin de l'introduction, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ désigne un espace de probabilité filtré Polonais satisfaisant les conditions habituelles. L'équation de Kac n'étant pas linéaire, nous considérons également l'espace de probabilité auxiliaire $([0, 1], \mathcal{B}([0, 1]), d\alpha)$. Par souci de clarté, nous utilisons les notations \mathbb{E} pour l'espérance et \mathcal{L} pour la loi de variables aléatoires ou processus définis sur $(\Omega, \mathcal{F}, \mathbb{P})$, et les notations \mathbb{E}_α et \mathcal{L}_α pour l'espérance et la loi de variables aléatoires ou processus définis sur $([0, 1], \mathcal{B}([0, 1]), d\alpha)$. On désigne par α -processus les processus sur $([0, 1], \mathcal{B}([0, 1]), d\alpha)$. Nous utilisons les mêmes notations pour les équations de Boltzmann et Landau dans la section suivante.

Nous nous intéressons dans un premier temps à l'asymptotique des collisions rasantes, puis, en s'inspirant des idées développées dans ce premier point, nous approchons la solution de l'équation de Kac dans le cas général, ce qui nous permet de construire un système de particules, facilement simulable, et approchant la solution de l'équation de Kac.

0.3.1 Asymptotique des collisions rasantes

Il s'agit du cas où il y a de plus en plus de collisions qui engendrent des déviations de plus en plus petites. Il est alors connu que la solution de l'équation de Boltzmann converge vers la solution de l'équation de Landau. Plus précisément, Degond, Lucquin-Desreux [17] et Desvillettes [20]

ont montré la convergence des opérateurs (mais pas des solutions), et Villani [70] a donné des résultats de compacité et de convergence de sous-suites. Le résultat d'unicité de Fournier et Guérin dans [32] a permis d'établir la convergence des solutions. Cependant aucun résultat donnant une vitesse pour cette convergence n'était connu il y a encore peu de temps. La même problématique se pose pour l'équation de Kac. L'équation limite est l'équation de Fokker-Planck suivante :

$$(0.3.2) \quad \frac{\partial}{\partial t} g_t(v) = \frac{\mathcal{E}}{2} \frac{\partial^2}{\partial v^2} g_t(v) + \frac{1}{2} \frac{\partial}{\partial v} (v g_t(v)),$$

où $\mathcal{E} := \int_{\mathbb{R}} v^2 g_0(v) dv$. On peut observer que cette équation limite ne devrait pas être linéaire, mais la non-linéarité n'apparaît qu'à travers l'énergie du système $\int_{\mathbb{R}} v^2 g_t(v) dv$, qui est constante en temps.

Nous donnons ici une vitesse de convergence explicite que l'on pense être optimale. Le même type de résultat a déjà été prouvé par Toscani [66], mais la vitesse qu'il donne n'est pas très explicite et pas optimale. Notre résultat est le suivant.

Théorème 0.3.1. *Soient $g_0 \in \mathcal{P}_4(\mathbb{R})$ et β une section efficace telle que $\int_0^\pi \theta^2 \beta(\theta) d\theta = 1$. Pour $\epsilon \in (0, 1)$, on pose $\beta_\epsilon(\theta) = \frac{\pi^3}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}$ et on considère la solution (faible) $(g_t^\epsilon)_{t \geq 0}$ de (0.3.1) avec g_0 pour donnée initiale et β_ϵ pour section efficace. Si $(g_t)_{t \geq 0}$ est la solution (faible) de (0.3.2) avec même donnée initiale g_0 , alors il existe une constante C telle que pour tout $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, \infty)} \mathcal{W}_2(g_t^\epsilon, g_t) \leq C\epsilon.$$

On peut remarquer que l'on obtient une majoration uniforme en temps. Pour obtenir un tel résultat, il a fallu être le plus précis possible dans nos différentes estimations et notamment utiliser au maximum tout les termes négatifs.

La démonstration s'appuie sur l'utilisation d'E.D.S. On considère un α -processus $(W_t^\epsilon)_{t \geq 0}$ tel que $\mathcal{L}_\alpha(W_t^\epsilon) = g_t^\epsilon$ pour tout $t \geq 0$. Soient N^ϵ une $(\mathcal{F}_t)_{t \geq 0}$ -mesure de Poisson sur $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ d'intensité $ds d\alpha \beta(\theta) d\theta$, $(B_t)_{t \geq 0}$ un $(\mathcal{F}_t)_{t \geq 0}$ -mouvement Brownien et V_0 une \mathcal{F}_0 -variable aléatoire ayant

pour loi f_0 . On considère $(V_t^\epsilon)_{t \geq 0}$ et $(Y_t)_{t \geq 0}$ solutions des E.D.S.s suivantes

$$\begin{aligned} V_t^\epsilon &= V_0 + \int_0^t \int_0^1 \int_{-\pi}^\pi \left[(\cos \theta - 1) V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha) \right] \tilde{N}^\epsilon(ds d\alpha d\theta) \\ &\quad - b_\epsilon \int_0^t V_s^\epsilon ds, \\ Y_t &= V_0 - \frac{1}{2} \int_0^t Y_s ds + \sqrt{\mathcal{E}} B_t, \end{aligned}$$

où $b_\epsilon = \int_{-\pi}^\pi (1 - \cos \theta) \beta_\epsilon(\theta) d\theta$. En appliquant la formule d'Itô à $\varphi(V_t^\epsilon)$ et $\varphi(Y_t)$ pour $\varphi \in C_b^2(\mathbb{R})$ et en utilisant des arguments d'unicité, on observe que $\mathcal{L}(V_t^\epsilon) = g_t^\epsilon$ et $\mathcal{L}(Y_t) = g_t$. On a donc $\mathcal{W}_2^2(g_t^\epsilon, g_t) \leq \mathbb{E}[|V_t^\epsilon - Y_t|^2]$. Pour estimer cette dernière quantité, on introduit les processus intermédiaires suivants

$$\begin{aligned} Y_t^\epsilon &= V_0 - b_\epsilon \int_0^t Y_s^\epsilon ds - \int_0^t \int_0^1 \int_{-\pi}^\pi \sin \theta W_{s-}^\epsilon(\alpha) \tilde{N}^\epsilon(ds d\alpha d\theta), \\ \tilde{Y}_t^\epsilon &= V_0 - b_\epsilon \int_0^t \tilde{Y}_s^\epsilon ds + \sqrt{\mathcal{E} \gamma_\epsilon} B_t, \end{aligned}$$

où $\gamma_\epsilon = \int_{-\pi}^\pi \sin^2 \theta \beta_\epsilon(\theta) d\theta$.

Le processus $(Y_t^\epsilon)_{t \geq 0}$ s'obtient à partir de l'E.D.S définissant $(V_t^\epsilon)_{t \geq 0}$ en retirant la partie aléatoire dans l'intégrale par rapport à la mesure de Poisson, afin de pouvoir ensuite utiliser le Corollaire 0.2.3 (on rappelle que $W^\epsilon(\alpha)$ est défini sur $[0, 1]$). On calcule facilement $\varphi(t) := \mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2]$: à l'aide de la formule d'Itô, on remarque que φ est solution d'une équation différentielle ordinaire que l'on résoud facilement, obtenant ainsi une majoration de $\varphi(t)$.

Le point clef de la démonstration est l'estimation de $\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t^\epsilon)^2]$. La constante γ_ϵ est choisie de telle sorte que

$$\mathbb{E} \left[\left(\int_0^t \int_0^1 \int_{-\pi}^\pi \sin \theta W_{s-}^\epsilon(\alpha) \tilde{N}^\epsilon(ds d\alpha d\theta) \right)^2 \right] = \mathbb{E}[(\sqrt{\mathcal{E} \gamma_\epsilon} B_t)^2].$$

En précisant que l'on peut donner une forme explicite à Y_t^ϵ et \tilde{Y}_t^ϵ , on peut ainsi facilement utiliser le Corollaire 0.2.3 et on obtient une majoration de $\mathcal{W}_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon), \mathcal{L}(Y_t^\epsilon))$ (on peut noter que c'est à ce moment précis que l'on a besoin d'une donnée initiale dans $\mathcal{P}_4(\mathbb{R})$). Il reste ensuite à coupler le mouvement Brownien et la mesure de Poisson de telle sorte que $\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t^\epsilon)^2] = \mathcal{W}_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon), \mathcal{L}(Y_t^\epsilon))$.

Enfin, pour conclure, il reste à contrôler $\mathbb{E}\left[(\tilde{Y}_t^\epsilon - Y_t)^2\right]$. On peut observer que $(\tilde{Y}_t^\epsilon)_{t \geq 0}$ et $(Y_t)_{t \geq 0}$ sont des processus de Ornstein-Uhlenbeck et que les quantités b_ϵ et γ_ϵ tendent vers 1/2 et 1 respectivement. Quelques lignes de calculs nous donnent ainsi le résultat souhaité.

0.3.2 Remplacement des petites collisions par une diffusion

On va maintenant chercher à approcher la solution de l'équation de Kac dans le cas général. Jusqu'ici l'idée la plus courante était de tronquer les *petites* collisions qui sont en nombre infini sur chaque intervalle de temps fini. Mais on vient de voir que les collisions rasantes sont bien approchées par un terme de type Fokker-Planck ou par une diffusion de type Ornstein-Uhlenbeck en terme probabiliste. On part ainsi de l'équation de Kac avec une section efficace β vérifiant $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$ et, pour un $\epsilon \in (0, 1)$ fixé, on remplace les collisions dont l'angle de la déviation θ est inférieur à ce ϵ par un terme de type Fokker-Planck. On obtient l'équation suivante que l'on peut qualifier d'hybride :

$$(0.3.3) \quad \frac{\partial}{\partial t} f_t^\epsilon(v) = b_\epsilon \frac{\partial}{\partial v} \left[v f_t^\epsilon(v) \right] + \mathcal{E} b_\epsilon \frac{\partial^2}{\partial v^2} f_t^\epsilon(v) \\ + \int_{v_* \in \mathbb{R}} \int_{|\theta| \geq \epsilon} \left(f_t^\epsilon(v') f_t^\epsilon(v'_*) - f_t^\epsilon(v) f_t^\epsilon(v_*) \right) \beta(\theta) d\theta dv_*,$$

où

$$b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta \quad \text{et} \quad \mathcal{E} = \int_{\mathbb{R}} v^2 f_0^\epsilon(v) dv.$$

L'existence et l'unicité d'une solution faible (avec une donnée initiale dans $\mathcal{P}_2(\mathbb{R})$) s'obtient en utilisant la distance de la variation totale. L'unicité repose sur des arguments de régularisation et l'existence se montre à l'aide d'une E.D.S. construite à partir de mesures de Poisson.

Le résultat est le suivant.

Théorème 0.3.2. *Soient $f_0 \in \mathcal{P}_4(\mathbb{R})$ et β une section efficace telle que $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$. Pour $\epsilon \in (0, 1)$, on considère $(f_t)_{t \geq 0}$ et $(f_t^\epsilon)_{t \geq 0}$ solutions (faibles) de (0.3.1) et (0.3.3) respectivement, ayant la même donnée initiale*

f_0 . Alors pour tout $T > 0$, tout $\epsilon \in (0, 1)$, on a

$$\sup_{t \in [0, T]} W_2^2(f_t, f_t^\epsilon) \leq C\epsilon^2 \min \left(1 + T, \frac{1}{\int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta} \right),$$

où C dépend uniquement de $\int_{\mathbb{R}} v^2 f_0(dv)$, $\int_{\mathbb{R}} v^4 f_0(dv)$ et de $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

Remarquons que pour obtenir la meilleure estimation en ϵ , il nous faut une dépendance (polynomiale) en temps. On obtient également une majoration uniforme, mais dans ce cas la vitesse en ϵ n'est plus optimale. Notons que là encore, il a fallu être le plus précis possible dans nos calculs pour éviter une dépendance exponentielle en temps.

On peut également observer que notre majoration ne dépend pas de la nature de la singularité de β en 0. Par analogie avec l'équation de Boltzmann, il est de coutume de considérer des sections efficace β telles que $\beta(\theta) \stackrel{\theta=0}{\approx} |\theta|^{-1-\nu}$ avec $\nu \in (0, 2)$. En se contentant de tronquer les petites collisions, la majoration obtenue est de l'ordre de $C_T \epsilon^{1-\nu/2}$ (cf [21]), alors qu'en remplaçant les petites collisions par une diffusion, on obtient une majoration de l'ordre de ϵ , indépendamment de ν . On peut expliquer ce phénomène par le fait que bien que l'on remplace plus de petites collisions lorsque ν est proche de 2, ces petites collisions sont dans le même temps mieux approchées par le terme de diffusion. On obtient ainsi une meilleure approximation qu'en se contentant de tronquer les petites collisions surtout si ν est proche de 2 et donc si la singularité est forte en 0.

La preuve du Théorème 0.3.2 utilise le même type d'arguments que celle du Théorème 0.3.1. Elle est cependant beaucoup plus compliquée, notamment du fait de la non-linéarité de l'équation *hybride* (alors que l'équation de Fokker-Planck est linéaire) et du fait que l'on va devoir traiter simultanément les *petites* et les *grandes* collisions. Sur l'espace de probabilité auxiliaire $([0, 1], \mathcal{B}([0, 1]), d\alpha)$, on considère, pour tout $t \geq 0$, des α -variables aléatoires W_t et W_t^ϵ ayant pour loi respective f_t et f_t^ϵ tel que $W_2^2(f_t, f_t^\epsilon) = \mathbb{E}_\alpha[(W_t - W_t^\epsilon)^2]$. On considère également les E.D.S. liées à (0.3.1) et (0.3.3)

suivantes

$$\begin{aligned}
V_t &= V_0 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[(\cos \theta - 1)V_{s-} - \sin \theta W_{s-}(\alpha) \right] \tilde{N}(dsd\alpha d\theta) - b \int_0^t V_s ds, \\
V_t^\epsilon &= V_0^\epsilon + \int_0^t \int_{|\theta| \geq \epsilon} \int_0^1 \left[(\cos \theta - 1)V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha) \right] N(d\theta d\alpha ds) \\
&\quad - b_\epsilon \int_0^t V_s^\epsilon ds + \sqrt{2\mathcal{E}b_\epsilon} B_t,
\end{aligned}$$

où le processus $(B_t)_{t \geq 0}$ est un $(\mathcal{F}_t)_{t \geq 0}$ -mouvement Brownien et N est une $(\mathcal{F}_t)_{t \geq 0}$ -mesure de Poisson sur $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ d'intensité $dsd\alpha\beta(\theta)d\theta$, soigneusement construits afin qu'une certaine distance de Wasserstein soit réalisée et qu'on puisse calculer cette dernière à l'aide du Corollaire 0.2.3 (il faut notamment considérer dans un premier temps deux espaces de probabilité distincts afin de séparer *petites* et *grandes* déviations).

On observe que, pour tout $t \geq 0$, $\mathcal{L}(V_t) = f_t$, $\mathcal{L}(V_t^\epsilon) = f_t^\epsilon$ et ainsi, par définition de la distance de Wasserstein, $\mathcal{W}_2^2(f_t, f_t^\epsilon) \leq \mathbb{E}[(V_t - V_t^\epsilon)^2]$. L'introduction d'un processus intermédiaire et de nombreuses lignes de calculs permettent ensuite de conclure (cf Section 1.5 pour plus de détails).

0.3.3 Système de particules

En s'inspirant de la partie précédente, on va construire un système de particules approchant la solution de l'équation de Kac en remplaçant les petites collisions par un terme de diffusion. Pour être exact, le système de particules construit approchera la solution de l'équation hybride (0.3.3), mais du fait du Théorème 0.3.2, il approchera également la solution de (0.3.1).

On se donne $f_0 \in \mathcal{P}(\mathbb{R})$ et une section efficace β telle que $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$. On fixe un entier n et on considère:

- une famille de variables aléatoires i.i.d. $(V_0^i)_{i \in \{1, \dots, n\}}$ de loi f_0 ,
- une famille de mesures de Poisson i.i.d. $(N^{i,n}(dsd\theta dj))_{i \in \{1, \dots, n\}}$ sur $[0, \infty) \times [-\pi, \pi] \times \{1, \dots, n\}$ d'intensité $ds\beta(\theta)d\theta \frac{1}{n} \sum_{k=1}^n \delta_k(dj)$,
- une famille de mouvements Browniens i.i.d. $(B_t^i)_{t \geq 0, i \in \{1, \dots, n\}}$.

Pour $\epsilon \in (0, 1)$, on considère $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ solution du système d'E.D.S. suivant : pour $i = 1, \dots, n$, pour tout $t \geq 0$,

$$(0.3.4) \quad \begin{aligned} V_t^{i,n,\epsilon} = & V_0^i + \int_0^t \int_{|\theta| > \epsilon} \int_{j \in \{1, \dots, n\}} \left[(\cos \theta - 1) V_{s-}^{i,n,\epsilon} - \sin \theta V_{s-}^{j,n,\epsilon} \right] N^{i,n}(ds d\theta dj) \\ & - b_\epsilon \int_0^t V_s^{i,n,\epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

où $\mathcal{E} = \int_{\mathbb{R}} v^2 f_0(dv)$ et $b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta$.

La variable $V_t^{i,n,\epsilon}$ représente la vitesse de la i -ème particule au temps t . Son comportement est le suivant : après un temps exponentiel τ de paramètre $\Lambda_\epsilon = \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$, elle se percute avec une autre particule d'indice j choisie aléatoirement et on pose $V_\tau^{i,n,\epsilon} = (\cos \Theta) V_{\tau-}^{i,n,\epsilon} - (\sin \Theta) V_{\tau-}^{j,n,\epsilon}$, où Θ est une variable aléatoire $\Lambda_\epsilon^{-1} \beta(\theta) \mathbb{1}_{|\theta| \geq \epsilon} d\theta$ -distribuée. Entre deux sauts, $V^{i,n,\epsilon}$ se comporte comme un processus d'Ornstein-Uhlenbeck

$$V_t^{i,n,\epsilon} = V_s^{i,n,\epsilon} - b_\epsilon \int_s^t V_u^{i,n,\epsilon} du + \sqrt{2\mathcal{E}b_\epsilon} (B_t^i - B_s^i).$$

On peut résoudre explicitement cette dernière E.D.S. et on obtient

$$V_t^{i,n,\epsilon} = V_s^{i,n,\epsilon} e^{-b_\epsilon(t-s)} + \sqrt{2\mathcal{E}b_\epsilon} e^{-b_\epsilon(t-s)} \int_s^t e^{b_\epsilon u} dB_u^i.$$

On observe que pour simuler notre système sur un intervalle $[0, T]$, il nous faut en moyenne $nT \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$ sauts. Le coût de simulation est ainsi de l'ordre de $nT \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$ (qui est au pire de l'ordre de $nT\epsilon^{-2}$). Le fait que l'on puisse résoudre explicitement la dernière E.D.S. est très important pour obtenir un tel coût de simulation (de fait pour faire évoluer le système entre deux sauts, il suffit de simuler des Gaussiennes).

Theorem 0.3.1. *Soient $f_0 \in \mathcal{P}_4(\mathbb{R})$ et β une section efficace telle que $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$. On considère $(f_t)_{t \geq 0}$ solution (faible) de l'équation de Kac (0.3.1) avec f_0 pour donnée initiale. Pour $n \in \mathbb{N}^*$ et $\epsilon \in (0, 1)$, on considère la solution $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ de (0.3.4). On pose $\mu_t^{n,\epsilon} = \frac{1}{n} \sum_{i=1}^n \delta_{V_t^{i,n,\epsilon}}$. Alors pour tout $T > 0$, tout $n \geq 2$ et tout $\epsilon \in (0, 1)$, on a*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n,\epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right),$$

où C dépend uniquement de $\int_{\mathbb{R}} v^2 f_0(dv)$, $\int_{\mathbb{R}} v^4 f_0(dv)$ et de $\int_0^\pi \theta^2 \beta(\theta) d\theta$, et où pour tout $t \geq 0$, $\mu_t^n = \frac{1}{n} \sum_1^n \delta_{V_t^i}$, avec $(V_t^i)_{i \in \{1, \dots, n\}}$ une famille de particules i.i.d. de loi f_t .

On remarque ainsi que notre système de particule approche aussi bien f_t que le système de particules usuel composé de particules i.i.d. de loi f_t , qui lui n'est pas simulable du fait de la non-linéarité de l'équation de Kac. Si on suppose que la donnée initiale f_0 admet des moments de tout ordre, on peut estimer $\mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right]$ et on obtient

$$\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \frac{1}{n^{(1/2)^-}} \right).$$

Dans [21], Desvillettes, Graham et Méléard ont considéré un système de particules en tronquant les petites collisions. Avec cette méthode, ils obtiennent en gros la majoration suivante $\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \tilde{\mu}_t^{n, \epsilon}) \right] \leq C_T \left(\epsilon^{2-\nu} + \frac{e^{C_T \Lambda_\epsilon}}{n} \right)$, où $\Lambda_\epsilon = \int_{|\theta| > \epsilon} \beta(\theta) d\theta \approx \epsilon^{-\nu}$ si β vérifie $\beta(\theta) \stackrel{\theta=0}{\approx} |\theta|^{-1-\nu}$ avec $\nu \in (0, 2)$. Si on compare avec notre résultat, on peut observer que

- dans le premier terme, on a une erreur de l'ordre de ϵ^2 au lieu de $\epsilon^{2-\nu}$. Cela vient du fait que l'on a remplacé les petites collisions par un terme de diffusion;
- dans le second terme, notre borne ne dépend pas de ϵ . Cela vient du fait qu'on utilise une distance de Wasserstein qui est bien adaptée à notre travail. Dans [21], le résultat est donné avec une distance de Wasserstein mais une distance de variation totale est utilisée dans les calculs;
- le coût de simulation est du même ordre pour les deux systèmes.

Nous terminons cette partie par quelques graphiques. On prend $\beta(\theta) = |\theta|^{-1-\nu}$ avec $0 < \nu < 2$ pour section efficace et $f_0 = (\delta_{-1} + \delta_1)/2$ pour donnée initiale. On considère deux systèmes de particules dont on va comparer l'efficacité. Le premier est le système $(V_t^{i, n, \epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ introduit dans (0.3.4) où l'on rappelle que l'on a remplacé les petites collisions par une diffusion (on parlera ainsi de système avec diffusion) et le système suivant où l'on

se contente de tronquer ces petites collisions (on parlera de système sans diffusion) : pour $i \in \{1, \dots, n\}$ et $t \geq 0$,

$$\tilde{V}_t^{i,n,\epsilon} = V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \tilde{V}_{s-}^{i,n,\epsilon} - \sin \theta \tilde{V}_{s-}^{j,n,\epsilon} \right] N^{i,n}(ds d\theta dj).$$

Pour nos simulations, on a pris un temps final $T^{final} = 0.1$. On a tracé une courbe de référence (qui sera en bleue) obtenue en simulant $n = 10^7$ particules avec $\epsilon = 0.03$ puis en utilisant une procédure de lissage.

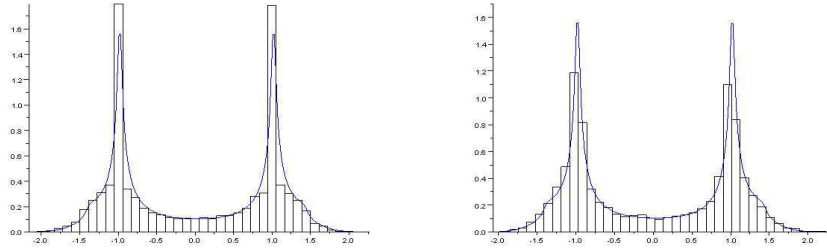


Figure 1: $\nu = 0.5$. Graphique de gauche: système sans diffusion, $n = 2.10^4$, $\epsilon = 0.1$. Graphique de droite: système avec diffusion, $n = 10^4$, $\epsilon = 0.1$. Même temps de simulation d'environ 0.05s.

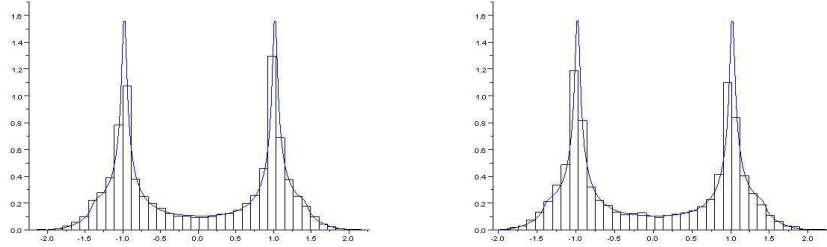


Figure 2: $\nu = 0.5$. Graphique de gauche: système sans diffusion, $n = 10^4$, $\epsilon = 0.02$. Graphique de droite: système avec diffusion, $n = 10^4$, $\epsilon = 0.1$. Même temps de simulation d'environ 0.05s.

On remarque que le système avec diffusion converge plus rapidement surtout si ν est proche de 2 (la différence étant moins flagrante pour ν plus

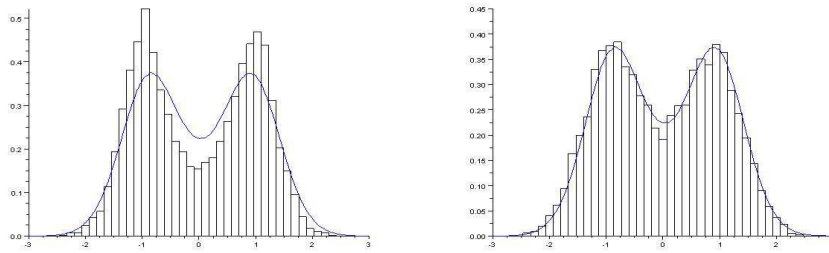


Figure 3: $\nu = 1.5$. Graphique de gauche: système sans diffusion, $n = 2 \cdot 10^4$, $\epsilon = 0.1$. Graphique de droite: système avec diffusion, $n = 10^4$, $\epsilon = 0.1$. Même temps de simulation d'environ 0.14s.

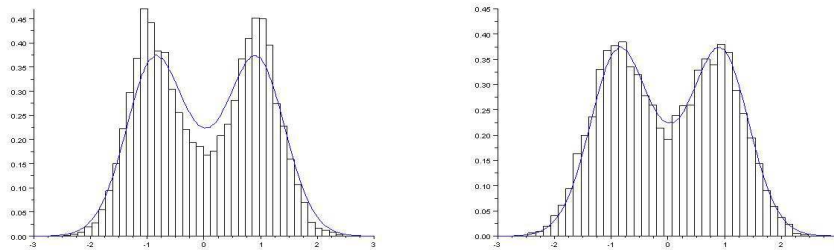


Figure 4: $\nu = 1.5$. Graphique de gauche: système sans diffusion, $n = 10^4$, $\epsilon = 0.06$. Graphique de droite: système avec diffusion, $n = 10^4$, $\epsilon = 0.1$. Même temps de simulation d'environ 0.14s.

petit). Ce phénomène n'est pas surprenant au vu des commentaires effectués après le Théorème 0.3.2.

0.4 Asymptotique des collisions rasantes pour l'équation de Boltzmann spatialement homogène pour des potentiels mous et de Coulomb

0.4.1 Présentation du modèle

Dans le deuxième chapitre de cette thèse, nous allons étudier l'équation de Boltzmann pour des potentiels mous et de Coulomb. Elle décrit la densité $f_t(v, x)$ de particules qui se trouvent en une position $x \in \mathbb{R}^3$ et se déplacent à une vitesse $v \in \mathbb{R}^3$ dans un gaz au temps $t \geq 0$. L'équation est la suivante

$$\partial_t f_t(v, x) + v \cdot \nabla_x f_t(v, x) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) [f_t(v', x) f_t(v'_*, x) - f_t(v, x) f_t(v_*, x)],$$

où les vitesses avant collision sont données par

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

et où θ est *l'angle de la déviation* défini par $\cos \theta = \frac{(v - v_*) \cdot \sigma}{|v - v_*|}$. La fonction $B = B(|v - v_*|, \theta) = B(|v' - v'_*|, \theta)$ est appelée le noyau de collision et ses caractéristiques dépendent de la nature des interactions entre les particules. Nous traitons ici le cas spatialement homogène où la vitesse des particules ne dépend pas de leur position (notons que si la donnée initiale ne dépend pas de la position x , il en est de même pour la solution de l'équation de Boltzmann). L'équation est alors la suivante

$$(0.4.1) \quad \partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)],$$

On peut l'interpréter de la façon suivante : pour chaque $v \in \mathbb{R}^3$, de nouvelles particules ayant pour vitesse v apparaissent du fait de collisions

entre des particules de vitesses v' et v'_* , à un taux $B(|v' - v'_*|, \theta)$, tandis que des particules ayant pour vitesse v disparaissent en se choquant avec d'autres particules de vitesse v_* , à un taux $B(|v - v_*|, \theta)$. On peut par exemple voir Alexandre [3], Cercignani [15], Desvillettes [23] et Villani [71] pour plus de détails.

Les collisions sont supposées élastiques et on a donc, au moins formellement, la conservation de la masse, du moment et de l'énergie cinétique pour les solutions de (0.4.1). On suppose ainsi sans perte de généralité que $\int_{\mathbb{R}^3} f_0(v) dv = 1$. Comme pour Kac, on étudie le cas sans cutoff où l'on se donne une section efficace $\beta : (0, \pi] \rightarrow [0, \infty)$ vérifiant

$$(0.4.2) \quad \int_0^\pi \theta^2 \beta(\theta) d\theta = \frac{4}{\pi},$$

ce qui correspond à la vraie situation physique. En général, on suppose uniquement $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$, mais afin d'alléger certains calculs, on peut supposer sans perte de généralité que cette quantité finie est égale à $\frac{4}{\pi}$. Pour l'équation de Kac, nous avons traité le cas Maxwellien et le taux de collisions des particules dépendait uniquement de la section efficace β . Nous traitons ici des potentiels mous et de Coulomb et le taux de collisions dépend donc également de la vitesse des particules. Pour les potentiels mous, on considère ainsi, pour $0 < \epsilon \leq \pi$ (on se concentre directement sur les collisions rasantes), le noyau de collision suivant

$$(0.4.3) \quad B_\epsilon(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta_\epsilon(\theta) \quad \text{avec} \quad \beta_\epsilon(\theta) = \frac{\pi^3}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon},$$

où $\gamma \in (-3, 0)$. On peut observer que l'on a toujours

$$(0.4.4) \quad \int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta = \frac{4}{\pi},$$

pour tout $\epsilon \in (0, \pi]$. On suppose également dans le cas des potentiels mous qu'il existe $\nu \in (0, 2)$ et $0 < c_1 < c_2$ tels que

$$(0.4.5) \quad c_1 \theta^{-1-\nu} \leq \beta(\theta) \leq c_2 \theta^{-1-\nu} \quad \text{pour tout} \quad \theta \in (0, \pi].$$

Pour le cas Coulomb, on considère le noyau de collision suivant

$$(0.4.6) \quad B_\epsilon(|v - v_*|, \theta) \sin \theta = (|v - v_*| + h_\epsilon)^{-3} \beta_\epsilon(\theta),$$

où $\epsilon \in (0, 1)$, $h_\epsilon \in (0, 1)$ décroît vers 0 quand ϵ tend vers 0 et pour $\theta \in (0, \pi]$,

$$(0.4.7) \quad \beta_\epsilon(\theta) = \frac{c_\epsilon}{\log \frac{1}{\epsilon}} \frac{\cos \theta/2}{\sin^3 \theta/2} \mathbb{1}_{\epsilon \leq \theta \leq \pi/2},$$

où c_ϵ est telle que (0.4.4) soit vérifiée. On observe que la section efficace est différente des sections efficaces que l'on a considérées jusqu'ici. Cela vient du fait que le cas Coulomb est un cas à part qui se traite d'une manière différente. Le lecteur intéressé pourra se rapporter à Villani [70, Section 7] pour plus de détails sur le sujet. Notre noyau de collision est presque identique à celui considéré dans [70]. Nous avons juste apporté deux petites modifications. La première est l'ajout du paramètre h_ϵ dans la partie dépendant de la vitesse. Le seul but de cet ajout est d'obtenir facilement l'existence et l'unicité d'une solution à (0.4.1). En effet, l'objectif de notre travail est l'obtention d'une vitesse de convergence dans le cadre de l'asymptotique des collisions rasantes, et l'ajout de ce paramètre ne nous est pas nécessaire pour atteindre cet objectif (on peut remarquer que nous demandons uniquement à h_ϵ de décroître vers 0 sans aucune précision sur la vitesse de cette convergence). La seconde modification est l'ajout de c_ϵ dans la section efficace afin que (0.4.4) soit satisfaite. Cela ne change pas la nature de la section efficace puisque c_ϵ est proche de $\frac{1}{2\pi}$ quand ϵ est petit.

On observe finalement sur cette section efficace que, du fait que l'on ait (0.4.4) pour tout $\epsilon > 0$ et que $\int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \leq \frac{C}{\log 1/\epsilon} \rightarrow 0$, elle se concentre bien sur les collisions rasantes.

L'équation limite pour l'asymptotique des collisions rasantes est l'équation de Landau. De fait, elle a été déduite de l'équation de Boltzmann par Landau en 1936 quand les collisions rasantes prédominent dans le gaz. Elle décrit la densité $g_t(v)$ de particules ayant pour vitesse $v \in \mathbb{R}^3$ au temps $t \geq 0$:

$$(0.4.8) \quad \partial_t g_t(v) = \frac{1}{2} \sum_{i,j=1}^3 \partial_i \left\{ \int_{\mathbb{R}^3} l_{ij}(v - v_*) \left[g_t(v_*) \partial_j g_t(v) - g_t(v) \partial_j g_t(v_*) \right] dv_* \right\},$$

où $l(z)$ est une matrice 3×3 symétrique positive pour tout $z \in \mathbb{R}^3$, dépendante du paramètre $\gamma \in [-3, 0)$, définie par

$$(0.4.9) \quad l_{ij}(z) = |z|^\gamma (|z|^2 \delta_{ij} - z_i z_j).$$

Comme pour l'équation de Boltzmann, on remarque que les solutions à (0.4.8) conservent au moins formellement la masse, le moment et l'énergie cinétique et on suppose ainsi sans perte de généralité que $\int_{\mathbb{R}^3} g_0(v) dv = 1$. On peut voir Villani [70, 71] pour plus de détails sur cette équation, plus particulièrement son sens physique et sa dérivation de l'équation de Boltzmann.

L'objet du Chapitre 2 est de donner une vitesse de convergence explicite de la solution de l'équation de Boltzmann vers celle de l'équation de Landau dans le cadre de l'asymptotique des collisions rasantes pour des potentiels mous et de Coulomb. Les vitesses obtenues sont moins bonnes que pour l'équation de Kac. Ceci n'est pas illogique car des difficultés supplémentaires apparaissent pour plusieurs raisons. Pour commencer, nous travaillons en dimension 3. Cela entraîne une complication dans la paramétrisation des vitesses avant collisions et surtout en dimension 3 le résultat permettant d'estimer la distance de Wasserstein entre l'intégrale par rapport à une mesure de Poisson et une Gaussienne est plus compliqué à obtenir. Une difficulté supplémentaire vient du fait que l'on traite des potentiels mous et de Coulomb et que le taux de collision dépend donc de la vitesse des particules. Enfin, on peut également noter qu'à l'inverse de Kac, l'équation limite (celle de Landau) n'est pas linéaire.

Le fait que l'on obtienne une vitesse de convergence explicite présente un double intérêt physique et numérique. Un intérêt physique car cela permet de justifier que l'équation de Landau est une bonne approximation de Boltzmann quand les collisions rasantes prédominent. Un intérêt numérique car on peut espérer montrer qu'il est préférable d'approcher les petites collisions par un terme de type Landau plutôt que de les négliger comme on l'a fait avec l'équation de Kac dans la section précédente.

0.4.2 Les potentiels mous

Nous donnons ici notre résultat sur l'asymptotique des collisions rasantes dans le cas des potentiels mous.

Théorème 0.4.1. *Soient $\gamma \in (-3, 0)$, $\nu \in (0, 2)$ et, pour $\epsilon \in (0, \pi]$, B_ϵ un noyau de collision satisfaisant (0.4.2-0.4.3-0.4.5).*

(i) On suppose que $\gamma \in (-1, 0)$ et $\nu \in (-\gamma, 1)$. Soient $p > \max(5, \gamma^2/(\nu + \gamma))$ et $f_0 \in \mathcal{P}_{p+2}(\mathbb{R}^3)$ telle que $H(f_0) < \infty$. Alors il existe une unique solution (faible) $(g_t)_{t \in [0, \infty)}$ à (0.4.8) avec $g_0 = f_0$, et pour tout $\epsilon \in (0, \pi]$, il existe une

unique solution (faible) $(f_t^\epsilon)_{t \in [0, \infty)}$ à (0.4.1) avec B_ϵ pour noyau de collision et $f_0^\epsilon = f_0$ pour donnée initiale. De plus, pour tout $T > 0$ et $\epsilon \in (0, 1)$,

$$\sup_{[0, T]} \mathcal{W}_2(f_t^\epsilon, g_t) \leq C \epsilon^{\frac{p}{2p+3}},$$

où C est une constante qui dépend de T, p, γ, f_0 .

(ii) On suppose que $\gamma \in (-3, 0)$. Soient $p \geq 5$ et $f_0 \in \mathcal{P}_{p+2}(\mathbb{R}^3)$ telle que $f_0 \in L^q(\mathbb{R}^3)$ pour un $q > \frac{3}{3+\gamma}$. Alors il existe $T_* = T_*(q, \|f_0\|_{L^q}) > 0$ tel qu'il existe une unique solution (faible) $(g_t)_{t \in [0, T_*]}$ à (0.4.8) avec $g_0 = f_0$, et pour tout $\epsilon \in (0, \pi]$, et tel qu'il existe une unique solution (faible) $(f_t^\epsilon)_{t \in [0, T_*]}$ à (0.4.1) avec B_ϵ pour noyau de collision et $f_0^\epsilon = f_0$ pour donnée initiale. De plus, pour tout $\epsilon \in (0, 1)$,

$$\sup_{[0, T_*]} \mathcal{W}_2(f_t^\epsilon, g_t) \leq \epsilon^{\frac{p}{2p+3}},$$

où C est une constante qui dépend de p, q, γ, f_0 .

Commençons par remarquer que les énoncés d'existence et d'unicité que nous donnons dans le résultat précédent sont des conséquences directes des papiers de Fournier-Mouhot [35] et Fournier-Guérin [32]-[33]. Il a juste fallu s'assurer que les estimées sur $(f_t^\epsilon)_{t \in [0, T_*]}$ ne dépendaient pas de ϵ . On peut également remarquer que c'est du fait de ces résultats d'existence et d'unicité que l'on a un résultat global en temps pour $\gamma \in (-1, 0)$ et local en temps sinon.

Notons également que comme annoncé, la vitesse obtenu n'est pas optimale : elle est de l'ordre de $\epsilon^{1/2-}$ si la donnée initiale admet des moments de tout ordre alors que l'on peut penser que comme pour Kac, la vitesse optimale est de ϵ . De plus le résultat est loin d'être uniforme en temps puisque les constantes en dépendent de manière exponentielle.

La démonstration s'appuie sur l'utilisation d'E.D.S. Avant d'introduire celle liée à l'équation de Boltzmann, il nous faut donner une paramétrisation de la vitesse des particules avant les collisions. Les notations qui suivent sont tirées de Fournier-Méléard [34]. Pour chaque $X \in \mathbb{R}^3$, on introduit $I(X), J(X) \in \mathbb{R}^3$ tel que $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ soit une base orthonormée de \mathbb{R}^3 . On demande également que $I(-X) = -I(X)$ et $J(-X) = -J(X)$. Pour

$X, v, v_* \in \mathbb{R}^3$, $\theta \in [0, \pi]$ et $\varphi \in [0, 2\pi)$, on pose

$$\begin{cases} \Gamma(X, \varphi) := (\cos \varphi)I(X) + (\sin \varphi)J(X), \\ v' := v'(v, v_*, \theta, \varphi) := v - \frac{1-\cos \theta}{2}(v - v_*) + \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ v'_* := v'_*(v, v_*, \theta, \varphi) := v_* + \frac{1-\cos \theta}{2}(v - v_*) - \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ a := a(v, v_*, \theta, \varphi) := (v' - v) = -(v'_* - v_*). \end{cases}$$

Il nous faut également retirer la dépendance en vitesse $|v - v_*|^\gamma$ dans le taux de collision. On considère ainsi

$$H_\epsilon(\theta) := \int_\theta^\epsilon \beta_\epsilon(x) dx \quad \text{et} \quad G_\epsilon(z) := H_\epsilon^{-1}(z),$$

et pour $z \in (0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$, on définit

$$c_\epsilon(v, v_*, z, \varphi) := a[v, v_*, G_\epsilon(z/|v - v_*|^\gamma), \varphi].$$

On se donne un α -processus $(\tilde{V}_t^\epsilon)_{t \in [0, T]}$ tel que $\mathcal{L}_\alpha(\tilde{V}_t^\epsilon) = f_t^\epsilon$ pour tout $t \in [0, T]$. On considère également une $(\mathcal{F}_t)_{t \in [0, T]}$ -mesure de Poisson N sur $[0, T] \times [0, \infty) \times [0, 2\pi] \times [0, 1]$ d'intensité $dsdzd\varphi d\alpha$, et V_0 une variable aléatoire \mathcal{F}_0 -mesurable de loi f_0 . On considère enfin $(V_t^\epsilon)_{t \in [0, T]}$ solution de l'E.D.S. suivante

$$\begin{aligned} V_t^\epsilon = & V_0 + \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 c_\epsilon(V_{s-}^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ & - k_\epsilon \int_0^t \int_0^1 |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^\gamma (V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha, \end{aligned}$$

où $k_\epsilon = \pi \int_0^\pi (1 - \cos \theta) \beta_\epsilon(\theta) d\theta$. On a alors $\mathcal{L}(V_t^\epsilon) = f_t^\epsilon$ pour tout $t \in [0, T]$.

L'équation de Landau n'étant pas linéaire, il va nous falloir considérer un bruit blanc tridimensionnelle $W(ds, d\alpha)$ sur $[0, T] \times [0, 1]$ de mesure de covariance $dsd\alpha$ (dans le sens défini par Walsh [73]). On se donne un α -processus $(\tilde{Y}_t)_{t \in [0, T]}$ tel que $\mathcal{L}_\alpha(\tilde{Y}_t) = g_t$ pour tout $t \in [0, T]$ et on considère une variable aléatoire Y_0 \mathcal{F}_0 -mesurable de loi g_0 . On considère enfin le processus $(Y_t)_{t \in [0, T]}$ solution de l'E.D.S. suivante

$$Y_t = Y_0 + \int_0^t \int_0^1 \sigma(Y_s - \tilde{Y}_s(\alpha)) W(ds, d\alpha) + \int_0^t \int_0^1 b(Y_s - \tilde{Y}_s(\alpha)) ds d\alpha,$$

où pour tout $z \in \mathbb{R}^3$,

$$b(z) = -2|z|^\gamma z \quad \text{et} \quad \sigma(z) = |z|^{\gamma/2} \begin{pmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{pmatrix}.$$

On a $\mathcal{L}(Y_t) = g_t$ pour tout $t \in [0, T]$ (on peut observer que $\sigma(z)\sigma^*(z) = l(z)$ avec $l(z)$ donné par (0.4.9)).

L'idée et le schéma de la preuve sont les mêmes que pour celle du Théorème 0.3.1. Cependant celle-ci est comme on pouvait s'y attendre beaucoup plus compliquée.

Le point clef est encore une fois de coupler la mesure de Poisson et le bruit blanc (qui joue le rôle du mouvement Brownien) afin qu'une certaine distance de Wasserstein soit réalisée, puis d'estimer cette distance à l'aide de la Proposition 0.2.5. Cependant ce résultat s'applique plus difficilement que le Corollaire 0.2.3. En premier lieu la fonction déterministe intervenant dans l'intégrale par rapport à la mesure de Poisson ne peut plus dépendre du temps, ce qui nous amène à discrétiser le temps avec une subdivision dont le pas est inférieur à $1/n$ pour un entier n . De plus les hypothèses d'applications de ce résultat sont plus contraignantes, nous obligeant à faire certaines considérations alourdissant les calculs.

Une autre différence avec la preuve utilisée pour Kac vient du terme en $|v - v_*|^\gamma$ dans le noyau de collision. De ce fait, il nous faudra traiter des termes du type $\int_0^t \int_0^1 \mathbb{E}[X_s | V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^\delta] d\alpha ds$ avec $\delta = \gamma$ ou $\gamma + 1$ (si $\gamma < -1$) ou $\gamma + 2$ (si $\gamma < -2$). Ces derniers seront contrôlés par $\int_0^t \mathbb{E}[X_s] \|f_s\|_{L^q} ds$ en utilisant le fait que pour $\delta \in (-3, 0)$ et pour $q \in (3/(3 + \delta), \infty]$, il existe une constante $C_{\delta, q}$ telle que pour tout $h \in \mathcal{P}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$,

$$\sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} h(v_*) |v - v_*|^\delta dv_* \leq C_{\delta, q} \|h\|_{L^q(\mathbb{R}^3)} + 1.$$

On peut se rapporter à la partie 2.5 pour tout les détails de la démonstration.

0.4.3 Le potentiel de Coulomb

On donne ici notre résultat principal sur le potentiel de Coulomb.

Théorème 0.4.2. *On prend $\gamma = -3$. Soient B_ϵ donnée par (0.4.6), $p \geq 7$ et $f_0 \in \mathcal{P}_p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Alors il existe $T_* = T_*(\|f_0\|_{L^\infty})$ tel qu'il existe*

une unique solution (faible) $(g_t)_{t \in [0, T_*]}$ à (0.4.8) avec $g_0 = f_0$, et pour tout $\epsilon \in (0, 1)$, il existe une unique solution (faible) $(f_t^\epsilon)_{t \in [0, T_*]}$ à (0.4.1) ayant B_ϵ pour noyau de collision et $f_0^\epsilon = f_0$ pour condition initiale. De plus, pour tout $\epsilon \in (0, 1)$,

$$\sup_{[0, T_*]} \mathcal{W}_2(f_t^\epsilon, g_t) \leq C \left(h_\epsilon^a + \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^a \right),$$

où C et $a > 0$ dépendent de p et de f_0 .

Comme pour les potentiels mous, les énoncés d'existence et d'unicité dans le théorème précédent sont des conséquences directes de résultats existants. Pour l'équation de Boltzmann (0.4.1), on a considéré un noyau borné et l'existence et l'unicité sont classiques et peuvent se trouver dans le papier de Fournier-Guérin [32]. Là encore, il a fallu s'assurer que les estimées intervenant pour la stabilité dans $L^\infty(\mathbb{R}^3)$ ne dépendent pas de ϵ . Pour Landau (0.4.8), l'existence se trouve dans Arsen'ev-Peskov [4] et l'unicité dans Fournier [29].

On observe deux termes d'erreur dans notre résultat. Le premier, (h_ϵ^a) , vient du fait que l'on a ajouté le paramètre h_ϵ dans le noyau de collision afin on le rappelle d'avoir facilement l'existence et l'unicité pour (0.4.1). Le second terme, $\left(\frac{1}{\log \frac{1}{\epsilon}} \right)^a$, est la vraie vitesse de convergence pour l'asymptotique des collisions rasantes dans le cas Coulomb. Nous sommes loin de la vitesse optimale que l'on pense être de $\sqrt{\frac{1}{\log \frac{1}{\epsilon}}}$ (la constante a est très petite). Il est important de noter que les deux termes d'erreur ne sont pas liés. En supposant l'existence et l'unicité pour (0.4.1), on pourrait ainsi prendre $h_\epsilon = 0$ et nos preuves resteraient valides. Cependant, comme h_ϵ peut tendre vers 0 aussi vite que l'on veut, on pense que le fait d'avoir ce premier terme d'erreur ne pose pas de problème.

Le schéma de la preuve est exactement le même que pour le cas des potentiels mous mais certaines étapes se démontrent de manière différente car il faut gérer des termes du type $\int_0^t \int_0^1 \mathbb{E}[X_s | V_s^\epsilon - \tilde{V}_s^\epsilon |^{-3}] d\alpha ds$. On ne peut pas traiter ces termes de la même façon que pour les potentiels mous car on travaille dans \mathbb{R}^3 et donc $\int_{|x| < 1} |x|^{-3} dx = \infty$. On utilise le fait qu'il existe une constante C tel que pour tout $h \in \mathcal{P}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, pour tout $\epsilon \in (0, 1]$,

$$\sup_{v \in \mathbb{R}^3} \int_{|v-v_*| \geq \epsilon} |v-v_*|^{-3} h(v_*) dv_* \leq 1 + C \|h\|_\infty \log(1/\epsilon).$$

Ceci nous amène à utiliser la généralisation du lemme de Grönwall suivante.

Lemme 0.4.1. *On considère la fonction croissante continue $\psi : [0, \infty) \rightarrow \mathbb{R}_+$ définie par*

$$\psi(x) = x(1 - \mathbb{1}_{x \leq 1} \log x).$$

Soient $T > 0$ et $\gamma : [0, T] \rightarrow \mathbb{R}_+$ satisfaisant $\int_0^T \gamma(s) ds < \infty$. On considère une fonction bornée $\rho : [0, T] \rightarrow \mathbb{R}_+$ tel que, pour un certain $a \geq 0$, pour tout $t \in [0, T]$, $\rho(t) \leq a + \int_0^t \gamma(s) \psi(\rho(s)) ds$. On pose $K := \int_0^T \gamma(s) ds$. Alors $\rho(t) \leq C(a e^{-K} + a)$ pour tout $t \in [0, T]$, où C dépend uniquement de K .

0.5 Propagation du chaos pour un modèle de Keller-Segel sous-critique

Dans le troisième et dernier chapitre, nous étudions l'équation de Keller-Segel dans un cadre sous-critique. L'équation est la suivante

$$(0.5.1) \quad \frac{\partial f_t(x)}{\partial t} = \chi \nabla_x \cdot ((K * f_t)(x)) f_t(x) + \Delta_x f_t(x),$$

où $f : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\chi > 0$ et

$$(0.5.2) \quad K(x) := \frac{x}{|x|^{\alpha+1}} \quad \alpha \in (0, 1),$$

est le noyau de champ de force. Dans le cas standard, le noyau considéré est $K(x) = x/|x|^2$. L'équation de Keller-Segel standard décrit un modèle de chimiotaxie : il s'agit de l'étude du mouvement de cellules (bactéries ou amibes) qui sont attirées par une substance chimique qu'elles produisent. Elle a été introduite pour la première fois par Keller et Segel dans [49, 50]. Blanchet, Dolbeault et Perthame ont donné dans [6] des résultats d'existence globale de solutions et de comportement en temps long si le paramètre χ (qui représente la sensibilité de la cellule à l'attracteur chimique) est plus petit que $8\pi/M$ où M est la masse initiale (qui est ici toujours égale à 1 puisque nous considérons uniquement des probabilités). On peut voir Horstmann [42, 43] pour plus de détails sur le sujet.

On considère l'E.D.S. non-linéaire suivante liée à (0.5.1)

$$(0.5.3) \quad X_t = X_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(dx) ds + \sqrt{2} B_t,$$

où $f_t = \mathcal{L}(X_t)$. On peut observer, en appliquant la formule d'Itô à $\varphi(X_t)$ pour $\varphi \in C_b^2(\mathbb{R}^2)$ et en prenant l'espérance, que si $(X_t)_{t \geq 0}$ est une solution de (0.5.3), alors en posant $f_t = \mathcal{L}(X_t)$ pour tout $t \geq 0$, $(f_t)_{t \geq 0}$ est solution de (0.5.1) (dans un sens faible).

Le but principal du Chapitre 3 est de montrer la propagation du chaos vers la solution $(X_t)_{t \geq 0}$ de (0.5.3) du système de particules suivant :

$$(0.5.4) \quad \forall i = 1, \dots, N, \quad X_t^{i,N} = X_0^{i,N} - \frac{\chi}{N} \sum_{j=1, j \neq i}^N \int_0^t K(X_s^{i,N} - X_s^{j,N}) ds + \sqrt{2} B_t^i,$$

où $(B^i)_{i=1, \dots, N}$ est une famille de mouvements Brownien i.i.d. dans \mathbb{R}^2 . Pour cela, on s'inspire du papier de Fournier, Hauray et Mischler [36] où ils démontrent la propagation du chaos pour un modèle de Navier-Stokes 2D. Notons dès maintenant que nous obtenons cette propagation du chaos à l'aide d'outils de type tension et que nous n'avons donc pas de vitesse de convergence du système de particules comme pour Kac dans le Chapitre 1.

0.5.1 Notations et propagation du chaos

Pour $N \geq 2$ et E un espace Polonais, on note $\mathcal{P}_{sym}(E^N)$ l'ensemble des probabilités symétriques sur E^N , i.e. l'ensemble des probabilités qui sont des lois de variables aléatoires échangeables à valeur dans E^N .

On considère pour tout $F \in \mathcal{P}_{sym}((\mathbb{R}^2)^N)$ avec une densité (et également un moment d'ordre positif fini pour définir l'entropie) l'entropie de Boltzmann et l'information de Fisher définies par

$$H(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} F(x) \log F(x) dx \quad \text{et} \quad I(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \frac{|\nabla F(x)|^2}{F(x)} dx.$$

Pour tout $k \geq 0$, on définit également ($x_i \in \mathbb{R}^2$ représente la i -ème coordonnée de $x \in (\mathbb{R}^2)^N$),

$$M_k(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \sum_{i=1}^N |x_i|^k F(dx).$$

On a normalisé par $1/N$ afin d'avoir pour tout $f \in \mathcal{P}(\mathbb{R}^2)$,

$$H(f^{\otimes N}) = H(f), \quad I(f^{\otimes N}) = I(f) \quad \text{et} \quad M_k(f^{\otimes N}) = M_k(f).$$

Donnons maintenant la définition de la propagation du chaos.

Définition 0.5.1. *Soit X une variable aléatoire à valeurs dans E . Une suite (X_1^N, \dots, X_N^N) de variables aléatoires échangeables à valeurs dans E est X -chaotique si l'une des trois conditions équivalentes suivantes est satisfaite*

(i) (X_1^N, X_2^N) converge en loi vers deux copies indépendantes de X quand $N \rightarrow +\infty$;

(ii) pour tout $j \geq 1$, (X_1^N, \dots, X_j^N) converge en loi vers j copies indépendantes de X quand $N \rightarrow +\infty$;

(iii) la mesure empirique $\mu_{X^N}^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N} \in \mathcal{P}(E)$ converge en loi vers la constante $\mathcal{L}(X)$ quand $N \rightarrow +\infty$.

On peut voir Sznitman [62] pour l'équivalence des trois conditions ou Hauray-Mischler [40, Theorem 1.2] où l'équivalence est établie de façon quantitative.

La propagation du chaos dans le sens de Sznitman a toujours un sens pour un système de N particules échangeables évoluant avec le temps si, quand les conditions initiales $(X_0^{1,N}, \dots, X_0^{N,N})$ sont X_0 -chaotique, les trajectoires $((X_t^{1,N})_{t \geq 0}, \dots, (X_t^{N,N})_{t \geq 0})$ sont $(X_t)_{t \geq 0}$ -chaotique, où $(X_t)_{t \geq 0}$ est (l'unique) solution du modèle limite attendu.

On rappelle finalement une notion plus forte de chaos (voir [40]) introduite par Kac dans [47] et formalisée récemment dans [14] : le chaos entropique.

Définition 0.5.2. *Soit f une probabilité sur E . Une suite (F^N) de probabilités symétriques sur E^N est entropiquement f -chaotique si*

$$F_1^N \rightarrow f \quad \text{faiblement dans } \mathcal{P}(E) \quad \text{et} \quad H(F^N) \rightarrow H(f) \quad \text{quand } N \rightarrow \infty,$$

où F_1^N désigne la première marginale de F^N .

On peut observer que comme l'entropie est semi continue inférieurement (de telle sorte que $H(f) \leq \liminf_N H(F^N)$) et est convexe, le chaos entropique (qui impose $\lim_N H(F^N) = H(f)$) est une notion plus forte de convergence qui implique que pour tout $j \geq 1$, la densité de la loi de (X_1^N, \dots, X_j^N) tend fortement dans L^1 vers $f^{\otimes j}$ quand $N \rightarrow \infty$ (voir Brezis [8]).

0.5.2 Résultats d'existence et d'unicité

La première étape pour montrer la propagation du chaos est de montrer l'existence et l'unicité pour le système de particules (0.5.4).

Théorème 0.5.1. *Soient $\alpha \in (0,1)$ et $N \geq 2$ fixés. On suppose que $M_1(F_0^N) < \infty$ et $H(F_0^N) < \infty$. Alors il existe une unique solution forte $(X_t^{i,N})_{t \geq 0, i=1, \dots, N}$ à (0.5.4). De plus, les particules ne se choquent jamais p.s., i.e. on a p.s. pour tout $t \geq 0$ et $i \neq j$, $X_t^{i,N} \neq X_t^{j,N}$.*

Pour démontrer ce résultat, on introduit un système de particules utilisant un noyau régulier K_ϵ avec un paramètre de cutoff au dénominateur. On contrôle uniformément en ϵ l'entropie, le premier moment et l'information de Fisher de la loi du système avec cutoff grâce à une formule de dissipation de l'entropie (ce qui nous permet d'obtenir le même type de contrôle pour la loi du système (0.5.4)). L'existence et l'unicité pour le vrai système de particules (sans cutoff) sont immédiats jusqu'à un certain temps d'arrêt qui correspond au premier instant où deux particules se percutent. Grâce notamment aux estimées uniformes en ϵ précédentes et au fait qu'une martingale locale positive est finie p.s., on montre que ce temps d'explosion est infinie p.s. et le résultat est prouvé.

On montre ensuite l'existence et l'unicité pour l'E.D.S. non linéaire (0.5.3).

Théorème 0.5.2. *Soient $\alpha \in (0,1)$ et $f_0 \in \mathcal{P}_1(\mathbb{R}^2)$ telle que $H(f_0) < \infty$. Il existe une unique solution forte $(X_t)_{t \geq 0}$ à (0.5.3) tel que pour un $p > 2/(1-\alpha)$,*

$$(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2)),$$

où f_t est la loi de X_t .

L'existence en loi repose sur des arguments de tension du système de particules (0.5.4) et du fait que toute limite de $Q^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N})_{t \geq 0}}$ appartienne à l'ensemble des mesures de probabilités $f \in \mathcal{P}(C((0, \infty), \mathbb{R}^2))$ tel que f soit la loi de $(X_t)_{t \geq 0}$ solution de (0.5.3) vérifiant (on note $f_t = \mathcal{L}(X_t)$)

$$(0.5.5) \quad \forall T > 0, \quad \int_0^T I(f_s) ds < \infty \quad \text{et} \quad \sup_{[0, T]} M_1(f_s) < \infty.$$

On utilisera ces mêmes arguments pour la propagation du chaos. Pour l'unicité trajectorielle, on prend deux solutions $(X_t)_{t \geq 0}$ et $(Y_t)_{t \geq 0}$ à (0.5.3), on montre que

$$\mathbb{E}[\sup_{[0,T]} |X_t - Y_t|] \leq C \int_0^T \mathbb{E}[|X_s - Y_s|](1 + \|f_s + g_s\|_{L^p}) ds,$$

pour tout $T > 0$ où $f_t := \mathcal{L}(X_t)$ et $g_t := \mathcal{L}(Y_t)$, et on conclut à l'aide du lemme de Grönwall. La singularité du noyau K est similaire à celle intervenant dans le noyau de collision de l'équation de Boltzmann. Pour obtenir la dernière majoration, on a ainsi repris le même type d'arguments utilisés par Fournier-Guérin dans [32] : on considère $R_s \in \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ tel que $\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| R_s(dx, dy) = \mathcal{W}_1(f_s, g_s)$ et on utilise la majoration suivante

$$\mathbb{E}\left[|X_s - Y_s| \int_{\mathbb{R}^2} \frac{1}{|X_s - x|^{\alpha+1}} f_s(dx)\right] \leq C(1 + \|f_s\|_{L^p}) \mathbb{E}[|X_s - Y_s|],$$

avec $p > \frac{2}{1-\alpha}$.

Il reste enfin à montrer l'existence et l'unicité pour l'équation de Keller-Segel sous-critique (0.5.1).

Théorème 0.5.3. *Soit $\alpha \in (0, 1)$. On suppose que $f_0 \in \mathcal{P}_1(\mathbb{R}^2)$ est telle que $H(f_0) < \infty$.*

(i) *Il existe une unique solution (faible) f à (0.5.1) telle que*

$$f \in L_{loc}^\infty([0, \infty), \mathcal{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2)) \quad \text{pour un } p > \frac{2}{1-\alpha}.$$

(ii) *Cette solution satisfait de plus pour tout $T > 0$,*

$$\int_0^T I(f_s) ds < \infty,$$

pour tout $q \in [1, 2)$ et pour tout $T > 0$,

$$\nabla_x f \in L^{2q/(3q-2)}(0, T; L^q(\mathbb{R}^2)),$$

pour tout $p \geq 1$,

$$f \in C([0, \infty); L^1(\mathbb{R}^2)) \cap C((0, \infty); L^p(\mathbb{R}^2)),$$

et pour tout $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2,\infty}(\mathbb{R})$ telle que β'' soit continue par morceaux à support compact,

$$\begin{aligned} \partial_t \beta(f) = & \chi (K * f) \cdot \nabla_x (\beta(f)) + \Delta_x \beta(f) \\ & - \beta''(f) |\nabla_x f|^2 + \chi \beta'(f_s) f_s (\nabla_x \cdot K * f_s), \end{aligned}$$

sur $[0, \infty) \times \mathbb{R}^2$ dans le sens des distributions.

L'existence et l'unicité reposent sur des arguments probabilistes. Le travail a déjà été fait pour l'existence. En effet, si on prend le processus $(X_t)_{t \geq 0}$ du Théorème 0.5.2 solution de (0.5.3), alors en posant $f_t = \mathcal{L}(X_t)$, $(f_t)_{t \geq 0}$ est solution de (0.5.1).

Pour l'unicité, on se donne deux solutions $(f_t)_{t \geq 0}$ et $(g_t)_{t \geq 0}$ et on considère

$$\begin{aligned} X_t &= X_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(dx) ds + \sqrt{2} B_t, \\ Y_t &= Y_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(Y_s - x) g_s(dx) ds + \sqrt{2} B_t. \end{aligned}$$

À l'aide d'un résultat de Bhatt-Karandikar [5] donnant des conditions pour avoir l'existence et l'unicité à un problème de martingale donné, on montre que $\mathcal{L}(X_t) = f_t$ et $\mathcal{L}(Y_t) = g_t$. Puis comme précédemment, on montre que

$$\mathbb{E}(|X_t - Y_t|) \leq \mathcal{W}_1(f_0, g_0) \exp \left(C \int_0^t (1 + \|f_s + g_s\|_{L^p}) ds \right),$$

ce qui permet de conclure puisque $\mathcal{W}_1(f_t, g_t) \leq \mathbb{E}(|X_t - Y_t|)$ par définition de la distance de Wasserstein.

La renormalisation de la solution se fait par des arguments d'E.D.P.

0.5.3 Propagation du chaos

Nous avons tous les outils en main pour montrer notre résultat principal.

Theorem 0.5.3. Soient $\alpha \in (0, 1)$ et $(X_0^{i,N})_{i=1,\dots,N}$ une famille de variables aléatoires de loi F_0^N telle que pour un $f_0 \in \mathcal{P}(\mathbb{R}^2)$,

$$\begin{cases} F_0^N \in \mathcal{P}_{sym}((\mathbb{R}^2)^N) & \text{est } f_0\text{-chaotique;} \\ \sup_{N \geq 2} M_1(F_0^N) < \infty, & \sup_{N \geq 2} H(F_0^N) < \infty. \end{cases}$$

Pour chaque $N \geq 2$, on considère l'unique solution $(X_t^{i,N})_{i=1,\dots,N,t \geq 0}$ de (0.5.4). Soit $(X_t)_{t \geq 0}$ l'unique solution de (0.5.3).

(i) La suite $(X_t^{i,N})_{i=1,\dots,N,t \geq 0}$ est $(X_t)_{t \geq 0}$ -chaotique. En particulier, la mesure empirique $Q^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N})_{t \geq 0}}$ converge en loi vers $\mathcal{L}((X_t)_{t \geq 0})$ dans $\mathcal{P}(C((0, \infty), \mathbb{R}^2))$.

(ii) Supposons de plus que $\lim_N H(F_0^N) = H(f_0)$. Pour tout $t \geq 0$, la suite $(X_t^{i,N})_{i=1,\dots,N}$ est alors entropiquement X_t -chaotique. En particulier, pour tout $j \geq 1$ et tout $t \geq 0$, en notant F_{tj}^N la densité de la loi de $(X_t^{1,N}, \dots, X_t^{j,N})$, on a

$$\lim_{N \rightarrow \infty} \|F_{tj}^N - f_t^{\otimes j}\|_{L^1((\mathbb{R}^2)^j)} = 0.$$

Notons que les hypothèses du résultat précédent sont vérifiées si la famille $(X_0^{i,N})_{i=1,\dots,N}$ est i.i.d. de loi $f_0 \in \mathcal{P}_1(\mathbb{R}^2)$ telle que $H(f_0) < \infty$. La démonstration du point (i) repose sur le fait que la famille $\{\mathcal{L}(Q^N), N \geq 2\}$ soit tendue dans $\mathcal{P}(\mathcal{P}(C([0, \infty), \mathbb{R}^2)))$ et que toute limite de Q^N appartienne p.s. à l'ensemble des mesures de probabilités $f \in \mathcal{P}(C((0, \infty), \mathbb{R}^2))$ tel que f soit la loi d'une solution à (0.5.3) vérifiant (0.5.5). Mais d'après le Théorème 0.5.2, cet ensemble est réduit à $f := (\mathcal{L}(X_t))_{t \geq 0}$. On en déduit que Q^N tend en loi vers f quand $N \rightarrow \infty$.

En utilisant la renormalisation de la solution de (0.5.1), on obtient la formule de dissipation de l'entropie suivante

$$H(f_t) + \int_0^t I(f_s) ds = H(f_0) + \chi(1 - \alpha) \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x - y|^{\alpha+1}} ds.$$

Cette formule et un petit travail supplémentaire nous permettent d'obtenir le chaos entropique.

0.6 Quelques perspectives

Dans le chapitre 2, les vitesses de convergences ne sont pas optimales. Il serait ainsi intéressant d'améliorer ces dernières. On pourrait alors adapter les idées du chapitre 1 en approchant la solution de l'équation de Boltzmann dans le cas général en remplaçant les petites collisions par un terme de type Landau.

Dans le chapitre 3, le résultat de propagation du chaos est obtenu à l'aide d'un argument de tension. En s'inspirant du chapitre 1, on pourrait chercher

à obtenir un résultat donnant une vitesse de convergence du système de particules. De plus on considère un modèle de Keller-Segel sous-critique. On pourrait essayer de considérer le modèle critique où $\alpha = 1$ ou du moins de s'en approcher en considérant un petit cutoff (avec par exemple un K tel que $|K(x)| = (|x| + n^{-p})^{-1}$ avec p grand).

Chapter 1

Asymptotic of grazing collisions and particle approximation for the Kac equation without cutoff

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Abstract. The subject of this article is the Kac equation without cutoff. We first show that in the asymptotic of grazing collisions, the Kac equation can be approximated by a Fokker-Planck equation. The convergence is uniform in time and we give an explicit rate of convergence. Next, we replace the small collisions by a small diffusion term in order to approximate the solution of the Kac equation and study the resulting error. We finally build a system of stochastic particles undergoing collisions and diffusion, that we can easily simulate, which approximates the solution of the Kac equation without cutoff. We give some estimates on the rate of convergence.

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1.1 Introduction

1.1.1 The model

The spatially homogeneous Boltzmann equation (see Cercignani [15], Villani [71]) describes the density $f_t(v)$ of particles in a gas, which move with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The Kac equation is a one-dimensional *caricature* of the Boltzmann equation. It writes

$$(1.1.1) \quad \frac{\partial f_t}{\partial t}(v) = \int_{v_* \in \mathbb{R}} \int_{\theta = -\pi}^{\pi} \left(f_t(v') f_t(v'_*) - f_t(v) f_t(v_*) \right) \beta(\theta) d\theta dv_*,$$

where $t \geq 0$, $v \in \mathbb{R}$ and where the pre-collisional velocities are given by

$$(1.1.2) \quad v' = v \cos \theta - v_* \sin \theta, \quad v'_* = v \sin \theta + v_* \cos \theta.$$

The function $\beta : [-\pi, \pi] - \{0\} \rightarrow \mathbb{R}_+$ is an even function called *cross section*. Let us interpret this equation: with a rate proportional to $\beta(\theta)$, new particles with velocity v appear due to a collision between two particles with velocities v' and v'_* , while particles with velocity v disappear because they collide with another particle with velocity v_* . See Kac [47, 48] and Desvillettes [19] for more precisions. If we have $\int_0^\pi \beta(\theta) d\theta = \infty$, then there is an infinite number of collisions for each particle during any time interval. The case where we assume $\int_0^\pi \beta(\theta) d\theta < \infty$ (case with cutoff) has been much studied. We will focus here on the real physical situation where we only assume $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$ (case without cutoff). By analogy with the 3d-Boltzmann equation, we will include the case where, for some $0 < \nu < 2$,

$$(1.1.3) \quad \beta(\theta) \stackrel{\theta=0}{\approx} |\theta|^{-1-\nu}.$$

We will use in this article Wasserstein distances. Let us recall that for $p \geq 1$, if f and g are two probability measures on \mathbb{R} with a moment of order p ,

$$W_p(f, g) = \inf \left\{ \mathbb{E}(|U - V|^p)^{1/p}, U \sim f, V \sim g \right\},$$

where the infimum is taken over all random variables U with law f and V with law g . See e.g. Villani [72] for many details on the subject. In particular, it is known that the infimum is reached : one can build $U \sim f$ and $V \sim g$ such that $W_p^p(f, g) = \mathbb{E}(|U - V|^p)$.

1.1.2 Asymptotic of grazing collisions

Assume that there are more and more collisions, but that these collisions generate smaller and smaller deviations. For example, consider $\beta_\epsilon(\theta) = \frac{1}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}$. Then, we have $\int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta = \text{const}$ and $\int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \rightarrow 0$. It is known that in this case, the solutions of Boltzmann's equation converge to the solution of the Fokker-Planck-Landau equation. To be more precise, Degond and Lucquin-Desreux [17] and Desvillettes [20] have shown the convergence of the operators (not of the solutions) and Villani [70] has shown some compactness results and the convergence of subsequences. The uniqueness results of [32] show the true convergence (under some more restrictive assumptions).

If we denote by $(g_t^\epsilon)_{t \geq 0}$ the solution of equation (1.1.1) with cross section β_ϵ and initial condition $g_0^\epsilon(v) = g_0(v)$ and if we assume $\int_{\mathbb{R}} v^4 g_0(v) dv < \infty$ we will show that $\sup_{t \in [0, \infty)} W_2(g_t^\epsilon, g_t) \leq C\epsilon$, with $(g_t)_{t \geq 0}$ starting from g_0 and solving

$$(1.1.4) \quad \frac{\partial}{\partial t} g_t(v) = \frac{\mathcal{E}}{2} \frac{\partial^2}{\partial v^2} g_t(v) + \frac{1}{2} \frac{\partial}{\partial v} (v g_t(v)),$$

where $\mathcal{E} := \int_{\mathbb{R}} v^2 g_0(v) dv$. This limit equation is *nonlinear*, but the nonlinearity appears only through $\int_{\mathbb{R}} v^2 g_t(v) dv$, which is constant in time.

A similar result has already been proved by Toscani [66] with a stronger distance but the rate of convergence is not very explicit. We believe that the present rate of convergence is optimal.

1.1.3 Replacing grazing collisions by a small diffusion term

We come back to the Kac equation (1.1.1) with fixed cross section β . Numerically, we must truncate small collisions, since they are in infinite number. There are two possibilities.

One may truncate roughly small collisions by replacing β by $\tilde{\beta}_\epsilon(\theta) = \beta(\theta) \mathbb{1}_{|\theta| > \epsilon}$. We denote by $(f_t^\epsilon)_{t \geq 0}$ the solution of (1.1.1) with this $\tilde{\beta}_\epsilon$.

One may replace small collisions by a small diffusion term in the spirit of

grazing collisions. We denote by $(f_t^\epsilon)_{t \geq 0}$ the solution to

$$(1.1.5) \quad \begin{aligned} \frac{\partial}{\partial t} f_t^\epsilon(v) = & b_\epsilon \frac{\partial}{\partial v} \left[v f_t^\epsilon(v) \right] + \mathcal{E} b_\epsilon \frac{\partial^2}{\partial v^2} f_t^\epsilon(v) \\ & + \int_{v_* \in \mathbb{R}} \int_{|\theta| \geq \epsilon} \left(f_t^\epsilon(v') f_t^\epsilon(v'_*) - f_t^\epsilon(v) f_t^\epsilon(v_*) \right) \beta(\theta) d\theta dv_*, \end{aligned}$$

where

$$(1.1.6) \quad b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta \quad \text{and} \quad \mathcal{E} = \int_{\mathbb{R}} v^2 f_0^\epsilon(v) dv.$$

We will show that $\sup_{t \in [0, T]} W_2(f_t, f_t^\epsilon) \leq C\epsilon(1 + \sqrt{T})$ if $\int_{\mathbb{R}} v^4 f_0(dv) < \infty$. We can observe that when neglecting roughly grazing collisions, we get $\sup_{t \in [0, T]} W_2(\tilde{f}_t^\epsilon, f_t) \leq C_T \epsilon^{1-\nu/2}$ (see Desvillettes-Graham-Méléard [21]) if β is as in (1.1.3). We can yet notice that there is no dependance on ν in our result. This is due to the fact that the more ν is close to 2, the more we neglect small collisions, but the more small collisions are well-approximated by the diffusion term. The proof is inspired by [30].

1.1.4 A finite system of stochastic particles

Let β be a given cross section and f_0 an initial datum with $\int_{\mathbb{R}} v^4 f_0(dv) < \infty$. We consider a solution $(f_t)_{t \geq 0}$ of (1.1.1).

For $\epsilon > 0$ fixed, we are going to build a system of n stochastic particles that we can simulate with a cost of order $Tn \int_{|\theta| > \epsilon} \beta(\theta) d\theta$ on $[0, T]$, which is at worst of order $T\epsilon^{-2}n$. If we denote by $\mu_t^{n, \epsilon}$ the empirical measure associated to this system of particles and by μ_t^n the empirical measure associated with a system of n i.i.d. particles with law f_t , we will show that

$$\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right).$$

Our system of particles is thus as efficient as the system of particles with *true i.i.d. particles* with law f_t , which cannot be simulated because of the nonlinearity. If we assume that f_0 has infinitely many moments, we will get

$$\sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \frac{1}{n^{(1/2)^-}} \right).$$

This system of particles uses the ideas of the previous section : we replace small collisions by a small diffusion term, which gives an error of order ϵ . In [60], Sundén and Wennberg give some numerical results using the same idea.

In Desvillettes-Graham-Méléard [21], they just cutoff small collisions and they get, roughly, something like $\sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t, \tilde{\mu}_t^{n,\epsilon}) \right] \leq C_T \left(\epsilon^{2-\nu} + \frac{e^{C_T \Lambda_\epsilon}}{n} \right)$, with $\Lambda_\epsilon = \int_{|\theta|>\epsilon} \beta(\theta) d\theta \approx \epsilon^{-\nu}$ if β is as in (1.1.3). If we compare this result with our result, we can observe the following.

- In the first term, we get an error of order ϵ^2 instead of $\epsilon^{2-\nu}$. It is due to the fact that we replace small collisions by a small diffusion term.
- In the second term, we get a bound which does not depend on ϵ . It is because we use a Wasserstein distance which is well-adapted for this study. In Desvillettes-Graham-Méléard [21], they give the final result with a Wasserstein distance, but to get this result they use a variation distance.
- The cost of simulation for the two systems of particles is similar.

1.1.5 Comments

We managed to obtain some bounds uniform in time for the asymptotic of grazing collisions. For our two other main results, we tried to limit the time dependance. We thus avoid getting bounds with exponential terms.

The bound we get for $\mathbb{E}[W_2^2(f_t, \mu_t^n)]$ is not very satisfactory. *A priori*, it is of order $n^{-(1/2)^-}$ (if the initial condition has infinitely many moments, see Lemma 1.8.4) which gives a bound for $\mathbb{E}[W_2(f_t, \mu_t^n)]$ of order $n^{-(1/4)^-}$. We expected to get a bound of order $n^{-1/2}$ as in the central limit theorem, but we cannot get it. See Peyre [55] for example to get more details. It seems to be the only defect of W_2 for this study.

Assuming that $\int_0^\pi \theta \beta(\theta) d\theta < \infty$ (e.g. if we assume (1.1.3) with $\nu \in (0, 1)$), we get a bound for $\mathbb{E}[W_1(f_t, \mu_t^{n,\epsilon})]$ which is of order $\epsilon + n^{-1/2}$ but with an exponential dependance in time. If $\int_0^\pi \theta^\gamma \beta(\theta) d\theta < \infty$ (e.g. if $\nu < \gamma$) for some $\gamma \in (1, 2)$, we also study $\mathbb{E}[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})]$.

In a future work, we will apply the same kind of methods to the homogeneous Boltzmann equation. We hope to get some results which will probably be less optimal.

1.1.6 Bibliographic comments

In this article, we use probabilistic methods. The first who used probabilistic methods to study the Kac (or Boltzmann) equation is McKean [51, 52]. He was investigating the convergence to equilibrium and he proposed some probabilistic representation of Wild's sums, using some tools now known as the McKean graphs. This idea has been used and refined in many papers, let us quote a few of them by Carlen-Carvalho-Gabetta [12, 11, 13] and more recently by Dolera-Gabetta-Regazzini [26, 27]. The exact rate of convergence to equilibrium for the Kac equation is now very precisely known.

Another pioneer of probabilistic methods for kinetic equations is Tanaka. In [64], he discovered the noticeable property that the Wasserstein distance with quadratic cost is non-increasing along solutions to the Kac equation. This idea was extended in [65] to the Boltzmann equation for Maxwell molecules. His study was based on the use of some nonlinear stochastic processes related to the Kac and Boltzmann equations. The paper of Graham-Méléard [37] (about the regularization properties of the Kac equation via Malliavin calculus) is based on the work of Tanaka, as well as [35, 32] dealing with the uniqueness of solutions to the Boltzmann equation. Our work is also strongly inspired by Tanaka's papers.

The present paper is also closely linked with the original problem of Kac [47, 48], concerning the propagation of chaos for the Kac equation. The idea is to make a step toward the rigorous justification of the Boltzmann equation, by showing that some simplified particle systems converge to a simplified Boltzmann equation. On this topic, some important results have been obtained by Sznitman [61] (Boltzmann equation for hard spheres), Desvillettes-Graham-Méléard [21] (Kac equation without cutoff) and more recently by Peyre [55] (large deviations for the Boltzmann equation for Maxwell molecules) and by Mischler-Mouhot [53] (quantitative propagation of chaos for the Boltzmann equation for hard spheres and Maxwell molecules). Let us mention that we provide here a very simple proof of the propagation of chaos for the Kac equation (in a slightly modified context) with very satisfying quantitative estimates, see Section 1.6 until Proposition 1.6.2.

To end with probabilistic references, let us mention that our results strongly rely on a recent paper of Rio [57] giving some precise rate of convergence for the standard central limit theorem in Wasserstein distance.

To conclude, we quote the article of Pareschi-Toscani-Villani [54]. This

paper is devoted to a numerical method, based on the use of Fourier transforms, for the homogeneous Boltzmann equation. It also uses an approximation of the grazing collisions by a small Landau-type diffusive term. Of course, our numerical method can be immediately extended to the true homogeneous Boltzmann equation. From a numerical point of view, we are not able to decide which method is better. In [54], periodic solutions are considered, which is not physical. One would need to study the error due to such an approximation. Furthermore, the theoretical results in [54] only concern the rate of convergence of the operators (when dealing with quite regular functions), not of the solutions. So the main advantage of our paper is that we really provide an (almost optimal) estimate of the numerical error, taking into account all the approximation parameters (number of particles and cutoff parameter). Unfortunately, we only study a simplified model. We believe that our estimates also hold true for the true homogeneous Boltzmann equation, but we are far from being able to prove it.

1.1.7 Plan of the paper

In the next section, we will state more precisely our three main results. In Section 1.3, we will give a probabilistic interpretation of the three equations. Sections 1.4, 1.5 and 1.6 are devoted to the proofs of our main results. Some numerical illustrations will be given in Section 1.7. At the end of the paper, we will give an appendix with some results about the Wasserstein distance between a compensated Poisson integral and a centered Gaussian law with same variance, the rate of convergence of an empirical measure using Wasserstein distances, the moments of the solution to (1.1.1) and the well-posedness for a certain kind of P.D.E.s.

1.2 Results

1.2.1 Weak solutions

Let β be a cross section satisfying

$$(1.2.1) \quad \int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta < \infty.$$

For $k \geq 0$, we denote by $\mathcal{P}_k(\mathbb{R})$ the set of probability measures on \mathbb{R} admitting a moment of order k and by $C_b^2(\mathbb{R})$ the space of real bounded func-

tions which are in $C^2(\mathbb{R})$ with first and second derivatives bounded. We say that a family of probability measures $(f_t)_{t \geq 0}$ is in $L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ if $\sup_{[0, T]} \int_{\mathbb{R}} v^2 f_t(dv) < \infty$ for all T . If $\varphi \in C_b^2(\mathbb{R})$ and $(v, v_*) \in \mathbb{R}^2$, we set

$$(1.2.2) \quad K_\beta^\varphi(v, v_*) = \int_{-\pi}^{\pi} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) - (v(\cos \theta - 1) - v_* \sin \theta) \varphi'(v) \right] \beta(\theta) d\theta - bv \varphi'(v),$$

with

$$(1.2.3) \quad b = \int_{-\pi}^{\pi} (1 - \cos \theta) \beta(\theta) d\theta.$$

If $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, then one easily checks, using that β is even, that

$$(1.2.4) \quad K_\beta^\varphi(v, v_*) = \int_{-\pi}^{\pi} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) \right] \beta(\theta) d\theta$$

We now define precisely the notion of solutions that we will use.

Definition 1.2.1. *Consider a cross section β satisfying (1.2.1).*

1. *We say that $(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ solves (1.1.1) if for any φ in $C_b^2(\mathbb{R})$, any $t \geq 0$,*

$$(1.2.5) \quad \int_{\mathbb{R}} \varphi(v) f_t(dv) = \int_{\mathbb{R}} \varphi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} K_\beta^\varphi(v, v_*) f_s(dv) f_s(dv_*) ds.$$

2. *We say that $(g_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ solves (1.1.4) if for any φ in $C_b^2(\mathbb{R})$, any $t \geq 0$,*

$$(1.2.6) \quad \int_{\mathbb{R}} \varphi(v) g_t(dv) = \int_{\mathbb{R}} \varphi(v) g_0(dv) + \frac{1}{2} \mathcal{E} \int_0^t \int_{\mathbb{R}} \varphi''(v) g_s(dv) ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} v \varphi'(v) g_s(dv) ds,$$

where $\mathcal{E} := \int v^2 g_0(dv)$.

3. For $\epsilon \in (0, 1)$ fixed, we say that $(f_t^\epsilon)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ solves (1.1.5) if for any φ in $C_b^2(\mathbb{R})$, any $t \geq 0$,

$$(1.2.7) \quad \int_{\mathbb{R}} \varphi(v) f_t^\epsilon(dv) = \int_{\mathbb{R}} \varphi(v) f_0^\epsilon(dv) - b_\epsilon \int_0^t \int_{\mathbb{R}} v \varphi'(v) f_s^\epsilon(dv) ds \\ + \mathcal{E} b_\epsilon \int_0^t \int_{\mathbb{R}} \varphi''(v) f_s^\epsilon(dv) ds + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\beta_\epsilon}^\varphi(v, v_*) f_s^\epsilon(dv) f_s^\epsilon(dv_*) ds,$$

where

$$(1.2.8) \quad \beta_\epsilon(\theta) = \beta(\theta) \mathbb{1}_{|\theta| > \epsilon}, \quad b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta \quad \text{and} \quad \mathcal{E} = \int_{\mathbb{R}} v^2 f_0^\epsilon(dv).$$

Observe that all the terms in the above equations are well-defined. For example in (1.2.5), the last term is well-defined because for $\varphi \in C_b^2(\mathbb{R})$, $|K_{\beta}^\varphi(v, v_*)| \leq C \int_0^\pi \theta^2 \beta(\theta) d\theta (|v|^2 + |v_*|^2) \|\varphi''\|_\infty + b|v| \|\varphi'\|_\infty$.

Proposition 1.2.2. *Let f_0, g_0 and f_0^ϵ be in $\mathcal{P}_2(\mathbb{R})$ and let β satisfy (1.2.1). There is existence and uniqueness of solutions $(f_t)_{t \geq 0}$, $(g_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ to equations (1.1.1), (1.1.4) and (1.1.5) starting from f_0, g_0 and f_0^ϵ respectively, in the sense of Definition 1.2.1. Furthermore, we have energy conservation: for any $t \geq 0$*

$$(1.2.9) \quad \int_{\mathbb{R}} v^2 f_t(dv) = \int_{\mathbb{R}} v^2 f_0(dv), \quad \int_{\mathbb{R}} v^2 g_t(dv) = \int_{\mathbb{R}} v^2 g_0(dv)$$

and

$$(1.2.10) \quad \int_{\mathbb{R}} v^2 f_t^\epsilon(dv) = \int_{\mathbb{R}} v^2 f_0^\epsilon(dv).$$

For the proof of the previous result, one can see Toscani-Villani [67] for (1.1.1). For (1.1.4), use Proposition 1.8.6 with $a = \frac{1}{2}\mathcal{E}$, $b = -\frac{1}{2}$ and $q = r = 0$. For (1.1.5), use Proposition 1.8.6 with $a = \mathcal{E}b_\epsilon$, $b = -b_\epsilon$, $q = 0$ and $r(t, v, v_*, dh)$ defined by $r(t, v, v_*, A) = \int_{-\pi}^\pi \mathbb{1}_{|\theta| > \epsilon} \mathbb{1}_A(v(\cos \theta - 1) - v_* \sin \theta) \beta(\theta) d\theta$ for all Borel subset A of \mathbb{R} . Observe that this indeed satisfies $\sup_{t, v, v_*} r(t, v, v_*, \mathbb{R}) = \int_{-\pi}^\pi \mathbb{1}_{|\theta| > \epsilon} \beta(\theta) d\theta < \infty$ and $\sup_{t \geq 0} \int_{\mathbb{R}} (h^2 + 2vh) r(t, v, v_*, dh) = \int_{|\theta| > \epsilon} \sin^2 \theta \beta(\theta) d\theta (v^2 + v_*^2) = C(v^2 + v_*^2)$. To get energy conservation, it suffices to apply (1.2.5), (1.2.6) and (1.2.7) with $\varphi(v) = v^2$.

1.2.2 Asymptotic of grazing collisions

Our first main result is the following.

Theorem 1.2.3. *Let $g_0 \in \mathcal{P}_4(\mathbb{R})$ and let $(\beta_\epsilon)_{\epsilon \in (0,1)}$ be a family of cross sections verifying $\int_{-\pi}^{\pi} \theta^2 \beta_\epsilon(\theta) d\theta = 1$ and $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta \xrightarrow{\epsilon \rightarrow 0} 0$. For $\epsilon \in (0, 1)$, let $(g_t^\epsilon)_{t \geq 0}$ be the solution of (1.1.1) with g_0 for initial datum and β_ϵ for cross section. If $(g_t)_{t \geq 0}$ is the solution of (1.1.4) with the same g_0 for initial datum, then for all $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, \infty)} W_2^2(g_t^\epsilon, g_t) \leq C \frac{\int_{\mathbb{R}} v^4 g_0(dv)}{\mathcal{E}} \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta,$$

where C is a universal constant.

This convergence result was already known (see for example Toscani [66]), but we get here an explicit and probably optimal rate of convergence, which, to our knowledge, had never been done so far.

Remark 1.2.4. *If we consider a cross section β with $\int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta = 1$ and if for any $\epsilon \in (0, 1)$, we set $\beta_\epsilon(\theta) = \frac{\pi^3}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}$, then $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta \leq \epsilon^2$.*

1.2.3 Error when we replace the small collisions by a small diffusion term

Let us explain briefly why (1.2.7) approximates (1.2.5): consider a cross section β satisfying (1.2.1) and $\varphi \in C_b^2(\mathbb{R})$. Write, using that β is even,

$$\begin{aligned} K_\beta^\varphi(v, v_*) &= \int_{-\epsilon}^{\epsilon} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) \right. \\ &\quad \left. - (v(\cos \theta - 1) - v_* \sin \theta) \varphi'(v) \right] \beta(\theta) d\theta + K_{\beta_\epsilon}^\varphi(v, v_*) - b_\epsilon v \varphi'(v) \\ &\approx \int_{-\epsilon}^{\epsilon} \left[v(\cos \theta - 1) - v_* \sin \theta \right]^2 \beta(\theta) d\theta \frac{\varphi''(v)}{2} + K_{\beta_\epsilon}^\varphi(v, v_*) - b_\epsilon v \varphi'(v) \\ &\approx \frac{\varphi''(v)}{2} v_*^2 \int_{-\epsilon}^{\epsilon} \sin^2 \theta \beta(\theta) d\theta + \frac{\varphi''(v)}{2} v^2 \int_{-\epsilon}^{\epsilon} (\cos \theta - 1)^2 \beta(\theta) d\theta \\ &\quad + K_{\beta_\epsilon}^\varphi(v, v_*) - b_\epsilon v \varphi'(v) \\ &\approx \varphi''(v) v_*^2 b_\epsilon + K_{\beta_\epsilon}^\varphi(v, v_*) - b_\epsilon v \varphi'(v). \end{aligned}$$

We decided to neglect the second term in the fourth line of this approximate equality, since it is much smaller than the other terms, because $\int_{-\epsilon}^{\epsilon} (\cos \theta - 1)^2 \beta(\theta) d\theta \leq \int_{-\epsilon}^{\epsilon} \theta^4 \beta(\theta) d\theta \leq \epsilon^2 \int_{-\epsilon}^{\epsilon} \theta^2 \beta(\theta) d\theta \approx \epsilon^2 \int_{-\epsilon}^{\epsilon} \sin^2 \theta \beta(\theta) d\theta$. In order to obtain an equation preserving the kinetic energy, we replaced $\frac{1}{2} \int_{-\epsilon}^{\epsilon} \sin^2 \theta \beta(\theta) d\theta$ by b_ϵ (both are approximately equal to $\frac{1}{2} \int_{-\epsilon}^{\epsilon} \theta^2 \beta(\theta) d\theta$). Our second main result is the following.

Theorem 1.2.5. *Let $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (1.2.1). For $\epsilon \in (0, 1)$, we consider $(f_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ solutions of (1.1.1) and (1.1.5) respectively, both starting from f_0 . Then for any $T > 0$, any $\epsilon \in (0, 1)$, we have*

$$\sup_{t \in [0, T]} W_2^2(f_t, f_t^\epsilon) \leq C \epsilon^2 \min \left(1 + T, \frac{1}{\int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta} \right),$$

where C depends only on $\int_{\mathbb{R}} v^2 f_0(dv)$, $\int_{\mathbb{R}} v^4 f_0(dv)$ and on $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

We can observe that we are not so far to get a bound uniform in time for ϵ^2 (we do not have exponential bounds).

Remark 1.2.6. *If β is as in (1.1.3), we get a bound in $C \min(\epsilon^2(1+T), \epsilon^\nu)$.*

1.2.4 System of particles

Let f_0 be a probability measure on \mathbb{R} and let β be a cross section satisfying (1.2.1). We fix an integer n and we consider:

- a family of i.i.d. random variables $(V_0^i)_{i \in \{1, \dots, n\}}$ with law f_0 ,
- a family of i.i.d. Poisson measures $(N^{i,n}(dsd\theta dj))_{i \in \{1, \dots, n\}}$ on $[0, \infty) \times [-\pi, \pi] \times \{1, \dots, n\}$ with intensity measure $ds\beta(\theta)d\theta \frac{1}{n} \sum_{k=1}^n \delta_k(dj)$,
- a family of i.i.d. Brownian motions $(B_t^i)_{t \geq 0, i \in \{1, \dots, n\}}$.

For $\epsilon \in (0, 1)$, we consider $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ solution of the following system of SDEs: for $i = 1, \dots, n$, for all $t \geq 0$,

(1.2.11)

$$\begin{aligned} V_t^{i,n,\epsilon} = & V_0^i + \int_0^t \int_{|\theta| > \epsilon} \int_{j \in \{1, \dots, n\}} \left[(\cos \theta - 1) V_{s-}^{i,n,\epsilon} - \sin \theta V_{s-}^{j,n,\epsilon} \right] N^{i,n}(dsd\theta dj) \\ & - b_\epsilon \int_0^t V_s^{i,n,\epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

where $\mathcal{E} = \int_{\mathbb{R}} v^2 f_0(dv)$ and $b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta$.

The quantity $V_t^{i,n,\epsilon}$ has to be thought as the velocity of the i -th particle at time t . The behavior of $(V_t^{i,n,\epsilon})_{t \geq 0}$ is the following: after an exponential time τ with parameter $\Lambda_\epsilon = \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$, it collides with another particle labelled j chosen at random and then we set $V_\tau^{i,n,\epsilon} = (\cos \Theta) V_{\tau-}^{i,n,\epsilon} - (\sin \Theta) V_{\tau-}^{j,n,\epsilon}$, where Θ is $\Lambda_\epsilon^{-1} \beta(\theta) \mathbb{1}_{|\theta| \geq \epsilon} d\theta$ -distributed. Between two jumps, $V^{i,n,\epsilon}$ behaves like an Ornstein-Uhlenbeck process

$$V_t^{i,n,\epsilon} = V_s^{i,n,\epsilon} - b_\epsilon \int_s^t V_u^{i,n,\epsilon} du + \sqrt{2\mathcal{E}b_\epsilon} (B_t^i - B_s^i).$$

We can solve explicitly this last SDE and we get

$$V_t^{i,n,\epsilon} = V_s^{i,n,\epsilon} e^{-b_\epsilon(t-s)} + \sqrt{2\mathcal{E}b_\epsilon} e^{-b_\epsilon(t-s)} \int_s^t e^{b_\epsilon u} dB_u^i.$$

Hence the strong existence and uniqueness of a solution $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ to (1.2.11) is straightforward.

We can observe that to simulate our system of particles on $[0, T]$, we need to simulate in mean $nT \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$ jumps. We thus have a cost of simulation of order $nT \int_{|\theta| \geq \epsilon} \beta(\theta) d\theta$. The fact that we can explicitly solve the previous SDE is fundamental in order to have such a cost of simulation.

Theorem 1.2.7. *Let $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (1.2.1). We consider $(f_t)_{t \geq 0}$ solution to the Kac equation (1.1.1) starting from f_0 . For $n \in \mathbb{N}^*$ and $\epsilon \in (0, 1)$, we consider the solution $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ to (1.2.11). We set $\mu_t^{n,\epsilon} = \frac{1}{n} \sum_1^n \delta_{V_t^{i,n,\epsilon}}$. Then for any $T > 0$, any $n \geq 2$ and any $\epsilon \in (0, 1)$, we have*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n,\epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right),$$

where C depends only on $\int_{\mathbb{R}} v^2 f_0(dv)$, $\int_{\mathbb{R}} v^4 f_0(dv)$ and on $\int_0^\pi \theta^2 \beta(\theta) d\theta$, and where for all $t \geq 0$, $\mu_t^n = \frac{1}{n} \sum_1^n \delta_{V_t^i}$, where $(V_t^i)_{i \in \{1, \dots, n\}}$ is a family of i.i.d. particles with law f_t .

Applying Lemma 1.8.4 of the appendix we will deduce the following consequence:

Corollary 1.2.8. *Under the same assumptions and notation as in Theorem 1.2.7, if f_0 has a moment of order $p \geq 4$ with p even, then for all $T > 0$, all $n \geq 2$ and all $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n, \epsilon}) \right] \leq C(1 + T)^3 \left(\epsilon^2 + \frac{1}{n^{\frac{p-2}{2p-2}}} \right),$$

where C depends only on p , f_0 and $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

We end this section with a result using another Wasserstein distance.

Proposition 1.2.9. *Under the same assumptions and notation as in Theorem 1.2.7, if the cross section β satisfies the stronger assumption $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, then for all $T > 0$, all $n \geq 2$ and all $\epsilon \in (0, 1)$,*

$$\sup_{t \in [0, T]} \mathbb{E} \left[W_1(f_t, \mu_t^{n, \epsilon}) \right] \leq C_T \left(\epsilon + \frac{1}{\sqrt{n}} \right),$$

where C_T depends only on T , $\int_{\mathbb{R}} v^4 f_0(dv)$ and on $\int_0^\pi \theta \beta(\theta) d\theta$.

We thus have a better dependence in n , but we get exponential bounds in time.

1.3 Probabilistic interpretation of the equations

This section is strongly inspired by Tanaka [65] and Desvillettes-Graham-Méléard [21]. Until the end of the article, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ will designate a Polish filtered probability space satisfying the usual conditions. Such a space is Borel isomorphic to the Lebesgue space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ which we will use as an auxiliary space. To be as clear as possible, we will use the notation \mathbb{E} for the expectation and \mathcal{L} for the law of a random variable or process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and we will use the notation \mathbb{E}_α and \mathcal{L}_α for the expectation and law of random variables or processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$. The processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ will be called α -processes.

We say that a \mathbb{R} -valued process $(V_t)_{t \geq 0}$ is a L^2 -process if it is càdlàg, adapted and if $\mathbb{E}(\sup_{[0, T]} V_t^2) < \infty$ for all $T \geq 0$.

Now, we introduce a nonlinear stochastic differential equation linked with (1.1.1).

Proposition 1.3.1. *Let β be a cross section satisfying (1.2.1). Let $f_0 \in \mathcal{P}_2(\mathbb{R})$ and let $(f_t)_{t \geq 0}$ be the solution to (1.1.1) starting from f_0 . Consider any α -process $(W_t)_{t \geq 0}$ such that $\mathcal{L}_\alpha(W_t) = f_t$ for all $t \geq 0$. Let also N be a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $dsd\alpha\beta(\theta)d\theta$, and V_0 a \mathcal{F}_0 -measurable random variable with law f_0 . Then there exists a unique L^2 -process $(V_t)_{t \geq 0}$ such that for all $t \geq 0$,*

$$(1.3.1) \quad V_t = V_0 + \int_0^t \int_0^1 \int_{-\pi}^\pi [(\cos \theta - 1)V_{s-} - \sin \theta W_{s-}(\alpha)] \tilde{N}(dsd\alpha d\theta) - b \int_0^t V_s ds,$$

with b given by (1.2.3). Furthermore, $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$.

Proof. While stated in a slightly different way, this result is almost contained in Desvillettes-Graham-Méléard [21, Theorem 3.4]. See the proof of Proposition 1.3.3 below for similar arguments. \square

Let us now write down a probabilistic interpretation of (1.1.4).

Proposition 1.3.2. *Let $g_0 \in \mathcal{P}_2(\mathbb{R})$ and set $\mathcal{E} = \int_{\mathbb{R}} v^2 g_0(dv)$. Consider a \mathcal{F}_0 -measurable random variable Y_0 with law g_0 and a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $(B_t)_{t \geq 0}$. Then there exists a unique L^2 -process $(Y_t)_{t \geq 0}$ such that for all $t \geq 0$,*

$$(1.3.2) \quad Y_t = Y_0 - \frac{1}{2} \int_0^t Y_s ds + \sqrt{\mathcal{E}} B_t.$$

Furthermore, $\mathcal{L}(Y_t) = g_t$ for all $t \geq 0$, where $(g_t)_{t \geq 0}$ is the unique solution to (1.1.4).

Proof. The existence and uniqueness of Y is classical since (1.3.2) is a S.D.E. with Lipschitz coefficients (we can observe that Y is an Ornstein-Uhlenbeck process). By Itô's formula, we have for any $\varphi \in C_b^2(\mathbb{R})$

$$\varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s) \left(-\frac{1}{2} Y_s ds + \sqrt{\mathcal{E}} dB_s \right) + \frac{\mathcal{E}}{2} \int_0^t \varphi''(Y_s) ds.$$

Taking expectations and setting $\mu_t = \mathcal{L}(Y_t)$, we get for any $\varphi \in C_b^2(\mathbb{R})$

$$\begin{aligned} \int_{\mathbb{R}} \varphi(v) \mu_t(dv) &= \int_{\mathbb{R}} \varphi(v) g_0(dv) - \frac{1}{2} \int_0^t \int_{\mathbb{R}} v \varphi'(v) \mu_s(dv) ds \\ &\quad + \frac{\mathcal{E}}{2} \int_0^t \int_{\mathbb{R}} \varphi''(v) \mu_s(dv). \end{aligned}$$

Thus $(\mu_t)_{t \geq 0}$ solves (1.1.4) in the sense of Definition 1.2.1. We get $(\mu_t)_{t \geq 0} = (g_t)_{t \geq 0}$ by uniqueness (see Proposition 1.2.2). \square

It remains to give a probabilistic interpretation of (1.1.5).

Proposition 1.3.3. *Let $\epsilon \in (0, 1)$ be fixed. Consider a cross-section β satisfying (1.2.1), a probability measure $f_0^\epsilon \in \mathcal{P}_2(\mathbb{R})$, and the corresponding unique solution $(f_t^\epsilon)_{t \geq 0}$ to (1.1.5). Consider any α -process $(W_t^\epsilon)_{t \geq 0}$ such that for all $t \geq 0$, $\mathcal{L}_\alpha(W_t^\epsilon) = f_t^\epsilon$. Let V_0^ϵ be a \mathcal{F}_0 -measurable random variable with law f_0^ϵ , let N be a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta(\theta) d\theta$ and let $(B_t)_{t \geq 0}$ be a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion independent of N . Then there exists a unique L^2 -process $(V_t^\epsilon)_{t \geq 0}$ such that for all $t \geq 0$,*

$$(1.3.3) \quad V_t^\epsilon = V_0^\epsilon + \int_0^t \int_{|\theta| \geq \epsilon} \int_0^1 \left[(\cos \theta - 1) V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha) \right] N(d\theta d\alpha ds) \\ - b_\epsilon \int_0^t V_s^\epsilon ds + \sqrt{2\mathcal{E}b_\epsilon} B_t,$$

with b_ϵ defined in (1.2.8). Furthermore, $\mathcal{L}(V_t^\epsilon) = f_t^\epsilon$ for all $t \geq 0$.

Proof. See Ikeda-Watanabe [44, Theorem 9.1] for existence and uniqueness of $(V_t^\epsilon)_{t \geq 0}$: (1.3.3) is a classical jumping S.D.E. with Lipschitz coefficients. Let $\varphi \in C_b^2(\mathbb{R})$. By Itô's formula for jump processes (see e.g. Ikeda-Watanabe [44, Theorem 5.1]), we have

$$\varphi(V_t^\epsilon) = \varphi(V_0^\epsilon) + \int_0^t \varphi'(V_s^\epsilon) (-b_\epsilon V_s^\epsilon ds + \sqrt{2\mathcal{E}b_\epsilon} dB_s) + \mathcal{E}b_\epsilon \int_0^t \varphi''(V_s^\epsilon) ds \\ + \int_0^t \int_{|\theta| \geq \epsilon} \int_0^1 \left[\varphi(\cos \theta V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha)) - \varphi(V_{s-}^\epsilon) \right] N(d\theta d\alpha ds).$$

Taking expectations and setting $\mu_t^\epsilon = \mathcal{L}(V_t^\epsilon)$, we get for any $\varphi \in C_b^2(\mathbb{R})$, using that $\mathcal{L}_\alpha(W_t^\epsilon) = f_t^\epsilon$,

$$\int_{\mathbb{R}} \varphi(v) \mu_t^\epsilon(dv) = \int_{\mathbb{R}} \varphi(v) f_0^\epsilon(dv) - b_\epsilon \int_0^t \int_{\mathbb{R}} v \varphi'(v) \mu_s^\epsilon(dv) ds \\ + \mathcal{E}b_\epsilon \int_0^t \int_{\mathbb{R}} \varphi''(v) \mu_s^\epsilon(dv) ds \\ + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\theta| \geq \epsilon} \left[\varphi(v \cos \theta - v_* \sin \theta) - \varphi(v) \right] \beta(\theta) d\theta \mu_s^\epsilon(dv) f_s^\epsilon(dv_*) ds.$$

But $(f_t^\epsilon)_{t \geq 0}$ solves the same equation since it solves (1.1.5) in the sense of Definition 1.2.1. Since $(f_s^\epsilon)_{s \geq 0}$ is given, this equation is linear and we have uniqueness of the solution. Indeed, we use Proposition 1.8.6 with $a = \mathcal{E}b_\epsilon$, $b = -b_\epsilon$, $r = 0$ and $q(t, v, A) = \int_{-\pi}^{\pi} \int_{\mathbb{R}} \mathbb{1}_{|\theta| > \epsilon} \mathbb{1}_A(v(\cos \theta - 1) - v_* \sin \theta) f_t^\epsilon(dv_*) \beta(\theta) d\theta$ for all Borel subset $A \subset \mathbb{R}$, which satisfies $\sup_{t,v} q(t, v, \mathbb{R}) = \int_{|\theta| > \epsilon} \beta(\theta) d\theta < \infty$ and $\sup_{t \geq 0} \int_{\mathbb{R}} (h^2 + 2vh) q(t, v, dh) = \int_{|\theta| > \epsilon} \sin^2 \theta \beta(\theta) d\theta \left(\int_{\mathbb{R}} v_*^2 f_0(dv_*) + v^2 \right) \leq C(1 + v^2)$. Finally, we get $(\mu_t^\epsilon)_{t \geq 0} = (f_t^\epsilon)_{t \geq 0}$. \square

1.4 The Grazing collisions limit

We consider a family of cross sections $(\beta_\epsilon)_{\epsilon \in (0,1)}$ with $\int_{-\pi}^{\pi} \theta^2 \beta_\epsilon(\theta) d\theta = 1$ and $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta \xrightarrow{\epsilon \rightarrow 0} 0$. Let $g_0 \in \mathcal{P}_4(\mathbb{R})$. For any $\epsilon \in (0,1)$, we consider $(g_t^\epsilon)_{t \geq 0}$ the unique solution of (1.1.1) with cross section β_ϵ starting from g_0 . We also consider $(g_t)_{t \geq 0}$ the unique solution of (1.1.4) starting from g_0 . For $\epsilon \in (0,1)$, we consider a \mathcal{F}_0 -measurable random variable V_0 with law g_0 , and a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure N^ϵ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta_\epsilon(\theta) d\theta$. We also consider an α -process $(W_t^\epsilon)_{t \geq 0}$ such that $\mathcal{L}_\alpha(W_t^\epsilon) = g_t^\epsilon$ for all $t \geq 0$. Let $(B_t)_{t \geq 0}$ be a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion. We consider $(V_t^\epsilon)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ solutions of the following S.D.E.s

$$\begin{aligned} V_t^\epsilon = & V_0 + \int_0^t \int_0^1 \int_{-\pi}^{\pi} \left[(\cos \theta - 1) V_{s-}^\epsilon - \sin \theta W_{s-}^\epsilon(\alpha) \right] \tilde{N}^\epsilon(ds d\alpha d\theta) \\ & - b_\epsilon \int_0^t V_s^\epsilon ds, \end{aligned}$$

where $b_\epsilon = \int_{-\pi}^{\pi} (1 - \cos \theta) \beta_\epsilon(\theta) d\theta$ and

$$(1.4.1) \quad Y_t = V_0 - \frac{1}{2} \int_0^t Y_s ds + \sqrt{\mathcal{E}} B_t.$$

Theorem 1.2.3 is a corollary of the following statement.

Theorem 1.4.1. *For any $t \geq 0$ and any $\epsilon \in (0, 1)$, we can couple the Poisson*

measure N^ϵ and the Brownian motion B in such a way that

$$\begin{aligned} \mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq & 4 \left[\frac{\mathcal{E} \int_{-\pi}^{\pi} (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta}{2b_\epsilon} \right. \\ & + C \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E} \gamma_\epsilon} \\ & \left. + \mathcal{E} \left(|\ln(2b_\epsilon)|^2 + \left(\frac{\gamma_\epsilon}{2b_\epsilon} + 1 \right) |2b_\epsilon - 1| + 2|\gamma_\epsilon - 1| \right) \right], \end{aligned}$$

where $b_\epsilon = \int_{-\pi}^{\pi} (1 - \cos \theta) \beta_\epsilon(\theta) d\theta$, $\gamma_\epsilon = \int_{-\pi}^{\pi} \sin^2 \theta \beta_\epsilon(\theta) d\theta$, $\mathcal{E} = \mathbb{E}[V_0^2]$ and C is a universal constant.

Let us insist on the fact that the coupling between N^ϵ and B depends on t . Assuming for a moment that this result holds true, we can prove Theorem 1.2.3.

Proof of Theorem 1.2.3. First recalling that $\mathcal{L}(V_t^\epsilon) = g_t^\epsilon$ by Proposition 1.3.1 and $\mathcal{L}(Y_t) = g_t$ by Proposition 1.3.2, we have $W_2^2(g_t^\epsilon, g_t) \leq \mathbb{E}[(V_t^\epsilon - Y_t)^2]$. If $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta > 1$, we have $\mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq 2\mathbb{E}[(V_t^\epsilon)^2] + 2\mathbb{E}[Y_t^2] = 4\mathcal{E} \leq 4\mathcal{E} \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta$. We now suppose that $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta < 1$. Using the Taylor-Lagrange inequality, we have

$$\begin{aligned} |1 - \cos \theta| &\leq \theta^2/2, & |2(1 - \cos \theta) - \theta^2| &\leq \theta^4/12 \\ |\sin \theta| &\leq |\theta| & \text{and} & \quad |\sin^2 \theta - \theta^2| &\leq \theta^4/3. \end{aligned}$$

Using these inequalities, we get

$$\int_{-\pi}^{\pi} (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta \leq \int_{-\pi}^{\pi} \frac{\theta^4}{4} \beta_\epsilon(\theta) d\theta, \quad \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta \leq \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta,$$

and, recalling that $\int_{-\pi}^{\pi} \theta^2 \beta_\epsilon(\theta) d\theta = 1$,

$$|2b_\epsilon - 1| \leq \int_{-\pi}^{\pi} (\theta^4/12) \beta_\epsilon(\theta) d\theta, \quad |\gamma_\epsilon - 1| \leq \int_{-\pi}^{\pi} (\theta^4/3) \beta_\epsilon(\theta) d\theta.$$

Since $\int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta < 1$ by assumption, we have $2b_\epsilon \geq 11/12$, $2/3 \leq \gamma_\epsilon \leq 4/3$ and $2b_\epsilon - 1 \in [-1/12, 1/12]$ which allows us to write $|\ln(2b_\epsilon)|^2 = |\ln(1 + (2b_\epsilon - 1))|^2 \leq 4|2b_\epsilon - 1|^2$. We thus get $\mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq C(\mathcal{E} + \frac{\mathbb{E}(V_0^4)}{\mathcal{E}}) \int_{-\pi}^{\pi} \theta^4 \beta_\epsilon(\theta) d\theta$, which concludes the proof, since $\mathcal{E}^2 = \mathbb{E}[V_0^2]^2 \leq \mathbb{E}[V_0^4]$ by the Cauchy-Schwarz inequality. \square

It remains to prove Theorem 1.4.1. Let us start with the following lemma.

Lemma 1.4.2. *For $\epsilon \in (0, 1)$, let Y^ϵ be the unique solution of*

$$(1.4.2) \quad Y_t^\epsilon = V_0 - b_\epsilon \int_0^t Y_s^\epsilon ds - \int_0^t \int_0^1 \int_{-\pi}^\pi \sin \theta W_{s-}^\epsilon(\alpha) \tilde{N}^\epsilon(ds d\alpha d\theta)$$

(since W^ϵ is a given α -process, this is a classical S.D.E. with Lipschitz coefficients). Then for all $t \geq 0$,

$$\mathbb{E}\left((V_t^\epsilon - Y_t^\epsilon)^2\right) \leq \frac{\mathcal{E} \int_{-\pi}^\pi (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta}{2b_\epsilon}.$$

Proof. Observing that

$$V_t^\epsilon - Y_t^\epsilon = \int_0^t \int_0^1 \int_{-\pi}^\pi (\cos \theta - 1) V_{s-}^\epsilon \tilde{N}^\epsilon(ds d\alpha d\theta) - b_\epsilon \int_0^t (V_s^\epsilon - Y_s^\epsilon) ds,$$

we get by Itô's formula

$$\begin{aligned} (V_t^\epsilon - Y_t^\epsilon)^2 &= \int_0^t \int_0^1 \int_{-\pi}^\pi \left[(V_{s-}^\epsilon - Y_{s-}^\epsilon + (\cos \theta - 1) V_{s-}^\epsilon)^2 \right. \\ &\quad \left. - (V_{s-}^\epsilon - Y_{s-}^\epsilon)^2 \right] \tilde{N}^\epsilon(ds d\alpha d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{-\pi}^\pi \left[(V_s^\epsilon - Y_s^\epsilon + (\cos \theta - 1) V_s^\epsilon)^2 - (V_s^\epsilon - Y_s^\epsilon)^2 \right. \\ &\quad \left. - 2(V_s^\epsilon - Y_s^\epsilon)(\cos \theta - 1) V_s^\epsilon \right] ds d\alpha \beta_\epsilon(\theta) d\theta \\ &\quad - 2b_\epsilon \int_0^t (V_s^\epsilon - Y_s^\epsilon)^2 ds \\ &= M_t^\epsilon + \int_0^t \int_{-\pi}^\pi (\cos \theta - 1)^2 (V_s^\epsilon)^2 ds \beta_\epsilon(\theta) d\theta - 2b_\epsilon \int_0^t (V_s^\epsilon - Y_s^\epsilon)^2 ds, \end{aligned}$$

where M_t^ϵ is a martingale with mean 0. So using that $\mathbb{E}[(V_t^\epsilon)^2] = \int_{\mathbb{R}} v^2 g_t^\epsilon(dv) = \int_{\mathbb{R}} v^2 g_0(dv) = \mathcal{E}$ for all $t \geq 0$ by (1.2.9), we have

$$\begin{aligned} \mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2] &= \int_0^t \int_{-\pi}^\pi (1 - \cos \theta)^2 \mathbb{E}[(V_s^\epsilon)^2] \beta_\epsilon(\theta) d\theta ds \\ &\quad - 2b_\epsilon \int_0^t \mathbb{E}[(V_s^\epsilon - Y_s^\epsilon)^2] ds \\ &= \mathcal{E} t \int_{-\pi}^\pi (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta - 2b_\epsilon \int_0^t \mathbb{E}[(V_s^\epsilon - Y_s^\epsilon)^2] ds. \end{aligned}$$

Differentiating this equality with respect to t , we find an O.D.E. that can be solved explicitly. This gives

$$\mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2] = \frac{\mathcal{E} \int_{-\pi}^{\pi} (1 - \cos \theta)^2 \beta_\epsilon(\theta) d\theta}{2b_\epsilon} (1 - e^{-2b_\epsilon t}).$$

The conclusion follows. \square

In the following lemma, using Corollary 1.8.2, we will find a suitable coupling between our Poisson measure N^ϵ and our Brownian motion B .

Lemma 1.4.3. *Let \tilde{Y}^ϵ be the unique solution of*

$$(1.4.3) \quad \tilde{Y}_t^\epsilon = V_0 - b_\epsilon \int_0^t \tilde{Y}_s^\epsilon ds + \sqrt{\mathcal{E}\gamma_\epsilon} B_t.$$

We consider the process Y^ϵ defined in Lemma 1.4.2. For any $\epsilon \in (0, 1)$ and for each $t \geq 0$, we can couple the Poisson measure N^ϵ and the Brownian motion B in such a way that

$$\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t^\epsilon)^2] \leq C \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E}\gamma_\epsilon},$$

where C is a universal constant and $\gamma_\epsilon = \int_{-\pi}^{\pi} \sin^2 \theta \beta_\epsilon(\theta) d\theta$.

Observe that for each t we need a suitable coupling. We are not able to find a coupling working simultaneously for all values of t .

Proof. Applying Itô's formula, we get $\tilde{Y}_t^\epsilon e^{b_\epsilon t} = V_0 + \sqrt{\mathcal{E}\gamma_\epsilon} \int_0^t e^{b_\epsilon s} dB_s$ and $Y_t^\epsilon e^{b_\epsilon t} = V_0 - \int_0^t \int_0^1 \int_{-\pi}^{\pi} e^{b_\epsilon s} \sin \theta W_{s-}^\epsilon(\alpha) \tilde{N}^\epsilon(ds d\alpha d\theta)$. We observe that the random variable $\sqrt{\mathcal{E}\gamma_\epsilon} \int_0^t e^{b_\epsilon s} dB_s$ follows a centered normal law with variance $\mathcal{E}\gamma_\epsilon \int_0^t e^{2b_\epsilon s}$ which is equal to $\int_0^t \int_0^1 \int_{-\pi}^{\pi} e^{2b_\epsilon s} \sin^2 \theta (W_s^\epsilon(\alpha))^2 \beta_\epsilon(\theta) d\theta d\alpha ds$ because $\mathcal{L}_\alpha(W_s^\epsilon) = g_s^\epsilon$ and due to (1.2.9). So using Corollary 1.8.2, we get

$$\begin{aligned} W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon e^{b_\epsilon t}), \mathcal{L}(Y_t^\epsilon e^{b_\epsilon t})) &\leq C_0 \frac{\int_0^t \int_0^1 \int_{-\pi}^{\pi} e^{4b_\epsilon s} \sin^4 \theta (W_s^\epsilon(\alpha))^4 \beta_\epsilon(\theta) ds d\alpha d\theta}{\mathcal{E}\gamma_\epsilon \int_0^t e^{2b_\epsilon s} ds} \\ &= C_0 \frac{\int_0^t e^{4b_\epsilon s} \mathbb{E}_\alpha((W_s^\epsilon)^4) ds \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E}\gamma_\epsilon \int_0^t e^{2b_\epsilon s} ds}. \end{aligned}$$

Using Lemma 1.8.5, since $\mathcal{L}(W_s^\epsilon) = g_s^\epsilon$ and since g^ϵ solves (1.1.1) (with the cross section β_ϵ), we deduce $\mathbb{E}_\alpha[(W_s^\epsilon)^2] \leq \int_{\mathbb{R}} v^4 g_0(dv) + 3\mathcal{E}^2 = \mathbb{E}[V_0^4] + 3\mathcal{E}^2$ for all $s \geq 0$. Hence, using that $\frac{\int_0^t e^{4b_\epsilon s} ds}{\int_0^t e^{2b_\epsilon s} ds} \leq e^{2b_\epsilon t}$, we have

$$W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon e^{b_\epsilon t}), \mathcal{L}(Y_t^\epsilon e^{b_\epsilon t})) \leq C_0 \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta e^{2b_\epsilon t}}{\mathcal{E}\gamma_\epsilon}.$$

Consequently,

$$W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon), \mathcal{L}(Y_t^\epsilon)) \leq C_0 \frac{(\mathbb{E}(V_0^4) + 3\mathcal{E}^2) \int_{-\pi}^{\pi} \sin^4 \theta \beta_\epsilon(\theta) d\theta}{\mathcal{E}\gamma_\epsilon}.$$

To conclude, it suffices to take N^ϵ and B in such a way that $\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t^\epsilon)^2] = W_2^2(\mathcal{L}(\tilde{Y}_t^\epsilon), \mathcal{L}(Y_t^\epsilon))$. \square

Let us now give the last lemma needed to prove Theorem 1.4.1.

Lemma 1.4.4. *Consider the unique solutions Y and \tilde{Y}^ϵ to (1.4.1) and (1.4.3) respectively, driven by the same Brownian motion B . Then for all $t \geq 0$ fixed and for all $\epsilon \in (0, 1)$,*

$$\mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t)^2] \leq \mathcal{E} \left(|\ln(2b_\epsilon)|^2 + \left(\frac{\gamma_\epsilon}{2b_\epsilon} + 1 \right) |2b_\epsilon - 1| + 2|\gamma_\epsilon - 1| \right).$$

Proof. We have $\tilde{Y}_t^\epsilon = V_0 e^{-b_\epsilon t} + \sqrt{\mathcal{E}\gamma_\epsilon} e^{-b_\epsilon t} \int_0^t e^{b_\epsilon s} dB_s$ and $Y_t = V_0 e^{-t/2} + \sqrt{\mathcal{E}} e^{-t/2} \int_0^t e^{s/2} dB_s$ as in the proof of Lemma 1.4.3. Since B and V_0 are independent, we have

$$\begin{aligned} \mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t)^2] &= \mathbb{E}(V_0^2)(e^{-b_\epsilon t} - e^{-t/2})^2 \\ &\quad + \mathbb{E} \left[\left(\int_0^t (\sqrt{\mathcal{E}\gamma_\epsilon} e^{-b_\epsilon t} e^{b_\epsilon s} - \sqrt{\mathcal{E}} e^{-t/2} e^{s/2}) dB_s \right)^2 \right] \\ &= \mathcal{E}(e^{-b_\epsilon t} - e^{-t/2})^2 + \mathcal{E} \int_0^t (\sqrt{\gamma_\epsilon} e^{-b_\epsilon(t-s)} - e^{-(t-s)/2})^2 ds \\ &= \mathcal{E}(e^{-b_\epsilon t} - e^{-t/2})^2 + \mathcal{E} \int_0^t (\sqrt{\gamma_\epsilon} e^{-b_\epsilon s} - e^{-s/2})^2 ds. \end{aligned}$$

We set $h(t) = (e^{-b_\epsilon t} - e^{-t/2})^2$. The function h reaches its maximum at $t_0 = \ln(2b_\epsilon)/(b_\epsilon - 1/2)$. Moreover, $|h(t_0)| = |e^{-b_\epsilon t_0} - e^{-t_0/2}|^2 \leq |b_\epsilon t_0 - t_0/2|^2 = |t_0|^2 |b_\epsilon - 1/2|^2 = |\ln(2b_\epsilon)|^2$. Next,

$$\begin{aligned}
\int_0^t (\sqrt{\gamma_\epsilon} e^{-b_\epsilon s} - e^{-s/2})^2 ds &\leq \int_0^\infty (\sqrt{\gamma_\epsilon} e^{-b_\epsilon s} - e^{-s/2})^2 ds \\
&= \frac{\gamma_\epsilon}{2b_\epsilon} + 1 - \frac{2\sqrt{\gamma_\epsilon}}{b_\epsilon + 1/2} \\
&= \frac{\gamma_\epsilon(b_\epsilon + 1/2) + 2b_\epsilon(b_\epsilon + 1/2) - 4\sqrt{\gamma_\epsilon}b_\epsilon}{2b_\epsilon(b_\epsilon + 1/2)} \\
&\leq \frac{1}{b_\epsilon} \left[\gamma_\epsilon(1/2 - b_\epsilon) + 2b_\epsilon(\gamma_\epsilon + b_\epsilon + 1/2 - 2\sqrt{\gamma_\epsilon}) \right] \\
&= \frac{\gamma_\epsilon}{2b_\epsilon} (1 - 2b_\epsilon) + 2 \left[(\sqrt{\gamma_\epsilon} - 1)^2 + (b_\epsilon - 1/2) \right] \\
&\leq \left(\frac{\gamma_\epsilon}{2b_\epsilon} + 1 \right) |2b_\epsilon - 1| + 2|\gamma_\epsilon - 1|,
\end{aligned}$$

the last inequality coming from $(\sqrt{x} - \sqrt{y})^2 \leq |x - y|$. The lemma is proved. \square

We can now conclude this section.

Proof of Theorem 1.4.1. For $\epsilon \in (0, 1)$ and $t \geq 0$ fixed, we take the Poisson measure N^ϵ and the Brownian motion B as in Lemma 1.4.3 and we consider the processes V^ϵ , Y , Y^ϵ and \tilde{Y}^ϵ build with this N^ϵ and this B . Then, writing

$$\mathbb{E}[(V_t^\epsilon - Y_t)^2] \leq 4 \left[\mathbb{E}[(V_t^\epsilon - Y_t^\epsilon)^2] + \mathbb{E}[(Y_t^\epsilon - \tilde{Y}_t^\epsilon)^2] + \mathbb{E}[(\tilde{Y}_t^\epsilon - Y_t)^2] \right],$$

and using Lemmas 1.4.2, 1.4.3 and 1.4.4, we immediately conclude. \square

1.5 Cutoff approximation with diffusion

The whole section is dedicated to the proof of Theorem 1.2.5. Let thus $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (1.2.1). We fix $\epsilon \in (0, 1)$, and we consider the solutions $(f_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ to (1.1.1) and (1.1.5) respectively, both starting from f_0 .

We will proceed as follows. We fix some $t_0 \geq 0$ for the whole proof. We will build some solutions $(V_t)_{t \geq 0}$ and $(V_t^\epsilon)_{t \geq 0}$ to (1.3.1) and (1.3.3), both

starting from some initial value V_0 with law f_0 , coupled in such a way that $\mathbb{E}[(V_{t_0} - V_{t_0}^\epsilon)^2]$ is as small as possible.

We divide the proof into five steps. In the first step, we introduce the (suitably coupled) processes $(V_t)_{t \geq 0}$, $(V_t^\epsilon)_{t \geq 0}$ as well as an intermediate process $(\tilde{V}_t^\epsilon)_{t \geq 0}$. In Step 2, we upperbound $\mathbb{E}[(V_{t_0}^\epsilon - \tilde{V}_{t_0}^\epsilon)^2]$. Step 3 is dedicated to the study of $\mathbb{E}[(\tilde{V}_{t_0}^\epsilon - V_{t_0})^2]$. In Step 4, we show that $\mathbb{E}[(V_{t_0}^\epsilon - \tilde{V}_{t_0}^\epsilon)(\tilde{V}_{t_0}^\epsilon - V_{t_0})] = 0$. We conclude in Step 5.

In the whole section, we will use the notation

$$(1.5.1) \quad b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta, \quad c_\epsilon = 2b_\epsilon + \int_{|\theta| \geq \epsilon} \sin^2 \theta \beta(\theta) d\theta, \\ d_\epsilon = \int_{|\theta| \geq \epsilon} \sin^2 \theta \beta(\theta) d\theta \quad \text{and} \quad \gamma_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta)^2 \beta(\theta) d\theta.$$

Step 1: the coupling.

- Let $(\Omega_i, \mathcal{F}^i, (\mathcal{F}_t^i)_{t \geq 0}, \mathbb{P}_i)$, $i = 1, 2$, be two Polish filtered probability spaces satisfying the usual conditions and consider the following filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, (\mathcal{F}_t^1 \otimes \mathcal{F}_t^2)_{t \geq 0}, \mathbb{P}_1 \otimes \mathbb{P}_2)$. We denote by \mathbb{E} the expectation under \mathbb{P} and by \mathbb{E}_i the expectation under \mathbb{P}_i .

- On $(\Omega_1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t \geq 0}, \mathbb{P}_1)$, we consider a f_0 -distributed random variable V_0 \mathcal{F}_0^1 -measurable, as well as a $(\mathcal{F}_t^1)_{t \geq 0}$ -Poisson measure $N_{|\theta| \geq \epsilon}$ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $ds d\alpha \beta(\theta) \mathbb{1}_{|\theta| \geq \epsilon} d\theta$. We set

$$(1.5.2) \quad X_t = \int_0^t \int_0^1 \int_{-\pi}^\pi (\cos \theta - 1) N_{|\theta| \geq \epsilon}(ds d\alpha d\theta) - b_\epsilon t,$$

We consider the Doléans-Dade exponential of X , see Jacod-Shiryaev [46, Theorem 4.61], defined by

$$(1.5.3) \quad Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

There holds

$$(1.5.4) \quad Z_t = e^{X_t} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} = e^{-b_\epsilon t} \prod_{i \geq 1} \cos \theta_i \mathbb{1}_{T_i \leq t},$$

where $(T_i, \theta_i, \alpha_i)_{i \geq 1}$ are the marks of the Poisson measure $N_{|\theta| \geq \epsilon}$. Observe that a.s., $Z_t \neq 0 \forall t \geq 0$, because $\beta(\theta)d\theta$ does not give weight to $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$.

Of course, the processes $(X_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ depend on ϵ but we do not write this dependence in order to lighten notations.

- For each $t \geq 0$, we consider some α -random variables W_t and W_t^ϵ with respective laws f_t and f_t^ϵ verifying

$$(1.5.5) \quad W_2^2(f_t, f_t^\epsilon) = \mathbb{E}_\alpha[(W_t - W_t^\epsilon)^2].$$

- Recall that $t_0 \geq 0$ is fixed. Fix also $\omega_1 \in \Omega_1$. On $(\Omega_2, \mathcal{F}^2, (\mathcal{F}_t^2)_{t \geq 0}, \mathbb{P}_2)$, we consider a $(\mathcal{F}_t^2)_{t \geq 0}$ -Poisson measure $N_{|\theta| < \epsilon}^{\omega_1}$ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $dsd\alpha\beta(\theta)\mathbb{1}_{|\theta| < \epsilon}d\theta$ and a Brownian motion $(B_t^{\omega_1})_{t \geq 0}$ (we do not write the dependence in t_0 and ϵ) such that:

$$(1.5.6) \quad \begin{aligned} & W_2^2(\mu_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \\ &= \mathbb{E}_2 \left[\left(\int_0^{t_0} \int_0^1 \int_{-\pi}^\pi (Z_{t_0} Z_{s-}^{-1})(\omega_1) \sin(-\theta) W_{s-}(\alpha) \tilde{N}_{|\theta| < \epsilon}^{\omega_1}(\omega_2, dsd\alpha d\theta) \right. \right. \\ & \quad \left. \left. - \int_0^{t_0} \sqrt{2\mathcal{E}b_\epsilon}(Z_{t_0} Z_s^{-1})(\omega_1) dB_s^{\omega_1}(\omega_2) \right)^2 \right], \end{aligned}$$

where

$$(1.5.7) \quad \mu_{t_0}^{\omega_1} = \mathcal{L}_2 \left(\int_0^{t_0} \int_0^1 \int_{-\pi}^\pi (Z_{t_0} Z_{s-}^{-1})(\omega_1) \sin(-\theta) W_{s-}(\alpha) \tilde{N}_{|\theta| < \epsilon}^{\omega_1}(\omega_2, dsd\alpha d\theta) \right),$$

$$(1.5.8) \quad \nu_{t_0}^{\omega_1} = \mathcal{L}_2 \left(\int_0^{t_0} \sqrt{2\mathcal{E}b_\epsilon}(Z_{t_0} Z_s^{-1})(\omega_1) dB_s^{\omega_1}(\omega_2) \right).$$

Here again we do not write the dependence in ϵ of $\mu_{t_0}^{\omega_1}$ and $\nu_{t_0}^{\omega_1}$.

- For $(\omega_1, \omega_2) \in \Omega$, we can now set $N(\omega_1, \omega_2) = N_{|\theta| \geq \epsilon}(\omega_1) + N_{|\theta| < \epsilon}^{\omega_1}(\omega_2)$ and $(B_t(\omega_1, \omega_2))_{t \geq 0} = (B_t^{\omega_1}(\omega_2))_{t \geq 0}$. Clearly, as random objects on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, the process $(B_t)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion and N is a $(\mathcal{F}_t)_{t \geq 0}$ -Poisson measure on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $dsd\alpha\beta(\theta)d\theta$.

- Setting $\mathcal{E} := \mathbb{E}[V_0^2]$, for $0 < \epsilon < 1$, we consider the processes $(V_t)_{t \geq 0}$, $(V_t^\epsilon)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ solutions to (1.3.1) and (1.3.3) with B ,

N , W , W^ϵ defined previously, both starting from V_0 . We also introduce the process $(\tilde{V}_t^\epsilon)_{t \geq 0}$ solution of the following S.D.E.:

$$(1.5.9) \quad \begin{aligned} \tilde{V}_t^\epsilon = & V_0 + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos \theta - 1) \tilde{V}_{s-}^\epsilon - \sin \theta W_{s-}(\alpha) \right] N(ds d\alpha d\theta) \\ & - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - b_\epsilon \int_0^t \tilde{V}_s^\epsilon ds. \end{aligned}$$

By Proposition 1.3.1 and Proposition 1.3.3, $\mathcal{L}(V_t)$ and $\mathcal{L}(V_t^\epsilon)$ are nothing but f_t and f_t^ϵ respectively. We set $\Delta_t^\epsilon = V_t - \tilde{V}_t^\epsilon$, $\tilde{\Delta}_t^\epsilon = \tilde{V}_t^\epsilon - V_t^\epsilon$ and $\delta_t^\epsilon(\alpha) = W_s(\alpha) - W_s^\epsilon(\alpha)$.

Step 2: the aim is here to prove that

$$(1.5.10) \quad \mathbb{E} \left((\tilde{\Delta}_{t_0}^\epsilon)^2 \right) \leq d_\epsilon e^{-c_\epsilon t_0} \int_0^{t_0} e^{c_\epsilon s} \mathbb{E}_\alpha(\delta_s^2) ds + C\epsilon^2,$$

where C depends only on \mathcal{E} and $\mathbb{E}[V_0^4]$ and where c_ϵ and d_ϵ are defined in (1.5.1). Making the difference between (1.5.9) and (1.3.3), we get

$$(1.5.11) \quad \begin{aligned} \tilde{\Delta}_t^\epsilon = & \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos \theta - 1) \tilde{\Delta}_{s-}^\epsilon - \sin \theta \delta_{s-}^\epsilon(\alpha) \right] N(ds d\alpha d\theta) - b_\epsilon \int_0^t \tilde{\Delta}_s^\epsilon ds \\ & - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - \sqrt{2\mathcal{E}b_\epsilon} B_t \\ = & H_t + \int_0^t \tilde{\Delta}_{s-}^\epsilon dX_s, \end{aligned}$$

with $(X_t)_{t \geq 0}$ defined in (1.5.2) and with

$$\begin{aligned} H_t = & - \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \sin \theta \delta_{s-}^\epsilon(\alpha) N(ds d\alpha d\theta) \\ & - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - \sqrt{2\mathcal{E}b_\epsilon} B_t. \end{aligned}$$

We do not write the dependence in ϵ for H . According to Jacod [45], $\tilde{\Delta}_t^\epsilon = (L_t + D_t)Z_t$, where Z_t was defined in (1.5.4) and where

$$(1.5.12) \quad D_t = - \int_0^t \int_0^1 \int_{|\theta| < \epsilon} Z_{s-}^{-1} \sin \theta W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) - \sqrt{2\mathcal{E}b_\epsilon} \int_0^t Z_s^{-1} dB_s$$

and

$$(1.5.13) \quad L_t = - \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} Z_{s-}^{-1} \frac{\sin \theta}{\cos \theta} \delta_{s-}^\epsilon(\alpha) N(ds d\alpha d\theta).$$

To verify this, it suffices to apply Itô's formula and observe that the process $((L_t + D_t)Z_t)_{t \geq 0}$ satisfies the same S.D.E. than $(\tilde{\Delta}_t^\epsilon)_{t \geq 0}$, i.e $(L_t + D_t)Z_t = H_t + \int_0^t (L_{s-} + D_{s-})Z_{s-} dX_s$. This S.D.E. has Lipschitz coefficients and thus has a unique solution. The processes $(D_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ depend on ϵ but we do not write this dependence.

Hence

$$(1.5.14) \quad \mathbb{E}[(\tilde{\Delta}_t^\epsilon)^2] = \mathbb{E}[L_t^2 Z_t^2] + \mathbb{E}[D_t^2 Z_t^2] + 2\mathbb{E}[L_t D_t Z_t^2].$$

- First,

$$(1.5.15) \quad \mathbb{E}[L_t D_t Z_t^2] = \mathbb{E}_1[\mathbb{E}_2(L_t D_t Z_t^2)] = \mathbb{E}_1[L_t Z_t^2 \mathbb{E}_2(D_t)] = 0,$$

because for $\omega = (\omega_1, \omega_2)$, we have $(L_t D_t Z_t^2)(\omega) = (L_t Z_t^2)(\omega_1) D_t(\omega_1, \omega_2)$ and because for ω_1 fixed, $\mathbb{E}_2[D_t(\omega_1, \omega_2)] = 0$. Indeed, recall that $(L_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ depend only on ω_1 and that for ω_1 fixed, $N_{|\theta| < \epsilon}(\omega_1, \omega_2)$ is a Poisson measure while $(B_t(\omega_1, \omega_2))_{t \geq 0}$ is a Brownian motion on $(\Omega_2, \mathcal{F}^2, \mathcal{F}_t^2, \mathbb{P}_2)$, so that $(D_t(\omega_1, \omega_2))_{t \geq 0}$ is a centered martingale (for ω_1 fixed).

- By Itô's formula, we have

$$\begin{aligned} Z_t^2 L_t^2 &= -2b_\epsilon \int_0^t Z_s^2 L_s^2 ds \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[\left(Z_{s-} + (\cos \theta - 1) Z_{s-} \right)^2 \left(L_{s-} - \frac{\sin \theta}{\cos \theta} Z_{s-}^{-1} \delta_{s-}^\epsilon(\alpha) \right)^2 \right. \\ &\quad \quad \left. - Z_{s-}^2 L_{s-}^2 \right] N(ds d\alpha d\theta) \\ &= -2b_\epsilon \int_0^t Z_s^2 L_s^2 ds \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos^2 \theta - 1) Z_{s-}^2 L_{s-}^2 + \sin^2 \theta (\delta_{s-}^\epsilon(\alpha))^2 \right. \\ &\quad \quad \left. - 2 \cos \theta \sin \theta Z_{s-} L_{s-} \delta_{s-}^\epsilon(\alpha) \right] N(ds d\alpha d\theta). \end{aligned}$$

Taking expectations and recalling (1.5.1), we get (use that $\cos \theta \sin \theta \beta(\theta) d\theta$ is odd)

$$\mathbb{E}(Z_t^2 L_t^2) = -c_\epsilon \int_0^t \mathbb{E}(Z_s^2 L_s^2) ds + d_\epsilon \int_0^t \mathbb{E}_\alpha[(\delta_s^\epsilon)^2] ds.$$

Solving this differential equation, we find

$$(1.5.16) \quad \mathbb{E}(Z_t^2 L_t^2) = d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \mathbb{E}_\alpha[(\delta_s^\epsilon)^2] ds.$$

- It remains to compute $\mathbb{E}(Z_{t_0}^2 D_{t_0}^2)$. Recalling (1.5.12), we directly obtain

$$\begin{aligned} \mathbb{E}(Z_{t_0}^2 D_{t_0}^2) &= \mathbb{E} \left[\left(Z_{t_0} \int_0^{t_0} \int_0^1 \int_{|\theta| < \epsilon} Z_{s-}^{-1} \sin(-\theta) W_{s-}(\alpha) \tilde{N}(ds d\alpha d\theta) \right. \right. \\ &\quad \left. \left. - Z_{t_0} \int_0^{t_0} Z_s^{-1} \sqrt{2\mathcal{E}b_\epsilon} dB_s \right)^2 \right] \\ &= \mathbb{E}_1 \left[\mathbb{E}_2 \left[\left(\int_0^{t_0} \int_0^1 \int_{-\pi}^\pi (Z_{t_0} Z_{s-}^{-1})(\omega_1) \sin(-\theta) W_{s-}(\alpha) \tilde{N}_{|\theta| < \epsilon}^{\omega_1}(\omega_2, ds d\alpha d\theta) \right. \right. \right. \\ &\quad \left. \left. - \int_0^{t_0} (Z_{t_0} Z_s^{-1})(\omega_1) \sqrt{2\mathcal{E}b_\epsilon} dB_s^{\omega_1}(\omega_2) \right)^2 \right] \right]. \end{aligned}$$

We thus obtain $\mathbb{E}(Z_{t_0}^2 D_{t_0}^2) = \mathbb{E}_1(W_2^2(\mu_{t_0}, \nu_{t_0}))$, recall (1.5.6). We consider

$$(1.5.17) \quad \eta_{t_0}^{\omega_1} := \mathcal{L}_2 \left(\sigma_\epsilon \int_0^{t_0} (Z_{t_0} Z_s^{-1})(\omega_1) dB_s^{\omega_1}(\omega_2) \right),$$

with

$$\sigma_\epsilon = \sqrt{\mathcal{E} \int_{|\theta| < \epsilon} \sin^2 \theta \beta(\theta) d\theta}.$$

Using the triangular inequality, we have

$$W_2^2(\mu_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \leq 2(W_2^2(\mu_{t_0}^{\omega_1}, \eta_{t_0}^{\omega_1}) + W_2^2(\eta_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1})).$$

By Corollary 1.8.2 and since $\mathbb{E}_\alpha(W_s^2) = \mathcal{E}$ for all $s \geq 0$ by the energy conservation, we have (recall that ω_1 is fixed)

$$W_2^2(\mu_{t_0}^{\omega_1}, \eta_{t_0}^{\omega_1}) \leq C_0 \frac{\int_0^{t_0} \int_{|\theta| < \epsilon} (Z_{t_0} Z_s^{-1})^4(\omega_1) \sin^4 \theta \mathbb{E}_\alpha(W_s^4) \beta(\theta) d\theta ds}{\int_0^{t_0} \int_{|\theta| < \epsilon} (Z_{t_0} Z_s^{-1})^2(\omega_1) \sin^2 \theta \mathcal{E} \beta(\theta) d\theta ds}.$$

But $\mathbb{E}_\alpha(W_s^4) \leq \mathbb{E}(V_0^4) + 3\mathcal{E}^2$ by Lemma 1.8.5. Furthermore, recalling (1.5.4), we have $|Z_{t_0}Z_s^{-1}| = |e^{-b_\epsilon(t_0-s)} \prod_{s < T_i \leq t} \cos \theta_i| \leq 1$. Since finally $\sin^2 \theta \leq \theta \leq \epsilon^2$ on $[-\epsilon, \epsilon]$, we easily deduce that for all ω_1 fixed,

$$W_2^2(\mu_{t_0}^{\omega_1}, \eta_{t_0}^{\omega_1}) \leq \frac{C_0(\mathbb{E}(V_0^4) + 3\mathcal{E}^2)}{\mathcal{E}} \epsilon^2.$$

Finally, it obviously holds, recall (1.5.8) and (1.5.17), that for all ω_1 fixed,

$$\begin{aligned} W_2^2(\eta_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) &\leq \int_0^{t_0} \left(\sqrt{2\mathcal{E}b_\epsilon} - \sqrt{\mathcal{E} \int_{|\theta| < \epsilon} \sin^2 \theta \beta(\theta) d\theta} \right)^2 (Z_s^{-1} Z_{t_0})^2(\omega_1) ds \\ &\leq \mathcal{E} \left| 2b_\epsilon - \int_{|\theta| < \epsilon} \sin^2 \theta \beta(\theta) d\theta \right| \int_0^{t_0} (Z_s^{-1} Z_{t_0})^2(\omega_1) ds. \end{aligned}$$

We used that $(\sqrt{x} - \sqrt{y})^2 \leq |x - y|$. Recalling that $|Z_s^{-1} Z_{t_0}| \leq e^{-b_\epsilon(t_0-s)}$, we easily get $\int_0^{t_0} (Z_s^{-1} Z_{t_0})^2(\omega_1) ds \leq \frac{1}{2b_\epsilon}$. Furthermore, $\left| 2b_\epsilon - \int_{|\theta| < \epsilon} \sin^2 \theta \beta(\theta) d\theta \right| = \left| \int_{|\theta| < \epsilon} (2(1 - \cos \theta) - \sin^2 \theta) \beta(\theta) d\theta \right| \leq \int_{|\theta| < \epsilon} \theta^4 \beta(\theta) d\theta$. Finally, it is easily checked that, since $\epsilon \in (0, 1)$, $b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta \geq \frac{1}{4} \int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta$. Hence it holds that for all ω_1 fixed,

$$W_2^2(\eta_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \leq 2\mathcal{E} \frac{\int_{|\theta| < \epsilon} \theta^4 \beta(\theta) d\theta}{\int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta} \leq 2\mathcal{E}\epsilon^2.$$

We conclude that $W_2^2(\mu_{t_0}^{\omega_1}, \nu_{t_0}^{\omega_1}) \leq C\epsilon^2$ (where C depends on \mathcal{E} and $\mathbb{E}[V_0^4]$), whence

$$(1.5.18) \quad \mathbb{E}[Z_{t_0}^2 D_{t_0}^2] \leq C\epsilon^2.$$

Gathering (1.5.14), (1.5.15), (1.5.16) and (1.5.18), we deduce (1.5.10).

Step 3: in this step, we check that

$$(1.5.19) \quad \mathbb{E}[(\Delta_t^\epsilon)^2] \leq \frac{\mathcal{E}}{4} \epsilon^2$$

for all $t \geq 0$. We first observe that (1.3.1) can be rewritten as

$$\begin{aligned} V_t = V_0 &+ \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos \theta - 1) V_{s-} - \sin \theta W_{s-}(\alpha) \right] N(ds d\alpha d\theta) \\ &+ \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[(\cos \theta - 1) V_{s-} - \sin \theta W_{s-}(\alpha) \right] \tilde{N}(ds d\alpha d\theta) - b_\epsilon \int_0^t V_s ds. \end{aligned}$$

Hence, making the difference with (1.5.9), we find

$$(1.5.20) \quad \begin{aligned} \Delta_t^\epsilon &= \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} (\cos \theta - 1) \Delta_{s-}^\epsilon N(ds d\alpha d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} (\cos \theta - 1) V_{s-} \tilde{N}(ds d\alpha d\theta) - b_\epsilon \int_0^t \Delta_s^\epsilon ds. \end{aligned}$$

Applying Itô's formula, we get

$$\begin{aligned} (\Delta_t^\epsilon)^2 &= -2b_\epsilon \int_0^t (\Delta_s^\epsilon)^2 ds + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} (\cos^2 \theta - 1) (\Delta_{s-}^\epsilon)^2 N(ds d\alpha d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[(\Delta_{s-}^\epsilon + (\cos \theta - 1) V_{s-})^2 - (\Delta_{s-}^\epsilon)^2 \right] \tilde{N}(ds d\alpha d\theta) \\ &\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[(\Delta_s^\epsilon + (\cos \theta - 1) V_s)^2 - (\Delta_s^\epsilon)^2 \right. \\ &\quad \quad \left. - 2\Delta_s^\epsilon (\cos \theta - 1) V_s \right] \beta(\theta) ds d\alpha d\theta \\ &= - \left(2b_\epsilon \int_0^t (\Delta_s^\epsilon)^2 ds + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \sin^2 \theta (\Delta_{s-}^\epsilon)^2 N(ds d\alpha d\theta) \right) + M_t \\ &\quad + \int_0^t \int_{|\theta| < \epsilon} V_s^2 (1 - \cos \theta)^2 \beta(\theta) d\theta ds, \end{aligned}$$

where $(M_t)_{t \geq 0}$ is a centered martingale. Taking expectations, this yields, recalling (1.5.1) and that $\mathbb{E}[V_s^2] = \mathcal{E}$ for all $s \geq 0$ by the energy conservation,

$$\mathbb{E}[(\Delta_t^\epsilon)^2] = -c_\epsilon \int_0^t \mathbb{E}[(\Delta_s^\epsilon)^2] ds + \gamma_\epsilon \mathcal{E} t.$$

Thus

$$\mathbb{E}[(\Delta_t^\epsilon)^2] = \frac{\gamma_\epsilon \mathcal{E}}{c_\epsilon} (1 - e^{-c_\epsilon t}) \leq \frac{\gamma_\epsilon \mathcal{E}}{c_\epsilon}.$$

But $\gamma_\epsilon \leq b_\epsilon \frac{\epsilon^2}{2}$ (because for $|\theta| < \epsilon$, $(1 - \cos \theta) \leq \theta^2/2 < \epsilon^2/2$) and $c_\epsilon \geq 2b_\epsilon$. We deduce that $\gamma_\epsilon/c_\epsilon \leq \epsilon^2/4$ and finally get (1.5.19).

Step 4: we now check that $\mathbb{E}[\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon] = 0$ for all $t \geq 0$. Applying Itô's

formula, using (1.5.20) and (1.5.11), we have

$$\begin{aligned}
\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon &= -\sqrt{2\mathcal{E}b_\epsilon} \int_0^t \Delta_s^\epsilon dB_s - 2b_\epsilon \int_0^t \Delta_s^\epsilon \tilde{\Delta}_s^\epsilon ds \\
&\quad + \int_0^t \int_0^1 \int_{|\theta| \geq \epsilon} \left[(\cos \theta \Delta_{s-}^\epsilon) (\cos \theta \tilde{\Delta}_{s-}^\epsilon - \sin \theta \delta_{s-}^\epsilon(\alpha)) \right. \\
&\quad \quad \left. - \Delta_{s-}^\epsilon \tilde{\Delta}_{s-}^\epsilon \right] N(ds d\alpha d\theta) \\
&\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[\left(\Delta_{s-}^\epsilon + (\cos \theta - 1)V_{s-} \right) \left(\tilde{\Delta}_{s-}^\epsilon - \sin \theta W_{s-}(\alpha) \right) \right. \\
&\quad \quad \left. - \Delta_{s-}^\epsilon \tilde{\Delta}_{s-}^\epsilon \right] \tilde{N}(ds d\alpha d\theta) \\
&\quad + \int_0^t \int_0^1 \int_{|\theta| < \epsilon} \left[\left(\Delta_s^\epsilon + (\cos \theta - 1)V_s \right) \left(\tilde{\Delta}_s^\epsilon - \sin \theta W_s(\alpha) \right) \right. \\
&\quad \quad \left. - \Delta_s^\epsilon \tilde{\Delta}_s^\epsilon - \tilde{\Delta}_s^\epsilon (\cos \theta - 1)V_s + \Delta_s^\epsilon \sin \theta W_s(\alpha) \right] \beta(\theta) ds d\alpha d\theta.
\end{aligned}$$

Taking expectation and using that β is even, we get

$$\mathbb{E}[\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon] = -2b_\epsilon \int_0^t \mathbb{E}[\Delta_s^\epsilon \tilde{\Delta}_s^\epsilon] ds + \int_0^t \int_{|\theta| \geq \epsilon} (\cos^2 \theta - 1) \mathbb{E}[\Delta_s^\epsilon \tilde{\Delta}_s^\epsilon] \beta(\theta) d\theta ds.$$

So the function $t \mapsto \mathbb{E}[\Delta_t^\epsilon \tilde{\Delta}_t^\epsilon]$ solves the O.D.E. $y' = -c_\epsilon y$, see (1.5.1). Since $y(0) = 0$, we easily conclude.

Step 5: conclusion. Using Steps 2, 3 and 4, we find that

$$\begin{aligned}
\mathbb{E}[(V_{t_0} - V_{t_0}^\epsilon)^2] &= \mathbb{E}[(\Delta_{t_0}^\epsilon)^2] + \mathbb{E}[(\tilde{\Delta}_{t_0}^\epsilon)^2] + 2\mathbb{E}[\Delta_{t_0}^\epsilon \tilde{\Delta}_{t_0}^\epsilon] \\
&\leq d_\epsilon e^{-c_\epsilon t_0} \int_0^{t_0} e^{c_\epsilon s} \mathbb{E}_\alpha[(\delta_s^\epsilon)^2] ds + K\epsilon^2,
\end{aligned}$$

where K depends only on \mathcal{E} and $\mathbb{E}[V_0^4]$. We set $u(t) = W_2^2(f_t, f_t^\epsilon) = \mathbb{E}_\alpha[(\delta_s^\epsilon)^2]$ by (1.5.5). Since $\mathcal{L}(V_{t_0}) = f_{t_0}$ and $\mathcal{L}(V_{t_0}^\epsilon) = f_{t_0}^\epsilon$, we have $u(t_0) \leq \mathbb{E}[(V_{t_0} - V_{t_0}^\epsilon)^2]$. Since $t_0 \geq 0$ is arbitrary, we get, for all $t \geq 0$,

$$u(t) \leq d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} u(s) ds + K\epsilon^2 =: v(t).$$

Consequently,

$$\begin{aligned} v'(t) &= -c_\epsilon \left(v(t) - K\epsilon^2 \right) + d_\epsilon u(t) \\ &\leq -c_\epsilon \left(v(t) - K\epsilon^2 \right) + d_\epsilon v(t) \\ &\leq (d_\epsilon - c_\epsilon)v(t) + c_\epsilon K\epsilon^2. \end{aligned}$$

We first observe that $d_\epsilon \leq c_\epsilon$, so that $v(t) \leq v(0) + Kc_\epsilon\epsilon^2 t \leq K\epsilon^2(1 + c_\epsilon)(1 + t) \leq C\epsilon^2(1 + t)$, because $c_\epsilon \leq \int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta$, see (1.5.1).

We can also obtain a uniform in time bound. Recall that $v'(t) \leq (d_\epsilon - c_\epsilon)v(t) + c_\epsilon K\epsilon^2 = -2b_\epsilon v(t) + c_\epsilon K\epsilon^2$. We observe in fact that $v'(t) \leq 0$ as soon as $v(t) \geq \frac{c_\epsilon K\epsilon^2}{2b_\epsilon}$. Since $v(0) = K\epsilon^2 \leq \frac{c_\epsilon K\epsilon^2}{2b_\epsilon}$, we classically deduce that $v(t) \leq \frac{c_\epsilon K\epsilon^2}{2b_\epsilon} \leq C\frac{\epsilon^2}{b_\epsilon}$ for all $t \geq 0$.

So we have $W_2^2(f_t, f_t^\epsilon) = u(t) \leq v(t) \leq C \min\left(\epsilon^2(1 + t), \frac{\epsilon^2}{b_\epsilon}\right)$ for all $t \geq 0$. To complete the proof of Theorem 1.2.5, it suffices to observe that $4b_\epsilon \geq \int_{|\theta| < \epsilon} \theta^2 \beta(\theta) d\theta$ for any $\epsilon \in (0, 1)$. \square

1.6 Convergence of the particle system

In this section, we prove the results about the approximation of the solution of the Kac equation by a system of particles. Let thus $f_0 \in \mathcal{P}_4(\mathbb{R})$ and let β be a cross section satisfying (1.2.1). We fix $\epsilon \in (0, 1)$, and we consider the solutions $(f_t)_{t \geq 0}$ and $(f_t^\epsilon)_{t \geq 0}$ to (1.1.1) and (1.1.5) respectively, both starting from f_0 .

In the first part, we will rewrite the system of particles (1.2.11) in a suitable way and in the second part, we will introduce a system of i.i.d. particles with law $(f_t^\epsilon)_{t \geq 0}$. Using these systems of particles, we will be able to prove Theorem 1.2.7 and its corollary. We will end this section with the proof of Proposition 1.2.9 and with an extension about the Wasserstein distance W_γ for $\gamma \in (1, 2)$.

We recall a useful result (see e.g. Villani [72, Remark 2.19 (iii)]).

Proposition 1.6.1. *If μ and ν are two probability measures on \mathbb{R} , for $\gamma \geq 1$, we have $W_\gamma^\gamma(\mu, \nu) = \int_0^1 (F_\mu^{-1}(\alpha) - F_\nu^{-1}(\alpha))^\gamma d\alpha$ where $F_\mu(x) = \mu((-\infty, x])$ and $F_\nu(x) = \nu((-\infty, x])$.*

1.6.1 Another way to write system (1.2.11)

We fix an integer n and we consider:

- a family of i.i.d. random variables $(V_0^i)_{i \in \{1, \dots, n\}}$ with law f_0 ,
- a family of i.i.d. Poisson measures $(N^i)_{i \in \{1, \dots, n\}}$ on $[0, \infty) \times [0, 1] \times [-\pi, \pi]$ with intensity measure $dt d\alpha \beta(\theta) d\theta$,
- a family of i.i.d. Brownian motions $(B_t^i)_{t \geq 0, i \in \{1, \dots, n\}}$.

For $\epsilon \in (0, 1)$, we consider $(V_t^{i, n, \epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ solution of the following system of S.D.E.s: for $i = 1, \dots, n$, for all $t \geq 0$,

$$\begin{aligned} V_t^{i, n, \epsilon} = & V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) V_{s-}^{i, n, \epsilon} - \sin \theta \left(F_{s-}^{n, \epsilon} \right)^{-1}(\alpha) \right] N^i(d\theta d\alpha ds) \\ & - b_\epsilon \int_0^t V_s^{i, n, \epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

where $F_t^{n, \epsilon} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{V_t^{i, n, \epsilon} \leq x}$, $\mathcal{E} = \int_{\mathbb{R}} v^2 f_0(dv)$ and $b_\epsilon = \int_{|\theta| < \epsilon} (1 - \cos \theta) \beta(\theta) d\theta$.

This particle system is identical (in law) to the one introduced in (1.2.11). Indeed, it suffices to note that given $(V_{s-}^{i, n, \epsilon})_{i \in \{1, \dots, n\}}$, the law of $(F_{s-}^{n, \epsilon})^{-1}(\alpha)$ (with α uniformly distributed on $[0, 1]$) is the same as that of $V_{s-}^{j, n, \epsilon}$ (with j uniformly distributed in $\{1, \dots, n\}$).

1.6.2 A system of i.i.d. particles

For $i \in \{1, \dots, n\}$ and $\epsilon > 0$, we consider the process $(\bar{V}_t^{i, \epsilon})_{t \geq 0}$ solution of the following S.D.E. (with the same random objects V_0^i , N^i and B^i as previously),

$$\begin{aligned} \bar{V}_t^{i, \epsilon} = & V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \bar{V}_{s-}^{i, \epsilon} - \sin \theta \left(F_{s-}^\epsilon \right)^{-1}(\alpha) \right] N^i(ds d\alpha d\theta) \\ & - b_\epsilon \int_0^t \bar{V}_s^{i, \epsilon} ds + \sqrt{2\mathcal{E}b_\epsilon} B_t^i, \end{aligned}$$

where $F_t^\epsilon(x) = \int_{-\infty}^x f_t^\epsilon(dv)$.

For each $s \geq 0$, it holds that $\mathcal{L}_\alpha((F_s^\epsilon)^{-1}) = f_s^\epsilon$. Hence we can apply Proposition 1.3.3 with $W_s^\epsilon = (F_s^\epsilon)^{-1}$ and deduce that for each $i \in \{1, \dots, n\}$, each $t \geq 0$, $\mathcal{L}(\bar{V}_t^{i,\epsilon}) = f_t^\epsilon$. Furthermore, the processes $(\bar{V}_t^i)_{t \geq 0}$ are obviously i.i.d. (for $i = 1, \dots, n$).

1.6.3 Proof of Theorem 1.2.7

We start with the following result.

Proposition 1.6.2. *We set $\mu_t^{n,\epsilon} = \frac{1}{n} \sum_1^n \delta_{V_t^{i,n,\epsilon}}$ and $\bar{\mu}_t^{n,\epsilon} = \frac{1}{n} \sum_1^n \delta_{\bar{V}_t^{i,\epsilon}}$. Then for any $T > 0$,*

$$\sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon}) \right] \leq C(1+T)^2 \sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right],$$

where C depends only on $\int_0^\pi \theta^2 \beta(\theta) d\theta$.

Proof. To lighten notation, we set $V_t^i = V_t^{i,n,\epsilon}$ and $\bar{V}_t^i = \bar{V}_t^{i,\epsilon}$ for the whole proof. By the triangular inequality, we have $W_2(f_t^\epsilon, \mu_t^{n,\epsilon}) \leq W_2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) + W_2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon})$. Hence, by squaring and taking expectations

$$(1.6.1) \quad \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon}) \right] \leq \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right] + \mathbb{E} \left[W_2^2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \\ + 2\mathbb{E} \left[W_2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) W_2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right].$$

Using the fact that $W_2^2\left(\frac{1}{n} \sum_1^n \delta_{x_i}, \frac{1}{n} \sum_1^n \delta_{y_i}\right) \leq \frac{1}{n} \sum_1^n |x_i - y_i|^2$, we have

$$\mathbb{E} \left[W_2^2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \leq \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n |\bar{V}_t^i - V_t^i|^2 \right) = \mathbb{E} \left(|\bar{V}_t^1 - V_t^1|^2 \right).$$

We set $\Delta_t = \bar{V}_t^1 - V_t^1$. It holds that

$$\Delta_t = \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \Delta_{s-} - \sin \theta \delta_{s-}(\alpha) \right] N^1(ds d\alpha d\theta) - b_\epsilon \int_0^t \Delta_s ds,$$

where $\delta_t(\alpha) = (F_t^{n,\epsilon})^{-1}(\alpha) - (F_t^\epsilon)^{-1}(\alpha)$. Applying Itô's formula, we get

$$\Delta_t^2 = \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\Delta_s + (\cos \theta - 1) \Delta_s - \sin \theta \delta_s(\alpha))^2 - \Delta_s^2 \right] N^1(ds d\alpha d\theta) \\ - 2b_\epsilon \int_0^t \Delta_s^2 ds.$$

Taking expectations and using Proposition 1.6.1, we get, with c_ϵ and d_ϵ defined in (1.5.1),

$$\begin{aligned} v(t) &:= \mathbb{E}(\Delta_t^2) = -c_\epsilon \int_0^t \mathbb{E}(\Delta_s^2) ds + d_\epsilon \int_0^t \mathbb{E} \left(\int_0^1 \delta_s^2(\alpha) d\alpha \right) ds \\ &= -c_\epsilon \int_0^t v(s) ds + d_\epsilon \int_0^t \mathbb{E}[W_2^2(f_s^\epsilon, \mu_s^{n,\epsilon})] ds \\ &= d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \mathbb{E}[W_2^2(f_s^\epsilon, \mu_s^{n,\epsilon})] ds, \end{aligned}$$

the last equality being obtained by solving the differential equation satisfied by v . If we set $u(t) := \mathbb{E}[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon})]$, $s_n := \sup_{[0,T]} \mathbb{E}[W_2^2(f_s^\epsilon, \bar{\mu}_s^{n,\epsilon})]$ and if we return to (1.6.1), we thus find, for all $t \in [0, T]$,

$$\begin{aligned} u(t) &\leq s_n + v(t) + 2\mathbb{E} \left[W_2(f_t^\epsilon, \bar{\mu}_t^n) W_2(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \\ &\leq s_n + v(t) + 2\sqrt{s_n} \sqrt{v(t)}, \end{aligned}$$

by the Cauchy-Schwarz inequality. We thus have

$$\begin{aligned} v(t) &= d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} u(s) ds \\ &\leq d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \left(s_n + v(s) + 2\sqrt{s_n} \sqrt{v(s)} \right) ds \\ &\leq s_n + d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} \left(v(s) + 2\sqrt{s_n} \sqrt{v(s)} \right) ds =: w(t). \end{aligned}$$

We used that $d_\epsilon e^{-c_\epsilon t} \int_0^t e^{c_\epsilon s} ds = \frac{d_\epsilon}{c_\epsilon} (1 - e^{-c_\epsilon t}) \leq 1$, since $d_\epsilon \leq c_\epsilon$, recall (1.5.1). Differentiating w , we get

$$\begin{aligned} w'(t) &= -c_\epsilon (w(t) - s_n) + d_\epsilon (v(t) + 2\sqrt{s_n} \sqrt{v(t)}) \\ &\leq c_\epsilon s_n + w(t) (-c_\epsilon + d_\epsilon) + 2d_\epsilon \sqrt{s_n} \sqrt{w(t)} \\ &\leq c_\epsilon s_n + 2d_\epsilon \sqrt{s_n} \sqrt{w(t)} \\ &\leq a s_n + 2a \sqrt{s_n} \sqrt{w(t)}, \end{aligned}$$

where $a = \int_{-\pi}^\pi \theta^2 \beta(\theta) d\theta$.

Putting $x(t) = w(t)/s_n$, we deduce that $x(0) = 1$ and $x'(t) \leq \frac{a(1 + 2\sqrt{x(t)})}{2a(1 + \sqrt{x(t)})} \leq 4a\sqrt{1 + x(t)}$, whence $\sqrt{1 + x(t)} - \sqrt{1 + x(0)} \leq 2at$, which gives $x(t) \leq C(1 + t)^2$, and so $w(t) \leq C(1 + t)^2 s_n$.

To summarize, we have $v(t) \leq w(t) \leq C s_n (1 + t)^2$ and, for all $t \in [0, T]$,

$$\begin{aligned} \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon}) \right] &= u(t) \leq s_n + v(t) + 2\sqrt{s_n} \sqrt{v(t)} \\ &\leq s_n + C s_n (1 + t)^2 + 2\sqrt{s_n} \sqrt{C} \sqrt{s_n} (1 + t) \\ &\leq C s_n (1 + t)^2 \\ &= C \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_s^\epsilon, \bar{\mu}_s^{n,\epsilon}) \right] (1 + t)^2. \end{aligned}$$

This concludes the proof. \square

Theorem 1.2.7 follows almost immediately.

Proof of Theorem 1.2.7. For each $t \geq 0$, we consider an i.i.d. sequence $(\hat{V}_t^i)_{i \in \{1, \dots, n\}}$ with law f_t such that for each i , $\mathbb{E}[(\hat{V}_t^i - \bar{V}_t^{i,\epsilon})^2] = W_2^2(f_t, f_t^\epsilon)$. Then we set $\mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{\hat{V}_t^i}$. Using the triangular inequality, Theorem 1.2.5 and Proposition 1.6.2, we have

$$\begin{aligned} \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n,\epsilon}) \right] &\leq 2 \sup_{[0, T]} W_2^2(f_t, f_t^\epsilon) + 2 \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \mu_t^{n,\epsilon}) \right] \\ (1.6.2) \quad &\leq C(1 + T)\epsilon^2 + C(1 + T)^2 \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right]. \end{aligned}$$

We use again the triangular inequality to obtain

$$\mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right] \leq 4 \left(W_2^2(f_t^\epsilon, f_t) + \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] + \mathbb{E} \left[W_2^2(\mu_t^n, \bar{\mu}_t^{n,\epsilon}) \right] \right).$$

But $\mathbb{E} \left[W_2^2(\mu_t^n, \bar{\mu}_t^{n,\epsilon}) \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left(|\hat{V}_t^i - \bar{V}_t^{i,\epsilon}|^2 \right) = W_2^2(f_t, f_t^\epsilon)$. So using Theorem 1.2.5, we get

$$(1.6.3) \quad \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon}) \right] \leq C \left((1 + T)\epsilon^2 + \sup_{[0, T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right).$$

Inserting (1.6.3) in (1.6.2), we obtain

$$\sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^{n,\epsilon}) \right] \leq C(1+T)^3 \left(\epsilon^2 + \sup_{[0,T]} \mathbb{E} \left[W_2^2(f_t, \mu_t^n) \right] \right),$$

which concludes the proof. \square

Finally, we give the proof of Corollary 1.2.8.

Proof of Corollary 1.2.8 It suffices to apply Theorem 1.2.7, Lemma 1.8.4 with $\gamma = 2$ and $q = p - \gamma$, and Lemma 1.8.5. \square

1.6.4 Other Wasserstein distances

The first part of the following result is Proposition 1.2.9 and in the second part, we give some estimates about $\mathbb{E}[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})]$ for $\gamma \in (1, 2)$.

Proposition 1.6.3. *Adopt the same notation as in Theorem 1.2.7.*

(i) *If we assume that $f_0 \in \mathcal{P}_4(\mathbb{R})$ and if $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, then*

$$\sup_{[0,T]} \mathbb{E}[W_1(f_t, \mu_t^{n,\epsilon})] \leq C_T \left(\epsilon + \frac{1}{\sqrt{n}} \right),$$

where C_T depends only on T , f_0 and β .

(ii) *If $f_0 \in \mathcal{P}_p(\mathbb{R})$ for some even $p \geq 4$ and if $\int_0^\pi \theta^\gamma \beta(\theta) d\theta < \infty$ for some $\gamma \in (1, 2)$, then*

$$\sup_{[0,T]} \mathbb{E}[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})] \leq C_T \left(\epsilon^\gamma + \frac{1}{n^{\frac{p-\gamma}{2p-2}}} \right),$$

where C_T depends only on T , f_0 , β , p and γ .

Proof. Let $\gamma \in [1, 2)$ be fixed. We assume that $\int_0^\pi \theta^\gamma \beta(\theta) d\theta < \infty$ and we set $V_t^i = V_t^{i,n,\epsilon}$ and $\bar{V}_t^i = \bar{V}_t^{i,\epsilon}$ for the whole proof to lighten notation.

Step 1: we first prove that

$$I_{\gamma,\epsilon}(x) := \int_{|\theta|>\epsilon} [|\cos \theta - x \sin \theta|^\gamma - 1] \beta(\theta) d\theta \leq C(1 + |x|^\gamma).$$

To this end, we set $J_{\gamma,\epsilon}(x) = \int_{|\theta|>\epsilon} [|1 - x\theta|^\gamma - 1] \beta(\theta) d\theta$. Using the inequality $|a^\gamma - b^\gamma| \leq C|a - b|(a^{\gamma-1} + b^{\gamma-1})$, we get

$$\begin{aligned} |I_{\gamma,\epsilon}(x) - J_{\gamma,\epsilon}(x)| &\leq C \int_{|\theta|>\epsilon} |(\cos \theta - 1) - x(\sin \theta - \theta)| \\ &\quad (|\cos \theta - x \sin \theta|^{\gamma-1} + |1 - x\theta|^{\gamma-1}) \beta(\theta) d\theta \\ &\leq C \int_{|\theta|>\epsilon} \theta^2 (1 + |x|)(1 + |x|^{\gamma-1}) \beta(\theta) d\theta \\ &\leq C(1 + |x|)(1 + |x|^{\gamma-1}) \\ &\leq C(1 + |x|^\gamma). \end{aligned}$$

Using the fact that β is even, we can write

$$J_{\gamma,\epsilon}(x) = \int_{|\theta|>\epsilon} [|1 - x\theta|^\gamma - 1 + \gamma x\theta] \beta(\theta) d\theta = J_{\gamma,\epsilon}^1(x) + J_{\gamma,\epsilon}^2(x)$$

with

$$J_{\gamma,\epsilon}^1(x) = \int_{|\theta|>\epsilon, |x\theta|<1/2} [|1 - x\theta|^\gamma - 1 + \gamma x\theta] \beta(\theta) d\theta$$

and

$$J_{\gamma,\epsilon}^2(x) = \int_{|\theta|>\epsilon, |x\theta|>1/2} [|1 - x\theta|^\gamma - 1] \beta(\theta) d\theta.$$

By Taylor's formula, we get, observing that $|x^2\theta^2| \leq |x|^\gamma |\theta|^\gamma$ if $|x\theta| < 1/2$,

$$J_{\gamma,\epsilon}^1(x) \leq C \int_{|\theta|>\epsilon, |x\theta|<1/2} x^2 \theta^2 \beta(\theta) d\theta \leq C|x|^\gamma \int_{-\pi}^{\pi} |\theta|^\gamma \beta(\theta) d\theta.$$

Next, since $|x\theta| > 1/2$ implies $1 + |x\theta|^\gamma \leq (1 + 2^\gamma)|x\theta|^\gamma$,

$$J_{\gamma,\epsilon}^2(x) \leq C \int_{|\theta|>\epsilon, |x\theta|>1/2} [1 + |x\theta|^\gamma] \beta(\theta) d\theta \leq C|x|^\gamma \int_{-\pi}^{\pi} |\theta|^\gamma \beta(\theta) d\theta.$$

We thus have $J_{\gamma,\epsilon}(x) \leq C|x|^\gamma$ and hence $I_{\gamma,\epsilon}(x) \leq C(1 + |x|^\gamma)$.

Step 2: using Step 1, we now prove

$$\mathbb{E} \left[W_\gamma^\gamma(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon}) \right] \leq C e^{Ct} \int_0^t \mathbb{E} [W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})] ds,$$

for all $t \geq 0$. We have $\mathbb{E}\left[W_\gamma^\gamma(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon})\right] \leq \mathbb{E}(|\Delta_t|^\gamma)$, where $\Delta_t = \bar{V}_t^1 - V_t^1$. It holds

$$\Delta_t = \int_0^t \int_0^1 \int_{|\theta|>\epsilon} \left[(\cos \theta - 1)\Delta_{s-} - \sin \theta \delta_{s-}(\alpha) \right] N^1(ds d\alpha d\theta) - b_\epsilon \int_0^t \Delta_s ds,$$

where $\delta_t(\alpha) = (F_t^{n,\epsilon})^{-1}(\alpha) - (F_t^\epsilon)^{-1}(\alpha)$. By Itô's formula,

$$\begin{aligned} \mathbb{E}\left[|\Delta_t|^\gamma\right] &= \mathbb{E}\left[\int_0^t \int_0^1 \int_{|\theta|>\epsilon} \left[|\cos \theta \Delta_s - \sin \theta \delta_s(\alpha)|^\gamma - |\Delta_s|^\gamma \right] \beta(\theta) ds d\alpha d\theta \right] \\ &\quad - \gamma b_\epsilon \int_0^t \mathbb{E}(|\Delta_s|^\gamma) ds \\ &\leq \mathbb{E}\left[\int_0^t \int_0^1 \int_{|\theta|>\epsilon} \left[|\cos \theta - \sin \theta \frac{\delta_s(\alpha)}{\Delta_s}|^\gamma - 1 \right] \beta(\theta) ds d\alpha d\theta \right]. \end{aligned}$$

Using Step 1 and then Proposition 1.6.1, we get

$$\begin{aligned} \mathbb{E}\left[|\Delta_t|^\gamma\right] &\leq C \mathbb{E}\left[\int_0^t \int_0^1 |\Delta_s|^\gamma \left(1 + \frac{|\delta_s(\alpha)|^\gamma}{|\Delta_s|^\gamma}\right) d\alpha ds\right] \\ &\leq C \int_0^t \mathbb{E}[|\Delta_s|^\gamma] ds + C \int_0^t \mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})\right] ds. \end{aligned}$$

We conclude by Grönwall's lemma.

Step 3: using very similar arguments as in the proof of Theorem 1.2.7 and observing that $W_\gamma(f_t, f_t^\epsilon) \leq W_2(f_t, f_t^\epsilon)$, we get

$$\mathbb{E}\left[W_\gamma^\gamma(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon})\right] \leq C(1+t)^{\gamma/2} \left[\epsilon^\gamma + \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] \right],$$

for all $t \geq 0$, where μ_t^n is the empirical measure of a sequence of i.i.d. random variables with law f_t .

Step 4: the aim of this step is to prove that

$$(1.6.4) \quad \sup_{[0,T]} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})\right] \leq C_T \left(\epsilon^\gamma + \sup_{[0,T]} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] \right).$$

Using the triangular inequality, Theorem 1.2.5, Step 2 and Step 3, we have

$$\begin{aligned} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})\right] &\leq C\left(W_\gamma^\gamma(f_t, f_t^\epsilon) + \mathbb{E}\left[W_\gamma^\gamma(f_t^\epsilon, \bar{\mu}_t^{n,\epsilon})\right] + \mathbb{E}\left[W_\gamma^\gamma(\bar{\mu}_t^{n,\epsilon}, \mu_t^{n,\epsilon})\right]\right) \\ &\leq C_T\left(\epsilon^\gamma + \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] + C_T \int_0^t \mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})\right] ds\right). \end{aligned}$$

Using again the triangular inequality and Theorem 1.2.5, we get

$$\begin{aligned} \mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, \mu_s^{n,\epsilon})\right] &\leq C\left(\mathbb{E}\left[W_\gamma^\gamma(f_s^\epsilon, f_s)\right] + \mathbb{E}\left[W_\gamma^\gamma(f_s, \mu_s^{n,\epsilon})\right]\right) \\ &\leq C(1+T)^{\gamma/2}\epsilon^\gamma + C\mathbb{E}\left[W_\gamma^\gamma(f_s, \mu_s^{n,\epsilon})\right]. \end{aligned}$$

We thus have

$$\mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^{n,\epsilon})\right] \leq C_T\left(\epsilon^\gamma + \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] + \int_0^t \mathbb{E}\left[W_\gamma^\gamma(f_s, \mu_s^{n,\epsilon})\right] ds\right),$$

and we conclude with the help of Grönwall's lemma.

Step 5: we can now prove (i). Since $f_0 \in \mathcal{P}_4(\mathbb{R})$, Lemma 1.8.5 implies that $\sup_{t \geq 0} \int_{\mathbb{R}} v^4 f_t(dv) < \infty$, whence $\sup_{t \geq 0} \mathbb{E}\left[W_1(f_t, \mu_t^n)\right] \leq \frac{C}{\sqrt{n}}$ by Lemma 1.8.3. Inserting this in (1.6.4), we easily conclude.

Step 6: we finally prove (ii). Since $f_0 \in \mathcal{P}_p(\mathbb{R})$, Lemma 1.8.5 implies that $\sup_{t \geq 0} \int_{\mathbb{R}} v^p f_t(dv) < \infty$, whence $\sup_{t \geq 0} \mathbb{E}\left[W_\gamma^\gamma(f_t, \mu_t^n)\right] \leq \frac{C}{n^{\frac{p-\gamma}{2p-2}}}$ by Lemma 1.8.4. Inserting this in (1.6.4), we easily conclude. \square

1.7 Numerical results

We consider here the cross section $\beta(\theta) = |\theta|^{-1-\nu}$, with $0 < \nu < 2$. Let f_0 be a probability measure admitting a moment of order 4. We fix an integer n , a small parameter $\epsilon > 0$, and we take the same notation as in Section 1.2.4.

We simulate two systems of particles : the system $(V_t^{i,n,\epsilon})_{t \geq 0, i \in \{1, \dots, n\}}$ described in Section 1.2.4 (system with diffusion) and the following system without diffusion: for $i \in \{1, \dots, n\}$ and $t \geq 0$,

$$\tilde{V}_t^{i,n,\epsilon} = V_0^i + \int_0^t \int_0^1 \int_{|\theta| > \epsilon} \left[(\cos \theta - 1) \tilde{V}_{s^-}^{i,n,\epsilon} - \sin \theta \tilde{V}_{s^-}^{j,n,\epsilon} \right] N^{i,n}(ds d\theta dj).$$

The algorithm is the following (we write in *italic* the parts which only concern the system with diffusion).

- We set $t = 0$, and for $i = 1, \dots, n$, we simulate $V(i) \sim f_0$ and set $T^{up}(i) = 0$.
- While $t < T^{fin}$ (where T^{fin} is the time that we want to reach), we simulate an exponential random variable T with parameter $n \int_{|\theta| > \epsilon} \beta(\theta) d\theta$ and we put $t = t + T$. We choose randomly two integers i and j in $\{1, \dots, n\}$. *For our system with diffusion, we update the particles i and j by setting*

$$V(i) = V(i) \exp(-b_\epsilon(t - T^{up}(i))) + G(i),$$

and

$$V(j) = V(j) \exp(-b_\epsilon(t - T^{up}(j))) + G(j),$$

where $G(i)$ (resp. $G(j)$) has a centered Gaussian law with variance $1 - \exp(-2b_\epsilon(t - T^{up}(i)))$ (resp. $1 - \exp(-2b_\epsilon(t - T^{up}(j)))$), where b_ϵ is defined in (1.5.1), and we set $T^{up}(i) = T^{up}(j) = t$.

Next, we simulate a random variable Θ with density $\beta_\epsilon / \|\beta_\epsilon\|_1$, where $\beta_\epsilon(\theta) = \beta(\theta) \mathbb{1}_{|\theta| > \epsilon}$, and then, for the two systems, we put $V(i) = \cos \Theta V(i) - \sin \Theta V(j)$.

- *Only for the system with diffusion, we update all particles with a Gaussian term: for $i \in \{1, \dots, n\}$,*

$$V(i) = V(i) \exp(-b_\epsilon(t - T^{up}(i))) + G(i),$$

where $G(i)$ has a centered Gaussian law with variance $1 - \exp(-2b_\epsilon(t - T^{up}(i)))$.

For our simulation, we take $T^{fin} = 0.1$. Our initial data is $f_0 = (\delta_{-1} + \delta_1)/2$. The goal here is to see what system is more efficient. For this, we need a reference curve. We obtain it by simulating $n = 10^7$ particles with $\epsilon = 0.03$, and by using a smoothing procedure.

We see that the system with diffusion term is much more efficient when ν is close to 2. For ν smaller, the difference not clear.

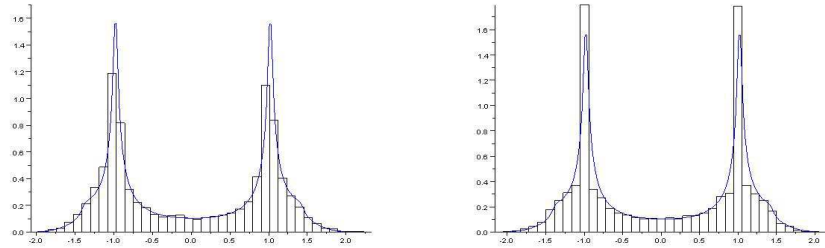


Figure 1.1: $\nu = 0.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 2 \cdot 10^4$, $\epsilon = 0.1$. Both simulations need approximately 0.05s.

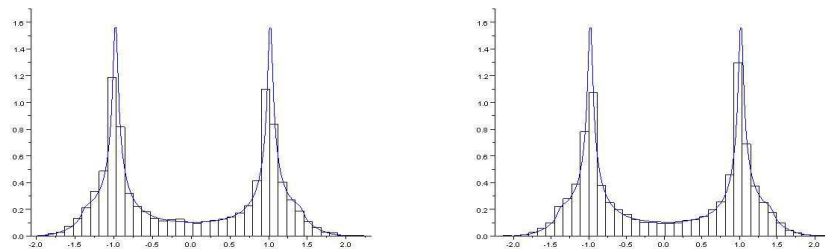


Figure 1.2: $\nu = 0.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 10^4$, $\epsilon = 0.02$. Both simulations need approximately 0.05s.

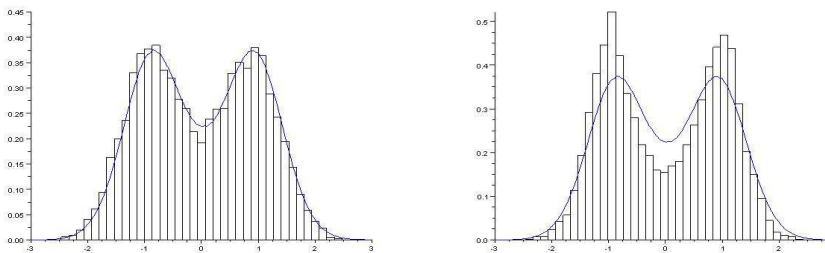


Figure 1.3: $\nu = 1.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 2 \cdot 10^4$, $\epsilon = 0.1$. Both simulations need approximately 0.14s.

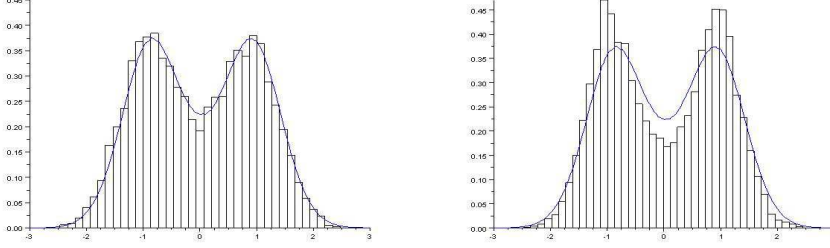


Figure 1.4: $\nu = 1.5$. Left graphic: system with diffusion, $n = 10^4$, $\epsilon = 0.1$. Right graphic: system without diffusion, $n = 10^4$, $\epsilon = 0.06$. Both simulations need approximately 0.14s.

1.8 Appendix

1.8.1 Wasserstein distance between a Poisson integral and a Gaussian law

We start with a result of Rio [57, Theorem 4.1], which gives some very precise rate of convergence for the standard central limit theorem in Wasserstein distance.

Theorem 1.8.1. *There exists a constant C_0 such that for any positive integer n , for any sequence $(X_i)_{i \geq 0}$ of real independent centered random variables in L^4 ,*

$$W_2^2(\eta_n, \mathcal{N}(0, 1)) \leq C_0 v_n^{-2} \sum_{i=1}^n \mathbb{E}(|X_i|^4),$$

where $\eta_n = \mathcal{L}(v_n^{-1/2} S_n)$, $S_n = \sum_{i=1}^n X_i$, $v_n = \text{Var}(S_n)$.

Using this result, we can estimate the Wasserstein distance between a compensated Poisson integral and a centered Gaussian law with the same variance. The following result is very close to [30, Corollary 6].

Corollary 1.8.2. *If E is a Polish space endowed with a non-negative σ -finite measure ν , if N is a Poisson measure on $[0, T] \times E$ with intensity measure $dt\nu(dz)$ and if $H : [0, T] \times E \mapsto \mathbb{R}$ is a deterministic function such that $\int_0^t \int_E (H^2(s, z) + H^4(s, z))\nu(dz)ds < \infty$, then setting*

$$X_t = \int_0^t \int_E H(s, z) \tilde{N}(ds, dz), \quad q_t = \int_0^t \int_E H^2(s, z) \nu(dz) ds,$$

we have

$$W_2^2(\mathcal{L}(X_t), \mathcal{N}(0, q_t)) \leq C_0 \frac{\int_0^t \int_E H^4(s, z) \nu(dz) ds}{q_t},$$

where C_0 is a universal constant (the same as in Theorem 1.8.1).

Proof. For $n \geq 1$, $i \in \{1, \dots, n\}$, we set

$$X_i^n = \sqrt{n} \int_{(i-1)t/n}^{it/n} \int_E H(s, z) \tilde{N}(ds, dz) \quad \text{and} \quad S_n = \sum_{i=1}^n X_i^n.$$

We have $X_t = \frac{S_n}{\sqrt{n}}$. The random variables X_i^n are independent, centered,

$$\begin{aligned} \mathbb{E}[(X_i^n)^2] &= n \int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds, \\ v_n = \text{Var}(S_n) &= \sum_{i=1}^n \mathbb{E}[(X_i^n)^2] = nq_t. \end{aligned}$$

It classically holds that

$$\begin{aligned} \mathbb{E}[(X_i^n)^4] &= n^2 \int_{(i-1)t/n}^{it/n} \int_E H^4(s, z) \nu(dz) ds \\ &\quad + 3n^2 \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}[(X_i^n)^4] &= n^2 \int_0^t \int_E H^4(s, z) \nu(dz) ds \\ &\quad + 3n^2 \sum_{i=1}^n \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2. \end{aligned}$$

By Theorem 1.8.1,

$$\begin{aligned}
W_2^2(\mathcal{L}(X_t), \mathcal{N}(0, q_t)) &= W_2^2\left(\mathcal{L}\left(\frac{1}{\sqrt{n}}S_n\right), \mathcal{N}(0, q_t)\right) \\
&= q_t W_2^2\left(\mathcal{L}\left(\frac{1}{\sqrt{v_n}}S_n\right), \mathcal{N}(0, 1)\right) \\
&\leq C_0 \frac{q_t}{v_n^2} \sum_{i=1}^n \mathbb{E}[(X_i^n)^4] \\
&\leq C_0 \frac{\int_0^t \int_E H^4(s, z) \nu(dz) ds}{q_t} \\
&\quad + 3 \frac{C_0}{q_t} \sum_{i=1}^n \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2.
\end{aligned}$$

Setting $F(t) = \int_0^t |f(s)| ds$ with $f(s) = \int_E H^2(s, z) \nu(dz)$ and observing that F is continuous (and so uniformly continuous on $[0, T]$ for all $T \geq 0$), we obtain that $\sum_{i=1}^n \left(\int_{(i-1)t/n}^{it/n} \int_E H^2(s, z) \nu(dz) ds \right)^2 \rightarrow 0$ when $n \rightarrow +\infty$. Since the last formula holds for all $n \geq 1$, we easily conclude. \square

1.8.2 Rate of convergence of empirical measures

We first give a classical result about the Wasserstein distance W_1 .

Lemma 1.8.3. *Let μ be a probability measure in $\mathcal{P}_4(\mathbb{R})$. We consider n i.i.d. random variables $(X_i)_{i \in \{1, \dots, n\}}$ with law μ and we set $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$. Then there exists a constant C depending only on $\int_{\mathbb{R}} x^4 \mu(dx)$ such that*

$$\mathbb{E}\left[W_1(\mu, \mu_n)\right] \leq \frac{C}{\sqrt{n}}.$$

Proof. If we set $F(x) = \mu((-\infty, x])$ and $F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_i \leq x}$, we have (see Villani [72, p 75])

$$\begin{aligned}
\mathbb{E}\left(W_1(\mu, \mu_n)\right) &= \mathbb{E}\left(\int_{-\infty}^{\infty} |F(x) - F_n(x)| dx\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty} \left|F(x) - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_i \leq x}\right| dx\right).
\end{aligned}$$

If $Y \sim \mathcal{B}(n, p)$, $\mathbb{E}\left(\left|\frac{Y}{n} - p\right|\right) \leq \sqrt{\mathbb{E}\left(\left(\frac{Y}{n} - p\right)^2\right)} = \frac{1}{\sqrt{n}}\sqrt{p(1-p)}$. Hence, since for each x , $\sum_{i=1}^n \mathbb{1}_{X_i \leq x} \sim \mathcal{B}(n, F(x))$,

$$\mathbb{E}\left(W_1(\mu, \mu_n)\right) \leq \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \sqrt{F(x)(1-F(x))} dx.$$

But $A := \int_{\mathbb{R}} x^4 \mu(dx) < \infty$ implies that for $x \geq 1$, $(1-F(x)) = \mu([x, +\infty)) \leq A/x^4$ and for $x \leq -1$, $F(x) = \mu((-\infty, x]) \leq A/x^4$, and we thus have $\int_{-\infty}^{\infty} \sqrt{F(x)(1-F(x))} dx < \infty$. \square

We now deduce similar estimates for other Wasserstein distances.

Lemma 1.8.4. *Let μ be a probability measure admitting a moment of order $q + \gamma$, with $\gamma > 1$ and $q > 0$. We consider n i.i.d. random variables $(X_i)_{i \in \{1, \dots, n\}}$ with law μ and we set $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$. There exists a constant C depending on γ , q and on the moment of μ of order $q + \gamma$ such that*

$$\mathbb{E}\left[W_\gamma^\gamma(\mu, \mu_n)\right] \leq \frac{C}{n^{\frac{q}{2(q+\gamma-1)}}}.$$

Proof. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which X_1, \dots, X_n are defined. For a fixed $\omega \in \Omega$, we consider two random variables X and Y^ω defined on the probability space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ with $\mathcal{L}_\alpha(X) = \mu$ and $\mathcal{L}_\alpha(Y^\omega) = \mu_n(\omega)$ such that $W_1(\mu, \mu_n(\omega)) = \mathbb{E}_\alpha(|X - Y^\omega|)$. Then we have, for any $A > 0$,

$$\begin{aligned} W_\gamma^\gamma(\mu, \mu_n(\omega)) &\leq \mathbb{E}_\alpha(|X - Y^\omega|^\gamma) \\ &\leq A^{\gamma-1} W_1(\mu, \mu_n(\omega)) + \mathbb{E}_\alpha(|X - Y^\omega|^\gamma \mathbb{1}_{|X - Y^\omega| > A}). \end{aligned}$$

We observe that

$$\mathbb{E}_\alpha(|X - Y^\omega|^\gamma \mathbb{1}_{|X - Y^\omega| > A}) \leq \frac{\mathbb{E}_\alpha(|X - Y^\omega|^{q+\gamma})}{A^q}.$$

But, setting $m_p(\mu) = \int_{\mathbb{R}} |x|^p \mu(dx)$,

$$\begin{aligned} \mathbb{E}_\alpha(|X - Y^\omega|^{q+\gamma}) &\leq C \left(\mathbb{E}_\alpha(|X|^{q+\gamma}) + \mathbb{E}_\alpha(|Y^\omega|^{q+\gamma}) \right) \\ &= C(m_{q+\gamma}(\mu) + m_{q+\gamma}(\mu_n(\omega))). \end{aligned}$$

One easily checks that $\mathbb{E}\left[m_{q+\gamma}(\mu_n)\right] = m_{q+\gamma}(\mu)$. Using Lemma 1.8.3, we finally get

$$\mathbb{E}\left[W_\gamma^\gamma(\mu, \mu_n)\right] \leq C\left(\frac{A^{\gamma-1}}{\sqrt{n}} + \frac{m_{q+\gamma}(\mu)}{A^q}\right).$$

Choosing $A = n^{\frac{1}{2(q+\gamma-1)}}$ completes the proof. \square

1.8.3 Moments of a solution to (1.1.1)

In many places of the proof, we need to upperbound $\int_{\mathbb{R}} v^4 f_t(dv)$ for any $t \geq 0$ where $(f_t)_{t \geq 0}$ solves (1.1.1). We also need to upperbound higher moments.

Lemma 1.8.5. *For $f_0 \in \mathcal{P}_4(\mathbb{R})$, consider the unique solution $(f_t)_{t \geq 0}$ to (1.1.1). For any $t \geq 0$, we have*

$$\int_{\mathbb{R}} v^4 f_t(dv) \leq \int_{\mathbb{R}} v^4 f_0(dv) + 3 \int_{\mathbb{R}} v^2 f_0(dv).$$

If $f_0 \in \mathcal{P}_p(\mathbb{R})$ with p even, then there exists a constant C depending on p , β and on $\int_{\mathbb{R}} v^p f_0(dv)$ such that for any $t \geq 0$,

$$\int_{\mathbb{R}} v^p f_t(dv) \leq C.$$

Proof. We only treat the case $p = 4$, see Truesdell [68] and Desvilletes [18] for the general case. If we take $\varphi(v) = v^4$, we find, recalling (1.2.2) and using that β is even,

$$K_\beta^\varphi(v, v_*) = \int_{-\pi}^{\pi} \left[(\cos^4 \theta - 1)v^4 + \sin^4 \theta v_*^4 + 6 \cos^2 \theta \sin^2 \theta v^2 v_*^2 \right] \beta(\theta) d\theta.$$

Setting $m_k(\mu) = \int_{\mathbb{R}} v^k \mu(dv)$ for μ a probability measure on \mathbb{R} and $k \in \mathbb{N}$, we thus get, using (1.2.5),

$$\begin{aligned} m_4(f_t) = m_4(f_0) + \int_0^t \int_{-\pi}^{\pi} & \left[-(1 - \cos^4 \theta - \sin^4 \theta) m_4(f_s) \right. \\ & \left. + 6 \cos^2 \theta \sin^2 \theta m_2^2(f_s) \right] \beta(\theta) d\theta ds. \end{aligned}$$

Recalling that $m_2(f_s) = m_2(f_0) =: \mathcal{E}$ for any $s \geq 0$, observing that $(\cos^2 \theta + \sin^2 \theta)^2 = 1$, which gives $2 \cos^2 \theta \sin^2 \theta = 1 - \cos^4 \theta - \sin^4 \theta$ and setting $c =$

$\int_{-\pi}^{\pi} (1 - \cos^4 \theta - \sin^4 \theta) \beta(\theta) d\theta$, we have

$$m_4(f_t) = m_4(f_0) - c \int_0^t m_4(f_s) ds + 3c\mathcal{E}^2 t,$$

whence

$$m_4(f_t) = (m_4(f_0) - 3\mathcal{E}^2) \exp(-ct) + 3\mathcal{E}^2 \leq m_4(f_0) + 3\mathcal{E}^2,$$

as desired. \square

1.8.4 Well-posedness for a P.D.E

To conclude this paper, we state the following result.

Proposition 1.8.6. *For $t \geq 0$ and $(v, v_*) \in \mathbb{R}^2$, we consider two finite non-negative measures $q(t, v, dh)$ and $r(t, v, v_*, dh)$ on \mathbb{R} such that $\Lambda_q := \sup_{t,v} q(t, v, \mathbb{R}) < \infty$, $\Lambda_r := \sup_{t,v,v_*} r(t, v, v_*, \mathbb{R}) < \infty$ and for all $T \geq 0$, all $(v, v_*) \in \mathbb{R}^2$,*

$$(1.8.1) \quad \sup_{[0,T]} \int_{\mathbb{R}} (h^2 + 2vh) q(t, v, dh) \leq C_T (1 + v^2),$$

and

$$(1.8.2) \quad \sup_{[0,T]} \int_{\mathbb{R}} (h^2 + 2vh) r(t, v, v_*, dh) \leq C_T (1 + v^2 + v_*^2),$$

Let also $a \geq 0$ and $b \in \mathbb{R}$ be fixed. Then, for any $f_0 \in \mathcal{P}_2(\mathbb{R})$, there exists a unique $(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ such that for all $\varphi \in C_b^2(\mathbb{R})$, all $t \geq 0$,

$$(1.8.3) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) f_t(dv) &= a \int_{\mathbb{R}} \varphi''(v) f_t(dv) + b \int_{\mathbb{R}} v \varphi'(v) f_t(dv) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] q(t, v, dh) f_t(dv) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] r(t, v, v_*, dh) f_t(dv) f_t(dv_*). \end{aligned}$$

Proof. We denote by $\mathcal{M}(\mathbb{R})$ the set of finite signed measures on \mathbb{R} . If $\mu \in \mathcal{M}(\mathbb{R})$, we set $|\mu|_{TV} = \sup_{\varphi \in L^\infty, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}} \varphi(v) \mu(dv)$. Using the Lusin Theorem (see e.g. [9, Theorem 9.11]), we have $|\mu|_{TV} = \sup_{\varphi \in C_b, \|\varphi\|_\infty \leq 1} \int_{\mathbb{R}} \varphi(v) \mu(dv)$.

We also have $|\mu|_{TV} = \int_{\mathbb{R}} |\mu|(dv)$ where $|\mu| = \mu_+ + \mu_-$ and if μ has a density f with respect to the Lebesgue measure, $|\mu|_{TV} = \int_{\mathbb{R}} |f(v)|dv$.

Preliminaries. For $\epsilon > 0$, we set $G_\epsilon(v) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{v^2}{2\epsilon}}$. Let $\mu \in \mathcal{M}(\mathbb{R})$. We claim that $\lim_{\epsilon \rightarrow 0} |\mu * G_\epsilon|_{TV} = |\mu|_{TV}$. Observe that this is not obvious, since it does not hold true, generally, that $\lim_{\epsilon \rightarrow 0} |\mu * G_\epsilon - \mu|_{TV} = 0$ (choose e.g. $\mu = \delta_0$). First, we have

$$\begin{aligned} |\mu * G_\epsilon|_{TV} &= \int_{\mathbb{R}} |\mu * G_\epsilon(v)|dv = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} G_\epsilon(v-w)\mu(dw) \right|dv \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} G_\epsilon(v-w)dv |\mu(dw)| = |\mu|_{TV}. \end{aligned}$$

Next, let $\alpha > 0$. There exists a function $\varphi \in C_b$ with $\|\varphi\|_\infty \leq 1$ such that $\int_{\mathbb{R}} \varphi(v)\mu(dv) \geq |\mu|_{TV} - \alpha$. We have, since $\mu * G_\epsilon$ clearly converges weakly (in the sense of measures) to μ ,

$$|\mu * G_\epsilon|_{TV} \geq \int_{\mathbb{R}} \varphi(v)(\mu * G_\epsilon)(v)dv \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}} \varphi(v)\mu(dv) \geq |\mu|_{TV} - \alpha.$$

Making α tend to zero, we get $\liminf_{\epsilon \rightarrow 0} |\mu * G_\epsilon|_{TV} \geq |\mu|_{TV}$.

Uniqueness. We consider two solutions $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$, with $f_0 = g_0$ and for $t \geq 0$ we set $\mu_t = f_t - g_t$. For any $\varphi \in C_b^2(\mathbb{R})$, any $t \geq 0$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v)\mu_t(dv) &= a \int_{\mathbb{R}} \varphi''(v)\mu_t(dv) + b \int_{\mathbb{R}} v\varphi'(v)\mu_t(dv) \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)]q(t, v, dh)\mu_t(dv) \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)]r(t, v, v_*, dh)[f_t(dv)f_t(dv_*) \\ &\quad \quad \quad - g_t(dv)g_t(dv_*)]. \end{aligned}$$

We observe that $f_t(dv)f_t(dv_*) - g_t(dv)g_t(dv_*) = f_t(dv)\mu_t(dv_*) + g_t(dv_*)\mu_t(dv)$.

We have

$$\begin{aligned}
\partial_t(\mu_t * G_\epsilon)(v) &= \frac{d}{dt} \int_{\mathbb{R}} G_\epsilon(v-w) \mu_t(dw) \\
&= a \int_{\mathbb{R}} G_\epsilon''(v-w) \mu_t(dw) - b \int_{\mathbb{R}} w G_\epsilon'(v-w) \mu_t(dw) \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} [G_\epsilon(v-w-h) - G_\epsilon(v-w)] \mu_t(dw) q(t, w, dh) \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [G_\epsilon(v-w-h) - G_\epsilon(v-w)] [f_t(dw) \mu_t(dw_*) \\
&\quad\quad\quad + g_t(dw_*) \mu_t(dw)] r(t, w, w_*, dh).
\end{aligned}$$

For $\eta > 0$, we consider a function Γ_η of class C^2 such that for any $x \in \mathbb{R}$, $(|x| - \eta)_+ \leq \Gamma_\eta(x) \leq |x|$, $\Gamma_\eta''(x) \geq 0$ and $\|\Gamma_\eta'\|_\infty \leq 1$. We also assume that $\Gamma_{\eta_1} \geq \Gamma_{\eta_2}$ if $\eta_1 \leq \eta_2$. Observing that $\int_{\mathbb{R}} G_\epsilon''(v-w) \mu_t(dw) = (\mu_t * G_\epsilon)''(v)$, we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \Gamma_\eta((\mu_t * G_\epsilon)(v)) dv &= \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) \partial_t(\mu_t * G_\epsilon)(v) dv \\
&= A_t + B_t + C_t + D_t,
\end{aligned}$$

where

$$A_t = a \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) (\mu_t * G_\epsilon)''(v) dv,$$

$$B_t = -b \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) w G_\epsilon'(v-w) \mu_t(dw) dv,$$

$$\begin{aligned}
C_t &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) [G_\epsilon(v-w-h) \\
&\quad\quad\quad - G_\epsilon(v-w)] \mu_t(dw) q(t, w, dh) dv,
\end{aligned}$$

and

$$\begin{aligned}
D_t &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) [G_\epsilon(v-w-h) - G_\epsilon(v-w)] [f_t(dw) \mu_t(dw_*) \\
&\quad\quad\quad + g_t(dw_*) \mu_t(dw)] r(t, w, w_*, dh) dv.
\end{aligned}$$

Using an integration by parts and recalling that $\Gamma_\eta''(x) \geq 0$ for any $x \in \mathbb{R}$, we have

$$A_t = -a \int_{\mathbb{R}} \Gamma_\eta''((\mu_t * G_\epsilon)(v)) ((\mu_t * G_\epsilon)'(v))^2 dv \leq 0.$$

First writing $w = v + w - v$ and then using an integration by parts (observe that $\int_{\mathbb{R}} G_\epsilon'(v - w) \mu_t(dw) = (\mu_t * G_\epsilon)'(v)$), we have

$$\begin{aligned} B_t &\leq -b \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma_\eta'((\mu_t * G_\epsilon)(v)) v G_\epsilon'(v - w) \mu_t(dw) dv \\ &\quad + \|\Gamma_\eta'\|_\infty |b| \int_{\mathbb{R}} \int_{\mathbb{R}} |w - v| |G_\epsilon'(v - w)| |\mu_t(dw)| dv \\ &\leq b \int_{\mathbb{R}} \Gamma_\eta((\mu_t * G_\epsilon)(v)) dv + |b| \int_{\mathbb{R}} |v| |G_\epsilon'(v)| dv \int_{\mathbb{R}} |\mu_t|(dw) \\ &\leq C \left(\int_{\mathbb{R}} |(\mu_t * G_\epsilon)(v)| dv + |\mu_t|_{TV} \right) \leq C |\mu_t|_{TV}. \end{aligned}$$

We used the preliminaries and the fact that $\int_{\mathbb{R}} |v| |G_\epsilon'(v)| dv \leq C$. Using next that $\int_{\mathbb{R}} G_\epsilon(v - w - h) dv = \int_{\mathbb{R}} G_\epsilon(v - w) dv = 1$, we have

$$\begin{aligned} C_t + D_t &\leq 2 \|\Gamma_\eta'\|_\infty \Lambda_q \int_{\mathbb{R}} |\mu_t|(dw) + 2 \|\Gamma_\eta'\|_\infty \Lambda_r \left(\int_{\mathbb{R}} |\mu_t|(dw_*) + \int_{\mathbb{R}} |\mu_t|(dw) \right) \\ &\leq C |\mu_t|_{TV}. \end{aligned}$$

We thus get

$$\frac{d}{dt} \int_{\mathbb{R}} \Gamma_\eta(\mu_t * G_\epsilon(v)) dv \leq C |\mu_t|_{TV}.$$

Using the monotone convergence Theorem (recall that $\Gamma_\eta(x)$ increases to $|x|$ as η decreases to 0) and recalling that $\mu_0 = 0$, we have

$$\begin{aligned} |\mu_t * G_\epsilon|_{TV} &= \lim_{\eta \rightarrow 0} \int_{\mathbb{R}} \Gamma_\eta((\mu_t * G_\epsilon)(v)) dv \\ &\leq \lim_{\eta \rightarrow 0} \int_{\mathbb{R}} \Gamma_\eta((\mu_0 * G_\epsilon)(v)) dv + C \int_0^t |\mu_s|_{TV} ds \\ &\leq C \int_0^t |\mu_s|_{TV} ds. \end{aligned}$$

Making ϵ tend to 0 and using the preliminaries, we get,

$$|\mu_t|_{TV} \leq C \int_0^t |\mu_s|_{TV} ds,$$

and we deduce that $|\mu_t|_{TV} = 0$ by Grönwall's lemma.

Existence. For $(Q_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$ given, we consider the following linear P.D.E. with unknown $(g_t^Q)_{t \geq 0}$: for all $\varphi \in C_b^2(\mathbb{R})$, all $t \geq 0$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) g_t^Q(dv) &= a \int_{\mathbb{R}} \varphi''(v) g_t^Q(dv) + b \int_{\mathbb{R}} v \varphi'(v) g_t^Q(dv) \\ (1.8.4) \quad &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] q(t, v, dh) g_t^Q(dv) \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(v+h) - \varphi(v)] r(t, v, v_*, dh) g_t^Q(dv) Q_t(dv_*). \end{aligned}$$

For $t \geq 0$ and $(v, v_*) \in \mathbb{R}^2$, we consider the following probability measures

$$\eta_{t,v}^q(dh) := \frac{q(t, v, dh)}{\Lambda_q} + \left(1 - \frac{q(t, v, \mathbb{R})}{\Lambda_q}\right) \delta_0(dh)$$

and

$$\eta_{t,v,v_*}^r(dh) := \frac{r(t, v, v_*, dh)}{\Lambda_r} + \left(1 - \frac{r(t, v, v_*, \mathbb{R})}{\Lambda_r}\right) \delta_0(dh),$$

and we set $F_{t,v}^q(x) := \eta_{t,v}^q((-\infty, x])$ and $F_{t,v,v_*}^r(x) := \eta_{t,v,v_*}^r((-\infty, x])$. We also set $H^q(t, v, u) := (F_{t,v}^q)^{-1}(u)$, $H^r(t, v, v_*, u) := (F_{t,v,v_*}^r)^{-1}(u)$ and we consider the following S.D.E.

$$\begin{aligned} V_t = V_0 &+ \int_0^t \int_0^1 H^q(s, V_{s-}, u) N^q(dsdu) \\ (1.8.5) \quad &+ \int_0^t \int_{\mathbb{R}} \int_0^1 H^r(s, V_{s-}, v_*, u) N^r(ds dv_* du) + b \int_0^t V_s ds + \sqrt{2a} B_t, \end{aligned}$$

where N^q is a Poisson measure with intensity measure $\Lambda_q dsdu$, N^r is a Poisson measure with intensity measure $\Lambda_r dsdu Q_s(dv_*)$ and B is a Brownian motion. There is existence and uniqueness for this S.D.E. because the Poisson measures N^q and N^r are finite, and because the drift and diffusion coefficients

are Lipschitz-continuous (see Ikeda-Watanabe [44]). Using Itô's formula and taking expectations, we get, for any $\varphi \in C_b^2(\mathbb{R})$,

$$\begin{aligned}
\mathbb{E}[\varphi(V_t)] &= \mathbb{E}[\varphi(V_0)] + \int_0^t \mathbb{E} \left[\int_0^1 [\varphi(V_s + H^q(s, V_s, u)) - \varphi(V_s)] \Lambda_q du \right] ds \\
&\quad + \int_0^t \mathbb{E} \left[\int_{\mathbb{R}} \int_0^1 [\varphi(V_s + H^r(s, V_s, v_*, u)) - \varphi(V_s)] \Lambda_r Q_s(dv_*) du \right] ds \\
&\quad + b \int_0^t \mathbb{E}[\varphi'(V_s) V_s] ds + a \int_0^t \mathbb{E}[\varphi''(V_s)] ds \\
&= \mathbb{E}[\varphi(V_0)] + \int_0^t \mathbb{E} \left[\int_{\mathbb{R}} [\varphi(V_s + h) - \varphi(V_s)] q(s, V_s, dh) \right] ds \\
&\quad + \int_0^t \mathbb{E} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} [\varphi(V_s + h) - \varphi(V_s)] r(s, V_s, v_*, dh) Q_s(dv_*) \right] ds \\
&\quad + b \int_0^t \mathbb{E}[\varphi'(V_s) V_s] ds + a \int_0^t \mathbb{E}[\varphi''(V_s)] ds.
\end{aligned}$$

Setting $g_t^Q = \mathcal{L}(V_t)$, we thus realize that $(g_t^Q)_{t \geq 0}$ solves (1.8.4).

If $(Q_t)_{t \geq 0}$ and $(R_t)_{t \geq 0}$ are in $L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$, then by the same kind of arguments as in the uniqueness proof, we have for any $t \in [0, T]$, denoting $\mu_t = g_t^Q - g_t^R$,

$$|\mu_t|_{TV} \leq C \int_0^t (|\mu_s|_{TV} + |Q_s - R_s|_{TV}) ds,$$

whence by Grönwall's Lemma,

$$\sup_{[0, T]} |\mu_t|_{TV} \leq C_T \int_0^T |Q_s - R_s|_{TV} ds.$$

We consider $f_0 \in \mathcal{P}_2(\mathbb{R})$. For $t \geq 0$, we set $f_t^0 = f_0$ and $f_t^{k+1} = f_t^{f^k}$. Then we have

$$\sup_{[0, T]} |f_t^{k+1} - f_t^k|_{TV} \leq C_T \int_0^T |f_s^k - f_s^{k-1}|_{TV} ds.$$

We classically conclude that $(f_t^k)_{t \geq 0}$ converges as k tends to infinity to some $(f_t)_{t \geq 0}$ solving (1.8.3). Using (1.8.1), (1.8.2) and (1.8.3) with $\varphi(v) = v^2$, we see that $(f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}))$.

Chapter 2

Asymptotic of grazing collisions for the spatially homogeneous Boltzmann equation for soft and Coulomb potentials

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Abstract. We give an explicit bound for the Wasserstein distance with quadratic cost between the solutions of Boltzmann's and Landau's equations in the case of soft and Coulomb potentials. This gives an explicit rate of convergence for the grazing collisions limit. Our result is local in time for very soft and Coulomb potentials and global in time for moderately soft potentials.

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Keywords: Kinetic Theory, Boltzmann equation, Landau equation, Grazing collisions.

2.1 Introduction and main result

2.1.1 The Boltzmann equation

If we denote by $f_t(v)$ the density of particles which move with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ in a spatially homogeneous dilute gas, then, under some assumptions, f solves the Boltzmann equation

$$(2.1.1) \quad \partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) [f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)],$$

where the pre-collisional velocities are given by

$$(2.1.2) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

and θ is the so-called *deviation angle* defined by $\cos \theta = \frac{(v - v_*) \cdot \sigma}{|v - v_*|}$. The function $B = B(|v - v_*|, \theta) = B(|v' - v'_*|, \theta)$ is called the collision kernel and depends on the nature of the interactions between particles.

Let us interpret this equation: for each $v \in \mathbb{R}^3$, new particles with velocity v appear due to a collision between two particles with velocities v' and v'_* , at rate $B(|v' - v'_*|, \theta)$, while particles with velocity v disappear because they collide with another particle with velocity v_* , at rate $B(|v - v_*|, \theta)$. See Cercignani [15], Desvillettes [23], Villani [71] and Alexandre [3] for much more details.

Since the collisions are assumed to be elastic, conservation of mass, momentum and kinetic energy hold at least formally for solutions to (2.1.1) and we will assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v) dv = 1$.

We will first assume that the collision kernel B has the following form

$$(A1(\gamma)) \quad B(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta(\theta),$$

where $\beta : (0, \pi] \mapsto [0, \infty)$ is a function and $\gamma \in \mathbb{R}$. We consider the case of particles which interact through repulsive forces following an inverse power law, which means that two particles apart from a distance r exert on each other a force proportional to $1/r^s$, with $s \in (2, \infty)$. In this case, we have

$$(2.1.3) \quad \beta(\theta) \stackrel{0}{\sim} c s \theta^{-1-\nu} \quad \text{with } \nu = \frac{2}{s-1} \in (0, 2), \quad \text{and } \gamma = \frac{s-5}{s-1} \in (-3, 1).$$

One classically names *hard potentials* the case where $\gamma \in (0, 1)$ (i.e. $s > 5$), *Maxwellian molecules* the case where $\gamma = 0$ (i.e. $s = 5$), *moderately soft potentials* the case where $\gamma \in (-1, 0)$ (i.e. $s \in (3, 5)$), *very soft potentials* the case where $\gamma \in (-3, -1]$ (i.e. $s \in (2, 3]$). We will study in this paper all soft potentials.

In all these cases, we have $\int_0^\pi \beta(\theta) d\theta = +\infty$, which means that there is an infinite number of *grazing collisions* (collisions with a very small deviation) for each particle during any time interval. We will consider the Boltzmann equation without cutoff where we assume

$$(A2) \quad \int_0^\pi \theta^2 \beta(\theta) d\theta = \frac{4}{\pi},$$

which corresponds to the real physical situation. The classical assumption is only $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$ but we can assume without loss of generality that it is equal to $\frac{4}{\pi}$ (it suffices to make a change of time).

In the case of soft potentials, we will suppose that for some $\nu \in (0, 2)$ and $0 < c_1 < c_2$,

$$(A3(\nu)) \quad c_1 \theta^{-1-\nu} \leq \beta(\theta) \leq c_2 \theta^{-1-\nu} \quad \text{for all } \theta \in (0, \pi].$$

In order to focus on grazing collisions for soft potentials, we also set, for $0 < \epsilon \leq \pi$,

$$(2.1.4) \quad B_\epsilon(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta_\epsilon(\theta) \quad \text{with} \quad \beta_\epsilon(\theta) = \frac{\pi^3}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}.$$

Observe that β_ϵ is concentrated on small deviation angles, but for all $\epsilon \in (0, \pi)$,

$$(2.1.5) \quad \int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta = \frac{4}{\pi}.$$

When the particles exert on each other a force proportional to $1/r^2$, we talk about Coulomb potential. As explained in Villani [70, Section 7], the Boltzmann equation does not make sense in this case because grazing collisions become preponderant over all other collisions. To treat the Coulomb case, we will consider the following collision kernel

$$(AC) \quad B_\epsilon(|v - v_*|, \theta) \sin \theta = (|v - v_*| + h_\epsilon)^{-3} \beta_\epsilon(\theta),$$

where $\epsilon \in (0, 1)$, $h_\epsilon \in (0, 1)$ decreases to 0 as ϵ tends to 0 and for $\theta \in (0, \pi]$,

$$(2.1.6) \quad \beta_\epsilon(\theta) = \frac{c_\epsilon}{\log \frac{1}{\epsilon}} \frac{\cos \theta/2}{\sin^3 \theta/2} \mathbb{1}_{\epsilon \leq \theta \leq \pi/2},$$

where c_ϵ is such that (2.1.5) is satisfied. We can compute explicitly c_ϵ and we get

$$c_\epsilon = \frac{4}{\pi} \frac{\log \frac{1}{\epsilon}}{\frac{\epsilon^2}{\sin^2 \epsilon/2} + \frac{4\epsilon \cos \epsilon/2}{\sin \epsilon/2} + 8 \log \frac{1}{\sqrt{2} \sin \epsilon/2} - \pi^2/2 - 2\pi},$$

which tends to $\frac{1}{2\pi}$ as $\epsilon \rightarrow 0$.

We thus take the same collision kernel as in Villani [70, Section 7] with two small modifications. We add h_ϵ in the velocity part only to get easily existence and uniqueness of solutions to (2.1.1). Indeed, we do not need it for the calculus of the rate of convergence in Theorem 2.1.2 (observe that we only ask to h_ϵ to decrease to 0 without asking any rate for this convergence). We use c_ϵ to get (2.1.5) for our convenience, but it does not change the nature of the cross section since c_ϵ is close to $\frac{1}{2\pi}$ when ϵ is small.

Since we have (2.1.5) for each $\epsilon > 0$ and since $\int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \leq \frac{C}{\log 1/\epsilon} \rightarrow 0$, this cross section indeed concentrates on grazing collisions.

2.1.2 The Landau equation

We consider the spatially homogeneous Landau equation in dimension 3 for soft and Coulomb potentials. This equation of kinetic physics, also called Fokker-Planck-Landau equation, has been derived from the Boltzmann equation by Landau in 1936 when the grazing collisions prevail in the gas. It describes the density $g_t(v)$ of particles having the velocity $v \in \mathbb{R}^3$ at time $t \geq 0$:

$$(2.1.7) \quad \partial_t g_t(v) = \frac{1}{2} \sum_{i,j=1}^3 \partial_i \left\{ \int_{\mathbb{R}^3} l_{ij}(v-v_*) \left[g_t(v_*) \partial_j g_t(v) - g_t(v) \partial_j g_t(v_*) \right] dv_* \right\},$$

where $l(z)$ is a symmetric nonnegative 3×3 matrix for each $z \in \mathbb{R}^3$, depending on a parameter $\gamma \in [-3, 0)$, defined by

$$(2.1.8) \quad l_{ij}(z) = |z|^\gamma (|z|^2 \delta_{ij} - z_i z_j).$$

As for the Boltzmann equation, we can observe that the solutions to (2.1.7) conserve at least formally the mass, the momentum and the kinetic energy and we assume without loss of generality that $\int_{\mathbb{R}^3} g_0(v) dv = 1$.

We refer to Villani [70, 71] for more details on this equation, especially its physical meaning and its derivation from the Boltzmann equation.

2.1.3 Notation

We denote by $C_b^2(\mathbb{R}^3)$ the set of real bounded functions which are in $C^2(\mathbb{R}^3)$ with first and second derivatives bounded and by $L^p(\mathbb{R}^3)$ the space of measurable functions f with $\|f\|_{L^p} := (\int_{\mathbb{R}^3} |f(v)|^p dv)^{1/p} < +\infty$.

For $k \geq 0$, we denote by $\mathcal{P}_k(\mathbb{R}^3)$ the set of probability measures on \mathbb{R}^3 admitting a moment of order k (i.e. such that $m_k(f) := \int_{\mathbb{R}^3} |v|^k f(dv) < \infty$) and for $\alpha \in (-3, 0]$, we introduce the space $\mathcal{J}_\alpha(\mathbb{R}^3)$ of probability measures f on \mathbb{R}^3 such that

$$(2.1.9) \quad J_\alpha(f) := \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\alpha f(dv_*) < \infty.$$

For any $T > 0$, we also denote by $L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$, $L^\infty([0, T], L^p(\mathbb{R}^3))$, $L^1([0, T], \mathcal{J}_\alpha(\mathbb{R}^3))$ and $L^1([0, T], L^p(\mathbb{R}^3))$ the set of measurable families $(f_t)_{t \in [0, T]}$ of probability measures on \mathbb{R}^3 with $\sup_{[0, T]} m_2(f_t) < +\infty$, $\sup_{[0, T]} \|f_t\|_{L^p} < +\infty$, $\int_0^T J_\alpha(f_t) dt < +\infty$ and $\int_0^T \|f_t\|_{L^p} dt < +\infty$ respectively. We finally denote the entropy of a nonnegative function $f \in L^1(\mathbb{R}^3)$ by

$$H(f) := \int_{\mathbb{R}^3} f(v) \log(f(v)) dv.$$

In this article, we will use the Wasserstein distance with quadratic cost for our results of convergence: if $f, g \in \mathcal{P}_2(\mathbb{R}^3)$,

$$\mathcal{W}_2(f, g) = \inf \left\{ \mathbb{E}[|U - V|^2]^{1/2}, U \sim f, V \sim g \right\},$$

where the infimum is taken over all \mathbb{R}^3 -valued random variables U with law f and V with law g . It is known that the infimum is reached and more precisely if we fix $U \sim f$, then there exists $V \sim g$ such that $\mathcal{W}_2^2(f, g) = \mathbb{E}[|U - V|^2]$. See e.g. Villani [72] for many details on the subject.

2.1.4 The main results

We first give an explicit rate of convergence for the asymptotic of grazing collisions for soft potentials. Observe that the existence and the uniqueness of solutions to (2.1.1) and (2.1.7) that we state in the following result are direct consequences of the papers of Fournier-Mouhot [35] and Fournier-Guérin [32]-[33]. The precise notion of weak solutions that we use is given in the next section.

Theorem 2.1.1. *Let $\gamma \in (-3, 0)$, $\nu \in (0, 2)$ and B be a collision kernel which satisfies **(A1)(γ)-A2-A3(ν)**. For $\epsilon \in (0, \pi]$, we consider B_ϵ as in (2.1.4).*

(i) If $\gamma \in (-1, 0)$ and $\nu \in (-\gamma, 1)$, let $p > \max(5, \gamma^2/(\nu + \gamma))$ and $f_0 \in \mathcal{P}_{p+2}(\mathbb{R}^3)$ such that $H(f_0) < \infty$. Then there exists a unique weak solution $(g_t)_{t \in [0, \infty)}$ to (2.1.7) with $g_0 = f_0$, and for any $\epsilon \in (0, \pi]$, there exists a unique weak solution $(f_t^\epsilon)_{t \in [0, \infty)}$ to (2.1.1) with collision kernel B_ϵ and initial condition $f_0^\epsilon = f_0$. Moreover, for any $T > 0$ and $\epsilon \in (0, 1)$,

$$\sup_{[0, T]} \mathcal{W}_2(f_t^\epsilon, g_t) \leq C \epsilon^{\frac{p}{2p+3}},$$

where C is a constant depending on T, p, γ, f_0 .

(ii) If $\gamma \in (-3, 0)$, let $f_0 \in \mathcal{P}_{p+2}(\mathbb{R}^3)$ for some $p \geq 5$ such that $f_0 \in L^q(\mathbb{R}^3)$ for some $q > \frac{3}{3+\gamma}$. Then there exists $T_ = T_*(q, \|f_0\|_{L^q}) > 0$ such that there exists a unique weak solution $(g_t)_{t \in [0, T_*]}$ to (2.1.7) with $g_0 = f_0$, and for any $\epsilon \in (0, \pi]$, there exists a unique weak solution $(f_t^\epsilon)_{t \in [0, T_*]}$ to (2.1.1) with collision kernel B_ϵ and initial condition $f_0^\epsilon = f_0$. Moreover, for any $\epsilon \in (0, 1)$,*

$$\sup_{[0, T_*]} \mathcal{W}_2(f_t^\epsilon, g_t) \leq C \epsilon^{\frac{p}{2p+3}},$$

where C is a constant depending on p, q, γ, f_0 .

Point (i) applies to the case of moderately soft potentials ($s \in (3, 5)$) and (ii) applies to the case of very soft potentials ($s \in (2, 3]$). The proof of this result is based on a more general inequality, see Theorem 2.3.1.

We now treat the case of Coulomb potential. The existence and the uniqueness of solutions to (2.1.1) and (2.1.7) stated below are direct consequences of the papers of Fournier-Guérin [32] for (2.1.1) and Arsen'ev-Peskov [4] and Fournier [29] for (2.1.7).

Theorem 2.1.2. *Let $\gamma = -3$, B_ϵ be given by (AC) and let $f_0 \in \mathcal{P}_p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p \geq 7$. Then there exists $T_* = T_*(\|f_0\|_{L^\infty})$ such that there exists a unique weak solution $(g_t)_{t \in [0, T_*]}$ to (2.1.7) with $g_0 = f_0$, and for any $\epsilon \in (0, 1)$, there exists a unique weak solution $(f_t^\epsilon)_{t \in [0, T_*]}$ to (2.1.1) with collision kernel B_ϵ and initial condition $f_0^\epsilon = f_0$. Moreover, for any $\epsilon \in (0, 1)$,*

$$\sup_{[0, T_*]} \mathcal{W}_2(f_t^\epsilon, g_t) \leq C \left(h_\epsilon^a + \left(\frac{1}{\log \frac{1}{\epsilon}} \right)^a \right),$$

where C and $a > 0$ depend on p and f_0 .

The constant a can be made explicit from the proof. We have two error terms. The first one (h_ϵ^a) comes from the fact that we introduce a parameter in the collision kernel in order to get easily existence and uniqueness of solutions to (2.1.1). The second one $\left(\left(\frac{1}{\log \frac{1}{\epsilon}} \right)^a \right)$ is the true rate of convergence that we get for the asymptotic of grazing collisions in the Coulomb case. These two terms are not linked, so that assuming existence and uniqueness for (2.1.1), we could take $h_\epsilon = 0$ (which still makes our proofs valid). Anyway, since we allow h_ϵ to decrease to 0 as fast as one wants, we believe that this is not really a limitation.

2.1.5 Comments and main difficulties

It was already known that in the limit of grazing collisions, the solution to Boltzmann's equation converges to the solution of the Landau equation. To be more precise, Degond and Lucquin-Desreux [17] and Desvillettes [20] have shown the convergence of the operators (not of the solutions) and Villani [70] has shown some compactness results and the convergence of subsequences. The uniqueness results of Fournier-Guérin [32] and Fournier [29] show the true convergence (under some more restrictive assumptions). In this article, we give an explicit rate for this convergence and we thus justify the fact that the Landau equation is a good approximation of the Boltzmann equation in the limit of grazing collisions.

In all cases (soft or Coulomb potentials), we expect to get a bound for $\mathcal{W}_2(f_t^\epsilon, g_t)$ of order $\sqrt{\int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta}$ as for the Kac equation (see [31]). For soft potentials, the rate of convergence that we get is $\epsilon^{1/2-}$ (if f_0 is nice)

instead of ϵ . For the Coulomb potential (which is the only case which has a real physical interest), we get a rate of order $\left(\frac{1}{\log \frac{1}{\epsilon}}\right)^a$ with $a > 0$ very small (if f_0 is nice) instead of $\sqrt{\frac{1}{\log \frac{1}{\epsilon}}}$. This last case is very complicated because of the huge singularity, and there may be underlying reasons for the slow convergence.

The results are local in time, except for moderately soft potentials, but this was expected since the uniqueness results for the Boltzmann and Landau equations are also local in time.

To our knowledge, the present paper is the first, with the one of He [41], which states an explicit rate of convergence. He obtains a better rate (ϵ instead of $\epsilon^{1/2-}$ for soft potentials) but considers much more regular solutions (lying in $\mathcal{P}_p(\mathbb{R}^3) \cap H_l^N(\mathbb{R}^3)$ for some $N \geq 6$, $l > 0$ and p which depends on N). Furthermore, for the Coulomb case, He uses a cross section which does not seem to correspond to the physical situation (it resembles more at the case of soft potentials).

Our result has two main interests. A physical one, since it gives a justification for the Landau equation, and a numerical one. Indeed, in a recent paper about the Kac equation [31], using the same kind of result for grazing collisions, we have shown numerically and theoretically that it is much more efficient to replace small collisions (which cannot be simulated) by a Landau-type term than to neglect them. Theorem 1.1 shows that this should also be the case for the Boltzmann equation for soft potentials.

Our proofs use probabilistic methods. The first who used probabilistic methods to study a Boltzmann-type equation (the Kac equation) is McKean [51, 52]. He was investigating the convergence to equilibrium and he proposed some probabilistic representation of Wild's sums, using some tools now known as the McKean graphs. The present article is strongly inspired by Tanaka [64]. He proved that the Wasserstein distance with quadratic cost between two solutions of Kac's equation is non-increasing. He extended the same ideas in [65] to the Boltzmann equation for Maxwell molecules. His study was based on the use of some nonlinear stochastic processes related to the Kac and Boltzmann equations. The same kind of ideas is also used in Desvillettes-Graham-Méléard [21].

In this article, we will also use a result of Zaitsev [74] in order to obtain a bound for the Wasserstein distance between a compensated Poisson integral

and a Gaussian random variable. Such an idea comes from the paper of Fournier [30] about the approximation of Lévy-driven stochastic differential equations in one dimension, see also [31]. Since we work here in dimension 3, such a result is much more difficult to obtain.

If we compare the present work to our similar result for the Kac equation, another difficulty is the fact that we treat the case of soft and Coulomb potentials ($\gamma \in [-3, 0)$) instead of the Maxwell case ($\gamma = 0$) where the velocity part of the collision kernel is constant. These reasons explain why we are not able to obtain an optimal rate of convergence.

2.1.6 Plan of the paper

In the next section, we precise the notion of weak solutions that we shall use, we give well-posedness results and some properties of the solutions to Boltzmann's and Landau's equations. In Section 2.3, we give a general result about the Wasserstein distance between solutions of Boltzmann's and Landau's equations for soft potentials and we deduce Theorem 2.1.1. In Section 2.4 we give a probabilistic interpretation of the equations (2.1.1) and (2.1.7). Section 2.5 is devoted to the proof of our general result for soft potentials. In Section 2.6, we study the Coulomb case. We end the paper with an appendix where we give a result about the distance between a compensated Poisson integral and a centered Gaussian law with the same variance, a result about the ellipticity of the diffusion matrix l (recall (2.1.8)), a generalized Grönwall Lemma and another technical result .

2.2 Weak solutions

2.2.1 Preliminary observations

Soft potentials

We consider a collision kernel which satisfies (**A1**(γ)-**A2**-**A3**(ν)) and we set, for $\theta \in (0, \pi]$,

$$(2.2.1) \quad H(\theta) := \int_{\theta}^{\pi} \beta(x) dx \quad \text{and} \quad G(z) := H^{-1}(z).$$

The function H is a continuous decreasing bijection from $(0, \pi]$ into $[0, +\infty)$ and $G : [0, +\infty) \rightarrow (0, \pi]$ is its inverse function. By Fournier-Guérin [32,

Lemma 1.1, (i)], Assumption **(A3)**(ν) implies that there exists $\kappa_1 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$(A4) \quad \int_0^\infty (G(z/x) - G(z/y))^2 dz \leq \kappa_1 \frac{(x-y)^2}{x+y}.$$

Lemma 2.2.1. For $\epsilon \in (0, \pi]$, we consider β_ϵ as in (2.1.4), and we set for $\theta \in (0, \epsilon]$

$$H_\epsilon(\theta) := \int_\theta^\epsilon \beta_\epsilon(x) dx \quad \text{and} \quad G_\epsilon(z) := H_\epsilon^{-1}(z).$$

The function H_ϵ is a continuous decreasing bijection from $(0, \epsilon]$ into $[0, +\infty)$ and $G_\epsilon : [0, +\infty) \rightarrow (0, \epsilon]$ is its inverse function. Then for all $\epsilon \in (0, \pi]$, G_ϵ satisfies **(A4)** with the same $\kappa_1 > 0$ as G .

Proof. Observing that $H_\epsilon(\theta) = \frac{\pi^2}{\epsilon^2} H(\frac{\pi\theta}{\epsilon})$ and $G_\epsilon(z) = \frac{\epsilon}{\pi} G(\frac{\epsilon^2 z}{\pi^2})$, we have, for all $x, y > 0$ and all $\epsilon \in (0, \pi]$,

$$\begin{aligned} \int_0^\infty \left(G_\epsilon\left(\frac{z}{x}\right) - G_\epsilon\left(\frac{z}{y}\right) \right)^2 dz &= \int_0^\infty \frac{\epsilon^2}{\pi^2} \left(G\left(\frac{\epsilon^2 z}{\pi^2 x}\right) - G\left(\frac{\epsilon^2 z}{\pi^2 y}\right) \right)^2 dz \\ &= \int_0^\infty \left(G\left(\frac{u}{x}\right) - G\left(\frac{u}{y}\right) \right)^2 du. \end{aligned}$$

That concludes the proof. \square

To deal with soft potentials, we will use that for $\alpha \in (-3, 0)$ and for $q \in (3/(3+\alpha), \infty]$, there exists a constant $C_{\alpha, q}$ such that for any $h \in \mathcal{P}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$,

$$(2.2.2) \quad \begin{aligned} J_\alpha(h) &= \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} h(v_*) |v - v_*|^\alpha dv_* \\ &\leq \sup_{v \in \mathbb{R}^3} \int_{|v_* - v| < 1} h(v_*) |v - v_*|^\alpha dv_* + \sup_{v \in \mathbb{R}^3} \int_{|v_* - v| \geq 1} h(v_*) dv_* \\ &\leq C_{\alpha, q} \|h\|_{L^q(\mathbb{R}^3)} + 1, \end{aligned}$$

where

$$C_{\alpha, q} = \left[\int_{|v_*| \leq 1} |v_*|^{\alpha q / (q-1)} dv_* \right]^{(q-1)/q} < \infty,$$

since by assumption $\alpha q / (q-1) > -3$. This computation will be useful in many proofs of this article.

Coulomb potential

We consider the collision kernel B_ϵ given by **(AC)** and we set, for $\epsilon \in (0, 1)$ and $\theta \in [\epsilon, \pi/2]$,

$$(2.2.3) \quad H_\epsilon(\theta) := \int_\theta^{\pi/2} \beta_\epsilon(x) dx \quad \text{and} \quad G_\epsilon(z) := H_\epsilon^{-1}(z).$$

The function H_ϵ is a continuous decreasing bijection from $[\epsilon, \pi/2]$ into $[0, H_\epsilon(\epsilon)]$ and we extend its inverse function $G_\epsilon : [0, H_\epsilon(\epsilon)] \rightarrow [\epsilon, \pi/2]$ on $[0, \infty)$ by setting $G_\epsilon(z) = 0$ for $z > H_\epsilon(\epsilon)$.

Lemma 2.2.2. *There exists $\kappa_2 > 0$ such that for all $x, y \in \mathbb{R}_+$, for all $\epsilon \in (0, 1)$,*

$$(A5) \quad \int_0^\infty (G_\epsilon(z/x) - G_\epsilon(z/y))^2 dz \leq \kappa_2 \left(\frac{(x-y)^2}{x+y} + \frac{\max(x,y)}{\log \frac{1}{\epsilon}} \log \frac{\max(x,y)}{\min(x,y)} \right).$$

Proof. We have, for $\theta \in [\epsilon, \pi/2]$ and $z \in [0, \infty)$,

$$H_\epsilon(\theta) = \frac{c_\epsilon}{\log \frac{1}{\epsilon}} (\sin^{-2} \frac{\theta}{2} - 2) \quad \text{and} \quad G_\epsilon(z) = 2 \arcsin \left(\frac{\log \frac{1}{\epsilon}}{c_\epsilon} z + 2 \right)^{-\frac{1}{2}} \mathbb{1}_{\{z < H_\epsilon(\epsilon)\}}.$$

We consider $0 < x < y$. We have

$$\begin{aligned} \int_0^\infty (G_\epsilon(z/x) - G_\epsilon(z/y))^2 dz &= \int_0^{xH_\epsilon(\epsilon)} (G_\epsilon(z/x) - G_\epsilon(z/y))^2 dz \\ &\quad + \int_{xH_\epsilon(\epsilon)}^{yH_\epsilon(\epsilon)} G_\epsilon^2(z/y) dz \\ &=: A + B. \end{aligned}$$

Using that for any $a, b > 2$,

$$\begin{aligned} \left(\arcsin \frac{1}{\sqrt{a}} - \arcsin \frac{1}{\sqrt{b}} \right)^2 &\leq 2 \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right)^2 \\ &= 2 \left(\frac{b-a}{\sqrt{ab}(\sqrt{a} + \sqrt{b})} \right)^2 \leq 2 \frac{(b-a)^2}{ab(a+b)}, \end{aligned}$$

and setting $K_\epsilon := \frac{\log \frac{1}{\epsilon}}{c_\epsilon}$, we have, recalling that $0 < x < y$,

$$\begin{aligned}
A &\leq C \int_0^{\frac{x}{K_\epsilon \sin^2 \frac{\epsilon}{2}}} K_\epsilon^2 \left| \frac{1}{x} - \frac{1}{y} \right|^2 \frac{z^2 dz}{\left(\frac{z}{x} K_\epsilon + 1 \right) \left(\frac{z}{y} K_\epsilon + 1 \right) \left(\frac{z}{x} K_\epsilon + \frac{z}{y} K_\epsilon + 1 \right)} \\
&\leq C \frac{(x-y)^2}{y} K_\epsilon^2 \int_0^{\frac{x}{K_\epsilon \sin^2 \frac{\epsilon}{2}}} \frac{z^2 dz}{\left(z K_\epsilon + x \right)^2 \left(z K_\epsilon + y \right)} \\
&\leq C \frac{(x-y)^2}{x+y} K_\epsilon^2 \int_0^{\frac{x}{K_\epsilon \sin^2 \frac{\epsilon}{2}}} \frac{z^2 dz}{\left(z K_\epsilon + x \right)^3} \\
&\leq C \frac{(x-y)^2}{x+y} \left(\int_0^{\frac{x}{K_\epsilon}} \frac{z^2 K_\epsilon^2 dz}{x^3} + \int_{\frac{x}{K_\epsilon}}^{\frac{x}{K_\epsilon \sin^2 \frac{\epsilon}{2}}} \frac{dz}{z K_\epsilon} \right) \\
&\leq C \frac{(x-y)^2}{x+y} \left(\frac{1}{K_\epsilon} + \frac{\log \frac{1}{\sin^2 \frac{\epsilon}{2}}}{K_\epsilon} \right) \\
&\leq C \frac{(x-y)^2}{x+y}.
\end{aligned}$$

We finally used that $K_\epsilon \sim \frac{1}{2\pi \log \frac{1}{\epsilon}}$ as $\epsilon \rightarrow 0$. Using that $\arcsin \frac{1}{\sqrt{a}} \leq \sqrt{2} \frac{1}{\sqrt{a}}$ for any $a > 2$, we get for B ,

$$\begin{aligned}
B &\leq 8 \int_{xH_\epsilon(\epsilon)}^{yH_\epsilon(\epsilon)} \frac{y dz}{K_\epsilon z + y} = 8 \frac{y}{K_\epsilon} \log \frac{K_\epsilon y H_\epsilon(\epsilon) + y}{K_\epsilon x H_\epsilon(\epsilon) + y} \leq 8 \frac{y}{K_\epsilon} \log \frac{K_\epsilon y H_\epsilon(\epsilon) + y}{K_\epsilon x H_\epsilon(\epsilon) + x} \\
&= 8 \frac{c_\epsilon y}{\log \frac{1}{\epsilon}} \log \frac{y}{x},
\end{aligned}$$

which ends the proof since $\sup_{\epsilon \in (0,1)} c_\epsilon < \infty$ (recall that $c_\epsilon \rightarrow \frac{1}{2\pi}$). \square

2.2.2 The Landau equation

We consider the operator L defined, for any $\phi \in C_b^2(\mathbb{R}^3)$, by

$$(2.2.4) \quad L\phi(v, v_*) = \frac{1}{2} \sum_{i,j=1}^3 l_{ij}(v - v_*) \partial_{ij}^2 \phi(v) + \sum_{i=1}^3 b_i(v - v_*) \partial_i \phi(v),$$

where l_{ij} is defined in (2.1.8) and

$$(2.2.5) \quad b_i(z) = \sum_{j=1}^3 \partial_j l_{ij}(z) = -2|z|^\gamma z_i, \quad \text{for } i = 1, 2, 3.$$

For any $\phi \in C_b^2$, we have

$$\begin{aligned} |L\phi(v, v_*)| &\leq C_\phi(|v - v_*|^{\gamma+1} + |v - v_*|^{\gamma+2}) \\ &\leq C_\phi\left(1 + |v|^2 + |v_*|^2 + |v - v_*|^{\gamma+1}\mathbb{1}_{\gamma \in [-3, -1]}\right). \end{aligned}$$

We can thus observe that all the terms in the following definition are well-defined.

Definition 2.2.3. *Let $\gamma \in [-3, 0)$. We say that $(g_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ is a weak solution to (2.1.7) if*

$$(2.2.6) \quad \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+1} g_t(dv) g_t(dv_*) dt < \infty,$$

(which is automatically satisfied if $\gamma \in [-1, 0)$) and if for any $\phi \in C_b^2(\mathbb{R}^3)$ and any $t \in [0, T]$,

$$(2.2.7) \quad \int_{\mathbb{R}^3} \phi(v) g_t(dv) = \int_{\mathbb{R}^3} \phi(v) g_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L\phi(v, v_*) g_s(dv) g_s(dv_*) ds.$$

We now recall a result of Fournier and Guérin [33] which gives existence and uniqueness of a weak solution for the Landau equation.

Theorem 2.2.4. *(i) Assume that $\gamma \in (-2, 0)$. Let $p(\gamma) := \gamma^2/(2 + \gamma)$. Let $g_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \mathcal{P}_p(\mathbb{R}^3)$ for some $p > p(\gamma)$ satisfy also $H(g_0) < \infty$. Consider $q \in (3/(3 + \gamma), (3p - 3\gamma)/(p - 3\gamma)) \subset (3/(3 + \gamma), 3)$. Then the Landau equation (2.1.7) has a unique weak solution $(g_t)_{t \geq 0}$ in $L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{loc}^1([0, \infty), L^q(\mathbb{R}^3))$.*

(ii) Assume that $\gamma \in (-3, 0)$, and let $q > 3/(3 + \gamma)$. Let $g_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$. Then there exists $T_ > 0$ depending on $q, \|g_0\|_{L^q}$ such that there exists a unique weak solution $(g_t)_{t \in [0, T_*]}$ to (2.1.7) lying in $L^\infty([0, T_*], \mathcal{P}_2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3))$.*

(iii) Assume that $\gamma = -3$. Let $g_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then there exists $T_ > 0$ depending on $\|g_0\|_{L^\infty}$ such that there exists a unique weak solution $(g_t)_{t \in [0, T_*]}$ to (2.1.7) lying in $L^\infty([0, T_*], \mathcal{P}_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$.*

(iv) For any $t \geq 0$ (case (i)) or $t \in [0, T_]$ (case (ii) and (iii)), we have*

$$(2.2.8) \quad \int_{\mathbb{R}^3} g_t(v) \phi(v) dv = \int_{\mathbb{R}^3} g_0(v) \phi(v) dv, \quad \phi(v) = 1, v, |v|^2.$$

We also have the decay of entropy: for all $t \geq 0$ (case (i)) or $t \in [0, T_*]$ (case (ii) and (iii)),

$$(2.2.9) \quad \int_{\mathbb{R}^3} g_t(v) \log g_t(v) dv \leq \int_{\mathbb{R}^3} g_0(v) \log g_0(v) dv.$$

Furthermore, if $m_p(g_0) < \infty$ for some $p \geq 2$, then $\sup_{[0, T]} m_p(g_s) < \infty$ for all $T \geq 0$ (case (i)) or all $T \in [0, T_*]$ (case (ii) and (iii)).

For Points (i) and (ii), one can see Fournier-Guérin [33, Corollary 1.4]. For Point (iii), one can see Arsen'ev-Peskov [4] for the existence and Fournier [29] for the uniqueness of $(g_t)_{t \in [0, T_*]}$. The conservation of mass, momentum and energy and the decay of entropy are classical in Point (iv). For the propagation of moments, one can see Villani [71, Section 2.4 p 73] for $\gamma \in (-2, 0)$ and [69, Appendix B p 193] for $\gamma \in [-3, -2]$.

2.2.3 The Boltzmann equation

We take here the notation of Fournier-Mélérard [34]. For each $X \in \mathbb{R}^3$, we introduce $I(X), J(X) \in \mathbb{R}^3$ such that $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is an orthonormal basis of \mathbb{R}^3 . We also require that $I(-X) = -I(X)$ and $J(-X) = -J(X)$ for convenience. For $X, v, v_* \in \mathbb{R}^3$, for $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$, we set

$$(2.2.10) \quad \begin{cases} \Gamma(X, \varphi) := (\cos \varphi)I(X) + (\sin \varphi)J(X), \\ v' := v'(v, v_*, \theta, \varphi) := v - \frac{1-\cos \theta}{2}(v - v_*) + \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ v'_* := v'_*(v, v_*, \theta, \varphi) := v_* + \frac{1-\cos \theta}{2}(v - v_*) - \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ a := a(v, v_*, \theta, \varphi) := (v' - v) = -(v'_* - v_*), \end{cases}$$

which is nothing but a suitable spherical parametrization of (2.1.2): we write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v-v_*}{|v-v_*|} \cos \theta + \frac{I(v-v_*)}{|v-v_*|} \sin \theta \cos \varphi + \frac{J(v-v_*)}{|v-v_*|} \sin \theta \sin \varphi$. We can now give the notion of weak solution of Boltzmann's equation.

Definition 2.2.5. Consider a collision kernel $B(|v - v_*|, \theta) \sin \theta = \Phi(|v - v_*|)\beta(\theta)$ with β satisfying **(A2)**. A family $(f_t)_{t \in [0, T]} \in L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ is said to be a weak solution to (2.1.1) if

$$(2.2.11) \quad \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^2 \Phi(|v - v_*|) f_t(dv) f_t(dv_*) dt < \infty,$$

and if for any $\phi \in C_b^2(\mathbb{R}^3)$ and any $t \in [0, T]$,

$$(2.2.12) \quad \int_{\mathbb{R}^3} \phi(v) f_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A\phi(v, v_*) f_s(dv) f_s(dv_*) ds,$$

where

$$(2.2.13) \quad A\phi(v, v_*) = \frac{\Phi(|v - v_*|)}{2} \int_0^\pi \int_0^{2\pi} [\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)] d\varphi \beta(\theta) d\theta.$$

For any $v, v_* \in \mathbb{R}^3$, $\theta \in [0, \pi]$ and $\phi \in C_b^2(\mathbb{R}^3)$, we have (see Villani [70, p 291])

$$(2.2.14) \quad \left| \int_0^{2\pi} [\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)] d\varphi \right| \leq C \|\phi''\|_\infty \theta^2 |v - v_*|^2,$$

so that **(A2)** and (2.2.11) ensure that all the terms in (2.2.12) are well-defined.

We now give a result of existence and uniqueness for the Boltzmann equation with soft potentials.

Theorem 2.2.6. *Let $\gamma \in (-3, 0)$, $\nu \in (0, 2)$ and B be a collision kernel which satisfies **(A1)**(γ)-**A2**-**A3**(ν). For $\epsilon \in (0, \pi]$, we consider B_ϵ as in (2.1.4).*

(i) *We assume that $\gamma \in (-1, 0)$ and $\nu \in (-\gamma, 1)$. For some $p > \gamma^2/(\nu + \gamma)$, let $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \mathcal{P}_p(\mathbb{R}^3)$ with $H(f_0) < \infty$. Then for any $\epsilon \in (0, \pi]$, there exists a unique weak solution $(f_t^\epsilon)_{t \in [0, \infty)}$ to (2.1.1) with collision kernel B_ϵ starting from f_0 lying in $L_{loc}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{loc}^1([0, \infty), L^q(\mathbb{R}^3))$ for some (explicit) $q \in (3/(3 + \gamma), 3/(3 - \nu))$ with estimates uniform in ϵ .*

(ii) *We next consider the general case. Let $q \in (3/(3 + \gamma), \infty)$. For any $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, there exists $T_* = T_*(\|f_0\|_{L^q}, q) > 0$ such that for any $\epsilon \in (0, \pi]$, there exists a unique weak solution $(f_t^\epsilon)_{t \in [0, T_*]}$ to (2.1.1) with collision kernel B_ϵ starting from f_0 lying in $L^\infty([0, T_*], \mathcal{P}_2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3))$, with estimates uniform in ϵ .*

(iii) *For any $t \geq 0$ (case (i)) or $t \in [0, T_*]$ (case (ii)), any $\epsilon \in (0, \pi)$,*

$$(2.2.15) \quad \int_{\mathbb{R}^3} f_t^\epsilon(v) \phi(v) dv = \int_{\mathbb{R}^3} f_0(v) \phi(v) dv, \quad \phi(v) = 1, v, |v|^2,$$

and

$$(2.2.16) \quad \int_{\mathbb{R}^3} f_t^\epsilon(v) \log f_t^\epsilon(v) dv \leq \int_{\mathbb{R}^3} f_0(v) \log f_0(v) dv.$$

Furthermore, if $\gamma \in (-2, 0)$ and $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ for some $p \geq 4$, then for any $\epsilon \in (0, \pi)$, any $T \geq 0$ (case (i)) or any $T \in [0, T_*]$ (case (ii)),

$$\sup_{[0, T]} m_p(f_t^\epsilon) \leq C_{p, T} m_p(f_0),$$

where $C_{p, T}$ is a constant which does not depend on ϵ .

To prove (i) and (ii), we follow the line of some proofs in Fournier-Mouhot [35] and Fournier-Guérin [32].

Proof. Point (ii) is a consequence of [32, Proof of Corollary 1.5, Step 2] (recall **(A4)**). More precisely, we only need to check in their proof that T_* does not depend on ϵ . For this, it suffices to prove that for any $\epsilon > 0$, there exists a constant C which does not depend on ϵ such that any weak solution to (2.1.1) (with cross section B_ϵ) *a priori* satisfies

$$(2.2.17) \quad \frac{d}{dt} \|f_t^\epsilon\|_{L^q} \leq C(1 + \|f_t^\epsilon\|_{L^q}^2).$$

This will guarantee that for $0 \leq t \leq T_* := \frac{1}{2C}(\pi/2 - \arctan \|f_0\|_{L^q})$, we have

$$\|f_t^\epsilon\|_{L^q} \leq \tan(\arctan \|f_0\|_{L^q} + Ct) \leq \tan\left(\frac{\pi}{4} + \frac{1}{2} \arctan \|f_0\|_{L^q}\right).$$

We classically may replace in $A\phi$ (recall (2.2.13)) $\beta_\epsilon(\theta)$ by $\hat{\beta}_\epsilon(\theta) = [\beta_\epsilon(\theta) + \beta_\epsilon(\pi - \theta)]\mathbb{1}_{\theta \in (0, \pi/2]}$, see e.g. Desvillettes-Mouhot [24, Section 2]. Following the line of [24, proof of Proposition 3.2], we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv &\leq (q-1) \int_{\mathbb{R}^3} f_t^\epsilon(v_*) dv_* \int_{\mathbb{R}^3} dv |v - v_*|^\gamma \int_0^{\pi/2} \hat{\beta}_\epsilon(\theta) d\theta \\ &\quad \int_0^{2\pi} d\varphi [(f_t^\epsilon)^q(v') - (f_t^\epsilon)^q(v)]. \end{aligned}$$

Using the cancellation Lemma of Alexandre-Desvillettes-Villani-Wennberg [1, Lemma 1] (with $N = 3$, f given by $(f_t^\epsilon)^q$, and $B(|v - v_*|, \cos \theta) \sin \theta = \hat{\beta}_\epsilon(\theta) |v - v_*|^\gamma$), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv &\leq 2\pi(q-1) \int_{\mathbb{R}^3} f_t^\epsilon(v_*) dv_* \int_{\mathbb{R}^3} (f_t^\epsilon)^q dv \int_0^{\pi/2} \hat{\beta}_\epsilon(\theta) d\theta \\ &\quad |\cos^{-3}(\theta/2) (|v - v_*| \cos^{-1}(\theta/2))^\gamma - |v - v_*|^\gamma|. \end{aligned}$$

One easily checks that $|\cos^{-3}(\theta/2)(|v - v_*| \cos^{-1}(\theta/2))^\gamma - |v - v_*|^\gamma| \leq C|v - v_*|^\gamma \theta^2$ for all $\theta \in (0, \pi/2]$ (where C depends only on γ). Since $\int_0^{\pi/2} \theta^2 \hat{\beta}_\epsilon(\theta) d\theta \leq \int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta = \frac{4}{\pi}$, we finally get with $C = C(\gamma, q)$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv &\leq C \int_{\mathbb{R}^3} (f_t^\epsilon)^q(v) dv \int_{\mathbb{R}^3} |v - v_*|^\gamma f_t^\epsilon(v_*) dv_* \\ &\leq C \int_{\mathbb{R}^3} (f_t^\epsilon)^q(v) dv + C_{\gamma,q} \left[\int_{\mathbb{R}^3} (f_t^\epsilon)^q(v) dv \right]^{1+1/q}, \end{aligned}$$

by (2.2.2) and since $q > 3/(3 + \gamma)$. This yields

$$\frac{d}{dt} \|f_t^\epsilon\|_{L^q} = \frac{1}{q} \|f_t^\epsilon\|_{L^q}^{1-q} \frac{d}{dt} \int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv \leq C \|f_t^\epsilon\|_{L^q} + C_{\gamma,q} \|f_t^\epsilon\|_{L^q}^2,$$

from which (2.2.17) immediately follows.

We now prove (iii). First observe that the conservation of mass, momentum and kinetic energy and the decay of entropy are classical.

Next let $\gamma \in (-2, 0)$ and $p \geq 4$. We want to apply (2.2.12) with $\phi(v) = |v|^p$. We set $\Delta = |v'|^p + |v'_*|^p - |v|^p - |v_*|^p$ (see (2.2.10)). Observing that $v' = v + a$, $v'_* = v_* - a$, and $\nabla \phi(v) = p|v|^{p-2}v$, $\phi''(v) = p|v|^{p-2}I_3 + p(p-2)|v|^{p-4}vv^*$ (where ϕ'' is the Hessian matrix of ϕ) and using Taylor's formula, we have

$$\begin{aligned} \Delta &= a \cdot (p|v|^{p-2}v - p|v_*|^{p-2}v_*) \\ &+ \frac{1}{2}a \cdot \left[p(|w_1|^{p-2} + |w_2|^{p-2})a + p(p-2) \left(|w_1|^{p-4}(w_1w_1^*)a + |w_2|^{p-4}(w_2w_2^*)a \right) \right] \\ &= pa \cdot (|v|^{p-2}(v - v_*) + (|v|^{p-2} - |v_*|^{p-2})v_*) \\ &+ \frac{p}{2} \left[(|w_1|^{p-2} + |w_2|^{p-2})|a|^2 + (p-2) \left(|w_1|^{p-4}(a \cdot w_1)^2 + |w_2|^{p-4}(a \cdot w_2)^2 \right) \right], \end{aligned}$$

where $w_1 = v + \lambda_1 a$ for some $\lambda_1 \in [0, 1]$ and $w_2 = v_* + \lambda_2 a$ for some $\lambda_2 \in [0, 1]$. We have $|w_1|^{p-2} + |w_2|^{p-2} \leq C_p(|v|^{p-2} + |v_*|^{p-2})$ where C_p is a constant which only depends on p . Observing that

$$\begin{aligned} \left| |v|^{p-2} - |v_*|^{p-2} \right| |v_*| &\leq C_p |v - v_*| (|v|^{p-3} + |v_*|^{p-3}) |v_*| \\ &\leq C_p |v - v_*| (|v|^{p-2} + |v_*|^{p-2}), \end{aligned}$$

that $|a|^2 = \frac{1-\cos\theta}{2} |v - v_*|^2$, $\int_0^{2\pi} a d\varphi = -\frac{1-\cos\theta}{2} (v - v_*)$, $\int_0^\pi \frac{1-\cos\theta}{2} \beta_\epsilon(\theta) d\theta \leq \frac{4}{\pi}$

by (2.1.5) and using (2.2.12) with ϕ , we get

$$\begin{aligned} \frac{d}{dt} m_p(f_t^\epsilon) &\leq C_p \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} (|v|^{p-2} + |v_*|^{p-2}) f_t^\epsilon(dv) f_t^\epsilon(dv_*) \\ &\leq C_p \left(1 + m_p(f_t^\epsilon) + m_2(f_t^\epsilon) m_{p-2}(f_t^\epsilon) \right) \\ &\leq C_p (1 + m_p(f_t^\epsilon)), \end{aligned}$$

with C depending on $p, \gamma, m_2(f_0)$ (we used that $x^{\gamma+2} \leq C_\gamma(1+x^2)$ for any $x \geq 0$). Point (iii) immediately follows.

The existence and the uniqueness in (i) are already proved in Fournier-Guérin [32]. We only have to check that the estimates are uniform in ϵ . For that, it suffices to show that for any $\alpha \in (0, \gamma)$

$$(2.2.18) \quad \int_0^T \|(1 + |v|^{\gamma-\alpha}) f_t^\epsilon\|_{L^{\frac{3}{3-\nu}}} dt \leq C(1+T),$$

with C independent of ϵ . Indeed, since we have $\sup_{[0,T]} m_p(f_t^\epsilon) \leq C$ for some $p > \frac{\gamma^2}{\gamma+\nu}$ (with C independent of ϵ) by (iii), we will get

$$\|f^\epsilon\|_{L^1([0,T], L^q(\mathbb{R}^3))} \leq C_{T,q},$$

for some $q \in (3/(3+\gamma), 3/(3-\nu))$ by Fournier-Mouhot [32, Step 3 of the proof of Corollary 2.4]. Looking at Desvillettes-Mouhot [24, paragraph before Equation (3.2)], we see that to prove (2.2.18), it suffices to check that

$$(2.2.19) \quad \int_0^T \|\sqrt{f_t^\epsilon}\|_{H^{\nu/2}(|v| \leq R)}^2 dt \leq CR^{|\gamma|}(1+T),$$

for some constant C which does not depend on ϵ . It remains to follow the line of Alexandre-Desvillettes-Villani-Wennberg [1, Theorem 1] to get (2.2.19). More precisely, we have to check that the constants which appear in the following inequality [1, Theorem 1] (observe that here $\Phi(|v|) = |v|^\gamma$ does not vanish at 0)

$$(2.2.20) \quad \|\sqrt{f_t^\epsilon}\|_{H^{\nu/2}(|v| < R)}^2 \leq 2c_{f^\epsilon}^{-1} R^{|\gamma|} \left(D(f_t^\epsilon) + (C_1 + C_2) \|(1 + |v|^2) f_t^\epsilon\|_{L^1}^2 \right),$$

do not depend on ϵ , where $D(f_t^\epsilon)$ is the functional of dissipation of entropy (see (2.2.22) below). The constant C_1 comes from [1, Corollary 2]. This constant is such that (observe that there is a misprint in the corollary)

$$\Lambda(|v - v_*|) + |v - v_*|\Lambda'(|v - v_*|) \leq C_1(|v - v_*|^\gamma + |v - v_*|^2),$$

where

$$\Lambda(|v - v_*|) = \int_0^\pi |v - v_*|^\gamma (1 - \cos \theta) \beta_\epsilon(\theta) d\theta,$$

and

$$\Lambda'(|v - v_*|) = \int_0^\pi \sup_{1 < \lambda \leq \sqrt{2}} \frac{|v - v_*|^\gamma (\lambda^\gamma - 1)}{|v - v_*|(\lambda - 1)} (1 - \cos \theta) \beta_\epsilon(\theta) d\theta.$$

We can thus take $C_1 = \frac{|\gamma|+1}{2} \int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta = 2 \frac{|\gamma|+1}{\pi}$. Then we deal with the constant C_2 which comes from [1, Lemma 2]. This constant depends on

$$\int_0^{\pi/2} \cos^{-4} \frac{\theta}{2} \sin^2 \frac{\theta}{2} \beta_\epsilon(\theta) d\theta \leq C \int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta$$

and since this last integral is equal to $\frac{4}{\pi}$, the constant C_2 does not depend on ϵ .

The constant c_{f^ϵ} comes from [1, Proposition 2]. It is of the form $C'_{f^\epsilon} K$. First $C'_{f^\epsilon} > 0$ is controlled (from below) by upperbounds of $m_1(f_t^\epsilon)$ and $\int_{\mathbb{R}^3} f_t^\epsilon \log(1 + f_t^\epsilon(v)) dv$, which are both classically controlled (uniformly in ϵ) by $m_2(f_0)$ and $H(f_0)$. Next, $K > 0$ is such that for all $|\xi| \geq 1$,

$$\int_0^{\pi/2} \left(\frac{|\xi|^2}{2} (1 - \cos \theta) \wedge 1 \right) \beta_\epsilon(\theta) d\theta \geq K |\xi|^\nu.$$

One easily deduces from **(A3)(ν)** that such an inequality holds uniformly in $\epsilon \in (0, 1)$.

Hence (2.2.20) holds uniformly in $\epsilon \in (0, 1)$, and we find that

$$(2.2.21) \quad \|\sqrt{f_t^\epsilon}\|_{H^{\nu/2}(|v| < R)}^2 \leq CR^{|\gamma|} \left[D(f_t^\epsilon) + (1 + m_2(f_t^\epsilon))^2 \right],$$

for some constant C depending only on f_0 (and on γ, β but not on ϵ). Integrating (2.2.21) in time and using that

$$(2.2.22) \quad \int_0^T D(f_t^\epsilon) dt = H(f_0) - H(f_t^\epsilon) \leq H(f_0) + Cm_2(f_0),$$

(because classically, $H(f) \geq -Cm_2(f)$), we finally deduce (2.2.19) and that concludes the proof. \square

We finally treat the Coulomb case.

Theorem 2.2.7. *Assume (AC) and let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. Then there exists a unique weak solution $(f_t^\epsilon)_{t \in [0, \infty)}$ to (2.1.1). Furthermore, if $f_0 \in L^\infty(\mathbb{R}^3)$, then there exists $T_* = T_*(\|f_0\|_{L^\infty}) > 0$ such that $\sup_{\epsilon \in (0, 1)} \sup_{[0, T_*]} \|f_t^\epsilon\|_{L^\infty} < \infty$.*

Proof. We observe that for $\epsilon > 0$ fixed, we consider a cutoff case with a bounded cross section: for any $v, v_* \in \mathbb{R}^3$ and $\theta \in [0, \pi/2]$, $B_\epsilon(|v - v_*|, \theta) \leq C_\epsilon$. The existence and the uniqueness of $(f_t^\epsilon)_{t \in [0, \infty)}$ are thus classical.

For the stability in $L^\infty(\mathbb{R}^3)$, like in the previous proof (there is no need to introduce $\hat{\beta}_\epsilon$ here since β_ϵ is supported in $[0, \pi/2]$), we have for all $q \geq 1$, all $\epsilon \in (0, 1)$,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv \\ & \leq (q-1) \int_{\mathbb{R}^3} f_t^\epsilon(v_*) dv_* \int_{\mathbb{R}^3} dv (|v - v_*| + h_\epsilon)^{-3} \int_0^{\pi/2} \beta_\epsilon(\theta) d\theta \\ & \quad \int_0^{2\pi} d\varphi [(f_t^\epsilon)^q(v') - (f_t^\epsilon)^q(v)] \\ & \leq (q-1) \int_{\mathbb{R}^3} f_t^\epsilon(v_*) dv_* \int_{\mathbb{R}^3} dv |v - v_*|^{-3} \int_0^{\pi/2} \beta_\epsilon(\theta) d\theta \\ & \quad \int_0^{2\pi} d\varphi [(f_t^\epsilon)^q(v') - (f_t^\epsilon)^q(v)]. \end{aligned}$$

Using now the cancellation Lemma of Alexandre-Villani [2, Proposition 3] (with $N = 3$, f given by $(f_t^\epsilon)^q$, and $B(|v - v_*|, \cos \theta) \sin \theta = \beta_\epsilon(\theta) |v - v_*|^{-3}$), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv \leq \lambda_\epsilon (q-1) \int_{\mathbb{R}^3} (f_t^\epsilon(v_*))^{q+1} dv_* \leq C(q-1) \|f_t^\epsilon\|_{L^\infty} \|f_t^\epsilon\|_{L^q}^q,$$

since

$$\begin{aligned} \lambda_\epsilon & := \frac{4\pi^2}{3} \int_0^{\pi/2} \log \frac{1}{\cos \theta/2} \beta_\epsilon(\theta) d\theta \leq \frac{4\pi^2}{3} \int_0^{\pi/2} \frac{1}{\cos \theta/2} (1 - \cos \theta/2) \beta_\epsilon(\theta) d\theta \\ & \leq \frac{\sqrt{2}\pi^2}{3} \int_0^{\pi/2} \theta^2 \beta_\epsilon(\theta) d\theta = \frac{4\sqrt{2}\pi}{3}. \end{aligned}$$

We thus get

$$\begin{aligned} \frac{d}{dt} \|f_t^\epsilon\|_{L^q} &\leq \frac{1}{q} \left(\int_{\mathbb{R}^3} |f_t^\epsilon(v)|^q dv \right)^{1/q-1} C(q-1) \|f_t^\epsilon\|_{L^\infty} \|f_t^\epsilon\|_{L^q}^q \\ &\leq C \|f_t^\epsilon\|_{L^\infty} \|f_t^\epsilon\|_{L^q}. \end{aligned}$$

Making q tend to infinity, we get

$$\frac{d}{dt} \|f_t^\epsilon\|_{L^\infty} \leq C \|f_t^\epsilon\|_{L^\infty}^2,$$

and thus taking $T_* < \frac{1}{C\|f_0\|_{L^\infty}}$, we have for any $t < T_*$

$$\|f_t^\epsilon\|_{L^\infty} \leq \frac{\|f_0\|_{L^\infty}}{1 - C\|f_0\|_{L^\infty}t}.$$

This concludes the proof. \square

2.3 A general estimate for soft potentials

In this section, we give a general estimate for the distance between a solution of Boltzmann's equation and a solution of Landau's equation (for soft potentials) from which Theorem 2.1.1 follows.

Theorem 2.3.1. *Let $\gamma \in (-3, 0)$ and let B be a collision kernel which satisfies **(A1(γ)-A2-A4)**. Let $T > 0$ and $p \geq 5$. Let $f = (f_t)_{t \in [0, T]}$ be a weak solution of (2.1.1) with collision kernel B and $g = (g_t)_{t \in [0, T]}$ be a weak solution of (2.1.7) with $H(g_0) < \infty$. We assume that $f \in L^1([0, T], J_\gamma)$, $g \in L^1([0, T], J_\gamma) \cap L^\infty([0, T], \mathcal{P}_{p+2}(\mathbb{R}^3))$ and if $\gamma \in (-3, -1)$, that f and g belong to $L^\infty([0, T], J_{\gamma+1})$. Assume furthermore that $\int_0^\pi \theta^4 \beta(\theta) d\theta \leq 1$. Then for any $n \geq 1$, $\eta \in (0, \pi)$ and $M > \sqrt{2m_2(g_0)}$,*

$$\begin{aligned} \sup_{[0, T]} \mathcal{W}_2^2(f_t, g_t) &\leq C \left[\mathcal{W}_2^2(f_0, g_0) + \frac{1}{n} + \int_0^\pi \theta^4 \beta(\theta) d\theta \right. \\ &\quad \left. + \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right) + \frac{1}{M^p} \right], \end{aligned}$$

where

$$(2.3.1) \quad r_\eta = \frac{\pi}{4} \int_0^\eta \theta^2 \beta(\theta) d\theta$$

and where C depends on $p, T, \kappa_1, \gamma, \int_0^T J_\gamma(f_s + g_s) ds, \sup_{[0, T]} m_{p+2}(g_s), H(g_0)$, and additionally on $\sup_{[0, T]} J_{\gamma+1}(f_s + g_s)$ if $\gamma \in (-3, -1)$.

This result is proved in Section 2.5. We can now deduce Theorem 2.1.1.

Proof of Theorem 2.1.1. We consider a collision kernel which satisfies **(A1)(γ)-A2-A3(ν)** and we set $\beta_\epsilon = \frac{\pi^3}{\epsilon^3} \beta\left(\frac{\pi\theta}{\epsilon}\right) \mathbb{1}_{|\theta| < \epsilon}$ and $B_\epsilon(|v - v_*|, \theta) \sin \theta = |v - v_*|^\gamma \beta_\epsilon(\theta)$. We first note that **(A2)** is satisfied by B_ϵ (see (2.1.5)) and that **(A3(ν))** implies **(A4)** (see Lemma 2.2.1).

We now prove point (i). We thus assume that $\gamma \in (-1, 0)$, $\nu \in (-\gamma, 1)$ and fix $T > 0$. Since $f_0 \in \mathcal{P}_{p+2}(\mathbb{R}^3)$ for some $p > \max(5, \gamma^2/(\nu + \gamma))$ and since $H(f_0) < \infty$, by Theorems 2.2.6 and 2.2.4, there exists $(f_t^\epsilon)_{t \in [0, T]}$ solution to (2.1.1) with collision kernel B_ϵ and $(g_t)_{t \in [0, T]}$ solution to (2.1.7) both starting from f_0 and lying in $L^\infty([0, T], \mathcal{P}_{p+2}(\mathbb{R}^3)) \cap L^1([0, T], L^q(\mathbb{R}^3))$ for some $q \in (3/(3+\gamma), 3/(3-\nu))$ (uniformly in $\epsilon \in (0, 1)$). Now using (2.2.2), we get that $(f_t^\epsilon)_{t \in [0, T]}$ and $(g_t)_{t \in [0, T]}$ belong to $L^1([0, T], J_\gamma(\mathbb{R}^3))$ (uniformly in $\epsilon \in (0, 1)$). We thus can use Theorem 2.3.1 with $\beta = \beta_\epsilon, \eta = \epsilon, n \approx \epsilon^{\frac{-2p}{2p+3}}$ and $M = \sqrt{2m_2(f_0)} \epsilon^{\frac{-2}{2p+3}}$ and we get (observe that $\int_\eta^\pi \theta^2 \beta_\epsilon(\theta) d\theta = 0, r_\eta = 1$ and that $\int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \leq C\epsilon^2$)

$$\begin{aligned} \sup_{[0, T]} \mathcal{W}_2^2(f_t, g_t) &\leq C \left[\epsilon^{\frac{2p}{2p+3}} + \epsilon^2 + \epsilon^{\frac{2p+2}{2p+3}} (\log^2 \epsilon + \epsilon^{\frac{-2}{2p+3}}) + \epsilon^{\frac{2p}{2p+3}} \right] \\ &\leq C \epsilon^{\frac{2p}{2p+3}}, \end{aligned}$$

since $\log^2 \epsilon \leq C \epsilon^{\frac{-2}{2p+3}}$ for any $\epsilon \in (0, 1)$. Point (i) is proved.

For point (ii), we consider $f_0 \in \mathcal{P}_{p+2}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $p \geq 5$ and $q > \frac{3}{3+\gamma}$ with $H(f_0) < \infty$. By Theorems 2.2.6 and 2.2.4, there exists $T_* > 0, (f_t^\epsilon)_{t \in [0, T_*]}$ solution to (2.1.1) with collision kernel B_ϵ and $(g_t)_{t \in [0, T_*]}$ solution to (2.1.7) both starting from f_0 and lying in $L^\infty([0, T_*], \mathcal{P}_2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3))$ (uniformly in $\epsilon \in (0, 1)$). We also have that $(g_t)_{t \in [0, T_*]}$ belongs to $L^\infty([0, T_*], \mathcal{P}_{p+2}(\mathbb{R}^3))$. Using again (2.2.2), we get that $(f_t^\epsilon)_{t \in [0, T_*]}$ and $(g_t)_{t \in [0, T_*]}$ belong to $L^1([0, T_*], J_\gamma(\mathbb{R}^3))$ and to $L^\infty([0, T_*], J_{\gamma+1}(\mathbb{R}^3))$ if $\gamma \in (-3, -1)$, all this uniformly in $\epsilon \in (0, 1)$. We conclude the proof as previously. \square

2.4 Probabilistic interpretation of the equations

We will use probabilistic tools in order to prove Theorems 2.1.1 and 2.1.2, like in the paper of Tanaka [65]. Until the end of the article, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ will designate a Polish filtered probability space satisfying the usual conditions. Such a space is Borel isomorphic to the Lebesgue space $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ which we will use as an auxiliary space. To be as clear as possible, we will use the notation \mathbb{E} for the expectation and \mathcal{L} for the law of a random variable or process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and we will use the notation \mathbb{E}_α and \mathcal{L}_α for the expectation and law of random variables or processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$. The processes on $([0, 1], \mathcal{B}([0, 1]), d\alpha)$ will be called α -processes.

2.4.1 The Boltzmann equation

We first need to rewrite the collision operator A defined in (2.2.13) as in Fournier-Guérin [32]. The goal of this operation is to make disappear the velocity-dependance $|v - v_*|^\gamma$ in the rate. One can find the following lemma and its proof in [32, Lemma 2.1].

Lemma 2.4.1. *Let $B(|v - v_*|, \theta) \sin \theta = \Phi(|v - v_*|)\beta(\theta)$ with β satisfying (A2). We set*

$$(2.4.1) \quad k := \pi \int_0^\pi (1 - \cos \theta) \beta(\theta) d\theta.$$

Recalling (2.2.1) and (2.2.10), we define for $z \in (0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$,

$$(2.4.2) \quad c(v, v_*, z, \varphi) := a[v, v_*, G(z/\Phi(|v - v_*|)), \varphi].$$

We have $A\phi(v, v_*) = \frac{1}{2}[A_1\phi(v, v_*) + A_1\phi(v_*, v)]$ for all $v, v_* \in \mathbb{R}^3$ and $\phi \in C_b^2(\mathbb{R}^3)$, where

$$(2.4.3) \quad \begin{aligned} A_1\phi(v, v_*) &= \int_0^\infty \int_0^{2\pi} \left(\phi[v + c(v, v_*, z, \varphi)] - \phi[v] - c[v, v_*, z, \varphi] \cdot \nabla \phi[v] \right) d\varphi dz \\ &\quad - k\Phi(|v - v_*|) \nabla \phi(v) \cdot (v - v_*) \\ &= \int_0^\infty \int_0^{2\pi} \left(\phi[v + c(v, v_*, z, \varphi + \varphi_0)] - \phi[v] \right. \\ &\quad \left. - c[v, v_*, z, \varphi + \varphi_0] \cdot \nabla \phi[v] \right) d\varphi dz - k\Phi(|v - v_*|) \nabla \phi(v) \cdot (v - v_*), \end{aligned}$$

the second equality holding for any $\varphi_0 \in [0, 2\pi)$ (which may depend on v, v_*, z). As a consequence, we may replace A by A_1 in (2.2.12).

We now recall a fundamental remark by Tanaka [65], slightly precised in Fournier-Méléard [34, Lemma 2.6].

Lemma 2.4.2. *There exists a measurable function $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow [0, 2\pi)$, such that for all $X, Y \in \mathbb{R}^3$, all $\varphi \in [0, 2\pi)$,*

$$(2.4.4) \quad |\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0(X, Y))| \leq 3|X - Y|,$$

where $\Gamma(X, Y)$ is defined in (2.2.10).

We now introduce a nonlinear stochastic differential equation linked with (2.1.1).

Proposition 2.4.3. *Let $B(|v - v_*|, \theta) \sin \theta = \Phi(|v - v_*|)\beta(\theta)$ satisfying (i) (A1(γ)) for some $\gamma \in (-3, 0)$, (A2) and (A4) or (ii) (AC). For some $T > 0$, let $f = (f_t)_{t \in [0, T]}$ be a solution to (2.1.1) lying in (i) $L^1([0, T], J_\gamma(\mathbb{R}^3)) \cap L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ or in (ii) $L^\infty([0, T], L^\infty(\mathbb{R}^3)) \cap L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$. Consider any α -process $(\tilde{V}_t)_{t \in [0, T]}$ such that $\mathcal{L}_\alpha(\tilde{V}_t) = f_t$ for all $t \in [0, T]$. Let also N be a $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure on $[0, T] \times [0, \infty) \times [0, 2\pi] \times [0, 1]$ with intensity measure $dsdzd\varphi d\alpha$, and V_0 a \mathcal{F}_0 -measurable random variable with law f_0 . Then there exists a unique process $(V_t)_{t \in [0, T]}$ such that for all $t \in [0, T]$,*

$$(2.4.5) \quad \begin{aligned} V_t = & V_0 + \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 c(V_{s-}, \tilde{V}_s(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ & - k \int_0^t \int_0^1 \Phi(|V_s - \tilde{V}_s(\alpha)|)(V_s - \tilde{V}_s(\alpha)) ds d\alpha, \end{aligned}$$

with k given by (2.4.1) and c given by (2.4.2). Furthermore, $\mathcal{L}(V_t) = f_t$ for all $t \in [0, T]$.

Proof. We start with case (i). In this case, the existence and the uniqueness of $(V_t)_{t \in [0, T]}$ are already proved in Fournier-Guérin [32, proof of Lemma 4.6, Steps 3 to 6]. We set $\mu_t = \mathcal{L}(V_t)$. Using Itô's formula for jump processes (see e.g. Ikeda-Watanabe [44, Theorem 5.1]) and taking expectations, we have for any $\phi \in C_b^2(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \phi(v) \mu_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A_1 \phi(v, v_*) \mu_s(dv) f_s(dv_*) ds.$$

We thus have $\mu_t = f_t$ for any $t \in [0, T]$ by [32, Lemma 4.6].

The case (ii) is easier since it is a cutoff case with bounded collision kernel and we leave it to the reader. \square

2.4.2 The Landau equation

To give a probabilistic interpretation of (2.1.7), we need to use a three-dimensional space-time white noise $W(ds, d\alpha)$ on $[0, T] \times [0, 1]$ with covariance measure $dsd\alpha$ (in the sense of Walsh [73]). Recall that W is an orthogonal martingale measure with covariance $dsd\alpha$.

Proposition 2.4.4. (i) *Let $\gamma \in [-3, 0)$. For some $T > 0$, let $g = (g_t)_{t \in [0, T]}$ be a solution to (2.1.7) lying in $L^1([0, T], J_\gamma(\mathbb{R}^3)) \cap L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ if $\gamma \in (-3, 0)$ and in $L^\infty([0, T], L^\infty(\mathbb{R}^3)) \cap L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ if $\gamma = -3$. Consider any α -process $(\tilde{Y}_t)_{t \in [0, T]}$ such that $\mathcal{L}_\alpha(\tilde{Y}_t) = g_t$ for all $t \in [0, T]$. Let also W be a three-dimensional space-time white noise on $[0, T] \times [0, 1]$ with covariance measure $dsd\alpha$, and Y_0 a \mathcal{F}_0 -measurable random variable with law g_0 . Then there exists a unique process $(Y_t)_{t \in [0, T]}$ such that for all $t \in [0, T]$,*

$$(2.4.6) \quad Y_t = Y_0 + \int_0^t \int_0^1 \sigma(Y_s - \tilde{Y}_s(\alpha)) W(ds, d\alpha) + \int_0^t \int_0^1 b(Y_s - \tilde{Y}_s(\alpha)) dsd\alpha,$$

with for any $z \in \mathbb{R}^3$, $b(z)$ given in (2.2.5) and

$$(2.4.7) \quad \sigma(z) = |z|^{\gamma/2} \begin{pmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{pmatrix}.$$

We observe that $\sigma(z)\sigma^*(z) = l(z)$ with $l(z)$ given by (2.1.8). Furthermore, $\mathcal{L}(Y_t) = g_t$ for all $t \in [0, T]$.

(ii) *It is possible to handle this construction in such a way that $\mathcal{L}((Y_t)_{t \in [0, T]}) = \mathcal{L}_\alpha((\tilde{Y}_t)_{t \in [0, T]})$.*

One can see Fournier-Guérin [33, Proposition 2.1] for the proof of point (i) when $\gamma \in (-3, 0)$ and Fournier [29, Proposition 10] when $\gamma = -3$.

Proof of point (ii). We first observe that the law of $(Y_t)_{t \in [0, T]}$ does not depend on the choice of $(\tilde{Y}_t)_{t \in [0, T]}$. To get convinced, use a substitution to rewrite $\int_0^t \int_0^1 \sigma(Y_s - \tilde{Y}_s(\alpha)) W(ds, d\alpha)$ as $\int_0^t \int_{\mathbb{R}^3} \sigma(Y_s - z) \hat{W}(ds, dz)$ where $\hat{W}(ds, dz)$ is a white noise with covariance $g_s(dz)ds$.

We thus consider some α -process $(\tilde{Z}_t)_{t \in [0, T]}$ such that $\mathcal{L}_\alpha(\tilde{Z}_t) = g_t$ for any $t \in [0, T]$ from which we build $(Z_t)_{t \in [0, T]}$ solution to (2.4.6). Next we consider an α -process $(\tilde{Y}_t)_{t \in [0, T]}$ such that $\mathcal{L}_\alpha((\tilde{Y}_t)_{t \in [0, T]}) = \mathcal{L}((Z_t)_{t \in [0, T]})$ from which we build $(Y_t)_{t \in [0, T]}$. Due to the previous observation, we have $\mathcal{L}((Y_t)_{t \in [0, T]}) = \mathcal{L}((Z_t)_{t \in [0, T]})$ and thus $\mathcal{L}((Y_t)_{t \in [0, T]}) = \mathcal{L}_\alpha((\tilde{Y}_t)_{t \in [0, T]})$. \square

2.5 Soft potentials

This section is devoted to the proof of Theorem 2.3.1. We fix $\gamma \in (-3, 0)$, $T > 0$ and we consider a collision kernel satisfying **(A1(γ)-A2-A4)**. We consider $(f_t)_{t \in [0, T]}$ and $(g_t)_{t \in [0, T]}$ solutions of (2.1.1) and (2.1.7) respectively.

2.5.1 Definition of the processes

We consider two random variables V_0 and Y_0 with law f_0 and g_0 respectively such that $\mathbb{E}[|V_0 - Y_0|^2] = \mathcal{W}_2^2(f_0, g_0)$. We fix a white noise W on $[0, T] \times [0, 1]$ with covariance measure $dsd\alpha$ and we consider a process $(Y_t)_{t \in [0, T]}$ and an α -process $(\tilde{Y}_t)_{t \in [0, T]}$ such that for any $t \in [0, T]$, $\mathcal{L}(Y_t) = \mathcal{L}_\alpha(\tilde{Y}_t) = g_t$, such that $\mathcal{L}_\alpha((\tilde{Y}_t)_{t \in [0, T]}) = \mathcal{L}((Y_t)_{t \in [0, T]})$ and such that (2.4.6) is satisfied. For any $t \in [0, T]$, we consider an α -random variable \tilde{V}_t with law f_t such that $\mathcal{W}_2^2(f_t, g_t) = \mathbb{E}_\alpha[|\tilde{V}_t - \tilde{Y}_t|^2]$ and we consider the solution $(V_t)_{t \in [0, T]}$ to (2.4.5) for some $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure N as in Proposition 2.4.3. We will precise later the dependence of N with the white noise W . We recall the equations satisfied by $(V_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$, and we introduce some intermediate processes (here $n \in \mathbb{N}^*$ is fixed)

$$\begin{aligned} V_t &= V_0 + \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 c(V_{s-}, \tilde{V}_s(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ &\quad - k \int_0^t \int_0^1 |V_s - \tilde{V}_s(\alpha)|^\gamma (V_s - \tilde{V}_s(\alpha)) ds d\alpha, \\ V_t^n &= \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 c(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ &\quad - k \int_0^t \int_0^1 |V_s - \tilde{V}_s(\alpha)|^\gamma (V_s - \tilde{V}_s(\alpha)) ds d\alpha + V_0, \end{aligned}$$

$$\begin{aligned}
I_t^n &= \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \tilde{N}(ds, dz, d\varphi, d\alpha) \\
&\quad + \int_0^t \int_0^1 b(V_s - \tilde{V}_s(\alpha)) ds d\alpha + V_0, \\
J_t^n &= V_0 + \int_{a_0}^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) W(ds, d\alpha) \\
&\quad + \int_0^t \int_0^1 b(V_s - \tilde{V}_s(\alpha)) ds d\alpha, \\
Y_t &= Y_0 + \int_0^t \int_0^1 \sigma(Y_s - \tilde{Y}_s(\alpha)) W(ds, d\alpha) + \int_0^t \int_0^1 b(Y_s - \tilde{Y}_s(\alpha)) ds d\alpha,
\end{aligned}$$

where (recall Lemma 2.4.2)

$$(2.5.1) \quad \Phi_n(s, \alpha) = \varphi_0(V_s - \tilde{V}_s(\alpha), Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha))$$

$$(2.5.2) \quad d(v, w, z, \varphi) = \frac{1}{2} G\left(\frac{z}{|v-w|^\gamma}\right) \Gamma(v-w, \varphi),$$

and where a_0 and ρ_n are defined as follows.

We consider the subdivision $0 < a_0^n < \dots < a_{\lfloor 2nT \rfloor - 1}^n < a_{\lfloor 2nT \rfloor}^n = T$ obtained using Proposition 2.7.5 with $h(s) = J_\gamma(g_s)$ on $[0, T]$. In order to lighten notation, we write $a_i = a_i^n$. For $s \in [0, T]$, we set

$$\rho_n(s) = \sum_{i=0}^{\lfloor 2nT \rfloor - 1} a_i \mathbb{1}_{s \in [a_i, a_{i+1})}.$$

By construction, we have $a_0 < 1/n$, $1/4n < a_{i+1} - a_i < 1/n$, whence $\sup_{[0, T]} |s - \rho_n(s)| \leq 1/n$, and

$$(2.5.3) \quad \int_{a_0}^T J_\gamma(g_{\rho_n(s)}) ds \leq 3 \int_0^T J_\gamma(g_s) ds + 3.$$

We end this subsection with the following lemma.

Lemma 2.5.1. *For any $v, v_* \in \mathbb{R}^3$, we have (recall (2.4.1), (2.4.2) and (2.5.2))*

$$(2.5.4) \quad \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)|^2 dz d\varphi = k |v - v_*|^{\gamma+2},$$

and

$$(2.5.5) \quad \int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi) - d(v, v_*, z, \varphi)|^2 dz d\varphi \leq \int_0^\pi \theta^4 \beta(\theta) d\theta |v - v_*|^{\gamma+2}.$$

Proof. We have for any $v, v_* \in \mathbb{R}^3$, $z \in [0, \infty)$ and $\varphi \in [0, 2\pi)$ (recall (2.2.10)),

$$\begin{aligned} |c(v, v_*, z, \varphi)|^2 &= \left| \frac{1 - \cos G\left(\frac{z}{|v-v_*|^\gamma}\right)}{2} (v - v_*) - \frac{\sin G\left(\frac{z}{|v-v_*|^\gamma}\right)}{2} \Gamma(v - v_*, \varphi) \right|^2 \\ &= \frac{\left(1 - \cos G\left(\frac{z}{|v-v_*|^\gamma}\right)\right)^2 + \sin^2 G\left(\frac{z}{|v-v_*|^\gamma}\right)}{4} |v - v_*|^2 \\ &= \frac{1 - \cos G\left(\frac{z}{|v-v_*|^\gamma}\right)}{2} |v - v_*|^2, \end{aligned}$$

since for any $X \in \mathbb{R}^3$, the vectors X and $\Gamma(X, \varphi)$ are orthogonal, and $|\Gamma(X, \varphi)| = |X|$. Using the change of variable $\theta = G\left(\frac{z}{|v-v_*|^\gamma}\right)$, we get

$$\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)|^2 dz d\varphi = \pi \int_0^\pi (1 - \cos \theta) \beta(\theta) d\theta |v - v_*|^{\gamma+2},$$

and (2.5.4) follows. Using the same arguments, we have

$$\begin{aligned} &\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi) - d(v, v_*, z, \varphi)|^2 dz d\varphi \\ &= \int_0^\pi \int_0^{2\pi} |v - v_*|^\gamma \left| \frac{\cos \theta - 1}{2} (v - v_*) + \frac{\sin \theta - \theta}{2} \Gamma(v - v_*, \varphi) \right|^2 \beta(\theta) d\varphi d\theta \\ &= 2\pi \int_0^\pi \frac{(\cos \theta - 1)^2 + (\sin \theta - \theta)^2}{4} \beta(\theta) d\theta |v - v_*|^{\gamma+2}. \end{aligned}$$

It suffices to observe that for $\theta \in [0, \pi]$

$$2\pi \frac{(\cos \theta - 1)^2 + (\sin \theta - \theta)^2}{4} \leq \theta^4$$

to conclude the proof. \square

2.5.2 The proof

We start with a preliminary lemma.

Lemma 2.5.2. (i) *There exists a constant C depending on $m_2(f_0)$ and additionally on $\sup_{s \in [0, T]} J_{\gamma+1}(f_s)$ if $\gamma \in (-3, -1)$ such that for $0 \leq t' \leq t \leq T$ with $t - t' < 1$,*

$$\mathbb{E} \left[|V_t - V_{t'}|^2 \right] \leq C(t - t').$$

The same bound holds for $\mathbb{E} \left[|Y_t - Y_{t'}|^2 \right]$ and $\mathbb{E}_\alpha \left[|\tilde{Y}_t - \tilde{Y}_{t'}|^2 \right]$ with C depending on $m_2(g_0)$ and additionally on $\sup_{s \in [0, T]} J_{\gamma+1}(g_s)$ if $\gamma \in (-3, -1)$.

(ii) *For all $t \in [0, T]$, we have*

$$\mathbb{E} \left[|V_t - V_{\rho_n(t)}|^2 \right] + \mathbb{E} \left[|Y_t - Y_{\rho_n(t)}|^2 \right] + \mathbb{E}_\alpha \left[|\tilde{Y}_t - \tilde{Y}_{\rho_n(t)}|^2 \right] \leq \frac{C}{n}.$$

Proof. Recalling that $t - \rho_n(t) \leq 1/n$, we observe that (ii) immediately follows from (i) taking $t' = \rho_n(t)$. Let's prove (i). Observing that

$$\begin{aligned} V_t - V_{t'} &= \int_{t'}^t \int_0^\infty \int_0^{2\pi} \int_0^1 c(V_{s-}, \tilde{V}_s(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ &\quad - k \int_{t'}^t \int_0^1 |V_s - \tilde{V}_s(\alpha)|^\gamma (V_s - \tilde{V}_s(\alpha)) ds d\alpha, \end{aligned}$$

and using (2.5.4), we get

$$\begin{aligned} \mathbb{E} \left[|V_t - V_{t'}|^2 \right] &\leq 2 \int_{t'}^t \int_0^\infty \int_0^{2\pi} \int_0^1 \mathbb{E} \left[|c(V_s, \tilde{V}_s(\alpha), z, \varphi)|^2 \right] ds dz d\varphi d\alpha \\ &\quad + 2k^2 \mathbb{E} \left[\left| \int_{t'}^t \int_0^1 |V_s - \tilde{V}_s(\alpha)|^\gamma (V_s - \tilde{V}_s(\alpha)) ds d\alpha \right|^2 \right] \\ &\leq 2k \int_{t'}^t \mathbb{E} \left[\mathbb{E}_\alpha [|V_s - \tilde{V}_s|^{\gamma+2}] \right] ds \\ &\quad + 2k^2 \mathbb{E} \left[\left(\int_{t'}^t \mathbb{E}_\alpha [|V_s - \tilde{V}_s|^{\gamma+1}] ds \right)^2 \right] \\ &=: A + B. \end{aligned}$$

We first deal with A . If $\gamma \in [-2, 0)$, using that $|a|^{\gamma+2} \leq 1 + |a|^2$ and recalling that $\mathbb{E}(|V_s|^2) = \mathbb{E}_\alpha(|\tilde{V}_s|^2) = m_2(f_0)$, we have

$$A \leq 4k \int_{t'}^t \mathbb{E} \left[\mathbb{E}_\alpha [1 + |V_s|^2 + |\tilde{V}_s|^2] \right] ds \leq 4k(1 + 2m_2(f_0))(t - t'),$$

and if $\gamma \in (-3, -2)$, then a.s., $\mathbb{E}_\alpha[|V_s - \tilde{V}_s|^{\gamma+2}] \leq 1 + \mathbb{E}_\alpha[|V_s - \tilde{V}_s|^{\gamma+1}] = 1 + \int_{\mathbb{R}^3} |V_s - v_*|^{\gamma+1} f_s(dv_*) \leq 1 + J_{\gamma+1}(f_s)$ (recall (2.1.9)), so that

$$A \leq 2k \int_{t'}^t (1 + J_{\gamma+1}(f_s)) ds \leq C(t - t'),$$

where C depends on $\sup_{s \in [0, T]} J_{\gamma+1}(f_s)$. We now deal with B . If $\gamma \in [-1, 0)$, using first the Cauchy-Schwarz inequality and then that $|a|^{2\gamma+2} \leq 1 + |a|^2$, we get

$$\begin{aligned} B &\leq 2k^2 \mathbb{E} \left[(t - t') \int_{t'}^t \mathbb{E}_\alpha[|V_s - \tilde{V}_s|^{2\gamma+2}] ds \right] \\ &\leq 4k^2 (t - t') \int_{t'}^t \mathbb{E} \left[\mathbb{E}_\alpha[1 + |V_s|^2 + |\tilde{V}_s|^2] \right] ds \leq 4k^2 (1 + 2m_2(f_0)) (t - t')^2, \end{aligned}$$

and if $\gamma \in (-3, -1)$, as previously, we have

$$B \leq 2k^2 \left(\int_{t'}^t J_{\gamma+1}(f_s) ds \right)^2 \leq C(t - t')^2.$$

This finally gives

$$\mathbb{E} \left[|V_t - V_{t'}|^2 \right] \leq C(t - t'),$$

where C depends on $m_2(f_0)$ and on $\sup_{s \in [0, T]} J_{\gamma+1}(f_s)$. The computation of $\mathbb{E} \left[|Y_t - Y_{t'}|^2 \right]$ is very similar and we leave it for the reader. Since $(\mathcal{L}_\alpha(\tilde{Y}_t)_{t \geq 0}) = (\mathcal{L}(Y_t)_{t \geq 0})$, we have $\mathbb{E}_\alpha \left[|\tilde{Y}_t - \tilde{Y}_{t'}|^2 \right] = \mathbb{E} \left[|Y_t - Y_{t'}|^2 \right]$ and that concludes the proof. \square

The following lemma states as follows.

Lemma 2.5.3. *There exists a constant C depending on $m_2(f_0)$, $m_2(g_0)$, $\int_0^T J_\gamma(f_s + g_s) ds$ and additionally on $\sup_{[0, T]} J_{\gamma+1}(f_s + g_s)$ if $\gamma \in (-3, -1)$, such that, if $t \in [a_0, T]$,*

$$\begin{aligned} \mathbb{E} \left[|V_t - V_t^n|^2 \right] &\leq C \left(\frac{1}{n} + \int_{a_0}^t J_\gamma(f_s + g_{\rho_n(s)}) \left(\mathbb{E}[|V_s - Y_s|^2] \right. \right. \\ &\quad \left. \left. + \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2] \right) ds \right). \end{aligned}$$

Proof. We have

$$\begin{aligned} V_t - V_t^n &= \int_0^{a_0} \int_0^\infty \int_0^{2\pi} \int_0^1 c(V_{s-}, \tilde{V}_s(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ &\quad + \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 \left[c(V_{s-}, \tilde{V}_s(\alpha), z, \varphi) \right. \\ &\quad \left. - c(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right] \tilde{N}(ds, dz, d\varphi, d\alpha). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{E}[|V_t - V_t^n|^2] &\leq 2 \int_0^{a_0} \int_0^\infty \int_0^{2\pi} \int_0^1 \mathbb{E}[|c(V_s, \tilde{V}_s(\alpha), z, \varphi)|^2] ds dz d\varphi d\alpha \\ &\quad + 2 \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 \mathbb{E}[|c(V_s, \tilde{V}_s(\alpha), z, \varphi) \\ &\quad - c(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha))|^2] ds dz d\varphi d\alpha. \end{aligned}$$

So using (2.5.4) and Fournier-Guérin [32, Lemma 2.3], we have

$$\begin{aligned} \mathbb{E}[|V_t - V_t^n|^2] &\leq C \int_0^{a_0} \int_0^1 \mathbb{E}[|V_s - \tilde{V}_s(\alpha)|^{\gamma+2}] ds d\alpha \\ &\quad + C \int_{a_0}^t \int_0^1 \mathbb{E} \left[(|V_s - Y_{\rho_n(s)}|^2 + |\tilde{V}_s(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2) \right. \\ &\quad \left. (|V_s - \tilde{V}_s(\alpha)|^\gamma + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma) \right] ds d\alpha. \end{aligned}$$

Since for any $x \geq 0$, $x^{\gamma+2} \leq C_\gamma(1+x^2)$ if $\gamma \in [-2, 0)$ and $\mathbb{E}_\alpha[|V_s - \tilde{V}_s|^{\gamma+2}] \leq 1 + \mathbb{E}_\alpha[|V_s - \tilde{V}_s|^{\gamma+1}] \leq 1 + \int_{\mathbb{R}^3} |V_s - v|^{\gamma+1} f_s(dv) \leq 1 + J_{\gamma+1}(f_s)$ a.s. if $\gamma \in (-3, -2)$, we get (recall that $a_0 < 1/n$)

$$\int_0^{a_0} \int_0^1 \mathbb{E}[|V_s - \tilde{V}_s(\alpha)|^{\gamma+2}] ds d\alpha = \int_0^{a_0} \mathbb{E}_\alpha[\mathbb{E}[|V_s - \tilde{V}_s|^{\gamma+2}]] ds \leq \frac{C}{n}.$$

We thus have

$$\begin{aligned}
\mathbb{E}\left[|V_t - V_t^n|^2\right] &\leq \frac{C}{n} \\
&+ C \int_{a_0}^t \mathbb{E}\left[|V_s - Y_{\rho_n(s)}|^2 \mathbb{E}_\alpha(|V_s - \tilde{V}_s|^\gamma + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^\gamma)\right] ds \\
&+ C \int_{a_0}^t \mathbb{E}_\alpha\left[|\tilde{V}_s - \tilde{Y}_{\rho_n(s)}|^2 \mathbb{E}(|V_s - \tilde{V}_s|^\gamma + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^\gamma)\right] ds \\
&\leq \frac{C}{n} + C \int_{a_0}^t \mathbb{E}[|V_s - Y_{\rho_n(s)}|^2] J_\gamma(f_s + g_{\rho_n(s)}) ds \\
&+ C \int_{a_0}^t \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_{\rho_n(s)}|^2] J_\gamma(f_s + g_{\rho_n(s)}) ds.
\end{aligned}$$

Using first that $\mathbb{E}[|V_s - Y_{\rho_n(s)}|^2] \leq 2\mathbb{E}[|V_s - Y_s|^2] + 2\mathbb{E}[|Y_s - Y_{\rho_n(s)}|^2]$, $\mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_{\rho_n(s)}|^2] \leq 2\mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2] + 2\mathbb{E}_\alpha[|\tilde{Y}_s - \tilde{Y}_{\rho_n(s)}|^2]$, next Lemma 2.5.2 and (2.5.3) concludes the proof. \square

We next estimate $V_t^n - I_t^n$.

Lemma 2.5.4. *There exists a constant C depending on T , γ , $\int_0^T J_\gamma(f_s) ds$, $m_2(f_0)$, $m_2(g_0)$ and additionally on $\sup_{[0,T]} J_{\gamma+1}(f_s + g_s)$ if $\gamma \in (-3, -1)$ such that, if $t \in [a_0, T]$,*

$$\mathbb{E}(|V_t^n - I_t^n|^2) \leq C \int_0^\pi \theta^4 \beta(\theta) d\theta.$$

Proof. We have (recall (2.2.5))

$$\begin{aligned}
V_t^n - I_t^n &= \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 \left[c(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right. \\
&\quad \left. - d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right] \tilde{N}(ds, dz, d\varphi, d\alpha) \\
&\quad - (k-2) \int_0^t \int_0^1 |V_s - \tilde{V}_s(\alpha)|^\gamma (V_s - \tilde{V}_s(\alpha)) ds d\alpha.
\end{aligned}$$

Recalling (2.4.1) and that $\int_0^\pi \theta^2 \beta(\theta) d\theta = \frac{4}{\pi}$, we first observe that

$$|k-2| = \pi \left| \int_0^\pi \left(1 - \cos \theta - \frac{\theta^2}{2}\right) \beta(\theta) d\theta \right| \leq \frac{\pi}{24} \int_0^\pi \theta^4 \beta(\theta) d\theta.$$

So recalling (2.5.5), we get (recall that $\int_0^\pi \theta^4 \beta(\theta) d\theta \leq 1$)

$$\begin{aligned} \mathbb{E}[|V_t^n - I_t^n|^2] &\leq 2 \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 \mathbb{E} \left[|c(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right. \\ &\quad \left. - d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha))|^2 \right] ds dz d\varphi d\alpha \\ &\quad + 2(k-2)^2 \mathbb{E} \left[\left(\int_0^t \int_0^1 |V_s - \tilde{V}_s(\alpha)|^{\gamma+1} ds d\alpha \right)^2 \right] \\ &\leq C \int_0^\pi \theta^4 \beta(\theta) d\theta \left(\int_{a_0}^t \mathbb{E} \left[\mathbb{E}_\alpha[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{\gamma+2}] \right] ds \right. \\ &\quad \left. + \mathbb{E} \left[\left(\int_0^t \mathbb{E}_\alpha[|V_s - \tilde{V}_s|^{\gamma+1}] ds \right)^2 \right] \right). \end{aligned}$$

We conclude using the same arguments as in the proof of Lemma 2.5.2 (recall that $Y_{\rho_n(s)} \sim g_{\rho_n(s)}$). \square

The following lemma is the key point of the proof of Theorem 2.3.1.

Lemma 2.5.5. *Assume that $m_{p+2}(g_0) < \infty$ for some $p \geq 5$. We can couple the Poisson measure N and the white noise W in such a way that there exists a constant C depending on $\gamma, T, m_{p+2}(g_0), H(g_0)$, and additionally on $\sup_{[0, T]} J_{\gamma+1}(f_s + g_s)$ if $\gamma \in (-3, -2)$ such that for any $M > \sqrt{2m_2(g_0)}$, any $\eta \in (0, \pi)$, any $t \in [a_0, T]$,*

$$\mathbb{E}[|I_t^n - J_t^n|^2] \leq C \left[\eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right) + \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \frac{1}{M^p} \right].$$

It is because of this lemma that we do not have an optimal rate of convergence (recall that we obtain here a bound in $\epsilon^{1/2-}$ for $\mathcal{W}_2(f_t^\epsilon, g_t)$ while we get a bound in ϵ for the Kac equation, see [31]). More precisely, it is due to the fact that we need to partition the interval $[0, T]$ in order to use Proposition 2.7.2.

Proof. We fix $\eta \in (0, \pi)$ and $M > \sqrt{2m_2(g_0)}$ for the whole proof, which we divide in several steps.

Step 1: For $0 < u < u'$ and y fixed, we set

$$\mu_u^{u'}(y) := \mathcal{L}\left(\int_u^{u'} \int_0^\infty \int_0^{2\pi} \int_0^1 d(y, \tilde{Y}_u(\alpha), z, \varphi) \mathbb{1}_{\{G(z/|y-\tilde{Y}_u(\alpha)|^\gamma) \leq \eta\}} \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} \tilde{N}(ds, dz, d\varphi, d\alpha)\right),$$

and

$$\nu_u^{u'}(y) := \mathcal{L}\left(\sqrt{r_\eta} \int_u^{u'} \int_0^1 \sigma(y - \tilde{Y}_u(\alpha)) \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} W(ds, d\alpha)\right).$$

We have

$$\begin{aligned} \text{Cov}(\nu_u^{u'}(y)) &= r_\eta \int_u^{u'} \int_0^1 \sigma(y - \tilde{Y}_u(\alpha)) \sigma^*(y - \tilde{Y}_u(\alpha)) \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} ds d\alpha \\ &= r_\eta (u' - u) \int_0^1 l(y - \tilde{Y}_u(\alpha)) \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} d\alpha \\ &= (u' - u) \zeta_u(y), \end{aligned}$$

where

$$\zeta_u(y) = r_\eta \int_0^1 l(y - \tilde{Y}_u(\alpha)) \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} d\alpha.$$

We thus observe that $\nu_u^{u'}(y) = \mathcal{N}(0, (u' - u)\zeta_u(y))$. Now in order to compute $\text{Cov}(\mu_u^{u'}(y))$, we first observe that for $v \in \mathbb{R}^3$ (recall (2.2.10))

$$\begin{aligned} \int_0^{2\pi} \Gamma(v, \varphi) \Gamma^*(v, \varphi) d\varphi &= \int_0^{2\pi} (\cos \varphi I(v) + \sin \varphi J(v)) \\ &\quad (\cos \varphi I(v)^* + \sin \varphi J(v)^*) d\varphi \\ &= \pi(I(v)I(v)^* + J(v)J(v)^*) \\ &= \pi(|v|^2 I_3 - vv^*) = \pi|v|^{-\gamma} l(v). \end{aligned}$$

Using that $\int_0^\infty G^2(z/x) \mathbb{1}_{\{G(z/x) \leq \eta\}} dz = x \int_0^\eta \theta^2 \beta(\theta) d\theta = \frac{4x}{\pi} r_\eta$, we have (recall

(2.5.2) and (2.1.8))

$$\begin{aligned}
& \text{Cov}(\mu_u^{u'}(y)) \\
&= \int_u^{u'} \int_0^\infty \int_0^{2\pi} \int_0^1 d(y, \tilde{Y}_u(\alpha), z, \varphi) d^*(y, \tilde{Y}_u(\alpha), z, \varphi) \mathbb{1}_{\{G(z/|y-\tilde{Y}_u(\alpha)|^\gamma) \leq \eta\}} \\
&\quad \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} d\alpha d\varphi dz ds \\
&= \frac{1}{4} \int_u^{u'} \int_0^\infty \int_0^{2\pi} \int_0^1 G^2(z/|y-\tilde{Y}_u(\alpha)|^\gamma) \Gamma(y-\tilde{Y}_u(\alpha), \varphi) \Gamma^*(y-\tilde{Y}_u(\alpha), \varphi) \\
&\quad \mathbb{1}_{\{G(z/|y-\tilde{Y}_u(\alpha)|^\gamma) \leq \eta\}} \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} d\alpha d\varphi dz ds \\
&= \frac{r_\eta}{\pi} (u' - u) \int_0^1 |y - \tilde{Y}_u(\alpha)|^\gamma \int_0^{2\pi} \Gamma(y - \tilde{Y}_u(\alpha), \varphi) \Gamma^*(y - \tilde{Y}_u(\alpha), \varphi) d\varphi \\
&\quad \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} d\alpha \\
&= (u' - u) r_\eta \int_0^1 l(y - \tilde{Y}_u(\alpha)) \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} d\alpha \\
&= (u' - u) \zeta_u(y).
\end{aligned}$$

Step 2: the aim of this step is to prove that

$$(2.5.6) \quad \mathcal{W}_2^2\left(\mu_u^{u'}(y), \nu_u^{u'}(y)\right) \leq C\eta^2 M^2 (1 + |y|^7) \left(\log^2 \frac{(u' - u)r_\eta}{\eta^2} + M\right).$$

We set

$$\kappa_u(y) := \sup_{\alpha, z, \varphi} |\zeta_u^{-1/2}(y) d(y, \tilde{Y}_u(\alpha), z, \varphi) \mathbb{1}_{\{G(z/|y-\tilde{Y}_u(\alpha)|^\gamma) \leq \eta\}} \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}}|,$$

where the supremum is taken over all $\alpha \in [0, 1]$, $z \in [0, \infty)$ and $\varphi \in [0, 2\pi]$. By Step 1 and Proposition 2.7.2, we have

$$\begin{aligned}
(2.5.7) \quad \mathcal{W}_2^2\left(\mu_u^{u'}(y), \nu_u^{u'}(y)\right) &\leq C\kappa_u^2(y) |\zeta_u(y)| \max\left(1, \log\left(\frac{u' - u}{\kappa_u^2(y)}\right)\right)^2 \\
&\leq C|\zeta_u(y)| \psi(\kappa_u^2(y)),
\end{aligned}$$

where $\psi(x) = x(1 + \log^2 \frac{u'-u}{x})$ for any $x > 0$. Let's first deal with $\zeta_u(y)$. Setting $\bar{l}^h(v) = \int_{\mathbb{R}^3} l(v - v_*) h(v_*) dv_*$ for a nonnegative function h , we observe that

$$\zeta_u(y) = \lambda_{M,u} r_\eta \bar{l}^{g_{M,u}}(y),$$

with $\lambda_{M,u} = \int_{\mathbb{R}^3} g_u(v) \mathbb{1}_{\{|v|<M\}} dv$ and $g_{M,u}(v) = \lambda_{M,u}^{-1} g_u(v) \mathbb{1}_{\{|v|<M\}}$ (recall that $\mathcal{L}_\alpha(\tilde{Y}_u) = g_u$). Observing that $\lambda_{M,u} \geq 1 - m_2(g_u)/M^2 = 1 - m_2(g_0)/M^2$, we have $\lambda_{M,u} > 1/2$ for any $u \in [0, T]$ since $M > \sqrt{2m_2(g_0)}$ by assumption. We thus have

$$m_2(g_{M,u}) = \lambda_{M,u}^{-1} \int_{\mathbb{R}^3} |v|^2 g_u(v) \mathbb{1}_{\{|v|<M\}} dv \leq 2 \int_{\mathbb{R}^3} |v|^2 g_0(v) dv =: E_0,$$

and

$$\begin{aligned} H(g_{M,u}) &= \lambda_{M,u}^{-1} \int_{\mathbb{R}^3} g_u(v) \mathbb{1}_{\{|v|<M\}} \log(\lambda_{M,u}^{-1} g_u(v)) dv \\ &= \lambda_{M,u}^{-1} \int_{\mathbb{R}^3} g_u(v) \mathbb{1}_{\{|v|<M\}} \left(\log(\lambda_{M,u}^{-1}) + \log(g_u(v)) \right) dv \\ &\leq \log(\lambda_{M,u}^{-1}) + 2 \int_{\mathbb{R}^3} g_u(v) \log(g_u(v)) \mathbb{1}_{\{|v|<M\}} dv \\ &\leq \log(2) + 2 \int_{\mathbb{R}^3} g_u(v) |\log g_u(v)| dv \\ &\leq \log(2) + 2H(g_u) + C(1 + m_2(g_u)) \\ &\leq \log(2) + 2H(g_0) + C(1 + m_2(g_0)) =: H_0. \end{aligned}$$

We first used that classically $\int_{\mathbb{R}^3} g(v) |\log g(v)| dv \leq H(g) + C(1 + m_2(g))$ for any $g \in \mathcal{P}_2(\mathbb{R}^3)$ and we then used (2.2.8)-(2.2.9). So using Proposition 2.7.3, there is $c = c(\gamma, E_0, H_0)$ such that for all $u \in [0, T]$, all $\xi \in \mathbb{R}^3$,

$$(\bar{l}^{g_{M,u}}(y)\xi) \cdot \xi \geq c(1 + |y|)^\gamma |\xi|^2,$$

and thus

$$(\zeta_u(y)\xi) \cdot \xi \geq cr_\eta(1 + |y|)^\gamma |\xi|^2.$$

This gives

$$|\zeta_u(y)^{-1/2}|^2 \leq Cr_\eta^{-1}(1 + |y|)^{|\gamma|},$$

and we thus get (recall that $d(y, \tilde{Y}_u(\alpha), z, \varphi) = \frac{1}{2}G(z/|y - \tilde{Y}_u(\alpha)|^\gamma)\Gamma(y - \tilde{Y}_u(\alpha), \varphi)$ and that $|\Gamma(X, \varphi)| = |X|$ for any $X \in \mathbb{R}^3$)

$$\begin{aligned} (2.5.8) \quad \kappa_u^2(y) &\leq Cr_\eta^{-1}(1 + |y|)^{|\gamma|} \eta^2 \sup_{\alpha \in [0,1]} |y - \tilde{Y}_u(\alpha)|^2 \mathbb{1}_{\{|\tilde{Y}_u(\alpha)|<M\}} \\ &\leq Cr_\eta^{-1}(1 + |y|)^{|\gamma|} \eta^2 (|y|^2 + M^2). \end{aligned}$$

We also have

$$\begin{aligned}
(2.5.9) \quad |\zeta_u(y)| &\leq r_\eta \int_0^1 |y - \tilde{Y}_u(\alpha)|^{\gamma+2} d\alpha \\
&= r_\eta \int_{\mathbb{R}^3} |y - v|^{\gamma+2} g_u(dv) \\
&\leq Cr_\eta (|y|^{\gamma+2} + m_2(g_u)) \mathbb{1}_{\gamma \in [-2,0)} + Cr_\eta J_{\gamma+2}(g_u) \mathbb{1}_{\gamma \in (-3,-2)} \\
&\leq Cr_\eta (1 + |y|^{\gamma+2} \mathbb{1}_{\{\gamma \in [-2,0)\}}),
\end{aligned}$$

where C depends on $\sup_{[0,T]} J_{\gamma+1}(g_s)$ if $\gamma \in (-3, -2)$ (of course, $J_{\gamma+2}(g_u)$ is controlled by $J_{\gamma+1}(g_u)$ since $\gamma + 1 < \gamma + 2 < 0$) or on $m_2(g_0)$ if $\gamma \in [-2, 0)$. Coming back to (2.5.7), observing that ψ is an increasing function of x and using (2.5.8) and (2.5.9), we get

$$\begin{aligned}
\mathcal{W}_2^2(\mu_u^{u'}(y), \nu_u^{u'}(y)) &\leq C(1 + |y|^{\gamma+2} \mathbb{1}_{\{\gamma \in [-2,0)\}}) (1 + |y|)^{|\gamma|} \eta^2 (|y|^2 + M^2) \\
&\quad \left(1 + \log^2 \frac{(u' - u)r_\eta}{(1 + |y|)^{|\gamma|} \eta^2 (|y|^2 + M^2)} \right) \\
&\leq C\eta^2 (1 + |y|^{\gamma+2} \mathbb{1}_{\{\gamma \in [-2,0)\}}) (1 + |y|)^{|\gamma|} (|y|^2 + M^2) \\
&\quad \left(1 + \log^2 \frac{(u' - u)r_\eta}{\eta^2} + \log^2 [(1 + |y|)^{|\gamma|} (|y|^2 + M^2)] \right),
\end{aligned}$$

since $\log^2(a/b) \leq 2\log^2(a) + 2\log^2(b)$. Observing that $x \log^2 x \leq C(1 + x^{1.1})$ for any $x \geq 0$, we have

$$\begin{aligned}
(1 + |y|)^{|\gamma|} (|y|^2 + M^2) \log^2 [(1 + |y|)^{|\gamma|} (|y|^2 + M^2)] \\
\leq C(1 + (1 + |y|)^{1.1|\gamma|} (|y|^2 + M^2)^{1.1}).
\end{aligned}$$

Using that

$$(1 + |y|^{\gamma+2} \mathbb{1}_{\{\gamma \in [-2,0)\}}) (1 + |y|)^{|\gamma|} \leq C(1 + |y|^3)$$

and

$$(1 + |y|^{\gamma+2} \mathbb{1}_{\{\gamma \in [-2,0)\}}) (1 + |y|)^{1.1|\gamma|} \leq C(1 + |y|^4)$$

(recall that $\gamma \in (-3, 0)$), we finally get

$$\begin{aligned}
\mathcal{W}_2^2\left(\mu_u^{u'}(y), \nu_u^{u'}(y)\right) &\leq C\eta^2 \left[(1 + |y|^3)(|y|^2 + M^2) \left(1 + \log^2 \frac{(u' - u)r_\eta}{\eta^2}\right) \right. \\
&\quad \left. + (1 + |y|^4)(|y|^2 + M^2)^{1.1} \right] \\
&\leq C\eta^2 \left[M^2(1 + |y|^3)(1 + |y|^2) \left(1 + \log^2 \frac{(u' - u)r_\eta}{\eta^2}\right) \right. \\
&\quad \left. + M^3(1 + |y|^4)(1 + |y|^3) \right] \\
&\leq C\eta^2 M^2(1 + |y|^7) \left(\log^2 \frac{(u' - u)r_\eta}{\eta^2} + M \right).
\end{aligned}$$

Step 3: recall that the white noise W is fixed. In this step we want to build the Poisson measure N in order to have $\mathbb{E}[|I_t^n - J_t^n|^2]$ as small as possible.

For any $i \in \{0, \dots, \lfloor 2nT \rfloor - 1\}$, we build a $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure $N^{*,i}$ on $[a_i, a_{i+1}) \times [0, \infty) \times [0, 2\pi] \times [0, 1]$ with intensity measure $d\sigma dz d\varphi d\alpha$ such that a.s.

(2.5.10)

$$\begin{aligned}
&\mathcal{W}_2^2\left(\mu_{a_i}^{a_{i+1}}(Y_{a_i}), \nu_{a_i}^{a_{i+1}}(Y_{a_i})\right) \\
&= \mathbb{E} \left[\left| \int_{a_i}^{a_{i+1}} \int_0^\infty \int_0^{2\pi} \int_0^1 d(Y_{a_i}, \tilde{Y}_{a_i}(\alpha), z, \varphi) \right. \right. \\
&\quad \left. \mathbb{1}_{\{G(z|Y_{a_i} - \tilde{Y}_{a_i}(\alpha)|^\gamma) < \eta\}} \mathbb{1}_{\{|\tilde{Y}_{a_i}(\alpha)| < M\}} \tilde{N}^{*,i}(ds, dz, d\varphi, d\alpha) \right. \\
&\quad \left. - \sqrt{r_\eta} \int_{a_i}^{a_{i+1}} \int_0^1 \sigma(Y_{a_i} - \tilde{Y}_{a_i}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{a_i}(\alpha)| < M\}} W(ds, d\alpha) \right|^2 \Big| \mathcal{F}_{a_i} \Big].
\end{aligned}$$

We are able to do this because Y_{a_i} is \mathcal{F}_{a_i} -measurable. We now consider a $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure N_{ini}^* on $[0, a_0) \times [0, \infty) \times [0, 2\pi] \times [0, 1]$ with intensity measure $d\sigma dz d\varphi d\alpha$. For any $[u, u'] \subset [0, T]$ and $A \subset [0, \infty) \times [0, 2\pi] \times [0, 1]$, we set $N^*([u, u'] \times A) := N_{ini}^*([u, u'] \cap [0, a_0) \times A) + \sum_{i=0}^{\lfloor 2nT \rfloor - 1} N^{*,i}([u, u'] \cap [a_i, a_{i+1}) \times A)$ (observe that N^* is a $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure on $[0, T] \times [0, \infty) \times [0, 2\pi] \times [0, 1]$ with intensity measure $d\sigma dz d\varphi d\alpha$).

Recalling that $\Phi_n(s, \alpha) \in [0, 2\pi)$, we set $(\varphi - \Phi_n(s, \alpha))$ should be inter-

puted modulo 2π)

$$\begin{aligned} \psi : [u, u'] \times [0, \infty) \times [0, 2\pi] \times [0, 1] &\rightarrow [u, u'] \times [0, \infty) \times [0, 2\pi] \times [0, 1] \\ (t, z, \varphi, \alpha) &\mapsto (t, z, \varphi - \Phi_n(s, \alpha), \alpha). \end{aligned}$$

We consider the image N of N^* by ψ . Using a remark of Tanaka [65], since $\Phi_n(s, \alpha) = \varphi_0(V_s - \tilde{V}_s(\alpha), Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha))$ is predictable, we get that N is also a $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure on $[0, T] \times [0, \infty) \times [0, 2\pi] \times [0, 1]$ with intensity measure $dsdzd\varphi d\alpha$.

Step 4: we set

$$\begin{aligned} A = \mathbb{E} \left[\left| \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right. \right. \\ \left. \left. \mathbb{1}_{\{G(z/|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma) \leq \eta\}} \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| < M\}} \tilde{N}(ds, dz, d\varphi, d\alpha) \right. \right. \\ \left. \left. - \sqrt{r_\eta} \int_{a_0}^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| < M\}} W(ds, d\alpha) \right|^2 \right]. \end{aligned}$$

The aim of this step is to show that

$$(2.5.11) \quad A \leq C\eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right),$$

where C depends on γ , T , $m_\gamma(g_0)$ and additionally on $\sup_{[0, T]} J_{\gamma+1}(g_s)$ if $\gamma \in (-3, -2)$. By construction of N , we have

$$\begin{aligned} A = \mathbb{E} \left[\left| \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi) \right. \right. \\ \left. \left. \mathbb{1}_{\{G(z/|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma) \leq \eta\}} \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| < M\}} \tilde{N}^*(ds, dz, d\varphi, d\alpha) \right. \right. \\ \left. \left. - \sqrt{r_\eta} \int_{a_0}^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| < M\}} W(ds, d\alpha) \right|^2 \right], \end{aligned}$$

and setting for any $0 < u < u' < T$ and $y \in \mathbb{R}^3$,

$$\begin{aligned} X_u^{u'}(y) := \int_u^{u'} \int_0^\infty \int_0^{2\pi} \int_0^1 d(y, \tilde{Y}_u(\alpha), z, \varphi) \mathbb{1}_{\{G(z/|y - \tilde{Y}_u(\alpha)|^\gamma) \leq \eta\}} \\ \mathbb{1}_{\{|\tilde{Y}_u(\alpha)| < M\}} \tilde{N}^*(ds, dz, d\varphi, d\alpha), \end{aligned}$$

we have

$$\begin{aligned}
A &\leq \mathbb{E} \left[\left| \sum_{i=0}^{\lfloor 2nT \rfloor - 1} \left(X_{a_i}^{a_{i+1}}(Y_{a_i}) \right. \right. \right. \\
&\quad \left. \left. - \sqrt{r_\eta} \int_{a_i}^{a_{i+1}} \int_0^1 \sigma(Y_{a_i} - \tilde{Y}_{a_i}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{a_i}(\alpha)| < M\}} W(ds, d\alpha) \right) \right|^2 \Big] \\
&= \sum_{i=0}^{\lfloor 2nT \rfloor - 1} \mathbb{E} \left[\left| X_{a_i}^{a_{i+1}}(Y_{a_i}) \right. \right. \\
&\quad \left. \left. - \sqrt{r_\eta} \int_{a_i}^{a_{i+1}} \int_0^1 \sigma(Y_{a_i} - \tilde{Y}_{a_i}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{a_i}(\alpha)| < M\}} W(ds, d\alpha) \right|^2 \right],
\end{aligned}$$

since for $i \neq j$, $(N_{[a_i, a_{i+1}]}^*, W_{[a_i, a_{i+1}]})$ and $(N_{[a_j, a_{j+1}]}^*, W_{[a_j, a_{j+1}]})$ are independent, which gives

$$\begin{aligned}
&\mathbb{E} \left[\left(X_{a_i}^{a_{i+1}}(Y_{a_i}) - \sqrt{r_\eta} \int_{a_i}^{a_{i+1}} \int_0^1 \sigma(Y_{a_i} - \tilde{Y}_{a_i}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{a_i}(\alpha)| < M\}} W(ds, d\alpha) \right) \right. \\
&\quad \left. \left(X_{a_j}^{a_{j+1}}(Y_{a_j}) - \sqrt{r_\eta} \int_{a_j}^{a_{j+1}} \int_0^1 \sigma(Y_{a_j} - \tilde{Y}_{a_j}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{a_j}(\alpha)| < M\}} W(ds, d\alpha) \right) \right] = 0.
\end{aligned}$$

First taking the conditional expectation with respect to \mathcal{F}_{a_i} for each term of the sum, and then using (2.5.10) and (2.5.6), we have

$$\begin{aligned}
A &\leq \sum_{i=0}^{\lfloor 2nT \rfloor - 1} \mathbb{E} \left[\mathbb{E} \left[\left| X_{a_i}^{a_{i+1}}(Y_{a_i}) \right. \right. \right. \\
&\quad \left. \left. - \sqrt{r_\eta} \int_{a_i}^{a_{i+1}} \int_0^1 \sigma(Y_{a_i} - \tilde{Y}_{a_i}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{a_i}(\alpha)| < M\}} W(ds, d\alpha) \right|^2 \Big| \mathcal{F}_{a_i} \right] \Big] \\
&= \sum_{i=0}^{\lfloor 2nT \rfloor - 1} \mathbb{E} \left[\mathcal{W}_2^2 \left(\mu_{a_i}^{a_{i+1}}(Y_{a_i}), \nu_{a_i}^{a_{i+1}}(Y_{a_i}) \right) \right] \\
&\leq C \sum_{i=0}^{\lfloor 2nT \rfloor - 1} \eta^2 M^2 \left(1 + \mathbb{E}[|Y_{a_i}|^7] \right) \left(\log^2 \left(\frac{r_\eta (a_{i+1} - a_i)}{\eta^2} \right) + M \right) \\
&\leq C \eta^2 M^2 n \left(\log^2 \left(\frac{r_\eta}{n \eta^2} \right) + M \right) \leq C \eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n \eta^2) + M \right),
\end{aligned}$$

where we used that $1/4n < a_{i+1} - a_i < 1/n$ by construction (recall Proposition 2.7.5) and that $\mathbb{E}[|Y_{a_i}|^7] \leq Cm_7(g_0)$.

Step 5: we finally compute $\mathbb{E}[|I_t^n - J_t^n|^2]$. We have

$$\begin{aligned} I_t^n - J_t^n &= \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ &\quad - \int_{a_0}^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) W(ds, d\alpha). \end{aligned}$$

This gives $\mathbb{E}[|I_t^n - J_t^n|^2] \leq 4(A + B + D)$ with

$$B = (\sqrt{r_\eta} - 1)^2 \mathbb{E} \left[\left| \int_{a_0}^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| < M\}} W(ds, d\alpha) \right|^2 \right],$$

and

$$\begin{aligned} D &= \mathbb{E} \left[\left| \int_{a_0}^t \int_0^\infty \int_0^{2\pi} \int_0^1 d(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right. \right. \\ &\quad \left. \left. \mathbb{1}_{\{G(z/|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma) > \eta\} \cup \{|\tilde{Y}_{\rho_n(s)}(\alpha)| > M\}} \tilde{N}(ds, dz, d\varphi, d\alpha) \right. \right. \\ &\quad \left. \left. - \int_{a_0}^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| > M\}} W(ds, d\alpha) \right|^2 \right]. \end{aligned}$$

Using that $\sum_{i,k=1}^3 \sigma_{ik}^2(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) = 2|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^{\gamma+2}$ (recall (2.4.7)),

(2.5.12)

$$\begin{aligned} B &= (\sqrt{r_\eta} - 1)^2 \int_{a_0}^t \int_0^1 \mathbb{E} \left[\sum_{i,k=1}^3 \sigma_{ik}^2(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) \right] \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| < M\}} ds d\alpha \\ &\leq 2(\sqrt{r_\eta} - 1)^2 \int_{a_0}^t \int_0^1 \mathbb{E} \left[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^{\gamma+2} \right] ds d\alpha \\ &\leq C \left(\int_\eta^\pi \theta^2 \beta(\theta) d\theta \right)^2 \left(t \mathbb{1}_{\{\gamma \in [-2, 0]\}} + \int_{a_0}^t J_{\gamma+2}(g_{\rho_n(s)}) ds \mathbb{1}_{\{\gamma \in (-3, -2)\}} \right) \\ &\leq Ct \int_\eta^\pi \theta^2 \beta(\theta) d\theta, \end{aligned}$$

where C depends on $m_2(g_0)$ and on $\sup_{[0,T]} J_{\gamma+2}(g_s)$ if $\gamma \in (-3, -2)$ (which is controlled by $\sup_{[0,T]} J_{\gamma+1}(g_s)$). We used that $|\sqrt{r_\eta} - 1| \leq C|r_\eta - 1|$ and that $r_\eta - 1 = -\frac{\pi}{4} \int_\eta^\pi \theta^2 \beta(\theta) d\theta$ by **(A2)**. We removed the square of $\int_\eta^\pi \theta^2 \beta(\theta) d\theta$ because it will appear without square in the computation of D . Using first that $|a - b|^2 \leq 2|a|^2 + 2|b|^2$, and then the substitution $\theta = G(z/|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma)$ for which $dz = |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma \beta(\theta) d\theta$ (recall (2.5.2)) for the Poisson integral, we get

$$\begin{aligned} D &\leq C \int_{a_0}^t \int_0^\pi \int_0^1 \theta^2 \beta(\theta) \mathbb{E}[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^{\gamma+2}] \mathbb{1}_{\{\theta > \eta\} \cup \{|\tilde{Y}_{\rho_n(s)}(\alpha)| > M\}} d\alpha d\theta ds \\ &\quad + C \int_{a_0}^t \int_0^1 \mathbb{E}[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^{\gamma+2}] \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| > M\}} d\alpha ds. \end{aligned}$$

If $\gamma \in [-2, 0)$, we have

$$\begin{aligned} D &\leq C \int_\eta^\pi \theta^2 \beta(\theta) d\theta \int_{a_0}^t \left(1 + \mathbb{E}[|Y_{\rho_n(s)}|^2] + \mathbb{E}_\alpha[|\tilde{Y}_{\rho_n(s)}|^2]\right) ds \\ &\quad + C \int_{a_0}^t \mathbb{E}_\alpha \left[\left(1 + \mathbb{E}[|Y_{\rho_n(s)}|^2] + |\tilde{Y}_{\rho_n(s)}(\alpha)|^2\right) \mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| > M\}} \right] ds \\ &\leq Ct \int_\eta^\pi \theta^2 \beta(\theta) d\theta + C \int_{a_0}^t \frac{1 + \mathbb{E}_\alpha[|\tilde{Y}_{\rho_n(s)}|^p] + \mathbb{E}_\alpha[|\tilde{Y}_{\rho_n(s)}|^{2+p}]}{M^p} ds \\ (2.5.13) \quad &\leq Ct \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \frac{Ct}{M^p}, \end{aligned}$$

where C depends on $m_{p+2}(g_0)$. If $\gamma \in (-3, -2)$

$$\begin{aligned} (2.5.14) \quad D &\leq C \int_\eta^\pi \theta^2 \beta(\theta) d\theta \int_{a_0}^t J_{\gamma+2}(g_{\rho_n(s)}) ds \\ &\quad + C \int_{a_0}^t J_{\gamma+2}(g_{\rho_n(s)}) \mathbb{E}_\alpha[\mathbb{1}_{\{|\tilde{Y}_{\rho_n(s)}(\alpha)| > M\}}] ds \\ &\leq C \int_\eta^\pi \theta^2 \beta(\theta) d\theta \int_{a_0}^t J_{\gamma+2}(g_{\rho_n(s)}) ds + \frac{C}{M^p} \int_{a_0}^t J_{\gamma+2}(g_{\rho_n(s)}) ds \\ &\leq Ct \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \frac{Ct}{M^p}, \end{aligned}$$

where C depends on $m_p(g_0)$ and on $\sup_{[0,T]} J_{\gamma+1}(g_s)$. It suffices to use (2.5.11), (2.5.12), (2.5.13) and (2.5.14) to conclude. \square

We finally state the last lemma needed to conclude the proof of Theorem 2.3.1.

Lemma 2.5.6. *There exists a constant C depending on γ , T , $\int_0^T J_\gamma(f_s + g_s)ds$, $m_2(g_0)$ and additionnally on $\sup_{[0,T]} J_{\gamma+1}(g_s)$ if $\gamma \in (-3, -1)$ such that, if $t \in [a_0, T]$,*

$$\begin{aligned} \mathbb{E}[|J_t^n - Y_t|^2] &\leq C \left(\mathbb{E}[|V_0 - Y_0|^2] + \frac{1}{n} \right. \\ &\quad \left. + \int_0^t \left(\mathbb{E}[|V_s - Y_s|^2] + \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2] \right) J_\gamma(f_s + g_s) ds \right). \end{aligned}$$

Proof. We have

$$\begin{aligned} J_t^n - Y_t &= V_0 - Y_0 - \int_0^{a_0} \int_0^1 \sigma(Y_s - \tilde{Y}_s(\alpha)) W(ds, d\alpha) \\ &\quad + \int_{a_0}^t \int_0^1 \left[\sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) - \sigma(Y_s - \tilde{Y}_s(\alpha)) \right] W(ds, d\alpha) \\ &\quad + \int_0^t \int_0^1 \left[b(V_s - \tilde{V}_s(\alpha)) - b(Y_s - \tilde{Y}_s(\alpha)) \right] ds d\alpha. \end{aligned}$$

Using Itô's formula and taking expectations, we get

$$\begin{aligned} \mathbb{E}[|J_t^n - Y_t|^2] &= \mathbb{E}[|V_0 - Y_0|^2] + \int_0^{a_0} \int_0^1 \mathbb{E} \left[\sum_{i,k=1}^3 \sigma_{ik}^2(Y_s - \tilde{Y}_s(\alpha)) \right] ds d\alpha \\ &\quad + \int_{a_0}^t \int_0^1 \sum_{i,k=1}^3 \mathbb{E} \left[\left(\sigma_{ik}(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) \right. \right. \\ &\quad \quad \left. \left. - \sigma_{ik}(Y_s - \tilde{Y}_s(\alpha)) \right)^2 \right] ds d\alpha \\ &\quad + 2 \int_0^t \int_0^1 \mathbb{E} \left[\left(b(V_s - \tilde{V}_s(\alpha)) - b(Y_s - \tilde{Y}_s(\alpha)) \right) \cdot (J_s^n - Y_s) \right] ds d\alpha \\ &=: \mathbb{E}[|V_0 - Y_0|^2] + 2 \int_0^{a_0} \int_0^1 \mathbb{E}[|Y_s - \tilde{Y}_s(\alpha)|^{\gamma+2}] ds d\alpha + A + B, \end{aligned}$$

since $\sum_{i,k=1}^3 \sigma_{ik}^2(Y_s - \tilde{Y}_s(\alpha)) = 2|Y_s - \tilde{Y}_s(\alpha)|^{\gamma+2}$ (recall (2.4.7)). Using that $|a|^{\gamma+2} \leq 1 + |a|^2$ if $\gamma \in [-2, 0)$ (recall also that $\mathbb{E}(|Y_s|^2) = \mathbb{E}_\alpha(|\tilde{Y}_s|^2) = m_2(g_0)$)

and that $\mathbb{E}[|Y_s - \tilde{Y}_s(\alpha)|^{\gamma+2}] \leq J_{\gamma+2}(g_s) \leq 1 + J_{\gamma+1}(g_s)$ if $\gamma \in (-3, -2)$, we have (recall that $a_0 \leq 1/n$)

$$\int_0^{a_0} \int_0^1 \mathbb{E}[|Y_s - \tilde{Y}_s(\alpha)|^{\gamma+2}] ds d\alpha \leq \frac{C}{n}.$$

Using Fournier-Guérin [33, Remark 2.2], we get

$$\begin{aligned} A &\leq C \int_{a_0}^t \int_0^1 \mathbb{E} \left[|Y_{\rho_n(s)} - Y_s + \tilde{Y}_s(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2 (|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^\gamma \right. \\ &\quad \left. + |Y_s - \tilde{Y}_s(\alpha)|^\gamma) \right] ds d\alpha \\ &\leq C \int_{a_0}^t \left(\mathbb{E}[|Y_{\rho_n(s)} - Y_s|^2] + \mathbb{E}_\alpha[|\tilde{Y}_s - \tilde{Y}_{\rho_n(s)}|^2] \right) J_\gamma(g_{\rho_n(s)} + g_s) ds \\ &\leq \frac{C}{n} \left(\int_0^T J_\gamma(g_s) ds + 1 \right), \end{aligned}$$

(recall Lemma 2.5.2 and (2.5.3)) and

$$\begin{aligned} B &\leq C \int_0^t \int_0^1 \mathbb{E} \left[|V_s - Y_s + \tilde{Y}_s(\alpha) - \tilde{V}_s(\alpha)| (|V_s - \tilde{V}_s(\alpha)|^\gamma \right. \\ &\quad \left. + |Y_s - \tilde{Y}_s(\alpha)|^\gamma) |J_s^n - Y_s| \right] ds d\alpha \\ &\leq C \int_0^t \mathbb{E} \left[(|V_s - Y_s|^2 + |J_s^n - Y_s|^2) \mathbb{E}_\alpha[|V_s - \tilde{V}_s(\alpha)|^\gamma + |Y_s - \tilde{Y}_s(\alpha)|^\gamma] \right] ds \\ &\quad + C \int_0^t \mathbb{E}_\alpha \left[|\tilde{V}_s - \tilde{Y}_s|^2 \mathbb{E}[|V_s - \tilde{V}_s(\alpha)|^\gamma + |Y_s - \tilde{Y}_s(\alpha)|^\gamma] \right] ds \\ &\leq C \int_0^t \left(\mathbb{E}[|V_s - Y_s|^2] + \mathbb{E}[|J_s^n - Y_s|^2] + \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2] \right) J_\gamma(f_s + g_s) ds. \end{aligned}$$

We thus get

$$\begin{aligned} \mathbb{E}[|J_t^n - Y_t|^2] &\leq \mathbb{E}[|V_0 - Y_0|^2] + \frac{C}{n} \\ &\quad + C \int_0^t \left(\mathbb{E}[|V_s - Y_s|^2] + \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2] + \mathbb{E}[|J_s^n - Y_s|^2] \right) J_\gamma(f_s + g_s) ds, \end{aligned}$$

and we conclude by Grönwall's lemma. \square

We can now prove Theorem 2.3.1.

Proof of Theorem 2.3.1. We couple the Poisson measure N and the white noise W as in Lemma 2.5.5. Recall that $\mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2] = \mathcal{W}_2^2(f_s, g_s) \leq \mathbb{E}[|V_s - Y_s|^2] =: u(s)$ and $\mathbb{E}[|V_0 - Y_0|^2] = \mathcal{W}_2^2(f_0, g_0)$. We first observe that if $t < a_0$,

$$\begin{aligned} u(t) &\leq 4\mathbb{E}[|V_t - V_0|^2] + 4\mathbb{E}[|V_0 - Y_0|^2] + 4\mathbb{E}[|Y_t - Y_0|^2] \\ &\leq C\left(\mathbb{E}[|V_0 - Y_0|^2] + a_0\right) \leq C\left(\mathcal{W}_2^2(f_0, g_0) + \frac{1}{n}\right), \end{aligned}$$

by Lemma 2.5.2 and the result is proved when $t < a_0$. Using Lemmas 2.5.3, 2.5.4, 2.5.5, 2.5.6 and (2.5.3), we have, for $t \in [a_0, T]$,

$$\begin{aligned} u(t) &\leq C\left(\frac{1}{n} + \int_{a_0}^t J_\gamma(f_s + g_{\rho_n(s)})\left(\mathbb{E}[|V_s - Y_s|^2] + \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2]\right)ds\right) \\ &\quad + C\int_0^\pi \theta^4 \beta(\theta) d\theta \\ &\quad + C\left[\eta^2 M^2 n\left(\log^2(r_\eta) + \log^2(n\eta^2) + M\right) + \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \frac{1}{M^p}\right] \\ &\quad + C\left(\mathbb{E}[|V_0 - Y_0|^2] + \frac{1}{n}\right. \\ &\quad \quad \left. + \int_0^t \left(\mathbb{E}[|V_s - Y_s|^2] + \mathbb{E}_\alpha[|\tilde{V}_s - \tilde{Y}_s|^2]\right) J_\gamma(f_s + g_s) ds\right) \\ &\leq C\left(\mathcal{W}_2^2(f_0, g_0) + \int_0^t J_\gamma(f_s + g_s + g_{\rho_n(s)} \mathbb{1}_{\{s \geq a_0\}}) u(s) ds + \int_0^\pi \theta^4 \beta(\theta) d\theta\right. \\ &\quad \left. + \frac{1}{n} + \eta^2 M^2 n\left(\log^2(r_\eta) + \log^2(n\eta^2) + M\right) + \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \frac{1}{M^p}\right), \end{aligned}$$

for all $n \in \mathbb{N}^*$, $\eta \in (0, \pi)$ and $M > \sqrt{2m_2(g_0)}$. Using the generalized Grönwall Lemma and (2.5.3), we get

$$\begin{aligned} u(t) &\leq C\left(\mathcal{W}_2^2(f_0, g_0) + \int_0^\pi \theta^4 \beta(\theta) d\theta + \frac{1}{n}\right. \\ &\quad \left. + \eta^2 M^2 n\left(\log^2(r_\eta) + \log^2(n\eta^2) + M\right) + \int_\eta^\pi \theta^2 \beta(\theta) d\theta + \frac{1}{M^p}\right). \end{aligned}$$

This concludes the proof since $\mathcal{W}_2^2(f_t, g_t) \leq \mathbb{E}[|V_t - Y_t|^2] = u(t)$. \square

2.6 The Coulomb Case

This section is devoted to the proof of Theorem 2.1.2. We thus assume **(AC)** and consider $f_0 \in \mathcal{P}_p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p \geq 7$. By Theorems 2.2.7 and 2.2.4 (iii), we can consider $T > 0$ and $(f_t^\epsilon)_{t \in [0, T]}$, $(g_t)_{t \in [0, T]}$ solutions to (2.1.1) and (2.1.7) respectively, both starting from f_0 and lying in $L^\infty([0, T], L^\infty(\mathbb{R}^3)) \cap L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ (uniformly in ϵ for f^ϵ) with g which additionally lies in $L^\infty([0, T], \mathcal{P}_p(\mathbb{R}^3))$.

2.6.1 Some preliminary results

The main difficulty of the Coulomb case is the fact that $\int_{|v| < 1} |v|^{-3} dv$ is not finite. We will use the following lemma stated in the paper of Fournier [29, Lemma 4] in order to deal with this difficulty.

Lemma 2.6.1. *Let $\alpha \in (-3, 0]$. There is a constant C_α such that for all $h \in \mathcal{P}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, all $\epsilon \in (0, 1]$,*

$$\begin{aligned} \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\alpha h(v_*) dv_* &\leq 1 + C_\alpha \|h\|_\infty, \\ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\alpha h(v) h(v_*) dv dv_* &\leq 1 + C_\alpha \|h\|_\infty, \\ \sup_{v, w \in \mathbb{R}^3} \int_{|v - v_*| \leq \epsilon} |w - v_*|^\alpha h(v_*) dv_* &\leq C_\alpha \|h\|_\infty \epsilon^{3+\alpha}. \end{aligned}$$

There is a constant C such that for all $h \in \mathcal{P}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, all $\epsilon \in (0, 1]$,

$$\sup_{v \in \mathbb{R}^3} \int_{|v - v_*| \geq \epsilon} |v - v_*|^{-3} h(v_*) dv_* \leq 1 + C \|h\|_\infty \log(1/\epsilon).$$

We will need to use a generalisation of the Grönwall Lemma. To this aim, we consider the increasing continuous function $\psi : [0, \infty) \rightarrow \mathbb{R}_+$ defined by

$$(2.6.1) \quad \psi(x) = x(1 - \mathbb{1}_{x \leq 1} \log x).$$

Setting $\tilde{\psi}(x) := x(1 - \log x)\mathbb{1}_{x \in [0, 1/2]} + (x \log 2 + 1/2)\mathbb{1}_{x \geq 1/2}$, we observe that $\psi(x)/2 \leq \tilde{\psi}(x) \leq 2\psi(x)$ for any $x \geq 0$. Since the function $\tilde{\psi} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is concave increasing, this last observation will almost allow us to apply the Jensen inequality to the function ψ .

As mentioned before, we only need the parameter h_ϵ in order to have easily existence and uniqueness of solutions of (2.1.1). In order to point out that we do not need this cutoff parameter in quite all calculus, we consider (recall (2.2.3) and (2.2.10))

$$(2.6.2) \quad c_\epsilon(v, v_*, z, \varphi) = a(v, v_*, G_\epsilon(z/|v - v_*|^{-3}), \varphi),$$

and

$$(2.6.3) \quad c_{h_\epsilon, \epsilon}(v, v_*, z, \varphi) = a(v, v_*, G_\epsilon(z/(|v - v_*| + h_\epsilon)^{-3}), \varphi).$$

Lemma 2.6.2. (i) For any $v, v_* \in \mathbb{R}^3$,

$$\int_0^\infty \int_0^{2\pi} |c_\epsilon(v, v_*, z, \varphi)|^2 dz d\varphi = k_\epsilon |v - v_*|^{-1},$$

where

$$(2.6.4) \quad k_\epsilon = \pi \int_0^\pi (1 - \cos \theta) \beta_\epsilon(\theta) d\theta.$$

We also have $k_\epsilon \leq 2$.

(ii) For any $v, v_* \in \mathbb{R}^3$,

$$\int_0^\infty \int_0^{2\pi} |(c_{h_\epsilon, \epsilon} - c_\epsilon)(v, v_*, z, \varphi)|^2 dz d\varphi \leq C h_\epsilon |v - v_*|^{-2}.$$

(iii) For any $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,

$$\begin{aligned} & \int_0^\infty \int_0^{2\pi} \left| c_\epsilon(v, v_*, z, \varphi) - c_\epsilon(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)) \right|^2 dz d\varphi \\ & \leq C \min \left\{ |v - v_*|^{-1} + |\tilde{v} - \tilde{v}_*|^{-1}, \right. \\ & \quad (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)(|v - v_*|^{-3} + |\tilde{v} - \tilde{v}_*|^{-3}) \\ & \quad \left. + \frac{1}{\log \frac{1}{\epsilon}} [|v - v_*|^{-2} + |\tilde{v} - \tilde{v}_*|^{-2} + |v - v_*|^2 + |\tilde{v} - \tilde{v}_*|^2] \right\}. \end{aligned}$$

Proof. We easily get the first part of Point (i) from (2.5.4). The fact that $k_\epsilon \leq 2$ comes from (2.1.5), just observing that $1 - \cos \theta \leq \theta^2/2$.

We now prove (ii). Recalling (2.2.10) and using that for any $X \in \mathbb{R}^3$, the vectors X and $\Gamma(X, \varphi)$ are orthogonal, and $|\Gamma(X, \varphi)| = |X|$, we have

$$\begin{aligned}
& |(c_{h_\epsilon, \epsilon} - c_\epsilon)(v, v_*, z, \varphi)|^2 \\
&= \left| \frac{1}{2} \left[1 - \cos G_\epsilon \left(\frac{z}{(|v - v_*| + h_\epsilon)^{-3}} \right) - \left(1 - \cos G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) \right) \right] (v - v_*) \right. \\
&\quad \left. + \frac{1}{2} \left[\sin G_\epsilon \left(\frac{z}{(|v - v_*| + h_\epsilon)^{-3}} \right) - \sin G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) \right] \Gamma(v - v_*, \varphi) \right|^2 \\
&= \frac{1}{4} \left[\cos G_\epsilon \left(\frac{z}{(|v - v_*| + h_\epsilon)^{-3}} \right) - \cos G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) \right]^2 |v - v_*|^2 \\
&\quad + \frac{1}{4} \left[\sin G_\epsilon \left(\frac{z}{(|v - v_*| + h_\epsilon)^{-3}} \right) - \sin G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) \right]^2 |v - v_*|^2 \\
&\leq \frac{1}{2} \left[G_\epsilon \left(\frac{z}{(|v - v_*| + h_\epsilon)^{-3}} \right) - G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) \right]^2 |v - v_*|^2,
\end{aligned}$$

since $(\cos \theta - \cos \theta')^2 + (\sin \theta - \sin \theta')^2 \leq 2(\theta - \theta')^2$, which gives

$$\begin{aligned}
& \int_0^\infty \int_0^{2\pi} |(c_{h_\epsilon, \epsilon} - c_\epsilon)(v, v_*, z, \varphi)|^2 dz d\varphi \\
&\leq \pi |v - v_*|^2 \int_0^\infty \left| G_\epsilon \left(\frac{z}{(|v - v_*| + h_\epsilon)^{-3}} \right) - G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) \right|^2 dz \\
&\leq \pi |v - v_*|^2 \kappa_2 \left(\frac{((|v - v_*| + h_\epsilon)^{-3} - |v - v_*|^{-3})^2}{(|v - v_*| + h_\epsilon)^{-3} + |v - v_*|^{-3}} \right. \\
&\quad \left. + \frac{|v - v_*|^{-3}}{\log \frac{1}{\epsilon}} \log \frac{|v - v_*|^{-3}}{(|v - v_*| + h_\epsilon)^{-3}} \right),
\end{aligned}$$

by **(A5)**. Now using that for any $x, h > 0$,

$$\begin{aligned}
x^2 \frac{((x+h)^{-3} - x^{-3})^2}{(x+h)^{-3} + x^{-3}} &= x^2 \frac{(x^3 - (x+h)^3)^2}{(x+h)^6 x^6} \frac{(x+h)^3 x^3}{x^3 + (x+h)^3} \\
&= \frac{h^2 (3x^2 + 3xh + h^2)^2}{x(x+h)^3 (x^3 + (x+h)^3)} \\
&\leq C \frac{h}{x^2} \frac{h(x^4 + x^2 h^2 + h^4)}{(x+h)^5} \\
&\leq C \frac{h}{x^2},
\end{aligned}$$

and

$$x^{-1} \log \frac{x^{-3}}{(x+h)^{-3}} \leq 3x^{-1} \log \left(1 + \frac{h}{x}\right) \leq 3hx^{-2},$$

we get

$$\begin{aligned} \int_0^\infty \int_0^{2\pi} |(c_{h_\epsilon, \epsilon} - c_\epsilon)(v, v_*, z, \varphi)|^2 dz d\varphi &\leq C \left(h_\epsilon + \frac{h_\epsilon}{\log \frac{1}{\epsilon}} \right) |v - v_*|^{-2} \\ &\leq Ch_\epsilon |v - v_*|^{-2}. \end{aligned}$$

We finally prove (iii). First, by Point (i), we have

$$\begin{aligned} \int_0^\infty \int_0^{2\pi} |c_\epsilon(v, v_*, z, \varphi)|^2 dz d\varphi \\ = \int_0^\infty \int_0^{2\pi} |c_\epsilon(v, v_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))|^2 dz d\varphi = k_\epsilon |v - v_*|^{-1}, \end{aligned}$$

with $k_\epsilon \leq 2$ and we thus get the first bound in the min. For the second bound, we set $\Delta := \left| c_\epsilon(v, v_*, z, \varphi) - c_\epsilon(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)) \right|^2$. Looking at the proof of Fournier-Guérin [32, Lemma 2.3], we get

$$\begin{aligned} \Delta \leq C \left[|(v - v_*) - (\tilde{v} - \tilde{v}_*)|^2 \left(G_\epsilon^2 \left(\frac{z}{|v - v_*|^{-3}} \right) + G_\epsilon^2 \left(\frac{z}{|\tilde{v} - \tilde{v}_*|^{-3}} \right) \right) \right. \\ \left. + \min(|v - v_*|^2, |\tilde{v} - \tilde{v}_*|^2) \left| G_\epsilon \left(\frac{z}{|v - v_*|^{-3}} \right) - G_\epsilon \left(\frac{z}{|\tilde{v} - \tilde{v}_*|^{-3}} \right) \right|^2 \right]. \end{aligned}$$

Using the substitution $\theta = G_\epsilon(z/\Phi(|v - v_*|))$ or $\theta = G_\epsilon(z/\Phi(|\tilde{v} - \tilde{v}_*|))$, we have

$$\begin{aligned} \int_0^\infty \int_0^{2\pi} \left(G_\epsilon^2 \left(\frac{z}{|v - v_*|^{-3}} \right) + G_\epsilon^2 \left(\frac{z}{|\tilde{v} - \tilde{v}_*|^{-3}} \right) \right) dz d\varphi \\ = 2\pi \int_0^\pi \theta^2 \beta_\epsilon(\theta) d\theta \left(|v - v_*|^{-3} + |\tilde{v} - \tilde{v}_*|^{-3} \right) = 8 \left(|v - v_*|^{-3} + |\tilde{v} - \tilde{v}_*|^{-3} \right). \end{aligned}$$

We set $a = |v - v_*|$ and $b = |\tilde{v} - \tilde{v}_*|$. Using **(A5)** (observe that $\log \frac{\max(x,y)}{\min(x,y)} \leq$

$|\log x| + |\log y|$), we get

$$\begin{aligned}
& \min(a^2, b^2) \int_0^\infty \int_0^{2\pi} \left| G_\epsilon\left(\frac{z}{a^{-3}}\right) - G_\epsilon\left(\frac{z}{b^{-3}}\right) \right|^2 dz d\varphi \\
& \leq C \min(a, b)^2 \left[\frac{(a^{-3} - b^{-3})^2}{a^{-3} + b^{-3}} + \frac{1}{\log \frac{1}{\epsilon}} \max(a^{-3}, b^{-3}) [|\log a^{-3}| + |\log b^{-3}|] \right] \\
& \leq C(a - b)^2 (a^{-3} + b^{-3}) + \frac{C}{\log \frac{1}{\epsilon}} \min(a, b)^{-1} [|\log a| + |\log b|] \\
& \leq C(a - b)^2 (a^{-3} + b^{-3}) + \frac{C}{\log \frac{1}{\epsilon}} [a^{-2} + b^{-2} + a^2 + b^2],
\end{aligned}$$

where we used that

$$\begin{aligned}
\min(a, b)^2 \frac{(a^{-3} - b^{-3})^2}{a^{-3} + b^{-3}} & \leq 9 \min(a, b)^2 (a - b)^2 \frac{\min(a, b)^{-8}}{a^{-3} + b^{-3}} \\
& \leq 9(a - b)^2 \min(a, b)^{-3} \leq 9(a - b)^2 (a^{-3} + b^{-3}),
\end{aligned}$$

and that (observe that $|\log a| \leq a^{-1} + a$)

$$\begin{aligned}
\min(a, b)^{-1} [|\log a| + |\log b|] & \leq \frac{1}{2} [a^{-1} + b^{-1}]^2 + \frac{1}{2} [|\log a| + |\log b|]^2 \\
& \leq C(a^{-2} + b^{-2} + a^2 + b^2).
\end{aligned}$$

We thus have

$$\begin{aligned}
\int_0^\infty \int_0^{2\pi} \Delta dz d\varphi & \leq C |(v - v_*) - (\tilde{v} - \tilde{v}_*)|^2 \left(|v - v_*|^{-3} - |\tilde{v} - \tilde{v}_*|^{-3} \right) \\
& \quad + \frac{C}{\log \frac{1}{\epsilon}} [|v - v_*|^{-2} + |\tilde{v} - \tilde{v}_*|^{-2} + |v - v_*|^2 + |\tilde{v} - \tilde{v}_*|^2],
\end{aligned}$$

which concludes the proof. \square

2.6.2 Definition of the processes

We consider a random variable V_0 with law f_0 . We fix a white noise W on $[0, T] \times [0, 1]$ with covariance measure $dsd\alpha$ and we consider a process $(Y_t)_{t \in [0, T]}$ and an α -process $(\tilde{Y}_t)_{t \in [0, T]}$ such that for any $t \in [0, T]$, $\mathcal{L}(Y_t) = \mathcal{L}_\alpha(\tilde{Y}_t) = g_t$, such that $\mathcal{L}_\alpha((\tilde{Y}_t)_{t \in [0, T]}) = \mathcal{L}((Y_t)_{t \in [0, T]})$ and such that (2.4.6) is satisfied with $\gamma = -3$ (see Proposition 2.4.4). For any $t \in [0, T]$, we consider

an α -random variable \tilde{V}_t^ϵ with law f_t^ϵ such that $\mathcal{W}_2^2(f_t^\epsilon, g_t) = \mathbb{E}_\alpha[|\tilde{V}_t^\epsilon - \tilde{Y}_t|^2]$ and we consider the solution $(V_t^\epsilon)_{t \in [0, T]}$ to (2.4.5) (with $\Phi(|v - v_*|) = (|v - v_*| + h_\epsilon)^{-3}$) for some $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson measure N as in Proposition 2.4.3. We will precise later the dependence of N with the white noise W . We recall the equations satisfied by $(V_t^\epsilon)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$, and we introduce some intermediate processes (here $n \in \mathbb{N}^*$ is fixed)

$$\begin{aligned}
V_t^\epsilon &= V_0 + \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 c_{h_\epsilon, \epsilon}(V_{s-}^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\
&\quad - k_\epsilon \int_0^t \int_0^1 (|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)| + h_\epsilon)^{-3} (V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha, \\
W_t^\epsilon &= V_0 + \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 c_\epsilon(V_{s-}^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi) \tilde{N}(ds, dz, d\varphi, d\alpha) \\
&\quad - k_\epsilon \int_0^t \int_0^1 |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-3} (V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha, \\
V_t^{n, \epsilon} &= \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 c_\epsilon(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \tilde{N}(ds, dz, d\varphi, d\alpha) \\
&\quad - k_\epsilon \int_0^t \int_0^1 |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-3} (V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha + V_0 A, \\
I_t^{n, \epsilon} &= \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 d_\epsilon(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \tilde{N}(ds, dz, d\varphi, d\alpha) \\
&\quad + \int_0^t \int_0^1 b(V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha + V_0, \\
J_t^{n, \epsilon} &= V_0 + \int_0^t \int_0^1 \sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) W(ds, d\alpha) \\
&\quad + \int_0^t \int_0^1 b(V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha, \\
Y_t &= V_0 + \int_0^t \int_0^1 \sigma(Y_s - \tilde{Y}_s(\alpha)) W(ds, d\alpha) + \int_0^t \int_0^1 b(Y_s - \tilde{Y}_s(\alpha)) ds d\alpha,
\end{aligned}$$

where (recall Lemma 2.4.2)

$$(2.6.5) \quad \Phi_n(s, \alpha) = \varphi_0(V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha), Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)),$$

d_ϵ is defined by replacing γ by -3 and G by G_ϵ in (2.5.2). Recall that $b(v) = -2|v|^{-3}v$ and that k_ϵ is defined in (2.6.4). Finally, ρ_n is defined as follows.

We consider some subdivision $0 = a_0^n < \dots < a_{[2nT]-1}^n < a_{[2nT]}^n = T$ of $[0, T]$ such that $1/4n < a_{i+1}^n - a_i^n < 1/n$. In order to lighten notation, we write $a_i = a_i^n$. For $s \in [0, T]$, we set

$$\rho_n(s) = \sum_{i=0}^{[2nT]-1} a_i \mathbb{1}_{s \in [a_i, a_{i+1})}.$$

Observe that by construction, we have $\sup_{[0, T]} |s - \rho_n(s)| \leq 1/n$.

2.6.3 The proof.

The ideas will be the same as for Theorem 2.3.1. The proofs will thus be very similar to those used for Lemmas 2.5.2, 2.5.3, 2.5.4, 2.5.5 and 2.5.6. So instead of rewriting all the proofs, we will only point out the modifications that we have to handle.

We start by a lemma where we compute the error due to the parameter h_ϵ in the collision kernel. Observe that after this lemma, we will use a collision kernel which corresponds to the real Coulomb case (without the parameter h_ϵ) for our computations. Furthermore, the errors that we will get after this lemma will not depend on h_ϵ . This confirms the fact that the parameter h_ϵ is not useful to get a rate of convergence for the grazing collisions limit for the Coulomb potential. Here again, we recall that we only introduce this parameter in order to get easily existence and uniqueness of $(f_t^\epsilon)_{t \in [0, T]}$ (and of the process $(V_t^\epsilon)_{t \in [0, T]}$).

Lemma 2.6.3. *There exists a constant C depending on $\sup_{[0, T]} \|f_t^\epsilon\|_\infty$ such that for any $t \in [0, T]$, any $\epsilon \in (0, 1)$,*

$$\mathbb{E}[|V_t^\epsilon - W_t^\epsilon|^2] \leq Ch_\epsilon^{e^{-C}}.$$

Proof. We have

$$\begin{aligned} V_t^\epsilon - W_t^\epsilon &= \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 (c_{h_\epsilon, \epsilon} - c_\epsilon) \left(V_s^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi \right) \tilde{N}(ds, dz, d\varphi, d\alpha) \\ &\quad - k_\epsilon \int_0^t \int_0^1 \left((|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)| + h_\epsilon)^{-3} \right. \\ &\quad \left. - |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-3} \right) (V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) ds d\alpha. \end{aligned}$$

Using Itô's formula and taking expectations, we thus get

$$\begin{aligned} \mathbb{E}[|V_t^\epsilon - W_t^\epsilon|^2] &= \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 \mathbb{E} \left[\left| (c_{h_\epsilon, \epsilon} - c_\epsilon) \left(V_s^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi \right) \right|^2 \right] ds dz d\varphi d\alpha \\ &\quad - 2k_\epsilon \int_0^t \int_0^1 \mathbb{E} \left[\left((|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)| + h_\epsilon)^{-3} - |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-3} \right) \right. \\ &\quad \left. (V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) \cdot (V_s^\epsilon - W_s^\epsilon) \right] ds d\alpha \\ &=: A + B. \end{aligned}$$

Using Point (i) of Lemma 2.6.2, we get

$$A \leq Ch_\epsilon \int_0^t \mathbb{E} \left[\mathbb{E}_\alpha [|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-2}] \right] ds \leq Ch_\epsilon \int_0^t (1 + \|f_s\|_\infty) ds,$$

by Lemma 2.6.1. For B , we first observe that for any $x, h, y > 0$

$$\begin{aligned} (x^{-3} - (x+h)^{-3})xy &\leq \mathbb{1}_{y \geq 1} x^{-3}xy + \mathbb{1}_{y \leq x} (x^{-3} - (x+h)^{-3})x^2 \\ &\quad + \mathbb{1}_{y^2 \leq x \leq y < 1} x^{-3}y^2 + \mathbb{1}_{x < y^2 < 1} x^{-3}xy \\ &\leq \mathbb{1}_{y \geq 1} x^{-2}y^2 + 3\mathbb{1}_{y \leq x} hx^{-2} \\ &\quad + \mathbb{1}_{y^2 \leq x \leq y < 1} x^{-3}y^2 + \mathbb{1}_{x < y^2 < 1} x^{-2}. \end{aligned}$$

We thus get (recall that $k_\epsilon \leq 2$)

$$\begin{aligned} B &\leq C \int_0^t \mathbb{E} \left[\mathbb{E}_\alpha [\mathbb{1}_{|V_s^\epsilon - W_s^\epsilon| \geq 1} |V_s^\epsilon - W_s^\epsilon|^2 |V_s^\epsilon - \tilde{V}_s^\epsilon|^{-2} \right. \\ &\quad + \mathbb{1}_{|V_s^\epsilon - W_s^\epsilon| \leq |V_s^\epsilon - \tilde{V}_s^\epsilon|} h_\epsilon |V_s^\epsilon - \tilde{V}_s^\epsilon|^{-2} \\ &\quad + \mathbb{1}_{|V_s^\epsilon - W_s^\epsilon|^2 \leq |V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |V_s^\epsilon - W_s^\epsilon| < 1} |V_s^\epsilon - W_s^\epsilon|^2 |V_s^\epsilon - \tilde{V}_s^\epsilon|^{-3} \\ &\quad \left. + \mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| < |V_s^\epsilon - W_s^\epsilon|^2 < 1} |V_s^\epsilon - \tilde{V}_s^\epsilon|^{-2} \right] ds. \end{aligned}$$

Using that $\mathcal{L}_\alpha(\tilde{V}_s^\epsilon) = f_s^\epsilon$ and Lemma 2.6.1,

$$\begin{aligned} B &\leq C \int_0^t \mathbb{E} \left[(1 + \|f_s^\epsilon\|_\infty) |V_s^\epsilon - W_s^\epsilon|^2 + (1 + \|f_s^\epsilon\|_\infty) h_\epsilon \right. \\ &\quad + \mathbb{1}_{|V_s^\epsilon - W_s^\epsilon| < 1} |V_s^\epsilon - W_s^\epsilon|^2 (1 - \|f_s^\epsilon\|_\infty \log(|V_s^\epsilon - W_s^\epsilon|^2)) \\ &\quad \left. + \|f_s^\epsilon\|_\infty |V_s^\epsilon - W_s^\epsilon|^2 \right] ds \\ &\leq C \int_0^t (1 + \|f_s^\epsilon\|_\infty) \mathbb{E} \left[|V_s^\epsilon - W_s^\epsilon|^2 + h_\epsilon + \psi(|V_s^\epsilon - W_s^\epsilon|^2) \right] ds, \end{aligned}$$

where ψ was defined in (2.6.1). Using that $x \leq \psi(x)$ for any $x \geq 0$ and the approximate Jensen inequality (recall the paragraph just after (2.6.1)), we thus get

$$\mathbb{E}[|V_t^\epsilon - W_t^\epsilon|^2] \leq Ch_\epsilon + C \int_0^t \psi(\mathbb{E}[|V_s^\epsilon - W_s^\epsilon|^2]) ds,$$

where C depends on $\sup_{[0,T]} \|f_t^\epsilon\|_\infty$. The conclusion follows by Lemma 2.7.4. \square

Lemma 2.6.4. (i) *There exists a constant C depending on $\sup_{s \in [0,T]} \|f_s^\epsilon\|_\infty$ and on $m_2(f_0)$ such that for $0 \leq t' \leq t \leq T$ with $t - t' < 1$, for any $\epsilon \in (0, 1)$,*

$$\mathbb{E}[|V_t^\epsilon - V_{t'}^\epsilon|^2] \leq C(t - t').$$

The same bound holds for $\mathbb{E}[|Y_t - Y_{t'}|^2]$ and $\mathbb{E}_\alpha[|\tilde{Y}_t - \tilde{Y}_{t'}|^2]$ with C depending on $m_2(g_0)$ and on $\sup_{[0,T]} \|g_t\|_\infty$.

(ii) *For all $t \in [0, T]$, we have*

$$\mathbb{E}[|V_t^\epsilon - V_{\rho_n(t)}^\epsilon|^2] + \mathbb{E}[|Y_t - Y_{\rho_n(t)}|^2] + \mathbb{E}_\alpha[|\tilde{Y}_t - \tilde{Y}_{\rho_n(t)}|^2] \leq \frac{C}{n}.$$

Proof. Since $t - \rho_n(t) \leq 1/n$, (ii) immediately follows from (i). To prove (i) (for example for $(V_t^\epsilon)_{t \in [0,T]}$), we follow the line of the proof of Lemma 2.5.2, and we get (observe that $(a + h)^{-3} \leq a^{-3}$)

$$\begin{aligned} \mathbb{E}[|V_t^\epsilon - V_{t'}^\epsilon|^2] &\leq 2k_\epsilon \int_{t'}^t \mathbb{E}[\mathbb{E}_\alpha[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-1}]] ds \\ &\quad + 2k_\epsilon^2 \mathbb{E}\left[\left(\int_{t'}^t \mathbb{E}_\alpha[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-2}] ds\right)^2\right] \\ &\leq 2k_\epsilon \int_{t'}^t (1 + \|f_s^\epsilon\|_\infty) ds + 2k_\epsilon^2 \left(\int_{t'}^t (1 + \|f_s^\epsilon\|_\infty) ds\right)^2 \\ &\leq C(t - t'), \end{aligned}$$

by Lemma 2.6.1 and we conclude the proof as for Lemma 2.5.2. \square

Lemma 2.6.5. *There exists a constant C depending on $\sup_{s \in [0,T]} \|f_s^\epsilon + g_s\|_\infty$ and on $m_2(f_0)$ such that, for any $n \geq 2$, $\epsilon \in (0, 1)$ and $t \in [0, T]$*

$$\begin{aligned} \mathbb{E}[|W_t^\epsilon - V_t^{n,\epsilon}|^2] &\leq \frac{C}{\log \frac{1}{\epsilon}} \\ &\quad + C \int_0^t \left(\frac{\log n}{n} + \psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) \right) ds. \end{aligned}$$

Proof. Observing that

$$W_t^\epsilon - V_t^{n,\epsilon} = \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 \left[c_\epsilon(V_{s-}^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi) - c_\epsilon(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right] \tilde{N}(ds, dz, d\varphi, d\alpha),$$

we have

$$I := \mathbb{E} \left[|W_t^\epsilon - V_t^{n,\epsilon}|^2 \right] = \int_0^t \int_0^\infty \int_0^{2\pi} \int_0^1 \mathbb{E} \left[\left| c_\epsilon(V_s^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi) - c_\epsilon(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right|^2 \right] ds dz d\varphi d\alpha.$$

We set

$$\delta = \int_0^\infty \int_0^{2\pi} \left| c_\epsilon(V_s^\epsilon, \tilde{V}_s^\epsilon(\alpha), z, \varphi) - c_\epsilon(Y_{\rho_n(s)}, \tilde{Y}_{\rho_n(s)}(\alpha), z, \varphi + \Phi_n(s, \alpha)) \right|^2 dz d\varphi.$$

Setting $a_s := |V_s^\epsilon - Y_{\rho_n(s)}| + |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|$, $v_s := |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|$, $y_s := |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|$ and using Lemma 2.6.2, we get

$$\begin{aligned} \delta &\leq C \mathbb{1}_{a_s \geq 1} \left(v_s^{-1} + y_s^{-1} \right) \\ &\quad + C \mathbb{1}_{a_s \leq 1} \mathbb{1}_{v_s \geq |V_s^\epsilon - Y_{\rho_n(s)}|^2, y_s \geq |V_s^\epsilon - Y_{\rho_n(s)}|^2} \left[|V_s^\epsilon - Y_{\rho_n(s)}|^2 \left(v_s^{-3} + y_s^{-3} \right) + \frac{1}{\log \frac{1}{\epsilon}} [v_s^{-2} + y_s^{-2} + v_s^2 + y_s^2] \right] \\ &\quad + C \mathbb{1}_{a_s \leq 1} \mathbb{1}_{v_s \geq |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2, y_s \geq |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2} \left[|\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2 \left(v_s^{-3} + y_s^{-3} \right) + \frac{1}{\log \frac{1}{\epsilon}} [v_s^{-2} + y_s^{-2} + v_s^2 + y_s^2] \right] \\ &\quad + C \mathbb{1}_{a_s \leq 1} \mathbb{1}_{v_s \leq |V_s^\epsilon - Y_{\rho_n(s)}|^2} \left(v_s^{-1} + y_s^{-1} \right) \\ &\quad + C \mathbb{1}_{a_s \leq 1} \mathbb{1}_{y_s \leq |V_s^\epsilon - Y_{\rho_n(s)}|^2} \left(v_s^{-1} + y_s^{-1} \right) \\ &\quad + C \mathbb{1}_{a_s \leq 1} \mathbb{1}_{v_s \leq |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2} \left(v_s^{-1} + y_s^{-1} \right) \\ &\quad + C \mathbb{1}_{a_s \leq 1} \mathbb{1}_{y_s \leq |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)|^2} \left(v_s^{-1} + y_s^{-1} \right) \\ &=: C \sum_{i=1}^7 \delta_i. \end{aligned}$$

We thus have $I \leq \sum_{i=1}^7 I_i$ where $I_i = \int_0^t \int_0^1 \mathbb{E}[\delta_i] ds d\alpha$. Using that $\mathbb{1}_{a_s \geq 1} \leq a_s^2$, we have

$$\begin{aligned} I_1 &\leq \int_0^t \int_0^1 \mathbb{E} \left[\left(|V_s^\epsilon - Y_{\rho_n(s)}| + |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_{\rho_n(s)}(\alpha)| \right)^2 \left(|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-1} \right. \right. \\ &\quad \left. \left. + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)|^{-1} \right) \right] ds d\alpha \\ &\leq 2 \int_0^t \mathbb{E} \left[|V_s^\epsilon - Y_{\rho_n(s)}|^2 \mathbb{E}_\alpha \left[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-1} + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-1} \right] \right] ds \\ &\quad + 2 \int_0^t \mathbb{E}_\alpha \left[|\tilde{V}_s^\epsilon - \tilde{Y}_{\rho_n(s)}|^2 \mathbb{E} \left[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-1} + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-1} \right] \right] ds \\ &\leq C \int_0^t \left(\mathbb{E} \left[|V_s^\epsilon - Y_{\rho_n(s)}|^2 \right] + \mathbb{E}_\alpha \left[|\tilde{V}_s^\epsilon - \tilde{Y}_{\rho_n(s)}|^2 \right] \right) \left(1 + \|f_s\|_\infty + \|g_{\rho_n(s)}\|_\infty \right) ds, \end{aligned}$$

by Lemma 2.6.1. We thus get, using the triangular inequality and Lemma 2.6.4,

$$(2.6.6) \quad I_1 \leq C \int_0^t \left(\frac{1}{n} + \mathbb{E} \left[|V_s^\epsilon - Y_s|^2 \right] + \mathbb{E}_\alpha \left[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2 \right] \right) ds,$$

where C depends on $\sup_{s \in [0, T]} \|f_s^\epsilon + g_s\|_\infty$. Using Lemma 2.6.1, we have

$$\begin{aligned} I_2 &\leq \int_0^t \mathbb{E} \left[|V_s^\epsilon - Y_{\rho_n(s)}|^2 \mathbb{1}_{|V_s^\epsilon - Y_{\rho_n(s)}| \leq 1} \right. \\ &\quad \left(\mathbb{E}_\alpha \left[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-3} \mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \geq |V_s^\epsilon - Y_{\rho_n(s)}|^2} \right. \right. \\ &\quad \left. \left. + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-3} \mathbb{1}_{|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}| \geq |V_s^\epsilon - Y_{\rho_n(s)}|^2} \right) \right] ds \\ &\quad + \frac{C}{\log \frac{1}{\epsilon}} \int_0^t \mathbb{E} \left[\mathbb{E}_\alpha \left[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-2} + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-2} \right. \right. \\ &\quad \left. \left. + |V_s^\epsilon - \tilde{V}_s^\epsilon|^2 + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^2 \right] \right] ds \\ &\leq \int_0^t \mathbb{E} \left[|V_s^\epsilon - Y_{\rho_n(s)}|^2 \mathbb{1}_{|V_s^\epsilon - Y_{\rho_n(s)}| \leq 1} \left(1 - \right. \right. \\ &\quad \left. \left. C \|f_s + g_{\rho_n(s)}\|_\infty \log |V_s^\epsilon - Y_{\rho_n(s)}|^2 \right) \right] ds \\ &\quad + \frac{C}{\log \frac{1}{\epsilon}} \int_0^t \left(1 + \|f_s + g_{\rho_n(s)}\|_\infty + m_2(f_0) \right) ds. \end{aligned}$$

Recalling (2.6.1) and using the (approximate) Jensen inequality for the function ψ , the fact that $\psi(a+b) \leq \psi(a) + \psi(b)$ and that the function ψ is increasing, and the Lemma 2.6.4 we get

$$\begin{aligned}
(2.6.7) \quad I_2 &\leq C \int_0^t \mathbb{E} \left[\psi(|V_s^\epsilon - Y_{\rho_n(s)}|^2) \right] ds + \frac{C}{\log \frac{1}{\epsilon}} \\
&\leq C \int_0^t \left(\psi\left(\frac{C}{n}\right) + \psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) \right) ds + \frac{C}{\log \frac{1}{\epsilon}} \\
&\leq C \int_0^t \left(\frac{\log n}{n} + \psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) \right) ds + \frac{C}{\log \frac{1}{\epsilon}},
\end{aligned}$$

where C depends on $\sup_{s \in [0, T]} \|f_s^\epsilon + g_s\|_\infty$. Using the same ideas, we also have

$$(2.6.8) \quad I_3 \leq C \int_0^t \left(\frac{\log n}{n} + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) \right) ds + \frac{C}{\log \frac{1}{\epsilon}}.$$

We now deal with I_4 .

$$\begin{aligned}
I_4 &\leq \int_0^t \mathbb{E} \left[\mathbb{E}_\alpha \left[\mathbb{1}_{a_s \leq 1} \mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |V_s^\epsilon - Y_{\rho_n(s)}|^2} (|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-1} \right. \right. \\
&\quad \left. \left. + |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-1}) \right] \right] ds.
\end{aligned}$$

Using Lemma 2.6.1, we first observe that

$$\begin{aligned}
\mathbb{E}_\alpha \left[\mathbb{1}_{a_s \leq 1} \mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |V_s^\epsilon - Y_{\rho_n(s)}|^2} |V_s^\epsilon - \tilde{V}_s^\epsilon|^{-1} \right] \\
\leq C \mathbb{1}_{|V_s^\epsilon - Y_{\rho_n(s)}| \leq 1} \|f_s\|_\infty |V_s^\epsilon - Y_{\rho_n(s)}|^4 \\
\leq C |V_s^\epsilon - Y_{\rho_n(s)}|^2.
\end{aligned}$$

Next, using the Hölder inequality with $p = 3$ and $q = 3/2$, and then Lemma 2.6.1, we get

$$\begin{aligned}
&\mathbb{E}_\alpha \left[\mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |V_s^\epsilon - Y_{\rho_n(s)}|^2} |Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-1} \right] \\
&\leq \mathbb{E}_\alpha \left[\mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |V_s^\epsilon - Y_{\rho_n(s)}|^2} \right]^{\frac{1}{3}} \mathbb{E}_\alpha \left[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{\frac{-3}{2}} \right]^{\frac{2}{3}} \\
&\leq \left(C \|f_s^\epsilon\|_\infty |V_s^\epsilon - Y_{\rho_n(s)}|^6 \right)^{\frac{1}{3}} \left(1 + C \|g_{\rho_n(s)}\|_\infty \right)^{\frac{2}{3}} \\
&\leq C (1 + \|f_s^\epsilon + g_{\rho_n(s)}\|_\infty) |V_s^\epsilon - Y_{\rho_n(s)}|^2.
\end{aligned}$$

We thus have

$$(2.6.9) \quad I_4 \leq C \int_0^t \mathbb{E}[|V_s^\epsilon - Y_{\rho_n(s)}|^2] ds \leq C \int_0^t \left(\frac{1}{n} + \mathbb{E}[|V_s^\epsilon - Y_s|^2] \right) ds$$

by Lemma 2.6.4. With the same arguments,

$$(2.6.10) \quad I_5 \leq C \int_0^t \left(\frac{1}{n} + \mathbb{E}[|V_s^\epsilon - Y_s|^2] \right) ds,$$

and

$$(2.6.11) \quad I_6 + I_7 \leq C \int_0^t \left(\frac{1}{n} + \mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2] \right) ds.$$

It suffices to use (2.6.6), (2.6.7), (2.6.8), (2.6.9), (2.6.10), (2.6.11) and to observe that $x \leq \psi(x)$ for any $x \geq 0$ to conclude the proof. \square

Lemma 2.6.6. *There exists a constant C depending on $\sup_{s \in [0, T]} \|f_s^\epsilon + g_s\|_\infty$ and on T such that for any $n \geq 2$, $\epsilon \in (0, 1)$ and $t \in [0, T]$,*

$$\mathbb{E}[|V_t^{n, \epsilon} - I_t^{n, \epsilon}|^2] \leq C \int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta.$$

Proof. As in the proof of Lemma 2.5.4, we have

$$\begin{aligned} \mathbb{E}[|V_t^{n, \epsilon} - I_t^{n, \epsilon}|^2] &\leq C \int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \left(\int_0^t \mathbb{E} \left[\mathbb{E}_\alpha[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-1}] ds \right. \right. \\ &\quad \left. \left. + \mathbb{E} \left[\left(\int_0^t \mathbb{E}_\alpha[|V_s^\epsilon - \tilde{V}_s^\epsilon|^{-2}] ds \right)^2 \right] \right) \right) \\ &\leq C \int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \left(\int_0^t (1 + \|g_{\rho_n(s)}\|_\infty) ds \right. \\ &\quad \left. + \mathbb{E} \left[\left(\int_0^t (1 + \|f_s^\epsilon\|_\infty) ds \right)^2 \right] \right) \\ &\leq C \int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta, \end{aligned}$$

by Lemma 2.6.1. \square

The following lemma states as follows.

Lemma 2.6.7. *Assume that $m_{p+2}(f_0) < \infty$ for some $p \geq 5$. We can couple the Poisson measure N and the white noise W in such a way that there exists a constant C depending on T , $m_{p+2}(f_0)$, $H(f_0)$, and $\sup_{s \in [0, T]} \|f_s^\epsilon + g_s\|_\infty$ such that for any $M > \sqrt{2m_2(f_0)}$, $\eta \in [\epsilon, \pi]$, $n \geq 2$ and $t \in [0, T]$,*

$$\mathbb{E}[|J_t^{n, \epsilon} - J_t^{n, \epsilon}|^2] \leq C \left[\eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right) + \int_\eta^\pi \theta^2 \beta_\epsilon(\theta) d\theta + \frac{1}{M^p} \right],$$

where

$$r_\eta = \frac{\pi}{4} \int_0^\eta \theta^2 \beta_\epsilon(\theta) d\theta.$$

Proof. It suffices to follow the line of the proof of Lemma 2.5.5, recalling that

$$\mathbb{E}[|Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}|^{-1}] \leq (1 + C \|g_{\rho_n(s)}\|_\infty) \leq C,$$

by Lemma 2.6.1. □

We now give the last lemma needed to prove Theorem 2.1.2.

Lemma 2.6.8. *There exists a constant C depending on $\sup_{s \in [0, T]} \|f_s^\epsilon + g_s\|_\infty$ and on T such that for any $n \geq 2$, $\epsilon \in (0, 1)$ and $t \in [0, T]$,*

$$\mathbb{E}[|J_t^{n, \epsilon} - Y_t|^2] \leq C \frac{\log n}{n} + C \int_0^t \left(\psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) + \psi(\mathbb{E}[|J_s^{n, \epsilon} - Y_s|^2]) \right) ds.$$

Proof. The Itô formula gives

$$\begin{aligned} \mathbb{E}[|J_t^{n, \epsilon} - Y_t|^2] &= \int_0^t \int_0^1 \mathbb{E} \left[|\sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}(\alpha)) - \sigma(Y_s - \tilde{Y}_s(\alpha))|^2 \right] ds d\alpha \\ &\quad + \int_0^t \int_0^1 \mathbb{E} \left[\left(b(V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) - b(Y_s - \tilde{Y}_s(\alpha)) \right) \cdot (J_s^{n, \epsilon} - Y_s) \right] ds d\alpha \\ &=: A + B. \end{aligned}$$

Using Fournier [29, Lemma 6], we get

$$\begin{aligned}
A &\leq 2 \int_0^t \mathbb{E}_\alpha \left[\mathbb{E} [|\sigma(Y_{\rho_n(s)} - \tilde{Y}_{\rho_n(s)}) - \sigma(Y_{\rho_n(s)} - \tilde{Y}_s)|^2] \right] ds \\
&\quad + 2 \int_0^t \mathbb{E} \left[\mathbb{E}_\alpha [|\sigma(Y_{\rho_n(s)} - \tilde{Y}_s) - \sigma(Y_s - \tilde{Y}_s)|^2] \right] ds \\
&\leq C \int_0^t (1 + \|g_{\rho_n(s)}\|_\infty) \mathbb{E}_\alpha \left[\psi(|\tilde{Y}_{\rho_n(s)} - \tilde{Y}_s|^2) \right] ds \\
&\quad + C \int_0^t (1 + \|g_s\|_\infty) \mathbb{E} \left[\psi(|Y_{\rho_n(s)} - Y_s|^2) \right] ds \\
&\leq C \int_0^t \psi(C/n) ds \\
&\leq C \frac{\log n}{n},
\end{aligned}$$

where we used the (approximate) Jensen inequality for ψ , Lemma 2.6.4 and the fact that ψ is increasing (recall (2.6.1)). For B , we first set $R = \left(b(V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) - b(Y_s - \tilde{Y}_s(\alpha)) \right) \cdot (J_s^{n,\epsilon} - Y_s)$ and $E_s = \{|J_s^{n,\epsilon} - Y_s| \geq |V_s^\epsilon - Y_s| + |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_s(\alpha)|\}$. Using [29, Lemma 3], we have

$$\begin{aligned}
R &\leq \mathbb{1}_{E_s^c} (|V_s^\epsilon - Y_s| + |\tilde{V}_s^\epsilon(\alpha) - \tilde{Y}_s(\alpha)|) |b(V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)) - b(Y_s - \tilde{Y}_s(\alpha))| \\
&\quad + C \mathbb{1}_{E_s} \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \geq 1} |J_s^{n,\epsilon} - Y_s|^2 (|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-2} + |Y_s - \tilde{Y}_s(\alpha)|^{-2}) \\
&\quad + C \mathbb{1}_{E_s} \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \leq 1} \mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)| > |J_s^{n,\epsilon} - Y_s|^4} \mathbb{1}_{|Y_s - \tilde{Y}_s(\alpha)| > |J_s^{n,\epsilon} - Y_s|^4} |J_s^{n,\epsilon} - Y_s|^2 \\
&\quad \quad (|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-3} + |Y_s - \tilde{Y}_s(\alpha)|^{-3}) \\
&\quad + C \mathbb{1}_{E_s} \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \leq 1} \mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)| \leq |J_s^{n,\epsilon} - Y_s|^4} (|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-2} \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + |Y_s - \tilde{Y}_s(\alpha)|^{-2}) \\
&\quad + C \mathbb{1}_{E_s} \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \leq 1} \mathbb{1}_{|Y_s - \tilde{Y}_s(\alpha)| \leq |J_s^{n,\epsilon} - Y_s|^4} (|V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-2} + |Y_s - \tilde{Y}_s(\alpha)|^{-2}) \\
&=: \sum_{i=1}^5 R_i.
\end{aligned}$$

We thus have $B \leq \sum_{i=1}^5 B_i$ where $B_i := \int_0^t \mathbb{E} [\mathbb{E}_\alpha [R_i]] ds$. Using [29, Lemma 7], we get

$$B_1 \leq C \int_0^t (1 + \|f_s^\epsilon + g_s\|_\infty) \left(\psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) \right) ds.$$

For B_2 , we easily get by Lemma 2.6.1,

$$B_2 \leq C \int_0^t \mathbb{E}[|J_s^{n,\epsilon} - Y_s|^2](1 + \|f_s^\epsilon\|_\infty + \|g_s\|_\infty) ds.$$

Using Lemma 2.6.1, we have

$$\begin{aligned} B_3 &\leq C \int_0^t \mathbb{E} \left[|J_s^{n,\epsilon} - Y_s|^2 \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| < 1} \mathbb{E}_\alpha [\mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| > |J_s^{n,\epsilon} - Y_s|^4} |V_s^\epsilon - \tilde{V}_s^\epsilon(\alpha)|^{-3}] \right. \\ &\quad \left. + \mathbb{1}_{|Y_s - \tilde{Y}_s| > |J_s^{n,\epsilon} - Y_s|^4} |Y_s - \tilde{Y}_s|^{-3} \right] ds \\ &\leq C \int_0^t \mathbb{E} \left[|J_s^{n,\epsilon} - Y_s|^2 \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| < 1} (1 + (\|f_s^\epsilon\|_\infty + \|g_s\|_\infty) \log \frac{1}{|J_s^{n,\epsilon} - Y_s|^4}) \right] ds. \end{aligned}$$

Recalling (2.6.1), observing that $\log(x^4) = 2 \log(x^2)$ and using the (approximate) Jensen inequality for ψ , we get

$$B_3 \leq C \int_0^t \psi(\mathbb{E}[|J_s^{n,\epsilon} - Y_s|^2]) ds.$$

We also have by Lemma 2.6.1,

$$\begin{aligned} B_4 &\leq C \int_0^t \|f_s^\epsilon\|_\infty \mathbb{E} \left[|J_s^{n,\epsilon} - Y_s|^4 \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \leq 1} \right] ds \\ &\quad + C \int_0^t \mathbb{E} \left[\mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \leq 1} \mathbb{E}_\alpha [\mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |J_s^{n,\epsilon} - Y_s|^4} |Y_s - \tilde{Y}_s|^{-2}] \right] ds. \end{aligned}$$

Using first the Hölder inequality with $p = 5$ and $q = 5/4$, and then Lemma 2.6.1, we get

$$\begin{aligned} &\mathbb{E}_\alpha [\mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |J_s^{n,\epsilon} - Y_s|^4} |Y_s - \tilde{Y}_s|^{-2}] \\ &\leq \mathbb{E}_\alpha [\mathbb{1}_{|V_s^\epsilon - \tilde{V}_s^\epsilon| \leq |J_s^{n,\epsilon} - Y_s|^4}]^{\frac{1}{5}} \mathbb{E}_\alpha [|Y_s - \tilde{Y}_s|^{-\frac{5}{2}}]^{\frac{4}{5}} \\ &\leq (C \|f_s^\epsilon\|_\infty |J_s^{n,\epsilon} - Y_s|^{12})^{\frac{1}{5}} (1 + C \|g_s\|_\infty)^{\frac{4}{5}} \\ &\leq C (1 + \|f_s^\epsilon + g_s\|_\infty) |J_s^{n,\epsilon} - Y_s|^{\frac{12}{5}}. \end{aligned}$$

We thus get

$$\begin{aligned} B_4 &\leq C \int_0^t \mathbb{E} \left[(|J_s^{n,\epsilon} - Y_s|^4 + |J_s^{n,\epsilon} - Y_s|^{\frac{12}{5}}) \mathbb{1}_{|J_s^{n,\epsilon} - Y_s| \leq 1} \right] ds \\ &\leq C \int_0^1 \mathbb{E}[|J_s^{n,\epsilon} - Y_s|^2] ds. \end{aligned}$$

We have the same bound for B_5 and thus (recalling that $x \leq \psi(x)$ for any $x \geq 0$)

$$B \leq C \int_0^t \left(\psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) + \psi(\mathbb{E}[|J_s^{n,\epsilon} - Y_s|^2]) \right) ds,$$

which concludes the proof. \square

2.6.4 Proof of Theorem 2.1.2

We set $u(t) := \mathbb{E}[|V_t^\epsilon - Y_t|^2]$ and $v(t) := \mathbb{E}[|V_t^\epsilon - W_t^\epsilon|^2] + \mathbb{E}[|W_t^\epsilon - V_t^{n,\epsilon}|^2] + \mathbb{E}[|V_t^{n,\epsilon} - I_t^{n,\epsilon}|^2] + \mathbb{E}[|I_t^{n,\epsilon} - J_t^{n,\epsilon}|^2] + \mathbb{E}[|J_t^{n,\epsilon} - Y_t|^2]$. We have $u(t) \leq Cv(t)$ and using Lemmas 2.6.3, 2.6.5, 2.6.6, 2.6.7 and 2.6.8, we get

$$\begin{aligned} v(t) &\leq Ch_\epsilon^{e-C} + \frac{C}{\log \frac{1}{\epsilon}} \\ &\quad + C \int_0^t \left(\frac{\log n}{n} + \psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) \right) ds \\ &\quad + C \int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \\ &\quad + C \left[\eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right) + \int_\eta^\pi \theta^2 \beta_\epsilon(\theta) d\theta + \frac{1}{M^p} \right] \\ &\quad + C \frac{\log n}{n} + C \int_0^t \left(\psi(\mathbb{E}[|V_s^\epsilon - Y_s|^2]) + \psi(\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2]) \right. \\ &\quad \quad \left. + \psi(\mathbb{E}[|J_s^{n,\epsilon} - Y_s|^2]) \right) ds. \end{aligned}$$

Since $\mathbb{E}[|J_s^{n,\epsilon} - Y_s|^2] \leq v(s)$ and $u(s) \leq Cv(s)$ for any $s \in (0, T]$, using that the function ψ (recall (2.6.1)) is increasing, we get (recall that $\mathbb{E}_\alpha[|\tilde{V}_s^\epsilon - \tilde{Y}_s|^2] = \mathcal{W}_2^2(f_s^\epsilon, g_s) \leq u(s)$ for any $s \in [0, T]$)

$$\begin{aligned} v(t) &\leq Ch_\epsilon^{e-C} + C \frac{\log n}{n} + \frac{C}{\log \frac{1}{\epsilon}} + C \int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta + C \int_0^t \psi(v(s)) ds \\ &\quad + C \left[\eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right) + \int_\eta^\pi \theta^2 \beta_\epsilon(\theta) d\theta + \frac{1}{M^p} \right]. \end{aligned}$$

Setting, for $\epsilon \in (0, 1)$, $\eta = \frac{1}{\log \frac{1}{\epsilon}}$, $n \approx \left(\log \frac{1}{\epsilon} \right)^{\frac{2p}{2p+3}}$, $M = \sqrt{2m_2(f_0)} \left(\log \frac{1}{\epsilon} \right)^{\frac{2}{2p+3}}$ and observing that $\int_0^\pi \theta^4 \beta_\epsilon(\theta) d\theta \leq \frac{C}{\log \frac{1}{\epsilon}}$, $\int_\eta^\pi \theta^2 \beta_\epsilon(\theta) d\theta \leq \frac{C \log \log \frac{1}{\epsilon}}{\log \frac{1}{\epsilon}}$, $\lim_{\epsilon \rightarrow 0} r_\eta =$

1 (whence $\log^2 r_\eta$ is bounded for $\epsilon \in (0, 1)$) and

$$\begin{aligned} & \eta^2 M^2 n \left(\log^2(r_\eta) + \log^2(n\eta^2) + M \right) \\ & \leq \frac{C}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p+2}{2p+3}}} \left(1 + \log^2 \left(\log \frac{1}{\epsilon} \right) + \left(\log \frac{1}{\epsilon} \right)^{\frac{2}{2p+3}} \right) \leq \frac{C}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p}{2p+3}}}, \end{aligned}$$

we get

$$\begin{aligned} v(t) & \leq Ch_\epsilon^{e^{-C}} + \frac{C \log \log \frac{1}{\epsilon}}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p}{2p+3}}} + \frac{C}{\log \frac{1}{\epsilon}} + \frac{C}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p}{2p+3}}} \\ & \quad + \frac{C \log \log \frac{1}{\epsilon}}{\log \frac{1}{\epsilon}} + C \int_0^t \psi(v(s)) ds \\ & \leq Ch_\epsilon^{e^{-C}} + \frac{C}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p-1}{2p+3}}} + C \int_0^t \psi(v(s)) ds. \end{aligned}$$

By Lemma 2.7.4, if ϵ is small enough (such that $Ch_\epsilon^{e^{-C}} + \frac{C}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p-1}{2p+3}}} \leq 1$)

we finally have

$$v(t) \leq C \left(h_\epsilon^{e^{-C}} + \frac{C}{\left(\log \frac{1}{\epsilon}\right)^{\frac{2p-1}{2p+3}}} \right)^{e^{-C}} \leq Ch_\epsilon^a + \left(\frac{C}{\log \frac{1}{\epsilon}} \right)^a,$$

for some $a > 0$. This concludes the proof since $W_2^2(f_t^\epsilon, g_t) \leq \mathbb{E}[|V_s^\epsilon - Y_s|^2] = u(t) \leq Cv(t)$ and since for ϵ greater, we have $W_2^2(f_t^\epsilon, g_t) \leq 2m_2(f_0)$. \square

2.7 Appendix

2.7.1 Distance between a compensated Poisson integral and a Gaussian variable

We first recall a result of Zaitsev [74]. For $\tau \geq 0$ and $d \in \mathbb{N}$, let $\mathcal{A}_d(\tau)$ be the class of probability distributions F on \mathbb{R}^d for which the function $\varphi(z) = \log \int_{\mathbb{R}^d} e^{z \cdot x} F(dx)$ is analytic on $\{z \in \mathbb{C}^d, |z|\tau < 1\}$ and $|d_u d_v^2 \varphi(z)| \leq |u|\tau \mathbb{D}v \cdot v$ for all $u, v \in \mathbb{R}^d$ and $|z|\tau < 1$, where \mathbb{D} is the covariance matrix of F , and $d_u \varphi$ is the derivative of φ in the direction u .

Theorem 2.7.1. (Zaitsev [74, Theorem 2]) Suppose that $\tau \geq 1$ and that ξ_1, \dots, ξ_n are independent random vectors with distributions $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E}(\xi_k) = 0$, $\text{Cov}(\xi_k) = I_d$, $k = 1, \dots, n$. Then one can build on some probability space a family of independent random vectors X_1, \dots, X_n such that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$ for any $k = 1, \dots, n$ and a family of independent random vectors $Y_1, \dots, Y_n \sim \mathcal{N}(0, I_d)$ such that

$$\mathbb{E} \left[\exp \left(\frac{a \Delta_n(X, Y)}{\tau} \right) \right] \leq \exp \left(b \max(1, \log n / \tau^2) \right),$$

where

$$\Delta_n(X, Y) = \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i \right|,$$

and a, b are positive quantities depending only on d .

Using this result, we estimate the distance between a compensated Poisson integral and a Gaussian variable.

Proposition 2.7.2. Let A be a measurable space endowed with a non negative σ -finite measure ν and N be a Poisson measure on $[0, \infty) \times A$ with intensity measure $dt\nu(dz)$. We consider $h : A \rightarrow \mathbb{R}^d$ and we set $Z_t = \int_0^t \int_A h(z) \tilde{N}(ds, dz)$, $\mu_t = \mathcal{L}(Z_t)$ and $\Gamma = \int_A h(z) h^*(z) \nu(dz)$. If $\kappa := \max_{z \in A} |\Gamma^{-1/2} h(z)| \in (0, \infty)$, then

$$\mathcal{W}_2^2(\mu_t, \mathcal{N}(0, t\Gamma)) \leq C \kappa^2 |\Gamma| \left[\max \left(1, \log \frac{t}{\kappa^2} \right) \right]^2,$$

where C depends only on d and where $\mathcal{N}(0, t\Gamma)$ is the Gaussian distribution on \mathbb{R}^d with mean 0 and covariance matrix $t\Gamma$.

Proof. For $n \in \mathbb{N}^*$ to be chosen later and $i \in \{1, \dots, n\}$, we consider

$$\xi_i = \sqrt{\frac{n}{t}} \Gamma^{-1/2} \int_{(i-1)t/n}^{it/n} \int_A h(z) \tilde{N}(ds, dz).$$

We want to use Theorem 2.7.1. We first observe that the random variables ξ_i are i.i.d., $\mathbb{E}(\xi_i) = 0$ and $\text{Cov}(\xi_i) = I_d$. We now prove that $\xi_1 \in \mathcal{A}_d(\tau)$ for some $\tau \geq 1$. For $u \in \mathbb{R}^d$, we have $\mathbb{E} \left(\exp(u \cdot \xi_1) \right) = \exp(\varphi(u))$, with

$$\varphi(u) = \frac{t}{n} \int_A \left[\exp \left(\sqrt{\frac{n}{t}} (\Gamma^{-1/2} h(z)) \cdot u \right) - 1 - \sqrt{\frac{n}{t}} (\Gamma^{-1/2} h(z)) \cdot u \right] \nu(dz).$$

For $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} & d_x d_y^2 \varphi(u) \\ &= \sqrt{\frac{n}{t}} \int_A \left[\exp \left(\sqrt{\frac{n}{t}} (\Gamma^{-1/2} h(z)) \cdot u \right) [(\Gamma^{-1/2} h(z)) \cdot y]^2 (\Gamma^{-1/2} h(z)) \cdot x \right] \nu(dz). \end{aligned}$$

We now search for $\tau > 0$ such that $|d_x d_y^2 \varphi(u)| \leq |x| \tau |y|^2$ for any u satisfying $|u| < \frac{1}{\tau}$. We have, recalling that $\kappa := \max_{z \in A} |\Gamma^{-1/2} h(z)|$,

$$\begin{aligned} |d_x d_y^2 \varphi(u)| &\leq \sqrt{\frac{n}{t}} \int_A \exp \left(\sqrt{\frac{n}{t}} |\Gamma^{-1/2} h(z)| |u| \right) |\Gamma^{-1/2} h(z)|^2 |y|^2 |\Gamma^{-1/2} h(z)| |x| \nu(dz) \\ &\leq \sqrt{\frac{n}{t}} \exp \left(\sqrt{\frac{n}{t}} \frac{\kappa}{\tau} \right) |y|^2 \kappa |x| \int_A |\Gamma^{-1/2} h(z)|^2 \nu(dz), \end{aligned}$$

since $|u| < \frac{1}{\tau}$. We have, observing that Γ is symmetric,

$$\begin{aligned} \int_A |\Gamma^{-1/2} h(z)|^2 \nu(dz) &= \int_A h^*(z) \Gamma^{-1} h(z) \nu(dz) \\ &= \sum_{i,j=1}^d \int_A h_i(z) (\Gamma^{-1})_{ij} h_j(z) \nu(dz) \\ &= \sum_{i,j=1}^d (\Gamma^{-1})_{ij} \int_A h_i(z) h_j(z) \nu(dz) \\ &= \sum_{i,j=1}^d (\Gamma^{-1})_{ij} \Gamma_{ij} = \sum_{i=1}^d \left(\sum_{j=1}^d (\Gamma^{-1})_{ij} \Gamma_{ji} \right) = \sum_{i=1}^d (\Gamma^{-1} \Gamma)_{ii} = d. \end{aligned}$$

Setting $\tau = 2d\kappa \sqrt{\frac{n}{t}}$, we thus have

$$|d_x d_y^2 \varphi(u)| \leq |x| |y|^2 \sqrt{\frac{n}{t}} \exp \left(\sqrt{\frac{n}{t}} \frac{\kappa}{\tau} \right) d\kappa = |x| |y|^2 \frac{\tau}{2} \exp \left(\frac{1}{2d} \right) \leq |x| |y|^2 \tau.$$

So we have $\xi_i \in \mathcal{A}_d(\tau)$ with $\tau = 2d\kappa \sqrt{\frac{n}{t}}$. Thus choosing $n \geq \frac{t}{4d^2\kappa^2}$ so that $\tau \geq 1$, we can apply Theorem 2.7.1: one can construct on some probability space a sequence of independent random vectors X_1, \dots, X_n such that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$ for any $k = 1, \dots, n$ and a sequence of independent random vectors $Y_1, \dots, Y_n \sim \mathcal{N}(0, I_d)$ such that

$$\mathbb{E} \left[\exp \left(\frac{a}{2d} \frac{\sqrt{t}}{\kappa} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^n X_i - \sum_{i=1}^n Y_i \right| \right) \right] \leq \exp \left(b \max(1, \log \frac{t}{4d^2\kappa^2}) \right).$$

Then setting $R_t := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ (observe that $\mathcal{L}(R_t) = \mathcal{L}((t\Gamma)^{-1/2}Z_t)$) and $Y := \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i$ (observe that $\mathcal{L}(Y) = \mathcal{N}(0, I_d)$), we get

$$\mathbb{E} \left[\exp \left(\frac{a}{2d} \frac{\sqrt{t}}{\kappa} |R_t - Y| \right) \right] \leq \exp \left(b \max(1, \log \frac{t}{\kappa^2}) \right).$$

For $x \geq 0$, we have

$$\begin{aligned} \mathbb{P}(|R_t - Y|^2 \geq x) &= \mathbb{P} \left(\exp \left(\frac{a}{2d} \frac{\sqrt{t}}{\kappa} |R_t - Y| \right) \geq \exp \left(\frac{a}{2d} \frac{\sqrt{t}}{\kappa} \sqrt{x} \right) \right) \\ &\leq \exp \left(- \frac{a}{2d} \frac{\sqrt{t}}{\kappa} \sqrt{x} \right) \exp \left(b \max(1, \log \frac{t}{\kappa^2}) \right). \end{aligned}$$

We consider x_0 verifying $\frac{a}{2d} \frac{\sqrt{t}}{\kappa} \sqrt{x_0} = b \max(1, \log \frac{t}{\kappa^2})$.

$$\begin{aligned} \mathbb{E}(|R_t - Y|^2) &= \int_0^\infty \mathbb{P}(|R_t - Y|^2 \geq x) dx \\ &\leq x_0 + \exp \left(b \max(1, \log \frac{t}{\kappa^2}) \right) \int_{x_0}^{+\infty} \exp \left(- \frac{a}{2d} \frac{\sqrt{t}}{\kappa} \sqrt{x} \right) dx \\ &= x_0 + \int_{x_0}^{+\infty} \exp \left(- \frac{a}{2d} \frac{\sqrt{t}}{\kappa} (\sqrt{x} - \sqrt{x_0}) \right) dx \\ &= x_0 + 2 \int_0^{+\infty} (y + \sqrt{x_0}) \exp \left(- \frac{a}{2d} \frac{\sqrt{t}}{\kappa} y \right) dy \\ &= x_0 + 2 \left(\frac{4d^2 \kappa^2}{a^2 t} + \frac{2d \sqrt{x_0} \kappa}{a \sqrt{t}} \right) \\ &\leq C \frac{\kappa^2}{t} \left[\max \left(1, \log \frac{t}{\kappa^2} \right) \right]^2. \end{aligned}$$

We thus have

$$\mathcal{W}_2^2(R_t, \mathcal{N}(0, I_d)) \leq C \frac{\kappa^2}{t} \left[\max \left(1, \log \frac{t}{\kappa^2} \right) \right]^2,$$

and finally, since Z_t has the same law as $\sqrt{t}\Gamma^{1/2}R_t$,

$$\mathcal{W}_2^2(Z_t, \mathcal{N}(0, t\Gamma)) \leq C \kappa^2 |\Gamma| \left[\max \left(1, \log \frac{t}{\kappa^2} \right) \right]^2.$$

□

2.7.2 Ellipticity of the diffusion matrix

In this article, we need some ellipticity hypothesis for the diffusion matrix l , recall (2.1.8). To this aim, we will extend some result stated in Desvillettes-Villani [22] for $\gamma \geq 0$.

Proposition 2.7.3. *Let $\gamma \in [-3, 0)$ and $E_0, H_0 > 0$ be two constants. Consider a nonnegative function f such that $\int_{\mathbb{R}^3} f(v)dv = 1$, $m_2(f) \leq E_0$ and $H(f) \leq H_0$. There exists a constant $c = c(\gamma, E_0, H_0)$ such that for any $v \in \mathbb{R}^3$ and any $\xi \in \mathbb{R}^3$,*

$$(\bar{l}^f(v)\xi) \cdot \xi \geq c(1 + |v|)^\gamma |\xi|^2,$$

where $\bar{l}^f(v) = \int_{\mathbb{R}^3} l(v - v_*) f(v_*) dv_*$.

Proof. For $\gamma \in [-2, 0)$, it is easy to check that in the proof of [22, Proposition 4], they only use that $\gamma + 2 \geq 0$. For $\gamma \in [-3, -2)$, we have to adapt a little bit their proof. In this case, estimate (44) of their proof still holds: for all $v \in \mathbb{R}^3$, $\theta \in (0, \pi/2)$ and $R_* > 0$

$$\begin{aligned} (\bar{l}^f(v)\xi) \cdot \xi &\geq \int_{\mathbb{R}^3 \setminus D_{\theta, \xi}(v)} dv_* \mathbb{1}_{|v_*| \leq R_*} |v - v_*|^{\gamma+2} f(v_*) \sin^2 \theta \\ &\geq (|v| + R_*)^{\gamma+2} \sin^2 \theta \int_{\mathbb{R}^3 \setminus D_{\theta, \xi}(v)} dv_* \mathbb{1}_{|v_*| \leq R_*} f(v_*), \end{aligned}$$

(recall that $\gamma + 2 < 0$) where $D_{\theta, \xi}(v) = \left\{ v_* \in \mathbb{R}^3, \left| \frac{v-v_*}{|v-v_*|} \cdot \xi \right| \geq \cos \theta \right\}$ is the cone centred at v , of axis directed by ξ , and of angle θ . Now following the scheme of their proof, we easily get that $(\bar{l}^f(v)\xi) \cdot \xi \geq K|v|^\gamma$ if $|v| \geq 2R_*$ and that $(\bar{l}^f(v)\xi) \cdot \xi \geq K$ if $|v| < 2R_*$ with $R_* = 2\sqrt{E_0}$, which concludes the proof. \square

2.7.3 Generalization of the Grönwall Lemma

In order to treat the Coulomb case, we need to use the following generalization of the Grönwall lemma.

Lemma 2.7.4. *Let $T > 0$ and $\gamma : [0, T] \rightarrow \mathbb{R}_+$ satisfy $\int_0^T \gamma(s)ds < \infty$. Let ψ be defined by (2.6.1). Consider a bounded function $\rho : [0, T] \rightarrow \mathbb{R}_+$ such that, for some $a \geq 0$, for all $t \in [0, T]$, $\rho(t) \leq a + \int_0^t \gamma(s)\psi(\rho(s))ds$. We set $K := \int_0^T \gamma(s)ds$. Then $\rho(t) \leq C(a^{e^{-K}} + a)$ for all $t \in [0, T]$, where C only depends on K .*

Proof. From Chemin [16, Lemme 5.2.1 p. 89], we get that $M(a) - M(\rho(t)) \leq \int_0^t \gamma(s) ds$ for all $t \in [0, T]$, where $M(x) := \int_x^1 (1/\psi(y)) dy$ for $x > 0$.

Recalling that $\psi(y) = y(1 - \mathbb{1}_{y \leq 1} \log y)$, we get that $M(x) = \log(1 - \log x)$ for $x \in [0, 1]$ and $M(x) = -\log x$ for $x > 1$. Let $t \in [0, T]$ be fixed.

If $a \leq 1$ and $\rho(t) \leq 1$, we have $\log\left(\frac{1 - \log a}{1 - \log \rho(t)}\right) \leq K$ which gives $\rho(t) \leq e^{1 - e^{-K}} a e^{-K}$.

If $a \leq 1$ and $\rho(t) > 1$, we have $\log((1 - \log a)\rho(t)) \leq K$ which gives $\rho(t) \leq \frac{e^K}{1 - \log a}$ and thus necessarily (since $\rho(t) > 1$) $a > e^{1 - e^K}$. Thus $\rho(t) \leq e^K \leq e^K e^{e^K - 1} a$.

If $a > 1$ and $\rho(t) > 1$, we have $\log \frac{\rho(t)}{a} \leq K$ which gives $\rho(t) \leq e^K a$.

If $a > 1$ and $\rho(t) \leq 1$, we have $\rho(t) \leq 1 < a$, which concludes the proof.

□

2.7.4 Construction of a subdivision

We end this paper with the following result.

Proposition 2.7.5. *For $T > 0$ fixed, we consider $h \in \mathcal{L}^1([0, T])$ with $h(s) \geq 0$ for any $s \in [0, T]$. For any $n \in \mathbb{N}^*$, there exist a subdivision $0 < a_0^n < \dots < a_{\lfloor 2nT \rfloor - 1}^n < a_{\lfloor 2nT \rfloor}^n = T$ such that $a_0^n < 1/n$ and for any $i \in \{0, \dots, \lfloor 2nT \rfloor - 1\}$, $1/4n < a_{i+1}^n - a_i^n < 1/n$ and*

$$\sum_{i=0}^{\lfloor 2nT \rfloor - 1} (a_{i+1}^n - a_i^n) h(a_i^n) \leq 3 \int_0^T h(s) ds + 3.$$

Proof. We take $a_i^n \in \left(\frac{i}{2n}, \frac{2i+1}{4n}\right]$ such that $h(a_i^n) \leq h(s) + 1/T$ for any $s \in \left(\frac{i}{2n}, \frac{2i+1}{4n}\right]$. We set $g(s) = \sum_{i=0}^{\lfloor 2nT \rfloor - 1} h(a_i^n) \mathbb{1}_{\left\{s \in \left(\frac{i}{2n}, \frac{2i+1}{4n}\right]\right\}}$. We have $g(s) \leq h(s) + 1/T$, $1/4n < a_{i+1}^n - a_i^n < 3/4n$ and thus

$$\sum_{i=0}^{\lfloor 2nT \rfloor - 1} (a_{i+1}^n - a_i^n) h(a_i^n) \leq \frac{3}{4n} \sum_{i=0}^{\lfloor 2nT \rfloor - 1} h(a_i^n) = 3 \int_0^T g(s) ds \leq 3 \int_0^T h(s) ds + 3,$$

which concludes the proof. □

Chapter 3

Propagation of chaos for a subcritical Keller-Segel model

D. Godinho and C. Quininao. Propagation of chaos for a subcritical Keller-Segel model.

This paper deals with a subcritical Keller-Segel equation. Starting from the stochastic particle system associated with it, we show well-posedness results and the propagation of chaos property. More precisely, we show that the empirical measure of the system tends towards the unique solution of the limit equation as the number of particles goes to infinity.

3.1 Introduction and main results

The subject of this paper is the convergence of a stochastic particle system to a nonlinear and nonlocal equation which can be seen as a subcritical version of the classical Keller-Segel equation.

3.1.1 The subcritical Keller-Segel Equation

Consider the equation:

$$(3.1.1) \quad \frac{\partial f_t(x)}{\partial t} = \chi \nabla_x \cdot ((K * f_t)(x)) f_t(x) + \Delta_x f_t(x),$$

where $f : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\chi > 0$. The force field kernel $K : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ comes from an attractive potential $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ and is defined by

$$(3.1.2) \quad K(x) := \frac{x}{|x|^{\alpha+1}} = -\nabla \underbrace{\left(\frac{1}{\alpha-1} |x|^{1-\alpha} \right)}_{\Phi(x)}, \quad \alpha \in (0, 1).$$

The standard Keller-Segel equation correspond to the critical case $K(x) = x/|x|^2$ (i.e., more singular at $x = 0$) and it describes a model of chemotaxis, i.e., the movement of cells (usually bacteria or amoebae) which are attracted by some chemical substance that they produce. This equation has been first introduced by Keller and Segel in [49, 50]. Blanchet-Dolbeault-Perthame showed in [6] some nice results on existence of global weak solutions if the parameter χ (which is the sensitivity of the bacteria to the chemo-attractant) is smaller than $8\pi/M$ where M is the initial mass (here M will always be 1 since we will deal with probability measures). For more details on the subject, see [42, 43].

3.1.2 The particle system

We consider the following system of particles

$$(3.1.3) \quad \forall i = 1, \dots, N, \quad X_t^{i,N} = X_0^{i,N} - \frac{\chi}{N} \sum_{j=1, j \neq i}^N \int_0^t K(X_s^{i,N} - X_s^{j,N}) ds + \sqrt{2} B_t^i,$$

where $(B^i)_{i=1, \dots, N}$ is an independent family of 2D standard Brownian motions and K is defined in (3.1.2). We will show in the sequel that there is propagation of chaos to the solution of the following nonlinear S.D.E linked with (3.1.1) (see the next paragraph)

$$(3.1.4) \quad X_t = X_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(dx) ds + \sqrt{2} B_t,$$

where $f_t = \mathcal{L}(X_t)$.

3.1.3 Weak solution for the P.D.E

For any Polish space E , we denote by $\mathbf{P}(E)$ the set of all probability measures on E which we endow with the topology of weak convergence defined by

duality against functions of $C_b(E)$. We give the notion of weak solution that we use in this paper.

Definition 3.1.1. *We say that $f = (f_t)_{t \geq 0} \in C([0, \infty), \mathbf{P}(\mathbb{R}^2))$ is a weak solution to (3.1.1) if*

$$(3.1.5) \quad \forall T > 0, \quad \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |K(x-y)| f_s(dx) f_s(dy) ds < \infty,$$

and if for all $\varphi \in C_b^2(\mathbb{R}^2)$, all $t \geq 0$,

$$(3.1.6) \quad \begin{aligned} \int_{\mathbb{R}^2} \varphi(x) f_t(dx) &= \int_{\mathbb{R}^2} \varphi(x) f_0(dx) + \int_0^t \int_{\mathbb{R}^2} \Delta_x \varphi(x) f_s(dx) ds \\ &\quad - \chi \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K(x-y) \cdot \nabla_x \varphi(x) f_s(dy) f_s(dx) ds. \end{aligned}$$

Remark 3.1.2. *We can see easily that if $(X_t)_{t \geq 0}$ is a solution to (3.1.4), then setting $f_t = \mathcal{L}(X_t)$ for any $t \geq 0$, $(f_t)_{t \geq 0}$ is a weak solution of (3.1.1) in the sense of Definition 3.1.1 provided it satisfies (3.1.5). Indeed, by Itô's formula, we find that for $\varphi \in C_b^2(\mathbb{R}^2)$,*

$$\begin{aligned} \varphi(X_t) &= \varphi(X_0) - \chi \int_0^t \nabla_x \varphi(X_s) \cdot \int_{\mathbb{R}^2} K(X_s - y) f_s(dy) ds \\ &\quad + \int_0^t \sqrt{2} \nabla_x \varphi(X_s) \cdot dB_s + \int_0^t \Delta_x \varphi(X_s) ds. \end{aligned}$$

Taking expectations, we get (3.1.6).

3.1.4 Notation and propagation of chaos

For $N \geq 2$, we denote by $\mathbf{P}_{sym}(E^N)$ the set of symmetric probability measures on E^N , i.e. the set of probability measures which are laws of exchangeable E^N -valued random variables.

We consider for any $F \in \mathbf{P}_{sym}((\mathbb{R}^2)^N)$ with a density (a finite moment of positive order is also required in order to define the entropy) the Boltzmann entropy and the Fisher information which are defined by

$$H(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} F(x) \log F(x) dx \quad \text{and} \quad I(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \frac{|\nabla F(x)|^2}{F(x)} dx.$$

We also define $(x_i \in \mathbb{R}^2$ stands for the i -th coordinate of $x \in (\mathbb{R}^2)^N$), for $k \geq 0$,

$$M_k(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \sum_{i=1}^N |x_i|^k F(dx).$$

Observe that we proceed to the normalization by $1/N$ in order to have, for any $f \in \mathbf{P}(\mathbb{R}^2)$,

$$H(f^{\otimes N}) = H(f), \quad I(f^{\otimes N}) = I(f) \quad \text{and} \quad M_k(f^{\otimes N}) = M_k(f).$$

We introduce the space $\mathbf{P}_1(\mathbb{R}^2) := \{f \in \mathbf{P}(\mathbb{R}^2), M_1(f) < \infty\}$ and we recall the definition of the Wasserstein distance: if $f, g \in \mathbf{P}_1(\mathbb{R}^2)$,

$$\mathcal{W}_1(f, g) = \inf \left\{ \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| R(dx, dy) \right\},$$

where the infimum is taken over all probability measures R on $\mathbb{R}^2 \times \mathbb{R}^2$ with f for first marginal and g for second marginal. It is known that the infimum is reached. See e.g. Villani [72] for many details on the subject.

We now define the notion of propagation of chaos.

Definition 3.1.3. *Let X be some E -valued random variable. A sequence (X_1^N, \dots, X_N^N) of exchangeable E -valued random variables is said to be X -chaotic if one of the three following equivalent conditions is satisfied:*

- (i) (X_1^N, X_2^N) goes in law to 2 independent copies of X as $N \rightarrow +\infty$;
- (ii) for all $j \geq 1$, (X_1^N, \dots, X_j^N) goes in law to j independent copies of X as $N \rightarrow +\infty$;
- (iii) the empirical measure $\mu_{X^N}^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N} \in \mathbf{P}(E)$ goes in law to the constant $\mathcal{L}(X)$ as $N \rightarrow +\infty$.

We refer to [62] for the equivalence of the three conditions or [40, Theorem 1.2] where the equivalence is established in a quantitative way.

Propagation of chaos in the sense of Sznitman holds for a system of N exchangeable particles evolving in time if when the initial conditions $(X_0^{1,N}, \dots, X_0^{N,N})$ are X_0 -chaotic, the trajectories $((X_t^{1,N})_{t \geq 0}, \dots, (X_t^{N,N})_{t \geq 0})$ are $(X_t)_{t \geq 0}$ -chaotic, where $(X_t)_{t \geq 0}$ is the (unique) solution of the expected (one-particle) limit model.

We finally recall a stronger (see [40]) sense of chaos introduced by Kac in [47] and formalized recently in [14]: the entropic chaos.

Definition 3.1.4. Let f be some probability measure on E . A sequence (F^N) of symmetric probability measures on E^N is said to be entropically f -chaotic if

$$F_1^N \rightarrow f \text{ weakly in } \mathbf{P}(E) \text{ and } H(F^N) \rightarrow H(f) \text{ as } N \rightarrow \infty,$$

where F_1^N stands for the first marginal of F^N .

We can observe that since the entropy is lower semi continuous (so that $H(f) \leq \liminf_N H(F^N)$) and is convex, the entropic chaos (which requires $\lim_N H(F^N) = H(f)$) is a stronger notion of convergence which implies that for all $j \geq 1$, the density of the law of (X_1^N, \dots, X_j^N) goes to $f^{\otimes j}$ strongly in L^1 as $N \rightarrow \infty$ (see [8]).

3.1.5 Main results

We first give a result of existence and uniqueness for (3.1.1).

Theorem 3.1.5. Let $\alpha \in (0, 1)$. Assume that $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ is such that $H(f_0) < \infty$.

(i) There exists a unique weak solution f to (3.1.1) such that

(3.1.7)

$$f \in L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2)) \text{ for some } p > \frac{2}{1-\alpha}.$$

(ii) This solution furthermore satisfies that for all $T > 0$,

$$(3.1.8) \quad \int_0^T I(f_s) ds < \infty,$$

for any $q \in [1, 2)$ and for all $T > 0$,

$$(3.1.9) \quad \nabla_x f \in L^{2q/(3q-2)}(0, T; L^q(\mathbb{R}^2)),$$

for any $p \geq 1$,

$$(3.1.10) \quad f \in C([0, \infty); L^1(\mathbb{R}^2)) \cap C((0, \infty); L^p(\mathbb{R}^2)),$$

and that for any $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2,\infty}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside a compact set,

$$(3.1.11) \quad \begin{aligned} \partial_t \beta(f) = & \chi(K * f) \cdot \nabla_x(\beta(f)) + \Delta_x \beta(f) \\ & - \beta''(f) |\nabla_x f|^2 + \chi \beta'(f_s) f_s (\nabla_x \cdot K * f_s), \end{aligned}$$

on $[0, \infty) \times \mathbb{R}^2$ in the distributional sense.

We denote by F_0^N the law of $(X_0^{i,N})_{i=1,\dots,N}$. We assume that for some $f_0 \in \mathbf{P}(\mathbb{R}^2)$,

$$(3.1.12) \quad \begin{cases} F_0^N \in \mathbf{P}_{sym}((\mathbb{R}^2)^N) & \text{is } f_0\text{-chaotic;} \\ \sup_{N \geq 2} M_1(F_0^N) < \infty, & \sup_{N \geq 2} H(F_0^N) < \infty. \end{cases}$$

Observe that this condition is satisfied if the random variables $(X_0^{i,N})_{i=1,\dots,N}$ are i.i.d. with law $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ such that $H(f_0) < \infty$. The next result states the well-posedness for the particle system (3.1.3).

Theorem 3.1.6. *Let $\alpha \in (0, 1)$.*

(i) *Let $N \geq 2$ be fixed and assume that $M_1(F_0^N) < \infty$ and $H(F_0^N) < \infty$. There exists a unique strong solution $(X_t^{i,N})_{t \geq 0, i=1,\dots,N}$ to (3.1.3). Furthermore, the particles a.s. never collapse i.e. it holds that a.s., for any $t \geq 0$ and $i \neq j$, $X_t^{i,N} \neq X_t^{j,N}$.*

(ii) *Assume (3.1.12). If for all $t \geq 0$, we denote by $F_t^N \in \mathbf{P}_{sym}((\mathbb{R}^2)^N)$ the law of $(X_t^{i,N})_{i=1,\dots,N}$, then there exists a constant C depending on χ , $\sup_{N \geq 2} H(F_0^N)$ and $\sup_{N \geq 2} M_1(F_0^N)$ such that for all $t \geq 0$ and $N \geq 2$*

$$H(F_t^N) \leq C(1+t), \quad M_1(F_t^N) \leq C(1+t), \quad \int_0^t I(F_s^N) ds \leq C(1+t).$$

Furthermore for any $T > 0$,

$$(3.1.13) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{1,N}| \right] \leq C(1+T).$$

We also have

$$(3.1.14) \quad H(F_t^N) + \int_0^t I(F_s^N) ds \leq H(F_0^N) + \frac{\chi}{N^2} \sum_{i \neq j} \int_0^t \mathbb{E}[\operatorname{div} K(X_s^{i,N} - X_s^{j,N})] ds.$$

We next state a well-posedness result for the nonlinear S.D.E. (3.1.4).

Theorem 3.1.7. *Let $\alpha \in (0, 1)$ and $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ such that $H(f_0) < \infty$. There exists a unique strong solution $(X_t)_{t \geq 0}$ to (3.1.4) such that for some $p > 2/(1-\alpha)$,*

$$(3.1.15) \quad (f_t)_{t \geq 0} \in L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2)),$$

where f_t is the law of X_t . Furthermore, $(f_t)_{t \geq 0}$ is the unique solution to (3.1.1) given in Theorem 3.1.5.

We finally give the result about propagation of chaos.

Theorem 3.1.8. *Let $\alpha \in (0, 1)$. Assume (3.1.12). For each $N \geq 2$, consider the unique solution $(X_t^{i,N})_{i=1,\dots,N,t \geq 0}$ to (3.1.3). Let $(X_t)_{t \geq 0}$ be the unique solution to (3.1.4).*

(i) *The sequence $(X_t^{i,N})_{i=1,\dots,N,t \geq 0}$ is $(X_t)_{t \geq 0}$ -chaotic.*

In particular, the empirical measure $Q^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N})_{t \geq 0}}$ goes in law to $\mathcal{L}((X_t)_{t \geq 0})$ in $\mathbf{P}(C((0, \infty), \mathbb{R}^2))$.

(ii) *Assume furthermore that $\lim_N H(F_0^N) = H(f_0)$. For all $t \geq 0$, the sequence $(X_t^{i,N})_{i=1,\dots,N}$ is then X_t -entropically chaotic. In particular, for any $j \geq 1$ and any $t \geq 0$, denoting by F_{tj}^N the density of the law of $(X_t^{1,N}, \dots, X_t^{j,N})$, it holds that*

$$\lim_{N \rightarrow \infty} \|F_{tj}^N - f_t^{\otimes j}\|_{L^1((\mathbb{R}^2)^j)} = 0.$$

We can observe that the condition $\lim_N H(F_0^N) = H(f_0)$ is satisfied if the random variables $(X_0^{i,N})_{i=1,\dots,N}$ are i.i.d. with law f_0 such that $H(f_0) < \infty$.

3.1.6 Comments

This paper is some kind of adaptation of the work of Fournier-Hauray-Mischler in [36] where they show the propagation of chaos of some particle system for the 2D viscous vortex model. We use the same methods for a subcritical Keller-Segel equation. The proofs are thus sometimes very similar to those in [36] but there are some differences due to the facts that i) there are no circulation parameter (\mathcal{M}_i^N in [36]): this simplify the situation since we thus deal with solutions which are probabilities and ii) the kernel is not the same: it is not divergence-free and we thus have to deal with some additional terms in our computations. We can also notice that due to this fact, we have no already known result for the existence and uniqueness of the particle system that we consider.

The proof of Theorem 3.1.5 follows the ideas of renormalisation solutions to a PDE introduced by Di Perna and Lions in [25] and developed since then. The key point is to be able to find good *a priori* estimates which allow us to approximate the weak solutions by regular functions, i.e., to use C^k functions instead of L^1 . Then, using these estimates, one can pass to the limit and go back to the initial problem. One can further see that the uniqueness result is proven based on coupling methods and the Wasserstein distance. This will

allow us to use more general initial conditions than we could use in a strictly deterministic framework.

The proof of existence and uniqueness for the particle system (3.1.3) (Theorem 3.1.6) use some nice arguments. Like for S.D.Es with locally Lipschitz coefficients, we show existence and uniqueness up to an explosion time and the interesting part of the proof is to show that this explosion time is infinite a.s.

To our knowledge, there is no other work that give a convergence result of some particle system for a chemotaxis model with a singular kernel K and without cutoff parameter. In [59], Stevens studies a particle system with two kinds of particles corresponding to bacteria and chemical substance. She shows convergence of the system for smooth initial data (lying in $C_b^3(\mathbb{R}^d)$) and for regular kernels (continuously differentiable and bounded together with their derivatives). In [39], Haskovec and Schmeiser consider a kernel with a cutoff parameter $K_\epsilon(x) = \frac{x}{|x|(|x|+\epsilon)}$. They get some well-posedness result for the particle system and they show the weak convergence of subsequences due to a tightness result (observe that here we have propagation of chaos and also entropic chaos). In a recent work [10], Calvez and Corrias work on some one-dimensional Keller-Segel model. They study a dynamical particle system for which they give a global existence result under some assumptions on the initial distribution of the particles that prevents collisions. They also give two blow-up criteria for the particle system but they do not state a convergence result for this system.

Finally, it is important to notice that the present method can not be directly adapted for the standard case $\alpha = 1$ because in this last situation the entropy and the Fisher information are not controlled.

3.1.7 Plan of the paper

In the next section, we give some preliminary results. In Section 3.3, we establish the well-posedness of the particle system (3.1.3). In Section 3.4, we prove the tightness of the particle system and we show that any limit point belongs to the set of solutions to the nonlinear S.D.E. (3.1.4). In Section 3.5, we show that the P.D.E. (3.1.1) and the nonlinear S.D.E. (3.1.4) are well-posed and we show the propagation of chaos. Finally, in the last section, we improve the regularity of the solution, give some renormalization results for the solution to (3.1.1) and we conclude with the entropic chaos.

3.2 Preliminaries

In this section, we recall some lemmas stated in [36] and [40] and we state a result on the regularity of the kernel K defined in (3.1.2). The first result tells us that pairs of particles which law have finite Fisher information cannot be too close.

Lemma 3.2.1. (*[36, Lemma 3.3]*) *Consider $F \in \mathbf{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ with finite Fisher information and (X_1, X_2) a random variable with law F . Then for any $\gamma \in (0, 2)$ and any $\beta > \gamma/2$ there exists $C_{\gamma, \beta}$ so that*

$$\mathbb{E}[|X_1 - X_2|^{-\gamma}] = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{F(x_1, x_2)}{|x_1 - x_2|^\gamma} dx_1 dx_2 \leq C_{\gamma, \beta} (I(F)^\beta + 1).$$

In the next lemma, we see that the Fisher information of the marginals of some $F \in \mathbf{P}_{sym}((\mathbb{R}^2)^N)$ is smaller than the Fisher information of F .

Lemma 3.2.2. (*[40, Lemma 3.7]*) *For any $F \in \mathbf{P}_{sym}((\mathbb{R}^2)^N)$ and $1 \leq l \leq N$, $I(F_l) \leq I(F)$, where $F_l \in \mathbf{P}_{sym}((\mathbb{R}^2)^l)$ denotes the marginal probability of F on the l -th block of variables.*

The following lemma allows us to control from below the entropy of some $F \in \mathbf{P}_k((\mathbb{R}^2)^N)$ by its moment of order k for any $k > 0$.

Lemma 3.2.3. (*[36, Lemma 3.1]*) *For any $k, \lambda \in (0, \infty)$, there is a constant $C_{k, \lambda} \in \mathbb{R}$ such that for any $N \geq 1$, any $F \in \mathbf{P}_k((\mathbb{R}^2)^N)$,*

$$H(F) \geq -C_{k, \lambda} - \lambda M_k(F).$$

The next result tells us that a probability measure on \mathbb{R}^2 with finite Fisher information belongs to L^p for any $p \geq 1$ and its derivatives, to L^q for any $q \in [1, 2)$.

Lemma 3.2.4. (*[36, Lemma 3.2]*) *For any $f \in \mathbf{P}(\mathbb{R}^2)$ with finite Fisher information, there holds*

$$\begin{aligned} \forall p \in [1, \infty), \quad \|f\|_{L^p(\mathbb{R}^2)} &\leq C_p I(f)^{1-1/p}, \\ \forall q \in [1, 2), \quad \|\nabla_x f\|_{L^q(\mathbb{R}^2)} &\leq C_q I(f)^{3/2-1/q}. \end{aligned}$$

We end this section with the following result on K .

Lemma 3.2.5. *Let $\alpha \in (0, 1)$. There exists a constant C_α such that for all $x, y \in \mathbb{R}^2$*

$$|K(x) - K(y)| \leq C_\alpha |x - y| \left(\frac{1}{|x|^{\alpha+1}} + \frac{1}{|y|^{\alpha+1}} \right).$$

Proof. We have

$$\begin{aligned} |K(x) - K(y)| &= \left| x \left(\frac{1}{|x|^{\alpha+1}} - \frac{1}{|y|^{\alpha+1}} \right) + \frac{x-y}{|y|^{\alpha+1}} \right| \\ &\leq |x| |x-y| (\alpha+1) \max \left(\frac{1}{|x|^{\alpha+2}}, \frac{1}{|y|^{\alpha+2}} \right) + \frac{|x-y|}{|y|^{\alpha+1}}. \end{aligned}$$

By symmetry, we also have

$$|K(x) - K(y)| \leq |y| |x-y| (\alpha+1) \max \left(\frac{1}{|x|^{\alpha+2}}, \frac{1}{|y|^{\alpha+2}} \right) + \frac{|x-y|}{|x|^{\alpha+1}}.$$

So we deduce that

$$\begin{aligned} |K(x) - K(y)| &\leq |x-y| \left[(\alpha+1) \min(|x|, |y|) \max \left(\frac{1}{|x|^{\alpha+2}}, \frac{1}{|y|^{\alpha+2}} \right) \right. \\ &\quad \left. + \frac{1}{|x|^{\alpha+1}} + \frac{1}{|y|^{\alpha+1}} \right] \\ &\leq |x-y| \left[(\alpha+1) \frac{1}{\min(|x|, |y|)^{\alpha+1}} + \frac{1}{|x|^{\alpha+1}} + \frac{1}{|y|^{\alpha+1}} \right] \\ &\leq 2(\alpha+2) |x-y| \left(\frac{1}{|x|^{\alpha+1}} + \frac{1}{|y|^{\alpha+1}} \right). \end{aligned}$$

which concludes the proof. \square

3.3 Well-posedness for the system of particles

Let's now introduce another particle system with a regularized kernel. We set, for $\epsilon \in (0, 1)$,

$$(3.3.1) \quad K_\epsilon(x) = \frac{x}{\max(|x|, \epsilon)^{\alpha+1}},$$

which obviously satisfies $|K_\epsilon(x) - K_\epsilon(y)| \leq C_{\alpha,\epsilon}|x - y|$ and we consider the following system of S.D.E.s

$$(3.3.2) \quad \forall i = 1, \dots, N, \quad X_t^{i,N,\epsilon} = X_0^{i,N} - \frac{\chi}{N} \sum_{j=1, j \neq i}^N \int_0^t K_\epsilon(X_s^{i,N,\epsilon} - X_s^{j,N,\epsilon}) ds + \sqrt{2} B_t^i,$$

for which strong existence and uniqueness thus holds.

The following result will be useful for the proof of Theorem 3.1.6. Its proof is very similar to the proof of [36, Proposition 5.1].

Proposition 3.3.1. *Let $\alpha \in (0, 1)$.*

(i) *Let $N \geq 2$ be fixed. Assume that $M_1(F_0^N) < \infty$ and $H(F_0^N) < \infty$. For all $t \geq 0$, we denote by $F_t^{N,\epsilon} \in \mathbf{P}_{\text{sym}}((\mathbb{R}^2)^N)$ the law of $(X_t^{i,N,\epsilon})_{i=1,\dots,N}$. Then*

$$(3.3.3) \quad H(F_t^{N,\epsilon}) = H(F_0^N) + \frac{\chi}{N^2} \sum_{i \neq j} \int_0^t \int_{(\mathbb{R}^2)^N} \text{div} K_\epsilon(x_i - x_j) F_s^{N,\epsilon}(x) ds dx \\ - \int_0^t I(F_s^{N,\epsilon}) ds.$$

(ii) *There exists a constant C which depends on χ , $H(F_0^N)$ and $M_1(F_0^N)$ such that for all $t \geq 0$ and $N \geq 2$,*

$$(3.3.4) \quad H(F_t^{N,\epsilon}) \leq C(1+t), \quad M_1(F_t^{N,\epsilon}) \leq C(1+t), \quad \int_0^t I(F_s^{N,\epsilon}) ds \leq C(1+t).$$

Furthermore,

$$(3.3.5) \quad \mathbb{E} \left[\sup_{[0,T]} |X_t^{1,N,\epsilon}| \right] \leq C(1+T).$$

Proof. Let $\varphi \in C_b^2((\mathbb{R}^2)^N)$, and $t \geq 0$ be fixed. Using Itô's formula, we compute the expectation of $\varphi(X_t^{1,N,\epsilon}, \dots, X_t^{N,N,\epsilon})$ and get (recall that $x_i \in \mathbb{R}^2$ stands for the i -th coordinate of $x \in (\mathbb{R}^2)^N$)

$$(3.3.6) \quad \frac{d}{dt} \int_{(\mathbb{R}^2)^N} \varphi(x) F_t^{N,\epsilon}(dx) = - \frac{\chi}{N} \int_{(\mathbb{R}^2)^N} \sum_{i \neq j} K_\epsilon(x_i - x_j) \cdot \nabla_{x_i} \varphi(x) F_t^{N,\epsilon}(dx) \\ + \int_{(\mathbb{R}^2)^N} \Delta_x \varphi(x) F_t^{N,\epsilon}(dx).$$

We deduce that $F^{N,\epsilon}$ is a weak solution to

$$(3.3.7) \quad \partial_t F_t^{N,\epsilon}(x) = \frac{\chi}{N} \sum_{i \neq j} \operatorname{div}_{x_i} (F_t^{N,\epsilon}(x) K_\epsilon(x_i - x_j)) + \Delta_x F_t^{N,\epsilon}(x).$$

We are now able to compute the evolution of the entropy.

$$\begin{aligned} \frac{d}{dt} H(F_t^{N,\epsilon}) &= \frac{1}{N} \int_{(\mathbb{R}^2)^N} \partial_t F_t^{N,\epsilon}(x) (1 + \log F_t^{N,\epsilon}(x)) dx \\ &= \frac{\chi}{N^2} \sum_{i \neq j} \int_{(\mathbb{R}^2)^N} \operatorname{div}_{x_i} (F_t^{N,\epsilon}(x) K_\epsilon(x_i - x_j)) (1 + \log F_t^{N,\epsilon}(x)) dx \\ &\quad + \frac{1}{N} \int_{(\mathbb{R}^2)^N} \Delta_x F_t^{N,\epsilon}(x) (1 + \log F_t^{N,\epsilon}(x)) dx. \end{aligned}$$

Performing some integrations by parts, we get

$$\begin{aligned} \frac{d}{dt} H(F_t^{N,\epsilon}) &= -\frac{\chi}{N^2} \sum_{i \neq j} \int_{(\mathbb{R}^2)^N} K_\epsilon(x_i - x_j) \cdot \nabla_{x_i} F_t^{N,\epsilon}(x) dx - I(F_t^{N,\epsilon}) \\ &= \frac{\chi}{N^2} \sum_{i \neq j} \int_{(\mathbb{R}^2)^N} \operatorname{div} K_\epsilon(x_i - x_j) F_t^{N,\epsilon}(x) dx - I(F_t^{N,\epsilon}), \end{aligned}$$

and (3.3.3) follows. Using that $\operatorname{div} K_\epsilon(x) = \frac{1-\alpha}{|x|^{\alpha+1}} \mathbb{1}_{\{|x| \geq \epsilon\}} + \frac{2}{\epsilon^{\alpha+1}} \mathbb{1}_{\{|x| < \epsilon\}} \leq \frac{2}{|x|^{\alpha+1}}$ and the exchangeability of the particles, we get

$$\begin{aligned} \frac{d}{dt} H(F_t^{N,\epsilon}) &\leq \frac{2\chi}{N^2} \sum_{i \neq j} \int_{(\mathbb{R}^2)^N} \frac{F_t^{N,\epsilon}(x)}{|x_i - x_j|^{\alpha+1}} dx - I(F_t^{N,\epsilon}) \\ &\leq 2\chi \int_{(\mathbb{R}^2)^N} \frac{F_t^{N,\epsilon}(x)}{|x_1 - x_2|^{\alpha+1}} dx - I(F_t^{N,\epsilon}). \end{aligned}$$

Since $\alpha \in (0, 1)$, we can use Lemma 3.2.1 with $\gamma = \alpha + 1$ and β such that $\frac{\alpha+1}{2} < \beta < 1$, which gives

$$\int_{(\mathbb{R}^2)^N} \frac{F_t^{N,\epsilon}(x) dx}{|x_1 - x_2|^{\alpha+1}} \leq C(I(F_{t_2}^{N,\epsilon})^\beta + 1),$$

where $F_{t_2}^{N,\epsilon}$ is the two-marginal of $F_t^{N,\epsilon}$. By Lemma 3.2.2, we have $I(F_{t_2}^{N,\epsilon}) \leq I(F_t^{N,\epsilon})$. Using that $Cx^\beta \leq C + \frac{x}{6\chi}$ (changing the value of the constant C), we thus get

$$\frac{d}{dt} H(F_t^{N,\epsilon}) \leq C - \frac{2}{3} I(F_t^{N,\epsilon}),$$

and thus

$$(3.3.8) \quad H(F_t^{N,\epsilon}) + \frac{2}{3} \int_0^t I(F_s^{N,\epsilon}) ds \leq H(F_0^N) + Ct.$$

We now compute $M_1(F_t^{N,\epsilon})$. We first observe that

$$M_1(F_t^{N,\epsilon}) = \frac{1}{N} \int_{(\mathbb{R}^2)^N} \sum_{i=1}^N |x_i| F_t^{N,\epsilon}(dx) = \mathbb{E}[|X_t^{1,N,\epsilon}|],$$

since the particles are exchangeable. We will need to control $\mathbb{E}[\sup_{[0,T]} |X_t^{1,N,\epsilon}|]$ in the sequel. We have

$$(3.3.9) \quad \begin{aligned} \mathbb{E} \left[\sup_{[0,T]} |X_t^{1,N,\epsilon}| \right] &\leq C \left(\mathbb{E}[|X_0^1|] + \mathbb{E} \left[\sup_{[0,T]} |B_t^1| \right] \right) \\ &\quad + \mathbb{E} \left[\sup_{t \in [0,T]} \left| \frac{1}{N} \sum_{j \neq 1} \int_0^t K_\epsilon(X_s^{1,N,\epsilon} - X_s^{j,N,\epsilon}) ds \right| \right] \\ &\leq C \left(\mathbb{E}[|X_0^1|] + T + \frac{1}{N} \sum_{j \neq 1} \int_0^T \mathbb{E}[|K_\epsilon(X_s^{1,N,\epsilon} - X_s^{j,N,\epsilon})|] ds \right) \\ &\leq C \left(\mathbb{E}[|X_0^1|] + T + \int_0^T \mathbb{E} \left[\frac{1}{|X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon}|^\alpha} \right] ds \right). \end{aligned}$$

Using Lemma 3.2.1 with $\gamma = \alpha$ and β such that $\frac{\alpha}{2} < \beta < 1$ and recalling that $I(F_{t_2}^{N,\epsilon}) \leq I(F_t^{N,\epsilon})$, we get

$$(3.3.10) \quad \begin{aligned} M_1(F_t^{N,\epsilon}) &\leq C \left(M_1(F_0^N) + T + \int_0^t I(F_s^{N,\epsilon})^\beta ds \right) \\ &\leq C \left(M_1(F_0^N) + T \right) + \frac{1}{3} \int_0^t I(F_s^{N,\epsilon}) ds, \end{aligned}$$

where we used that $Cx^\beta \leq C + \frac{x}{3}$ (changing the value of C). Summing (3.3.8) and (3.3.10), we thus find

$$H(F_t^{N,\epsilon}) + M_1(F_t^{N,\epsilon}) + \frac{1}{3} \int_0^t I(F_s^{N,\epsilon}) ds \leq H(F_0^N) + Ct + C(1 + M_1(F_0^N)).$$

Since the quantities M_1 and I are positive, we immediately get $H(F_t^{N,\epsilon}) \leq C(1+t)$. Using Lemma 3.2.3, we have $H(F_t^{N,\epsilon}) \geq -C - M_1(F_t^{N,\epsilon})/2$, so that

$$M_1(F_t^{N,\epsilon}) + \frac{1}{3} \int_0^t I(F_s^{N,\epsilon}) ds \leq C(1+t) + M_1(F_t^{N,\epsilon})/2.$$

Using again the positivity of M_1 and I , we easily get (3.3.4). Coming back to (3.3.9), we finally observe that

$$\mathbb{E} \left[\sup_{[0,T]} |X_t^{1,N,\epsilon}| \right] \leq C \left(\mathbb{E}[|X_0^1|] + T + \int_0^T I(F_s^{N,\epsilon}) ds \right) \leq C(1 + \mathbb{E}[|X_0^1|] + T),$$

which gives (3.3.5) and concludes the proof. \square

We can now give the proof of existence and uniqueness for the particle system (3.1.3).

Proof of Theorem 3.1.6. Like in [63], the key point of the proof is to show that particles of the system (3.1.3) a.s. never collide. We divide the proof in three steps. The first step consists in showing that a.s. there are no collisions between particles for the system (3.3.2). In the second step, we deduce that the particles of the system (3.1.3) also never collide, which ensures global existence and uniqueness for (3.1.3). In the last step, we establish the estimates about the entropy, Fisher information and the first moment. We fix $N \geq 2$ and for all $\epsilon \in (0, 1)$, we consider $(X_t^{i,N,\epsilon})_{i=1,\dots,N,t \geq 0}$ the unique solution to (3.3.2).

Step 1. Let $\tau_\epsilon := \inf\{t \geq 0, \exists i \neq j, |X_t^{i,N,\epsilon} - X_t^{j,N,\epsilon}| \leq \epsilon\}$. The aim of this step is to prove that $\lim_{\epsilon \rightarrow 0} \mathbb{P}[\tau_\epsilon < T] = 0$ for all $T > 0$. We fix $T > 0$ and introduce

$$(3.3.11) \quad S_t^\epsilon := \frac{1}{N^2} \sum_{i \neq j} \log |X_t^{i,N,\epsilon} - X_t^{j,N,\epsilon}|.$$

For any $A > 1$, we have

$$\begin{aligned}
(3.3.12) \quad \mathbb{P}[\tau_\epsilon < T] &\leq \mathbb{P}\left[\inf_{[0,T]} S_{t \wedge \tau_\epsilon}^\epsilon \leq S_{\tau_\epsilon}^\epsilon\right] \\
&\leq \mathbb{P}[\exists i, \exists t \in [0, T], |X_t^{i,N,\epsilon}| > A] \\
&\quad + \mathbb{P}\left[\forall i, \forall t \in [0, T], |X_t^{i,N,\epsilon}| \leq A, \inf_{[0,T]} S_{t \wedge \tau_\epsilon}^\epsilon \leq S_{\tau_\epsilon}^\epsilon\right] \\
&\leq \frac{N\mathbb{E}\left[\sup_{[0,T]} |X_t^{1,N,\epsilon}|\right]}{A} + \mathbb{P}\left[\inf_{[0,T]} S_{t \wedge \tau_\epsilon}^\epsilon \leq \frac{\log \epsilon}{N^2} + \log 2A\right] \\
&\leq \frac{C(1+T)N}{A} + \mathbb{P}\left[\inf_{[0,T]} S_{t \wedge \tau_\epsilon}^\epsilon \leq \frac{\log \epsilon}{N^2} + \log 2A\right],
\end{aligned}$$

where we used (3.3.5). We thus want to compute $\mathbb{P}\left[\inf_{[0,T]} S_{t \wedge \tau_\epsilon}^\epsilon \leq -M\right]$ for all (large) $M > 0$. Using Itô's formula, that $K_\epsilon(x) = K(x)$ for any $|x| \geq \epsilon$ (see (3.3.1)) and that $\Delta(\log|x|) = 0$ on $\{x \in \mathbb{R}^2, |x| > \epsilon\}$, we have

$$\begin{aligned}
\log |X_{t \wedge \tau_\epsilon}^{i,N,\epsilon} - X_{t \wedge \tau_\epsilon}^{j,N,\epsilon}| &= \log |X_0^{i,N} - X_0^{j,N}| + M_{t \wedge \tau_\epsilon}^{i,j,\epsilon} \\
&\quad - \frac{\chi}{N} \int_0^{t \wedge \tau_\epsilon} \left[\sum_{k \neq i,j} \left(K(X_s^{i,N,\epsilon} - X_s^{k,N,\epsilon}) - K(X_s^{j,N,\epsilon} - X_s^{k,N,\epsilon}) \right) \right. \\
&\quad \left. + 2K(X_s^{i,N,\epsilon} - X_s^{j,N,\epsilon}) \right] \cdot \frac{X_s^{i,N,\epsilon} - X_s^{j,N,\epsilon}}{|X_s^{i,N,\epsilon} - X_s^{j,N,\epsilon}|^2} ds \\
&=: \log |X_0^{i,N} - X_0^{j,N}| + M_{t \wedge \tau_\epsilon}^{i,j,\epsilon} + R_{t \wedge \tau_\epsilon}^{i,j,\epsilon},
\end{aligned}$$

where $M_t^{i,j,\epsilon}$ is a martingale. Setting $S_0 := \frac{1}{N^2} \sum_{i \neq j} \log |X_0^{i,N} - X_0^{j,N}|$, $M_t^\epsilon := \frac{1}{N^2} \sum_{i \neq j} M_{t \wedge \tau_\epsilon}^{i,j,\epsilon}$ and $R_t^\epsilon := \frac{1}{N^2} \sum_{i \neq j} R_{t \wedge \tau_\epsilon}^{i,j,\epsilon}$, we thus have

$$S_{t \wedge \tau_\epsilon}^\epsilon = S_0 + M_t^\epsilon + R_t^\epsilon,$$

so that

$$\begin{aligned}
(3.3.13) \quad \mathbb{P}\left(\inf_{[0,T]} S_{t \wedge \tau_\epsilon}^\epsilon \leq -M\right) &\leq \mathbb{P}(S_0 \leq -M/3) + \mathbb{P}\left(\inf_{[0,T]} M_t^\epsilon \leq -M/3\right) \\
&\quad + \mathbb{P}\left(\inf_{[0,T]} R_t^\epsilon \leq -M/3\right).
\end{aligned}$$

Using exchangeability and that $|K(x)| = |x|^{-\alpha}$, we clearly have for some

constant C independent of N and ϵ ,

$$\begin{aligned}
\mathbb{E}[\sup_{[0,T]} |R_t^\epsilon|] &\leq C\chi \int_0^T \mathbb{E}\left[\frac{1}{|X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon}|^{\alpha+1}}\right] ds \\
&\leq C\chi \int_0^T (1 + I(F_{s^2}^{N,\epsilon})) ds \\
(3.3.14) \qquad &\leq C(1 + T),
\end{aligned}$$

where we used Lemma 3.2.1, the fact that $I(F_{t^2}^{N,\epsilon}) \leq I(F_t^{N,\epsilon})$ by Lemma 3.2.2, and finally Proposition 3.3.1. We thus get

$$(3.3.15) \quad \mathbb{P}(\inf_{[0,T]} R_t^\epsilon \leq -M/3) \leq \mathbb{P}(\sup_{[0,T]} |R_t^\epsilon| \geq M/3) \leq \frac{C(1+T)}{M}.$$

We now want to compute $\mathbb{P}(\inf_{[0,T]} M_t^\epsilon \leq -M/3)$. Using that $\log|x| \leq |x|$, we have

$$S_t^\epsilon \leq \frac{1}{N^2} \sum_{i \neq j} (|X_t^{i,N,\epsilon}| + |X_t^{j,N,\epsilon}|) \leq \frac{2}{N} \sum_i |X_t^{i,N,\epsilon}|.$$

Consequently,

$$\begin{aligned}
M_t^\epsilon &\leq S_{t \wedge \tau_\epsilon}^\epsilon + \sup_{s \in [0,T]} |R_s^\epsilon| - S_0 \\
&\leq \frac{2}{N} \sum_i \sup_{s \in [0,T]} |X_s^{i,N,\epsilon}| + \sup_{s \in [0,T]} |R_s^\epsilon| - S_0 =: K^\epsilon - S_0 =: Z^\epsilon.
\end{aligned}$$

We have

$$(3.3.16) \quad \mathbb{P}(\inf_{[0,T]} M_t^\epsilon \leq -M/3) \leq \mathbb{P}(Z^\epsilon \geq \sqrt{M/3}) + \mathbb{P}(\inf_{[0,T]} M_t^\epsilon \leq -M/3, Z^\epsilon < \sqrt{M/3}).$$

Since $(M_t^\epsilon)_{t \geq 0}$ is a continuous local martingale, there exists a Brownian Motion β such that $M_t^\epsilon = \beta_{\langle M^\epsilon \rangle_t}$. For $x \in \mathbb{R}$, we set $\sigma_x := \inf\{t \geq 0, \beta_t = x\}$. Using that $\sup_{[0,T]} M_t^\epsilon \leq Z^\epsilon$ a.s.,

$$\begin{aligned}
\mathbb{P}(\inf_{[0,T]} M_t^\epsilon \leq -M/3, Z^\epsilon < \sqrt{M/3}) &\leq \mathbb{P}(\inf_{[0,T]} M_t^\epsilon \leq -M/3, \sup_{[0,T]} M_t^\epsilon < \sqrt{M/3}) \\
&\leq \mathbb{P}(\sigma_{-M/3} \leq \sigma_{\sqrt{M/3}}) \\
(3.3.17) \qquad &= \frac{\sqrt{M/3}}{M/3 + \sqrt{M/3}} \leq \sqrt{\frac{3}{M}},
\end{aligned}$$

by classical results on the Brownian Motion. Using (3.3.5) and (3.3.14), we get that $\mathbb{E}[K^\epsilon] \leq C(1 + T)$ where C does not depend on ϵ . So using the Markov inequality,

$$\begin{aligned}
 \mathbb{P}(Z^\epsilon \geq \sqrt{M/3}) &= \mathbb{P}(K^\epsilon - S_0 \geq \sqrt{M/3}) \\
 &\leq \mathbb{P}(K^\epsilon \geq \sqrt{M/12}) + \mathbb{P}(-S_0 \geq \sqrt{M/12}) \\
 (3.3.18) \quad &\leq \frac{C(1 + T)}{\sqrt{M}} + \mathbb{P}(-S_0 \geq \sqrt{M/12}).
 \end{aligned}$$

Gathering (3.3.16), (3.3.17) and (3.3.18), we find that

$$(3.3.19) \quad \mathbb{P}(\inf_{[0,T]} M_t^\epsilon \leq -M/3) \leq \frac{C(1 + T)}{\sqrt{M}} + \mathbb{P}(-S_0 \geq \sqrt{M/12}).$$

Coming back to (3.3.12) and (3.3.13), using (3.3.15) and (3.3.19) with $M = -\frac{\log \epsilon}{N^2} - \log 2A$, we finally get that for any $\epsilon \in (0, 1)$, any $A > 1$ such that $\frac{\log \epsilon}{N^2} + \log 2A < 0$,

$$\begin{aligned}
 \mathbb{P}(\tau_\epsilon < T) &\leq \frac{C(1 + T)N}{A} + \mathbb{P}\left(S_0 \leq \left(\frac{\log \epsilon}{N^2} + \log 2A\right)/3\right) \\
 &\quad + \frac{C(1 + T)}{-\frac{\log \epsilon}{N^2} - \log 2A} + \frac{C(1 + T)}{\sqrt{-\frac{\log \epsilon}{N^2} - \log 2A}} \\
 &\quad + \mathbb{P}\left(S_0 \leq -\sqrt{\left(-\frac{\log \epsilon}{N^2} - \log 2A\right)/12}\right).
 \end{aligned}$$

Observe finally that $S_0 > -\infty$ a.s. (because F_0^N has a density since $H(F_0^N) < \infty$) so that $\lim_{M \rightarrow +\infty} \mathbb{P}(S_0 < -M) = 0$. Letting $\epsilon \rightarrow 0$ in the above formula, we get that for all $A > 1$,

$$\limsup_{\epsilon} \mathbb{P}(\tau_\epsilon < T) \leq \frac{C(1 + T)N}{A}.$$

It only remains to make A go to ∞ to conclude this step.

Step 2. Since K is Lipschitz-continuous outside 0, classical arguments give existence and uniqueness of a solution to (3.1.3) until the explosion time $\tau = \inf\{t \geq 0, \exists i \neq j, X_t^{i,N} = X_t^{j,N}\}$. We can observe that since $K_\epsilon(x) = K(x)$ for any $|x| \geq \epsilon$, $(X^{i,N,\epsilon})_{i=1,\dots,N}$ is solution to (3.1.3) on $[0, \tau_\epsilon]$

so that for any $i = 1, \dots, N$, $X_t^{i,N} = X_t^{i,N,\epsilon}$ on $[0, \tau_\epsilon]$. We thus have $\tau_\epsilon < \tau$ for any $\epsilon \in (0, 1)$ a.s. so that, using Step 1, we have for any $T > 0$

$$\mathbb{P}(\tau < T) \leq \mathbb{P}(\tau_\epsilon < T) \xrightarrow{\epsilon \rightarrow 0} 0.$$

Thus $\tau = \infty$ a.s. which proves global existence and uniqueness for (3.1.3).

Step 3. Using that the functionals H , I and M_1 are lower semi-continuous and Proposition 3.3.1, we have

$$(3.3.20) \quad \begin{aligned} H(F_t^N) &\leq \liminf_{\epsilon} H(F_t^{N,\epsilon}) \leq C(1+t), \\ \int_0^t I(F_s^N) ds &\leq \liminf_{\epsilon} \int_0^t I(F_s^{N,\epsilon}) ds \leq C(1+t), \end{aligned}$$

and

$$M_1(F_t^N) \leq \liminf_{\epsilon} M_1(F_t^{N,\epsilon}) \leq C(1+t).$$

Using the Fatou lemma and (3.3.5), we get

$$\mathbb{E} \left[\sup_{[0,T]} |X_t^{1,N}| \right] \leq \liminf_{\epsilon} \mathbb{E} \left[\sup_{[0,T]} |X_t^{1,N,\epsilon}| \right] \leq C(1+T),$$

and (3.1.13) is proven. It remains to prove (3.1.14). Using again that the functionals H and I are lower semi-continuous and using (3.3.3), we get

$$\begin{aligned} H(F_t^N) + \int_0^t I(F_s^N) ds &\leq \liminf_{\epsilon} \left[H(F_t^{N,\epsilon}) + \int_0^t I(F_s^{N,\epsilon}) ds \right] \\ &\leq H(F_0^N) \\ &\quad + \liminf_{\epsilon} \frac{\chi}{N^2} \int_0^t \sum_{i \neq j} \mathbb{E}[\operatorname{div} K_{\epsilon}(X_s^{i,N,\epsilon} - X_s^{j,N,\epsilon})] ds. \end{aligned}$$

By exchangeability, it suffices to prove that, as $\epsilon \rightarrow 0$,

$$D_{\epsilon} := \int_0^t \mathbb{E}[\operatorname{div} K_{\epsilon}(X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon})] ds \rightarrow \int_0^t \mathbb{E}[\operatorname{div} K(X_s^{1,N} - X_s^{2,N})] ds =: D.$$

By Step 2, we have $X_s^{i,N} = X_s^{i,N,\epsilon}$ for any i and $s \leq \tau_{\epsilon}$ and thus recalling that $K_{\epsilon}(x) = K(x)$ for any $|x| \geq \epsilon$, we get that a.s. for any $s < \tau_{\epsilon}$

$$\operatorname{div} K_{\epsilon}(X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon}) = \operatorname{div} K(X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon}) = \operatorname{div} K(X_s^{1,N} - X_s^{2,N}).$$

So using that $\operatorname{div}K(x) \leq 2|x|^{-\alpha-1}$ and $\operatorname{div}K_\epsilon(x) \leq 2|x|^{-\alpha-1}$, we get

$$|D - D_\epsilon| \leq C \int_0^t \mathbb{E} \left[\mathbb{1}_{\{\tau_\epsilon < s\}} \left(\frac{1}{|X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon}|^{\alpha+1}} + \frac{1}{|X_s^{1,N} - X_s^{2,N}|^{\alpha+1}} \right) \right] ds.$$

Let $a \in \left(0, \frac{1-\alpha}{1+\alpha}\right)$ (in order to have $(1+a)(\alpha+1) < 2$). Using first the Hölder inequality with $p = 1+a$ and q such that $1/p + 1/q = 1$, and then Lemma 3.2.1 with $\beta = 1$, we get

$$\begin{aligned} |D - D_\epsilon| &\leq C \int_0^t \mathbb{P}(\tau_\epsilon < s)^{1/q} \mathbb{E} \left[\left(\frac{1}{|X_s^{1,N,\epsilon} - X_s^{2,N,\epsilon}|^{(\alpha+1)(1+a)}} \right. \right. \\ &\quad \left. \left. + \frac{1}{|X_s^{1,N} - X_s^{2,N}|^{(\alpha+1)(1+a)}} \right) \right]^{1/p} ds \\ &\leq C \mathbb{P}(\tau_\epsilon < t)^{1/q} \int_0^t [1 + I(F_s^{N,\epsilon}) + I(F_s^N)] ds \\ &\leq C(1+t) \mathbb{P}(\tau_\epsilon < t)^{1/q}, \end{aligned}$$

by (3.3.4) and (3.3.20). This tends to 0 as $\epsilon \rightarrow 0$ by Step 1 and concludes the proof. \square

3.4 Convergence of the particle system

We start this section with a tightness result for the particle system (3.1.3).

Lemma 3.4.1. *Let $\alpha \in (0, 1)$. Assume (3.1.12). For each $N \geq 2$, let $(X_t^{i,N})_{i=1,\dots,N}$ be the unique solution to (3.1.3) and $Q^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N})_{t \geq 0}}$.*

(i) *The family $\{\mathcal{L}((X_t^{1,N})_{t \geq 0}), N \geq 2\}$ is tight in $\mathbf{P}(C([0, \infty), \mathbb{R}^2))$.*

(ii) *The family $\{\mathcal{L}(Q^N), N \geq 2\}$ is tight in $\mathbf{P}(\mathbf{P}(C([0, \infty), \mathbb{R}^2)))$.*

Proof. Since the system is exchangeable, we deduce (ii) from (i) by [62, Proposition 2.2]. Let's prove (i). Let thus $\eta > 0$ and $T > 0$ be fixed. To prove the tightness of $\{\mathcal{L}((X_t^{1,N})_{t \geq 0}), N \geq 2\}$ in $\mathbf{P}(C([0, \infty), \mathbb{R}^2))$, we have to find a compact subset $\mathcal{K}_{\eta,T}$ of $C([0, T], \mathbb{R}^2)$ such that $\sup_N \mathbb{P}[(X_t^{1,N})_{t \in [0, T]} \notin \mathcal{K}_{\eta,T}] \leq \eta$. We first set $Z_T := \sup_{0 < s < t < T} \sqrt{2} |B_t^1 - B_s^1| / |t - s|^{1/3}$. This random variable is a.s. finite since the paths of a Brownian motion are a.s. Hölder continuous with index 1/3. We can also notice that the law of Z_T does not

depend on N . Using the Hölder inequality with $p = 3$ and $q = 3/2$, we get that for all $0 < s < t < T$,

$$\begin{aligned}
\left| \frac{\chi}{N} \sum_{j=2}^N \int_s^t K(X_u^{1,N} - X_u^{j,N}) du \right| &\leq \frac{\chi}{N} \sum_{j=2}^N \int_s^t \frac{du}{|X_u^{1,N} - X_u^{j,N}|^\alpha} \\
&\leq \frac{\chi}{N} (t-s)^{1/3} \sum_{j=2}^N \left(\int_0^T \frac{du}{|X_u^{1,N} - X_u^{j,N}|^{3\alpha/2}} \right)^{2/3} \\
&\leq (t-s)^{1/3} \left(\chi + \frac{\chi}{N} \sum_{j=2}^N \int_0^T \frac{du}{|X_u^{1,N} - X_u^{j,N}|^{3\alpha/2}} \right) \\
&=: (t-s)^{1/3} U_T^N.
\end{aligned}$$

Using Lemma 3.2.1 with $\gamma = 3\alpha/2$ and $\beta = 1$, the exchangeability of the system of particles, and denoting by $F_{u_2}^N$ the two-marginal of F_u^N , we have

$$\begin{aligned}
\mathbb{E}(U_T^N) &= \chi + \chi \frac{N-1}{N} \int_0^T \mathbb{E} \left(\frac{1}{|X_u^{1,N} - X_u^{2,N}|^{3\alpha/2}} \right) du \\
&\leq \chi + C \int_0^T (1 + I(F_{u_2}^N)) du \\
&\leq \chi + C \int_0^T (1 + I(F_u^N)) du \\
&\leq C(1 + T),
\end{aligned}$$

where we used that $I(F_{t_2}^N) \leq I(F_t^N)$ by Lemma 3.2.2 and Theorem 3.1.6. We thus have $\sup_{N \geq 2} \mathbb{E}(U_T^N) < \infty$. Furthermore, Z_T is also a.s. finite so that we can find $R > 0$ such that $\mathbb{P}(Z_T + U_T^N > R) \leq \eta/2$ for all $N \geq 2$. Recalling (3.1.12), we can also find $a > 0$ such that $\sup_{N \geq 2} \mathbb{P}(X_0^{1,N} > a) \leq \eta/2$. We now consider

$$\begin{aligned}
\mathcal{K}_{\eta,T} &:= \{f \in C([0, T], \mathbb{R}^2), \\
&\quad |f(0)| \leq a, |f(t) - f(s)| \leq R(t-s)^{1/3} \forall 0 < s < t < T\},
\end{aligned}$$

which is a compact subset of $C([0, T], \mathbb{R}^2)$ by Ascoli's theorem. Observing that for all $0 < s < t < T$, $|X_t^{1,N} - X_s^{1,N}| \leq (Z_T + U_t^N)(t-s)^{1/3}$, we get

$$\mathbb{P}[(X_t^{1,N})_{t \in [0, T]} \notin \mathcal{K}_{\eta,T}] \leq \mathbb{P}(|X_0^{1,N}| > a) + \mathbb{P}(Z_T + U_T^N > R) \leq \eta,$$

which concludes the proof. \square

We define \mathcal{S} as the set of all probability measures $f \in \mathbf{P}(C([0, \infty), \mathbb{R}^2))$ such that f is the law of $(X_t)_{t \geq 0}$ solution to (3.1.4) satisfying (setting $f_t = \mathcal{L}(X_t)$)

$$(3.4.1) \quad \forall T > 0, \quad \int_0^T I(f_s) ds < \infty \quad \text{and} \quad \sup_{[0, T]} M_1(f_s) < \infty.$$

Observe that by Lemma 3.2.4, (3.4.1) implies (3.1.7).

Proposition 3.4.2. *Let $\alpha \in (0, 1)$ and assume (3.1.12). For each $N \geq 2$, let $(X_0^{i, N})_{i=1, \dots, N}$ be F_0^N -distributed and consider the solution $(X_t^{i, N})_{i=1, \dots, N, t \geq 0}$ to (3.1.3). Assume that there is a subsequence of $\mathcal{Q}^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i, N})_{t \geq 0}}$ going in law to some $\mathbf{P}(C([0, \infty), \mathbb{R}^2))$ -valued random variable \mathcal{Q} . Then \mathcal{Q} a.s. belongs to \mathcal{S} .*

Proof. We consider a (not relabelled) subsequence of \mathcal{Q}^N going in law to some \mathcal{Q} and we introduce the identity map $\psi : C([0, \infty); \mathbb{R}^2) \rightarrow C([0, \infty); \mathbb{R}^2)$. Using the arguments of [36, Proposition 6.1], we have to prove that \mathcal{Q} a.s. satisfies

- (a) $\mathcal{Q} \circ (\psi(0))^{-1} = f_0$;
- (b) setting $\mathcal{Q}_t = \mathcal{Q} \circ (\psi(t))^{-1}$, $(\mathcal{Q}_t)_{t \geq 0}$ satisfies (3.4.1);
- (c) for all $0 < t_0 < \dots < t_k < s < t$, $\varphi_1, \dots, \varphi_k \in C_b(\mathbb{R}^2)$, $\varphi \in C_b^2(\mathbb{R}^2)$, $\mathcal{F}(\mathcal{Q}) = 0$ where, for $f \in \mathbf{P}(C([0, \infty), \mathbb{R}^2))$,

$$\begin{aligned} \mathcal{F}(f) := & \iint f(d\gamma) f(d\tilde{\gamma}) \varphi_1(\gamma_{t_1}) \dots \varphi_k(\gamma_{t_k}) \\ & \left[\varphi(\gamma_t) - \varphi(\gamma_s) + \chi \int_s^t \nabla_x \varphi(\gamma_u) \cdot K(\gamma_u - \tilde{\gamma}_u) du - \int_s^t \Delta_x \varphi(\gamma_u) du \right]. \end{aligned}$$

For simplicity, we split the proof in many steps.

Step 1. By assumption (3.1.12), we have that F_0^N is f_0 -chaotic which implies that $\mathcal{Q}_0^N = \mathcal{Q}^N \circ \psi(0)^{-1}$ goes weakly to f_0 in law, and, since f_0 is deterministic, also in probability. Hence $\mathcal{Q}_0 = f_0$ a.s. and thus $f \circ \psi(0)^{-1} = f_0$. Thus \mathcal{Q} a.s. satisfies (a).

Step 2. Since $\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}$ goes weakly to \mathcal{Q}_t , for all $j \geq 1$, F_{tj}^N goes weakly to π_{tj} , where $\pi_t := \mathcal{L}(\mathcal{Q}_t)$ and $\pi_{tj} := \int_{\mathbf{P}(\mathbb{R}^2)} f^{\otimes j} \pi_t(df)$. We can thus apply [40, Theorem 5.7] (and then the Fatou Lemma) to get

$$\begin{aligned} \mathbb{E} \left[\int_0^T I(\mathcal{Q}_s) ds \right] &= \int_0^T \mathbb{E}[I(\mathcal{Q}_s)] ds \leq \int_0^T \liminf_N I(F_s^N) ds \\ &\leq \liminf_N \int_0^T I(F_s^N) ds, \end{aligned}$$

which is finite by Theorem 3.1.6. We conclude that $\int_0^T I(\mathcal{Q}_s) ds < \infty$ a.s. We also have, using the Fatou lemma and the exchangeability of the particles,

$$\begin{aligned} \mathbb{E} \left[\sup_{[0,T]} M_1(\mathcal{Q}_t) \right] &\leq \mathbb{E} \left[\liminf_N \sup_{[0,T]} M_1(\mathcal{Q}_t^N) \right] \\ &\leq \liminf_N \mathbb{E} \left[\sup_{[0,T]} \frac{1}{N} \sum_{i=1}^N |X_t^{i,N}| \right] \\ &\leq \liminf_N \mathbb{E} \left[\sup_{[0,T]} |X_t^{1,N}| \right] \leq C(1+T), \end{aligned}$$

by (3.1.13), so that $\sup_{[0,T]} M_1(\mathcal{Q}_t) < \infty$ a.s. Consequently, \mathcal{Q} a.s. satisfies (b).

Step 3.1. Using Itô's formula

$$\begin{aligned} O_t^i &:= \varphi(X_s^{i,N}) + \frac{\chi}{N} \sum_{j \neq i} \int_0^t \nabla_x \varphi(X_s^{i,N}) \cdot K(X_s^{i,N} - X_s^{j,N}) ds \\ &\quad - \int_0^t \Delta_x \varphi(X_s^{i,N}) ds \\ &= \varphi(X_0^{i,N}) + \sqrt{2} \int_0^t \nabla_x \varphi(X_s^{i,N}) \cdot dB_s^i. \end{aligned}$$

But, using the last equality, we see that

$$\begin{aligned} \mathcal{F}(\mathcal{Q}^N) &= \frac{1}{N} \sum_{i=1}^N \varphi_1(X_{t_1}^{i,N}) \dots \varphi_k(X_{t_k}^{i,N}) [O_t^i - O_s^i] \\ &= \frac{\sqrt{2}}{N} \sum_{i=1}^N \varphi_1(X_{t_1}^{i,N}) \dots \varphi_k(X_{t_k}^{i,N}) \int_s^t \nabla_x \varphi(X_u^{i,N}) \cdot dB_u^i. \end{aligned}$$

From there, and thanks to the independence of the Brownian motions we conclude that (recall that the functions $\varphi_1, \dots, \varphi_k, \nabla_x \varphi$ are bounded)

$$\mathbb{E} [(\mathcal{F}(\mathcal{Q}^N))^2] \leq \frac{C_{\mathcal{F}}}{N}.$$

Step 3.2. We also introduce the regularized version of \mathcal{F} . For $\varepsilon \in (0, 1)$, we define \mathcal{F}_ε replacing K by K_ε defined by (3.3.1). Since $f \mapsto \mathcal{F}_\varepsilon(f)$ is continuous and bounded from $\mathbf{P}(C([0, \infty); \mathbb{R}^2))$ to \mathbb{R} and since \mathcal{Q}^N goes in law to \mathcal{Q} , we deduce that for any $\varepsilon \in (0, 1)$,

$$\mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q})|] = \lim_N \mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q}^N)|].$$

Step 3.3. Using that all the functions and their derivatives involved in \mathcal{F} are bounded and that $|K_\varepsilon(x) - K(x)| \leq |x|^{-\alpha} \mathbb{1}_{0 \leq |x| \leq \varepsilon}$, we get

$$\begin{aligned} |\mathcal{F}(f) - \mathcal{F}_\varepsilon(f)| &\leq \chi C_{\mathcal{F}} \iiint_0^t |\gamma(u) - \tilde{\gamma}(u)|^{-\alpha} \mathbb{1}_{0 < |\gamma(u) - \tilde{\gamma}(u)| < \varepsilon} du f(d\gamma) f(d\tilde{\gamma}) \\ &\leq C_{\mathcal{F}} \varepsilon^{3/2-\alpha} \iiint_0^t |\gamma(u) - \tilde{\gamma}(u)|^{-3/2} \mathbb{1}_{\gamma(u) \neq \tilde{\gamma}(u)} du f(d\gamma) f(d\tilde{\gamma}). \end{aligned}$$

Thus,

$$|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)| \leq \frac{C_{\mathcal{F}} \varepsilon^{3/2-\alpha}}{N^2} \sum_{i \neq j} \int_0^t |X_u^{i,N} - X_u^{j,N}|^{-3/2} du,$$

and by exchangeability

$$\mathbb{E} [|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \leq C_{\mathcal{F}} \varepsilon^{3/2-\alpha} \int_0^t \mathbb{E} [|X_u^{1,N} - X_u^{2,N}|^{-3/2}] du.$$

Using Lemma 3.2.1 with $\gamma = 3/2$ and $\beta = 1$ and denoting by F_{u2}^N the two-marginal of F_u^N , we have

$$\mathbb{E} [|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \leq C_{\mathcal{F}} \varepsilon^{3/2-\alpha} \int_0^t I(F_{u2}^N) du.$$

Using that $I(F_{t2}^N) \leq I(F_t^N)$ by Lemma 3.2.2 and Theorem 3.1.6 we conclude that

$$\mathbb{E} [|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \leq C_{\mathcal{F}} \varepsilon^{3/2-\alpha}.$$

Step 3.4. Now we see that

$$|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})| \leq C_{\mathcal{F}} \varepsilon^{3/2-\alpha} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x-y|^{-3/2} \mathcal{Q}_s(dx) \mathcal{Q}_s(dy) ds.$$

Step 2 says that (3.4.1) holds true for \mathcal{Q}_s , then thanks to Lemma 3.2.4 we get that a.s., $\nabla_x \mathcal{Q}_s \in L^{2q/(3q-2)}(0, T; L^q(\mathbb{R}^2))$ for all $q \in [1, 2)$. Then using [36, Lemma 3.5] for $\gamma = 3/2$ we deduce that a.s.

$$\lim_{\varepsilon \rightarrow 0} |\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})| = 0.$$

Step 3.5. Using Steps 3.1, 3.2 and 3.3, we finally observe, using the same arguments as in [36, Proposition 6.1, Step 4.5], that

$$\mathbb{E}[|\mathcal{F}(\mathcal{Q})| \wedge 1] \leq C_{\mathcal{F}} \varepsilon^{3/2-\alpha} + \mathbb{E}[|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})| \wedge 1],$$

so that $\mathcal{F}(\mathcal{Q}) = 0$ a.s. by Step 3.4 thanks to dominated convergence and \mathcal{Q} a.s. satisfies (c) which concludes the proof. \square

3.5 Well-posedness and propagation of chaos

We start this section with the proof of existence and uniqueness for the nonlinear S.D.E. (3.1.4). We will use that for $\gamma \in (-2, 0)$, for $q \in (2/(2 + \gamma), \infty]$ and for any $h \in \mathbf{P}(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$,

$$\begin{aligned} \sup_{v \in \mathbb{R}^2} \int_{\mathbb{R}^2} h(v_*) |v - v_*|^\gamma dv_* &\leq \sup_{v \in \mathbb{R}^2} \int_{|v_* - v| < 1} h(v_*) |v - v_*|^\gamma dv_* \\ &\quad + \sup_{v \in \mathbb{R}^2} \int_{|v_* - v| \geq 1} h(v_*) dv_* \\ (3.5.1) \qquad \qquad \qquad &\leq C_{\gamma, q} \|h\|_{L^q(\mathbb{R}^2)} + 1, \end{aligned}$$

where

$$C_{\gamma, q} = \left[\int_{|v_*| \leq 1} |v_*|^{\gamma q / (q-1)} dv_* \right]^{(q-1)/q} < \infty,$$

since by assumption $\gamma q / (q - 1) > -2$.

Proof of Theorem 3.1.7. The existence in law follows from Proposition 3.4.2 and Lemma 3.4.1 (see the comment after (3.4.1)). We now prove

pathwise uniqueness which will also imply the strong existence. To this aim, we consider $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ two solutions of (3.1.4) such that, setting $f_s := \mathcal{L}(X_s)$ and $g_s := \mathcal{L}(Y_s)$, $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ are in $L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2))$ for some $p > \frac{2}{1-\alpha}$. For any $s > 0$, we consider the probability measure R_s on $\mathbb{R}^2 \times \mathbb{R}^2$ with first (respectively second) marginal equal to f_s (resp. g_s) such that

$$\mathcal{W}_1(f_s, g_s) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| R_s(dx, dy).$$

We have

$$\begin{aligned} X_t - Y_t &= -\chi \left(\int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(dx) ds - \int_0^t \int_{\mathbb{R}^2} K(Y_s - y) g_s(dy) ds \right) \\ &= -\chi \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} [K(X_s - x) - K(Y_s - y)] R_s(dx, dy). \end{aligned}$$

Using Lemma 3.2.5 and recalling that $\mathcal{L}(X_t) = f_t$, $\mathcal{L}(Y_t) = g_t$, and that R_t has marginals f_t and g_t , this gives

$$\begin{aligned} \mathbb{E}[\sup_{[0, T]} |X_t - Y_t|] &\leq C_\alpha \chi \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbb{E} \left[(|X_s - Y_s| + |x - y|) \left(\frac{1}{|X_s - x|^{\alpha+1}} \right. \right. \\ &\quad \left. \left. + \frac{1}{|Y_s - y|^{\alpha+1}} \right) \right] R_s(dx, dy) ds \\ &\leq C_\alpha \chi \int_0^T \mathbb{E} \left[|X_s - Y_s| \left(\int_{\mathbb{R}^2} \frac{1}{|X_s - x|^{\alpha+1}} f_s(dx) \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{R}^2} \frac{1}{|Y_s - y|^{\alpha+1}} g_s(dy) \right) \right] ds \\ &\quad + C_\alpha \chi \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| \mathbb{E} \left[\frac{1}{|X_s - x|^{\alpha+1}} \right. \\ &\quad \left. + \frac{1}{|Y_s - y|^{\alpha+1}} \right] R_s(dx, dy) ds. \end{aligned}$$

Using (3.5.1), we thus have, since $\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| R_s(dx, dy) = \mathcal{W}_1(f_s, g_s) \leq$

$\mathbb{E}[|X_s - Y_s|]$ by definition of \mathcal{W}_1 ,

$$\begin{aligned} \mathbb{E}[\sup_{[0,T]} |X_t - Y_t|] &\leq C \int_0^T \mathbb{E}[|X_s - Y_s|](1 + \|f_s\|_{L^p} + \|g_s\|_{L^p}) ds \\ &\quad + C \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|(1 + \|f_s\|_{L^p} + \|g_s\|_{L^p}) R_s(dx, dy) ds \\ &\leq C \int_0^T \mathbb{E}[|X_s - Y_s|](1 + \|f_s + g_s\|_{L^p}) ds. \end{aligned}$$

By Grönwall's Lemma, we thus get $\mathbb{E}(\sup_{[0,T]} |X_t - Y_t|) = 0$ and pathwise uniqueness is proven. \square

The following lemma is useful for the uniqueness of (3.1.1).

Lemma 3.5.1. *Let $p > 2/(1 - \alpha)$ and consider a weak solution $(f_t)_{t \geq 0}$ to (3.1.1) lying in $L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2))$. Assume that for some $h = (h_t)_{t \geq 0}$ lying in $L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2))$, for all $\varphi \in C_c^2(\mathbb{R}^2)$, all $t \geq 0$,*

$$(3.5.2) \quad \begin{aligned} \int_{\mathbb{R}^2} \varphi(x) h_t(dx) &= \int_{\mathbb{R}^2} \varphi(x) f_0(dx) + \int_0^t \int_{\mathbb{R}^2} \Delta_x \varphi(x) h_s(dx) ds \\ &\quad - \chi \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K(x - y) \cdot \nabla_x \varphi(x) f_s(dy) h_s(dx) ds. \end{aligned}$$

Then $h = f$.

Proof. For any $\varphi \in C_c^2(\mathbb{R}^2)$ and any $t \geq 0$, we set

$$\mathcal{A}_t \varphi(x) = \Delta_x \varphi(x) - \chi \int_{\mathbb{R}^2} K(x - y) \cdot \nabla_x \varphi(x) f_t(dy).$$

We will prove that for any $\mu \in \mathbf{P}_1(\mathbb{R}^2)$, there exists at most one h lying in $L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2))$ such that for all $t \geq 0$, $\varphi \in C_c^2(\mathbb{R}^2)$,

$$(3.5.3) \quad \int_{\mathbb{R}^2} \varphi(x) h_t(dx) = \int_{\mathbb{R}^2} \varphi(x) \mu(dx) + \int_0^t \int_{\mathbb{R}^2} \mathcal{A}_s \varphi(x) h_s(dx) ds.$$

This will conclude the proof since f and h solve this equation with $\mu = f_0$ by assumption.

Step 1. Let $\mu \in \mathbf{P}_1(\mathbb{R}^2)$. A continuous adapted \mathbb{R}^2 -valued process $(X_t)_{t \geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is said to solve the martingale problem $MP((\mathcal{A}_t)_{t \geq 0}, \mu)$ if $P \circ X_0^{-1} = \mu$ and if for all $\varphi \in C_c^2(\mathbb{R}^2)$, $(M_t^\varphi)_{t \geq 0}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ -martingale, where

$$M_t^\varphi = \varphi(X_t) - \int_0^t \mathcal{A}_s \varphi(X_s) ds.$$

Using Bhatt-Karandikar [5, Theorem 5.2] (see also Remark 3.1 in [5]), uniqueness for (3.5.3) holds if

- (i) there exists a countable subset $(\varphi_k)_{k \geq 1} \subset C_c^2$ such that for all $t \geq 0$, the closure (for the bounded pointwise convergence) of $\{(\varphi_k, \mathcal{A}_t \varphi_k), k \geq 1\}$ contains $\{(\varphi, \mathcal{A}_t \varphi), \varphi \in C_c^2\}$,
- (ii) for each $x_0 \in \mathbb{R}^2$, there exists a solution to $MP((\mathcal{A}_t)_{t \geq 0}, \delta_{x_0})$,
- (iii) for each $x_0 \in \mathbb{R}^2$, uniqueness (in law) holds for $MP((\mathcal{A}_t)_{t \geq 0}, \delta_{x_0})$.

Step 2. We first prove (i). Consider thus some countable $(\varphi_k)_{k \geq 1} \subset C_c^2$ dense in C_c^2 , in the sense that for $\psi \in C_c^2$, there exists a subsequence φ_{k_n} such that $\lim_{n \rightarrow \infty} (\|\psi - \varphi_{k_n}\|_\infty + \|\psi' - \varphi'_{k_n}\|_\infty + \|\psi'' - \varphi''_{k_n}\|_\infty) = 0$. We then have to prove that, for $t \geq 0$,

- (a) $\mathcal{A}_t \varphi_{k_n}(x)$ tends to $\mathcal{A}_t \psi(x)$ for all $x \in \mathbb{R}^2$,
- (b) $\sup_n \|\mathcal{A}_t \varphi_{k_n}\|_\infty < \infty$.

Let $x \in \mathbb{R}^2$. By Lemma 3.2.5, we have

$$\begin{aligned} |\mathcal{A}_t \varphi_{k_n}(x) - \mathcal{A}_t \psi(x)| &\leq \|\psi'' - \varphi''_{k_n}\|_\infty + \chi \|\psi' - \varphi'_{k_n}\|_\infty \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} f_t(dy) \\ &\rightarrow 0, \end{aligned}$$

since $\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} f_t(dy) \leq C(1 + \|f_t\|_{L^p})$ by (3.5.1). For (b), we can observe that setting $A := \sup_n (\|\varphi_{k_n}\|_\infty + \|\varphi'_{k_n}\|_\infty + \|\varphi''_{k_n}\|_\infty)$

$$|\mathcal{A}_t \varphi_{k_n}| \leq A + \chi A \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} f_t(dy) \leq A + CA(1 + \|f_t\|_{L^p}),$$

which concludes this step.

Step 3. Using classical arguments, we observe that a process $(X_t)_{t \geq 0}$ is a solution to $MP((\mathcal{A}_t)_{t \geq 0}, \delta_{x_0})$ if and only if there exists, on a possibly enlarged probability space, a $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $(B_t)_{t \geq 0}$ such that

$$(3.5.4) \quad X_t = x_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(dx) ds + \sqrt{2} B_t.$$

It thus suffices to prove existence and uniqueness in law for solutions to (3.5.4) to get (ii) and (iii).

Step 4. The proof of (pathwise) uniqueness for (3.5.4) is very similar with the proof of uniqueness for (3.1.4) which has already been done and we leave it to the reader.

Step 5. It remains to check (ii) to conclude. We thus have to prove the existence of a solution to (3.5.4). To this aim, we use a Picard iteration. We thus consider the constant process $X_t^0 = x_0$ and define recursively

$$X_t^{n+1} = x_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s^n - x) f_s(dx) ds + \sqrt{2} B_t.$$

Using the same kind of arguments as in the proof of Theorem 3.1.7, we get

$$\mathbb{E}(\sup_{[0,T]} |X_t^{n+1} - X_t^n|) \leq C \int_0^T \mathbb{E}[|X_s^n - X_s^{n-1}|] (1 + \|f_s\|_{L^p}) ds.$$

Since $\int_0^T (1 + \|f_s\|_{L^p}) ds < \infty$, we classically deduce that $\sum_n \mathbb{E}(\sup_{[0,T]} |X_t^{n+1} - X_t^n|) < \infty$, so that there is a continuous adapted process $(X_t)_{t \geq 0}$ such that for all $T > 0$, $\lim_n \mathbb{E}[\sup_{[0,T]} |X_t - X_t^n|] = 0$. This L^1 convergence implies that $(X_t)_{t \geq 0}$ is solution to (3.5.4), which concludes the proof. \square

The following result ensures that uniqueness holds for (3.1.1).

Theorem 3.5.2. *Let f_0 and g_0 be two probability measures with finite first moment. Let $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ be two solutions to (3.1.1) starting from f_0 and g_0 respectively and lying in $L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2))$ for some $p > 2/(1 - \alpha)$. Then*

$$\mathcal{W}_1(f_t, g_t) \leq \mathcal{W}_1(f_0, g_0) \exp\left(C \int_0^t (1 + \|f_s + g_s\|_{L^p}) ds\right).$$

Proof. Let thus $p > 2/(1 - \alpha)$, $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ be two solutions to (3.1.1) lying in $L_{loc}^\infty([0, \infty), \mathbf{P}_1(\mathbb{R}^2)) \cap L_{loc}^1([0, \infty); L^p(\mathbb{R}^2))$. For any $s \geq 0$, we consider the probability measure R_s on $\mathbb{R}^2 \times \mathbb{R}^2$ with first (respectively second) marginal equal to f_s (resp. g_s) such that

$$\mathcal{W}_1(f_s, g_s) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| R_s(dx, dy),$$

and we consider (X_0, Y_0) with law R_0 . We finally set

$$\begin{aligned} X_t &= X_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(dx) ds + \sqrt{2} B_t, \\ Y_t &= Y_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(Y_s - x) g_s(dx) ds + \sqrt{2} B_t. \end{aligned}$$

Using Itô's formula, we see that h defined by $h_t := \mathcal{L}(X_t)$ satisfies (3.5.2) and Lemma 3.5.1 ensures us that $\mathcal{L}(X_t) = f_t$. Similarly, we also have $\mathcal{L}(Y_t) = g_t$. Using the same arguments as in the proof of Theorem 3.1.7, we easily get

$$\mathbb{E}(|X_t - Y_t|) \leq \mathbb{E}[|X_0 - Y_0|] + C \int_0^t \mathbb{E}[|X_s - Y_s|] (1 + \|f_s + g_s\|_{L^p}) ds.$$

Using the Grönwall's Lemma and recalling that $\mathbb{E}[|X_0 - Y_0|] = \mathcal{W}_1(f_0, g_0)$, we get

$$\mathbb{E}(|X_t - Y_t|) \leq \mathcal{W}_1(f_0, g_0) \exp\left(C \int_0^t (1 + \|f_s + g_s\|_{L^p}) ds\right),$$

which concludes the proof since $\mathcal{W}_1(f_t, g_t) \leq \mathbb{E}(|X_t - Y_t|)$. \square

We can now give the proof of our well-posedness result for (3.1.1).

Proof of Theorem 3.1.5 (i). The existence follows by Theorem 3.1.7. Indeed consider $(X_t)_{t \geq 0}$ the unique solution of (3.1.4) with initial law f_0 and set for $t \geq 0$ $f_t := \mathcal{L}(X_t)$. Thanks to the Remark 3.1.2, f_t is a weak solution to (3.1.1) in the sense given by Definition 3.1.1 and (3.1.15) is exactly (3.1.7).

For uniqueness, consider two weak solutions $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ of (3.1.1) satisfying (3.1.7) with the same initial condition $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$. Then Theorem 3.5.2 ensures that $\mathcal{W}_1(f_t, g_t) = 0$ for any $t \geq 0$ which concludes the proof. \square

We end this section with the proof of our propagation of chaos result.

Proof of Theorem 3.1.8 (i). We consider $Q^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N})_{t \geq 0}}$. By Lemma 3.4.1, the family $\{\mathcal{L}(Q^N), N \geq 2\}$ is tight in $\mathbf{P}(\mathbf{P}(C([0, \infty), \mathbb{R}^2)))$. Furthermore, by proposition 3.4.2, any limit point of Q^N belongs a.s. to the set of all probability measures $f \in \mathbf{P}(C([0, \infty), \mathbb{R}^2))$ such that f is the law of a solution to (3.1.4) satisfying (3.1.9). But by Theorem 3.1.7, this set is reduced to $\mathcal{L}((X_t)_{t \geq 0}) =: f$. We thus deduce that Q^N goes in law to f as $N \rightarrow \infty$ which concludes the proof of (i).

3.6 Renormalization and entropic chaos

In this section, we first deal with the renormalization which will give us the dissipation of entropy for the solution to (3.1.1). From this, we will be able to show the entropic chaos for the system (3.1.3), which will conclude this paper.

Proof of Theorem 3.1.5 (ii). We adapt the ideas used in [36] for the 2D vortex model to our case, which in particular has a non divergence free kernel. We split the proof in four steps plus a Step 0 which is nothing but direct results of what we have already done. We consider the unique weak solution $f = (f_t)_{t \geq 0}$ of (3.1.1). In step 1 we deal with the necessary estimates on $K * f$ and $\nabla \cdot (K * f)$ to regularize f . In step 2 we show the convergence of a regular version of f towards f . In step 3, we improve the regularity of the solution using a well-known bootstrap argument. Finally, in step 4 we prove the renormalization property.

We first observe that by construction, f satisfies (3.1.8). Indeed, for any $t \geq 0$, we considered f_t as the law of X_t , where $(X_t)_{t \geq 0}$ is the unique solution to (3.1.4), obtained by Proposition 3.4.2 and Lemma 3.4.1, so that (3.4.1) (which englobes (3.1.8)) is satisfied.

Step 0. Direct Estimates. We start by noticing that (3.1.8) and Lemma 3.2.4 imply directly (3.1.9) and also that for any $p \in [1, \infty)$ and all $T > 0$,

$$(3.6.1) \quad f \in L^{p/(p-1)}(0, T; L^p(\mathbb{R}^2)).$$

Step 1. First Estimates. The aim of this step is to prove that for any $q > 2/\alpha$ and all $T > 0$:

$$(3.6.2) \quad (K * f) \in L^{2q/(\alpha q - 2)}(0, T; L^q(\mathbb{R}^2)),$$

and

$$(3.6.3) \quad \nabla_x \cdot (K * f) = K * (\nabla_x \cdot f) \in L^{2q/(q(1+\alpha) - 2)}(0, T; L^q(\mathbb{R}^2)).$$

Let us remember the Hardy-Littlewood-Sobolev inequality in 2D: for $1 \leq p < 2/(2 - \alpha)$,

$$\left\| \int_{\mathbb{R}^2} \frac{f(y)}{|\cdot - y|^{2-(2-\alpha)}} dy \right\|_{2p/(2-(2-\alpha)p)} \leq C_{\alpha,p} \|f\|_p.$$

Using (3.6.1) we get that for any $p \in (1, 2/(2 - \alpha))$ and all $T > 0$,

$$(K * f) \in L^{p/(p-1)}(0, T; L^{2p/(2-(2-\alpha)p)}(\mathbb{R}^2)),$$

and under the change of variables $q = 2p/(2 - (2 - \alpha)p)$ we easily deduce (3.6.2).

Similarly, but using (3.1.9) instead of (3.6.1), we get that for any $p \in (1, 2/(2 - \alpha))$ and all $T > 0$,

$$\nabla_x \cdot (K * f) \in L^{2p/(3p-2)}(0, T; L^{2p/(2-(2-\alpha)p)}(\mathbb{R}^2)),$$

applying the same change of variables $q = 2p/(2 - (2 - \alpha)p)$ we get (3.6.3).

Step 2. Continuity. Consider $T > 0$ fixed. For $q > 2/\alpha$ we have that $2q/(q(1 + \alpha) - 2) > q/(q - 1)$, then using (3.6.1) with $p = q/(q - 1) > 1$, and (3.6.3), we get that $f \nabla_x \cdot (K * f)$ belongs to $L^1(0, T; L^1(\mathbb{R}^2))$. The following lemma follows directly:

Lemma 3.6.1. *Consider a mollifier sequence (ρ_n) on \mathbb{R}^2 and introduce the mollified function $f_t^n := f_t * \rho_n$. Clearly, $f_t^n \in C([0, \infty), L^1(\mathbb{R}^2))$. For all $T > 0$, there exists $r^n \in L^1(0, T; L^1_{loc}(\mathbb{R}^2))$ that goes to 0 when $n \rightarrow \infty$, and such that*

$$(3.6.4) \quad \partial_t f^n - \chi \nabla_x \cdot ((K * f) f^n) - \Delta_x f^n = r^n.$$

Remark 3.6.2. *The proof of the previous lemma is a modification of [25, Lemma II.1.(ii) and Remark 4]. In fact, for all $T > 0$, $f \in L^\infty(0, T; L^1(\mathbb{R}^2))$ and for any $p > 2/\alpha$, $(K * f) \in L^1(0, T; L^p(\mathbb{R}^2))$. That suffices for the existence of r^n given by*

$$r^n := \chi [(\nabla \cdot ((K * f) f)) * \rho^n - \nabla \cdot ((K * f) f^n)],$$

which goes to 0 if $n \rightarrow \infty$ in $L^1(0, T; L^1_{loc}(\mathbb{R}^2))$.

As a consequence of Lemma 3.6.1, the chain rule applied to the smooth f^n reads

$$(3.6.5) \quad \begin{aligned} \partial_t \beta(f^n) = & \chi [(K * f) \cdot \nabla_x \beta(f^n) + \beta'(f^n) f^n \nabla_x \cdot (K * f)] \\ & + \Delta_x \beta(f^n) - \beta''(f^n) |\nabla_x f^n|^2 + \beta'(f^n) r^n, \end{aligned}$$

for any $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2,\infty}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Since the equation (3.6.4) with $(K * f)$ fixed is linear in f^n , the difference $f^{n,k} := f^n - f^k$ satisfies (3.6.4) with r^n replaced by $r^{n,k} := r^n - r^k \rightarrow 0$ in $L^1(0, T; L_{loc}^1(\mathbb{R}^2))$ and then also (3.6.5) (with again f^n and r^n changed in $f^{n,k}$ and $r^{n,k}$).

Now, choosing $\beta(s) = \beta_1(s)$ where $\beta_1(s) = s^2/2$ for $|s| \leq 1$ and $\beta_1(s) = |s| - 1/2$ for $|s| \geq 1$. It is clear that $\beta \in C^1(\mathbb{R})$, that $\beta', \beta'' \in L^\infty(\mathbb{R})$ and that the second derivative has compact support. For any nonnegative $\psi \in C_c^2(\mathbb{R}^2)$, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t, x)) \psi(x) dx \\ &= \int_{\mathbb{R}^2} \chi [(K * f) \cdot \nabla_x \beta_1(f^{n,k}) + \beta_1'(f^{n,k}) f^{n,k} \nabla_x \cdot (K * f)] \psi(x) dx \\ & \quad + \int_{\mathbb{R}^2} [\Delta_x \beta_1(f^{n,k}) - \beta_1''(f^{n,k}) |\nabla_x f^{n,k}|^2 + \beta_1'(f^{n,k}) r^{n,k}] \psi(x) dx \\ & \leq \int_{\mathbb{R}^2} |r^{n,k}(t, x)| \psi(x) dx + \int_{\mathbb{R}^2} \beta_1(f^{n,k}) \Delta_x \psi dx \\ & \quad + \chi \int_{\mathbb{R}^2} |f^{n,k} \nabla_x \cdot (K * f)| \psi(x) dx - \chi \int_{\mathbb{R}^2} \beta_1(f^{n,k}) \nabla_x \cdot ((K * f) \psi(x)) dx, \end{aligned}$$

where we have used that $|\beta_1'| \leq 1$ and that $\beta_1'' \geq 0$. We know that $f_0 \in L^1(\mathbb{R}^2)$ then $f^{n,k}(0) \rightarrow 0$ in $L^1(\mathbb{R}^2)$, also that $r^{n,k} \rightarrow 0$ in $L^1(0, T; L_{loc}^1(\mathbb{R}^2))$. It is not difficult to see that $\beta_1(f^{n,k})(K * f) \rightarrow 0$ in $L^1(0, T; L_{loc}^1(\mathbb{R}^2))$, (because β_1 is sub-linear, and for all $0 < \alpha < 1$ there is $q := p/(p-1) > 2/\alpha$, then using (3.6.1) and (3.6.2): $f^{n,k} \rightarrow 0$ in $L^{p/(p-1)}(0, T; L^p(\mathbb{R}^2))$, and $(K * f) \in L^{q/(q-1)}(0, T; L^q(\mathbb{R}^2))$).

The same arguments apply to $\beta_1(f^{n,k}) \nabla_x \cdot (K * f)$ and $|f^{n,k} \nabla_x \cdot (K * f)|$, and then both goes to 0 as $n, k \rightarrow \infty$ in $L^1(0, T; L_{loc}^1(\mathbb{R}^2))$. Finally, we get

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t, x)) \psi(x) dx \xrightarrow{n, k \rightarrow \infty} 0.$$

Since ψ is arbitrary, we deduce that there exists $\bar{f} \in C([0, \infty); L_{loc}^1(\mathbb{R}^2))$ so that $f^n \rightarrow \bar{f}$ in $C([0, \infty); L_{loc}^1(\mathbb{R}^2))$ with the topology of the uniform convergence on any compact subset in time. Together with the convergence $f^n \rightarrow f$ in $C([0, \infty); \mathbf{P}(\mathbb{R}^2))$ we get that $f = \bar{f}$. We end this Step by concluding that, with the same convention for the notion of convergence on $[0, \infty)$: $f^n \rightarrow f$

in $C([0, \infty); L^1(\mathbb{R}^2))$.

Step 3. Additional estimates. From (3.6.5), we know that for all $0 < t_0 < t_1$, all $\psi \in C_c^2(\mathbb{R}^2)$,

$$\begin{aligned}
(3.6.6) \quad & \int_{\mathbb{R}^2} \beta(f_{t_1}^n) \psi(x) dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s^n) |\nabla_x f_s^n|^2 \psi(x) dx ds \\
&= \int_{\mathbb{R}^2} \beta(f_{t_0}^n) \psi(x) dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta'(f_s^n) r^n \psi(x) dx ds \\
&+ \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta(f_s^n) [\Delta_x \psi(x) - \chi(K * f) \nabla_x \psi(x)] dx ds \\
&+ \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^2} [\beta'(f_s^n) f_s^n - \beta(f_s^n)] \psi(x) \nabla_x \cdot (K * f) dx ds.
\end{aligned}$$

Let us choose $0 \leq \psi \in C_c^2(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2,\infty}(\mathbb{R})$ convex such that β'' is nonnegative and vanishes outside of a compact set (notice that, there is a constant $C > 0$ such that $s\beta'(s) \leq C\beta(s)$). We can pass to the limit as $n \rightarrow \infty$ (for details see Step 2) to get

$$\begin{aligned}
\int_{\mathbb{R}^2} \beta(f_{t_1}) \psi(x) dx &\leq \int_{\mathbb{R}^2} \beta(f_{t_0}) \psi(x) dx \\
&+ \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta(f_s) [\Delta_x \psi(x) - \chi(K * f) \nabla_x \psi(x)] dx ds \\
&+ \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^2} [-\beta(f_s) + \beta'(f_s) f_s] \psi(x) \nabla_x \cdot (K * f) dx ds.
\end{aligned}$$

It is not hard to deduce, by approximating $\psi \equiv 1$ by a well-chosen sequence ψ_R that

$$\int_{\mathbb{R}^2} \beta(f_{t_1}) dx \leq \int_{\mathbb{R}^2} \beta(f_{t_0}) dx + \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^2} [-\beta(f_s) + \beta'(f_s) f_s] \nabla_x \cdot (K * f) dx ds.$$

whenever β is admissible.

Now we deal with the regularity in space of (3.1.10). Let us start by noticing that taking $p > 2/(1 - \alpha)$:

$$(3.6.7) \quad \nabla_x (K * f)(x) = \int_{\mathbb{R}^2} \frac{(1 - \alpha) f(y)}{|x - y|^{1+\alpha}} dy,$$

so that using (3.5.1),

$$\int_0^T \|\nabla_x(K * f_s)\|_{L^\infty(\mathbb{R}^2)} \leq C(\alpha, p) \int_0^T (\|f_s\|_{L^p(\mathbb{R}^2)} + 1) < \infty,$$

and due to the fact that $s\beta'(s) \leq C\beta(s)$, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \beta(f_{t_1}) dx &\leq \int_{\mathbb{R}^2} \beta(f_{t_0}^n) dx \\ &\quad + (C+1)\chi \int_{t_0}^{t_1} \|\nabla_x(K * f)(x)\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \beta(f_s) dx ds. \end{aligned}$$

Then Grönwall's lemma implies that for all $0 < t_0 < t_1 < T$,

$$\int_{\mathbb{R}^2} \beta(f_{t_1}) dx \leq C(\alpha, T) \int_{\mathbb{R}^2} \beta(f_{t_0}^n) dx.$$

Finally letting $\beta(s) \rightarrow |s|^q/q$, we get that for all $q \geq 1$ and all $0 < t_0 < t_1 < T$,

$$(3.6.8) \quad \|f(t_1, \cdot)\|_{L^q(\mathbb{R}^2)} \leq C(q, \alpha, T) \|f(t_0, \cdot)\|_{L^q(\mathbb{R}^2)}.$$

Coming back to (3.6.6) and using $\beta_M(s) = s^2/2$ for $|s| \leq M$ and $\beta_M(s) = M|s| - M^2/2$ for $|s| \geq M$, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \beta_M(f_{t_1}^n) \psi dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \mathbb{1}_{|f_s| \leq M} |\nabla_x f_s^n|^2 \psi dx ds \\ &= \int_{\mathbb{R}^2} \beta_M(f_{t_0}^n) \psi dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta'_M(f_s^n) r^n \psi(x) dx ds \\ &\quad + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta_M(f_s^n) [\Delta \psi(x) - \chi(K * f) \nabla_x \psi(x)] dx ds \\ &\quad + \chi \int_{t_0}^{t_1} \int_{\mathbb{R}^2} [\beta'_M(f_s^n) f_s^n - \beta_M(f_s^n)] \psi(x) \nabla_x \cdot (K * f) dx ds, \end{aligned}$$

similarly as above we first make $n \rightarrow \infty$, then we approximate $\psi \equiv 1$ by a well-chosen sequence ψ_R and make $R \rightarrow \infty$, and finally make the limit $M \rightarrow \infty$ to find that for every $T \geq t_1 \geq t_0 \geq 0$:

$$\begin{aligned} &\int_{\mathbb{R}^2} |f_{t_1}|^2 dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla_x f_s|^2 dx ds \\ &\leq \int_{\mathbb{R}^2} |f_{t_0}|^2 dx + \chi \int_{t_0}^{t_1} \|\nabla_x(K * f)(x)\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |f_s|^2 dx ds. \end{aligned}$$

We conclude, using (3.6.8), that for all $0 < t_0 < T$ and any $q \in [1, \infty)$:

$$(3.6.9) \quad f \in L^\infty(t_0, T; L^q(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x f \in L^2((t_0, T) \times \mathbb{R}^2).$$

To get the continuity in time of (3.1.10), we need to improve even more the estimates on f which will be achieved using a bootstrap argument. First, fixing $p > 2/(2 - \alpha)$ we notice that for all $t_0 > 0$

$$\|K * f_t\|_{L^\infty} \leq C(p)(1 + \|f_t\|_{L^p}) \Rightarrow K * f_t \in L^\infty(t_0, T; L^\infty(\mathbb{R}^2)),$$

and thanks to (3.6.7) and (3.6.9):

$$\|\nabla_x(K * f_t)\|_{L^\infty} \leq C(p)(1 + \|f_t\|_{L^p}) \Rightarrow \nabla_x(K * f_t) \in L^\infty(t_0, T; L^\infty(\mathbb{R}^2)),$$

we thus have

$$\partial_t f - \Delta_x f = [\chi f \nabla_x \cdot (K * f) + (K * f) \cdot \nabla_x f] \in L^2((t_0, T) \times \mathbb{R}^2),$$

and [7, Theorem X.11] provides the maximal regularity in L^2 spaces for the heat equation, in other words: for all $t_0 > 0$

$$f \in L^\infty(t_0, T; H^1(\mathbb{R}^2)) \cap L^2(t_0, T; H^2(\mathbb{R}^2)).$$

Remark 3.6.3. *We emphasize that the previous bound is true for all t_0 . In fact, when $f_{t_0} \in H^1(\mathbb{R}^2)$, the maximal regularity implies the above bound in the time interval $[t_0, \infty)$. But thanks to (3.6.9), we can find t_0 arbitrary close to 0 such that $f_{t_0/2} \in H^1(\mathbb{R}^2)$, then we get the conclusion.*

Using now the interpolation inequality, there exists a constant $C > 0$ such that

$$\|\nabla_x f\|_{L^3(\mathbb{R}^2)} \leq C \|D^2 f\|_{L^2(\mathbb{R}^2)}^{2/3} \|f\|_{L^2(\mathbb{R}^2)}^{1/3},$$

which implies

$$\int_{t_0}^T \|\nabla_x f\|_{L^3(\mathbb{R}^2)}^3 ds \leq C \int_{t_0}^T \|D^2 f\|_{L^2(\mathbb{R}^2)}^2 \|f\|_{L^2(\mathbb{R}^2)} ds < \infty.$$

Thanks to the previous calculus and again [7, Theorem X.12] we conclude that $\partial_t f, \nabla_x f \in L^3((t_0, T) \times \mathbb{R}^2)$ and then the Morrey's inequality implies that for all $t_0 > 0$

$$f \in C^0((t_0, T) \times \mathbb{R}^2),$$

all together allow us to deduce that

$$f \in C([0, T]; L^1(\mathbb{R}^2)) \cap C((0, T); L^2(\mathbb{R}^2)).$$

We can go even further iterating this argument, using the interpolation inequality and the Sobolev inequality, to deduce that $\nabla_x f \in L^p((t_0, T) \times \mathbb{R}^2)$ for any $1 < p < \infty$, $[\chi f \nabla_x \cdot (K * f) + (K * f) \cdot \nabla_x f] \in L^p((t_0, T) \times \mathbb{R}^2)$ for all $t_0 > 0$. Then the maximal regularity of the heat equation in L^p spaces (see [7, Theorem X.12]) implies that for all $t_0 > 0$

$$\partial_t f, \nabla_x f \in L^p((t_0, T) \times \mathbb{R}^2),$$

and then using again the Morrey's inequality: $f \in C^{0, \alpha}((t_0, T) \times \mathbb{R}^2)$ for any $0 < \alpha < 1$, and any $t_0 > 0$. All together allow us to conclude (3.1.10).

Step 4. Renormalization. To end the proof we show (3.1.11). Let thus $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2, \infty}(\mathbb{R})$ sub-linear, such that β'' is piecewise continuous and vanishes outside of a compact set. Thanks to (3.6.9), we can pass to the limit in the similar identity as (3.6.6) obtained for time dependent test functions $\psi \in C_c^2([0, \infty) \times \mathbb{R}^2)$ to get

$$(3.6.10) \quad \begin{aligned} \int_{t_0}^{\infty} \int_{\mathbb{R}^2} \beta''(f_s) |\nabla_x f_s|^2 \psi_s \, dx \, ds &= \int_{\mathbb{R}^2} \beta(f_{t_0}) \psi_{t_0} \, dx \\ &+ \chi \int_{t_0}^{\infty} \int_{\mathbb{R}^2} \psi_s(x) \nabla_x \cdot (K * f)(f_s \beta'(f_s) - \beta(f_s)) \, dx \, ds \\ &+ \int_{t_0}^{\infty} \int_{\mathbb{R}^2} \beta(f_s) (\Delta_x \psi_s(x) - \chi(K * f) \nabla_x \psi_s(x) + \partial_t \psi_s(x)) \, dx \, ds. \end{aligned}$$

In the case $\psi \geq 0$ and $\beta'' \geq 0$ we can pass to the limit $t_0 \rightarrow 0$ thanks to monotonous convergence in the first term, the continuity property obtained in Step 2 in the second term, and the monotonous convergence in the other terms (recall that $s\beta'(s) \leq \beta(s)$, β is sub-linear and $|f|(1 + |K * f| + |\nabla \cdot (K * f)|)$ belongs to $L^1(0, T; L^1(\mathbb{R}^2))$ thanks to (3.6.2) and (3.6.3)). We get

$$(3.6.11) \quad \begin{aligned} \int_0^{\infty} \int_{\mathbb{R}^2} \beta''(f_s) |\nabla_x f_s|^2 \psi_s \, dx \, ds &= \int_{\mathbb{R}^2} \beta(f_0) \psi_{t_0} \, dx \\ &+ \int_0^{\infty} \int_{\mathbb{R}^2} \beta(f_s) [\Delta_x \psi_s - \chi \nabla_x \cdot (K * f) \cdot \psi_s + \partial_t \psi_s] \, dx \, ds \\ &+ \chi \int_0^{\infty} \int_{\mathbb{R}^2} \beta'(f_s) f_s \psi_s(x) \nabla_x \cdot (K * f) \, dx \, ds, \end{aligned}$$

and the bound given by (3.6.11) implies directly that we can pass to the limit $t_0 \rightarrow 0$ in the general case for ψ in (3.6.10) which is nothing but (3.1.11) in the distributional sense. \square

We now give a useful lemma for the entropic chaos.

Lemma 3.6.4. *Let $\alpha \in (0, 1)$ and $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ such that $H(f_0) < \infty$. Let $(f_t)_{t \geq 0}$ be the unique solution of (3.1.1) satisfying (3.1.7). Then*

$$(3.6.12) \quad H(f_t) + \int_0^t I(f_s) ds = H(f_0) + \chi(1 - \alpha) \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x - y|^{\alpha+1}} ds.$$

Proof. For $m > 1$, let us take $\beta_m \in C^1(\mathbb{R}) \cap W_{loc}^{2,\infty}(\mathbb{R})$ given by

$$\beta_m(s) = \begin{cases} s \log(s) + (1 - s)/m & \text{for } m^{-1} \leq s \leq m, \\ \beta_m(m_-) + \beta'_m(m_-)(s - m) & \text{for } s > m, \\ \beta_m(m_+^{-1}) + \beta'_m(m_+^{-1})(s - m^{-1}) & \text{for } s < m^{-1}, \end{cases}$$

so that $\beta_m(s) \leq Cs$ and $\beta_m \rightarrow s \log(s)$ for any $s > 0$.

Since β_m is admissible (in the sense of Theorem 3.1.5), then using (3.1.11) we get that for any $\psi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned} \int \beta_m(f_t) \psi dx - \int \beta_m(f_0) \psi dx &= \chi \int_0^t \int \nabla_x \cdot (K * f) (f \beta'_m(f) - \beta_m(f)) \psi dx ds \\ &+ \int_0^t \int \beta_m(f) (\Delta_x \psi - \chi(K * f) \cdot \nabla_x \psi) dx ds \\ &- \int_0^t \int \beta''_m(f) |\nabla_x f|^2 \psi dx ds, \end{aligned}$$

using that $\beta''_m(s)$ is nonnegative, that β_m grows linearly at $+\infty$ and that $(f_s)_{s \geq 0}$ is nonnegative we can make $\psi \rightarrow 1$ to get

$$\begin{aligned} \int \beta_m(f_t) dx - \int \beta_m(f_0) dx &= \chi \int_0^t \int \nabla_x \cdot (K * f) (f \beta'_m(f) - \beta_m(f)) dx ds \\ &- \int_0^t \int \beta''_m(f) |\nabla_x f|^2 dx ds. \end{aligned}$$

In fact, the first and the second terms converge thanks to monotonous convergence and that $|\beta_m(s)| \leq C|s|$. The third term is a consequence of the monotonous convergence, that $\beta'_m(s)$ is bounded, and that $f \nabla \cdot (K * f)$ (resp. $|f(K * f)|$ for the fourth term) is integrable by (3.6.3) (resp. (3.6.2)). The last term is a consequence of (3.4.1).

Finally, we notice that in the interval $(0, 1]$ the function $-\beta_m$ increases to $-s \log(s)$ while in the interval $[1, \infty)$, $\beta_m(s)$ increases to $s \log(s)$. Thanks to the monotonous convergence we can make $m \rightarrow \infty$ and using the integrability of all the limits we get (3.6.12). \square

It remains to conclude with the proof of the entropic chaos.

Proof of Theorem 3.1.8 (ii). We only have to prove that for each $t \geq 0$, $H(F_t^N)$ tends to $H(f_t)$. To this aim, we first show that for any $t \geq 0$

$$(3.6.13) \quad L := \limsup_N \left[H(F_t^N) + \int_0^t I(F_s^N) ds \right] \leq H(f_t) + \int_0^t I(f_s) ds.$$

Let $t \geq 0$ be fixed. Using (3.1.14) and recalling that $H(F_0^N) \rightarrow H(f_0)$ by assumption, we have

$$L \leq H(f_0) + \limsup_N \frac{\chi(1-\alpha)}{N^2} \sum_{i \neq j} \int_0^t \mathbb{E} \left[\frac{1}{|X_s^{i,N} - X_s^{j,N}|^{\alpha+1}} \right] ds,$$

so that using that $H(f_t) + \int_0^t I(f_s) ds = H(f_0) + \chi(1-\alpha) \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x-y|^{\alpha+1}} ds$ by Lemma 3.6.4, we only have to prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \int_0^t \mathbb{E} \left[\sum_{i \neq j} \frac{1}{|X_s^{i,N} - X_s^{j,N}|^{\alpha+1}} \right] ds = \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x-y|^{\alpha+1}} ds.$$

By exchangeability, it suffices to prove that, as $N \rightarrow \infty$,

$$D_N := \int_0^t \mathbb{E} \left[\frac{1}{|X_s^{1,N} - X_s^{2,N}|^{\alpha+1}} \right] ds \rightarrow \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x-y|^{\alpha+1}} ds =: D.$$

For any $\epsilon > 0$, we have

$$|D - D_N| \leq |D - D_\epsilon| + |D_\epsilon - D_{N,\epsilon}| + |D_{N,\epsilon} - D_N|,$$

where $D_{N,\epsilon} = \int_0^t \mathbb{E} \left[\frac{1}{(|X_s^{1,N} - X_s^{2,N}| \vee \epsilon)^{\alpha+1}} \right] ds$ and $D_\epsilon = \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{(|x-y| \vee \epsilon)^{\alpha+1}} ds$. Using that for any $\epsilon > 0$ fixed, the function $(x, y) \mapsto (|x-y| \vee \epsilon)^{-\alpha-1}$ is bounded continuous and that $\mathcal{L}(X_s^{1,N}, X_s^{2,N})$ goes weakly to $f_s \otimes f_s$ for any $s \geq 0$, we have $\lim_N \mathbb{E} \left[\frac{1}{(|X_s^{1,N} - X_s^{2,N}| \vee \epsilon)^{\alpha+1}} \right] = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{(|x-y| \vee \epsilon)^{\alpha+1}}$. By dominated convergence, we thus get that $\lim_N |D_\epsilon - D_{N,\epsilon}| = 0$. We thus have

$$\limsup_N |D - D_N| \leq |D - D_\epsilon| + \limsup_N |D_{N,\epsilon} - D_N| \quad \forall \epsilon > 0.$$

Let $\tilde{\alpha}$ be such that $\alpha + 1 < \tilde{\alpha} < 2$. We have

$$\begin{aligned} |D - D_\epsilon| &\leq 2 \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x-y|^{\alpha+1}} \mathbb{1}_{\{|x-y| < \epsilon\}} ds \\ &\leq 2\epsilon^{\tilde{\alpha}-\alpha-1} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f_s(dx) f_s(dy)}{|x-y|^{\tilde{\alpha}}} ds \\ &\leq C\epsilon^{\tilde{\alpha}-\alpha-1} \int_0^t (1 + I(f_s)) ds \leq C(1+t)\epsilon^{\tilde{\alpha}-\alpha-1}, \end{aligned}$$

by Lemma 3.2.1 (applied with $F = f_s \otimes f_s$, for which $I(F_s) = I(f_s)$) and (3.1.8). Using the same arguments, we also have for any $N \geq 2$,

$$|D_{N,\epsilon} - D_N| \leq C\epsilon^{\tilde{\alpha}-\alpha-1} \int_0^t (1 + I(F_s^N)) ds \leq C(1+t)\epsilon^{\tilde{\alpha}-\alpha-1}.$$

We thus get that $\limsup_N |D - D_N| = 0$ and (3.6.13) is proven.

Using [40, Theorem 3.4 and Theorem 5.7], we have

$$(3.6.14) \quad \liminf_N H(F_t^N) \geq H(f_t) \quad \text{and} \quad \liminf_N \int_0^t I(F_s^N) ds \geq \int_0^t I(f_s) ds.$$

Using (3.6.13) and (3.6.14), we easily conclude that

$$\lim_N H(F_t^N) = H(f_t) \quad \text{and} \quad \lim_N \int_0^t I(F_s^N) ds = \int_0^t I(f_s) ds,$$

which concludes the proof. \square

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