



Infinite Dimensional Singularly Perturbed Dynamical Systems: Theory and Applications

Ousmane Seydi

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THÈSE

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L'UNIVERSITÉ BORDEAUX 1

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Par **Ousmane SEYDI**

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DOCTEUR

SPÉCIALITÉ: MATHÉMATIQUES APPLIQUÉES ET CALCUL SCIENTIFIQUE

Perturbations singulières des systèmes dynamiques en dimension infinie: théorie et applications

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Résumé

L'objectif de cette thèse est d'étudier et de donner des outils pour la compréhension des problèmes de perturbations singulières pour des modèles épidémiques et des problèmes de dynamiques de populations. Les modèles considérés sont des équations structurées en âge qui peuvent dans certains cas se réécrire comme des équations à retard.

L'étude de ces classes d'exemples s'est faite avec succès et a permis de comprendre et de mettre en évidence toute la complexité et l'étendue de ces problèmes.

Comme on peut le remarquer dans la littérature, l'une des clés fondamentales à la compréhension de ces problèmes est l'étude des variétés normalement hyperboliques en dimension infinie que nous avons largement étudiées dans cette thèse.

L'approche utilisée est la méthode de Lyapunov-Perron. Ce qui nous a amené à étudier les problèmes de persistance et d'existence de trichotomie (dichotomie) exponentielle qui sont des éléments fondamentaux dans l'utilisation de cette méthode.

Mots-clés:

Perturbations singulières, variétés normalement hyperboliques, trichotomie exponentielle, dichotomie exponentielle, équations à retard, équations structurées en âge.

Infinite Dimensional Singularly Perturbed Dynamical Systems : Theory and Applications

Abstract

In this thesis we aim to give tools to understand singular perturbations in epidemic models and population dynamic models. We study some singularly perturbed delay differential equation which does not enter into the class frame work of geometric singular perturbation for delay differential equations. An example of singularly perturbed age structured model is also studied.

The study of these examples allowed us to understand and highlight some complexities of these problems. One of the main tools in understanding such questions is the normally hyperbolic manifolds theory which is our central focus in this thesis. The approach used here is the Lyapunov-Perron method. Therefore the problems of persistence and existence of exponential trichotomy (dichotomy) are also stressed since there are one of the main ingredients of this method.

Keywords:

Singular perturbations, normally hyperbolic manifolds, exponential trichotomy, exponential dichotomy, delay equations, age structured models.

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Introduction

1. Introduction

La théorie des variétés invariantes a pris un rôle prépondérant ces dernières décennies dans la compréhension des systèmes dynamiques et s'est avérée indispensable dans l'étude des problèmes de perturbations singulières. Elle regroupe deux grands axes: la théorie de la variété centrale et la théorie des *variétés normalement hyperboliques* (voir Définition 5). Cette dernière citée constitue l'objet de cette thèse. Plus précisément nous nous intéressons à la persistance de variétés normalement hyperboliques et au comportement asymptotique des solutions au voisinage de la variété persistante. De manière intuitive, une variété invariante peut être vue comme étant une variété incluse dans l'espace de phase du système dynamique et sur laquelle toute orbite qui y démarre y reste pour tout temps [166]. Cette théorie présente un intérêt certain car elle est applicable dans plusieurs classes de systèmes intervenant dans la modélisation de phénomènes biologiques. Plus particulièrement lorsque le système modélisant le phénomène présente des entités évoluant à des échelles de temps différentes induisant ce qui est communément appelé système lent-rapide. Elle permet entre autres la réduction des systèmes par élimination d'une composante sur laquelle on n'a pas de données précises ou de ramener l'étude à un système en dimension finie.

2. Motivation

Le point de départ de cette thèse était de donner une justification mathématique d'une réduction d'un système d'équations structurées en âge décrivant un problème d'infections nosocomiales. Le système d'équations considéré est le suivant

$$\left\{ \begin{array}{l} \frac{dS(t)}{dt} = \nu_R N_P - \nu_R S(t) - \frac{\nu_V P_I}{N_H} \beta_V S(t) H_C(t) \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -\nu_R i(t, a), \\ i(t, 0) = \frac{\nu_V P_I}{N_H} \beta_V S(t) H_C(t) \\ S(0) = S_0 \geq 0 \\ i(0, \cdot) = i_0 \in L^1_+(0, +\infty) \end{array} \right. \quad (1)$$

et

$$\begin{cases} \frac{dH_U(t)}{dt} = \nu_H N_H - \nu_H H_U(t) - \frac{\nu_V P_C}{N_P} H_U(t) \int_0^\infty \gamma(a) i(t, a) da \\ \frac{dH_C(t)}{dt} = \frac{\nu_V P_C}{N_P} H_U(t) \int_0^\infty \gamma(a) i(t, a) da - \nu_H H_C(t) \\ H_U(0) = H_{U0} \geq 0 \\ H_C(0) = H_{C0} \geq 0. \end{cases} \quad (2)$$

Le système (1)-(2) décrit un processus de contamination par des bactéries résistantes dans un hôpital. Dans ce modèle on suppose qu'on a une transmission croisée c'est-à-dire que le personnel soignant peut transmettre les bactéries aux patients et réciproquement. La variable $S(t)$ représente le nombre de patients sensibles c'est-à-dire susceptible d'être contaminés par un personnel soignant au temps t tandis que la variable $i(t, a)$ représente le nombre de patients infectés par une bactérie résistante au temps t avec un âge d'infection a . C'est-à-dire $\int_{a_-}^{a_+} i(t, a) da$ est le nombre de patients infecté ayant un âge d'infection compris entre a_- et a_+ . La variable $H_U(t)$ représente le nombre de personnel soignant non porteur de bactéries à l'instant t et $H_C(t)$ le nombre de personnel soignant porteur de bactéries à l'instant t .

Cette version particulière du modèle introduit dans les travaux de Magal and McCluskey [94, Section 7] a pour but de décrire la propagation des bactéries résistantes sur plusieurs mois induisant ainsi des échelles de temps différentes car le processus de contamination des patients est à l'ordre de la semaine tandis que le processus de colonisation et de décolonisation du personnel soignant se fait à l'échelle de l'heure. La question posée est donc de savoir si le processus de colonisation et de décolonisation du personnel soignant peut se réduire à l'état quasi stationnaire c'est-à-dire si le système (1)-(2) peut être approximé par

$$\begin{cases} \frac{dS(t)}{dt} = \nu_R N_P - \nu_R S(t) - \frac{\nu_V P_I}{N_H} \beta_V S(t) H_C(t) \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -\nu_R i(t, a), \\ i(t, 0) = \frac{\nu_V P_I}{N_H} \beta_V S(t) H_C(t) \\ S(0) = S_0 \geq 0 \\ i(0, \cdot) = i_0 \in L_+^1(0, +\infty), \end{cases} \quad (3)$$

avec $H_C(t)$ et $H_U(t)$ les solutions du système

$$\begin{cases} 0 = \nu_H N_H - \nu_H H_U(t) - \frac{\nu_V P_C}{N_P} H_U(t) \int_0^\infty \gamma(a) i(t, a) da \\ 0 = \frac{\nu_V P_C}{N_P} H_U(t) \int_0^\infty \gamma(a) i(t, a) da - \nu_H H_C(t). \end{cases} \quad (4)$$

Cette approximation a été utilisée dans D'Agata et al. [48] en faisant une hypothèse de quasi stationnarité pour le système (2). Il faut noter que le fait de pouvoir faire

l'approximation (3)-(4) permet d'étudier le problème en ne considérant que des données sur les patients ce qui très pratique car en général on a pas de données concernant le personnel soignant.

Pour donner une justification mathématique rigoureuse de l'approximation (3)-(4) il semble claire qu'il faut faire appel à la théorie des perturbations singulières pour les équations structurées en âge. Cependant ces problèmes ont été assez peu étudiés dans la littérature notamment quelques exemples assez généraux ont été traités par Arino et al. [5] et Greiner et al. [69]. Nous référons aussi aux travaux de Banasiak et al. [14], Nguyen-Huu et al. [108] et Ducrot, Magal and Seydi [55] pour plus de résultats dans ce sujet.

Une approche pour comprendre les problèmes de perturbations singulières pour les équations structurées en âge consiste à réécrire ces systèmes sous forme d'équations différentielles à retard. Pour ce faire en supposant que la fonction $\gamma(a)$ qui décrit le temps nécessaire pour devenir infectieux est une fonction qui vaut 0 sur un intervalle $[0, \tau]$ et 1 pour $a \geq \tau$ et en supposant de plus que $H_C(t) + H_U(t) = N_H$ et que $S(t) + I(t) = N_P$ (avec N_H et N_P des constantes) nous avons réécrit le problème (1)-(2) sous forme de système d'équations différentielles à retard de la forme

$$\begin{cases} \frac{dx^\varepsilon(t)}{dt} = -\mu x^\varepsilon(t) + \alpha(1 - x^\varepsilon(t)) y^\varepsilon(t), & t \geq 0, \\ \varepsilon \frac{dy^\varepsilon(t)}{dt} = -\nu y^\varepsilon(t) + \beta(1 - y^\varepsilon(t)) x^\varepsilon(t - \tau), & t \geq 0, \\ x^\varepsilon(\theta) = \varphi(\theta), & \forall \theta \in [-\tau, 0] \text{ et } y^\varepsilon(0) = y_0, \end{cases} \quad (5)$$

où μ, α, ν, β et $\varepsilon \ll 1$ sont des constantes strictement positives (voir Ducrot, Magal and Seydi [57]). Là aussi subsiste un problème pour utiliser les résultats classiques de la littérature car en faisant le changement de variables $\hat{x}(t) := x^\varepsilon(\varepsilon t)$ et $\hat{y}(t) := y^\varepsilon(\varepsilon t)$ on obtient le système

$$\begin{cases} \frac{d\hat{x}(t)}{dt} = \varepsilon [-\mu \hat{x}(t) + \alpha(1 - \hat{x}(t)) \hat{y}(t)], & t \geq 0, \\ \frac{d\hat{y}(t)}{dt} = -\nu \hat{y}(t) + \beta(1 - \hat{y}(t)) \hat{x}(t - \tau \varepsilon^{-1}), & t \geq 0, \\ \hat{x}(\theta) = \hat{\varphi}(\theta) \text{ et } \hat{y}(0) = y_0 \end{cases} \quad (6)$$

où l'on peut choisir de définir $\hat{\varphi}$ soit par $\hat{\varphi}(\theta) = \varphi(\varepsilon\theta)$ pour $\theta \in [-\tau \varepsilon^{-1}, 0]$ soit par $\hat{\varphi}(\theta) = \varphi(\frac{\theta}{\varepsilon})$ pour $\theta \in [-\tau, 0]$. Le système (6) reste singulier en ε à cause du terme $\beta(1 - \hat{y}(t)) \hat{x}(t - \tau \varepsilon^{-1})$ car le retard $\tau \varepsilon^{-1}$ tend vers $+\infty$ quand $\varepsilon \rightarrow 0^+$. On ne peut donc utiliser les théories classiques car elles sont faites sous l'hypothèse d'un système régulier en ε après le changement de variables $\hat{x}(t) := x^\varepsilon(\varepsilon t)$ et $\hat{y}(t) := y^\varepsilon(\varepsilon t)$.

L'étude de perturbations singulières pour les équations différentielles à retard est assez complexe en générale et reste à ce jour assez peu comprise. Néanmoins il existe des travaux de Artstein and Slemrod [9] sur la convergence locale en temps quand $\varepsilon \rightarrow 0^+$ via des arguments de compacité et de mesures invariantes. Des auteurs comme Mallet-Paret and Nussbaum [100], Chow, Lin and Mallet-Paret [43] ont aussi montré la complexité des

ces problèmes en étudiant la limite du système

$$\varepsilon \frac{dx^\varepsilon(t)}{dt} = -x^\varepsilon(t) + f(x^\varepsilon(t - \tau)) \in \mathbb{R}^n, \quad t \geq 0 \text{ avec } n \in \mathbb{N},$$

quand $\varepsilon \rightarrow 0^+$ vers l'équation

$$x(t) = f(x(t - \tau)) \in \mathbb{R}^n, \quad t \geq 0 \text{ avec } n \in \mathbb{N}.$$

C'est ainsi que nous nous sommes naturellement intéressés aux problèmes de perturbations singulières et donc aux problèmes de persistance de variétés invariantes dans cette thèse. Comme nous l'avons souligné ci-dessus les problèmes de perturbations singulières pour les équations structurées en âge restent vastes et compliqués en général. En vue d'une future étude plus approfondie et d'une possible généralisation de la théorie des variétés normalement hyperboliques pour les équations structurées en âge, nous nous proposons dans cette thèse d'apporter une contribution à la compréhension de cette théorie pour les équations aux différences en dimension infinie. Cet intérêt pour les équations aux différences en dimension infinie vient du fait que les résultats obtenus peuvent être transposés au cas des systèmes d'équations différentielles. Ce type d'approche a déjà été utilisé dans la littérature pour traiter notamment des systèmes d'équations différentielles pour lesquelles on a pas de formule de variation de la constante. Nous référons par exemple à Chen and Hale [33], Kristin [86] et Magal [93].

3. Historique

La théorie des variétés invariantes occupe une place importante dans l'étude des systèmes dynamiques du fait de son vaste domaine d'application notamment en biologie, physique, chimie... Il existe deux grands axes dans cette théorie: **La théorie de la variété centrale** et **la théorie des variétés normalement hyperboliques**. Les questions principalement traitées par les chercheurs dans ce domaine concernent *l'existence, la régularité, la persistance* de ces variétés et aussi l'existence des variétés *stable, instable, centre-stable, centre-instable* et les *fibrés invariants* qui leur sont associés. Les pionniers dans ce domaine ont été Poincaré [125, 1892], Hadamard [71, 1901], Lyapunov [92, 1907] et Perron [117, 118, 116, respectivement en 1928, 1929 and 1930]. Ils ont d'ailleurs donné leurs noms aux deux méthodes les plus utilisées dans cette théorie à savoir **la méthode d'Hadamard** et **la méthode de Lyapunov-Perron**.

La méthode d'Hadamard est une approche à caractère géométrique qui utilise les propriétés de la variété afin de construire des systèmes de coordonnées et de faire un point fixe sur un ensemble de fonctions Lipschitziennes tandis que **la méthode de Lyapunov-Perron** qui sera amplement développée dans cette thèse est à caractère analytique et fait appelle à la formule de *variation de la constante* pour définir des points fixes avec des équations intégrales.

La grande difficulté liée à la méthode d'Hadamard est de trouver l'ensemble de fonctions lipschitziennes pour lequel le théorème de point fixe de contraction peut s'appliquer. L'avantage avec cette méthode est que l'on n'a pas besoin de formule de variation de la constante pour le système dynamique considéré ce qui permet de couvrir beaucoup de cas tandis que la méthode de Lyapunov-Perron qui est la méthode utilisée dans ce manuscrit est elle basée sur l'utilisation de la formule de variation de la constante, ce qui limite très souvent le cadre d'application car dans certains cas nous n'avons pas de formule de variation de la constante. Néanmoins comme l'a été noté dans Bates, Lu and Zeng [19], une utilisation de la méthode de Lyapunov-Perron peut donner plus d'informations sur la régularité des variétés invariantes. Il faut noter aussi qu'il existe d'autres méthodes comme la transformée de Lie et la méthode d'Irwin développées respectivement par Marsden and Scheuler [105, 1987] et Irwin [79, 1970] dans le contexte de la théorie des variétés invariantes. Nous référons à Wiggins [166] pour une discussion plus détaillée sur ces méthodes.

La théorie de la variété centrale concerne l'existence de variété invariante pour un système dynamique autour d'un point d'équilibre non hyperbolique. Lorsqu'elle est globale et dans le cadre des systèmes dynamiques en temps continue, la variété centrale autour d'un point d'équilibre non hyperbolique est une variété invariante par le système dynamique et tangente aux sous-espaces propres associés aux valeurs propres à parties réelles nulles. Il est évidemment connu que la variété centrale est à caractère locale mais nous donnons ici une définition dans le cas où elle est globale car les propriétés locales se déduisent généralement de propriétés globales par troncature. Nous suggérons au lecteur les travaux de Vanderbauwhede [159, 161] et Chow, Li and Wang [36] pour une bonne introduction et nous référons à Pliss [124], Kelley [84], Carr [29], Sijbrand [151], Vanderbauwhede and Iooss [160], Magal and Ruan [95], Bates and Jones [21], etc. pour plus de détails à ce sujet.

la théorie des variétés normalement hyperboliques peut être comprise comme une extension de la théorie connue pour les points d'équilibres hyperboliques à un ensemble invariant par exemple un ensemble de points d'équilibres, une orbite hétérocline ou homocline etc. Cependant cet extension induit des difficultés liées à la dynamique dans l'ensemble invariant qui peut s'avérer complexe. Il est donc nécessaire d'avoir une bonne connaissance de la dynamique sur l'ensemble. Intuitivement l'idée des variétés normalement hyperboliques est de considérer des variétés dans lesquelles les solutions du système dynamique se comportent comme sur une variété centrale globale tandis que sur les directions normales à la variété les solutions convergent vers celle-ci ou divergent de celle-ci. Cette théorie a suscité l'intérêt de plusieurs chercheurs ces dernières décennies et a connu un grand bond en avant avec les travaux de Fenichel [64, 65, 66] et Hirsch, Pugh and Shub [77] sur les variétés compactes normalement hyperboliques dans le cadre des systèmes dynamiques en dimension finie. Un exposé détaillé des théorèmes de Fenichel, dans le cas où la variété est compacte et donnée par le graphe d'une application, a été présenté par Jones [81] (voir aussi Wiggins [166]) avec des preuves très différentes basées sur des considérations géométriques et le principe de Wazewski. Toujours dans le même cadre de graphe d'une application et en dimension finie, Sakamoto [144] a étendu les résultats de

Fenichel en ne faisant pas d'hypothèse de compacité et utilisant la méthode de Lyapunov-Perron. Chow, Liu and Yi [40, 41] ont aussi démontré un résultat très général concernant l'existence et la régularité de variété centrale pour des ensembles invariants normalement hyperboliques d'équations différentielles ordinaires. Récemment Pliss and Sell [122] ont introduit la notion de variété *faiblement normalement hyperbolique* et étudié la robustesse de ces variétés. Il est certain qu'au vue de cette précédente discussion, la question qui arrive naturellement est: Est ce qu'une variété persistante pour un système dynamique est normalement hyperbolique? La réponse à cette question a été apportée par Măné [104] seulement dans le cadre des systèmes dynamiques en dimension finie.

Contrairement à la dimension finie, il existe très peu de travaux pour les systèmes dynamiques en dimension infinie. Henry [76, p.278, Theorem 9.1.2] a étudié la persistance et la régularité de variétés compactes pour des problèmes paraboliques semi linéaires sous la condition que la variété étudiée induit un système de coordonnées simple. Son résultat est assez général avec une démonstration basée sur la méthode de Lyapunov-Perron et la présentation de son théorème principal permet de couvrir certains résultats obtenus par Sakamoto [144]. Magalhães [98] a étudié les problèmes de persistance et de régularité de variété normalement hyperboliques pour certaines classes d'équations différentielles à retard. À ce jour nous pouvons noter la très importante contribution de Bates, Lu and Zeng [19] concernant les variétés compactes normalement hyperboliques pour les semi-flots dans des espaces de Banach. Ils ont par la suite généralisé leur résultat dans [20] en enlevant l'hypothèse de compacité pour les remplacer par des estimation uniformes. La particularité dans [20] est que les auteurs ont aussi étudié les cas où la variété n'est que positivement (ou négativement) invariante. Les démonstrations dans [19, 20] sont basées sur une extension de la méthode d'Hadamard. Jones and Shkoller [82] ont aussi apporté leur contribution pour certaines classes d'équations aux dérivées partielles. Récemment Eldering [63] a étudié le problème de persistance de variétés non compactes pour des systèmes dynamiques sur des variétés à géométrie bornée. Enfin basés aussi sur la méthode de Lyapunov-Perron il y a les résultats Pötzsche [130] pour les équations aux différences non autonomes.

4. Quelques exemples de perturbations singulières en dimension infinie

Exemple 1:

Dans Ducrot, Magal and Seydi [55] (voir Chapitre 5), nous avons montré que le système

$$\left\{ \begin{array}{l} \frac{\partial u_\varepsilon}{\partial t}(t, a) + \frac{\partial u_\varepsilon}{\partial a}(t, a) = - \left[\underbrace{\mu(a) + m \left(\int_0^{+\infty} \gamma_1(a) u_\varepsilon(t, a) da \right) h(a)}_{\text{competition entre espèces}} \right] u_\varepsilon(t, a) \\ \quad - \left[\underbrace{\frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) \int_0^{+\infty} \gamma_2(a) u_\varepsilon(t, a) da}_{\text{competition rapide entre espèces}} \right] u_\varepsilon(t, a), \quad a \geq 0, \\ u_\varepsilon(t, 0) = \int_0^{+\infty} \beta(a) u_\varepsilon(t, a) da, \\ u_\varepsilon(0, \cdot) = \varphi \in L_+^p((0, +\infty); \mathbb{R}), \end{array} \right. \quad (7)$$

converge, pour $\varepsilon \rightarrow 0^+$, localement uniformément en temps vers le système

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t, a) + \frac{\partial u}{\partial a}(t, a) = - \mu(a) u(t, a) \\ \quad - \underbrace{m \left(\int_0^{+\infty} \gamma_1(a) u(t, a) da \right) h(a)}_{\text{competition entre espèces}} u(t, a), \quad a \geq 0, \\ u(t, 0) = \exp \left(- \underbrace{\int_0^{+\infty} \gamma_2(a) u(t, a) da}_{\text{limitation des naissances}} \right) \int_0^{+\infty} \beta(a) u(t, a) da, \\ u(0, \cdot) = \varphi \in L_+^p((0, +\infty); \mathbb{R}), \text{ with } 1 \leq p < +\infty. \end{array} \right. \quad (8)$$

La fonction $m : [0, +\infty) \rightarrow [0, +\infty)$ est continue et localement lipschitzienne et

$$\mu, h \in L_+^\infty([0, +\infty), \mathbb{R}),$$

$$\beta, \gamma_1, \gamma_2 \in L_+^p([0, +\infty), \mathbb{R}),$$

avec $1 < q \leq +\infty$ et $\frac{1}{p} + \frac{1}{q} = 1$.

L'étude de cet exemple montre que nous pouvons avoir des équations aux limites surprenantes pour des EDPs singulières structurées en âge. Rapelons que le modèle (8) communément appelé modèle de Ricker [135, 136] a été considéré par Liu and Cohen [89] pour

$$m(x) = x, \quad \forall x \geq 0.$$

Nous référons aussi à [7, 28, 24, 25, 67, 70, 91, 134, 137, 138, 139, 158] pour plus de résultats sur le système (8) et des applications dans le domaine de l'écologie.

Exemple 2:

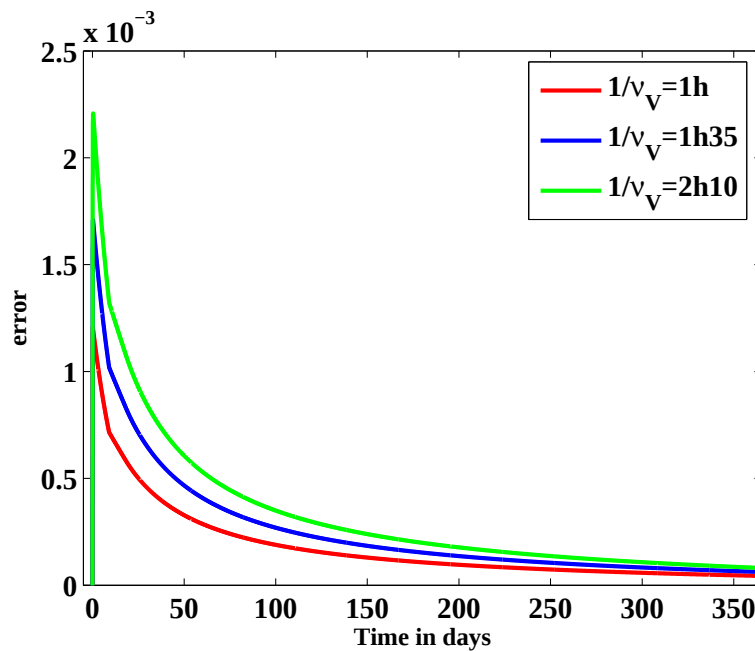
Dans Ducrot, Magal and Seydi [57] (voir Chapitre 6) nous avons étudié la convergence uniforme sur $[0, +\infty)$ de $x^\varepsilon(t)$ vers $x(t)$ et $y^\varepsilon(t)$ vers $h(x(t - \tau))$ quand $\varepsilon \rightarrow 0^+$ (voir system 5) avec $t \in [0, +\infty) \mapsto x(t)$ solution de

$$\begin{cases} \frac{dx(t)}{dt} = -\mu x(t) + \alpha(1 - x(t))h(x(t - \tau)), & t \geq 0, \\ x(\theta) = \varphi(\theta), & \forall \theta \in [-\tau, 0], \end{cases} \quad (9)$$

et l'application $x \in [0, +\infty) \mapsto h(x)$ donnée par

$$h(x) = \frac{\beta x}{\nu + \beta x}.$$

Le résultat de convergence locale sur $[0, +\infty)$ est un résultat classique de Artstein and Slemrod [9] tandis que le résultat de convergence uniforme sur $[0, +\infty)$ est un peu plus délicat car il n'est pas toujours vrai. En effet dans [57] nous avons obtenu des résultats de convergence uniforme sur $[0, +\infty)$ (voir Chapitre 6) et nous avons obtenu de plus que pour $\varphi \equiv 0$ et $y_0 \neq 0$ dans (5) l'application $t \in [0, +\infty) \mapsto x^\varepsilon(t)$ ne converge pas uniformément vers la solution de (9) avec $\varphi \equiv 0$ mais vers une **unique orbite hétérocline** du système (9). Le graphique ci-dessous illustre l'erreur commise en approximant $t \in [0, +\infty) \mapsto x^\varepsilon(t)$ par $t \in [0, +\infty) \mapsto x(t)$. Ici la simulation numérique a été faite en utilisant des données réelles avec $\varepsilon = \frac{1}{\nu_V}$ le temps de visite d'un personnel soignant à un patient (voir Chapitre 6 pour plus de détails).



5. Analyse abstraite des perturbations singulières géométriques

Soit $(X, \|\cdot\|)$ un espace de Banach. On note $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)})$ l'espace de Banach des applications linéaires bornées définies de X à valeurs dans X avec

$$\|A\|_{\mathcal{L}(X)} := \sup \{\|Ax\| : x \in X \text{ et } \|x\| = 1\}, \forall A \in \mathcal{L}(X).$$

Dans la suite nous allons donner brièvement les résultats principaux de chaque chapitre.

Chapitre 1:

Ce chapitre fait l'objet d'un article soumis (voir Ducrot, Magal and Seydi [56]). Il est consacré à l'étude de persistance de la trichotomie exponentielle (voir Définition 1 ci-dessous). Soit $A \in \mathcal{L}(X)$ un opérateur linéaire borné sur $(X, \|\cdot\|)$. Rappelons que le rayon spectral de A est défini par

$$r(A) := \lim_{n \rightarrow +\infty} \|A^n\|_{\mathcal{L}(X)}^{1/n}.$$

La définition de la trichotomie exponentielle ci-dessous est inspirée de la définition introduite dans Hale and Lin [73].

Définition 1 Soit $A \in \mathcal{L}(X)$ un opérateur linéaire borné sur $(X, \|\cdot\|)$. On dira que A est exponentiellement trichotomique s'il existe trois projecteurs $\Pi_s, \Pi_c, \Pi_u \in \mathcal{L}(X)$ tels que

$$X = X_s \oplus X_c \oplus X_u, \quad (10)$$

et

$$A(X_k) \subset X_k, \forall k = s, c, u, \quad (11)$$

avec

$$X_k := \Pi_k(X), \forall k = s, c, u,$$

et

$$X_c \oplus X_u = (I - \Pi_s)(X), X_s \oplus X_u = (I - \Pi_c)(X) \text{ et } X_s \oplus X_c = (I - \Pi_u)(X).$$

De plus on suppose qu'il existe une constante $\alpha \in (0, 1)$ telle que les propriétés suivantes sont satisfaites:

(i) Soit $A_s \in \mathcal{L}(X_s)$ la part de A dans X_s (i.e. $A_s(x) = A(x), \forall x \in X_s$). On suppose que

$$r(A_s) \leq \alpha; \quad (12)$$

(ii) Soit $A_u \in \mathcal{L}(X_u)$ la part de A dans X_u (i.e. $A_u(x) = A(x), \forall x \in X_u$). On suppose que A_u est inversible et que

$$r(A_u^{-1}) \leq \alpha; \quad (13)$$

(iii) Soit $A_c \in \mathcal{L}(X_c)$ la part de A dans X_c (i.e. $A_c(x) = A(x), \forall x \in X_u$). On suppose que A_c est inversible et que

$$r(A_c) < \alpha^{-1} \text{ et } r(A_c^{-1}) < \alpha^{-1}. \quad (14)$$

Remarque 2 Les propriétés (12)-(14) sont équivalentes c'est-à-dire qu'il existe trois constante $\kappa \geq 1$ et $0 < \rho_0 < \rho$ telles que

$$\|A_c^n\|_{\mathcal{L}(X_c)} \leq \kappa e^{\rho_0|n|}, \forall n \in \mathbb{Z}, \quad (15)$$

$$\|A_s^n\|_{\mathcal{L}(X_s)} \leq \kappa e^{-\rho n}, \forall n \in \mathbb{N}, \quad (16)$$

et

$$\|A_u^{-n}\|_{\mathcal{L}(X_u)} \leq \kappa e^{-\rho n}, \forall n \in \mathbb{N}. \quad (17)$$

Dans la suite nous dirons que A est exponentiellement trichotomique avec les exposants $\rho_0 < \rho$, la constante κ et les projecteurs $\{\Pi^\alpha\}_{\alpha=s,c,u} \subset \mathcal{L}(X)$ pour signifier que les conditions (15)-(17) sont satisfaites.

Cette précédente Définition 1 nous permet de donner la définition de la dichotomie exponentielle comme suit:

Définition 3 Soit $A \in \mathcal{L}(X)$ un opérateur linéaire borné sur $(X, \|\cdot\|)$. On dira que A est **exponentiellement dichotomique** si A est exponentiellement trichotomique avec $X_c = \{0\}$.

Le résultat principal du Chapitre 1 est le suivant.

Théorème 1 (Perturbation) Soit $A \in \mathcal{L}(X)$ un opérateur linéaire borné sur X . Supposons que A est exponentiellement trichotomique avec les exposants $\rho_0 < \rho$, une constante κ et des projecteurs $\{\Pi^\alpha\}_{\alpha=s,c,u} \subset \mathcal{L}(X)$ (voir Remarque 2). Soient $\hat{\rho}_0, \hat{\rho}$ et $\hat{\kappa}$ des constantes telles que

$$0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho \text{ et } \hat{\kappa} > \kappa.$$

Il existe $\delta_0 = \delta_0(\rho_0, \hat{\rho}_0, \hat{\rho}, \rho, \kappa, \hat{\kappa}) \in (0, \sqrt{2} - 1)$ tel que pour tout $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ si $\|B\|_{\mathcal{L}(X)} \leq \delta$, alors $(A + B) \in \mathcal{L}(X)$ est exponentiellement trichotomique avec les exposants $\hat{\rho}_0$ et $\hat{\rho}$, la constante $\hat{\kappa}$ et les projecteurs $\hat{\Pi}_s, \hat{\Pi}_c, \hat{\Pi}_u \in \mathcal{L}(X)$.

De plus les trois projecteurs $\hat{\Pi}_s, \hat{\Pi}_c, \hat{\Pi}_u \in \mathcal{L}(X)$ satisfont les estimations

$$\|\hat{\Pi}_k - \Pi_k\|_{\mathcal{L}(X)} < \frac{\kappa\delta}{\delta_0 - \delta} \leq \delta_0 < \sqrt{2} - 1, \forall k = s, c, u,$$

et le sous espace $\hat{X}_k := \hat{\Pi}_k(X)$ est isomorphique au sous espaces $X_k = \Pi_k(X)$, pour tout $k = s, c, u$.

Pour tout $n \in \mathbb{N}$ on a

$$\begin{aligned} \left\| (A+B)_s^n \widehat{\Pi}_s - A_s^n \Pi_s \right\|_{\mathcal{L}(X)} &\leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho} n}, \\ \left\| (A+B)_u^{-n} \widehat{\Pi}_u - A_u^{-n} \Pi_u \right\|_{\mathcal{L}(X)} &\leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho} n}, \end{aligned}$$

et pour tout $n \in \mathbb{Z}$ on a

$$\left\| (A+B)_c^n \widehat{\Pi}_c - A_c^n \Pi_c \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{\widehat{\rho}_0 |n|}.$$

Dans le chapitre 1 nous avons obtenu comme conséquence du Théorème 1 des résultats de persistance de trichotomie exponentielle pour des équations aux différences non autonomes (i.e. A et B dépendent de $n \in \mathbb{Z}$) ainsi que de persistance de la trichotomie exponentielle pour des systèmes dynamiques linéaires stochastiques (i.e. A et B dépendent de variables aléatoires). Ces conséquences ont été obtenues à l'aide des semigroupes de Howland (voir Chicone and Latushkin [35]) en rendant autonomes les systèmes non autonomes.

Chapitre 2:

Le Chapitre 2 fait l'objet d'un article soumis (voir Ducrot, Magal and Seydi [58]). Dans ce chapitre nous considérons des équations aux différences non autonomes et traitons des problèmes de trichotomie exponentielle sur des intervalles de longueur finie.

Soit $\{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ une famille d'opérateurs linéaires bornés. Rappelons que le **semigroupe d'évolution** associé à $\mathbf{A}$ est donné par

$$U_{\mathbf{A}}(n, p) := \begin{cases} A_{n-1} \dots A_p, & \text{si } n > p \\ I_{\mathcal{L}(X)}, & \text{si } n = p. \end{cases}$$

Définition 4 Soit $\mathbb{I}$ un intervalle de $\mathbb{Z}$ et soit $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{I} \rightarrow \mathcal{L}(X)$ une famille d'opérateurs linéaires bornés sur X . Alors $U_{\mathbf{A}}$ admet une trichotomie exponentielle sur $\mathbb{I}$ avec les exposants $0 < \rho_0 < \rho$ et la constante κ s'il existe trois familles de projecteurs $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{I}} : \mathbb{I} \rightarrow \mathcal{L}(X)$ avec $\alpha = u, s, c$ satisfaisant les propriétés suivantes:

(i) Pour tout $n \in \mathbb{I}$ et $\alpha, \beta \in \{u, s, c\}$ on a

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ si } \alpha \neq \beta,$$

et

$$\Pi_n^s + \Pi_n^u + \Pi_n^c = I.$$

(ii) Pour tout $n, m \in \mathbb{I}$ avec $n \geq m$ on a

$$U_{\mathbf{A}}^\alpha(n, m) := \Pi_n^\alpha U_{\mathbf{A}}(n, m) = U_{\mathbf{A}}(n, m) \Pi_m^\alpha, \text{ avec } \alpha = u, s, c.$$

(iii) $U_{\mathbf{A}}^{\alpha}(n, m)$ est inversible de $\Pi_m^{\alpha}(X)$ à valeurs dans $\Pi_n^{\alpha}(X)$ pour tout $n, m \in \mathbb{I}$, $n \geq m$, $\alpha = u, c$ et on note $U_{\mathbf{A}}^{\alpha}(m, n) : \Pi_n^{\alpha}(X) \rightarrow \Pi_m^{\alpha}(X)$ son inverse.

(iv) Pour tout $x \in X$, pour tout $n, m \in \mathbb{I}$ on a

$$\|U_{\mathbf{A}}^c(n, m) \Pi_m^c x\| \leq \kappa e^{\rho_0|n-m|} \|x\|,$$

et si $n \geq m$ on a

$$\|U_{\mathbf{A}}^s(n, m) \Pi_m^s x\| \leq \kappa e^{-\rho(n-m)} \|x\|,$$

$$\|U_{\mathbf{A}}^u(m, n) \Pi_n^u x\| \leq \kappa e^{-\rho(n-m)} \|x\|.$$

Le théorème principal du Chapitre 2 est le suivant.

Théorème 2 Soient $\rho > \hat{\rho} > \hat{\rho}_0 > \rho_0 > 0$ et $\kappa \geq 1$. Soit $\{\theta_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z}$ et $T_0 \in \mathbb{N} \setminus \{0\}$ tels que

$$\mathbb{Z} = \bigcup_{i \in \mathbb{Z}} [\theta_i, \theta_i + T_0].$$

Soit $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ une famille d'opérateurs linéaires bornés sur X . Supposons que

$$\|A_n\|_{\mathcal{L}(X)} \leq K, \quad \forall n \in \mathbb{Z},$$

avec $K > 0$.

Il existe des constantes $\hat{T} := \hat{T}(T_0, K, \kappa, \rho, \hat{\rho}, \rho_0, \hat{\rho}_0) > 0$ et $\hat{\kappa} := \hat{\kappa}(K, \kappa, \rho) \geq \kappa$ telles que pour tout $T \geq \hat{T}$ si $\mathbf{A}$ est exponentiellement trichotomique sur chaque intervalle $[\theta_i, \theta_i + T]$ de $\mathbb{Z}$ (avec les exposants ρ et ρ_0 , la constante κ) alors $\mathbf{A}$ est exponentiellement trichotomique sur $\mathbb{Z}$ (avec les exposants $\hat{\rho}$ et $\hat{\rho}_0$, la constante $\hat{\kappa}$).

Chapitres 3 et 4:

Le Chapitre 3 est consacré à l'étude de persistance de variétés normalement hyperboliques.

Soit $T : X \rightarrow X$ une application continument différentiable sur X . On suppose qu'il existe deux sous espaces vectoriels fermés X^c et X^h tels que

$$X = X^c \oplus X^h,$$

et que X^c est invariant par l'application $x \rightarrow T(x)$. Dans la suite on notera $\Pi^h \in \mathcal{L}(X)$ l'opérateur linéaire de projection satisfaisant

$$\Pi^h(X) = X^h \text{ et } (I - \Pi^h)(X) = X^c.$$

On pose

$$\Pi^c := I - \Pi^h.$$

Supposons de plus que

$$DT(x)X^c \subset X^c \text{ et } DT(x)X^h \subset X^h, \quad \forall x \in X^c,$$

avec de bonnes estimations sur la croissances des systèmes linéarisés autour des orbites sur X^c (c'est la normale hyperbolicité voir Définition 5 ci-dessous).

Pour étudier un problème avec un petit paramètre on regardera une application $G : \mathbb{R} \times X \rightarrow X$ telle que

$$G(0, \cdot) = T.$$

On reformule alors le problème perturbé

$$x_{n+1} = G(\varepsilon, x_n)$$

où ε est supposé proche de 0, comme

$$x_{n+1} = T(x_n) + \tilde{T}(x_n), \quad (18)$$

avec $\tilde{T}(x) = G(\varepsilon, x) - T(x)$, $x \in X$ une petite perturbation de T .

Tout d'abord rappelons la définition de normale hyperbolicité et faisons le lien avec la trichotomie exponentielle étudiée aux Chapitre 1 et Chapitre 2.

Définition 5 (Normale hyperbolicité) Soit $T : X \rightarrow X$ une application continument différentiable. Soit M un ensemble invariant par T i.e.

$$T(M) = M.$$

On dira que M est **normalement hyperbolique** pour T si les propriétés suivantes sont vérifiées:

(i) Pour tout $x \in M$, il existe trois sous espaces vectoriels X_x^u , X_x^s et X_x^c tels que

$$X = X_x^u \oplus X_x^s \oplus X_x^c, \quad (19)$$

(ii) Pour tout $\alpha = s, c, u$ on a

$$DT(x)(X_x^\alpha) \subset X_{T(x)}^\alpha,$$

et pour $\alpha = u, c$, l'application $DT(x)|_{X_x^\alpha}$ est un isomorphisme de X_x^α à valeurs dans $X_{T(x)}^\alpha$;

(iii) Il existe une constante $\kappa \geq 1$, des taux de croissance $0 < \rho_0 < \rho$ tels que pour tout $n \in \mathbb{N}$ et tout $x \in M$ on ait

$$\left\{ \begin{array}{l} \|DT^n(x)|_{X_x^s}\| \leq \kappa e^{-\rho n} \\ \frac{1}{\kappa} e^{\rho n} \leq \inf \{\|DT^n(x)x^u\| : x^u \in X_x^u \text{ et } \|x^u\| = 1\}, \end{array} \right. \quad (20)$$

et

$$\frac{1}{\kappa} e^{-\rho_0 n} \leq \inf \{\|DT^n(x)x^c\| : x^c \in X_x^c \text{ et } \|x^c\| = 1\} \leq \|DT^n(x)|_{X_x^c}\| \leq \kappa e^{\rho_0 n}. \quad (21)$$

$$\|DT^n(x)|_{X_x^s}\| \leq \kappa e^{-\rho n},$$

avec

$$\|DT^n(x)|_{X_x^\alpha}\|_{\mathcal{L}(X)} := \sup \{\|DT^n(x)x^\alpha\| : x^\alpha \in X_x^\alpha \text{ et } \|x^\alpha\| = 1\}, \quad \alpha = s, c.$$

Cette précédente définition est inspirée de la définition de Bates, Lu and Zeng [19, p.12]. Il faut noter que nous avons donné cette définition pour des systèmes dynamiques discrets et que nous avons utilisé le terme ensemble au lieu de variété. Mais il est clair que dans le cas où M est une variété on dira tout simplement que T possède une variété normalement hyperbolique. Cependant la définition donnée dans [19] pour les variétés est un peu plus générale que celle proposée ici. La différence principale entre les deux définitions est la condition (iii) qui est donnée dans [19] qui peut être interprété pour les systèmes dynamiques discrets sous la forme suivante:

(iii') Il existe une constante $\lambda \in (0, 1)$ telle que pour tout $x \in M$ on a

$$\lambda \inf \{\|DT(x)x^u\| : x^u \in X_x^u \text{ et } \|x^u\| = 1\} > \max \{1, \|DT(x)|_{X_x^c}\| \}, \quad (22)$$

et

$$\lambda \min \{1, \inf \{\|DT(x)x^c\| : x^c \in X_x^c \text{ et } \|x^c\| = 1\}\} > \|DT(x)|_{X_x^s}\|. \quad (23)$$

À noter aussi que dans le cas où M est une variété, X_x^c désigne l'espace tangent à M au point x . Nous référons à [19, 64, 77] pour les définitions les plus courantes dans la littérature.

Le lien entre la trichotomie exponentielle et la normale hyperbolicité est donné par le lemme suivant.

Lemme 6 Soit $T : X \rightarrow X$ une application continument différentiable sur X . Soit M un ensemble invariant par T tel que T définit une bijection de M à valeurs dans M . Alors les propriétés suivantes sont équivalentes:

- (i) M est normalement hyperbolique pour T avec les taux de croissances $0 < \rho_0 < \rho$ et les projecteurs associés à la décomposition

$$X = X_x^c \oplus X_x^u \oplus X_x^s, \quad x \in M,$$

sont uniformément bornés par rapport à $x \in M$.

- (ii) Il existe une constante $\hat{\kappa} > 0$ telle que pour toute orbite complète $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ de T dans M (i.e. $x_{n+1} = T(x_n) \in M, \forall n \in \mathbb{Z}$) le semigroupe d'évolution associé à $DF(\mathbf{x}) = \{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ a une trichotomie exponentielle avec la constante $\hat{\kappa} > 0$, et les taux de croissances $0 < \rho_0 < \rho$.

La preuve de la condition (i) $\implies$ (ii) est une simple observation tandis que la preuve de (ii) $\implies$ (i) fait appel au Lemme 1.1.7 du Chapitre 1 (voir aussi Ducrot, Magal and Seydi [56]) avec une simple vérification. Ce résultat apparait comme une généralisation du résultat de Palmer [113] concernant le lien entre la dichotomie exponentielle et les variétés hyperboliques. Il devient donc évident que l'étude des variétés normalement hyperboliques est fortement liée à l'étude de la trichotomie ou dichotomie exponentielle.

Hypothèse 1 *On suppose que $T : X \rightarrow X$ est continument différentiable sur X . On suppose de plus qu'il existe deux sous-espaces vectoriels fermés X^c (espace central) et X^h (espace hyperbolique) tels que*

$$X = X^c \oplus X^h.$$

On suppose aussi que $M := X^c$ est normalement hyperbolique pour T et invariant par T (i.e. $T(M) = M$). On supposera de plus que

(i) *Pour tout $x \in M$ on a: $X_x^c = M = X^c$.*

(ii) *Pour tout $x \in M$ on a: $X^h = X_x^u \oplus X_x^s$.*

On remarque que l'hypothèse ci-dessus consiste en plus de la normale hyperbolicité à supposer que l'espace central X^c et l'espace hyperbolique X^h sont indépendants du point $x \in M$ et on suppose de plus que M coïncide avec l'espace central X^c . Notons que l'on regarde une variété normalement hyperbolique un peu moins générale qui sera suffisante pour traiter bon nombre d'exemples. On peut aussi voir qu'en projetant sur X^c parallèlement à X^h (respectivement sur X^h parallèlement à X^c) le système (18) se met sous la forme plus explicite suivante (voir Chapitre 3).

$$\begin{cases} x_{n+1} = F(x_n) + R_c(x_n, y_n), & n \in \mathbb{N}, \\ y_{n+1} = A(x_n)y_n + R_h(x_n, y_n), & n \in \mathbb{N}, \end{cases} \quad (24)$$

avec $F : M \rightarrow M$, $R_c : M \times X^h \rightarrow M$, $R_h : M \times X^h \rightarrow X^h$ et $A(\cdot) : M \rightarrow \mathcal{L}(X^h)$.

Dans toute la suite de cette section nous supposons que l'Hypothèse 1 est satisfaite. Afin d'énoncer le résultat principal du chapitre 3 pour le système (24) nous avons besoin de l'hypothèse suivante.

Rappelons d'abord que pour une application $H : Z_1 \rightarrow Z_2$ avec $(Z_1, \|\cdot\|_{Z_1})$ et $(Z_2, \|\cdot\|_{Z_2})$ des espaces vectoriels normés on note

$$\|H\|_{\text{Lip}(Z_1, Z_2)} := \sup_{x, y \in Z_1, x \neq y} \frac{\|H(x) - H(y)\|_{Z_2}}{\|x - y\|_{Z_1}}.$$

Si $Z_1 = Z_2$ on écrira $\|H\|_{\text{Lip}(Z_1)}$ au lieu de $\|H\|_{\text{Lip}(Z_1, Z_1)}$.

Hypothèse 2 *On suppose que:*

(H1) F est continument différentiable sur M et que

- (i) $DF(\cdot) : M \rightarrow \mathcal{L}(M)$ est continue et globalement lipschitzienne sur M ,
- (ii) $F(M) = M$.

(H2) Il existe des constantes $\kappa \geq 1$ et $0 < \rho_0 < \rho$ telles que

- (i) Pour toute orbite complète $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ de F dans M (i.e. $x_{n+1} = F(x_n) \in M, \forall n \in \mathbb{Z}$) la famille d'opérateurs linéaires $\{A(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ est exponentiellement dichotomique sur $\mathbb{Z}$ avec la constante κ , l'exposant ρ et la famille de projecteurs $\{\Pi_{x_n}^\alpha\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$, $\alpha = u, s$.
- (ii) Pour tout $x \in M$, $DF(x) : M \rightarrow M$ est inversible sur M et on a pour tout $n \in \mathbb{N}$

$$\frac{1}{\kappa} e^{-\rho_0 n} \leq \inf \{ \|DF^n(x)w\| : w \in M, \|w\| = 1 \} \leq \kappa e^{\rho_0 n}.$$

(H3) Pour tout $i = c, h$, l'application R_i est bornée et globalement lipschitzienne sur $M \times X^h$ i.e.

$$\sup_{(x,y) \in M \times X^h} \|R_i(x,y)\| + \|R_i\|_{Lip(M \times X^h, X^h)} < +\infty.$$

(H4) $A(\cdot) : M \rightarrow \mathcal{L}(X^h)$ est bornée et globalement lipschitzienne sur M i.e.

$$\sup_{x \in M} \|A(x)\|_{\mathcal{L}(X^h)} + \|A\|_{Lip(M, \mathcal{L}(X^h))} < +\infty.$$

(H5) Il existe une orbite complète $\hat{\mathbf{x}} = \{\hat{x}_n\}_{n \in \mathbb{Z}}$ de F dans M (i.e. $x_{n+1} = F(x_n) \in M, \forall n \in \mathbb{Z}$) telle que

$$\sup_{n \in \mathbb{Z}} e^{-\rho_0 |n|} \|\hat{x}_n\| < +\infty.$$

Le résultat principal du Chapitre 3 est le suivant.

Théorème 3 (Persistance) Supposons que l'Hypothèse 2 est satisfaite. Soient $\hat{\rho}_0, \hat{\rho}$ des constantes positives telles que $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$. Soit $\eta \in (\hat{\rho}_0, \hat{\rho})$ donné et fixé. Il existe $\delta > 0$ (assez petit) tel que si

$$\sup_{(x,y) \in M \times X^h} \|R_i(x,y)\| + \|R_i\|_{Lip(M \times X^h, X^h)} \leq \delta,$$

alors il existe une fonction $\Psi : M \rightarrow X^h$ tel que sont graphe

$$\widehat{M} := \{x + \Psi(x) : x \in M\},$$

satisfait les propriétés suivantes:

- (i) Pour tout $x + \Psi(x) \in \widehat{M}$ il existe une unique solution $(\mathbf{x}, \Psi(\mathbf{x})) = \{(x_n, \Psi(x_n))\}_{n \in \mathbb{Z}}$ de (24) avec $x_0 = x$ et

$$\sup_{n \in \mathbb{Z}} e^{-\eta|n|} [\|x_n\| + \|\Psi(x_n)\|] < +\infty.$$

- (ii) Si $(\mathbf{x}, \mathbf{y}) = \{(x_n, y_n)\}_{n \in \mathbb{Z}}$ est une orbite complète de (24) et

$$\sup_{n \in \mathbb{Z}} e^{-\eta|n|} \|y_n\| < +\infty,$$

alors

$$x_0 + y_0 \in \widehat{M} \Leftrightarrow y_0 = \Psi(x_0).$$

- (iii) La fonction $\Psi : M \rightarrow X^h$ est bornée et globalement lipschitzienne sur M .

Pour des soucis de simplicité dans l'énoncé du Théorème 3 nous avons omis les estimations obtenues pour la borne supérieure et la constante de Lipschitz de Ψ ainsi que la dépendance du paramètre δ par rapport aux paramètres. Nous référons aux Chapitre 3 pour plus de détails sur le Théorème 3. Notons que ce dernier théorème permet d'obtenir les résultats de persistance de variétés normalement hyperboliques pour l'équation aux différences (18). Un résultat de persistance de variétés normalement hyperboliques a été aussi obtenu pour un semiflot $\{T(t)\}_{t \geq 0}$ défini sur l'espace de Banach X (voir Chapitre 3) comme conséquence du Théorème 3.

Le Chapitre 4 est consacré à l'étude du comportement asymptotique des orbites de (18) au voisinage de la variété persistante obtenue au chapitre 3. Nous ne donnerons ici qu'une propriété pour illustrer le comportement asymptotique des orbites au voisinage de la variété persistante pour (18).

Hypothèse 3 On suppose en plus de l'Hypothèse 1 que

- (i) L'application $DT(\cdot) : X \rightarrow \mathcal{L}(X)$ est continue, bornée et globalement lipschitzienne sur X i.e.,

$$\sup_{x \in X} \|DT(x)\|_{\mathcal{L}(X)} + \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} < +\infty.$$

- (ii) Il existe une orbite complète $\widehat{\mathbf{x}} = \{\widehat{x}_n\}_{n \in \mathbb{Z}}$ de T dans M (i.e. $x_{n+1} = T(x_n) \in M, \forall n \in \mathbb{Z}$) telle que $\sup_{n \in \mathbb{Z}} e^{-\rho_0|n|} \|\widehat{x}_n\| < +\infty$.

L'Hypthèse 3 permet de vérifier l'Hypthèse 2. Le théorème suivant donne un résultat de stabilité de la variété persistante pour (18).

Théorème 4 (Stabilité) Supposons que les Hypothèses 1 et 3 sont satisfaites. Supposons de plus que $\tilde{T} : X \rightarrow X$ est continue, bornée et globalement Lipschitzienne sur X . Soient $\widehat{\rho}_0, \widehat{\rho}$

des constantes positives telles que $0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$. Soit $\eta \in (\widehat{\rho}_0, \widehat{\rho})$ donné et fixé. Alors il existe des constantes $\delta > \delta_0 > 0$ (assez petites) et $\gamma > 0$ telles que si

$$\left\| \tilde{T} \right\|_{\infty} + \left\| \tilde{T} \right\|_{Lip(X)} < \delta,$$

alors il existe une application $\Psi : M \rightarrow X^h$ telle que son graphe

$$\widehat{M} = \{x + \Psi(x) : x \in M\},$$

satisfait les propriétés suivantes:

(i) $\widehat{M}$ est invariant par $T + \tilde{T}$ c'est-à-dire $(T + \tilde{T})(\widehat{M}) = \widehat{M}$.

(ii) Il existe une application $\Theta : V(M, \delta_0) \rightarrow \widehat{M}$ (où $V(M, \delta_0) := \{y \in X : \|\Pi^h y\| \leq \delta_0\}$) telle que pour tout $y \in X$ satisfaisant

$$\left\| \Pi^h (T + \tilde{T})^n(y) \right\| \leq \delta_0, \quad \forall n \in \mathbb{N},$$

on a

$$\left\| (T + \tilde{T})^n(\Theta(y)) - (T + \tilde{T})^n(y) \right\| \leq \gamma e^{-\eta n}, \quad \forall n \in \mathbb{N}.$$

5. Résumé, discussions et plan de la thèse

Ce travail consiste à apporter une contribution dans la compréhension de la théorie des variétés normalement hyperboliques via la méthode de Lyapunov-Perron. Nous avons donc été amenés dans cette thèse à développer ou démontrer au fur et à mesure des résultats pour l'aboutissement de notre objectif. Nous avons donc eu besoin de démontrer des résultats de persistances de trichotomie exponentielle et de dichotomie exponentielle dans les chapitres 1 et 2 pour ensuite les utiliser dans les chapitres 3 et 4 afin de démontrer les résultats de persistance de variétés normalement hyperboliques, d'existence de fibrés invariants et de stabilité.

Dans le **Chapitre 1**, nous avons revisité le problème de persistance de la trichotomie (ou dichotomie) exponentielle (voir Définition 1 ou Définition 4 et Remarque 2) pour des équations aux différences autonomes, non autonomes en dimension infinie et les systèmes dynamiques linéaires stochastiques en dimension infinie. Le résultat obtenu dans ce chapitre n'est pas nouveau car il a été obtenu dans Pötschze [127], Pliss and Sell [121]. Cependant l'originalité de ce chapitre vient de l'approche utilisée et les avantages que peuvent procurer une telle approche. La méthode utilisée habituellement dans la littérature consiste à faire un décalage du spectre pour utiliser les résultats connus sur la dichotomie tandis que dans ce chapitre nous utilisons une méthode directe via une formulation en terme de problème de point fixe. Nous avons commencé par démontrer un résultat de persistance de la trichotomie pour des équations aux différences autonomes. Ensuite pour

obtenir nos résultats sur les équations aux différences non autonomes nous avons rendu autonome le système non autonome à l'aide des semigroupes d'Howland (voir Chicone and Latushkin [35]) et appliqué le résultat obtenu pour les systèmes autonomes. Enfin le résultat sur les systèmes dynamiques linéaires stochastiques est obtenu à l'aide des résultats obtenus sur les équations aux différences non autonomes.

Il faut noter que la clef fondamentale de cette approche est la formulation du problème de point fixe qui permet d'avoir une formule qui lie directement le semigroupe d'évolution perturbé au semigroupe d'évolution d'origine. Par conséquent on obtient aussi une formule qui lie les projecteurs d'origines à ceux obtenus par perturbation. L'avantage de cet approche est que via les formules proposées on peut aussi étudier la dépendance des perturbations par rapport à des paramètres comme la continuité, différentiabilité ou même étudier la trichotomie exponentielle non uniforme (voir Barreira and Valls [16]) mais ces trois derniers points ne sont pas traités dans cette thèse et sont laissés pour un futur travail.

Les problèmes de persistance de la dichotomie ou trichotomie exponentielle pour les équations aux différences non autonomes ont été largement étudiés dans la littérature. Plusieurs travaux successifs menés par différents auteurs ont permis d'arriver aux résultats les plus généraux. Parmi ces travaux nous avons les résultats sur la dichotomie dans le cas où chaque élément de la famille d'opérateurs considérée est inversible sur l'espace de phase [142, 143, 115, 13, 45]. Toujours dans le cas de la dichotomie nous avons les cas où l'inversibilité des opérateurs est restreinte sur la partie instable [76, 121, 127, 169, 38] ainsi qu'une généralisation des taux de croissance [127, 16, 169].

Quant à la trichotomie exponentielle, les résultats les moins généraux ont été obtenus dans le cas où on a une contraction du semigroupe d'évolution suivant la partie centrale et chaque élément de la famille d'opérateurs considérée est inversible sur l'espace de phase [61, 62, 114]. Une généralisation a été apportée en remplaçant la contraction suivant la partie centrale par une borne uniforme [143, 12]. Enfin nous avons les cas plus généraux qui consistent à restreindre l'inversibilité des opérateurs suivant les parties centrale et instable et autoriser une croissance sous exponentielle suivant la partie centrale [127, 121].

Le **Chapitre 2** constitue l'une des clefs fondamentales dans l'utilisation de la méthode de Lyapunov-Perron dans le contexte de la théorie des variétés normalement hyperboliques. Il s'agit de donner une condition suffisante sur une famille d'intervalles de longueur finie pour avoir la trichotomie (ou dichotomie) exponentielle sur $\mathbb{Z}$ d'une équation aux différences non autonome en dimension infinie. Ce chapitre est une généralisation des travaux assez récents de Palmer [110, 2011] sur les équations aux différences non autonomes en dimension finie. Dans [110] l'auteur suppose que chaque élément de la famille d'opérateurs considérée est inversible sur l'espace de phase tandis que nous restreignons l'inversibilité sur les parties centrale et instable. Ce résultat a permis de donner comme application l'existence de la trichotomie le long des pseudo orbites dans un voisinage d'un ensemble normalement hyperbolique. Notons aussi que ce résultat a plusieurs applications pour les systèmes dynamiques discrets notamment sur la robustesse des ensembles hyperboliques, l'hyperbolicité le long des pseudo orbites, les systèmes presque

périodiques...(voir [110] pour plus d'applications).

Le but du **Chapitre 3** est de démontrer un résultat de persistance de variétés normalement hyperboliques pour des équations aux différences en dimension infinie et de faire une application pour les semiflats dans les espaces de Banach. Nous avons étudié les équations aux différences car sous certaines hypothèses nous pouvons toujours utiliser une formule de variation de la constante en faisant des itérations. L'avantage est que nous pouvons appliquer ces résultats à des semiflats qui ne permettent pas l'utilisation de la formule de variations de la constante. Ces idées ont été utilisées par exemple dans Chen and Hale [33] (voir aussi Magal [93]) pour l'étude de l'existence d'une variété centre-instable autour d'un point d'équilibre non hyperbolique. Comparée à la littérature sur les systèmes dynamiques en temps continu, la littérature sur les systèmes dynamiques en temps discret n'est pas très vaste. Nous pouvons citer notamment [1, 44, 83, 154, 130]. Il faut noter que dans ce chapitre nous considérons des perturbations Lipschitziennes pour le système (24) et que les références précédemment citées considèrent toutes des perturbations qui sont au moins C^1 . Enfin notons qu'en dehors de l'aspect dimension infinie, la contribution majeure apportée dans le système (24) est la nonlinéarité F . Une discussion plus détaillée est faite dans la Section 3.2 du Chapitre 3.

Le **Chapitre 4** est consacré à l'étude du comportement asymptotique des orbites au voisinage de la variété persistante obtenue au Chapitre 3. Nous avons notamment montré l'existence de fibré stable et invariant autour de la variété persistante et un résultat de stabilité de cette variété dans son voisinage. Une application pour les semiflats dans les espaces de Banach est aussi donnée.

Finalement les **Chapitres 5** et **6** sont des études d'exemples de perturbations singulières en dimension infinie via des méthodes directes. Le **Chapitres 5** fait l'objet d'un article publié et **Chapitres 6** un article à paraître dans un journal.

Chapitre 1

Persistence of exponential trichotomy for bounded linear operators: A Lyapunov-Perron approach

1.1 Introduction

Let $A \in \mathcal{L}(X)$ be a bounded linear operator on a Banach space $(X, \|\cdot\|)$. Recall that the spectral radius of A is defined by

$$r(A) := \lim_{n \rightarrow +\infty} \|A^n\|_{\mathcal{L}(X)}^{1/n}.$$

Assume that A has a **state space decomposition**, whenever A is regarded as the following discrete time dynamical system

$$\begin{cases} x(n+1) = Ax(n), & \text{for } n \in \mathbb{N}, \\ x(0) = x_0 \in X. \end{cases} \quad (1.1.1)$$

Namely, we can find three closed subspaces X_s the **stable subspace**, X_c the **central subspace**, and X_u the **unstable subspace** such that

$$X = X_s \oplus X_c \oplus X_u \text{ and } A(X_k) \subset X_k, \forall k = s, c, u.$$

Moreover if we define for each $k = s, c, u$ $A_k \in \mathcal{L}(X_k)$ the part of A in X_k (i.e. $A_k(x) = A(x), \forall x \in X_k$). Then there exists a constant $\alpha \in (0, 1)$ such that

$$r(A_s) \leq \alpha < 1$$

the linear operator A_u on X_u is invertible and

$$r(A_u^{-1}) \leq \alpha < 1$$

and the operator A_c on X_c is invertible and

$$r(A_c) < \alpha^{-1} \text{ and } r(A_c^{-1}) < \alpha^{-1}.$$

We summarize the notion of state space decomposition into the following definition. In the context of linear dynamical systems (or linear skew-product semiflows) this notion also corresponds to the notion of exponential trichotomy. The following definition corresponds to the one introduced by Hale and Lin in [73].

Definition 1.1.1 Let $A \in \mathcal{L}(X)$ be a bounded linear operator on a Banach space $(X, \|\cdot\|)$. We will say that (1.1.1) has an **exponential trichotomy** (or A is **exponentially trichotomic**) if there exist three bounded linear projectors $\Pi_s, \Pi_c, \Pi_u \in \mathcal{L}(X)$ such that

$$X = X_s \oplus X_c \oplus X_u, \quad (1.1.2)$$

and

$$A(X_k) \subset X_k, \forall k = s, c, u, \quad (1.1.3)$$

where

$$X_k := \Pi_k(X), \forall k = s, c, u,$$

and

$$X_c \oplus X_u = (I - \Pi_s)(X), X_s \oplus X_u = (I - \Pi_c)(X) \text{ and } X_s \oplus X_c = (I - \Pi_u)(X).$$

Moreover we assume that there exists a constant $\alpha \in (0, 1)$ satisfying the following properties:

(i) Let $A_s \in \mathcal{L}(X_s)$ be the part of A in X_s (i.e. $A_s(x) = A(x), \forall x \in X_s$) we assume that

$$r(A_s) \leq \alpha; \quad (1.1.4)$$

(ii) Let $A_u \in \mathcal{L}(X_u)$ be the part of A in X_u (i.e. $A_u(x) = A(x), \forall x \in X_u$) we assume that A_u is invertible and

$$r(A_u^{-1}) \leq \alpha; \quad (1.1.5)$$

(iii) Let $A_c \in \mathcal{L}(X_c)$ be the part of A in X_c (i.e. $A_c(x) = A(x), \forall x \in X_c$) we assume that A_c is invertible and

$$r(A_c) < \alpha^{-1} \text{ and } r(A_c^{-1}) < \alpha^{-1}. \quad (1.1.6)$$

Remark 1.1.2 The above properties (1.1.4)-(1.1.6) are also equivalent to say that there exist three constants $\kappa \geq 1$ and $0 < \rho_0 < \rho$ such that

$$\|A_c^n\|_{\mathcal{L}(X_c)} \leq \kappa e^{\rho_0 |n|}, \forall n \in \mathbb{Z}, \quad (1.1.7)$$

$$\|A_s^n\|_{\mathcal{L}(X_s)} \leq \kappa e^{-\rho n}, \forall n \in \mathbb{N}, \quad (1.1.8)$$

and

$$\|A_u^{-n}\|_{\mathcal{L}(X_u)} \leq \kappa e^{-\rho n}, \forall n \in \mathbb{N}. \quad (1.1.9)$$

In the sequel, the above estimates will be referred to as exponential trichotomy with exponents $\rho_0 < \rho$, constant κ and associated to the projectors $\{\Pi^\alpha\}_{\alpha=s,c,u}$.

By using the definition of exponential trichotomy we may also define the notion of exponential dichotomy.

Definition 1.1.3 *Let $A \in \mathcal{L}(X)$ be a bounded linear operator on a Banach space $(X, \|\cdot\|)$. We will say that A has an **exponential dichotomy** if A has an exponential trichotomy with $X_c = \{0\}$.*

The notion of exponential dichotomy was introduced by Perron [116] to study the existence of bounded solutions for some non-autonomous differential equations. This has been used and further developed by Coppel in [45] for finite dimensional ODE where the author studied, in particular, the persistence of exponential dichotomy under small perturbation using Perron fixed point formulation. Such a persistence result has been successfully used by Palmer [111, 110] to get finite time conditions for exponential dichotomy for finite dimensional ODE. This notion has been further investigated and developed in the 70's by Sacker and Sell (see [143, 142, 140] and the references therein) for linear skew-product semiflows in Banach spaces. In this series of works the unstable projector is assumed to have a finite rank. The finite dimensional assumption for the unstable projector has been removed in the definition of exponential dichotomy given by Henry in [76]. The characterisation of exponential dichotomy in term of admissible spaces (namely in term of solvability of the non-homogeneous equation in some suitable spaces) is proved and allows the author to prove the persistence of such an invariant splitting by using a fixed point argument on suitable solutions of the inhomogeneous equation. (we also refer to Chow and Leiva [38, 39] for discussions and new results in this direction). Let us mention that Latushkin and Schnaubelt in [87] obtained persistence of exponential dichotomy of cocycle using spectral properties of Howland semigroup. Let us also note that Pötzsche in [127] considered an other kind of difference equations (degenerate like) in Banach spaces and proved the persistence of exponential dichotomy using Perron approach and without any finite dimensional assumption. Let us also mention [129] where the lack of invertibility of infinite dimensional semiflows is stressed. More recently the concept of exponential dichotomy (or trichotomy) has been extended to the so-called non-uniform exponential dichotomy (or trichotomy) by Barreira and Valls in a series of papers (we refer to the monograph [18] and the references therein). Using Perron approach, the authors proved the persistence of non-uniform dichotomy under small weighted linear perturbations. Let us also mention the recent work of Zhou et al [169] who deal with the perturbation of tempered exponential dichotomy for linear random difference equations in Banach spaces using a new characterisation with admissible spaces as well as Perron formulation.

Note that exponential dichotomy is a basic tool to study stability for non-autonomous dynamical systems. It is also a powerful ingredient to construct suitable invariant manifolds for non-linear hyperbolic problems. In order to take into account non-hyperbolic dynamics one needs to take care of the center subspace behaviour leading to the notion of exponential trichotomy. Such a notion has been introduced by Sacker and Sell in [143]

in order to take into account uniformly bounded entire solutions for some linear dynamical systems. Latter Hale and Lin in [73] introduced a weaker definition of exponential trichotomy allowing small exponential growth of the solutions on the center subspace. In this paper a persistence result of exponential trichotomy is proved under some finite dimensional assumption for the center part. This has been successfully used in the study of persistence of heteroclinic orbits for some functional differential equations. Further generalisations have been investigated by Pliss and Sell in [121] where the finite dimensional condition is removed. In this work the authors stated some equivalence between exponential trichotomy and exponential dichotomy for some shifted operators (to shift the spectrum). Based on the admissible space characterisation of dichotomy, the authors proved a very general persistence result for exponential trichotomy. Let us also mention that a similar definition of exponential trichotomy has been used by Chicone and Latushkin in [34] to construct center manifold for some non-autonomous infinite dimensional dynamical systems. Throughout this work we shall use this notion and we refer to Definitions 1.1.1, 1.1.6 and 1.1.11 for precise definitions of exponential trichotomy in several contexts (namely autonomous, non-autonomous and random dynamical systems). In the context of non-uniform trichotomy, let us mention the work of Barreira and Valls in [16] who used this shifted technique to prove the persistence of non-uniform exponential trichotomy for invertible time continuous semiflows. We also refer to [15] for another characterisation of (non-uniform) exponential trichotomy in term of Lyapunov function and robustness result for finite dimensional systems. (We refer to [107, 126, 148] and the references therein for other results on this topic). We also refer to Lian and Lu [88] and the monograph of Arnold [8] for the construction of stable and unstable manifold for some infinite dimensional random dynamical systems and to Barreira and Valls [17, 18] for a center manifold using non-uniform exponential trichotomy. Note that these constructions are based on the Lyapunov-Perron method. (we refer to [116, 159, 95] and the references cited therein) and to Bates et al [19, 20], Chow et al [36, 40, 41] and the references therein for the construction of invariant manifolds using Hadamard method.

In this work we propose to revisit the problem of persistence of exponential trichotomy by dealing with a direct proof based on Perron fixed point argument for the perturbed semiflows and projectors. More specifically if $B \in \mathcal{L}(X)$ (with $\|B\|_{\mathcal{L}(X)}$ small enough) we aim at investigating the persistence of such state space decomposition for a bounded linear perturbation of system (1.1.1) that reads as

$$\begin{cases} x(n+1) = (A+B)x(n), \text{ for } n \in \mathbb{N}, \\ x(0) = x_0 \in X. \end{cases} \quad (1.1.10)$$

The main result of the manuscript is the following theorem.

Theorem 1.1.4 (Perturbation) *Let $A \in \mathcal{L}(X)$ be a bounded linear operator on a Banach space X , and assume that A has exponential trichotomy with exponents $\rho_0 < \rho$, constant κ and associated to the projectors $\{\Pi^\alpha\}_{\alpha=s,c,u}$ (see Remark 1.1.2). Then for each $B \in \mathcal{L}(X)$ with $\|B\|_{\mathcal{L}(X)}$ small enough the linear operator $(A+B)$ has an exponential trichotomy, which*

corresponds to the following state space decomposition

$$X = \widehat{X}_s \oplus \widehat{X}_c \oplus \widehat{X}_u,$$

and which corresponds to the bounded linear projectors $\widehat{\Pi}_s, \widehat{\Pi}_c, \widehat{\Pi}_u \in \mathcal{L}(X)$ satisfying

$$\widehat{X}_k := \widehat{\Pi}_k(X), \forall k = s, c, u,$$

and

$$\widehat{X}_c \oplus \widehat{X}_u = (I - \widehat{\Pi}_s)(X), \widehat{X}_s \oplus \widehat{X}_u = (I - \widehat{\Pi}_c)(X) \text{ and } \widehat{X}_s \oplus \widehat{X}_c = (I - \widehat{\Pi}_u)(X).$$

More precisely, let three constants $\widehat{\rho}_0, \widehat{\rho} \in (0, +\infty)$ and $\widehat{\kappa}$ be given such that

$$0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho \text{ and } \widehat{\kappa} > \kappa.$$

There exists $\delta_0 = \delta_0(\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \kappa, \widehat{\kappa}) \in (0, \sqrt{2} - 1)$ such that for each $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ if $\|B\|_{\mathcal{L}(X)} \leq \delta$, then $(A + B)$ has an exponential trichotomy with exponent $\widehat{\rho}_0$ and $\widehat{\rho}$ and with constant $\widehat{\kappa}$.

Moreover, the three associated projectors $\widehat{\Pi}_s, \widehat{\Pi}_c, \widehat{\Pi}_u \in \mathcal{L}(X)$ satisfy

$$\|\widehat{\Pi}_k - \Pi_k\|_{\mathcal{L}(X)} < \frac{\kappa\delta}{\delta_0 - \delta} \leq \delta_0 < \sqrt{2} - 1, \forall k = s, c, u,$$

and as a consequence the subspace $\widehat{X}_k := \widehat{\Pi}_k(X)$ is isomorphic to the subspace $X_k = \Pi_k(X)$.

Furthermore the following estimates hold true for each $n \in \mathbb{N}$,

$$\begin{aligned} \|(A + B)_s^n \widehat{\Pi}_s - A_s^n \Pi_s\|_{\mathcal{L}(X)} &\leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\widehat{\rho}n}, \\ \|(A + B)_u^{-n} \widehat{\Pi}_u - A_u^{-n} \Pi_u\|_{\mathcal{L}(X)} &\leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\widehat{\rho}n}, \end{aligned}$$

and for each $n \in \mathbb{Z}$

$$\|(A + B)_c^n \widehat{\Pi}_c - A_c^n \Pi_c\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{\widehat{\rho}_0|n|}.$$

Remark 1.1.5 The last inequality shows that (by choosing the norm of B small enough) the growth on the central part for the perturbed system satisfies

$$\sup_{n \in \mathbb{Z}} e^{-\widehat{\rho}_0|n|} \|(A + B)_c^n \widehat{\Pi}_c\|_{\mathcal{L}(X)} < +\infty.$$

This result has several consequences for non-autonomous discrete time linear equations and linear random difference equations by using Howland semigroup procedure to reformulate these problems as autonomous systems.

In the next subsection we will state the consequences of Theorem 1.1.4 on the above mentioned topics. Section 2 is devoted to the proof of Theorem 1.1.4. Sections 3 and 4 are respectively concerned with an application of Theorem 1.1.4 for non-autonomous dynamical system (see Theorem 1.1.8) and for random linear difference equation (see Theorem 1.1.13).

1.1.1 Consequences of Theorem 1.1.4 for discrete time non-autonomous dynamical system

As mentioned above, exponential trichotomy or dichotomy plays an important role in the study of the asymptotic behaviour of non-autonomous dynamical systems. Roughly speaking exponential trichotomy generalizes the usual spectral theory of linear semigroups to linear evolution operators. It ensures an invariant state space decomposition at each time into three sub-spaces: a stable, an unstable and a central space in which the the evolution operator has different exponential behaviours. Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be a given sequence of bounded linear operators on the Banach space $(Y, \|\cdot\|)$. Consider the linear non-autonomous system

$$\begin{cases} x(n+1) = A_n x(n), & \text{for } n \geq m, \\ x(m) = x_m \in Y. \end{cases} \quad (1.1.11)$$

Let us introduce the discrete evolution semigroup associated to $\mathbf{A}$ defined as the 2-parameters linear operator on $\Delta_+ := \{(n, m) \in \mathbb{Z}^2 : n \geq m\}$ by

$$U_{\mathbf{A}}(n, m) := \begin{cases} A_{n-1} \dots A_m, & \text{if } n > m, \\ I_Y, & \text{if } n = m, \end{cases}$$

wherein I_Y denotes the identity operator in Y . In the following we will always use the notation $n \geq m$ as well as $U_{\mathbf{A}}(n, m)$ for the evolution semigroup. Whenever $U_{\mathbf{A}}(m, n)$ is considered this will mean that $U_{\mathbf{A}}(n, m)$ is invertible and

$$U_{\mathbf{A}}(m, n) = U_{\mathbf{A}}(n, m)^{-1}.$$

Let us observe that $U_{\mathbf{A}}$ satisfies

$$U_{\mathbf{A}}(n, k) U_{\mathbf{A}}(k, m) = U_{\mathbf{A}}(n, m) \text{ for each } n \geq k \geq m.$$

Then let us recall the following definition taken from Hale and Lin [73].

Definition 1.1.6 (Exponential trichotomy) Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be given. Then $U_{\mathbf{A}}$ has an **exponential trichotomy** (or $\mathbf{A}$ is **exponentially trichotomic**) on $\mathbb{Z}$ with constant κ , exponents $0 < \rho_0 < \rho$ if there exist three families of projectors $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$, with $\alpha = u, s, c$ satisfying the following properties:

(i) For all $n \in \mathbb{Z}$ and $\alpha, \beta \in \{u, s, c\}$ we have

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ if } \alpha \neq \beta,$$

and

$$\Pi_n^s + \Pi_n^u + \Pi_n^c = I_Y.$$

(ii) For all $n, m \in \mathbb{Z}$ with $n \geq m$ we have

$$U_{\mathbf{A}}^{\alpha}(n, m) := \Pi_n^{\alpha} U_{\mathbf{A}}(n, m) = U_{\mathbf{A}}(n, m) \Pi_m^{\alpha}, \text{ for } \alpha = u, s, c.$$

(iii) $U_{\mathbf{A}}^{\alpha}(n, m)$ is invertible from $\Pi_m^{\alpha}(Y)$ into $\Pi_n^{\alpha}(Y)$ for all $n \geq m$ in $\mathbb{Z}$, $\alpha = u, c$ and its inverse is denoted by $U_{\mathbf{A}}^{\alpha}(m, n) : \Pi_n^{\alpha}(Y) \rightarrow \Pi_m^{\alpha}(Y)$.

(iv) For each $y \in Y$ we have for all $n, m \in \mathbb{Z}$

$$\|U_{\mathbf{A}}^c(n, m) \Pi_m^c y\| \leq \kappa e^{\rho_0|n-m|} \|y\|, \quad (1.1.12)$$

and if $n \geq m$

$$\|U_{\mathbf{A}}^s(n, m) \Pi_m^s y\| \leq \kappa e^{-\rho(n-m)} \|y\|, \quad (1.1.13)$$

$$\|U_{\mathbf{A}}^u(m, n) \Pi_n^u y\| \leq \kappa e^{-\rho(n-m)} \|y\|. \quad (1.1.14)$$

Let us observe that the operators $U_{\mathbf{A}}^{\alpha}(n, p) \in \mathcal{L}(Y)$, for $n \geq p$ in $\mathbb{Z}$ and $\alpha = u, s, c$ (resp. $U_{\mathbf{A}}^{\alpha}(p, n)$, for $n \geq p$ in $\mathbb{Z}$ and $\alpha = u, c$) inherit the evolution property of $U_{\mathbf{A}}$ that reads as

$$U_{\mathbf{A}}^{\alpha}(n, p) U_{\mathbf{A}}^{\alpha}(p, m) = U_{\mathbf{A}}^{\alpha}(n, m), \quad \forall n \geq p \geq m \text{ in } \mathbb{Z} \text{ and } \alpha = u, s, c,$$

respectively

$$U_{\mathbf{A}}^{\alpha}(m, p) U_{\mathbf{A}}^{\alpha}(p, n) = U_{\mathbf{A}}^{\alpha}(m, n), \quad \forall n \geq p \geq m \text{ in } \mathbb{Z} \text{ and } \alpha = u, c.$$

Before stating our result, let us note that since $\Pi_n^{\alpha} = U_{\mathbf{A}}^{\alpha}(n, n)$, for $\alpha = u, s, c$ and $n \in \mathbb{Z}$, property (iv) in Definition 1.1.6 implies that the projectors are uniformly bounded by the constant κ .

The following lemma is well known in the context of exponential dichotomy (see for instance [128]).

Lemma 1.1.7 *Let $\mathbf{A} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be given such that $U_{\mathbf{A}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant κ , exponents $0 < \rho_0 < \rho$ and projectors $\Pi^{\alpha} = \{\Pi_n^{\alpha}\}_{n \in \mathbb{Z}}$ for $\alpha = u, s, c$. Then the state decomposition is uniquely determined by*

$$\begin{aligned} \Pi_n^s(Y) &= \left\{ y \in Y : \sup_{k \geq n} e^{\rho(k-n)} \|U_{\mathbf{A}}(k, n) y\| < +\infty \right\}, \\ \Pi_n^u(Y) &= \left\{ \begin{array}{l} y \in Y : \exists \{y_k\}_{k \leq n} \subset Y \text{ with } y_k = A_{k-1} y_{k-1}, \ y_n = y \\ \text{and } \sup_{k \leq n} e^{\rho(n-k)} \|y_k\| < +\infty \end{array} \right\}, \end{aligned}$$

and

$$\Pi_n^c(Y) = \left\{ \begin{array}{l} y \in Y : \exists \{y_k\}_{k \in \mathbb{Z}} \subset Y \text{ with } y_{k+1} = A_k y_k, \ y_n = y \\ \text{and } \sup_{k \in \mathbb{Z}} e^{-\rho_0|n-k|} \|y_k\| < +\infty \end{array} \right\}.$$

The above lemma ensures the uniqueness of the sequence of projectors appearing in the splitting of an exponential trichotomy in $\mathbb{Z}$. Furthermore whenever $U_{\mathbf{A}}$ has an exponential trichotomy on $\mathbb{Z}$, the range of the center projector Π_0^c corresponds to the **global center-manifold** of the linear discrete non-autonomous equation

$$y_{n+1} = A_n y_n, \quad n \geq 0.$$

Using Howland's semigroups like procedure (see Chicone and Latushkin [35]), as a consequence of Theorem 1.1.4, we obtain the following version for non-autonomous dynamical systems.

Theorem 1.1.8 (Perturbation) *Let $\mathbf{A} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be a given uniformly bounded sequence such that $U_{\mathbf{A}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three families of projectors $\{\Pi^\alpha : \mathbb{Z} \rightarrow \mathcal{L}(Y)\}_{\alpha=s,c,u}$. Let $\rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and $\hat{\kappa} > \kappa$ be given. Then there exists $\delta_0 := \delta_0(\rho_0, \hat{\rho}_0, \hat{\rho}, \rho, \kappa, \hat{\kappa}) \in (0, \sqrt{2} - 1)$ such that for each $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ and each $\mathbf{B} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ with*

$$\sup_{n \in \mathbb{Z}} \|B_n\| \leq \delta,$$

the evolution semigroup $U_{\mathbf{A}+\mathbf{B}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant $\hat{\kappa}$, exponents $\hat{\rho}$, $\hat{\rho}_0$ and projectors $\{\hat{\Pi}^\alpha\}_{\alpha=s,c,u}$. For each $n \in \mathbb{Z}$ and $\alpha = u, s, c$, the spaces $\mathcal{R}(\Pi_n^\alpha)$ and $\mathcal{R}(\hat{\Pi}_n^\alpha)$ are isomorphic. Moreover the following perturbation estimates hold true: we have for all $n \geq p$,

$$\|U_{\mathbf{A}+\mathbf{B}}^s(n, p) - U_{\mathbf{A}}^s(n, p)\| \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\hat{\rho}(n-p)}, \quad (1.1.15)$$

$$\|U_{\mathbf{A}+\mathbf{B}}^u(p, n) - U_{\mathbf{A}}^u(p, n)\| \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\hat{\rho}(n-p)}, \quad (1.1.16)$$

and for all $(n, p) \in \mathbb{Z}$

$$\|U_{\mathbf{A}+\mathbf{B}}^c(n, p) - U_{\mathbf{A}}^c(n, p)\| \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{\hat{\rho}_0|n-p|}. \quad (1.1.17)$$

1.1.2 Consequences of Theorem 1.1.4 for discrete time random dynamical system

The aim of this section is to apply Theorem 1.1.4 and Theorem 1.1.8 to study the perturbation of trichotomy for some linear random difference equations in Banach space. (we refer to the monograph of Arnold [8] for the notion of dichotomy for random dynamical systems and Zhou et al [169] for recent results on this topics).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space and let $(Y, \|\cdot\|)$ be a Banach space. Let us consider a strongly measurable map $A : \Omega \rightarrow \mathcal{L}(Y)$ as well as the linear random difference equation

$$x_{n+1} = A(\theta^n \omega) x_n, \quad n \geq m, \quad x_m \in Y,$$

wherein $\{\theta^n\}_{n \in \mathbb{Z}}$ is a measurable dynamical system on Ω . Here we recall the following definition:

Definition 1.1.9 A map $\theta : \mathbb{Z} \times \Omega \rightarrow \Omega$ is said to be a measurable dynamical system on Ω if

(i) The map θ is measurable;

(ii) $\theta(0, \cdot) = I_\Omega$ and $\theta(n + m, \cdot) = \theta(n, \cdot) \circ \theta(m, \cdot)$ for all $n, m \in \mathbb{Z}$;

Definition 1.1.10 Let $(Z, \|\cdot\|)$ be a given Banach space and a map $A : \Omega \rightarrow \mathcal{L}(Z)$ is said to be strongly measurable if for each $z \in Z$ the map $A(\cdot)z : \Omega \rightarrow Z$ is measurable from $(\Omega, \mathcal{F})$ to $(Z, \mathcal{B}(Z))$. Here $\mathcal{B}(Z)$ denotes the Borel σ -algebra generated by the open sets in Z .

Similarly to the case of non-autonomous linear difference equation, one introduces the random evolution operation associated to the strongly measurable operator $A : \Omega \rightarrow \mathcal{L}(Y)$ defined by

$$U_A(n, m, \omega) = \begin{cases} A(\theta^{n-1} \omega) \dots A(\theta^m \omega) & \text{if } n > m \\ I_Y & \text{if } n = m \end{cases}.$$

Setting $\Phi(n, \omega) := U_A(n, 0, \omega)$ for all $n \geq 0$ note that Φ is a linear random dynamical system (see Arnold [8]) and it enjoys the well-known cocycle property, namely

$$\Phi(0, \omega) = I_Y \text{ and } \Phi(n + m, \omega) = \Phi(n, \theta^m \omega) \circ \Phi(m, \omega), \quad \forall n, m \geq 0, \quad \omega \in \Omega.$$

Let us also note that

$$U_A(n, m, \theta^k \omega) = U_A(n + k, m + k, \omega), \quad \forall n \geq m, \quad k \in \mathbb{Z}, \quad \omega \in \Omega.$$

Let us also recall the following definition extended from the notion of dichotomy taken from Arnold [8]

Definition 1.1.11 The linear random evolution operator U_A has an exponential trichotomy if there exists a θ -invariant set $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ and three strongly measurable projectors $\Pi^\alpha : \tilde{\Omega} \rightarrow \mathcal{L}(Y)$ such that

(i) For all $n \in \mathbb{N}$, $\omega \in \tilde{\Omega}$ and $\alpha, \beta \in \{u, s, c\}$ we have

$$\Pi^\alpha(\omega) \Pi^\beta(\omega) = 0, \quad \text{if } \alpha \neq \beta,$$

and

$$\Pi^s(\omega) + \Pi^u(\omega) + \Pi^c(\omega) = I_Y.$$

(ii) For all $n, m \in \mathbb{Z}$ with $n \geq m$ we have

$$U_A^\alpha(n, m, \omega) := \Pi^\alpha(\theta^n \omega) U_A(n, m, \omega) = U_A(n, m, \omega) \Pi^\alpha(\theta^m \omega), \text{ for } \alpha = u, s, c.$$

(iii) For each $\omega \in \tilde{\Omega}$, $U_A^\alpha(n, m, \omega)$ is invertible from $\Pi^\alpha(\theta^m \omega)(Y)$ into $\Pi^\alpha(\theta^n \omega)(X)$ for all $n \geq m$ in $\mathbb{Z}$, $\alpha = u, c$ and its inverse is denoted by $U_A^\alpha(m, n, \omega) : \Pi^\alpha(\theta^n \omega)(Y) \rightarrow \Pi^\alpha(\theta^m \omega)(Y)$. Furthermore for each $\alpha = s, c, u$ the operators $\omega \rightarrow U_A^\alpha(n, m, \omega) \Pi^\alpha(\theta^m \omega)$ are strongly measurable for each $n \geq m$ if $\alpha = s$, each $n \leq m$ if $\alpha = u$ and each $(n, m) \in \mathbb{Z}^2$ for $\alpha = c$.

(iv) For each $y \in Y$ we have for all $n, m \in \mathbb{Z}$

$$\|U^c(n, m, \omega) \Pi^c(\theta^m \omega) y\| \leq \kappa(\omega) e^{\rho_0(\omega)|n-m|} \|y\|, \quad (1.1.18)$$

and if $n \geq m$

$$\|U^s(n, m, \omega) \Pi^s(\theta^m \omega) y\| \leq \kappa(\omega) e^{-\rho(\omega)(n-m)} \|y\|, \quad (1.1.19)$$

$$\|U^u(m, n, \omega) \Pi^u(\theta^n \omega) y\| \leq \kappa(\omega) e^{-\rho(\omega)(n-m)} \|y\|. \quad (1.1.20)$$

Here the maps ρ, ρ_0 are θ -invariant random variables from $\tilde{\Omega} \rightarrow (0, \infty)$ while κ is a random variable from $\tilde{\Omega} \rightarrow (0, \infty)$.

Remark 1.1.12 Let us note that the above definition shows that for $\alpha = s, c, u$:

$$\sup_{m \in \mathbb{Z}} \|\Pi^\alpha(\theta^m \omega)\|_{\mathcal{L}(Y)} \leq \kappa(\omega), \quad \forall \omega \in \tilde{\Omega}.$$

Then our result reads as:

Theorem 1.1.13 Let $A : \Omega \rightarrow \mathcal{L}(Y)$ be a given strongly linear operator. Let us assume that U_A has an exponential trichotomy according to Definition 1.1.11 with set $\tilde{\Omega}$, exponent $\rho_0(\omega) < \rho(\omega)$ and constant $\kappa(\omega)$. Assume furthermore that

$$\sup_{n \in \mathbb{Z}} \|A(\theta^n \omega)\|_{\mathcal{L}(Y)} < \infty, \quad \forall \omega \in \tilde{\Omega}.$$

Let $\hat{\rho}_0, \hat{\rho} : \tilde{\Omega} \rightarrow (0, \infty)$ be two θ -invariant random variables and $\hat{\kappa} : \tilde{\Omega} \rightarrow (0, \infty)$ be a random variable such that

$$\begin{cases} \rho_0(\omega) < \hat{\rho}_0(\omega) < \hat{\rho}(\omega) < \rho(\omega), \quad \forall \omega \in \tilde{\Omega}, \\ \kappa(\omega) < \hat{\kappa}(\omega). \end{cases}$$

There exists a random variable $\delta_0 : \tilde{\Omega} \rightarrow (0, \infty)$ such that if $\delta : \tilde{\Omega} \rightarrow (0, \infty)$ is a given random variable with $\delta(\omega) < \frac{\delta_0^2(\omega)}{\kappa(\omega) + \delta_0(\omega)}$ and $B : \Omega \rightarrow \mathcal{L}(Y)$ is a strongly measurable linear operator with

$$\sup_{n \in \mathbb{Z}} \|B(\theta^n \omega)\|_{\mathcal{L}(Y)} < \delta(\omega), \quad \forall \omega \in \tilde{\Omega};$$

then the evolution operator U_{A+B} has an exponential trichotomy with exponents $\hat{\rho}_0 < \hat{\rho}$ and constant $\hat{\kappa}$. We furthermore have the following perturbation estimates for each $\omega \in \tilde{\Omega}$: for all $n \geq p$,

$$\|U_{A+B}^s(n, p, \omega) - U_A^s(n, p, \omega)\| \leq \frac{\kappa(\omega)\delta(\omega)}{\delta_0(\omega) - \delta(\omega)} e^{-\hat{\rho}(\omega)(n-p)}, \quad (1.1.21)$$

$$\|U_{A+B}^u(p, n, \omega) - U_A^u(p, n, \omega)\| \leq \frac{\kappa(\omega)\delta(\omega)}{\delta_0(\omega) - \delta(\omega)} e^{-\hat{\rho}(\omega)(n-p)}, \quad (1.1.22)$$

and for all $(n, p) \in \mathbb{Z}$

$$\|U_{A+B}^c(n, p, \omega) - U_A^c(n, p, \omega)\| \leq \frac{\kappa(\omega)\delta(\omega)}{\delta_0(\omega) - \delta(\omega)} e^{\hat{\rho}_0(\omega)|n-p|}, \quad \forall (n, p). \quad (1.1.23)$$

1.2 Proof of Theorem 1.1.4

1.2.1 A continuity projector lemma

The following lemma is inspired from [19, Lemma 4.1].

Lemma 1.2.1 *Let $\Pi : X \rightarrow X$ and $\hat{\Pi} : X \rightarrow X$ be two bounded linear projectors on a Banach space X . Assume that*

$$\|\Pi - \hat{\Pi}\|_{\mathcal{L}(X)} < \delta \text{ with } 0 < \delta < \sqrt{2} - 1, \quad (1.2.1)$$

Then Π is invertible from $\hat{\Pi}(X)$ into $\Pi(X)$ and

$$\left\| \left(\Pi|_{\hat{\Pi}(X)} \right)^{-1} x \right\| \leq \frac{1}{1-\delta} \|x\|, \quad \forall x \in \Pi(X). \quad (1.2.2)$$

Remark 1.2.2 *By symmetry, the bounded linear projector $\hat{\Pi}$ is also invertible from $\Pi(X)$ into $\hat{\Pi}(X)$ and*

$$\left\| \left(\hat{\Pi}|_{\Pi(X)} \right)^{-1} x \right\| \leq \frac{1}{1-\delta} \|x\|, \quad \forall x \in \hat{\Pi}(X) \quad (1.2.3)$$

Proof. We will first prove two claims.

Claim 1.2.3 *If $\hat{\Pi}\Pi$ is invertible from $\hat{\Pi}(X)$ into $\hat{\Pi}(X)$ then $\hat{\Pi}$ is onto from $\Pi(X)$ into $\hat{\Pi}(X)$.*

Proof of Claim 1.2.3 Let $y \in \widehat{\Pi}(X)$. Since the map $\widehat{\Pi}\Pi$ is invertible on $\widehat{\Pi}(X)$, there exists a unique $x \in \widehat{\Pi}(X)$ such that

$$\widehat{\Pi}\Pi x = y.$$

Therefore by setting $\bar{x} = \Pi x \in \Pi(X)$ we have

$$\widehat{\Pi}\bar{x} = y,$$

which implies the surjectivity of $\widehat{\Pi}$ from $\Pi(X)$ into $\widehat{\Pi}(X)$.

Claim 1.2.4 *If $\Pi\widehat{\Pi}$ is invertible from $\Pi(X)$ into $\Pi(X)$ then $\widehat{\Pi}$ is one to one from $\Pi(X)$ into $\widehat{\Pi}(X)$.*

Proof of Claim 1.2.4 Let $x \in \Pi(X)$ such that $\widehat{\Pi}x = 0$. Then we have

$$\widehat{\Pi}x = 0 \implies \Pi\widehat{\Pi}x = 0.$$

Since $\Pi\widehat{\Pi}$ is invertible from $\Pi(X)$ into $\Pi(X)$ we deduce that $x = 0$.

Let us now prove that $\widehat{\Pi}\Pi$ is invertible from $\widehat{\Pi}(X)$ into $\widehat{\Pi}(X)$. Indeed we have

$$\widehat{\Pi}\Pi = I - (I - \widehat{\Pi}\Pi)$$

hence it is sufficient to prove that

$$\|I - \widehat{\Pi}\Pi\|_{\mathcal{L}(\widehat{\Pi}(X))} < 1. \quad (1.2.4)$$

Let $x \in \widehat{\Pi}(X)$. We have

$$\begin{aligned} x - \widehat{\Pi}\Pi x &= \widehat{\Pi}x - \widehat{\Pi}\Pi x \\ &= \widehat{\Pi}x - \Pi x + \Pi\Pi x - \widehat{\Pi}\Pi x \\ &= [\widehat{\Pi} - \Pi]x + [\widehat{\Pi} - \Pi]\Pi x. \end{aligned}$$

thus

$$\|x - \widehat{\Pi}\Pi x\| \leq \|\widehat{\Pi} - \Pi\|_{\mathcal{L}(X)} \|x\| + \|\widehat{\Pi} - \Pi\|_{\mathcal{L}(X)} \|\Pi x\|$$

and

$$\|x - \widehat{\Pi}\Pi x\| \leq \delta \|x\| + \delta \|\Pi x\|.$$

Since $x \in \widehat{\Pi}(X)$ we have

$$\begin{aligned} \Pi x &= \Pi x - x + x \\ &= \Pi x - \widehat{\Pi}x + x \end{aligned}$$

hence

$$\|\Pi x\| \leq \left\| \Pi - \widehat{\Pi} \right\|_{\mathcal{L}(X)} \|x\| + \|x\|$$

and

$$\|\Pi x\| < (1 + \delta) \|x\|.$$

We obtain

$$\left\| x - \widehat{\Pi} \Pi x \right\| < \delta (2 + \delta) \|x\|.$$

and since $\delta \in (0, \sqrt{2} - 1)$, we have

$$\delta (2 + \delta) < 1,$$

we deduce that $\widehat{\Pi} \Pi$ is invertible from $\widehat{\Pi}(X)$ into $\widehat{\Pi}(X)$. By symmetry it follows that $\Pi \widehat{\Pi}$ is also invertible from $\Pi(X)$ into $\Pi(X)$.

To conclude the proof let us estimate the norm of the inverse of $\Pi|_{\widehat{\Pi}(X)}$. Let $x \in \widehat{\Pi}(X)$. We have

$$\begin{aligned} \Pi x &= \Pi x - x + x \\ &= \Pi x - \widehat{\Pi} x + x \end{aligned}$$

thus

$$\begin{aligned} \|\Pi x\| &\geq \|x\| - \left\| \widehat{\Pi} x - \Pi x \right\| \\ &\geq \|x\| - \left\| \widehat{\Pi} - \Pi \right\|_{\mathcal{L}(X)} \|x\| \\ &\geq (1 - \delta) \|x\|, \end{aligned}$$

and the result follows. ■

1.2.2 Derivation of the fixed point problem

Recall the discrete time variation of constant formula. Namely we have

$$\begin{aligned} (A + B)^n &= A(A + B)^{n-1} + B(A + B)^{n-1} \\ &= A^2(A + B)^{n-2} + AB(A + B)^{n-2} + B(A + B)^{n-1} \end{aligned}$$

thus by induction

$$(A + B)^n = A^n + A^{n-1}B(A + B) + \dots + AB(A + B)^{n-2} + B(A + B)^{n-1}, \quad (1.2.5)$$

so that for each $n \geq p$ we obtain

$$(A + B)^{n-p} = A^{n-p} + \sum_{m=p}^{n-1} A^{n-m-1} B (A + B)^{m-p}. \quad (1.2.6)$$

In the sequel and throughout this work we shall use the following summation convention:

$$\sum_n^m = 0 \text{ if } m < n.$$

This notational convention is similar to the one used by Vanderbauwhede in [159] who specified this using the symbol $\sum^{(+)}$.

Then using the above constant variation formula, one obtains the following fixed point formulation for a perturbed trichotomic semiflows:

Lemma 1.2.5 *Let $A \in \mathcal{L}(X)$ be given such that it has an exponential trichotomy with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three projectors Π_k , $k = s, c, u$. Let $B \in \mathcal{L}(X)$ be given such that $A + B$ has an exponential trichotomy with constant $\hat{\kappa}$, exponents $0 < \hat{\rho}_0 < \hat{\rho}$ such that $\rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and associated to the three projectors $\hat{\Pi}_k$, $k = s, c, u$. Then one has for each $n \in \mathbb{N}$,*

$$\begin{aligned} (A + B)_s^n \hat{\Pi}_s &= A_s^n \Pi_s \hat{\Pi}_s + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_s^m \hat{\Pi}_s \\ &\quad - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B (A + B)_s^{n+m} \hat{\Pi}_s, \end{aligned} \quad (1.2.7)$$

$$\begin{aligned} (A + B)_u^{-n} \hat{\Pi}_u &= A_u^{-n} \Pi_u \hat{\Pi}_u \\ &\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B (A + B)_u^{m-n} \hat{\Pi}_u \\ &\quad + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B (A + B)_u^{-m-1-n} \hat{\Pi}_u, \end{aligned} \quad (1.2.8)$$

$$\begin{aligned} (A + B)_c^n \hat{\Pi}_c &= A_c^n \Pi_c \hat{\Pi}_c + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B (A + B)_c^m \hat{\Pi}_c \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m+n} \hat{\Pi}_c \\ &\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1+n} \hat{\Pi}_c. \end{aligned} \quad (1.2.9)$$

$$\begin{aligned}
(A+B)_c^{-n} \widehat{\Pi}_c &= A_c^{-n} \Pi_c \widehat{\Pi}_c - \sum_{m=0}^{n-1} A_c^{-m-1} \Pi_c B (A+B)_c^{m-n} \widehat{\Pi}_c \\
&\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A+B)_c^{m-n} \widehat{\Pi}_c \\
&\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A+B)_c^{-m-1-n} \widehat{\Pi}_c.
\end{aligned} \tag{1.2.10}$$

$$\begin{aligned}
\widehat{\Pi}_s &= \Pi_s - \sum_{m=0}^{+\infty} A_s^m \Pi_s B \left[(A+B)_u^{-m-1} \widehat{\Pi}_u + (A+B)_c^{-m-1} \widehat{\Pi}_c \right] \\
&\quad - \sum_{m=0}^{+\infty} \left[A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c \right] B (A+B)_s^m \widehat{\Pi}_s,
\end{aligned} \tag{1.2.11}$$

$$\begin{aligned}
\widehat{\Pi}_u &= \Pi_u + \sum_{m=0}^{+\infty} \left[A_c^m \Pi_c + A_s^m \Pi_s \right] B (A+B)_u^{-m-1} \widehat{\Pi}_u \\
&\quad + \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B \left[(A+B)_s^m \widehat{\Pi}_s + (A+B)_c^m \widehat{\Pi}_c \right],
\end{aligned} \tag{1.2.12}$$

and

$$\begin{aligned}
\widehat{\Pi}_c &= I - \widehat{\Pi}_s - \widehat{\Pi}_u \\
&= \Pi_c - \sum_{m=0}^{+\infty} \left[A_c^m \Pi_c + A_s^m \Pi_s \right] B (A+B)_u^{-m-1} \widehat{\Pi}_s \\
&\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B \left[(A+B)_s^m \widehat{\Pi}_s + (A+B)_c^m \widehat{\Pi}_c \right] \\
&\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B \left[(A+B)_u^{-m-1} \widehat{\Pi}_u + (A+B)_c^{-m-1} \widehat{\Pi}_c \right] \\
&\quad + \sum_{m=0}^{+\infty} \left[A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c \right] B (A+B)_s^m \widehat{\Pi}_s
\end{aligned} \tag{1.2.13}$$

Proof. Derivation of formula (1.2.11) for $\widehat{\Pi}_s$: By applying $\widehat{\Pi}_s$ on the right side of (1.2.6) we obtain

$$(A+B)_s^{n-p} \widehat{\Pi}_s = A^{n-p} \widehat{\Pi}_s + \sum_{m=p}^{n-1} A^{n-m-1} B (A+B)_s^{m-p} \widehat{\Pi}_s, \quad \forall n \geq p. \tag{1.2.14}$$

By fixing $p = 0$ and by applying $A_u^{-n}\Pi_u$ on the left side of the above formula we obtain

$$A_u^{-n}\Pi_u (A+B)_s^n \widehat{\Pi}_s = \Pi_u \widehat{\Pi}_s + \sum_{m=0}^{n-1} A_u^{-m-1}\Pi_u B (A+B)_s^m \widehat{\Pi}_s. \quad (1.2.15)$$

Since for each $n \geq 0$ one has

$$\|A_u^{-n}\Pi_u\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho n} \|\Pi_u\|_{\mathcal{L}(X)} \quad \text{and} \quad \|(A+B)_s^n \widehat{\Pi}_s\|_{\mathcal{L}(X)} \leq \kappa e^{-\widehat{\rho} n} \|\widehat{\Pi}_s\|_{\mathcal{L}(X)},$$

by letting n go to $+\infty$ in (1.2.15) it follows that

$$\Pi_u \widehat{\Pi}_s = - \sum_{m=0}^{+\infty} A_u^{-m-1}\Pi_u B (A+B)_s^m \widehat{\Pi}_s. \quad (1.2.16)$$

By fixing $p = 0$ and by applying $A_c^{-n}\Pi_c$ (instead of $A_u^{-n}\Pi_u$) on the left side of (1.2.14) we obtain

$$\Pi_c \widehat{\Pi}_s = - \sum_{m=0}^{+\infty} A_c^{-m-1}\Pi_c B (A+B)_s^m \widehat{\Pi}_s. \quad (1.2.17)$$

Then combining (1.2.16) and (1.2.17) leads us to

$$\begin{aligned} \widehat{\Pi}_s &= \Pi_s \widehat{\Pi}_s + \Pi_u \widehat{\Pi}_s + \Pi_c \widehat{\Pi}_s \\ &= \Pi_s \widehat{\Pi}_s - \sum_{m=0}^{+\infty} [A_u^{-m-1}\Pi_u + A_c^{-m-1}\Pi_c] B (A+B)_s^m \widehat{\Pi}_s. \end{aligned} \quad (1.2.18)$$

It thus remains to reformulate $\Pi_s \widehat{\Pi}_s$ by using

$$\Pi_s \widehat{\Pi}_s = \Pi_s \left[I - \widehat{\Pi}_u - \widehat{\Pi}_c \right].$$

Therefore we will compute $\Pi_s \widehat{\Pi}_u$ and $\Pi_s \widehat{\Pi}_c$.

Computation of $\Pi_s \widehat{\Pi}_u$: By applying $\widehat{\Pi}_u$ on the right side of (1.2.6) we have

$$(A+B)_u^{n-p} \widehat{\Pi}_u = A^{n-p} \widehat{\Pi}_u + \sum_{m=p}^{n-1} A^{n-m-1} B (A+B)_u^{m-p} \widehat{\Pi}_u, \quad \forall n \geq p. \quad (1.2.19)$$

By applying Π_s on the left side of the above formula we obtain

$$\Pi_s (A+B)_u^{n-p} \widehat{\Pi}_u = A_s^{n-p} \Pi_s \widehat{\Pi}_u + \sum_{m=p}^{n-1} A_s^{n-m-1} \Pi_s B (A+B)_u^{m-p} \widehat{\Pi}_u, \quad \forall n \geq p, \quad (1.2.20)$$

and by applying $(A+B)_u^{p-n} \widehat{\Pi}_u$ on the right side of (1.2.20) we have

$$\Pi_s \widehat{\Pi}_u = A_s^{n-p} \Pi_s (A+B)_u^{p-n} \widehat{\Pi}_u + \sum_{m=p}^{n-1} A_s^{n-m-1} \Pi_s B (A+B)_u^{m-n} \widehat{\Pi}_u, \quad \forall n \geq p, \quad (1.2.21)$$

and since

$$\|A_s^{n-p}\Pi_s\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)} \|\Pi_c\|_{\mathcal{L}(X)}$$

and

$$\|(A+B)_u^{p-n}\widehat{\Pi}_u\|_{\mathcal{L}(X)} \leq \kappa e^{-\widehat{\rho}(n-p)} \|\widehat{\Pi}_u\|_{\mathcal{L}(X)},$$

by taking the limit when p go to $-\infty$ in (1.2.21) yields

$$\Pi_s \widehat{\Pi}_u = \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A+B)_u^{-m-1} \widehat{\Pi}_u. \quad (1.2.22)$$

Computation of $\Pi_s \widehat{\Pi}_c$: Starting from the equality

$$(A+B)_c^{n-p} \widehat{\Pi}_c = A^{n-p} \widehat{\Pi}_c + \sum_{m=p}^{n-1} A^{n-m-1} B (A+B)_c^{m-p} \widehat{\Pi}_c, \quad \forall n \geq p.$$

and applying $(A+B)_c^{p-n} \widehat{\Pi}_u$ on the right side of this formula we obtain for each $n \geq p$

$$\Pi_s \widehat{\Pi}_c = A_s^{n-p} \Pi_s (A+B)_c^{p-n} \widehat{\Pi}_u + \sum_{k=p}^{n-1} A_s^{n-m-1} \Pi_s B (A+B)_c^{m-n} \widehat{\Pi}_c, \quad \forall n \geq p. \quad (1.2.23)$$

and since

$$\|A_s^{n-p}\Pi_s\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)} \|\Pi_c\|_{\mathcal{L}(X)},$$

and

$$\|(A+B)_u^{p-n}\widehat{\Pi}_u\|_{\mathcal{L}(X)} \leq \kappa e^{\widehat{\rho}_0(n-p)} \|\widehat{\Pi}_u\|_{\mathcal{L}(X)},$$

with $\widehat{\rho}_0 < \rho$, by letting p go to $-\infty$ into (1.2.23) we derive

$$\Pi_s \widehat{\Pi}_c = \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A+B)_c^{-m-1} \widehat{\Pi}_c. \quad (1.2.24)$$

Computation of $\Pi_s \widehat{\Pi}_s$: By summing (1.2.22) and (1.2.24) it follows that

$$\Pi_s [\widehat{\Pi}_c + \widehat{\Pi}_u] = \sum_{m=0}^{+\infty} A_s^m \Pi_s B \left[(A+B)_c^{-m-1} \widehat{\Pi}_c + (A+B)_u^{-m-1} \widehat{\Pi}_u \right], \quad (1.2.25)$$

and since $\widehat{\Pi}_c + \widehat{\Pi}_u = I - \widehat{\Pi}_s$ it follows that

$$\Pi_s \widehat{\Pi}_s = \Pi_s - \sum_{m=0}^{+\infty} A_s^m \Pi_s B \left[(A+B)_c^{-m-1} \widehat{\Pi}_c + (A+B)_u^{-m-1} \widehat{\Pi}_u \right]. \quad (1.2.26)$$

Finally the expression of $\widehat{\Pi}_s$ in (1.2.11) follows by combining (1.2.18) and (1.2.26).

Computation of $\widehat{\Pi}_u$ and $\widehat{\Pi}_c$: The derivation of the formula (1.2.12) for $\widehat{\Pi}_u$ uses the same arguments as for $\widehat{\Pi}_s$. The formula (1.2.13) for $\widehat{\Pi}_c$ is obtained by using $\widehat{\Pi}_c = I - \widehat{\Pi}_s - \widehat{\Pi}_u$.
Computation of $(A + B)_s^n \widehat{\Pi}_s$: Next we derive (1.2.7). By applying $(A + B)_s^n \widehat{\Pi}_s$ on the right side of (1.2.11) we obtain

$$(A + B)_s^n \widehat{\Pi}_s = \Pi_s (A + B)_s^n \widehat{\Pi}_s - \sum_{m=0}^{+\infty} [A_u^{-m-1} + A_c^{-m-1}] B (A + B)_s^{m+n} \widehat{\Pi}_s. \quad (1.2.27)$$

In order to determine $\Pi_s (A + B)_s^n \widehat{\Pi}_s$, we apply Π_s on the left side of (1.2.14) and we obtain

$$\Pi_s (A + B)_s^n \widehat{\Pi}_s = A_s^{n-p} \Pi_s \widehat{\Pi}_s + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_s^m \widehat{\Pi}_s, \quad \forall n \in \mathbb{N}, \quad (1.2.28)$$

and (1.2.7) follows.

Computation of $(A + B)_c^n \widehat{\Pi}_c$ for $n \geq 0$: By applying $(A + B)_c^n \widehat{\Pi}_c$ on the right side of (1.2.13) we obtain for each $n \in \mathbb{N}$

$$\begin{aligned} (A + B)_c^n \widehat{\Pi}_c &= \Pi_c (A + B)_c^n \widehat{\Pi}_c - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m+n} \widehat{\Pi}_c \\ &\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1+n} \widehat{\Pi}_c. \end{aligned} \quad (1.2.29)$$

Next we compute $\Pi_c (A + B)_c^n \widehat{\Pi}_c$. By using the variation of constant formula (1.2.6) with $p = 0$, and applying $\widehat{\Pi}_c$ on the right side and Π_c on the left side we obtain

$$\Pi_c (A + B)_c^n \widehat{\Pi}_c = A_c^n \Pi_c \widehat{\Pi}_c + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B (A + B)_c^m \widehat{\Pi}_c, \quad (1.2.30)$$

and (1.2.9) follows.

Computation of $(A + B)_c^n \widehat{\Pi}_c$ for $n \leq 0$: By applying $(A + B)_c^{-n} \widehat{\Pi}_c$ on the right side of (1.2.13) we obtain

$$\begin{aligned} (A + B)_c^{-n} \widehat{\Pi}_c &= \Pi_c (A + B)_c^{-n} \widehat{\Pi}_c \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m-n} \widehat{\Pi}_c \\ &\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1-n} \widehat{\Pi}_c. \end{aligned} \quad (1.2.31)$$

Next we compute $\Pi_c (A + B)_c^{-n} \widehat{\Pi}_c$. By applying $(A + B)_c^{-n} \widehat{\Pi}_c$ on the right side of the variation of constant formula (1.2.6) (with $p = 0$) we obtain

$$\widehat{\Pi}_c = A^n (A + B)_c^{-n} \widehat{\Pi}_c + \sum_{m=0}^{n-1} A^{n-m-1} B (A + B)_c^{m-n} \widehat{\Pi}_c. \quad (1.2.32)$$

By applying $A_c^{-n} \Pi_c$ on the left side of the above formula we get

$$A_c^{-n} \Pi_c \widehat{\Pi}_c = \Pi_c (A + B)_c^{-n} \widehat{\Pi}_c + \sum_{m=0}^{n-1} A_c^{-m-1} \Pi_c B (A + B)_c^{m-n} \widehat{\Pi}_c, \quad (1.2.33)$$

and the result follows. ■

1.2.3 Abstract reformulation of the fixed point problem

In this section we reformulate the fixed point problem (1.2.7)-(1.2.12) by using an abstract fixed point formulation. Let $\eta > 0$. Define

$$\mathbb{L}_{-\eta}(\mathbb{N}, \mathcal{L}(X)) := \left\{ \mathbf{u} : \mathbb{N} \rightarrow \mathcal{L}(X) : \sup_{n \in \mathbb{N}} e^{\eta n} \|u_n\| < +\infty \right\},$$

which is a Banach space endowed with the norm

$$\|\mathbf{u}\|_{\mathbb{L}_{-\eta}} := \sup_{n \in \mathbb{N}} e^{\eta n} \|u_n\|.$$

Define

$$\mathbb{L}_{\eta}(\mathbb{Z}, \mathcal{L}(X)) := \left\{ \mathbf{v} : \mathbb{Z} \rightarrow \mathcal{L}(X) : \sup_{n \in \mathbb{Z}} e^{-\eta |n|} \|v_n\|_{\mathcal{L}(X)} < +\infty \right\}$$

which is a Banach space endowed with the norm

$$\|\mathbf{v}\|_{\mathbb{L}_{\eta}} := \sup_{n \in \mathbb{Z}} e^{-\eta |n|} \|v_n\|_{\mathcal{L}(X)}.$$

Consider S_- the shift operators on $\mathbb{L}_{\pm\eta}(\mathbb{N}, \mathcal{L}(X))$

$$S_- (\mathbf{u})_n = u_{n+1} \text{ whenever } n \in \mathbb{Z} \text{ or } n \in \mathbb{N}.$$

Let $C \in \mathcal{L}(X)$. In the following we will use the linear operators

$$\Phi_C(\mathbf{u})_n = \sum_{m=0}^{n-1} C^m u_{n-1-m}, \text{ and } \Theta_C(\mathbf{u})_n = \sum_{m=0}^{+\infty} C^m u_{n+m}.$$

Reformulations of equation (1.2.7) on $\widehat{X}_s$: Set

$$E_n^s := (A + B)_s^n \widehat{\Pi}_s, \quad n \in \mathbb{N},$$

we require

$$E^s \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

where $\hat{\rho}$ is the constant introduced in Theorem 1.1.4. Consider the linear operator $\Phi_s : \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$

$$\Phi_s = \Phi_{A_s \Pi_s}$$

and

$$\Theta_{cu} = \Theta_{(A_c^{-1} \Pi_c + A_u^{-1} \Pi_u)}.$$

We observe that

$$\Phi_s \circ B \circ (E^s)_n = \sum_{l=0}^{n-1} A_s^l \Pi_s B E_n^s (A + B)_s^{n-1-l} \hat{\Pi}_s = \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_s^m \hat{\Pi}_s$$

therefore the equation (1.2.7) can be rewritten for $n \in \mathbb{N}$ as

$$E_n^s = A_s^n \Pi_s \hat{\Pi}_s + \Phi_s \circ B \circ (E^s)_n - \Theta_{cu}((A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) B E^s)_n. \quad (1.2.34)$$

In order to solve the fixed point problem we will use the following lemma.

Lemma 1.2.6 *The operators Φ_s and Θ_{cu} map $\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ into itself and are bounded linear operators on $\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$. More precisely we have*

$$\|\Phi_s(\mathbf{u})\|_{\mathbb{L}_{-\hat{\rho}}} \leq \frac{\kappa e^{\hat{\rho}}}{1 - e^{\hat{\rho} - \rho}} \|\mathbf{u}\|_{\mathbb{L}_{-\hat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

and

$$\|\Theta_{cu}(\mathbf{u})\|_{\mathbb{L}_{-\hat{\rho}}} \leq \left[\frac{\kappa}{1 - e^{\rho_0 - \hat{\rho}}} + \frac{\kappa}{1 - e^{-(\rho + \hat{\rho})}} \right] \|\mathbf{u}\|_{\mathbb{L}_{-\hat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)).$$

Reformulation of equation (1.2.8) on $\hat{X}_u$: Set

$$E_n^u := (A + B)_u^{-n} \hat{\Pi}_u, \quad n \in \mathbb{N},$$

we require

$$E^u \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

where $\hat{\rho}$ is the constant introduced in Theorem 1.1.4. Consider the linear operator $\Phi_u : \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$

$$\Phi_u := \Phi_{A_u^{-1} \Pi_u}$$

and $\Theta_{sc} : \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$

$$\Theta_{sc} := \Theta_{(A_s \Pi_s + A_c \Pi_c)}.$$

We observe that

$$\begin{aligned}\Phi_u \circ (A_u^{-1} \Pi_u B) \circ S_- (E^u)_n &= \sum_{m=0}^{n-1} A_u^{-m} \Pi_u (A_u^{-1} \Pi_u B) E_{n-m}^u \\ &= \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B (A + B)_u^{m-n} \widehat{\Pi}_u\end{aligned}$$

therefore equation (1.2.8) can be rewritten for each $n \in \mathbb{N}$ as

$$E_n^u = A_u^{-n} \Pi_u \widehat{\Pi}_u - \Phi_u \circ (A_u^{-1} \Pi_u B) \circ S_- (E^u)_n + \Theta_{sc} \circ B \circ S_- (E^u)_n. \quad (1.2.35)$$

Lemma 1.2.7 *The operators Φ_u and Θ_{sc} map $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ into itself and are bounded linear operators on $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$. More precisely we have*

$$\|\Phi_u(\mathbf{u})\|_{\mathbb{L}_{-\widehat{\rho}}} \leq \frac{\kappa e^{\widehat{\rho}}}{1 - e^{\widehat{\rho}-\rho}} \|\mathbf{u}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

and

$$\|\Theta_{sc}(\mathbf{u})\|_{\mathbb{L}_{-\widehat{\rho}}} \leq \left[\frac{\kappa}{1 - e^{-(\rho+\widehat{\rho})}} + \frac{\kappa}{1 - e^{\rho_0-\widehat{\rho}}} \right] \|\mathbf{u}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)).$$

Reformulation of equation (1.2.9)-(1.2.10) on $\widehat{X}_c$: Set

$$E_n^c := (A + B)_c^n \widehat{\Pi}_c, \quad n \in \mathbb{Z}$$

we require

$$E^c \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)).$$

Define

$$\Phi_c(\mathbf{u})_n := \begin{cases} \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c u_m, & \text{if } n \geq 0 \\ - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c u_{m+n}, & \text{if } n \leq 0 \end{cases}$$

and

$$\Theta_{su}(\mathbf{u})_n := - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u u_{m+n} + \sum_{m=0}^{+\infty} A_s^m \Pi_s u_{n-1-m}, \quad \text{for } n \in \mathbb{Z},$$

therefore equations (1.2.9)-(1.2.10) can be rewritten for each $n \in \mathbb{Z}$ as

$$E_n^c := A_c^n \Pi_c \left[I - \left(\widehat{\Pi}_s + \widehat{\Pi}_u \right) \right] + \Phi_c(BE^c)_n + \Theta_{su}(BE^c)_n. \quad (1.2.36)$$

Lemma 1.2.8 *The operators Φ_c and Θ_{su} map $\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ into itself and are bounded linear operators on $\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. More precisely we have*

$$\|\Phi_c(\mathbf{v})\|_{\mathbb{L}_{\hat{\rho}_0}} \leq \frac{\kappa}{1 - e^{\rho_0 - \hat{\rho}_0}} \|\mathbf{v}\|_{\mathbb{L}_{\hat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)),$$

and

$$\|\Theta_{su}(\mathbf{v})\|_{\mathbb{L}_{\hat{\rho}_0}} \leq \left[\frac{\kappa e^{\hat{\rho}}}{1 - e^{\hat{\rho}_0 - \hat{\rho}}} + \frac{\kappa e^{-\hat{\rho}}}{1 - e^{\hat{\rho}_0 - \hat{\rho}}} \right] \|\mathbf{v}\|_{\mathbb{L}_{\hat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)).$$

Reformulation of equation (1.2.11)-(1.2.12) for the projectors on $\hat{X}_s$ and $\hat{X}_u$: Define the linear operator

$$\Theta_s(u) := \Theta_{A_s \Pi_s}(u)_0 = \sum_{m=0}^{+\infty} A_s^m \Pi_s u_m$$

then equation (1.2.11) becomes

$$\hat{\Pi}_s = \Pi_s - \Theta_s \circ B \circ S_-(E^u + \chi_-(E^c)) + \Theta_{cu}((A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) B E^s)_0 \quad (1.2.37)$$

where $\chi_- : \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ is defined by

$$\chi_-(E^c)_n = E_{-n}^c \text{ for } n \geq 0.$$

Define

$$\Theta_u(u) := \Theta_{A_u^{-1} \Pi_u}(u)_0 = \sum_{m=0}^{+\infty} A_u^{-m} \Pi_u u_m,$$

$$\hat{\Pi}_u = \Pi_u + \Theta_{sc} \circ B \circ S_-(E^u)_0 + \Theta_u(A_u^{-1} \Pi_u B (E^s + \chi_+(E^c))) \quad (1.2.38)$$

where $\chi_+ : \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ is defined by

$$\chi_+(E^c)_n := E_n^c \text{ for } n \geq 0.$$

Lemma 1.2.9 *The operators Θ_s and Θ_u have the following properties:*

(i) Θ_s and Θ_u map $\mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ into $\mathcal{L}(X)$ with

$$\|\Theta_s(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{\hat{\rho}_0 - \rho}} \|\mathbf{v}\|_{\mathbb{L}_{\hat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X)),$$

and

$$\|\Theta_u(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{\hat{\rho}_0 - \rho}} \|\mathbf{v}\|_{\mathbb{L}_{\hat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X));$$

(ii) Θ_s and Θ_u map $\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \subset \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ into $\mathcal{L}(X)$ with

$$\|\Theta_s(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{-\hat{\rho}-\rho}} \|\mathbf{v}\|_{\mathbb{L}_{-\hat{\rho}}}, \quad \forall \mathbf{v} \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

$$\|\Theta_u(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{-\hat{\rho}-\rho}} \|\mathbf{v}\|_{\mathbb{L}_{-\hat{\rho}}}, \quad \forall \mathbf{v} \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)).$$

By using the expressions of $\hat{\Pi}_s$ and $\hat{\Pi}_u$ obtained in (1.2.37) and (1.2.38), and by replacing those expressions into $A_s^n \Pi_s \hat{\Pi}_s$ (respectively into $A_u^{-n} \Pi_u \hat{\Pi}_u$ and $A_c^n \Pi_c [I - (\hat{\Pi}_s + \hat{\Pi}_u)]$) in equation (1.2.34) (respectively in (1.2.35) and (1.2.36)) we will derive a new fixed point problem only for E^s , E^u and E^c given explicitly as follow

$$\begin{aligned} E_n^s &= A_s^n \Pi_s - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B [E_{m+1}^u + E_{-m-1}^c] \\ &\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B E_m^s - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_{n+m}^s, \end{aligned} \quad (1.2.39)$$

$$\begin{aligned} E_n^u &= A_u^{-n} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s + E_m^c] \\ &\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B E_{n+m+1}^u, \end{aligned} \quad (1.2.40)$$

and for each $n \in \mathbb{Z}$

$$\begin{aligned} E_n^c &= A_c^n \Pi_c - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B E_{m+1}^u + \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B E_m^s \\ &\quad + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B E_m^c - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B E_{n+m}^c \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_{n+m}^c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1+n}^c. \end{aligned} \quad (1.2.41)$$

Moreover with this notation the explicit formulas for $\hat{\Pi}_k$, $k = s, c, u$ read as

$$\begin{aligned} \hat{\Pi}_s &= \Pi_s - \sum_{m=0}^{+\infty} A_s^m \Pi_s B [E_{m+1}^u + E_{-m-1}^c] \\ &\quad - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_m^s, \end{aligned} \quad (1.2.42)$$

$$\begin{aligned}\widehat{\Pi}_u &= \Pi_u + \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B [E_m^s + E_m^c] \\ &\quad + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] E_{m+1}^u,\end{aligned}\tag{1.2.43}$$

and

$$\begin{aligned}\widehat{\Pi}_c &= \Pi_c - \sum_{m=0}^{+\infty} A_c^m \Pi_c B E_{m+1}^u + \sum_{m=0}^{+\infty} A_c^{-m-1} \Pi_c B E_m^s \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_m^c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1}^c.\end{aligned}\tag{1.2.44}$$

Observe that we have the following relation

$$E_0^k = \widehat{\Pi}_k, \quad k = s, c, u,\tag{1.2.45}$$

and the following identity

$$E_0^s + E_0^c + E_0^u = \widehat{\Pi}_s + \widehat{\Pi}_c + \widehat{\Pi}_u = I.\tag{1.2.46}$$

Furthermore the system (1.2.39)-(1.2.41) also re-writes as the following compact form

$$\begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} = \begin{pmatrix} A_s \Pi_s \\ A_u^{-1} \Pi_u \\ A_c \Pi_c \end{pmatrix} + \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{pmatrix} \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix},\tag{1.2.47}$$

wherein the linear operators $\{\mathcal{J}_{ij}\}_{1 \leq i,j \leq 3}$ are given by

$$\begin{aligned}\mathcal{J}_{11} &:= A_s \Pi_s \circ \Theta_{cu}(\cdot)_0 \circ (A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) \circ B + \Phi_s \circ B \\ &\quad - \Theta_{cu} \circ (A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) \circ B \\ \mathcal{J}_{12} &:= -A_s \Pi_s \circ \Theta_s \circ B \circ S_- \\ \mathcal{J}_{13} &:= -A_s \Pi_s \circ \Theta_s \circ B \circ S_- \circ \chi_- \\ \mathcal{J}_{21} &:= A_u^{-1} \Pi_u \circ \Theta_u \circ A_u^{-1} \Pi_u \circ B \\ \mathcal{J}_{22} &:= A_u^{-1} \Pi_u \circ \Theta_{sc}(\cdot)_0 \circ B \circ S_- - \Phi_u \circ A_u^{-1} \Pi_u \circ B \circ S_- \\ &\quad + \Theta_{sc} \circ B \circ S_- \\ \mathcal{J}_{23} &:= A_u^{-1} \Pi_u \circ \Theta_u \circ A_u^{-1} \Pi_u \circ B \circ \chi_+ \\ \mathcal{J}_{31} &:= -A_c \Pi_c \circ \Theta_{cu}(\cdot)_0 \circ (A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) \circ B \\ &\quad - A_c \Pi_c \circ \Theta_u(\cdot) \circ A_u^{-1} \Pi_u \circ B \\ \mathcal{J}_{32} &:= A_c \Pi_c \circ \Theta_s \circ B \circ S_- - A_c \Pi_c \circ \Theta_{sc}(\cdot)_0 \circ B \circ S_- \\ \mathcal{J}_{33} &:= A_c \Pi_c \circ \Theta_s \circ B \circ S_- \circ \chi_- \\ &\quad A_c \Pi_c \circ \Theta_u \circ A_u^{-1} \Pi_u \circ B \circ \chi_+ + \Phi_c \circ B + \Theta_{su} \circ B.\end{aligned}$$

Therefore one can define a matrix of bounded linear operators $\mathcal{J}$ on $\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ by

$$\mathcal{J} := \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{pmatrix}. \quad (1.2.48)$$

In the sequel we define a norm in $\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ by

$$\left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\| = \max \left\{ \|E^s\|_{\mathbb{L}_{-\hat{\rho}}}, \|E^u\|_{\mathbb{L}_{-\hat{\rho}}}, \|E^c\|_{\mathbb{L}_{\hat{\rho}_0}} \right\}.$$

Lemma 1.2.10 *Let the conditions of Theorem 1.1.4 be satisfied. Then the linear operator $\mathcal{J}$ defined in (1.2.48) is bounded on $\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. More precisely there exists a constant $C := C(\kappa, \rho, \rho_0, \hat{\rho}, \hat{\rho}_0)$ such that for each $E^s \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, $E^u \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ and $E^c \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ we have*

$$\left\| \mathcal{J} \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\| \leq C \|B\|_{\mathcal{L}(X)} \left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\|.$$

Proof. Let us note that χ_+ and χ_- are bounded linear operators defined from $\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ into $\mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$. Furthermore we have

$$\|\chi_+\|_{\mathcal{L}(\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)), \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X)))} \leq 1, \quad (1.2.49)$$

and

$$\|\chi_-\|_{\mathcal{L}(\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)), \mathbb{L}_{\hat{\rho}_0}(\mathbb{N}, \mathcal{L}(X)))} \leq 1. \quad (1.2.50)$$

We also note that $S_- \in \mathcal{L}(\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)))$ and $S_- \in \mathcal{L}(\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)))$ with

$$\|S_-\|_{\mathcal{L}(\mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)))} \leq e^{\hat{\rho}_0} \text{ and } \|S_-\|_{\mathcal{L}(\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)))} \leq 1. \quad (1.2.51)$$

Therefore by combining (1.2.49)-(1.2.51) with lemmas (1.2.6)-(1.2.9) the result follows easily by simple computations. $\blacksquare$

Remark 1.2.11 *Similarly to the definition of operator $\mathcal{J}$ (see (1.2.39)-(1.2.41)), one may define the linear operator $\hat{\mathcal{J}} \in \mathcal{L}(\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, X))$ by*

$$\hat{\mathcal{J}} \begin{pmatrix} f^s \\ f^u \\ f^c \end{pmatrix} = \begin{pmatrix} g^s \\ g^u \\ g^c \end{pmatrix},$$

such that for $n \geq 0$

$$\begin{aligned}
g_n^s &= - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B [f_{m+1}^u + f_{-m-1}^c] - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B f_{n+m}^s \\
&\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B f_m^s, \\
g_n^u &= \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [f_m^s + f_m^c] + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B f_{n+m+1}^u \\
&\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B f_{n-m}^u,
\end{aligned}$$

and for each $n \in \mathbb{Z}$

$$\begin{aligned}
g_n^c &= - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B f_{m+1}^u + \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B f_m^s \\
&\quad + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B f_m^c - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B f_{n+m}^c \\
&\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B f_{n+m}^c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B f_{-m-1+n}^c.
\end{aligned}$$

Note that one has for each $(f^s, f^u, f^c)^T \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, X)$:

$$\left\| \hat{\mathcal{J}} \begin{pmatrix} f^s \\ f^u \\ f^c \end{pmatrix} \right\|_{\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, X)} \leq C \|B\|_{\mathcal{L}(X)} \left\| \begin{pmatrix} f^s \\ f^u \\ f^c \end{pmatrix} \right\|_{\mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, X) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, X)},$$

and wherein the constant $C > 0$ is provided by Lemma 1.2.10.

As a consequence of the above lemma we obtain the following result:

Proposition 1.2.12 *Let the conditions of Theorem 1.1.4 be satisfied. Then there exists $\delta_0 := \delta_0(\kappa, \rho, \rho_0, \hat{\rho}, \hat{\rho}_0) \in (0, C^{-1})$ such that for each $\delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$ and each $B \in \mathcal{L}(X)$ with*

$$\|B\|_{\mathcal{L}(X)} \leq \delta,$$

there exists a unique $E^s \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, $E^u \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ and $E^c \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ such that (1.2.47) (or equivalently (1.2.39)-(1.2.41)) holds true. Moreover we have the following properties

(i) For each $n \in \mathbb{N}$

$$\|E_n^s\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta_0}{\delta_0 - \delta} e^{-\hat{\rho}n} \text{ and } \|E_n^u\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta_0}{\delta_0 - \delta} e^{-\hat{\rho}n},$$

and for each $n \in \mathbb{Z}$

$$\|E_n^c\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta_0}{\delta_0 - \delta} e^{\hat{\rho}_0|n|}.$$

(ii) The following estimates hold:

$$\|E_n^s - A_s^n \Pi_s\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\hat{\rho}n}, \quad n \in \mathbb{N},$$

$$\|E_n^u - A_u^{-n} \Pi_u\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\hat{\rho}n}, \quad n \in \mathbb{N},$$

and

$$\|E_n^c - A_c^n \Pi_c\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{\hat{\rho}_0|n|}, \quad n \in \mathbb{Z}.$$

Proof. Let $\delta_0 \in (0, C^{-1})$ be given. Assume that

$$\|B\|_{\mathcal{L}(X)} \leq \delta \text{ with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right). \quad (1.2.52)$$

Then since $\frac{\delta_0^2}{\kappa + \delta_0} \leq \delta_0$, the existence and the uniqueness of a fixed point of (1.2.47) (or equivalently (1.2.39)-(1.2.41)) follows from Lemma 1.2.10.

In order to obtain the properties (i) and (ii) we will make use of (1.2.47). First of all let us observe that $A_s \Pi_s \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, $A_u^{-1} \Pi_u \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ and $A_c \Pi_c \in \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ and by using (1.1.7)-(1.1.9) we obtain

$$\left\| \begin{pmatrix} A_s \Pi_s \\ A_u^{-1} \Pi_u \\ A_c \Pi_c \end{pmatrix} \right\| \leq \kappa. \quad (1.2.53)$$

Proof of (i): By using the fixed point problem (1.2.47) combined with Lemma 1.2.10 and (1.2.52) we obtain that

$$\left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\| \leq \kappa + C\delta \left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\|,$$

so that

$$\left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\| \leq \frac{\kappa}{1 - C\delta} \leq \frac{\kappa\delta_0}{\delta_0 - \delta}. \quad (1.2.54)$$

This prove (i).

Proof of (ii): By using the fixed point problem (1.2.47) combined with Lemma 1.2.10 and (1.2.52) we obtain that

$$\left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} - \begin{pmatrix} A_s^+ \Pi_s \\ A_u^- \Pi_u \\ A_c^+ \Pi_c \end{pmatrix} \right\| \leq C \delta \left\| \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \right\|, \quad (1.2.55)$$

so that plugging (1.2.54) into (1.2.55) yields

$$\left\| \begin{pmatrix} E^s - A_s^+ \Pi_s \\ E^u - A_u^- \Pi_u \\ E^c - A_c^+ \Pi_c \end{pmatrix} \right\| \leq \frac{\kappa C \delta}{1 - C \delta} \leq \frac{\kappa \delta}{\delta_0 - \delta}.$$

This prove (ii). ■

1.2.4 Regularized semigroup property and orthogonality property

Definition 1.2.13 A family of bounded linear operators $\{W_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X)$ is a discrete time regularized semigroup if

$$W_n W_p = W_{n+p}, \quad \forall n, p \in \mathbb{N}. \quad (1.2.56)$$

Remark 1.2.14 If $\{W_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X)$ is a regularized semigroup then W_0 is a bounded linear projector on X . In fact we have

$$W_0 W_0 = W_{0+0} = W_0.$$

Remark 1.2.15 Observe that if $\{W_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X)$ is a discrete time regularized semigroup then trivially by setting

$$C := W_0, \quad (1.2.57)$$

and using the property (1.2.56) we obtain

$$W_n W_p = C W_{n+p}. \quad (1.2.58)$$

The property (1.2.57) and (1.2.58) correspond to the notion of *C-regularized semigroup* given in [51, Definition 3.1 p.13] for discrete time.

In the next lemmas we will show that $\{E_n^k\}_{n \in \mathbb{N}}$, $k = s, c, u$, are regularized semigroups and that we have the orthogonality property namely for each $n \in \mathbb{N}$

$$E_n^k E_n^l = 0_{\mathcal{L}(X)} \text{ if } k, l = s, c, u \text{ with } k \neq l.$$

The latter equality will allows us to obtain that the bounded linear projectors $\widehat{\Pi}_k = E_0^k$, $k = s, c, u$ satisfy the orthogonality property

$$\widehat{\Pi}_k \widehat{\Pi}_l = 0_{\mathcal{L}(X)} \text{ if } k, l = s, c, u \text{ with } k \neq l.$$

Lemma 1.2.16 *Let the conditions of Theorem 1.1.4 be satisfied. If*

$$\|B\|_{\mathcal{L}(X)} \leq \delta, \text{ with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$$

where δ_0 is given in Proposition 1.2.12 then the following properties hold:

(i) *for each $n, p \in \mathbb{N}$ we have*

$$E_n^u E_p^u = E_{n+p}^u \text{ and } E_n^s E_p^u = 0_{\mathcal{L}(X)},$$

and for each $n \in \mathbb{Z}, p \in \mathbb{N}$ we have

$$E_n^c E_p^u = 0_{\mathcal{L}(X)}.$$

(ii) $\widehat{\Pi}_u$ *is a bounded linear projector on X .*

Proof. First of all let us note that since we have $\widehat{\Pi}_u = E_0^u$ the property (ii) is a direct consequence of the property (i). Therefore we will focus on the property (i).

The idea of this proof is to derive a suitable closed system of equations for the following three quantities (wherein $p \in \mathbb{N}$ is fixed)

$$\{E_n^u E_p^u - E_{n+p}^u\}_{n \in \mathbb{N}}, \{E_n^s E_p^u\}_{n \in \mathbb{N}} \text{ and } \{E_n^c E_p^u\}_{n \in \mathbb{Z}}.$$

Equation for $\{E_n^u E_p^u - E_{n+p}^u\}_{n \in \mathbb{N}}$: Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$ be given. Multiplying the right side of E_n^u given in (1.2.40) by E_p^u leads us to

$$\begin{aligned} E_n^u E_p^u &= A_u^{-n} \Pi_u E_p^u + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s E_p^u + E_m^c E_p^u] \\ &\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u E_p^u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B E_{n+m+1}^u E_p^u. \end{aligned} \quad (1.2.59)$$

Next by using also (1.2.40) and replacing n with $n + p$ we obtain

$$\begin{aligned} E_{n+p}^u &= A_u^{-n-p} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] \\ &\quad - \sum_{m=0}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B E_{n+p+m+1}^u. \end{aligned} \quad (1.2.60)$$

Therefore by subtracting (1.2.60) from (1.2.59) we get

$$\begin{aligned}
E_n^u E_p^u - E_{n+p}^u &= A_u^{-n} \Pi_u E_p^u - A_u^{-n-p} \Pi_u \\
&\quad - \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] \\
&\quad + \sum_{m=0}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u E_p^u \\
&\quad + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s E_p^u + E_m^c E_p^u] \\
&\quad + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B [E_{n+m+1}^u E_p^u - E_{n+p-m}^u].
\end{aligned} \tag{1.2.61}$$

Now note that by using (1.2.40), replacing n with p in order to obtain E_p^u and multiplying its left side by $A_u^{-n} \Pi_u$ we obtain

$$\begin{aligned}
A_u^{-n} \Pi_u E_p^u &= A_u^{-n-p} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] \\
&\quad - \sum_{m=0}^{p-1} [A_u^{-n-m-1} \Pi_u] B E_{p-m}^u,
\end{aligned}$$

and since we have

$$- \sum_{m=0}^{p-1} [A_u^{-n-m-1} \Pi_u] B E_{p-m}^u = - \sum_{m=n}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u,$$

it follows that

$$\begin{aligned}
A_u^{-n} \Pi_u E_p^u &= A_u^{-n-p} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] \\
&\quad - \sum_{m=n}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u.
\end{aligned} \tag{1.2.62}$$

Therefore by plugging the expression of $A_u^{-n}\Pi_u E_p^u$ given by (1.2.62) into (1.2.61) we obtain that

$$\begin{aligned} E_n^u E_p^u - E_{n+p}^u &= - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B [E_{n-m}^u E_p^u - E_{n+p-m}^u] \\ &\quad + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s E_p^u + E_m^c E_p^u] \\ &\quad + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B [E_{n+m+1}^u E_p^u - E_{n+p+m+1}^u]. \end{aligned} \quad (1.2.63)$$

Equation for $\{E_n^s E_p^u\}_{n \in \mathbb{N}}$: Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$ be given. Then by using (1.2.39) and multiplying the right side of E_n^s by E_p^u we obtain

$$\begin{aligned} E_n^s E_p^u &= A_s^n \Pi_s E_p^u - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B [E_{m+1}^u E_p^u + E_{-m-1}^c E_p^u] \\ &\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B E_m^s E_p^u - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_{n+m}^s E_p^u. \end{aligned} \quad (1.2.64)$$

Next by replacing n by p in (1.2.40) we obtain E_p^u and by multiplying its left side by $A_s^n \Pi_s$ we get

$$A_s^n \Pi_s E_p^u = \sum_{m=0}^{+\infty} A_s^{n+m} \Pi_s E_{p+m+1}^u. \quad (1.2.65)$$

Then plugging (1.2.65) into (1.2.64) yields

$$\begin{aligned} E_n^s E_p^u &= - \sum_{m=0}^{+\infty} A_s^{n+m} \Pi_s [E_{m+1}^u E_p^u - E_{p+m+1}^u] \\ &\quad - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B E_{-m-1}^c E_p^u \\ &\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B E_m^s E_p^u \\ &\quad - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_{n+m}^s E_p^u. \end{aligned} \quad (1.2.66)$$

Equation for $\{E_n^c E_p^u\}_{n \in \mathbb{Z}}$: Let $n \in \mathbb{Z}$ and $p \in \mathbb{N}$ be given. By multiplying the right side of (1.2.41) by E_p^u we get

$$\begin{aligned}
E_n^c E_p^u &= A_c^n \Pi_c E_p^u - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B E_{m+1}^u E_p^u + \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B E_m^s E_p^u \\
&\quad + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B E_m^c E_p^u - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B E_{n+m}^c E_p^u \\
&\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_{n+m}^c E_p^u + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1+n}^c E_p^u.
\end{aligned} \tag{1.2.67}$$

Next by replacing n by p in (1.2.40) we obtain E_p^u and by multiplying its left side by $A_c^n \Pi_c$ we get

$$A_c^n \Pi_c E_p^u = \sum_{m=0}^{+\infty} A_c^{n+m} \Pi_c E_{p+m+1}^u, \tag{1.2.68}$$

Therefore by plugging (1.2.68) into (1.2.67) we get

$$\begin{aligned}
E_n^c E_p^u &= - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B [E_{m+1}^u E_p^u - E_{p+m+1}^u] \\
&\quad + \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B E_m^s E_p^u + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B E_m^c E_p^u \\
&\quad - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B E_{n+m}^c E_p^u - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_{n+m}^c E_p^u \\
&\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1+n}^c E_p^u.
\end{aligned} \tag{1.2.69}$$

Now observe that by letting $p \in \mathbb{N}$ fixed and setting

$$\widehat{E}_n^u = E_n^u E_p^u - E_{n+p}^u, \quad n \in \mathbb{N},$$

$$\widehat{E}_n^s = E_n^s E_p^u, \quad n \in \mathbb{N},$$

and

$$\widehat{E}_n^c := E_n^c E_p^u, \quad n \in \mathbb{Z},$$

we obtain that $\widehat{E}^s \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, $\widehat{E}^u \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ and $\widehat{E}^c \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. We also observe by using (1.2.63), (1.2.66) and (1.2.69) that the sequences $\widehat{E}^s$, $\widehat{E}^u$, $\widehat{E}^c$ satisfy

$$\begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix} = \mathcal{J} \begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix},$$

where the bounded linear operator $\mathcal{J}$ is defined in (1.2.48). Hence we infer from Lemma 1.2.10 that

$$\left\| \begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix} \right\| \leq C \|B\|_{\mathcal{L}(X)} \left\| \begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix} \right\|$$

and since we have

$$C \|B\|_{\mathcal{L}(X)} \leq C\delta < 1,$$

it follows that

$$\widehat{E}_n^s = \widehat{E}_n^u = 0_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{N},$$

and

$$\widehat{E}_n^c = 0_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{Z}.$$

This completes the proof of this lemma. ■

Remark 1.2.17 *The arguments for the proof of the next two lemmas are similar to the arguments for the proof of Lemma 1.2.16. Therefore, in the sequel, we will only sketch the important steps.*

Lemma 1.2.18 *Let the conditions of Theorem 1.1.4 be satisfied. If*

$$\|B\|_{\mathcal{L}(X)} \leq \delta, \text{ with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$$

where δ_0 is given in Proposition 1.2.12 then the following properties hold:

(i) *for each $n, p \in \mathbb{N}$ we have*

$$E_n^s E_p^s = E_{n+p}^s \text{ and } E_n^u E_p^s = 0_{\mathcal{L}(X)},$$

and for each $n \in \mathbb{Z}, p \in \mathbb{N}$ we have

$$E_n^c E_p^s = 0_{\mathcal{L}(X)}.$$

(ii) $\widehat{\Pi}_s$ *is a bounded linear projector on X .*

Proof. First of all let us note that since we have $\widehat{\Pi}_s = E_0^s$ the property (ii) is a direct consequence of the property (i). Therefore we will focus on the property (i).

The idea of this proof is to derive a suitable closed system of equations for the following three quantities (wherein $p \in \mathbb{N}$ is fixed):

$$\{E_n^s E_p^s - E_{n+p}^s\}_{n \in \mathbb{N}}, \{E_n^u E_p^s\}_{n \in \mathbb{N}} \text{ and } \{E_n^c E_p^s\}_{n \in \mathbb{Z}}.$$

By proceeding as in the proof of Lemma 1.2.16 we obtain the following closed system of equations:

for each $n \in \mathbb{N}$

$$\begin{aligned} E_n^s E_p^s - E_{n+p}^s &= - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B [E_{m+1}^u E_p^s + E_{-m-1}^c E_p^s] \\ &\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B [E_m^s E_p^s - E_m^s] \\ &\quad - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B [E_{n+m}^s E_p^s - E_{n+p+m}^s], \end{aligned} \quad (1.2.70)$$

$$\begin{aligned} E_n^u E_p^s &= \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s E_p^s - E_{p+m}^s] \\ &\quad + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B E_m^c E_p^s, \\ &\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u E_p^s + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] E_{n+m+1}^u E_p^s, \end{aligned} \quad (1.2.71)$$

and for each $n \in \mathbb{Z}$

$$\begin{aligned} E_n^c E_p^s &= \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B [E_m^s E_p^s - E_{p+m}^s] \\ &\quad + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B E_m^c E_p^s - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B E_{n+m}^c E_p^s \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_{n+m}^c E_p^s + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1+n}^c E_p^s. \end{aligned} \quad (1.2.72)$$

Thus by letting $p \in \mathbb{N}$ fixed and setting

$$\widehat{E}_n^s := E_n^s E_p^s - E_{n+p}^s, \quad n \in \mathbb{N},$$

$$\widehat{E}_n^u := E_n^u E_p^s, \quad n \in \mathbb{N},$$

$$\widehat{E}_n^c := E_n^c E_p^s, \quad n \in \mathbb{Z},$$

we obtain that $\widehat{E}^s \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, $\widehat{E}^u \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ and $\widehat{E}^c \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. We also observe that by using (1.2.70)-(1.2.72) the sequences $\widehat{E}^s$, $\widehat{E}^u$, $\widehat{E}^c$ satisfy

$$\begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix} = \mathcal{J} \begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix}.$$

Hence it follows that

$$\widehat{E}_n^s = \widehat{E}_n^u = 0_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{N},$$

and

$$\widehat{E}_n^c = 0_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{Z}.$$

This completes the proof of this lemma. ■

Lemma 1.2.19 *Let the conditions of Theorem 1.1.4 be satisfied. If*

$$\|B\|_{\mathcal{L}(X)} \leq \delta, \text{ with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$$

where δ_0 is given in Proposition 1.2.12 then the following properties hold:

(i) for each $n, p \in \mathbb{Z}$ we have

$$E_n^c E_p^c = E_{n+p}^c,$$

and for each $n \in \mathbb{N}, p \in \mathbb{Z}$ we have

$$E_n^s E_p^c = E_n^u E_p^c = 0_{\mathcal{L}(X)}.$$

(ii) $\widehat{\Pi}_c$ is a bounded linear projector on X .

Proof. First of all let us note that since we have $\widehat{\Pi}_c = E_0^c$ the property (ii) is a direct consequence of the property (i). Therefore we will focus on the property (ii). The idea of this proof is to derive a suitable closed system of equations for the following three quantities (wherein $p \in \mathbb{N}$ is fixed):

$$\{E_n^c E_p^c - E_{n+p}^c\}_{n \in \mathbb{Z}}, \{E_n^u E_p^c\}_{n \in \mathbb{N}} \text{ and } \{E_n^s E_p^c\}_{n \in \mathbb{N}}.$$

By proceeding as in the proof of Lemma 1.2.16 we obtain the following closed system of equations:

for each $n \in \mathbb{Z}$

$$\begin{aligned}
E_n^c E_p^c - E_{n+p}^c &= - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B E_{m+1}^u E_p^c \\
&\quad + \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B E_m^s E_p^c \\
&\quad + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B [E_m^c E_p^c - E_m^c] \\
&\quad - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B [E_{n+m}^c E_p^c - E_{n+p+m}^c] \\
&\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B [E_{n+m}^c E_p^c - E_{n+p+m}^c] \\
&\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B [E_{-m-1+n}^c E_p^c - E_{-m-1+n+p}^c].
\end{aligned} \tag{1.2.73}$$

for each $n \in \mathbb{N}$

$$\begin{aligned}
E_n^u E_p^c &= \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B E_m^s E_p^c \\
&\quad + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^c E_p^c - E_{p+m}^c] \\
&\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u E_p^c + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] E_{n+m+1}^u E_p^c,
\end{aligned} \tag{1.2.74}$$

and

$$\begin{aligned}
E_n^s E_p^c &= - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B E_{m+1}^u E_p^c \\
&\quad - \sum_{m=0}^{+\infty} A_s^{n+m} \Pi_s B [E_{-m-1}^c E_p^c - E_{-m-1+p}^c] \\
&\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B E_m^s E_p^c - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_{n+m}^s E_p^c.
\end{aligned} \tag{1.2.75}$$

Thus by letting $p \in \mathbb{Z}$ fixed and setting

$$\widehat{E}_n^c := E_n^c E_p^c - E_{n+p}^c, \quad n \in \mathbb{Z},$$

$$\widehat{E}_n^u := E_n^u E_p^c, \quad n \in \mathbb{N},$$

$$\widehat{E}_n^s := E_n^s E_p^c, \quad n \in \mathbb{N},$$

we obtain that $\widehat{E}^s \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, $\widehat{E}^u \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ and $\widehat{E}^c \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. We also observe that by using (1.2.74)-(1.2.75) the sequences $\widehat{E}^s$, $\widehat{E}^u$, $\widehat{E}^c$ satisfy

$$\begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix} = \mathcal{J} \begin{pmatrix} \widehat{E}^s \\ \widehat{E}^u \\ \widehat{E}^c \end{pmatrix},$$

and it follows that

$$\widehat{E}_n^s = \widehat{E}_n^u = 0_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{N},$$

and

$$\widehat{E}_n^c = 0_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{Z}.$$

This completes the proof of this lemma. ■

1.2.5 Proof of Theorem 1.1.4

In this section we complete the proof of Theorem 1.1.4. The main points are summarized in the following lemma.

Lemma 1.2.20 *Let us assume that the conditions of Theorem 1.1.4 are satisfied. Up to reduce the value of δ_0 provided by Proposition 1.2.12 so that $\delta_0 < \min \left\{ \frac{1}{6\kappa^3+1}, \sqrt{2} - 1 \right\}$, if $B \in \mathcal{L}(X)$ satisfies*

$$\|B\|_{\mathcal{L}(X)} \leq \delta, \quad \text{with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0} \right)$$

then the following properties hold:

- (i) *The three bounded linear projectors $\widehat{\Pi}_s, \widehat{\Pi}_u$ and $\widehat{\Pi}_c$ provided by Lemmas 1.2.16, 1.2.18 and 1.2.19 satisfy*

$$\widehat{\Pi}_k \widehat{\Pi}_l = 0_{\mathcal{L}(X)} \quad \text{if } k \neq l, \quad \text{with } k, l = s, u, c, \quad (1.2.76)$$

and

$$\left\| \widehat{\Pi}_k - \Pi_k \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} \leq \delta_0. \quad (1.2.77)$$

- (ii) *For each $n \in \mathbb{N}$ and $k = s, c$ we have $E_n^k = (A + B)^n \widehat{\Pi}_k$.*

- (iii) *For each $n \in \mathbb{N}$ and $k = u, c$ $(A + B)^n \widehat{\Pi}_k$ is invertible from $\widehat{\Pi}_k(X)$ into $\widehat{\Pi}_k(X)$ with*

$$E_{-n}^c (A + B)^n \widehat{\Pi}_c = (A + B)^n \widehat{\Pi}_c E_{-n}^c = \widehat{\Pi}_c, \quad (1.2.78)$$

and

$$E_n^u (A + B)^n \widehat{\Pi}_u = (A + B)^n \widehat{\Pi}_u E_n^u = \widehat{\Pi}_u. \quad (1.2.79)$$

(iv) The projectors $\widehat{\Pi}_k$, $k = s, u, c$ satisfy

$$(A + B) \widehat{\Pi}_k = \widehat{\Pi}_k (A + B). \quad (1.2.80)$$

Proof. Proof of (i): By recalling that $E_0^k = \widehat{\Pi}_k$ it follows from Lemmas 1.2.16, 1.2.18 and 1.2.19 that (1.2.76) holds true. Moreover the condition (ii) of Proposition 1.2.12 together with $\delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$ provide that

$$\left\| \widehat{\Pi}_k - \Pi_k \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} \leq \delta_0 \in \left(0, \sqrt{2} - 1\right). \quad (1.2.81)$$

This completes the proof of (i).

Proof of (ii): Let $n \in \mathbb{N} \setminus \{0\}$ be given. We will first prove that $E_n^s = (A + B)^n \widehat{\Pi}_s$. By replacing n by $n - 1$ in (1.2.39) and multiplying the left side of E_{n-1}^s by A it follows that

$$A E_{n-1}^s = E_n^s - B E_{n-1}^s \iff E_n^s = (A + B) E_{n-1}^s,$$

so that

$$E_n^s = (A + B)^n E_0^s = (A + B)^n \widehat{\Pi}_s.$$

Next we prove that $E_n^c = (A + B)^n \widehat{\Pi}_c$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N} \setminus \{0\}$ be given. By replacing n by $n - 1$ in (1.2.41) and multiplying the left side of E_{n-1}^c by A we obtain

$$A E_{n-1}^c = E_n^c - B E_{n-1}^c \iff E_n^c = (A + B) E_{n-1}^c,$$

providing that

$$E_n^c = (A + B)^n \widehat{\Pi}_c. \quad (1.2.82)$$

This completes the proof of (ii).

Proof of (iii): The proof of this point is split into two parts: the central case and the unstable case.

Invertibility of the central part:

Let us prove that for each $n \in \mathbb{N}$ the operator $(A + B)^n \widehat{\Pi}_c$ is invertible from $\widehat{\Pi}_c(X)$ into $\widehat{\Pi}_c(X)$.

In fact each $n \in \mathbb{N}$ by using Lemma 1.2.19 combined with (1.2.82) we obtain

$$E_{-n}^c (A + B)^n \widehat{\Pi}_c = E_{-n}^c E_n^c = E_0^c = \widehat{\Pi}_c = E_n^c E_{-n}^c = (A + B)^n \widehat{\Pi}_c E_{-n}^c.$$

This prove that $(A + B)^n \widehat{\Pi}_c$ is invertible from $\widehat{\Pi}_c(X)$ into $\widehat{\Pi}_c(X)$ and (1.2.78) holds true.

Invertibility of the unstable part:

we will prove that $(A + B)^n \widehat{\Pi}_u$ is invertible from $\widehat{\Pi}_u(X)$ into $\widehat{\Pi}_u(X)$ or equivalently (1.2.79) holds true. To prove this result we will proceed into two steps.

Step 1: Let $n \in \mathbb{N} \setminus \{0\}$ be given. We first prove that $(A + B)^n \widehat{\Pi}_u E_n^u = \widehat{\Pi}_u$. To do so we multiply the left side of E_n^u given in (1.2.40) by A to obtain

$$A E_n^u = E_{n-1}^u - B E_n^u \iff (A + B) E_n^u = E_{n-1}^u,$$

providing that

$$(A + B)^n E_n^u = E_0^u = \widehat{\Pi}_u, \quad (1.2.83)$$

and since by Lemma 1.2.19 we have $\widehat{\Pi}_u E_0^u = E_0^u E_n^u = E_n^u$ it follows from (1.2.83) that

$$(A + B)^n \widehat{\Pi}_u E_n^u = \widehat{\Pi}_u. \quad (1.2.84)$$

Step 2: It remains to show that $E_n^u (A + B)^n \widehat{\Pi}_u = \widehat{\Pi}_u$. Because of the semiflows property for E^u provided in Lemma 1.2.16, in order to prove the above equality it is sufficient to show that it holds true for $n = 1$, namely $E_1^u (A + B) \widehat{\Pi}_u = \widehat{\Pi}_u$ or equivalently

$$E_1^u \widehat{\Pi}_u (A + B) \widehat{\Pi}_u = \widehat{\Pi}_u. \quad (1.2.85)$$

Hence in what follows we will prove that (1.2.85) holds true. To do so, let us note that due to (1.2.84) we have

$$\widehat{\Pi}_u (A + B) \widehat{\Pi}_u E_1^u = \widehat{\Pi}_u \widehat{\Pi}_u = \widehat{\Pi}_u. \quad (1.2.86)$$

Thus because of the left invertibility provided by the above equality, in order to prove (1.2.85) we only need to prove that $\widehat{\Pi}_u (A + B) \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} : \widehat{\Pi}_u(X) \rightarrow \widehat{\Pi}_u(X)$ is invertible.

Before proceeding to the proof of this statement let us note that since we have $E_0^k = \widehat{\Pi}_k$, $k = s, u, c$ it follows from the condition (i) of Proposition 1.2.12 that

$$\left\| \widehat{\Pi}_k \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta_0}{\delta_0 - \delta}, \quad k = s, u, c. \quad (1.2.87)$$

Now note that

$$\begin{aligned} \widehat{\Pi}_u (A + B) \widehat{\Pi}_u &= \widehat{\Pi}_u A \widehat{\Pi}_u + \widehat{\Pi}_u B \widehat{\Pi}_u \\ &= \widehat{\Pi}_u A [\Pi_u + \Pi_s + \Pi_c] \widehat{\Pi}_u + \widehat{\Pi}_u B \widehat{\Pi}_u \\ &= \widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u + \widehat{\Pi}_u A \Pi_s \widehat{\Pi}_u + \widehat{\Pi}_u A \Pi_c \widehat{\Pi}_u \\ &\quad + \widehat{\Pi}_u B \widehat{\Pi}_u, \end{aligned}$$

that is

$$\widehat{\Pi}_u (A + B) \widehat{\Pi}_u = \widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u + L_u$$

wherein we have set

$$L_u := \widehat{\Pi}_u A \Pi_s \widehat{\Pi}_u + \widehat{\Pi}_u A \Pi_c \widehat{\Pi}_u + \widehat{\Pi}_u B \widehat{\Pi}_u.$$

Next observe that due to (1.2.81) Lemma 1.2.1 applies to Π_u and $\widehat{\Pi}_u$ and provides that $\Pi_u|_{\widehat{\Pi}_u(X)}$ is an isomorphism from $\widehat{\Pi}_u(X)$ onto $\Pi_u(X)$ while $\widehat{\Pi}_u|_{\Pi_u(X)}$ is an isomorphism from $\Pi_u(X)$ onto $\widehat{\Pi}_u(X)$. One furthermore has

$$\left\| \left(\widehat{\Pi}_u|_{\Pi_u(X)} \right)^{-1} x \right\| \leq \frac{1}{1 - \delta_0} \|x\|, \quad \forall x \in \Pi_u(X),$$

and

$$\left\| \left(\Pi_u|_{\widehat{\Pi}_u(X)} \right)^{-1} x \right\| \leq \frac{1}{1 - \delta_0} \|x\|, \quad \forall x \in \widehat{\Pi}_u(X).$$

Therefore since $A\Pi_u$ is an isomorphism from $\Pi_u(X)$ onto itself one deduces that $\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)}$ is an isomorphism from $\widehat{\Pi}_u(X)$ onto itself and

$$\begin{aligned} \left\| \left(\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \right)^{-1} x \right\| &\leq \frac{1}{(1 - \delta_0)} \|A_u^{-1} \Pi_u\|_{\mathcal{L}(X)} \|x\|, \\ &\leq \frac{1}{(1 - \delta_0)} \kappa e^{-\rho} \|x\|, \quad \forall x \in \widehat{\Pi}_u(X). \end{aligned} \quad (1.2.88)$$

Hence in order to prove that $\widehat{\Pi}_u(A + B)\widehat{\Pi}_u$ is an isomorphism of $\widehat{\Pi}_u(X)$ let us observe that

$$\widehat{\Pi}_u(A + B)\widehat{\Pi}_u = \widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \left[\widehat{\Pi}_u + \left(\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \right)^{-1} L_u \right],$$

and let us prove that

$$\left\| \left(\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \right)^{-1} L_u \right\|_{\mathcal{L}(\widehat{\Pi}_u(X))} < 1.$$

To do so we will first estimate $\|L_u\|_{\mathcal{L}(\widehat{\Pi}_u(X))}$. Note that we have

$$\begin{aligned} L_u &= \widehat{\Pi}_u A \Pi_s \widehat{\Pi}_u + \widehat{\Pi}_u A \Pi_c \widehat{\Pi}_u + \widehat{\Pi}_u B \widehat{\Pi}_u \\ &= \widehat{\Pi}_u A_s \Pi_s \left[\widehat{\Pi}_u - \Pi_u \right] + \widehat{\Pi}_u A_c \Pi_c \left[\widehat{\Pi}_u - \Pi_u \right] + \widehat{\Pi}_u B \widehat{\Pi}_u. \end{aligned}$$

Then by using (1.2.77) and (1.2.87) one gets

$$\|L_u\|_{\mathcal{L}(\widehat{\Pi}_u(X))} \leq (\kappa + \kappa\delta_0) \kappa e^{-\rho} \delta_0 + (\kappa + \kappa\delta_0) \kappa e^{\rho_0} \delta_0 + (\kappa + \kappa\delta_0) \|B\|_{\mathcal{L}(X)},$$

and recalling that

$$\|B\|_{\mathcal{L}(X)} \leq \delta \leq \delta_0 \text{ and } \delta_0 \in (0, 1),$$

it follows that

$$\begin{aligned} \|L_u\|_{\mathcal{L}(\widehat{\Pi}_u(X))} &\leq 2\kappa^2 \delta_0 + 2\kappa^2 e^{\rho_0} + 2\kappa \delta_0 \\ &\leq 6\kappa^2 e^{\rho_0} \delta_0. \end{aligned} \quad (1.2.89)$$

Now combining (1.2.88) with (1.2.89) provides that

$$\begin{aligned} \left\| \left(\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \right)^{-1} L_u \right\|_{\mathcal{L}(\widehat{\Pi}_u(X))} &\leq \left\| \left(\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \right)^{-1} \right\|_{\mathcal{L}(\widehat{\Pi}_u(X))} \|L_u\|_{\mathcal{L}(\widehat{\Pi}_u(X))} \\ &\leq \frac{1}{(1 - \delta_0)} \kappa e^{-\rho} 6\kappa^2 e^{\rho_0} \delta_0 \\ &\leq \frac{6\kappa^3}{(1 - \delta_0)} \delta_0, \end{aligned}$$

so that up to reduce δ_0 such that

$$\delta_0 < \min \left\{ C^{-1}, \frac{1}{6\kappa^3 + 1} \right\}, \quad (1.2.90)$$

with C the constant provided by Lemma 1.2.10 we obtain that

$$\left\| \left(\widehat{\Pi}_u A \Pi_u \widehat{\Pi}_u|_{\widehat{\Pi}_u(X)} \right)^{-1} L_u \right\|_{\mathcal{L}(\widehat{\Pi}_u(X))} < 1,$$

and the result follows. This completes the proof of (iii).

Proof of (iv): Recall that by (ii) we have

$$E_1^k = (A + B) \widehat{\Pi}_k, \quad \forall k = s, c,$$

so that

$$\widehat{\Pi}_k (A + B) \widehat{\Pi}_k = \widehat{\Pi}_k E_1^k = E_0^k E_1^k = E_1^k = (A + B) \widehat{\Pi}_k,$$

that is

$$(A + B) \widehat{\Pi}_k = \widehat{\Pi}_k (A + B) \widehat{\Pi}_k, \quad k = s, c. \quad (1.2.91)$$

Moreover the property (iii) implies that $(A + B) \widehat{\Pi}_u$ maps $\widehat{\Pi}_u(X)$ into itself that is

$$(A + B) \widehat{\Pi}_u = \widehat{\Pi}_u (A + B) \widehat{\Pi}_u. \quad (1.2.92)$$

Therefore for each $k = s, c, u$ by using (1.2.91) and (1.2.92) combined with the orthogonality property in (1.2.76) we obtain

$$\begin{aligned} \widehat{\Pi}_k (A + B) &= \widehat{\Pi}_k (A + B) \left[\widehat{\Pi}_s + \widehat{\Pi}_u + \widehat{\Pi}_c \right] \\ &= \widehat{\Pi}_k (A + B) \widehat{\Pi}_k = (A + B) \widehat{\Pi}_k. \end{aligned}$$

This completes the proof of this lemma. ■

Together with the above material we are now able to complete the proof of Theorem 1.1.4.

Proof of Theorem 1.1.4. By observing that for each $k = s, u, c$, $(A + B)_k \in \mathcal{L}(X_k)$ the part of $A + B$ in $\widehat{\Pi}_k(X)$ (i.e. $(A + B)_k x = (A + B)x, \forall x \in \widehat{\Pi}_k(X)$) satisfy

$$(A + B)_k \widehat{\Pi}_k = (A + B) \widehat{\Pi}_k,$$

the proof of Theorem 1.1.4 becomes a direct consequence of Proposition 1.2.12 and Lemma 1.2.20. ■

1.3 Proof of Theorem 1.1.8

The aim of this section is to complete the proof of Theorem 1.1.8. The proof of this result is based on the reformulation of the non-autonomous linear difference equation as an autonomous one using Howland semigroup approach (see Chicone and Latushkin [35]). We will then use Theorem 1.1.4 to perturb the Howland semigroup and complete the proof of Theorem 1.1.8.

Let $(Y, \|\cdot\|)$ be a given Banach space. Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be a given sequence of bounded linear operators such that $U_{\mathbf{A}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three families of projectors $\{\Pi^\alpha : \mathbb{Z} \rightarrow \mathcal{L}(X)\}_{\alpha=s,c,u}$.

In order to prove the above result let us introduce some notations and operator framework. Let $q \in [1, \infty]$ be given and let us introduce the Banach space $X = l^q(\mathbb{Z}, Y)$. Let us consider the bounded linear operator $\mathcal{A} : l^q(\mathbb{Z}; Y) \rightarrow l^q(\mathbb{Z}; Y)$ defined by

$$(\mathcal{A}u)_k = A_{k-1}u_{k-1}, \quad \forall k \in \mathbb{Z}, \quad \forall u \in l^q(\mathbb{Z}; Y).$$

Next let us consider the three bounded linear operators $\mathcal{P}_\alpha \in \mathcal{L}(X)$ defined for $\alpha = s, c, u$ by

$$(\mathcal{P}_\alpha u)_k = \Pi_k^\alpha u_k, \quad \forall k \in \mathbb{Z}, \quad \forall u \in X.$$

Using the above notations let us note that for each $\alpha = s, c, u$, $\mathcal{P}_\alpha$ is a projector on X that satisfies

- $\mathcal{P}_\alpha \mathcal{P}_\beta = 0_{\mathcal{L}(X)}$ for all $\alpha \neq \beta$.
- $\mathcal{P}_s + \mathcal{P}_c + \mathcal{P}_u = I_{\mathcal{L}(X)}$.
- for each $\alpha = s, c, u$, one has

$$\mathcal{P}_\alpha \mathcal{A} = \mathcal{A} \mathcal{P}_\alpha.$$

Next we set $X_\alpha = \mathcal{P}_\alpha(X)$ for $\alpha = s, c, u$ and the following straightforward lemma holds true:

Lemma 1.3.1 *The following holds true:*

- (i) *The part $\mathcal{A}_s$ of $\mathcal{A}$ in X^s satisfies $r(\mathcal{A}_s) \leq e^\rho$. We furthermore have for each $u \in X^s$ and each $(n, k) \in \mathbb{N} \times \mathbb{Z}$:*

$$(\mathcal{A}_s^n u)_k = U_{\mathbf{A}}^s(k, k-n) \Pi_{k-n}^s u_{k-n}.$$

- (ii) *The part $\mathcal{A}_u$ of $\mathcal{A}$ in X^u is invertible and it satisfies $r(\mathcal{A}_u^{-1}) \leq e^{-\rho}$. We furthermore have for each $u \in X^u$ and each $(n, k) \in \mathbb{N} \times \mathbb{Z}$:*

$$(\mathcal{A}_u^{-n} u)_k = U_{\mathbf{A}}^u(k, k+n) \Pi_{k+n}^u u_{k+n}.$$

(iii) The part $\mathcal{A}_c$ of $\mathcal{A}$ in X^c is invertible and it satisfies:

$$r(\mathcal{A}_c) \leq e^{\rho_0} \text{ and } r(\mathcal{A}_c^{-1}) \leq e^{\rho_0}.$$

We furthermore have for each $u \in X^u$ and each $(n, k) \in \mathbb{Z} \times \mathbb{Z}$:

$$(\mathcal{A}_c^n u)_k = U_{\mathbf{A}}^c(k, k-n) \Pi_{k-n}^c u_{k-n}$$

Remark 1.3.2 The above discussion and the above lemma imply that the operator $\mathcal{A}$ has an exponential trichotomy according to Definition 1.1.1.

Let $\mathbf{B} = \{B_n\}_{n \in \mathbb{Z}}$ be a bounded sequence in $\mathcal{L}(Y)$. Then let us consider the bounded linear operator $\mathcal{B} \in \mathcal{L}(X)$ defined by

$$(\mathcal{B}u)_k = B_{k-1}u_{k-1}, \quad \forall k \in \mathbb{Z}, \quad \forall u \in X.$$

Then note that one has:

$$\|\mathcal{B}\|_{\mathcal{L}(X)} \leq \sup_{k \in \mathbb{Z}} \|B_k\|_{\mathcal{L}(Y)}. \quad (1.3.1)$$

We are now interesting in the spectral properties of $\mathcal{A} + \mathcal{B}$ by applying Theorem 1.1.4. We fix $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and $\hat{\kappa} > \kappa$. Using the constant $\delta_0 > 0$ provided by Theorem 1.1.4, we fix a bounded sequence $\mathbf{B} = \{B_n\}_{n \in \mathbb{Z}}$ in $\mathcal{L}(Y)$ such that

$$\sup_{n \in \mathbb{Z}} \|B_n\|_{\mathcal{L}(Y)} \leq \frac{\delta_0^2}{\kappa + \delta_0}.$$

In view of (1.3.1), Theorem 1.1.4 applies to the perturbation problem $\mathcal{A} + \mathcal{B}$ and operator $(\mathcal{A} + \mathcal{B})$ has an exponential trichotomy with exponent $\hat{\rho}_0$ and $\hat{\rho}$ and with constant $\hat{\kappa}$. If we denote the three corresponding projectors by $\hat{\mathcal{P}}_s, \hat{\mathcal{P}}_c, \hat{\mathcal{P}}_u \in \mathcal{L}(X)$ and $\hat{X}^\alpha = \hat{\mathcal{P}}_s(X)$ we have:

$$\begin{pmatrix} (\mathcal{A} + \mathcal{B})_s^\cdot \hat{\mathcal{P}}_s \\ (\mathcal{A} + \mathcal{B})_u^\cdot \hat{\mathcal{P}}_u \\ (\mathcal{A} + \mathcal{B})_c^\cdot \hat{\mathcal{P}}_c \end{pmatrix} = (I - \mathcal{J})^{-1} \begin{pmatrix} \mathcal{A}_s^\cdot \mathcal{P}_s \\ \mathcal{A}_u^\cdot \mathcal{P}_u \\ \mathcal{A}_c^\cdot \mathcal{P}_c \end{pmatrix}, \quad (1.3.2)$$

wherein the bounded linear operator $\mathcal{J}$ acting on the Banach space $\mathcal{X} := \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\hat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\hat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ is defined in (1.2.48). One furthermore has the following estimates

$$\left\| (\mathcal{A} + \mathcal{B})_c^n \hat{\mathcal{P}}_c \right\|_{\mathcal{L}(X)} \leq \hat{\kappa} e^{\hat{\rho}_0 |n|}, \quad \forall n \in \mathbb{Z}, \quad (1.3.3)$$

$$\left\| (\mathcal{A} + \mathcal{B})_s^n \hat{\mathcal{P}}_s \right\|_{\mathcal{L}(X)} \leq \hat{\kappa} e^{-\hat{\rho} n}, \quad \forall n \in \mathbb{N}, \quad (1.3.4)$$

and

$$\left\| (\mathcal{A} + \mathcal{B})_u^{-n} \hat{\mathcal{P}}_u \right\|_{\mathcal{L}(X)} \leq \hat{\kappa} e^{-\hat{\rho} n}, \quad \forall n \in \mathbb{N}, \quad (1.3.5)$$

as well as the following estimates for each $n \in \mathbb{N}$,

$$\left\| (\mathcal{A} + \mathcal{B})_s^n \widehat{\mathcal{P}}_s - \mathcal{A}_s^n \mathcal{P}_s \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho} n}, \quad (1.3.6)$$

$$\left\| (\mathcal{A} + \mathcal{B})_u^{-n} \widehat{\mathcal{P}}_u - \mathcal{A}_u^{-n} \mathcal{P}_u \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho} n}, \quad (1.3.7)$$

and for each $n \in \mathbb{Z}$

$$\left\| (\mathcal{A} + \mathcal{B})_c^n \widehat{\mathcal{P}}_c - \mathcal{A}_c^n \mathcal{P}_c \right\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{\widehat{\rho}_0 |n|}. \quad (1.3.8)$$

In order to prove our perturbation result, namely Theorem 1.1.8, we will show that the perturbed projectors exhibit a suitable structure that inherited from the one of the shift operators $\mathcal{A}$ and $\mathcal{B}$ that read as

$$\left(\widehat{\mathcal{P}}_\alpha u \right)_k = \widehat{\Pi}_k^\alpha u_k, \quad \forall u \in X, \quad \alpha = s, c, u,$$

and wherein for each $k \in \mathbb{Z}$ and $\alpha = s, c, u$, $\widehat{\Pi}_k^\alpha$ denotes a projector of Y . To do so, let us introduce for each $p \in \mathbb{Z}$ the linear bounded operator $\mathcal{D}_p \in \mathcal{L}(X)$ defined for each $u \in X$ and $k \in \mathbb{Z}$ by

$$(\mathcal{D}_p u)_k = \begin{cases} u_p & \text{if } k = p \\ 0 & \text{if } k \neq p \end{cases}$$

Together with this notation, let us note that for each $p \in \mathbb{Z}$ the following commutativity properties hold true:

$$\begin{aligned} \mathcal{D}_p \mathcal{A}_s^n \mathcal{P}_s &= \mathcal{A}_s^n \mathcal{P}_s \mathcal{D}_{p-n}, \quad \forall n \geq 0, \quad \forall p \in \mathbb{Z}, \\ \mathcal{D}_p \mathcal{A}_u^{-n} \mathcal{P}_u &= \mathcal{A}_u^{-n} \mathcal{P}_u \mathcal{D}_{p+n}, \quad \forall n \geq 0, \quad \forall p \in \mathbb{Z}, \\ \mathcal{D}_p \mathcal{A}_c^n \mathcal{P}_u &= \mathcal{A}_c^n \mathcal{P}_u \mathcal{D}_{p-n}, \quad \forall n \in \mathbb{Z}, \quad \forall p \in \mathbb{Z}. \end{aligned} \quad (1.3.9)$$

One may also note that $\mathcal{B}$ satisfies:

$$\mathcal{D}_p \mathcal{B} = \mathcal{B} \mathcal{D}_{p-1}, \quad \forall p \in \mathbb{Z}. \quad (1.3.10)$$

If one considers the closed subspace $\mathcal{Z} \subset \mathcal{X}$ defined by

$$\mathcal{Z} = \left\{ \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \in \mathcal{X} : \begin{pmatrix} \mathcal{D}_p E^s - E^s \mathcal{D}_{p-} \\ \mathcal{D}_p E^u - E^u \mathcal{D}_{p+} \\ \mathcal{D}_p E^c - E^c \mathcal{D}_{p-} \end{pmatrix} = 0_{\mathcal{X}}, \quad \forall p \in \mathbb{Z} \right\},$$

then we claim that

Claim 1.3.3 *The linear bounded operator $\mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ satisfies:*

$$\mathcal{J} \mathcal{Z} \subset \mathcal{Z}.$$

We postpone the proof of this claim and complete the proof of Theorem 1.1.8. Using the above claim and recalling that $\mathcal{Z}$ is a closed subspace of $\mathcal{X}$ lead us to

$$(I - \mathcal{J})^{-1} \mathcal{Z} \subset \mathcal{Z}.$$

Indeed since $\|\mathcal{J}\|_{\mathcal{L}(\mathcal{X})} < 1$ then $(I - \mathcal{J})^{-1} = \sum_{k=0}^{\infty} \mathcal{J}^k$. Hence due to (1.3.9) one obtains that

$$\begin{pmatrix} \mathcal{A}_s \mathcal{P}_s \\ \mathcal{A}_u^- \mathcal{P}_u \\ \mathcal{A}_c \mathcal{P}_c \end{pmatrix} \in \mathcal{Z},$$

while (1.3.2) ensures that

$$\begin{pmatrix} (\mathcal{A} + \mathcal{B})_s \widehat{\mathcal{P}}_s \\ (\mathcal{A} + \mathcal{B})_u^- \widehat{\mathcal{P}}_u \\ (\mathcal{A} + \mathcal{B})_c \widehat{\mathcal{P}}_c \end{pmatrix} \in \mathcal{Z}.$$

The above statement completes the proof of Theorem 1.1.8. Indeed let us first note that the above result implies that for each $p \in \mathbb{Z}$ and $\alpha = s, c, u$,

$$\mathcal{D}_p \widehat{\mathcal{P}}_\alpha = \widehat{\mathcal{P}}_\alpha \mathcal{D}_p.$$

This means that for each $p \in \mathbb{Z}$ there exists three projectors $\widehat{\Pi}_p^\alpha$ for $\alpha = s, c, u$ such that:

$$\left(\widehat{\mathcal{P}}_\alpha u \right)_p = \widehat{\Pi}_p^\alpha u_p \text{ for all } p \in \mathbb{Z}, u \in X \text{ and } \alpha = s, c, u.$$

Note that the properties $\widehat{\mathcal{P}}_\alpha \widehat{\mathcal{P}}_\beta = 0$ for $\alpha \neq \beta$ and $\widehat{\mathcal{P}}_s + \widehat{\mathcal{P}}_c + \widehat{\mathcal{P}}_u = I_X$ directly re-write as for each $p \in \mathbb{Z}$:

$$\widehat{\Pi}_k^\alpha \widehat{\Pi}_k^\beta = 0 \text{ for } \alpha \neq \beta \text{ and } \widehat{\Pi}_k^s + \widehat{\Pi}_k^c + \widehat{\Pi}_k^u = I_Y.$$

It remains to check that $\mathbf{A} + \mathbf{B}$ has an exponential trichotomy with constant $\widehat{\kappa}$, exponents $\widehat{\rho}_0 < \widehat{\rho}$ and associated to the projectors $\left\{ \widehat{\Pi}_k^\alpha \right\}_{k \in \mathbb{Z}}$ with $\alpha = s, c, u$.

Property (ii) of Definition 1.1.6:

For each $k \in \mathbb{Z}$ and $n \geq 0$ and $\alpha = s, c, u$ one has for each $u \in X$:

$$\left[\widehat{\mathcal{P}}_\alpha (\mathcal{A} + \mathcal{B})^n u \right]_k = \widehat{\Pi}_k^\alpha [(\mathcal{A} + \mathcal{B})^n u]_k = \widehat{\Pi}_k^\alpha U_{\mathbf{A}+\mathbf{B}}(k, k-n) u_{k-n}.$$

Since $\widehat{\mathcal{P}}_\alpha (\mathcal{A} + \mathcal{B})^n = (\mathcal{A} + \mathcal{B})^n \widehat{\mathcal{P}}_\alpha$ one obtains that

$$\widehat{\Pi}_k^\alpha U_{\mathbf{A}+\mathbf{B}}(k, k-n) u_{k-n} = U_{\mathbf{A}+\mathbf{B}}(k, k-n) \left[\widehat{\mathcal{P}}_\alpha u \right]_{k-n} = U_{\mathbf{A}+\mathbf{B}}(k, k-n) \widehat{\Pi}_{k-n}^\alpha u_{k-n}.$$

This proves statement (ii).

Proof of (iii) in Definition 1.1.6:

Let us set for each $k \in \mathbb{Z}$ the subspaces $\widehat{Y}_k^\alpha = \widehat{\Pi}_k^\alpha(Y)$. Let $n \geq 0$ be given. Recall that

$(\mathcal{A} + \mathcal{B})^n$ is invertible from $\widehat{X}_u$ onto itself. Hence for each $v \in \widehat{X}_u$ there exists a unique $u \in \widehat{X}_u$ such that $(\mathcal{A} + \mathcal{B})^n u = v$. Rewrite this as for each $k \in \mathbb{Z}$:

$$U_{\mathbf{A}+\mathbf{B}}(k, k-n)u_{k-n} = v_k.$$

This proves that for each $n \geq 0$ and each $k \in \mathbb{Z}$ the linear operator $U_{\mathbf{A}+\mathbf{B}}(k, k-n)$ is invertible from $\widehat{Y}_{k-n}^u$ onto $\widehat{Y}_k^u$. Furthermore one has for each $n \geq 0$ and $p \in \mathbb{Z}$:

$$\left[(\mathcal{A} + \mathcal{B})^{-n} \widehat{\mathcal{P}}_u v \right]_p = U_{\mathbf{A}+\mathbf{B}}(p, p+n) \widehat{\Pi}_{p+n}^u v_{p+n}.$$

The same arguments hold true for the central part and one obtains that for each $k \in \mathbb{Z}$ and $n \geq 0$ the linear operator $U_{\mathbf{A}+\mathbf{B}}(k, k-n)$ is invertible from $\widehat{Y}_{k-n}^c$ onto $\widehat{Y}_k^c$. Furthermore one has for each $n \in \mathbb{Z}$ and $p \in \mathbb{Z}$:

$$\left[(\mathcal{A} + \mathcal{B})^n \widehat{\mathcal{P}}_c v \right]_p = U_{\mathbf{A}+\mathbf{B}}(p, p-n) \widehat{\Pi}_{p-n}^c v_{p-n}.$$

This proves that (iii) is true.

Proof of (iv) in Definition 1.1.6:

The proof of the growth estimates directly follow from the trichotomy estimates for $(\mathcal{A} + \mathcal{B})_\alpha$ recalled in (1.3.3)-(1.3.5).

Finally the perturbed estimates for projected evolution semiflows stated in Theorem 1.1.8 directly follows from (1.3.6)-(1.3.8). This completes the proof of the result.

To complete the proof of Theorem 1.1.8 it remains to prove Claim 1.3.3.

Proof of Claim 1.3.3. Let $(E^s, E^u, E^c)^T \in \mathcal{X}$ be given. Let us set $(F^s, F^u, F^c)^T = \mathcal{J}(E^s, E^u, E^c)^T$. Then according to the definition of $\mathcal{J}$ (see (1.2.48) and (1.2.39)-(1.2.41)) for each $n \geq 0$ one has

$$\begin{aligned} F_n^s &= \mathcal{A}_s^n \mathcal{P}_s - \sum_{m=0}^{+\infty} \mathcal{A}_s^{m+n} \mathcal{P}_s \mathcal{B} [E_{m+1}^u + E_{-m-1}^c] \\ &\quad + \sum_{m=0}^{n-1} \mathcal{A}_s^{n-m-1} \mathcal{P}_s \mathcal{B} E_m^s - \sum_{m=0}^{+\infty} [\mathcal{A}_u^{-m-1} \mathcal{P}_u + \mathcal{A}_c^{-m-1} \mathcal{P}_c] \mathcal{B} E_{n+m}^s. \end{aligned}$$

Recalling (1.3.9) and (1.3.10) one directly check that

$$\mathcal{D}_p F_n^s = F_n^s \mathcal{D}_{p-n}, \quad \forall n \geq 0, p \in \mathbb{Z}.$$

Using the formula described in (1.2.40) and (1.2.41) one may directly check the claim. ■

1.4 Proof of Theorem 1.1.13

The aim of this section is to complete the proof of Theorem 1.1.13. Similarly to the proof of Theorem 1.1.8 we shall use Howland semigroup formulation.

In order to prove this result let us consider for each $\omega \in \tilde{\Omega}$ the sequence of bounded linear operators $\mathbf{A}(\omega) = \{A(\theta^n \omega)\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}, \mathcal{L}(Y))$. Using the notations of Theorem 1.1.8, note that for each $\omega \in \tilde{\Omega}$:

$$U_A(n, m, \omega) = U_{\mathbf{A}(\omega)}(n, m), \quad \forall n \geq m.$$

Furthermore if we set for each $\omega \in \tilde{\Omega}$, $n \in \mathbb{Z}$ and $\alpha = s, c, u$, $\Pi_n^\alpha(\omega) = \Pi^\alpha(\theta^n \omega)$, one obtains that for each fixed $\omega \in \tilde{\Omega}$ the linear evolution operator $U_{\mathbf{A}(\omega)}$ has an exponential trichotomy with exponents $\rho_0(\omega) < \rho(\omega)$, constant $\kappa(\omega)$ and projectors $\{\Pi_n^\alpha(\omega)\}_{n \in \mathbb{Z}}$.

Let $\hat{\rho}_0$, $\hat{\rho}$ and $\hat{\kappa}$ be three given random variables from $\tilde{\Omega}$ to $(0, \infty)$ such that for each $\omega \in \tilde{\Omega}$:

$$\begin{cases} \rho_0(\omega) < \hat{\rho}_0(\omega) < \hat{\rho}(\omega) < \rho(\omega), \\ \kappa(\omega) < \hat{\kappa}(\omega). \end{cases}$$

For each $\omega \in \tilde{\Omega}$ let $\delta_0(\omega) = \delta_0(\rho(\omega), \hat{\rho}_0(\omega), \hat{\rho}(\omega), \rho(\omega), \hat{\kappa}(\omega), \kappa(\omega)) > 0$ be the constant provided by Theorem 1.1.8. Using the explicit formula of δ_0 (see (1.2.90) and Lemme 1.2.10), one can observe that $\omega \rightarrow \delta_0(\omega)$ can be chosen as a measurable function. Next we choose a strongly measurable $\omega \rightarrow B(\omega)$ from $\tilde{\Omega} \rightarrow \mathcal{L}(Y)$ such that

$$\sup_{n \in \mathbb{Z}} \|B(\theta^n \omega)\|_{\mathcal{L}(Y)} < \frac{\delta_0^2(\omega)}{\kappa(\omega) + \delta_0(\omega)}, \quad \forall \omega \in \tilde{\Omega}.$$

Next if we set for each $\omega \in \tilde{\Omega}$, $\mathbf{B}(\omega) = \{B(\theta^n \omega)\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}, \mathcal{L}(Y))$ then Theorem 1.1.8 applies for each given and fixed $\omega \in \tilde{\Omega}$. Hence for each $\omega \in \tilde{\Omega}$, the evolution operator $U_{(\mathbf{A}+\mathbf{B})(\omega)}$ has an exponential trichotomy with exponents $\hat{\rho}_0(\omega) < \hat{\rho}(\omega)$ and constant $\hat{\kappa}(\omega)$. We furthermore denote by $\{\hat{\Pi}_n^\alpha(\omega)\}_{n \in \mathbb{Z}}$ with $\alpha = s, c, u$, the perturbed sequence of projectors.

To complete the proof of Theorem 1.1.13 two points remain to be proved: The structure of the perturbed projectors; The measurability of the regularized semiflows.

Structure of the projectors:

Let us show that

$$\hat{\Pi}_n^\alpha(\omega) = \hat{\Pi}_0^\alpha(\theta^n \omega), \quad \forall \omega \in \tilde{\Omega}, \quad n \in \mathbb{Z}, \quad \alpha = s, c, u.$$

To do so let us note that for each $k \in \mathbb{Z}$, each $n \geq m$, $\omega \in \tilde{\Omega}$ one has:

$$U_{(\mathbf{A}+\mathbf{B})(\theta^k \omega)}(n, m) = U_{A+B}(n, m, \theta^k \omega) = U_{A+B}(k+n, k+m, \omega) = U_{(\mathbf{A}+\mathbf{B})(\omega)}(k+n, k+m).$$

Hence $U_{A+B}(n, m, \theta^k \omega)$ has an exponential trichotomy with exponent $\hat{\rho}_0(\omega) < \hat{\rho}(\omega)$ and associated to the projectors $\hat{\Pi}_n^\alpha(\theta^k \omega)$ and $\hat{\Pi}_{n+k}^\alpha(\omega)$. As a consequence of the uniqueness of the trichotomic projectors (see Lemma 1.1.7) one gets for $\alpha = s, c, u$:

$$\hat{\Pi}_n^\alpha(\theta^k \omega) = \hat{\Pi}_{n+k}^\alpha(\omega), \quad \forall (k, n) \in \mathbb{Z}^2, \quad \omega \in \tilde{\Omega}.$$

The result follows.

In the sequel we shall denote $\widehat{\Pi}^\alpha(\omega) = \widehat{\Pi}_0^\alpha(\omega)$ for each $\alpha = s, c, u$ and $\omega \in \widetilde{\Omega}$.

Measurability:

We will prove that the map $\omega \rightarrow U_{A+B}^s(n, m, \omega)$ (resp. $\omega \rightarrow U_{A+B}^u(n, m, \omega)$, resp. $\omega \rightarrow U_{A+B}^c(n, m, \omega)$) is strongly measurable for each $n \geq m$ (resp. $n \leq m$, resp. $(n, m) \in \mathbb{Z}^2$). To do that we will come back to the fixed point formulation using Howland semigroup. Let us introduce $X = l^1(\mathbb{Z}; Y)$ as well as for each $\omega \in \widetilde{\Omega}$ the bounded linear operators on X , $\mathcal{A}(\omega)$ and $\mathcal{B}(\omega)$ defined for all $u \in X$ by

$$(\mathcal{A}(\omega)u)_k = A(\theta^{k-1}\omega) u_{k-1}, \quad (\mathcal{B}(\omega)u)_k = B(\theta^{k-1}\omega) u_{k-1}, \quad \forall k \in \mathbb{Z}.$$

Then our first lemma reads as:

Lemma 1.4.1 *Let $E : \widetilde{\Omega} \rightarrow \mathcal{L}(Y)$ be a strongly measurable map such that*

$$\sup_{n \in \mathbb{Z}} \|E(\theta^n \omega)\|_{\mathcal{L}(Y)} < \infty \quad \forall \omega \in \widetilde{\Omega}.$$

Then the shift operator $\mathcal{E} : \widetilde{\Omega} \rightarrow \mathcal{L}(l^1(\mathbb{Z}, Y))$ is strongly measurable. Here the shift operator is defined by

$$(\mathcal{E}(\omega)u)_k = E(\theta^{k-1}\omega) u_{k-1}$$

Proof. Let us consider the sequence $\{v^N\}_{N \geq 1} \subset l^1(\mathbb{Z}; Y)$ defined by

$$v_k^N = v_k \text{ if } k \leq N \text{ else } 0.$$

One the other hand note that $\|\mathcal{E}(\omega)v^N - \mathcal{E}(\omega)v\|_{l^1(\mathbb{Z}; Y)} \rightarrow 0$ as $N \rightarrow \infty$ for each $\omega \in \widetilde{\Omega}$. Hence since for each $N \geq 1$, the map $\omega \rightarrow \mathcal{E}(\omega)v^N$ is measurable from $\widetilde{\Omega}$ into $l^1(\mathbb{Z}; Y)$ the result follows. We refer for instance to the textbook of Dudley [59] for metric space valued measurability. ■

Let us define for each $\omega \in \widetilde{\Omega}$ the three linear operators $\mathcal{P}_\alpha(\omega) \in \mathcal{L}(X)$ defined for $\alpha = s, c, u$ by

$$(\mathcal{P}_\alpha(\omega)u)_k = \Pi^\alpha(\theta^k \omega) u_k, \quad \forall k \in \mathbb{Z}, \quad \forall u \in X.$$

Let us introduce for each $\omega \in \widetilde{\Omega}$ the linear operator $\mathcal{J}_\omega$ defined on the Banach space $\mathbb{L}_{-\widehat{\rho}(\omega)}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\widehat{\rho}(\omega)}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\widehat{\rho}_0(\omega)}(\mathbb{Z}, \mathcal{L}(X))$ defined in (1.2.48). Hence one gets for each $\omega \in \widetilde{\Omega}$:

$$\begin{pmatrix} (\mathcal{A} + \mathcal{B})_s^+(\omega) \widehat{\mathcal{P}}_s(\omega) \\ (\mathcal{A} + \mathcal{B})_u^-(\omega) \widehat{\mathcal{P}}_u(\omega) \\ (\mathcal{A} + \mathcal{B})_c^-(\omega) \widehat{\mathcal{P}}_c(\omega) \end{pmatrix} = \sum_{k=0}^{\infty} \mathcal{J}_\omega^k \begin{pmatrix} \mathcal{A}_s^+(\omega) \mathcal{P}_s(\omega) \\ \mathcal{A}_u^-(\omega) \mathcal{P}_u(\omega) \\ \mathcal{A}_c^-(\omega) \mathcal{P}_c(\omega) \end{pmatrix}, \quad (1.4.1)$$

We fix $v \in X$. Then one has for each $\omega \in \tilde{\Omega}$:

$$\begin{pmatrix} e^{\hat{\rho}(\omega) \cdot} (\mathcal{A} + \mathcal{B})_s^{\cdot}(\omega) \hat{\mathcal{P}}_s(\omega) v \\ e^{\hat{\rho}(\omega) \cdot} (\mathcal{A} + \mathcal{B})_u^{\cdot}(\omega) \hat{\mathcal{P}}_u(\omega) v \\ e^{\hat{\rho}_0(\omega) |\cdot|} (\mathcal{A} + \mathcal{B})_c^{\cdot}(\omega) \hat{\mathcal{P}}_c(\omega) v \end{pmatrix} \in \mathcal{X} := l^\infty(\mathbb{N}; X) \times l^\infty(\mathbb{N}; X) \times l^\infty(\mathbb{Z}; X).$$

Let us introduce for each $\omega \in \tilde{\Omega}$ the linear operator $\mathcal{H}(\omega) : \mathcal{X} \rightarrow \mathbb{L}_{-\hat{\rho}(\omega)}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\hat{\rho}(\omega)}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\hat{\rho}_0(\omega)}(\mathbb{Z}, \mathcal{L}(X))$ by

$$\mathcal{H}(\omega) \begin{pmatrix} f^s \\ f^u \\ f^c \end{pmatrix} = \begin{pmatrix} e^{-\hat{\rho}(\omega) \cdot} f^s \\ e^{-\hat{\rho}(\omega) \cdot} f^u \\ e^{\hat{\rho}_0(\omega) |\cdot|} f^c \end{pmatrix}.$$

Recalling the definition of operator $\hat{\mathcal{J}}$ in Remark 1.2.11, one may introduce for each $\omega \in \tilde{\Omega}$ the linear operator $\tilde{\mathcal{J}}(\omega) \in \mathcal{L}(\mathcal{X})$ defined by

$$\tilde{\mathcal{J}}(\omega) = \mathcal{H}(\omega)^{-1} \circ \hat{\mathcal{J}}(\omega) \circ \mathcal{H}(\omega).$$

Note that for each $\omega \in \tilde{\Omega}$, $\|\tilde{\mathcal{J}}(\omega)\|_{\mathcal{L}(\mathcal{X})} < 1$ (see estimate of $\hat{\mathcal{J}}$ in Remark 1.2.11). Furthermore using this notation note that one has for each $\omega \in \tilde{\Omega}$

$$\begin{pmatrix} e^{\hat{\rho}(\omega) \cdot} (\mathcal{A} + \mathcal{B})_s^{\cdot}(\omega) \hat{\mathcal{P}}_s(\omega) v \\ e^{\hat{\rho}(\omega) \cdot} (\mathcal{A} + \mathcal{B})_u^{\cdot}(\omega) \hat{\mathcal{P}}_u(\omega) v \\ e^{\hat{\rho}_0(\omega) |\cdot|} (\mathcal{A} + \mathcal{B})_c^{\cdot}(\omega) \hat{\mathcal{P}}_c(\omega) v \end{pmatrix} = \sum_{k=0}^{\infty} [\tilde{\mathcal{J}}(\omega)]^k \begin{pmatrix} e^{-\hat{\rho}(\omega) \cdot} \mathcal{A}_s^{\cdot}(\omega) \mathcal{P}_s(\omega) v \\ e^{-\hat{\rho}(\omega) \cdot} \mathcal{A}_u^{\cdot}(\omega) \mathcal{P}_u(\omega) v \\ e^{\hat{\rho}_0(\omega) |\cdot|} \mathcal{A}_c^{\cdot}(\omega) \mathcal{P}_c(\omega) v \end{pmatrix}. \quad (1.4.2)$$

Here for each $\omega \in \tilde{\Omega}$ the above series converges for the topology of $\mathcal{X}$. To complete the proof of this result we require to introduce the following definition:

Definition 1.4.2 A map $f = (f^s, f^u, f^c) : \tilde{\Omega} \rightarrow \mathcal{X}$ is said to be component measurable if the map $\omega \rightarrow f_n^\alpha(\omega)$ is measurable from $\tilde{\Omega}$ into $(X, \mathcal{B}(X))$ for any $n \geq 0$ for $\alpha = s, u$ and any $n \in \mathbb{Z}$ for $\alpha = c$.

Then we claim that:

Claim 1.4.3 For each component measurable map $f : \tilde{\Omega} \rightarrow \mathcal{X}$ the map $\omega \rightarrow \tilde{\mathcal{J}}(\omega)f(\omega)$ is component measurable.

Before proving this claim let us observe that due to Lemma 1.4.1 the following map is component measurable from $\tilde{\Omega}$ to $\mathcal{X}$:

$$\omega \rightarrow \begin{pmatrix} e^{-\hat{\rho}(\omega) \cdot} \mathcal{A}_s^{\cdot}(\omega) \mathcal{P}_s(\omega) v \\ e^{-\hat{\rho}(\omega) \cdot} \mathcal{A}_u^{\cdot}(\omega) \mathcal{P}_u(\omega) v \\ e^{\hat{\rho}_0(\omega) |\cdot|} \mathcal{A}_c^{\cdot}(\omega) \mathcal{P}_c(\omega) v \end{pmatrix}$$

Hence Claim 1.4.3 coupled with (1.4.2) ensures that the the following map is component measurable

$$\omega \rightarrow \begin{pmatrix} e^{\widehat{\rho}(\omega) \cdot} (\mathcal{A} + \mathcal{B})_s^{\cdot}(\omega) \widehat{\mathcal{P}}_s(\omega) v \\ e^{\widehat{\rho}(\omega) \cdot} (\mathcal{A} + \mathcal{B})_u^{\cdot}(\omega) \widehat{\mathcal{P}}_u(\omega) v \\ e^{\widehat{\rho}_0(\omega) \cdot} (\mathcal{A} + \mathcal{B})_c^{\cdot}(\omega) \widehat{\mathcal{P}}_c(\omega) v \end{pmatrix}.$$

To finally complete the proof of Theorem 1.1.13 it remains to prove Claim 1.4.3.

Proof of Claim 1.4.3. Let $f = (f^s, f^u, f^c) : \widetilde{\Omega} \rightarrow \mathcal{X}$ be a component measurable map. Let us set

$$g(w) = \widehat{\mathcal{J}}(w) f(w) \text{ with } g = (g^s, g^u, g^c).$$

Then due the definition of $\widehat{\mathcal{J}}$ in Remark 1.2.11 one has for each $n \geq 0$

$$\begin{aligned} g_n^s &= -e^{\widehat{\rho}(\omega)n} \sum_{m=0}^{\infty} \mathcal{A}_s^{m+n}(\omega) \mathcal{P}_s(\omega) \mathcal{B}(\omega) [e^{-\widehat{\rho}(\omega)(m+1)} f_{m+1}^u(\omega) + e^{\widehat{\rho}_0(\omega)|m+1|} f_{-m-1}^c(\omega)] \\ &\quad + e^{\widehat{\rho}(\omega)n} \sum_{m=0}^{n-1} \mathcal{A}_s^{n-m-1}(\omega) \mathcal{P}_s(\omega) \mathcal{B}(\omega) e^{-\widehat{\rho}(\omega)m} f_m^s(\omega) \\ &\quad - e^{\widehat{\rho}(\omega)n} \sum_{m=0}^{+\infty} [\mathcal{A}_u^{-m-1}(\omega) \mathcal{P}_u(\omega) + \mathcal{A}_c^{-m-1}(\omega) \mathcal{P}_c(\omega)] \mathcal{B}(\omega) e^{-\widehat{\rho}(\omega)(n+m)} f_{n+m}^s(\omega). \end{aligned}$$

Here the series converges for the topology of X so that, using Lemma 1.4.1, g_n^s is measurable for each $n \geq 0$. Using the definition of the other components (see Remark 1.2.11), one may similarly conclude that g_n^u is measurable for each $n \geq 0$ while g_n^c is measurable for any $n \in \mathbb{Z}$. This completes the proof of Claim 1.4.3. $\blacksquare$

Chapitre 2

A finite-time condition for exponential trichotomy in infinite dynamical systems

2.1 Introduction

Consider the following linear non-autonomous discrete time dynamical system

$$\begin{cases} x_{n+1} = A_n x_n, \text{ for each } n \geq m, \\ \text{and} \\ x_m = x \in X, \end{cases} \quad (2.1.1)$$

where $x = \{x_n\}_{n \geq m}$ is a sequence in a Banach space $(X, \|\cdot\|)$ and $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}}$ is a sequence in $\mathcal{L}(X)$ the space of bounded linear operators on X . The discrete time evolution semigroup associated to the system (2.1.1) or equivalently associated to the sequence of bounded linear operator $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is defined as $\{U_{\mathbf{A}}(n, p)\}_{n \geq p} \subset \mathcal{L}(X)$ which is a parametrized family of bounded linear operators on X defined on

$$\Delta_+ := \{(n, p) \in \mathbb{Z}^2 : n \geq p\},$$

by

$$U_{\mathbf{A}}(n, p) := \begin{cases} A_{n-1} \dots A_p, & \text{if } n > p, \\ I, & \text{if } n = p. \end{cases}$$

In this work we will use the following notion of exponential dichotomy taken from Hale and Lin [73].

Definition 2.1.1 Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{I}} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ be given with $\mathbb{I}$ an interval of $\mathbb{Z}$. Then $U_{\mathbf{A}}$ has an **exponential dichotomy** (or $\mathbf{A}$ is **exponentially dichotomic**) on $\mathbb{I}$ with constant κ , exponents $\rho > 0$ if there exist two families of projectors $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{I}} : \mathbb{I} \rightarrow \mathcal{L}(X)$, with $\alpha = u, s$ satisfying the following properties:

(i) For all $n \in \mathbb{I}$ and $\alpha, \beta \in \{u, s\}$ we have

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ if } \alpha \neq \beta,$$

and

$$\Pi_n^s + \Pi_n^u = I.$$

(ii) For all $n, m \in \mathbb{I}$ with $n \geq m$ we have

$$U_{\mathbf{A}}^\alpha(n, m) := \Pi_n^\alpha U_{\mathbf{A}}(n, m) = U_{\mathbf{A}}(n, m) \Pi_m^\alpha, \text{ for } \alpha = u, s.$$

(iii) $U_{\mathbf{A}}^u(n, m)$ is invertible from $\Pi_m^u(X)$ into $\Pi_n^u(X)$ for all $n \geq m$ in $\mathbb{I}$ and its inverse is denoted by $U_{\mathbf{A}}^\alpha(m, n) : \Pi_n^\alpha(X) \rightarrow \Pi_m^\alpha(X)$.

(iv) For each $x \in X$ we have for all $n \geq m$ in $\mathbb{I}$

$$\|U_{\mathbf{A}}^s(n, m) \Pi_m^s x\| \leq \kappa e^{-\rho(n-m)} \|x\|, \quad (2.1.2)$$

$$\|U_{\mathbf{A}}^u(m, n) \Pi_n^u x\| \leq \kappa e^{-\rho(n-m)} \|x\|. \quad (2.1.3)$$

We also introduce the following notion of relatively dense subset of integers taken from Palmer [110].

Definition 2.1.2 Let $\mathbb{D} = \{\theta_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z}$ be a non decreasing sequence of integers and let $T_0 \in \mathbb{N} \setminus \{0\}$.

(i) We will say that $\mathbb{D}$ is a **relatively dense subset of integers for T_0** if every interval (in $\mathbb{Z}$) of length T_0 contains at least one point of $\mathbb{D}$.

(ii) We will say that $\mathbb{D}$ is a **T_0 -covering of $\mathbb{Z}$** if

$$\mathbb{Z} = \bigcup_{i \in \mathbb{Z}} [\theta_i, \theta_i + T_0],$$

where $[\theta_i, \theta_i + T_0]$ is understood as an interval in $\mathbb{Z}$.

Lemma 2.1.3 The properties (i) and (ii) in Definition 2.1.2 are equivalent.

Proof. **(i) $\implies$ (ii).** Let $n \in \mathbb{Z}$ be given. Then $[n - T_0, n]$ is an interval of length T_0 and there exists $\theta_i \in [n - T_0, n]$ so that $n \in [\theta_i, \theta_i + T_0]$. **(ii) $\implies$ (i).** Let $[n, p]$ be an interval of $\mathbb{Z}$ of length T_0 . Then one has $[n, p] = [n, n + T_0]$. By using (ii) and since $p = n + T_0 \in \mathbb{Z}$, we can find $\theta_i \in \mathbb{D}$ such that $n + T_0 \in [\theta_i, \theta_i + T_0]$. Therefore $\theta_i \in [n, n + T_0]$. The proof is completed. $\blacksquare$

In continuation of a recent work by Palmer [110] on the finite time condition for exponential dichotomy, here we will prove the following theorem.

Theorem 2.1.4 *Let $\rho > \hat{\rho} > 0$ and $\kappa \geq 1$. Let $\mathbb{D} = \{\theta_i\}_{i \in \mathbb{Z}}$ be a T_0 -covering of $\mathbb{Z}$. Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ be a given sequence of bounded linear operators on a Banach space X . Assume that*

$$\|A_n\|_{\mathcal{L}(X)} \leq K,$$

for some positive real constant $K > 0$.

There exist two constants $\hat{T} := \hat{T}(T_0, K, \kappa, \rho, \hat{\rho}) > 0$ and $\hat{\kappa} := \hat{\kappa}(K, \kappa, \rho) \geq \kappa$ such that for each $T \geq \hat{T}$ if $\mathbf{A}$ is exponentially dichotomic on each intervals $[\theta_i, \theta_i + T]$ of $\mathbb{Z}$ (with constant $\kappa \geq 1$ and exponent $\rho > 0$) then $\mathbf{A}$ is exponentially dichotomic on $\mathbb{Z}$ (with constant $\hat{\kappa}$ and exponent $\hat{\rho}$).

Compared to Theorem 2.1 in Palmer [110] one may observe that in Theorem 2.1.4 the invertibility of the bounded linear operators A_n is not required. We also refer to Palmer [113, Lemma 2.17] for early results on this topic. In [113, Lemma 2.17], Palmer considers intervals of the form $[(i-1)m, im]$ (where m a fixed positive integer) and assume that the projectors

$$\{\Pi_n^{i\alpha}, n \in [(i-1)m, im]\},$$

for $\alpha = u, s$ are close at the endpoints. That is to say that the norm of $\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}$ is small enough. In Palmer [110] this condition is not required anymore. Actually the closeness of the projectors follows from the exponential dichotomy property on intervals large enough (see Lemma 2.4.8). Nevertheless the proof of Lemma 2.4.8 will be different compared to Palmer [110] due to the non invertibility of the linear operators.

We should also mention that there are various perturbation results using finite time conditions. We refer to Henry [76, Theorem 7.6.8 p. 234], Sakamoto [145, Theorem 4], Palmer [110, 111, 112, 113] and Pötzsche [129] for more results on this subject.

Theorem 2.1.4 has many consequences, namely shadowing in dynamical systems, robustness of hyperbolic sets (see also Sacker and Sell [140, Theorem 6] for spectral theory approach), hyperbolicity along pseudo orbits, slowly varying systems, almost periodic systems... (see for instance [110, Section 3] and [45]). This concept is also the main tools in the theory of invariant hyperbolic sets and invariant normally hyperbolic sets using Lyapunov-Perron approach. We refer the reader to Sakamoto [144, 145], Henry [76] for more results on this topic.

The main goal of this chapter is to extend Palmer's [110] results from the exponential dichotomy to the exponential trichotomy (see section 2.2) for infinite dimensional dynamical systems.

The plan of this chapter is the following. In section 2.2 we will present Theorem 2.2.3 which is the main result of this chapter. In section 2.3 we present an application of this theorem to study the persistence of exponential trichotomy along pseudo orbits. The last part of this chapter is mainly devoted to the proof of Theorem 2.2.3.

2.2 Main results

Before presenting the main results of this chapter, we first need to define the notion of exponential trichotomy. The notion used here is taken from Hale and Lin [73].

Definition 2.2.1 Let $\mathbb{I}$ be an interval of $\mathbb{Z}$ and let $\mathbf{A} = \{A_n\}_{n \in \mathbb{I}} : \mathbb{I} \rightarrow \mathcal{L}(X)$ be a map. Then $U_{\mathbf{A}}$ has an **exponential trichotomy** (or $\mathbf{A}$ is **exponentially trichotomic**) on $\mathbb{I}$ with constant κ , exponents $0 < \rho_0 < \rho$ if there exist three families of projectors $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{I}} : \mathbb{I} \rightarrow \mathcal{L}(X)$, with $\alpha = u, s, c$ satisfying the following properties:

(i) For all $n \in \mathbb{I}$ and $\alpha, \beta \in \{u, s, c\}$ we have

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ if } \alpha \neq \beta,$$

and

$$\Pi_n^s + \Pi_n^u + \Pi_n^c = I.$$

(ii) For all $n, m \in \mathbb{I}$ with $n \geq m$ we have

$$U_{\mathbf{A}}^\alpha(n, m) := \Pi_n^\alpha U_{\mathbf{A}}(n, m) = U_{\mathbf{A}}(n, m) \Pi_m^\alpha, \text{ for } \alpha = u, s, c.$$

(iii) $U_{\mathbf{A}}^\alpha(n, m)$ is invertible from $\Pi_m^\alpha(X)$ into $\Pi_n^\alpha(X)$ for all $n \geq m$ in $\mathbb{I}$, $\alpha = u, c$ and its inverse is denoted by $U_{\mathbf{A}}^\alpha(m, n) : \Pi_m^\alpha(X) \rightarrow \Pi_n^\alpha(X)$.

(iv) For each $x \in X$ we have for all $n, m \in \mathbb{I}$

$$\|U_{\mathbf{A}}^c(n, m) \Pi_m^c x\| \leq \kappa e^{\rho_0|n-m|} \|x\|, \quad (2.2.1)$$

and if $n \geq m$

$$\|U_{\mathbf{A}}^s(n, m) \Pi_m^s x\| \leq \kappa e^{-\rho(n-m)} \|x\|, \quad (2.2.2)$$

$$\|U_{\mathbf{A}}^u(m, n) \Pi_n^u x\| \leq \kappa e^{-\rho(n-m)} \|x\|. \quad (2.2.3)$$

Remark 2.2.2 The above definition coincides with the Definition 2.1.1 whenever $\Pi_n^c = 0_{\mathcal{L}(X)}, \forall n \in \mathbb{I}$.

The main result of this chapter is the following.

Theorem 2.2.3 Let $\rho > \hat{\rho} > \hat{\rho}_0 > \rho_0 > 0$ and $\kappa \geq 1$. Let $\mathbb{D} = \{\theta_i\}_{i \in \mathbb{Z}}$ be a T_0 -covering of $\mathbb{Z}$. Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ be a given sequence of bounded linear operators on a Banach space X . Assume that

$$\|A_n\|_{\mathcal{L}(X)} \leq K,$$

for some positive real constant $K > 0$.

There exists $\hat{T} := \hat{T}(T_0, K, \kappa, \rho, \hat{\rho}, \rho_0, \hat{\rho}_0) > 0$ and $\hat{\kappa} := \hat{\kappa}(K, \kappa, \rho) \geq \kappa$ such that for each $T \geq \hat{T}$ if $\mathbf{A}$ is exponentially trichotomic on each intervals $[\theta_i, \theta_i + T]$ of $\mathbb{Z}$ (with constant κ exponents ρ and ρ_0) then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ (with constant $\hat{\kappa}$ exponents $\hat{\rho}$ and $\hat{\rho}_0$).

2.3 Application

In this section, we present an application of Theorem 2.2.3 to study the persistence of the normal hyperbolicity along pseudo orbits for the discrete time dynamical system

$$x_{n+1} = F(x_n), \forall n \geq 0, \text{ with } x_0 = x \in X, \quad (2.3.1)$$

whenever $F : X \rightarrow X$ is a continuously differentiable map on the Banach space X .

Recall that a sequence $x = \{x_n\}_{n \in \mathbb{Z}} \subset X$ is called a δ -**pseudo-orbit** for F if

$$\|x_{n+1} - F(x_n)\| \leq \delta, \forall n \in \mathbb{Z}, \quad (2.3.2)$$

while we will say that $x = \{x_n\}_{n \in \mathbb{Z}}$ is a **complete orbit** for F if

$$x_{n+1} = F(x_n), \forall n \in \mathbb{Z}.$$

So a complete orbit is nothing but a δ -pseudo-orbit with $\delta = 0$ in (2.3.2).

Next we recall the definition of normally hyperbolic invariant sets inspired by Bates, Lu and Zeng [19]. This notion plays a crucial role in the context of the theory of geometric singular perturbation and we refer to [19, 64, 77] (and reference therein) for more results.

Definition 2.3.1 (Normally hyperbolic set) Let $F : X \rightarrow X$ be a continuously differentiable map on the Banach space X . Let $M \subset X$ be an invariant subset for F , that is to say that

$$F(M) = M.$$

Then we will say that $M \subset X$ is **normally hyperbolic** for F if the following properties are satisfied:

(i) For each $x \in M$, there exists three closed subspaces X_x^α with $\alpha = s, c, u$ such that

$$X = X_x^u \oplus X_x^s \oplus X_x^c, \quad (2.3.3)$$

for each $\alpha = s, c, u$, $DF(x)(X_x^\alpha) \subset X_{F(x)}^\alpha$ and for $\alpha = u, c$, the map $DF(x)|_{X_x^\alpha}$ is invertible from X_x^α onto $X_{F(x)}^\alpha$;

(ii) There exist a constant $\kappa \geq 1$, rates $0 < \rho_0 < \rho$ such that for each $n \geq 0$ and each $x \in M$,

$$\begin{cases} \|DF^n(x)|_{X_x^s}\| \leq \kappa e^{-\rho n}, \\ \text{and} \\ \frac{1}{\kappa} e^{\rho n} \leq \inf \{\|DF^n(x)x^u\| : x^u \in X_x^u \text{ and } \|x^u\| = 1\}, \end{cases} \quad (2.3.4)$$

and

$$\begin{aligned} \frac{1}{\kappa} e^{-\rho_0 n} &\leq \inf \{\|DF^n(x)x^c\| : x^c \in X_x^c \text{ and } \|x^c\| = 1\} \\ &\leq \|DF^n(x)|_{X_x^c}\| \leq \kappa e^{\rho_0 n}, \end{aligned} \quad (2.3.5)$$

where

$$\|DF^n(x)|_{X_x^\alpha}\| = \sup \{\|DF^n(x)x\| : x \in X_x^\alpha \text{ and } \|x\| = 1\} \text{ for } \alpha = s, c.$$

By using the property (i), for each $x \in M$, we can define three projectors Π_x^α for $\alpha = u, s, c$ associated to the state decomposition (2.3.3). For each $x \in M$ and each $\alpha \in \{s, c, u\}$ these projectors are uniquely determined by

$$\mathcal{R}(\Pi_x^\alpha) = X_x^\alpha \text{ and } \mathcal{N}(\Pi_x^\alpha) = \bigoplus_{\alpha' \in \{s, c, u\} \setminus \{\alpha\}} X_x^{\alpha'}.$$

Define an (open) ε -neighborhood of a subset $M \subset X$

$$\mathcal{V}(M, \varepsilon) := \{x \in X : d(x, M) < \varepsilon\}$$

where

$$d(x, M) := \inf_{y \in M} \|x - y\|.$$

One may equivalently define $\mathcal{V}(M, \varepsilon)$ as

$$\mathcal{V}(M, \varepsilon) := \bigcup_{x \in M} B(x, \varepsilon)$$

where

$$B(x, \varepsilon) := \{y \in X : \|x - y\| < \varepsilon\}.$$

Now we make the following assumption.

Assumption 2.3.2 *Let $F : X \rightarrow X$ be a continuously differentiable map on the Banach space X . Let $M \subset X$ be an invariant subset for F . We assume that:*

(i) *M is normally hyperbolic for F with the constant $\kappa \geq 1$ and the rates $0 < \rho_0 < \rho$.*

(ii) *There exists $\kappa_0 \geq 1$ such that*

$$\sup_{x \in M} \|\Pi_x^\alpha\|_{\mathcal{L}(X)} \leq \kappa_0, \quad \alpha = s, c, u. \quad (2.3.6)$$

(iii) *There exist $K > 0$ and $\varepsilon_0 > 0$ such that the map $x \rightarrow DF(x)$ is uniformly continuous from $\mathcal{V}(M, \varepsilon_0)$ into $\mathcal{L}(X)$ and*

$$\begin{cases} \|F(x) - F(y)\| \leq K \|x - y\|, \forall x, y \in \mathcal{V}(M, \varepsilon_0), \\ \text{and} \\ \sup_{x \in M} \|DF(x)\|_{\mathcal{L}(X)} \leq K. \end{cases} \quad (2.3.7)$$

Remark 2.3.3 *Whenever $\mathcal{V}(M, \varepsilon_0)$ is convex, assertion (iii) can be replaced by*

$$\sup_{x \in \mathcal{V}(M, \varepsilon_0)} \|DF(x)\|_{\mathcal{L}(X)} \leq K.$$

The main result of this section is the following Proposition.

Proposition 2.3.4 *Let Assumption 2.3.2 be satisfied. Let $0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$. There exist two constants*

$$\delta_0 := \delta_0(K, \kappa, \kappa_0, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}) > 0 \text{ and } \varepsilon_0 := \varepsilon_0(K, \kappa, \kappa_0, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}) > 0$$

such that for each $\delta \in [0, \delta_0)$, $\varepsilon \in [0, \varepsilon_0)$ if $x = \{x_n\}_{n \in \mathbb{Z}} \subset \mathcal{V}(M, \varepsilon)$ is a δ -pseudo-orbit of F in the neighborhood $\mathcal{V}(M, \varepsilon)$ of M then $DF(\mathbf{x}) = \{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on $\mathbb{Z}$ with a constant $\widehat{\kappa} := \widehat{\kappa}(K, \kappa, \rho_0, \rho)$ and exponents $\widehat{\rho}_0, \widehat{\rho}$.

Remark 2.3.5 *Under Assumption 2.3.2, for each complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of F in M the family of bounded linear operators $DF(\mathbf{x}) = \{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\widetilde{\kappa} := \kappa \kappa_0$ and exponents ρ_0, ρ .*

2.4 Proof of Theorem 2.2.3

We will start this section by considering the case where $\mathbf{A} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ is exponentially trichotomic on intervals of the form

$$[(i-1)m, im] \text{ for each } i \in \mathbb{Z},$$

where $m > 0$ is some integer. We observe that $\mathbb{Z}$ is a disjoint union of the intervals $\{[(i-1)m, im]\}_{i \in \mathbb{Z}}$. Therefore

$$\mathbb{Z} = \bigcup_{i \in \mathbb{Z}} [(i-1)m, im[= \bigcup_{i \in \mathbb{Z}} [(i-1)m, im-1].$$

As we will see at the end of this section, this is not a real restriction for the proof of Theorem 2.2.3. In order to clarify the notion of exponential trichotomy on a family of intervals, we introduce the following definition.

Definition 2.4.1 *Let $m > 0$ be an integer. We will say that a sequence of bounded linear operators $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on the family of intervals $\{[(i-1)m, im] : i \in \mathbb{Z}\}$ with uniform constant κ and uniform exponents $\rho > 0$ and $\rho_0 \in (0, \rho)$ if on each interval $[(i-1)m, im]$ the following properties are satisfied:*

- (i) *There exist three families of projectors $\{\Pi_n^{i\alpha}, n \in [(i-1)m, im]\} \subset \mathcal{L}(X)$, with $\alpha = u, s, c$ satisfying for each $n \in [(i-1)m, im]$*

$$\Pi_n^{i\alpha} \Pi_n^{i\beta} = 0, \text{ if } \alpha \neq \beta,$$

and

$$\Pi_n^{is} + \Pi_n^{ic} + \Pi_n^{iu} = I.$$

- (ii) *For all $n, p \in [(i-1)m, im]$ with $n \geq p$ we have*

$$U_{\mathbf{A}}^{i\alpha}(n, p) := \Pi_n^{i\alpha} U_{\mathbf{A}}(n, p) = U_{\mathbf{A}}(n, p) \Pi_p^{i\alpha}, \text{ for } \alpha = u, s, c. \quad (2.4.1)$$

(iii) The map $U_{\mathbf{A}}^{i\alpha}(n, p)$ is invertible from $\Pi_p^{i\alpha}(X)$ into $\Pi_n^{i\alpha}(X)$ for all $n \geq p$ in $[(i-1)m, im]$ and $\alpha = u, c$ and its inverse is denoted by

$$U_{\mathbf{A}}^{i\alpha}(p, n) : \Pi_n^\alpha(X) \rightarrow \Pi_p^\alpha(X). \quad (2.4.2)$$

(iv) For each $x \in X$ we have for all $n, p \in [(i-1)m, im]$

$$\|U_{\mathbf{A}}^{ic}(n, p) \Pi_p^{ic} x\| \leq \kappa e^{\rho_0|n-p|} \|x\|, \quad (2.4.3)$$

and if $n \geq p$

$$\|U_{\mathbf{A}}^{is}(n, p) \Pi_p^{is} x\| \leq \kappa e^{-\rho(n-p)} \|x\|, \quad (2.4.4)$$

$$\|U_{\mathbf{A}}^{iu}(p, n) \Pi_n^{iu} x\| \leq \kappa e^{-\rho(n-p)} \|x\|. \quad (2.4.5)$$

Remark 2.4.2 Let us notice that due to the condition (iv), we have for each $i \in \mathbb{Z}$ and each $n \in [(i-1)m, im]$

$$\|\Pi_n^{i\alpha}\|_{\mathcal{L}(X)} \leq \kappa, \quad \alpha = u, s, c. \quad (2.4.6)$$

Let us recall a perturbation theorem proved in Ducrot, Magal and Seydi [56, Theorem 1.1.8].

Theorem 2.4.3 (Perturbation) Let $\mathbf{A} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ be a given uniformly bounded sequence such that $U_{\mathbf{A}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three families of projectors $\{\Pi^\alpha : \mathbb{Z} \rightarrow \mathcal{L}(X)\}_{\alpha=s,c,u}$. Let $\rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and $\hat{\kappa} > \kappa$ be given. Then there exists $\delta_0 := \delta_0(\rho_0, \hat{\rho}_0, \hat{\rho}, \rho, \kappa, \hat{\kappa}) \in (0, \sqrt{2} - 1)$ such that for each $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ and each $\mathbf{B} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ with

$$\sup_{n \in \mathbb{Z}} \|B_n\|_{\mathcal{L}(X)} \leq \delta,$$

the evolution semigroup $U_{\mathbf{A}+\mathbf{B}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant $\hat{\kappa}$, exponents $\hat{\rho}$, $\hat{\rho}_0$ and projectors $\{\hat{\Pi}^\alpha\}_{\alpha=s,c,u}$. For each $n \in \mathbb{Z}$ and each $\alpha = u, s, c$, the spaces $R(\Pi_n^\alpha)$ and $R(\hat{\Pi}_n^\alpha)$ are isomorphic. Moreover the following perturbation estimates hold true: we have for all $n \geq p$,

$$\|U_{\mathbf{A}+\mathbf{B}}^s(n, p) - U_{\mathbf{A}}^s(n, p)\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\hat{\rho}(n-p)}, \quad (2.4.7)$$

$$\|U_{\mathbf{A}+\mathbf{B}}^u(p, n) - U_{\mathbf{A}}^u(p, n)\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\hat{\rho}(n-p)}, \quad (2.4.8)$$

and for all $(n, p) \in \mathbb{Z}^2$

$$\|U_{\mathbf{A}+\mathbf{B}}^c(n, p) - U_{\mathbf{A}}^c(n, p)\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{\hat{\rho}_0|n-p|}. \quad (2.4.9)$$

In the next lemma we will show that if $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on the family of intervals $\{[i-1m, im]\}_{i \in \mathbb{Z}}$ (m a positive integer large enough) and that the norm of $\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}$, $i \in \mathbb{Z}$ is small then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$. This lemma generalizes Lemma 2.3 in [110].

Lemma 2.4.4 *Let $\rho > \hat{\rho} > \hat{\rho}_0 > \rho_0 > 0$, $\kappa \geq 1$ and $K \geq 1$ be fixed. Define*

$$\begin{cases} \tilde{\kappa} := \max \{2\kappa^3 e^{\rho+\rho_0}, (3\kappa K + K) \kappa\} \\ \text{and} \\ m_0 := \max \left\{ 2, \frac{2}{\rho-\hat{\rho}} \ln \tilde{\kappa}, \frac{2}{\hat{\rho}_0-\rho_0} \ln \tilde{\kappa} \right\}. \end{cases} \quad (2.4.10)$$

Let $A = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ be a sequence of bounded linear operators on X . Assume that

(i)

$$\sup_{n \in \mathbb{Z}} \|A_n\|_{\mathcal{L}(X)} \leq K. \quad (2.4.11)$$

(ii) *There exists an integer $m \geq m_0$ such that $A = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on the family of intervals $\{[(i-1)m, im] : i \in \mathbb{Z}\}$ with uniform constant κ and uniform exponents ρ and ρ_0 .*

Then there exist two constants $\hat{\kappa} := \hat{\kappa}(K, \kappa, \rho, \rho_0) > \tilde{\kappa}^2$ and $\eta_0 := \eta_0(\rho_0, \hat{\rho}_0, \hat{\rho}, \rho, \kappa, \hat{\kappa}, K) \in (0, \sqrt{2} - 1)$ such that

$$\left\| \Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} \right\|_{\mathcal{L}(X)} \leq \eta_0, \forall i \in \mathbb{Z}, \forall \alpha = u, s, c, \quad (2.4.12)$$

implies that $A = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\hat{\kappa}$ and exponents $\hat{\rho}$ and $\hat{\rho}_0$.

In order to prove Lemma 2.4.4, we will need the following auxiliary lemma (see [19] or [56, Lemma 1.2.1]).

Lemma 2.4.5 *Let $\Pi : X \rightarrow X$ and $\hat{\Pi} : X \rightarrow X$ be two bounded linear projectors on a Banach space X . Assume that*

$$\left\| \Pi - \hat{\Pi} \right\|_{\mathcal{L}(X)} < \delta \text{ with } 0 < \delta < \sqrt{2} - 1, \quad (2.4.13)$$

Then Π is invertible from $\hat{\Pi}(X)$ into $\Pi(X)$ and

$$\left\| \left(\Pi|_{\hat{\Pi}(X)} \right)^{-1} x \right\| \leq \frac{1}{1-\delta} \|x\|, \forall x \in \Pi(X). \quad (2.4.14)$$

Remark 2.4.6 *By symmetry, the bounded linear projector $\hat{\Pi}$ is also invertible from $\Pi(X)$ into $\hat{\Pi}(X)$ and*

$$\left\| \left(\hat{\Pi}|_{\Pi(X)} \right)^{-1} x \right\| \leq \frac{1}{1-\delta} \|x\|, \forall x \in \hat{\Pi}(X) \quad (2.4.15)$$

Proof of Lemma 2.4.4. The principle of the proof is to construct an auxiliary sequence of bounded linear operators $\bar{A} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ and three families of projectors $\{\bar{\Pi}^\alpha\}_{\alpha=u,s,c} \subset \mathcal{L}(X)$, such that $\bar{A}$ is exponentially trichotomic on $\mathbb{Z}$, and $\bar{A}$ is close to A .

To do so let $m \geq m_0$ be the positive integer defined in the assumption of Lemma 2.4.4. Recall that for each $i \in \mathbb{Z}$, A is exponentially trichotomic on $[(i-1)m, im]$, with uniform constant κ and uniform exponents $\rho > 0$, $\rho_0 \in (0, \rho)$ and projectors $\{\Pi_n^{i\alpha}, n \in [(i-1)m, im]\} \subset \mathcal{L}(X)$, $\alpha = u, s, c$ satisfying the properties (i)-(iv) stated in Definition 2.4.1.

We define a family of bounded linear projectors $\{\bar{\Pi}_n^\alpha\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$, $\alpha = u, s, c$ given on each interval $[(i-1)m, im[= [(i-1)m, im-1]$ of $\mathbb{Z}$ by

$$\bar{\Pi}_n^\alpha := \Pi_n^{i\alpha}, \text{ for } (i-1)m \leq n \leq im-1. \quad (2.4.16)$$

It follows that

$$\bar{\Pi}_{im}^\alpha = \Pi_{im}^{(i+1)\alpha}, \forall i \in \mathbb{Z}. \quad (2.4.17)$$

We define $\bar{A} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ on each intervals $[(i-1)m, im[= [(i-1)m, im-1]$ of $\mathbb{Z}$ by

$$\bar{A}_n := \begin{cases} A_n, & \text{if } n \in [(i-1)m, im-2], \\ \sum_{\alpha=u,s,c} \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} & \text{if } n = im-1. \end{cases} \quad (2.4.18)$$

Next we will prove that $\bar{A} = \{\bar{A}_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic with projectors $\{\bar{\Pi}^\alpha\}_{\alpha=u,s,c}$ and that $\bar{A}$ is close to A .

To do so, we verify the properties (i)-(iv) stated in Definition 2.2.1. Without loss of generality, we can assume that $\eta_0 \in (0, 1/2)$, and that

$$\left\| \Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} \right\|_{\mathcal{L}(X)} \leq \eta_0, \forall i \in \mathbb{Z} \text{ and } \alpha = u, s, c, \quad (2.4.19)$$

Proof of (i). From (2.4.16) for each $n \in \mathbb{Z}$ we have

$$\bar{\Pi}_n^u + \bar{\Pi}_n^s + \bar{\Pi}_n^c = I_{\mathcal{L}(X)} \text{ and } \bar{\Pi}_n^\beta \bar{\Pi}_n^\alpha = 0_{\mathcal{L}(X)} \text{ for } \alpha, \beta \in \{u, s, c\} \text{ with } \alpha \neq \beta,$$

and property (i) is satisfied.

Proof of (ii). We will prove that

$$\bar{\Pi}_{n+1}^\alpha \bar{A}_n = \bar{A}_n \bar{\Pi}_n^\alpha, \forall n \in \mathbb{Z} \text{ and } \alpha = u, s, c, \quad (2.4.20)$$

or equivalently

$$U_{\bar{A}}^\alpha(n, p) := U_{\bar{A}}(n, p) \bar{\Pi}_p^\alpha = \bar{\Pi}_n^\alpha U_{\bar{A}}(n, p), \forall n \geq p \text{ and } \alpha = u, s, c. \quad (2.4.21)$$

Let $n \in \mathbb{Z}$ be given. Let $i \in \mathbb{Z}$ be given such that $n \in [(i-1)m, im[$. Then if $(i-1)m \leq n+1 < im$ that is $n \neq im-1$ by using (2.4.16) and (2.4.18), property (2.4.20) is clearly verified.

If $n + 1 = im$ (i.e. $n = im - 1$) one has from (2.4.17)-(2.4.18) that for each $\alpha = u, s, c$

$$\begin{aligned}
 \bar{\Pi}_{n+1}^\alpha \bar{A}_n &= \bar{\Pi}_{im}^\alpha \bar{A}_{im-1} \\
 &= \Pi_{im}^{(i+1)\alpha} \sum_{\beta=u,s,c} \Pi_{im}^{(i+1)\beta} A_{im-1} \Pi_{im-1}^{i\beta} \\
 &= \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} \\
 &= \left[\sum_{\beta=u,s,c} \Pi_{im}^{(i+1)\beta} A_{im-1} \Pi_{im-1}^{i\beta} \right] \Pi_{im-1}^{i\alpha} \\
 &= \bar{A}_{im-1} \Pi_{im-1}^{i\alpha},
 \end{aligned}$$

so we obtain

$$\bar{\Pi}_{n+1}^\alpha \bar{A}_n = \bar{A}_{im-1} \Pi_{im-1}^{i\alpha} = \bar{A}_{im-1} \bar{\Pi}_{im-1}^\alpha = \bar{A}_n \bar{\Pi}_n^\alpha, \quad \forall \alpha = u, s, c.$$

Proof of (iii). We need to prove that for each $n \geq p$ and each $\alpha = u, c$ the linear operator $U_{\bar{\mathbf{A}}}^\alpha(n, p)$ is invertible from $\bar{\Pi}_p^\alpha(X)$ into $\bar{\Pi}_n^\alpha(X)$. Due to the definition of the evolution semigroup $U_{\bar{\mathbf{A}}}^\alpha$ in (2.4.21) it is sufficient to prove that for each $n \in \mathbb{Z}$ and each $\alpha = u, c$, the operator $\bar{A}_n \bar{\Pi}_n^\alpha$ is invertible from $\bar{\Pi}_n^\alpha(X)$ into $\bar{\Pi}_{n+1}^\alpha(X)$. But on each interval $[(i-1)m, im[$, $i \in \mathbb{Z}$ the operator

$$\bar{A}_n \bar{\Pi}_n^\alpha = A_n \Pi_n^{i\alpha}, \quad \text{with } n \in [(i-1)m, im[\text{ and } n \neq im-1,$$

is invertible from $\bar{\Pi}_n^\alpha(X) = \Pi_n^{i\alpha}(X)$ into $\bar{\Pi}_{n+1}^\alpha(X) = \Pi_{n+1}^{i\alpha}(X)$. Therefore, it is sufficient to prove that $\bar{A}_{im-1} \bar{\Pi}_{im-1}^\alpha$ is invertible from $\bar{\Pi}_{im-1}^\alpha(X) = \Pi_{im-1}^{i\alpha}(X)$ into $\bar{\Pi}_{im}^\alpha(X) = \Pi_{im}^{(i+1)\alpha}(X)$. But due to (2.4.18), we have

$$\bar{A}_{im-1} \bar{\Pi}_{im-1}^\alpha = \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, s, c,$$

hence

$$\bar{A}_{im-1} \bar{\Pi}_{im-1}^\alpha = \Pi_{im}^{(i+1)\alpha} \bar{A}_{im-1}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, s, c. \quad (2.4.22)$$

But by assumption A_{im-1} is invertible from $\bar{\Pi}_{im-1}^\alpha(X) = \Pi_{im-1}^{i\alpha}(X)$ into $\Pi_{im}^{i\alpha}(X)$ and since

$$\left\| \Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} \right\|_{\mathcal{L}(X)} < \sqrt{2} - 1,$$

we deduce from Lemma 2.4.5, that $\Pi_{im}^{(i+1)\alpha}$ is invertible from $\Pi_{im}^{i\alpha}(X)$ into $\Pi_{im}^{(i+1)\alpha}(X)$. Therefore $\bar{A}_{im-1}$ is invertible from $\Pi_{im-1}^{i\alpha}(X)$ into $\Pi_{im}^{(i+1)\alpha}(X)$ and

$$(\bar{A}_{im-1} \bar{\Pi}_{im-1}^\alpha)^{-1} = (A_{im-1} \Pi_{im-1}^{i\alpha})^{-1} \left[\Pi_{im}^{(i+1)\alpha} \right]^{-1}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, c.$$

So we deduce that for each $n \geq p$ and each $\alpha = u, c$, the operator $U_{\bar{\mathbf{A}}}^\alpha(n, p)$ is invertible from $\mathcal{R}(\bar{\Pi}_p^\alpha)$ into $\mathcal{R}(\bar{\Pi}_n^\alpha)$ and its inverse is defined by

$$U_{\bar{\mathbf{A}}}^\alpha(p, n) : \mathcal{R}(\bar{\Pi}_n^\alpha) \rightarrow \mathcal{R}(\bar{\Pi}_p^\alpha). \quad (2.4.23)$$

More precisely, for each $i \in \mathbb{Z}$ and each $\alpha = u, c$ the inverse of $U_{\bar{\mathbf{A}}}^\alpha(im, im-1)$ is given by

$$U_{\bar{\mathbf{A}}}^\alpha(im-1, im) : \mathcal{R}(\bar{\Pi}_{im}^\alpha) \rightarrow \mathcal{R}(\bar{\Pi}_{im-1}^\alpha).$$

which is defined by

$$U_{\bar{\mathbf{A}}}^\alpha(im-1, im) = U_{\mathbf{A}}^{i\alpha}(im-1, im) \left[\Pi_{im}^{(i+1)\alpha} \right]^{-1}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, c, \quad (2.4.24)$$

and by using again Lemma 2.4.5 combined with (2.4.19), we also deduce that for each $i \in \mathbb{Z}$ and each $\alpha = u, c$

$$\|U_{\bar{\mathbf{A}}}^\alpha(im-1, im) \bar{\Pi}_{im}^\alpha\|_{\mathcal{L}(X)} \leq \frac{\kappa^2 e^{\rho_0}}{1 - \eta_0} \leq 2\kappa^2 e^{\rho_0}, \quad \forall \eta_0 \in (0, 1/2). \quad (2.4.25)$$

Closeness of A and $\bar{A}$: Let us now give some estimate of the supremum norm of $A - \bar{A}$. By using (2.4.18) we have

$$\sup_{n \in \mathbb{Z}} \|\bar{A}_n - A_n\|_{\mathcal{L}(X)} = \sup_{i \in \mathbb{Z}} \|\bar{A}_{im-1} - A_{im-1}\|_{\mathcal{L}(X)}.$$

But

$$\begin{aligned} \|A_{im-1} - \bar{A}_{im-1}\|_{\mathcal{L}(X)} &= \left\| \sum_{\alpha=u,s,c} \Pi_{im}^{i\alpha} A_{im-1} - \sum_{\alpha=u,s,c} \Pi_{im}^{(i+1)\alpha} \bar{A}_{im-1} \right\|_{\mathcal{L}(X)} \\ &= \left\| \sum_{\alpha=u,s,c} \left[\Pi_{im}^{i\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} \right] \right\|_{\mathcal{L}(X)} \\ &\leq \sum_{\alpha=u,s,c} \left\| \Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} \right\|_{\mathcal{L}(X)} \|A_{im-1} \Pi_{im-1}^{i\alpha}\|_{\mathcal{L}(X)} \\ &\leq \sum_{\alpha=u,s,c} \left\| \Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} \right\|_{\mathcal{L}(X)} \|A_{im-1}\|_{\mathcal{L}(X)} \|\Pi_{im-1}^{i\alpha}\|_{\mathcal{L}(X)}. \end{aligned}$$

So by using (2.4.6), (2.4.11) and (2.4.19) we deduce that

$$\sup_{n \in \mathbb{Z}} \|A_n - \bar{A}_n\|_{\mathcal{L}(X)} \leq 3\kappa K \eta_0, \quad \forall \eta_0 \in (0, \sqrt{2} - 1), \quad (2.4.26)$$

and by using (2.4.11) and (2.4.26) we obtain

$$\sup_{n \in \mathbb{Z}} \|\bar{A}_n\|_{\mathcal{L}(X)} \leq \sup_{n \in \mathbb{Z}} \|A_n - \bar{A}_n\|_{\mathcal{L}(X)} + \sup_{n \in \mathbb{Z}} \|A_n\|_{\mathcal{L}(X)} \leq 3\kappa K + K. \quad (2.4.27)$$

Proof of (iv). We will proceed into three steps to provide some growth rate estimates for $U_{\bar{\mathbf{A}}}$.

Step 1. Let $i \in \mathbb{Z}$ be given. Let $n, p \in [(i-1)m, im]$ be such that

$$(i-1)m \leq p \leq n < im.$$

Then we have

$$U_{\mathbf{A}}^{\alpha}(n, p) = U_{\bar{\mathbf{A}}}(n, p) \bar{\Pi}_p^{\alpha} = U_{\mathbf{A}}(n, p) \Pi_p^{i\alpha} = U_{\mathbf{A}}^{i\alpha}(n, p), \text{ for } \alpha = u, s, c.$$

Hence we deduce from (2.4.3), (2.4.4) and (2.4.5) that for $(i-1)m \leq p \leq n < im$

$$\begin{aligned} \|U_{\mathbf{A}}^s(n, p)\|_{\mathcal{L}(X)} &\leq \kappa e^{-\rho(n-p)}, \quad \|U_{\mathbf{A}}^u(p, n)\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)}, \\ \|U_{\mathbf{A}}^c(n, p)\|_{\mathcal{L}(X)} &\leq \kappa e^{\rho_0(n-p)}, \quad \|U_{\mathbf{A}}^c(p, n)\|_{\mathcal{L}(X)} \leq \kappa e^{\rho_0(n-p)}. \end{aligned} \quad (2.4.28)$$

Step 2. Let $i \in \mathbb{Z}$ be given. Let $n, p \in [(i-1)m, im]$ be such that $(i-1)m \leq p \leq n = im$. First note that we have

$$U_{\mathbf{A}}^{\alpha}(im, p) = \bar{\Pi}_{im}^{\alpha} = \bar{\Pi}_{im}^{(i+1)\alpha} \text{ if } p = im$$

and

$$U_{\mathbf{A}}^{\alpha}(im, p) = U_{\mathbf{A}}^{\alpha}(im, im-1) U_{\mathbf{A}}^{i\alpha}(im-1, p) = \bar{A}_{im-1} U_{\mathbf{A}}^{i\alpha}(im-1, p), \text{ if } p < im.$$

Therefore it follows from the Step 1 and (2.4.27) that

$$\|U_{\mathbf{A}}^s(n, p)\|_{\mathcal{L}(X)} \leq (3\kappa K + K) e^{\rho} \kappa e^{-\rho(im-p)},$$

and

$$\|U_{\mathbf{A}}^c(n, p)\| \leq (3\kappa K + K) \kappa e^{-\rho_0} e^{\rho_0(im-p)}.$$

Note that one has for $\alpha = u, c$

$$U_{\mathbf{A}}^{\alpha}(p, im) = \bar{\Pi}_{im}^{\alpha} = \bar{\Pi}_{im}^{(i+1)\alpha} \text{ if } p = im$$

and

$$U_{\mathbf{A}}^{\alpha}(p, im) = U_{\mathbf{A}}^{i\alpha}(p, im-1) U_{\mathbf{A}}^{\alpha}(im-1, im), \text{ if } p < im,$$

so that we deduce from Step 1 and (2.4.25) that

$$\|U_{\mathbf{A}}^u(p, n)\| \leq 2\kappa^2 e^{\rho_0} \kappa e^{\rho} e^{-\rho(im-p)} \text{ and } \|U_{\mathbf{A}}^u(p, n)\| \leq 2\kappa^2 e^{\rho_0} \kappa e^{-\rho_0} e^{-\rho(im-p)}.$$

Step 3. We can summarize Step 1 and Step 2 as follows for each $i \in \mathbb{Z}$ and each $n, p \in [(i-1)m, im]$ with $n \geq p$ we have

$$\begin{aligned} \|U_{\mathbf{A}}^s(n, p)\|_{\mathcal{L}(X)} &\leq \tilde{\kappa} e^{-\rho(n-p)}, \quad \|U_{\mathbf{A}}^u(p, n)\|_{\mathcal{L}(X)} \leq \tilde{\kappa} e^{-\rho(n-p)}, \\ \|U_{\mathbf{A}}^c(n, p)\|_{\mathcal{L}(X)} &\leq \tilde{\kappa} e^{\rho_0(n-p)}, \quad \|U_{\mathbf{A}}^c(p, n)\|_{\mathcal{L}(X)} \leq \tilde{\kappa} e^{\rho_0(n-p)}, \end{aligned}$$

where $\tilde{\kappa}$ is defined in (2.4.10).

Now let $n, p \in \mathbb{Z}$ with $n > p$ be given. Then there exists $j \leq i$ such that

$$(j-1)m \leq p \leq jm \text{ and } (i-1)m \leq n \leq im.$$

Since the case $j = i$ is already studied in Step 1 and Step 2, it is sufficient to study the case $j \leq i-1$. In fact if $j \leq i-1$ one has for $\alpha = s$

$$\begin{aligned} \|U_{\bar{\mathbf{A}}}^s(n, p)\| &= \|U_{\bar{\mathbf{A}}}^s(n, (i-1)m) U_{\bar{\mathbf{A}}}^s((i-1)m, jm) U_{\bar{\mathbf{A}}}^s(jm, p)\| \\ &\leq \|U_{\bar{\mathbf{A}}}^s(n, (i-1)m)\| \|U_{\bar{\mathbf{A}}}^s((i-1)m, jm)\| \|U_{\bar{\mathbf{A}}}^s(jm, p)\| \\ &\leq \tilde{\kappa} e^{-\rho(n-(i-1)m)} (\tilde{\kappa} e^{-\rho})^{(i-1)m-jm} \tilde{\kappa} e^{-\rho(jm-p)} \\ &\leq \tilde{\kappa}^2 \tilde{\kappa}^{i-1-j} e^{-\rho(n-p)} \end{aligned}$$

and since $(i-1-j)m \leq n-p$ and $\tilde{\kappa} \geq 1$ one obtains

$$\|U_{\bar{\mathbf{A}}}^s(n, p)\| \leq \tilde{\kappa}^2 \left(e^{-\rho} \tilde{\kappa}^{\frac{1}{m}} \right)^{n-p} \leq \tilde{\kappa}^2 e^{-(\rho - \frac{1}{m} \ln \tilde{\kappa})(n-p)}. \quad (2.4.29)$$

Similarly we also obtain

$$\|U_{\bar{\mathbf{A}}}^c(n, p)\| \leq \tilde{\kappa}^2 e^{(\rho_0 + \frac{1}{m} \ln \tilde{\kappa})(n-p)}, \quad (2.4.30)$$

$$\|U_{\bar{\mathbf{A}}}^c(p, n)\| \leq \tilde{\kappa}^2 e^{(\rho_0 + \frac{1}{m} \ln \tilde{\kappa})(n-p)}, \quad (2.4.31)$$

$$\|U_{\bar{\mathbf{A}}}^u(n, p)\| \leq \tilde{\kappa}^2 e^{-(\rho - \frac{1}{m} \ln \tilde{\kappa})(n-p)}. \quad (2.4.32)$$

Now note that since $m \geq m_0$ with m_0 defined in (2.4.10) one has the following inequality

$$\rho_0 + \frac{1}{m} \ln \tilde{\kappa} < \frac{\hat{\rho}_0 + \rho_0}{2} < \frac{\hat{\rho} + \rho}{2} < \rho - \frac{1}{m} \ln \tilde{\kappa},$$

Therefore one can claim that $\bar{\mathbf{A}}$ is exponentially trichotomic with constant $\tilde{\kappa}^2$, exponents $\frac{\hat{\rho}_0 + \rho_0}{2}$ and $\frac{\hat{\rho} + \rho}{2}$. The proof is completed. $\blacksquare$

Remark 2.4.7 Let us notice that in Lemma 2.4.4, condition (2.4.11) can be replaced by

$$\|A_n \Pi_n^{iu}\|_{\mathcal{L}(X)} \leq K, \quad \forall i \in \mathbb{Z} \text{ and } n \in [(i-1)m, im[. \quad (2.4.33)$$

In fact by using (2.4.3), (2.4.5) and (2.4.33), we deduce that for each $i \in \mathbb{Z}$ and $n \in [(i-1)m, im[$

$$\begin{aligned} \|A_n\|_{\mathcal{L}(X)} &\leq \|A_n \Pi_n^{iu}\|_{\mathcal{L}(X)} + \|A_n \Pi_n^{is}\|_{\mathcal{L}(X)} + \|A_n \Pi_n^{ic}\|_{\mathcal{L}(X)} \\ &\leq K + \kappa e^{-\rho} + \kappa e^{\rho_0}. \end{aligned}$$

The next lemma will allow to derive the closeness of the projectors. This lemma generalizes Lemma 2.2 in [110], but this proof is different since the linear operators are not invertible.

Lemma 2.4.8 *Let $\kappa > 0$, $0 < \rho_0 < \rho$ be given and let $l > 0$ be an integer. Let $a, b \in \mathbb{Z}$ such that $b - a \geq 2l$. Assume that there exist two families of projectors $\{\Pi^\alpha\}_{\alpha=u,s,c}$ and $\{\bar{\Pi}^\alpha\}_{\alpha=u,s,c}$ such that $A = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ has two exponentially trichotomies on $[a, b] \cap \mathbb{Z}$, with constant κ , exponents $0 < \rho_0 < \rho$, with respect to both families of projectors. Then we have*

$$\sup_{n \in [a+l, b-l] \cap \mathbb{Z}} \|\bar{\Pi}_n^\alpha - \Pi_n^\alpha\|_{\mathcal{L}(X)} \leq 6\kappa^3 e^{-(\rho-\rho_0)l}. \quad (2.4.34)$$

Proof. Let $n \in [a+l, b-l]$ be given. Denote by $U_{\mathbf{A}}$ the evolution semigroup associated to $\mathbf{A}$. For each $\alpha = u, c$ and each $n, p \in [a, b]$ with $n \geq p$ we defined $\bar{U}_{\mathbf{A}}^\alpha(p, n) : \bar{\Pi}_n^\alpha(X) \rightarrow \bar{\Pi}_p^\alpha(X)$ the inverse of the bounded linear operator $U_{\mathbf{A}}(n, p) \bar{\Pi}_p^\alpha : \bar{\Pi}_p^\alpha(X) \rightarrow \bar{\Pi}_n^\alpha(X)$. We claim that

$$\|\bar{\Pi}_n^s x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|, \quad \forall x \in \Pi_n^\alpha(X), \forall \alpha = u, c. \quad (2.4.35)$$

Let $x \in \Pi_n^\alpha(X)$ with $\alpha = u, c$. We have

$$x = \Pi_n^\alpha x = U_{\mathbf{A}}(n, n-l) U_{\mathbf{A}}^\alpha(n-l, n) x,$$

which implies

$$\bar{\Pi}_n^s x = U_{\mathbf{A}}(n, n-l) \bar{\Pi}_{n-l}^s U_{\mathbf{A}}^\alpha(n-l, n) x$$

hence

$$\|\bar{\Pi}_n^s x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|.$$

Similarly by using Π_n^s and $\bar{U}_{\mathbf{A}}^\alpha(n-l, n)$ instead of $\bar{\Pi}_n^s$ and $U_{\mathbf{A}}^\alpha(n-l, n)$ we obtain that

$$\|\Pi_n^s x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|, \quad \forall x \in \bar{\Pi}_n^\alpha(X), \forall \alpha = u, c. \quad (2.4.36)$$

Next we claim that

$$\|\Pi_n^u x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|, \quad \forall x \in \bar{\Pi}_n^\alpha(X), \forall \alpha = s, c. \quad (2.4.37)$$

Let $x \in \bar{\Pi}_n^\alpha(X)$ be given with $\alpha = s, c$. Then

$$\Pi_n^u x = U_{\mathbf{A}}^u(n, n+l) U_{\mathbf{A}}(n+l, n) x = U_{\mathbf{A}}^u(n, n+l) U_{\mathbf{A}}(n+l, n) \bar{\Pi}_n^\alpha x$$

so that

$$\|\Pi_n^u x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|,$$

and we obtain in a similar way that

$$\|\bar{\Pi}_n^u x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|, \quad \forall x \in \Pi_n^\alpha(X), \forall \alpha = u, c. \quad (2.4.38)$$

Finally we claim that

$$\|\Pi_n^c x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|, \quad \forall x \in \bar{\Pi}_n^\alpha(X), \forall \alpha = u, c. \quad (2.4.39)$$

Let $x \in \bar{\Pi}_n^u(X)$ be given. Then one has

$$x = \Pi_n^u x = U_{\mathbf{A}}(n, n-l) \bar{U}_{\mathbf{A}}^u(n-l, n) x,$$

which implies that

$$\Pi_n^c x = U_{\mathbf{A}}^c(n, n-l) \bar{U}_{\mathbf{A}}^u(n-l, n) x,$$

thus

$$\|\Pi_n^c x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|.$$

Let $x \in \bar{\Pi}_n^s(X)$ be given. Then

$$\Pi_n^c x = U_{\mathbf{A}}^c(n, n+l) U_{\mathbf{A}}(n+l, n) x = U_{\mathbf{A}}^c(n, n+l) U_{\mathbf{A}}(n+l, n) \bar{\Pi}_n^s x$$

and we obtain

$$\|\Pi_n^c x\| \leq \kappa^2 e^{-(\rho-\rho_0)l} \|x\|.$$

Now we prove the inequality (2.4.34).

Let $x \in X$ be given. Then we have

$$\begin{aligned} \|\bar{\Pi}_n^s - \Pi_n^s\| x &= \|[I - \Pi_n^s] \bar{\Pi}_n^s x - \Pi_n^s [I - \bar{\Pi}_n^s] x\| \\ &= \|[\Pi_n^c + \Pi_n^u] \bar{\Pi}_n^s x - \Pi_n^s [\bar{\Pi}_n^c + \bar{\Pi}_n^u] x\| \\ &\leq \|\Pi_n^c \bar{\Pi}_n^s x\| + \|\Pi_n^u \bar{\Pi}_n^s x\| + \|\Pi_n^s \bar{\Pi}_n^c x\| + \|\Pi_n^s \bar{\Pi}_n^u x\|. \end{aligned}$$

Hence by (2.4.36), (2.4.37) and (2.4.39) we obtain

$$\begin{aligned} \|\bar{\Pi}_n^s - \Pi_n^s\| x &\leq \kappa^2 e^{-(\rho-\rho_0)l} [\|\bar{\Pi}_n^s x\| + \|\bar{\Pi}_n^u x\| + \|\bar{\Pi}_n^c x\| + \|\bar{\Pi}_n^u x\|] \\ &\leq 3\kappa^3 e^{-(\rho-\rho_0)l} \|x\|. \end{aligned} \quad (2.4.40)$$

Similarly we have

$$\begin{aligned} \|\bar{\Pi}_n^u - \Pi_n^u\| x &= \|\Pi_n^c \bar{\Pi}_n^u x + \Pi_n^s \bar{\Pi}_n^u x - \Pi_n^u \bar{\Pi}_n^c x - \Pi_n^u \bar{\Pi}_n^s x\| \\ &\leq \|\Pi_n^c \bar{\Pi}_n^u x\| + \|\Pi_n^s \bar{\Pi}_n^u x\| + \|\Pi_n^u \bar{\Pi}_n^c x\| + \|\Pi_n^u \bar{\Pi}_n^s x\| \\ &\leq \kappa^2 e^{-(\rho-\rho_0)l} [\|\bar{\Pi}_n^u x\| + \|\bar{\Pi}_n^u x\| + \|\bar{\Pi}_n^c x\| + \|\bar{\Pi}_n^s x\|] \\ &\leq 3\kappa^3 e^{-(\rho-\rho_0)l} \|x\|. \end{aligned} \quad (2.4.41)$$

Since $\bar{\Pi}_n^c = I - \bar{\Pi}_n^u - \bar{\Pi}_n^s$ and $\Pi_n^c = I - \Pi_n^u - \Pi_n^s$ we obtain from (2.4.40) and (2.4.41) that

$$\|\bar{\Pi}_n^c - \Pi_n^c\| x \leq 6\kappa^3 e^{-(\rho-\rho_0)l} \|x\|. \quad (2.4.42)$$

The proof is complete. ■

Proof of Theorem 2.2.3. Let $\mathbb{D} = \{\theta_i\}_{i \in \mathbb{Z}}$ be a T_0 -covering of $\mathbb{Z}$. Therefore from Definition 2.1.2 one knows that for each interval of $\mathbb{Z}$ of length T_0 there exists some θ_i in this interval. Let $T > 3T_0$ be an integer which is assumed to be divisible by 6. Assume that $\mathbf{A}$ is exponentially trichotomic on each interval $[\theta_j, \theta_j + T]$.

Let $n \in \mathbb{Z}$ be given. Since $T > 3T_0$, it follows that the interval $\left[n - \frac{T}{3}, n\right]$ contains at least one $\theta_j \in \mathbb{D}$, and

$$\begin{aligned} \left[n, n + \frac{T}{2}\right] &\subset [\theta_j, \theta_j + T], \\ \left[n + \frac{T}{6}, n + \frac{2T}{6}\right] &\subset [\theta_j, \theta_j + T]. \end{aligned}$$

Since n is an arbitrary integer, by choosing $n = i\frac{T}{6}$ for some integer $i \in \mathbb{Z}$ there exists an integer $j \in \mathbb{Z}$ such that

$$\left[i\frac{T}{6}, i\frac{T}{6} + \frac{T}{2}\right] = \left[i\frac{T}{6}, i\frac{T}{6} + \frac{3T}{6}\right] \subset [\theta_j, \theta_j + T].$$

Assuming that $\mathbf{A}$ is exponentially trichotomic on $[\theta_j, \theta_j + T]$. It follows that $\mathbf{A}$ is also exponentially trichotomic on

$$\left[(i+1)\frac{T}{6}, (i+2)\frac{T}{6}\right], \forall i \in \mathbb{Z}$$

and the exponential trichotomy can be extended to the interval

$$\left[i\frac{T}{6}, (i+2)\frac{T}{6} + \frac{T}{6}\right].$$

Now since $T > 3T_0$ can be chosen arbitrary large, the result follows from Lemma 2.4.4 and Lemma 2.4.8. ■

2.5 Proof of Proposition 2.3.4

Before proving Proposition 2.3.4 we first prove Remark 2.3.5. This remark will be useful during the proof of Proposition 2.3.4.

Claim 2.5.1 *We claim that if $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ is a complete orbit of F in M then the evolution semigroup $U_{\mathbf{x}} := U_{DF(\mathbf{x})}$ associated to $DF(\mathbf{x}) = \{DF(x_n)\}_{n \in \mathbb{Z}}$ has an exponential trichotomy with constant $\kappa\kappa_0$ and exponents ρ_0, ρ .*

Let $\mathbf{x}$ be a complete orbit of F in M . By (2.4.4) and condition (i) of Definition 2.3.1 we can find three families of projectors $\{\Pi_{x_n}^\alpha\}_{n \in \mathbb{Z}}$, $\alpha = u, s, c$ satisfying

$$\Pi_{x_n}^\alpha \Pi_{x_n}^{\alpha'} = 0 \text{ if } \alpha' \neq \alpha \text{ and } \Pi_{x_n}^u + \Pi_{x_n}^s + \Pi_{x_n}^c = I. \quad (2.5.1)$$

Recall that the evolution semigroup $U_{\mathbf{x}}$ associated to $DF(\mathbf{x})$ is defined by

$$U_{\mathbf{x}}(n, p) := \begin{cases} DF(x_{n-1}) \dots DF(x_p), & \text{if } n > p, \\ I, & \text{if } n = p. \end{cases}$$

Remark that since $\mathbf{x}$ is a complete orbit of F , by condition (i) of Definition 2.3.1, one has that for each $n \in \mathbb{Z}$

$$\begin{aligned} \Pi_{F(x_n)}^\alpha DF(x_n) \Pi_{x_n}^\alpha &= DF(x_n) \Pi_{x_n}^\alpha \\ \Pi_{x_{n+1}}^\alpha DF(x_n) \Pi_{x_n}^\alpha &= DF(x_n) \Pi_{x_n}^\alpha. \end{aligned} \quad (2.5.2)$$

Hence (2.5.2) combined with (2.5.1) implies that for each $n \in \mathbb{Z}$ and each $\alpha = u, s, c$

$$\Pi_{x_{n+1}}^\alpha DF(x_n) = \Pi_{x_{n+1}}^\alpha DF(x_n) \Pi_{x_n}^\alpha \quad (2.5.3)$$

and (2.5.2) combined with (2.5.3) yields

$$\Pi_{x_{n+1}}^\alpha DF(x_n) = DF(x_n) \Pi_{x_n}^\alpha, \quad \forall n \in \mathbb{Z}. \quad (2.5.4)$$

Therefore by using (2.5.4) one can deduce that for each $n \geq p$ and each $\alpha = u, s, c$

$$U_{\mathbf{x}}^\alpha(n, p) := \Pi_{x_n}^\alpha U_{\mathbf{x}}(n, p) = U_{\mathbf{x}}(n, p) \Pi_{x_p}^\alpha = DF(x_{n-1}) \Pi_{x_{n-1}}^\alpha \dots DF(x_p) \Pi_{x_p}^\alpha.$$

By using again Definition 2.3.1-(i) we have for each $n \geq p$ and each $\alpha = u, c$ that the bounded linear operator $U_{\mathbf{x}}^\alpha(n, p) : \Pi_{x_p}^\alpha(X) \rightarrow \Pi_{x_n}^\alpha(X)$ is invertible. Define its inverse as

$$U_{\mathbf{x}}^\alpha(p, n) : \Pi_{x_n}^\alpha(X) \rightarrow \Pi_{x_p}^\alpha(X).$$

By observing that

$$U_{\mathbf{x}}(n, p) = DF^{n-p}(x_p), \quad \forall n \geq p,$$

we obtain from Definition 2.3.1-(ii) and (2.3.6) that for each $n \geq p$

$$\begin{aligned} \|U_{\mathbf{x}}^s(n, p)\|_{\mathcal{L}(X)} &\leq \kappa \kappa_0 e^{-\rho(n-p)}, \\ \|U_{\mathbf{x}}^u(p, n)\|_{\mathcal{L}(X)} &\leq \kappa \kappa_0 e^{-\rho(n-p)}, \end{aligned}$$

and for each $(n, p) \in \mathbb{Z}^2$

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(X)} \leq \kappa \kappa_0 e^{\rho_0 |n-p|}.$$

This prove the exponential trichotomy for $U_{\mathbf{x}}$.

We now turn to the proof of Proposition 2.3.4.

Claim 2.5.2 *Let $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ be given. We claim that if $\mathbf{z} = \{z_n\}_{n \in \mathbb{Z}}$ is a δ -pseudo-orbit of F in a neighborhood $\mathcal{V}(M, \varepsilon)$ of M for some ε and δ small enough then the evolution semigroup $U_{\mathbf{z}} := U_{DF(\mathbf{z})}$ associated to $DF(\mathbf{z}) = \{DF(z_n)\}_{n \in \mathbb{Z}}$ has an exponential trichotomy with constant $\hat{\kappa}$ and exponents $\hat{\rho}_0$ and $\hat{\rho}$.*

In order to prove Claim 2.5.2 we will need the following corollary of Theorem 2.4.3.

Corollary 2.5.3 *Let $\mathbb{I}$ be a given interval in $\mathbb{Z}$. Let $\mathbf{A} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ be a given uniformly bounded sequence such that $U_{\mathbf{A}}$ has an exponential trichotomy on $\mathbb{Z}$ with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three families of projectors $\{\Pi^\alpha : \mathbb{Z} \rightarrow \mathcal{L}(X)\}_{\alpha=s,c,u}$. Let $\hat{\rho}_0$, $\hat{\rho}$ and $\hat{\kappa}$ be three real numbers such that*

$$\rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho \text{ and } \hat{\kappa} > \kappa.$$

Then there exists $\delta_0 := \delta_0(\rho_0, \hat{\rho}_0, \hat{\rho}, \rho, \kappa, \hat{\kappa}) \in (0, \sqrt{2} - 1)$ such that for each $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ and each $\mathbf{B} : \mathbb{I} \rightarrow \mathcal{L}(X)$ with

$$\sup_{n \in \mathbb{I}} \|B_n\|_{\mathcal{L}(X)} \leq \delta,$$

the evolution semigroup $U_{\mathbf{A}+\mathbf{B}}$ has an exponential trichotomy on $\mathbb{I}$ with constant $\hat{\kappa}$, exponents $\hat{\rho}$, $\hat{\rho}_0$ and projectors $\{\hat{\Pi}^\alpha\}_{\alpha=s,c,u}$. Moreover for each $n \in \mathbb{I}$ and $\alpha = u, s, c$, the spaces $R(\Pi_n^\alpha)$ and $R(\hat{\Pi}_n^\alpha)$ are isomorphic.

Proof. To prove the Corollary we define the sequences $\chi : \mathbb{Z} \rightarrow \{0, 1\}$

$$\chi_n = \begin{cases} 1 & \text{if } n \in \mathbb{I}, \\ 0 & \text{if } n \notin \mathbb{I}, \end{cases}$$

and $\hat{\mathbf{B}} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ by

$$\hat{\mathbf{B}} := \chi \mathbf{B}.$$

Theorem 2.4.3 applies with $\mathbf{A}$ and $\hat{\mathbf{B}}$ and the result follows since $\mathbf{A} + \mathbf{B} = \mathbf{A} + \hat{\mathbf{B}}$ on $\mathbb{I}$. ■

Proof of Claim 2.5.2. First recall that by Claim 2.5.1 for each complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of F in M , the evolution semigroup $U_{\mathbf{x}}$ associated to $DF(\mathbf{x})$ has an exponential trichotomy with constant $\kappa \geq 0$ and exponents $0 < \rho_0 < \rho$. Let $\bar{\mathbf{z}} = \{\bar{z}_n\}_{n \in \mathbb{Z}}$ be a δ -pseudo orbit of F lying in a neighborhood $\mathcal{V}(M, \varepsilon)$ of M . Then there exists a sequences $\mathbf{z} = \{z_n\}_{n \in \mathbb{Z}} \subset M$ such that

$$\|\bar{z}_n - z_n\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}.$$

Let us proof that $\mathbf{z} = \{z_n\}_{n \in \mathbb{Z}} \subset M$ is a pseudo orbit of F in M . In fact by using Assumption 2.3.2-(iii) we have

$$\begin{aligned} \|z_{n+1} - F(z_n)\| &\leq \|z_{n+1} - \bar{z}_{n+1}\| + \|\bar{z}_{n+1} - F(\bar{z}_n)\| + \|F(\bar{z}_n) - F(z_n)\| \\ &\leq \varepsilon + \delta + K\varepsilon, \end{aligned}$$

so that $\mathbf{z} = \{z_n\}_{n \in \mathbb{Z}}$ is a $\varepsilon(1 + K) + \delta$ -pseudo orbit of F in M . Next let $\theta \in \mathbb{Z}$ and T be a positive integer. Consider the complete orbit $\mathbf{x}$ given by $x_n = F^{n-\theta}(z_\theta)$ for each $n \in \mathbb{Z}$ with

the notation $z_\theta = F^0(z_\theta)$. Then since $\mathbf{z}$ is a $\varepsilon(1+K) + \delta$ -pseudo orbit of F in M one has

$$\begin{aligned} \|x_{\theta+2} - z_{\theta+2}\| &= \|F^2(z_\theta) - z_{\theta+2}\| \\ &\leq \|F(F(z_\theta)) - F(z_{\theta+1})\| + \|F(z_{\theta+1}) - z_{\theta+2}\| \\ &\leq K \|F(z_\theta) - z_{\theta+1}\| + \|F(z_{\theta+1}) - z_{\theta+1}\| \\ &\leq K [\varepsilon(1+K) + \delta] + \varepsilon(1+K) + \delta. \end{aligned}$$

By induction one can easily derive that for each $k \in [0, T]$

$$\begin{aligned} \|x_{\theta+k} - z_{\theta+k}\| &\leq (1+K + \dots + K^{k-1}) [\varepsilon(1+K) + \delta] \\ &\leq (1+K + \dots + K^{T-1}) [\varepsilon(1+K) + \delta]. \end{aligned}$$

Since by Assumption 2.3.2 the map $x \rightarrow DF(x)$ is uniformly continuous on $\mathcal{V}(M, \varepsilon_0)$, we can define the modulus of continuity of $x \rightarrow DF(x)$ which is a map $\omega : [0, \varepsilon_0] \rightarrow [0, +\infty)$ defined by

$$\omega(\varepsilon) := \sup_{x, y \in \mathcal{V}(M, \varepsilon_0) : \|x-y\| \leq \varepsilon} \|DF(x) - DF(y)\|.$$

Thus one gets for each $k \in [0, T]$

$$\|DF(x_{\theta+k}) - DF(z_{\theta+k})\| \leq \omega((1+K + \dots + K^{T-1}) [\varepsilon(1+K) + \delta]).$$

Therefore we obtain that

$$\begin{aligned} \|DF(x_{\theta+k}) - DF(\bar{z}_{\theta+k})\| &\leq \|DF(x_{\theta+k}) - DF(z_{\theta+k})\| + \|DF(z_{\theta+k}) - DF(\bar{z}_{\theta+k})\| \\ &\leq \omega(\varepsilon) + \omega((1+K + \dots + K^{T-1}) [\varepsilon(1+K) + \delta]). \end{aligned} \tag{2.5.5}$$

Next observe that

$$DF(\bar{\mathbf{z}}) = DF(\mathbf{x}) + [DF(\bar{\mathbf{z}}) - DF(\mathbf{x})].$$

Recalling that the evolution semigroup $U_{\mathbf{x}}$ has an exponential trichotomy on $\mathbb{Z}$ and writing (2.5.5) as

$$\sup_{k \in [\theta, \theta+T]} \|DF(x_k) - DF(z_k)\| \leq \omega(\varepsilon) + \omega((1+K + \dots + K^{T-1}) [\varepsilon(1+K) + \delta]),$$

Corollary 2.5.3 applies and ensures that for δ and ε sufficiently small, depending only on $K, \kappa, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}$ and T , that $U_{\mathbf{z}}$ has an exponential trichotomy on $[\theta, \theta+T]$ with constant $\bar{\kappa} = \kappa + 1$, exponents $\frac{\rho + \widehat{\rho}}{2}, \frac{\rho_0 + \widehat{\rho}_0}{2} \in (0, \frac{\rho + \widehat{\rho}}{2})$ and projectors $\{\bar{\Pi}_n^\alpha\}_{n \in \mathbb{Z}}$ with $\alpha = u, s, c$.

We now complete the proof by applying Theorem 2.2.3. To do so first note that since $U_{\mathbf{z}}$ has exponential trichotomy on each $[\theta, \theta+T]$ with $\theta \in \mathbb{Z}$ whenever δ and ε sufficiently small, one can use $\mathbb{Z}$ as a relative dense subset of integers or equivalently a 1-covering. Furthermore we also note that

$$\frac{\rho_0 + \widehat{\rho}_0}{2} < \widehat{\rho}_0 < \widehat{\rho} < \frac{\rho + \widehat{\rho}}{2},$$

and since the choices of δ and ε depend only on $K, \kappa, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}$ and T , one can choose T large enough (depending only on $K, \kappa, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}$) in the previous lines such that Theorem [2.2.3](#) holds for T , with the constant of trichotomy $\kappa + 1$ and exponents $\frac{\rho_0 + \widehat{\rho}_0}{2}$ and $\frac{\rho + \widehat{\rho}}{2}$. The proof is complete. ■

Chapitre 3

Persistence of a class of normally hyperbolic invariant manifolds in Banach spaces

3.1 Introduction

The invariant manifolds theory plays an important role in the study of dynamical systems due to its various subjects. The pioneer works on such a topic have been considered by Poincaré [125, 1892], Hadamard [71, 1901], Lyapunov [92, 1907] and Perron [117, 118, 116, respectively in 1928, 1929 and 1930]. There are two main axes in the theory of invariant manifold: the *center manifold theory* and the *hyperbolic* or *normally hyperbolic invariant manifolds theory*. The main questions addressed in both theories are the existence, the smoothness and the persistence of such manifolds as well as their associated stable, unstable, center-stable and center-unstable manifolds. Two main approaches have been extensively used in the theory of invariant manifolds: the so-called Lyapunov-Perron method and the Hadamard's graph transform method.

The Hadamard's graph transform method is based on geometric ideas and relies on a fixed point formulation over a metric space of Lipschitz graphs while the Lyapunov-Perron method is more analytic and is based on a fixed point formulation in term of integral equations using constant variation formula. Let us also mention that the so-called Lie Transform's Method (or Deformation method) and the Irwin's method have been developed respectively by Marsden and Scheuler [105, 1987] and Irwin [79, 1970] in the context of invariant manifolds theory. We refer to Wiggins [166] for more details on these two methodologies.

The center manifold theory is concerned with the existence and smoothness of an invariant manifold for a dynamical system around a nonhyperbolic equilibrium point. There are several results and huge literature on this subject. We refer the reader to Pliss [124], Kelley [84], Carr [29], Sijbrand [151], Vanderbauwhede and Iooss [160], Vanderbauwhede [159, 161], Magal and Ruan [95], Bates and Jones [21], Chow, Li and Wang

[36], etc.

The normally hyperbolic invariant manifold theory can be viewed as an extension of the persistence of hyperbolic equilibrium points to invariant sets (for example a set of equilibrium points, heteroclinic orbit, homoclinic orbit). However this extension induces important difficulties due to the fact that the dynamics on the invariant sets can be very complicated. Therefore it becomes crucial to have a good knowledge of the dynamic on the invariant manifold as well as on the normal directions. Roughly speaking the orbits lying on the normally hyperbolic invariant manifolds behave like in a center manifold while the orbits lying on the normal directions expand or contract toward the manifold.

This theory has a long and rich history. Let us mention that great progresses have been obtained in the remarkable and independent works of Fenichel in the series of papers [64, 65, 66] and of Hirsch, Pugh and Shub [77], for compact normally hyperbolic manifolds of finite dimensional dynamical systems. Since then many authors have worked on this subject and the literature in the finite dimensional case becomes very large so that it is impossible to give an exhaustive list. However let us mention that in the case of the compact invariant manifold given as a graph of a function, a detailed exposition of Fenichel's theorems [66] has been given by Jones in [81] (see also Wiggins [166] for further discussion on Fenichel's theorems). In his work [81] the author provided fairly different proofs based on geometric considerations and topological arguments such as Wazewski Principle. Sakamoto in [144] extended the results of Fenichel in the noncompact invariant graph case for ODEs (Ordinary Differential Equations). His proof is essentially based on the Lyapunov-Perron approach. Recently a very general results for invariant sets of ODEs has been prove in Chow, Liu and Yi in [40, 41] while Pliss and Sell [122] generalize the notion of normally hyperbolic set into the so-called *weakly normally hyperbolic sets* and investigate on the persistence of such manifolds. Finally let us mention an important result proved by Măné in [104] that states that persistent invariant manifold for finite dimensional dynamical systems are normally hyperbolic.

By contrast of the finite dimensional case, there are only few references for infinite dynamical systems as far as we know. Henry [76] obtained one of the ealiest results on this context by considering semi linear parabolic equations with a proof based on Lyapunov-Perron approach. He proved the persistence and smoothness of a compact invariant manifolds with trivial normal bundle. Note that Henry's theorem [76, p.278, Theorem 9.1.2] applies to ODEs and covers some of the results obtained by Sakamoto [144]. We also note that Magalhães [98] proved persistence and smoothness of hyperbolic manifold for some class of delay differential equations. The more important progress were made by Bates, Lu and Zeng [19] for compact normally hyperbolic manifolds. In their manuscript the authors considered semiflows in Banach spaces and deal with existence, persistence and smoothness of compact normally hyperbolic manifolds as well as their associated center-stable and center-unstable manifolds. Further generalization to noncompact manifold which is inflowing or (and) overflowing have been made by the same authors in [20]. A closely related result to [19] can be be found in Jones and Shkoller [82] where the authors have investigated invariant manifolds for PDEs. We also refer to Eldering [63] for recent results

on the persistence of noncompact normally hyperbolic manifolds in bounded geometry.

In this chapter we will deal with persistence of normally hyperbolic manifolds for discrete dynamical systems under a small Lipschitz continuous perturbation. Such perturbations in the context of normally hyperbolic manifolds have been hardly considered in the literature. We will give further discussion and comparison with the existing literature after the statement of our main result.

To put our work into this context, let us consider

$$u_{n+1} = T(u_n), \quad n \in \mathbb{N}, \quad u_0 \in X, \quad (3.1.1)$$

where $(X, \|\cdot\|)$ is a Banach space and $T : X \rightarrow X$ a continuously differentiable map on X . The aim of this work is to provide a persistence result of invariant manifolds for (3.1.1) under a small Lipschitz continuous perturbation. The sense of the Lipschitz continuous perturbation of (3.1.1) as well as the precise context will be made clear throughout the following discussions. Let $\tilde{T} : X \rightarrow X$ be a continuous map on X . Consider the following perturbed system

$$u_{n+1} = T(u_n) + \tilde{T}(u_n), \quad n \in \mathbb{N}, \quad u_0 \in X. \quad (3.1.2)$$

Following the definition given in [19], let us give the definition of what we mean by normally hyperbolic set. For the usual definition of a normally hyperbolic manifold we refer to [19, 64, 77] and the references therein.

Definition 3.1.1 *A set $M \subset X$ is said to be **normally hyperbolic invariant** for T if the following properties are satisfied:*

(i) *The set M is invariant under T , namely $T(M) = M$.*

(ii) *For each $x \in M$, there exists three closed subspaces X_x^α with $\alpha = s, c, u$ such that*

$$X = X_x^u \oplus X_x^s \oplus X_x^c, \quad (3.1.3)$$

and for each $\alpha = s, c, u$, $DT(x)(X_x^\alpha) \subset X_{T(x)}^\alpha$ and for $\alpha = u, c$, the map $DT(x)|_{X_x^\alpha}$ is an isomorphism from X_x^α onto $X_{T(x)}^\alpha$.

(iii) *There exist a constant $\kappa \geq 1$, rates $0 < \rho_0 < \rho$ such that for each $n \in \mathbb{N}$ and each $x \in M$,*

$$\begin{cases} \|DT^n(x)|_{X_x^s}\| \leq \kappa e^{-\rho n} \\ \inf \{\|DT^n(x)x^u\| : x^u \in X_x^u \text{ and } \|x^u\| = 1\} \geq \frac{1}{\kappa} e^{\rho n}, \end{cases} \quad (3.1.4)$$

and

$$\frac{1}{\kappa} e^{-\rho_0 n} \leq \inf \{\|DT^n(x)x^c\| : x^c \in X_x^c \text{ and } \|x^c\| = 1\} \leq \|DT^n(x)|_{X_x^c}\| \leq \kappa e^{\rho_0 n}, \quad (3.1.5)$$

where

$$\|DF^n(x)|_{X_x^\alpha}\| := \sup \{\|DF^n(x)x\| : x \in X_x^\alpha \text{ and } \|x\| = 1\} \text{ for } \alpha = s, c \text{ and } n \in \mathbb{N}.$$

Observe that due to (ii), one can define for each $x \in M$, the projectors Π_x^α for $\alpha = u, s, c$ associated to the state decomposition (3.1.3) that satisfy for each $x \in M$ and $\alpha \in \{s, c, u\}$:

$$\mathcal{R}(\Pi_x^\alpha) = X_x^\alpha \text{ and } \mathcal{N}(\Pi_x^\alpha) = \bigoplus_{\alpha' \in \{s, c, u\} \setminus \{\alpha\}} X_x^{\alpha'}, \quad (3.1.6)$$

where $\mathcal{R}(\Pi_x^\alpha)$ is the range of Π_x^α and $\mathcal{N}(\Pi_x^\alpha)$ its kernel. Let $(Y, \|\cdot\|_Y)$ be a normed space and $E \subseteq Y$. We denote by $Lip(E, Y)$ the space of Lipschitz continuous maps defined from E into Y . We set

$$\|H\|_{Lip(E, Y)} := \sup_{x, y \in E: x \neq y} \|x - y\|_Y^{-1} \|H(x) - H(y)\|_Y, \quad \forall H \in Lip(E, Y).$$

In the sequel, when $E = Y$ we will simply write $\|H\|_{Lip(E)}$ instead of $\|H\|_{Lip(E, E)}$.

In order to make clear the context of our work let us make the following assumption.

Assumption 3.1.2 Assume that the map $T : X \rightarrow X$ is continuously differentiable on X . Assume that there exist two closed subspaces X^c (center space) and X^h (hyperbolic space) such that

$$X = X^c \oplus X^h.$$

Assume moreover that $M := X^c$ is a normally hyperbolic space for T (i.e. $T(M) = M$). Furthermore we will assume in addition that

(i) For each $x \in M$ we have: $X_x^c = M = X^c$.

(ii) For each $x \in M$ we have: $X^h = X_x^u \oplus X_x^s$.

From now on we will always consider that Assumption 3.1.2 is satisfied. One may observe that due to the conditions (i) and (ii) of Assumption 3.1.2, for each $x \in X$ the projector Π_x^c defined in (3.1.6) does not depend on x as well as the projector $\Pi_x^u + \Pi_x^s$. Therefore we set

$$\Pi^c := \Pi_x^c \text{ and } \Pi^h := \Pi_x^u + \Pi_x^s, \quad \forall x \in X. \quad (3.1.7)$$

Note that one also has

$$X = M \oplus X^h, \quad \mathcal{R}(\Pi^c) = M \text{ and } \mathcal{R}(\Pi^h) = X^h.$$

If $\{u_n\}_{n \in \mathbb{Z}}$ is a solution of (3.1.2), let us define for each $n \in \mathbb{Z}$

$$x_n = \Pi^c u_n \in M \text{ and } y_n = \Pi^h u_n \in X^h.$$

Recalling that

$$\mathcal{R}(\Pi^c) = M = T(M) \text{ and } DT(x)X^h \subset X^h, \quad \forall x \in M,$$

and projecting system (3.1.2) along $X^c = M$ and X^h , we obtain that the sequences $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ and $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}}$ satisfy the following system

$$\begin{cases} x_{n+1} = T(x_n) + R_c(x_n, y_n), & n \in \mathbb{N}, \\ y_{n+1} = DT(x_n)y_n + R_h(x_n, y_n), & n \in \mathbb{N}. \end{cases} \quad (3.1.8)$$

wherein the maps $R_c : M \times X^h \rightarrow M$ and $R_h : M \times X^h \rightarrow X^h$ are given by

$$R_c(x, y) := \Pi^c \left[T(x + y) - T(x) + \tilde{T}(x + y) \right], \quad (3.1.9)$$

and

$$R_h(x, y) := \Pi^h \left[T(x + y) + \tilde{T}(x + y) \right] - DT(x)y. \quad (3.1.10)$$

One may observe that system (3.1.8) takes the following general form

$$\begin{cases} x_{n+1} = F(x_n) + R_c(x_n, y_n), & n \in \mathbb{N}, \\ y_{n+1} = A(x_n)y_n + R_h(x_n, y_n), & n \in \mathbb{N}, \end{cases} \quad (3.1.11)$$

where $F : M \rightarrow M$, $R_c : M \times X^h \rightarrow M$, $R_h : M \times X^h \rightarrow X^h$ and $A(\cdot) : M \rightarrow \mathcal{L}(X^h)$. In order to provide a more general result we will study the system (3.1.11) and derive consequences for system (3.1.8) (or equivalently (3.1.2)). System (3.1.11) is the discrete analogy of the *slow-fast system*. This kind of systems has been extensively studied in the context of continuous time dynamical systems. All the references cited above are mainly focused upon continuous time dynamical system while the literature on discrete time dynamical systems is not as extensive. The main advantage using discrete time formulation is that one can deduce results for continuous time semiflows (see for instance to [33, 86] where this technique was successfully used). Without pretending to give an exhaustive list we refer the reader to [1, 44, 83, 154, 130] and the references therein. Comments, improvements and comparison with this literature will be discussed in Section 3.2.

In Section 3.2 we will present the main results of this work concerning system (3.1.11) and derive some consequences for the map $T + \tilde{T}$ as well as a consequence for continuous time semiflows. In Section 3.3.1 we present some preliminary lemmas and finally the Section 3.3 is devoted to the proof of the main theorem.

3.2 Main results and consequences

Before going to the main results of this section, let us give the definitions and notations that will be used throughout this chapter. Let $\mathbf{u} := \{u_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow X$ be a given sequence and ε a non negative real value. Then $\mathbf{u}$ is called a ε -**pseudo-orbit** (resp. **complete orbit** or **0-pseudo-orbit**) of a map $H : X \rightarrow X$ if

$$\|u_{n+1} - H(u_n)\| \leq \varepsilon, \forall n \in \mathbb{Z} \text{ (resp. } u_{n+1} = H(u_n), \forall n \in \mathbb{Z}).$$

Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ be a given sequence of operators. Then we denote by $U_{\mathbf{A}}$ its associated evolution semigroup defined by

$$U_{\mathbf{A}}(n, p) := \begin{cases} A_{n-1} \dots A_p, & \text{if } n > p, \\ I, & \text{if } n = p, \end{cases} \quad (3.2.1)$$

wherein I denotes the identity operator in X . In what follows we will always use the notation $n \geq p$ as well as $U_{\mathbf{A}}(n, p)$ for the evolution semigroup. Whenever $U_{\mathbf{A}}(p, n)$ is considered this will mean that $U_{\mathbf{A}}(n, p)$ is invertible and

$$U_{\mathbf{A}}(p, n) = U_{\mathbf{A}}(n, p)^{-1}.$$

Let us observe that $U_{\mathbf{A}}$ satisfies

$$U_{\mathbf{A}}(n, p) U_{\mathbf{A}}(p, m) = U_{\mathbf{A}}(n, m) \text{ for } n \geq p \geq m.$$

With this notation we recall the following definition of **exponential trichotomy** taken from [73].

Definition 3.2.1 Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(X)$ be given. Let $\mathbb{I}$ be an interval of $\mathbb{Z}$. Then $U_{\mathbf{A}}$ has an **exponential trichotomy** on $\mathbb{I}$ with constant κ , exponents $0 < \rho_0 < \rho$ if there exist three families of projectors $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{I}} : \mathbb{I} \rightarrow \mathcal{L}(X)$, with $\alpha = u, s, c$ satisfying the following properties:

(i) For all $n \in \mathbb{I}$ and $\alpha, \beta \in \{u, s, c\}$ we have

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ if } \alpha \neq \beta,$$

and

$$\Pi_n^s + \Pi_n^u + \Pi_n^c = I.$$

(ii) For all $n, p \in \mathbb{I}$ with $n \geq p$ we have

$$U_{\mathbf{A}}^\alpha(n, p) := \Pi_n^\alpha U_{\mathbf{A}}(n, p) = U_{\mathbf{A}}(n, p) \Pi_p^\alpha, \text{ for } \alpha = u, s, c.$$

(iii) $U_{\mathbf{A}}^\alpha(n, p)$ is invertible from $\mathcal{R}(\Pi_p^\alpha)$ into $\mathcal{R}(\Pi_n^\alpha)$ for all $n \geq p$ in $\mathbb{I}$, $\alpha = u, c$ and its inverse is denoted by $U_{\mathbf{A}}^\alpha(p, n) : \mathcal{R}(\Pi_n^\alpha) \rightarrow \mathcal{R}(\Pi_p^\alpha)$.

(iv) For each $y \in X$ we have for all $n, p \in \mathbb{I}$

$$\|U_{\mathbf{A}}^c(n, p) \Pi_p^c y\| \leq \kappa e^{\rho_0 |n-p|} \|y\|,$$

and if $n \geq p$

$$\|U_{\mathbf{A}}^s(n, p) \Pi_p^s y\| \leq \kappa e^{-\rho(n-p)} \|y\|,$$

$$\|U_{\mathbf{A}}^u(p, n) \Pi_n^u y\| \leq \kappa e^{-\rho(n-p)} \|y\|.$$

Remark 3.2.2 When $\Pi_n^c = 0_{\mathcal{L}(X)}$ for all $n \in \mathbb{I}$ in the above definition, we say that $U_{\mathbf{A}}$ has an **exponential dichotomy** on $\mathbb{I}$ with constant κ and exponent ρ . Note that this notion corresponds to the notion given in [72] and [127].

Let us observe that the operators $U_{\mathbf{A}}^\alpha(n, p)$, for $n \geq p$ in $\mathbb{I}$ and $\alpha = u, s, c$ (resp. $U_{\mathbf{A}}^\alpha(p, n)$, for $n \geq p$ in $\mathbb{I}$ and $\alpha = u, c$) inherit the evolution property of $U_{\mathbf{A}}$ that reads as

$$U_{\mathbf{A}}^\alpha(n, p) U_{\mathbf{A}}^\alpha(p, m) = U_{\mathbf{A}}^\alpha(n, m), \quad \forall n \geq p \geq m \text{ in } \mathbb{I} \text{ and } \alpha = u, s, c,$$

respectively

$$U_{\mathbf{A}}^\alpha(m, p) U_{\mathbf{A}}^\alpha(p, n) = U_{\mathbf{A}}^\alpha(m, n), \quad \forall n \geq p \geq m \text{ in } \mathbb{I} \text{ and } \alpha = u, c.$$

Before stating the assumption for this section, let us define for each $\eta \geq 0$ and $W = M, X^h$ the following weighted Banach space

$$\mathbb{L}_\eta(\mathbb{Z}, W) := \left\{ \mathbf{w} : \mathbb{Z} \rightarrow W \text{ such that } \|\mathbf{w}\|_\eta := \sup_{n \in \mathbb{Z}} e^{-\eta|n|} \|w_n\| < +\infty \right\}. \quad (3.2.2)$$

Observe that for $\eta = 0$ we have

$$\mathbb{L}_0(\mathbb{Z}, W) = l^\infty(\mathbb{Z}, W) = \left\{ \mathbf{w} : \mathbb{Z} \rightarrow W \text{ such that } \|\mathbf{w}\|_0 := \sup_{n \in \mathbb{Z}} \|w_n\| < +\infty \right\}. \quad (3.2.3)$$

Note that $\{\mathbb{L}_\eta(\mathbb{Z}, W), \eta \geq 0\}$ form a **scale of Banach spaces** that is for each $0 \leq \eta \leq \hat{\eta}$ we have

$$\mathbb{L}_\eta(\mathbb{Z}, W) \subseteq \mathbb{L}_{\hat{\eta}}(\mathbb{Z}, W),$$

and the embedding is continuous namely

$$\|\mathbf{w}\|_{\hat{\eta}} \leq \|\mathbf{w}\|_\eta, \quad \forall \mathbf{w} \in \mathbb{L}_\eta(\mathbb{Z}, W).$$

The assumption for this section is the following

Assumption 3.2.3 We assume that

(H1) F is continuously differentiable on M and

- (i) $DF(\cdot) : M \rightarrow \mathcal{L}(M)$ is Lipschitz continuous on M ,
- (ii) $F(M) = M$.

(H2) There exists a constant $\kappa \geq 1$, rates $\rho > 0$ and $\rho_0 \in (0, \rho)$ such that

- (i) For each complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of F the sequences $A(\mathbf{x}) = \{A(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ is exponentially dichotomic on $\mathbb{Z}$ with constant $\kappa \geq 1$ and exponent $\rho > 0$ associated to the families of projectors $\{\Pi_{x_n}^\alpha\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$, $\alpha = u, s$.

(ii) For each $x \in M$, the operator $DF(x) : M \rightarrow M$ is invertible. Furthermore for each $n \in \mathbb{N}$ we have

$$\frac{1}{\kappa} e^{-\rho_0 n} \leq \inf \{ \|DF^n(x) w\| : w \in M, \|w\| = 1 \} \leq \|DF^n(x)\|_{\mathcal{L}(M)} \leq \kappa e^{\rho_0 n}. \quad (3.2.4)$$

(H3) For each $i = c, h$ the map R_i is bounded and Lipschitz continuous on $M \times X^h$ that is

$$\|R_i\|_{C(M \times X^h, X^h)} + \|R_i\|_{Lip(M \times X^h, X^h)} < +\infty,$$

with

$$\|R_i\|_{C(M \times X^h, X^h)} := \sup_{(x,y) \in M \times X^h} \|R_i(x, y)\|.$$

(H4) $A(\cdot)$ is bounded and Lipschitz continuous on M that is

$$\|A\|_{Lip(M, \mathcal{L}(X^h))} + \|A\|_{C(M, \mathcal{L}(X^h))} < +\infty, \quad (3.2.5)$$

with

$$\|A\|_{C(M, \mathcal{L}(X^h))} := \sup_{x \in M} \|A(x)\|_{\mathcal{L}(X^h)}.$$

(H5) There exists a complete orbit $\hat{\mathbf{x}} = \{\hat{x}_n\}_{n \in \mathbb{Z}} \subset M$ of F such that $\hat{\mathbf{x}} \in \mathbb{L}_{\rho_0}(\mathbb{Z}, M)$.

Our main result is stated as follow

Theorem 3.2.4 Let Assumption 3.2.3 be satisfied. Let $\hat{\rho}_0, \hat{\rho}$ be given such that $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$. Let $\eta \in (\hat{\rho}_0, \hat{\rho})$ be given. Let us define

$$K_F := \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))}, \text{ and } K_A := \|A\|_{Lip(M \times \mathcal{L}(X^h))} + \|A\|_{C(M, \mathcal{L}(X^h))}.$$

Then there exist four constants $\varepsilon := \varepsilon(\kappa, \rho, \rho_0, \hat{\rho}, \hat{\rho}_0, K_A, K_F) > 0$, $\hat{\kappa} := \hat{\kappa}(\kappa, \rho, \rho_0, K_A) \geq \kappa$, $C := C(\kappa, \rho_0, \hat{\rho}_0, K_A, \eta)$ and $\gamma := \gamma(\kappa, \rho, \hat{\rho}, \rho_0, \hat{\rho}_0, K_A, \eta)$ such that if

$$\|R_c\|_{C(M \times X^h, X^h)} \leq \min(1, \varepsilon) \text{ and } \|R_c\|_{Lip(M \times X^h, M)} \leq \min\left(1, \frac{1}{2C}\right),$$

and

$$\max \left\{ \|R_h\|_{C(M \times X^h, M)}, \|R_h\|_{Lip(M \times X^h, M)} \right\} < \min \left\{ \frac{1}{6\gamma}, \frac{1}{12C\gamma}, \frac{1}{12C\gamma^2 \|A\|_{Lip(M, \mathcal{L}(M))}} \right\},$$

then there exists some function $\Psi : M \rightarrow X^h$ such that the graph

$$\widehat{M} = \{(x, \Psi(x)) : x \in M\},$$

has the following properties:

- (i) For each $(x, \Psi(x)) \in \widehat{M}$ there exists a unique solution $(\mathbf{x}, \Psi(\mathbf{x})) = \{(x_n, \Psi(x_n))\}_{n \in \mathbb{Z}}$ of (3.1.11) with $x_0 = x$ and $(\mathbf{x}, \Psi(\mathbf{x})) \in \mathbb{L}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$.
- (ii) If $(\mathbf{x}, \mathbf{y}) = \{(x_n, y_n)\}_{n \in \mathbb{Z}}$ is a solution of (3.1.11) and $\mathbf{y} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ then $(x_0, y_0) \in \widehat{M}$.
- (iii) The map $\Psi : M \rightarrow X^h$ is bounded and Lipschitz continuous on M with the following estimates.

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \sup_{x \in M} \|R_h(x, 0)\|,$$

and

$$\|\Psi(x) - \Psi(\bar{x})\| \leq 4\widehat{\kappa} \left[1 + \gamma^2 \|A\|_{\text{Lip}(M, \mathcal{L}(M))} \right] \|R_h\|_{\text{Lip}(M \times X^h, X^h)} \|x - \bar{x}\|, \quad \forall x, \bar{x} \in M.$$

It is important to point out the fact that, in the system (3.1.11), when F is a bounded linear operator and A does not depend on the variable x , the above theorem corresponds to the classical center manifold theorem for discrete time dynamical systems. We refer for instance to Bravo de la Parra and Sánchez [23] for an application to population dynamics discrete time models with different time scales (see also Bouyekhf et al. [27]). Such systems (3.1.11) has been studied in Pötzsche [130] when $F = I_{\mathcal{L}(X)}$ in a very general manner considering explicit dependence of the discrete time and where the perturbations R_c and R_h were considered to be at least continuously differentiable with respect to the state space. See also Praly [132] who investigates a spacial case of [130]. However the result in [130, 132] cannot be applied to our system due to the non linearity F and the Lipschitz regularity. For dynamical systems like (3.1.11) considering bounded and Lipschitz continuous perturbations, the literature is significantly small. We refer to Sakamoto [145] and Yi [167] for ODEs and to [132] for finite dimensional discrete time system.

3.2.1 Consequence for $T + \tilde{T}$

Before stating the main result of this section, we will make in addition to Assumption 3.1.2, the following assumption.

Assumption 3.2.5 *We assume that:*

- (i) There exists a complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of T in M with $\mathbf{x} \in \mathbb{L}_{\rho_0}(\mathbb{Z}, M)$.
- (ii) $DT(\cdot) : X \rightarrow \mathcal{L}(X)$ is uniformly bounded and Lipschitz continuous; that is

$$\|DT(\cdot)\|_{C(X, \mathcal{L}(X))} + \|DT(\cdot)\|_{\text{Lip}(X, \mathcal{L}(X))} < +\infty; \quad (3.2.6)$$

- (iii) We have the following uniform bound

$$\sup_{x \in M} \|\Pi_x^\alpha\|_{\mathcal{L}(X)} \leq \kappa, \quad \alpha = u, s. \quad (3.2.7)$$

The main theorem for this section is concerned to the persistence of the normally hyperbolic manifold M .

Theorem 3.2.6 *Let Assumption 3.1.2 and Assumption 3.2.5 be satisfied. Let $\widehat{\rho}_0, \widehat{\rho}$ be given such that $0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$. Let $\eta \in (\widehat{\rho}_0, \widehat{\rho})$ be given. Then there exist $\delta > 0$ and a constant $\widehat{\gamma} > 0$ such that if*

$$\left\| \tilde{T} \right\|_{C(X)} + \left\| \tilde{T} \right\|_{Lip(X)} \leq \delta^2,$$

then there exists some $\Psi : M \rightarrow X^h$ such that the manifold

$$\tilde{M} = \{x + \Psi(x) : x \in M\},$$

has the following properties:

(i) *If $\mathbf{x} + \mathbf{y} = \{x_n + y_n\}_{n \in \mathbb{Z}} \subset M \oplus X^h$ is a complete orbit of $T + \tilde{T}$ with*

$$\|y_n\| \leq \delta, \forall n \in \mathbb{Z},$$

then $x_0 + y_0 \in \tilde{M}$.

(ii) *For each $x + \Psi(x) \in \tilde{M}$ there exists a unique solution $\mathbf{x} + \Psi(\mathbf{x}) = \{x_n + \Psi(x_n)\}_{n \in \mathbb{Z}} \subset \tilde{M}$ of $T + \tilde{T}$ with $x_0 = x$ and $(\mathbf{x}, \Psi(\mathbf{x})) \in \mathbb{L}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$.*

(iii) *Ψ is a bounded and Lipschitz continuous map on M with*

$$\|\Psi(x) - \Psi(\bar{x})\| \leq \widehat{\gamma} \delta \|x - \bar{x}\|, \forall x, \bar{x} \in M, \text{ and}$$

and

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \left\| \tilde{T} \right\|_{C(X)},$$

where γ is the constant defined in Theorem 3.2.4.

Remark 3.2.7 *Observe that the property (ii) stated in Theorem 3.2.6 ensures that the map $T + \tilde{T} : X \rightarrow X$ is a bijection from $\tilde{M}$ into $\tilde{M}$.*

Proof of Theorem 3.2.6. As in the classical center manifold theorem, we will need the cut off function $\chi \in C^\infty(\mathbb{R}, [0, 1])$ defined by

$$\chi(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| \geq 2, \end{cases} \quad (3.2.8)$$

in order to modify the system

$$\begin{cases} x_{n+1} = T(x_n) + R_c(x_n, y_n), & n \in \mathbb{N}, \\ y_{n+1} = DT(x_n)y_n + R_h(x_n, y_n), & n \in \mathbb{N}. \end{cases} \quad (3.2.9)$$

(or equivalently (3.1.2)) so that it satisfies Assumption 3.2.3. Let us firstly note that conditions (H1), (H2), (H4) and (H5) stated in Assumption 3.2.3 are direct consequences of Assumption 3.1.2 and Assumption 3.2.5 combined with (3.1.7) and Remark 2.3.5 in Ducrot, Magal and Seydi [58] (see Chapter 2).

Next we claim that

$$\|R_c(x, y)\| \leq \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|y\|^2 + \kappa \|\tilde{T}\|_{C(X)}, \quad \forall (x, y) \in M \times X^h, \quad (3.2.10)$$

and

$$\|R_h(x, y)\| \leq 2\kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|y\|^2 + 2\kappa \|\tilde{T}\|_{C(X)}, \quad \forall (x, y) \in M \times X^h, \quad (3.2.11)$$

while for each $(x, y), (\bar{x}, \bar{y}) \in M \times X^h$ we have

$$\begin{aligned} \|R_c(x, y) - R_c(\bar{x}, \bar{y})\| &\leq \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} [\|x - \bar{x}\| + \|y - \bar{y}\|] \\ &\quad + \kappa \|\tilde{T}\|_{Lip(X)} [\|x - \bar{x}\| + \|y - \bar{y}\|]. \end{aligned} \quad (3.2.12)$$

and

$$\begin{aligned} \|R_h(x, y) - R_h(\bar{x}, \bar{y})\| &\leq 2\kappa \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} [\|x - \bar{x}\| + \|y - \bar{y}\|] \\ &\quad + 2\kappa \|\tilde{T}\|_{Lip(X)} [\|x - \bar{x}\| + \|y - \bar{y}\|] \\ &\quad + \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} [\|y - \bar{y}\| + \|x - \bar{x}\| \|\bar{y}\|]. \end{aligned} \quad (3.2.13)$$

We will only give details for the estimates of $\|R_c(x, y)\|$ and $\|R_h(x, y) - R_h(\bar{x}, \bar{y})\|$ since the arguments are quite similar and even easier for $\|R_c(x, y) - R_c(\bar{x}, \bar{y})\|$ and $\|R_h(x, y)\|$. Before proving this claim recall that due to (3.2.7) we have

$$\|\Pi^c\|_{\mathcal{L}(X)} \leq \kappa \text{ and } \|\Pi^h\|_{\mathcal{L}(X)} \leq \|\Pi_x^u\|_{\mathcal{L}(X)} + \|\Pi_x^s\|_{\mathcal{L}(X)} \leq 2\kappa, \quad \forall x \in M. \quad (3.2.14)$$

Estimate of $\|R_c(x, y)\|$: Let $(x, y) \in M \times X^h$ be given. Recalling the definition of R_c in (3.1.9), observe that we have

$$\begin{aligned} R_c(x, y) &= \Pi^c \left[T(x + y) - T(x) + \tilde{T}(x + y) \right] \\ &= \Pi^c \int_0^1 DT(x + ly) dl.y + \Pi^c \tilde{T}(x + y). \end{aligned} \quad (3.2.15)$$

Hence since we have $\Pi^c DT(x)y = 0$ by using (3.2.15) one can re-write $R_c(x, y)$ as

$$R_c(x, y) = \Pi^c \int_0^1 [DT(x + ly) - DT(x)] dl.y + \Pi^c \tilde{T}(x + y), \quad \forall u \in X, \quad (3.2.16)$$

and (3.2.10) follows by combining (3.2.16) and (3.2.6) with (3.2.14).

Estimate of $\|R_h(x, y) - R_h(\bar{x}, \bar{y})\|$: Let $(x, y), (\bar{x}, \bar{y}) \in M \times X^h$ be given. Then recalling the definition of R_c in (3.1.9) one has

$$\begin{aligned} R_h(x, y) - R_h(\bar{x}, \bar{y}) &= \Pi^h [T(x + y) - T(\bar{x} + \bar{y})] + \Pi^h [\tilde{T}(x + y) - \tilde{T}(\bar{x} + \bar{y})] \\ &\quad - DT(x)y + DT(\bar{x})\bar{y} \\ &= \Pi^h \int_0^1 DT(l(x + y) + (1 - l)(\bar{x} + \bar{y})) dl (x - \bar{x} + y - \bar{y}) \\ &\quad + \Pi^h [\tilde{T}(x + y) - \tilde{T}(\bar{x} + \bar{y})] \\ &\quad - DT(x)(y - \bar{y}) + (DT(\bar{x}) - DT(x))\bar{y} \end{aligned}$$

so that using (3.2.14) and (3.2.6) one obtains

$$\begin{aligned} \|R_h(x, y) - R_h(\bar{x}, \bar{y})\| &\leq 2\kappa \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} [\|x - \bar{x}\| + \|y - \bar{y}\|] \\ &\quad + 2\kappa \left\| \tilde{T} \right\|_{Lip(X)} [\|x - \bar{x}\| + \|y - \bar{y}\|] \\ &\quad + \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \|y - \bar{y}\| \\ &\quad + \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|x - \bar{x}\| \|\bar{y}\|, \end{aligned}$$

and (3.2.13) follows. This completes the proof of (3.2.13).

Then it is clear that the maps R_i , $i = c, h$ do not satisfy the condition **(H3)** stated in Assumption 3.2.3. Therefore to obtain this condition we consider, as in the classical center manifold theory, the following truncated system

$$\begin{cases} x_{n+1} = T(x_n) + R_c(\delta, x_n, y_n), & n \in \mathbb{N}, \\ y_{n+1} = DT(x_n)y_n + R_h(\delta, x_n, y_n), & n \in \mathbb{N}, \end{cases} \quad (3.2.17)$$

with $\delta > 0$ and the maps $R_c(\delta, \cdot) : M \times X^h \rightarrow M$ and $R_h(\delta, \cdot) : M \times X^h \rightarrow X^h$ defined by

$$R_i(\delta, x, y) := \chi(\|y\| \delta^{-1}) R_i(x, y), \quad \forall (x, y) \in M \times X^h, \text{ for } i = c, h. \quad (3.2.18)$$

Hence using the definition of χ in (3.2.8) one may observe that the system (3.2.17) agrees with the system (3.2.9) when $\|y_n\| \leq \delta$. Therefore by using (3.2.10)-(3.2.13) it becomes easy to show that there exists a positive constant $C_0 > 0$ depending only on $\kappa, \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))}, \|DT(\cdot)\|_{C(X, \mathcal{L}(X))}$ and $\|\chi\|_{Lip(\mathbb{R})}$ such that if

$$\left\| \tilde{T} \right\|_{C(X)} + \left\| \tilde{T} \right\|_{Lip(X)} \leq \delta^2,$$

then

$$\sup_{(x, y) \in M \times X^h} \|R_i(\delta, x, y)\| \leq C_0 \delta^2, \text{ for } i = c, h, \quad (3.2.19)$$

and

$$\|R_h(\delta, \cdot)\|_{Lip(M \times X^h, M)} \leq C_0 \delta \text{ and } \|R_c(\delta, \cdot)\|_{Lip(M \times X^h, M)} \leq C_0 \delta. \quad (3.2.20)$$

The idea is to prove the persistence property for system (3.2.17) and deduce properties of (3.2.9) (or equivalently (3.1.2)). In fact letting δ go to 0 ensure that there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ the condition in Theorem 3.2.4 are satisfied so that the properties (i)-(iii) stated in Theorem 3.2.4 are satisfied for system (3.2.17). Thus the property (i) stated in Theorem 3.2.6 follows by using the fact that the system (3.2.17) agrees with system (3.2.9) when $\|y_n\| \leq \delta$ as well as the Lipschitz property of Ψ in the property (iii). Let us now show that the property (ii) stated in Theorem 3.2.6 and the uniform bound of Ψ in the property (iii) holds true. To do so we make use of the property (iii) in Theorem 3.2.4 that is

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \sup_{x \in M} \|R_h(\delta, x, 0)\|,$$

with $\gamma > 0$ given in Theorem 3.2.4. In fact since we have

$$R_h(\delta, x, 0) = R_h(x, 0) = \tilde{T}(x), \quad \forall x \in M,$$

it follows that

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \sup_{x \in M} \|\tilde{T}(x)\| \leq \frac{6}{5} \gamma \delta^2 \leq \delta \text{ (up to reduce } \delta),$$

so that solutions $(\mathbf{x}, \mathbf{y}) = \{(x_n, y_n)\}_{n \in \mathbb{Z}}$ of (3.2.17) starting from $\widehat{M}$ satisfies

$$(\mathbf{x}, \mathbf{y}) \in \mathbb{L}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h),$$

and

$$y_n = \Psi(x_n), \quad \forall n \in \mathbb{Z} \text{ and } \|y_n\| = \|\Psi(x_n)\| \leq \delta.$$

This means that $(\mathbf{x}, \mathbf{y}) = \{(x_n, y_n)\}_{n \in \mathbb{Z}} = \{(x_n, \Psi(x_n))\}_{n \in \mathbb{Z}}$ is also a solution of (3.2.9). Therefore we deduce that $\widehat{M}$ is invariant by (3.2.9). The proof is complete. ■

3.2.2 Consequence for time continuous maps

In this section we will derive a consequence of Theorem 3.2.6 for time continuous maps. Recall that a family $\{T(t)\}_{t \geq 0}$ defines a continuous semiflow on X if

$$(i) \quad T(0) = I_{\mathcal{L}(X)} \text{ and } T(t+l) = T(t)T(l), \quad \forall t, l \geq 0,$$

$$(ii) \quad \text{The map } (t, u) \rightarrow T(t)u \text{ is continuous from } [0, +\infty) \times X \text{ into } X.$$

We also recall that $v \in C(\mathbb{R}, X)$ is a **complete orbit** of $\{T(t)\}_{t \geq 0}$ if

$$v(t) = T(t-l)v(l), \quad \forall t, l \in \mathbb{R} \text{ with } t \geq l.$$

The main result of this section is the following

Theorem 3.2.8 Let $\{T(t)\}_{t \geq 0}$ be a continuous semiflow on X . Let $T_0 : X \rightarrow X$ be a C^1 map satisfying Assumption 3.1.2 and Assumption 3.2.5. Assume that there exists $t_0 > 0$ such that

$$\gamma(t_0) := \sup_{l \in [0, t_0]} \|T(l)\|_{Lip(X)} < +\infty, \quad (3.2.21)$$

and

$$T(l)M \subset M, \quad \forall l \in [0, t_0).$$

Then for each $\hat{\rho}_0$ and $\hat{\rho}$ with $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and each $\eta \in (\hat{\rho}_0, \hat{\rho})$ there exist two constants $\delta > 0$ and $\hat{\gamma} > 0$ such that if

$$\|T(t_0) - T_0\|_{C(X)} + \|T(t_0) - T_0\|_{Lip(X)} \leq \delta^2,$$

then there exists some function $\Psi : M \rightarrow X^h$ such that the manifold

$$\tilde{M} = \{x + \Psi(x) : x \in M\},$$

has the following properties:

(i) The map Ψ is bounded and Lipschitz continuous on M with

$$\|\Psi(x) - \Psi(\bar{x})\| \leq \hat{\gamma}\delta \|x - \bar{x}\|, \quad \forall x, \bar{x} \in M,$$

and

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5}\gamma \|T(t_0) - T_0\|_{C(X)},$$

where γ is the constant defined in Theorem 3.2.4.

(ii) For each $t \geq 0$ the map $T(t)$ is a bijection from $\tilde{M}$ into $\tilde{M}$.

(iii) For each $x + \Psi(x) \in \tilde{M}$, there exists a unique complete orbit $v : \mathbb{R} \rightarrow \tilde{M}$ of $\{T(t)\}_{t \geq 0}$ with $v(0) = x + \Psi(x)$.

(iv) If $v \in C(\mathbb{R}, X)$ is a complete orbit of $\{T(t)\}_{t \geq 0}$ such that

$$\|\Pi^h v(t)\| \leq \delta, \quad \forall t \in \mathbb{R},$$

then $v(t) \in \tilde{M}$ for all $t \in \mathbb{R}$.

Proof. It is clear that under conditions of Theorem 3.2.8, Theorem 3.2.6 holds true for $T = T_0$ and $\tilde{T} = T(t_0) - T_0$ with $\delta > 0$ small enough. Let $\Psi : M \rightarrow X^h$ be the map such that the manifold

$$\tilde{M} = \{x + \Psi(x) : x \in M\},$$

satisfies the properties (i)-(vi) stated in Theorem 3.2.6 for $T = T_0$, $\tilde{T} = T(t_0) - T_0$ and $T + \tilde{T} = T(t_0)$. Therefore the condition (i) holds true. Before proceeding to the proof of

properties (ii)-(vi) let us assume without loss of generality that δ satisfies also the inequality

$$\delta < \frac{1}{\kappa\gamma(t_0)\frac{6}{5}\gamma}, \quad (3.2.22)$$

wherein $\gamma(t_0)$ is defined in (3.2.21).

Proof of (ii): Let us first prove that

$$T(t)\tilde{M} = \tilde{M}, \quad \forall t \geq 0. \quad (3.2.23)$$

First of all we infer from Remark 3.2.7 that

$$T(t_0)\tilde{M} = \tilde{M}. \quad (3.2.24)$$

Now let us prove that $T(t)\tilde{M} = \tilde{M}$ for all $t \in [0, t_0]$.

Let $t \in [0, t_0]$ be given and fixed. Let $x_0 + \Psi(x_0) \in \tilde{M}$. Then there exists a unique complete orbit $\mathbf{x} + \Psi(\mathbf{x}) = \{x_n + \Psi(x_n)\}_{n \in \mathbb{Z}} \subset \tilde{M}$ of $T(t_0)$. Observe that the sequence $\{T(t)(x_n + \Psi(x_n))\}_{n \in \mathbb{Z}}$ is a complete orbit of $T(t_0)$. Hence in order to obtain that $T(t)(x_0 + \Psi(x_0)) \in \tilde{M}$ we only need to prove that

$$\|\Pi^h T(t)(x_n + \Psi(x_n))\| \leq \delta, \quad \forall n \in \mathbb{Z}, \quad (3.2.25)$$

and apply the property (ii) stated in Theorem 3.2.6 for $T + \tilde{T} = T(t_0)$. In fact since for each $t \in [0, t_0]$ we have $T(t)M \subset M$ it follows that

$$\Pi^h T(t)(x_n + \Psi(x_n)) = \Pi^h [T(t)(x_n + \Psi(x_n)) - T(t)x_n]. \quad (3.2.26)$$

Next recalling that

$$\sup_{x \in \tilde{M}} \|\Psi(x)\| \leq \frac{6}{5}\gamma \|T(t_0) - T_0\|_{C(X)} \leq \frac{6}{5}\gamma\delta^2,$$

we infer from (3.2.22) and (3.2.26) that for each $n \in \mathbb{Z}$

$$\begin{aligned} \|\Pi^h T(t)(x_n + \Psi(x_n))\| &\leq \|\Pi^h\|_{\mathcal{L}(X)} \sup_{l \in [0, t_0]} \|T(l)\|_{Lip(\mathcal{V}(M, \varepsilon^*), X)} \sup_{x \in \tilde{M}} \|\Psi(x)\| \\ &\leq \kappa\gamma(t_0) \frac{6}{5}\gamma\delta^2 \\ &\leq \delta, \end{aligned}$$

so that (3.2.25) holds true and we conclude that

$$T(t)\tilde{M} \subset \tilde{M}, \quad \forall t \in [0, t_0]. \quad (3.2.27)$$

In order to obtain the inverse inclusion in (3.2.27) we just observe that by using (3.2.24) we have

$$\tilde{M} = T(t_0)\tilde{M} = T(t)T(t_0 - t)\tilde{M} \subset T(t)\tilde{M}, \quad \forall t \in [0, t_0].$$

We have proved that

$$T(t) \tilde{M} = \tilde{M}, \quad \forall t \in [0, t_0]. \quad (3.2.28)$$

Finally for each $t > t_0$ there exist $n \in \mathbb{N}$ and $l \in [0, t_0]$ such that $t = nt_0 + l$ and we obtain by the semiflows property combined with (3.2.24) and (3.2.28) that

$$T(t) \tilde{M} = T(nt_0 + l) \tilde{M} = T^n(t_0) T(l) \tilde{M} = \tilde{M}, \quad (3.2.29)$$

so that $T(t) \tilde{M} = \tilde{M}$ for all $t \geq 0$.

It remains to prove that the map $T(t) : X \rightarrow X$ is one to one on $\tilde{M}$ for each $t \geq 0$. To do so let us recall that due to Remark 3.2.7 the map $T(t_0)$ is a bijection from $\tilde{M}$ into $\tilde{M}$. Therefore by using the semiflows property we have

$$T(t_0) = T(t) T(t_0 - t), \quad \forall t \in [0, t_0],$$

and since $T(t_0)$ is one to one on $\tilde{M}$ we obtain that for each $t \in [0, t_0]$ the map $T(t_0 - t) : X \rightarrow X$ is one to one on $\tilde{M}$ or equivalently for $t \in [0, t_0]$ the map $T(t) : X \rightarrow X$ is one to one on $\tilde{M}$. The conclusion for $t > t_0$ follows from the semiflows property as is (3.2.29). The proof of (ii) is complete.

Proof of (iii): Let $u \in \tilde{M}$ be given. Then due to the property (ii) one can define

$$v(t) := T(t) u \in \tilde{M} \text{ and } v(-t) := T(t)^{-1} u \in \tilde{M}, \quad \forall t \geq 0,$$

and the result follows by simple verification.

Proof of (iv): Let $v \in C(\mathbb{R}, X)$ be a complete orbit of $\{T(t)\}_{t \geq 0}$ such that

$$\|\Pi^h v(t)\| \leq \delta, \quad \forall t \in \mathbb{R}.$$

Then we have in particular

$$\|\Pi^h v(mt_0)\| \leq \delta, \quad \forall m \in \mathbb{Z}. \quad (3.2.30)$$

Next we first prove that $v(pt_0) \in \tilde{M}$ for each $p \in \mathbb{Z}$. To do so we will make use of the property (ii) stated in Theorem 3.2.6. Let $p \in \mathbb{Z}$ be given. Observe that since $v \in C(\mathbb{R}, X)$ is a complete orbit of $\{T(t)\}_{t \geq 0}$, the sequence $\{v(nt_0 + pt_0)\}_{n \in \mathbb{Z}}$ satisfies

$$v((n+1)t_0 + pt_0) = T(t_0) v(nt_0 + pt_0), \quad \forall n \in \mathbb{Z},$$

so that $\{v(nt_0 + pt_0)\}_{n \in \mathbb{Z}}$ is a complete orbit of $T(t_0)$ and we infer from (3.2.30) and the property (ii) stated in Theorem 3.2.6 that $v(pt_0) \in \tilde{M}$.

In order to finish the proof of (iv) we note that in general for $t \in \mathbb{R}$ one can write that $t = pt_0 + l$ with $p \in \mathbb{Z}$ and $l \in [0, t_0]$. Therefore since we have

$$v(pt_0 + l) = T(l) v(pt_0) \text{ and } T(l) \tilde{M} = \tilde{M},$$

by recalling that $v(pt_0) \in \tilde{M}$ for all $p \in \mathbb{Z}$ it follows that $v(pt_0 + l) = v(t) \in \tilde{M}$. The proof of (iv) is complete. $\blacksquare$

3.3 Proof of the global center-like manifold theorem

The goal of this section is to provide a proof of the global center-like manifold theorem for the system (3.1.11).

Next we fix the notations that will be used throughout this section. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ be a given sequence in M . For notational simplicity we denote in the sequel by $\{U_{\mathbf{x}}^c(n, p)\}_{n \geq p} \subset \mathcal{L}(M)$ the evolution semigroup associated to $\{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ that is

$$U_{\mathbf{x}}^c(n, p) := \begin{cases} DF(x_{n-1}) \dots DF(x_p), & \text{if } n > p, \\ I_{\mathcal{L}(M)}, & \text{if } n = p, \end{cases} \quad (3.3.1)$$

and by $\{U_{\mathbf{x}}(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ the evolution semigroup associated to $\{A(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ that is

$$U_{\mathbf{x}}(n, p) := \begin{cases} A(x_{n-1}) \dots A(x_p), & \text{if } n > p, \\ I_{\mathcal{L}(X^h)}, & \text{if } n = p. \end{cases} \quad (3.3.2)$$

3.3.1 Preliminary lemmas

In this section we will give some technical lemmas that will be used in the sequel. Let us firstly make the following remark

Remark 3.3.1 *Observe that if $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ is a complete orbit of F then one has for each $n \geq p$*

$$U_{\mathbf{x}}^c(n, p) = DF^{n-p}(x_p),$$

so that $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ is invertible with inverse

$$U_{\mathbf{x}}^c(p, n) := U_{\mathbf{x}}^c(n, p)^{-1} : M \rightarrow M$$

and condition (3.2.4) ensures that

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \kappa e^{\rho_0|n-p|}, \quad \forall (n, p) \in \mathbb{Z}^2.$$

Next note that a direct consequence of condition **(H1)** and condition **(H2)-(ii)** in Assumption 3.2.3 is the following

Lemma 3.3.2 *F is a bijection from M onto M and F^{-1} is Lipschitz continuous on M with*

$$\|F^{-1}\|_{Lip(M)} \leq \kappa e^{\rho_0}. \quad (3.3.3)$$

Proof. Recalling that $F(M) = M$ it is sufficient to prove that F is one to one on M . To do so let $x, \bar{x} \in M$ be given. Then by using (3.2.4) one has

$$\|F(x) - F(\bar{x})\| = \left\| \int_0^1 DF(lx + (1-l)\bar{x}) dl \cdot (x - \bar{x}) \right\| \geq \frac{1}{\kappa} e^{-\rho_0} \|x - \bar{x}\|. \quad (3.3.4)$$

since $lx + (1-l)\bar{x} \in M$ for each $l \in [0, 1]$. This prove that F is one to one on M and that (3.3.3) holds true. $\blacksquare$

The next proposition is a perturbation result of exponential dichotomy along pseudo orbits of F .

Proposition 3.3.3 *Let conditions (H1)-(H4) in Assumption 3.2.3 be satisfied. Let $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ be given. There exist $\varepsilon := \varepsilon(\kappa, \rho, \rho_0, K_A, \hat{\rho}_0, K_F, \hat{\rho}) > 0$ and $\hat{\kappa} := \hat{\kappa}(\kappa, \rho, \rho_0, K_A) \geq \kappa$ with*

$$K_A := \|A\|_{Lip(M \times \mathcal{L}(X^h))} + \|A\|_{C(M, \mathcal{L}(X^h))} \text{ and } K_F := \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))},$$

such that for each sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ with

$$\|x_{n+1} - F(x_n)\| \leq \varepsilon,$$

the following properties hold true:

(i) For each $n, p \in \mathbb{Z}$ with $n \geq p$ the operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ is invertible with inverse

$$U_{\mathbf{x}}^c(p, n) := U_{\mathbf{x}}^c(n, p)^{-1} : M \rightarrow M \quad (3.3.5)$$

and

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \hat{\kappa} e^{\hat{\rho}_0 |n-p|}, \quad \forall (n, p) \in \mathbb{Z}^2.$$

(ii) The family of operators $\{U_{\mathbf{x}}(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ has an exponential dichotomy on $\mathbb{Z}$ with constant $\hat{\kappa}$, exponent $\hat{\rho}$ and projectors $\Pi^\alpha(\mathbf{x}) = \{\Pi_n^\alpha(\mathbf{x})\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$, $\alpha = u, s$.

Proof. Proof of (i): Firstly note that one can say from Remark 3.3.1 that for each complete orbit $\bar{\mathbf{x}} = \{\bar{x}_n\}_{n \in \mathbb{Z}} \subset M$ of F , $\{DF(\bar{x}_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ , exponents ρ and $\rho_0 \in (0, \rho)$ associated to the three families of projectors $\{\Pi_n^\alpha(\bar{\mathbf{x}})\}_{n \in \mathbb{Z}}$, $\alpha = u, s, c$ such that

$$\Pi_n^c(\bar{\mathbf{x}}) = I_{\mathcal{L}(M)} \text{ and } \Pi_n^u(\bar{\mathbf{x}}) = \Pi_n^s(\bar{\mathbf{x}}) = 0_{\mathcal{L}(M)}, \quad \forall n \in \mathbb{Z}.$$

Let $m > 0$ be a positive integer satisfying

$$\frac{\ln(\kappa + 1)}{m} < \frac{\hat{\rho}_0 - \rho_0}{2}, \quad (3.3.6)$$

Let $\varepsilon > 0$ be given. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ be given satisfying

$$\|x_{n+1} - F(x_n)\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}. \quad (3.3.7)$$

Define the family of intervals $\{\mathbb{I}_i\}_{i \in \mathbb{Z}}$ of $\mathbb{Z}$ as

$$\mathbb{I}_i := [im, (i+1)m], \quad \forall i \in \mathbb{Z}.$$

Next on each interval $\mathbb{I}_i, i \in \mathbb{Z}$, we will rewrite $\{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ as a perturbation of a sequence $\{B_n^i\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ which is exponentially trichotomic on $\mathbb{Z}$.

To be more precise recall that F is invertible on M and define for each $i \in \mathbb{Z}$

$$B_n^i := DF(F^{n-im}(x_{im})), \quad \forall n \in \mathbb{Z}.$$

Since for each $i \in \mathbb{Z}$

$$\mathbf{x}^i := \{x_n^i\}_{n \in \mathbb{Z}} = \{F^{n-im}(x_{im})\}_{n \in \mathbb{Z}},$$

is a complete orbit of F we know that $\{B_n^i\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ , exponents ρ and $\rho_0 \in (0, \rho)$ associated to three families of projectors $\{\Pi_n^\alpha(\mathbf{x}^i)\}_{n \in \mathbb{Z}}, \alpha = u, s, c$ such that

$$\Pi_n^c(\mathbf{x}^i) = I_{\mathcal{L}(M)} \text{ and } \Pi_n^u(\mathbf{x}^i) = \Pi_n^s(\mathbf{x}^i) = 0_{\mathcal{L}(M)}, \quad \forall n \in \mathbb{Z}.$$

Let us now show that for each $i \in \mathbb{Z}$ the sequence of operators $\{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is close to $\{B_n^i\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ on $\mathbb{I}_i$.

Let us first observe that for each $i \in \mathbb{Z}$ one has

$$\begin{aligned} x_{im+2}^i - x_{im+2} &= F^2(x_{im}) - x_{im+2} \\ &= [F(F(x_{im})) - F(x_{im+1})] + [F(x_{im+1}) - x_{im+2}] \\ &= \int_0^1 DF(lF(x_{im}) + (1-l)x_{im+1}) dl \cdot (F(x_{im}) - x_{im+1}) \\ &\quad + F(x_{im+1}) - x_{im+2} \end{aligned}$$

and since $DF(\cdot)$ is uniformly bounded on M and

$$lT(x_{im}) + (1-l)x_{im+1} \in M, \quad \forall l \in [0, 1],$$

we infer from (3.3.7) that

$$\|\tilde{x}_{im+2} - x_{im+2}\| \leq \sup_{x \in M} \|DF(x)\|_{\mathcal{L}(M)} \varepsilon + \varepsilon.$$

By observing that from (3.2.4) one has

$$\sup_{x \in M} \|DF(x)\|_{\mathcal{L}(M)} \leq \kappa e^{\rho_0},$$

on can easily obtain by induction that for each $k \in [0, m]$

$$\|x_{im+k}^i - x_{im+k}\| \leq (1 + \kappa e^{\rho_0} + \dots + \kappa^{k-1} e^{\rho_0(k-1)}) \varepsilon \leq (1 + \kappa e^{\rho_0} + \dots + \kappa^m e^{\rho_0 m}) \varepsilon.$$

Thus one gets for each $k \in [0, m]$

$$\begin{aligned} \|B_{im+k}^i - DF(x_{im+k})\|_{\mathcal{L}(M)} &= \|DF(x_{im+k}^i) - DF(x_{im+k})\|_{\mathcal{L}(M)} \\ &\leq \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))} (1 + \kappa e^{\rho_0} + \dots + \kappa^m e^{\rho_0 m}) \varepsilon, \end{aligned}$$

so that

$$\sup_{k \in [im, (i+1)m]} \|B_k^i - DF(x_k)\|_{\mathcal{L}(M)} \leq \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))} (1 + \kappa e^{\rho_0} + \dots + \kappa^m e^{\rho_0 m}) \varepsilon. \quad (3.3.8)$$

Now recalling that we have

$$0 < \rho_0 < \frac{\rho_0 + \widehat{\rho}_0}{2} < \frac{\rho + \widehat{\rho}}{2} < \rho,$$

one can chose ε small enough depending only on $\kappa, m, \rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))}$ so that Corollary 2.5.3 in Ducrot, Magal and Seydi [58] (see Chapter 2) applies on each interval $\mathbb{I}_i := [im, (i+1)m]$, $i \in \mathbb{Z}$ and provides that for each $i \in \mathbb{Z}$, $\{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is exponentially trichotomic on each $\mathbb{I}_i$ with constant $\kappa + 1$, exponents $\frac{\rho_0 + \widehat{\rho}_0}{2}$ and $\frac{\rho + \widehat{\rho}}{2}$ associated to three families of projectors $\{\Pi_n^{i\alpha}(\mathbf{x})\}_{n \in \mathbb{I}_i}$, $\alpha = u, s, c$. Moreover since we have the isomorphism properties between the projectors $\{\Pi_n^{i\alpha}(\mathbf{x})\}_{n \in \mathbb{I}_i}$ and $\{\Pi_n^\alpha(\mathbf{x}^i)\}_{n \in \mathbb{I}_i}$, $\alpha = u, s, c$ we get for each $n \in \mathbb{I}_i$

$$\Pi_n^{i\alpha}(\mathbf{x}) = \Pi_n^c(\mathbf{x}^i) = I_{\mathcal{L}(M)},$$

and

$$\Pi_n^u(\mathbf{x}^i) = \Pi_n^s(\mathbf{x}^i) = \Pi_n^{iu}(\mathbf{x}) = \Pi_n^{is}(\mathbf{x}) = 0_{\mathcal{L}(M)}.$$

So one can summarize what we have proved as follow. On each interval $\mathbb{I}_i = [im, (i+1)m]$, $i \in \mathbb{Z}$ we have

$$U_{\mathbf{x}}^c(n, p) : M \rightarrow M,$$

is invertible for each $n \geq p$ in $\mathbb{I}_i$ with inverse defined as in (3.3.5) and

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq (\kappa + 1) e^{-\frac{\rho_0 + \widehat{\rho}_0}{2}|n-p|}, \quad \forall n, p \in \mathbb{I}_i = [im, (i+1)m]. \quad (3.3.9)$$

Therefore it is clear that for each $n, p \in \mathbb{Z}$ with $n \geq p$ the operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ is invertible with inverse define as in (3.3.5). It remains now to give an estimate of $\{U_{\mathbf{x}}^c(n, p)\}_{n \geq p} \subset \mathcal{L}(M)$ for n, p varying in $\mathbb{Z}$.

Now let $n, p \in \mathbb{Z}$ such that $n > p$. Then there exists $i, j \in \mathbb{Z}$ with $i > j$ such that $n \in [im, (i+1)m]$ and $p \in [jm, (j+1)m]$ so that by using (3.3.9) we obtain

$$\begin{aligned} \|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} &= \|U_{\mathbf{x}}^c(n, im) U_{\mathbf{x}}^c(im, jm) U_{\mathbf{x}}^c(jm, p)\|_{\mathcal{L}(M)} \\ &\leq \|U_{\mathbf{x}}^c(n, im)\|_{\mathcal{L}(X^c)} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(X^c)} \|U_{\mathbf{x}}^c(jm, p)\|_{\mathcal{L}(M)} \\ &\leq (\kappa + 1) e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-im)} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(X^c)} (\kappa + 1) e^{\frac{\rho_0 + \hat{\rho}_0}{2}(jm-p)} \\ &\leq (\kappa + 1)^2 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p)} e^{\frac{\rho_0 + \hat{\rho}_0}{2}(jm-im)} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(M)}, \end{aligned}$$

and since we have

$$\begin{aligned} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(X^c)} &= \|U_{\mathbf{x}}^c(im, (i-1)m) \dots U_{\mathbf{x}}^c((j-1)m, jm)\|_{\mathcal{L}(M)} \\ &\leq (\kappa + 1)^{(i-j)} e^{\frac{\rho_0 + \hat{\rho}_0}{2}(im-jm)}, \end{aligned}$$

it follows that

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq (\kappa + 1) e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p)} (\kappa + 1)^{(i-j+1)}. \quad (3.3.10)$$

Furthermore by observing that

$$im - jm + m \leq n - p,$$

we obtain from (3.3.10) that

$$\begin{aligned} \|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} &\leq (\kappa + 1) e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p)} (\kappa + 1)^{\frac{n-p}{m}} \\ &\leq (\kappa + 1) e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p) + (n-p) \frac{\ln(\kappa+1)}{m}}. \end{aligned}$$

Finally we infer from (3.3.6) that

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(X^c)} \leq (\kappa + 1) e^{\hat{\rho}_0(n-p)}. \quad (3.3.11)$$

We also have in a similar manner that for each $n, p \in \mathbb{Z}$ with $n \geq p$

$$\|U_{\mathbf{x}}^c(p, n)\|_{\mathcal{L}(M)} \leq (\kappa + 1) e^{\hat{\rho}_0(n-p)}. \quad (3.3.12)$$

This completes the proof of (i).

Proof of (ii): The idea of the proof is the same as in the proof of (i). By proceeding as in the proof of (i) one can easily prove that for a given positive integer $m > 0$ large enough and ε small enough depending only on $\kappa, m, \rho_0, \hat{\rho}_0, \hat{\rho}, \rho$ and upon the quantity

$$\max \left\{ \sup_{x \in M} \|A(x)\|_{\mathcal{L}(X^h)}, \|A\|_{\text{Lip}(M, \mathcal{L}(X^h))} \right\},$$

for any given sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$

$$\|x_{n+1} - F(x_n)\| \leq \varepsilon, \quad \forall n \in \mathbb{Z},$$

the sequence $\{A(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ is exponentially dichotomic on each interval $[im, (i+1)m]$, $i \in \mathbb{Z}$ with uniform constant $\kappa+1$, and exponents $\frac{\rho_0 + \hat{\rho}_0}{2}$ and $\frac{\rho + \hat{\rho}}{2}$. Then Theorem 2.2.3 in Ducrot, Magal and Seydi [58] (see Chapter 2) applies and allows us to conclude to the proof of (ii). ■

From now on, recalling that ρ_0 and ρ are given and fixed by Assumption 3.2.3 we fix $\hat{\rho}_0, \hat{\rho}$ and η such that

$$0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho \text{ and } \eta \in (\hat{\rho}_0, \hat{\rho}). \quad (3.3.13)$$

As a consequence of the above lemma one can deduce the following technical lemma.

Lemma 3.3.4 *Let conditions (H1)-(H4) in Assumption 3.2.3 be satisfied. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ be a sequence satisfying*

$$\|x_{n+1} - F(x_n)\| \leq \varepsilon,$$

with ε provided by Proposition 3.3.3. Then for each pair of sequences $\mathbf{u} = \{u_n\}_{n \in \mathbb{Z}} \subset M$ and $\mathbf{w} = \{w_n\}_{n \in \mathbb{Z}} \subset M$ satisfying

$$u_{n+1} = DF(x_n)u_n + w_n, \quad \forall n \in \mathbb{Z}, \quad (3.3.14)$$

we have (recalling (3.3.13))

$$\sup_{k \in [-n, n]} e^{-\eta|k|} \|u_k\| \leq \kappa \|u_0\| + \frac{\hat{\kappa} e^{\hat{\rho}_0}}{1 - e^{\rho_0 - \eta}} \sup_{k \in [-n, n]} e^{-\eta|k|} \|w_k\|, \quad \forall n \geq 0, \quad (3.3.15)$$

and where $\hat{\kappa}$ is provided by Proposition 3.3.3.

Proof. Let $\mathbf{u} = \{u_n\}_{n \in \mathbb{Z}} \subset M$ and $\mathbf{w} = \{w_n\}_{n \in \mathbb{Z}} \subset X$ satisfying (3.3.14). By using the variation of constants formula one has

$$u_n = U_{\mathbf{x}}^c(n, p) u_p + \sum_{k=p}^{n-1} U_{\mathbf{x}}^c(n, k+1) w_k, \quad \forall n \geq p. \quad (3.3.16)$$

Hence for $p = 0$ in (3.3.16) we obtain

$$u_n = U_{\mathbf{x}}^c(n, 0) u_0 + \sum_{k=0}^{n-1} U_{\mathbf{x}}^c(n, k+1) w_k, \quad \forall n \in \mathbb{N}. \quad (3.3.17)$$

Next multiplying the left side of (3.3.16) by $U_{\mathbf{x}}^c(p, n)$ we obtain

$$U_{\mathbf{x}}^c(p, n) u_n = u_p + \sum_{k=p}^{n-1} U_{\mathbf{x}}^c(p, k+1) w_k, \quad \forall n \geq p, \quad (3.3.18)$$

so that letting $n = 0$ in (3.3.18) provides

$$u_p = U_{\mathbf{x}}^c(p, 0) u_0 - \sum_{k=p}^{-1} U_{\mathbf{x}}^c(p, k+1) w_k, \quad \forall p \leq 0. \quad (3.3.19)$$

It remains to prove the estimates in (3.3.15).

Let $n \in \mathbb{N}$ be given. Then on one hand, by using (3.3.17), we have

$$\begin{aligned} e^{-\eta n} \|u_n\| &\leq e^{-\eta n} \|U_{\mathbf{x}}^c(n, 0) u_0\| + \sum_{k=0}^{n-1} e^{-\eta n} \|U_{\mathbf{x}}^c(n, k+1) w_k\| \\ &\leq \kappa \|u_0\| + \sum_{k=0}^{n-1} \widehat{\kappa} e^{-\eta(n-k)} e^{\widehat{\rho}_0(n-1-k)} \sup_{k \in [0, n]} e^{-\eta k} \|w_k\| \\ &\leq \widehat{\kappa} \|u_0\| + \frac{\widehat{\kappa} e^{-\eta}}{1 - e^{\widehat{\rho}_0 - \eta}} \sup_{k \in [0, n]} e^{-\eta k} \|w_k\|, \end{aligned}$$

so that

$$\sup_{k \in [0, n]} e^{-\eta k} \|u_k\| \leq \widehat{\kappa} \|u_0\| + \frac{\widehat{\kappa} e^{-\eta}}{1 - e^{\widehat{\rho}_0 - \eta}} \sup_{k \in [0, n]} e^{-\eta k} \|w_k\|. \quad (3.3.20)$$

On the other hand by using (3.3.19) we have

$$\begin{aligned} e^{-\eta n} \|u_{-n}\| &\leq e^{-\eta n} \|U_{\mathbf{x}}^c(-n, 0) u_0\| + \sum_{k=-n}^{-1} e^{-\eta n} \|U_{\mathbf{x}}^c(-n, k+1) w_k\| \\ &\leq \widehat{\kappa} \|u_0\| + \frac{\widehat{\kappa} e^{\widehat{\rho}_0}}{1 - e^{\widehat{\rho}_0 - \eta}} \sup_{k \in [-n, -1]} e^{\eta k} \|w_k\|, \end{aligned}$$

so that

$$\sup_{k \in [-n, -1]} e^{\eta k} \|u_k\| \leq \widehat{\kappa} \|u_0\| + \frac{\widehat{\kappa} e^{-\eta}}{1 - e^{\widehat{\rho}_0 - \eta}} \sup_{k \in [-n, -1]} e^{\eta k} \|w_k\|. \quad (3.3.21)$$

The result follows by combining (3.3.20) and (3.3.21). $\blacksquare$

In the next lemma we prove a perturbation result for the x -equation of system (3.1.11).

Lemma 3.3.5 *Let conditions (H1), (H2)-(ii) and (H3) in Assumption 3.2.3 be satisfied. Let $H : M \times X^h \rightarrow M$ be a Lipschitz continuous map on M such that*

$$\|H\|_{Lip(M \times X^h, X^h)} \|F^{-1}\|_{Lip(M)} < 1. \quad (3.3.22)$$

Then for each sequence $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \subset X^h$ and $x_0 \in M$, there exists a unique complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ of

$$x_{n+1} = F(x_n) + H(x_n, y_n), \quad n \in \mathbb{Z}.$$

Proof. To prove this lemma we only need to prove that for each $y \in X^h$ the map $F(\cdot) + H(\cdot, y) : M \rightarrow M$ is invertible on M . To do so we will prove that for each $\tilde{x}$ there exists a unique $x \in M$ such that $F(x) + H(x, y) = \tilde{x}$ by using a fixed point argument.

Let $\tilde{x} \in M$ be given.

Recalling that F is invertible from M into M one can define a map $J : M \rightarrow M$ by

$$J(x) := F^{-1}(\tilde{x} - H(x, y)),$$

and observe that

$$J(x) = x \iff F(x) + H(x, y) = \tilde{x}.$$

Now note that by using Lemma 3.3.2 one obtains for each $x, \bar{x} \in M$

$$\begin{aligned} \|J(x) - J(\bar{x})\| &\leq \sup_{n \in \mathbb{Z}} \|F^{-1}\|_{Lip(M)} \|H(x, y) - H(\bar{x}, y)\| \\ &\leq \sup_{n \in \mathbb{Z}} \|F^{-1}\|_{Lip(M)} \sup_{n \in \mathbb{Z}} \|H\|_{Lip(M \times X^h)} \|x - \bar{x}\|, \end{aligned}$$

and we infer from (3.3.22) that J is a strict contraction on the Banach space M . The result follows. $\blacksquare$

In order to state our next lemma we define for a given continuous function $g : [0, 1] \rightarrow X$ the **regularized pointwise Lipschitz constant** at $l \in [0, 1]$ as

$$Lip[g](l) := \overline{\lim}_{l' \rightarrow l, h \rightarrow 0} \frac{1}{|h|} \|g(l' + h) - g(l')\|. \quad (3.3.23)$$

This definition is inspired from the definition of the **pointwise Lipschitz constant** at $l \in [0, 1]$ given in [60] namely

$$Lipg(l) := \overline{\lim}_{h \rightarrow 0} \frac{1}{|h|} \|g(l + h) - g(l)\|. \quad (3.3.24)$$

The following lemma reads as:

Lemma 3.3.6 *Let $g : [0, 1] \rightarrow X$ be a globally Lipschitz continuous function on $[0, 1]$. Then we have*

$$\|g(1) - g(0)\| \leq \int_0^1 Lip[g](l) \lambda(dl),$$

with λ the Lebesgue measure on $[0, 1]$.

In order to prove Lemma 3.3.6 we will need a generalized version of Fatou's lemma due to Serfozo [150] and generalized in [80]. The version stated here is inspired from [80, Lemma 3.2 and Lemma 3.3]. Let us define as in [150] and [80] the following “generalized liminf” (respectively “generalized limsup”)

$$f_*(l) := \inf \left\{ \underline{\lim}_{n \rightarrow +\infty} f_n(l_n) : l_n \rightarrow l \right\}, \quad (3.3.25)$$

and

$$f^*(l) := \sup \left\{ \overline{\lim}_{n \rightarrow +\infty} f_n(l_n) : l_n \rightarrow l \right\}, \quad (3.3.26)$$

where $\{f_n\}_{n \in \mathbb{N}}$ is a family of Borel measurable functions on $[0, 1]$.

The generalized Fatou's lemma is recalled in the following lemma.

Lemma 3.3.7 Let $\mathbb{P}([0, 1])$ be the set of probability measures on $[0, 1]$. Let $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{P}([0, 1])$ and $\mu \in \mathbb{P}([0, 1])$ such that

$$\mu_n \rightarrow \mu \text{ weakly,}$$

that is for any bounded continuous function f on $[0, 1]$

$$\int_0^1 f(l) \mu_n(dl) \rightarrow \int_0^1 f(l) \mu(dl) \text{ as } n \rightarrow +\infty.$$

If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of non negative Borel measurable functions on $[0, 1]$ then

$$\int_0^1 f_*(l) \mu(dl) \leq \liminf_{n \rightarrow +\infty} \int_0^1 f_n(l) \mu_n(dl), \quad (3.3.27)$$

and if in addition the sequence $\{f_n\}_{n \in \mathbb{N}}$ is bounded above we have

$$\overline{\lim}_{n \rightarrow +\infty} \int_0^1 f_n(l) \mu_n(dl) \leq \int_0^1 f^*(l) \mu(dl). \quad (3.3.28)$$

Remark 3.3.8 We should point out the fact that the usual version of Fatou's lemma fails in general for varying probability measures. In order to see this we consider

$$f(l) = \mathbf{1}_{\mathbb{R} \setminus \mathbb{Q} \cap [0, 1]}(l), \quad \forall l \in [0, 1], \quad (3.3.29)$$

and observe that

$$f_*(l) = 0, \quad \forall l \in [0, 1].$$

Next consider the family of probability measures $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{P}([0, 1])$ defined by

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\frac{k}{n}},$$

where $\delta_{\frac{k}{n}}$, $k = 0, 1, \dots, n-1$ denotes the Dirac measure on $[0, 1]$ concentrated on $\{\frac{k}{n}\}$. It is well known that $\mu_n \rightarrow \lambda$ weakly where λ denotes the Lebesgue measure on $[0, 1]$. Furthermore we have

$$1 = \int_0^1 f(l) \lambda(dl) \geq \liminf_{n \rightarrow +\infty} \int_0^1 f(l) \mu_n(dl) = 0,$$

while

$$0 = \int_0^1 f_*(l) \lambda(dl) \leq \liminf_{n \rightarrow +\infty} \int_0^1 f(l) \mu_n(dl) = 0.$$

Now by using Lemma 3.3.7 we are now able to prove Lemma 3.3.6.

Proof of Lemma 3.3.6. To prove our lemma we will make use of Lemma 3.3.7. To do so we will construct a sequence of bounded Borel non negative measurable functions $\{f_n\}_{n \in \mathbb{N}}$ on $[0, 1]$ and a family of probability measures $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{P}([0, 1])$ such that

$$\|g(1) - g(0)\| \leq \int_0^1 f_n(l) \mu_n(dl), \quad \forall n \geq 1. \quad (3.3.30)$$

To do so we observe that

$$g(1) - g(0) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{g\left(\frac{k}{n} + \frac{1}{n}\right) - g\left(\frac{k}{n}\right)}{\frac{1}{n}}, \quad \forall n \geq 1.$$

so that

$$\|g(1) - g(0)\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \left\| \frac{g\left(\frac{k}{n} + \frac{1}{n}\right) - g\left(\frac{k}{n}\right)}{\frac{1}{n}} \right\|, \quad \forall n \geq 1. \quad (3.3.31)$$

Therefore by setting

$$f_n(l) := \left\| \frac{g\left(l + \frac{1}{n}\right) - g(l)}{\frac{1}{n}} \right\| \times \mathbf{1}_{[0, 1 - \frac{1}{n}]}(l), \quad \forall n \geq 1,$$

and considering the probability measures $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{P}([0, 1])$ defined by

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\frac{k}{n}},$$

where $\delta_{\frac{k}{n}}$, $k = 0, 1, \dots, n-1$ denotes the Dirac measure on $[0, 1]$ concentrated on $\{\frac{k}{n}\}$ it becomes clear from (3.3.31) that (3.3.30) holds true. Furthermore the sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is bounded above due to the fact that g is globally Lipschitz continuous on $[0, 1]$. Thus since $\mu_n \rightarrow \lambda$ weakly with λ the Lebesgue measure on $[0, 1]$ it follows from Lemma 3.3.7 that

$$\overline{\lim}_{n \rightarrow +\infty} \int_0^1 f_n(l) \mu_n(dl) \leq \int_0^1 f^*(l) \lambda(dl),$$

so that (3.3.30) ensures that

$$\|g(1) - g(0)\| \leq \int_0^1 f^*(l) \lambda(dl).$$

Next we observe that due to Definitions in (3.3.23) and (3.3.26) we have

$$f^*(l) \leq \text{Lip}[g](l), \quad \forall l \in [0, 1],$$

and the result follows. ■

3.3.2 Proof of Theorem 3.2.4

The main idea of this proof is to re-write system (3.1.11) as the following fixed point problem

$$\mathbf{x} = \mathcal{K}^c(x_0, \mathbf{y}) \text{ and } \mathbf{y} = \mathcal{K}^h(\mathbf{x}, \mathcal{G}(\mathbf{x}, \mathbf{y})), \quad (3.3.32)$$

where $\mathcal{K}^c, \mathcal{K}^h, \mathcal{G}$ are operators that will be precisely defined in the following lemmas.

Roughly speaking we will prove that for any given sequence $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \subset X^h$ and $x_0 \in M$ there exists a unique sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ such that

$$x_{n+1} = F(x_n) + R_c(x_n, y_n), \quad \forall n \in \mathbb{Z} \Leftrightarrow \mathbf{x} = \mathcal{K}^c(x_0, \mathbf{y}). \quad (3.3.33)$$

Hence we decouple the system (3.3.32) into a unique fixed point problem on $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \subset X^h$ as

$$\mathbf{y} = \mathcal{K}^h(\cdot, \mathcal{G}(\cdot, \mathbf{y})) \circ \mathcal{K}^c(x_0, \mathbf{y}). \quad (3.3.34)$$

This section is divided into two parts. The first part is devoted to the construction of the fixed point problem (3.3.34) by defining explicitly the operators and studying their properties while in the second part we will complete the proof of Theorem 3.2.4 by using the results obtained in the first part.

3.3.3 Fixed point problem

In order to define the operator $\mathcal{K}^c$, we first prove the following lemma. For technical reasons that will appear in the proof of Lemma 3.3.12 we will explicitly write down the dependence of the solutions of (3.3.33) with respect to the map R_c , $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \subset X^h$ and the initial value.

Proposition 3.3.9 *Let conditions (H1) and (H2) in Assumption 3.2.3 be satisfied. Let $\varepsilon > 0$ be the constant provided by Proposition 3.3.3. Let us set (recalling (3.3.13))*

$$C := \max \left\{ \frac{4\widehat{\kappa}e^{\widehat{\rho}_0}}{1 - e^{\widehat{\rho}_0 - \eta}}, 4\kappa e^{\rho_0} \right\}. \quad (3.3.35)$$

Then for each bounded and Lipschitz continuous map $H : M \times X^h \rightarrow M$ with

$$\|H\|_{C(M \times X^h, X^h)} \leq \varepsilon \text{ and } \|H\|_{Lip(M \times X^h, M)} \leq C^{-1}, \quad (3.3.36)$$

for each sequence $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \subset X^h$ and $x_0 \in M$, there exists a unique solution $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ denoted by $\mathbf{x} := \mathbf{x}(x_0, \mathbf{y}, H)$ of the problem

$$x_{n+1} = F(x_n) + H(x_n, y_n), \quad n \in \mathbb{Z}. \quad (3.3.37)$$

Moreover for each $(x_0, \mathbf{y}), (\bar{x}_0, \bar{\mathbf{y}}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$, each $H, \bar{H} : M \times X^h \rightarrow M$ satisfying (3.3.36) we have

$$\mathbf{x}(x_0, \mathbf{y}, H) - \mathbf{x}(x_0, \bar{\mathbf{y}}, \bar{H}) \in \mathbb{L}_\eta(\mathbb{Z}, M),$$

and

$$\begin{aligned} \|\mathbf{x}(x_0, \mathbf{y}, H) - \mathbf{x}(x_0, \bar{\mathbf{y}}, \bar{H})\|_\eta &\leq 2\hat{\kappa} \|x_0 - \bar{x}_0\| \\ &+ C \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta \\ &+ 2 \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|. \end{aligned} \quad (3.3.38)$$

Proof. Firstly let us observe that if $H : M \times X^h \rightarrow M$ satisfies condition (3.3.36) then one has

$$\|H\|_{C(M \times X^h, X^h)} \leq \varepsilon, \quad (3.3.39)$$

and

$$\frac{\hat{\kappa}e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \|H\|_{Lip(M \times X^h, M)} < \frac{1}{4} \text{ and } \kappa e^{\rho_0} \|H\|_{Lip(M \times X^h, M)} < \frac{1}{4}. \quad (3.3.40)$$

Thus by using (3.3.40) we obtain from (3.3.3) that $\|H\|_{Lip(M \times X^h, X^h)} \|F^{-1}\|_{Lip(M)} < 1$ and we infer from Lemma 3.3.5 that for each $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \in X^h$, $x_0 \in M$ and $H : M \times X^h \rightarrow M$ satisfying (3.3.36) there exists a unique solution $\mathbf{x} := \mathbf{x}(x_0, \mathbf{y}, H) = \{x_n\}_{n \in \mathbb{Z}}$ of (3.3.37). We will now prove further properties of this solution and in particular the dependence with respect to x_0 , $\mathbf{y}$ and H .

Let $(x_0, \mathbf{y}), (\bar{x}_0, \bar{\mathbf{y}}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ be given. Let $H, \bar{H} : M \times X^h \rightarrow M$ satisfying (3.3.36) be given and let us denote

$$\mathbf{x} := \mathbf{x}(x_0, \mathbf{y}, H) \text{ and } \bar{\mathbf{x}} := \mathbf{x}(x_0, \bar{\mathbf{y}}, \bar{H}),$$

the unique solution of

$$x_{n+1} = F(x_n) + H(x_n, y_n), \quad n \in \mathbb{Z}, \quad (3.3.41)$$

and

$$\bar{x}_{n+1} = F(\bar{x}_n) + \bar{H}(\bar{x}_n, \bar{y}_n), \quad n \in \mathbb{Z}.$$

Next define for each $l \in [0, 1]$

$$a_0(l) := lx_0 + (1-l)\bar{x}_0 \text{ and } b_n(l) := ly_n + (1-l)\bar{y}_n, \quad \forall n \in \mathbb{Z}, \quad (3.3.42)$$

and

$$G(l, \cdot) := lH(\cdot) + (1-l)\bar{H}(\cdot). \quad (3.3.43)$$

Observe that for each $l \in [0, 1]$ the map $G(l, \cdot) : M \times X^h \rightarrow M$ is bounded and Lipschitz continuous on $M \times X^h$ and satisfies (3.3.36).

Moreover for each $l \in [0, 1]$, $a_0(l) \in M$ and $\mathbf{b}(l) = \{b_n(l)\}_{n \in \mathbb{Z}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ so that due to the existence part, there exists a unique sequence solution of the system

$$a_{n+1}(l) = F(a_n(l)) + G(l, a_n(l), b_n(l)), \quad n \in \mathbb{Z}. \quad (3.3.44)$$

Note that since $a_0(1) = x_0$, $a_0(0) = \bar{x}_0$ and

$$G(1, \cdot) = H(\cdot) \text{ and } G(0, \cdot) = \bar{H},$$

by the uniqueness of the solutions one has

$$a_n(1) = x_n \text{ and } a_n(0) = \bar{x}_n, \forall n \in \mathbb{Z}. \quad (3.3.45)$$

Furthermore recalling that F and F^{-1} are both Lipschitz continuous on M and using the fact that $a_0(\cdot) : [0, 1] \rightarrow X$ as well as $b_n(\cdot) : [0, 1] \rightarrow X$, $n \in \mathbb{Z}$ are all Lipschitz continuous on $[0, 1]$, it is clear that for each $n \in \mathbb{Z}$, $a_n(\cdot) : [0, 1] \rightarrow X$ is Lipschitz continuous on $[0, 1]$. Therefore one can define for each $n \in \mathbb{Z}$, $Lip[a_n](l)$, $l \in [0, 1]$ as in (3.3.23).

We claim that for each $l \in [0, 1]$

$$\begin{aligned} \sup_{n \in \mathbb{Z}} e^{-\eta|n|} \|Lip[a_n](l)\| &\leq 2\hat{\kappa} \|x_0 - \bar{x}_0\| \\ &+ \frac{2\hat{\kappa}e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta \\ &+ \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|. \end{aligned} \quad (3.3.46)$$

In order to prove that (3.3.46) holds true we will make use of Lemma 3.3.6 and Lemma 3.3.4. To do so we will first estimate the quantity $\frac{1}{h} [a_n(l+h) - a_n(l)]$. Set

$$u_n(l, h) := \frac{1}{h} [a_n(l+h) - a_n(l)], \forall n \in \mathbb{Z} \text{ and } l+h, l \in [0, 1], \quad (3.3.47)$$

and note that by using (3.3.44) $\{u_n\}_{n \in \mathbb{Z}}$ satisfies for each $n \in \mathbb{Z}$, $l+h, l \in [0, 1]$

$$u_{n+1}(l, h) = DF(a_n(l)) u_n(l, h) + f_n(l, h), \quad (3.3.48)$$

where we have set for each $n \in \mathbb{Z}$, $l+h, l \in [0, 1]$

$$\begin{aligned} f_n(l, h) &:= \frac{1}{h} [F(a_n(l+h)) - F(a_n(l))] - DF(a_n(l)) \frac{1}{h} [a_n(l+h) - a_n(l)] \\ &+ \frac{1}{h} [G(l+h, a_n(l+h), b_n(l+h)) - G(l, a_n(l), b_n(l))]. \end{aligned} \quad (3.3.49)$$

Next we note that due to (3.3.44) and (3.3.43) one gets

$$\|a_{n+1}(l) - F(a_n(l))\| \leq \varepsilon, \forall n \in \mathbb{Z}, \quad (3.3.50)$$

so that by using (3.3.48) we infer from Lemma 3.3.4 that

$$\sup_{k \in [-n, n]} e^{-\eta|k|} \|u_k(l, h)\| \leq \hat{\kappa} \|u_0(l, h)\| + \frac{\hat{\kappa}e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \sup_{k \in [-n, n]} e^{-\eta|k|} \|f_k(l, h)\|, \forall n \in \mathbb{N}. \quad (3.3.51)$$

Next on the one hand we derive from (3.3.49) that for each $n \in \mathbb{Z}$, $l + h, l \in [0, 1]$

$$\begin{aligned} f_n(l, h) &= \frac{1}{h} \int_0^1 [DF(sa_n(l+h) + (1-s)a_n(l)) - DF(a_n(l))] ds [a_n(l+h) - a_n(l)] \\ &\quad + \frac{1}{h} [G(l+h, a_n(l+h), b_n(l+h)) - G(l, a_n(l), b_n(l))]. \end{aligned}$$

On the other hand by setting for each $n \in \mathbb{Z}$, $l + h, l \in [0, 1]$

$$g_n(l, h) := G(l+h, a_n(l+h), b_n(l+h)) - G(l, a_n(l), b_n(l)), \quad (3.3.52)$$

one has

$$\|f_n(l, h)\| \leq \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))} h \|u_n(l, h)\|^2 + \frac{1}{h} \|g_n(l, h)\|. \quad (3.3.53)$$

Let us now derive an estimate for the norm of $g_n(l, h)$. To do so, recalling (3.3.43), we obtain that for each $n \in \mathbb{Z}$, $l + h, l \in [0, 1]$:

$$\begin{aligned} g_n(l, h) &= (l+h)H(a_n(l+h), b_n(l+h)) + (1-l-h)\bar{H}(a_n(l+h), b_n(l+h)) \\ &\quad - lH(a_n(l), b_n(l)) - (1-l)\bar{H}(a_n(l), b_n(l)) \\ &= l[H(a_n(l+h), b_n(l+h)) - H(a_n(l), b_n(l))] \\ &\quad + h[H(a_n(l+h), b_n(l+h)) - \bar{H}(a_n(l+h), b_n(l+h))] \\ &\quad + (1-l)[\bar{H}(a_n(l+h), b_n(l+h)) - \bar{H}(a_n(l), b_n(l))]. \end{aligned}$$

Hence we get for each $n \in \mathbb{Z}$, $l + h, l \in [0, 1]$

$$\begin{aligned} \|g_n(l, h)\| &\leq \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|a_n(l+h) - a_n(l)\| \\ &\quad + \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|b_n(l+h) - b_n(l)\| \\ &\quad + h \sup_{(x,y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|. \end{aligned} \quad (3.3.54)$$

Plugging (3.3.54) into (3.3.53) yields for each $n \in \mathbb{Z}$, $l + h, l \in [0, 1]$

$$\begin{aligned} \|f_n(l, h)\| &\leq \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))} h \|u_n(l, h)\|^2 \\ &\quad + \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|u_n(l, h)\| \\ &\quad + \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \frac{1}{h} \|b_n(l+h) - b_n(l)\| \\ &\quad + \sup_{(x,y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|. \end{aligned} \quad (3.3.55)$$

Now observe that due to the definition of $a_0(l)$ and $\mathbf{b}(l) = \{b_n(l)\}_{n \in \mathbb{Z}}$ in (3.3.42) we have for $n \in \mathbb{Z}$, $l + h, l \in [0, 1]$

$$u_0(l, h) = x_0 - \bar{x}_0 \text{ and } \frac{1}{h} [b_n(l+h) - b_n(l)] = y_n - \bar{y}_n, \quad (3.3.56)$$

so that

$$\begin{aligned}
\|f_n(l, h)\| &\leq \|DF(\cdot)\|_{Lip(M, \mathcal{L}(M))} h \|u_n(l, h)\|^2 \\
&\quad + \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|u_n(l, h)\| \\
&\quad + \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|y_n - \bar{y}_n\| \\
&\quad + \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|.
\end{aligned}$$

Recalling that

$$\overline{\lim}_{l' \rightarrow l, h \rightarrow 0} u_n(l', h) = Lip[a_n](l) < +\infty,$$

one obtains that for each $n \in \mathbb{Z}$

$$\begin{aligned}
\overline{\lim}_{l' \rightarrow l, h \rightarrow 0} \|f_n(l', h)\| &\leq \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) Lip[a_n](l) \quad (3.3.57) \\
&\quad + \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|y_n - \bar{y}_n\| \\
&\quad + \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|.
\end{aligned}$$

Therefore combining (3.3.51) with (3.3.56) and (3.3.57) we obtain that for each $n \in \mathbb{N}$ and $l \in [0, 1]$:

$$\begin{aligned}
\sup_{k \in [-n, n]} e^{-\eta|k|} \|Lip[a_k](l)\| &\leq \hat{\kappa} \|x_0 - \bar{x}_0\| \quad (3.3.58) \\
&\quad + \frac{\hat{\kappa} e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \\
&\quad \times \sup_{k \in [-n, n]} e^{-\eta|k|} \|Lip[a_k](l)\| \\
&\quad + \frac{\hat{\kappa} e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \\
&\quad \times \sup_{k \in [-n, n]} e^{-\eta|k|} \|y_k - \bar{y}_k\| \\
&\quad + \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|.
\end{aligned}$$

Hence by using the fact that

$$\sup_{k \in [-n, n]} e^{-\eta|k|} \|y_k - \bar{y}_k\| \leq \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta, \quad \forall n \in \mathbb{N}, \quad (3.3.59)$$

we infer from (3.3.58) and (3.3.40) that for each $n \in \mathbb{N}$ and $l \in [0, 1]$

$$\begin{aligned} \sup_{k \in [-n, n]} e^{-\eta|k|} \|Lip[a_k](l)\| &\leq 2\hat{\kappa} \|x_0 - \bar{x}_0\| \\ &+ \frac{2\hat{\kappa}e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta \\ &+ 2 \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|, \end{aligned}$$

and (3.3.46) follows.

Next we infer from Lemma 3.3.6 and the equalities in (3.3.45) that for each $n \in \mathbb{Z}$

$$e^{-\eta|n|} \|x_n - \bar{x}_n\| = e^{-\eta|n|} \|a_n(1) - a_n(0)\| \leq e^{-\eta|n|} \int_0^1 Lip[a_n](l) \lambda(dl) \leq \int_0^1 Lip[a_n](l) \lambda(dl),$$

and since we have

$$\begin{aligned} \int_0^1 Lip[a_n](l) \lambda(dl) &\leq 2\hat{\kappa} \|x_0 - \bar{x}_0\| \\ &+ \frac{2\hat{\kappa}e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}} \left(\|H\|_{Lip(M \times X^h, M)} + \|\bar{H}\|_{Lip(M \times X^h, M)} \right) \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta \\ &+ 2 \sup_{(x, y) \in M \times X^h} \|H(x, y) - \bar{H}(x, y)\|, \end{aligned}$$

estimate (3.3.38) follows and completes the proof of Proposition 3.3.9. $\blacksquare$

Notation 3.3.10 For convenience for the next argument we introduce for each $\eta \geq 0$ the set

$$\mathbb{E}_\eta(\mathbb{Z}, M) := \{\mathbf{x} \in \mathbb{L}_\eta(\mathbb{Z}, M) : \text{(A-1) and (A-2) are satisfied}\},$$

where

(A-1) for each $n \geq p$, the operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ is invertible with inverse $U_{\mathbf{x}}^c(p, n)$ and

$$\|U_{\mathbf{x}}^c(n, p)y\| \leq \hat{\kappa}e^{\hat{\rho}_0|n-p|} \|y\|, \quad \forall y \in M, \quad \forall (n, p) \in \mathbb{Z}^2. \quad (3.3.60)$$

(A-2) $\{U_{\mathbf{x}}(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ has an exponential dichotomy on $\mathbb{Z}$ with constant $\hat{\kappa}$ and exponents $\hat{\rho} > \hat{\rho}_0$ associated with projectors $\{\Pi_n^\alpha(\mathbf{x})\}_{n \in \mathbb{Z}}$, $\alpha = u, s$.

Recall that condition **(A-2)** implies that

$$\Pi_n^u(\mathbf{x}) + \Pi_n^s(\mathbf{x}) = I_{\mathcal{L}(X^h)}, \quad \forall n \in \mathbb{Z},$$

$$U_{\mathbf{x}}^\alpha(n, p) := \Pi_n^\alpha(\mathbf{x}) U_{\mathbf{x}}(n, p) = U_{\mathbf{x}}(n, p) \Pi_p^\alpha(\mathbf{x}), \quad \forall n \geq p, \quad \alpha = u, s,$$

and $U_{\mathbf{x}}^u(n, p)$ is invertible from $\mathcal{R}(\Pi_p^u(\mathbf{x}))$ into $\mathcal{R}(\Pi_n^u(\mathbf{x}))$ with inverse

$$U_{\mathbf{x}}^u(p, n) : \mathcal{R}(\Pi_n^u(\mathbf{x})) \rightarrow \mathcal{R}(\Pi_p^u(\mathbf{x})),$$

while the growth estimates are given as

$$\|U_{\mathbf{x}}^s(n, p) \Pi_n^u(\mathbf{x}) y\| \leq \widehat{\kappa} e^{-\widehat{\rho}(n-p)} \|y\|, \quad \forall y \in X^h, \quad (3.3.61)$$

$$\|U_{\mathbf{x}}^u(p, n) \Pi_n^u(\mathbf{x}) y\| \leq \widehat{\kappa} e^{-\widehat{\rho}(n-p)} \|y\|, \quad \forall y \in X^h. \quad (3.3.62)$$

Remark 3.3.11 Observe if $\mathbf{x} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ then due to (3.3.61) and (3.3.62) one has

$$\|U_{\mathbf{x}}^u(n, n) \Pi_n^\alpha(\mathbf{x})\|_{\mathcal{L}(X^h)} = \|\Pi_n^\alpha(\mathbf{x})\|_{\mathcal{L}(X^h)} \leq \widehat{\kappa}, \quad \forall n \in \mathbb{Z} \text{ and } \alpha = u, s.$$

In the next lemma we give the precise definition of $\mathcal{K}^c$ stated in (3.3.32) as well as some properties that will be used in the sequel.

Lemma 3.3.12 Let Assumption 3.2.3 be satisfied. Assume moreover that

$$\|R_c\|_{C(M \times X^h, X^h)} \leq \varepsilon \text{ and } \|R_c\|_{Lip(M \times X^h, M)} \leq C^{-1},$$

where ε is defined in Proposition 3.3.3 and C given in Proposition 3.3.9. Then the operator

$$\mathcal{K}^c : M \times \mathbb{L}_\eta(\mathbb{Z}, X^h) \rightarrow \mathbb{E}_\eta(\mathbb{Z}, M),$$

defined by $\mathbf{x} = \mathcal{K}^c(x_0, \mathbf{y})$ with $\mathbf{x}$ given by the solution of

$$x_{n+1} = F(x_n) + R_c(x_n, y_n), \quad n \in \mathbb{Z}, \quad (3.3.63)$$

is well defined and satisfies the following Lipschitz estimate:

$$\|\mathcal{K}^c(x_0, \mathbf{y}) - \mathcal{K}^c(\bar{x}_0, \bar{\mathbf{y}})\|_\eta \leq 2\widehat{\kappa} \|x_0 - \bar{x}_0\| + 2C \|R_c\|_{Lip(M \times X^h, M)} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta, \quad (3.3.64)$$

for each $(x_0, \mathbf{y}), (\bar{x}_0, \bar{\mathbf{y}}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$.

Proof. Let us notice that under conditions of Lemma 3.3.12 one obtains from Proposition 3.3.9 that for each $(x_0, \mathbf{y}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ there exists a unique solution $\mathbf{x} = \mathbf{x}(x_0, \mathbf{y}, R_c) = \{x_n\}_{n \in \mathbb{Z}} \subset M$ of (3.3.63). Furthermore we have

$$\|x_{n+1} - F(x_n)\| \leq \|R_c\|_{C(M \times X^h, X^h)} \leq \varepsilon, \quad \forall n \in \mathbb{Z}. \quad (3.3.65)$$

Next we prove that for each $(x_0, \mathbf{y}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ the solution $\mathbf{x}(x_0, \mathbf{y}, R_c)$ of (3.3.63) belongs to $\mathbb{L}_\eta(\mathbb{Z}, M)$.

Let $(x_0, \mathbf{y}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ be given. Recall that due to condition (H5) there exists a unique complete orbit $\widehat{\mathbf{x}} = \{\widehat{x}_n\}_{n \in \mathbb{Z}}$ of F such that $\widehat{\mathbf{x}} \in \mathbb{L}_{\rho_0}(\mathbb{Z}, M) \subset \mathbb{L}_\eta(\mathbb{Z}, M)$ ($\rho_0 < \eta$).

Note that according to the notations of Proposition 3.3.9 one also has $\hat{\mathbf{x}} = \mathbf{x}(\hat{x}_0, \mathbf{y}, 0)$ where 0 is understood as the null function from $M \times X^h$ into X^h . Therefore one has

$$\mathbf{x}(x_0, \mathbf{y}, R_c) - \mathbf{x}(\hat{x}_0, \mathbf{y}, 0) \in \mathbb{L}_\eta(\mathbb{Z}, M).$$

Writing $\mathbf{x}(x_0, \mathbf{y}, R_c) = [\mathbf{x}(x_0, \mathbf{y}, R_c) - \mathbf{x}(\hat{x}_0, \mathbf{y}, 0)] + \mathbf{x}(\hat{x}_0, \mathbf{y}, 0)$ it follows that $\mathbf{x}(x_0, \mathbf{y}, R_c) \in \mathbb{L}_\eta(\mathbb{Z}, M)$.

Thus by setting

$$\mathcal{K}^c(x_0, \mathbf{y}) := \mathbf{x}(x_0, \mathbf{y}, R_c) \in \mathbb{L}_\eta(\mathbb{Z}, M), \quad \forall (x_0, \mathbf{y}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h),$$

the inequality (3.3.64) follows from (3.3.38). Finally we infer from (3.3.65) and Proposition 3.3.3 that

$$\mathcal{K}^c(x_0, \mathbf{y}) \in \mathbb{E}_\eta(\mathbb{Z}, M), \quad \forall (x_0, \mathbf{y}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h).$$

The proof is complete. ■

Now it remains to state a precise definition of operators $\mathcal{K}^h$ and $\mathcal{G}$ (see (3.3.32)). To do so let us first prove the following integrale representation for the resolution of the y -equation.

Lemma 3.3.13 *Let conditions (H1)-(H4) in Assumption 3.2.3 be satisfied. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ be given. Then a sequence $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ satisfies*

$$y_{n+1} = A(x_n) y_n + R_h(x_n, y_n), \quad n \in \mathbb{Z}, \quad (3.3.66)$$

if and only if

$$y_n = \sum_{k=-\infty}^{n-1} U_{\mathbf{x}}^s(n, k+1) R_h(x_k, y_k) - \sum_{k=n}^{+\infty} U_{\mathbf{x}}^u(n, k+1) R_h(x_k, y_k), \quad \forall n \in \mathbb{Z}. \quad (3.3.67)$$

Proof. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ be given. Let $\mathbf{y} = \{y_n\}_{n \in \mathbb{Z}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ satisfying (3.3.66).

Thus due to the variation of constants formula one has

$$y_n = U_{\mathbf{x}}(n, p) y_p + \sum_{k=p}^{n-1} U_{\mathbf{x}}(n, k+1) R_h(x_k, y_k), \quad \forall n \geq p. \quad (3.3.68)$$

On the one hand, multiplying the left side of (3.3.68) by $\Pi_n^s(\mathbf{x})$ yields

$$\Pi_n^s(\mathbf{x}) y_n = U_{\mathbf{x}}^s(n, p) y_p + \sum_{k=p}^{n-1} U_{\mathbf{x}}^s(n, k+1) R_h(x_k, y_k), \quad \forall n \geq p. \quad (3.3.69)$$

Thus since $\eta \in (\hat{\rho}_0, \hat{\rho})$ and

$$\|U_{\mathbf{x}}^s(n, p) y_p\| \leq \hat{\kappa} e^{(\eta - \hat{\rho})(n-p)} \|\mathbf{y}\|_\eta, \quad \forall n \geq p,$$

by letting p go to $-\infty$ in (3.3.69) one gets

$$\Pi_n^s(\mathbf{x}) y_n = \sum_{k=-\infty}^{n-1} U_{\mathbf{x}}^s(n, k+1) R_h(x_k, y_k), \quad \forall n \in \mathbb{Z}. \quad (3.3.70)$$

On the other hand multiplying the left side of (3.3.68) by $U_{\mathbf{x}}^u(p, n) \Pi_p^u(\mathbf{x})$ we obtain

$$U_{\mathbf{x}}^u(p, n) y_n = \Pi_p^u(\mathbf{x}) y_p + \sum_{k=p}^{n-1} U_{\mathbf{x}}^u(p, k+1) R_h(x_k, y_k), \quad \forall n \geq p. \quad (3.3.71)$$

Thus since $\eta \in (\hat{\rho}_0, \hat{\rho})$ and

$$\|U_{\mathbf{x}}^u(p, n) y_n\| \leq \hat{\kappa} e^{(\eta - \hat{\rho})(n-p)} \|\mathbf{y}\|_{\eta}, \quad \forall n \geq p,$$

by letting n go to $+\infty$ in (3.3.71) one gets

$$\Pi_p^u(\mathbf{x}) y_p = - \sum_{k=p}^{+\infty} U_{\mathbf{x}}^u(p, k+1) R_h(x_k, y_k), \quad \forall p \in \mathbb{Z}. \quad (3.3.72)$$

Since $y_n = \Pi_n^u(\mathbf{x}) y_n + \Pi_n^s(\mathbf{x}) y_n \in X^h$ for all $n \in \mathbb{Z}$, adding up (3.3.70) to (3.3.72) (with p replaced by n), yields to (3.3.67).

Conversely let $\mathbf{y} \in \mathbb{L}_{\eta}(\mathbb{Z}, X^h)$ satisfy (3.3.67). Then one easily obtains that (3.3.66) holds true by multiplying the left side of (3.3.67) by $A(x_n)$. ■

The foregoing lemma allows us to define the operator $\mathcal{K}^h$ stated in (3.3.32).

Lemma 3.3.14 *Let conditions (H1)-(H4) in Assumption 3.2.3 be satisfied. Then the operator*

$$\mathcal{K}^h : \mathbb{E}_{\eta}(\mathbb{Z}, M) \times \mathbb{L}_{\eta}(\mathbb{Z}, X^h) \rightarrow \mathbb{L}_{\eta}(\mathbb{Z}, X^h),$$

given by

$$\mathcal{K}^h(\mathbf{x}, \mathbf{y})_n := \sum_{k=-\infty}^{n-1} U_{\mathbf{x}}^s(n, k+1) y_k - \sum_{k=n}^{+\infty} U_{\mathbf{x}}^u(n, k+1) y_k, \quad \forall n \in \mathbb{Z}, \quad (3.3.73)$$

is well defined. Moreover if one sets $\gamma > 0$ defined by

$$\gamma := \max \left\{ \frac{\hat{\kappa} e^{\eta}}{1 - e^{\eta - \hat{\rho}}} + \frac{\hat{\kappa} e^{-\hat{\rho}}}{1 - e^{\eta - \hat{\rho}}}; \frac{\hat{\kappa}}{1 - e^{-\hat{\rho}}} + \frac{\hat{\kappa} e^{-\hat{\rho}}}{1 - e^{-\hat{\rho}}} \right\},$$

then the following properties hold true:

i) *For each $\mathbf{x} \in \mathbb{E}_{\eta}(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_{\eta}(\mathbb{Z}, X^h)$ we have*

$$\|\mathcal{K}^h(\mathbf{x}, \mathbf{y})\|_{\eta} \leq \gamma \|\mathbf{y}\|_{\eta}. \quad (3.3.74)$$

ii) For each $\mathbf{x} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_0(\mathbb{Z}, X^h)$ (see (3.3.75)), $\mathcal{K}^h(\mathbf{x}, \mathbf{y}) \in \mathbb{L}_0(\mathbb{Z}, X^h)$ and

$$\|\mathcal{K}^h(\mathbf{x}, \mathbf{y})\|_0 \leq \gamma \|\mathbf{y}\|_0, \quad (3.3.75)$$

iii) For each $\mathbf{x}, \bar{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_0(\mathbb{Z}, X^h)$ we have

$$\|\mathcal{K}^h(\mathbf{x}, \mathbf{y}) - \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})\|_\eta \leq \gamma^2 \|A\|_{Lip(M, \mathcal{L}(M))} \|\mathbf{y}\|_0 \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta. \quad (3.3.76)$$

Proof. Note that since $\mathcal{K}^h(\mathbf{x}, \mathbf{0}) = \mathbf{0}$, property (i) will imply that $\mathcal{K}^h$ is well defined and maps into $\mathbb{L}_\eta(\mathbb{Z}, X^h)$. Therefore we will directly focus on properties (i), (ii) and (iii).

Proof of (i) and (ii): In order to prove (i) and (ii) let us fix $\hat{\eta} \in [0, \hat{\rho}]$. Let $\mathbf{x} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_{\hat{\eta}}(\mathbb{Z}, X^h)$ be given. Then by using (3.3.61)-(3.3.62) one has for each $n \in \mathbb{Z}$

$$\begin{aligned} e^{-\hat{\eta}|n|} \|\mathcal{K}^h(\mathbf{x}, \mathbf{y})_n\| &\leq \sum_{k=-\infty}^{n-1} \hat{\kappa} e^{-\hat{\eta}|n|} e^{-\hat{\rho}(n-k-1)} \|y_k\| + \sum_{k=n}^{+\infty} \hat{\kappa} e^{-\hat{\eta}|n|} e^{-\hat{\rho}(k+1-n)} \|y_k\| \\ &\leq \sum_{k=-\infty}^{n-1} \hat{\kappa} e^{-\hat{\eta}|n|} e^{-\hat{\rho}(n-k-1)} e^{\hat{\eta}|k|} \|\mathbf{y}\|_{\hat{\eta}} + \sum_{k=n}^{+\infty} \hat{\kappa} e^{-\hat{\eta}|n|} e^{-\hat{\rho}(k+1-n)} e^{\hat{\eta}|k|} \|\mathbf{y}\|_{\hat{\eta}} \\ &\leq \sum_{k=-\infty}^{n-1} \hat{\kappa} e^{\hat{\eta}(n-k)} e^{-\hat{\rho}(n-k-1)} \|\mathbf{y}\|_{\hat{\eta}} + \sum_{k=n}^{+\infty} \hat{\kappa} e^{\hat{\eta}(n-k)} e^{-\hat{\rho}(k+1-n)} \|\mathbf{y}\|_{\hat{\eta}}, \end{aligned}$$

so that

$$e^{-\hat{\eta}|n|} \|\mathcal{K}^h(\mathbf{x}, \mathbf{y})_n\| \leq \frac{\hat{\kappa} e^{\hat{\eta}}}{1 - e^{\hat{\eta} - \hat{\rho}}} \|\mathbf{y}\|_{\hat{\eta}} + \frac{\hat{\kappa} e^{-\hat{\rho}}}{1 - e^{\hat{\eta} - \hat{\rho}}} \|\mathbf{y}\|_{\hat{\eta}}, \quad \forall n \in \mathbb{Z}. \quad (3.3.77)$$

Hence choosing $\hat{\eta} = 0$ and $\hat{\eta} = \eta$ in (3.3.77) we obtain

$$\|\mathcal{K}^h(\mathbf{x}, \mathbf{y})\|_\eta \leq \gamma \|\mathbf{y}\|_\eta \quad \text{and} \quad \|\mathcal{K}^h(\mathbf{x}, \mathbf{y})\|_0 \leq \gamma \|\mathbf{y}\|_0,$$

where we have set

$$\gamma := \max \left\{ \frac{\hat{\kappa} e^\eta}{1 - e^{\eta - \hat{\rho}}} + \frac{\hat{\kappa} e^{-\hat{\rho}}}{1 - e^{\eta - \hat{\rho}}}; \frac{\hat{\kappa}}{1 - e^{-\hat{\rho}}} + \frac{\hat{\kappa} e^{-\hat{\rho}}}{1 - e^{-\hat{\rho}}} \right\}.$$

This completes the proof of (i) and (ii).

Proof of (iii):

Let $\mathbf{x}, \bar{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_0(\mathbb{Z}, X^h)$ be given. Firstly note that multiplying the left side of $\mathcal{K}^h(\mathbf{x}, \mathbf{y})_n$ (resp. $\mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_n$) by $A(x_n)$ (resp. $A(\bar{x}_n)$) yields

$$A(x_n) \mathcal{K}^h(\mathbf{x}, \mathbf{y})_n + y_n = \mathcal{K}^h(\mathbf{x}, \mathbf{y})_{n+1}, \quad \forall n \in \mathbb{Z}, \quad (3.3.78)$$

and

$$A(\bar{x}_n) \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_n + y_n = \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_{n+1}, \quad \forall n \in \mathbb{Z}. \quad (3.3.79)$$

Therefore by setting

$$a_n := \mathcal{K}^h(\mathbf{x}, \mathbf{y})_n - \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_n \in X^h, \quad \forall n \in \mathbb{Z}, \quad (3.3.80)$$

and

$$b_n := [A(x_n) - A(\bar{x}_n)] \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_n \in X^h, \quad \forall n \in \mathbb{Z}, \quad (3.3.81)$$

one has

$$a_{n+1} = A(x_n) a_n + b_n, \quad \forall n \in \mathbb{Z}. \quad (3.3.82)$$

Hence since $\mathbf{y} \in \mathbb{L}_0(\mathbb{Z}, X^h) \subset \mathbb{L}_\eta(\mathbb{Z}, X^h)$ by using (3.3.80) and property (ii) one can easily see that $\mathbf{a} \in \mathbb{L}_0(\mathbb{Z}, X^h) \subset \mathbb{L}_\eta(\mathbb{Z}, X^h)$. Therefore by using similar arguments as in the proof of Lemma 3.3.13 we obtain that the sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{Z}}$ satisfies

$$a_n = \sum_{k=-\infty}^{n-1} U_{\mathbf{x}}^s(n, k+1) b_k - \sum_{k=n}^{+\infty} U_{\mathbf{x}}^u(n, k+1) b_k, \quad \forall n \in \mathbb{Z}. \quad (3.3.83)$$

Furthermore recalling that $A(\cdot)$ is Lipschitz continuous on M we obtain

$$\|b_n\| = \|[A(x_n) - A(\bar{x}_n)] \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_n\| \leq \|A\|_{\text{Lip}(M, \mathcal{L}(M))} \|x_n - \bar{x}_n\| \|\mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})_n\|, \quad \forall n \in \mathbb{Z}. \quad (3.3.84)$$

Thus since $\mathbf{y} \in \mathbb{L}_0(\mathbb{Z}, X^h)$ and $\mathbf{x}, \bar{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M) \subset \mathbb{L}_\eta(\mathbb{Z}, M)$, (3.3.84) combined with property (ii) provide that $\mathbf{b} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ and

$$\|\mathbf{b}\|_\eta \leq \|A\|_{\text{Lip}(M, \mathcal{L}(M))} \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta \gamma \|\mathbf{y}\|_0. \quad (3.3.85)$$

Now since $\mathbf{b} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ it follows from (3.3.83) that $\mathbf{a} = \mathcal{K}^h(\mathbf{x}, \mathbf{b})$ and due to property (i) we obtain

$$\|\mathbf{a}\|_\eta = \|\mathcal{K}^h(\mathbf{x}, \mathbf{b})\|_\eta \leq \gamma \|\mathbf{b}\|_\eta.$$

Finally (3.3.80) and (3.3.85) combined with the above inequality imply that

$$\|\mathbf{a}\|_\eta = \|\mathcal{K}^h(\mathbf{x}, \mathbf{y}) - \mathcal{K}^h(\bar{\mathbf{x}}, \mathbf{y})\|_\eta \leq \gamma \|\mathbf{b}\|_\eta \leq \gamma \|A\|_{\text{Lip}(M, \mathcal{L}(M))} \gamma \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta \|\mathbf{y}\|_0. \quad (3.3.86)$$

The proof of (iii) is complete. ■

In the next lemma we deal with operator $\mathcal{G}$ stated in (3.3.32).

Lemma 3.3.15 *Let condition (H3) in Assumption 3.2.3 be satisfied. Then the operator*

$$\mathcal{G} : \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h) \rightarrow \mathbb{L}_\eta(\mathbb{Z}, X^h),$$

given by

$$\mathcal{G}(\mathbf{x}, \mathbf{y})_n = R_h(x_n, y_n), \quad \forall n \in \mathbb{Z}.$$

is well defined. Moreover the following properties hold true:

i) for each $\mathbf{x} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ we have

$$\mathcal{G}(\mathbf{x}, \mathbf{y}) \in \mathbb{L}_0(\mathbb{Z}, X^h) \text{ and } \|\mathcal{G}(\mathbf{x}, \mathbf{y})\|_0 \leq \|R_h\|_{C(M \times X^h, X^h)}. \quad (3.3.87)$$

ii) for each $\mathbf{x}, \bar{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{y}, \bar{\mathbf{y}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ we have

$$\|\mathcal{G}(\mathbf{x}, \mathbf{y}) - \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\|_\eta \leq \|R_h\|_{Lip(M \times X^h, X^h)} \left[\|\mathbf{y} - \bar{\mathbf{y}}\|_\eta + \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta \right]. \quad (3.3.88)$$

Proof. The proof of this lemma is straightforward and thus omitted. $\blacksquare$

3.3.4 Proof of Theorem 3.2.6

Now with the above material we are able to give the proof of Theorem 3.2.6.

Proof of Theorem 3.2.6. The proof of this theorem is divided into three steps. Assume that

$$\|R_c\|_{C(M \times X^h, X^h)} \leq \min(1, \varepsilon) \text{ and } \|R_c\|_{Lip(M \times X^h, M)} \leq \min\left(1, \frac{1}{2C}\right), \quad (3.3.89)$$

with ε given in Proposition 3.3.3 and C given in Proposition 3.3.9. Assume moreover that

$$\max\left\{\|R_h\|_{C(M \times X^h, M)}, \|R_h\|_{Lip(M \times X^h, M)}\right\} < \min\left\{\frac{1}{6\gamma}, \frac{1}{12C\gamma}, \frac{1}{12C\gamma^2 \|A\|_{Lip(M, \mathcal{L}(M))}}\right\}. \quad (3.3.90)$$

Step 1: Existence of an invariant manifold.

Let us notice that according to Lemmas 3.3.12-3.3.14 we know that sequences $\mathbf{x} \in \mathbb{L}_\eta(\mathbb{Z}, M)$ and $\mathbf{y} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ satisfy the system

$$\begin{cases} x_{n+1} = T(x_n) + R_c(x_n, y_n), & n \in \mathbb{Z}, \\ y_{n+1} = DT(x_n)y_n + R_h(x_n, y_n), & n \in \mathbb{Z}, \end{cases} \quad (3.3.91)$$

if and only if

$$\mathbf{x} = \mathcal{K}^c(x_0, \mathbf{y}) \in \mathbb{E}_\eta(\mathbb{Z}, M), \quad (3.3.92)$$

and

$$\mathbf{y} = \mathcal{K}^h(., \mathcal{G}(., \mathbf{y})) \circ \mathcal{K}^c(x_0, \mathbf{y}) \in \mathbb{L}_\eta(\mathbb{Z}, X^h). \quad (3.3.93)$$

Therefore the above system ((3.3.91) or equivalently (3.3.92)-(3.3.93)) decouples and it is sufficient to solve the fixed point problem (3.3.93) for each $x_0 \in M$.

Now we will prove that for each given $x_0 \in M$ there exists a unique $\mathbf{y} := \mathbf{y}(x_0) \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ satisfying the fixed point problem (3.3.93). To do so we define the operator

$$\mathcal{J} : M \times \mathbb{L}_\eta(\mathbb{Z}, X^h) \rightarrow \mathbb{L}_\eta(\mathbb{Z}, X^h),$$

by

$$\mathcal{J}(x_0, \mathbf{y}) := \mathcal{K}^h(., \mathcal{G}(., \mathbf{y})) \circ \mathcal{K}^c(x_0, \mathbf{y}), \quad \forall (x_0, \mathbf{y}) \in M \times \mathbb{L}_\eta(\mathbb{Z}, X^h).$$

which is well defined due to Lemmas 3.3.12-3.3.14.

Claim 3.3.16 *We claim that there exists a constant $C_0 > 0$ such that for each $\mathbf{y}, \bar{\mathbf{y}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ and $x_0, \bar{x}_0 \in M$*

$$\begin{aligned} \|\mathcal{J}(x_0, \mathbf{y}) - \mathcal{J}(\bar{x}_0, \bar{\mathbf{y}})\|_\eta &\leq C_0 \|R_h\|_{\text{Lip}(M \times X^h, X^h)} \|x_0 - \bar{x}_0\| \\ &\quad + \frac{1}{2} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta. \end{aligned} \quad (3.3.94)$$

In order to prove our claim let us define for a given $\mathbf{y}, \bar{\mathbf{y}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$ and $x_0, \bar{x}_0 \in M$

$$\mathbf{x} := \mathcal{K}^c(x_0, \mathbf{y}) \in \mathbb{E}_\eta(\mathbb{Z}, M) \text{ and } \bar{\mathbf{x}} := \mathcal{K}^c(\bar{x}_0, \bar{\mathbf{y}}) \in \mathbb{E}_\eta(\mathbb{Z}, M). \quad (3.3.95)$$

Then one has

$$\begin{aligned} \mathcal{J}(x_0, \mathbf{y}) - \mathcal{J}(\bar{x}_0, \bar{\mathbf{y}}) &= \mathcal{K}^h(\mathbf{x}, \mathcal{G}(\mathbf{x}, \mathbf{y})) - \mathcal{K}^h(\bar{\mathbf{x}}, \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})) \\ &= [\mathcal{K}^h(\mathbf{x}, \mathcal{G}(\mathbf{x}, \mathbf{y})) - \mathcal{K}^h(\mathbf{x}, \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))] \\ &\quad + [\mathcal{K}^h(\mathbf{x}, \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})) - \mathcal{K}^h(\bar{\mathbf{x}}, \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}}))] \end{aligned}$$

so that by using (3.3.74) and (3.3.76) we obtain

$$\begin{aligned} \|\mathcal{J}(x_0, \mathbf{y}) - \mathcal{J}(\bar{x}_0, \bar{\mathbf{y}})\|_\eta &\leq \gamma \|\mathcal{G}(\mathbf{x}, \mathbf{y}) - \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\|_\eta \\ &\quad + \gamma^2 \|A\|_{\text{Lip}(M, \mathcal{L}(M))} \|\mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\|_0 \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta. \end{aligned}$$

Hence since

$$\|\mathcal{G}(\mathbf{x}, \mathbf{y}) - \mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\|_\eta \leq \|R_h\|_{\text{Lip}(M \times X^h, X^h)} \left[\|\mathbf{y} - \bar{\mathbf{y}}\|_\eta + \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta \right],$$

and

$$\|\mathcal{G}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\|_0 \leq \|R_h\|_{C(M \times X^h, X^h)},$$

it follows that

$$\begin{aligned} \|\mathcal{J}(x_0, \mathbf{y}) - \mathcal{J}(\bar{x}_0, \bar{\mathbf{y}})\|_\eta &\leq \gamma \|R_h\|_{\text{Lip}(M \times X^h, X^h)} \left[\|\mathbf{y} - \bar{\mathbf{y}}\|_\eta + \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta \right] \\ &\quad + \gamma^2 \|A\|_{\text{Lip}(M, \mathcal{L}(M))} \|R_h\|_{C(M \times X^h, X^h)} \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta. \end{aligned} \quad (3.3.96)$$

Recalling the definition of $\mathbf{x}$ and $\bar{\mathbf{x}}$ stated in (3.3.95) we infer from (3.3.64) that

$$\begin{aligned} \|\mathbf{x} - \bar{\mathbf{x}}\|_\eta &= \|\mathcal{K}^c(x_0, \mathbf{y}) - \mathcal{K}^c(\bar{x}_0, \bar{\mathbf{y}})\|_\eta \\ &\leq 2\hat{\kappa} \|x_0 - \bar{x}_0\| + 2C \|R_c\|_{\text{Lip}(M \times X^h, M)} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta. \end{aligned} \quad (3.3.97)$$

Therefore by plugging (3.3.97) into (3.3.96) one gets

$$\begin{aligned}
\|\mathcal{J}(x_0, \mathbf{y}) - \mathcal{J}(\bar{x}_0, \bar{\mathbf{y}})\|_\eta &\leq \gamma \|R_h\|_{Lip(M \times X^h, X^h)} 2\widehat{\kappa} \|x_0 - \bar{x}_0\| \\
&\quad + \gamma^2 \|A\|_{Lip(M, \mathcal{L}(M))} \|R_h\|_{C(M \times X^h, X^h)} 2\widehat{\kappa} \|x_0 - \bar{x}_0\| \\
&\quad + \gamma \|R_h\|_{Lip(M \times X^h, X^h)} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta \\
&\quad + 2C\gamma \|R_h\|_{Lip(M \times X^h, X^h)} \|R_c\|_{Lip(M \times X^h, M)} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta \\
&\quad + 2C\gamma^2 \|A\|_{Lip(M, \mathcal{L}(M))} \|R_h\|_{C(M \times X^h, X^h)} \|R_c\|_{Lip(M \times X^h, M)} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta.
\end{aligned}$$

Setting

$$C_0 := 2\widehat{\kappa} \left[\gamma + \gamma^2 \|A\|_{Lip(M, \mathcal{L}(M))} \right], \quad (3.3.98)$$

we conclude from (3.3.89) and (3.3.90) that Claim 3.3.16 holds true.

Next we infer from (3.3.94) that for each $x_0 \in M$ and $\mathbf{y}, \bar{\mathbf{y}} \in \mathbb{L}_\eta(\mathbb{Z}, X^h)$

$$\|\mathcal{J}(x_0, \mathbf{y}) - \mathcal{J}(x_0, \bar{\mathbf{y}})\|_\eta \leq \frac{1}{2} \|\mathbf{y} - \bar{\mathbf{y}}\|_\eta, \quad (3.3.99)$$

so that the uniform Banach fixed point theorem ensures that for each $x_0 \in M$ there exists a unique fixed point $\mathbf{y} := \psi(x_0) = \{\psi(x_0)_n\}_{n \in \mathbb{Z}}$ of $\mathcal{J}(x_0, \cdot)$ in $\mathbb{L}_\eta(\mathbb{Z}, X^h)$ such that

$$\psi(x_0) = \mathcal{K}^h(\cdot, \mathcal{G}(\cdot, \psi(x_0))) \circ \mathcal{K}^c(x_0, \psi(x_0)).$$

Then we can define a map $\varphi : M \rightarrow X^h$ as

$$\Psi(x_0) := \psi(x_0)_0, \quad \forall x_0 \in M, \quad (3.3.100)$$

and we consider the manifold

$$\widehat{M} := \{(x_0, \Psi(x_0)) : x_0 \in M\}. \quad (3.3.101)$$

Step 2: Invariance of $\widehat{M}$.

In this step we will show that $\widehat{M}$ is invariant under system (3.3.91). To do so let $(x_0, \Psi(x_0)) \in \widehat{M}$ then there exists a unique $(\mathbf{x}, \mathbf{y}) \in \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ satisfying (3.3.91). Now let $p \in \mathbb{Z}$ be fixed and consider $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in \mathbb{L}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ defined by

$$\widehat{x}_n = x_{n+p} \text{ and } \widehat{y}_n = y_{n+p}, \quad \forall n \in \mathbb{Z}.$$

Then it is easy to verify that $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ and satisfies (3.3.91) so that

$$\widehat{y}_0 = \Psi(\widehat{x}_0) \iff y_p = \Psi(x_p).$$

That proves the invariance of $\widehat{M}$ since $p \in \mathbb{Z}$ is arbitrary.

Step 3: Properties of Ψ .

Let us prove that Ψ is uniformly bounded on M . To do so we know due to the invariance of $\widehat{M}$ that for each $x \in M$ there exists a unique $(\mathbf{x}, \Psi(\mathbf{x})) = \{(x_n, \Psi(x_n))\}_{n \in \mathbb{Z}} \in \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ with $x_0 = x$ satisfying (3.3.91) and $\Psi(\mathbf{x})$ satisfies

$$\begin{aligned}\Psi(\mathbf{x}) &= \mathcal{K}^h(., \mathcal{G}(., \Psi(\mathbf{x}))) \circ \mathcal{K}^c(x_0, \Psi(\mathbf{x})) \\ &= \mathcal{K}^h(\mathbf{x}, \mathcal{G}(\mathbf{x}, \Psi(\mathbf{x}))),\end{aligned}$$

with $\mathbf{x} = \mathcal{K}^c(x_0, \Psi(\mathbf{x}))$. Recalling (3.3.74) we obtain

$$\|\Psi(\mathbf{x})\|_\eta = \|\mathcal{K}^h(\mathbf{x}, \mathcal{G}(\mathbf{x}, \Psi(\mathbf{x})))\|_\eta \leq \gamma \|\mathcal{G}(\mathbf{x}, \Psi(\mathbf{x}))\|_\eta,$$

so that

$$\|\Psi(\mathbf{x})\|_\eta \leq \gamma \|\mathcal{G}(\mathbf{x}, \Psi(\mathbf{x}))\|_\eta \leq \gamma \|\mathcal{G}(\mathbf{x}, \Psi(\mathbf{x})) - \mathcal{G}(\mathbf{x}, \mathbf{0})\|_\eta + \gamma \|\mathcal{G}(\mathbf{x}, \mathbf{0})\|_\eta.$$

Then by using (3.3.88) we obtain

$$\|\Psi(\mathbf{x})\|_\eta \leq \gamma \|R_h\|_{\text{Lip}(M \times X^h, X^h)} \|\Psi(\mathbf{x})\|_\eta + \gamma \|\mathcal{G}(\mathbf{x}, \mathbf{0})\|_\eta.$$

This re-writes due to (3.3.90) as

$$\|\Psi(\mathbf{x})\|_\eta \leq \frac{\gamma}{1 - \gamma \|R_h\|_{\text{Lip}(M \times X^h, X^h)}} \|\mathcal{G}(\mathbf{x}, \mathbf{0})\|_\eta \leq \frac{6}{5} \gamma \|\mathcal{G}(\mathbf{x}, \mathbf{0})\|_\eta. \quad (3.3.102)$$

Now we use the explicit expression of the operator $\mathcal{G}$ to deduce that

$$\|\mathcal{G}(\mathbf{x}, \mathbf{0})_n\| = \|R_h(x_n, 0)\| \leq \sup_{x \in M} \|R_h(x, 0)\|, \quad \forall n \in \mathbb{Z}. \quad (3.3.103)$$

Then we infer from (3.3.102) and (3.3.103) that

$$\|\Psi(\mathbf{x})\|_\eta \leq \frac{6}{5} \gamma \sup_{x \in M} \|R_h(x, 0)\|,$$

and it follows that

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \sup_{x \in M} \|R_h(x, 0)\|. \quad (3.3.104)$$

Next we prove that the map $\Psi : M \rightarrow X^h$ is Lipschitz continuous on M . Let $x, \bar{x} \in M$ be given. Then we know that there exists $(\mathbf{x}, \Psi(\mathbf{x})) = \{(x_n, \Psi(x_n))\}_{n \in \mathbb{Z}} \in \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ and $(\bar{\mathbf{x}}, \Psi(\bar{\mathbf{x}})) = \{(\bar{x}_n, \Psi(\bar{x}_n))\}_{n \in \mathbb{Z}} \in \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{Z}, X^h)$ with $x_0 = x$ and $\bar{x}_0 = \bar{x}$ such that

$$\begin{cases} \mathbf{x} = \mathcal{K}^c(x_0, \Psi(\mathbf{x})) \\ \Psi(\mathbf{x}) = \mathcal{J}(x_0, \Psi(\mathbf{x})) = \mathcal{K}^h(., \mathcal{G}(., \Psi(\mathbf{x}))) \circ \mathcal{K}^c(x_0, \Psi(\mathbf{x})), \end{cases} \quad (3.3.105)$$

and

$$\begin{cases} \bar{\mathbf{x}} = \mathcal{K}^c(\bar{x}_0, \Psi(\bar{\mathbf{x}})) \\ \Psi(\bar{\mathbf{x}}) = \mathcal{J}(\bar{x}_0, \Psi(\bar{\mathbf{x}})) = \mathcal{K}^h(., \mathcal{G}(., \Psi(\bar{\mathbf{x}}))) \circ \mathcal{K}^c(\bar{x}_0, \Psi(\bar{\mathbf{x}})). \end{cases} \quad (3.3.106)$$

Hence due to (3.3.94) one gets

$$\begin{aligned} \|\Psi(\mathbf{x}) - \Psi(\bar{\mathbf{x}})\|_{\eta} &\leq C_0 \|R_h\|_{Lip(M \times X^h, X^h)} \|x_0 - \bar{x}_0\| \\ &\quad + \frac{1}{2} \|\Psi(\mathbf{x}) - \Psi(\bar{\mathbf{x}})\|_{\eta}, \end{aligned} \quad (3.3.107)$$

(with C_0 defined in (3.3.98)) so that

$$\|\Psi(\mathbf{x}) - \Psi(\bar{\mathbf{x}})\|_{\eta} \leq 2C_0 \|R_h\|_{Lip(M \times X^h, X^h)} \|x_0 - \bar{x}_0\|. \quad (3.3.108)$$

One deduces that

$$\|\Psi(x_0) - \Psi(\bar{x}_0)\| \leq 2C_0 \|R_h\|_{Lip(M \times X^h, X^h)} \|x_0 - \bar{x}_0\|. \quad (3.3.109)$$

This completes the proof of Theorem 3.2.6. ■

Chapitre 4

Asymptotic behaviour near the persistent invariant manifolds

4.1 Introduction

The invariant foliations theory is a very important tool in understanding asymptotic behavior of solutions of dynamical systems near invariant manifolds. Special interest is to understand how solutions are moving toward or away from the invariant manifold at a specific rates. Specifically one would like to know if the solutions moving toward are attracted by a solution lying in the manifold. This is the case in the global center manifold theory for non hyperbolic equilibrium points. In fact it is a well known that a solution starting outside the global center manifold is exponentially attracted by a unique complete orbit lying in the global center-unstable manifold. Such a question is well known and extensively studied in the literature. We refer for instance to Vanderbauwhede [159] and Chow, Li and Wang [36] for a nice introduction in the context of ordinary differential equations and for example to Chen, Hale and Tan [33], Liu, Magal and Ruan [90] as well as the references therein for the infinite dimensional dynamical systems.

In the context of normally hyperbolic (or hyperbolic) invariant manifolds existence and smoothness of invariant foliations was proved in Hirsch, Pugh and Shub [77], Fenichel [65, 66] for finite dimensional dynamical systems (see also Jones [81] and Wiggins [166]). We also refer to Kirchgraber and Palmer [85], Sakamoto [146] for applications of invariant foliations to linearization of vector fields near invariant manifolds for finite dimensional systems and to Sakamoto [144] for application to heteroclinic connection. Bates, Lu and Zeng [22] give an extension of the foliations theory to compact normally hyperbolic invariant manifolds for semiflows in Banach spaces where the authors prove a very general result. They later generalize their result in [20] to the so called approximately normally hyperbolic manifolds with no compactness assumption. All the previous cited references are concerned to continuous time dynamical systems while for the discrete time cases we only refer to Vanderbauwhede [162], Aulbach and Pötzsche [12], Pötzsche [131], Aulbach and Wanner [11] for related results.

In order to clarify our work and put it in the context of invariant foliations theory, we consider the system

$$u_{n+1} = T(u_n), \quad u_0 \in X, \quad (4.1.1)$$

where $(X, \|\cdot\|)$ is a Banach space and $T : X \rightarrow X$ a continuously differentiable map on X . Let $\tilde{T} : X \rightarrow X$ be a Lipschitz continuous map and consider the following perturbed system

$$u_{n+1} = T(u_n) + \tilde{T}(u_n), \quad u_0 \in X. \quad (4.1.2)$$

Then we will make the following assumptions.

Assumption 4.1.1 *We assume that the map $T : X \rightarrow X$ is continuously differentiable and*

(H1) *There exists a normally hyperbolic invariant subspace $M \subset X$ of T with rates $0 < \rho_0 < \rho$ and constant κ (see Definition (3.1.1) in Chapter 3 for more general definition), this means that there exists a closed vector space X^h such that*

(i) *$T(M) = M$ and for each $x \in M$, there exists two closed subspaces X_x^α , $\alpha = u, s$ with*

$$X^h = X_x^s \oplus X_x^u, \text{ and } X = M \oplus X^h, \quad (4.1.3)$$

and

$$DT(x)M \subset M, \quad DT(x)X_x^\alpha \subset X_x^\alpha, \text{ for } \alpha = u, s.$$

(ii) *for each $x \in M$, $DT(x)$ is invertible from M (resp. X_x^u) into M (resp. X_x^u) with*

$$\begin{cases} \|DT^n(x)\|_{\mathcal{L}(X_x^s)} \leq \kappa e^{-\rho n} \\ \inf \{\|DT^n(x)x^u\| : x^u \in X_x^u \text{ and } \|x^u\| = 1\} \geq \frac{1}{\kappa} e^{\rho n}, \end{cases} \quad (4.1.4)$$

$$\frac{1}{\kappa} e^{-\rho_0 n} \leq \inf \{\|DT^n(x)x^c\| : x^c \in M \text{ and } \|x^c\| = 1\} \leq \|DT^n(x)\|_{\mathcal{L}(M)} \leq \kappa e^{\rho_0 n}. \quad (4.1.5)$$

(iii) *The projectors Π_x^α , $\alpha = u, s$ and Π^c associated to the state decomposition (4.1.3) that satisfy for each $x \in M$*

$$\mathcal{R}(\Pi_x^\alpha) = X_x^\alpha, \quad \alpha = u, s \text{ and } \mathcal{R}(\Pi^c) = M, \quad (4.1.6)$$

with

$$\mathcal{N}(\Pi_x^u) = X_x^s \oplus M, \quad \mathcal{N}(\Pi_x^s) = X_x^u \oplus M \text{ and } \mathcal{N}(\Pi^c) = \mathcal{N}(\Pi_x^u) = X_x^u \oplus X_x^s,$$

are uniformly bounded on M , that reads as

$$\sup_{x \in M} \|\Pi_x^\alpha\|_{\mathcal{L}(X)} \leq \kappa, \quad \alpha = u, s \text{ and } \|\Pi^c\|_{\mathcal{L}(X)} \leq \kappa. \quad (4.1.7)$$

(H3) There exists a complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of T in M with $\sup_{n \in \mathbb{Z}} e^{-\rho_0 |n|} \|x_n\| < +\infty$.

(H4) $DT(\cdot) : X \rightarrow \mathcal{L}(X)$ is uniformly bounded and Lipschitz continuous on X that is

$$\|DT(\cdot)\|_{C(X, \mathcal{L}(X))} + \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} < +\infty. \quad (4.1.8)$$

Let us notice that in the foregoing Assumption we take, without loss of generality, the same constant κ in **(H1)**-(ii) and **(H1)**-(iii) in order to simplify the computations in the sequel. In fact if we have two different constants κ_0 and κ_1 respectively in **(H1)**-(ii) and **(H1)**-(iii) one can replace them by $\kappa := \max\{\kappa_0, \kappa_1\}$ so that the conditions **(H1)**-(ii) and **(H1)**-(iii) remain true.

For the sake of completeness, let us now recall the result obtained in the previous Chapter 3 for the perturbed system (4.1.2).

Theorem 4.1.2 *Let Assumption 4.1.1 be satisfied. Let $\hat{\rho}_0, \hat{\rho}$ be given such that $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$. Let $\eta \in (\hat{\rho}_0, \hat{\rho})$ be given. Then there exist $\delta > 0$ and constants $\hat{\gamma} > 0$, $\gamma > 0$ such that if*

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \delta^2,$$

then there exists some $\Psi : M \rightarrow X^h$ such that the manifold

$$\tilde{M} = \{x + \Psi(x) : x \in M\},$$

as the following properties:

(i) *If $\mathbf{x} + \mathbf{y} = \{x_n + y_n\}_{n \in \mathbb{Z}} \subset M \oplus X^h$ is a complete orbit of $T + \tilde{T}$ with $\|y_n\| \leq \delta$ for all $n \in \mathbb{Z}$ then $x_0 + y_0 \in \tilde{M}$, that is $y_0 = \Psi(x_0)$.*

(ii) *For each $x + \Psi(x) \in \tilde{M}$ there exists a unique solution $\mathbf{x} + \Psi(\mathbf{x}) = \{x_n + \Psi(x_n)\}_{n \in \mathbb{Z}} \subset \tilde{M}$ of $T + \tilde{T}$ with $x_0 = x$ and*

$$\sup_{n \in \mathbb{Z}} e^{-\eta |n|} \|x_n\| < +\infty \text{ and } \sup_{n \in \mathbb{Z}} e^{-\eta |n|} \|\Psi(x_n)\| < +\infty.$$

(iii) *Ψ is bounded and Lipschitz continuous on M with*

$$\|\Psi(x) - \Psi(\bar{x})\| \leq \hat{\gamma} \delta \|x - \bar{x}\|, \quad \forall x, \bar{x} \in M \text{ and } \sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \|\tilde{T}\|_{C(X)}.$$

The aim of this chapter is to study the asymptotic behavior of the orbits of the perturbed system (4.1.2) lying outside of the persistent invariant manifold $\tilde{M}$ stated in Theorem 4.1.2. We will prove in particular existence of invariant foliations for system (4.1.2) and an attraction property for the persistent manifold $\tilde{M}$. We also give a consequence for

semiflows in Banach spaces. It should be pointed out the fact that the result proved by Bates, Lu and Zeng [20] may be applied to our system with some modifications since their proof uses essentially the time-t map. However we have to mention that the main difference between the result in [22] and our result is that we prove invariant foliations for the perturbed system with Lipschitz continuous small perturbations while in [20] the authors proved invariant foliations for C^1 semiflows.

Note that the invariant manifold considered here is non compact and the center part associated to the state decomposition of the normal hyperbolicity is allowed to be infinite dimensional. Our result is a generalization of the result proved by Sakamoto [144] in the context of singularly perturbed ordinary differential equations with C^1 small perturbations.

The chapter is organized as follow: In Section 4.2 we give our main result and a consequence for semiflows in Banach spaces. Section 4.3 is devoted to the proof of the main result.

4.2 Main result and consequences

Similarly to Chapter 3, in what follows our functional framework will be the following weighted Banach spaces for $\mathbb{I} = \mathbb{N}, \mathbb{N}_-, \mathbb{Z}$ and $Y = M, X^h, X$

$$\mathbb{L}_\eta(\mathbb{I}, Y) := \left\{ \mathbf{y} : \mathbb{I} \rightarrow Y \text{ such that } \|\mathbf{y}\|_{\eta,+} := \sup_{n \in \mathbb{I}} e^{\eta n} \|y_n\| < +\infty \right\}, \quad \eta \in \mathbb{R},$$

with

$$\mathbb{L}_0(\mathbb{I}, Y) = l^\infty(\mathbb{I}, Y) = \left\{ \mathbf{y} : \mathbb{I} \rightarrow Y \text{ such that } \|\mathbf{y}\|_{\infty,+} := \sup_{n \in \mathbb{I}} \|y_n\| < +\infty \right\},$$

where $\mathbb{N}_- = \{0, -1, -2, \dots\}$. Note that $\{\mathbb{L}_\eta(\mathbb{N}, Y), \eta \geq 0\}$ form a **scale of Banach spaces** that is for each $0 \leq \eta \leq \hat{\eta}$ we have

$$\mathbb{L}_{\hat{\eta}}(\mathbb{N}, Y) \subseteq \mathbb{L}_\eta(\mathbb{N}, Y),$$

with continuous embedding, namely

$$\|\mathbf{y}\|_\eta \leq \|\mathbf{y}\|_{\hat{\eta}}, \quad \forall \mathbf{y} \in \mathbb{L}_{\hat{\eta}}(\mathbb{N}, Y).$$

Similarly to Chapter 3, let us introduce some notations that will be used in the sequel. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ be a given sequence in M . Note that due to the property **(H1)-(i)** in Assumption 4.1.1 one has

$$\Pi^c DT(x) = DT(x) \Pi^c \text{ and } \Pi^h DT(x) = DT(x) \Pi^h, \quad \forall x \in M,$$

so that we define the evolution semigroup $\{U_{\mathbf{x}}^c(n, p)\}_{n \geq p} \subset \mathcal{L}(M)$ associated to $\{DT(x_n) \Pi^c\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ by

$$U_{\mathbf{x}}^c(n, p) := \begin{cases} DT(x_{n-1}) \dots DT(x_p) \Pi^c, & \text{if } n > p, \\ I_{\mathcal{L}(M)}, & \text{if } n = p, \end{cases} \quad (4.2.1)$$

and we define the evolution semigroup $\{U_{\mathbf{x}}^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ associated to $\{DT(x) \Pi^h\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ by

$$U_{\mathbf{x}}^h(n, p) := \begin{cases} DT(x_{n-1}) \dots DT(x_p) \Pi^h, & \text{if } n > p, \\ I_{\mathcal{L}(X^h)}, & \text{if } n = p. \end{cases} \quad (4.2.2)$$

Observe that with this definition we have

$$U_{\mathbf{x}}^c(n, p) = \Pi^c U_{\mathbf{x}}^c(n, p) = U_{\mathbf{x}}^c(n, p) \Pi^c, \quad \forall n \geq p,$$

and

$$U_{\mathbf{x}}^h(n, p) = \Pi^h U_{\mathbf{x}}^h(n, p) = U_{\mathbf{x}}^h(n, p) \Pi^h.$$

Let us now consider the following discrete dynamical system

$$x_{n+1} = \Pi^c \left(T + \tilde{T} \right) (x_n + \Psi(x_n)), \quad \forall n \in \mathbb{N} \text{ and } x_0 = x \in M, \quad (4.2.3)$$

where $\Psi : M \rightarrow X^h$ is the map given in Theorem 4.1.2.

Then since for each $x \in M$, $x + \Psi(x) \in \tilde{M}$ it follows from the property (ii) stated in Theorem 4.1.2 and the property

$$\Pi^c \Psi(x) = 0, \quad \forall x \in M,$$

that for each $x \in M$ there exists a unique complete orbit of (4.2.3). Hence for simplicity in the notations and for future references it will be denoted by $\sigma(x) = \{\sigma_n(x)\}_{n \in \mathbb{Z}} \subset M$ the unique complete orbit of (4.2.3) with

$$\sigma_0(x) = x, \quad \forall x \in M.$$

Furthermore remark that for each $x \in M$, the sequence $\{\sigma_n(x) + \Psi(\sigma_n(x))\}_{n \in \mathbb{Z}}$ is the complete orbit of $T + \tilde{T}$ with

$$\sigma_0(x) + \Psi(\sigma_0(x)) = x + \Psi(x).$$

Hence according to the above introduced notations we are able to introduce some vector spaces that will be used in the statement of the main theorem. Let $\hat{\rho}_0$ and $\hat{\rho}$ be given such that $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$. We define for each $x \in M$ the following vector spaces

$$E_x^s := \left\{ y \in X^h : \{U_{\sigma(x)}^h(n, 0) y\}_{n \geq 0} \in \mathbb{L}_{\hat{\rho}}(\mathbb{N}, X^h) < +\infty \right\}, \quad (4.2.4)$$

and

$$E_x^u := \left\{ \begin{array}{l} y \in X^h : \exists \{y_k\}_{k \in \mathbb{N}_-} \subset X^h \\ \text{with } y_n = DT(\sigma_{n-1}(x)) y_{n-1}, \quad y_0 = y \text{ and } \mathbf{y} \in \mathbb{L}_{-\hat{\rho}}(\mathbb{N}_-, X^h) \end{array} \right\}. \quad (4.2.5)$$

We also define the **stable bundle** E^s by

$$E^s := \{x + y \in X : x \in M \text{ and } y \in E_x^s\}, \quad (4.2.6)$$

and the **unstable bundle** E^u by

$$E^u := \{x + y \in X : x \in M \text{ and } y \in E_x^u\}. \quad (4.2.7)$$

The main result of this section is stated as follow

Theorem 4.2.1 *Let Assumption 4.1.1 be satisfied. Let $\hat{\rho}_0, \hat{\rho}$ be given such that $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$. Let $\eta \in (\hat{\rho}_0, \hat{\rho})$ be given. Then there exists $\delta_1 > \delta_2 > \delta_3 > \delta_4 > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$ such that if*

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \delta_1^2,$$

then $X^h = E_x^u \oplus E_x^s$, for all $x \in M$ and there exist some functions

$$\Psi^c, \Psi^u : E^s(\delta_2) \rightarrow M \oplus X^h = X,$$

with

$$E^s(\delta_2) := \{x + z \in X : x \in M, z \in E_x^s \text{ and } \|z\| \leq \delta_2\},$$

and a family of leaves $\{\mathcal{W}_\eta^s(x, \delta_2)\}_{x \in M}$ with

$$\mathcal{W}_\eta^s(x, \delta_2) = \left\{ \begin{array}{l} x_0 + y_0 \in M \oplus X^h : x_0 = x + \Psi^c(x + z^s) \\ y_0 = \Psi(x_0) + z^s + \Psi^u(x + z^s) \\ z^s \in E_x^s \text{ and } \|z^s\| \leq \delta_2 \end{array} \right\}, \quad \forall x \in M,$$

that are Lipschitz stable manifolds with the following properties:

(i) *For each $x \in M$*

$$\Psi^\alpha(x) = 0, \quad \alpha = u, c,$$

and for each $x + z^s \in E^s(\delta_2)$ we have

$$\Psi^u(x + z^s) \in E_x^u, \quad \Psi^c(x + z^s) \in M,$$

and

$$\|z^s + \Psi^u(x + z^s)\| \leq \delta_1 \text{ and } \|\Psi^c(x + z^s)\| \leq \delta_1.$$

(ii) *For each $x + z^s, x + \bar{z}^s \in E^s(\delta_2)$ we have*

$$\|\Psi^u(x + z^s) - \Psi^u(x + \bar{z}^s)\| \leq \gamma_1 \|z^s - \bar{z}^s\|,$$

and

$$\|\Psi^c(x + z^s) - \Psi^c(x + \bar{z}^s)\| \leq \gamma_1 \|z^s - \bar{z}^s\|.$$

(iii) For each $u_0 \in \mathcal{W}_\eta^s(x, \delta_2)$, $x \in M$, the positive orbit $\mathbf{u} = \{u_n\}_{n \in \mathbb{N}}$ of $T + \tilde{T}$ satisfies

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) + \Psi(\sigma_n(x)) - u_n\| \leq \gamma_2 \delta_1.$$

(iv) If $\mathbf{u} = \{u_n\}_{n \in \mathbb{N}} \subset X$ is a positive orbit of $T + \tilde{T}$ with

$$\|\Pi^h u_0 - \Psi(\Pi^c u_0)\| \leq \delta_3,$$

and

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\Pi^h u_n - \Psi(\Pi^c u_n)\| \leq \delta_1 \text{ and } \sup_{n \in \mathbb{N}} e^{\eta n} \|\Pi^c u_n - \sigma_n(x)\| \leq \delta_1,$$

for some $x \in M$ then $u_0 \in \mathcal{W}_\eta^s(x, \delta_2)$.

(v) If $\mathbf{u} = \{u_n\}_{n \in \mathbb{N}} \subset X$ is a positive orbit of $T + \tilde{T}$ with

$$\|\Pi^h u_0\| \leq \delta_4 \text{ and } \sup_{n \in \mathbb{N}} \|\Pi^h u_n\| \leq \frac{\delta_1}{2},$$

then there exists a unique $x \in M$ such that $u_0 \in \mathcal{W}_\eta^s(x, \delta_1)$.

(vi) The manifolds $\mathcal{W}_\eta^s(x, \delta_2)$, $x \in M$ are disjoint or equal.

(vii) The union of the manifolds

$$\mathcal{W}_\eta^s(\delta_2) = \bigcup_{x \in M} \mathcal{W}_\eta^s(x, \delta_2),$$

is forward invariant for $T + \tilde{T}$ in the following sense: There exists $p \in \mathbb{N}$ such that if $u \in \mathcal{W}_\eta^s(x, \delta_2)$ for some $x \in M$ then

$$\left((T + \tilde{T})^n(u) \right) \in \mathcal{W}_\eta^s(\sigma_n(x), \delta_2), \quad \forall n \geq p.$$

4.2.1 Consequence for continuous time maps

In this section we will derive a consequence of Theorem 4.2.1 for time continuous maps. Recall that a family $\{T(t)\}_{t \geq 0}$ defines a continuous semiflow on X if

(i) $T(0) = I_{\mathcal{L}(X)}$ and $T(t+l) = T(t)T(l)$, $\forall t, l \geq 0$,

(ii) The map $(t, u) \rightarrow T(t)u$ is continuous from $[0, +\infty) \times X$ into X .

We also recall that $v \in C(\mathbb{R}_+, X)$ (resp. $C(\mathbb{R}, X)$) is a **positive orbit** (resp. **complete orbit**) of $\{T(t)\}_{t \geq 0}$ if

$$v(t) = T(t-l)v(l), \quad \forall t, l \in \mathbb{R}_+ \text{ (resp. } \forall t, l \in \mathbb{R} \text{) with } t \geq l.$$

Before stating the main result of this section, let us recall the following result obtained in Chapter 3 for the semiflow $\{T(t)\}_{t \geq 0}$ on X . It reads as:

Theorem 4.2.2 Let $\{T(t)\}_{t \geq 0}$ be a continuous semiflow on X . Let $T_0 : X \rightarrow X$ be a C^1 map satisfying Assumption 4.1.1. Assume that there exists $t_0 > 0$ such that

$$\gamma(t_0) := \sup_{l \in [0, t_0]} \|T(l)\|_{Lip(X)} < +\infty, \quad (4.2.8)$$

and

$$T(l)M \subset M, \quad \forall l \in [0, t_0].$$

Then for each $\hat{\rho}_0$ and $\hat{\rho}$ with $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and each $\eta \in (\hat{\rho}_0, \hat{\rho})$ there exist three constants $\delta > 0$, $\hat{\gamma} > 0$ and $\gamma > 0$ such that if

$$\|T(t_0) - T_0\|_{C(X)} + \|T(t_0) - T_0\|_{Lip(X)} \leq \delta^2,$$

then there exists some function $\Psi : M \rightarrow X^h$ such that the manifold

$$\tilde{M} = \{x + \Psi(x) : x \in M\},$$

has the following properties:

(i) The map Ψ is bounded and Lipschitz continuous on M with

$$\|\Psi(x) - \Psi(\bar{x})\| \leq \hat{\gamma}\delta \|x - \bar{x}\|, \quad \forall x, \bar{x} \in M,$$

and

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5}\gamma \|T(t_0) - T_0\|_{C(X)}.$$

(ii) For each $t \geq 0$ the map $T(t)$ is a bijection from $\tilde{M}$ into $\tilde{M}$.

(iii) For each $x + \Psi(x) \in \tilde{M}$, there exists a unique complete orbit $v : \mathbb{R} \rightarrow \tilde{M}$ of $\{T(t)\}_{t \geq 0}$ with $v(0) = x + \Psi(x)$.

(iv) If $v \in C(\mathbb{R}, X)$ is a complete orbit of $\{T(t)\}_{t \geq 0}$ such that

$$\|\Pi^h v(t)\| \leq \delta, \quad \forall t \in \mathbb{R},$$

then $v(t) \in \tilde{M}$ for all $t \in \mathbb{R}$.

The main result of this section is the following:

Theorem 4.2.3 (Stability) Let $\{T(t)\}_{t \geq 0}$ be a continuous semiflow on X . Let $T_0 : X \rightarrow X$ be a C^1 map satisfying Assumption 4.1.1. Assume that there exists $t_0 > 0$ such that

$$\gamma(t_0) := \sup_{l \in [0, t_0]} \|T(l)\|_{Lip(X)} < +\infty, \quad (4.2.9)$$

and

$$T(l)M \subset M, \forall l \in [0, t_0).$$

Then for each $\hat{\rho}_0$ and $\hat{\rho}$ with $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$ and each $\eta \in (\hat{\rho}_0, \hat{\rho})$ there exist $\delta_1 > \delta_2 > \delta_3 > \delta_4 > 0$ (small enough) and constants $\gamma_1 > 0, \gamma_2 > 0, \hat{\gamma} > 0$ and $\gamma > 0$ such that if

$$\|T(t_0) - T_0\|_{C(X)} + \|T(t_0) - T_0\|_{Lip(X)} \leq \delta_1^2,$$

then the properties of Theorem 4.1.2 and Theorem 4.2.1 holds true for $T = T_0$ and $\tilde{T} = T(t_0) - T_0$. Moreover if $u \in C(\mathbb{R}_+, X)$ is a positive orbit of $\{T(t)\}_{t \geq 0}$ with

$$\|\Pi^h u(0)\| \leq \delta_4 \text{ and } \sup_{t \geq 0} \|\Pi^h u(t)\| \leq \frac{\delta_1}{2},$$

then there exists a unique complete orbit $v : \mathbb{R} \rightarrow \tilde{M}$ of $\{T(t)\}_{t \geq 0}$ such that

$$\|u(t) - v(t)\| \leq \gamma(t_0) \gamma_2 \delta_1 e^\eta e^{-\frac{\eta}{t_0} t}, \forall t \geq 0. \quad (4.2.10)$$

Proof. It is clear that under conditions of Theorem 4.2.3, Theorem 4.2.2 holds true for the semiflow $\{T(t)\}_{t \geq 0}$ on X . Furthermore Theorem 4.2.1 applies to $T = T_0$ and $\tilde{T} = T(t_0) - T_0$ provided that the conditions (i)-(vii) stated in Theorem 4.2.1 hold true for $T = T_0$ and $\tilde{T} = T(t_0) - T_0$. In the sequel the constants $\delta_1, \delta_2, \delta_3, \delta_4, \gamma_1$ and $\gamma_2 > 0$ are provided by the application of Theorem 4.2.1 as mentioned above.

In order to complete the proof we will make use of the property (v) stated in Theorem 4.2.1 for $T = T_0$ and $\tilde{T} = T(t_0) - T_0$.

Let $u \in C(\mathbb{R}_+, X)$ be a given positive orbit of $\{T(t)\}_{t \geq 0}$ such that

$$\|\Pi^h u(0)\| \leq \delta_4 \text{ and } \sup_{t \geq 0} \|\Pi^h u(t)\| \leq \frac{\delta_1}{2}. \quad (4.2.11)$$

The remaining part of the proof will be split into two steps. In the first step we prove the existence of a positive orbit of $\{T(t)\}_{t \geq 0}$ lying in $\tilde{M}$ such that (4.2.10) holds true. In the second step we prove its uniqueness.

Existence: Let $u \in C(\mathbb{R}_+, X)$ be a positive orbit of $\{T(t)\}_{t \geq 0}$ satisfying (4.2.11). Then by setting

$$u_0 := u(0)$$

one has

$$u(t) = T(t)u_0, \forall t \geq 0,$$

and due to (4.2.11) we have in particular that

$$\|\Pi^h u(0)\| \leq \delta_4 \text{ and } \sup_{n \geq 0} \|\Pi^h T(nt_0)u_0\| \leq \frac{\delta_1}{2}. \quad (4.2.12)$$

Hence since $\{T(nt_0)u_0\}_{n \in \mathbb{N}}$ is a positive orbit of $T(t_0)$, we infer from properties (v) and (iii) stated in Theorem 4.2.1 that there exists a unique $x \in M$ such that

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) + \Psi(\sigma_n(x)) - T(nt_0)u_0\| \leq \gamma_2 \delta_1, \quad (4.2.13)$$

where $\{\sigma_n(x)\}_{n \in \mathbb{Z}} \subset M$ solve of the system

$$x_{n+1} = \Pi^c T(t_0)(x_n + \Psi(x_n)), \quad \forall n \in \mathbb{Z}, \quad x_0 = x.$$

In particular $\{\sigma_n(x) + \Psi(\sigma_n(x))\}_{n \in \mathbb{Z}} \subset \tilde{M}$ is a complete orbit of $T(t_0)$ lying in $\tilde{M}$. Note that by setting

$$v_0 := \sigma_0(x) + \Psi(\sigma_0(x)) = x + \Psi(x) \in \tilde{M},$$

one obtains

$$\sigma_n(x) + \Psi(\sigma_n(x)) = T(nt_0)v_0, \quad \forall n \in \mathbb{N}. \quad (4.2.14)$$

Moreover since $v_0 = x + \Psi(x) \in \tilde{M}$ due to the condition (iii) stated in Theorem 4.2.2 there exists a unique complete orbit $v \in C(\mathbb{R}, \tilde{M})$ of $\{T(t)\}_{t \geq 0}$ with

$$v(0) = v_0 = x + \Psi(x) \in \tilde{M},$$

so that

$$v(t) = T(t)v_0 \in \tilde{M}, \quad t \geq 0 \implies v(nt_0) = T(nt_0)v_0 \in \tilde{M}, \quad \forall n \in \mathbb{N}.$$

Let us now show that the complete orbit $v \in C(\mathbb{R}, \tilde{M})$ of $\{T(t)\}_{t \geq 0}$ satisfies (4.2.10). To do so we first infer from (4.2.13) and (4.2.14) that

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|T(nt_0)v_0 - T(nt_0)u_0\| = \sup_{n \in \mathbb{N}} e^{\eta n} \|v(nt_0) - u(nt_0)\| \leq \gamma_2 \delta_1. \quad (4.2.15)$$

Then on the one hand for $l \in [0, t_0]$

$$\begin{aligned} \|u(l) - v(l)\| &= \|T(l)u_0 - T(l)v_0\| \\ &\leq \sup_{l \in [0, t_0]} \|T(l)\|_{Lip(X)} \|v_0 - u_0\| \\ &\leq \gamma(t_0) \|v_0 - u_0\|, \end{aligned}$$

so that due to (4.2.15) one obtains $\|v(0) - u(0)\| \leq \gamma_2 \delta_1$ and

$$\|u(l) - v(l)\| \leq \gamma(t_0) \gamma_2 \delta_1, \quad \forall l \in [0, t_0] \implies \sup_{l \in [0, t_0]} e^{\eta \frac{l}{t_0}} \|u(l) - v(l)\| \leq \gamma(t_0) \gamma_2 \delta_1 e^{\eta}. \quad (4.2.16)$$

On the other hand for each $t > t_0$ by writing that $t = nt_0 + l$ for some $n \in \mathbb{N}$ and $l \in [0, t_0]$ we infer from (4.2.15) that

$$\begin{aligned} \|u(t) - v(t)\| &= \|T(t)v_0 - T(t)u_0\| \\ &= \|T(l)T(nt_0)v_0 - T(l)T(nt_0)u_0\| \\ &\leq \sup_{l \in [0, t_0]} \|T(l)\|_{Lip(X)} \gamma_2 \delta_1 e^{-\eta n} \\ &\leq \gamma(t_0) \gamma_2 \delta_1 e^{-\eta \frac{t-l}{t_0}}, \end{aligned}$$

so that

$$\sup_{t > t_0} e^{\eta \frac{t}{t_0}} \|u(t) - v(t)\| \leq \gamma(t_0) \gamma_2 \delta_1 e^\eta. \quad (4.2.17)$$

Therefore (4.2.10) follows by combining (4.2.16) with (4.2.17).

Uniqueness: Let $v, \bar{v} \in C(\mathbb{R}, \tilde{M})$ be two complete orbits of $\{T(t)\}_{t \geq 0}$ with $v(0) = v_0$ and $\bar{v}(0) = \bar{v}_0$ such that for each $t \geq 0$

$$\|u(t) - v(t)\| \leq \gamma(t_0) \gamma_2 \delta_1 e^\eta e^{-\eta \frac{t}{t_0}} \text{ and } \|u(t) - \bar{v}(t)\| \leq \gamma(t_0) \gamma_2 \delta_1 e^\eta e^{-\eta \frac{t}{t_0}}, \quad (4.2.18)$$

where $u \in C(\mathbb{R}_+, X)$ satisfies (4.2.11).

Since for each $t \geq 0$, $T(t)$ is a bijection from $\tilde{M}$ into $\tilde{M}$, in order to prove that $v = \bar{v}$ we only need to prove that there exists $p \in \mathbb{N}$ such that $v(pt_0) = \bar{v}(pt_0)$.

Let $p \in \mathbb{N}$ be given. Consider the sequences $\{v(nt_0 + pt_0)\}_{n \in \mathbb{Z}}$, $\{\bar{v}(nt_0 + pt_0)\}_{n \in \mathbb{Z}}$ and $\{u(nt_0 + pt_0)\}_{n \in \mathbb{N}}$ parametrized by p . Observe that the sequences $\{v(nt_0 + pt_0)\}_{n \in \mathbb{Z}}$, $\{\bar{v}(nt_0 + pt_0)\}_{n \in \mathbb{Z}}$ are complete orbits of $T(t_0)$ lying in $\tilde{M}$ while the sequence $\{u(nt_0 + pt_0)\}_{n \in \mathbb{N}}$ is a positive orbit of $T(t_0)$.

Next for more convenience and to get back to the notations used in Theorem 4.2.1 we define

$$v(nt_0 + pt_0) := x_n + \Psi(x_n) = \sigma_n(x_0) + \Psi(\sigma_n(x_0)), \quad n \in \mathbb{Z},$$

and

$$\bar{v}(nt_0 + pt_0) := \bar{x}_n + \Psi(\bar{x}_n) = \sigma_n(\bar{x}_0) + \Psi(\sigma_n(\bar{x}_0)), \quad n \in \mathbb{Z}.$$

and observe that one has

$$v(pt_0) = \bar{v}(pt_0) \Leftrightarrow \sigma_0(x_0) = \bar{x}_0 = \bar{x}_0 = \sigma_0(\bar{x}_0).$$

The idea of the proof is to show that there exists $p \in \mathbb{N}$ such that $u(pt_0) \in \mathcal{W}_\eta^s(x_0, \delta_2) \cap \mathcal{W}_\eta^s(\bar{x}_0, \delta_2)$ and since the manifolds $\{\mathcal{W}_\eta^s(x, \delta_2)\}_{x \in M}$ are disjoint or equal this would lead us to $\bar{x}_0 = x_0$.

To do so we will make use of the property (iv) stated in Theorem 4.2.1. In fact by using (4.2.18) it follows that for each $n \geq 0$

$$\begin{aligned} \|u(nt_0 + pt_0) - \sigma_n(x_0) - \Psi(\sigma_n(x_0))\| &= \|u(nt_0 + pt_0) - v(nt_0 + pt_0)\| \\ &\leq \gamma(t_0) \gamma_2 \delta_1 e^{\eta e^{-\eta p}} e^{-\eta n}. \end{aligned} \quad (4.2.19)$$

Then since

$$\begin{aligned} \Pi^c u(nt_0 + pt_0) - \sigma_n(x_0) &= \Pi^c [u(nt_0 + pt_0) - \sigma_n(x_0) - \Psi(\sigma_n(x_0))], \\ \Pi^h u(nt_0 + pt_0) - \Psi(\sigma_n(x_0)) &= \Pi^h [u(nt_0 + pt_0) - \sigma_n(x_0) - \Psi(\sigma_n(x_0))] \end{aligned}$$

and

$$\begin{aligned} \Pi^h u(nt_0 + pt_0) - \Psi(\Pi^c u(nt_0 + pt_0)) &= \Pi^h [u(nt_0 + pt_0) - \Psi(\sigma_n(x_0))] + \\ &\quad \Psi(\sigma_n(x_0)) - \Psi(\Pi^c u(nt_0 + pt_0)), \end{aligned}$$

by using the fact that Ψ is Lipschitz continuous on M combined with (4.2.19) it can be easily obtained that for $p \in \mathbb{N}$ large enough the condition of (iv) stated in Theorem 4.2.1 are satisfied so that $u(pt_0) \in \mathcal{W}_\eta^s(x_0, \delta_2)$. By changing the role $x_0, \bar{x}_0$ one can similarly obtains that $u(pt_0) \in \mathcal{W}_\eta^s(\bar{x}_0, \delta_1)$. The result follows. ■

4.3 Asymptotic behavior

This section is devoted to the proof of Theorem 4.2.1. We decompose it into two subsections. In the first subsection we prove some perturbations results of exponential trichotomy. In the second subsection we provide the proof of Theorem 4.2.1.

4.3.1 Exponential trichotomy along pseudo-orbits

Recall that a sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ is called an ε -pseudo orbit of T with $\varepsilon > 0$ a non negative real value if

$$\|x_{n+1} - T(x_n)\| \leq \varepsilon, \forall n \in \mathbb{Z}.$$

Next we deal with the existence of exponential trichotomy, according to the Definition 3.2.1 in Chapter 3, along the pseudo-orbit of T . This result is very similar to that of Proposition 3.3.3 in Chapter 3 but we give here the proof for the sake of clarity.

Proposition 4.3.1 *Let Assumption 4.1.1 be satisfied. Let $\hat{\rho}_0, \hat{\rho}$ be given such that $0 < \rho_0 < \hat{\rho}_0 < \hat{\rho} < \rho$. There exists $\varepsilon > 0$ depending only on $\kappa, \rho, \rho_0, \hat{\rho}_0, \hat{\rho}, K_T$ with*

$$K_T := \max \left\{ \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))}, \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \right\},$$

and $\hat{\kappa} := \hat{\kappa}(\kappa, \rho, \rho_0, K_T) > \kappa^2$ with $\frac{\hat{\rho}}{\ln \hat{\kappa}} \in [0, 1]$ such that if $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ satisfies

$$\|x_{n+1} - T(x_n)\| \leq \varepsilon, \forall n \in \mathbb{Z},$$

then the following properties hold true:

- (i) For each $n \geq p$, the bounded linear operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ defined in (4.2.1) is invertible on M with inverse

$$U_{\mathbf{x}}^c(p, n) := U_{\mathbf{x}}^c(n, p)^{-1} : M \rightarrow M,$$

and

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \hat{\kappa} e^{\hat{\rho}_0 |n-p|}, \forall (n, p) \in \mathbb{Z}^2.$$

- (ii) The evolution semigroup $\{U_{\mathbf{x}}^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ has an exponential dichotomy on $\mathbb{Z}$ with constant $\hat{\kappa}$, exponent $\hat{\rho}$ and projectors $\Pi^\alpha(\mathbf{x}) = \{\Pi_n^\alpha(\mathbf{x})\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$, $\alpha = u, s$.

In order to prove Proposition 4.3.1, we will need the following result on exponential trichotomy along complete orbits of T .

Lemma 4.3.2 *Let Assumption 4.1.1 be satisfied. Then for each complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of T the following properties hold true:*

- (i) *For each $n \geq p$, the bounded linear operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ defined in (4.2.1) is invertible on M with inverse*

$$U_{\mathbf{x}}^c(p, n) := U_{\mathbf{x}}^c(n, p)^{-1} : M \rightarrow M,$$

and

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \kappa e^{\rho_0|n-p|}, \quad \forall (n, p) \in \mathbb{Z}^2.$$

- (ii) *The evolution semigroup $\{U_{\mathbf{x}}^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ has an exponential dichotomy on $\mathbb{Z}$ with constant κ^2 and exponent $\rho > \rho_0$ and projectors $\{\bar{\Pi}_{x_n}^\alpha\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$, $\alpha = u, s$.*

Proof. First remark that since $\mathbf{x}$ is a complete orbit of T , by the condition (H1) stated in Assumption 4.1.1 one has for each $n \in \mathbb{Z}$

$$\Pi_{T(x_n)}^\alpha DT(x_n) = DT(x_n) \Pi_{x_n}^\alpha \Leftrightarrow \Pi_{x_{n+1}}^\alpha DT(x_n) = DT(x_n) \Pi_{x_n}^\alpha, \quad \text{for } \alpha = u, s, \quad (4.3.1)$$

and

$$\Pi^c DT(x_n) = DT(x_n) \Pi^c. \quad (4.3.2)$$

On one hand note that by using the definition of $\{U_{\mathbf{x}}^c(n, p)\}_{n \geq p} \subset \mathcal{L}(M)$ in (4.2.1) one obtains

$$U_{\mathbf{x}}^c(n, p) = DT^{n-p}(x_p), \quad \forall n \geq p,$$

so that the property (i) stated in this lemma follows directly from the condition (H1)-(ii) stated in Assumption 4.1.1.

On the other hand recalling the definition of $\{U_{\mathbf{x}}^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ in (4.2.2) yields

$$U_{\mathbf{x}}^h(n, p) = DT^{n-p}(x_p), \quad \forall n \geq p,$$

so that by setting

$$\bar{\Pi}_{x_n}^\alpha y := \Pi_{x_n}^\alpha y, \quad \forall n \in \mathbb{Z}, \forall y \in X^h,$$

the properties for the dichotomy condition can be easily verified using the condition (H1)-(ii) stated in Assumption 4.1.1. ■

Now with the above lemma we are ready to complete the proof of Proposition 4.3.1.

Proof of Proposition 4.3.1. Before proceeding to our proof, let $\hat{\kappa}_0 > 0$ be the smallest positive value with

$$\hat{\kappa}_0 > \kappa^2 \geq \kappa \text{ and } \frac{\hat{\rho}}{\ln \hat{\kappa}_0} \in [0, 1].$$

Let us fix $m \in \mathbb{N}$ a positive integer large enough such that Theorem 2.2.3 in Ducrot, Magal and Seydi [58] (see Chapter 2) holds true with m and such that

$$0 < \frac{\ln \widehat{\kappa}_0}{m} < \frac{\widehat{\rho}_0 - \rho_0}{2}. \quad (4.3.3)$$

Proof of (i): Firstly note that one gets from Lemma 4.3.2 that for each complete orbit $\bar{x} = \{\bar{x}_n\}_{n \in \mathbb{Z}} \subset M$ of T , $\{DT(\bar{x}_n) \Pi^c\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ , exponents ρ and $\rho_0 \in (0, \rho)$ associated to the three families of projectors $\{\bar{\Pi}_{\bar{x}_n}^\alpha\}_{n \in \mathbb{Z}}$, $\alpha = u, s, c$ such that

$$\bar{\Pi}_{\bar{x}_n}^c = I_{\mathcal{L}(M)} \text{ and } \bar{\Pi}_{\bar{x}_n}^u = \bar{\Pi}_{\bar{x}_n}^s = 0_{\mathcal{L}(M)}, \quad \forall n \in \mathbb{Z}.$$

Let $\varepsilon > 0$ be given. Let $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ be given satisfying

$$\|x_{n+1} - T(x_n)\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}. \quad (4.3.4)$$

Define the family of intervals $\{\mathbb{I}_i\}_{i \in \mathbb{Z}}$ of $\mathbb{Z}$ as

$$\mathbb{I}_i := [im, (i+1)m], \quad \forall i \in \mathbb{Z}.$$

Next on each interval $\mathbb{I}_i$, we will rewrite $\{DT(x_n) \Pi^c\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ as a small perturbation of a sequence $\{B_n^i\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ which is exponentially trichotomic on $\mathbb{Z}$. To do so recall that T is invertible on M and define for each $i \in \mathbb{Z}$

$$B_n^i := DT(T^{n-im}(x_{im})) \Pi^c, \quad \forall n \in \mathbb{Z}.$$

Since for each $i \in \mathbb{Z}$

$$\mathbf{x}^i := \{x_n^i\}_{n \in \mathbb{Z}} = \{T^{n-im}(x_{im})\}_{n \in \mathbb{Z}},$$

is a complete orbit of T we know that for any $i \in \mathbb{Z}$, $\{B_n^i\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ , exponents ρ and $\rho_0 \in (0, \rho)$ associated to the three families of projectors $\{\Pi_n^\alpha(\mathbf{x}^i)\}_{n \in \mathbb{Z}}$, $\alpha = u, s, c$ defined by

$$\Pi_n^c(\mathbf{x}^i) = I_{\mathcal{L}(M)} \text{ and } \Pi_n^u(\mathbf{x}^i) = \Pi_n^s(\mathbf{x}^i) = 0_{\mathcal{L}(M)}, \quad \forall n \in \mathbb{Z}.$$

Let us now show that for each $i \in \mathbb{Z}$ the sequence of operators $\{DF(x_n) \Pi^c\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is close to $\{B_n^i\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ on $\mathbb{I}_i$.

We first observe that for each $i \in \mathbb{Z}$ one has

$$\begin{aligned} x_{im+2}^i - x_{im+2} &= T^2(x_{im}) - x_{im+2} \\ &= [T(T(x_{im})) - T(x_{im+1})] + [T(x_{im+1}) - x_{im+2}] \\ &= \int_0^1 DT(lF(x_{im}) + (1-l)x_{im+1}) dl \cdot (T(x_{im}) - x_{im+1}) \\ &\quad + T(x_{im+1}) - x_{im+2}. \end{aligned}$$

Next recalling the notation

$$K_T := \max \left\{ \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))}, \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \right\}, \quad (4.3.5)$$

we infer from (4.3.4) that

$$\|x_{im+2}^i - x_{im+2}\| \leq \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \varepsilon + \varepsilon \leq K_T \varepsilon + \varepsilon.$$

By induction one obtains for each $k \in [0, m]$, $i \in \mathbb{Z}$:

$$\|x_{im+k}^i - x_{im+k}\| \leq (1 + K_T + \dots + K_T^{k-1}) \varepsilon \leq (1 + K_T + \dots + K_T^m) \varepsilon.$$

Thus one gets for each $k \in [0, m]$

$$\begin{aligned} \|B_{im+k}^i - DT(x_{im+k})\|_{\mathcal{L}(M)} &= \|DT(x_{im+k}^i) \Pi^c - DT(x_{im+k}) \Pi^c\|_{\mathcal{L}(M)} \\ &\leq \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|\Pi^c\|_{\mathcal{L}(M)} (1 + K_T + \dots + K_T^m) \varepsilon \\ &\leq K_T (1 + K_T + \dots + K_T^m) \varepsilon, \end{aligned}$$

so that for each $i \in \mathbb{Z}$

$$\sup_{k \in [im, (i+1)m]} \|B_k^i - DT(x_k)\|_{\mathcal{L}(X)} \leq K_T (1 + K_T + \dots + K_T^m) \varepsilon. \quad (4.3.6)$$

Now by observing that we have

$$\widehat{\kappa}_0 > \kappa \text{ and } 0 < \rho_0 < \frac{\rho_0 + \widehat{\rho}_0}{2} < \frac{\rho + \widehat{\rho}}{2} < \rho,$$

one can choose ε small enough depending only on $\kappa, m, \rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, K_T$ so that Corollary 2.5.3 in Ducrot, Magal and Seydi [58] (see Chapter 2) applies on each interval $\mathbb{I}_i := [im, (i+1)m]$, $i \in \mathbb{Z}$ and provides that for each $i \in \mathbb{Z}$, $\{DT(x_n) \Pi^c\}_{n \in \mathbb{Z}} \subset \mathcal{L}(M)$ is exponentially trichotomic on each $\mathbb{I}_i$ with constant $\widehat{\kappa}_0$, exponents $\frac{\rho_0 + \widehat{\rho}_0}{2}$ and $\frac{\rho + \widehat{\rho}}{2}$ associated to the three families of projectors $\{\Pi_n^{i\alpha}(\mathbf{x})\}_{n \in \mathbb{I}_i}$, $\alpha = u, s, c$. Moreover since we have the isomorphism properties between the projectors $\{\Pi_n^{i\alpha}(\mathbf{x})\}_{n \in \mathbb{I}_i}$ and $\{\Pi_n^\alpha(\mathbf{x}^i)\}_{n \in \mathbb{I}_i}$, $\alpha = u, s, c$ we get for each $n \in \mathbb{I}_i$

$$\Pi_n^{i\alpha}(\mathbf{x}) = \Pi_n^c(\mathbf{x}^i) = I_{\mathcal{L}(M)},$$

and

$$\Pi_n^u(\mathbf{x}^i) = \Pi_n^s(\mathbf{x}^i) = \Pi_n^{iu}(\mathbf{x}) = \Pi_n^{is}(\mathbf{x}) = 0_{\mathcal{L}(M)}.$$

One can summarize what we have proved as follow: On each interval $\mathbb{I}_i = [im, (i+1)m]$, $i \in \mathbb{Z}$, we have

$$U_{\mathbf{x}}^c(n, p) : M \rightarrow M,$$

is invertible for each $n \geq p$ in $\mathbb{I}_i$ with inverse $U_{\mathbf{x}}^c(p, n) : M \rightarrow M$ and

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \widehat{\kappa}_0 e^{-\frac{\rho_0 + \widehat{\rho}_0}{2}|n-p|}, \quad \forall n, p \in \mathbb{I}_i = [im, (i+1)m]. \quad (4.3.7)$$

Therefore it is clear that for each $n, p \in \mathbb{Z}$ with $n \geq p$ the operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ is invertible with inverse $U_{\mathbf{x}}^c(p, n) : M \rightarrow M$. It remains now to give an estimates of $\{U_{\mathbf{x}}^c(n, p)\}_{n \geq p} \subset \mathcal{L}(M)$ for n, p varying in $\mathbb{Z}$.

Now let $n, p \in \mathbb{Z}$ be given such that $n > p$. Then there exist $i, j \in \mathbb{Z}$ with $i > j$ such that $n \in [im, (i+1)m]$ and $p \in [jm, (j+1)m]$ so that by using (4.3.7) we obtain

$$\begin{aligned} \|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} &= \|U_{\mathbf{x}}^c(n, im) U_{\mathbf{x}}^c(im, jm) U_{\mathbf{x}}^c(jm, p)\|_{\mathcal{L}(M)} \\ &\leq \|U_{\mathbf{x}}^c(n, im)\|_{\mathcal{L}(M)} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(M)} \|U_{\mathbf{x}}^c(jm, p)\|_{\mathcal{L}(M)} \\ &\leq \hat{\kappa}_0 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-im)} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(M)} \hat{\kappa}_0 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(jm-p)} \\ &\leq \hat{\kappa}_0^2 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p)} e^{\frac{\rho_0 + \hat{\rho}_0}{2}(jm-im)} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(M)}, \end{aligned}$$

and since we have

$$\begin{aligned} \|U_{\mathbf{x}}^c(im, jm)\|_{\mathcal{L}(M)} &= \|U_{\mathbf{x}}^c(im, (i-1)m) \dots U_{\mathbf{x}}^c((j-1)m, jm)\|_{\mathcal{L}(M)} \\ &\leq \hat{\kappa}_0^{(i-j)} e^{\frac{\rho_0 + \hat{\rho}_0}{2}(im-jm)}, \end{aligned}$$

it follows that

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \hat{\kappa}_0 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p)} \hat{\kappa}_0^{(i-j+1)}. \quad (4.3.8)$$

Furthermore by observing that

$$im - jm + m \leq n - p,$$

we obtain from (4.3.8) that

$$\begin{aligned} \|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} &\leq \hat{\kappa}_0 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p)} \hat{\kappa}_0^{\frac{n-p}{m}} \\ &\leq \hat{\kappa}_0 e^{\frac{\rho_0 + \hat{\rho}_0}{2}(n-p) + (n-p) \frac{\ln \hat{\kappa}_0}{m}}. \end{aligned}$$

Finally we infer from (4.3.3) that

$$\|U_{\mathbf{x}}^c(n, p)\|_{\mathcal{L}(M)} \leq \hat{\kappa}_0 e^{\hat{\rho}_0(n-p)}. \quad (4.3.9)$$

We also have in a similar manner that for each $n, p \in \mathbb{Z}$ with $n \geq p$

$$\|U_{\mathbf{x}}^c(p, n)\|_{\mathcal{L}(M)} \leq \hat{\kappa}_0 e^{\hat{\rho}_0(n-p)}. \quad (4.3.10)$$

This completes the proof of (i).

Proof of (ii): The idea of the proof is the same as in the proof of (i). By proceeding as in the proof of (i) one can easily prove that for ε small enough depending only on $\kappa, m, \rho_0, \hat{\rho}_0, \hat{\rho}, \rho$ and K_T (defined in (4.3.5)) we have for each sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}} \subset M$ satisfying

$$\|x_{n+1} - T(x_n)\| \leq \varepsilon, \quad \forall n \in \mathbb{Z},$$

the sequence $\{DF(x_n)\Pi^h\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ is exponentially dichotomic on each interval $[im, (i+1)m]$, $i \in \mathbb{Z}$ with uniform constant $\kappa + 1$ and exponent $\frac{\rho + \hat{\rho}}{2}$. Then observing that

$$0 < \hat{\rho} < \frac{\rho + \hat{\rho}}{2},$$

we apply Theorem 2.2.3 in Ducrot, Magal and Seydi [58] (see Chapter 2) to obtain that $\{DF(x_n)\Pi^h\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ is exponentially dichotomic on $\mathbb{Z}$ with constant $\tilde{\kappa} := \tilde{\kappa}(K_T, \kappa, \rho, \hat{\rho})$ and exponent $\hat{\rho}$. This completes the proof of (ii).

To finish the proof of our result we just note that by setting

$$\hat{\kappa} := \max\{\tilde{\kappa}, \hat{\kappa}_0\} \implies \frac{\hat{\rho}}{\ln \hat{\kappa}} \in [0, 1],$$

the estimates (4.3.9)-(4.3.10) remain true and $\{DF(x_n)\Pi^h\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ is again exponentially dichotomic on $\mathbb{Z}$ with constant $\hat{\kappa}$ and exponent $\hat{\rho}$. ■

4.3.2 Proof of Theorem 4.2.1

The proof of Theorem 4.2.1 will be given along several technical lemmas. Before proceeding, let us introduce the following notation:

Notation 4.3.3 For convenience for the next argument we introduce for each $\eta \geq 0$ the set

$$\mathbb{E}_\eta(\mathbb{Z}, M) := \{\mathbf{x} \in \mathbb{L}_\eta(\mathbb{Z}, M) : \text{(A-1) and (A-2) are satisfied}\},$$

where

(A-1) for each $n \geq p$, the operator $U_{\mathbf{x}}^c(n, p) : M \rightarrow M$ is invertible with inverse $U_{\mathbf{x}}^c(p, n)$ and

$$\|U_{\mathbf{x}}^c(n, p)y\| \leq \hat{\kappa}e^{\hat{\rho}_0|n-p|}\|y\|, \quad \forall y \in M, \quad \forall (n, p) \in \mathbb{Z}^2. \quad (4.3.11)$$

(A-2) $\{U_{\mathbf{x}}^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ has an exponential dichotomy on $\mathbb{Z}$ with constant $\hat{\kappa}$ and exponents $\hat{\rho} > \hat{\rho}_0$ associated with projectors $\{\Pi_n^\alpha(\mathbf{x})\}_{n \in \mathbb{Z}}$, $\alpha = u, s$.

Recall that the condition **(A-2)** implies that

$$\Pi_n^u(\mathbf{x}) + \Pi_n^s(\mathbf{x}) = I_{\mathcal{L}(X^h)}, \quad \forall n \in \mathbb{Z},$$

$$U_{\mathbf{x}}^\alpha(n, p) := \Pi_n^\alpha(\mathbf{x})U_{\mathbf{x}}^h(n, p) = U_{\mathbf{x}}^h(n, p)\Pi_p^\alpha(\mathbf{x}), \quad \forall n \geq p, \quad \alpha = u, s,$$

and $U_{\mathbf{x}}^u(n, p)$ is invertible from $\mathcal{R}(\Pi_p^u(\mathbf{x}))$ into $\mathcal{R}(\Pi_n^u(\mathbf{x}))$ with inverse

$$U_{\mathbf{x}}^u(p, n) : \mathcal{R}(\Pi_n^u(\mathbf{x})) \rightarrow \mathcal{R}(\Pi_p^u(\mathbf{x})),$$

while the growth estimates are given as

$$\|U_{\mathbf{x}}^s(n, p)\Pi_n^u(\mathbf{x})y\| \leq \hat{\kappa}e^{-\hat{\rho}(n-p)}\|y\|, \quad \forall y \in X^h, \quad (4.3.12)$$

and

$$\|U_{\mathbf{x}}^u(p, n)\Pi_n^u(\mathbf{x})y\| \leq \hat{\kappa}e^{-\hat{\rho}(n-p)}\|y\|, \quad \forall y \in X^h. \quad (4.3.13)$$

Remark 4.3.4 Observe if $\mathbf{x} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ then due to (4.3.12) and (4.3.13) one has

$$\|U_{\mathbf{x}}^u(n, n) \Pi_n^\alpha(\mathbf{x})\|_{\mathcal{L}(X^h)} = \|\Pi_n^\alpha(\mathbf{x})\|_{\mathcal{L}(X^h)} \leq \hat{\kappa}, \quad \forall n \in \mathbb{Z} \text{ and } \alpha = u, s. \quad (4.3.14)$$

Remark 4.3.5 From now on we will fix the exponents $\hat{\rho}_0 > 0, \hat{\rho} > 0$ and $\eta > 0$ such that

$$0 < \rho_0 < \hat{\rho}_0 < \eta < \hat{\rho} < \rho. \quad (4.3.15)$$

Lemma 4.3.6 Let Assumption 4.1.1 be satisfied. Then there exists $\hat{\delta} \in (0, \min(\delta, \varepsilon))$ with ε given in Proposition 4.3.1 and δ given in Theorem 4.1.2 such that if

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \hat{\delta}^2,$$

then for each $x \in M$, the complete orbit $\sigma(x) = \{\sigma_n(x)\}_{n \in \mathbb{Z}} \subset M$ of (4.2.3) belongs to $\mathbb{E}_\eta(\mathbb{Z}, M)$.

Proof. In order to prove this lemma we will make use of Proposition 4.3.1. Let $x \in M$ be given. Let us set for notational simplicity

$$x_n := \sigma_n(x), \quad \forall n \in \mathbb{Z}.$$

Thus recalling that the sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ satisfies

$$x_{n+1} = \Pi^c T(x_n + \Psi(x_n)) + \Pi^c \tilde{T}(x_n + \Psi(x_n)), \quad \forall n \in \mathbb{Z},$$

we obtain for each $n \in \mathbb{Z}$

$$\begin{aligned} x_{n+1} - T(x_n) &= \Pi^c [T(x_n + \Psi(x_n)) - T(x_n)] + \Pi^c \tilde{T}(x_n + \Psi(x_n)) \\ &= \Pi^c \int_0^1 DT(x_n + l\Psi(x_n)) dl \cdot \Psi(x_n) \\ &\quad + \Pi^c \tilde{T}(x_n + \Psi(x_n)) \end{aligned}$$

so that

$$\begin{aligned} \|x_{n+1} - T(x_n)\| &\leq \|\Pi^c\|_{\mathcal{L}(X)} \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \sup_{x \in M} \|\Psi(x_n)\| \\ &\quad + \|\Pi^c\|_{\mathcal{L}(X)} \|\tilde{T}\|_{C(X)}. \end{aligned}$$

Since we have from Theorem 4.1.2 that

$$\sup_{x \in M} \|\Psi(x_n)\| \leq \frac{6}{5} \gamma \|\tilde{T}\|_{C(X)},$$

with $\gamma > 0$ a positive constant, we obtain that for each $n \in \mathbb{Z}$

$$\begin{aligned} \|x_{n+1} - T(x_n)\| &\leq \|\Pi^c\|_{\mathcal{L}(X)} \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \frac{6}{5} \gamma \|\tilde{T}\|_{C(X)} \\ &\quad + \|\Pi^c\|_{\mathcal{L}(X)} \|\tilde{T}\|_{C(X)}. \end{aligned}$$

The result follows by taking $\hat{\delta}$ small enough such that

$$\left[\|\Pi^c\|_{\mathcal{L}(X)} \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \frac{6}{5} \gamma + \|\Pi^c\|_{\mathcal{L}(X)} \right] \hat{\delta} < 1 \text{ and } \hat{\delta} < \varepsilon.$$

■

We also make the following remark.

Remark 4.3.7 Under the condition of the above Lemma 4.3.6, the projectors $\{\Pi_n^\alpha(\sigma(x))\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X^h)$ associated to the exponential dichotomy of $\{U_{\sigma(x)}^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ satisfy

$$\Pi_n^\alpha(\sigma(\sigma_p(x))) = \Pi_{n+p}^\alpha(\sigma(x)), \quad \forall n, p \in \mathbb{Z} \text{ and } \alpha = u, s, c. \quad (4.3.16)$$

Indeed by using the fact that

$$\sigma_n(\sigma_p(x)) = \sigma_{n+p}(x), \quad \forall n, p \in \mathbb{Z},$$

the equality (4.3.16) follows directly from Lemma 1.1.7 in Chapter 1 on the uniqueness of the projectors. Furthermore let us also notice that again by Lemma 1.1.7 in Chapter 1 we have

$$E_x^\alpha = \mathcal{R}[\Pi_0^\alpha(\sigma(x))], \quad \forall x \in M \text{ and } \alpha = u, s. \quad (4.3.17)$$

and

$$E^\alpha = \{x + y : y \in \mathcal{R}[\Pi_0^\alpha(\sigma(x))]\}, \quad \alpha = u, s,$$

where $\mathcal{R}[\Pi_0^\alpha(\sigma(x))]$ is the range of $\Pi_0^\alpha(\sigma(x))$ while E_x^α , $\alpha = u, s$ and E^α are defined respectively in (4.2.4)-(4.2.5) and in (4.2.6)-(4.2.7).

With the above material we are now ready to argue for the more technical part of the proof of Theorem 4.1.2.

To do so we need to bring the system (4.1.2) into a suitable system of coordinates. In fact by projecting (4.1.2) successively along M and X^h we obtain the following equivalent system

$$\begin{cases} x_{n+1} = \Pi^c(T + \tilde{T})(x_n + y_n), & n \in \mathbb{N} \\ y_{n+1} = \Pi^h(T + \tilde{T})(x_n + y_n), & n \in \mathbb{N}, \end{cases} \quad (4.3.18)$$

wherein $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}} \subset M$ and $\mathbf{y} = \{y_n\}_{n \in \mathbb{N}} \subset X^h$. Next we make the following change of variables

$$z_n = y_n - \Psi(x_n), \quad \forall n \in \mathbb{N},$$

so that the sequence $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ satisfies the system

$$\begin{cases} x_{n+1} = \Pi^c \left(T + \tilde{T} \right) (x_n + z_n + \Psi(x_n)) \\ z_{n+1} = \Pi^h \left(T + \tilde{T} \right) (x_n + z_n + \Psi(x_n)) \\ \quad - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (x_n + z_n + \Psi(x_n)) \right). \end{cases} \quad (4.3.19)$$

Now let $\eta > 0$ defined in (4.3.15). We define a **stable leaf** with respect to the complete orbit $\sigma(x) = \{\sigma_n(x)\}_{n \in \mathbb{Z}} \subset M$, $\sigma_0(x) = x \in M$ of (4.2.3) as follow

$$W^s(x, r) := \left\{ (x_0, z_0) \in M \times X^h : (\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}} \text{ is the solution of (4.3.19)} \right. \\ \left. \begin{array}{l} \text{with } \|\mathbf{x} - \sigma(x)\|_{\eta,+} = \sup_{n \in \mathbb{N}} e^{\eta n} \|x_n - \sigma_n(x)\| < r \\ \text{and } \|\mathbf{z}\|_{\eta,+} = \sup_{n \in \mathbb{N}} e^{\eta n} \|z_n\| < r \end{array} \right\} \quad (4.3.20)$$

wherein $r > 0$ is a positive real parameter.

Remark 4.3.8 Note that the invariance property of $\tilde{M}$ stated in Theorem 4.1.2 can be characterized by the equality

$$\Pi^h \left(T + \tilde{T} \right) (x + \Psi(x)) = \Psi \left(\Pi^c \left(T + \tilde{T} \right) (x + \Psi(x)) \right), \quad \forall x \in M, \quad (4.3.21)$$

so that $W^s(x, r)$ is not empty since it always contains the point $(x, 0)$ due to the fact that the sequence $(\sigma(x), \mathbf{0}) = \{(\sigma_n(x), 0)\}_{n \in \mathbb{Z}}$ solves the system (4.3.19).

In order to characterize the elements $W^s(x, r)$, $x \in M$, $r > 0$, let us now notice that $(x_0, z_0) \in W^s(x, r)$ with $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ the associated solution of (4.3.19) if and only if there exists

$$\mathbf{w} = \{w_n\}_{n \in \mathbb{N}} \subset M, \quad \|\mathbf{w}\|_{\eta,+} = \sup_{n \in \mathbb{N}} e^{\eta n} \|w_n\| \leq r,$$

and

$$\mathbf{z} = \{z_n\}_{n \in \mathbb{N}} \subset X^h, \quad \|\mathbf{z}\|_{\eta,+} = \sup_{n \in \mathbb{N}} e^{\eta n} \|z_n\| < r,$$

such that $\mathbf{x} = \sigma(x) + \mathbf{w}$ and $(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}}$ satisfies the system

$$\begin{cases} w_{n+1} = DT(\sigma_n(x)) w_n + G_1(\sigma_n(x), w_n, z_n), \quad n \in \mathbb{N}, \\ z_{n+1} = DT(\sigma_n(x)) z_n + G_2(\sigma_n(x), w_n, z_n), \quad n \in \mathbb{N}, \end{cases} \quad (4.3.22)$$

where the maps $G_1 : M \times M \times X^h \rightarrow M$ and $G_2 : M \times M \times X^h \rightarrow X^h$ are given for each $(v, w, z) \in M \times M \times X^h$ by

$$\begin{aligned} G_1(v, w, z) &:= \Pi^c \left(T + \tilde{T} \right) (w + v + z + \Psi(w + v)) \\ &\quad - \Pi^c \left(T + \tilde{T} \right) (v + \Psi(v)) - DT(v)w, \end{aligned} \quad (4.3.23)$$

and

$$\begin{aligned} G_2(v, w, z) &:= \Pi^h \left(T + \tilde{T} \right) (w + v + z + \Psi(v)) \\ &\quad - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (w + v + z + \Psi(w + v)) \right) - DT(v)z. \end{aligned} \quad (4.3.24)$$

In order to simplify the notation in the sequel we define for $Y = M, X^h$ the closed ball of $\mathbb{L}_\eta(\mathbb{N}, Y)$, $\eta \geq 0$, centered at $\mathbf{0} \in \mathbb{L}_\eta(\mathbb{N}, Y)$, with radius $r > 0$ by

$$\mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, Y)}(\mathbf{0}, r) := \left\{ \mathbf{y} \in \mathbb{L}_\eta(\mathbb{N}, Y) : \|\mathbf{y}\|_{\eta,+} \leq r \right\}.$$

The next lemma will allow us to rewrite the above system (4.3.22) in term of a fixed point problem.

Lemma 4.3.9 *Let Assumption 4.1.1 be satisfied. Let $\hat{\eta} \in [0, \eta]$ with η defined in (4.3.15). Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}} \in \mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)$ be given. Then $(\mathbf{w}, \mathbf{z})$ satisfies the system*

$$\begin{cases} w_{n+1} = DT(v_n)w_n + G_1(v_n, w_n, z_n), & n \in \mathbb{N}, \\ z_{n+1} = DT(v_n)z_n + G_2(v_n, w_n, z_n), & n \in \mathbb{N}, \end{cases} \quad (4.3.25)$$

if and only if there exists $z_0^s \in \mathcal{R}(\Pi_0^s(\mathbf{v}))$ with $z_0^s = \Pi_0^s(\mathbf{v})z_0$ such that

$$w_n = - \sum_{k=n}^{+\infty} U_v^c(n, k+1) G_1(v_k, w_k, z_k), \quad \forall n \in \mathbb{N}, \quad (4.3.26)$$

and

$$z_n = U_v^s(n, 0)z_0^s + \sum_{k=0}^{n-1} U_v^s(n, k+1) G_2(v_k, w_k, z_k) - \sum_{k=n}^{+\infty} U_v^u(n, k+1) G_1(v_k, w_k, z_k), \quad \forall n \in \mathbb{N}. \quad (4.3.27)$$

Proof. Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}} \in \mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)$ be given. Assume that $(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}}$ satisfies (4.3.25). By using the variation of constants formula combined with the fact that $\{(w_n, z_n)\}_{n \in \mathbb{N}} \subset M \times X^h$ one has from the definition of $\{U_v^c(n, p)\}_{n \geq p} \subset \mathcal{L}(M)$ and $\{U_v^h(n, p)\}_{n \geq p} \subset \mathcal{L}(X^h)$ that

$$w_n = U_v^c(n, p)w_p + \sum_{k=p}^{n-1} U_v^c(n, k+1) G_1(v_k, w_k, z_k), \quad \forall n \geq p \geq 0, \quad (4.3.28)$$

and

$$z_n = U_v^h(n, p) z_p + \sum_{k=p}^{n-1} U_v^h(n, k+1) G_1(v_k, w_k, z_k), \quad \forall n \geq p \geq 0. \quad (4.3.29)$$

Next on the one hand by multiplying the left side of (4.3.28) by $U_v^c(p, n)$, $n \geq p$ we deduce that

$$U_v^c(p, n) w_n = w_p + \sum_{k=p}^{n-1} U_v^c(p, k+1) G_1(w_k, z_k, x_k), \quad \forall n \geq p \geq 0. \quad (4.3.30)$$

On the other hand by multiplying the left side of (4.3.29) by $U_v^u(p, n)$ we deduce that

$$U_v^u(p, n) z_n = \Pi_p^u(v) z_p + \sum_{k=p}^{n-1} U_v^u(p, k+1) G_1(v_k, w_k, z_k), \quad \forall n \geq p \geq 0. \quad (4.3.31)$$

Now observe that due to (4.3.11) and (4.3.13) we have

$$\|U_v^c(p, n) w_n\| \leq \hat{\kappa} e^{\hat{\rho}_0(n-p)} e^{-\eta n} \|\mathbf{w}\|_{\eta,+}, \quad \forall n \geq p \geq 0,$$

and

$$\|U_v^u(p, n) z_n\| \leq \hat{\kappa} e^{-\hat{\rho}(n-p)} e^{-\hat{\eta} n} \|\mathbf{z}\|_{\hat{\eta},+}, \quad \forall n \geq p \geq 0,$$

so that by letting n goes to $+\infty$ in (4.3.30)-(4.3.31) it follows that

$$w_p = - \sum_{k=p}^{+\infty} U_v^c(p, k+1) G_1(v_k, w_k, z_k), \quad \forall p \geq 0, \quad (4.3.32)$$

and

$$\Pi_p^u(v) z_p = - \sum_{k=p}^{+\infty} U_v^u(p, k+1) G_1(v_k, w_k, z_k), \quad \forall p \geq 0. \quad (4.3.33)$$

Finally by multiplying the left side of (4.3.29) by $\Pi_n^s(v)$ and let $p = 0$ we obtain

$$\Pi_n^s(v) z_n = U_v^s(n, 0) z_0 + \sum_{k=0}^{n-1} U_v^s(n, k+1) G_1(v_k, w_k, z_k), \quad \forall n \in \mathbb{N}. \quad (4.3.34)$$

Therefore since

$$\Pi_n^s(v) z_n + \Pi_n^u(v) z_n = z_n, \quad \forall n \in \mathbb{N}$$

the system (4.3.26)-(4.3.27) follows from (4.3.32)-(4.3.34) with $z_0^s = \Pi_0^s(v) z_0$. The converse property follows directly by multiplying the left side of (4.3.26) and (4.3.27) respectively by $DT(v_n) \Pi^c$ and $DT(v_n) \Pi^h$, $n \in \mathbb{N}$. ■

Inspired by the above Lemma 4.3.9 we define for each $v \in \mathbb{E}_\eta(\mathbb{Z}, M)$, $\mathbf{w} = \{w_n\}_{n \in \mathbb{N}} \subset M$ and $\mathbf{z} = \{z_n\}_{n \in \mathbb{N}} \subset X^h$ the operator $\mathcal{K}^c$ by

$$\mathcal{K}^c(v, \mathbf{w})_n := - \sum_{k=n}^{+\infty} U_v^c(n, k+1) w_k, \quad \forall n \in \mathbb{N}, \quad (4.3.35)$$

the operators $\mathcal{K}^h$ by

$$\mathcal{K}^h(\mathbf{v}, \mathbf{z})_n := \sum_{k=0}^{n-1} U_{\mathbf{v}}^s(n, k+1) z_k - \sum_{k=n}^{+\infty} U_{\mathbf{v}}^u(n, k+1) z_k, \quad \forall n \in \mathbb{N}, \quad (4.3.36)$$

and the operators $\mathcal{S}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$\mathcal{S}(\mathbf{v}, y)_n := U_{\mathbf{v}}^s(n, 0) \Pi_0^s(\mathbf{v}) y, \quad \forall n \in \mathbb{N} \text{ and } y \in X^h, \quad (4.3.37)$$

and

$$\mathcal{G}_1(\mathbf{v}, \mathbf{z}, \mathbf{w})_n := G_1(v_n, w_n, z_n) \text{ and } \mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n := G_2(v_n, w_n, z_n), \quad \forall n \in \mathbb{N}. \quad (4.3.38)$$

As a consequence of the above Lemma 4.3.9 we have

Lemma 4.3.10 (fixed point formulation) *Let Assumption 4.1.1 be satisfied. Assume moreover that*

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \hat{\delta}^2,$$

with $\hat{\delta}$ given in Lemma 4.3.6. Let $x \in M$ be given. Then $(x_0, z_0) \in W^s(x, r)$, $r > 0$, with $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ the associated solution of (4.3.19) if and only if there exists

$$(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, r)$$

such that

(i) The sequence $\mathbf{x}$ is given by $\mathbf{x} = \boldsymbol{\sigma}(x) + \mathbf{w}$

(ii) $(\mathbf{w}, \mathbf{z})$ satisfies the fixed point problem

$$\begin{cases} \mathbf{w} = \mathcal{K}^c(\boldsymbol{\sigma}(x), \mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w})) \\ \mathbf{z} = \mathcal{S}(\boldsymbol{\sigma}(x), z_0^s) + \mathcal{K}^h(\boldsymbol{\sigma}(x), \mathcal{G}_2(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w})) \\ \Pi_0^s(\boldsymbol{\sigma}(x)) z_0 = z_0^s. \end{cases} \quad (4.3.39)$$

In the sequel we will investigate on the fixed point problem (4.3.39) in $\mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, r)$ for some $r > 0$ small enough. To do so we will first prove some properties related to the operators $\mathcal{K}^c, \mathcal{K}^h, \mathcal{S}, \mathcal{G}_1, \mathcal{G}_2$ defined in (4.3.35)-(4.3.38).

Lemma 4.3.11 *Let Assumption 4.1.1. Then the operators $\mathcal{K}^c$ is a well defined from $\mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_\eta(\mathbb{N}, M)$ into $\mathbb{L}_\eta(\mathbb{N}, M)$. Moreover there exists a constant*

$$C_c := \hat{\kappa} \frac{e^{\hat{\rho}_0}}{1 - e^{\hat{\rho}_0 - \eta}}, \quad (4.3.40)$$

(with $\hat{\rho}_0, \hat{\rho}, \eta$ defined in (4.3.15)) such that for each $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{L}_\eta(\mathbb{N}, M)$ we have

$$\|\mathcal{K}^c(\mathbf{v}, \mathbf{w}) - \mathcal{K}^c(\mathbf{v}, \bar{\mathbf{w}})\|_{\eta, +} \leq C_c \|\mathbf{w} - \bar{\mathbf{w}}\|_{\eta, +}. \quad (4.3.41)$$

Proof. Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{w} = \{w_n\}_{n \in \mathbb{N}} \subset M$ be given. Then for each $n \in \mathbb{N}$ one has

$$\begin{aligned} e^{\eta n} \|\mathcal{K}_n^c(\mathbf{v}, \mathbf{w})\| &\leq \sum_{k=n}^{+\infty} e^{\eta n} \|U_{\mathbf{v}}^c(n, k+1) w_k\| \\ &\leq \sum_{k=n}^{+\infty} e^{\eta n} \widehat{\kappa} e^{\widehat{\rho}_0(k+1-n)} \|\mathbf{w}\|_{\widehat{\eta},+} e^{-\eta k} \\ &\leq \widehat{\kappa} \frac{e^{\widehat{\rho}_0}}{1 - e^{\widehat{\rho}_0 - \eta}} \|\mathbf{w}\|_{\eta,+} \end{aligned}$$

The result follows by observing that $\mathcal{K}^c(\mathbf{v}, \cdot)$ is a bounded linear operator. $\blacksquare$

Lemma 4.3.12 *Let Assumption 4.1.1 be satisfied. Let $\widehat{\eta} \in [0, \eta]$ with η defined in (4.3.15). Then the operators $\mathcal{K}^h$ is well defined from $\mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{L}_{\widehat{\eta}}(\mathbb{N}, X^h)$ into $\mathbb{L}_{\widehat{\eta}}(\mathbb{N}, X^h)$. Moreover there exists a constant*

$$C_h := \widehat{\kappa} \frac{e^\eta}{1 - e^{\eta - \widehat{\rho}}} + \widehat{\kappa} \frac{e^{-\widehat{\rho}}}{1 - e^{-\widehat{\rho}}}, \quad (4.3.42)$$

(with $\widehat{\rho}, \eta$ defined in (4.3.15)) such that for each $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{z}, \bar{\mathbf{z}} \in \mathbb{L}_{\widehat{\eta}}(\mathbb{N}, X^h)$ we have

$$\|\mathcal{K}^h(\mathbf{v}, \mathbf{z}) - \mathcal{K}^h(\mathbf{v}, \bar{\mathbf{z}})\|_{\widehat{\eta},+} \leq C_h \|\mathbf{z} - \bar{\mathbf{z}}\|_{\widehat{\eta},+}. \quad (4.3.43)$$

Proof. Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $\mathbf{z} \in \mathbb{L}_{\widehat{\eta}}(\mathbb{N}, X^h)$ be given. Then for each $n \in \mathbb{N}$ one has

$$\begin{aligned} e^{\widehat{\eta} n} \|\mathcal{K}^h(\mathbf{v}, \mathbf{z})_n\| &\leq \sum_{k=0}^{n-1} e^{\widehat{\eta} n} \|U_{\mathbf{v}}^s(n, k+1) z_k\| + \sum_{k=n}^{+\infty} e^{\widehat{\eta} n} \|U_{\mathbf{v}}^u(n, k+1) z_k\| \\ &\leq \sum_{k=0}^{n-1} e^{\widehat{\eta} n} \widehat{\kappa} e^{-\widehat{\rho}(n-k-1)} \|\mathbf{z}\|_{\widehat{\eta},+} e^{-\widehat{\eta} k} + \sum_{k=n}^{+\infty} e^{\widehat{\eta} n} \widehat{\kappa} e^{-\widehat{\rho}(k+1-n)} \|\mathbf{z}\|_{\widehat{\eta},+} e^{-\widehat{\eta} k} \\ &\leq \widehat{\kappa} \frac{e^{\widehat{\eta}}}{1 - e^{\widehat{\eta} - \widehat{\rho}}} \|\mathbf{z}\|_{\widehat{\eta},+} + \widehat{\kappa} \frac{e^{-\widehat{\rho}}}{1 - e^{-\widehat{\eta} - \widehat{\rho}}} \|\mathbf{z}\|_{\widehat{\eta},+} \\ &\leq \widehat{\kappa} \frac{e^\eta}{1 - e^{\eta - \widehat{\rho}}} \|\mathbf{z}\|_{\widehat{\eta},+} + \widehat{\kappa} \frac{e^{-\widehat{\rho}}}{1 - e^{-\widehat{\rho}}} \|\mathbf{z}\|_{\widehat{\eta},+}. \end{aligned}$$

The result follows by observing that $\mathcal{K}^h(\mathbf{v}, \cdot)$ is a bounded linear operator. $\blacksquare$

Lemma 4.3.13 *Let Assumption 4.1.1 be satisfied. Let $\widehat{\eta} \in [0, \eta]$ with η defined in (4.3.15). Then the operators $\mathcal{S}$ is well defined from $\mathbb{E}_\eta(\mathbb{Z}, M) \times X^h$ into $\mathbb{L}_{\widehat{\eta}}(\mathbb{N}, X^h)$. Moreover for each $(\mathbf{v}, y) \in M \times X^h$ we have*

$$\|\mathcal{S}(\mathbf{v}, y)\|_{\eta,+} \leq \widehat{\kappa} \|y\|. \quad (4.3.44)$$

Proof. The proof of this lemma is trivial and thus omitted. $\blacksquare$

Lemma 4.3.14 *Let Assumption 4.1.1 be satisfied. Assume moreover that*

$$\left\| \tilde{T} \right\|_{C(X)} + \left\| \tilde{T} \right\|_{Lip(X)} \leq \delta^2,$$

with δ given in Theorem 4.1.2. Let $\hat{\eta} \in [0, \eta]$ with η defined in (4.3.15) and let $r > 0$ be given. Then the operators

$$\mathcal{G}_1 : \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, r) \rightarrow \mathbb{L}_{\hat{\eta}}(\mathbb{N}, M),$$

and

$$\mathcal{G}_2 : \mathbb{E}_\eta(\mathbb{Z}, M) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, r) \rightarrow \mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h),$$

are well defined with the following properties

(i) *For each $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$*

$$\mathcal{G}_1(\mathbf{v}, \mathbf{0}, \mathbf{0}) = \mathbf{0} \text{ and } \mathcal{G}_2(\mathbf{v}, \mathbf{0}, \mathbf{0}) = \mathbf{0}.$$

(ii) *There exists a constant C_1 depending only on κ , $\|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))}$, $\|DT(\cdot)\|_{C(X, \mathcal{L}(X))}$, $\hat{\gamma}$ and γ (γ and $\hat{\gamma}$ given in Theorem 4.1.2) such that for each $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $(\mathbf{z}, \mathbf{w}), (\bar{\mathbf{z}}, \bar{\mathbf{w}}) \in \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, r)$ we have*

$$\|\mathcal{G}_1(\mathbf{v}, \mathbf{z}, \mathbf{w}) - \mathcal{G}_1(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})\|_{\eta,+} \leq C_1(r + \delta) \left[\|\mathbf{w} - \bar{\mathbf{w}}\|_{\eta,+} + \|\mathbf{z} - \bar{\mathbf{z}}\|_{\eta,+} \right] \quad (4.3.45)$$

(iii) *There exists a constant C_2 depending only on κ , $\|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))}$, $\|DT(\cdot)\|_{C(X, \mathcal{L}(X))}$, $\hat{\gamma}$ and γ (γ and $\hat{\gamma}$ given in Theorem 4.1.2) such that for each $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $(\mathbf{z}, \mathbf{w}), (\bar{\mathbf{z}}, \bar{\mathbf{w}}) \in \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, r)$ we have*

$$\|\mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_2(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n\| \leq C_2(r + \delta) \left[\|\mathbf{w} - \bar{\mathbf{w}}\|_{\eta,+} + \|\mathbf{z} - \bar{\mathbf{z}}\|_{\eta,+} \right] \quad (4.3.46)$$

Proof. Firstly observe that the conditions (i) and (ii) ensure that the maps $\mathcal{G}_1$ and $\mathcal{G}_2$ are well defined. Therefore we will directly focus on the proof of (i) and (ii).

Proof of (i): Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ be given. By using the explicit definition of $\mathcal{G}_1$ and $\mathcal{G}_2$ one gets for each $n \in \mathbb{N}$

$$\mathcal{G}_1(\mathbf{v}, \mathbf{0}, \mathbf{0})_n = G_1(v_n, 0, 0) = 0,$$

and due to Remark 4.3.8 we also obtain

$$\begin{aligned} \mathcal{G}_2(\mathbf{v}, \mathbf{0}, \mathbf{0})_n &= G_1(v_n, 0, 0) \\ &= \Pi^h \left(T + \tilde{T} \right) (v_n + \Psi(v_n)) - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (v_n + \Psi(v_n)) \right) \\ &= 0. \end{aligned}$$

The proof of (i) is complete.

Proof of (ii): Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $(\mathbf{z}, \mathbf{w}), (\bar{\mathbf{z}}, \bar{\mathbf{w}}) \in \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, r)$ be given. Note that we have

$$\|z_n\| \leq r, \|\bar{z}_n\| \leq r, \|w_n\| \leq r, \|\bar{w}_n\| \leq r, \forall n \in \mathbb{N}. \quad (4.3.47)$$

Recall that for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \mathcal{G}_1(\mathbf{v}, \mathbf{z}, \mathbf{w})_n &= \Pi^c \left(T + \tilde{T} \right) (v_n + w_n + z_n + \Psi(w_n + v_n)) \\ &\quad - \Pi^c \left(T + \tilde{T} \right) (v_n + \Psi(v_n)) - \Pi^c DT(v_n) w_n, \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_1(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n &= \Pi^c \left(T + \tilde{T} \right) (v_n + \bar{w}_n + \bar{z}_n + \Psi(\bar{w}_n + v_n)) \\ &\quad - \Pi^c \left(T + \tilde{T} \right) (v_n + \Psi(v_n)) - \Pi^c DT(v_n) \bar{w}_n. \end{aligned}$$

Then by setting for each $n \in \mathbb{N}$

$$a_n := w_n + z_n + \Psi(w_n + v_n), \quad \forall n \in \mathbb{N}, \quad (4.3.48)$$

and

$$\bar{a}_n := \bar{w}_n + \bar{z}_n + \Psi(\bar{w}_n + v_n), \quad \forall n \in \mathbb{N}, \quad (4.3.49)$$

we obtain that

$$\begin{aligned} \mathcal{G}_1(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_1(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n &= \Pi^c [T(v_n + a_n) - T(v_n + \bar{a}_n)] \\ &\quad + \Pi^c [\tilde{T}(v_n + a_n) - \tilde{T}(v_n + \bar{a}_n)] \\ &\quad - \Pi^c DT(v_n) (w_n - \bar{w}_n) \\ &= \Pi^c \int_0^1 [DT(v_n + la_n + (1-l)\bar{a}_n) - DT(v_n)] (w_n - \bar{w}_n) \\ &\quad + \Pi^c [\tilde{T}(v_n + a_n) - \tilde{T}(v_n + \bar{a}_n)] \\ &\quad - \Pi^c DT(v_n) (w_n - \bar{w}_n). \end{aligned}$$

Therefore for each $n \in \mathbb{N}$ we get

$$\begin{aligned} \|\mathcal{G}_1(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_1(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n\| &\leq \|\Pi^c\|_{\mathcal{L}(X)} \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \\ &\quad \times \int_0^1 \|la_n + (1-l)\bar{a}_n\| \|w_n - \bar{w}_n\| dl \\ &\quad + \|\Pi^c\|_{\mathcal{L}(X)} \|\tilde{T}\|_{Lip(X)} \|a_n - \bar{a}_n\| \end{aligned} \quad (4.3.50)$$

Next recalling that

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \left\| \tilde{T} \right\|_{C(X)}, \quad (4.3.51)$$

and

$$\|\Psi(x) - \Psi(\bar{x})\| \leq \hat{\gamma} \delta \|x - \bar{x}\|, \quad \forall x, \bar{x} \in M, \quad (4.3.52)$$

with $\gamma, \hat{\gamma}$ provided by Theorem 4.1.2 we obtain that for each $n \in \mathbb{N}$

$$\begin{aligned} \|a_n - \bar{a}_n\| &= \|w_n - \bar{w}_n + z_n - \bar{z}_n + \Psi(w_n + v_n) - \Psi(\bar{w}_n + v_n)\| \\ &\leq \|w_n - \bar{w}_n\| + \|z_n - \bar{z}_n\| + \hat{\gamma} \delta \|w_n - \bar{w}_n\| + \hat{\gamma} \delta \|z_n - \bar{z}_n\|, \end{aligned} \quad (4.3.53)$$

and

$$\|\bar{a}_n\| \leq \|\bar{w}_n\| + \|\bar{z}_n\| + \frac{6}{5} \gamma \left\| \tilde{T} \right\|_{C(X)}. \quad (4.3.54)$$

By plugging (4.3.53) and (4.3.54) into (4.3.50) we obtain

$$\begin{aligned} \|\mathcal{G}_1(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_1(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n\| &\leq \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|\bar{w}_n\| \|w_n - \bar{w}_n\| \\ &\quad + \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|\bar{z}_n\| \|w_n - \bar{w}_n\| \\ &\quad + \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \frac{6}{5} \gamma \left\| \tilde{T} \right\|_{C(X)} \|w_n - \bar{w}_n\| \\ &\quad + \kappa \left\| \tilde{T} \right\|_{Lip(X)} \|w_n - \bar{w}_n\| \\ &\quad + \kappa \left\| \tilde{T} \right\|_{Lip(X)} \|z_n - \bar{z}_n\| \\ &\quad + \kappa \left\| \tilde{T} \right\|_{Lip(X)} \hat{\gamma} \delta \|w_n - \bar{w}_n\| \\ &\quad + \kappa \left\| \tilde{T} \right\|_{Lip(X)} \hat{\gamma} \delta \|z_n - \bar{z}_n\|. \end{aligned}$$

Thus by using (4.3.47), (4.3.46) follows.

Proof of (iii): Let $\mathbf{v} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ and $(\mathbf{z}, \mathbf{w}), (\bar{\mathbf{z}}, \bar{\mathbf{w}}) \in \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, M)}(\mathbf{0}, r) \times \mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, r)$ be given and observe that

$$\|z_n\| \leq r, \|\bar{z}_n\| \leq r, \|w_n\| \leq r, \|\bar{w}_n\| \leq r, \quad \forall n \in \mathbb{N}.$$

Recall that for each $n \in \mathbb{N}$ one has

$$\begin{aligned} \mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n &= \Pi^h \left(T + \tilde{T} \right) (w_n + v_n + z_n + \Psi(v_n)) - \Pi^h DT(v_n) z_n \\ &\quad - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (w_n + v_n + z_n + \Psi(w_n + v_n)) \right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_2(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n &= \Pi^h \left(T + \tilde{T} \right) (\bar{w}_n + v_n + \bar{z}_n + \Psi(v_n)) - \Pi^h DT(v_n) \bar{z}_n \\ &\quad - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (\bar{w}_n + v_n + \bar{z}_n + \Psi(\bar{w}_n + v_n)) \right). \end{aligned}$$

Then by setting

$$b_n := w_n + z_n \text{ and } \bar{b}_n := \bar{w}_n + \bar{z}_n, \quad (4.3.55)$$

we obtain for each $n \in \mathbb{N}$

$$\begin{aligned} \mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_2(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n &= \Pi^h [T(b_n + v_n + \Psi(v_n)) - T(\bar{b}_n + v_n + \Psi(v_n))] \\ &\quad + \Pi^h [\tilde{T}(b_n + v_n + \Psi(v_n)) - \tilde{T}(\bar{b}_n + v_n + \Psi(v_n))] \\ &\quad - \Psi\left(\Pi^c\left(T + \tilde{T}\right)(b_n + v_n + \Psi(w_n + v_n))\right) \\ &\quad + \Psi\left(\Pi^c\left(T + \tilde{T}\right)(\bar{b}_n + v_n + \Psi(\bar{w}_n + v_n))\right) \\ &\quad - \Pi^h DT(v_n)(z_n - \bar{z}_n) \\ &= \Pi^h \int_0^1 DT(lb_n + (1-l)\bar{b}_n + v_n + \Psi(v_n)) dl (b_n - \bar{b}_n) \\ &\quad - \Pi^h DT(v_n)(z_n - \bar{z}_n) \\ &\quad - \Psi\left(\Pi^c\left(T + \tilde{T}\right)(b_n + v_n + \Psi(w_n + v_n))\right) \\ &\quad + \Psi\left(\Pi^c\left(T + \tilde{T}\right)(\bar{b}_n + v_n + \Psi(\bar{w}_n + v_n))\right). \end{aligned}$$

Next observe that since $\{w_n\}_{n \in \mathbb{N}} \subset M$ and $\{\bar{w}_n\}_{n \in \mathbb{N}} \subset M$ we have for each $n \in \mathbb{N}$

$$\Pi^h DT(v_n)(z_n - \bar{z}_n) = \Pi^h DT(v_n)(z_n - \bar{z}_n + w_n - \bar{w}_n) = \Pi^h DT(v_n)(b_n - \bar{b}_n),$$

so that

$$\begin{aligned} \mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_2(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n &= \Pi^h \int_0^1 DT(lb_n + (1-l)\bar{b}_n + v_n + \Psi(v_n)) dl (b_n - \bar{b}_n) \\ &\quad - \Pi^h DT(v_n)(b_n - \bar{b}_n) \\ &\quad - \Psi\left(\Pi^c\left(T + \tilde{T}\right)(b_n + v_n + \Psi(w_n + v_n))\right) \\ &\quad + \Psi\left(\Pi^c\left(T + \tilde{T}\right)(\bar{b}_n + v_n + \Psi(\bar{w}_n + v_n))\right). \end{aligned}$$

Hence by using the Lipschitz property of $\tilde{T}, T$ combined with (4.3.51) and (4.3.52) we obtain for each $n \in \mathbb{N}$

$$\begin{aligned} \|\mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_2(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n\| &\leq 2\kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \\ &\quad \times \int_0^1 \|lb_n + (1-l)\bar{b}_n + \Psi(v_n)\| dl \|b_n - \bar{b}_n\| \\ &\quad + \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \kappa \hat{\gamma} \delta \|b_n - \bar{b}_n\| \\ &\quad + \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \kappa \hat{\gamma}^2 \delta^2 \|w_n - \bar{w}_n\| \\ &\quad + \kappa \hat{\gamma} \delta \left\| \tilde{T} \right\|_{Lip(X)} [\|b_n - \bar{b}_n\| + \hat{\gamma} \delta \|w_n - \bar{w}_n\|], \end{aligned}$$

and since we have

$$\begin{aligned} \|b_n - \bar{b}_n\| &\leq \|w_n - \bar{w}_n\| + \|z_n - \bar{z}_n\|, \\ \|\bar{b}_n\| &\leq \|\bar{w}_n\| + \|\bar{z}_n\| \quad \text{and} \quad \|b_n\| \leq \|w_n\| + \|z_n\|, \end{aligned}$$

by setting

$$\widehat{C}_2 := \max \left\{ 2\kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))}, \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \kappa \widehat{\gamma}, \|DT(\cdot)\|_{C(X, \mathcal{L}(X))} \kappa \widehat{\gamma}^2 \right\}$$

we obtain

$$\begin{aligned} \|\mathcal{G}_2(\mathbf{v}, \mathbf{z}, \mathbf{w})_n - \mathcal{G}_2(\mathbf{v}, \bar{\mathbf{z}}, \bar{\mathbf{w}})_n\| &\leq \widehat{C}_2 \left[\int_0^1 l \|b_n\| + (1-l) \|\bar{b}_n\| dl + \|\Psi(v_n)\| \right] \\ &\quad \times [\|w_n - \bar{w}_n\| + \|z_n - \bar{z}_n\|] \\ &\quad + \widehat{C}_2 \delta [\|w_n - \bar{w}_n\| + \|z_n - \bar{z}_n\|] \\ &\quad + \widehat{C}_2 \delta^2 \|w_n - \bar{w}_n\| \\ &\quad + \kappa \widehat{\gamma} \delta^3 [\|w_n - \bar{w}_n\| + \|z_n - \bar{z}_n\|] + \widehat{\gamma} \delta \|w_n - \bar{w}_n\|, \end{aligned}$$

and (4.3.46) follows with simple computations.

The proof is complete. ■

Now we are in a position to give the following proposition which is the main step for the proof of Theorem 4.2.1.

Proposition 4.3.15 *Let Assumption 4.1.1 be satisfied. There exists $\delta_0 \in (0, \min(\delta, \widehat{\delta}))$ with δ given in Theorem 4.1.2, $\widehat{\delta}$ given in Lemma 4.3.6 such that if*

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \delta_0^2,$$

then there exist some

$$\Psi^c, \Psi^u : E^s(\delta_0/2\widehat{\kappa}) \rightarrow M \oplus X^h = X,$$

with

$$E^s(\delta_0/2\widehat{\kappa}) := \{x + z_0^s : z_0^s \in E_x^s \text{ and } \|z_0^s\| \leq \delta_0/2\widehat{\kappa}\},$$

such that the following properties hold true:

(i) For each $x \in M$ the subset $\widehat{W}^s(x, \delta_0/2\widehat{\kappa})$ of $W^s(x, \delta_0)$ defined as

$$\widehat{W}^s(x, \delta_0/2\widehat{\kappa}) := \{(x_0, z_0) \in W^s(x, \delta_0) : \|\Pi_0^s(\sigma(x)) z_0\| \leq \delta_0/2\widehat{\kappa}\},$$

satisfies:

$$\widehat{W}^s(x, \delta_0/2\widehat{\kappa}) = \{(x + \Psi^c(x + z_0^s), z_0^s + \Psi^u(x + z_0^s)) : x + z_0^s \in E^s(\delta_0/2\widehat{\kappa})\}. \quad (4.3.56)$$

(ii) If $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ is a solution of (4.3.19) with $\|z_0\| \leq \delta_0/2\widehat{\kappa}^2$ and

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|x_n - \sigma_n(x)\| \leq \delta_0 \text{ and } \sup_{n \in \mathbb{N}} e^{\eta n} \|z_n\| \leq \delta_0,$$

for some $x \in M$ then $(x_0, z_0) \in \widehat{W}^s(x, \delta_0/2\widehat{\kappa})$.

(iii) For each $x + z_0^s \in E^s(\delta_0/2\widehat{\kappa})$ we have

$$\Psi^u(x + z_0^s) \in E_x^u, \quad \Psi^c(x + z_0^s) \in M,$$

and

$$\|z_0^s + \Psi^u(x + z_0^s)\| \leq \delta_0 \text{ and } \|\Psi^c(x + z_0^s)\| \leq \delta_0.$$

(iv) For each $x \in M$ we have

$$\Psi^\alpha(x + 0) = 0, \quad \alpha = u, c.$$

(v) For each $x \in M$, $z_0^s, \bar{z}_0^s \in E_x^s$ with $\|z_0^s\| \leq \delta_0/2\widehat{\kappa}$ and $\|\bar{z}_0^s\| \leq \delta_0/2\widehat{\kappa}$ we have

$$\|\Psi^u(x + z_0^s) - \Psi^u(x + \bar{z}_0^s)\| \leq \|z_0^s - \bar{z}_0^s\| + 2\widehat{\kappa} \|z_0^s - \bar{z}_0^s\|,$$

and

$$\|\Psi^c(x + z_0^s) - \Psi^c(x + \bar{z}_0^s)\| \leq 2\widehat{\kappa} \|z_0^s - \bar{z}_0^s\|.$$

(vi) The set

$$\begin{aligned} \widehat{W}^s(\delta_0/2\widehat{\kappa}) &: = \bigcup_{x \in M} \widehat{W}^s(x, \delta_0/2\widehat{\kappa}) \\ &= \{(x + \Psi^c(x + z_0^s), z_0^s + \Psi^u(x + z_0^s)) : x + z_0^s \in E^s(\delta_0/2\widehat{\kappa})\}, \end{aligned} \quad (4.3.57)$$

is forward invariant with respect to the system (4.3.19) in the following sense: there exists $p \in \mathbb{N}$ such that if $(x_0, z_0) \in \widehat{W}^s(x, \delta_0/2\widehat{\kappa})$ for some $x \in M$ then the associated solution $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ of (4.3.19) satisfies

$$(x_n, z_n) \in \widehat{W}^s(\sigma_n(x), \delta_0/2\widehat{\kappa}), \quad \forall n \geq p.$$

Proof. Let $\delta_0 \in (0, \min(\delta, \widehat{\delta}))$ be small enough such that all the previous lemmas as well as Theorem 4.1.2 remain true with δ and $\widehat{\delta}$ replaced by δ_0 and such that

$$[2C_c C_1 + 2C_h C_2] \delta_0 \leq \frac{1}{2}, \quad (4.3.58)$$

where the constant C_c and C_h are defined respectively in (4.3.40) and (4.3.42) while C_1 and C_2 are defined in Lemma 4.3.14.

Let $x \in M$ be given. Note that under the condition of this proposition one has from Lemma 4.3.10 that $(x_0, z_0) \in \widehat{W}^s(x, \delta_0/2\widehat{\kappa})$ with $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ the associated solution of (4.3.19) if and only if there exists

$$(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$$

such that

(a) The sequence $\mathbf{x}$ is given by $\mathbf{x} = \boldsymbol{\sigma}(x) + \mathbf{w}$

(b) $(\mathbf{w}, \mathbf{z})$ satisfies the fixed point problem

$$\begin{cases} \mathbf{w} = \mathcal{K}^c(\boldsymbol{\sigma}(x), \mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w})) \\ \mathbf{z} = \mathcal{S}(\boldsymbol{\sigma}(x), z_0^s) + \mathcal{K}^h(\boldsymbol{\sigma}(x), \mathcal{G}_2(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w})) \\ \Pi_0^s(\boldsymbol{\sigma}(x)) z_0 = z_0^s \in E_x^s = \mathcal{R}[\Pi_0^s(\boldsymbol{\sigma}(x))]. \end{cases}$$

(c) $\|z_0^s\| \leq \delta_0/2\widehat{\kappa}$

Hence according to the foregoing observation we consider for each $x \in M$ and $z_0^s \in E_x^s$ with $\|z_0^s\| \leq \delta_0/2\widehat{\kappa}$ the operator

$$\mathcal{J}(x, z_0^s, \cdot) : \mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_\eta(\mathbb{N}, X^h) \rightarrow \mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_\eta(\mathbb{N}, X^h),$$

defined for each $(\mathbf{z}, \mathbf{w}) \in \mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_\eta(\mathbb{N}, X^h)$ by

$$\mathcal{J}(x, z_0^s, \mathbf{z}, \mathbf{w}) := (\mathcal{K}^c(\boldsymbol{\sigma}(x), \mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w})) ; \mathcal{S}(\boldsymbol{\sigma}(x), z_0^s) + \mathcal{K}^h(\boldsymbol{\sigma}(x), \mathcal{G}_2(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w}))).$$

Next we investigate the existence of fixed points of the operator $\mathcal{J}(x, z_0^s, \cdot)$ in $\mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$. To do so let us endow $\mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_\eta(\mathbb{N}, X^h)$ with the norm

$$\|(\mathbf{z}, \mathbf{w})\|_{\eta,+} := \|\mathbf{z}\|_{\eta,+} + \|\mathbf{w}\|_{\eta,+}, \quad \forall (\mathbf{w}, \mathbf{z}) \in \mathbb{L}_\eta(\mathbb{N}, M) \times \mathbb{L}_\eta(\mathbb{N}, X^h).$$

Now note that by using the properties of $\mathcal{K}^c$, $\mathcal{S}$, $\mathcal{K}^h$, $\mathcal{G}_1$ and $\mathcal{G}_2$ combined with the fact that

$$\mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{0}, \mathbf{0}) = \mathcal{G}_2(\boldsymbol{\sigma}(x), \mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad (4.3.59)$$

and

$$\|(\mathbf{z}, \mathbf{w})\|_{\eta,+} \leq 2\delta_0,$$

one gets from Lemma 4.3.11 and Lemma 4.3.14

$$\begin{aligned} \|\mathcal{K}^c(\boldsymbol{\sigma}(x), \mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w}))\|_{\eta,+} &\leq C_c \|\mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w})\|_{\eta,+} \\ &\leq C_c \|\mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w}) - \mathcal{G}_1(\boldsymbol{\sigma}(x), \mathbf{0}, \mathbf{0})\|_{\eta,+} \\ &\leq C_c C_1 \delta_0 \|(\mathbf{z}, \mathbf{w})\|_{\eta,+} \\ &\leq 2C_c C_1 \delta_0^2 \end{aligned}$$

and similarly (see Lemma 4.3.12 and Lemma 4.3.14)

$$\|\mathcal{K}^h(\boldsymbol{\sigma}(x), \mathcal{G}_2(\boldsymbol{\sigma}(x), \mathbf{z}, \mathbf{w}))\|_{\eta,+} \leq 2C_h C_2 \delta_0^2,$$

while we have from Lemma 4.3.13

$$\|\mathcal{S}(\sigma(x), z_0^s)\|_{\eta,+} \leq \widehat{\kappa} \|z_0^s\| \leq \delta_0/2.$$

Hence for each $(z, w) \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ we obtain

$$\begin{aligned} \|\mathcal{J}(x, z_0^s, z, w)\|_{\eta,+} &\leq \|\mathcal{K}^c(\sigma(x), \mathcal{G}_1(\sigma(x), z, w))\|_{\eta,+} \\ &\quad + \|\mathcal{K}^h(\sigma(x), \mathcal{G}_2(\sigma(x), z, w))\|_{\eta,+} \\ &\quad + \|\mathcal{S}(\sigma(x), z_0^s)\|_{\eta,+} \\ &\leq [2C_c C_1 + 2C_h C_2] \delta_0^2 + \delta_0/2. \end{aligned}$$

Thus we infer from (4.3.58) that the operator $\mathcal{J}(x, z_0^s, \cdot)$ maps $\mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ into itself. Furthermore by using the same arguments, one has for each $(z, w), (\bar{z}, \bar{w}) \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$

$$\begin{aligned} \|\mathcal{J}(x, z_0^s, z, w) - \mathcal{J}(x, z_0^s, \bar{z}, \bar{w})\|_{\eta,+} &\leq [2C_c C_1 + 2C_h C_2] \delta_0^2 \|(z, w) - (\bar{z}, \bar{w})\|_{\eta,+} \\ &\leq \frac{1}{2} \|(z, w) - (\bar{z}, \bar{w})\|_{\eta,+}, \end{aligned}$$

that is $\mathcal{J}(x, z_0^s, \cdot)$ is a $\frac{1}{2}$ -contraction on $\mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ uniformly with respect to $x \in M$ and $z_0^s \in E_x^s$ with $\|z_0^s\| \leq \delta_0/2\widehat{\kappa}$. Therefore we infer from the uniform Banach fixed point theorem that for each $x \in M$ and $z_0^s \in E_x^s$ with $\|z_0^s\| \leq \delta_0/2\widehat{\kappa}$ the operator $\mathcal{J}(x, z_0^s, \cdot)$ admits a unique fixed point that we will denote by

$$z := z(x, z_0^s) \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0) \text{ and } w := w(x, z_0^s) \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0),$$

Moreover by using the explicit formula of the operators one gets

$$z(x, z_0^s)_0 = z_0^s + \Pi_0^u(\sigma(x)) z(x, z_0^s)_0.$$

Therefore one can define $\Psi^\alpha : E^s(\delta_0/2\widehat{\kappa}) \rightarrow X$, $\alpha = u, c$ by

$$\Psi^u(x + z_0^s) := \Pi_0^u(\sigma(x)) z(x, z_0^s)_0 \text{ and } \Psi^c(x + z_0^s) := w(x, z_0^s)_0.$$

Then the characterization of $\left\{ \widehat{W}^s(x, \delta_0/2\widehat{\kappa}) \right\}_{x \in M}$ stated in (i) of the proposition as well as the property (ii)-(iii) stated in the proposition follows easily.

In order to see that the property (iv) holds true we just observe that for each $x \in M$ by using (4.3.59) we have

$$\mathcal{J}(x, 0, \mathbf{0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0}),$$

and the uniqueness of the fixed point of $\mathcal{J}(x, 0, \cdot)$ implies that

$$z(x, 0) = 0 \text{ and } w(x, 0) = 0, \forall x \in M,$$

so that

$$\Psi^u(x+0) = \Psi^c(x+0) = 0.$$

Next we prove the property (v).

To do so since $(\mathbf{z}(x, z_0^s), \mathbf{w}(x, z_0^s))$ is a fixed point of $\mathcal{J}(x, z_0^s, \cdot)$ one can deduce easily that for each $z_0^s, \bar{z}_0^s \in E_x^s$ with $x \in M$ and $\|z_0^s\| \leq \delta_0/2\hat{\kappa}$, $\|\bar{z}_0^s\| \leq \delta_0/2\hat{\kappa}$ we have

$$\begin{aligned} \|(\mathbf{z}(x, z_0^s), \mathbf{w}(x, z_0^s)) - (\mathbf{z}(x, \bar{z}_0^s), \mathbf{w}(x, \bar{z}_0^s))\|_{\eta,+} &\leq \hat{\kappa} \|z_0^s - \bar{z}_0^s\| \\ &+ \frac{1}{2} \|(\mathbf{z}(x, z_0^s), \mathbf{w}(x, z_0^s)) - (\mathbf{z}(x, \bar{z}_0^s), \mathbf{w}(x, \bar{z}_0^s))\|_{\eta,+}, \end{aligned} \quad (4.3.60)$$

so that

$$\|(\mathbf{z}(x, z_0^s), \mathbf{w}(x, z_0^s)) - (\mathbf{z}(x, \bar{z}_0^s), \mathbf{w}(x, \bar{z}_0^s))\|_{\eta,+} \leq 2\hat{\kappa} \|z_0^s - \bar{z}_0^s\|$$

Then we have in particular for any $x \in M$ and $z_0^s, \bar{z}_0^s \in E_x^s$ with $\|z_0^s\| \leq \delta_0/2\hat{\kappa}$ and $\|\bar{z}_0^s\| \leq \delta_0/2\hat{\kappa}$

$$\|\mathbf{z}(x, z_0^s)_0 - \mathbf{z}(x, \bar{z}_0^s)_0\| + \|\mathbf{w}(x, z_0^s)_0 - \mathbf{w}(x, \bar{z}_0^s)_0\| \leq 2\hat{\kappa} \|z_0^s - \bar{z}_0^s\|, \quad (4.3.61)$$

that is due to the definition of Ψ^u and Ψ^c

$$\|z_0^s + \Psi^u(x + z_0^s) - \bar{z}_0^s - \Psi^u(x + \bar{z}_0^s)\| + \|\Psi^c(x + z_0^s) - \Psi^c(x + \bar{z}_0^s)\| \leq 2\hat{\kappa} \|z_0^s - \bar{z}_0^s\|. \quad (4.3.62)$$

Hence we deduce from (4.3.62) that for any $x \in M$ and $z_0^s, \bar{z}_0^s \in E_x^s$ with $\|z_0^s\| \leq \delta_0/2\hat{\kappa}$ and $\|\bar{z}_0^s\| \leq \delta_0/2\hat{\kappa}$

$$\|\Psi^u(x + z_0^s) - \Psi^u(x + \bar{z}_0^s)\| \leq \|z_0^s - \bar{z}_0^s\| + 2\hat{\kappa} \|z_0^s - \bar{z}_0^s\|,$$

and

$$\|\Psi^c(x + z_0^s) - \Psi^c(x + \bar{z}_0^s)\| \leq 2\hat{\kappa} \|z_0^s - \bar{z}_0^s\|.$$

This completes the proof of (v).

Next we prove the property (vi).

To do so we first recall that

$$\widehat{W}^s(\delta_0/2\hat{\kappa}) = \bigcup_{x \in M} \widehat{W}^s(x, \delta_0/2\hat{\kappa}).$$

Let $(x_0, z_0) \in \widehat{W}^s(x, \delta_0/2\hat{\kappa})$ for some $x \in M$ and denote by $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ the associated solution of (4.3.19). Then by using the definition of $\widehat{W}^s(x, \delta_0/2\hat{\kappa}) \subset \widehat{W}^s(x, \delta_0)$ we know that there exists a unique $\mathbf{w} = \{w_n\}_{n \in \mathbb{N}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0)$ and $\mathbf{z} = \{z_n\}_{n \in \mathbb{N}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ such that $\mathbf{x} = \boldsymbol{\sigma}(x) + \mathbf{w}$ and $(\mathbf{w}, \mathbf{z}) = \{(w_n, z_n)\}_{n \in \mathbb{N}}$ satisfies

$$\begin{cases} w_{n+1} = DT(\sigma_n(x))w_n + G_1(\sigma_n(x), w_n, z_n), & n \in \mathbb{N}, \\ z_{n+1} = DT(\sigma_n(x))z_n + G_2(\sigma_n(x), w_n, z_n), & n \in \mathbb{N}. \end{cases} \quad (4.3.63)$$

Let $p \geq 1$ be given such that $\hat{\kappa}e^{-\hat{\rho}p} \leq 1$. Let us set

$$\bar{x} := \sigma_p(x).$$

Then recalling that

$$\sigma_{n+p}(x) = \sigma_n(\sigma_p(x)) = \sigma_n(\bar{x}), \quad \forall n \in \mathbb{N}, \quad (4.3.64)$$

by using (4.3.63) and (4.3.64) we obtain that

$$\begin{cases} w_{n+p+1} = DT(\sigma_n(\bar{x})) w_{n+p} + G_1(\sigma_n(\bar{x}), w_{n+p}, z_{n+p}), & n \in \mathbb{N}, \\ z_{n+p+1} = DT(\sigma_n(\bar{x})) z_{n+p} + G_2(\sigma_n(\bar{x}), w_{n+p}, z_{n+p}), & n \in \mathbb{N}. \end{cases} \quad (4.3.65)$$

Next note that due to Lemma 4.3.6 and the fact that $(\mathbf{w}, \mathbf{z}) \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ one has

$$\begin{cases} \sigma(\bar{x}) \in \mathbb{E}_\eta(\mathbb{Z}, M) \\ (\mathbf{w}_{p+}, \mathbf{z}_{p+}) = \{(w_{n+p}, z_{n+p})\}_{n \in \mathbb{N}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(\mathbf{0}, \delta_0) \times \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0). \end{cases} \quad (4.3.66)$$

On the one hand, using system (4.3.65) and the same arguments as in the proof of Lemma 4.3.9, one gets for each $n \in \mathbb{N}$,

$$w_{p+n} = - \sum_{k=n}^{+\infty} U_{\sigma(\bar{x})}^c(n, k+1) G_1(\sigma_k(\bar{x}), z_{p+k}, w_{p+k}), \quad (4.3.67)$$

and

$$\begin{aligned} z_{p+n} &= U_{\sigma(\bar{x})}^s(n, 0) \Pi_0^s(\sigma(\bar{x})) z_p \\ &\quad + \sum_{k=0}^{n-1} U_{\sigma(\bar{x})}^s(n, k+1) G_2(\sigma_k(\bar{x}), z_{p+k}, w_{p+k}) \\ &\quad - \sum_{k=n}^{+\infty} U_{\sigma(\bar{x})}^u(n, k+1) G_2(\sigma_k(\bar{x}), z_{p+k}, w_{p+k}). \end{aligned} \quad (4.3.68)$$

On the other hand using system (4.3.63) we obtain from Lemma 4.3.9 that for each $n \in \mathbb{N}$,

$$w_{p+n} = - \sum_{k=p+n}^{+\infty} U_{\sigma(x)}^c(p+n, k+1) G_1(w_k, z_k, x_k), \quad (4.3.69)$$

and

$$\begin{aligned} z_{p+n} &= U_{\sigma(x)}^s(n+p, 0) z_0^s + \sum_{k=0}^{p+n-1} U_{\sigma(x)}^s(n+p, k+1) G_2(\sigma_k(x), z_k, w_k) \\ &\quad - \sum_{k=p+n}^{+\infty} U_{\sigma(x)}^u(n+p, k+1) G_2(\sigma_k(x), z_k, w_k). \end{aligned} \quad (4.3.70)$$

Moreover note that using the Remark 4.3.7 and (4.3.64) one has

$$\Pi_n^s(\sigma(\bar{x})) = \Pi_n^s(\sigma(\sigma_p(x))) = \Pi_{n+p}^s(\sigma(x)), \quad \forall n \in \mathbb{N}, \quad (4.3.71)$$

so that

$$U_{\sigma(\bar{x})}^{\alpha}(n, l) = U_{\sigma(x)}^{\alpha}(p + n, p + l), \quad \forall l, n \in \mathbb{N}, l \geq n \text{ and } \alpha = u, c, \quad (4.3.72)$$

and

$$U_{\sigma(\bar{x})}^s(n, r) = U_{\sigma(x)}^s(p + n, p + r), \quad \forall r, n \in \mathbb{N}, n \geq r. \quad (4.3.73)$$

So the change of variable $k \leftarrow k - p$ in the sums in (4.3.69)-(4.3.70) yields for each $n \in \mathbb{N}$ to

$$w_{p+n} = - \sum_{k=n}^{+\infty} U_{\sigma(\bar{x})}^c(n, k+1) G_1(\sigma_k(x_p), z_{p+k}, w_{p+k}), \quad (4.3.74)$$

and

$$\begin{aligned} z_{p+n} &= U_{\sigma(x)}^s(n+p, 0) z_0^s + \sum_{k=0}^{n-1} U_{\sigma(\bar{x})}^s(n, k+1) G_2(\sigma_k(x_p), z_{p+k}, w_{p+k}) \\ &\quad - \sum_{k=n}^{+\infty} U_{\sigma(\bar{x})}^u(n, k+1) G_2(\sigma_k(x_p), z_{p+k}, w_{p+k}). \end{aligned} \quad (4.3.75)$$

Next we infer from (4.3.72) that

$$U_{\sigma(x)}^s(n+p, 0) z_0^s = U_{\sigma(x)}^s(n+p, p) U_{\sigma(x)}^s(p, 0) z_0^s = U_{\sigma(\bar{x})}^s(n, 0) U_{\sigma(x)}^s(p, 0) z_0^s, \quad \forall n \in \mathbb{N}.$$

Finally recalling that $\widehat{\kappa} e^{-\widehat{\rho} p} \leq 1$ we obtain

$$\|U_{\sigma(x)}^s(p, 0) z_0^s\| \leq \widehat{\kappa} e^{-\widehat{\rho} p} \|z_0^s\| \leq \frac{\delta_0}{2\widehat{\kappa}},$$

so that using the system (4.3.74)-(4.3.75) it follows that the sequence $(\mathbf{z}_{p+}, \mathbf{w}_{p+})$ is a fixed point of $\mathcal{J}(\bar{x}, z_p^s, \cdot, \cdot)$ with $z_p^s = U_{\sigma(x)}^s(p, 0) z_0^s$ and the uniqueness ensures that

$$z_p = z_p^s + \Psi^u(\bar{x}, z_p^s) \text{ and } w_p = \Psi^c(\bar{x}, z_p^s) \implies z_p = z_p^s + \Psi^u(\bar{x}, z_p^s) \text{ and } x_p = x_p + \Psi^c(\bar{x}, z_p^s).$$

The proof of (vi) is complete. $\blacksquare$

In order to complete the proof of Theorem 4.2.1 we will also need the following lemma that will allow us to know when a initial value (x_0, z_0) of a solution $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}}$ of (4.3.19) lies in some $W^s(x, \delta_0)$, with $x \in M$ and δ_0 the positive value defined in Proposition 4.3.15.

Lemma 4.3.16 *Let Assumption 4.1.1 be satisfied. Assume moreover that*

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \delta_0^2,$$

where $\delta_0 > 0$ is given in Proposition 4.3.15. If $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}} \subset M \times X^h$ is a solution of (4.3.19) with

$$\|z_0\| \leq \delta_0/2\widehat{\kappa}^2 \text{ and } \mathbf{z} \in \mathbb{B}_{\mathbb{L}_0(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0),$$

then there exists a unique $x \in M$ such that $(x_0, z_0) \in \widehat{W}^s(x, \delta_0/2\widehat{\kappa})$.

Proof. Let $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}} \subset M \times X^h$ be a solution of (4.3.19) that is for each $n \in \mathbb{N}$

$$\begin{cases} x_{n+1} = \Pi^c \left(T + \tilde{T} \right) (x_n + z_n + \Psi(x_n)), & n \in \mathbb{N} \\ z_{n+1} = \Pi^h \left(T + \tilde{T} \right) (x_n + z_n + \Psi(x_n)) \\ \quad - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (x_n + z_n + \Psi(x_n)) \right), & n \in \mathbb{N}. \end{cases} \quad (4.3.76)$$

In order to prove the lemma we will firstly prove that

$$\mathbf{z} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0) \quad (4.3.77)$$

and secondly that there exists a unique $x \in M$ such that

$$\|\Pi_0^s(\boldsymbol{\sigma}(x)) z_0\| \leq \delta_0/2\hat{\kappa} \text{ and } \|\mathbf{x} - \boldsymbol{\sigma}(x)\|_{\eta,+} = \sup_{n \in \mathbb{N}} e^{\eta n} \|x_n - \sigma_n(x)\| \leq \delta_0. \quad (4.3.78)$$

The arguments that will be used for the proof of conditions (4.3.77) and (4.3.78) are similar to those used in the proof Proposition 4.3.15. We will need in particular to extend the sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}} \subset M$ into a sequence defined in $\mathbb{Z}$ and such that the extended sequence belongs to $\mathbb{E}_\eta(\mathbb{Z}, M)$. To do so consider the extended sequence $\tilde{\mathbf{z}} = \{\tilde{z}_n\}_{n \in \mathbb{Z}}$ defined by

$$\tilde{z}_n = z_n, \text{ if } n \in \mathbb{N} \text{ and } \tilde{z}_n = 0, \text{ if } n \in \mathbb{N}_- \setminus \{0\},$$

and note that

$$\sup_{n \in \mathbb{Z}} \|\tilde{z}_n\| = \sup_{n \in \mathbb{N}} \|z_n\| \leq \delta_0. \quad (4.3.79)$$

Moreover since $x_0 \in M$ we know from Theorem 4.1.2 that there exists a unique sequence $\bar{\mathbf{x}} + \Psi(\bar{\mathbf{x}}) = \{\bar{x}_n + \Psi(\bar{x}_n)\}_{n \in \mathbb{N}}$ with $\bar{x}_0 = x_0$ satisfying

$$\bar{x}_{n+1} + \Psi(\bar{x}_{n+1}) = \left(T + \tilde{T} \right) (\bar{x}_n + \Psi(\bar{x}_n)), \quad \forall n \in \mathbb{Z}. \quad (4.3.80)$$

Hence by projecting (4.3.80) along M we obtain that

$$\bar{x}_{n+1} = \Pi^c \left(T + \tilde{T} \right) (\bar{x}_n + \Psi(\bar{x}_n)), \quad \forall n \in \mathbb{Z} \Leftrightarrow \bar{\mathbf{x}} = \boldsymbol{\sigma}(x_0). \quad (4.3.81)$$

Consider the extended sequence $\tilde{\mathbf{x}} = \{\tilde{x}_n\}_{n \in \mathbb{Z}}$ defined by

$$\tilde{x}_n = x_n, \text{ if } n \in \mathbb{N} \text{ and } \tilde{x}_n = \bar{x}_n, \text{ if } n \in \mathbb{N}_- \setminus \{0\},$$

so that due to (4.3.76) and (4.3.81) the sequence $\tilde{\mathbf{x}} = \{\tilde{x}_n\}_{n \in \mathbb{Z}}$ satisfies

$$\tilde{x}_{n+1} = \Pi^c \left(T + \tilde{T} \right) (\tilde{x}_n + \tilde{z}_n + \Psi(\tilde{x}_n)), \quad n \in \mathbb{Z}. \quad (4.3.82)$$

Now we proceed to the proof in three steps. In the first step we prove that the extended sequence $\tilde{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$, in the second step we prove that the condition (4.3.77) is satisfied

and finally in the last step we prove that there exists a unique $x \in M$ such that condition (4.3.78) holds true.

Step 1: In this step we will prove that $\tilde{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$. To do so we will make use of Proposition 4.3.1. Indeed by using (4.3.82) one gets for each $n \in \mathbb{Z}$

$$\begin{aligned} \tilde{x}_{n+1} - T(\tilde{x}_n) &= \Pi^c \left[\tilde{T}(\tilde{x}_n + \tilde{z}_n + \Psi(\tilde{x}_n)) + T(\tilde{x}_n + \tilde{z}_n + \Psi(\tilde{x}_n)) - T(\tilde{x}_n) \right] \\ &= \int_0^1 \Pi^c DT(\tilde{x}_n + l\tilde{z}_n + l\Psi(\tilde{x}_n)) dl \cdot (\tilde{z}_n + \Psi(\tilde{x}_n)) \\ &\quad + \Pi^c \tilde{T}(\tilde{x}_n + \tilde{z}_n + \Psi(\tilde{x}_n)) \end{aligned}$$

Therefore by using the fact that

$$\Pi^c DT(\tilde{x}_n)(\tilde{z}_n + \Psi(\tilde{x}_n)) = 0, \quad \forall n \in \mathbb{Z},$$

it follows that for each $n \in \mathbb{Z}$

$$\begin{aligned} \tilde{x}_{n+1} - T(\tilde{x}_n) &= \Pi^c \int_0^1 [DT(\tilde{x}_n + l\tilde{z}_n + l\Psi(\tilde{x}_n)) - DT(\tilde{x}_n)] dl \cdot (\tilde{z}_n + \Psi(\tilde{x}_n)) \\ &\quad + \Pi^c \tilde{T}(\tilde{x}_n + \tilde{z}_n + \Psi(\tilde{x}_n)). \end{aligned}$$

Hence

$$\begin{aligned} \|\tilde{x}_{n+1} - T(\tilde{x}_n)\| &\leq \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \|\tilde{z}_n + \Psi(\tilde{x}_n)\|^2 \\ &\quad + \kappa \|\tilde{T}\|_{C(X)}, \quad \forall n \in \mathbb{Z}, \end{aligned}$$

and since

$$\sup_{x \in M} \|\Psi(x)\| \leq \frac{6}{5} \gamma \|\tilde{T}\|_{C(X)} \quad \text{and} \quad \|\tilde{T}\|_{C(X)} \leq \delta_0^2,$$

we infer from (4.3.79) that

$$\|\tilde{x}_{n+1} - T(\tilde{x}_n)\| \leq \kappa \|DT(\cdot)\|_{Lip(X, \mathcal{L}(X))} \left(1 + \frac{6}{5} \gamma \delta_0\right)^2 \delta_0^2, \quad \forall n \in \mathbb{Z}.$$

Then Proposition 4.3.1 applies and ensures that $\tilde{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ up to reduce δ_0 . The proof of Step 1 is complete.

Step 2: Let us now prove that (4.3.77) holds true. First of all let us notice that due to the definition of $\tilde{\mathbf{x}} = \{\tilde{x}_n\}_{n \in \mathbb{Z}}$ one has for each $n \in \mathbb{N}$:

$$\begin{aligned} z_{n+1} &= \Pi^h \left(T + \tilde{T} \right) (\tilde{x}_n + z_n + \Psi(\tilde{x}_n)) \\ &\quad - \Psi \left(\Pi^c \left(T + \tilde{T} \right) (\tilde{x}_n + z_n + \Psi(\tilde{x}_n)) \right). \end{aligned} \tag{4.3.83}$$

Then we re-write (4.3.83) as the following equivalent system

$$z_{n+1} = DT(\tilde{x}_n) z_n + G_2(\tilde{x}_n, 0, z_n), \quad \forall n \in \mathbb{N}, \quad (4.3.84)$$

where G_2 is defined in (4.3.24). Then since $\tilde{\mathbf{x}} = \{\tilde{x}_n\}_{n \in \mathbb{Z}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ one can use the same arguments as in the proof of Lemma 4.3.9 to obtain that $\mathbf{z} \in \mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)$ for every $\hat{\eta} \in [0, \eta]$ satisfies (4.3.84) if and only if for each $n \in \mathbb{N}$

$$\begin{aligned} z_n = & U_{\tilde{\mathbf{x}}}^s(n, 0) \Pi_0^s(\tilde{\mathbf{x}}) z_0 + \sum_{k=0}^{n-1} U_{\tilde{\mathbf{x}}}^s(n, k+1) G_2(\tilde{x}_k, 0, z_k) \\ & - \sum_{k=n}^{+\infty} U_{\tilde{\mathbf{x}}}^u(n, k+1) G_2(\tilde{x}_k, 0, z_k), \end{aligned} \quad (4.3.85)$$

or equivalently, by recalling (4.3.36)-(4.3.38)

$$\mathbf{z} = \mathcal{S}(\tilde{\mathbf{x}}, \Pi_0^s(\tilde{\mathbf{x}}) z_0) + \mathcal{K}^h(\tilde{\mathbf{x}}, \mathcal{G}_2(\tilde{\mathbf{x}}, \mathbf{z}, 0)). \quad (4.3.86)$$

Thus as for the proof of Proposition 4.3.1 by using Lemma 4.3.14 .it is easy to see that for δ_0 small enough if $\|z_0\| \leq \delta_0/2\hat{\kappa}^2$ that is

$$\|\Pi_0^s(\tilde{\mathbf{x}}) z_0\| \leq \hat{\kappa} \|z_0\| \leq \frac{\delta_0}{2\hat{\kappa}},$$

then for each $\hat{\eta} \in [0, \eta]$ the operator $\mathcal{S}(\tilde{\mathbf{x}}, \Pi_0^s(\tilde{\mathbf{x}}) z_0) + \mathcal{K}^h(\tilde{\mathbf{x}}, \mathcal{G}_2(\tilde{\mathbf{x}}, \cdot, 0))$ maps $\mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ into itself and is a strict contraction. Therefore for each $\hat{\eta} \in [0, \eta]$, (4.3.86) admits a unique solution in $\mathbb{B}_{\mathbb{L}_{\hat{\eta}}(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$. Finally by using the fact that

$$\mathbb{B}_{\mathbb{L}_0(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0) = \mathbb{B}_{l^\infty(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0) \supset \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0),$$

it follows from the uniqueness of solutions that $\mathbf{z} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$. The proof of Step 2 is complete.

Step 3: Now we will make of use the fact that $\mathbf{z} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ in order to prove that there exists a unique x such that the condition (4.3.78) hold true.

Define the sequence $\mathbf{w} = \{w_n\}_{n \in \mathbb{N}} \subset M$ by

$$w_n = \sigma_n(x) - \tilde{x}_n = \sigma_n(x) - x_n, \quad \forall n \in \mathbb{N},$$

for some $x \in M$ that will be determined latter. Then $\mathbf{w} = \{w_n\}_{n \in \mathbb{N}}$ satisfies for each $n \in \mathbb{N}$,

$$\begin{aligned} w_{n+1} = & \Pi^c(T + \tilde{T})(w_n + \tilde{x}_n + \Psi(w_n + \tilde{x}_n)) \\ & - \Pi^c(T + \tilde{T})(\tilde{x}_n + z_n + \Psi(\tilde{x}_n)) \end{aligned}$$

or equivalently

$$w_{n+1} = DT(\tilde{x}_n) w_n + \tilde{G}_1(\tilde{x}_n, z_n, w_n), \quad \forall n \in \mathbb{N}. \quad (4.3.87)$$

Here we have set

$$\begin{aligned} \tilde{G}_1(\tilde{x}_n, z_n, w_n) := & \Pi^c(T + \tilde{T})(w_n + \tilde{x}_n + \Psi(w_n + \tilde{x}_n)) \\ & - \Pi^c(T + \tilde{T})(\tilde{x}_n + z_n + \Psi(\tilde{x}_n)) - DT(\tilde{x}_n) w_n, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (4.3.88)$$

Then since $\tilde{\mathbf{x}} \in \mathbb{E}_\eta(\mathbb{Z}, M)$ as for the proof of Lemma 4.3.9 we obtain that $\mathbf{w} \in \mathbb{L}_\eta(\mathbb{N}, M)$ satisfies (4.3.87) if and only if

$$w_n = - \sum_{k=n}^{+\infty} U_{\tilde{\mathbf{x}}}^c(n, k+1) \tilde{G}_1(\tilde{x}_k, z_k, w_k), \quad \forall n \in \mathbb{N}.$$

Next we define the operator

$$\tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})_n := \tilde{G}_1(\tilde{x}_n, z_n, w_n), \quad \forall n \in \mathbb{N}.$$

Then we obtain that $\mathbf{w} \in \mathbb{L}_\eta(\mathbb{N}, M)$ satisfies (4.3.87) if and only if $\mathbf{w}$ satisfies the fixed point problem

$$\mathbf{w} = \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})). \quad (4.3.89)$$

Now remark that for each $\mathbf{w} \in \mathbb{L}_\eta(\mathbb{N}, M)$ we have for each $n \in \mathbb{N}$

$$\begin{aligned} \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})_n &= \mathcal{G}_1(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{w})_n + \Pi^c(T + \tilde{T})(\tilde{x}_n + \Psi(\tilde{x}_n)) \\ &\quad - \Pi^c \hat{T}(\tilde{x}_n + z_n + \Psi(\tilde{x}_n)), \end{aligned} \quad (4.3.90)$$

with $\mathcal{G}_1$ defined in (4.3.38). Thus we infer from (4.3.90) and Lemma 4.3.14 that for each $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(0, \delta_0)$:

$$\begin{aligned} \left\| \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w}) - \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \bar{\mathbf{w}}) \right\|_{\eta,+} &\leq \left\| \mathcal{G}_1(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{w}) - \mathcal{G}_1(\tilde{\mathbf{x}}, \mathbf{0}, \bar{\mathbf{w}}) \right\|_{\eta,+} \\ &\leq C_1 \delta_0 \left\| \mathbf{w} - \bar{\mathbf{w}} \right\|_{\eta,+}. \end{aligned} \quad (4.3.91)$$

Moreover since $\mathbf{z} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, X^h)}(\mathbf{0}, \delta_0)$ and since for each $n \in \mathbb{N}$ one has:

$$\begin{aligned} \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{0})_n &= \Pi^c T(\tilde{x}_n + \Psi(\tilde{x}_n)) - \Pi^c T(\tilde{x}_n + z_n + \Psi(\tilde{x}_n)) \\ &\quad + \Pi^c T(\tilde{x}_n + \Psi(\tilde{x}_n)) - \Pi^c T(\tilde{x}_n + z_n + \Psi(\tilde{x}_n)) \\ &= \int_0^1 \Pi^c DT(\tilde{x}_n + \Psi(\tilde{x}_n) + lz_n) dl \cdot z_n \\ &\quad + \Pi^c T(\tilde{x}_n + \Psi(\tilde{x}_n)) - \Pi^c T(\tilde{x}_n + z_n + \Psi(\tilde{x}_n)), \end{aligned}$$

by observing that

$$\int_0^1 \Pi^c DT(\tilde{x}_n + \Psi(\tilde{x}_n) + lz_n) dl.z_n = \int_0^1 \Pi^c [DT(\tilde{x}_n + \Psi(\tilde{x}_n) + lz_n) - DT(\tilde{x}_n)] dl.z_n,$$

it follows that there exists a constant $\tilde{C}_1$ such that

$$\left\| \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{0}) \right\|_{\eta,+} \leq \tilde{C}_1 \delta_0^2. \quad (4.3.92)$$

Thus for each $\mathbf{w} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(0, \delta_0)$ we obtain from Lemma 4.3.11 and (4.3.92) that

$$\begin{aligned} \left\| \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})) \right\|_{\eta,+} &\leq \left\| \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})) - \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{0})) \right\|_{\eta,+} \\ &\quad + \left\| \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{0})) \right\|_{\eta,+} \\ &\leq C_c \left\| \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w}) - \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{0}) \right\|_{\eta,+} + C_c \left\| \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{0}) \right\|_{\eta,+} \end{aligned}$$

with C_c defined in (4.3.40) and we infer from (4.3.91) and (4.3.92) that

$$\begin{aligned} \left\| \tilde{\mathcal{K}}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})) \right\|_{\eta,+} &\leq C_c \tilde{C}_1 \delta_0 \|\mathbf{w}\|_{\eta,+} + C_c \tilde{C}_1 \delta_0^2 \\ &\leq C_c \tilde{C}_1 \delta_0^2 + C_c \tilde{C}_1 \delta_0^2 \end{aligned}$$

Furthermore for each $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(0, \delta_0)$ we obtain similarly that

$$\left\| \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})) - \mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \bar{\mathbf{w}})) \right\|_{\eta,+} \leq C_c \tilde{C}_1 \delta_0 \|\mathbf{w} - \bar{\mathbf{w}}\|_{\eta,+}.$$

Hence up to reduce δ_0 , the operator $\mathcal{K}^c(\tilde{\mathbf{x}}, \tilde{\mathcal{G}}_1(\tilde{\mathbf{x}}, \mathbf{z}, \cdot))$ maps $\mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(0, \delta_0)$ into itself and is a strict contraction uniformly with respect to $\tilde{x}$ and $\mathbf{z}$. So by using the fact that $\tilde{x}_n = x_n$ for $n \in \mathbb{N}$ one can claim that there exists a unique $\mathbf{w} \in \mathbb{B}_{\mathbb{L}_\eta(\mathbb{N}, M)}(0, \delta_0)$ such that for each $n \in \mathbb{N}$

$$\begin{aligned} w_{n+1} &= \Pi^c(T + \tilde{T})(w_n + x_n + \Psi(w_n + x_n)) \\ &\quad - \Pi^c(T + \tilde{T})(x_n + z_n + \Psi(x_n)). \end{aligned}$$

Furthermore note that by using the fact that $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}}$ satisfies

$$x_{n+1} = \Pi^c(T + \tilde{T})(x_n + z_n + \Psi(x_n)), \quad \forall n \in \mathbb{N},$$

we obtain that

$$w_{n+1} + x_{n+1} = \Pi^c(T + \tilde{T})(w_n + x_n + \Psi(w_n + x_n)), \quad \forall n \in \mathbb{N},$$

that is

$$\mathbf{w} + \mathbf{x} = \boldsymbol{\sigma}(w_0 + x_0),$$

and we set $x = w_0 + x_0$ which is unique since w_0 is unique. The proof is complete. ■

Now we are able to give the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. Without loss of generality let us choose δ_0 small enough such that

$$\frac{6}{5}\gamma\delta_0^2 \leq \frac{\delta_0}{4\widehat{\kappa}^2} \leq \frac{\delta_0}{2}, \quad (\widehat{\kappa} \geq 1) \quad (4.3.93)$$

and

$$\|\tilde{T}\|_{C(X)} + \|\tilde{T}\|_{Lip(X)} \leq \delta_0^2,$$

so that Theorem 4.1.2 as well as Proposition 4.3.15 and Lemma 4.3.16 hold true with such choice of δ_0 . Thus we set

$$\begin{aligned} \delta_1 &:= \delta_0, \quad \delta_2 := \delta_0/2\widehat{\kappa}, \quad \delta_3 := \delta_0/2\widehat{\kappa}^2 \text{ and } \delta_4 := \delta_0/4\widehat{\kappa}^2 \\ \gamma_1 &:= \widehat{\gamma} + 2 \text{ and } \gamma_2 := 1 + 2\widehat{\kappa}, \end{aligned} \quad (4.3.94)$$

and define

$$\mathcal{W}_\eta^s(x, \delta_2) = \left\{ \begin{array}{l} \bar{x} + \bar{y} \in M \oplus X^h : \bar{x} = x + \Psi^c(x + z^s) \\ \bar{y} = \Psi(\bar{x}) + z^s + \Psi^u(x + z^s) \\ \text{with } z^s \in E_x^s \text{ and } \|z^s\| \leq \delta_2 \end{array} \right\},$$

and

$$\mathcal{W}_\eta^s(\delta_2) := \bigcup_{x \in M} \mathcal{W}_\eta^s(x, \delta_2).$$

Then all the properties (i), (ii), (iv), (vi) and (vii) stated in Theorem 4.2.1 follow directly from Proposition 4.3.15 and Lemma 4.3.16. Let us now prove the properties (iii), (v) and (vi) that need a little more arguments.

Proof of (iii): Let $u_0 \in \mathcal{W}_\eta^s(x, \delta_2)$ for some $x \in M$. Then using the definition of $\mathcal{W}_\eta^s(x, \delta_2)$ it is clear that there exists $z^s \in E_x^s$ with $\|z^s\| \leq \delta_2$ such that

$$\Pi^c u_0 = x + \Psi^c(x + z^s) \text{ and } \Pi^h u_0 = \Psi(\Pi^c u_0) + z^s + \Psi^u(x + z^s),$$

that is

$$(\Pi^c u_0, \Pi^h u_0 - \Psi(\Pi^c u_0)) \in \widehat{W}^s(x, \delta_2).$$

Then by using the definition of $\left\{ \widehat{W}^s(x, \delta_2) \right\}_{x \in M}$ in (4.3.20) we know that there exists a unique solution $(\mathbf{x}, \mathbf{z}) = \{(x_n, z_n)\}_{n \in \mathbb{N}} \subset M \times X^h$ of (4.3.19) satisfying

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) - x_n\| \leq \delta_1 \text{ and } \sup_{n \in \mathbb{N}} e^{\eta n} \|z_n\| \leq \delta_1, \quad (4.3.95)$$

with

$$x_0 = \Pi^c u_0 \text{ and } z_0 = \Pi^h u_0 - \Psi(\Pi^c u_0). \quad (4.3.96)$$

Now note by using (4.3.19) it follows easily that the sequence $\{x_n + z_n + \Psi(x_n)\}_{n \in \mathbb{N}}$ is a positive orbit of $T + \tilde{T}$. Furthermore since we have from (4.3.96) that

$$x_0 + z_0 + \Psi(x_0) = \Pi^c u_0 + \Pi^h u_0 - \Psi(\Pi^c u_0) + \Psi(\Pi^c u_0) = u_0,$$

the uniqueness of the positive orbit ensures that

$$u_n = x_n + z_n + \Psi(x_n), \quad \forall n \in \mathbb{N}.$$

Then we infer from (4.3.95) and (4.3.94) that

$$\begin{aligned} \sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) + \Psi(\sigma_n(x)) - u_n\| &= \sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) + \Psi(\sigma_n(x)) - [x_n + z_n + \Psi(x_n)]\| \\ &\leq \sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) - x_n\| \\ &\quad + \sup_{n \in \mathbb{N}} e^{\eta n} \|\Psi(\sigma_n(x)) - \Psi(x_n)\| \\ &\quad + \sup_{n \in \mathbb{N}} e^{\eta n} \|z_n\| \\ &\leq \delta_1 + \hat{\gamma} \delta_1 + \delta_1 \leq \gamma_1 \delta_1. \end{aligned}$$

where $\hat{\gamma} \delta_1$ is the corresponding Lipschitz constant of Ψ .

Proof of (v): To prove this property we will make use of Lemma 4.3.16. Let $\mathbf{u} = \{u_n\}_{n \in \mathbb{N}} \subset X$ be a positive orbit of $T + \tilde{T}$ with

$$\|\Pi^h u_0\| \leq \delta_4 = \frac{\delta_0}{4\hat{\kappa}^2} \text{ and } \sup_{n \in \mathbb{N}} \|\Pi^h u_n\| \leq \frac{\delta_1}{2} = \frac{\delta_0}{2}.$$

In fact since the sequence $\{(\Pi^c u_n, \Pi^h u_n - \Psi(\Pi^c u_n))\}_{n \in \mathbb{N}}$ is a positive orbit of (4.3.19) we just need to verify the conditions stated in Lemma 4.3.16. To do so we infer from Theorem 4.1.2 and (4.3.93) that

$$\begin{aligned} \|\Pi^h u_0 - \Psi(\Pi^h u_0)\| &\leq \delta_4 + \sup_{x \in M} \|\Psi(x)\| \\ &\leq \delta_4 + \frac{6}{5} \gamma \|\tilde{T}\|_{C(X)} \\ &\leq \delta_4 + \frac{6}{5} \gamma \delta_0^2 \leq \frac{\delta_0}{2\hat{\kappa}^2}. \end{aligned}$$

Similarly we also have

$$\|\Pi^h u_n - \Psi(\Pi^h u_n)\| \leq \frac{\delta_0}{2} + \frac{6}{5} \gamma \delta_0^2 \leq \delta_0.$$

Thus Lemma 4.3.16 applies to the sequence $\{(\Pi^c u_n, \Pi^h u_n - \Psi(\Pi^c u_n))\}_{n \in \mathbb{N}}$ and provides that there exists a unique $x \in M$ such that $(\Pi^c u_0, \Pi^h u_0 - \Psi(\Pi^c u_0)) \in \widehat{W}^s(x, \delta_2)$. Thus

$$\Pi^c u_0 = x + \Psi^c(x + z^s) \text{ and } \Pi^h u_0 - \Psi(\Pi^c u_0) = z_0^s + \Psi^u(x + z^s),$$

for some $x \in M$ and $z^s \in E_x^s$ with $\|z^s\| \leq \delta_2$. The result follows.

Proof of (vi): Let us prove that the manifolds $\mathcal{W}_\eta^s(x, \delta_1)$, $x \in M$ are disjoint or equal. Let $u_0 \in \mathcal{W}_\eta^s(x, \delta_1) \cap \mathcal{W}_\eta^s(\bar{x}, \delta_1)$ with $x, \bar{x} \in M$. We will prove that $x = \bar{x}$. To do so note that since $u_0 \in \mathcal{W}_\eta^s(x, \delta_1)$, by the property (iii) stated in Theorem 4.2.1 one has

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(x) + \Psi(\sigma_n(x)) - u_n\| \leq \gamma_2 \delta_1. \quad (4.3.97)$$

Then since

$$\sigma_n(x) - \Pi^c u_n = \Pi^c[\sigma_n(x) + \Psi(\sigma_n(x)) - u_n], \quad \forall n \in \mathbb{N},$$

we have on the one hand for each $n \in \mathbb{N}$

$$\begin{aligned} \|\sigma_n(x) - \Pi^c u_n\| &\leq \|\Pi^h[\sigma_n(x) + \Psi(\sigma_n(x)) - u_n]\| \\ &\leq \|\Pi^h\|_{\mathcal{L}(X)} \|\sigma_n(x) + \Psi(\sigma_n(x)) - u_n\| \\ &\leq \kappa \gamma_1 \delta_1 e^{-\eta n} \end{aligned} \quad (4.3.98)$$

on the other hand for each $n \in \mathbb{N}$

$$\begin{aligned} \|\Pi^h u_n - \Psi(\Pi^c u_n)\| &\leq \|\Pi^h u_n - \Psi(\sigma_n(x))\| + \|\Psi(\sigma_n(x)) - \Psi(\Pi^c u_n)\| \\ &\leq \kappa \gamma_1 \delta_1 e^{-\eta n} + \widehat{\gamma} \delta_1 \|\sigma_n(x) - \Pi^c u_n\| \\ &\leq \kappa \gamma_1 \delta_1 e^{-\eta n} + \kappa \gamma_1 \delta_1 e^{-\eta n} \end{aligned} \quad (4.3.99)$$

Therefore by choosing $p \in \mathbb{N}$ large enough such that

$$\kappa \gamma_1 \delta_1 e^{-\eta p} + \widehat{\gamma} \delta_1 \kappa \gamma_1 \delta_1 e^{-\eta p} \leq \delta_3 = \frac{\delta_0}{2\widehat{\kappa}^2} \leq \delta_1 \text{ and } \gamma_2 \delta_1 e^{-\eta p} \leq \delta_3 \leq \delta_1,$$

we obtain from (4.3.97)-(4.3.99) that

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_{n+p}(x) + \Psi(\sigma_{n+p}(x)) - u_{n+p}\| \leq \delta_3,$$

and

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\Pi^h u_{n+p} - \Psi(\Pi^c u_{n+p})\| \leq \delta_3 \text{ and } \sup_{n \in \mathbb{N}} e^{\eta n} \|\Pi^c u_{n+p} - \sigma_{n+p}(x)\| \leq \delta_3.$$

Hence since $\sigma_{n+p}(x) = \sigma_n(\sigma_p(x))$ by setting $\widehat{u}_n := u_{n+p}$, $n \in \mathbb{N}$ we obtain that

$$\sup_{n \in \mathbb{N}} e^{\eta n} \|\sigma_n(\sigma_p(x)) + \Psi(\sigma_n(\sigma_p(x))) - \widehat{u}_n\| \leq \delta_3,$$

and

$$\sup_{n \in \mathbb{N}} e^{\eta n} \left\| \Pi^h \hat{u}_n - \Psi(\Pi^c \hat{u}_n) \right\| \leq \delta_3 \text{ and } \sup_{n \in \mathbb{N}} e^{\eta n} \left\| \Pi^c \hat{u}_n - \sigma_{n+p}(x) \right\| \leq \delta_3,$$

we infer from the conditions (iv) stated in Theorem 4.2.1 that $\hat{u}_0 = u_p \in \mathcal{W}_\eta^s(\sigma_p(x), \delta_2)$. By proceeding similarly with x replaced by $\bar{x}$ one obtains (up to increase p) that $\hat{u}_0 = u_p \in \mathcal{W}_\eta^s(\sigma_p(\bar{x}), \delta_2)$. Furthermore since the sequence $\left\{ (\Pi_n^c \hat{u}, \Pi_n^h \hat{u}_n - \Psi(\Pi_n^c \hat{u})) \right\}_{n \in \mathbb{N}}$ is a positive orbit of (4.3.19) and

$$\sup_{n \in \mathbb{N}} e^{\eta n} \left\| \Pi^h \hat{u}_n - \Psi(\Pi^c \hat{u}_n) \right\| \leq \delta_3 \implies \left\| \Pi^h \hat{u}_n - \Psi(\Pi^c \hat{u}_n) \right\| \leq \delta_3 = \frac{\delta_0}{2\hat{\kappa}^2}, \quad \forall n \in \mathbb{N}.$$

we infer from Lemma 4.3.16 that $\sigma_p(\bar{x}) = \sigma_p(x)$ so that $x = \bar{x}$. The proof of (vi) is complete. ■

Chapitre 5

Nonlinear boundary conditions derived by singular perturbations in age structured population dynamics models

5.1 Introduction

Age structured models have been extensively used in the context of population dynamics mainly to take care of the history of individuals. We refer to the books [2, 46, 78, 119, 157, 163] for a nice overview and more results on this topics. In this chapter, we consider a singular perturbation problem for a class of nonlinear age structured model. As far as we know, the only work going in that direction is due to Arino et al. [5]. In [5] they investigate the singular limit of patch and age structured model, and by assuming that the spatial motion is fast, and they derive a limit model without space. Here we will focus on another modelling issue. Our goal is to derive (to understand) Ricker's [135, 136] type nonlinear boundary condition as a singular limit. The main question addressed in this chapter is to understand in which sense the limit exists.

The class of age structured model considered in this chapter is the following

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t, a) + \frac{\partial u}{\partial a}(t, a) = -\mu(a)u(t, a) \\ \quad - \underbrace{m \left(\int_0^{+\infty} \gamma_1(a)u(t, a)da \right) h(a)}_{\text{intraspecific competition}}]u(t, a), a \geq 0, \\ \\ u(t, 0) = \exp \left(- \underbrace{\int_0^{+\infty} \gamma_2(a)u(t, a)da}_{\text{limitation of births}} \right) \int_0^{+\infty} \beta(a)u(t, a)da \\ \\ u(0, \cdot) = \varphi \in L_+^p((0, +\infty); \mathbb{R}), \text{ with } 1 \leq p < +\infty. \end{array} \right. \quad (5.1.1)$$

In order to define properly this semiflow generated by (5.1.1) we make the following assumption.

Assumption 5.1.1 We assume that $m : [0, +\infty) \rightarrow [0, +\infty)$ is locally Lipschitz continuous, and

$$\mu, h \in L_+^\infty((0, +\infty); \mathbb{R}), \quad (5.1.2)$$

$$\beta, \gamma_1, \gamma_2 \in L_+^q((0, +\infty); \mathbb{R}), \quad (5.1.3)$$

where $1 < q \leq +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In this context of ecology this class of model corresponds to density dependent age structured population dynamics models. The model (5.1.1) has been considered previously by Liu and Cohen [89] for

$$m(x) = x, \quad \forall x \geq 0.$$

We also refer to [7, 28, 24, 25, 67, 70, 91, 134, 137, 138, 139, 158] for more information and results on this topic in the context of the ecology.

Here the distribution $a \rightarrow u(t, a)$ is the density of a population at time $t \geq 0$. This means that

$$\int_{a_1}^{a_2} u(t, a) da$$

is the number of individuals at time t with an age between a_1 and a_2 (with $0 \leq a_1 < a_2 \leq +\infty$). Therefore the total number of individuals in the population is given by

$$\int_0^{+\infty} u(t, a) da.$$

The term $-\mu(a)u(t, a)$ describes the mortality of individuals, and $\mu(a)$ is the age-specific mortality rate of individuals. The term $\int_0^{+\infty} \beta(a)u(t, a) da$ is the flux of new born individuals while $\beta(a)$ is the age-specific fertility rate. The term $-m\left(\int_0^{+\infty} \gamma_1(a)u(t, a) da\right) h(a)u(t, a)$ describes an intra-specific competition between individuals. Namely this term is introduced to describe the limitations for resources (food, space, etc...). Here we focus on the last term arising in the boundary condition of (5.1.1) that reads as

$$\exp\left(-\int_0^{+\infty} \gamma_2(a)u(t, a) da\right),$$

and that describes a birth limitation. In the context of fisheries, Ricker [135, 136] introduced this term to describe a cannibalism phenomenon of larva by adults. In the context forests, this term can be regarded as a term of competition for light between small trees with large trees.

In order to understand this term, we consider the following system

$$\left\{ \begin{array}{l} \frac{\partial u_\varepsilon}{\partial t}(t, a) + \frac{\partial u_\varepsilon}{\partial a}(t, a) = - \left[\underbrace{\mu(a) + m \left(\int_0^{+\infty} \gamma_1(a) u_\varepsilon(t, a) da \right) h(a)}_{\text{intraspecific competition}} \right] u_\varepsilon(t, a) \\ \quad - \left[\underbrace{\frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) \int_0^{+\infty} \gamma_2(a) u_\varepsilon(t, a) da}_{\text{fast intraspecific competition}} \right] u_\varepsilon(t, a), \quad a \geq 0, \\ u_\varepsilon(t, 0) = \int_0^{+\infty} \beta(a) u_\varepsilon(t, a) da \\ u_\varepsilon(0, \cdot) = \varphi \in L_+^p((0, +\infty); \mathbb{R}). \end{array} \right. \quad (5.1.4)$$

where

$$\mathbf{1}_{[0, \varepsilon]}(a) := \begin{cases} 1, & \text{if } a \in [0, \varepsilon], \\ 0, & \text{otherwise.} \end{cases}$$

One may observe that system (5.1.4) has a linear boundary condition. From a modelling point of view the term $\int_0^{+\infty} \gamma_2(a) u(t, a) da$ appears in system (5.1.4) as a fast predation or competition process between young individuals and older individuals.

Since the population is usually assumed to have a finite number of individuals at each time. Hence, one usually imposes

$$\int_0^{+\infty} u(t, a) da < +\infty, \forall t \geq 0,$$

so that the natural state space for the age structured model is L^1 (i.e. $p = 1$). Here in order to derive some convergence results when $\varepsilon \rightarrow 0$ we will rather consider the general case $p \in [1, +\infty)$. We refer to Magal and Ruan [96, 97, 95] and Thieme [156] for results on this topic. A direct consequence of the results in [97, (see section 3)] combined together with the application part in [96, (section 6)] we obtain the following result.

Theorem 5.1.2 *Let Assumption 5.1.1 be satisfied. Let $p \in [1, \infty)$ and $\varepsilon > 0$. There exists a unique continuous semiflow $\{\widehat{U}_\varepsilon(t)\}_{t \geq 0}$ on $L_+^p(0, \infty)$ (respectively $\{\widehat{U}(t)\}_{t \geq 0}$ on $L_+^p(0, \infty)$) such that for each $\varphi \in L_+^p(0, \infty)$ the map $t \rightarrow \widehat{U}_\varepsilon(t)\varphi = u_\varepsilon^\varphi(t)$ (respectively the map $t \rightarrow \widehat{U}(t)\varphi$ is an integrated (or mild) solution of (5.1.4) (respectively of (5.1.1)).*

In order to derive a convergence result of the solution of system (5.1.4) to the solution of system (5.1.1) when $\varepsilon \rightarrow 0$, we need to impose some extra conditions on the map γ_2 .

Assumption 5.1.3 *We assume that*

$$\gamma_2 \in W^{1, q}((0, +\infty), \mathbb{R}), \quad (5.1.5)$$

and

$$\widehat{\kappa} := \sup_{\varepsilon \in (0,1]} \frac{1}{\varepsilon} \|\gamma_2\|_{L^q([0,\varepsilon])} < +\infty. \quad (5.1.6)$$

wherein $q \in (1, \infty]$ is defined by $\frac{1}{p} + \frac{1}{q} = 1$.

The condition (5.1.6) reads as

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^q} \int_0^\varepsilon \gamma_2(a)^q da < +\infty, \text{ if } q \in (1, +\infty),$$

and

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup_{a \in [0,\varepsilon]} \gamma_2(a) < +\infty, \text{ if } q = +\infty.$$

Therefore the condition (5.1.6) combined with (5.1.5) implies

$$\gamma_2(0) = 0, \quad (5.1.7)$$

and the condition (5.1.6) is satisfied if (for example) there exists a constant $\delta > 0$, such that

$$\gamma_2(a) \leq \delta a,$$

for all $a > 0$ small enough.

Under the above conditions we obtain the following convergence result.

Theorem 5.1.4 *Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $\tau > 0$. Then for each $\varphi \in L_+^p((0, +\infty); \mathbb{R})$*

$$\sup_{t \in [0, \tau]} \left\| \widehat{U}_\varepsilon(t)\varphi - \widehat{U}(t)\varphi \right\|_{L^p} \rightarrow 0 \text{ as } \varepsilon(> 0) \rightarrow 0.$$

Remark 5.1.5 *Beyond this convergence result, this class of examples provides a class of non-densely defined Cauchy problem that can be approximated by a singular limit of densely defined semi-linear Cauchy problem. Indeed, it is well known that problem (5.1.4) has a densely defined Cauchy problem formulation, and system (5.1.1) has "only" a non-densely defined Cauchy problem formulation (see Proposition 5.2.6 for more presicions).*

Theorem 5.1.4 shows that Ricker's type boundary conditions can be interpreted as a fast predation (or limitation) process. But it seems that some extra-conditions on γ_2 (see Assumption 5.1.3) are needed to derive such a result.

We now turn to convergence results uniformly with respect to the initial distribution φ in a bounded set. In order consider this problem, we will need to introduce the following definition.

Definition 5.1.6 Let $p \in [1, \infty)$ be given. Let $\mathcal{B}$ be a subset of $L_+^p((0, +\infty); \mathbb{R})$. We define the quantity $\kappa_p(\mathcal{B}) \in [0, \infty)$ as

$$\kappa_p(\mathcal{B}) := \lim_{\delta \searrow 0^+} \sup_{\varphi \in \mathcal{B}} \left(\int_0^\delta |\varphi(a)|^p da \right)^{\frac{1}{p}}.$$

We will say that $\mathcal{B}$ is a $(p-)$ **non-atomic set at 0** if

$$\kappa_p(\mathcal{B}) = 0.$$

The first main result of this chapter is the following theorem.

Theorem 5.1.7 Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $p \in [1, \infty)$ and $\tau > 0$. Then for each $\mathcal{B}$ bounded subset of $L_+^p(0, \infty)$ there exists a constant $\tilde{C} = \tilde{C}(\gamma_1, \gamma_2, \beta, \tau, \mathcal{B}) > 0$ such that

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \in [0, \tau]} \sup_{\varphi \in \mathcal{B}} \left\| \widehat{U}_\varepsilon(t)\varphi - \widehat{U}(t)\varphi \right\|_{L^p} \leq \tilde{C} \kappa_p(\mathcal{B}).$$

Set

$$\mathcal{B}_{|(0, \delta)} := \{ \varphi_{|(0, \delta)} : \varphi \in \mathcal{B} \} \subset L^p(0, \delta),$$

for each constant $\delta > 0$.

In section 2.1, we will see that the functional $\kappa_p(\mathcal{B})$ satisfies most of the properties of a measure of non-compactness. In particular, if $\mathcal{B}_{|(0, \delta)}$ is compact in $L^p(0, \delta)$ for some $\delta > 0$, then

$$\kappa_p(\mathcal{B}) = 0.$$

Of course the converse implication is false. This question will be studied in section 2.1.

As a direct consequence of Theorem 5.1.7 we obtain the following corollary.

Corollary 5.1.8 (Strong uniform convergence) Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $\tau > 0$. Assume that

$$p \in [1, \infty),$$

and $\mathcal{B}$ is a bounded subset of $L_+^p(0, \infty)$ satifying

$$\kappa_p(\mathcal{B}) = 0.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, \tau]} \sup_{\varphi \in \mathcal{B}} \left\| \widehat{U}_\varepsilon(t)\varphi - \widehat{U}(t)\varphi \right\|_{L^p} = 0.$$

Remark 5.1.9 From this corollary we deduce that if $\mathcal{B}$ is a compact subset of $L_+^p(0, \infty)$, then

$$\widehat{U}_\varepsilon(t)\varphi \rightarrow \widehat{U}(t)\varphi \text{ as } \varepsilon \rightarrow 0 \text{ in } L^p(0, \infty),$$

and the convergence is uniform with respect to t in bounded intervals and uniform with respect to φ in $\mathcal{B}$.

Note that when $\mathcal{B}$ consists of a single point $\mathcal{B} = \{\varphi\}$, it is compact and Corollary 5.1.8 implies Theorem 5.1.4.

The second main result of this chapter is the following theorem.

Theorem 5.1.10 (Weak uniform convergence) *Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $\tau > 0$ and $M > 0$. Assume that*

$$p \in (1, \infty).$$

Then

$$\widehat{U}_\varepsilon(t)\varphi \rightharpoonup \widehat{U}(t)\varphi \text{ as } \varepsilon \rightarrow 0 \text{ for the weak topology in } L^p$$

uniformly with respect to $t \in [0, \tau]$, and $\varphi \in L^p_+(0, \infty)$ with $\|\varphi\|_{L^p} \leq M$. That is to say that, for each $\psi \in L^q(0, +\infty)$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, \tau]} \sup_{\varphi \in L^p_+(0, \infty): \|\varphi\|_{L^p} \leq M} \int_0^{+\infty} \psi(a) \left(\widehat{U}_\varepsilon(t)\varphi(a) - \widehat{U}(t)\varphi(a) \right) da = 0.$$

The plan of this chapter is the following. The section 2 is splitted into two parts. The section 2.1 summarizes some properties of κ_p . The section 2.2 is devoted to the existence of a continuous semiflow by applying integrated semigroups theory. Section 2.3 is focusing on Volterra's integral equations formulations. In section 3, we will prove Theorem 5.1.7 and Theorem 5.1.10.

5.2 Preliminary

5.2.1 Pseudo measure of non compactness

Let $\mathcal{B}$ be a bounded subset of $L^p((0, +\infty), \mathbb{R})$ for some $p \in [1, \infty)$. For each $\delta_0 > 0$ we set

$$\mathcal{B}^p_{|(0, \delta_0)} := \left\{ \|\varphi\|_{|(0, \delta_0)}^p : \varphi \in \mathcal{B} \right\} \subset L^1_+(0, \delta_0).$$

We summarize some properties of κ_p in the following proposition.

Proposition 5.2.1 *Let $p \in [1, \infty)$ and $\delta_0 > 0$. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bounded subsets of $L^p((0, +\infty); \mathbb{R})$. Then the following properties are satisfied:*

- (i) *If $\mathcal{B}^p_{|(0, \delta_0)}$ is relatively compact for the weak topology $\sigma(L^1, L^\infty)$, then $\kappa_p(\mathcal{B}) = 0$;*
- (ii) *If there exists $p' \in (p, +\infty)$, such that $\mathcal{B}_{|(0, \delta_0)}$ is a bounded in $L^{p'}_+(0, \delta_0)$ then $\kappa_p(\mathcal{B}) = 0$;*
- (iii) *If $\mathcal{B}_{|(0, \delta_0)}$ is compact for the strong topology of $L^p(0, \delta_0)$ then $\kappa_p(\mathcal{B}) = 0$;*
- (iv) $\kappa_p(\mathcal{B} + \mathcal{B}') \leq \kappa_p(\mathcal{B}) + \kappa_p(\mathcal{B}')$;
- (v) $\mathcal{B} \subset \mathcal{B}' \implies \kappa_p(\mathcal{B}) \leq \kappa_p(\mathcal{B}')$;

- (vi) $\kappa_p(\overline{\mathcal{B}}) = \kappa_p(\mathcal{B})$;
- (vii) $\kappa_p(\text{conv}(\mathcal{B})) = \kappa_p(\mathcal{B})$;
- (viii) $\kappa_p(\mathcal{B} \cup \mathcal{B}') = \max\{\kappa_p(\mathcal{B}), \kappa_p(\mathcal{B}')\}$;
- (ix) $\kappa_p(c\mathcal{B}) = |c| \kappa_p(\mathcal{B}), \forall c \in \mathbb{R}$.

Proof. The proof for (i) and (ii) are respectively direct consequences of Dunford-Pettis's Theorem (see Brezis [26, Theorem 4.30 p.115]) and De la Vallée-Poussin's theorem (see Brezis [26, Problem 23-D p.468]). To prove (iii), let $\eta > 0$ be given. Then since $\mathcal{B}|_{(0, \delta_0)}$ is compact without loss of generality we may assume that

$$\mathcal{B}|_{(0, \delta_0)} \subset \bigcup_{k=1}^m B(\varphi_k, \eta),$$

where $B(\varphi_k, \eta)$ is an open ball in $L_+^p(0, \delta_0)$. Hence for $\psi \in \mathcal{B}|_{(0, \delta_0)}$, there exists $k_0 \in \{1, \dots, m\}$ such that $\psi \in B(\varphi_{k_0}, \eta)$ and we have for each $\delta \in (0, \delta_0)$

$$\begin{aligned} \left(\int_0^\delta |\psi(a)|^p da \right)^{\frac{1}{p}} &\leq \left(\int_0^\delta |\varphi_{k_0}(a) - \psi(a)|^p da \right)^{\frac{1}{p}} + \left(\int_0^\delta |\varphi_{k_0}(a)|^p da \right)^{\frac{1}{p}}, \\ &\leq \eta + \left(\int_0^\delta |\varphi_{k_0}(a)|^p da \right)^{\frac{1}{p}} \\ &\leq \eta + \max_{k=1, \dots, m} \left(\int_0^\delta |\varphi_k(a)|^p da \right)^{\frac{1}{p}}, \quad \forall \delta \in (0, \delta_0). \end{aligned}$$

Thus letting $\delta \searrow 0$ leads us to

$$\kappa_p(\mathcal{B}) \leq \eta, \quad \forall \eta > 0$$

and the results follows. The proofs of properties (iv)-(ix) are similar to the proofs for measure of non-compactness (see Deimling [53], Martin [106], Sell and You [149]). ■

To conclude this subsection, we will characterize the fact that $\mathcal{B}$ is atomic set at 0, that is to say that

$$\kappa_1(\mathcal{B}) > 0.$$

To do so, let us first recall some definitions and basic facts on bounded variation functions.

Definition 5.2.2 A map $\eta : [0, 1] \rightarrow \mathbb{R}$ has a **bounded variation on** $[0, 1]$ if

$$V(\eta, [0, 1]) = \sup \sum_{i=1}^n |\eta(x_{i+1}) - \eta(x_i)| < +\infty$$

where the supremum is taken over all the partitions $x_1 = 0 < x_2 < \dots < x_{n+1} = 1$.

Recall that by the Riesz's representation theorem, for each $\varphi^* \in C([0, 1], \mathbb{R})^*$ the dual space of $C([0, 1], \mathbb{R})$, we can find $\eta : [0, 1] \rightarrow \mathbb{R}$ a function with bounded variation on $[0, 1]$ such that for each $\chi \in C([0, 1], \mathbb{R})$,

$$\langle \varphi^*, \chi \rangle_{C^*, C} = \int_0^1 d\eta(x) \chi(x),$$

where the last integral is a Stieltjes integral which is defined as follows

$$\int_0^1 d\eta(x) \chi(x) = \lim_{\Delta(\Gamma) \rightarrow 0} \sum_{i=1}^n [\eta(x_{i+1}) - \eta(x_i)] \chi(y_i)$$

where the limit is taken over all the partition $\Gamma = (x_1, x_2, \dots, x_{n+1})$ satisfying

$$x_1 = 0 < x_2 < \dots < x_{n+1} = 1,$$

with

$$y_i \in [x_i, x_{i+1}],$$

and

$$\Delta(\Gamma) = \max_{i=1, \dots, n} x_{i+1} - x_i.$$

As a direct consequence of the definition of the Stieltjes integral, one deduces that

$$\langle \varphi^*, \chi \rangle_{C^*, C} \leq V(\eta, [0, 1]) \|\chi\|_\infty.$$

Definition 5.2.3 An element of φ^* of $C([0, 1], \mathbb{R})^*$ is said to have a **non null mass at 0**, if and only if

$$\lim_{\varepsilon \rightarrow 0} \sup_{\chi \in C \text{ with } \text{Support}(\chi) \subset [0, \varepsilon]} \langle \varphi^*, \chi \rangle_{C^*, C} > 0.$$

We also recall that

$$\varphi^* \geq 0 \Leftrightarrow \langle \varphi^*, \chi \rangle_{C^*, C} \geq 0, \forall \chi \in C_+([0, 1], \mathbb{R}).$$

We remark that if $\chi \in C([0, 1], \mathbb{R})$ with $\text{Support}(\chi) \subset [0, \varepsilon]$ then

$$\langle \varphi^*, \chi \rangle_{C^*, C} = \int_0^\varepsilon d\eta(x) \chi(x).$$

Furthermore, if $\varphi^* \in C_+([0, 1], \mathbb{R})^*$, then the map $x \rightarrow \eta(x)$ is a non-negative and increasing function from $[0, 1]$ into $\mathbb{R}$. Hence one deduces that

$$V(\eta, [0, \varepsilon]) \geq \sup_{\chi \in C \text{ with } \text{Support}(\chi) \subset [0, \varepsilon]} \langle \varphi^*, \chi \rangle_{C^*, C} \geq \lim_{\widehat{\varepsilon} \nearrow \varepsilon} V(\varphi^*, [0, \widehat{\varepsilon}]).$$

As a consequence, φ^* has a non null mass at 0 if and only if

$$c := \lim_{\varepsilon(>0) \rightarrow 0} V(\eta, [0, \varepsilon]) > 0.$$

This implies that

$$\eta(x) - \eta(0) > c, \forall x > 0.$$

therefore

$$\langle \varphi^*, \chi \rangle_{C^*, C} \geq c \delta_0(\chi), \forall \chi \in C_+([0, 1], \mathbb{R}),$$

where δ_0 is the Dirac mass at 0.

The main theorem of this section is the following.

Theorem 5.2.4 *Let $\mathcal{B}$ be a bounded subset of $L_+^1((0, 1), \mathbb{R})$. Then*

$$\kappa_1(\mathcal{B}) > 0$$

if and only if the weak closure of $\mathcal{B}$ considered as a subset of the dual space of $C([0, 1], \mathbb{R})$ contains at least one element with non zero mass at 0. More precisely $\kappa_1(\mathcal{B}) > 0$ if and only if $\mathcal{B}$ contains a sequence φ_n such that*

$$\varphi_n \xrightarrow{weak*} \varphi^* \text{ in } C([0, 1], \mathbb{R})^*$$

and there exists a positive constant $c > 0$ such that

$$\varphi^* \geq c \delta_0 \Leftrightarrow \langle \varphi^*, \chi \rangle_{C^*, C} \geq c \delta_0(\chi), \forall \chi \in C_+([0, 1], \mathbb{R}).$$

Proof. Let $\mathcal{B} \subset L_+^1((0, +\infty), \mathbb{R})$ be a given bounded subset. Assume that it satisfies

$$\kappa_1(\mathcal{B}) = \lim_{\delta \searrow 0^+} \sup_{\varphi \in \mathcal{B}} \int_0^\delta \varphi(a) da > 0.$$

Set

$$\rho := \kappa_1(\mathcal{B}).$$

Then we can find a decreasing sequence of positive numbers $\{\delta_n\}_{n \in \mathbb{N}}$ such that

$$\delta_n \searrow 0, \text{ as } n \rightarrow +\infty,$$

and a sequence

$$\varphi_n \in \mathcal{B}$$

such that

$$\int_0^{\delta_n} \varphi_n(a) da > \rho/2.$$

Let $\varepsilon > 0$ be given. Let $\chi \in C([0, 1], \mathbb{R})$ be given such that

$$\begin{cases} \chi \geq 0, \\ \chi(x) = 1 \text{ if } x \in [0, \varepsilon/2] \\ \text{Support}(\chi) \subset [0, \varepsilon]. \end{cases}$$

Then recall that

$$\lim_{n \rightarrow \infty} \int_0^1 \varphi_n(a) \chi(a) da = \int_0^\varepsilon \chi(a) d\varphi^*(a).$$

Since $\chi \geq 0$ then one obtains that

$$\int_0^1 \varphi_n(a) \chi(a) da \geq \int_0^{\varepsilon/2} \varphi_n(a) da, \quad \forall n \geq 0.$$

Now since $\delta_n \rightarrow 0$, there exists $N = N_\varepsilon$ such that for all $n \geq N_\varepsilon$ one has $\delta_n \leq \varepsilon/2$ and therefore

$$\frac{\rho}{2} < \int_0^{\delta_n} \varphi_n(a) da \leq \int_0^{\varepsilon/2} \varphi_n(a) da$$

This implies that

$$\frac{\rho}{2} \leq \int_0^1 \chi(a) d\varphi^*(a),$$

and the result follows. The converse easily holds true. ■

5.2.2 Integrated semigroup formulation

In order to explain the meaning of mild solutions for systems (5.1.1) and (5.1.4) we first introduce a suitable functional framework. Let $p \in [1, \infty)$. Consider the Banach space X as well as its positive cone X_+ respectively defined by

$$X = \mathbb{R} \times L^p(0, \infty), \quad X_+ = \mathbb{R}_+ \times L_+^p(0, \infty),$$

and endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \right\| = |\alpha| + \|\psi\|_{L^p}, \quad \forall \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \in X_p.$$

Next consider the linear operator $A : D(A) \subset X \rightarrow X$ defined by

$$A \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} -\psi(0) \\ -\psi' - \mu(\cdot)\psi \end{pmatrix},$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{1,p}(0, \infty).$$

Set

$$X_0 := \overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p(0, \infty).$$

Consider the bounded linear operator $B : X_0 \rightarrow X$ defined by

$$B \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \beta(a) \psi(a) da \\ 0_{L^p} \end{pmatrix}.$$

Set

$$X_{0+} := X_0 \cap X_+ = \{0_{\mathbb{R}}\} \times L_+^p(0, \infty).$$

Recall that, $(A + B)_0 : D((A + B)_0) \subset X_0 \rightarrow X_0$ the part of $A + B : D(A) \subset X \rightarrow X$ in X_0 is defined by

$$(A + B)_0 \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ -\psi' - \mu(\cdot)\psi \end{pmatrix}$$

and

$$D((A + B)_0) = \left\{ \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(A) : \psi(0) = \int_0^{+\infty} \beta(a) \psi(a) da \right\}.$$

By combining the bounded perturbation result of section 3 in [96], and by applying the results of section 6 in [96] we obtain the following result.

Lemma 5.2.5 *Let Assumption 5.1.1 be satisfied. Then the linear operator $(A + B)_0 : D((A + B)_0) \subset X_0 \rightarrow X_0$ is the infinitesimal generator a strongly continuous semigroup $\{T_{(A+B)_0}(t)\}_{t \geq 0}$ of bounded linear operators on X_0 .*

Next consider the map $F : X_0 \rightarrow X$ by

$$F \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \exp \left(- \int_0^{+\infty} \gamma_2(a) \psi(a) da \right) \int_0^{+\infty} \beta(a) \psi(a) da \\ -m \left(\int_0^{+\infty} \gamma_1(a) \psi(a) da \right) h(a) \psi(a) \end{pmatrix}.$$

For each $\varepsilon > 0$, we consider the map $F_\varepsilon : X_0 \rightarrow X$

$$F_\varepsilon \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \beta(a) \psi(a) da \\ - \left[m \left(\int_0^{+\infty} \gamma_1(a) \psi(a) da \right) h(a) - \frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) \int_0^{+\infty} \gamma_2(a) \psi(a) da \right] \psi(a) \end{pmatrix}$$

and $G_\varepsilon : X_0 \rightarrow X_0$ defined by

$$G_\varepsilon \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ - \left[m \left(\int_0^{+\infty} \gamma_1(a) \psi(a) da \right) h(a) - \frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) \int_0^{+\infty} \gamma_2(a) \psi(a) da \right] \psi(a) \end{pmatrix}.$$

By identifying $u(t, \cdot)$ to $v(t) = \begin{pmatrix} 0 \\ u(t, \cdot) \end{pmatrix}$, the problem (5.1.1) can be reformulated as the following abstract non-densely Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)), \quad v(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}, \quad (5.2.1)$$

and $u_\varepsilon(t, \cdot)$ to $v_\varepsilon(t) = \begin{pmatrix} 0 \\ u_\varepsilon(t, \cdot) \end{pmatrix}$, the problem (5.1.4) can be reformulated as the following abstract non-densely Cauchy problem

$$\frac{dv_\varepsilon(t)}{dt} = Av_\varepsilon(t) + F_\varepsilon(v_\varepsilon(t)), \quad v_\varepsilon(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}. \quad (5.2.2)$$

This former problem is also equivalent to the following densely defined Cauchy problem

$$\frac{dv_\varepsilon(t)}{dt} = (A + B)_0 v_\varepsilon(t) + G_\varepsilon(v_\varepsilon(t)), \quad v_\varepsilon(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}. \quad (5.2.3)$$

Now in order to derive a global existence result for the semiflow generated by (5.2.1) and (5.2.2) it is sufficient to use the following arguments combined together with the results in [97]. First since F and F_ε are Lipschitz continuous on bounded sets, the existence and uniqueness of a maximal semiflow follows. Secondly, the positivity of solutions is obtained by observing

$$(\lambda I - A)^{-1} X_+ \subseteq X_+ \text{ for each } \lambda > 0 \text{ large enough,}$$

and that for each constant $M > 0$ there exists $\lambda = \lambda(M) > 0$ such that

$$(F + \lambda I) \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}, \quad (5.2.4)$$

whenever $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}$ and $\left\| \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\| \leq M$. Finally, the global existence of positive solutions follows from the sub-linearity argument

$$F \begin{pmatrix} 0 \\ \psi \end{pmatrix} \leq B \begin{pmatrix} 0 \\ \psi \end{pmatrix} \text{ and } F_\varepsilon \begin{pmatrix} 0 \\ \psi \end{pmatrix} \leq B \begin{pmatrix} 0 \\ \psi \end{pmatrix}$$

whenever $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}$.

Now by applying the results in [97] we obtain the following theorem.

Proposition 5.2.6 *Let Assumption 5.1.1 be satisfied. Then there exists a unique continuous semiflow $\{U(t)\}_{t \geq 0}$ on X_{0+} such that for each $x \in X_{0+}$ the map $t \rightarrow U(t)x$ is the unique integrated solution (or mild solution) of (5.2.1) that is to say that $U(\cdot)x \in C([0, +\infty), X_{0+})$ and satisfies the following properties*

$$\int_0^t U(s)x ds \in D(A), \forall t \geq 0,$$

and

$$U(t)x = x + A \int_0^t U(s)x ds + \int_0^t F(U(s)x) ds, \quad \forall t \geq 0.$$

Similarly, for each $\varepsilon \in (0, 1]$, there exists a unique continuous semiflow $\{U_\varepsilon(t)\}_{t \geq 0}$ on X_{0+} such that for each $x \in X_{0+}$ the map $t \rightarrow U_\varepsilon(t)x$ is a mild solution of (5.2.2), that is to say that $U_\varepsilon(\cdot)x \in C([0, +\infty), X_{0+})$ and satisfies the following properties

$$\int_0^t U_\varepsilon(s)x ds \in D(A), \quad \forall t \geq 0,$$

and

$$U_\varepsilon(t)x = x + A \int_0^t U_\varepsilon(s)x ds + \int_0^t F_\varepsilon(U_\varepsilon(s)x) ds, \quad \forall t \geq 0.$$

Moreover $U_\varepsilon(\cdot)x \in C([0, +\infty), X_{0+})$ is a mild solution of the densely defined Cauchy problem of (5.2.3), that is to say that

$$U_\varepsilon(t)x = T_{(A+B)_0}(t)x + \int_0^t T_{(A+B)_0}(t-s)G_\varepsilon(U_\varepsilon(s)x) ds, \quad \forall t \geq 0.$$

5.2.3 Volterra's integral equation formulation

In this subsection, we present a Volterra's integral equation formulation of the age-structured systems (5.1.1) and (5.1.4). We refer to the book of Webb [163] and Iannelli [78] for more results and informations on this subject. Set

$$\Gamma_\gamma^\varphi(t) := \int_0^{+\infty} \gamma(a)u^\varphi(t, a)da, \quad \forall t \geq 0, \text{ for } \gamma = \gamma_1, \gamma_2, \beta. \quad (5.2.5)$$

for $\gamma = \gamma_1, \gamma_2$ or β and each initial distribution $\varphi \in L_+^p((0, +\infty), \mathbb{R})$.

The solution of system (5.1.1) integrated along the characteristics is given by

$$u^\varphi(a, t) = \begin{cases} I^\varphi(t, a, a-t)\varphi(a-t), & \text{if } a \geq t, \\ I^\varphi(t, a, 0) \exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) \Gamma_\beta^\varphi(t-a), & \text{if } a \leq t. \end{cases} \quad (5.2.6)$$

where

$$I^\varphi(t, a, s) = \exp\left(-\int_s^a \left[\mu(r) + m\left(\Gamma_{\gamma_1}^\varphi(r+t-a)\right)h(r)\right] dr\right), \quad (5.2.7)$$

for each $a \geq s \geq 0$. Combining (5.2.5) and (5.2.6) we deduce that $t \rightarrow \Gamma_{\gamma_1}^\varphi(t)$, $t \rightarrow \Gamma_{\gamma_2}^\varphi(t)$ and $t \rightarrow \Gamma_\beta^\varphi(t)$ are the unique continuous solutions of the system of Volterra's integral equations:

$$\begin{aligned} \Gamma_{\gamma_1}^\varphi(t) &= \int_t^{+\infty} \gamma_1(a)I^\varphi(t, a, a-t)\varphi(a-t)da + \int_0^t \gamma_1(a)I^\varphi(t, a, 0)\Gamma_\beta^\varphi(t-a)e^{-\Gamma_{\gamma_2}^\varphi(t-a)}da, \\ \Gamma_{\gamma_2}^\varphi(t) &= \int_t^{+\infty} \gamma_2(a)I^\varphi(t, a, a-t)\varphi(a-t)da + \int_0^t \gamma_2(a)I^\varphi(t, a, 0)\Gamma_\beta^\varphi(t-a)e^{-\Gamma_{\gamma_2}^\varphi(t-a)}da, \\ \Gamma_\beta^\varphi(t) &= \int_t^{+\infty} \beta(a)I^\varphi(t, a, a-t)\varphi(a-t)da + \int_0^t \beta(a)I^\varphi(t, a, 0)\Gamma_\beta^\varphi(t-a)e^{-\Gamma_{\gamma_2}^\varphi(t-a)}da. \end{aligned} \quad (5.2.8)$$

Similarly, set

$$\Gamma_{\gamma,\varepsilon}^\varphi(t) := \int_0^{+\infty} \gamma(a) u_\varepsilon^\varphi(t, a) da, \quad (5.2.9)$$

for $\gamma = \gamma_1, \gamma_2$ or β and each initial distribution $\varphi \in L_+^p((0, +\infty), \mathbb{R})$. As before we have

$$u_\varepsilon^\varphi(t, a) = \begin{cases} I_\varepsilon^\varphi(t, a, a-t) \varphi(a-t), & \text{if } a \geq t, \\ I_\varepsilon^\varphi(t, a, 0) \Gamma_{\beta,\varepsilon}^\varphi(t-a), & \text{if } a \leq t, \end{cases} \quad (5.2.10)$$

wherein

$$I_\varepsilon^\varphi(t, a, s) := \exp \left(- \int_s^a \left[\mu(r) + m \left(\Gamma_{\gamma_1,\varepsilon}^\varphi(r+t-a) \right) h(r) \right] dr - \int_{\min(\varepsilon,s)}^{\min(\varepsilon,a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) dr \right), \quad (5.2.11)$$

for each $a \geq s \geq 0$, and $t \rightarrow \Gamma_{\gamma_1,\varepsilon}^\varphi(t)$, $t \rightarrow \Gamma_{\gamma_2,\varepsilon}^\varphi(t)$ and $t \rightarrow \Gamma_{\beta,\varepsilon}^\varphi(t)$ are the unique continuous solutions of the system of Volterra's integral equations:

$$\begin{aligned} \Gamma_{\gamma_1,\varepsilon}^\varphi(t) &= \int_t^{+\infty} \gamma_1(a) I_\varepsilon^\varphi(t, a, a-t) \varphi(a-t) da \\ &\quad + \int_0^t \gamma_1(a) I_\varepsilon^\varphi(t, a, 0) \Gamma_{\beta,\varepsilon}^\varphi(t-a) da \\ \Gamma_{\gamma_2,\varepsilon}^\varphi(t) &= \int_t^{+\infty} \gamma_2(a) I_\varepsilon^\varphi(t, a, a-t) \varphi(a-t) da \\ &\quad + \int_0^t \gamma_2(a) I_\varepsilon^\varphi(t, a, 0) \Gamma_{\beta,\varepsilon}^\varphi(t-a) da \\ \Gamma_{\beta,\varepsilon}^\varphi(t) &= \int_t^{+\infty} \beta(a) I_\varepsilon^\varphi(t, a, a-t) \varphi(a-t) da \\ &\quad + \int_0^t \beta(a) I_\varepsilon^\varphi(t, a, 0) \Gamma_{\beta,\varepsilon}^\varphi(t-a) da. \end{aligned} \quad (5.2.12)$$

5.3 Proofs of the main results

We start this section with a preliminary estimate.

Lemma 5.3.1 *Let Assumption 5.1.1 be satisfied. Let $p \in [1, +\infty)$. Then for each $\varphi \in L_+^p(0, \infty)$ we have the following upper bound*

$$\left\| \widehat{U}(t)\varphi \right\|_{L^p} \leq \|\varphi\|_{L^p(0,\infty)} e^{\frac{\|\beta\|_{L^q}^p}{p} t}, \quad \forall t \geq 0, \quad (5.3.1)$$

and

$$\left\| \widehat{U}_\varepsilon(t)\varphi \right\|_{L^p} \leq \|\varphi\|_{L^p(0,\infty)} e^{\frac{\|\beta\|_{L^q}^p}{p} t}, \quad \forall t \geq 0. \quad (5.3.2)$$

As a consequence we deduce that for each $\gamma \in L_+^q((0, \infty); \mathbb{R})$ we have for each $\varepsilon > 0$, $\varphi \in L_+^p((0, \infty); \mathbb{R})$

$$0 \leq \Gamma_{\gamma,\varepsilon}^\varphi(t) \leq \|\gamma\|_{L^q} \|\varphi\|_{L^p} e^{\frac{\|\beta\|_{L^q}^p}{p} t}, \quad \forall t \geq 0. \quad (5.3.3)$$

Proof. The proof of the two inequalities are similar. Let us for example consider inequality (5.3.2). By using (5.2.10) we obtain

$$\begin{aligned} \int_0^{+\infty} |u_\varepsilon^\varphi(t, a)|^p da &\leq \int_0^{+\infty} |\varphi(l - t)|^p dl + \int_0^t |\Gamma_{\beta, \varepsilon}^\varphi(t - r)|^p dr \\ &\leq \|\varphi\|_{L^p}^p + \int_0^t \left| \int_0^{+\infty} \beta(\sigma) u^\varphi(t - r, \sigma) d\sigma \right|^p dr \\ &\leq \|\varphi\|_{L^p}^p + \|\beta\|_{L^q}^p \int_0^t \|u^\varphi(r, \cdot)\|_{L^p}^p dr, \end{aligned}$$

and the result follows from Gronwall's lemma. ■

5.3.1 Preliminary estimates

To derive the main results of this chapter it will be important to give some estimates of the quantity

$$\int_0^{+\infty} \psi(a) (u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)) da,$$

for some given test function $\psi \in L^q(0, \infty)$. To do so we shall re-write the above quantity as follows

$$\begin{aligned} \int_0^{+\infty} \psi(a) (u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)) da &= \int_0^t \psi(a) \mathcal{H}_\varepsilon^\varphi(a, t - a) da \\ &\quad + \int_t^{+\infty} \psi(a) \mathcal{J}_\varepsilon^\varphi(a, a - t) \varphi(a - t) da, \end{aligned} \quad (5.3.4)$$

where we have set for each $t \in [0, \tau]$,

$$\mathcal{H}_\varepsilon^\varphi(a, t - a) := I^\varphi(t, a, 0) e^{-\Gamma_{\gamma_2}^\varphi(t-a)} \Gamma_{\beta}^\varphi(t - a) - I_\varepsilon^\varphi(t, a, 0) \Gamma_{\beta, \varepsilon}^\varphi(t - a), \quad \forall a \leq t, \quad (5.3.5)$$

and

$$\mathcal{J}_\varepsilon^\varphi(a, a - t) := I^\varphi(t, a, a - t) - I_\varepsilon^\varphi(t, a, a - t), \quad \forall a \geq t. \quad (5.3.6)$$

Let us first derive some Lipschitz estimates independent of ε for some useful linear form, namely of the form $\Gamma_{\gamma_2, \varepsilon}^\varphi$.

Lemma 5.3.2 *Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $\tau > 0$ and $M > 0$. Then for each $\varepsilon \in (0, 1]$ and each $\varphi \in L_+^p((0, \infty); \mathbb{R})$ with $\|\varphi\|_{L^p} \leq M$, the map $t \mapsto \Gamma_{\gamma_2, \varepsilon}^\varphi(t)$ is Lipschitz continuous on $[0, \tau]$, and we have the following estimate*

$$\left\| \Gamma_{\gamma_2, \varepsilon}^\varphi \right\|_{Lip, [0, \tau]} \leq \kappa_{\gamma_2}(\tau, M) M e^{\frac{\|\beta\|_{L^q}^p}{p} \tau} \quad (5.3.7)$$

where

$$\left\| \Gamma_{\gamma_2, \varepsilon}^\varphi \right\|_{Lip, [0, \tau]} := \sup_{t, s \in [0, \tau]: t \neq s} \frac{\left| \Gamma_{\gamma_2, \varepsilon}^\varphi(t) - \Gamma_{\gamma_2, \varepsilon}^\varphi(s) \right|}{|t - s|},$$

and the constant

$$\begin{aligned}\kappa_{\gamma_2}(\tau, M) &:= \|\gamma_2'\|_{L^q} + \|\mu\|_\infty \|\gamma_2\|_{L^q} \\ &\quad + m_\infty \|h\|_\infty \|\gamma_2\|_{L^q} + \|\gamma_2\|_{L^q} \hat{\kappa} M e^{\frac{\|\beta\|_{L^q}^p}{p} \tau}\end{aligned}$$

where $\hat{\kappa}$ is defined in (5.1.6) and

$$m_\infty := \sup_{x \in \left[0, \|\gamma_1\|_{L^q} M e^{\frac{\|\beta\|_{L^q}^p}{p} \tau}\right]} m(x).$$

Proof. By choosing the initial distribution φ smooth enough that is to say that

$$D = \left\{ \varphi \in W^{1,p}(0, +\infty) : \varphi(0) = \int_0^{+\infty} \beta(a) \varphi(a) da \right\}$$

then it is well known that $t \rightarrow u_\varepsilon^\varphi(t, \cdot)$ is from $[0, +\infty)$ into $L^p(0, +\infty)$, and for $t \geq 0$,

$$u_\varepsilon^\varphi(t, \cdot) \in W^{1,p}(0, +\infty) \text{ with } u_\varepsilon^\varphi(t, 0) = \int_0^{+\infty} \beta(a) u_\varepsilon^\varphi(t, a) da.$$

Hence for such an initial distribution $t \mapsto \Gamma_{\gamma_2, \varepsilon}^\varphi(t)$ is continuously differentiable and we have

$$\begin{aligned}\frac{d\Gamma_{\gamma_2, \varepsilon}^\varphi(t)}{dt} &= \int_0^{+\infty} \gamma_2(a) \frac{\partial u_\varepsilon^\varphi(t, a)}{\partial t} da \\ &= - \int_0^{+\infty} \gamma_2(a) \frac{\partial u_\varepsilon^\varphi(t, a)}{\partial a} da - \int_0^{+\infty} \gamma_2(a) \mu(a) u_\varepsilon^\varphi(t, a) da \\ &\quad - m\left(\Gamma_{\gamma_1, \varepsilon}^\varphi(t)\right) \int_0^{+\infty} \gamma_2(a) h(a) u_\varepsilon^\varphi(t, a) da \\ &\quad - \Gamma_{\gamma_2, \varepsilon}^\varphi(t) \int_0^{+\infty} \gamma_2(a) \frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) u_\varepsilon^\varphi(t, a) da,\end{aligned}$$

since $\gamma_2(0) = 0$, by integrating by parts we obtain

$$\begin{aligned}
\frac{d\Gamma_{\gamma_2, \varepsilon}^\varphi(t)}{dt} &= \gamma_2(0) \int_0^{+\infty} \beta(a) u_\varepsilon^\varphi(t, a) da + \int_0^{+\infty} \gamma_2'(a) u_\varepsilon^\varphi(t, a) da \\
&\quad - \int_0^{+\infty} \gamma_2(a) \mu(a) u_\varepsilon^\varphi(t, a) da \\
&\quad - m(\Gamma_{1, \varepsilon}^\varphi(t)) \int_0^{+\infty} \gamma_2(a) h(a) u_\varepsilon^\varphi(t, a) da \\
&\quad - \Gamma_{\gamma_2, \varepsilon}^\varphi(t) \int_0^{+\infty} \gamma_2(a) \frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) u_\varepsilon^\varphi(t, a) da, \\
&= \int_0^{+\infty} \gamma_2'(a) u_\varepsilon^\varphi(t, a) da \\
&\quad - \int_0^{+\infty} \gamma_2(a) \mu(a) u_\varepsilon^\varphi(t, a) da \\
&\quad - m \left(\int_0^{+\infty} \gamma_1(a) u_\varepsilon^\varphi(t, a) da \right) \int_0^{+\infty} \gamma_2(a) h(a) u_\varepsilon^\varphi(t, a) da \\
&\quad - \left(\int_0^{+\infty} \gamma_2(a) u_\varepsilon^\varphi(t, a) da \right) \left(\int_0^{+\infty} \gamma_2(a) \frac{1}{\varepsilon} \mathbf{1}_{[0, \varepsilon]}(a) u_\varepsilon^\varphi(t, a) da \right).
\end{aligned}$$

Using Holder's inequality and Lemma 5.3.1, we obtain (5.3.7).

Now let $\varphi \in L_+^p(0, +\infty)$. Since $D \cap L_+^p(0, +\infty)$ is dense in $L_+^p(0, +\infty)$, we can find a sequence $\{\varphi_n\}_{n \geq 0} \subset D \cap L_+^p(0, +\infty)$ such that

$$\varphi_n \xrightarrow{n \rightarrow +\infty} \varphi \text{ in } L^p(0, +\infty).$$

Therefore from the fact that

$$|\Gamma_{\gamma, \varepsilon}^{\varphi_n}(s) - \Gamma_{\gamma, \varepsilon}^{\varphi_n}(l)| \leq \kappa_\gamma(t, M) |s - l|, \quad \forall s, l \in [0, t], \quad \forall n \geq 0.$$

Passing to the limit $n \rightarrow \infty$ and using the continuity of the semiflow with respect to the initial value, we obtain

$$|\Gamma_{\gamma, \varepsilon}^\varphi(s) - \Gamma_{\gamma, \varepsilon}^\varphi(l)| \leq \kappa_\gamma(t, M) |s - l|, \quad \forall s, l \in [0, t], \quad \forall \varphi \in X_{0+},$$

this completes the proof of Lemma 5.3.2. ■

Now we are ready to give the estimates which are the key point to prove our main results.

Lemma 5.3.3 *Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $\tau > 0$ and $M > 0$. Then there exists a constant $C_0 = C_0(\tau, M) > 0$ such that for each $\varepsilon \in (0, 1]$,*

$$\begin{aligned}
&\int_0^t |\psi(a)| |\mathcal{H}_\varepsilon^\varphi(a, t-a)| da \\
&\leq C_0 \left[\|\psi\|_{L^q} \varepsilon^{\frac{1}{p}} + \int_0^t (|\psi(t-r)| + \|\psi\|_{L^q}) F_\varepsilon(\varphi)(r) dr \right]
\end{aligned} \tag{5.3.8}$$

whenever $\varphi \in L_+^p((0, +\infty), \mathbb{R})$ with $\|\varphi\|_{L^p} \leq M$, $t \in [0, \tau]$, $\psi \in L^q((0, +\infty), \mathbb{R})$, and

$$F_\varepsilon(\varphi)(t) = \sum_{\gamma=\gamma_1, \gamma_2, \beta} |\Gamma_{\gamma, \varepsilon}^\varphi(t) - \Gamma_\gamma^\varphi(t)|, \quad \forall t \in [0, \tau]. \quad (5.3.9)$$

Proof. Let $\varphi \in L_+^p((0, \infty), \mathbb{R})$ be such that $\|\varphi\|_{L^p} \leq M$ and $\psi \in L^q((0, +\infty), \mathbb{R})$. Let $\varepsilon \in (0, 1]$. Then for each $t \in [0, \tau]$ and each $a \in [0, t]$,

$$\mathcal{H}_\varepsilon^\varphi(a, t-a) = \left[I^\varphi(t, a, 0) e^{-\Gamma_{\gamma_2}^\varphi(t-a)} - I_\varepsilon^\varphi(a, 0) \right] \Gamma_\beta^\varphi(t-a) + I_\varepsilon^\varphi(t, a, 0) [\Gamma_\beta^\varphi(t-a) - \Gamma_{\beta, \varepsilon}^\varphi(t-a)].$$

Recall that

$$|I_\varepsilon^\varphi(t, a, 0)| \leq 1, \quad \forall a \in [0, t], \quad \forall t \leq \tau,$$

hence

$$\begin{aligned} \int_0^t |\psi(a)| |\mathcal{H}_\varepsilon^\varphi(a, t-a)| da &\leq \|\Gamma_\beta^\varphi\|_{\infty, [0, \tau]} \int_0^t |\psi(a)| \left| I^\varphi(t, a, 0) \exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) - I_\varepsilon^\varphi(t, a, 0) \right| da \\ &\quad + \int_0^t |\psi(a)| |\Gamma_\beta^\varphi(t-a) - \Gamma_{\beta, \varepsilon}^\varphi(t-a)| da. \end{aligned}$$

It remains to evaluate the integral

$$I_1 = \int_0^t |\psi(a)| \left| I^\varphi(t, a, 0) \exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) - I_\varepsilon^\varphi(t, a, 0) \right| da.$$

Since

$$I^\varphi(t, a, 0) \exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) = \exp\left(-\int_0^a \left[\mu(r) + m\left(\Gamma_{\gamma_1}^\varphi(r+t-a)\right) h(r)\right] dr - \Gamma_{\gamma_2}^\varphi(t-a)\right),$$

and

$$I_\varepsilon^\varphi(t, a, 0) = \exp\left(-\int_0^a \left[\mu(r) + m\left(\Gamma_{\gamma_1, \varepsilon}^\varphi(r+t-a)\right) h(r)\right] dr - \int_0^{\min(\varepsilon, a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(r+t-a) dr\right)$$

we deduce that

$$\begin{aligned} &\left| I^\varphi(t, a, 0) \exp\left(-\Gamma_{\gamma_2}^\varphi(t-a)\right) - I_\varepsilon^\varphi(t, a, 0) \right| \\ &\leq \|m\|_{Lip, [0, C_1]} \|h\|_{L^\infty} \int_0^a \left| \Gamma_{\gamma_1}^\varphi(r+t-a) - \Gamma_{\gamma_1, \varepsilon}^\varphi(r+t-a) \right| dr \\ &\quad + \left| \Gamma_{\gamma_2}^\varphi(t-a) - \int_0^{\min(\varepsilon, a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(r+t-a) dr \right| \\ &\leq \|m\|_{Lip, [0, C_1]} \|h\|_{L^\infty} \int_0^a \left| \Gamma_{\gamma_1}^\varphi(t-l) - \Gamma_{\gamma_1, \varepsilon}^\varphi(t-l) \right| dl \\ &\quad + \left| \Gamma_{\gamma_2}^\varphi(t-a) - \Gamma_{\gamma_2, \varepsilon}^\varphi(t-a) \right| \\ &\quad + \left| \Gamma_{\gamma_2, \varepsilon}^\varphi(t-a) - \int_0^{\min(\varepsilon, a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(r+t-a) dr \right| \end{aligned}$$

where $C_1 = \|\gamma_1\|_{L^q} M e^{\frac{\|\beta\|_{L^q}^p}{p}\tau}$. Next let us set

$$C_2 := \|m\|_{Lip,[0,C_1]} \|h\|_{L^\infty} \tau^{1/p}.$$

Then by using Holder's inequality, we obtain

$$\begin{aligned} I_1 &\leq C_2 \int_0^t \|\psi\|_{L^q} \left| \Gamma_{\gamma_1}^\varphi(t-l) - \Gamma_{\gamma_1,\varepsilon}^\varphi(t-l) \right| dl da \\ &\quad + \int_0^t |\psi(a)| \left| \Gamma_{\gamma_2}^\varphi(t-a) - \Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) \right| da \\ &\quad + \int_0^t |\psi(a)| \left| \Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) - \int_0^{\min(\varepsilon,a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) dr \right| da. \end{aligned}$$

Now it remains to evaluate

$$I_2 := \int_0^t |\psi(a)| \left| \Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) - \int_0^{\min(\varepsilon,a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) dr \right| da.$$

If $t \leq \varepsilon$, we have

$$\begin{aligned} I_2 &\leq \int_0^t |\psi(a)| \left[\Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) + \int_0^a \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) dr \right] da \\ &\leq C_3 \int_0^\varepsilon |\psi(a)| da \end{aligned}$$

with $C_3 := 2 \|\gamma_2\|_{L^q} M e^{\frac{\|\beta\|_{L^q}^p}{p}\tau}$. Hence by using Holder's inequality, we obtain

$$I_2 \leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q}.$$

If $t \geq \varepsilon$, we have

$$\begin{aligned} I_2 &: = \int_0^\varepsilon |\psi(a)| \left| \Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) - \int_0^a \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) dr \right| da \\ &\quad + \int_\varepsilon^t |\psi(a)| \left| \Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) - \int_0^\varepsilon \frac{1}{\varepsilon} \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) dr \right| da \\ &\leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q} \\ &\quad + \int_\varepsilon^t |\psi(a)| \frac{1}{\varepsilon} \int_0^\varepsilon \left| \Gamma_{\gamma_2,\varepsilon}^\varphi(t-a) - \Gamma_{\gamma_2,\varepsilon}^\varphi(r+t-a) \right| dr da \\ &\leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q} \\ &\quad + \left\| \Gamma_{\gamma_2,\varepsilon}^\varphi \right\|_{Lip,[0,\tau]} \int_\varepsilon^t |\psi(a)| \frac{1}{\varepsilon} \int_0^\varepsilon r dr da \\ &= C_3 \varepsilon^{1/p} \|\psi\|_{L^q} + \left\| \Gamma_{\gamma_2,\varepsilon}^\varphi \right\|_{Lip,[0,\tau]} \frac{\varepsilon}{2} \int_0^\tau |\psi(a)| da \\ &\leq C_3 \varepsilon^{1/p} \|\psi\|_{L^q} + \left\| \Gamma_{\gamma_2,\varepsilon}^\varphi \right\|_{Lip,[0,\tau]} \frac{\varepsilon}{2} \tau^{1/p} \|\psi\|_{L^q}. \end{aligned}$$

the result follows. ■

Lemma 5.3.4 *Let Assumptions 5.1.1 and 5.1.3 be satisfied. Let $\tau > 0$ and $M > 0$. Then there exists a constant $\widehat{C}_0 = \widehat{C}_0(\tau, M) > 0$ such that for each $\varepsilon \in (0, 1]$,*

$$\begin{aligned} & \int_t^{+\infty} |\psi(a)| |\mathcal{J}_\varepsilon^\varphi(a, a-t)| |\varphi(a-t)| da \\ & \leq \widehat{C}_0 \left[\|\psi\|_{L^q} \int_0^t \left| \Gamma_{\gamma_1}^\varphi(r) - \Gamma_{\gamma_1, \varepsilon}^\varphi(r) \right| dr + \int_0^\varepsilon |\psi(a+t)| |\varphi(a)| da \right], \end{aligned} \quad (5.3.10)$$

whenever $\varphi \in L_+^p((0, +\infty); \mathbb{R})$ with $\|\varphi\|_{L^p} \leq M$, $t \in [0, \tau]$, and $\psi \in L^q((0, +\infty); \mathbb{R})$.

Proof. Let $M > 0$ and $\tau > 0$ be given. Let $\varphi \in L_+^p((0, +\infty); \mathbb{R})$ be given such that $\|\varphi\|_{L^p} \leq M$. Then we have for each $a \geq t$ and for each $t \in [0, \tau]$,

$$\begin{aligned} \mathcal{J}_\varepsilon^\varphi(a, a-t) &= I_\varepsilon^\varphi(t, a, a-t) - I^\varphi(t, a, a-t) \\ &= \exp \left(- \int_{a-t}^a \left[\mu(r) + m \left(\Gamma_{\gamma_1, \varepsilon}^\varphi(r+t-a) \right) h(r) \right] dr - \int_{\min(\varepsilon, a-t)}^{\min(\varepsilon, a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(r+t-a) dr \right) \\ &\quad - \exp \left(- \int_{a-t}^a \left[\mu(r) + m \left(\Gamma_{\gamma_1}^\varphi(r+t-a) \right) h(r) \right] dr \right) \\ &\leq \left| \int_{a-t}^a \left[m \left(\Gamma_{\gamma_1, \varepsilon}^\varphi(r+t-a) \right) - m \left(\Gamma_{\gamma_1}^\varphi(r+t-a) \right) \right] h(r) dr \right| \\ &\quad + \left| \int_{\min(\varepsilon, a-t)}^{\min(\varepsilon, a)} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(r+t-a) dr \right| \end{aligned}$$

Let $\psi \in L^q((0, +\infty), \mathbb{R})$. Then

$$\begin{aligned} I_1 &:= \int_t^{+\infty} |\psi(a)| |J_\varepsilon^\varphi(a, a-t)| |\varphi(a-t)| da \\ &\leq \|m\|_{Lip, [0, C_1]} \|h\|_{L^\infty} \int_t^{+\infty} |\psi(a)| \left(\int_0^t \left| \Gamma_{\gamma_1}^\varphi(l) - \Gamma_{\gamma_1, \varepsilon}^\varphi(l) \right| dl \right) |\varphi(a-t)| da \\ &\quad + \int_t^{+\infty} |\psi(a)| \left| \int_{\min(\varepsilon, a-t)+t-a}^{\min(\varepsilon, a)+t-a} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(l) dl \right| |\varphi(a-t)| da \\ &= \|m\|_{Lip, [0, C_1]} \|h\|_{L^\infty} \int_t^{+\infty} |\psi(a)| \left(\int_0^t \left| \Gamma_{\gamma_1}^\varphi(l) - \Gamma_{\gamma_1, \varepsilon}^\varphi(l) \right| dl \right) |\varphi(a-t)| da \\ &\quad + \int_t^{t+\varepsilon} |\psi(a)| \left| \int_{\min(\varepsilon, a-t)+t-a}^{\min(\varepsilon, a)+t-a} \frac{1}{\varepsilon} \Gamma_{\gamma_2, \varepsilon}^\varphi(l) dl \right| |\varphi(a-t)| da \end{aligned}$$

therefore we obtain

$$\begin{aligned} I_1 &\leq \|m\|_{Lip, [0, C_1]} \|h\|_{L^\infty} \int_t^{+\infty} |\psi(a)| \left(\int_0^t \left| \Gamma_{\gamma_1}^\varphi(l) - \Gamma_{\gamma_1, \varepsilon}^\varphi(l) \right| dl \right) |\varphi(a-t)| da \\ &\quad + \left\| \Gamma_{\gamma_2, \varepsilon}^\varphi \right\|_{\infty, [0, \tau]} \int_t^{t+\varepsilon} |\psi(a)| |\varphi(a-t)| da \end{aligned}$$

and the result follows. ■

5.3.2 Proof of Theorem 5.1.7

Lemma 5.3.5 (Gronwall's like inequality) Let $\tau > 0$ be fixed and $f \in L^p_+((0, \tau), \mathbb{R})$. Assume in addition that there exist two constants $\alpha \geq 0$ and $\beta \geq 0$ such that

$$0 \leq f(t) \leq \alpha + \beta \left(\int_0^t f(s)^p ds \right)^{\frac{1}{p}}, \text{ for almost every } t \in (0, \tau). \quad (5.3.11)$$

Then

$$f(t) \leq 2\alpha \exp\left(\frac{2^p \beta^p}{p} t\right), \text{ for almost every } t \in (0, \tau).$$

Proof. We first observe that

$$(a + b)^p \leq (2 \max(a, b))^p \leq 2^p (a^p + b^p) \quad (5.3.12)$$

whenever $a \geq 0, b \geq 0$ and $p \in [1, \infty)$. The inequality (5.3.11) implies

$$f(t)^p \leq \left(\alpha + \beta \left(\int_0^t f(s)^p ds \right)^{\frac{1}{p}} \right)^p$$

and by using (5.3.12) we obtain

$$f(t)^p \leq 2^p \left(\alpha^p + \beta^p \int_0^t f(s)^p ds \right)$$

and by using Gronwall inequality in L^1 the result follows. ■

Proof. (of Theorem 5.1.7) Let $\mathcal{B} \subset L^p_+((0, +\infty); \mathbb{R})$ be a given bounded set. Let $\tau > 0$ be given and fixed. Since $\mathcal{B}$ is bounded, we set

$$M := \sup_{\varphi \in \mathcal{B}} \|\varphi\|_{L^p}.$$

Recalling that

$$F_\varepsilon(\varphi)(t) := \sum_{\gamma=\gamma_1, \gamma_2, \beta} |\Gamma_{\gamma, \varepsilon}^\varphi(t) - \Gamma_\gamma^\varphi(t)|,$$

we first observe that

$$F_\varepsilon(\varphi)(t) \leq \tilde{C}_0 \|u_\varepsilon^\varphi(t, \cdot) - u^\varphi(t, \cdot)\|_{L^p},$$

with

$$\tilde{C}_0 := (\|\gamma_1\|_{L^q} + \|\gamma_2\|_{L^q} + \|\beta\|_{L^q}).$$

Moreover it is well known that

$$\|u_\varepsilon^\varphi(t, \cdot) - u^\varphi(t, \cdot)\|_{L^p} = \sup_{\substack{\psi \in L^q((0, +\infty); \mathbb{R}) \\ \|\psi\|_{L^q} \leq 1}} \int_0^{+\infty} \psi(a) (u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)) da.$$

Let $\psi \in L^q((0, +\infty); \mathbb{R})$ with $\|\psi\|_{L^q} \leq 1$. We have

$$\int_0^{+\infty} \psi(a) (u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)) da = \int_0^t \psi(a) \mathcal{H}_\varepsilon^\varphi(a, t-a) da \\ + \int_t^{+\infty} \psi(a) \mathcal{J}_\varepsilon^\varphi(a, a-t) \varphi(a-t) da.$$

therefore due to Lemma 5.3.4 and Lemma 5.3.3, we obtain for each $\varepsilon \in (0, 1]$, each $t \in [0, \tau]$ and each $\varphi \in \mathcal{B}$ that

$$F_\varepsilon(\varphi)(t) \leq C_1 \left[\varepsilon^{\frac{1}{p}} + \int_0^t (|\psi(t-r)| + 2) F_\varepsilon(\varphi)(r) dr + \int_0^\varepsilon |\psi(a+t)| |\varphi(a)| da \right],$$

where

$$C_1 := \tilde{C}_0 (C_0 + \widehat{C}_0).$$

So by using Holder's inequality, we deduce that

$$F_\varepsilon(\varphi)(t) \leq C_1 \left[\left[\varepsilon^{\frac{1}{p}} + \left(\int_0^\varepsilon |\varphi(a)|^p da \right)^{1/p} \right] + [\|\psi\| + 2]_{L^q(0, \tau)} \left(\int_0^t F_\varepsilon(\varphi)(r)^p dr \right)^{1/p} \right],$$

and this implies the following:

$$F_\varepsilon(\varphi)(t) \leq C_1 \left[\left[\varepsilon^{\frac{1}{p}} + \left(\int_0^\varepsilon |\varphi(a)|^p da \right)^{1/p} \right] + [1 + 2\tau^{1/q}] \left(\int_0^t F_\varepsilon(\varphi)(r)^p dr \right)^{1/p} \right].$$

By applying Gronwall's like inequality given in Lemma 5.3.5, we obtain

$$F_\varepsilon(\varphi)(t) \leq 2C_1 \left[\varepsilon^{\frac{1}{p}} + \left(\int_0^\varepsilon |\varphi(a)|^p da \right)^{1/p} \right] \exp \left(\frac{2^p C_1^p [1 + 2\tau^{1/q}]^p}{p} \tau \right),$$

the proof of Theorem 5.1.7 is completed. ■

5.3.3 Proof of Theorem 5.1.10

By definition we have

$$F_\varepsilon(\varphi)(t) = \sum_{\gamma=\gamma_1, \gamma_2, \beta} \left| \int_0^\infty \gamma(a) (u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)) da \right| \\ \leq \int_0^\infty \left(\sum_{\gamma=\gamma_1, \gamma_2, \beta} |\gamma(a)| \right) |u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)| da.$$

Set

$$\chi := \sum_{\gamma=\gamma_1, \gamma_2, \beta} |\gamma(\cdot)| \in L_+^q(0, +\infty).$$

By using the same argument as in the proof of Theorem 5.1.7 with $\psi = \chi$, we obtain

$$F_\varepsilon(\varphi)(t) \leq \widehat{C}_1 \left[\varepsilon^{\frac{1}{p}} + \int_0^t (|\chi(t-r)| + 2) F_\varepsilon(\varphi)(r) dr + \int_0^\varepsilon |\chi(a+t)| |\varphi(a)| da \right],$$

for some constant $\widehat{C}_1 = \widehat{C}_1(\tau, M, \|\chi\|_{L^q}) > 0$.

Now Holder's inequality leads us to

$$F_\varepsilon(\varphi)(t) \leq \widehat{C}_1 \left[\left(\varepsilon^{\frac{1}{p}} + M \left(\int_0^\varepsilon |\chi(a+t)|^q da \right)^{1/q} \right) + [\|\chi\| + 2\|_{L^q(0,\tau)}] \left(\int_0^t F_\varepsilon(\varphi)(r)^p dr \right)^{1/p} \right].$$

Hence Gronwall's inequality (see Lemma 5.3.5) provides

$$F_\varepsilon(\varphi)(t) \leq 2\widehat{C}_1 \left[\varepsilon^{\frac{1}{p}} + M \left(\int_t^{t+\varepsilon} |\chi(a)|^q da \right)^{1/q} \right] \exp \left(\frac{2^p \widehat{C}_1^p [\|\chi\| + 2\|_{L^q(0,\tau)}]^p}{p} \tau \right).$$

Next, we infer from the continuity of the map $t \rightarrow \int_0^t |\chi(a)|^q da$ that

$$\sup_{t \in [0, \tau] \text{ and } \varphi \in \mathcal{B}} F_\varepsilon(\varphi)(t) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Let $\psi \in L^p(0, +\infty)$ be given and fixed. We have

$$\begin{aligned} I(t) &:= \int_0^{+\infty} \psi(a) (u_\varepsilon^\varphi(t, a) - u^\varphi(t, a)) da = \int_0^t \psi(a) \mathcal{H}_\varepsilon^\varphi(a, t-a) da \\ &\quad + \int_t^{+\infty} \psi(a) \mathcal{J}_\varepsilon^\varphi(a, a-t) \varphi(a-t) da. \end{aligned}$$

And by using the same arguments as in the proof of Theorem 5.1.7 we obtain

$$|I(t)| \leq \widehat{C}_1 \left[\varepsilon^{\frac{1}{p}} + \int_0^t (|\psi(t-r)| + 2) F_\varepsilon(\varphi)(r) dr + \int_0^\varepsilon |\psi(a+t)| |\varphi(a)| da \right],$$

that completes the proof of the result.

Chapitre 6

A singularly perturbed Delay Differential Equation modeling nosocomial infections

6.1 Introduction

In this chapter, we consider a model describing bacterial nosocomial infections (i.e. hospital acquired infections). In such a problem the pathogens (bacteria) are assumed to be transmitted from the patients to HealthCare Workers (HCW) and from the HCWs to the patients. A Susceptible (S) patients may become newly Infected (I) patient by contact with a colonized HCW. Typically, the colonization of HCWs is of a superficial form such as dirty hands that carry the pathogen. The HCWs are decomposed into the Uncolonized (H_U) and the Colonized (H_C). The fluxes of patients and HCWs are summarized in Figure 6.1.

The time scales for the process of colonization for HCWs and the process of infection for patients are fairly different. HCWs may recover from the colonization due to hygiene or due to the turn over in the medical unit (i.e. a shifts of 8 Hours). When a HCW becomes colonized, the HCW is assumed to be immediately capable to transmit the pathogen to a patient. The average time during which the HCW stays colonised is approximatively one or two hours. For a patient the infection process is much longer, and a patient needs several days to be capable to transmit the pathogen to HCWs. Therefore when a patient become infected, the period of time necessary to transmit the pathogen from patients to an HCW is much longer. In this sense there is (at least) one order of magnitude between the time scale for HCWs and the time scale for patients.

In this chapter, we will consider a special version of a model presented in Magal and McCluskey [94, Section 7]. By using the usual idea coming from slow-fast systems, we will cancel out the HCWs component of the system. Similar idea was already used in D'Agata et al. [48] (without mathematical justification), and as in [48] we will endup with a single equation for patients. The model derived turns to be similar (but different) to the one

introduced in Webb et al. in [164]. A practical motivation for this study comes from the fact that (usually) no data are available for the colonized HCWs. Therefore it also makes sense to try to get rid of the HCWs component in such a problem.

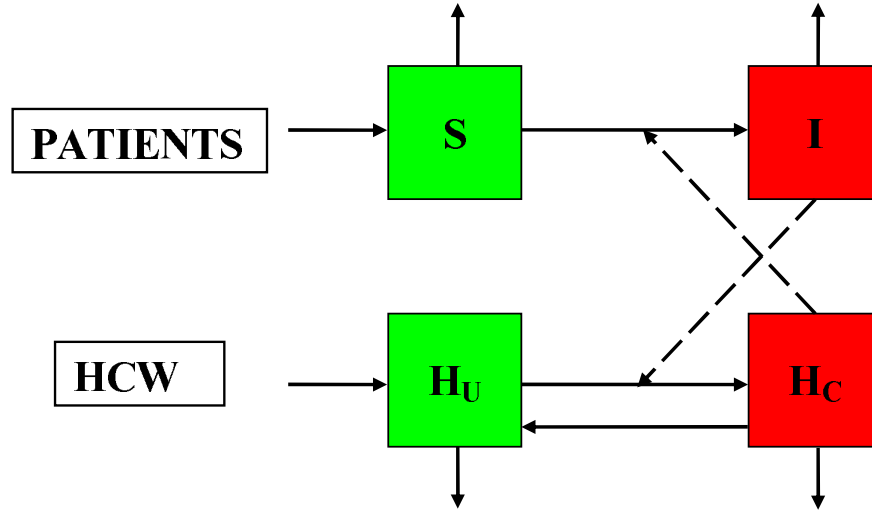


Figure 6.1: The figure represents a diagram of the individuals fluxes used to describe hospital acquired infections. In this diagram each solid arrow represents a flux of individuals, while the dashed arrows represent the influence of either infected patients or colonized HCWs on the pathogen acquisition.

Let $S(t)$ be the number of susceptible patients at time t , and $i(t, a)$ be the density of infected patients who have been infected for a duration a at time t . This means that

$$\int_{a_-}^{a_+} i(t, a) da$$

is the number of infected patients having an age of infection (i.e. the time since infection) $0 \leq a_- \leq a \leq a_+$. The age of infection is introduced in such a context to account for antibiotic treatment in the model. Let $H_U(t)$ be the number of un-colonized HCWs, $H_C(t)$ be the number of colonized HCWs. Assume that the number of patients and HCWs is constant in the hospital (or the intensive care unit), therefore we must have

$$S(t) + \int_0^{+\infty} i(t, a) da = N_P \quad \text{and} \quad H_U(t) + H_C(t) = N_H. \quad (6.1.1)$$

$$\text{Patient equation} \quad \begin{cases} \frac{dS(t)}{dt} = \nu_R N_P - \nu_R S(t) - \frac{\nu_V P_I}{N_H} \beta_V S(t) H_C(t) \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -\nu_R i(t, a), \\ i(t, 0) = \frac{\nu_V P_I}{N_H} \beta_V S(t) H_C(t) \\ S(0) = S_0 \geq 0 \\ i(0, \cdot) = i_0 \in L^1_+(0, +\infty), \end{cases} \quad (6.1.2)$$

The rate ν_V at which contacts between staff and patients occur is taken to be constant. The probability for a patient to have contact with a HCW is $\beta_V := N_H/N_p$ and when a contact occurs the probability that is with a contaminated HCW is the fraction $\frac{H_C}{N_H}$ of HCWs that are colonized, where N_H is the total number of HCWs and N_p is the total number of patients. Finally, given a contact between a susceptible patient and a contaminated HCW, the probability that the patient becomes infected is $P_I \in (0, 1]$. Thus, the rate at which incidence of new infections in the patient population is $\frac{\nu_V P_I}{N_H} \beta_V S H_C$. All newly infected patients enter the infected population with infection age 0.

Next, we determine equations for the HCWs, beginning with the incidence. while the system describing the HCWs colonization is the following

$$\text{HCW equation} \quad \begin{cases} \frac{dH_U(t)}{dt} = \nu_H N_H - \nu_H H_U(t) - \frac{\nu_V P_C}{N_P} H_U(t) \int_0^\infty \gamma(a) i(t, a) da \\ \frac{dH_C(t)}{dt} = \frac{\nu_V P_C}{N_P} H_U(t) \int_0^\infty \gamma(a) i(t, a) da - \nu_H H_C(t) \\ H_U(0) = H_{U0} \geq 0 \\ H_C(0) = H_{C0} \geq 0. \end{cases} \quad (6.1.3)$$

As in the patient equations, contacts occur at rate ν_V . Let $P_C \in (0, 1]$ be the maximum probability that a contact between an infected patient and a uncontaminated HCW leads to a new contamination. The relative infectivity of patients of infection age a is $\gamma(a)$ and the density of contacts with patients of infection age a is $\frac{i(t, a)}{N_P}$, where N_P is the total number of patients. Thus, the incidence of new contaminations in the HCW population is $\frac{\nu_V P_C}{N_P} H_U \int_0^{+\infty} \gamma(a) i(t, a) da$. The decontamination rate for HCWs is ν_H .

The meaning of the parameters, as well as the values used in simulations, are listed in Table 1.

Symbol	Interpretation	Value	Units
N_P	total number of patients	400*	-
N_H	total number of HCWs	100*	-
$T_H = \frac{1}{\nu_H}$	average time during which an HCW stays colonized	1*	hours
$T_V = \frac{1}{\nu_V}$	average duration of visit to a patient by a HCW plus time to the next visit	1.58***	hours
$T_R = \frac{1}{\nu_R}$	average time spent in the hospital for an infected patient	28*	days
P_I	probability for a patient to be infected by a HCW per visit	0.06**	-
P_C	probability for a HCW to be colonized by a patient per visit	0.4**	-
$\gamma(a)$	relative infectivity of patients of infection age a		
τ	time necessary to become infectious	9.86	days

Table 1: The parameter values are taken from [48], and are used in numerical simulations. Values marked with * were estimated for Beth Israel Deaconess Medical Center, Boston.

Values marked with ** were estimated for Cook County Hospital, Chicago. The parameter value τ is estimated in this work.

By using (6.1.1), the system (6.1.2)-(6.1.3) can be reduced to the following system

$$\begin{cases} \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -\nu_R i(t, a), \\ i(t, 0) = \frac{\nu_V P_I \beta_V}{N_H} \left(N_P - \int_0^{+\infty} i(t, a) da \right) H_C(t) \\ \frac{dH_C(t)}{dt} = \frac{\nu_V P_C}{N_P} (N_H - H_C(t)) \int_0^\infty \gamma(a) i(t, a) da - \nu_H H_C(t) \\ i(0, \cdot) = i_0 \in L_+^1(0, +\infty), H_C(0) = H_{C0} \geq 0. \end{cases} \quad (6.1.4)$$

Assuming for simplicity that

$$\gamma(a) = \begin{cases} 1, & a \in [\tau, +\infty) \\ 0, & \text{otherwise.} \end{cases} \quad (6.1.5)$$

By setting

$$I(t) := \int_0^{+\infty} i(t, a) da, \quad (6.1.6)$$

the system (6.1.4) can be rewritten for $t \geq \tau$,

$$\begin{cases} \frac{dI(t)}{dt} = \frac{\nu_V P_I \beta_V}{N_H} (N_P - I(t)) H_C(t) - \nu_R I(t) \\ \frac{dH_C(t)}{dt} = \frac{\nu_V P_C}{N_P} (N_H - H_C(t)) e^{-\nu_R \tau} I(t - \tau) - \nu_H H_C(t) \\ I(t, \cdot) = I_0(t) \geq 0, \forall t \in [-\tau, 0], H_C(0) = H_{C0} \geq 0. \end{cases} \quad (6.1.7)$$

The global asymptotic behavior of system (6.1.2)-(6.1.3) has been studied in [94]. For example the basic reproductive number for system (6.1.7) is given by

$$\mathcal{R}_0 = \sqrt{\frac{\nu_V^2}{\nu_H \nu_R} P_I \beta_V P_C e^{-\nu_R \tau}}. \quad (6.1.8)$$

The above formula suggests that the parameters τ play a crucial role for the persistence (or the invasion) of resistant pathogens. Clearly, these parameters are related to antibiotic treatment (see D'Agata et al. [48]). At the level of a single patient, antibiotic treatment provides an in-host environment that selects in favour of the resistant strain. As a consequence, due to antibiotic treatment, patients may become more likely to transmit resistant pathogens. But the effects of treatments for a single patient is a fairly complex system. Some mechanisms involved in such problems have been described in [49, 3] (see also references therein).

As far as we know no singular perturbation results are known for such age structured systems. Moreover relatively few examples have been considered in the literature. We refer

to Arino et al. [5] and Ducrot et al. [55] for two examples of singularly perturbed age structured systems. One may also observe that for the functional differential equations (6.1.7) (as far as we know) the usual theory does not apply (see Hale and Verduyn Lunel [74], Diekmann et al. [54], Arino et al. [6], and Smith [153]). We also refer to Magalhães [99, 98] and Artstein and Slemrod [9] for more result on singular perturbations theory in the context of delay differential equations.

In order to introduce the singularly perturbed system a discussion of the processes is in order. First the goal of the model is to describe the spread of the hospital epidemic over several months or a year. We observe that on the scale of one month year, a HCW visit of an average period $\frac{1}{\nu_V} \approx 1.5$ hours is very short. Thus, we should use the idea of slow-fast systems which as been successfully used for several classes of bio-medical problems (see Auger et al. [10], Hek [75]). The fast process corresponds here to HCW visit during which that contamination may happen while the slow processes correspond to patient infection, admission, and exit. Here we set $\frac{1}{\nu_V} = \varepsilon \ll 1$. In order to re-scale (6.1.7) with respect to ε , let us first notice that parameter $\frac{1}{\nu_H}$, the average time during which an HCW stays colonized is also related to ε . Indeed the longer the visit, the larger the bacterial load, and therefore the longer the time during which an HCW stays colonized. Here we shall assume a simple proportional law, that is

$$\nu_H = \gamma_H \nu_V = \frac{\gamma_H}{\varepsilon}.$$

Let us also mention that the probability P_I for a patient to become infected during a HCW visit also depends on $\frac{1}{\nu_V}$. Indeed since patients are motionless, the contamination process arises to due manipulation of the material, the patients themselves. As a consequence, the probability P_I can be decomposed as $P_I = \hat{P}_I \times \frac{1}{\nu_V}$ where $\hat{P}_I$ denotes the probability for a patient to become infected during a unit time of HCW visit. Here we assume that $\hat{P}_I$ is fixed so that

$$P_I = \hat{P}_I \varepsilon.$$

On the other hand the contamination process of an HCW by the contaminated patient and described by P_C is rather different. Indeed, the contamination of the environment holds as soon as the patient is contaminated. This environmental contamination occurs due to the bacterial spread as well as the manipulation of the material by the HCW. As a consequence, a contaminated patient and his environment ensure a rather strong probability of HCW colonisation even if the visit time is short. Hence we decompose the probability P_C into two terms $P_C = P_C^0 + \hat{P}_C \times \frac{1}{\nu_V}$ wherein $P_C^0 > 0$ corresponds to the initial probability of an HWC to become colonized as soon as he enters the contaminated environment while $\hat{P}_C$ corresponds to an additional probability to become colonized per unit time of visit. As a consequence one has

$$P_C := P_C(\varepsilon) = P_C^0 + \hat{P}_C \varepsilon.$$

Thus, the above system (6.1.7) becomes

$$\begin{cases} \frac{dI(t)}{dt} = \frac{\hat{P}_I \beta_V}{N_H} \left(N_P - \int_0^{+\infty} i(t, a) da \right) H_C(t) - \nu_R I(t) \\ \varepsilon \frac{dH_C(t)}{dt} = \frac{P_C(\varepsilon)}{N_P} (N_H - H_C(t)) e^{-\nu_R \tau} I(t - \tau) - \gamma_H H_C(t) \\ I(t, \cdot) = I_0(t) \geq 0, \forall t \in [-\tau, 0], H_C(0) = H_{C0} \geq 0. \end{cases} \quad (6.1.9)$$

Formally, when $\varepsilon = 0$ the second equation of the above system gives

$$H_C(t) = h(I(t - \tau)) \quad (6.1.10)$$

where $h : [0, +\infty) \rightarrow [0, +\infty)$

$$h(x) := \frac{\beta N_H x}{\gamma_H + \beta x}, \text{ with } \beta := \frac{P_C^0}{N_P} e^{-\nu_R \tau}.$$

The so called reduced system corresponds to the first equation of (6.1.9) (i.e. the slow equation of (6.1.9)) in which $H_C(t)$ is replaced by $h(I(t - \tau))$. Therefore the results model is nothing but the following single delay differential equation

$$\frac{dI(t)}{dt} = \frac{\hat{P}_I \beta_V}{N_H} (N_P - I(t)) h(I(t - \tau)) - \nu_R I(t). \quad (6.1.11)$$

In section 2, we will provide a careful comparison between the solutions of system (6.1.9) and the solution of system (6.1.11). A question left for future investigation is the comparison of the original model with age of infection with the following model

$$\begin{cases} \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -\nu_R i(t, a), \\ i(t, 0) = \frac{\hat{P}_I \beta_V}{N_H} \left(N_P - \int_0^{+\infty} i(t, a) da \right) h \left(\int_0^{+\infty} \gamma(a) i(t, a) da \right) \\ i(0, \cdot) = i_0 \in L_+^1(0, +\infty). \end{cases} \quad (6.1.12)$$

One may observe that this reduced model also corresponds to the model introduced in Webb et al. in [164]. We refer to [32, 50, 68, 109] (and the references therein) for more results on nosocomial infections modelling.

The plan of this chapter is the following. In section 2, we summarize the main results of this chapter. Section 3 is devoted to deriving preliminary result that will be used to the proof of Theorem 6.2.1 in Section 4. Finally Section 5 is devoted to the study of the convergence as $\varepsilon \rightarrow 0$ to the unique heteroclinic solution of the reduced system.

6.2 Main results

For simplicity we fix $\hat{P}_C = 0$, so we assume that $P_C(\varepsilon) \equiv P_C^0$. Then by introducing the prevalence $x^\varepsilon = \frac{I}{N_P}$ and $y^\varepsilon = \frac{H_C}{N_H}$, system (6.1.9) can be rewritten as the following delay

differential equations

$$\begin{cases} \frac{dx^\varepsilon(t)}{dt} = -\mu x^\varepsilon(t) + \alpha y^\varepsilon(t)(1 - x^\varepsilon(t)), \quad \forall t \geq 0, \\ \varepsilon \frac{dy^\varepsilon(t)}{dt} = -\nu y^\varepsilon(t) + \beta x^\varepsilon(t - \tau)(1 - y^\varepsilon(t)), \quad \forall t \geq 0, \\ y^\varepsilon(0) = y_0 \in \mathbb{R}, \\ x^\varepsilon(\theta) = \varphi(\theta), \quad \forall \theta \in [-\tau, 0], \end{cases} \quad (6.2.1)$$

wherein we have set

$$\mu = \nu_R, \quad \alpha = \frac{\widehat{P}_I \beta_V}{N_H}, \quad \nu = \gamma_H \text{ and } \beta = \frac{P_C^0}{N_P} e^{-\nu_R \tau}, \quad (6.2.2)$$

while $\varepsilon \in (0, 1)$ is a small parameter. Note that using the above notations, R_0 defined in (6.1.8) re-writes as

$$R_0 := \sqrt{\frac{\alpha \beta}{\mu \nu}}. \quad (6.2.3)$$

Let $C := C([- \tau, 0], \mathbb{R})$ be the Banach space of continuous functions from $[- \tau, 0]$ to $\mathbb{R}$ endowed with the supremum norm

$$\|\varphi\|_C := \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|.$$

By taking $\varepsilon = 0$ in equation (6.2.1) and solving the second equation in y , we obtain $y(t) = \frac{\beta x(t - \tau)}{\beta x(t - \tau) + \nu}$. By replacing y by this expression in the first equation of system (2.4.29), we obtain the reduced equation of (6.2.1)

$$\begin{cases} \frac{dx(t)}{dt} = -\mu x(t) + \alpha h(x(t - \tau))(1 - x(t)), \quad \forall t \geq 0, \\ x(\theta) = \varphi(\theta), \quad \forall \theta \in [-\tau, 0], \end{cases} \quad (6.2.4)$$

where the function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$h(x) := \frac{\beta x}{\beta x + \nu}, \quad \forall x \geq 0. \quad (6.2.5)$$

Set

$$M := C([- \tau, 0], [0, 1]) \times [0, 1]. \quad (6.2.6)$$

The main results are stated as follows.

Theorem 6.2.1 *Let $\tau, \mu, \alpha, \nu, \beta > 0$ be given positive constants and let $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$. Let $(x^\varepsilon, y^\varepsilon)$ (resp. x) be the solution of (6.2.1) with initial data $(\varphi, y_0) \in M$ (resp. of (6.2.4) with initial data φ) with*

$$\varphi \neq 0_C.$$

Then the following properties are satisfied

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \geq 0} |x^\varepsilon(t) - x(t)| = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \geq \varepsilon |\ln \varepsilon|} |y^\varepsilon(t) - h(x(t - \tau))| = 0.$$

Remark 6.2.2 If $R_0 \leq 1$ and $\varphi \equiv 0$ then the above uniform convergence holds true.

Remark 6.2.3 By using the classical change of time scale

$$x(t) = x^\varepsilon(\varepsilon t) \text{ and } y(t) = y^\varepsilon(\varepsilon t),$$

system (6.2.1) becomes

$$\begin{aligned} \frac{dx(t)}{dt} &= \varepsilon [-\mu x(t) + \alpha y(t)(1 - x(t))], \\ \frac{dy(t)}{dt} &= -\nu y(t) + \beta x(t - \tau_\varepsilon)(1 - y(t)). \end{aligned} \quad (6.2.7)$$

where $\tau_\varepsilon := \frac{\tau}{\varepsilon} \rightarrow +\infty$ as $\varepsilon(>0) \rightarrow 0$. One may observe that the equations remain singular after this change of time scale since the delay τ_ε goes to infinity as $\varepsilon \rightarrow 0$. To the best of our knowledge the only available nonlinear theory is concerned with convergence local in time towards the reduced system. We refer to Artstein and Slemrod [9] and the references therein for general results on this topic.

Remark 6.2.4 Roughly speaking the proof of the above result shows that in a very fast time t_ε of order $\varepsilon |\ln \varepsilon|$, $y^\varepsilon(t_\varepsilon)$ becomes very close to $h(\varphi(t_\varepsilon - \tau))$. Next $y^\varepsilon(t)$ stays close to $h(x^\varepsilon(t - \tau))$ with the following kind of estimate for $t \geq \tau$ and ε small enough:

$$y^\varepsilon(t) = h(x^\varepsilon(t - \tau)) + O(\varepsilon) + O\left(e^{-\frac{\nu(t-\tau)}{\varepsilon}}\right).$$

It is important to point out the fact that the above theorem is established in the context that the same initial condition φ is taken for the system (6.2.1) and (6.2.4). When φ is the zero function we do not have a global uniform convergence of x^ε to x whenever $y_0 \neq 0$ and $R_0 > 1$. The study of the convergence of x^ε to x is much more delicate. The result obtained is the following

Theorem 6.2.5 Assume that $R_0 > 1$. Then the reduced system (6.2.4) has a unique (up to time shift) heteroclinic orbit x^∞ such that

$$\lim_{t \rightarrow -\infty} x^\infty(t) = 0 \text{ and } \lim_{t \rightarrow +\infty} x^\infty(t) = \bar{x} := \frac{\alpha\beta - \mu\nu}{\alpha\beta + \mu\beta}. \quad (6.2.8)$$

Furthermore x^∞ is increasing on $\mathbb{R}$. Let $y_0 \in (0, 1]$ be given and let us denote by $(x^\varepsilon, y^\varepsilon)$ the solution of (6.2.1) with initial data $(0_C, y_0)$. Define

$$t_\varepsilon := \sup \left\{ t \geq 0 : x^\varepsilon(t) = \frac{\bar{x}}{2} \right\} < +\infty.$$

Then we have

$$\lim_{\varepsilon \rightarrow 0} t_\varepsilon = +\infty$$

and

$$\lim_{\varepsilon \rightarrow 0} x^\varepsilon(t + t_\varepsilon) = x^\infty(t),$$

uniformly in t on each interval of the form $[-T, +\infty)$ with $T \geq 0$ and where $x^\infty \equiv x^\infty(t)$ is the unique heteroclinic orbit of the reduced system (6.2.4) satisfying

$$x^\infty(0) = \frac{\bar{x}}{2}.$$

In order to illustrate the latter results and more specifically Theorem 6.2.1 with realistic parameters, we shall use the values described in Table 1. Notice that the average time necessary to become infectious, namely τ , is unknown and needs to be estimated. This is performed by using the expression of the endemic prevalence equilibrium $\bar{x}$ given in (6.2.8). Using the parameters of Table 1, to reach 10% prevalence of patient we obtain $\tau = 9.86 \text{ days}$. Figure 6.2-(a) illustrates how the equilibrium prevalence of patients varies with respect to the parameters τ and $\frac{1}{\nu_V}$. Note that the prevalence is very sensitive with respect to the average time of HCW visit. Indeed for the value $\tau = 9.86 \text{ days}$ the prevalence at equilibrium varies from 10% to 18% when the length of visit varies from 95 min to 90 min. Figure 6.2-(b) illustrates the dependence on the basic reproduction number R_0 with respect to τ and $\frac{1}{\nu_V}$. An increasing of the length of visit $\frac{1}{\nu_V}$ leads to a decrease of the basic reproduction number and thus on the bacteria's spread.

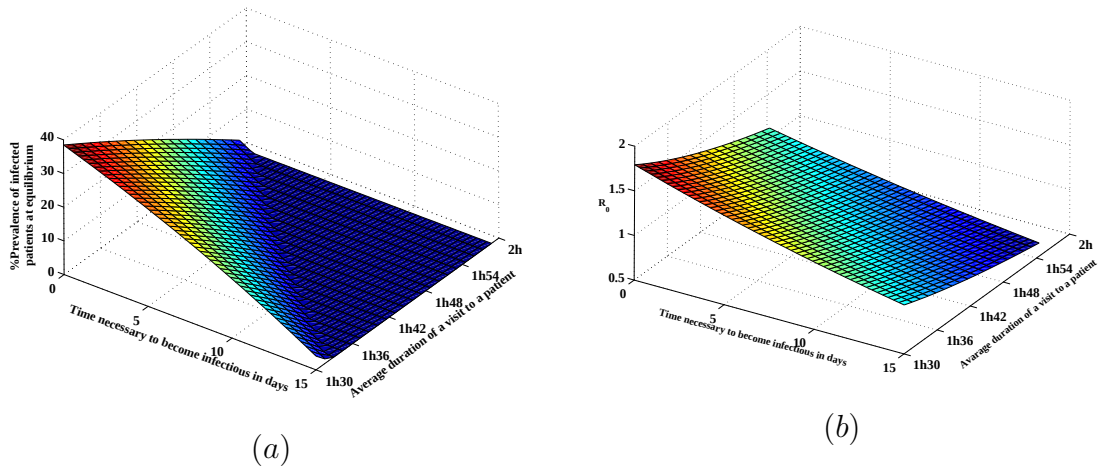


Figure 6.2: Figures (a) and (b) describe respectively the evolution of the prevalence of infected patients at the equilibrium and R_0 with respect to $1/\nu_V$ and τ .

Finally the convergence result stated in Theorem 6.2.1 is illustrated in Figure 6.3. The error between the prevalence for the full and reduced system is plotted for different values for the time of HCW visit. Together with the parameters of Table 1 and the different values of ν_V recalled in Figure 6.3, we obtain a maximal error of order 10^{-3} over one year's computation time.

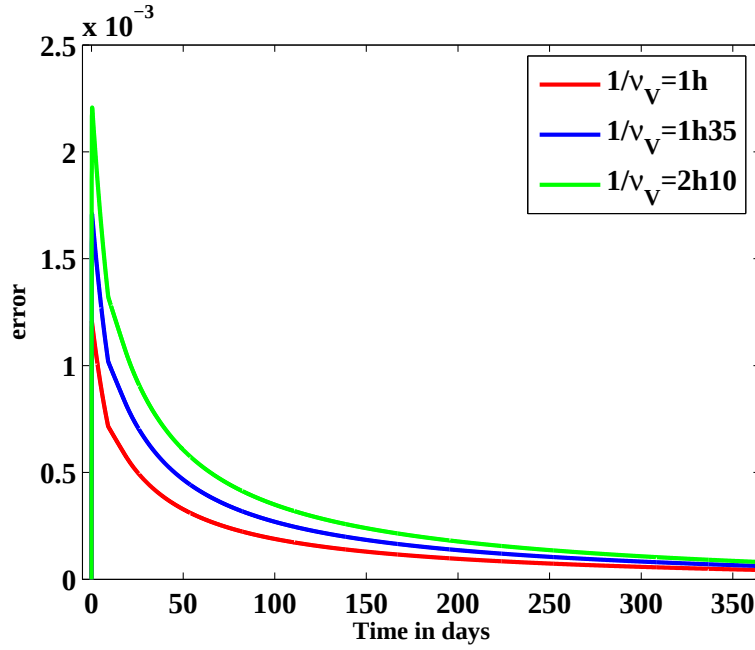


Figure 6.3: Errors between the full and the reduced systems for the same non zero initial data and different ν_V . Precisely $error(t) = |x^\varepsilon(t) - x(t)|$, the parameters ν , μ , α , β are computed using the relation (6.2.2) with the approximation $P_C^0 = P_C$ for the parameter value of the Table 1. The initial data for y is $y_0 = 0.5$ and the initial data for x and x^ε is $\varphi(t) = 0.6$ for $t \in [-9.86, 0]$.

6.3 Preliminaries

The aim of this section is to derive preliminary results for (6.2.1) and (6.2.4). We shall more specifically focus one existence and uniqueness of the solution as well as asymptotic behavior. We shall use the usual history function to deal with the delay differential equation, namely for each continuous function $x : [-\tau, T) \rightarrow \mathbb{R}$ for some given $T > 0$ we write $t \in [0, T) \mapsto x_t \in C$ defined by $x_t(\theta) = x(t + \theta)$ for each $\theta \in [-\tau, 0]$ and $t \in [0, T)$. We first state a preliminary result for the reduced system (2.4.31).

Lemma 6.3.1 *Consider the set*

$$\hat{M} := \{\varphi \in C : 0_C \leq \varphi \leq 1_C\}. \quad (6.3.1)$$

Then $\hat{M}$ is positively invariant with respect to the semiflow generated by (6.2.4). If we denote by $\{U(t)\}_{t \geq 0}$ the strongly continuous semiflow on $\hat{M}$ generated by (6.2.4) defined by $U(t)\varphi = x_t$ the following holds true:

(i) for each $(\varphi, \psi) \in (\hat{M})^2$

$$\varphi \leq \psi \Rightarrow U(t)\varphi \leq U(t)\psi, \forall t \geq 0. \quad (6.3.2)$$

(ii) When $R_0 \leq 1$ then the semiflow U only has the trivial equilibrium 0_C . When $R_0 > 1$ the semiflow admits exactly two equilibrium points: the trivial one and the constant $\bar{x}$ defined by

$$\bar{x} := \frac{\alpha\beta - \mu\nu}{\alpha\beta + \mu\beta}. \quad (6.3.3)$$

(iii) When $R_0 \leq 1$ then the trivial equilibrium 0_C is globally asymptotically stable in $\hat{M}$. When $R_0 > 1$ then the positive equilibrium $\bar{x}$ is globally asymptotically stable in $\hat{M} \setminus \{0_C\}$.

Proof. The proof of the forward invariance of $\hat{M}$ as well as (i) directly follows from the results of Smith [152]. Indeed if we define $g : C \rightarrow \mathbb{R}$ by

$$g(\psi) := -\mu\psi(0) + \alpha h(\psi(-\tau))(1 - \psi(0)).$$

Then one has $g(1_C) \leq 0$ and $g(0_C) = 0$ so that $\hat{M}$ is forward invariant and on $\hat{M}$ function g is quasi monotone. Now the proof (ii) comes from straightforward computations. It remains to prove (iii). To do so let us first notice that

$$\lim_{\delta \rightarrow 0} \frac{g(\delta 1_C)}{\delta} = -\mu + \frac{\alpha\beta}{\nu} = \mu [R_0^2 - 1] > 0.$$

Then using the results of Smith [152] for each $\delta \in (0, 1)$ small enough we have $U(t)(\delta 1_C) \rightarrow \bar{x}$ as $t \rightarrow +\infty$. On the other hand let us also notice that $g(1_C) < 0$ so that we deduce using (i) that $U(t)1_C \rightarrow \bar{x}$ as $t \rightarrow \infty$. To complete the proof (iii) it remains to show that for each $\varphi \in \hat{M} \setminus \{0_C\}$ the solution $t \rightarrow x_t = U(t)\varphi$ of the system (6.2.4) satisfies $x(t) > 0$ for all $t \geq \tau$. Since $\varphi \in \hat{M} \setminus \{0_C\}$ there exists $t_0 \in [0, \tau]$ such that $x(t_0) > 0$. Hence one gets for each

$$\begin{aligned} x(t) &= e^{-\mu(t-t_0)}x(t_0) + \int_{t_0}^t e^{-\mu(t-s)}\alpha h(x(s-\tau))(1-x(s))ds \\ &\geq e^{-\mu(t-t_0)}x(t_0) > 0, \forall t \geq t_0, \end{aligned}$$

and the result follows. ■

Let us now state a similar preliminary result for System (6.2.1).

Lemma 6.3.2 *Let $\varepsilon > 0$ be given. Then the subset $M \subset C \times \mathbb{R}$ (defined in (6.2.6)) is positively invariant by the semiflow generated by (6.2.1). If we denote by $\{U^\varepsilon(t)\}_{t \geq 0}$ the continuous semiflow on M generated by (6.2.1) defined by $U^\varepsilon(t)\varphi = (x_t^\varepsilon, y^\varepsilon(t))^T$ then the following holds true:*

(i) *for each $(\varphi, \psi) \in (M)^2$:*

$$\varphi \leq_{C \times \mathbb{R}} \psi \Rightarrow U(t)\varphi \leq_{C \times \mathbb{R}} U(t)\psi, \quad \forall t \geq 0, \quad (6.3.4)$$

where the partial order $\leq_{C \times \mathbb{R}}$ is defined by the usual positive cone $C_+ \times \mathbb{R}^+ \subset C \times \mathbb{R}$.

- (ii) *When $R_0 \leq 1$ then the only equilibrium of the semiflow U^ε is the trivial equilibrium $(0_C, 0)^T$. When $R_0 > 1$ the semiflow admits exactly two equilibrium points: the trivial one and the constant $(\bar{x}, \bar{y})^T$ where $\bar{x}$ is defined in (6.3.3) while $\bar{y} = h(\bar{x})$.*
- (iii) *When $R_0 \leq 1$ then the trivial equilibrium $(0_C, 0)^T$ is globally asymptotically stable in M . When $R_0 > 1$ then the interior equilibrium $(\bar{x}, \bar{y})^T$ is globally asymptotically stable in $M \setminus \{(0_C, 0)\}$.*

The proof of this results is straightforward and follows by the same steps and arguments as the one of Lemma 6.3.1.

Our next preliminary result relies on some property of the entire solutions of the reduced system (6.2.4). This will be needed in the proof of Theorem 6.2.1 as well as Theorem 6.2.5.

Lemma 6.3.3 *Assume that $R_0 > 1$. Then $\{x(t)\}_{t \in \mathbb{R}}$ is a complete orbit in $\hat{M}$ of (6.2.4) if and only if one of the following properties are satisfied:*

(i) *x is an equilibrium point of the system (6.2.4), namely $x(t) \equiv 0_C$ or $x(t) \equiv \bar{x}$.*

(ii) *x is a heteroclinic orbit of the system (6.2.4) satisfying the following properties*

(a) *$0 < x(t) \leq \bar{x}$ for all $t \in \mathbb{R}$.*

(b) *$\lim_{t \rightarrow +\infty} x(t) = \bar{x}$ and $\lim_{t \rightarrow -\infty} x(t) = 0$.*

Proof. Let us first notice that (i) or (ii) implies that x is an complete orbit of (6.2.4) in $\hat{M}$. Let $\{x(t)\}_{t \in \mathbb{R}}$ be a given complete orbit of the system (6.2.4) in $\hat{M}$ such that $x \not\equiv 0_C$ and $x \not\equiv \bar{x}$.

Let us first prove that x satisfies (ii)-(a). Since $x(t) \in \hat{M}$ for each $t \in \mathbb{R}$ one has $0_C \leq x_{t-s} \leq 1_C$ for each $t \in \mathbb{R}$ and $s \in \mathbb{R}$. Lemma 6.3.1-(i) yields that $0_C \leq x_t \leq U(s)1_C$ for each $s \geq 0$ and $t \in \mathbb{R}$. Lemma 6.3.1-(iii) implies that $U(s)1 \rightarrow \bar{x}$ as $s \rightarrow +\infty$ that ensures that $0 \leq x(t) \leq \bar{x}$ for all $t \in \mathbb{R}$. To complete the proof of (ii)-(a) it remains to prove that $0 < x(t)$ for all $t \in \mathbb{R}$. To prove this property let us argue by contradiction by assuming

that there exists $\tilde{t} \in \mathbb{R}$ such that $x(\tilde{t}) = 0$. Let us first notice that from the reduced system, one gets

$$\frac{d[e^{\mu t} x(t)]}{dt} = e^{\mu t} h(x(t - \tau))(1 - x(t)) \geq 0, \quad \forall t \in \mathbb{R},$$

so that $t \mapsto e^{\mu t} x(t)$ is non-decreasing. Hence $x(t) = 0$ for all $t \leq \tilde{t}$. Since $x(t) \equiv 0$ on $[\tilde{t} - \tau, \tilde{t}]$ one concludes that $x(t) = 0$ for all $t \geq \tilde{t}$. We obtain that $x(t) \equiv 0$, a contradiction that completes the proof of (ii)-(a).

It remains to prove (ii)-(b). First since $x \not\equiv 0_C$, Lemma 6.3.1-(iii) yields that $x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$. As a consequence we only need to show that $x(t) \rightarrow 0$ as $t \rightarrow -\infty$. This property is related to the following functional

$$V(x_t) := x(t) + \mu \int_{t-\tau}^t x(s) ds, \quad \forall t \in \mathbb{R}. \quad (6.3.5)$$

Straightforward computations yields that

$$\frac{dV(x_t)}{dt} = h(x(t - \tau)) [\alpha(\bar{x} - x(t)) + \mu(\bar{x} - x(t - \tau))], \quad \forall t \in \mathbb{R}. \quad (6.3.6)$$

Then due to (ii)-(a), $x(t) \leq \bar{x}$ for all $t \in \mathbb{R}$ and $t \mapsto V(x_t)$ is non-decreasing. To conclude let us consider a decreasing sequence $\{t_n\}_{n \geq 0}$ such that $t_n \rightarrow -\infty$ as $n \rightarrow +\infty$. Let us define the uniformly bounded sequence of shifted maps $\{x^n\}_{n \geq 0}$ be

$$x^n(t) = x(t + t_n), \quad \forall t \in \mathbb{R}.$$

Since x^n is an entire solution of (6.2.4) and since $\{x^n\}$ is uniformly bounded, one concludes that $\{\frac{dx^n}{dt}\}_{n \geq 0}$ is also uniformly bounded. As a consequence, possibly along a subsequence, one may assume that $x^n(t) \rightarrow x^\infty(t)$ as $n \rightarrow \infty$ locally uniformly in $t \in \mathbb{R}$ and wherein x^∞ is also an entire solution in $\hat{M}$ of (6.2.4). Next for each $n \geq 0$ and $K > 0$, integrating (6.3.6) over $[t_n - K, t_n + K]$ yields

$$\begin{aligned} V(x_{t_n+K}) &= \int_{-K}^K h(x^n(t - \tau)) [\alpha(\bar{x} - x^n(t)) + \mu(\bar{x} - x^n(t - \tau))] dt \\ &\quad + V(x)(t_n - K). \end{aligned}$$

Since $t \mapsto V(x_t)$ is non-increasing and bounded from below one obtains when $n \rightarrow +\infty$ that

$$\int_{-K}^K h(x^\infty(t - \tau)) [\alpha(\bar{x} - x^\infty(t)) + \mu(\bar{x} - x^\infty(t - \tau))] dt = 0, \quad \forall K > 0.$$

This implies that

$$h(x^\infty(t - \tau)) [\alpha(\bar{x} - x^\infty(t)) + \mu(\bar{x} - x^\infty(t - \tau))] \equiv 0,$$

so that $x^\infty(t) \equiv 0$ or $x^\infty(t) \equiv \bar{x}$. To conclude the proof we need to prove that $x^\infty(t) \equiv 0$. Let us argue by contradiction by assuming that $x^\infty(t) \equiv \bar{x}$. Then the functional $\varphi \rightarrow V(\varphi)$ is monotone increasing therefore

$$x_t \leq \bar{x}1_C \Rightarrow V(x_t) \leq V(\bar{x})$$

since $t \in \mathbb{R} \mapsto V(x_t)$ is non-decreasing we also have

$$V(\bar{x}) \leq V(x_t).$$

Therefore

$$V(x_t) = V(\bar{x}), \forall t \in \mathbb{R},$$

As a consequence $\frac{V(x_t)}{dt} \equiv 0$ that re-writes as

$$h(x(t - \tau)) [\alpha(\bar{x} - x(t)) + \mu(\bar{x} - x(t - \tau))] \equiv 0,$$

so that $x(t) \equiv 0$ or $x(t) \equiv \bar{x}$, a contradiction. The proof is completed. ■

6.4 Proof of Theorem 6.2.1

The aim of this section is to prove Theorem 6.2.1. This proof is divided into two parts. The first part is devoted to the convergence $x_t^\varepsilon \rightarrow x_t$ as $\varepsilon(> 0) \rightarrow 0$. The second part is related the behavior of $t \rightarrow y^\varepsilon(t)$.

6.4.1 Convergence of $t \mapsto x^\varepsilon(t)$

In order to investigate the uniform convergence of x^ε let us first prove the following local uniform convergence:

Lemma 6.4.1 (Local uniform convergence) *Let $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ be given. Let x be the solution of (6.2.4) with initial data φ . Then for each $\hat{\tau} > 0$ we have*

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [-\tau, \hat{\tau}]} |x^\varepsilon(t) - x(t)| = 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^{\hat{\tau}} y^\varepsilon(t) \psi(t) dt = \int_0^{\hat{\tau}} h(x(t - \tau)) \psi(t) dt, \quad \forall \psi \in L^1(0, \hat{\tau}; \mathbb{R}).$$

Note that the proof of the above result can be directly obtained using the theory of Artstein and Slemrod in [9]. For the sake of completeness we provide a direct and easy proof that takes into account the particular structure of our system to conclude to the local weak star convergence for the y -component. Let us also notice that since the work of Artstein

and Slemrod [9] deals with Young measure narrow convergence for the y -component, it allows to conclude to the (local) strong L^1 -convergence of $y^\varepsilon(t)$ to $h(x(t-\tau))$. Such a strong convergence will be derived latter on by deriving direct uniform estimates as well as layer time estimates.

Proof. The proof of the above result also relies on Arzela-Ascoli's theorem. Since $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon \in (0,1)} \subset C([0, \infty), M)$ is uniformly bounded, one gets by using (6.2.1) that $\{\frac{dx^\varepsilon}{dt}\}_{\varepsilon \in (0,1)}$ is also uniformly bounded in $C([0, \infty), \mathbb{R})$. Since $x_0^\varepsilon = \varphi$ for all $\varepsilon \in (0, 1)$ we infer from Arzela-Ascoli's theorem that $\{x^\varepsilon\}_{\varepsilon \in (0,1)}$ is relatively compact in $C_{loc}([-\tau, \infty), \hat{M})$ while due to Banach-Alaoglu-Bourbaki's theorem $\{y^\varepsilon\}_{\varepsilon \in (0,1)}$ is relatively compact for the weak-* topology of $\sigma(L_{loc}^\infty((0, \infty), \mathbb{R}), L_{loc}^1((0, \infty), \mathbb{R}))$.

Let $\hat{\tau} > 0$ be given and let $\{\varepsilon_n\}_{n \geq 1} \subset (0, 1)$ be a given sequence tending to 0 as $n \rightarrow \infty$. Up to a sub-sequence, one may assume that $x^{\varepsilon_n} \rightarrow x^0 \in C([-\tau, \hat{\tau}], \hat{M})$ uniformly on $[-\tau, \hat{\tau}]$ with $x^0(\theta) = \varphi(\theta)$ for each $\theta \in [-\tau, 0]$ and $y^{\varepsilon_n} \xrightarrow{*} y_0 \in L^\infty((0, \hat{\tau}), \mathbb{R})$ for the weak-* topology of $L^\infty((-\tau, \hat{\tau}), \mathbb{R})$. That is to say that for each $\hat{\tau} \in (0, +\infty)$

$$\lim_{n \rightarrow +\infty} \int_0^{\hat{\tau}} y^{\varepsilon_n}(t) \phi(t) dt = \int_0^{\hat{\tau}} y^0(t) \phi(t) dt, \forall \phi \in L^1((0, \hat{\tau}), \mathbb{R}).$$

It follows that

$$\int_0^{\hat{\tau}} y^0(t) \phi(t) dt \geq 0 \text{ and } \int_0^{\hat{\tau}} [1 - y^0(t)] \phi(t) dt \geq 0, \forall \phi \in L_+^1((0, \hat{\tau}), \mathbb{R}).$$

Since $\hat{\tau} < +\infty$, we deduce that

$$y^0 \in L^1((0, \hat{\tau}), \mathbb{R})$$

and

$$\int_0^{\hat{\tau}} y^0(t) \phi(t) dt \geq 0 \text{ and } \int_0^{\hat{\tau}} [1 - y^0(t)] \phi(t) dt \geq 0, \forall \phi \in L_+^\infty((0, \hat{\tau}), \mathbb{R}).$$

Now by applying the Hahn-Banach in $L^1((0, \hat{\tau}), \mathbb{R})$, it follows that

$$0 \leq y_0 \leq 1.$$

On the one hand, let $\psi \in C^1([0, \hat{\tau}], \mathbb{R})$ be a given test function. Multiplying the y^{ε_n} -equation in (6.2.1) by ψ and integrating over $(0, \hat{\tau})$ yields for each $n \geq 0$

$$\begin{aligned} & \varepsilon_n [y^{\varepsilon_n}(\hat{\tau}) \psi(\hat{\tau}) - y_0 \psi(0)] - \varepsilon_n \int_0^{\hat{\tau}} y^{\varepsilon_n}(t) \psi'(t) dt \\ &= \int_0^{\hat{\tau}} [\beta x^{\varepsilon_n}(t-\tau) (1 - y^{\varepsilon_n}(t)) - \nu y^{\varepsilon_n}(t)] \psi(t) dt. \end{aligned}$$

Letting $n \rightarrow +\infty$ provides

$$\int_0^{\hat{\tau}} [\beta x^0(t - \tau)(1 - y^0(t)) - \nu y^0(t)] \psi(t) dt = 0, \quad \forall \psi \in C^1([0, \hat{\tau}], \mathbb{R}),$$

so that

$$y^0(t) = h(x^0(t - \tau)) \quad \text{a.e. for } t \in [0, \hat{\tau}]. \quad (6.4.1)$$

On the other hand, from the x^{ε_n} -equation in (6.2.1) one has for each $n \geq 0$:

$$x^{\varepsilon_n}(t) = \varphi(0) + \int_0^t [\alpha(1 - x^{\varepsilon_n}(s))y^{\varepsilon_n}(s) - \mu x^{\varepsilon_n}(s)] ds, \quad \forall t \in [0, \hat{\tau}].$$

Letting $n \rightarrow +\infty$ provides that

$$x^0(t) = \varphi(0) + \int_0^t [\alpha(1 - x^0(s))y^0(s) - \mu x^0(s)] ds, \quad \forall t \in [0, \hat{\tau}].$$

Recalling (6.4.1) and that x^0 satisfies $x^0(\theta) = \varphi(\theta)$ for each $\theta \in [-\tau, 0]$ we obtain that $x^0 = x$ on $[-\tau, \hat{\tau}]$. This completes the proof of the result. $\blacksquare$

Before proving Theorem 6.2.1 we need some preliminary lemmas. First we have an estimation from below of solutions independent of the parameter $\varepsilon > 0$.

Lemma 6.4.2 Assume that $R_0 > 1$. Then for all $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ with $\varphi \neq 0_C$, the map $t \mapsto w^\varepsilon(t)$ defined by

$$w^\varepsilon(t) = x^\varepsilon(t) + \frac{\varepsilon\mu}{\beta}y^\varepsilon(t) + \mu \int_{t-\tau}^t x^\varepsilon(s) ds, \quad \forall t \geq 0, \quad (6.4.2)$$

satisfies the following properties:

(i) For all $t \geq 0$ and $\varepsilon > 0$

$$\frac{dw^\varepsilon(t)}{dt} = \alpha y^\varepsilon(t)(\bar{x} - x^\varepsilon(t)) + \mu y^\varepsilon(t)(\bar{x} - x^\varepsilon(t - \tau)).$$

(ii) There exist $\eta > 0$ and $\varepsilon_0 > 0$ such that

$$w^\varepsilon(t) \geq \eta, \quad \forall t \geq \tau, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Proof. The proof of (i) follows from straightforward computations. In order to prove (ii), let's observe that by integrating the x -equation in system (6.2.1) between $t - \tau$ and t we obtain that

$$x^\varepsilon(t) + \mu \int_{t-\tau}^t x^\varepsilon(s) ds = x^\varepsilon(t - \tau) + \alpha \int_{t-\tau}^t y^\varepsilon(s)(1 - x^\varepsilon(s)) ds, \quad \forall t \geq \tau.$$

Thus

$$w^\varepsilon(t) = \frac{\varepsilon\mu}{\beta}y^\varepsilon(t) + x^\varepsilon(t - \tau) + \alpha \int_{t-\tau}^t y^\varepsilon(s) (1 - x^\varepsilon(s)) ds, \quad \forall t \geq \tau.$$

Since $w^\varepsilon(t) \geq x^\varepsilon(t)$ for all $t \geq 0$, one obtains

$$w^\varepsilon(t) \geq \max\{x^\varepsilon(t), x^\varepsilon(t - \tau)\}, \quad \forall t \geq \tau. \quad (6.4.3)$$

If one sets $x_t = U(t)\varphi$ then since $\varphi \not\equiv 0$ one has $x(\tau) = [U(\tau)\varphi](0) > 0$. On the other hand due to Lemma 6.4.1 we know that $x^\varepsilon(\tau) \rightarrow x(\tau)$ as $\varepsilon \rightarrow 0$. Thus there exists $\varepsilon_0 > 0$ such that

$$x^\varepsilon(\tau) \geq \frac{x(\tau)}{2} > 0, \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (6.4.4)$$

To conclude the proof of (ii) we will use the following claim.

Claim 6.4.3 *Let $\varepsilon \in (0, \varepsilon_0)$ be given. Then for each $\delta \in (0, 1)$ such that $\frac{\delta}{2}x(\tau) < \bar{x}$ we have*

$$w^\varepsilon(t) \geq \frac{\delta}{2}x(\tau), \quad \forall t \geq \tau.$$

To prove this claim, let us notice that by (6.4.3) and (6.4.4) we have

$$w^\varepsilon(\tau) \geq x^\varepsilon(\tau) > \frac{\delta}{2}x(\tau).$$

Let us consider

$$t_0 := \sup \left\{ t > \tau : w^\varepsilon(l) \geq \frac{\delta}{2}x(\tau), \quad \forall l \in [\tau, t] \right\}.$$

Then let us prove that

$$t_0 = +\infty.$$

Assume that $t_0 < +\infty$, then one has

$$w^\varepsilon(t_0) = \frac{\delta}{2}x(\tau) < \bar{x}.$$

One can therefore introduce $t_1 > t_0$ defined by

$$t_1 = \sup \{ t > t_0 : w^\varepsilon(l) \leq \bar{x}, \quad \forall l \in [t_0, t] \}.$$

We infer from (6.4.3) that

$$x^\varepsilon(t) \leq \bar{x} \text{ and } x^\varepsilon(t - \tau) \leq \bar{x}, \quad \forall t \in [t_0, t_1].$$

As a consequence (i) the map $t \mapsto w^\varepsilon(t)$ is non-decreasing on $[t_0, t_1]$, that implies

$$w^\varepsilon(t) \geq w^\varepsilon(t_0) \geq \frac{\delta}{2}x(\tau), \quad \forall t \in [t_0, t_1].$$

This contradicts the definition of t_0 and completes the proof of (ii). ■

Coupling Lemma 6.3.3 and Lemma 6.4.2 lead to the following lemma.

Lemma 6.4.4 *Let us assume that $R_0 > 1$. Let $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ be given such that $\varphi \not\equiv 0_C$. Then for each sequence $\{\varepsilon_n\}_{n \geq 0} \subset (0, 1)$ and $\{t_n\}_{n \geq 0} \subset (0, \infty)$ such that $\varepsilon_n \rightarrow 0$ and $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ we have*

$$\lim_{n \rightarrow +\infty} x^{\varepsilon_n}(t + t_n) = \bar{x}, \text{ locally uniformly for } t \in \mathbb{R}.$$

Proof. Let $\{\varepsilon_n\}_{n \geq 0}$ and $\{t_n\}_{n \geq 0}$ be given sequences such that $\varepsilon_n \rightarrow 0$ and $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Define the sequences of shifted maps

$$x^n(t) := x^{\varepsilon_n}(t + t_n) \in [0, 1] \text{ and } y^n(t) := y^{\varepsilon_n}(t + t_n) \in [0, 1],$$

with $n \geq 0$ and $t \in (-t_n, +\infty)$, that satisfy the system of equations:

$$\begin{cases} \frac{dx^n(t)}{dt} = -\mu x^n(t) + \alpha(1 - x^n(t))y^n(t), \quad \forall t \geq -t_n, \\ \varepsilon_n \frac{dy^n(t)}{dt} = -\nu y^n(t) + \beta x^n(t - \tau)(1 - y^n(t)), \quad \forall t \geq -t_n. \end{cases}$$

Thus by using the same techniques as in the proof of Lemma 6.4.1, up to a sub-sequence, one may assume that $x^n \rightarrow x^\infty$ locally uniformly for $t \in \mathbb{R}$ wherein x^∞ is a complete orbit of (6.2.4) in $\hat{M}$. It remains to prove that $x^\infty \equiv \bar{x}$ that is a consequence of the uniform persistence result stated in Lemma 6.4.2-(ii). Indeed, since $\varphi \not\equiv 0$, there exists $\eta > 0$ and $N > 0$ such that for each $n \geq N$ and each $t \geq \tau - t_n$:

$$x^{\varepsilon_n}(t + t_n) + \frac{\varepsilon_n \mu}{\beta} y^{\varepsilon_n}(t + t_n) + \mu \int_{t-\tau}^t x^{\varepsilon_n}(s + t_n) ds \geq \eta.$$

Letting $n \rightarrow \infty$ yields

$$x^\infty(t) + \mu \int_{t-\tau}^t x^\infty(s) ds \geq \eta, \quad \forall t \in \mathbb{R}.$$

The classification of complete orbits of (6.2.4) provided by Lemma 6.3.3 allows us to conclude that $x^\infty(t) \equiv \bar{x}$ and the result follows. $\blacksquare$

We are now ready to prove the first part of Theorem 6.2.1.

Theorem 6.4.5 *Let $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ be given such that*

$$\text{either } \varphi \neq 0 \text{ or } \begin{pmatrix} y_0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ 0_C \end{pmatrix}.$$

Let x be the solution of (6.2.4) with initial data φ . Then we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{t \geq 0} |x^\varepsilon(t) - x(t)| = 0. \quad (6.4.5)$$

Remark 6.4.6 Using similar argument as in the proof of Theorem 6.4.5, the conclusion remains true whenever $R_0 \leq 1$ and $\varphi = 0$. However when $R_0 > 1$ then Theorem 6.4.5 is no longer true when $\varphi \equiv 0$ and $y_0 > 0$. The question will be studied in Theorem 6.2.5.

Proof. Let us first remark that when $\varphi = 0_C$ and $y_0 = 0$ then (6.4.5) is trivial verified since

$$x^{\varepsilon_n}(t) = x(t) = 0, \forall t \geq 0, \forall \varepsilon > 0.$$

Let $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ with $\varphi \neq 0$. Assume that (6.4.5) is not satisfied. Then there exist $\eta > 0$ and two sequences $\{\varepsilon_n\}_{n \geq 0} \rightarrow 0$ and $\{t_n\}_{n \geq 0}$ such that

$$|x^{\varepsilon_n}(t_n) - x(t_n)| > \eta, \forall n \geq 0. \quad (6.4.6)$$

Moreover by Lemma 6.4.1 we must have

$$\{t_n\}_{n \geq 0} \rightarrow +\infty.$$

Define the shifted maps

$$x^n(t) := x^{\varepsilon_n}(t + t_n) \text{ and } y^n(t) := y^{\varepsilon_n}(t + t_n),$$

for all $n \geq 0$ and all $t \in (-t_n, +\infty)$. Then we have

$$0 \leq x^n(t) \leq 1 \text{ and } 0 \leq y^n(t) \leq 1,$$

for all $n \geq 0$ and $t \in (-t_n, +\infty)$.

By using the same techniques as in the proof of Lemma 6.4.1, one may assume that $x^n(t) \rightarrow x^\infty(t)$ locally uniformly where x^∞ is a complete orbit of (6.2.4) in $\hat{M}$ such that

$$|x^\infty(0) - L| \geq \eta \quad (6.4.7)$$

where

$$L := \lim_{t \rightarrow +\infty} x(t).$$

So either

$$L = 0 \text{ or } L = \bar{x}.$$

According to the classification provided by Lemma 6.3.1-(iii) we will now split the proof into two parts: a) $R_0 \leq 1$ and $L = 0$; b) $R_0 > 1$ and $L = \bar{x}$.

a) If $R_0 \leq 1$ then x^∞ is an entire solution of (6.2.4) in $\hat{M}$ so that one can deduce that $x^\infty(t) \equiv 0$. This is a direct consequence of Lemma 6.3.1 (ii) and (iii). Since $L = 0$ we obtain a contradiction with (6.4.7).

b) If we consider the case when $R_0 > 1$. Then by Lemma 6.4.4 we deduce that $x^\infty \equiv \bar{x}$. But $\varphi \neq 0$ we also have $L = \bar{x}$ and we obtain a contradiction with (6.4.7). This completes the proof of the result. ■

6.4.2 Convergence of y^ε

The aim of this section is to study the convergence property of y^ε as $\varepsilon \rightarrow 0$ in order to complete the proof of Theorem 6.2.1. Let's start with an estimation of $y^\varepsilon(t) - h(x^\varepsilon(t - \tau))$ for $t \in [\tau, +\infty)$.

Lemma 6.4.7 *For each $\varepsilon > 0$ and each initial datum $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ we have*

$$|y^\varepsilon(t) - h(x^\varepsilon(t - \tau))| \leq e^{-\frac{\nu}{\varepsilon}(t-\tau)} |y^\varepsilon(\tau) - h(\varphi(0))| + \kappa\varepsilon, \quad \forall t \geq \tau,$$

with

$$\kappa := \frac{\beta(\mu + \alpha)}{\nu^2}.$$

Proof. Let us first notice that the integration of the y -equation in (6.2.1) yields for each $t \geq \tau$ to:

$$y^\varepsilon(t) = e^{-\frac{1}{\varepsilon} \int_\tau^t (\nu + \beta x^\varepsilon(l-\tau)) dl} y^\varepsilon(\tau) + \int_\tau^t e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta x^\varepsilon(l-\tau)) dl} \frac{\beta}{\varepsilon} x^\varepsilon(s - \tau) ds. \quad (6.4.8)$$

Equation (6.4.8) may of course be re-written for each $t \geq \tau$ as

$$y^\varepsilon(t) = e^{-\frac{1}{\varepsilon} \int_\tau^t (\nu + \beta x^\varepsilon(l-\tau)) dl} y^\varepsilon(\tau) + v^\varepsilon(t),$$

where the map $v^\varepsilon : [\tau, \infty) \rightarrow \mathbb{R}^+$ is defined by

$$v^\varepsilon(t) := \int_\tau^t e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta x^\varepsilon(l-\tau)) dl} \frac{\beta}{\varepsilon} x^\varepsilon(s - \tau) ds$$

then we observe that

$$\begin{aligned} v^\varepsilon(t) &= \int_\tau^t \frac{d}{ds} \left[e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta x^\varepsilon(l-\tau)) dl} \right] h(x^\varepsilon(s - \tau)) ds \\ &= \left[e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta x^\varepsilon(l-\tau)) dl} h(x^\varepsilon(s - \tau)) \right]_{s=\tau}^{s=t} \\ &\quad - \int_\tau^t \frac{d}{ds} \left[e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta x^\varepsilon(l-\tau)) dl} \right] h'(x^\varepsilon(s - \tau)) \frac{dx^\varepsilon}{dt}(s - \tau) ds \end{aligned}$$

Therefore for each $t \geq \tau$ one has

$$v^\varepsilon(t) - h(x^\varepsilon(t - \tau)) = -e^{-\frac{1}{\varepsilon} \int_\tau^t (\nu + \beta x^\varepsilon(l-\tau)) dl} h(\varphi(0)) - w^\varepsilon(t),$$

with

$$w^\varepsilon(t) = \int_\tau^t e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta x^\varepsilon(l-\tau)) dl} h'(x^\varepsilon(s - \tau)) \frac{dx^\varepsilon}{dt}(s - \tau) ds.$$

Together with these notations, one gets for each $t \geq \tau$

$$|y^\varepsilon(t) - h(x^\varepsilon(t - \tau))| \leq e^{-\frac{\nu}{\varepsilon}(t-\tau)} |y^\varepsilon(\tau) - h(\varphi(0))| + |w^\varepsilon(t)|. \quad (6.4.9)$$

It remains to obtain an estimate for the last term in the above inequality. But by using the x -equation in (6.2.1) we have

$$\left| \frac{dx^\varepsilon(t)}{dt} \right| \leq (\alpha + \mu), \forall t \geq 0.$$

Therefore

$$|w^\varepsilon(t)| \leq \int_\tau^t e^{-\frac{\nu}{\varepsilon}(t-s)} \frac{\beta(\mu + \alpha)}{\nu} ds, \forall t \geq \tau,$$

and the estimate follows from (6.4.9). ■

Next we evaluate $y^\varepsilon(t) - h(x(t - \tau))$ for $t \in [0, \tau]$. Set

$$\|h'\|_{\infty, [0,1]} := \sup_{x \in [0,1]} |h'(x)|.$$

Lemma 6.4.8 *Let $(\varphi, y_0)^T \in M$ be given. Then for each $\delta > 0$ there exists $\eta := \eta(\delta) > 0$ such that for each $\varepsilon \in (0, 1)$ and $t \in [0, \tau]$*

$$|y^\varepsilon(t) - h(\varphi(t - \tau))| \leq e^{-\frac{\nu t}{\varepsilon}} + \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\nu} \left[2e^{-\frac{\nu \eta}{\varepsilon}} + \delta \right].$$

Proof. Let $\delta > 0$ be given. Since φ is uniformly continuous on $[-\tau, 0]$, there exists $\eta := \eta(\delta) > 0$ such that for each $\theta_1, \theta_2 \in [-\tau, 0]$,

$$|\theta_1 - \theta_2| < \eta \implies |\varphi(\theta_1) - \varphi(\theta_2)| \leq \delta. \quad (6.4.10)$$

By using similar arguments as in the proof of Lemma 6.4.7 we obtain

$$y^\varepsilon(t) = e^{-\frac{1}{\varepsilon} \int_0^t (\nu + \beta \varphi(l - \tau)) dl} y_0 + v^\varepsilon(t), \forall t \in [0, \tau]$$

with $v^\varepsilon : [0, \tau] \rightarrow \mathbb{R}^+$ is defined by

$$v^\varepsilon(t) := \int_0^t \frac{d}{ds} \left[e^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta \varphi(l - \tau)) dl} \right] h(\varphi(s - \tau)) ds, \forall t \in [0, \tau].$$

Therefore we obtain

$$|y^\varepsilon(t) - h(\varphi(t - \tau))| \leq e^{-\frac{\nu t}{\varepsilon}} + |v^\varepsilon(t) - h(\varphi(t - \tau))|, \forall t \in [0, \tau]. \quad (6.4.11)$$

In order to provide a suitable estimate of the second term of the right hand side of the above inequality let us notice that

$$\begin{aligned}
 v^\varepsilon(t) &= \int_0^t \frac{de^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta \varphi(l-\tau)) dl}}{ds} [h(\varphi(s-\tau)) - h(\varphi(t-\tau))] ds \\
 &\quad + \int_0^t \frac{de^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta \varphi(l-\tau)) dl}}{ds} h(\varphi(t-\tau)) ds \\
 &= \int_0^t \frac{de^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta \varphi(l-\tau)) dl}}{ds} [h(\varphi(s-\tau)) - h(\varphi(t-\tau))] ds \\
 &\quad + h(\varphi(t-\tau)) - h(\varphi(t-\tau)) e^{-\frac{1}{\varepsilon} \int_0^t (\nu + \beta \varphi(l-\tau)) dl},
 \end{aligned}$$

thus

$$\begin{aligned}
 |v^\varepsilon(t) - h(\varphi(t-\tau))| &\leq \int_0^t \left| \frac{de^{-\frac{1}{\varepsilon} \int_s^t (\nu + \beta \varphi(l-\tau)) dl}}{ds} \right| |h(\varphi(s-\tau)) - h(\varphi(t-\tau))| ds \\
 &\leq \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\varepsilon} \int_0^t e^{-\frac{\nu}{\varepsilon}(t-s)} |\varphi(s-\tau) - \varphi(t-\tau)| ds \\
 &\leq \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\varepsilon} \left[\int_{t-\eta}^t e^{-\frac{\nu}{\varepsilon}(t-s)} |\varphi(s-\tau) - \varphi(t-\tau)| ds \right. \\
 &\quad \left. + \int_0^{t-\eta} e^{-\frac{\nu}{\varepsilon}(t-s)} |\varphi(s-\tau) - \varphi(t-\tau)| ds \right] \\
 &\leq 2 \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\varepsilon} \int_0^{t-\eta} e^{-\frac{\nu}{\varepsilon}(t-s)} ds \\
 &\quad + \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\varepsilon} \int_0^\eta e^{-\frac{\nu}{\varepsilon}l} |\varphi(t-\tau-l) - \varphi(t-\tau)| dl.
 \end{aligned}$$

Due to (6.4.10) one obtains

$$\int_0^\eta e^{-\frac{\nu}{\varepsilon}l} |\varphi(t-\tau-l) - \varphi(t-\tau)| dl \leq \int_0^\eta e^{-\frac{\nu}{\varepsilon}l} \delta dl, \quad \forall t \in [0, \tau],$$

that implies that for all $t \in [0, \tau]$,

$$|v^\varepsilon(t) - h(\varphi(t-\tau))| \leq \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\varepsilon} \left[2 \int_0^{t-\eta} e^{-\frac{\nu}{\varepsilon}(t-s)} ds + \delta \int_0^\eta e^{-\frac{\nu}{\varepsilon}l} dl \right],$$

that completes the proof. ■

We are now able to complete the proof of Theorem 6.2.5 by investigating the limit behavior of y^ε as $\varepsilon \rightarrow 0$.

Theorem 6.4.9 (Almost global uniform convergence) *Let $\begin{pmatrix} y_0 \\ \varphi \end{pmatrix} \in M$ be given such that $\varphi \neq 0_C$. Then the following holds true for each $K > 0$*

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \geq K\varepsilon |\ln \varepsilon|} |y^\varepsilon(t) - h(x(t-\tau))| = 0.$$

Proof. Let $K > 0$ be given. Let $\delta > 0$ be given. Due to Lemma 6.4.8 that there exists $\eta > 0$ such that for all $\varepsilon > 0$ small enough and $t \in [K\varepsilon|\ln \varepsilon|, \tau]$ one has

$$|y^\varepsilon(t) - h(\varphi(t - \tau))| \leq e^{-K\nu|\ln \varepsilon|} + \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\nu} \left[2e^{-\frac{\nu\eta}{\varepsilon}} + \delta \right]. \quad (6.4.12)$$

On the other hand from Lemma 6.4.7 we have

$$|y^\varepsilon(t) - h(x^\varepsilon(t - \tau))| \leq \kappa\varepsilon + e^{-\frac{\nu}{\varepsilon}(t-\tau)} |y^\varepsilon(\tau) - h(\varphi(0))|, \quad \forall t \geq \tau.$$

Now using (6.4.12) with $t = \tau$ to estimate $|y^\varepsilon(\tau) - h(\varphi(0))|$, one obtains that for all $\varepsilon > 0$ small enough and each $t \geq \varepsilon|\ln \varepsilon|$

$$|y^\varepsilon(t) - h(x^\varepsilon(t - \tau))| \leq \kappa\varepsilon + e^{-K\nu|\ln \varepsilon|} + \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\nu} \left[2e^{-\frac{\nu\eta}{\varepsilon}} + \delta \right].$$

As a consequence one obtains

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \geq K\varepsilon|\ln \varepsilon|} |y^\varepsilon(t) - h(x^\varepsilon(t - \tau))| \leq \|h'\|_{\infty, [0,1]} \frac{\nu + \beta}{\nu} \delta, \quad \forall \delta > 0$$

and the result follows. ■

6.5 Heteroclinic orbits

The aim of this section is to prove Theorem 6.2.5. To be more specific, in this section we consider the case where $\varphi \equiv 0_C$ and $y_0 \in (0, 1]$ and we are interested by the convergence of x^ε whenever $R_0 > 1$. In such a case, due to Lemma 6.3.2-(iii), the uniform convergence on the half line toward the solution of the reduced problem cannot hold true. Instead of that we will prove the convergence to the unique heteroclinic orbit of the reduced system. We conclude this chapter with a convergence result which achieves the proof of Theorem 6.2.5.

6.5.1 Existence and uniqueness of heteroclinic orbits for the reduced system

Our first result deals with the existence of heteroclinic orbits for the reduced system and the result reads as follows:

Proposition 6.5.1 *Assume that $R_0 > 1$. Then there exists an heteroclinic orbit x of the reduced system (6.2.4) that satisfies*

$$0 < x(t) \leq \bar{x}, \quad \forall t \in \mathbb{R}; \quad \lim_{t \rightarrow -\infty} x(t) = 0 \text{ and } \lim_{t \rightarrow +\infty} x(t) = \bar{x}.$$

Proof. Let $\varphi = 0_C$ and $y_0 \in (0, 1]$ be given. Due to Lemma 6.3.2 (iii) we know that for each $\varepsilon > 0$, $x^\varepsilon(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$. Next since $x^\varepsilon(0) = \varphi(0) = 0$, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $x^\varepsilon(t_\varepsilon) = \frac{\bar{x}}{2}$. Moreover due to Lemma 6.4.1 the family of maps $t \mapsto x_t^\varepsilon$ converges locally uniformly to the equilibrium 0_C , so that $t_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Hence one can define the family of shifted maps

$$\hat{x}_t^\varepsilon = x_{t+t_\varepsilon}^\varepsilon \text{ and } \hat{y}^\varepsilon(t) = y^\varepsilon(t + t_\varepsilon), \forall t \geq -t_\varepsilon.$$

Similarly to the proof of Lemma 6.4.1, there exists a sequence $\{\varepsilon_n\}_{n \geq 0} \subset (0, 1]$, tending to 0 as $n \rightarrow \infty$ such that $\hat{x}^{\varepsilon_n} \rightarrow x^0$ locally uniformly and where x^0 is an entire solution of (6.2.4) such that

$$x^\infty(0) = \frac{\bar{x}}{2}, \quad 0 \leq x^\infty(t) \leq 1, \quad \forall t \in \mathbb{R}.$$

As a consequence of the first constraint, x^∞ cannot be identically equal to an equilibrium point of (6.2.4), namely 0_C or $\bar{x}$. Then Lemma 6.3.3 applies and completes the proof of the result. ■

The next result of this section is related to the uniqueness of the heteroclinic orbit constructed in Proposition 6.5.1. Our precise result reads as follows:

Theorem 6.5.2 *Assume that $R_0 > 1$. The reduced system (6.2.4) has a unique (up to time shift) heteroclinic orbit x such that*

$$\lim_{t \rightarrow -\infty} x(t) = 0 \text{ and } \lim_{t \rightarrow +\infty} x(t) = \bar{x}.$$

The proof of this result will be related to Ikehara's theorem (see Carr and Chmaj [30] and the references cited therein) and Laplace transform (see Widder [165]). Our proof is inspired by the one by Carr and Chmaj [30] and Yu and Mei [168]. Before proving the above result, several lemmas are necessary. The uniqueness of this orbit is related to a suitable description of its behavior as $t \rightarrow -\infty$, when the function is approaching 0_C . We will therefore consider the linearized equation associated to (6.2.4) around 0_C , namely

$$\begin{cases} \frac{du(t)}{dt} = -\mu u(t) + \alpha h'(0) u(t - \tau), \\ u_0 = \varphi \in C. \end{cases} \quad (6.5.1)$$

The characteristic equation of the above delay differential equation is

$$\Delta(\lambda) := \lambda + \mu - \frac{\alpha\beta}{\nu} e^{-\lambda\tau}. \quad (6.5.2)$$

Then our first result is related to the location of the roots of the characteristic function Δ .

Lemma 6.5.3 *Assume that $R_0 > 1$. Then the following properties are satisfied*

(i) There exists a unique $\lambda_0 > 0$ such that $\Delta(\lambda_0) = 0$ and

$$\Delta'(\lambda_0) \neq 0 \text{ and } \Delta(\lambda) < 0, \forall \lambda \in [0, \lambda_0).$$

(ii) For all $z \in \mathbb{C}$ we have

$$\Delta(z) = 0 \text{ and } \operatorname{Re}(z) = \lambda_0 \iff z = \lambda_0.$$

Proof. The proof of (i) is obvious and thus omitted. Now let us prove (ii). Let $z \in \mathbb{C}$ be given such that $\Delta(z) = 0$ and $\operatorname{Re}(z) = \lambda_0$. Then we have $\Delta(\lambda_0 + i\operatorname{Im}(z)) = 0$ which implies that

$$\begin{cases} \lambda_0 + \mu = \frac{\alpha\beta}{\nu} e^{-\tau\lambda_0} \cos(\tau\operatorname{Im}(z)), \\ \operatorname{Im}(z) = -\frac{\alpha\beta}{\nu} e^{-\tau\lambda_0} \sin(\tau\operatorname{Im}(z)). \end{cases} \quad (6.5.3)$$

Since $\Delta(\lambda_0) = 0$, namely $\lambda_0 + \mu = \frac{\alpha\beta}{\nu} e^{-\tau\lambda_0}$, we infer from (6.5.3) that

$$\frac{\alpha\beta}{\nu} e^{-\tau\lambda_0} = \frac{\alpha\beta}{\nu} e^{-\tau\lambda_0} \cos(\tau\operatorname{Im}(z)) \implies \cos(\tau\operatorname{Im}(z)) = 1,$$

thus

$$\sin(\tau\operatorname{Im}(z)) = 0.$$

The result follows by using the second equation of (6.5.3). ■

In the sequel, we always assume that $R_0 > 1$ and let x be a given heteroclinic orbit of the reduced system (6.2.4) such that

$$0 < x(t) < 1 \text{ and } \lim_{t \rightarrow -\infty} x(t) = 0 \text{ and } \lim_{t \rightarrow \infty} x(t) = \bar{x}.$$

The aim of the next lemma is to prove that the convergence to 0 as $t \rightarrow -\infty$ is exponential. In the sequel we will prove that we have in fact $x(t) = O(e^{\lambda_0 t})$ as $t \rightarrow -\infty$ where λ_0 is described in Lemma 6.5.3.

Lemma 6.5.4 Assume that $R_0 > 1$. There exists $\rho > 0$ such that $x(t) = O(e^{\rho t})$ as $t \rightarrow -\infty$.

The proof of this result is split into three steps. In step 1, we show that $\int_{-\infty}^t x(s) ds < +\infty$ for all $t \in \mathbb{R}$. Step 2 is devoted to show that there exists $\rho > 0$ such that

$$\sup_{t \leq 0} e^{-\rho t} \int_{-\infty}^t x(s) ds < +\infty.$$

Finally step 3 completes the proof of the lemma.

Proof. Note that

$$\|h'\|_{\infty, [0,1]} = h'(0) = \frac{\beta}{\nu} > 0. \quad (6.5.4)$$

Since $R_0 > 1$ we can find $\eta \in (0, 1)$ such that

$$\frac{\alpha\beta}{\nu} (1 - \eta) > \mu. \quad (6.5.5)$$

Moreover due to (6.5.4), we can find $\delta > 0$ small enough such that

$$0 < x < \delta \implies h(x) > \frac{\beta}{\nu} (1 - \eta) x, \quad (6.5.6)$$

and

$$\frac{\alpha\beta}{\nu} (1 - \eta) (1 - \delta) > \mu. \quad (6.5.7)$$

Step 1: Let us prove that for each $t \in \mathbb{R}$, $\int_{-\infty}^t x(s) ds < +\infty$. Integrating (6.2.4) from t_0 to t yields

$$x(t) - x(t_0) = -\mu \int_{t_0}^t x(s) ds + \alpha \int_{t_0}^t h(x(s - \tau)) (1 - x(s)) ds. \quad (6.5.8)$$

Recalling that $x(t) \rightarrow 0$ as $t \rightarrow -\infty$, there exists $T > 0$ large enough such that for all $t \leq -T$

$$0 < x(t) < \delta,$$

where $\delta > 0$ is defined in (6.5.6). Hence we obtain

$$h(x(s - \tau)) \geq \frac{\beta}{\nu} (1 - \eta) x(s - \tau), \quad \forall s \leq -T, \quad (6.5.9)$$

and by combining (6.5.8) and (6.5.9) we obtain for all $t_0 \leq t \leq -T$, that

$$x(t) - x(t_0) \geq -\mu \int_{t_0}^t x(s) ds + \int_{t_0}^t \frac{\alpha\beta}{\nu} (1 - \eta) x(s - \tau) (1 - x(s)) ds.$$

But due to Lemma 6.3.3, we have

$$0 < x(t) \leq \bar{x}, \quad \forall t \in \mathbb{R},$$

therefore for all $t_0 \leq t \leq -T$,

$$x(t) - x(t_0) \geq -\mu \int_{t_0}^t x(s) ds + \int_{t_0}^t \frac{\alpha\beta}{\nu} (1 - \eta) (1 - \delta) x(s - \tau) ds,$$

thus

$$x(t) - x(t_0) \geq A \int_{t_0}^t [x(s - \tau) - x(s)] ds + B \int_{t_0}^t x(s) ds, \quad (6.5.10)$$

where

$$A := \frac{\alpha\beta}{\nu} (1 - \eta) (1 - \delta) > 0 \text{ and } B := \left[\frac{\alpha\beta}{\nu} (1 - \eta) (1 - \delta) - \mu \right] > 0.$$

Let us notice that

$$\int_{t_0}^t [x(s - \tau) - x(s)] ds = - \int_{t_0}^t \int_{-\tau}^0 \frac{dx(s+l)}{dl} dl ds = - \int_{-\tau}^0 [x(l+t) - x(l+t_0)] dl.$$

Due to the above reformulation and $B > 0$, recalling that $x(t) \rightarrow 0$ as $t \rightarrow -\infty$, allow us to let $t_0 \rightarrow -\infty$ into (6.5.10) yielding that for all $t \leq -T$,

$$x(t) + A \int_{-\tau}^0 x(l+t) dl \geq B \int_{-\infty}^t x(s) ds, \quad (6.5.11)$$

that completes the proof of Step 1.

Step 2: Let us prove that there exists $\rho > 0$ and some constant $\kappa > 0$ such that $e^{-\rho t} \int_{-\infty}^t x(s) ds \leq \kappa$, for all $t \in (-\infty, 0]$. To do so let us define $X : \mathbb{R} \rightarrow \mathbb{R}^+$ by

$$X(t) := \int_{-\infty}^t x(r) dr.$$

Note that due to (6.5.11), $X \in L^1(-\infty, -T)$. Since X is non-decreasing, one has for each $t \leq -T$

$$\int_{-\infty}^t \int_{-\tau}^0 x(l+s) ds dl = \int_{-\tau}^0 X(t+l) dl \leq \tau X(t),$$

therefore by integrating (6.5.11) over $(-\infty, t]$ we obtain

$$B \int_{-\infty}^t X(s) ds \leq (1 + \tau A) X(t), \quad \forall t \leq -T. \quad (6.5.12)$$

Now let $t_1 > 0$ be given large enough such that

$$\rho := \frac{1}{t_1} \ln \left(\frac{Bt_1}{1 + \tau A} \right) > 0.$$

Then note that since X is increasing then

$$X(t - t_1) \leq X(t + s), \quad \forall s \in [-t_1, 0], \quad \forall t \in (-\infty, -T].$$

This implies that for each $t \leq -T$

$$X(t - t_1) \leq \frac{1}{t_1} \int_{t-t_1}^t X(s) ds, \quad (6.5.13)$$

and this latter inequality combined together with (6.5.12) provides that for all $t \leq -T$:

$$X(t - t_1) \leq \frac{1}{t_1} \int_{-\infty}^t X(s) ds \leq \frac{1 + \tau A}{Bt_1} X(t).$$

Due to the definition of ρ , one obtains that

$$\sup_{t \leq -T} e^{-\rho t} X(t) < \infty,$$

that completes the proof of Step 2.

Step 3: This step will conclude the proof of Lemma 6.5.4. Integrating (6.2.4) over $(-\infty, t)$ for some given $t \leq 0$ yields to

$$x(t) \leq \int_{-\infty}^t \alpha h(x(s - \tau))(1 - x(s)) ds \leq \int_{-\infty}^t \alpha h(x(s - \tau)) ds \leq \int_{-\infty}^t \frac{\alpha \beta}{\nu} x(s - \tau) ds,$$

Step 2 applies and provides that the right hand side of this inequality is bounded by $Ke^{\rho t}$ on $(-\infty, 0]$ for some constant $K > 0$ and the result follows. ■

Define the *Laplace transform* of u

$$\mathcal{L}(u)(\lambda) := \int_0^{+\infty} u(t) e^{-\lambda t} dt$$

whenever the integral exists. We will say that the Laplace transform *converges* if the limit

$$\lim_{\tau \rightarrow +\infty} \int_0^\tau e^{-\lambda t} u(t) dt$$

exists, and we will say that the Laplace transform *diverges* otherwise.

For convenience let us recall the following theorem which can be found in Carr and Chmaj [30].

Theorem 6.5.5 (Ikehara's) *Let $u : [0, +\infty) \rightarrow [0, +\infty)$ a positive decreasing locally integrable function. Assume that there exists a function H which is analytic in the strip $\Sigma := \{\lambda \in \mathbb{C} : -\zeta \leq \operatorname{Re}(\lambda) < 0\}$ and there exists an integer $k > -1$ such that*

$$\mathcal{L}(u)(\lambda) := \frac{H(\lambda)}{(\lambda + \zeta)^{k+1}}, \forall \lambda \in \Sigma.$$

Then

$$\lim_{t \rightarrow +\infty} \frac{u(t)}{t^k e^{-\zeta t}} \text{ exists}$$

and this limit is equal to

$$\frac{H(-\zeta)}{\Gamma(\zeta + 1)}$$

where $\Gamma(x)$ is the gamma function.

Before recalling Widder's theorem, let us recall that for a function $u : [0, +\infty) \rightarrow \mathbb{R}$, we call *abscissa of convergence* of u ,

$$\text{abs}(u) := \inf \{ \text{Re}(\lambda) : \text{there exists } \lambda \in \mathbb{C} \text{ for which } \mathcal{L}(u)(\lambda) \text{ exists} \}.$$

Recall also that the *abscissa of absolute convergence* of u is $\text{abs}(|u|)$.

We refer to the proof of Proposition 1.4.1 p. 28 in Arendt et al. [4] of the following lemma.

Lemma 6.5.6 *Let $u : [0, +\infty) \rightarrow [0, +\infty)$ be a locally integrable map. Assume that $\mathcal{L}(u)(\lambda_0)$ converges for some complex number $\lambda_0 \in \mathbb{C}$. Then $\mathcal{L}(u)(\lambda)$ converges for each $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \text{Re}(\lambda_0)$.*

Remark 6.5.7 *By using this lemma we deduce that the Laplace transform of u converges for each $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \text{abs}(u)$ and diverges for each $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) < \text{abs}(u)$. This last property sometimes serves as a definition for the abscissa of convergence of u .*

The following Theorem is due to Widder [165, p.58] (see also Arendt et al. [4, Theorem 1.5.3. p. 34]).

Theorem 6.5.8 (Widder's) *Let $u : [0, +\infty) \rightarrow [0, +\infty)$ be a non-negative and locally integrable map. Assume that $\text{abs}(u) < +\infty$. Then $\mathcal{L}(u)(\lambda)$ is holomorphic in $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > \text{abs}(u)\}$. If in addition $\text{abs}(u) > -\infty$, then $\mathcal{L}(u)(\lambda)$ has a singularity at $\text{abs}(u)$.*

Now let us set

$$v(t) := x(-t), \quad \forall t \in \mathbb{R},$$

that is an entire solution of the equation

$$\frac{dv(t)}{dt} = \mu v(t) - \alpha h(v(t+\tau))(1-v(t)), \quad t \in \mathbb{R}. \quad (6.5.14)$$

Due to Lemma 6.5.4, we have

$$v(t) = O(e^{-\rho t}) \quad \text{as } t \rightarrow \infty.$$

Therefore the Laplace transform $\mathcal{L}(v)(\lambda)$ converges for each $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > -\rho$, and we must have

$$\text{abs}(v) \leq -\rho.$$

By applying the Laplace transform to (6.5.14) yields to

$$\left(\lambda - \mu + \frac{\alpha\beta}{\nu} e^{\lambda\tau} \right) \mathcal{L}(v)(\lambda) = v(0) + \frac{\alpha\beta}{\nu} e^{\lambda\tau} \int_0^\tau v(t) e^{-\lambda t} dt + \mathcal{L}(R)(\lambda),$$

where

$$R(t) := \frac{\alpha\beta}{\nu} v(t+\tau) - \alpha h(v(t+\tau))(1-v(t)).$$

Recalling the definition of Δ in (6.5.2), the latter equation rewrites as

$$-\Delta(-\lambda) \mathcal{L}(v)(\lambda) = v(0) + \frac{\alpha\beta}{\nu} e^{\lambda\tau} \int_0^\tau v(t) e^{-\lambda t} dt + \mathcal{L}(R)(\lambda). \quad (6.5.15)$$

Remark 6.5.9 Note that for all $t \in \mathbb{R}$

$$\begin{aligned} \alpha h(v(t+\tau))(1-v(t)) &= \alpha \frac{\beta v(t+\tau)}{\beta v(t+\tau) + \nu} (1-v(t)) \\ &= \frac{\alpha \beta v(t+\tau)}{\nu} \frac{\nu}{\beta v(t+\tau) + \nu} (1-v(t)) \\ &\leq \frac{\alpha \beta v(t+\tau)}{\nu} (1-v(t)), \end{aligned}$$

thus for all $t \in \mathbb{R}$

$$R(t) \geq \frac{\alpha \beta}{\nu} v(t+\tau) - \frac{\alpha \beta v(t+\tau)}{\nu} (1-v(t)) \geq \frac{\alpha \beta}{\nu} v(t+\tau) v(t) > 0,$$

and we deduce that

$$v(0) + \frac{\alpha \beta}{\nu} e^{\lambda \tau} \int_0^{\tau} v(t) e^{-\lambda t} dt + \int_0^{+\infty} R(t) e^{-\lambda t} dt > 0,$$

whenever $\lambda \in \mathbb{R}$ and $\mathcal{L}(v)(\lambda) < +\infty$.

In the next lemma we investigate to the analyticity of $\mathcal{L}(v)(\lambda)$.

Lemma 6.5.10 Let $\lambda_0 \in \mathbb{R}$ be the real number defined in Lemma 6.5.3. Then the Laplace transform $\mathcal{L}(v)(\lambda)$ is well defined and analytic in the strip $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\lambda_0\}$. Moreover we have

$$\lim_{\lambda(>-\lambda_0) \rightarrow -\lambda_0} \mathcal{L}(v)(\lambda) = +\infty,$$

while

$$\operatorname{abs}(R) < -\lambda_0.$$

Proof. Let us first prove that

$$\lim_{\lambda(>-\lambda_0^+) \rightarrow -\lambda_0^+} \mathcal{L}(v)(\lambda) = +\infty.$$

Let us assume that

$$\lim_{\lambda(>-\lambda_0^+) \rightarrow -\lambda_0^+} \mathcal{L}(v)(\lambda) < +\infty. \quad (6.5.16)$$

Next note that since $v(t) > 0$ for all $t \in \mathbb{R}$, we have for each $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\lambda_1 \geq \lambda_2 \implies \mathcal{L}(v)(\lambda_1) \leq \mathcal{L}(v)(\lambda_2).$$

Therefore (6.5.16) implies that

$$\mathcal{L}(v)(\lambda) < +\infty, \forall \lambda \in \mathbb{R} \text{ with } \lambda > -\lambda_0. \quad (6.5.17)$$

But by using Fatou's Lemma we obtain

$$\mathcal{L}(v)(-\lambda_0) = \int_0^{+\infty} \lim_{\lambda(>-\lambda_0) \rightarrow -\lambda_0} e^{\lambda t} v(t) dt \leq \lim_{\lambda(>-\lambda_0) \rightarrow -\lambda_0} \int_0^{+\infty} e^{\lambda t} v(t) dt < +\infty.$$

We conclude from (6.5.17) that

$$\mathcal{L}(v)(\lambda) < +\infty, \quad \forall \lambda \in [-\lambda_0, +\infty).$$

Now by using (6.5.15), it follows that

$$0 < \mathcal{L}(R)(\lambda) < +\infty, \quad \forall \lambda \in [-\lambda_0, +\infty),$$

and since $\Delta(\lambda_0) = 0$ by taking the limit when λ goes to $-\lambda_0^+$ (with $\lambda \in \mathbb{R}$) into (6.5.15) we obtain

$$\lim_{\lambda(>-\lambda_0) \rightarrow -\lambda_0} \mathcal{L}(R)(\lambda) = \mathcal{L}(R)(-\lambda_0) = -v(0) - \frac{\alpha\beta}{\nu} e^{-\lambda_0\tau} \int_0^\tau v(t) e^{\lambda_0 t} dt < 0,$$

that is a contradiction with the fact that $R(t) > 0$ for each $t \geq 0$ (see Remark 6.5.9).

The contradiction proves that $\mathcal{L}(v)$ has a singularity at $-\lambda_0$ and

$$\lim_{\lambda(>-\lambda_0^+) \rightarrow -\lambda_0^+} \mathcal{L}(v)(\lambda) = +\infty.$$

As a consequence of Lemma 6.5.6, we deduce that

$$-\lambda_0 \leq \text{abs}(v) \leq -\rho < 0.$$

Next we will prove that $\mathcal{L}(v)$ is analytic on the strip $\{\lambda \in \mathbb{C} : -\lambda_0 < \text{Re}(\lambda)\}$. Due to Theorem 6.5.8 it is sufficient to show that

$$\text{abs}(v) = -\lambda_0.$$

Assume by contradiction that

$$-\lambda_0 < \text{abs}(v).$$

Since $\lambda^* := \text{abs}(v) < 0$, we have

$$-\lambda_0 < \lambda^* < 0,$$

therefore by Lemma 6.5.3-(i) we obtain

$$\Delta(-\lambda^*) < 0. \tag{6.5.18}$$

Let $\eta \in (0, \rho)$ (where $\rho > 0$ is defined above). We also have for each $t \in \mathbb{R}$

$$\begin{aligned} 0 &< R(t) = \frac{\alpha\beta}{\nu} v(t+\tau) - \alpha \frac{\beta v(t+\tau)}{\beta v(t+\tau) + \nu} (1 - v(t)) \\ &= \frac{\alpha\beta}{\nu} v(t+\tau) \left[1 - \nu \frac{(1 - v(t))}{\beta v(t+\tau) + \nu} \right] \\ &= \frac{\alpha\beta}{\nu} v(t+\tau) \left[\frac{\beta v(t+\tau) + \nu v(t)}{\beta v(t+\tau) + \nu} \right] \\ &\leq \frac{\alpha\beta}{\nu} v(t+\tau) v(t). \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{+\infty} R(t) e^{-(\lambda^* - \eta/2)t} dt &\leq \frac{\alpha\beta}{\nu} \int_0^{+\infty} v(t + \tau) v(t) e^{-(\lambda^* - \eta/2)t} dt \\ &\leq \frac{\alpha\beta}{\nu} \int_0^{+\infty} v(t) e^{-(\lambda^* + \eta/2)t} dt \sup_{t \geq 0} e^{\eta t} v(t + \tau). \end{aligned}$$

Recalling Lemma 6.5.4 and the definition of v , due to the choice of $\eta \in (0, \rho)$ one has

$$\sup_{t \geq 0} e^{\eta t} v(t + \tau) < +\infty,$$

while since $\text{abs}(v) + \eta/2 > \text{abs}(v)$ we obtain that

$$\int_0^{+\infty} v(t) e^{-(\lambda^* + \eta/2)t} dt < \infty$$

so by (6.5.15)

$$\int_0^{+\infty} R(t) e^{-(\lambda^* - \eta/2)t} dt < \infty.$$

Thus

$$\text{abs}(R) \leq \text{abs}(v) - \eta/2, \quad \forall \eta \in (0, \rho).$$

Moreover since $\text{abs}(R) < \text{abs}(v)$ and since $-\lambda_0 < \text{abs}(v)$ there exists $\kappa > 0$ small enough such that the map

$$\lambda \mapsto \frac{1}{-\Delta(\lambda)} \left[v(0) + \frac{\alpha\beta}{\nu} e^{\lambda\tau} \int_0^\tau v(t) e^{-\lambda t} dt + \mathcal{L}(R)(\lambda) \right],$$

is analytic on the strip $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > \text{abs}(v) - \kappa\}$ and it is an extension of $\mathcal{L}(v)$, a contradiction with Widder's theorem, namely Theorem 6.5.8. As a consequence $\text{abs}(v) = -\lambda_0$. To complete the proof of the Lemma let us notice that using the same arguments as before, one has $\text{abs}(R) < -\lambda_0 (= \text{abs}(v))$ and the result follows. ■

Before proving Theorem 6.5.2 we need to derive the precise behavior of $x(t)$ when t goes to $-\infty$. This will be achieved in the next lemma. Before let us introduce, due to Lemma 6.5.10 the analytic function H acting from the strip $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > -\lambda_0\}$ into $\mathbb{C}$ defined by

$$H(\lambda) := (\lambda + \lambda_0) \mathcal{L}(v)(\lambda) \quad (6.5.19)$$

or equivalently

$$H(\lambda) := \frac{(\lambda + \lambda_0)}{-\Delta(-\lambda)} \left[v(0) + \frac{\alpha\beta}{\nu} e^{\lambda\tau} \int_0^\tau v(t) e^{-\lambda t} dt + \mathcal{L}(R)(\lambda) \right]. \quad (6.5.20)$$

Using this function, our next lemma reads as

Lemma 6.5.11 *The following holds true*

$$\lim_{t \rightarrow -\infty} \frac{x(t)}{e^{\lambda_0 t}} = \frac{H(-\lambda_0 - \mu)}{\Gamma(1 + \lambda_0 + \mu)} > 0, \quad (6.5.21)$$

with λ_0 defined in Lemma 6.5.3.

Proof. Since we have defined $v(t) = x(-t)$ for all $t \in \mathbb{R}$, (6.5.21) is equivalent to

$$\lim_{t \rightarrow +\infty} \frac{v(t)}{e^{-\lambda_0 t}} = \frac{H(-\lambda_0 - \mu)}{\Gamma(1 + \lambda_0 + \mu)}.$$

But equation (6.5.14) implies that

$$\frac{d[e^{-\mu t} v(t)]}{dt} = -e^{-\mu t} \alpha h(v(t + \tau))(1 - v(t)) \leq 0, \quad \forall t \in \mathbb{R},$$

therefore the map $t \in [0, +\infty) \rightarrow e^{-\mu t} v(t)$ is decreasing. Set

$$\hat{v}(t) := e^{-\mu t} v(t), \quad \forall t \geq 0.$$

Next notice that for each $\lambda \in \{\hat{\lambda} \in \mathbb{C} : -\lambda_0 - \mu \leq \operatorname{Re}(\hat{\lambda}) < 0\}$ one has

$$\int_0^{+\infty} \hat{v}(t) e^{-\lambda t} dt = \frac{H(\lambda + \mu)}{\lambda + \lambda_0 + \mu}.$$

Therefore since $\hat{v}$ is positive and decreasing, Ikehara's theorem implies

$$\lim_{t \rightarrow +\infty} \frac{\hat{v}(t)}{e^{-(\lambda_0 + \mu)t}} = \frac{H(-\lambda_0 - \mu)}{\Gamma(1 + \lambda_0 + \mu)} \Leftrightarrow \lim_{t \rightarrow -\infty} \frac{v(t)}{e^{\lambda_0 t}} = \frac{H(-\lambda_0 - \mu)}{\Gamma(1 + \lambda_0 + \mu)}.$$

That completes the proof. ■

Corollary 6.5.12 *Function x is increasing on $\mathbb{R}$.*

Proof. According to Lemma 6.5.11, there exists $\alpha_x > 0$ such that $e^{-\lambda_0 t} x(t) \rightarrow \alpha_x$ as $t \rightarrow -\infty$. Now from (6.2.4) one obtains that

$$\lim_{t \rightarrow -\infty} e^{-\lambda_0 t} x'(t) = \alpha_x \left[-\mu + \frac{\alpha \beta}{\nu} e^{-\lambda_0 \tau} \right] = \alpha_x \lambda_0 > 0.$$

The result follows from the results of Smith [152]. ■

We now have all the necessary ingredient to complete the proof of Theorem 6.5.2.

Proof of Theorem 6.5.2. Let x and y be two heteroclinic orbits of the reduced system (6.2.4). From Lemma 6.5.11 there exists $\alpha_x > 0$ and $\alpha_y > 0$ such that

$$\lim_{t \rightarrow -\infty} e^{-\lambda_0 t} x(t) = \alpha_x \quad \text{and} \quad \lim_{t \rightarrow -\infty} e^{-\lambda_0 t} y(t) = \alpha_y.$$

Hence there exists $h \in \mathbb{R}$ such that

$$\lim_{t \rightarrow -\infty} e^{-\lambda_0 t} x(t) = \lim_{t \rightarrow -\infty} e^{-\lambda_0 t} y(t+h).$$

Up to change $y(t)$ by $y(t+h)$, one may assume that $h=0$, that is

$$\lim_{t \rightarrow -\infty} e^{-\lambda_0 t} x(t) = \lim_{t \rightarrow -\infty} e^{-\lambda_0 t} y(t).$$

Next let us define

$$w(t) := \frac{x(t) - y(t)}{e^{\lambda_0 t}}, \quad \forall t \in \mathbb{R}.$$

We aim to show that $w(t) \equiv 0$, so that $x(t) \equiv y(t)$. To do so note that Lemma 6.5.11 ensures that $w(t) \rightarrow 0$ as $t \rightarrow -\infty$ and one can also notice that since x and y are bounded, one has $w(t) \rightarrow 0$ as $t \rightarrow \infty$. We conclude that w is bounded on $\mathbb{R}$. Assume by contradiction that $w(t) \not\equiv 0$. Then, replacing eventually $x - y$ by $y - x$, we can assume, without loss of generality, there exists $t_0 \in \mathbb{R}$ such that

$$w(t_0) = \sup_{t \in \mathbb{R}} |w(t)| > 0. \quad (6.5.22)$$

We claim that $w(t_0) = w(t_0 - \tau)$. Indeed since $w(t_0)$ is a maximum, we have

$$\begin{aligned} \frac{dw(t_0)}{dt} &= 0 = -(\lambda_0 + \mu) w(t_0) \\ &\quad + e^{-\lambda_0 t_0} [\lambda_0 \alpha h(x(t_0 - \tau))(1 - x(t_0)) - \alpha h(y(t_0 - \tau))(1 - y(t_0))], \end{aligned}$$

thus

$$\begin{aligned} (\lambda_0 + \mu) w(t_0) &= \alpha \frac{h(x(t_0 - \tau)) - h(y(t_0 - \tau))}{e^{\lambda_0 t_0}} (1 - x(t_0)) - \alpha w(t_0) h(y(t_0 - \tau)) \\ &\leq \alpha \frac{h(x(t_0 - \tau)) - h(y(t_0 - \tau))}{e^{\lambda_0 t}} \\ &\leq \alpha \int_0^1 h'(sx(t_0 - \tau) + (1-s)y(t_0 - \tau)) ds w(t_0 - \tau) \\ &\leq \alpha h'(0) w(t_0 - \tau) \\ &\leq \frac{\alpha \beta}{\nu} e^{-\lambda_0 \tau} w(t_0 - \tau). \end{aligned}$$

Here recalling that $\lambda_0 + \mu = \frac{\alpha \beta}{\nu} e^{-\lambda_0 \tau}$ it follows that

$$w(t_0) \leq w(t_0 - \tau).$$

Therefore since $w(t_0)$ is a maximum point we also have $w(t_0) \geq w(t_0 - \tau)$ so that $w(t_0) = w(t_0 - \tau)$. By induction one concludes $w(t_0) = w(t_0 - n\tau)$ for all $n \in \mathbb{N}$ which implies that

$$w(t_0) = \lim_{n \rightarrow +\infty} w(t_0 - n\tau) = \lim_{t \rightarrow -\infty} w(t) = 0.$$

That contradict the fact that $w(t_0) > 0$. Therefore $w(t) \equiv 0$ and the result follows. $\blacksquare$

6.5.2 Convergence to the heteroclinic orbits

In this subsection we study the convergence of x^ε whenever the initial conditions $\varphi = 0_C$ and $y_0 \neq 0$ and we complete the convergence part stated in Theorem 6.2.5. In the sequel we denote x^∞ the unique heteroclinic orbit of the reduced system provided by Theorem 6.5.2 such that $x^\infty(0) = \frac{\bar{x}}{2}$.

Lemma 6.5.13 *Assume that $R_0 > 1$. Let $y_0 \in (0, 1]$ be given and let us denote by $(x^\varepsilon, y^\varepsilon)$ the solution of (6.2.1) with initial data $(0_C, y_0)$. Then for each $\varepsilon > 0$ one has*

$$t_\varepsilon := \sup \left\{ t \geq 0 : x^\varepsilon(t) = \frac{\bar{x}}{2} \right\} < \infty \text{ and } \lim_{\varepsilon \rightarrow 0} t_\varepsilon = \infty,$$

and the following convergence holds true

$$\lim_{\varepsilon \rightarrow 0} x^\varepsilon(t + t_\varepsilon) = x^\infty(t),$$

converges uniformly on any intervals of the form $[-T, +\infty)$ with $T \geq 0$.

Proof. By using the same arguments as in the proof of Proposition 6.5.1 we obtain that there exists a family $\{t_\varepsilon\}_{\varepsilon > 0}$ such that for each $\varepsilon > 0$:

$$x^\varepsilon(t_\varepsilon) = \frac{\bar{x}}{2} \text{ and } \lim_{\varepsilon \rightarrow 0} t_\varepsilon = \infty. \quad (6.5.23)$$

and such that the family of functions $\hat{x}^\varepsilon(t) := x^\varepsilon(t + t_\varepsilon)$ converges locally uniformly to the unique heteroclinic orbit x^∞ . Now let $T > 0$ be given. We claim that $\hat{x}^\varepsilon$ converges uniformly to x^∞ on $[-T, +\infty)$. Indeed assume that the convergence is not uniform on $[-T, +\infty)$. Then there exists a sequence $\{\varepsilon_n\}$ tending to 0 as $n \rightarrow \infty$, $\eta > 0$ and a sequence $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that

$$|\hat{x}^{\varepsilon_n}(t_n) - x^\infty(t_n)| > \eta, \quad \forall n \geq 0. \quad (6.5.24)$$

Consider now the sequence of maps $x^n(t) := \hat{x}^{\varepsilon_n}(t_n + t)$. Then since $\hat{x}^{\varepsilon_n}(0) = \frac{\bar{x}}{2}$, Lemma 6.4.4 applies and provides that

$$\lim_{n \rightarrow +\infty} \hat{x}^{\varepsilon_n}(t + t_n) = \bar{x}, \text{ locally uniformly.}$$

Since $x^\infty(t_n) \rightarrow \bar{x}$ as $n \rightarrow \infty$ we reach a contradiction with (6.5.24). This completes the proof of the lemma and therefore completes the proof of Theorem (6.2.4). ■

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