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Zhicong Lin. Eulerian calculus arising from permutation statistics. General Mathematics [math.GM].
Université Claude Bernard - Lyon I, 2014. English. NNT : 2014LYO10064 . tel-00996105

HAL Id: tel-00996105

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N° d'ordre : 64-2014

Année 2014

THÈSE

présentée devant

l'**UNIVERSITÉ CLAUDE BERNARD-LYON 1**

pour l'obtention

du **DIPLÔME DE DOCTORAT**

(arrêté du 7 août 2006)

présentée et soutenue publiquement le 29 Avril 2014

par

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**EULERIAN CALCULUS ARISING FROM
PERMUTATION STATISTICS**

JURY

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Résumé

En 2010 Chung, Graham et Knuth ont démontré une remarquable identité symétrique sur les nombres eulériens et posé le problème de trouver un q -analogue de leur identité. En utilisant les q -polynômes eulériens introduits par Shareshian–Wachs, nous avons pu obtenir une telle q -identité. La preuve bijective que nous avons imaginée, nous a permis ensuite de démontrer d’autres q -identités symétriques, en utilisant un modèle combinatoire dû à Foata–Han. Entre temps, Hyatt a introduit les fonctions quasisymétriques eulériennes colorées afin d’étudier la distribution conjointe du nombre d’excédances et de l’indice majeur sur les permutations colorées. En appliquant le *Decrease Value Theorem* de Foata–Han, nous donnons d’abord une nouvelle preuve de sa formule principale sur la fonction génératrice des fonctions quasisymétriques eulériennes colorées, puis généralisons certaines identités eulériennes symétriques, en les exprimant comme des identités sur les fonctions quasisymétriques eulériennes colorées.

D’autre part, en prolongeant les travaux récents de Savage–Visontai et Beck–Braun, nous considérons plusieurs q -polynômes de descente des mots signés. Leurs fonctions génératrices factorielles et multivariées sont explicitement calculées. Par ailleurs, nous montrons que certains de ces polynômes n’ont que des zéros réels.

Enfin, nous étudions la fonction génératrice diagonale des nombres de Jacobi–Stirling de deuxième espèce, en généralisant des résultats analogues pour les nombres de Stirling et Legendre–Stirling de deuxième espèce. Il s’avère que cette fonction génératrice est une série rationnelle dont le numérateur est un polynôme à coefficients entiers positifs. En appliquant la théorie des P -partitions de Stanley nous trouvons des interprétations combinatoires de ces coefficients.

Mots-clés: statistiques de permutations, descentes, excédances, l’indice majeur, inversions, inversions admissibles, q -polynômes eulériens, factorisation de crochet, problèmes bijectifs, fonctions quasi-symétriques, fonctions quasi-symétriques eulériennes, P -partitions, multipermutations, nombres de Jacobi–Stirling, permutations de Stirling.

Abstract

In 2010 Chung–Graham–Knuth proved an interesting symmetric identity for the Eulerian numbers and asked for a q -analog version. Using the q -Eulerian polynomials introduced by Shareshian–Wachs we find such a q -identity. Moreover, we provide a bijective proof that we further generalize to prove other symmetric q -identities using a combinatorial model due to Foata–Han. Meanwhile, Hyatt has introduced the colored Eulerian quasisymmetric functions to study the joint distribution of the excedance number and major index on colored permutations. Using the *Decrease Value Theorem* of Foata–Han we give a new proof of his main generating function formula for the colored Eulerian quasisymmetric functions. Furthermore, certain symmetric q -Eulerian identities are generalized and expressed as identities involving the colored Eulerian quasisymmetric functions. Next, generalizing the recent works of Savage–Visontai and Beck–Braun we investigate some q -descent polynomials of general signed multipermutations. The factorial and multivariate generating functions for these q -descent polynomials are obtained and the real rootedness results of some of these polynomials are given. Finally, we study the diagonal generating function of the Jacobi–Stirling numbers of the second kind by generalizing the analogous results for the Stirling and Legendre–Stirling numbers of the second kind. It turns out that the generating function is a rational function, whose numerator is a polynomial with nonnegative integral coefficients. By applying Stanley’s theory of P -partitions we find combinatorial interpretations of those coefficients.

Keywords: permutation statistics, descents, excedances, major index, inversions, admissible inversions, q -Eulerian polynomials, bijective problems, hook factorization, quasisymmetric functions, Eulerian quasisymmetric functions, P -partitions, multipermutations, Jacobi–Stirling numbers, Stirling permutations

Acknowledgements

First of all, I would like to thank my advisor, professor Jiang Zeng, for all his guidance for this thesis. I am extremely grateful for his availability and time whenever I needed to discuss with him during my doctorate years.

Next, I am grateful to professors Dominique Foata, Florent Hivert and Dongsu Kim for accepting to be the rapporteurs of my thesis. In particular, Monsieur Foata had read carefully a previous version of this manuscript and made many valuable comments and suggestions, which have permitted to improve the clarity of the dissertation substantially. M. Frédéric Chapoton and Mme. Sylvie Corteel have kindness for being members of my jury. I am very honored for the presence of all of them at my defense.

It is a pleasure for me to thank my coauthors Ira M. Gessel and Guo-Niu Han, from whom I have learned much on combinatorics.

I am grateful to Masao Ishikawa for his encouragement and valuable discussions during his visits to Lyon in the last three years.

I wish to thank Péter Csikvári for sharing his precious experience of doing mathematics with me.

My thanks also go to many colleagues and supportive staff of Institut Camille Jordan. Especially, I would like to express my gratitude to Theresia Eisenkoelbl and Bodo Lass for their interests in my research.

Many friends in Lyon deserve my thanks:

- my fellows in bureau 219, Amélie, Adel, Alexis, Alain, Jean-Baptiste, Elodie, Chang, Mathias, Corentin, Benjamin, they were always willing to help me when I have trouble in French or questions in Latex and Maths;
- John and Evrad for their friendship;
- Ange, for many coffee-times and conversations;
- my chinese friends, Xiangwen, Zhenzhong, Ding Xiao, Yanping, Xiaodong, Huagui, Wang Xi, Xu Rui, Jiu Mingyuan, Ren Hui, Zeng Xiaolin, Ji Mingchao, for many enjoyable time.

Thanks to my wife, for the time she has spent with me in Lyon, and for her constant encouragement, wisdom, support, and love — I owe her much.

Finally, I am very grateful to China Scholarship Council (CSC) for providing me a scholarship for four years to study in France.

*À mes parents et ma soeur
À Wenqin, ma femme et à notre fils*

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Introduction

The classical *Eulerian polynomials* $A_n(t)$ are defined by

$$\sum_{k \geq 0} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}} \quad (1)$$

for integers $n \geq 0$. These polynomials were introduced by Euler [25] himself in his desire to find a closed expression for $\sum_{1 \leq i \leq k} i^n (-1)^i$. The Eulerian polynomials arise in a variety of contexts in mathematics and have many remarkable properties [32, 41]. For instance, one can derive easily from (1) the exponential generating function for the Eulerian polynomials

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{zt} - te^z}. \quad (2)$$

A *permutation* of $[n] := \{1, 2, \dots, n\}$ is a bijection $\pi : [n] \rightarrow [n]$. Let \mathfrak{S}_n denote the set of all permutations of $[n]$. A *statistic* on \mathfrak{S}_n is a mapping $st : \mathfrak{S}_n \rightarrow \mathbb{N}$. The study of statistics on permutations is a recurrent topic in Enumerative Combinatorics and its history can at least trace back to MacMahon [60, 61].

For each permutation $\pi \in \mathfrak{S}_n$ the *descent number* of π , denoted $\text{des}(\pi)$, is the number of positions i , $1 \leq i \leq n-1$, such that $\pi(i) > \pi(i+1)$. A classical result in Enumerative Combinatorics, discovered two centuries after Euler [70], is that the generating function for the descent statistic on permutations is the Eulerian polynomial:

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} = \sum_{k=0}^{n-1} A_{n,k} t^k. \quad (3)$$

So we usually call each statistic equidistributed with the descent number on permutations a *Eulerian statistic*. Another classical Eulerian statistic on permutations is the so-called *excedance number*. The bijection [30] which proves the equidistribution of the decent number and the excedance number on permutations (or more generally on words) is known as *Foata's first fundamental transformation*.

The set \mathfrak{S}_n has a group structure by composition. The resulting group is called the symmetric group, or the type A Coxeter group of order n . One important concept in Coxeter groups is the *length* of its elements. A well-known result in the theory of Coxeter groups is that, in the type A case, the length of an element equals its *inversion number* [6]. Recall that the inversion number of a permutation $\pi \in \mathfrak{S}_n$ is the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. The sum of all the descent positions of a permutation is called the *major index* of this permutation, which was introduced by MacMahon [61]. It is known [62] that the inversion number and the major index are identically distributed on permutations, and in memory of MacMahon, every statistic equidistributed with the inversion number and the major index is called a *Mahonian statistic*. There is also a bijection [31], usually called *Foata's second fundamental transformation*, that proves the equidistribution of the inversion number and the major index on permutations (or more generally on words).

The above four classical permutation statistics give rise to four kinds of q -Eulerian polynomials:

$$\begin{aligned} A_n^{\text{des,maj}}(t, q) &:= \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}, \\ A_n^{\text{des,inv}}(t, q) &:= \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} q^{\text{inv}(\pi)}, \\ A_n^{\text{exc,inv}}(t, q) &:= \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc}(\pi)} q^{\text{inv}(\pi)}, \\ A_n^{\text{exc,maj}}(t, q) &:= \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc}(\pi)} q^{\text{maj}(\pi)}, \end{aligned}$$

where $\text{exc}(\pi)$, $\text{inv}(\pi)$ and $\text{maj}(\pi)$ denote the excedance number, the inversion number and the major index of π , respectively. The generating functions for the first three q -Eulerian polynomials were derived by Carlitz [11], Stanley [83] and Clarke–Steingrímsson–Zeng [21], respectively. For a long time, the fourth q -Eulerian polynomial had not been exploited until in 2007, Shareshian and Wachs [74, 76] introduced some *Eulerian quasisymmetric functions* and proved an elegant q -analog of (2) for the exponential generating function for the polynomials $A_n^{\text{exc,maj}}(t, q)$. It is worth noting that Foata and Han [34] studied various statistics on words and calculated the quadruple distribution of the number of *fixed points*, descents and excedances, together with the major index over permutations, in the form of a factorial generating formula, which implies the result of Shareshian–Wachs. After that, the q -Eulerian polynomials $A_n^{\text{exc,maj}}(t, q)$ have attracted the attention of several authors [19, 35, 34, 33, 39, 51, 52, 75, 77]. In particular, Foata and Han [34, 35] derived another combinatorial interpretation of $A_n^{\text{exc,maj}}(t, q)$ by making use of Gessel's *hook factorization* [45] of permutations.

In Chapter 1 we continue along this line of research. The coefficients $A_{n,k}$ appearing in (3) are traditionally called *Eulerian numbers* [70]. In 2010 Chung–Graham–Knuth [20] found the following symmetric identity involving both Eulerian numbers and binomial coefficients

$$\sum_{k \geq 1} \binom{a+b}{k} A_{k,a-1} = \sum_{k \geq 1} \binom{a+b}{k} A_{k,b-1} \quad \text{for } a, b \geq 1 \quad (4)$$

and asked for a q -analog version of it. We find such a q -analog which involves $A_n^{\text{exc,maj}}(t, q)$ and the q -binomial coefficients, and thereby answer their question. Chung and Graham [19] further proved some similar symmetric q -Eulerian identities and asked for bijective proofs. Using Foata and Han’s combinatorial model, we provide bijective proofs for all the above symmetric q -Eulerian identities. We also study some restricted q -Eulerian polynomials and obtain a generalized symmetric identity for them. Moreover, a new recurrence formula for the q -Eulerian polynomials $A_n^{\text{exc,maj}}(t, q)$ is found.

Let l be a fixed positive integer and let $C_l \wr \mathfrak{S}_n$ be the wreath product of the cyclic group C_l of order l by the symmetric group \mathfrak{S}_n of order n . The group $C_l \wr \mathfrak{S}_n$ is also known as the *colored permutation group*. In the case $l = 1$ (resp. $l = 2$) the wreath product $C_l \wr \mathfrak{S}_n$ is the symmetric group \mathfrak{S}_n (resp. the group of the signed permutations or the *Type B Coxeter group* B_n). Various statistics on colored permutation groups have been studied in the literature and several q -analogs of colored Eulerian polynomials have been proposed [18, 28, 29, 36, 37, 39, 52, 85]. In particular, Foata and Han [39] derived from their *Decrease Value Theorem* [33] a factorial generating formula for the quadruple distribution of the number of *fixed points*, *flag descents* [1, 85], *flag excedances* [5] and of the *flag major index* [1, 85] on colored permutations. Generalizing the Eulerian quasisymmetric functions of Shareshian and Wachs, Hyatt [52] introduced some *colored Eulerian quasisymmetric functions* which enable him to calculate another quadruple distribution involving the so-called *starred descents* instead of the flag descents on colored permutations. The starting point of Chapter 2 is the attempt to obtain a symmetric function generalization of (1.1) for colored permutation groups.

In Chapter 2, we show how Hyatt’s generating function formula for the *fixed point colored Eulerian quasisymmetric functions* can be deduced from the Decrease Value Theorem. Using this generating function formula, we prove two symmetric function generalizations of the symmetric Eulerian identity (4) for some *flag Eulerian quasisymmetric functions*, which are specialized to the flag excedance numbers on colored permutations. Combinatorial proofs of those symmetric identities are also constructed. We also study some other properties of the flag Eulerian quasisymmetric functions. In particular, we confirm a recent conjecture of Mongelli [64] about the unimodality of the generating function for the flag excedances

over the type B derangements. Moreover, colored versions of the hook factorization and the *admissible inversions* [74, 58] of permutations are found. Finally, we introduce a colored analog of *Rawlings major index* [68] for colored permutations. We obtain an interpretation of the colored Eulerian quasisymmetric functions as a sum of some fundamental quasisymmetric functions related to the colored Rawlings major index, by applying Stanley's P -partition theory and a decomposition of the chromatic quasisymmetric functions due to Shareshian and Wachs [77].

The rest of this dissertation is devoted to the generalizations and extensions of identity (1), which, by (3), can be written as

$$\sum_{k \geq 0} (k+1)^n t^k = \frac{1}{(1-t)^{n+1}} \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)}. \quad (5)$$

Carlitz [11] gave the following q -analog of (5):

$$\sum_{k \geq 0} ([k+1]_q)^n t^k = \frac{1}{\prod_{j=0}^n (1-tq^j)} \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}, \quad (6)$$

where $[n]_q := 1 + q + \dots + q^{n-1}$.

Recently, Savage and Schuster [72] studied the Ehrhart series of lecture hall polytopes and introduced a generalization of the inversion tables called *inversion sequences* as follows. For a sequence of positive integers $\mathbf{s} = (s_i)_{i \geq 1}$, let $\mathfrak{I}_n^{\mathbf{s}}$ be the set of \mathbf{s} -inversion sequences of length n defined as

$$\mathfrak{I}_n^{(\mathbf{s})} := \{(e_1, \dots, e_n) \in \mathbb{Z}^n : 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

The *ascent number* of an \mathbf{s} -inversion sequence $\mathbf{e} = (e_1, \dots, e_n) \in \mathfrak{I}_n^{\mathbf{s}}$ is defined to be

$$\text{asc}(\mathbf{e}) := |\{0 \leq i < n : e_i/s_i < e_{i+1}/s_{i+1}\}|,$$

with the convention that $e_0 = 0$ and $s_0 = 1$. When $\mathbf{s} = (1, 2, 3, \dots)$, the set of \mathbf{s} -inversion sequences $\mathfrak{I}_n^{\mathbf{s}}$ is the set of inversion tables of length n . As shown in [72, Lemma 1], the ascent numbers on inversion tables of length n has the same distribution as the descent numbers on \mathfrak{S}_n .

In Chapter 3, motivated by a conjecture of Savage and Visontai [73] about the equidistribution of the descent statistic on signed permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ and the ascent statistic on $(1, 4, 3, 8, \dots, 2n-1, 4n)$ -inversion sequences, we investigate the descent polynomial over the signed permutations of a general multiset (multipermutations). We obtain a generalization of (6) for a q -analog of these descent polynomials and apply it to show that they have only real roots. Two proofs of the conjecture of Savage and Visontai are provided. Furthermore, multivariate generalizations of (6) that enumerate two different Euler–Mahonian distributions on type B Coxeter groups due to Beck and Braun [7] are generalized to signed multipermutations.

The *Stirling number* [69] of the second kind $S(n, k)$ enumerates the number of ways to partition $[n]$ into k non-empty subsets. Inspired by identity (5), Gessel and Stanley [48] proved that

$$\sum_{k \geq 0} S(n+k, k)t^k = \frac{1}{(1-t)^{2n+1}} \sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)}, \quad (7)$$

where \mathcal{S}_n is the set of permutations of $\{1, 1, 2, 2, \dots, n, n\}$ such that for each i , $1 \leq i \leq n$, the values between the two occurrences of i are larger than i . These restricted permutations are called *Stirling permutations*. In the study of Spectral theory Everitt et al. [27] introduced the *Jacobi–Stirling numbers* $\text{JS}(n, k; z)$. Note that the leading coefficient of $\text{JS}(n, k; z)$ (viewed as a polynomial in z) is the Stirling number $S(n, k)$. Many recent works [3, 24, 27, 43, 63] have been devoted to the study of Jacobi–Stirling numbers. In particular, Egge [24] has obtained a result similar to (7) for the *Legendre–Stirling numbers* [26] $\text{JS}(n, k; 1)$.

In Chapter 4, we study the descent statistic on generalized Stirling permutations of the multiset

$$\{1, 1, \bar{1}, 2, 2, \bar{2}, \dots, n, n, \bar{n}\}$$

that we call *Jacobi–Stirling permutations*. Generalizing the Gessel–Stanley identity (7) and Egge’s result concerning the Legendre–Stirling numbers, we give a combinatorial interpretation of some polynomials arising from the diagonal generating functions of Jacobi–Stirling numbers.

Chapter 1

Generalized q -Eulerian polynomials

1.1 Introduction and results

For each $\pi \in \mathfrak{S}_n$ a value i , $1 \leq i \leq n-1$, is an *excedance* (resp. *descent*) of π if $\pi(i) > i$ (resp. $\pi(i) > \pi(i+1)$). Let $\text{exc}(\pi)$ and $\text{des}(\pi)$ denote the number of excedances and descents of π , respectively. Recall that the Eulerian number $A_{n,k}$ counts the permutations in \mathfrak{S}_n with k descents (or k excedances), that is,

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc}(\pi)} = \sum_{0 \leq k \leq n-1} A_{n,k} t^k,$$

where $A_n(t)$ is the Eulerian polynomial appearing in (3). Also, see [32, 67] for some leisurely historical introductions to Eulerian polynomials and Eulerian numbers. In 2010 Chung, Graham and Knuth [20] proved the following symmetric identity involving Eulerian numbers and binomial coefficients:

$$\sum_{k \geq 1} \binom{a+b}{k} A_{k,a-1} = \sum_{k \geq 1} \binom{a+b}{k} A_{k,b-1} \quad (1.1)$$

for $a, b \geq 1$. At the end of [20], the authors asked for, among other unsolved problems, a q -analog version of (1.1).

Several q -analogs of Eulerian polynomials with combinatorial meanings have been introduced in the literature (see [11, 21, 83, 74]). Recall that the *major index*, denoted by $\text{maj}(\pi)$, of a permutation $\pi \in \mathfrak{S}_n$ is the sum of all the descents of π , i.e., $\text{maj}(\pi) := \sum_{\pi(i) > \pi(i+1)} i$. An element $i \in [n]$ is a *fixed point* of $\pi \in \mathfrak{S}_n$ if $\pi(i) = i$ and the number of fixed points of π is denoted by $\text{fix}(\pi)$. Define the (q, r) -Eulerian polynomials $A_n(t, r, q)$ by the following extension of (2):

$$\sum_{n \geq 0} A_n(t, r, q) \frac{z^n}{(q; q)_n} = \frac{(1-t)e(rz; q)}{e(tz; q) - te(z; q)}, \quad (1.2)$$

where

$$(t; q)_n := \prod_{i=0}^{n-1} (1 - tq^i)$$

is the q -shifted factorial and $e(z; q)$ is the following q -exponential function

$$e(z; q) := \sum_{n \geq 0} \frac{z^n}{(q; q)_n}.$$

The following interpretation for $A_n(t, r, q)$ was given by Shareshian and Wachs [74, 76]:

$$A_n(t, r, q) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc}(\pi)} r^{\text{fix}(\pi)} q^{(\text{maj} - \text{exc})\pi}. \quad (1.3)$$

These polynomials have attracted the attention of several authors (cf. [35, 34, 33, 39, 50, 52, 51, 58, 75, 77]).

Let $A_n(t, q) = A_n(t, 1, q)$. Define the q -Eulerian numbers $A_{n,k}(q)$ and the fixed point q -Eulerian numbers $A_{n,k,j}(q)$:

$$A_n(t, q) = \sum_k A_{n,k}(q) t^k \quad \text{and} \quad A_n(t, r, q) = \sum_{j,k} A_{n,k,j}(q) r^j t^k.$$

By (1.3), we have the following interpretations

$$A_{n,k}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \text{exc}(\pi)=k}} q^{(\text{maj} - \text{exc})\pi} \quad \text{and} \quad A_{n,k,j}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \text{exc}(\pi)=k \\ \text{fix}(\pi)=j}} q^{(\text{maj} - \text{exc})\pi}. \quad (1.4)$$

Recall that the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k}$$

for $0 \leq k \leq n$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$.

As developed in Section 1.3, we have got, and then proved the following q -analog of (1.1) involving both the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and the q -Eulerian numbers $A_{n,k}(q)$.

Theorem 1.1.1. *For any integers $a, b \geq 1$,*

$$\sum_{k \geq 1} \begin{bmatrix} a+b \\ k \end{bmatrix}_q A_{k,a-1}(q) = \sum_{k \geq 1} \begin{bmatrix} a+b \\ k \end{bmatrix}_q A_{k,b-1}(q). \quad (1.5)$$

Recently, Chung and Graham [19] derived from (1.2) the following two further symmetric q -Eulerian identities through some analytical arguments. They also asked for bijective proofs.

Theorem 1.1.2 (Chung–Graham). *Let a, b, j be integers with $a, b \geq 1$ and $j \geq 0$. Then*

$$\sum_{k \geq 1} (-1)^k \begin{bmatrix} a+b \\ k \end{bmatrix}_q q^{\binom{a+b-k}{2}} A_{k,a}(q) = \sum_{k \geq 1} (-1)^k \begin{bmatrix} a+b \\ k \end{bmatrix}_q q^{\binom{a+b-k}{2}} A_{k,b}(q), \quad (1.6)$$

$$\sum_{k \geq 1} \begin{bmatrix} a+b+j+1 \\ k \end{bmatrix}_q A_{k,a}^{(j)}(q) = \sum_{k \geq 1} \begin{bmatrix} a+b+j+1 \\ k \end{bmatrix}_q A_{k,b}^{(j)}(q). \quad (1.7)$$

In this chapter we will apply another interpretation of $A_n(t, r, q)$ introduced by Foata and Han [34] to construct bijective proofs of Theorems 1.1.1 and 1.1.2.

Next, for $1 \leq j \leq n$ we define the *restricted q -Eulerian polynomial* $B_n^{(j)}(t, q)$ by the exponential generating function:

$$\sum_{n \geq j} B_n^{(j)}(t, q) \frac{z^{n-1}}{(q; q)_{n-1}} = \left(\frac{A_{j-1}(t, q)(qz)^{j-1}}{(q; q)_{j-1}} \right) \frac{e(tz; q) - te(tz; q)}{e(tz; q) - te(z; q)}. \quad (1.8)$$

and the *restricted q -Eulerian number* $B_{n,k}^{(j)}(q)$ by $B_n^{(j)}(t, q) = \sum_k B_{n,k}^{(j)}(q)t^k$.

Remark 1.1. Originally, the *restricted Eulerian number* $B_{n,k}^{(j)}$ in [19] was defined to be the number of permutations $\pi \in \mathfrak{S}_n$ with $\text{des}(\pi) = k$ and $\pi(j) = n$. According to Lemma 1.5.1, $B_{n,k}^{(j)}(q)$ is really a q -analogue of $B_{n,k}^{(j)}$. This justifies the names “restricted q -Eulerian number” and “restricted q -Eulerian polynomial”.

Those q -analogues enable us to obtain the following generalized symmetric identity for the restricted q -Eulerian polynomials.

Theorem 1.1.3. *Let a, b, j be integers with $a, b \geq 1$ and $j \geq 2$. Then*

$$\sum_{k \geq 1} \begin{bmatrix} a+b+1 \\ k-1 \end{bmatrix}_q B_{k,a}^{(j)}(q) = \sum_{k \geq 1} \begin{bmatrix} a+b+1 \\ k-1 \end{bmatrix}_q B_{k,b}^{(j)}(q). \quad (1.9)$$

When $q = 1$, the above identity was proved by Chung and Graham [19], who also asked for a bijective proof. We shall give a bijective proof and an analytical proof of Theorem 1.1.3, the latter leading to a new recurrence formula for $A_n(t, r, q)$.

Theorem 1.1.4. *The (q, r) -Eulerian polynomials satisfy the following recurrence formula:*

$$A_{n+1}(t, r, q) = rA_n(t, r, q) + t \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q q^j A_j(t, r, q) A_{n-j}(t, q) \quad (1.10)$$

for $n \geq 1$ and initial conditions $A_0(t, r, q) = 1, A_1(t, r, q) = r$.

1.2 Hook factorization of permutations

A word $w = w_1 w_2 \dots w_m$ on \mathbb{N} is called a *hook* if $w_1 > w_2$ and either $m = 2$, or $m \geq 3$ and $w_2 < w_3 < \dots < w_m$. As shown in [45], each permutation $\pi \in \mathfrak{S}_n$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2\dots\tau_r$, where p is an increasing word and each factor $\tau_1, \tau_2, \dots, \tau_k$ is a hook. To derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. Denote by $\text{inv}(w)$ the numbers of *inversions* of a word $w = w_1 w_2 \dots w_m$, i.e., the number of pairs (w_i, w_j) such that $i < j$ and $w_i > w_j$. Foata and Han [34] introduced the statistics

$$\text{lec}(\pi) := \sum_{1 \leq i \leq k} \text{inv}(\tau_i) \quad \text{and} \quad \text{pix}(\pi) := \text{length}(p).$$

For example, the hook factorization of $\pi = 1\ 3\ 4\ 14\ 12\ 2\ 5\ 11\ 15\ 8\ 6\ 7\ 13\ 9\ 10$ is

$$1\ 3\ 4\ 14 \mid 12\ 2\ 5\ 11\ 15 \mid 8\ 6\ 7 \mid 13\ 9\ 10.$$

Hence $p = 1\ 3\ 4\ 14$, $\tau_1 = 12\ 2\ 5\ 11\ 15$, $\tau_2 = 8\ 6\ 7$, $\tau_3 = 13\ 9\ 10$, $\text{pix}(\pi) = 4$ and

$$\text{lec}(\pi) = \text{inv}(12\ 2\ 5\ 11\ 15) + \text{inv}(8\ 6\ 7) + \text{inv}(13\ 9\ 10) = 7.$$

Let $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_r$ be a series of sets on \mathbb{N} . Define

$$\text{inv}(\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_r) := |\{(k, j) : k \in \mathcal{A}_i, l \in \mathcal{A}_j, k > l \text{ and } i < j\}|.$$

We usually write $\text{cont}(\mathcal{A})$ for the set of all letters in a word \mathcal{A} , so that

$$(\text{inv} - \text{lec})\pi = \text{inv}(\text{cont}(p), \text{cont}(\tau_1), \dots, \text{cont}(\tau_r))$$

if $p\tau_1\tau_2\dots\tau_r$ is the hook factorization of π .

From Foata and Han [34, Theorem 1.4], we have the following combinatorial interpretations of the (q, r) -Eulerian polynomials

$$A_n(t, r, q) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{lec}(\pi)} r^{\text{pix}(\pi)} q^{(\text{inv} - \text{lec})\pi}. \quad (1.11)$$

Therefore

$$A_{n,k}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \text{lec}(\pi) = k}} q^{(\text{inv} - \text{lec})\pi} \quad \text{and} \quad A_{n,k,j}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \text{lec}(\pi) = k \\ \text{pix}(\pi) = j}} q^{(\text{inv} - \text{lec})\pi}. \quad (1.12)$$

It is known [80, Proposition 1.3.17] that the q -binomial coefficient has the interpretation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{(\mathcal{A}, \mathcal{B})} q^{\text{inv}(\mathcal{A}, \mathcal{B})}, \quad (1.13)$$

where the sum is over all ordered partitions $(\mathcal{A}, \mathcal{B})$ of $[n]$ such that $|\mathcal{A}| = k$. We will give bijective proofs of Theorem 1.1.1 and 1.1.2 using the interpretations in (1.12) and (1.13).

Remark 1.2. In [34], a bijection on \mathfrak{S}_n that carries the triplet $(\text{fix}, \text{exc}, \text{maj})$ to $(\text{pix}, \text{lec}, \text{inv})$ was constructed without being specified. This bijection consists of two steps. The first step (see [34, Section 6]) uses the word analogue of *Kim–Zeng decomposition* [55] and an updated version of *Gessel–Reutenauer standardization* [47] to construct a bijection on \mathfrak{S}_n that transforms the triplet $(\text{fix}, \text{exc}, \text{maj})$ to $(\text{pix}, \text{lec}, \text{imaj})$, where $\text{imaj}(\pi) := \text{maj}(\pi^{-1})$ for each permutation π . The second step (see [34, Section 7]) uses Foata’s second fundamental transformation (see [59]) to carry the triplet $(\text{pix}, \text{lec}, \text{imaj})$ to $(\text{pix}, \text{lec}, \text{inv})$. In view of this bijection, one can construct bijective proofs of Theorems 1.1.1 and 1.1.2 using the original interpretations in (1.4), through our bijective proofs.

To construct our bijective proofs, we need two elementary transformations. Let τ be a hook with $\text{inv}(\tau) = k$ and $\text{cont}(\tau) = \{x_1, \dots, x_m\}$, where $x_1 < \dots < x_m$. Define

$$d(\tau) = \begin{cases} x_m x_1 \cdots x_{m-1}, & \text{if } k = 1, \\ x_{m-k+1} x_1 \cdots x_{m-k} x_{m-k+2} \cdots x_m, & \text{if } 2 \leq k \leq m-1. \end{cases} \quad (1.14)$$

Clearly, $d(\tau)$ is the unique hook with $\text{cont}(d(\tau)) = \text{cont}(\tau)$ satisfying

$$\text{inv}(d(\tau)) = m - k = |\text{cont}(\tau)| - \text{inv}(\tau).$$

Let τ be a hook or an increasing word with $\text{inv}(\tau) = k$ and $\text{cont}(\tau) = \{x_1, \dots, x_m\}$, where $x_1 < \dots < x_m$. Define

$$d'(\tau) = x_{m-k} x_1 \cdots x_{m-k-1} x_{m-k+1} \cdots x_m. \quad (1.15)$$

Clearly, $d'(\tau)$ is the unique hook (when $k < m-1$) or increasing word (when $k = m-1$) with $\text{cont}(d(\tau)) = \text{cont}(\tau)$ satisfying

$$\text{inv}(d(\tau)) = m - k - 1 = |\text{cont}(\tau)| - 1 - \text{inv}(\tau).$$

1.3 Proofs of Theorem 1.1.1 and 1.1.2

A generating function proof of Theorem 1.1.1. Setting $r = 1$ in (1.2) and subtracting both sides by 1 we obtain

$$\sum_{n \geq 1} A_n(t, q) \frac{z^n}{(q; q)_n} = \frac{e(z; q) - e(tz; q)}{e(tz; q) - te(z; q)}. \quad (1.16)$$

Replacing t by t^{-1} and z by tz yields

$$t^n A_n(t^{-1}, q) = t A_n(t, q).$$

Thus, we have the symmetry property

$$A_{n,k}(q) = A_{n,n-k-1}(q). \quad (1.17)$$

Multiplying both sides of (1.16) by $e(tz; q) - te(z; q)$ we get

$$(e(tz; q) - te(z; q)) \sum_{n \geq 0} A_n(t, q) \frac{z^n}{(q; q)_n} = \sum_k \frac{(1-t^k)z^k}{(q; q)_k}. \quad (1.18)$$

Now,

$$\begin{aligned} e(tz; q) \sum_{n \geq 1} A_n(t, q) \frac{z^n}{(q; q)_n} &= \sum_k \frac{(tz)^k}{(q; q)_k} \sum_{n,i} A_{n,i}(q) t^i \frac{z^n}{(q; q)_n} \\ &= \sum_{k,n,i} \begin{bmatrix} n+k \\ k \end{bmatrix}_q A_{n,i}(q) t^{i+k} \frac{z^{n+k}}{(q; q)_{n+k}} \\ &= \sum_{k,n,i} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-k}(q) t^i \frac{z^n}{(q; q)_n}, \end{aligned}$$

and

$$\begin{aligned} te(z; q) \sum_{n \geq 1} A_n(t, q) \frac{z^n}{(q; q)_n} &= t \sum_k \frac{z^k}{(q; q)_k} \sum_{n,i} A_{n,i}(q) t^i \frac{z^n}{(q; q)_n} \\ &= \sum_{k,n,i} \begin{bmatrix} n+k \\ k \end{bmatrix}_q A_{n,i}(q) t^{i+1} \frac{z^{n+k}}{(q; q)_{n+k}} \\ &= \sum_{k,n,i} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-1}(q) t^i \frac{z^n}{(q; q)_n}. \end{aligned}$$

Substituting the last two expressions in (1.18) and identifying the coefficients of $t^i z^n / (q; q)_n$ on both sides yields

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-k}(q) - \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,i-1}(q) = \begin{cases} 1, & \text{if } i = 0 \neq n, \\ -1, & \text{if } i = n \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By setting $i = a$, $n = a + b$, and using (1.17), we obtain (1.5). \square

In the following, we will construct bijective proofs of Theorem 1.1.1 and 1.1.2 by making use of the two involutions d and d' defined in (1.14) and (1.15), respectively. As a warm-up, we first prove the symmetry property (1.17) by constructing an explicit involution on permutations.

Lemma 1.3.1. *There is an involution $\pi \mapsto \sigma$ on \mathfrak{S}_n satisfying*

$$\text{lec}(\pi) = n - 1 - \text{lec}(\sigma), \quad \text{and} \quad (\text{inv} - \text{lec})\pi = (\text{inv} - \text{lec})\sigma.$$

Proof. Let $\pi = p\tau_1\tau_2 \dots \tau_r$ be the hook factorization of $\pi \in \mathfrak{S}_n$.

- If $p \neq \emptyset$, let $\sigma = d'(p)d(\tau_1)d(\tau_2), \dots, d(\tau_r)$.
- If $p = \emptyset$, let $\sigma = d'(\tau_1)d(\tau_2)d(\tau_3) \dots d(\tau_r)$.

Since d and d' are two involutions, it is routine to check that such a mapping is an involution with the required properties. \square

For each fixed positive integer n , a *two-pix-permutation of $[n]$* is a sequence of words

$$\mathbf{v} = (p_1, \tau_1, \tau_2, \dots, \tau_{r-1}, \tau_r, p_2) \tag{1.19}$$

satisfying the following conditions:

- (C1) p_1 and p_2 are two increasing words, possibly empty;
- (C2) τ_1, \dots, τ_r are hooks for some positive integer r ;
- (C3) The concatenation $p_1\tau_1\tau_2 \dots \tau_{r-1}\tau_r p_2$ of all components of \mathbf{v} is a permutation of $[n]$.

We also extend the two statistics to the two-pix-permutations by

$$\begin{aligned} \text{lec}(\mathbf{v}) &= \text{inv}(\tau_1) + \text{inv}(\tau_2) + \dots + \text{inv}(\tau_r), \\ \text{inv}(\mathbf{v}) &= \text{inv}(p_1\tau_1\tau_2 \dots \tau_{r-1}\tau_r p_2). \end{aligned}$$

It follows that

$$(\text{inv} - \text{lec})\mathbf{v} = \text{inv}(\text{cont}(p_1), \text{cont}(\tau_1), \text{cont}(\tau_2), \dots, \text{cont}(\tau_r), \text{cont}(p_2)). \tag{1.20}$$

Lemma 1.3.2. *The generating function for all two-pix-permutations \mathbf{v} of n such that $\text{lec}(\mathbf{v}) = s$ by the statistic $\text{inv} - \text{lec}$ is*

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,s}(q). \tag{1.21}$$

Proof. By the hook factorization, the two-pix-permutation \mathbf{v} in (1.19) is in bijection with the pair (σ, p_2) , where $\sigma = p_1\tau_1\tau_2\dots\tau_{r-1}\tau_r$ is a permutation on $[n] \setminus \text{cont}(p_2)$ and p_2 is an increasing word. Thus, by (1.12), (1.13), and (1.20), the generating function for the two-pix-permutations \mathbf{v} of $[n]$ such that $\text{lec}(\mathbf{v}) = s$ and $|p_2| = n - k$ with respect to the weight $q^{(\text{inv} - \text{lec})(\mathbf{v})}$ is $[\begin{smallmatrix} n \\ k \end{smallmatrix}]_q A_{k,s}(q)$. \square

Lemma 1.3.3. *There is a bijection $\mathbf{v} \mapsto \mathbf{u}$ on the set of all two-pix-permutations of $[n]$ satisfying*

$$\text{lec}(\mathbf{v}) = n - 2 - \text{lec}(\mathbf{u}), \quad \text{and} \quad (\text{inv} - \text{lec})\mathbf{v} = (\text{inv} - \text{lec})\mathbf{u}.$$

Proof. We give an explicit construction of the bijection. Let \mathbf{v} be a two-pix-permutation and write

$$\mathbf{v} = (\tau_0, \tau_1, \tau_2, \dots, \tau_{r-1}, \tau_r, \tau_{r+1}),$$

where $\tau_0 = p_1$ and $\tau_{r+1} = p_2$. If τ_i (respectively τ_j) is the leftmost (respectively rightmost) non-empty word (clearly $i = 0, 1$ and $j = r, r + 1$), we can write \mathbf{v} in the following compact way by removing the empty words at the beginning or at the end:

$$\mathbf{v} = (\tau_i, \tau_{i+1}, \dots, \tau_{j-1}, \tau_j). \quad (1.22)$$

It is easy to see that the above procedure is reversible by adding some necessary empty words at the two ends of the compact form (1.22). Now we work with the compact form. Recall that

$$(\text{inv} - \text{lec})\mathbf{v} = \text{inv}(\text{cont}(\tau_i), \text{cont}(\tau_{i+1}), \dots, \text{cont}(\tau_{j-1}), \text{cont}(\tau_j)) \quad (1.23)$$

and $\text{lec}(\mathbf{v}) = \sum_{k=i}^j \text{inv}(\tau_k)$.

If $i = j$, then only one word τ_i is in the sequence \mathbf{v} . We define $\mathbf{u} = (\emptyset, \sigma_i, \emptyset)$, where σ_i is the unique word (hook) with content $[n]$ such that $\text{inv}(\sigma_i) = n - 2 - \text{inv}(\tau_i)$.

If $j > i$, we define the two-pix-permutation \mathbf{u} by

$$\mathbf{u} = (d'(\tau_i), d(\tau_{i+1}), d(\tau_{i+2}), \dots, d(\tau_{j-1}), d'(\tau_j)),$$

where d and d' are two involutions defined in (1.14) and (1.15).

Since $\text{inv}(d'(\tau_i)) = |\text{cont}(\tau_i)| - 1 - \text{inv}(\tau_i)$, $\text{inv}(d'(\tau_j)) = |\text{cont}(\tau_j)| - 1 - \text{inv}(\tau_j)$ and $\text{inv}(d(\tau_k)) = |\text{cont}(\tau_k)| - \text{inv}(\tau_k)$ for $k \neq i, j$, we have

$$\text{lec}(\mathbf{u}) = \sum |\text{cont}(\tau_k)| - 2 - \text{lec}(\mathbf{v}) = n - 2 - \text{lec}(\mathbf{v}).$$

Finally, it follows from (1.23) that $(\text{inv} - \text{lec})\mathbf{u} = (\text{inv} - \text{lec})\mathbf{v}$. \square

We give an example to illustrate the bijection. Let $\mathbf{v} = (27, 6389, 514, \emptyset)$. Then \mathbf{v} is a two-pix-permutation of $[9]$ and $\text{inv}(\mathbf{v}) = 19, \text{lec}(\mathbf{v}) = 3, (\text{inv} - \text{lec})\mathbf{v} = 16$. The compact form is $(27, 6389, 514)$, so that

$$\mathbf{u} = (d'(27), d(6389), d'(514)) = (72, 9368, 145).$$

Since the first word 72 is not increasing, we obtain the standard form by adding the empty word at the beginning, so that $\mathbf{u} = (\emptyset, 72, 9368, 145)$. Hence $\text{inv}(\mathbf{u}) = 20, \text{lec}(\mathbf{u}) = 4$, and $(\text{inv} - \text{lec})\mathbf{u} = 16$.

Bijjective proof of Theorem 1.1.1. Combining Lemmas 1.3.1, 1.3.2 and 1.3.3, we obtain a bijective proof of Theorem 1.1.1. \square

Bijjective proof of (1.6). Let $\mathfrak{S}_n(k) = \{\pi \in \mathfrak{S}_n : \text{pix}(\pi) = k\}$ and $\mathcal{D}_n = \mathfrak{S}_n(0)$. We first note that the left-hand side of (1.6) has the following interpretation:

$$\sum_{\substack{\pi \in \mathcal{D}_n \\ \text{lec } \pi = a}} q^{(\text{inv} - \text{lec})\pi} = \sum_{k \geq 1} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} A_{k,a}(q). \quad (1.24)$$

This interpretation follows immediately from [76, Corollary 4.4] and (1.11). One can also give a direct combinatorial proof of (1.24). Actually, by (1.12) and (1.13) we have

$$\begin{aligned} A_{n,a}(q) &= \sum_{\substack{\pi \in \mathfrak{S}_n \\ \text{lec } \pi = a}} q^{(\text{inv} - \text{lec})\pi} \\ &= \sum_k \sum_{\substack{\pi \in \mathfrak{S}_n(k) \\ \text{lec } \pi = a}} q^{\text{inv}(\text{cont}(p), \text{cont}(\tau_1 \dots \tau_r)) + \text{inv}(\text{cont}(\tau_1), \text{cont}(\tau_2), \dots, \text{cont}(\tau_r))} \\ &= \sum_k \sum_{\substack{\mathcal{A} \subseteq [n] \\ |\mathcal{A}| = k}} q^{\text{inv}(\mathcal{A}, [n] \setminus \mathcal{A})} \sum_{\substack{\pi \in \mathcal{D}_{n-k} \\ \text{lec } \pi = a}} q^{(\text{inv} - \text{lec})\pi} \\ &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{\pi \in \mathcal{D}_k \\ \text{lec } \pi = a}} q^{(\text{inv} - \text{lec})\pi}. \end{aligned}$$

Applying Gaussian inversion (or q -binomial inversion) to the above identity we obtain (1.24).

Lemma 1.3.4. For $0 \leq j \leq n$, there is an involution $\mathbf{v} \mapsto \mathbf{u}$ on $\mathfrak{S}_n(j)$ satisfying

$$\text{lec}(\mathbf{u}) = n - j - \text{lec}(\mathbf{v}) \quad \text{and} \quad (\text{inv} - \text{lec})\mathbf{u} = (\text{inv} - \text{lec})\mathbf{v}.$$

Proof. Let $\mathbf{v} = p\tau_1\tau_2 \dots \tau_r$ be the hook factorization of $\mathbf{v} \in \mathfrak{S}_n(j)$, where p is an increasing word and each factor $\tau_1, \tau_2, \dots, \tau_r$ is a hook. We define \mathbf{u} to be $pd(\tau_1) \dots d(\tau_r)$, where d is defined in (1.14). It is easy to check that this mapping is an involution on $\mathfrak{S}_n(j)$ with the desired properties. \square

Now, by (1.24), the symmetric identity (1.6) is equivalent to the $j = 0$ case of the above Lemma. \square

By (1.12), Lemma 1.3.4 gives a simple bijective proof of the following known [19, 76] symmetry property of the fixed point q -Eulerian numbers.

Corollary 1.3.5. *For $n, k, j \geq 0$,*

$$A_{n,k,j}(q) = A_{n,n-j-k}^{(j)}(q). \quad (1.25)$$

Bijective proof of (1.7). Let $\mathcal{W}_n(j)$ denote the set of all two-pix-permutations of $[n]$ with $|p_1| = j$.

Lemma 1.3.6. *Let a, j be fixed nonnegative integers. Then,*

$$\sum_{\substack{\mathbf{v} \in \mathcal{W}_n(j) \\ \text{lec } \mathbf{v} = a}} q^{(\text{inv} - \text{lec})\mathbf{v}} = \sum_{k \geq 1} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,a}^{(j)}(q). \quad (1.26)$$

Proof. By the hook factorization, the two-pix-permutation in (1.19) is in bijection with the pair (σ, p_2) , where $\sigma = p_1 \tau_1 \tau_2 \dots \tau_{r-1} \tau_r$ is a permutation on $[n] \setminus \text{cont}(p_2)$ and p_2 is an increasing word. Thus, by (1.12), (1.13) and (1.20), the generating function for all two-pix-permutations \mathbf{v} of $[n]$ with $|p_1| = j$ such that $\text{lec}(\mathbf{v}) = a$ and $|p_2| = n - k$ with respect to the weight $q^{(\text{inv} - \text{lec})\mathbf{v}}$ is $\begin{bmatrix} n \\ n-k \end{bmatrix}_q A_{k,a}^{(j)}(q)$. \square

Lemma 1.3.7. *Let j be a fixed nonnegative integer. Then, there is an involution $\mathbf{v} \mapsto \mathbf{u}$ on $\mathcal{W}_n(j)$ satisfying*

$$\text{lec}(\mathbf{v}) = n - j - 1 - \text{lec}(\mathbf{u}), \quad \text{and} \quad (\text{inv} - \text{lec})\mathbf{v} = (\text{inv} - \text{lec})\mathbf{u}.$$

Proof. We give an explicit construction of the bijection using the involutions d and d' defined in (1.14) and (1.15). Let $\mathbf{v} = (p_1, \tau_1, \tau_2, \dots, \tau_{r-1}, \tau_r, p_2)$ be a two-pix-permutation of $[n]$ with $|p_1| = j$. As previously, we write \mathbf{v} in compact form by removing the possible empty word at the end:

$$\mathbf{v} = (p_1, \tau_1, \dots, \tau_{i-1}, \tau_i),$$

where $i = r$ or $r + 1$. Define

$$\mathbf{u} = (p_1, d(\tau_1), d(\tau_2), \dots, d(\tau_{i-1}), d'(\tau_i)).$$

As d and d' are involutions, this mapping is an involution on $\mathcal{W}_n(j)$.

Since we have $\text{lec}(d(\tau_k)) = |\text{cont}(\tau_k)| - \text{lec}(\tau_k)$ for $1 \leq k \leq i-1$ and $\text{lec}(d'(\tau_i)) = |\text{cont}(\tau_i)| - 1$, it follows that

$$\text{lec}(\mathbf{u}) = \sum_{k=1}^{i-1} |\text{cont}(\tau_k)| + |\text{cont}(\tau_i)| - 1 - \text{lec}(\mathbf{v}) = n - j - 1 - \text{lec}(\mathbf{v}).$$

Finally, it follows from (1.20) that $(\text{inv} - \text{lec})\mathbf{u} = (\text{inv} - \text{lec})\mathbf{v}$. This finishes the proof of the lemma. \square

Combining Lemmas 1.3.6 and 1.3.7 we obtain a bijective proof of (1.7). \square

1.4 A new recurrence formula for (q, r) -Eulerian polynomials

The *Eulerian differential operator* δ_x is defined by

$$\delta_x(f(x)) := \frac{f(x) - f(qx)}{x},$$

for any $f(x) \in \mathbb{Q}[q][[x]]$ in the ring of formal power series in x over $\mathbb{Q}[q]$ (instead of the traditional $(f(x) - f(qx))/((1 - q)x)$, see [2, 15]). We need the following elementary properties of δ_x .

Lemma 1.4.1. *For any $f(x), g(x) \in \mathbb{Q}[q][[x]]$, we have*

$$\delta_x(f(x)g(x)) = f(qx)\delta(g(x)) + \delta(f(x))g(x)$$

and

$$\delta_x\left(\frac{1}{f(x)}\right) = \frac{-\delta_x(f(x))}{f(qx)f(x)} \quad (f(x) \neq 0).$$

Proof of Theorem 1.1.4. It is not difficult to verify that, for any variable a

$$\delta_z(e(az; q)) = ae(az; q). \tag{1.27}$$

Now, applying δ_z to both sides of (1.2) and using the above property and Lemma 1.4.1, we obtain

$$\begin{aligned} & \sum_{n \geq 0} A_{n+1}(t, r, q) \frac{z^n}{(q; q)_n} \\ &= \delta_z\left(\frac{(1-t)e(rz; q)}{e(tz; q) - te(z; q)}\right) \\ &= \delta_z((1-t)e(rz; q))(e(tz; q) - te(z; q))^{-1} + \delta_z((e(tz; q) - te(z; q))^{-1})(1-t)e(rz; q) \\ &= \frac{r(1-t)e(rz; q)}{e(tz; q) - te(z; q)} + \frac{(1-t)e(rz; q)(e(tz; q) - te(z; q))}{(e(tz; q) - te(z; q))(e(tz; q) - te(z; q))} \\ &= r \sum_{n \geq 0} A_n(t, r, q) \frac{z^n}{(q; q)_n} + t \left(\sum_{n \geq 0} A_n(t, r, q) \frac{(qz)^n}{(q; q)_n} \right) \left(\sum_{n \geq 1} A_n(t, q) \frac{z^n}{(q; q)_n} \right). \end{aligned}$$

Taking the coefficient of $\frac{z^n}{(q; q)_n}$ in both sides of the above equality, we get (1.10). \square

Remark 1.3. A different recurrence formula for $A_n(t, r, q)$ was obtained in [76, Corollary 4.3]. Eq. (1.10) is similar to two recurrence formulas in the literature: one for the (inv, des)- q -Eulerian polynomials in [66, Corollary 2.22] (see also [15]) and the other one for the (maj, des)- q -Eulerian polynomials in [66, Corollary 3.6].

We shall give another interpretation of $A_n(t, r, q)$ as follows. Let $\pi \in \mathfrak{S}_n$. Recall that an *inversion* of π is a pair $(\pi(i), \pi(j))$ such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. An *admissible inversion* of π is an inversion $(\pi(i), \pi(j))$ that satisfies either

- $1 < i$ and $\pi(i-1) < \pi(i)$ or
- there is some l such that $i < l < j$ and $\pi(i) < \pi(l)$.

We write $\text{ai}(\pi)$ the number of admissible inversions of π . Define the statistic $\text{aid}(\pi) := \text{ai}(\pi) + \text{des}(\pi)$. For example, if $\pi = 42153$ then there are 5 inversions, but only (4, 3) and (5, 3) are admissible. So $\text{inv}(\pi) = 5$, $\text{ai}(\pi) = 2$ and $\text{aid}(\pi) = 2 + 3 = 5$. The statistics “ai” and “aid” were first studied by Shareshian and Wachs [74] in the context of Poset Topology. Here we follow the definitions in [58]. The curious result that the pairs (aid, des) and (maj, exc) are equidistributed on \mathfrak{S}_n was proved in [58] using techniques from Poset Topology.

Let \mathcal{W} be the set of all the words on \mathbb{N} . We define a new statistic, denoted by “rix”, on \mathcal{W} recursively. Let $W = w_1 w_2 \cdots w_n$ be a word in \mathcal{W} and w_i be the rightmost maximum element of W . We define $\text{rix}(W)$ by (with convention that $\text{rix}(\emptyset) = 0$)

$$\text{rix}(W) := \begin{cases} 0, & \text{if } i = 1 \neq n, \\ 1 + \text{rix}(w_1 \cdots w_{n-1}), & \text{if } i = n, \\ \text{rix}(w_{i+1} w_{i+2} \cdots w_n), & \text{if } 1 < i < n. \end{cases}$$

For example, we have $\text{rix}(1524335) = 1 + \text{rix}(152433) = 1 + \text{rix}(2433) = 1 + \text{rix}(33) = 2 + \text{rix}(3) = 3$. As every permutation can be viewed as a word on \mathbb{N} , this statistic is well-defined on permutations.

We write $\mathfrak{S}_n^{(j)}$ the set of permutations $\pi \in \mathfrak{S}_n$ with $\pi(j) = n$. For $n \geq 1$ and $1 \leq j \leq n$, we define $B_n(t, r, q) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} r^{\text{rix}(\pi)} q^{\text{ai}(\pi)}$ and its restricted version by

$$B_n^{(j)}(t, r, q) := \sum_{\pi \in \mathfrak{S}_n^{(j)}} t^{\text{des}(\pi)} r^{\text{rix}(\pi)} q^{\text{ai}(\pi)}. \quad (1.28)$$

We should note here that the restricted q -Eulerian polynomial $B_n^{(j)}(t, q)$ is not equal to $B_n^{(j)}(t, 1, q)$ but to some modification of it, as will be shown in the next section.

Theorem 1.4.2. *We have the following interpretation for (q, r) -Eulerian polynomials:*

$$A_n(t, r, q) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} r^{\text{rix}(\pi)} q^{\text{ai}(\pi)}. \quad (1.29)$$

Proof. We will show that $B_n(t, r, q)$ satisfies the same recurrence formula and initial condition as $A_n(t, r, q)$. For $n \geq 1$ it is clear from the definition of $B_n(t, r, q)$ that

$$B_{n+1}(t, r, q) = \sum_{1 \leq j \leq n+1} B_{n+1}^{(j)}(t, r, q). \quad (1.30)$$

It is easy to see that

$$B_{n+1}^{(1)}(t, r, q) = tB_n(t, 1, q) \quad \text{and} \quad B_{n+1}^{(n+1)}(t, r, q) = rB_n(t, r, q). \quad (1.31)$$

We then consider $B_{n+1}^{(j)}(t, r, q)$ for the case of $1 < j < n + 1$.

For a set X let $\binom{X}{m}$ denote the m -element subsets of X and \mathfrak{S}_X the set of permutations of X . Also let $\mathcal{W}(n, j)$ be the set of all triples (W, π_1, π_2) such that $W \in \binom{[n]}{j}$ and $\pi_1 \in \mathfrak{S}_W, \pi_2 \in \mathfrak{S}_{[n] \setminus W}$. Clearly, the mapping $\pi \mapsto (W, \pi_1, \pi_2)$ defined by

- $W = \{\pi(i) : 1 \leq i \leq j - 1\}$,
- $\pi_1 = \pi(1)\pi(2) \cdots \pi(j - 1)$ and $\pi_2 = \pi(j + 1)\pi(j + 2) \cdots \pi(n)$

is a bijection between $\mathfrak{S}_n^{(j)}$ and $\mathcal{W}(n - 1, j - 1)$ and satisfies

$$\text{des}(\pi) = \text{des}(\pi_1) + \text{des}(\pi_2) + 1, \quad \text{rix}(\pi) = \text{rix}(\pi_2)$$

and

$$\text{ai}(\pi) = \text{ai}(\pi_1) + \text{ai}(\pi_2) + \text{inv}(W, [n - 1] \setminus W) + n - j.$$

Thus, for $1 < j < n + 1$ we have

$$\begin{aligned} B_{n+1}^{(j)}(t, r, q) &= \sum_{\pi \in \mathfrak{S}_{n+1}^{(j)}} t^{\text{des}(\pi)} r^{\text{rix}(\pi)} q^{\text{ai}(\pi)} \\ &= tq^{n+1-j} \sum_{(W, \pi_1, \pi_2) \in \mathcal{W}(n, j-1)} q^{\text{inv}(W, [n] \setminus W)} q^{\text{ai}(\pi_1)} t^{\text{des}(\pi_1)} r^{\text{rix}(\pi_2)} q^{\text{ai}(\pi_2)} t^{\text{des}(\pi_2)} \\ &= tq^{n+1-j} \sum_{W \in \binom{[n]}{j-1}} q^{\text{inv}(W, [n] \setminus W)} \sum_{\pi \in \mathfrak{S}_W} q^{\text{ai}(\pi)} t^{\text{des}(\pi)} \sum_{\pi_2 \in \mathfrak{S}_{[n] \setminus W}} r^{\text{rix}(\pi_2)} q^{\text{ai}(\pi_2)} t^{\text{des}(\pi_2)} \\ &= tq^{n+1-j} \left[\begin{matrix} n \\ j-1 \end{matrix} \right]_q B_{j-1}(t, 1, q) B_{n+1-j}(t, r, q), \end{aligned} \quad (1.32)$$

where we apply (1.13) to the last equality. Substituting (1.31) and (1.32) into (1.30) we obtain

$$B_{n+1}(t, r, q) = rB_n(t, r, q) + tB_n(t, 1, q) + t \sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q q^j B_j(t, r, q) B_{n-j}(t, 1, q).$$

By Theorem 1.1.4, $B_n(t, r, q)$ and $A_n(t, r, q)$ satisfy the same recurrence formula and initial condition, thus $B_n(t, r, q) = A_n(t, r, q)$. This finishes the proof of the theorem. \square

Corollary 1.4.3. *The three triples (rix, des, aid), (fix, exc, maj) and (pix, lec, inv) are equidistributed on \mathfrak{S}_n .*

Remark 1.4. At the *Permutation Patterns 2012* conference, A. Burstein [10] gave a direct bijection on \mathfrak{S}_n that transforms the triple (rix, des, aid) to (pix, lec, inv). The new statistic “rix” was introduced independently therein under the name “aix”. Actually, the definitions of both are slightly different, but they are the same up to an easy transformation. It would be very interesting to find a similar bijective proof of the equidistribution of (rix, des, aid) and (fix, exc, maj). See also Remark 1.2.

1.5 Two proofs of Theorem 1.1.3

1.5.1 An interpretation of $B_{n,k}^{(j)}(q)$ and a proof of Theorem 1.1.3

It follows from (1.2) and (1.8) that $B_{1,0}^{(1)}(q) = 1$ and $B_{n,k}^{(1)}(q) = A_{n-1,k-1}(q)$ for $k \geq 1$. For $j \geq 2$, we have the following interpretation for $B_{n,k}^{(j)}(q)$.

Lemma 1.5.1. *For $2 \leq j \leq n$, $B_{n,k}^{(j)}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n^{(j)} \\ \text{des}(\pi)=k}} q^{\text{ai}(\pi)+2j-n-1}$.*

Proof. Let $j \geq 2$. By means of the recurrence relation (1.32) one can compute without difficulty that the factorial generating function

$$\sum_{n \geq j} \frac{q^{2j-n-1} B_n^{(j)}(t, 1, q) z^{n-1}}{(q; q)_{n-1}}$$

is exactly the right side of (1.8) using (1.2) and (1.29). \square

Lemma 1.5.2. *For $1 < j < n$, we have*

$$B_{n,k}^{(j)}(q) = B_{n,n-1-k}^{(j)}(q).$$

Proof. We first construct an involution $f : \pi \mapsto \pi'$ on \mathfrak{S}_n satisfying

$$\text{ai}(\pi) = \text{ai}(\pi') \quad \text{and} \quad \text{des}(\pi) = n - 1 - \text{des}(\pi'). \quad (1.33)$$

For $n = 1$ define $f(\text{id}) = \text{id}$. For $n \geq 2$ suppose that $\pi = \pi_1 \cdots \pi_n$ is a permutation of $\{\pi_1, \dots, \pi_n\}$ and π_j is the maximum element in $\{\pi_1, \dots, \pi_n\}$. We construct f recursively as follows

$$f(\pi) = \begin{cases} f(\pi_2\pi_3 \cdots \pi_n) \pi_1, & \text{if } j = 1, \\ \pi_n f(\pi_1\pi_2 \cdots \pi_{n-1}), & \text{if } j = n, \\ f(\pi_1\pi_2 \cdots \pi_{j-1}) \pi_j f(\pi_{j+1}\pi_{j+2} \cdots \pi_n), & \text{otherwise.} \end{cases}$$

For example, if $\pi = 3257641$, then

$$f(\pi) = f(325)7f(641) = 5f(32)7f(41)6 = 5237146.$$

Clearly, $\text{ai}(\pi) = 7 = \text{ai}(\pi')$ and $\text{des}(\pi) = 4 = 7 - 1 - \text{des}(\pi')$. It is easy to check that f is an involution. Moreover, we can show that f satisfies (1.33) by induction on n .

For each $\pi = \pi_1 \cdots \pi_{j-1} n \pi_{j+1} \cdots \pi_n$ in $\mathfrak{S}_n^{(j)}$ we then define

$$g(\pi) = f(\pi_1 \cdots \pi_{j-1}) n f(\pi_{j+1} \cdots \pi_n).$$

As f is an involution, g is an involution on $\mathfrak{S}_n^{(j)}$. It follows from (1.33) that $\text{ai}(g(\pi)) = \text{ai}(\pi)$ and $\text{des}(\pi) = n - 1 - \text{des}(g(\pi))$, which completes the proof in view of Lemma 1.5.1. \square

Remark 1.5. A bijective proof of Lemma 1.5.2 when $q = 1$ was given in [19]. But their bijection does not preserve the admissible inversions. Supposing that $\pi = \pi_1 \cdots \pi_n$ is a permutation of $\{\pi_1, \dots, \pi_n\}$ and π_j is the maximum element in $\{\pi_1, \dots, \pi_n\}$, we modify f introduced above and make use of f' defined as follows:

$$f'(\pi) = \begin{cases} f'(\pi_2\pi_3 \cdots \pi_n) \pi_1, & \text{if } j = 1, \\ \pi, & \text{if } j = n, \\ f'(\pi_1\pi_2 \cdots \pi_{j-1}) \pi_j f'(\pi_{j+1}\pi_{j+2} \cdots \pi_n), & \text{otherwise.} \end{cases}$$

Then, f' provides another bijective proof of Corollary 1.3.5 using $(\text{des}, \text{rix}, \text{ai})$.

Now we are in position to give a generating function proof of Theorem 1.1.3.

Proof of Theorem 1.1.3. We start with the generating function given in (1.8). Multiplying both sides by $e(tz; q) - te(z; q)$, we obtain

$$\sum_{n,k} B_{n,k}^{(j)}(q) t^k \frac{z^{n-1}}{(q; q)_{n-1}} (e(tz; q) - te(z; q)) = \frac{(qz)^{j-1} A_{j-1}(t, q)}{(q; q)_{j-1}} (e(tz; q) - te(tz; q)).$$

Expanding the exponential functions, we have

$$\begin{aligned} \sum_{n,k,i} B_{n,k}^{(j)}(q) \frac{t^{k+i} z^{n+i-1}}{(q; q)_i (q; q)_{n-1}} - \sum_{n,k,i} B_{n,k}^{(j)}(q) \frac{t^{k+1} z^{n+i-1}}{(q; q)_i (q; q)_{n-1}} \\ = \frac{(qz)^{j-1} A_{j-1}(t, q)}{(q; q)_{j-1}} \sum_{n \geq 0} \frac{(1-t)t^n z^n}{(q; q)_n}. \end{aligned}$$

Identifying the coefficient of $t^l z^{m-1}$ gives

$$\begin{aligned} \sum_k \frac{B_{m+k-l,k}^{(j)}(q)}{(q; q)_{l-k} (q; q)_{m+k-l-1}} - \sum_i \frac{B_{m-i,l-1}^{(j)}(q)}{(q; q)_i (q; q)_{m-i-1}} \\ = \frac{q^{j-1} (A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q))}{(q; q)_{j-1} (q; q)_{m-j}}. \end{aligned}$$

Multiplying both sides by $(q; q)_{m-1}$, we get

$$\begin{aligned} \sum_k B_{m+k-l,k}^{(j)}(q) \begin{bmatrix} m-1 \\ l-k \end{bmatrix}_q - \sum_i B_{m-i,l-1}^{(j)}(q) \begin{bmatrix} m-1 \\ i \end{bmatrix}_q \\ = (A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q)) q^{j-1} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q. \end{aligned}$$

Changing the variables of the two summations on the left-hand side gives

$$\begin{aligned} \sum_k B_{k,k+l-m}^{(j)}(q) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q - \sum_k B_{k,l-1}^{(j)}(q) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \\ = (A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q)) q^{j-1} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q. \quad (1.34) \end{aligned}$$

Applying the symmetry property in Lemma 1.5.2 to the first summation on the left-hand side of (1.34) we obtain

$$\begin{aligned} \sum_k B_{k,k+l-m}^{(j)}(q) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \\ = B_{j,j+l-m}^{(j)}(q) \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q + \sum_{k \neq j} B_{k,m-1-l}^{(j)}(q) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q. \quad (1.35) \end{aligned}$$

It follows from Lemma 1.5.1 and Theorem 1.4.2 that

$$B_{n,k}^{(n)}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n^{(n)} \\ \text{des}(\pi)=k}} q^{\text{ai}(\pi)+n-1} = q^{n-1} A_{n-1,k}(q).$$

Using the symmetry property of $A_{n,k}(q)$, that is, $A_{n,k}(q) = A_{n,n-1-k}(q)$, and the above property, the right-hand side of (1.34) can be treated as follows:

$$\begin{aligned}
& (A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q))q^{j-1} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q \\
&= B_{j,j+l-m}^{(j)}(q) \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q - A_{j-1,m-1-l}(q)q^{j-1} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q \\
&= B_{j,j+l-m}^{(j)}(q) \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q - B_{j,m-1-l}^{(j)}(q) \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q.
\end{aligned} \tag{1.36}$$

Now, substituting (1.35), (1.36) into (1.34) we obtain

$$\sum_k B_{k,m-1-l}^{(j)}(q) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q = \sum_k B_{k,l-1}^{(j)}(q) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q,$$

which becomes (1.9) after setting $m = a + b + 2$ and $l - 1 = b$. \square

Remark 1.6. The only case left out in Theorem 1.1.3 is the case $j = 1$. However, as $B_{n,k}^{(1)}(q) = A_{n-1,k-1}(q)$, the corresponding symmetric identity for this case is (1.5).

1.5.2 A bijective proof of Theorem 1.1.3

Let

$$\tilde{\mathfrak{S}}_n^{(j)} := \{\pi \in \mathfrak{S}_n : \pi(j+1) = 1\} \quad \text{for } 1 \leq j < n$$

and

$$\tilde{\mathfrak{S}}_n^{(n)} := \{\pi' \square 1 : \pi' \in \mathfrak{S}_{[n] \setminus \{1\}}\}.$$

The “ \square ” in $\pi = \pi_1 \pi_2 \cdots \pi_{n-1} \square 1 \in \tilde{\mathfrak{S}}_n^{(n)}$ means that the n -th position of π is empty and the hook factorization of π is defined to be $p\tau_1 \cdots \tau_r \square 1$, where $p\tau_1 \cdots \tau_r$ is the hook factorization of $\pi_1 \cdots \pi_{n-1}$ and “ $\square 1$ ” is viewed as a hook. We also define the statistics

$$\text{lec}(\pi_1 \pi_2 \cdots \pi_{n-1} \square 1) = \sum_{i=1}^r \text{lec}(\tau_i), \quad \text{inv}(\pi_1 \pi_2 \cdots \pi_{n-1} \square 1) = \text{inv}(\pi_1 \pi_2 \cdots \pi_{n-1} 1).$$

For example, we have $\tilde{\mathfrak{S}}_3^{(3)} = \{32\square 1, 23\square 1\}$ with $\text{lec}(32\square 1) = 1$, $\text{lec}(23\square 1) = 0$, $\text{inv}(32\square 1) = 3$, and $\text{inv}(23\square 1) = 2$.

Lemma 1.5.3. For $1 \leq j \leq n$, $B_{n,k}^{(j)}(q) = \sum_{\substack{\pi \in \tilde{\mathfrak{S}}_n^{(j)} \\ \text{lec}(\pi) = k}} q^{(\text{inv} - \text{lec})\pi}$.

Proof. Let $\bar{B}_n^{(j)}(t, q) := \sum_{\pi \in \bar{\mathfrak{S}}_n^{(j)}} q^{(\text{inv} - \text{lec})\pi} t^{\text{lec} \pi}$. We recall that, to derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. Therefore, the hook factorization of $\pi = \pi_1 \cdots \pi_{j-1} \pi_j 1 \pi_{j+2} \cdots \pi_n$ in $\pi \in \bar{\mathfrak{S}}_n^{(j)}$ is $p\tau_1 \cdots \tau_s \tau'_1 \cdots \tau'_r$, where $p\tau_1 \cdots \tau_s$ and $\tau'_1 \cdots \tau'_r$ are hook factorizations of $\pi_1 \cdots \pi_{j-1}$ and $\pi_j 1 \pi_{j+2} \cdots \pi_n$, respectively. When $n > j$, from the structure of hook factorization we see that

$$\text{lec}(\pi_j 1 \pi_{j+2} \cdots \pi_n) = 1 + \text{lec}(\pi_j \pi_{j+2} \cdots \pi_n)$$

and

$$(\text{inv} - \text{lec})(\pi_j 1 \pi_{j+2} \cdots \pi_n) = (\text{inv} - \text{lec})(\pi_j \pi_{j+2} \cdots \pi_n).$$

Thus, by (1.13) we have

$$\bar{B}_n^{(j)}(t, q) = A_{j-1}(t, q) q^{j-1} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q t A_{n-j}(t, q) \quad (1.37)$$

for $n > j$. Clearly, $\bar{B}_j^{(j)}(t, q) = A_{j-1}(t, q) q^{j-1}$. So, by (1.2) the exponential generating function $\sum_{n \geq j} \bar{B}_n^{(j)}(t, q) z^{n-1} / (q; q)_{n-1}$ is the right-hand side of (1.8). This finishes the proof of the lemma. \square

Remark 1.7. This interpretation can also be deduced directly from the interpretation in Lemma 1.5.1 using Burstein's bijection [10].

For $X \subset [n]$ with $|X| = m$ and $1 \in X$, we can define $\bar{\mathfrak{S}}_X^{(j)}$ for $1 \leq j \leq m$ similarly as $\bar{\mathfrak{S}}_m^{(j)}$ like this:

$$\bar{\mathfrak{S}}_X^{(j)} := \{\pi \in \mathfrak{S}_X : \pi(j+1) = 1\} \text{ for } 1 \leq j < m \quad \text{and} \quad \bar{\mathfrak{S}}_X^{(m)} := \{\pi' \square 1 : \pi' \in \mathfrak{S}_{X \setminus \{1\}}\}.$$

For $1 \leq j \leq n$ we define a j -restricted two-pix-permutation of $[n]$ to be a pair $\mathbf{v} = (\pi, p_2)$ satisfying the following two properties:

- p_2 (possibly empty) is an increasing word on $[n]$;
- $\pi \in \bar{\mathfrak{S}}_X^{(j)}$ with $X = [n] \setminus \text{cont}(p_2)$.

Similarly, we define $\text{lec}(\mathbf{v}) = \text{lec}(\pi)$ and $\text{inv}(\mathbf{v}) = \text{inv}(\pi) + \text{inv}(\text{cont}(\pi), \text{cont}(p_2))$. Let $\mathcal{W}_n^{(j)}$ denote the set of all j -restricted two-pix-permutations of $[n]$.

Lemma 1.5.4. *Let a, j be positive integers. Then,*

$$\sum_{\substack{\mathbf{v} \in \mathcal{W}_n^{(j)} \\ \text{lec} \mathbf{v} = a}} q^{(\text{inv} - \text{lec})\mathbf{v}} = \sum_{k \geq 1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q B_{k,a}^{(j)}(q). \quad (1.38)$$

Proof. It follows from Lemma 1.5.3 and some similar arguments as in the proof of Lemma 1.3.6. \square

Lemma 1.5.5. *Let $2 \leq j \leq n$. Then, there is an involution $\mathbf{v} \mapsto \mathbf{u}$ on $\mathcal{W}_n^{(j)}$ satisfying*

$$\text{lec}(\mathbf{v}) = n - 2 - \text{lec}(\mathbf{u}), \quad \text{and} \quad (\text{inv} - \text{lec})\mathbf{v} = (\text{inv} - \text{lec})\mathbf{u}. \quad (1.39)$$

Proof. Suppose $\mathbf{v} = (\pi, p_2) \in \mathcal{W}_n^{(j)}$ and $\pi = \tau_0 \tau_1 \cdots \tau_r$ is the hook factorization of π such that τ_0 is a hook or an increasing word and τ_i ($1 \leq i \leq r$) are hooks. We also assume that $p_2 = x_1 \cdots x_l$ if p_2 is not empty. Note that $1 \notin \text{cont}(\tau_0)$ since $j \neq 1$. We will use the involutions d and d' defined in (1.14) and (1.15). There are several cases to be considered:

(i) $\tau_r = \square 1$. Then,

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})x_l 1 x_1 x_2 \cdots x_{l-1}, \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})\square 1, \emptyset), & \text{otherwise.} \end{cases}$$

(ii) $\tau_r = y_s 1 y_1 \cdots y_{s-1}$. Then,

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})d(\tau_r)d'(p_2), \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})\square 1, y_1 \cdots y_s), & \text{if } p_2 = \emptyset \text{ and } y_s > y_{s-1}; \\ (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})d'(\tau_r), \emptyset), & \text{otherwise.} \end{cases}$$

(iii) $1 \notin \text{cont}(\tau_r)$. Then,

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})d(\tau_r)d'(p_2), \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1}), d'(\tau_r)), & \text{if } p_2 = \emptyset \text{ and } \text{lec}(\tau_r) = |\tau_r| - 1; \\ (d'(\tau_0)d(\tau_1) \cdots d(\tau_{r-1})d'(\tau_r), \emptyset), & \text{otherwise.} \end{cases}$$

First, one can check that $\mathbf{u} \in \mathcal{W}_n^{(j)}$. Second, as d, d' are involutions, the above mapping is an involution. Finally, this involution satisfies (1.39) in all cases. This completes the proof of the lemma. \square

Combining Lemmas 1.5.4 and 1.5.5 we obtain a bijective proof of Theorem 1.1.3.

1.6 Concluding remarks

For a fixed $k \geq 1$, recall that the *Rawlings major index* $\text{rmaj}_k(\pi)$ of $\pi \in \mathfrak{S}_n$ can be defined as follows:

$$\begin{aligned} \text{DES}_{\geq k}(\pi) &:= \{i \in [n-1] : \pi_i - \pi_{i+1} \geq k\}, \\ \text{inv}_{<k}(\pi) &:= |\{(i, j) \in [n] \times [n] : i < j, \text{ and } 0 < \pi_i - \pi_j < k\}|, \\ \text{maj}_{\geq k}(\pi) &:= \sum_{i \in \text{DES}_{\geq k}(\pi)} i, \\ \text{rmaj}_k(\pi) &:= \text{inv}_{<k}(\pi) + \text{maj}_{\geq k}(\pi). \end{aligned}$$

It is clear that $\text{rmaj}_1(\pi) = \text{maj}(\pi)$ and $\text{rmaj}_n(\pi) = \text{inv}(\pi)$ for $\pi \in \mathfrak{S}_n$. Rawlings [68] showed that $\text{rmaj}_k(\pi)$ is a Mahonian statistic for each $k \geq 1$. Shareshian and Wachs [77] studied some *chromatic quasisymmetric functions* and proved the following equidistribution.

Theorem 1.6.1 ([77, Theorem 4.19]). *The two pairs (exc, maj) and $(\text{ides}, \text{rmaj}_2)$ are equidistributed on \mathfrak{S}_n , where $\text{ides}(\pi) := \text{des}(\pi^{-1})$.*

In view of the above result and Corollary 1.4.3, the following problem is interesting. This problem was also posed by Burstein [10] at the *Permutation Patterns 2012* conference.

Problem 1.1. *Can we describe a statistic “st” on \mathfrak{S}_n so that the two triples*

$$(\text{fix}, \text{exc}, \text{maj}) \text{ and } (\text{st}, \text{ides}, \text{rmaj}_2)$$

are equidistributed on \mathfrak{S}_n ?

Chapter 2

Colored Eulerian quasisymmetric functions

2.1 Introduction and preliminaries

Let l be a fixed positive integer throughout this chapter. Consider the wreath product $C_l \wr \mathfrak{S}_n$ of the cyclic group C_l of order l by the symmetric group \mathfrak{S}_n of order n . The group $C_l \wr \mathfrak{S}_n$ is also known as the *colored permutation group*. In case $l = 1, 2$, $C_l \wr \mathfrak{S}_n$ are respectively the symmetric group \mathfrak{S}_n and the *Type B Coxeter group* B_n . Various statistics on colored permutation groups have been studied in the literature and several kinds of q -analog of colored Eulerian polynomials are proposed [18, 28, 29, 37, 39, 52, 85]. It is worth noting that Foata and Han [34] studied various statistics on words and calculated the quadruple distribution of the number of fixed points, descents and excedances, together with the major index over permutations, in the form of a factorial generating function formula, which implies (1.2). They further extended their results to colored permutations [39]. Recently, in order to generalize (1.2) to colored permutation groups, Hyatt [52] introduced the *colored Eulerian quasisymmetric functions*, which generalize the Shareshian–Wachs Eulerian quasisymmetric functions [76]. The starting point for this chapter is the attempt to obtain a symmetric function generalization of (1.1) for colored permutation groups.

In [52], Hyatt introduced the *cv-cycle type colored Eulerian quasisymmetric functions* $Q_{\check{\lambda}, k}$, where $\check{\lambda}$ is a particular cv-cycle type. They are defined by first associating a fundamental quasisymmetric function with each colored permutation and then summing these fundamental quasisymmetric functions over colored permutations with cv-cycle type $\check{\lambda}$ and k excedances. The precise definition of $Q_{\check{\lambda}, k}$ is given in Section 2.2.1. It was announced in [52] that $Q_{\check{\lambda}, k}$ is in fact a symmetric function. This follows from the colored ornament interpretation of $Q_{\check{\lambda}, k}$ and the

plethysm inversion formula [87, Section 2.4]. But more importantly, we will give a combinatorial proof of this fact, which is needed in the bijective proof of the generalized symmetric identity in Theorem 2.4.12.

Another interesting Eulerian quasisymmetric function is the *fixed point colored Eulerian quasisymmetric function* $Q_{n,k,\vec{\alpha},\vec{\beta}}$, for $\vec{\alpha} \in \mathbb{N}^l$ and $\vec{\beta} \in \mathbb{N}^{l-1}$, which can be defined as a certain sum of $Q_{\check{\lambda},k}$. The main result in [52] is a generating function formula (see Theorem 2.2.6) for $Q_{n,k,\vec{\alpha},\vec{\beta}}$, which when applying the stable principal specialization would yield an extension of (1.2) for the joint distribution of excedance number and major index on colored permutations. This generating function formula was obtained through three main steps:

- (1) a colored analog of the Gessel–Reutenauer bijection [47] is used to give the colored ornaments characterization of $Q_{\check{\lambda},k}$;
- (2) the *Lyndon decomposition* is used to give the colored banners characterization of $Q_{\check{\lambda},k}$;
- (3) the generating function formula is derived by establishing a recurrence formula using the interpretation of $Q_{\check{\lambda},k}$ as colored banners.

The recurrence formula in step (3) was established through a complicated generalization of a bijection of Shareshian–Wachs [76], so it would be reasonable to expect a simpler approach. In Section 2.3, we will show how step (3) can be deduced directly from the *Decrease Value Theorem* developed by Foata–Han [33].

Section 2.4 deals with the symmetric function generalizations of (1.1) for colored permutation groups. We modify the fixed point Eulerian quasisymmetric functions to some $Q_{n,k,j}$ that we call *flag Eulerian quasisymmetric functions*, which are also a kind of generalization of the Eulerian quasisymmetric functions and would specialize to the *flag excedance numbers* studied in [5, 39]. The generating function formula for $Q_{n,k,j}$ follows easily from Hyatt’s generating function formula for $Q_{n,k,\vec{\alpha},\vec{\beta}}$. By making use of this generating function formula, we study the symmetry and unimodality of the flag Eulerian quasisymmetric functions and prove two symmetric function generalizations of (1.1), which involve both the *complete homogeneous symmetric functions* h_n and $Q_{n,k,j}$. We will construct bijective proofs of those two generalized symmetric identities, one of them leading to a new interesting approach to step 3 in the proof of [76, Theorem 1.2]. As one benefit of the study of the flag Eulerian quasisymmetric functions, we prove symmetry and unimodality of some colored q -Eulerian polynomials related to the number of fixed points, the flag excedances and the *flag major index* [1, 39] on colored permutations. Moreover, it motivates us to find colored versions of Gessel’s hook factorizations (introduced in Section 1.2) and the Linusson–Shareshian–Wachs admissible inversions (introduced in Section 1.4) for colored permutations.

Recently, Shareshian and Wachs [77] introduced the *chromatic quasisymmetric functions* of graphs and proved a surprising result that the two pairs (exc, maj) and $(\text{ides}, \text{rmaj}_2)$ are equidistributed on \mathfrak{S}_n ; see Theorem 1.6.1. The main purpose of Section 2.5 is to extend this result from permutations to colored permutations. More precisely, let $Q_{n,k,\vec{\beta}}$ be the colored Eulerian quasisymmetric function (the fixed points is not taken into account) defined by $Q_{n,k,\vec{\beta}} := \sum_{\vec{\alpha}} Q_{n,k,\vec{\alpha},\vec{\beta}}$. We introduce a colored analog of *Rawlings major index* [68] for colored permutations and prove a new interpretation of $Q_{n,k,\vec{\beta}}$ as a sum of some fundamental quasisymmetric functions related with these colored Rawlings major index. This is established by applying the P -partition theory and a decomposition of the chromatic quasisymmetric functions due to Shareshian and Wachs [77].

2.1.1 Quasisymmetric functions

We collect here the definitions and some facts about Gessel's quasisymmetric functions that will be used in the rest of this chapter; a good reference is [81, Chapter 7].

The quasisymmetric functions were originally introduced by Gessel [44] in the combinatorial interpretation of inner products of Skew Schur Functions. Let $f(\mathbf{x})$ be a formal power series of bounded degree in the variables $\mathbf{x} := \{x_1, x_2, \dots\}$ with rational coefficients. The function $f(\mathbf{x})$ is called a *symmetric function* if for any permutation w of the positive integers \mathbb{P} we have

$$f(x_{w(1)}, x_{w(2)}, \dots) = f(x_1, x_2, \dots);$$

it is called a *quasisymmetric function* if for any $a_1, \dots, a_k \in \mathbb{P}$ we have

$$[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Here, $[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f$ means the coefficient of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ in f . Clearly, every symmetric function is a quasisymmetric function, but not conversely. For instance, the series $\sum_{i < j} x_i^2 x_j$ is quasisymmetric but not symmetric.

Given a subset S of $[n-1]$, define the *fundamental quasisymmetric function* $F_{n,S}$ by

$$F_{n,S} = F_{n,S}(\mathbf{x}) := \sum_{\substack{i_1 \geq \cdots \geq i_n \geq 1 \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}. \quad (2.1)$$

If $S = \emptyset$, then $F_{n,S}$ is the *complete homogeneous symmetric function* h_n and if $S = [n-1]$, then $F_{n,S}$ is the *elementary symmetric function* e_n . The set of all quasisymmetric functions forms a ring and has the fundamental quasisymmetric functions as a basis. Define ω to be the involution on the ring of quasisymmetric

functions that maps $F_{n,S}$ to $F_{n,[n-1]\setminus S}$, which extends the involution on the ring of symmetric functions that takes h_n to e_n .

The *stable principal specialization* \mathbf{ps} is the ring homomorphism from the ring of symmetric functions to the ring of formal power series in the variable q , defined by

$$\mathbf{ps}(x_i) = q^{i-1}.$$

The following property of \mathbf{ps} is known [47].

Lemma 2.1.1 ([47, Lemma 5.2]). *For any $n \geq 1$ and $S \subseteq [n-1]$ we have*

$$\mathbf{ps}(F_{n,S}) = \frac{q^{\sum_{i \in S} i}}{(q; q)_n}. \quad (2.2)$$

In particular,

$$\mathbf{ps}(h_n) = \frac{1}{(q; q)_n} \quad \text{and} \quad \mathbf{ps}(e_n) = \frac{q^{n(n-1)/2}}{(q; q)_n}.$$

Proof. To any sequence (i_1, \dots, i_n) such that $i_1 \geq \dots \geq i_n \geq 0$ and $j \in S \Rightarrow i_j > i_{j+1}$ we can associate a sequence (i'_1, \dots, i'_n) with $i'_k = i_k - |\{j \in S : j \geq k\}|$ for $k = 1, 2, \dots, n$. It is clear that $\sum_j i_j = \sum_j i'_j + \sum_{j \in S} j$. Applying \mathbf{ps} to (2.1), we have

$$\begin{aligned} \mathbf{ps}(F_{n,S}) &= \sum_{\substack{i_1 \geq \dots \geq i_n \geq 0 \\ j \in S \Rightarrow i_j > i_{j+1}}} q^{i_1 + i_2 + \dots + i_n} \\ &= q^{\sum_{j \in S} j} \sum_{i'_1 \geq \dots \geq i'_n \geq 0} q^{i'_1 + \dots + i'_n}, \end{aligned}$$

which is equivalent to (2.2). □

2.2 Hyatt's colored Eulerian quasisymmetric functions

2.2.1 Statistics on colored permutation groups

We recall the definition of the colored Eulerian quasisymmetric functions introduced in [52]. Consider the following set of l -colored integers from 1 to n

$$[n]^l := \{1^0, 1^1, \dots, 1^{l-1}, 2^0, 2^1, \dots, 2^{l-1}, \dots, n^0, n^1, \dots, n^{l-1}\}.$$

If π is a word over $[n]^l$, we use π_i and $\epsilon_i \in \{0, 1, \dots, l-1\}$ to denote the i -th letter of π and the color of the i -th letter of π , respectively. We let $|\pi_i|$ denote the

positive integer obtained from π_i by removing the superscript. If π is a word of length m over $[n]^l$, we denote by $|\pi|$ the word

$$|\pi| := |\pi_1| |\pi_2| \cdots |\pi_m|.$$

In one-line notation the *colored permutation group* $C_l \wr \mathfrak{S}_n$ can be viewed as the set of the words π over $[n]^l$ such that $|\pi| \in \mathfrak{S}_n$.

Now, the *descent number*, $\text{des}(\pi)$, the *excedance number*, $\text{exc}(\pi)$, and the *major index*, $\text{maj}(\pi)$, of a colored permutation $\pi \in C_l \wr \mathfrak{S}_n$ are defined as follows:

$$\begin{aligned} \text{DES}(\pi) &:= \{j \in [n-1] : \pi_j > \pi_{j+1}\}, \\ \text{des}(\pi) &:= |\text{DES}(\pi)|, \quad \text{maj}(\pi) := \sum_{j \in \text{DES}(\pi)} j, \\ \text{EXC}(\pi) &:= \{j \in [n] : \pi_j > j^0\}, \quad \text{exc}(\pi) := |\text{EXC}(\pi)|, \end{aligned}$$

where we use the following *color order*

$$\mathcal{E} := \{1^{l-1} < \cdots < n^{l-1} < 1^{l-2} < \cdots < n^{l-2} < \cdots < 1^0 < \cdots < n^0\}.$$

Also, for $0 \leq k \leq l-1$, the *k-th color fixed point number* $\text{fix}_k(\pi)$ and the *k-th color number* $\text{col}_k(\pi)$ are defined by

$$\text{fix}_k(\pi) := |\{j \in [n] : \pi_j = j^k\}| \tag{2.3}$$

and

$$\text{col}_k(\pi) := |\{j \in [n] : \epsilon_j = k\}|. \tag{2.4}$$

The *fixed point vector* $\vec{\text{fix}}(\pi) \in \mathbb{N}^l$ and the *color vector* $\vec{\text{col}}(\pi) \in \mathbb{N}^{l-1}$ are then defined as

$$\vec{\text{fix}}(\pi) := (\text{fix}_0(\pi), \text{fix}_1(\pi), \dots, \text{fix}_{l-1}(\pi)), \quad \vec{\text{col}}(\pi) := (\text{col}_1(\pi), \dots, \text{col}_{l-1}(\pi))$$

respectively. For example, if $\pi = 5^2 2^1 4^0 3^2 1^2 6^0 \in C_3 \wr \mathfrak{S}_6$, then $\text{DES}(\pi) = \{3, 4\}$, $\text{des}(\pi) = 2$, $\text{EXC}(\pi) = \{3\}$, $\text{exc}(\pi) = 1$, $\text{maj}(\pi) = 7$, $\vec{\text{fix}}(\pi) = (1, 1, 0)$ and $\vec{\text{col}}(\pi) = (1, 3)$.

A colored permutation π can also be written in cycle form such that j^{ϵ_j} follows i^{ϵ_i} means that $\pi_i = j^{\epsilon_j}$. Continuing with the previous example, we can write it in cycle form as

$$\pi = (1^2, 5^2)(2^1)(3^2, 4^0)(6^0). \tag{2.5}$$

For each vector $\vec{\beta} = (\beta_1, \dots, \beta_{l-1}) \in \mathbb{N}^{l-1}$ let $|\vec{\beta}| := \beta_1 + \cdots + \beta_{l-1}$. Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_i)$ of n with $\lambda_i \geq 1$, a *cv-cycle type* is defined to be a multiset of pairs

$$\check{\lambda} = \{(\lambda_1, \vec{\beta}^1), \dots, (\lambda_i, \vec{\beta}^i)\}, \tag{2.6}$$

where $\vec{\beta}_1, \dots, \vec{\beta}^i$ are vectors from $\mathbb{N}^{\ell-1}$ with the property that $|\vec{\beta}_j| \leq \lambda_j$ for $j = 1, \dots, i$. Now, if γ is a cyclic colored permutation, written as a cycle $(c_1, \dots, c_{\ell(\gamma)})$ of length $\ell(\gamma)$, define the *cv-cycle type* of γ to be the singleton $\check{\lambda} := \{\ell(\gamma), \vec{\gamma}\}$, so that the inequality $|\vec{\text{col}}(\gamma)| \leq \ell(\gamma)$ holds. Next, if a permutation π from $C_l \wr \mathfrak{S}_n$ is the product of the disjoint cycles $\pi = \gamma_1 \cdots \gamma_i$, written in non-increasing order of their lengths, define the *cv-cycle type* of π to be the multiset of pairs

$$\check{\lambda}(\pi) := \{(\ell(\gamma_1), \vec{\text{col}}(\gamma_1)), \dots, (\ell(\gamma_i), \vec{\text{col}}(\gamma_i))\}.$$

With the above convention the sequence $(\ell(\gamma_1), \dots, \ell(\gamma_i))$ is a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_i)$ of n . Moreover, $|\vec{\text{col}}(\gamma_j)| \leq \lambda_j = \ell(\gamma_j)$ for $j = 1, \dots, i$ and $\vec{\text{col}}(\pi) = \vec{\text{col}}(\gamma_1) + \dots + \vec{\text{col}}(\gamma_i)$ by using component-wise addition. Thus, $\check{\lambda}$ is a well-defined cv-cycle type. For example, $\vec{\text{col}}(1^2, 5^2) = (0, 2)$, $\vec{\text{col}}(3^2, 4^0) = (0, 1)$, $\vec{\text{col}}(2^1) = (1, 0)$, $\vec{\text{col}}(6^0) = (0, 0)$, so that the cv-cycle type of the permutation π in (2.5) reads

$$\check{\lambda}(\pi) = \{(2, (0, 2)), (2, (0, 1)), (1, (1, 0)), (1, (0, 0))\}.$$

Next, we define a set value statistic DEX on $C_l \wr \mathfrak{S}_n$ that will be used in the definition of colored Eulerian quasisymmetric functions. Let

$$\mathcal{A} := \{\tilde{1}^0 < \tilde{2}^0 < \dots < \tilde{n}^0\} < \mathcal{E},$$

where \mathcal{E} has the same order as above, but now the letters with a tilde are less than the letters in \mathcal{E} . Given a colored permutation $\pi \in C_l \wr \mathfrak{S}_n$, construct a word $\tilde{\pi}$ of length n over \mathcal{A} as follows:

if $i \in \text{EXC}(\pi)$, then replace π_i by $\tilde{\pi}_i$, otherwise leave π_i alone.

For example, if $\pi = 4^2 6^0 2^0 5^0 3^2 1^2 \in C_3 \wr \mathfrak{S}_6$, then $\tilde{\pi} = 4^2 \tilde{6}^0 2^0 \tilde{5}^0 3^2 1^2$. Then $\text{DEX}(\pi)$ is defined by

$$\text{DEX}(\pi) := \text{DES}(\tilde{\pi}).$$

Using the example above we have

$$\text{DEX}(4^2 6^0 2^0 5^0 3^2 1^2) = \text{DES}(4^2 \tilde{6}^0 2^0 \tilde{5}^0 3^2 1^2) = \{1, 3, 5\}.$$

We are now ready to give the definition of the main object of this chapter.

Definition 2.1 ([52, Definition 2.1]). For any particular cv-cycle type $\check{\lambda}$ define the *cv-cycle type colored Eulerian quasisymmetric functions* $Q_{\check{\lambda}, k}$ by

$$Q_{\check{\lambda}, k} := \sum_{\pi} F_{n, \text{DEX}(\pi)},$$

summed over $\pi \in C_l \wr \mathfrak{S}_n$ with $\check{\lambda}(\pi) = \check{\lambda}$ and $\text{exc}(\pi) = k$. Given $\vec{\alpha} \in \mathbb{N}^l, \vec{\beta} \in \mathbb{N}^{l-1}$, the *fixed point colored Eulerian quasisymmetric functions* are then defined as

$$Q_{n,k,\vec{\alpha},\vec{\beta}} = \sum_{\pi} F_{n,\text{DEX}(\pi)} \quad (2.7)$$

summed over all $\pi \in C_l \wr \mathfrak{S}_n$ such that $\text{exc}(\pi) = k, \vec{\text{fix}}(\pi) = \vec{\alpha}$ and $\vec{\text{col}}(\pi) = \vec{\beta}$.

Lemma 2.2.1 ([52, Lemma 2.2]). *For every $\pi \in C_l \wr \mathfrak{S}_n$ we have*

$$\sum_{i \in \text{DEX}(\pi)} i = \text{maj}(\pi) - \text{exc}(\pi).$$

The following specialization of the fixed point colored Eulerian quasisymmetric functions follows from the above lemma and Eq. (2.2).

Lemma 2.2.2. *For all $n, k, \vec{\alpha}$ and $\vec{\beta}$,*

$$\text{ps}(Q_{n,k,\vec{\alpha},\vec{\beta}}) = (q; q)_n^{-1} \sum_{\pi} q^{(\text{maj} - \text{exc})\pi} \quad (2.8)$$

summed over all $\pi \in C_l \wr \mathfrak{S}_n$ such that $\text{exc}(\pi) = k, \vec{\text{fix}}(\pi) = \vec{\alpha}$ and $\vec{\text{col}}(\pi) = \vec{\beta}$.

2.2.2 Colored ornaments

We will use the colored ornament interpretation in [52] to prove combinatorially that $Q_{\check{\lambda},k}$ is a symmetric function.

Let \mathcal{B} be the infinite ordered alphabet given by

$$\mathcal{B} := \{1^0 < 1^1 < \dots < 1^{l-1} < \overline{1^0} < 2^0 < 2^1 < \dots < 2^{l-1} < \overline{2^0} < 3^0 < 3^1 < \dots\}. \quad (2.9)$$

A letter of the form u^m from \mathcal{B} is said to be m -colored, and each letter $\overline{u^0}$ is also said to be 0-colored. If w is a word over \mathcal{B} , we define the *color vector* $\vec{\text{col}}(w) \in \mathbb{N}^{l-1}$ of w to be

$$\vec{\text{col}}(w) := (\text{col}_1(w), \text{col}_2(w), \dots, \text{col}_{l-1}(w)),$$

where $\text{col}_m(w)$ is the number of m -colored letters in w for $m = 1, \dots, l-1$. The *absolute value of a letter* is the positive integer obtained by removing any colors or bars, so $|u^m| = |\overline{u^0}| = u$. The *weight of a letter* u^m or $\overline{u^0}$ is x_u .

We consider the circular word over \mathcal{B} . If w is a word on \mathcal{B} , we denote (w) the *circular word* obtained by placing the letters of w around a circle in a clockwise direction. A circular word (w) is said to be *primitive* if the word w can not be written as $w = w'w' \dots w'$ where w' is some proper subword of w . For example, $(\overline{1^0}, 2^1, 1^0, 2^1)$ is primitive but $(1^0, 2^1, 1^0, 2^1)$ is not because $1^0 2^1 1^0 2^1 = w'w'$ with $w' = 1^0 2^1$.

Definition 2.2 ([52, Definition 3.1]). A *colored necklace* is a circular primitive word (w) over the alphabet \mathcal{B} such that

- (1) Every barred letter is followed by a letter of lesser or equal absolute value.
- (2) Every 0-colored unbarred letter is followed by a letter of greater or equal absolute value.
- (3) Words of length one may not consist of a single barred letter.

A *colored ornament* is a multiset of colored necklaces.

The *weight* $\text{wt}(R)$ of a ornament R is the product of the weights of the letters of R . Similar to the *cv-cycle type* of a colored permutation, the cv-cycle type $\check{\lambda}(R)$ of a colored ornament R is the multiset

$$\check{\lambda}(R) = \{(\lambda_1, \vec{\beta}^1), \dots, (\lambda_i, \vec{\beta}^i)\},$$

where each pair $(\lambda_j, \vec{\beta}^j)$ corresponds to precisely one colored necklace in the ornament R with length λ_j and color vector $\vec{\beta}^j$. For example, if $l = 3$ and

$$R = (2^1, \overline{3^0}, 1^0, 1^2, 2^1, \overline{5^0}, \overline{3^0})(2^1, 2^2)(6^0, \overline{7^0})(2^0)(3^1),$$

then

$$\check{\lambda}(R) = \{(7, (2, 1)), (2, (1, 1)), (2, (0, 0)), (1, (0, 0)), (1, (1, 0))\}.$$

For a cv-cycle type $\check{\lambda}$ and a nonnegative integer k let $\mathcal{R}(\check{\lambda}, k)$ be the set of all colored ornaments of cv-cycle type $\check{\lambda}$ and exactly k barred letters. Using a colored analog of the Gessel–Reutenauer bijection [47], Hyatt [52, Corollary 3.3] proved the following colored ornament interpretation of $Q_{\check{\lambda}, k}$.

Theorem 2.2.3 (Hyatt’s Colored ornament interpretation).

$$Q_{\check{\lambda}, k} = \sum_{R \in \mathcal{R}(\check{\lambda}, k)} \text{wt}(R).$$

Theorem 2.2.4. *The cv-cycle type Eulerian quasisymmetric function $Q_{\check{\lambda}, k}$ is a symmetric function.*

Proof. We will extend the bijective proof of [76, Theorem 5.8] involving ornaments to the colored ornaments. For each $j \in \mathbb{P}$ we will construct a bijection ψ between colored necklaces that exchanges the number of occurrences of the value j and $j+1$ in a colored necklace, but preserves the number of occurrences of all other values, the total number of bars and the color vector. Since such a ψ can be extended

to colored ornaments by applying ψ to each colored necklace, the results will then follow from Theorem 2.2.3.

Case 1: The necklace R contains only the letters with values j and $j + 1$. Without loss of generality, we assume that $j = 1$. First replace all 1's with 2's and all 2's with 1's, leaving the bars and colors in their original positions. Now the problem is that each 0-colored 1 that is followed by a 2 has a bar but each 0-colored 2 that is followed by a 1 lacks a bar. We call a 1 that is followed by a 2 a rising 1 and a 2 that is followed by a 1 a falling 2. Since the number of rising 1 equals the number of falling 2 and they appear alternately, we can switch the color of each rising 1 with the color of its closest (in clockwise direction) falling 2. If, in addition, the rising 1 has a bar, then we also move the bar to its closest falling 2, thereby obtaining a colored necklace R' with the same number of bars and the same color vector as R , but with the number of 1's and 2's exchanged. Let $\psi(R) = R'$. Clearly, ψ is reversible. For example if $R = (2^2 \overline{2^0} 1^1 \overline{1^0} 1^0 \overline{2^0} 2^3 \overline{2^0} 2^1 1^0 1^0 \overline{2^0} 1^2 \overline{1^0} 1^0)$ then we get $(1^2 \overline{1^0} 2^1 \overline{2^0} 2^0 \overline{1^0} 1^3 \overline{1^0} 1^1 2^0 2^0 \overline{1^0} 2^2 \overline{2^0} 2^0)$ before the colors and bars are adjusted. After the colors and bars are adjusted, we have

$$\psi(R) = (1^2 1^0 2^1 \overline{2^0} \overline{2^0} \overline{1^0} 1^3 \overline{1^0} 1^0 2^0 2^1 1^0 2^2 \overline{2^0} \overline{2^0}).$$

Case 2: The necklace R has letters with values j and $j + 1$, and other letters which we will call intruders. The intruders enable us to form linear segments of R consisting only of letters with value j or $(j + 1)$. To obtain such a linear segment start with a letter of value j or $j + 1$ that follows an intruder and read the letters of R in a clockwise direction until another intruder is encountered. For example if

$$R = (\overline{5^0} 3^1 3^0 4^2 \overline{4^0} \overline{3^0} 3^1 \overline{3^0} 3^2 6^2 \overline{6^0} \overline{3^0} 3^0 3^1 \overline{4^0} 2^0 4^3 4^0) \quad (2.10)$$

and $j = 3$, then the segments are $3^1 3^0 4^2 \overline{4^0} \overline{3^0} 3^1 \overline{3^0} 3^2$, $\overline{3^0} 3^0 3^1 \overline{4^0}$ and $4^3 4^0$.

There are two types of segments, even segments and odd segments. An even (odd) segment contains an even (odd) number of switches, where a switch is a letter of value j followed by one of value $j + 1$ (call a rising j) or a letter of value $j + 1$ followed by one of value j (call a falling $j + 1$). We treat the even and odd segments separately.

Subcase 2.1: Even segments. In an even segment, we replace all j 's with $(j + 1)$'s and all $(j + 1)$'s with j 's. Again, this may produce problems on rising j or falling $j + 1$. So we switch the color of the i -th rising j with the color of the i -th falling $j + 1$ and move the bar (if it really has) from the i -th rising j to the i -th falling $j + 1$ to obtain a good segment, where rising j 's and falling $(j + 1)$'s are counted from left to right. This preserves the number of bars and color vector, and exchanges the number of j 's and $(j + 1)$'s. For example, the even segment $3^1 3^0 4^2 \overline{4^0} \overline{3^0} 3^1 \overline{3^0} 3^2$ gets replaced by $4^1 4^0 3^2 \overline{3^0} \overline{4^0} 4^1 \overline{4^0} 4^2$. After the bars and colors are adjusted we obtain $4^1 \overline{4^0} 3^2 3^0 \overline{4^0} 4^1 \overline{4^0} 4^2$.

Subcase 2.2: Odd segments. An odd segment either starts with a j and ends with a $j + 1$, or vice versa. Both cases are handled similarly. So we suppose we have an odd segment of the form

$$j^{m_1}(j+1)^{n_1}j^{m_2}(j+1)^{n_2}\cdots j^{m_r}(j+1)^{n_r},$$

where each $m_i, n_i > 0$ and the bars and colors have been suppressed. The number of switches is $2r - 1$. We replace it with the odd segment

$$j^{n_1}(j+1)^{m_1}j^{n_2}(j+1)^{m_2}\cdots j^{n_r}(j+1)^{m_r},$$

and put bars and colors in their original positions. Again, we may have created problems on rising j 's (but not on falling $(j + 1)$'s); so we need to adjust bars and colors around. Note that the positions of the rising j 's are in the set $\{N_1 + n_1, N_2 + n_2, N_3 + n_3, \dots, N_r + n_r\}$, where $N_i = \sum_{t=1}^{i-1} (n_t + m_t)$. Now we switch the color in position $N_i + n_i$ with the color in position $N_i + m_i$ and move the bar (if it really has) to position $N_i + m_i$, thereby obtain a good segment. For example, the odd segment $\overline{3^0} 3^0 3^1 \overline{4^0}$ gets replaced by $\overline{3^0} 4^0 4^1 \overline{4^0}$ before the bars and colors are adjusted. After the bars and colors are adjusted we have $3^1 4^0 \overline{4^0} \overline{4^0}$.

Let $\psi(R)$ be the colored necklace obtained by replacing all the segments in the way described above. For example if R is the colored necklace given in (2.10) then

$$\psi(R) = (\overline{5^0} 4^1 \overline{4^0} 3^2 3^0 \overline{4^0} 4^1 \overline{4^0} 4^2 6^2 \overline{6^0} 3^1 4^0 \overline{4^0} \overline{4^0} 2^0 3^3 3^0).$$

It is routine to check that ψ is reversible in all cases and thus is a bijection of colored necklaces. This completes the proof of the theorem. \square

2.2.3 Colored banners

We shall first give a brief review of Hyatt's colored banner interpretation [52] for $Q_{\lambda,k}$ and then present a slightly different version of his interpretation that will be used in the next section.

Definition 2.3 ([52, Definition 4.2]). A *colored banner* is a word $B = B(1)B(2)\cdots$ over the alphabet \mathcal{B} such that

- (1) if $B(i)$ is barred then $|B(i)| \geq |B(i + 1)|$,
- (2) if $B(i)$ is 0-colored and unbarred, then $|B(i)| \leq |B(i + 1)|$ or i equals the length of B ,
- (3) the last letter of B is unbarred.

Recall that a *Lyndon word* over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements. It is a result of Lyndon (cf. [59, Theorem 5.1.5]) that every word has a unique factorization into a lexicographically weakly increasing sequence of Lyndon words, called *Lyndon factorization*. We say that a word of length n has Lyndon type λ (where λ is a partition of n) if the parts of λ are equal to the lengths of the factors in the Lyndon factorization.

We make use of the Lyndon factorization in our colored banners study. The *cv-cycle type* of a colored banner B is defined to be the multiset

$$\check{\lambda}(B) = \left\{ (\lambda_1, \vec{\beta}^1), \dots, (\lambda_k, \vec{\beta}^k) \right\},$$

where B has Lyndon type λ , and the corresponding word of length λ_i in the Lyndon factorization has color vector $\vec{\beta}^i$. The *weight* $\text{wt}(B)$ of a banner is defined to be the product of the weights of all letters in B . For example, if $l = 3$ and

$$B = 2^0 2^2 2^1 3^1 \overline{5^0} \overline{3^0} 2^1 \overline{3^0} 1^0 1^2 2^1 \overline{7^0} 6^0,$$

then the Lyndon factorization of B is

$$B = (2^0)(2^2, 2^1)(3^1)(\overline{5^0}, \overline{3^0}, 2^1, \overline{3^0}, 1^0, 1^2, 2^1)(\overline{7^0}, 6^0)$$

and

$$\check{\lambda}(B) = \{(7, (2, 1)), (2, (1, 1)), (2, (0, 0)), (1, (0, 0)), (1, (1, 0))\}.$$

Let $K(\check{\lambda}, k)$ be the set of all colored banners with k barred letters whose cv-cycle type (with respect to the order of \mathcal{B} in (2.9)) is $\check{\lambda}$.

Theorem 2.2.5 (New colored banner interpretation). *There is a weight-preserving bijection from $\mathcal{R}(\check{\lambda}, k)$ to $K(\check{\lambda}, k)$. Consequently,*

$$Q_{\check{\lambda}, k} = \sum_{B \in K(\check{\lambda}, k)} \text{wt}(B).$$

Proof. The proof applies Lyndon factorization to the colored banners and is identical to the proof of [76, Theorem 3.6]. \square

Remark 2.1. Consider the following order $<_B$ on the alphabet \mathcal{B} :

$$\begin{aligned} 1^1 <_B \cdots <_B 1^{l-1} <_B 2^1 <_B \cdots <_B 2^{l-1} <_B \cdots <_B n^1 <_B \cdots <_B n^{l-1} <_B \\ <_B 1^0 <_B \overline{1^0} <_B 2^0 <_B \overline{2^0} <_B 3^0 <_B \overline{3^0} <_B \cdots <_B n^0 <_B \overline{n^0}. \end{aligned}$$

Hyatt [52, Theorem 4.3] applied the Lyndon factorization to the colored banners with the above order $<_B$ on \mathcal{B} to give a different colored banner interpretation

of $Q_{\lambda,k}$, which we should call the *original colored banner interpretation*. Our new colored banner interpretation stated here is closer to the word interpretation in Lemma 2.3.1, while the original colored banner interpretation will be used in the proof of Theorem 2.4.14.

The following generating function for $Q_{n,k,\vec{\alpha},\vec{\beta}}$ was computed in [52] by establishing a recurrence formula based on the original colored banner interpretation of $Q_{\lambda,k}$.

Theorem 2.2.6 (Hyatt). *Fix $l \in \mathbb{P}$ and let $r^{\vec{\alpha}} = r_0^{\alpha_0} \cdots r_{l-1}^{\alpha_{l-1}}$ and $s^{\vec{\beta}} = s_1^{\beta_1} \cdots s_{l-1}^{\beta_{l-1}}$. Then*

$$\sum_{\substack{n,k \geq 0 \\ \vec{\alpha} \in \mathbb{N}^l, \vec{\beta} \in \mathbb{N}^{l-1}}} Q_{n,k,\vec{\alpha},\vec{\beta}} z^n t^k r^{\vec{\alpha}} s^{\vec{\beta}} = \frac{H(r_0 z)(1-t) \left(\prod_{m=1}^{l-1} E(-s_m z) H(r_m s_m z) \right)}{\left(1 + \sum_{m=1}^{l-1} s_m \right) H(tz) - \left(t + \sum_{m=1}^{l-1} s_m \right) H(z)}, \quad (2.11)$$

where $H(z) := \sum_{i \geq 0} h_i z^i$ and $E(z) := \sum_{i \geq 0} e_i z^i$.

2.3 An application of the Decrease Value Theorem

The main objective of this section is to show how (2.11) can be deduced from the Decrease Value Theorem directly.

2.3.1 Decrease values in words

We begin with some word statistics studied in [33, 34]. Let $w = w_1 w_2 \cdots w_n$ be an arbitrary word over \mathbb{N} . Recall that an integer $i \in [n-1]$ is a *descent* of w if $w_i > w_{i+1}$; it is a *decrease* of w if $w_i = w_{i+1} = \cdots = w_j > w_{j+1}$ for some j such that $i \leq j \leq n-1$. The letter w_i is said to be a *decrease value* of w . The set of all decreases (resp. descents) of w is denoted by $\text{DEC}(w)$ (resp. $\text{DES}(w)$). Each descent is a decrease, but not conversely. Hence $\text{DES}(w) \subset \text{DEC}(w)$.

In parallel with the notions of descent and decrease, an integer $i \in [n]$ is a *rise* of w if $w_i < w_{i+1}$ (By convention that $w_{n+1} = \infty$, and thus n is always a rise); it is an *increase* of w if $i \notin \text{DEC}(w)$. The letter w_i is said to be an *increase value* of w . The set of all increases (resp. rises) of w is denoted by $\text{INC}(w)$ (resp. $\text{RISE}(w)$). Clearly, each rise is an increase, but not conversely. Hence $\text{RISE}(w) \subset \text{INC}(w)$.

Furthermore, a position i is said to be a *record* if $w_i \geq w_j$ for all j such that $1 \leq j \leq i-1$ and the letter w_i is called a *record value*. Denote by $\text{REC}(w)$ the set of all records of w .

Now, we define a mapping f from words on \mathbb{N} to colored banners as follows

$$f : w = w_1 w_2 \dots w_n \mapsto B = B(1)B(2) \dots B(n),$$

where

- $B(i) = \overline{u^0}$, if w_i is a decrease value such that $w_i = ul$ for some $u \in \mathbb{P}$;
- otherwise $B(i) = (u + 1)^m$, where $w_i = ul + m$ for some $u, m \in \mathbb{N}$ satisfies $0 \leq m \leq l - 1$.

For example, if $l = 3$, then

$$f(12, 12, 10, 1, 9, 12, 8, 12, 16, 2, 13, 19) = \overline{4^0} \overline{4^0} 4^1 1^1 4^0 \overline{4^0} 3^2 5^0 6^1 1^2 5^1 7^1.$$

We should check that such a word B over \mathcal{B} is a colored banner. In the definition of a colored banner, condition (3) is satisfied since the last letter of a word is always an increase value. If $B(i)$ is barred, then w_i is a decrease value and so $w_i \geq w_{i+1}$, which would lead $|B(i)| \geq |B(i+1)|$, and thus condition (1) is satisfied. Similarly, condition (2) is also satisfied. This shows that f is well defined.

A letter $k \in \mathbb{N}$ is called a m -colored letter (or value) if it is congruent to m ($0 \leq m \leq l - 1$) modulo l . For a word $w = w_1 \dots w_n$ over \mathbb{N} , we define the *colored vector* $\vec{\text{mod}}(w) \in \mathbb{N}^{l-1}$ of w to be

$$\vec{\text{mod}}(w) := (\text{mod}_1(w), \dots, \text{mod}_{l-1}(w)),$$

where $\text{mod}_m(w)$ is the number of m -colored letters in w for $m = 1, \dots, l - 1$. Supposing that $w_i = u_i l + m_i$ for some $0 \leq m_i \leq l - 1$, we then define the weight $\text{wt}(w)$ of w to be the monomial $x_{d(w_1)} \dots x_{d(w_n)}$, where $d(w_i) = u_i$ if w_i is a decrease value and $m_i = 0$, otherwise $d(w_i) = u_i + 1$. We also define the *cv-cycle type* of w to be the multiset

$$\check{\lambda}(w) = \left\{ (\lambda_1, \vec{\alpha}^1), \dots, (\lambda_k, \vec{\alpha}^k) \right\}$$

if w has Lyndon type λ (with respect to the order of \mathbb{N}), and the corresponding word of length λ_i in the Lyndon factorization has color vector $\vec{\alpha}^i$.

Lemma 2.3.1. *Let $W(\check{\lambda}, k)$ be the set of all words over \mathbb{N} with cv-cycle type $\check{\lambda}$ and exactly k 0-colored decrease values. Then*

$$Q_{\check{\lambda}, k} = \sum_{w \in W(\check{\lambda}, k)} \text{wt}(w).$$

Proof. Clearly, the mapping f is a bijection which maps 0-colored decrease values to 0-colored barred letters and preserves the color of letters. It is also weight preserving $\text{wt}(w) = \text{wt}(f(w))$. It is not hard to check that if the Lyndon factorization of a word w over \mathbb{N} is

$$w = (w_1)(w_2) \cdots (w_k),$$

then the Lyndon factorization (with respect to the order of \mathcal{B} in (2.9)) of the banner $f(w)$ is

$$f(w) = (f(w_1))(f(w_2)) \cdots (f(w_k)).$$

Thus f also keeps the Lyndon factorization type, which would complete the proof in view of Theorem 2.2.5. \square

2.3.2 Combinatorics of the Decrease Value Theorem

Let $[0, r]^*$ be the set of all finite words whose letters are taken from the alphabet $[0, r] := \{0, 1, \dots, r\}$. Introduce six sequences of commuting variables $(X_i), (Y_i), (Z_i), (T_i), (Y'_i), (T'_i)$ ($i = 0, 1, 2, \dots$), and for each word $w = w_1 w_2 \dots w_n$ from $[0, r]^*$ define the *weight* $\psi(w)$ of w to be

$$\begin{aligned} \psi(w) := & \prod_{i \in \text{DES}} X_{w_i} \prod_{i \in \text{RISE} \setminus \text{REC}} Y_{w_i} \prod_{i \in \text{DEC} \setminus \text{DES}} Z_{w_i} \\ & \times \prod_{i \in (\text{INC} \setminus \text{RISE}) \setminus \text{REC}} T_{w_i} \prod_{i \in \text{RISE} \cap \text{REC}} Y'_{w_i} \prod_{i \in (\text{INC} \setminus \text{RISE}) \cap \text{REC}} T'_{w_i}. \end{aligned}$$

The following generating function for the set $[0, r]^*$ by the weight ψ was calculated by Foata and Han [33, Theorem 1.2] using the properties of *Foata's first fundamental transformation* on words (see [59, Chap. 10]) and a noncommutative version of *MacMahon Master Theorem* (see [12, Chap. 4]).

Theorem 2.3.2 (Decrease Value Theorem). *We have:*

$$\sum_{w \in [0, r]^*} \psi(w) = \frac{\prod_{1 \leq j \leq r} \frac{1 - Z_j}{1 - Z_j + X_j} \prod_{0 \leq j \leq r} \frac{1 - T'_j}{1 - T'_j + Y'_j}}{1 - \sum_{1 \leq k \leq r} \frac{\prod_{1 \leq j \leq k-1} \frac{1 - Z_j}{1 - Z_j + X_j} \frac{X_k}{1 - Z_k + X_k}}{\prod_{0 \leq j \leq k-1} \frac{1 - T'_j}{1 - T'_j + Y'_j}}}. \quad (2.12)$$

We show in the following that one can also use the *Kim–Zeng decomposition of multiderangement* [55] (but not the word-analog of the Kim–Zeng decomposition developed in [34, Theorem 3.4]) instead of MacMahon Master Theorem to prove the Decrease Value Theorem combinatorially.

A letter w_i which is a record and also a rise value is called a *riserec value*. A word $w \in [0, r]^*$ having no equal letters in succession is called *horizontal derangement*. Denote by $[0, r]_d^*$ the set of all the horizontal derangement words in $[0, r]^*$ without riserec value. It was shown in [33] that the decrease value theorem is equivalent to

$$\sum_{w \in [0, r]_d^*} \psi(w) = \frac{1}{\prod_{1 \leq j \leq r} (1 + X_j) - \sum_{1 \leq i \leq r} \left(\prod_{0 \leq j \leq i-1} (1 + Y_j) \prod_{i+1 \leq j \leq r} (1 + X_j) \right) X_i},$$

which again can be rewritten as

$$\sum_{w \in [0, r]_d^*} \psi(w) = \frac{1}{1 - \sum_{1 \leq i \leq r} \left(\left(\prod_{0 \leq j \leq i-1} (1 + Y_j) - 1 \right) \prod_{i+1 \leq j \leq r} (1 + X_j) \right) X_i}. \quad (2.13)$$

A word $\sigma = s_1 s_2 \cdots s_k$ of k distinct nonnegative integers is called a *cycle* of length k if $s_1 = \min\{s_1, s_2, \dots, s_k\}$. The rises of a cycle σ are called the *excedances* of σ . Using Foata's first fundamental transformation on words, we can factorize each word in $[0, r]_d^*$ as a product of cycles of length at least 2, where the rises of the word are transformed into the excedances of the cycles. A cycle $\sigma = s_1 s_2 \cdots s_k$ is called a *prime cycle* if there exists i , $2 \leq i \leq k$, such that

$$s_1 < \cdots < s_{i-1} < s_k < s_{k-1} < \cdots < s_{i+1} < s_i.$$

By the two decompositions in [55], a product of cycles of length at least 2 admits a decomposition to some components of prime cycles preserving excedances, from which we can see Eq. (2.13) directly.

2.3.3 A new proof of Hyatt's result

Introduce three sequences of commuting variables $(\xi_i), (\eta_i), (\zeta_i)$, ($i = 0, 1, 2, \dots$) and make the following substitutions:

$$X_i = \xi_i, \quad Z_i = \xi_i, \quad Y_i = \eta_i, \quad T_i = \eta_i, \quad Y'_i = \zeta_i, \quad T'_i = \zeta_i \quad (i = 0, 1, 2, \dots).$$

The new weight $\psi'(w)$ attached to each word $w = y_1 y_2 \cdots y_n$ is then

$$\psi'(w) = \prod_{i \in \text{DEC}(w)} \xi_{y_i} \prod_{i \in (\text{INC} \setminus \text{REC})(w)} \eta_{y_i} \prod_{i \in (\text{INC} \cap \text{REC})(w)} \zeta_{y_i}, \quad (2.14)$$

and identity (2.12) becomes:

$$\sum_{w \in [0, r]^*} \psi'(w) = \frac{\frac{\prod_{1 \leq j \leq r} (1 - \xi_j)}{\prod_{0 \leq j \leq r} (1 - \zeta_j)}}{1 - \sum_{1 \leq k \leq r} \frac{\prod_{1 \leq j \leq k-1} (1 - \xi_j)}{\prod_{0 \leq j \leq r} (1 - \eta_j)} \zeta_k}. \quad (2.15)$$

Let η denote the homomorphism defined by the following substitutions of variables:

$$\eta := \begin{cases} \xi_j \leftarrow tY_{i-1}, \zeta_j \leftarrow r_0Y_i, \eta_j \leftarrow Y_i, & \text{if } j = li; \\ \xi_j \leftarrow s_mY_i, \zeta_j \leftarrow r_ms_mY_i, \eta_j \leftarrow s_mY_i, & \text{if } j = li + m \text{ for some } 1 \leq m \leq l-1. \end{cases}$$

Lemma 2.3.3. *We have*

$$\frac{\prod_{j \geq 0} (1 - sY_j) - \prod_{j \geq 0} (1 - Y_j)}{\prod_{j \geq 0} (1 - Y_j)} = (1 - s) \sum_{i \geq 0} Y_i \frac{\prod_{0 \leq j \leq i-1} (1 - sY_j)}{\prod_{0 \leq j \leq i} (1 - Y_j)}.$$

Proof. First, we may check that

$$\begin{aligned} & \prod_{0 \leq j \leq r} (1 - sY_j) - \prod_{0 \leq j \leq r} (1 - Y_j) \\ &= \sum_{0 \leq i \leq r} \prod_{0 \leq j \leq i} (1 - sY_j) \prod_{i+1 \leq j \leq r} (1 - Y_j) - \sum_{0 \leq i \leq r} \prod_{0 \leq j \leq i-1} (1 - sY_j) \prod_{i \leq j \leq r} (1 - Y_j) \\ &= (1 - s) \sum_{0 \leq i \leq r} Y_i \prod_{0 \leq j \leq i-1} (1 - sY_j) \prod_{i+1 \leq j \leq r} (1 - Y_j). \end{aligned}$$

Multiplying both sides by $\frac{1}{\prod_{0 \leq j \leq r} (1 - Y_j)}$ yields

$$\frac{\prod_{0 \leq j \leq r} (1 - sY_j) - \prod_{0 \leq j \leq r} (1 - Y_j)}{\prod_{0 \leq j \leq r} (1 - Y_j)} = (1 - s) \sum_{0 \leq i \leq r} Y_i \frac{\prod_{0 \leq j \leq i-1} (1 - sY_j)}{\prod_{0 \leq j \leq i} (1 - Y_j)}.$$

Letting r tend to infinity, we get the desired formula. \square

Theorem 2.3.4. *We have*

$$\lim_{r \rightarrow \infty} \sum_{w \in [0, r]^*} \eta \psi'(w) = \frac{H(r_0Y)(1-t)(\prod_{m=1}^{l-1} E(-s_mY)H(r_ms_mY))}{(1 + \sum_{m=1}^{l-1} H(tY) - (t + \sum_{m=1}^{l-1} H(Y))),} \quad (2.16)$$

where $H(tY) = \prod_{i \geq 0} (1 - tY_i)^{-1}$ and $E(sY) = \prod_{i \geq 0} (1 + sY_i)$.

Proof. To apply the homomorphism η , we write (2.15) as

$$\sum_{w \in [0, r]^*} \psi'(w) = \frac{N_r}{D_r}, \quad (2.17)$$

where

$$N_r = \frac{\prod_{1 \leq j \leq r} (1 - \xi_j)}{\prod_{0 \leq j \leq r} (1 - \zeta_j)}$$

and

$$D_r = 1 - \sum_{1 \leq k \leq r} \frac{\prod_{1 \leq j \leq k-1} (1 - \xi_j)}{\prod_{0 \leq j \leq r} (1 - \eta_j)} \xi_k.$$

Applying η to both sides of (2.17) we have

$$\sum_{w \in [0, r]^*} \eta \psi'(w) = \frac{\eta(N_r)}{\eta(D_r)}, \quad (2.18)$$

where

$$\eta(N_r) = \frac{\prod_{1 \leq i \leq \lfloor r/l \rfloor} (1 - tY_{i-1}) \prod_{m=1}^{l-1} \left(\prod_{0 \leq i \leq \lfloor (r-m)/l \rfloor} (1 - s_m Y_i) \right)}{\prod_{0 \leq i \leq \lfloor r/l \rfloor} (1 - r_0 Y_{i-1}) \prod_{m=1}^{l-1} \left(\prod_{0 \leq i \leq \lfloor (r-m)/l \rfloor} (1 - r_m s_m Y_i) \right)}$$

and

$$\begin{aligned} \eta(D_r) &= 1 - \sum_{1 \leq k \leq r} \frac{\prod_{1 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - tY_{i-1}) \prod_{m=1}^{l-1} \left(\prod_{0 \leq i \leq \lfloor (k-1-m)/l \rfloor} (1 - s_m Y_i) \right)}{\prod_{0 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - Y_i) \prod_{m=1}^{l-1} \left(\prod_{0 \leq i \leq \lfloor (k-1-m)/l \rfloor} (1 - s_m Y_i) \right)} \eta(\xi_k) \\ &= 1 - \sum_{1 \leq k \leq r} \frac{\prod_{1 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - tY_{i-1})}{\prod_{0 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - Y_i)} \eta(\xi_k). \end{aligned}$$

Letting r tend to infinity in (2.18) we obtain

$$\lim_{r \rightarrow \infty} \sum_{w \in [0, r]^*} \eta \psi'(w) = \frac{\frac{\prod_{i \geq 0} (1 - tY_i) \prod_{m=1}^{l-1} \prod_{i \geq 0} (1 - s_m Y_i)}{\prod_{i \geq 0} (1 - r_0 Y_{i-1}) \prod_{m=1}^{l-1} \prod_{i \geq 0} (1 - r_m s_m Y_i)}}{1 - \sum_{k \geq 1} \frac{\prod_{1 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - tY_{i-1})}{\prod_{0 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - Y_i)} \eta(\xi_k)}. \quad (2.19)$$

By the definition of η ,

$$\begin{aligned}
& 1 - \sum_{k \geq 1} \frac{\prod_{1 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - tY_{i-1})}{\prod_{0 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - Y_i)} \eta(\xi_k) \\
&= 1 - \prod_{i \geq 0} \frac{\prod_{0 \leq j \leq i-1} (1 - tY_j)}{\prod_{0 \leq j \leq i} (1 - tY_j)} tY_i - \sum_{m=1}^{l-1} \left(\prod_{i \geq 0} \frac{\prod_{0 \leq j \leq i-1} (1 - tY_j)}{\prod_{0 \leq j \leq i} (1 - tY_j)} s_m Y_i \right) \\
&= 1 - \left(t + \sum_{m=1}^{l-1} s_m \right) \prod_{i \geq 0} \frac{\prod_{0 \leq j \leq i-1} (1 - tY_j)}{\prod_{0 \leq j \leq i} (1 - tY_j)} Y_i.
\end{aligned}$$

By Lemma 2.3.3 the above identity becomes

$$\begin{aligned}
& 1 - \sum_{k \geq 1} \frac{\prod_{1 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - tY_{i-1})}{\prod_{0 \leq i \leq \lfloor (k-1)/l \rfloor} (1 - Y_i)} \eta(\xi_k) \\
&= 1 - \frac{t + \sum_{m=1}^{l-1} s_m}{1 - t} \times \frac{\prod_{j \geq 0} (1 - tY_j) - \prod_{j \geq 0} (1 - Y_j)}{\prod_{j \geq 0} (1 - Y_j)}.
\end{aligned}$$

Substituting this expression into (2.19), we get (2.16). \square

Combining the above theorem with Lemma 2.3.1 we get a Decrease Value Theorem approach to Hyatt's generating function (2.11).

2.4 Flag Eulerian quasisymmetric functions

Definition 2.4 (Flag Eulerian quasisymmetric functions). For $\vec{\beta} = (\beta_1, \dots, \beta_{l-1}) \in \mathbb{N}^{l-1}$ let

$$\text{csum}(\vec{\beta}) := \sum_{i=1}^{l-1} i \times \beta_i.$$

Define the *Flag Eulerian quasisymmetric functions* $Q_{n,k,j}$ as

$$Q_{n,k,j} := \sum_{i, \vec{\alpha}, \vec{\beta}} Q_{n,i, \vec{\alpha}, \vec{\beta}},$$

where the sum is over all integers i , vectors $\vec{\alpha} \in \mathbb{N}^l$ and $\vec{\beta} \in \mathbb{N}^{l-1}$ such that $li + \text{csum}(\vec{\beta}) = k$ and $\alpha_0 = j$.

We will show later that the flag Eulerian quasisymmetric functions have many analog (or generalized) properties of the Shareshian–Wachs Eulerian quasisymmetric functions [76].

Corollary 2.4.1 (of Theorem 2.2.6). *We have*

$$\sum_{n,k,j \geq 0} Q_{n,k,j} t^k r^j z^n = \frac{(1-t)H(rz)}{H(t^l z) - tH(z)}, \quad (2.20)$$

where $Q_{0,0,0} = 1$.

For each positive integer n the polynomial $[n]_q$ is defined as

$$[n]_q := 1 + q + \cdots + q^{n-1}.$$

By convention, $[0]_q = 0$.

Corollary 2.4.2. *Let $Q_n(t, r) = \sum_{j,k \geq 0} Q_{n,k,j} t^k r^j$. Then, $Q_n(t, r)$ satisfies the following recurrence relation:*

$$Q_n(t, r) = r^n h_n + \sum_{k=0}^{n-1} Q_k(t, r) h_{n-k} t[l(n-k) - 1]_t. \quad (2.21)$$

Moreover,

$$Q_n(t, r) = \sum_m \sum_{\substack{k_0 \geq 0 \\ l k_1, \dots, l k_m \geq 2 \\ \sum k_i = n}} r^{k_0} h_{k_0} \prod_{i=1}^m h_{k_i} t[l k_i - 1]_t. \quad (2.22)$$

Proof. By (2.20), we have

$$\sum_{n,k,j \geq 0} Q_n(t, r) z^n = \frac{H(rz)}{1 - \sum_{n \geq 1} t[l n - 1]_t h_n z^n},$$

which is equivalent to (2.21). It is not hard to show that the right-hand side of (2.22) satisfies the recurrence relation (2.21). This proves (2.22). \square

For each colored permutation $\pi \in C_l \wr \mathfrak{S}_n$ define the *flag excedance* statistic $\text{fexc}(\pi)$ studied in [5, 38, 39] as

$$\text{fexc}(\pi) := l \cdot \text{exc}(\pi) + \sum_{i=1}^n \epsilon_i.$$

Note that when $l = 1$, flag excedances are excedances on permutations. Define the number of fixed points of π , $\text{fix}(\pi)$, by

$$\text{fix}(\pi) := |\{j \in [n] : \pi_j = j^0\}|.$$

Clearly, $\text{fix}(\pi) = \text{fix}_0(\pi)$ defined in (2.3). The colored (q, r) -Eulerian polynomials $A_n^{(l)}(t, r, q)$ are then defined as

$$A_n^{(l)}(t, r, q) := \sum_{\pi \in \text{Cyl}\mathfrak{S}_n} t^{\text{fexc}(\pi)} r^{\text{fix}(\pi)} q^{(\text{maj} - \text{exc})\pi}.$$

In particular, the polynomial

$$A_n^{(l)}(t, q) := A_n^{(l)}(t, 1, q)$$

is called *colored q -Eulerian polynomial*. The following specialization follows immediately from Lemma 2.2.2.

Lemma 2.4.3. *Let $Q_n(t, r) = \sum_{j, k \geq 0} Q_{n, k, j} t^k r^j$. Then we have*

$$\mathbf{ps}(Q_n(t, r)) = (q; q)_n^{-1} A_n^{(l)}(t, r, q).$$

For nonnegative integers k_0, \dots, k_m such that $k_0 + \dots + k_m = n$ define the q -multinomial coefficient

$$\left[\begin{matrix} n \\ k_0, \dots, k_m \end{matrix} \right]_q := \frac{(q; q)_n}{(q; q)_{k_0} \cdots (q; q)_{k_m}}.$$

Applying the specialization \mathbf{ps} to both sides of (2.20), (2.21) and (2.22) yields the following formulas for $A_n^{(l)}(t, r, q)$.

Corollary 2.4.4. *We have*

$$\sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{z^n}{(q; q)_n} = \frac{(1-t)e(rz; q)}{e(t^l z; q) - te(z; q)}. \quad (2.23)$$

Remark 2.2. The above generalization of (1.2) can also be deduced from [39, Theorem 1.3] through some calculations; see the proof of [39, Theorem 5.2] for details.

Corollary 2.4.5. *We have*

$$A_n^{(l)}(t, r, q) = r^n + \sum_{k=0}^{n-1} \left[\begin{matrix} n \\ k \end{matrix} \right]_q A_k^{(l)}(t, r, q) t[l(n-k) - 1]_t \quad (2.24)$$

and

$$A_n^{(l)}(t, r, q) = \sum_m \sum_{\substack{k_0 \geq 0 \\ l k_1, \dots, l k_m \geq 2 \\ \sum k_i = n}} \left[\begin{matrix} n \\ k_0, \dots, k_m \end{matrix} \right]_q r^{k_0} \prod_{i=1}^m [l k_i - 1]_t.$$

2.4.1 Symmetry and unimodality

Let $A(t) = a_r t^r + a_{r+1} t^{r+1} + \cdots + a_s t^s$ be a nonzero polynomial in t whose coefficients come from a partially ordered ring R . Then $A(t)$ is t -symmetric with center of symmetry $\frac{s+r}{2}$ if $a_{r+k} = a_{s-k}$ for all $k = 0, 1, \dots, s-r$; is t -unimodal if for some $s \leq k \leq r$,

$$a_r \leq_R a_{r+1} \leq_R \cdots \leq_R a_{k-1} \leq a_i \geq a_{k+1} \cdots \geq_R a_{s-1} \geq_R a_s;$$

is \log -concave if $a_k^2 \geq a_{k-1} a_{k+1}$ for all $k = r+1, r+2, \dots, s-1$. The following result is classical (see [22, p. 270]).

Lemma 2.4.6. *A polynomial with positive coefficients and with only real roots is log-concave and that log-concavity implies unimodality.*

It is well known (see [22, p. 292]) that the Eulerian polynomial $A_n(t)$ is symmetric and has only real roots and therefore unimodal. The following fact [84, Proposition 1] is useful.

Lemma 2.4.7. *The product of two symmetric unimodal polynomials with respective centers of symmetry c_1 and c_2 is symmetric and unimodal with center of symmetry $c_1 + c_2$.*

Let $A_n^{(l)}(t)$ be the *colored Eulerian polynomials* defined as

$$A_n^{(l)}(t) := A_n^{(l)}(t, 1, 1) = \sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{\text{fexc}(\pi)}.$$

Foata and Han [39, Eq. (5.15)] showed that

$$A_n^{(l)}(t) = A_n(t)(1 + t + t^2 + \cdots + t^{l-1})^n. \quad (2.25)$$

Recently, Mongelli [64, Proposition 3.3] rediscovered the $l = 2$ case of (2.25), which implies that $A_n^{(2)}(t)$ is symmetry and has only real roots and therefore unimodal.

Remark 2.3. We can give a multivariate extension of (2.25) as follows. For each $\pi \in C_l \wr \mathfrak{S}_n$ we define the function

$$s(\pi, i) := \begin{cases} t^l, & \text{if } \epsilon_i = 0 \text{ and } i \in \text{EXC}(|\pi|); \\ 1, & \text{if } \epsilon_i = 0 \text{ and } i \notin \text{EXC}(|\pi|); \\ s_{\epsilon_i}, & \text{if } \epsilon_i \neq 0, \end{cases}$$

and set $w(\pi) = t^{l \cdot \text{exc}(\pi)} s_1^{\text{col}_1(\pi)} \cdots s_{l-1}^{\text{col}_{l-1}(\pi)}$ (see (2.4) for the definition of $\text{col}_i(\pi)$). It is easy to see that $w(\pi) = \prod_{i=1}^n s(\pi, i)$. For a fix $\sigma \in \mathfrak{S}_n$ with $\text{exc}(\sigma) = h$ we have

$$\begin{aligned} \sum_{\substack{\pi \in C_l \wr \mathfrak{S}_n \\ |\pi| = \sigma}} w(\pi) &= \sum_{\substack{\pi \in C_l \wr \mathfrak{S}_n \\ |\pi| = \sigma}} \prod_{i=1}^n s(\pi, i) \\ &= \prod_{i \in \text{EXC}(\sigma)} (t^l + s_1 + \cdots + s_{l-1}) \prod_{i \notin \text{EXC}(\sigma)} (1 + s_1 + \cdots + s_{l-1}) \\ &= (t^l + s_1 + s_2 + \cdots + s_{l-1})^h (1 + s_1 + s_2 + \cdots + s_{l-1})^{n-h}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{l \cdot \text{exc}(\pi)} s_1^{\text{col}_1(\pi)} \cdots s_{l-1}^{\text{col}_{l-1}(\pi)} &= \sum_{\sigma \in \mathfrak{S}_n} \sum_{\substack{\pi \in C_l \wr \mathfrak{S}_n \\ |\pi| = \sigma}} w(\pi) \\ &= \sum_{h=0}^{n-1} A_{n,h} (t^l + s_1 + s_2 + \cdots + s_{l-1})^h (1 + s_1 + s_2 + \cdots + s_{l-1})^{n-h} \\ &= A_n \left(\frac{t^l + s_1 + s_2 + \cdots + s_{l-1}}{1 + s_1 + s_2 + \cdots + s_{l-1}} \right) (1 + s_1 + s_2 + \cdots + s_{l-1})^n. \end{aligned}$$

Setting $s_i = t^i$ in the above equation yields (2.25). Note that (2.25) can also be deduced from (2.23) directly.

From Lemma 2.4.7 and (2.25) we see that $A_n^{(l)}(t)$ is symmetric and unimodal with center of symmetry $(ln - 1)/2$, although not real-rootedness when $l > 2$. A sequence a_0, a_1, \dots, a_n is said to have *no interval zero* if there do not exist integers $0 \leq i < j < k \leq n$ satisfying $a_i \neq 0, a_j = 0, a_k \neq 0$. It is known [84, Proposition 2] that the product of two log-concave polynomials with nonnegative coefficients and no interval zero coefficients is again log-concave, thus by (2.25) we have the following stronger result.

Proposition 2.4.8. *For $l \geq 1$ the polynomial $A_n^{(l)}(t)$ is symmetric and log-concave. In particular it is unimodal.*

Let $d_n^B(t)$ be the generating function for the flag excedances on the derangements in $C_2 \wr \mathfrak{S}_n$, i.e.

$$d_n^B(t) := \sum_{\substack{\pi \in C_2 \wr \mathfrak{S}_n \\ \text{fix}(\pi) = 0}} t^{\text{exc}(\pi)} = A_n^{(2)}(t, 0, 1). \quad (2.26)$$

At the end of [64], Mongelli noticed that $d_5^B(t)$ is not real-rootedness and conjectured that $d_n^B(t)$ is unimodal for any $n \geq 1$. It is this conjecture that motivates us

to study the symmetry and unimodality of the coefficients of t^k in the flag Eulerian quasisymmetric functions and the colored (q, r) -Eulerian polynomials.

We first recall some necessary definitions. Let Par be the set of all partitions of all nonnegative integers. Let $b = \{b_\lambda : \lambda \in \text{Par}\}$ be a basis of the space of symmetric functions. We can define a partial order relation on the ring of symmetric functions by

$$f \leq_b g \Leftrightarrow g - f \text{ is } b\text{-positive,}$$

where a symmetric function is said to be b -positive if it is a nonnegative linear combination of elements of the b -basis. Here we are concerned with h -basis $\{h_\lambda : \lambda \in \text{Par}\}$ and the Schur basis $\{s_\lambda : \lambda \in \text{Par}\}$. The following theorem reduces to [76, Theorem 5.1] when $l = 1$.

Theorem 2.4.9. *Let $Q_{n,k} = \sum_{j=0}^n Q_{n,k,j}$. Using the h -basis to partially order the ring of symmetric functions, we have for all n, j, k ,*

- (1) *the flag Eulerian quasisymmetric function $Q_{n,k,j}$ is a h -positive symmetric function;*
- (2) *the polynomial $\sum_{k=0}^{ln-1} Q_{n,k,j} t^k$ is t -symmetric and t -unimodal with center of symmetry $l(n-j)/2$;*
- (3) *the polynomial $\sum_{k=0}^{ln-1} Q_{n,k} t^k$ is t -symmetric and t -unimodal with center of symmetry $(ln-1)/2$.*

Proof. Part (1) follows from (2.22). We will use the fact in Lemma 2.4.7 to show Part (2) and (3). By (2.22) we have

$$\sum_{k=0}^{ln-1} Q_{n,k,j} t^k = \sum_m \sum_{\substack{lk_1, \dots, lk_m \geq 2 \\ \sum k_i = n-j}} h_j \prod_{i=1}^m h_{k_i} t[lk_i - 1]_t.$$

As each term $h_j \prod_{i=1}^m h_{k_i} t[lk_i - 1]_t$ is t -symmetric and t -unimodal with center of symmetry $\sum_i lk_i/2 = l(n-j)/2$, the sum of the terms on the right-hand side of the above equation has the same property, which shows Part (2).

In the following, we show that Part (3) also follows from (2.22). For any sequence of positive integers (k_1, \dots, k_m) we define

$$G_{k_1, \dots, k_m}^{(l)} := \prod_{i=1}^m h_{k_i} t[lk_i - 1]_t.$$

Then, by (2.22) we have

$$\sum_{k=0}^{ln-1} Q_{n,k,0} t^k = \sum_m \sum_{\substack{lk_1, \dots, lk_m \geq 2 \\ \sum k_i = n}} G_{k_1, \dots, k_m}^{(l)}$$

and

$$\sum_{j \geq 1} \sum_{k=0}^{ln-1} Q_{n,k,j} t^k = \sum_m \sum_{\substack{lk_1, \dots, lk_m \geq 2 \\ \sum k_i = n}} h_{k_1} G_{k_2, \dots, k_m}^{(l)}$$

assuming $l \geq 2$. We claim that $G_{k_1, \dots, k_m}^{(l)} + h_{k_1} G_{k_2, \dots, k_m}^{(l)}$ is t -symmetric and t -unimodal with center of symmetry $(ln - 1)/2$. Note that

$$G_{k_1, \dots, k_m}^{(l)} + h_{k_1} G_{k_2, \dots, k_m}^{(l)} = h_{k_1} (t[lk_1 - 1]_t + 1) G_{k_2, \dots, k_m}^{(l)}.$$

Clearly, $t[lk_1 - 1]_t + 1 = 1 + t + \dots + t^{lk_1 - 1}$ is t -symmetric and t -unimodal with center of symmetry $(lk_1 - 1)/2$, and $G_{k_2, \dots, k_m}^{(l)}$ is t -symmetric and t -unimodal with center of symmetry $l(n - k_1)/2$. Therefore, our claim holds and the proof of Part (3) is complete because of

$$\sum_{k=0}^{ln-1} Q_{n,k} t^k = \sum_{k=0}^{ln-1} Q_{n,k,0} t^k + \sum_{j \geq 1} \sum_{k=0}^{ln-1} Q_{n,k,j} t^k.$$

□

Remark 2.4. We can give a bijective proof of the symmetric property

$$Q_{n,k,j} = Q_{n,l(n-j)-k,j} \quad (2.27)$$

using the colored ornament interpretation of $Q_{n,k,j}$. We construct an involution φ on colored ornaments such that if the cv-cycle type of a colored banner R is

$$\check{\lambda}(R) = \{(\lambda_1, \vec{\beta}^1), \dots, (\lambda_r, \vec{\beta}^r)\},$$

then the cv-cycle type of $\varphi(R)$ is

$$\check{\lambda}(\varphi(R)) = \{(\lambda_1, \vec{\beta}^{1\perp}), \dots, (\lambda_r, \vec{\beta}^{r\perp})\},$$

where $\vec{\beta}^\perp := (\beta_{l-1}, \beta_{l-2}, \dots, \beta_1)$ for each $\vec{\beta} = (\beta_1, \dots, \beta_{l-1}) \in \mathbb{N}^{l-1}$. Let R be a colored banner. To obtain $\varphi(R)$, we first bar each unbarred 0-colored letter of each nonsingleton colored necklace of R and unbar each barred 0-colored letter. Next, we change the color of each m -colored letter of R to color $l - m$ for all $m = 1, \dots, l - 1$. Finally, for each i we replace each occurrence of the i -th smallest value in R with the i -th largest value leaving the bars and colors intact. This proves (2.27) because $Q_{n,k,j}$ is a symmetric function by Theorem 2.2.4.

For the ring of polynomials $\mathbb{Q}[q]$, where q is an indeterminate, we use the partial order relation:

$$f(q) \leq_q g(q) \Leftrightarrow g(q) - f(q) \text{ has nonnegative coefficients.}$$

We will use the following key fact from [76, Lemma 5.2].

Lemma 2.4.10. *If f is a Schur positive homogeneous symmetric function of degree n , then $(q; q)_n \mathbf{ps}(f)$ is a polynomial in q with nonnegative coefficients.*

Theorem 2.4.11. *We have for $n \geq 1$ and $j \geq 0$,*

(1) *the polynomial $\sum_{\substack{\pi \in C_l^l \mathfrak{S}_n \\ \text{fix}(\pi)=j}} t^{\text{exc}(\pi)} q^{(\text{maj} - \text{exc})\pi}$ is t -symmetric and t -unimodal with center of symmetry $\frac{l(n-j)}{2}$,*

(2) *the polynomial $A_n^{(l)}(t, q)$ is t -symmetric and t -unimodal with center of symmetry $\frac{ln-1}{2}$.*

Proof. Let f and g be two homogeneous symmetric functions of degree n with $f \leq_h g$. Since h -positivity implies Schur positivity [81, Corollary 7.12.4], by Lemma 2.4.10, we have

$$(q; q)_n \mathbf{ps}(f) \leq_q (q; q)_n \mathbf{ps}(g).$$

By Lemma 2.4.3, Part (1) and (2) are obtained by specializing Parts (2) and (3) of Theorem 2.4.9, respectively. \square

Remark 2.5. When $l = 1$, Parts (1) and (2) of the above theorem reduce to Parts (3) and (4) of Theorem 5.3 in [76], respectively. Part (1) of the above theorem implies the unimodality of $d_n^B(t)$ as conjectured by Mongelli [64, Conjecture 8.1]. Actually, he also conjectured that $d_n^B(t)$ is log-concave.

2.4.2 Generalized symmetric Eulerian identities

In this section we prove two generalized symmetric Eulerian identities involving the flag Eulerian quasisymmetric functions $Q_{n,k,j}$.

Theorem 2.4.12. *For $a, b \geq 1$ and $j \geq 0$ such that $a + b + 1 = l(n - j)$,*

$$\sum_{i \geq 0} h_i Q_{n-i,a,j} = \sum_{i \geq 0} h_i Q_{n-i,b,j}. \quad (2.28)$$

Proof. Cross-multiplying and expanding all the functions $H(z)$ in (2.20), we obtain

$$\sum_{n \geq 0} h_n (t^l z)^n \sum_{n, j, k \geq 0} Q_{n, k, j} r^j t^k z^n - t \sum_{n \geq 0} h_n z^n \sum_{n, j, k \geq 0} Q_{n, k, j} r^j t^k z^n = (1 - t) \sum_{n \geq 0} h_n (rz)^n.$$

Now, identifying the coefficients of z^n yields

$$\sum_{i, j, k} h_i Q_{n-i, k-l i, j} r^j t^k - \sum_{i, j, k} h_i Q_{n-i, k-1, j} r^j t^k = (1 - t) h_n r^n.$$

Hence, for $j < n$ we can identify the coefficients of $r^j t^k$ and obtain

$$\sum_{i \geq 0} h_i Q_{n-i, k-l i, j} = \sum_{i \geq 0} h_i Q_{n-i, k-1, j}. \quad (2.29)$$

Applying the symmetry property (2.27) to the left-hand side of the above equation yields

$$\sum_{i \geq 0} h_i Q_{n-i, l(n-j)-k, j} = \sum_{i \geq 0} h_i Q_{n-i, k-1, j},$$

which becomes (2.28) after setting $a = k - 1$ and $b = l(n - j) - k$ since now $n > j$. \square

A bijective proof of Theorem 2.4.12. We will give a bijective proof of

$$Q_{n, k} = Q_{n, l n - k - 1} \quad (2.30)$$

by means of colored banners, using Theorem 2.2.5. Construct an involution θ on colored banners as follows. Let B be a colored banner. To obtain $\theta(B)$, first bar each unbarred 0-colored letter of B , except the last letter; also unbar each barred letter. Next, change the color of each m -colored letter of B to color $l - m$ for all $m = 1, \dots, l - 1$, except the last letter; if the color of the last letter is a , change it to $l - 1 - a$. Finally, for each i replace each occurrence of the i -th smallest value in B with the i -th largest value, leaving the bars and colors intact.

Since $a + b = l(n - j) - 1$, by (2.30) we have $Q_{n-j, a} = Q_{n-j, b}$, which is equivalent to

$$\sum_{i \geq 0} Q_{n-j, a, i} = \sum_{i \geq 0} Q_{n-j, b, i}. \quad (2.31)$$

For any m, k, i , it follows from Theorem 2.2.3 that

$$Q_{m, k, i} = h_i Q_{m-i, k, 0}. \quad (2.32)$$

Thus, Eq. (2.31) becomes

$$\sum_{i \geq 0} h_i Q_{n-j-i, a, 0} = \sum_{i \geq 0} h_i Q_{n-j-i, b, 0}.$$

Multiplying both sides by h_j then gives

$$\sum_{i \geq 0} h_j h_i Q_{n-j-i,a,0} = \sum_{i \geq 0} h_j h_i Q_{n-j-i,b,0}.$$

Applying (2.32) once again, we obtain (2.28). \square

Remark 2.6. As the analytical proof of Theorem 2.4.12 is reversible, the above bijective proof together with the bijective proof of (2.27) provides a different proof of Corollary 2.4.1 using interpretations of $Q_{n,k,j}$ as colored ornaments and colored banners. In particular, this gives an alternative approach to Step 3 in the proof of Theorem 1.2 in [76].

Theorem 2.4.13. *Let $Q_{n,k} = \sum_j Q_{n,k,j}$. For $a, b \geq 1$ such that $a + b = ln$ we have:*

$$\sum_{i=0}^{n-1} h_i Q_{n-i,a-1} = \sum_{i=0}^{n-1} h_i Q_{n-i,b-1}. \quad (2.33)$$

Proof. With $r = 1$ identity (2.20) becomes:

$$\sum_{n,k \geq 0} Q_{n,k} t^k z^n = \frac{(1-t)H(z)}{H(t^l z) - tH(z)}.$$

Subtracting both sides by $Q_{0,0} = 1$ gives

$$\sum_{n \geq 1, k \geq 0} Q_{n,k} t^k z^n = \frac{H(z) - H(t^l z)}{H(t^l z) - tH(z)}.$$

Cross-multiplying and then identifying the coefficients of $t^k z^n$ ($1 \leq k \leq ln - 1$) yields

$$\sum_{i=0}^{n-1} h_i Q_{n-i,k-li} = \sum_{i=0}^{n-1} h_i Q_{n-i,k-1}.$$

Applying the symmetry property (2.30) to the left-hand side we get

$$\sum_{i=0}^{n-1} h_i Q_{n-i,ln-1-k} = \sum_{i=0}^{n-1} h_i Q_{n-i,k-1},$$

which is (2.33) when $a-1 = k-1$ and $b-1 = ln-1-k$, since now $1 \leq k \leq ln-1$. \square

To construct a bijective proof of Theorem 2.4.13, we need a refinement of the decomposition of the colored banners from [52].

A *0-colored marked sequence*, denoted by $(\omega, b, 0)$, is a weakly increasing sequence ω of positive integers, together with a positive integer b , which we call the mark, such that $1 \leq b < \text{length}(\omega)$. Let $M(n, b, 0)$ denote the set of all 0-colored marked sequences with $\text{length}(\omega) = n$ and mark equal to b .

For $m \in [l-1]$, a *m-colored marked sequence*, denoted by (ω, b, m) , is a weakly increasing sequence ω of positive integers, together with a integer b such that $0 \leq b < \text{length}(\omega)$. Let $M(n, b, m)$ denote the set of all m -colored marked sequences with $\text{length}(\omega) = n$ and mark equal to b .

Here we will use the original colored banner interpretation from Remark 2.1. Let $K_0(n, j, \vec{\beta})$ denote the set of all colored banners of length n , with Lyndon type having no parts of size one formed by a 0-colored letter, color vector equal to $\vec{\beta}$ and j bars. For $m \in [l-1]$ and $\beta_m > 0$, define

$$X_m := \bigsqcup_{\substack{0 \leq i \leq n-1 \\ j-n+i < k \leq j}} K_0(i, k, \vec{\beta}(\hat{m})) \times M(n-i, j-k, m),$$

where $\vec{\beta}(\hat{m}) = (\beta_1, \dots, \beta_{m-1}, \beta_m - 1, \dots, \beta_{l-1})$ and let $X_m := 0$ if $\beta_m = 0$. We also define

$$X_0 := \bigsqcup_{\substack{0 \leq i \leq n-2 \\ j-n+i < k < j}} K_0(i, k, \vec{\beta}) \times M(n-i, j-k, 0).$$

Theorem 2.4.14. *There is a bijection*

$$\Upsilon : K_0(n, j, \beta) \rightarrow \bigsqcup_{m=0}^{l-1} X_m$$

such that if $\Upsilon(B) = (B', (\omega, b, m))$, then $\text{wt}(B) = \text{wt}(B') \text{wt}(\omega)$ and $\vec{\beta}(\hat{m}) = \vec{\text{col}}(B')$ if $m \geq 1$ otherwise $\vec{\beta} = \vec{\text{col}}(B')$.

Proof. By [23, lemma 4.3], every banner $B \in K_0(n, j, \beta)$ has a unique factorization, that we also called increasing factorization (here we admit parts of size one formed by a letter with positive color), $B = B_1 \cdot B_2 \cdots B_d$ where each B_i has the form

$$B_i = \underbrace{(a_i, \dots, a_i)}_{p_i \text{ times}} \cdot u_i,$$

where $a_i \in \mathcal{B}$, $p_i > 0$ and u_i is a word (possibly empty) over the alphabet \mathcal{B} whose letters are all strictly less than a_i with respect to $<_B$, $a_1 \leq_B a_2 \leq_B \cdots \leq_B a_d$ and if u_i is empty then $B_i = a_i$ and for each $k \geq i$ with $a_k = a_i$ we has $B_k = B_i = a_i$. Note that the increasing factorization is a refinement of the Lyndon factorization.

For example, the Lyndon factorization of the banner

$$6^1, 1^2, 5^1, 6^1, 6^1, \overline{4^0}, \overline{4^0}, 4^1, 4^0, \overline{4^0}, 3^2, 5^0, 7^1$$

is

$$(6^1, 1^2, 5^1) \cdot (6^1) \cdot (6^1) \cdot (\overline{4^0}, \overline{4^0}, 4^1, 4^0, \overline{4^0}, 3^2) \cdot (5^0, 7^1),$$

and its increasing factorization is

$$(6^1, 1^2, 5^1) \cdot (6^1) \cdot (6^1) \cdot (\overline{4^0}, \overline{4^0}, 4^1, 4^0) \cdot (\overline{4^0}, 3^2) \cdot (5^0, 7^1).$$

First, we take the increasing factorization of B , say $B = B_1 \cdot B_2 \cdots B_d$. Let

$$B_d = \underbrace{(a, \dots, a)}_{p \text{ times}} \cdot u,$$

where $a \in \mathcal{B}$, $p > 0$ and u is a word (possibly empty) over \mathcal{B} whose letters are all strictly less than a with respect to the order $<_B$. Let γ be the bijection defined in [52, Theorem 4.5]. Now we describe the map Υ .

Case 1: a is 0-colored. Define $\Upsilon(B) = \gamma(B)$.

Case 2: a has positive color and u is not empty. Suppose that $u = i_1, i_2, \dots, i_k$.

Case 2.1: If $k \geq 2$, then define $\omega = |i_1|$, $b = 0$, m is the color of i_1 and $B' = B_1 \cdots B_{d-1} \cdot \widetilde{B}_d$, where

$$\widetilde{B}_d = \underbrace{a, \dots, a}_{p \text{ times}}, i_2, \dots, i_k.$$

Case 2.2: If $k = 1$, then define $\omega = |i_1|$, $b = 0$, m is the color of i_1 and

$$B' = B_1 \cdots B_{d-1} \cdot \underbrace{a \cdot a \cdots a}_{p \text{ times}},$$

where each a is a factor.

Case 3: a has positive color and u is empty. In this case $B_d = a$. Define $\omega = |a|$, $b = 0$, m is the color of a and $B' = B_1 \cdots B_{d-1}$.

This complete the description of the map Υ . Next we describe Υ^{-1} . Suppose we are given a banner B with increasing factorization $B = B_1 \cdots B_d$ where

$$B_d = \underbrace{a, \dots, a}_{p \text{ times}}, j_1, \dots, j_k,$$

and a m -colored marked sequence (ω, b, m) .

Case A: (inverse of Case 1) a is 0-colored or $\text{length}(\omega) \geq 2$. Define

$$\Upsilon^{-1}((B, (\omega, b, m))) = \gamma^{-1}((B, (\omega, b, m))).$$

Case B: (inverse of Case 2.1) a has positive color, $\omega = j_0$ is a letter with positive color m and j_1, \dots, j_k is not empty. Then let $\Upsilon^{-1}((B, (\omega, b, m))) = B_1 \cdots B_{d-1} \cdot \widetilde{B}_d$, where

$$\widetilde{B}_d = \underbrace{a, \dots, a}_{p \text{ times}}, j_0, j_1, \dots, j_k.$$

Case C: a has positive color, $\omega = j_0$ is a letter with positive color m and $B_d = a$. In this case, there exists a nonnegative integer k such that $B_{d-k} = B_{d-k+1} = \dots = B_d = a$ but $B_{d-k-1} \neq a$.

Case C1: (inverse of Case 3) If $j_0 \geq_B a$, then define

$$\Upsilon^{-1}((B, (\omega, b, m))) = B_1 \cdots B_d \cdot j_0,$$

where j_0 is a factor.

Case C2: (inverse of Case 2.2) Otherwise, j_0 is strictly less than a with respect to $<_B$ and we define $\Upsilon^{-1}((B, (\omega, b, m))) = B_1 \cdots B_{d-k-1} \cdot \widetilde{B}_{d-k}$, where

$$\widetilde{B}_{d-k} = \underbrace{a, \dots, a}_{k+1 \text{ times}}, j_0.$$

This completes the description of Υ^{-1} . One can check case by case that both maps are well defined and in fact inverses of each other. \square

For any nonnegative integers i, j , let $K_j(n, i, \vec{\beta})$ denote the set of all colored banners of length n , with Lyndon type having j parts of size one formed by a 0-colored letter, color vector equal to $\vec{\beta}$ and i bars. Let $\text{Com}_j(n, i, \vec{\beta})$ be the set of all compositions

$$\sigma = (\omega_0, (\omega_1, b_1, m_1), \dots, (\omega_r, b_r, m_r))$$

for some integer r , where ω_0 is a weakly increasing word of positive integers of length j and each (ω_i, b_i, m_i) is a m_i -colored marked sequence and satisfying

$$\sum_{j=0}^r \text{length}(\omega_j) = n, \quad \sum_{j=1}^r b_j = i \quad \text{and} \quad \vec{\beta} = (\beta_1, \dots, \beta_{l-1}),$$

where β_k equals the number of m_i such that $m_i = k$. Define the weight of σ by

$$\text{wt}(\sigma) := \text{wt}(\omega_0) \cdots \text{wt}(\omega_r).$$

By Theorem 2.4.14, we can construct a weight preserving bijection between $K_j(n, i, \vec{\beta})$ and $\text{Com}_j(n, i, \vec{\beta})$ by first factoring out the j parts of size one formed by a 0-colored letter in the Lyndon factorization of a banner and then factoring out marked sequences step by step in the increasing factorization of the remaining banner. Thus we have the following interpretation of $Q_{n,k,j}$.

Corollary 2.4.15. *We have*

$$Q_{n,k,j} = \sum_{\substack{i \in \mathbb{N}, \beta \in \mathbb{N}^{l-1} \\ \sigma \in \text{Com}_j(n, i, \beta) \\ li + \text{csum}(\beta) = k}} \text{wt}(\sigma).$$

Definition 2.5 (Two-fix-banners). For each fixed positive integer n , a *two-fix-banner* of length n is a sequence

$$\mathbf{v} = (\omega_0, (\omega_1, b_1, m_1), \dots, (\omega_r, b_r, m_r), \omega'_0) \quad (2.34)$$

satisfying the following conditions:

- (C1) ω_0 and ω'_0 are two weakly increasing sequences of positive integers, possibly empty;
- (C2) each (ω_i, b_i, m_i) is a m_i -colored marked sequence;
- (C3) $\text{length}(\omega_0) + \text{length}(\omega_1) + \dots + \text{length}(\omega_r) + \text{length}(\omega'_0) = n$.

Define the *flag excedance* statistic of \mathbf{v} by

$$\text{fexc}(\mathbf{v}) := l \sum_{i=1}^r b_i + \sum_{i=1}^r m_r.$$

A bijective proof of Theorem 2.4.13. The two-fix-banner \mathbf{v} in (2.34) is in bijection with the pair (σ, ω) , where $\omega = \omega'_0$ is a weakly increasing sequence of positive integers with length i for some nonnegative integer i and

$$\sigma = (\omega_0, (\omega_1, b_1, m_1), \dots, (\omega_r, b_r, m_r))$$

is a composition with $\sum_{j=0}^r \text{length}(\omega_j) = n - i$. Thus, by Corollary 2.4.15 we obtain the following interpretation.

Lemma 2.4.16. *Let TB_n denote the set of all two-fix-banners of length n . For any nonnegative integer a , we have*

$$\sum_{\substack{\mathbf{v} \in \text{TB}_n \\ \text{fexc}(\mathbf{v}) = a}} \text{wt}(\mathbf{v}) = \sum_{i=0}^{n-1} h_i Q_{n-i,a}.$$

By the above lemma, it suffices to construct an involution $\Phi : \text{TB}_n \rightarrow \text{TB}_n$ satisfying $\text{fexc}(\mathbf{v}) + \text{fexc}(\Phi(\mathbf{v})) = ln - 2$ for each $\mathbf{v} \in \text{TB}_n$. First we need to define

two local involutions. For a weakly increasing sequence of positive integers ω with $\text{length}(\omega) = k$, we define

$$d'(\omega) = (\omega, k - 1, l - 1),$$

which is a $(l-1)$ -colored marked sequence. For a m -colored mark sequence (ω, b, m) with $\text{length}(\omega) = k$, we define

$$d((\omega, b, m)) = \begin{cases} (\omega, k - b, 0) & \text{if } m = 0; \\ (\omega, k - 1 - b, l - m), & \text{otherwise.} \end{cases}$$

We also define

$$d'((\omega, b, m)) = \begin{cases} \omega, & \text{if } b = k - 1 \text{ and } m = l - 1; \\ (\omega, k - 1 - b, l - 1 - m), & \text{otherwise.} \end{cases}$$

One can check that d and d' are well-defined involutions.

Let \mathbf{v} be a two-fix-banner and write

$$\mathbf{v} = (\tau_0, \tau_1, \tau_2, \dots, \tau_{r-1}, \tau_r, \tau_{r+1}),$$

where $\tau_0 = \omega_0$ and $\tau_{r+1} = \omega'_0$. If τ_i (respectively τ_j) is the leftmost (respectively rightmost) non-empty sequence (clearly $i = 0, 1$ and $j = r, r + 1$), we can write \mathbf{v} in the following compact way by removing the empty sequences at the beginning or at the end:

$$\mathbf{v} = (\tau_i, \tau_{i+1}, \dots, \tau_{j-1}, \tau_j). \quad (2.35)$$

It is easy to see that the above procedure is reversible by adding some necessary empty words at the two ends of the compact form (2.35). Now we work with the compact form.

If $i = j$ (\mathbf{v} has only one sequence), we define

$$\Phi(\mathbf{v}) = \begin{cases} (\emptyset, (\tau_i, n - 1, l - 2), \emptyset), & \text{if } \tau_i \text{ is a weakly increasing sequence;} \\ (\omega, \emptyset), & \text{if } \tau_i = (\omega, n - 1, l - 2) \text{ is a marked sequence;} \\ (\emptyset, (\omega, n - 1 - b, l - 2 - m), \emptyset), & \text{otherwise, suppose } \tau_i = (\omega, b, m). \end{cases}$$

If $j > i$ (\mathbf{v} has at least two sequences), we define the two-fix-banner $\Phi(\mathbf{v})$ by

$$\Phi(\mathbf{v}) = (d'(\tau_i), d(\tau_{i+1}), d(\tau_{i+2}), \dots, d(\tau_{j-1}), d'(\tau_j)).$$

As d and d' are involutions, Φ is also an involution and one can check that in both cases Φ satisfy the desired property. This completes our bijective proof. \square

By Lemma 2.4.3, if we apply \mathbf{ps} to both sides of (2.28) and (2.33), we obtain the following two symmetric q -Eulerian identities.

Corollary 2.4.17. *For $a, b \geq 1$ and $j \geq 0$ such that $a + b + 1 = l(n - j)$,*

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,a,j}^{(l)}(q) = \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,b,j}^{(l)}(q), \quad (2.36)$$

where $A_n^{(l)}(t, r, q) = \sum_{k,j} A_{n,k,j}^{(l)}(q) t^k r^j$.

Corollary 2.4.18. *For $a, b \geq 1$ such that $a + b = ln$,*

$$\sum_{k \geq 1} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,a-1}^{(l)}(q) = \sum_{k \geq 1} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,b-1}^{(l)}(q), \quad (2.37)$$

where $A_n^{(l)}(t, q) = \sum_k A_{n,k}^{(l)}(q) t^k$.

Remark 2.7. When $l = 1$, symmetric identities (2.36) and (2.37) reduces to (1.7) and (1.5), respectively.

2.4.3 Two interpretations of colored (q, r) -Eulerian polynomials

We will introduce the colored hook factorization of a colored permutation. Recall that a word $w = w_1 w_2 \dots w_m$ over \mathbb{N} is called a *hook* if $w_1 > w_2$ and either $m = 2$, or $m \geq 3$ and $w_2 < w_3 < \dots < w_m$. We can extend the hooks to colored hooks. Let

$$[n]^l \subset \mathbb{N}^l := \{1^0, 1^1, \dots, 1^{l-1}, 2^0, 2^1, \dots, 2^{l-1}, \dots, i^0, i^1, \dots, i^{l-1}, \dots\}.$$

Definition 2.6 (Colored hook). A word $w = w_1 w_2 \dots w_m$ over \mathbb{N}^l is called a *colored hook* if

- $m \geq 2$ and $|w|$ is a hook and only w_1 may have positive color;
- or $m \geq 1$ and $|w|$ is an increasing word and only w_1 has positive color.

Clearly, each colored permutation $\pi = \pi_1 \pi_2 \dots \pi_n \in C_l \wr \mathfrak{S}_n$ admits a unique factorization, called its *colored hook factorization*, $p \tau_1 \tau_2 \dots \tau_r$, where p is a word formed by 0-colored letters, $|p|$ is an increasing word over \mathbb{N} and each factor $\tau_1, \tau_2, \dots, \tau_k$ is a colored hook. To derive the colored hook factorization of a colored permutation, one can start from the right and factor out each colored hook step by step. When $l = 1$, colored hook factorization is the hook factorization introduced

by Gessel [45] that we have already used in Section 1.2. For example, the colored hook factorization of

$$2^0 4^0 5^1 8^0 3^0 7^0 10^1 1^0 9^0 6^1 \in C_2 \wr \mathfrak{S}_{10} \quad (2.38)$$

is

$$2^0 4^0 | 5^1 | 8^0 3^0 7^0 | 10^1 1^0 9^0 | 6^1.$$

Let $w = w_1 w_2 \dots w_m$ be a word over \mathbb{N} . Define

$$\text{inv}(w) := |\{(i, j) : i < j, w_i > w_j\}|.$$

For a colored permutation $\pi \in C_l \wr \mathfrak{S}_n$ with colored hook factorization $p\tau_1\tau_2\dots\tau_r$, we define

$$\text{inv}(\pi) := \text{inv}(|\pi|) \quad \text{and} \quad \text{lec}(\pi) := \sum_{i=1}^r \text{inv}(|\tau_i|).$$

We also define

$$\text{flec}(\pi) := l \cdot \text{lec}(\pi) + \sum_{i=1}^n \epsilon_i \quad \text{and} \quad \text{pix}(\pi) := \text{length}(p).$$

For example, if π is the colored permutation in (2.38), then $\text{inv}(\pi) = 16$, $\text{lec}(\pi) = 4$, $\text{flec}(\pi) = 11$ and $\text{pix}(\pi) = 2$.

Theorem 2.4.19. *For $n \geq 1$, we have*

$$A_n^{(l)}(t, r, q) = \sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{\text{flec}(\pi)} r^{\text{pix}(\pi)} q^{\text{inv}(\pi) - \text{lec}(\pi)}. \quad (2.39)$$

Proof. Note that a colored hook of length k may contribute $1, 2, \dots, lk - 1$ to the statistic “flec” of a colored permutation. Consider the last colored hook (possibly empty) of each colored permutation then gives

$$\begin{aligned} & \sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{\text{flec}(\pi)} r^{\text{pix}(\pi)} q^{\text{inv}(\pi) - \text{lec}(\pi)} \\ &= \sum_{k=0}^n \sum_{\substack{\pi \in C_l \wr \mathfrak{S}_n \\ \pi = p\tau_1 \dots \tau_r \\ \#\tau_r = n-k}} t^{\text{flec}(\pi)} r^{\text{pix}(\pi)} q^{\text{inv}(\pi) - \text{lec}(\pi)} \\ &= r^n + \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q t^{[l(n-k) - 1]_t} \sum_{\pi \in C_l \wr \mathfrak{S}_k} t^{\text{flec}(\pi)} r^{\text{pix}(\pi)} q^{\text{inv}(\pi) - \text{lec}(\pi)}, \end{aligned}$$

where we apply (1.13) to the last equality. This shows that the right-hand side of (2.39) satisfies recurrence (2.24), which finishes the proof of the theorem. \square

Remark 2.8. The above theorem reduces to [35, Theorem 4] when $l = 1$.

The recurrence in Theorem 1.1.4 can be generalized to the colored (q, r) -Eulerian polynomials as follows.

Theorem 2.4.20. *The colored (q, r) -Eulerian polynomials satisfy the following recurrence formula:*

$$A_{n+1}^{(l)}(t, r, q) = (r + t[l-1]_t q^n) A_n^{(l)}(t, r, q) + t[l]_t \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_k^{(l)}(t, r, q) A_{n-k}^{(l)}(t, q) \quad (2.40)$$

with $A_0^{(l)}(t, r, q) = 1$ and $A_1^{(l)}(t, r, q) = r$.

Proof. Applying the Eulerian differential operator δ_z to both sides of (2.23) and using the property (1.27) and Lemma 1.4.1, we obtain

$$\begin{aligned} \sum_{n \geq 0} A_{n+1}^{(l)}(t, r, q) \frac{z^n}{(q; q)_n} &= \delta_z \left(\frac{(1-t)e(rz; q)}{e(t^l z; q) - te(z; q)} \right) \\ &= \frac{\delta_z((1-t)e(rz; q))}{e(t^l z; q) - te(z; q)} + \delta_z((e(t^l z; q) - te(z; q))^{-1}) (1-t)e(rzq^l; q) \\ &= B + C, \end{aligned}$$

where

$$B = \frac{r(1-t)e(rz; q)}{e(t^l z; q) - te(z; q)} = r \sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{z^n}{(q; q)_n},$$

and

$$\begin{aligned} C &= \frac{(1-t)e(rzq; q)(te(z; q) - t^l e(t^l z; q))}{(e(t^l qz; q) - te(qz; q))(e(t^l z; q) - te(z; q))} \\ &= \frac{(1-t)e(rzq; q)}{e(t^l qz; q) - te(qz; q)} \left(\frac{t^l(e(z; q) - t^l e(t^l z; q))}{e(t^l z; q) - te(z; q)} + \frac{te(z; q) - t^l e(z; q)}{e(t^l z; q) - te(z; q)} \right) \\ &= \left(\sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{(qz)^n}{(q; q)_n} \right) \left(t^l \sum_{n \geq 1} A_n^{(l)}(t, q) \frac{z^n}{(q; q)_n} + t[t]_{l-1} \sum_{n \geq 0} A_n^{(l)}(t, q) \frac{z^n}{(q; q)_n} \right). \end{aligned}$$

Taking the coefficient of $\frac{z^n}{(q; q)_n}$ in both sides of the equality

$$\sum_{n \geq 0} A_{n+1}^{(l)}(t, r, q) \frac{z^n}{(q; q)_n} = B + C,$$

we get (2.40). □

Remark 2.9. Once again, the derangement polynomial $d_n^B(t)$ defined in (2.26) is t -symmetric and t -unimodal with center of symmetry n follows from the recurrence (2.40) by induction on n using Lemma 2.4.7.

Recurrence relation (2.40) enables us to obtain another interpretation of the colored (q, r) -Eulerian polynomials $A_n^{(l)}(t, r, q)$. First, we define the *zero-descent number* of a colored permutation $\pi \in C_l \wr \mathfrak{S}_n$, denoted $\text{des}^z(\pi)$, by

$$\text{des}^z(\pi) := |\{i \in [n-1] : \epsilon_i = 0 \text{ and } |\pi_i| > |\pi_{i+1}|\}|.$$

We also define the *flag zero-descent number* by

$$\text{fdes}^z(\pi) := l \cdot \text{des}^z(\pi) + \sum_{i=1}^n \epsilon_i.$$

Definition 2.7 (Colored admissible inversion). A *colored admissible inversion* of π is a pair (i, j) with $1 \leq i < j \leq n$ that satisfies any one of the following three conditions

- $1 < i$ and $|\pi_{i-1}| < |\pi_i| > |\pi_j|$;
- there is some k such that $i < k < j$ and $|\pi_j| < |\pi_i| < |\pi_k|$;
- $\epsilon_j > 0$ and for any k such that $i \leq k < j$, we have $|\pi_k| < |\pi_j| < |\pi_{j+1}|$, where we take the convention $|\pi_{n+1}| = +\infty$.

We write $\text{ai}(\pi)$ the number of colored admissible inversions of π . For example, if $\pi = 4^0 1^0 2^1 5^0 3^1$ in $C_2 \wr \mathfrak{S}_5$, then $\text{ai}(\pi) = 3$. When $l = 1$, the colored admissible inversions agree with admissible inversions defined in Section 1.4.

Finally, we extend the statistic “rix” defined in Section 1.4 to the set of all words over \mathbb{N}^l . Let $w = w_1 \cdots w_n$ be a word over \mathbb{N}^l . Suppose that w_i is the unique rightmost letter of w such that $|w_i| = \max\{|w_1|, |w_2|, \dots, |w_n|\}$. We define $\text{rix}(w)$ by (with convention that $\text{rix}(\emptyset) = 0$)

$$\text{rix}(w) := \begin{cases} 0, & \text{if } i = 1 \neq n, \\ 1 + \text{rix}(w_1 \cdots w_{n-1}), & \text{if } i = n \text{ and } \epsilon_n = 0, \\ \text{rix}(w_{i+1} w_{i+2} \cdots w_n), & \text{if } 1 < i < n. \end{cases}$$

As a colored permutation can be viewed as a word over \mathbb{N}^l , the statistic “rix” is well-defined on colored permutations. For example, if $\pi = 1^0 6^1 2^0 5^1 3^0 4^1 7^0 \in C_2 \wr \mathfrak{S}_7$, then $\text{rix}(\pi) = 1 + \text{rix}(1^0 6^1 2^0 5^1 3^0 4^1) = 1 + \text{rix}(2^0 5^1 3^0 4^1) = 1 + \text{rix}(3^0 4^1) = 1 + \text{rix}(3^0) = 2$. We have the following colored version of Theorem 1.4.2.

Corollary 2.4.21. *For $n \geq 1$, we have*

$$A_n^{(l)}(t, r, q) = \sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{\text{fdes}^z(\pi)} r^{\text{rix}(\pi)} q^{\text{ai}(\pi)}. \quad (2.41)$$

Proof. By considering the position of the letter of π with maximal absolute value, we can show that the right-hand side of (2.41) satisfies the same recurrence relation (2.40) and initial condition as $A_n^{(l)}(t, r, q)$. The discussion is quite similar to the proof of Theorem 1.4.2 and is omitted. \square

By setting $r = 1$ in (2.41), we have $A_n^{(l)}(t, q) = \sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{\text{fdes}^z(\pi)} q^{\text{ai}(\pi)}$. Another statistic whose joint distribution with fdes^z is the same as that of ai will be discussed in next section (see Corollary 2.5.2).

2.5 Rawlings major index for colored permutations

2.5.1 Rawlings major index and colored Eulerian quasisymmetric functions

Definition 2.8 (Colored Rawlings major index). For $\pi \in C_l \wr \mathfrak{S}_n$ and $k \in [n]$, we define

$$\begin{aligned} \text{DES}_{\geq k}(\pi) &:= \{i \in [n] : |\pi_i| > |\pi_{i+1}| \text{ and either } \epsilon_i \neq 0 \text{ or } |\pi_i| - |\pi_{i+1}| \geq k\}, \\ \text{inv}_{<k}(\pi) &:= |\{(i, j) \in [n] \times [n] : i < j, \epsilon_i = 0 \text{ and } 0 < |\pi_i| - |\pi_j| < k\}|, \\ \text{maj}_{\geq k}(\pi) &:= \sum_{i \in \text{DES}_{\geq k}(\pi)} i. \end{aligned}$$

Then the *Rawlings major index* of π is defined as

$$\text{rmaj}_k(\pi) := \text{maj}_{\geq k}(\pi) + \text{inv}_{<k}(\pi).$$

For example, if $\pi = 2^0 6^1 1^0 5^0 4^1 3^1 7^0 \in C_2 \wr \mathfrak{S}_7$, then $\text{DES}_{\geq 2}(\pi) = \{2, 4\}$, $\text{inv}_{<2}(\pi) = 2$, $\text{maj}_{\geq 2}(\pi) = 2 + 4 = 6$ and so $\text{rmaj}_2(\pi) = 6 + 2 = 8$. Note that when $l = 1$, rmaj_k is the k -major index studied by Rawlings [68].

Let $Q_{n,k,\vec{\beta}}$ be the *colored Eulerian quasisymmetric functions* defined by

$$Q_{n,k,\vec{\beta}} := \sum_{\vec{\alpha}} Q_{n,k,\vec{\alpha},\vec{\beta}}.$$

The main result of this section is the following interpretation of $Q_{n,k,\vec{\beta}}$.

Theorem 2.5.1. *We have*

$$Q_{n,k,\vec{\beta}} = \sum_{\substack{\text{inv}_{<2}(\pi)=k \\ \text{c\ddot{o}l}(\pi)=\vec{\beta}}} F_{n,\text{DES}_{\geq 2}(\pi)}.$$

It follows from Theorem 2.5.1 and Eq. (2.7) that

$$\sum_{\substack{\text{inv}_{<2}(\pi)=k \\ \text{c\ddot{o}l}(\pi)=\vec{\beta}}} F_{n,\text{DES}_{\geq 2}(\pi)} = \sum_{\substack{\text{exc}(\pi)=k \\ \text{c\ddot{o}l}(\pi)=\vec{\beta}}} F_{n,\text{DEX}(\pi)}.$$

By Lemma 2.2.2 and Eq. (2.2), if we apply \mathbf{ps} to both sides of the above equation, we will obtain the following new interpretation of the colored q -Eulerian polynomial $A_n^{(l)}(t, q)$.

Corollary 2.5.2. *Let $s^{\vec{\beta}} = s_1^{\beta_1} \cdots s_{l-1}^{\beta_{l-1}}$ for $\vec{\beta} \in \mathbb{N}^{l-1}$. Then*

$$\sum_{\pi \in C_{l|\mathfrak{S}_n}} t^{\text{exc}(\pi)} q^{\text{maj}(\pi)} s^{\text{c\ddot{o}l}(\pi)} = \sum_{\pi \in C_{l|\mathfrak{S}_n}} t^{\text{inv}_{<2}(\pi)} q^{\text{rmaj}_2(\pi)} s^{\text{c\ddot{o}l}(\pi)}.$$

Consequently,

$$A_n^{(l)}(t, q) = \sum_{\pi \in C_{l|\mathfrak{S}_n}} t^{l \cdot \text{inv}_{<2}(\pi) + \sum_{i=1}^n \epsilon_i} q^{\text{maj}_{\geq 2}(\pi)} = \sum_{\pi \in C_{l|\mathfrak{S}_n}} t^{\text{fdes}^z(\pi)} q^{\text{maj}_{\geq 2}(\pi^{-1})}.$$

Remark 2.10. When $l = 1$, the above result reduces to Theorem 1.6.1.

In view of interpretation (2.41), we have the following colored version of Problem 1.1.

Problem 2.1. *Can we describe a statistic, denoted “fix₂”, equidistributed with “fix” on colored permutations, so that*

$$A_n^{(l)}(t, r, q) = \sum_{\pi \in C_{l|\mathfrak{S}_n}} t^{\text{fdes}^z(\pi)} r^{\text{fix}_2(\pi)} q^{\text{maj}_{\geq 2}(\pi^{-1})}?$$

2.5.2 Proof of Theorem 2.5.1: Chromatic quasisymmetric functions

Let G be a graph with vertex set $[n]$ and edge set $E(G)$. A *coloring* of a graph G is a function $\kappa : [n] \rightarrow \mathbb{P}$ such that whenever $\{i, j\} \in E(G)$ we have $\kappa(i) \neq \kappa(j)$. Given a function $\kappa : [n] \rightarrow \mathbb{P}$, set

$$\mathbf{x}_\kappa := \prod_{i \in [n]} x_{\kappa(i)}.$$

Shareshian and Wachs [77] generalized Stanley's Chromatic symmetric function of G to the *Chromatic quasisymmetric function* of G as

$$X_G(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}_G(\kappa)} \mathbf{x}_{\kappa},$$

where the sum is over all colorings κ and

$$\text{asc}_G(\kappa) := |\{\{i, j\} \in E(G) : i < j \text{ and } \kappa(i) > \kappa(j)\}|.$$

Recall that an *orientation* of G is a directed graph \mathfrak{o} with the same vertices, so that for every edge $\{i, j\}$ of G , exactly one of (i, j) or (j, i) is an edge of \mathfrak{o} . An orientation is often regarded as giving a direction to each edge of an undirected graph. Let P be a poset. Define

$$X_P = X_P(\mathbf{x}) := \sum_{\sigma} \mathbf{x}_{\sigma},$$

summed over all strict order-reversing maps $\sigma : P \rightarrow \mathbb{P}$ (i.e. if $s <_P t$, then $\sigma(s) > \sigma(t)$). Let \mathfrak{o} be an acyclic orientation of G and κ a coloring. We say that κ is *\mathfrak{o} -compatible* if $\kappa(i) < \kappa(j)$ whenever (j, i) is an edge of \mathfrak{o} . Every proper coloring is compatible with exactly one acyclic orientation \mathfrak{o} , viz., if $\{i, j\}$ is an edge of G with $\kappa(i) < \kappa(j)$, then let (j, i) be an edge of \mathfrak{o} . Thus, if $K_{\mathfrak{o}}$ denotes the set of \mathfrak{o} -compatible colorings of G , and if K_G denotes the set of all colorings of G , then we have a disjoint union $K_G = \cup_{\mathfrak{o}} K_{\mathfrak{o}}$. Hence, $X_G = \sum_{\mathfrak{o}} X_{\mathfrak{o}}$, where $X_{\mathfrak{o}} = \sum_{\kappa \in K_{\mathfrak{o}}} \mathbf{x}_{\kappa}$. Since \mathfrak{o} is acyclic, it induces a poset $\bar{\mathfrak{o}}$: make i less than j if (i, j) is an edge of \mathfrak{o} and then take the transitive closure of this relation. By the definition of X_P for a poset and of $X_{\mathfrak{o}}$ for an acyclic orientation, we have $X_{\bar{\mathfrak{o}}} = X_{\mathfrak{o}}$. Also, according to the definition of $\text{asc}_G(\kappa)$ for a *\mathfrak{o} -compatible* coloring κ , $\text{asc}_G(\kappa)$ depends only on \mathfrak{o} , that is,

$$\text{asc}_G(\kappa) = \text{asc}_G(\kappa') \quad \text{for any } \kappa, \kappa' \in K_{\mathfrak{o}}.$$

Thus, we can define $\text{asc}_G(\mathfrak{o})$ of an acyclic orientation \mathfrak{o} by

$$\text{asc}_G(\mathfrak{o}) := \text{asc}_G(\kappa) \quad \text{for any } \kappa \in K_{\mathfrak{o}}.$$

So

$$X_G(\mathbf{x}, t) = \sum_{\mathfrak{o}} t^{\text{asc}_G(\mathfrak{o})} X_{\bar{\mathfrak{o}}}, \tag{2.42}$$

summed over all acyclic orientations of G .

We have the following reciprocity theorem for chromatic quasisymmetric functions, which is a refinement of Stanley [79, Theorem 4.2].

Theorem 2.5.3 (Reciprocity theorem). *Let G be a graph on $[n]$. Define*

$$\overline{X}_G(\mathbf{x}, t) = \sum_{(\mathfrak{o}, \kappa)} t^{\text{asc}_G(\mathfrak{o})} \mathbf{x}_\kappa,$$

summed over all pairs (\mathfrak{o}, κ) where \mathfrak{o} is an acyclic orientation of G and κ is a function $\kappa : [n] \rightarrow \mathbb{P}$ satisfying $\kappa(i) \leq \kappa(j)$ if (i, j) is an edge of \mathfrak{o} . Then

$$\overline{X}_G(\mathbf{x}, t) = \omega X_G(\mathbf{x}, t),$$

where ω is the involution defined in Section 2.1.1.

Proof. For a poset P define

$$\overline{X}_P = \sum_{\sigma} x_{\sigma},$$

summed over all order-preserving mappings $\sigma : P \rightarrow \mathbb{P}$, i.e., if $s <_P t$ then $\sigma(s) \leq \sigma(t)$. The reciprocity theorem for P -partitions [80, Theorem 4.5.4] implies that

$$\omega X_P = \overline{X}_P.$$

Now, apply ω to Eq. (2.42). We get

$$\omega X_G(\mathbf{x}, t) = \sum_{\mathfrak{o}} t^{\text{asc}_G(\mathfrak{o})} \omega X_{\overline{\mathfrak{o}}} = \sum_{\mathfrak{o}} t^{\text{asc}_G(\mathfrak{o})} \overline{X}_{\overline{\mathfrak{o}}},$$

where \mathfrak{o} summed over all acyclic orientations of G . Hence $\overline{X}_G(\mathbf{x}, t) = \omega X_G(\mathbf{x}, t)$, as desired. \square

For $\pi \in \mathfrak{S}_n$, the G -inversion number of π is

$$\text{inv}_G(\pi) := |\{(i, j) : i < j, \pi(i) > \pi(j) \text{ and } \{\pi(i), \pi(j)\} \in E(G)\}|.$$

For $\pi \in \mathfrak{S}_n$ and P a poset on $[n]$, the P -descent set of π is

$$\text{DES}_P(\pi) := \{i \in [n-1] : \pi(i) >_P \pi(i+1)\}.$$

Define the *incomparability graph* $\text{inc}(P)$ of a poset P on $[n]$ to be the graph with vertex set $[n]$ and edge set $\{\{a, b\} : a \not\leq_P b \text{ and } b \not\leq_P a\}$.

Shareshian and Wachs [77, Theorem 4.15] stated the following fundamental quasisymmetric function basis decomposition of the chromatic quasisymmetric functions, which refines the result of Chow [16, Corollary 2].

Theorem 2.5.4 (Shareshian–Wachs). *Let G be the incomparability graph of a poset P on $[n]$. Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{inv}_G(\pi)} F_{n, \text{DES}_P(\pi)}.$$

Proof of Theorem 2.5.1. Let $\text{bar}(B)$ be the number of barred letters of a colored banner B . By Lemma 2.2.5 we have

$$Q_{n,k,\vec{\beta}} = \sum_B \text{wt}(B), \quad (2.43)$$

where the sum is over all colored banners B with $\text{bar}(B) = k$ and $\vec{\text{col}}(B) = \vec{\beta}$.

Let $c = c_1c_2 \dots c_n$ be a word of length n over $\{0\} \cup [l-1]$. Define $P_{n,k}^c$ to be the poset on vertex set $[n]$ such that $i <_P j$ in $P_{n,k}^c$ if and only if $i < j$ and either $c_i \neq 0$ or $j - i \geq k$. Let $G_{n,k}^c$ be the incomparability graph of $P_{n,k}^c$ (see Fig. 2.1). Clearly,

$$\overline{X}_{G_{n,2}^c}(\mathbf{x}, t) = \sum_B \text{wt}(B)t^{\text{bar}(B)}, \quad (2.44)$$

where the sum is over all colored banners B such that $B(i)$ is c_i -colored. Theorems 2.5.3



Figure 2.1: The graph $G_{9,2}^c$ with $c = 010020121$.

and 2.5.4 together gives

$$\overline{X}_{G_{n,2}^c}(\mathbf{x}, t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{inv}_{G_{n,2}^c}(\pi)} F_{n, \text{DES}_{P_{n,2}^c}(\pi)}.$$

Comparing with (2.44) we get

$$\sum_{\pi \in \mathfrak{S}_n} t^{\text{inv}_{G_{n,2}^c}(\pi)} F_{n, \text{DES}_{P_{n,2}^c}(\pi)} = \sum_B \text{wt}(B)t^{\text{bar}(B)}, \quad (2.45)$$

where the sum is over all colored banners B such that $B(i)$ is c_i -colored.

For each $\pi = \pi_1 \dots \pi_n \in C_l \wr \mathfrak{S}_n$, if $c = c_1c_2 \dots c_n$ is defined by the identification

$$\{1^{c_1}, 2^{c_2}, \dots, n^{c_n}\} = \{\pi_1, \pi_2, \dots, \pi_n\},$$

then we can verify that

$$\text{inv}_{<k}(\pi) = \text{inv}_{G_{n,k}^c}(|\pi|) \quad \text{and} \quad \text{DES}_{\geq k}(\pi) = \text{DES}_{P_{n,k}^c}(|\pi|). \quad (2.46)$$

Therefore, by (2.45) we have

$$\sum_{\substack{\pi \in C_l \wr \mathfrak{S}_n \\ \vec{\text{col}}(\pi) = \vec{\beta}}} t^{\text{inv}_{<k}(\pi)} F_{n, \text{DES}_{\geq k}(\pi)} = \sum_B \text{wt}(B)t^{\text{bar}(B)},$$

where the sum is over all colored banners B with $\vec{\text{col}}(B) = \vec{\beta}$. This finishes the proof of the theorem in view of the interpretation of $Q_{n,k,\vec{\beta}}$ in (2.43). \square

2.5.3 New Mahonian statistics on colored permutations

A statistic “st” on the colored permutation group $C_l \wr \mathfrak{S}_n$ is called *Mahonian* if

$$\sum_{\pi \in C_l \wr \mathfrak{S}_n} q^{\text{st}(\pi)} = [l]_q [2l]_q \cdots [nl]_q.$$

Adin and Roichman [1] introduced the *flag major index* of a colored permutation π , denoted $\text{fmaj}(\pi)$ and defined as

$$\text{fmaj}(\pi) := l \cdot \text{maj}(\pi) + \sum_{i=1}^n \epsilon_i.$$

It is known (cf. [28]) that the flag major index is Mahonian. Note that

$$\sum_{\pi \in C_l \wr \mathfrak{S}_n} t^{\text{fexc}(\pi)} r^{\text{fix}(\pi)} q^{\text{fmaj}(\pi)} = A_n^{(l)}(tq, r, q^l). \quad (2.47)$$

When $l = 1$, Rawlings [68] proved that rmaj_k is Mahonian for each k . Define the *flag Rawlings major index* of $\pi \in C_l \wr \mathfrak{S}_n$, $\text{fmaj}_k(\pi)$, by

$$\text{fmaj}_k(\pi) := l \cdot \text{rmaj}_k(\pi) + \sum_{i=1}^n \epsilon_i.$$

We should note that $\text{fmaj} \neq \text{fmaj}_1$ if $l \geq 2$. It follows from Eq. (2.47) and Corollary 2.5.2 that fmaj_2 is equidistributed with fmaj on colored permutation groups and thus is also Mahonian. More general, we have the following result.

Theorem 2.5.5. *For any $l, k \geq 1$ the flag Rawlings major index fmaj_k is Mahonian.*

Proof. A poset P on $[n]$ satisfying the following conditions:

- (1) $x <_P y$ implies $x <_{\mathbb{N}} y$;
- (2) if the disjoint union (or direct sum) $\{x <_P z\} + \{y\}$ is an induced subposet of P then $x <_{\mathbb{N}} y <_{\mathbb{N}} z$;

is called a *natural unit interval order*. It was observed in [77] that if P is a natural unit interval order, then by a result of Kasraoui [54, Theorem 1.8],

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}_{\text{inc}(P)}(\pi) + \text{maj}_P(\pi)} = [1]_q \cdots [n]_q \quad (2.48)$$

with $\text{maj}_P(\pi) := \sum_{i \in \text{DES}_P(\pi)} i$. Denote by W_n^l the set of all words of length n over $\{0\} \cup [l-1]$. For each $c = c_1 \cdots c_n$ in W_n^l , let $P_{n,k}^c$ and $G_{n,k}^c$ be the poset and the graph defined in the proof of Theorem 2.5.1, respectively. Note that $P_{n,k}^c$ is a natural unit interval order. Thus

$$\begin{aligned} \sum_{\pi \in C_l \wr \mathfrak{S}_n} q^{\text{fmaj}_k} &= \sum_{\pi \in C_l \wr \mathfrak{S}_n} q^{l(\text{rmaj}_{\geq k}(\pi) + \text{inv}_{<k}(\pi)) + \sum_{i=1}^n \epsilon_i} \\ &= \sum_{c \in W_n^l} \sum_{\pi \in \mathfrak{S}_n} q^{l(\text{inv}_{G_{n,k}^c}(\pi) + \text{maj}_{P_{n,k}^c}(\pi)) + \sum_i c_i} \quad (\text{by (2.46)}) \\ &= \sum_{c \in W_n^l} q^{\sum_i c_i} [1]_{q^l} \cdots [n]_{q^l} \quad (\text{by (2.48)}) \\ &= (1 + q + \cdots + q^{l-1})^n [1]_{q^l} \cdots [n]_{q^l} = [l]_q [2l]_q \cdots [nl]_q, \end{aligned}$$

which completes the proof of the theorem. \square

Remark 2.11. For each $\pi \in C_l \wr \mathfrak{S}_n$

$$\text{fmaj}_n(\pi) = l \left(\text{inv}_{<n}(\pi) + \sum_{i \in \text{DES}_{\geq n}(\pi)} i \right) + \sum_i \epsilon_i,$$

where

$$\text{inv}_{<n}(\pi) = |\{(i, j) : \epsilon_i = 0, i < j, \text{ and } |\pi_i| > |\pi_j|\}|$$

and

$$\text{DES}_{\geq n}(\pi) = \{i : \epsilon_i \neq 0 \text{ and } |\pi_i| > |\pi_{i+1}|\}.$$

Foata–Han [36] introduced a colored version of the inversion number called *flag inversion number* as:

$$\text{finv}(\pi) := \sum_{\substack{1 \leq i < j \\ 0 \leq \epsilon \leq l-1}} \chi(|\pi_i|^{\epsilon_i + \epsilon} > \pi_j) + \sum_i \epsilon_i,$$

where the operation $\epsilon_i + \epsilon$ is in the cyclic group C_l . The statistic “finv” is also a Mahonian statistic (see [28]). Clearly, the statistic “fmaj_n” is exactly the statistic of inversion number on \mathfrak{S}_n . However, for $l \geq 2$ the statistic “fmaj_n” is different with the flag inversion number on $C_l \wr \mathfrak{S}_n$.

Chapter 3

Descent polynomials of signed multipermutations

3.1 Introduction

For each sequence of positive integers $\mathbf{s} = \{s_i\}_{i \geq 1}$ let $\mathfrak{J}_n^{\mathbf{s}}$ be the set of \mathbf{s} -inversion sequences of length n defined as

$$\mathfrak{J}_n^{(\mathbf{s})} := \{(e_1, \dots, e_n) \in \mathbb{Z}^n : 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

The *ascent set* of an \mathbf{s} -inversion sequence $\mathbf{e} = (e_1, \dots, e_n) \in \mathfrak{J}_n^{\mathbf{s}}$ is the set

$$\text{Asc}(\mathbf{e}) := \left\{ 0 \leq i < n : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

with the convention that $e_0 = 0$ and $s_0 = 1$. Let $\text{asc}(\mathbf{e}) := |\text{Asc}(\mathbf{e})|$ be the *ascent statistic* on $\mathbf{e} \in \mathfrak{J}_n^{\mathbf{s}}$. The \mathbf{s} -inversion sequences and the ascent statistic were introduced by Savage and Schuster in [72]. When $\mathbf{s} = (1, 2, 3, \dots)$, the set of \mathbf{s} -inversion sequences $\mathfrak{J}_n^{\mathbf{s}}$ is known as the set of *inversion tables* (or *Lehmer codes*) of length n .

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a word of length n with letters from \mathbb{Z} . A *descent* of π is an index $i \in \{0, 1, \dots, n-1\}$ such that $\pi_i > \pi_{i+1}$ (with the convention that $\pi_0 = 0$). Denote by $\text{DES}(\pi)$ the set of descents of π and by $\text{des}(\pi)$ the number of descents of π . The *major index* of π , denoted $\text{maj}(\pi)$, is defined as

$$\text{maj}(\pi) := \sum_{i \in \text{DES}(\pi)} i.$$

In this chapter, permutations or signed permutations of a multiset are viewed as words on \mathbb{Z} .

It is well known that the number of descents on permutations of $[n]$ has the same distribution as the number of ascents on inversion tables of length n ; see [72, Lemma 1] for multivariate equidistribution. Let

$$P(\{1, 1, 2, 2, \dots, n, n\})$$

be the set of permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$. The following connection between multiset permutations and special inversion sequences was shown in [73, Theorem 3.23].

Theorem 3.1.1 (Savage–Visontai [73]).

$$\sum_{\pi \in P(\{1, 1, 2, 2, \dots, n, n\})} t^{\text{des}(\pi)} = \sum_{\mathbf{e} \in \mathfrak{I}_{2n}^{(1, 1, 3, 2, \dots, 2n-1, n)}} t^{\text{asc}(\mathbf{e})}.$$

Now we introduce the signed multiset permutations as follows: let

$$P^\pm(\{1, 1, 2, 2, \dots, n, n\})$$

be the set of all *signed* permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$, whose elements are those of the form $\pm\pi_1 \pm \pi_2 \cdots \pm \pi_{2n}$ with

$$\pi_1 \pi_2 \cdots \pi_{2n} \in P(\{1, 1, 2, 2, \dots, n, n\}).$$

For convenience, write $-n$ by \bar{n} for each positive integer n . For example,

$$P^\pm(\{1, 1\}) = \{1\,1, 1\,\bar{1}, \bar{1}\,1, \bar{1}\,\bar{1}\}.$$

Clearly,

$$|P^\pm(\{1, 1, 2, 2, \dots, n, n\})| = 2^n (2n)! = |\mathfrak{I}_{2n}^{(1, 4, 3, 8, \dots, 2n-1, 4n)}|.$$

Savage and Visontai [73, Conjecture 3.25] further conjectured the following equidistribution, which was proved very recently (and independently) by Chen et al. [13] using type B P -Partitions.

Theorem 3.1.2. *For any $n \geq 1$, we have*

$$\sum_{\pi \in P^\pm(\{1, 1, 2, 2, \dots, n, n\})} t^{\text{des}(\pi)} = \sum_{\mathbf{e} \in \mathfrak{I}_{2n}^{(1, 4, 3, 8, \dots, 2n-1, 4n)}} t^{\text{asc}(\mathbf{e})}. \quad (3.1)$$

For $n \leq 2$, we have

$$\sum_{\pi \in P^\pm(\{1, 1\})} t^{\text{des}(\pi)} = 1 + 3t = \sum_{\mathbf{e} \in \mathfrak{I}_2^{(1, 4)}} t^{\text{asc}(\mathbf{e})}$$

and

$$\sum_{\pi \in P^\pm(\{1,1,2,2\})} t^{\text{des}(\pi)} = 1 + 31t + 55t^2 + 9t^3 = \sum_{\mathbf{e} \in \mathfrak{J}_4^{(1,4,3,8)}} t^{\text{asc}(\mathbf{e})}.$$

In Section 3.2, we will give a simple proof of Theorem 3.1.2 by verifying that both sides of (3.1) satisfy the same recurrence formula. Motivated by this conjecture, our purpose is to study the descent polynomial of signed permutations of a general multiset (or called general signed multipermutations for brevity). In Section 3.3, we prove a factorial generating function formula for the $(\text{des}, \text{fmaj})$ -enumerator of general signed multipermutations, which generalizes a result of Chow and Gessel [17]. In particular, a different proof of Theorem 3.1.2 is also provided. In Section 3.4, we use the factorial formula to show that the descent polynomial of the signed multipermutations is real-rooted. Moreover, another different descent polynomial of the signed multipermutations is also shown to be real-rooted. In Section 3.5, we extend some multivariate identities due to Beck and Braun [7] from permutations to (signed) multipermutations. We end this chapter with some related open questions.

3.2 Proof of Theorem 3.1.2

Lemma 3.2.1. *Let $\mathbf{s} = (1, 4, 3, 8, \dots, 2n - 1, 4n)$ and*

$$E_n(t) = \sum_{i=0}^{2n-1} E_{n,i} t^i := \sum_{\mathbf{e} \in \mathfrak{J}_{2n}^{\mathbf{s}}} t^{\text{asc}(\mathbf{e})}.$$

Then,

$$E_{n+1,i} = (2i^2 + 3i + 1)E_{n,i} + (2i(4n - 2i + 3) + 2n + 1)E_{n,i-1} + (2n + 2 - i)(4n - 2i + 5)E_{n,i-2}, \quad (3.2)$$

with boundary conditions $E_{n,i} = 0$ for $i < 0$ or $i > 2n - 1$.

Proof. The following formula was established in [72, Theorem 13] using Ehrhart theory:

$$\frac{E_n(t)}{(1-t)^{2n+1}} = \sum_{k \geq 0} ((k+1)(2k+1))^n t^k. \quad (3.3)$$

Thus we have

$$\begin{aligned}
\frac{E_{n+1}(t)}{(1-t)^{2n+3}} &= \sum_{k \geq 0} ((k+1)(2k+1))^{n+1} t^k \\
&= \sum_{k \geq 0} ((k+1)(2k+1))^n (k+1)(2k+1) t^k \\
&= \sum_{k \geq 0} ((k+1)(2k+1))^n (2k(k-1) + 5k + 1) t^k \\
&= 2t^2 (E_n(t)(1-t)^{-2n-1})'' + 5t (E_n(t)(1-t)^{-2n-1})' + \frac{E_n(t)}{(1-t)^{2n+1}}.
\end{aligned}$$

Multiplying both sides by $(1-t)^{2n+3}$ gives

$$\begin{aligned}
E_{n+1}(t) &= 2t^2 [E_n''(t) + [(8n+4)t^2(1-t) + 5t(1-t)^2] E_n'(t) \\
&\quad + [(8n^2+2n)t^2 + (10n+3)t + 1] E_n(t)].
\end{aligned}$$

Extracting the coefficients of t^i in both sides we have

$$\begin{aligned}
E_{n+1,i} &= 2i(i-1)E_{n,i} - 4(i-1)(i-2)E_{n,i-1} + 2(i-2)(i-3)E_{n,i-2} \\
&\quad + 5iE_{n,i} + (8n-6)(i-1)E_{n,i-1} + (1-8n)(i-2)E_{n,i-2} \\
&\quad + E_{n,i} + (10n+3)E_{n,i-1} + (8n^2+2n)E_{n,i-2},
\end{aligned}$$

which becomes (3.2) after simplification. \square

Lemma 3.2.2. *Let*

$$P_n(t) = \sum_{i=0}^{2n-1} P_{n,i} t^i := \sum_{\pi \in P^\pm(\{1,1,2,2,\dots,n,n\})} t^{\text{des}(\pi)}.$$

Then,

$$\begin{aligned}
P_{n+1,i} &= (2i^2 + 3i + 1)P_{n,i} + (2i(4n - 2i + 3) + 2n + 1)P_{n,i-1} \\
&\quad + (2n + 2 - i)(4n - 2i + 5)P_{n,i-2}, \quad (3.4)
\end{aligned}$$

with boundary conditions $P_{n,i} = 0$ for $i < 0$ or $i > 2n - 1$.

Proof. Denote by $\mathcal{P}_{n,i}$ the set of signed permutations of $\{1, 1, 2, 2, \dots, n, n\}$ with i descents. Clearly, every signed permutation in $\mathcal{P}_{n+1,i}$ can be obtained from a signed permutation in $\mathcal{P}_{n,i}$, $\mathcal{P}_{n,i-1}$ or $\mathcal{P}_{n,i-2}$ by inserting $\{n+1, n+1\}$, $\{\overline{n+1}, \overline{n+1}\}$ or $\{n+1, \overline{n+1}\}$.

For each $\pi \in \mathcal{P}_{n,i}$, there are $i+1$ positions in π that inserting $n+1$ will not produce an extra descent. These are the i descent positions of π plus the position

to the right of π . Differently, we can only insert $\overline{n+1}$ to the i descent positions of π that may not produce an extra descent. Therefore, there are $\binom{i+1}{2}$ ways to insert $\{n+1, n+1\}$, $\binom{i}{2}$ ways to insert $\{\overline{n+1}, \overline{n+1}\}$ and $(i+1)i$ ways to insert $\{n+1, \overline{n+1}\}$ into π to let it become a signed permutation in $\mathcal{P}_{n+1,i}$. So there are $(2i^2 + 3i + 1)P_{n,i}$ signed permutations in $\mathcal{P}_{n+1,i}$ constructed from $\mathcal{P}_{n,i}$.

Similarly, there are $(2i(4n-2i+3)+2n+1)P_{n,i-1}$ and $(2n+2-i)(4n-2i+5)P_{n,i-2}$ signed permutations in $\mathcal{P}_{n+1,i}$ constructed from $\mathcal{P}_{n,i-1}$ and $\mathcal{P}_{n,i-2}$, respectively.

Summarizing all the above three cases, we obtain (3.4). \square

By Lemma 3.2.1 and 3.2.2, we see that $E_{n,i}$ and $P_{n,i}$ satisfy the same recurrence relation and boundary conditions, so they are equal. This finishes the proof of Theorem 3.1.2.

3.3 General signed multipermutations

In this section, we consider the descent polynomial on signed permutations of the general multiset $M_{\mathbf{m}} := \{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$ for any $\mathbf{m} := (m_1, m_2, \dots, m_n) \in \mathbb{P}^n$. Let $P(\mathbf{m})$ and $P^\pm(\mathbf{m})$ denote the set of all permutations and signed permutations of the multiset $M_{\mathbf{m}}$, respectively.

The following result is attributed to MacMahon [60, Volme 2, p. 211], whose proof can be found in [40, Section 7].

Theorem 3.3.1. *For every $\mathbf{m} \in \mathbb{P}^n$ with $m_1 + \dots + m_n = m$, we have*

$$\frac{\sum_{\pi \in P(\mathbf{m})} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{(t; q)_{m+1}} = \sum_{k \geq 0} \begin{bmatrix} m_1 + k \\ m_1 \end{bmatrix}_q \dots \begin{bmatrix} m_n + k \\ m_n \end{bmatrix}_q t^k. \quad (3.5)$$

Our signed version is:

Theorem 3.3.2. *For every $\mathbf{m} \in \mathbb{P}^n$ with $m_1 + \dots + m_n = m$, we have*

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)} q^{\text{fmaj}(\pi)} z^{\text{neg}(\pi)}}{(t; q^2)_{m+1}} = \sum_{k \geq 0} \prod_{r=1}^n \left(\sum_{i=0}^{m_r} (zq)^i \begin{bmatrix} m_r - i + k \\ m_r - i \end{bmatrix}_{q^2} \begin{bmatrix} i + k - 1 \\ i \end{bmatrix}_{q^2} \right) t^k, \quad (3.6)$$

where $\text{fmaj}(\pi) := 2 \text{maj}(\pi) + \text{neg}(\pi)$ and $\text{neg}(\pi)$ is the number of negative signs in π . In particular,

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)} q^{\text{fmaj}(\pi)}}{(t; q^2)_{m+1}} = \sum_{k \geq 0} \prod_{r=1}^n \begin{bmatrix} m_r + 2k \\ m_r \end{bmatrix}_q t^k. \quad (3.7)$$

Remark 3.1. Setting $z = 0$, we recover Theorem 3.3.1. It is worth noticing that Foata and Han [37, Theorem 1.2] have calculated (another signed version of Theorem 3.3.1) the factorial generating function formula for

$$\sum_{\pi \in P^{\pm}(\mathbf{m})} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)} z^{\text{neg}(\pi)},$$

involving the so-called *flag descent statistic* fdes , $\text{fdes}(\pi) := 2 \text{des}(\pi) - \chi(\pi_1 < 0)$, on signed multipermutations (or words).

Corollary 3.3.3 (Chow–Gessel [17]).

$$\frac{\sum_{\pi \in P^{\pm}(\{1, 2, \dots, n\})} t^{\text{des}(\pi)} q^{\text{fmaj}(\pi)}}{(t; q^2)_{n+1}} = \sum_{k \geq 0} ([2k + 1]_q)^n t^k.$$

Proof. Setting $m_1 = \dots = m_n = 1$ and $z = 1$ in Theorem 3.3.2. \square

Corollary 3.3.4.

$$\frac{\sum_{\pi \in P^{\pm}(\{1, 1, 2, 2, \dots, n, n\})} t^{\text{des}(\pi)}}{(1 - t)^{2n+1}} = \sum_{k \geq 0} ((k + 1)(2k + 1))^n t^k. \quad (3.8)$$

Proof. Setting $m_1 = \dots = m_n = 2$ and $z = q = 1$ in Theorem 3.3.2. \square

Remark 3.2. Comparing (3.8) with (3.3) we get another proof of Theorem 3.1.2.

We will prove Theorem 3.3.2 by using the technique of barred permutations inspired by Gessel and Stanley [48]. For each $\pi = \pi_1 \cdots \pi_m \in P^{\pm}(\mathbf{m})$ we call the space between π_i and π_{i+1} the i -th space of π for $0 < i < m$. We also call the space before π_1 and the space after π_m the 0-th space and the m -th space of π , respectively. If $i \in \text{DES}(\pi)$, then we call the i -th space a descent space.

Definition 3.1 (Barred permutations). A *barred permutation* on $\pi \in P^{\pm}(\mathbf{m})$ is obtained by inserting one or more vertical bars into some spaces of π such that there is at least one bar in every descent space of π . For example, $|\bar{1}|2\bar{2}|1|$ is a barred permutation on $\pi = \bar{1}221$ but $\bar{1}|2\bar{2}|1|$ is not, since $0 \in \text{DES}(\pi)$ and there is no bar in the 0-th space (i.e. before $\pi_1 = \bar{1}$).

Proof of Theorem 3.3.2. Let $B(\mathbf{m})$ be the set of barred permutations on $P^{\pm}(\mathbf{m})$. Let $\sigma \in B(\mathbf{m})$ be a barred permutation on π with b_i bars in the i -th space of π . We define the weight $\text{wt}(\sigma)$ to be

$$\text{wt}(\sigma) := t^{\sum b_i} q^{\text{neg}(\pi) + 2 \sum i b_i} z^{\text{neg}(\pi)}.$$

For example, $\text{wt}(|\bar{1}|2\bar{2}|1|) = t^6 q^{26} z^2$. Now, we count the barred permutations in $B(\mathbf{m})$ by the weight “wt” in two different ways. First, fix a permutation $\pi \in P^\pm(\mathbf{m})$, and sum over all barred permutations on π . Then, fix the number of bars k , and sum over all barred permutations with k bars.

Fix a permutation π a barred permutation on π can be obtained by inserting one bar in each descent space and then inserting any number of bars in all spaces. So counting all the barred permutations on π by the weight “wt” gives

$$t^{\text{des}(\pi)} q^{\text{fmaj}(\pi)} z^{\text{neg}(\pi)} \sum_{i \geq 0} t^i \sum_{i \geq 0} (tq^2)^i \cdots \sum_{i \geq 0} (tq^{2m})^i,$$

which is equal to $\frac{t^{\text{des}(\pi)} q^{\text{fmaj}(\pi)} z^{\text{neg}(\pi)}}{(t; q^2)_{m+1}}$. This shows that

$$\sum_{\sigma \in B(\mathbf{m})} \text{wt}(\sigma) = \frac{\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)} q^{\text{fmaj}(\pi)} z^{\text{neg}(\pi)}}{(t; q^2)_{m+1}}. \tag{3.9}$$

For a fixed integer $k \geq 0$ let $B_k(\mathbf{m})$ be the set of all barred permutations in $B(\mathbf{m})$ with k bars. Now, each barred permutation in $B_k(\mathbf{m})$ can be constructed by putting k bars in one line and then inserting m_r integers from $\{r, -r\}$, for $1 \leq r \leq n$, to the $k + 1$ spaces between each two adjacency bars (including the space in the left side and the right side), with the rule that

- all integers between two adjacency bars are in increasing order;
- negative integers can not be inserted to the left side of all bars.

Observe that inserting a positive (resp. negative) integer to the $(j+1)$ th space (from right to left) of two adjacency bars would contribute the factor q^{2j} (resp. $(qz)q^{2j}$) to the weight of the resulting barred permutation. Thus, by the well-known interpretation (cf. [40, Proposition 4.1]) of the q -binomial coefficient

$$\begin{bmatrix} n+r \\ n \end{bmatrix}_q = \sum_{0 \leq c_1 \leq c_2 \leq \dots \leq c_n \leq r} q^{\sum c_i}, \tag{3.10}$$

we have

$$\sum_{\sigma \in B_k(\mathbf{m})} \text{wt}(\sigma) = \prod_{r=1}^n \left(\sum_{i=0}^{m_r} (zq)^i \begin{bmatrix} m_r - i + k \\ m_r - i \end{bmatrix}_{q^2} \begin{bmatrix} i + k - 1 \\ i \end{bmatrix}_{q^2} \right) t^k.$$

Summing over all k in the above equation and comparing with Eq. (3.9) we get (3.6). By setting $z = 1$ in (3.6) and using the q -binomial theorem

$$\frac{1}{(t; q)_k} = \sum_{i \geq 0} \begin{bmatrix} k - 1 + i \\ i \end{bmatrix}_q t^i,$$

we obtain (3.7). □

3.4 Signed versions of Simion's result about real-rootedness

It was shown in [73, Theorem 1.1] that the ascent polynomial $\sum_{\mathbf{e} \in \mathcal{J}_{2n}^{(\mathbf{s})}} t^{\text{asc}(\mathbf{e})}$ has only real roots for each $\mathbf{s} \in \mathbb{P}^n$. In view of Theorem 3.1.2 we have:

Corollary 3.4.1. *The polynomial*

$$\sum_{\pi \in P^{\pm}(\{1,1,2,2,\dots,n,n\})} t^{\text{des}(\pi)}$$

has only real roots for any positive integer n .

Simion [78, Section 2] proved that the descent polynomial on the permutations of a general multiset has only real roots.

Theorem 3.4.2 (Simion [78]). *The descent polynomial $\sum_{\pi \in P(\mathbf{m})} t^{\text{des}(\pi)}$ has only real roots for every $\mathbf{m} \in \mathbb{P}^n$.*

We have the following signed version of Simion's result, which generalizes the $m_1 = m_2 = \dots = m_n = 1$ (i.e., the type B Coxeter group) case of Brenti [8] and Corollary 3.4.1.

Theorem 3.4.3. *The descent polynomial $\sum_{\pi \in P^{\pm}(\mathbf{m})} t^{\text{des}(\pi)}$ has only real roots for every $\mathbf{m} \in \mathbb{P}^n$.*

The key point of the proof of the above result lies in the following simple lemma.

Lemma 3.4.4. *Let*

$$\frac{F_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} f(k)(ak+b)t^k \quad \text{and} \quad \frac{F_{n-1}(t)}{(1-t)^n} = \sum_{k \geq 0} f(k)t^k. \quad (3.11)$$

If $a > 0$, $n > \frac{b}{a}$, $F_{n-1}(t)$ is a real-rooted polynomial with nonnegative coefficients and $F_n(t)$ is a polynomial with nonnegative coefficients, then $F_n(t)$ has only real roots.

Proof. Clearly, by (3.11) we have

$$\frac{F_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} f(k)(ak+b)t^k = b \frac{F_{n-1}(t)}{(1-t)^n} + a \left(\frac{F_{n-1}(t)}{(1-t)^n} \right)' t.$$

From the above equation we deduce that

$$F_n(t) = ((an-b)t+b) F_{n-1}(t) + at(1-t)F'_{n-1}(t).$$

The lemma then follows by applying a result of Brenti [8, Theorem 2.4.5], which was established through some standard argument by using Rolle's theorem. \square

Proof of Theorem 3.4.3. By setting $q = 1, z = 1$ in Eq. (3.7), we have

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)}}{(1-t)^{m+1}} = \sum_{k \geq 0} \prod_{r=1}^n \frac{(2k+1)(2k+2) \cdots (2k+m_r)}{m_r!} t^k. \quad (3.12)$$

We proceed by induction on $m = m_1 + m_2 + \cdots + m_n$ using the above formula.

When $m = 1$, then $\sum_{\pi \in P^\pm(\{1\})} t^{\text{des}(\pi)} = 1 + t$ and the result is true. Suppose that $m \geq 2$ and the result is true for $m - 1$. Let $\mathbf{m} - \mathbf{e}_1 := \{1^{m_1-1}, 2^{m_2}, \dots, n^{m_n}\}$. By Eq. (3.12) we have

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)}}{(1-t)^{m+1}} = \sum_{k \geq 0} f(k) \frac{(2k+m_1)}{m_1} t^k \quad \text{and} \quad \frac{\sum_{\pi \in P^\pm(\mathbf{m}-\mathbf{e}_1)} t^{\text{des}(\pi)}}{(1-t)^m} = \sum_{k \geq 0} f(k) t^k,$$

where $f(k) = \frac{(2k+1)(2k+2) \cdots (2k+m_1-1)}{(m_1-1)!} \prod_{r=2}^n \frac{(2k+1)(2k+2) \cdots (2k+m_r)}{m_r!}$. Now we can check that all the conditions of Lemma 3.4.4 are satisfied (with $n = m, a = \frac{2}{m_1}, b = 1$):

$$a = \frac{2}{m_1} > 0, \quad n = m > \frac{m_1}{2} = \frac{b}{a},$$

the polynomial $\sum_{\pi \in P^\pm(\mathbf{m}-\mathbf{e}_1)} t^{\text{des}(\pi)}$ is real-rooted with nonnegative coefficients by induction hypothesis and $\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)}$ is a polynomial with nonnegative coefficients. This implies that $\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)}$ is real-rooted, which completes the proof of the theorem. \square

Remark 3.3. Note that the above approach is also available for Theorem 3.4.2 about the real-rootedness of $\sum_{\pi \in P(\mathbf{m})} t^{\text{des}(\pi)}$.

The following interesting properties of the descent polynomial of signed multi-permutations can be obtained by combining Theorem 3.4.3 and Lemma 2.4.6.

Corollary 3.4.5. *The descent polynomial $\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)}$ is log-concave and unimodal for each $\mathbf{m} \in \mathbb{P}^n$.*

The following is another signed version of Simion's result.

Theorem 3.4.6. *The flag descent polynomial $\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{fdes}(\pi)}$ has only real roots for every $\mathbf{m} \in \mathbb{P}^n$. In particular, every flag descent polynomial of signed multipermutations is log-concave and unimodal.*

Proof. By [37, Theorem 1.2], we can derive that

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{fdes}(\pi)}}{(1-t)(1-t^2)^m} = \sum_{k \geq 0} \binom{m_1+k}{m_1} \cdots \binom{m_n+k}{m_n} t^k. \quad (3.13)$$

On the other hand, by setting $q = 1$ in Theorem 3.3.1 we have

$$\frac{\sum_{\pi \in P(\mathbf{m})} t^{\text{des}(\pi)}}{(1-t)^{m+1}} = \sum_{k \geq 0} \binom{m_1+k}{m_1} \cdots \binom{m_n+k}{m_n} t^k.$$

Comparing with Eq. (3.13), we obtain

$$\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{fdes}(\pi)} = (1+t)^m \sum_{\pi \in P(\mathbf{m})} t^{\text{des}(\pi)}.$$

It follows from the above relation and Theorem 3.4.2 that the flag descent polynomial $\sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{fdes}(\pi)}$ is real-rooted. \square

Remark 3.4. For $m_1 = m_2 = \cdots = m_n = 1$, the above result was already known to Mongelli [64].

3.5 Multivariate identities

The following well-known identity is due to Carlitz [11]:

$$\frac{\sum_{\pi \in \mathfrak{S}_m} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{(t; q)_{m+1}} = \sum_{k \geq 0} ([k+1]_q)^m t^k, \quad (3.14)$$

where \mathfrak{S}_m is the symmetric group of order m . Let z_0, z_1, z_2, \dots be a sequence of variables. Recently, Beck and Braun [7, Theorem 4.1] obtained the following multivariate identity that generalizes Carlitz's identity:

$$\frac{\sum_{\pi \in \mathfrak{S}_m} \prod_{i \in \text{DES}(\pi)} z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i})} = \sum_{k \geq 0} \prod_{i=1}^m [k+1]_{z_i} z_0^k. \quad (3.15)$$

This identity, as well as many type B versions, were established via the machine of polyhedral geometry.

Remark 3.5. As was pointed out in [7], the approach of polyhedral geometry is related to Stanley's theory of P -partitions and Eq. (3.15) resembles Corollary 7.1 in [82]. The technique of barred permutations is also closely related to the theory of P -partitions, as both were foreshadowed by work of MacMahon [60].

Note that Eq. (3.14) is the special case $m_1 = m_2 = \cdots = m_r = 1$ of MacMahon's identity in Theorem 3.3.1. Our technique of barred permutations used in the proof of Theorem 3.3.2 has the advantage that enables us to extend multivariate identity (3.15) to the multipermutations easily as follows.

Theorem 3.5.1.

$$\frac{\sum_{\pi \in P(\mathbf{m})} \prod_{i \in \text{DES}(\pi)} z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i})} = \sum_{k \geq 0} \begin{bmatrix} m_1 + k \\ m_1 \end{bmatrix}_{z_1} \cdots \begin{bmatrix} m_n + k \\ m_n \end{bmatrix}_{z_n} z_0^k. \quad (3.16)$$

The proof of the above identity, together with other multivariate generalizations for signed multipermutations involving the descents and the flag descents, will be given in this section.

Proof of Theorem 3.5.1. As $P(\mathbf{m}) \subseteq P^\pm(\mathbf{m})$, the definition of barred permutations on $\pi \in P(\mathbf{m})$ is valid. We will count the set of all barred permutations on $P(\mathbf{m})$, denoted $A(\mathbf{m})$, according to the weight

$$\text{wt}_A(\sigma) = \prod_{i=0}^m (z_0 z_{\pi_1} \cdots z_{\pi_i})^{b_i}$$

for any barred permutation σ on π with b_i bars in the i -th space of π , in two different ways similarly as in the proof of Theorem 3.3.2.

Fix a permutation π , a barred permutation on π can be obtained by inserting one bar in each descent space and then inserting any number of bars in all spaces. So counting all the barred permutations on π by the weight wt_A gives

$$\left(\sum_{k \geq 0} z_0^k \right) \left(\sum_{k \geq 0} (z_0 z_{\pi_1})^k \right) \cdots \left(\sum_{k \geq 0} (z_0 z_{\pi_1} \cdots z_{\pi_m})^k \right) \prod_{i \in \text{DES}(\pi)} z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i},$$

that is

$$\frac{\prod_{i \in \text{DES}(\pi)} z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i})}.$$

Therefore, we have

$$\sum_{\sigma \in A(\mathbf{m})} \text{wt}_A(\sigma) = \frac{\sum_{\pi \in P(\mathbf{m})} \prod_{i \in \text{DES}(\pi)} z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1} z_{\pi_2} \cdots z_{\pi_i})}. \quad (3.17)$$

Fix an integer $k \geq 0$ and let $A_k(\mathbf{m})$ be the set of barred permutations in $A(\mathbf{m})$ with k bars. Now each barred permutation in $A_k(\mathbf{m})$ can be constructed by putting k bars in one line and then inserting m_r copies of r , for $1 \leq r \leq n$, to the $k+1$ spaces between each two adjacency bars (including the left and right spaces). Note that all integers inserted to the same space must in increasing order and inserting an integer i in the $(j+1)$ th spaces (from right to left) would contribute the factor

z_i^j to the weight of the resulting barred permutation. Thus by the interpretation of q -binomial coefficients (3.10) we have

$$\sum_{\sigma \in A(\mathbf{m})} \text{wt}_A(\sigma) = \begin{bmatrix} m_1 + k \\ m_1 \end{bmatrix}_{z_1} \cdots \begin{bmatrix} m_n + k \\ m_n \end{bmatrix}_{z_n} z_0^k.$$

Comparing with (3.17) we get (3.16). \square

Setting $z_0 = t$ and $z_1 = z_2 = \cdots = z_m = q$ in (3.16) we get (3.5). Also, when $m_1 = m_2 = \cdots = m_r = 1$ identity (3.16) becomes the Beck–Braun identity (3.15). We continue to generalize other Beck–Braun identities of type B to signed multi-permutations. For any $\pi = \pi_1 \cdots \pi_m \in P^\pm(\mathbf{m})$, introduce the *sign change function* $\delta^\pi : \{1, 2, \dots, m\} \rightarrow \{0, 1\}^m$ as

$$\delta^\pi(i) = \delta_i^\pi := 1 \text{ if } \pi_i \pi_{i+1} < 0 \text{ or } i = m \text{ and } \pi_m < 0, \text{ otherwise } 0.$$

Define $\text{ch}(\pi) := \sum_j \delta_j^\pi$ to be the *total sign change* in π . For example, if $\pi = 1\bar{3}\bar{2}\bar{2}\bar{3}\bar{1}\bar{1}$ then $\delta^\pi = 1011001$ and $\text{ch}(\pi) = 4$. Recall that the flag descent of π is

$$\text{fdes}(\pi) = 2 \text{des}(\pi) - \chi(\pi_1 < 0).$$

We begin with the following relationships between the sign function and the statistics of flag descent, descent and flag major index.

Lemma 3.5.2. *For each $\pi \in P^\pm(\mathbf{m})$ we have*

$$\text{fdes}(\pi) = \sum_{\substack{j \in \text{DES}(\pi) \\ \delta_j^\pi = 0}} 2 + \text{ch}(\pi), \quad (3.18)$$

$$\text{des}(\pi) = \sum_{\substack{j \in \text{DES}(\pi) \\ \delta_j^\pi = 0}} 1 + \lceil \text{ch}(\pi)/2 \rceil, \quad (3.19)$$

$$\text{fmaj}(\pi) = \sum_{\substack{j \in \text{DES}(\pi) \\ \delta_j^\pi = 0}} 2j + \sum_{j=1}^m \delta_j^\pi j. \quad (3.20)$$

Proof. A descent j of π can arise in three cases:

- *zero descent:* $j = 0$ and $\pi_1 < 0$, or
- *sign change descent:* $\delta_j^\pi = 1$ and $\pi_j > \pi_{j+1}$, or
- *standard descent:* $\delta_j^\pi = 0$ and $\pi_j > \pi_{j+1}$.

Regarding sign change descents, consider the partial sums $\sum_{j=k}^m \delta_j^\pi$ of sign changes. By induction on $(m - k)$, we can show that

$$\sum_{j=k}^m \delta_j^\pi = \chi(\pi_k < 0) + 2|\{i \geq k : i \text{ is a sign change descent of } \pi\}|, \quad (3.21)$$

from which we obtain (3.18) and (3.19). Also, we have

$$\begin{aligned} \sum_{j=1}^m \delta_j^\pi j &= \sum_{k=1}^m \sum_{j=k}^m \delta_j^\pi \\ &= \sum_{k=1}^m \left(\chi(\pi_k < 0) + 2|\{i \geq k : i \text{ is a sign change descent of } \pi\}| \right) \\ &= \text{neg}(\pi) + \sum_{k=1}^m 2|\{i \geq k : i \text{ is a sign change descent of } \pi\}| \\ &= \text{neg}(\pi) + \sum_{\substack{j \in \text{DES}(\pi) \\ \delta_j^\pi = 1}} 2j, \end{aligned}$$

which shows (3.20). □

In the following, we deal with the multivariate identities involving flag descents and descents separately. Let $z_0, z_1, z_{\bar{1}}, z_2, z_{\bar{2}}, \dots$ be a sequence of variables.

3.5.1 An identity involving flag descents

We introduce the flag version of barred permutations.

Definition 3.2 (Flag barred permutations). For $\pi \in P^\pm(\mathbf{m})$ a *flag barred permutation* on π is obtained from π by inserting bars such that

- every j -th descent space with $\delta_j^\pi = 0$ of π receives at least 2 bars;
- the parity of the number of bars in the j -th space ($1 \leq j \leq m$) of π has the same parity as δ_j^π .

For example, $|\bar{1}|\bar{2}2|1|$ is a barred permutation on $\pi = \bar{1}\bar{2}21$ but not a flag barred permutation, the flag barred permutation on π with the least number of bars is $\bar{1}||2|2||1$.

Lemma 3.5.3. Let $\text{NDW}_m(r)$ be the set of all non-decreasing words of length m with letters from $\{0, 1, \dots, r\}$. For each $b \in \text{NDW}_m(r)$ let $\text{tot}(b)$, $\text{tot}^e(b)$, $\text{tot}^o(b)$

and $\text{odd}(b)$ denote the sum of all letters, the sum of all even letters, the sum of all odd letters and the number of odd letters in b , respectively. Then,

$$\sum_{b \in \text{NDW}_m(r)} p^{\text{tot}^e(b)} q^{\text{tot}^o(b)} z^{\text{odd}(b)} = \sum_{i=0}^m (zq)^i \begin{bmatrix} \lfloor (r-1)/2 \rfloor + i \\ i \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor r/2 \rfloor + m - i \\ m - i \end{bmatrix}_{p^2}.$$

Proof. Let $b = b_1 \dots b_m \in \text{NDW}_m(r)$. Let $1 \leq i_1 < \dots < i_k \leq m$ (resp. $1 \leq j_1 < \dots < j_l \leq m$) be the sequence of all integers i (resp. j) such that b_i is even (resp. odd). Then, b is completely characterized by the pair (b^e, b^o) , where $b^e := (b_{i_1}/2) \dots b_{i_k}/2$ and $b^o := ((b_{j_1} - 1)/2) \dots ((b_{j_l} - 1)/2)$. Moreover, $\text{tot}^e(b) = 2\text{tot}(b^e)$, $\text{tot}^o(b) = 2\text{tot}(b^o) + |b^o|$ and $\text{odd}(b) = |b^o|$. Hence, by the interpretation of q -binomial coefficients (3.10) we have

$$\begin{aligned} & \sum_{b \in \text{NDW}_m(r)} p^{\text{tot}^e(b)} q^{\text{tot}^o(b)} z^{\text{odd}(b)} \\ &= \sum_{i=0}^m \left((zq)^i \sum_{b^o \in \text{NDW}_i(\lfloor (r-1)/2 \rfloor)} q^{2\text{tot}(b^o)} \right) \left(\sum_{b^e \in \text{NDW}_{m-i}(\lfloor r/2 \rfloor)} p^{2\text{tot}(b^e)} \right) \\ &= \sum_{i=0}^m (zq)^i \begin{bmatrix} \lfloor (r-1)/2 \rfloor + i \\ i \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor r/2 \rfloor + m - i \\ m - i \end{bmatrix}_{p^2}, \end{aligned}$$

as desired. \square

Theorem 3.5.4.

$$\begin{aligned} & \frac{\sum_{\pi \in P^\pm(\mathbf{m})} z^{\text{neg}(\pi)} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0^2 z_{\pi_1}^2 \dots z_{\pi_i}^2 \prod_{i=1}^m z_0^{\delta_i^\pi} z_{\pi_1}^{\delta_i^\pi} \dots z_{\pi_i}^{\delta_i^\pi}}{(1 - z_0) \prod_{i=1}^m (1 - z_0^2 z_{\pi_1}^2 \dots z_{\pi_i}^2)} \\ &= \sum_{k \geq 0} z_0^k \prod_{r=1}^n \sum_{i=0}^{m_r} (z z_{\bar{r}})^i \begin{bmatrix} \lfloor (k-1)/2 \rfloor + i \\ i \end{bmatrix}_{z_{\bar{r}}^2} \begin{bmatrix} \lfloor k/2 \rfloor + m_r - i \\ m_r - i \end{bmatrix}_{z_{\bar{r}}^2}. \end{aligned} \quad (3.22)$$

In particular, if $z_r = z_{\bar{r}}$ for all $r \geq 1$, then

$$\begin{aligned} & \frac{\sum_{\pi \in P^\pm(\mathbf{m})} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0^2 z_{\pi_1}^2 \dots z_{\pi_i}^2 \prod_{i=1}^m z_0^{\delta_i^\pi} z_{\pi_1}^{\delta_i^\pi} \dots z_{\pi_i}^{\delta_i^\pi}}{(1 - z_0) \prod_{i=1}^m (1 - z_0^2 z_{\pi_1}^2 \dots z_{\pi_i}^2)} = \sum_{k \geq 0} z_0^k \prod_{r=1}^n \begin{bmatrix} m_r + k \\ m_r \end{bmatrix}_{z_r}. \end{aligned} \quad (3.23)$$

Proof. For each flag barred permutation σ on π with b_i bars in i -th space of π , we define the weight

$$\text{wt}_F(\sigma) = z^{\text{neg}(\pi)} \prod_{i=0}^m (z_0 z_{\pi_1} \cdots z_{\pi_i})^{b_i}.$$

Let $F(\mathbf{m})$ be the set of all flag barred permutations on $P^\pm(\mathbf{m})$. As previously, we count $F(\mathbf{m})$ by the weight wt_F twice.

Fix $\pi \in P^\pm(\mathbf{m})$. The flag barred permutation on π with the least number of bars, denoted by $\bar{\sigma}$, has the weight

$$\text{wt}_F(\bar{\sigma}) = z^{\text{neg}(\pi)} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0^2 z_{\pi_1}^2 z_{\pi_2}^2 \cdots z_{\pi_i}^2 \prod_{i=1}^m z_0^{\delta_i^\pi} z_{\pi_1}^{\delta_i^\pi} z_{\pi_2}^{\delta_i^\pi} \cdots z_{\pi_i}^{\delta_i^\pi}.$$

As every flag barred permutation on π can be obtained from $\bar{\sigma}$ by further inserting any number of bars in the 0-th space of π and an even number of bars in the i -th (for $1 \leq i \leq m$) space of π , we see that counting all the flag barred permutations on π according to the weight wt_F gives

$$\frac{z^{\text{neg}(\pi)} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0^2 z_{\pi_1}^2 z_{\pi_2}^2 \cdots z_{\pi_i}^2 \prod_{i=1}^m z_0^{\delta_i^\pi} z_{\pi_1}^{\delta_i^\pi} z_{\pi_2}^{\delta_i^\pi} \cdots z_{\pi_i}^{\delta_i^\pi}}{(1 - z_0) \prod_{i=1}^m (1 - z_0^2 z_{\pi_1}^2 z_{\pi_2}^2 \cdots z_{\pi_i}^2)}.$$

Therefore, the enumerative polynomial $\sum_{\sigma \in F(\mathbf{m})} \text{wt}_F(\sigma)$ equals the left-hand side of (3.22).

Next, for a fixed integer $k \geq 0$, let $F_k(\mathbf{m})$ be the set of flag barred permutations in $F(\mathbf{m})$ with k bars. Now, each flag barred permutation from $F_k(\mathbf{m})$ can be obtained in two steps. Firstly, we put k bars in one line and insert m_r copies of r , for $1 \leq r \leq n$, into the spaces between bars. This yields a permutation from $P(\mathbf{m})$ with k bars, which is not necessarily a flag barred permutation. Secondly, we determine the signs of all the integers and the order of the integers between each pair of two adjacency bars in the resulting object in such a way that it becomes a flag barred permutation. The signs and the orders of integers are unique according to the definition of a flag barred permutation, since

- all integers between two adjacency bars have the same sign and are in increasing order;
- the sign of an integer in the $(j + 1)$ th space (from right to left) of the $k + 1$ spaces between the k bars is “+” (resp. “−”) if j is even (resp. odd).

Thus, each way of inserting in the first step induces one and only one flag barred permutation. Moreover, inserting an integer r in the $(j+1)$ -th space (from right to left) of the $k+1$ spaces between the k bars will contribute the factor z_r^j (resp. $z_{\bar{r}}^j$) to the weight of the resulting flag barred permutation if j is even (resp. odd). Therefore, by Lemma 3.5.3 we have

$$\sum_{\sigma \in F_k(\mathbf{m})} \text{wt}_F(\sigma) = z_0^k \prod_{r=1}^n \sum_{i=0}^{m_r} (z z_{\bar{r}})^i \begin{bmatrix} \lfloor (k-1)/2 \rfloor + i \\ i \end{bmatrix}_{z_r^2} \begin{bmatrix} \lfloor k/2 \rfloor + m_r - i \\ m_r - i \end{bmatrix}_{z_r^2}.$$

Summing over all k and comparing with the fact that $\sum_{\sigma \in F(\mathbf{m})} \text{wt}_F(\sigma)$ equals the left-hand side of (3.22), we get (3.22). \square

Remark 3.6. For $z = 0$, identity (3.22) yields (3.16). Setting $m_1 = \dots = m_n = 1$ in (3.23), we recover Corollary 6.4 in [7]. By (3.18) and (3.20), if we set $z_0 = t$ and $z_1 = z_2 = \dots = z_n = q$ in (3.23), then we obtain

$$\frac{(1+t) \sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)}}{(t^2; q^2)_m} = \sum_{k \geq 0} \prod_{r=1}^n \begin{bmatrix} m_r + k \\ m_r \end{bmatrix}_q t^k,$$

which appears as identity (7.3) in [37].

3.5.2 An identity involving descents

Theorem 3.5.5.

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} z_0^{\text{neg}(\pi)} z_0^{\lceil \frac{\text{ch}(\pi)}{2} \rceil} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2 \prod_{i=1}^m z_{\pi_1}^{\delta_i^\pi} \cdots z_{\pi_i}^{\delta_i^\pi}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2)} = \sum_{k \geq 0} z_0^k \prod_{r=1}^n \left(\sum_{i=0}^{m_r} (z z_{\bar{r}})^i \begin{bmatrix} i + k - 1 \\ i \end{bmatrix}_{z_r^2} \begin{bmatrix} m_r - i + k \\ m_r - i \end{bmatrix}_{z_r^2} \right). \quad (3.24)$$

In particular, if $z_r = z_{\bar{r}}$ for all $r \geq 1$, then

$$\frac{\sum_{\pi \in P^\pm(\mathbf{m})} z_0^{\lceil \frac{\text{ch}(\pi)}{2} \rceil} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2 \prod_{i=1}^m z_{\pi_1}^{\delta_i^\pi} \cdots z_{\pi_i}^{\delta_i^\pi}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2)} = \sum_{k \geq 0} z_0^k \prod_{r=1}^n \begin{bmatrix} m_r + 2k \\ m_r \end{bmatrix}_{z_r}. \quad (3.25)$$

Proof. For each barred permutation σ on $\pi \in P^\pm(\mathbf{m})$ with b_i bars in i -th space of π , we define the weight

$$\text{wt}_B(\sigma) = z^{\text{neg}(\pi)} \prod_{i=0}^m (z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2)^{b_i} \prod_{\pi_i < 0} z_{\pi_i}.$$

Fix $\pi \in P^\pm(\mathbf{m})$. Then through similar discussion as in the second paragraph of the proof of Theorem 3.3.2, we can see that counting all barred permutations on π according to the weight wt_B gives

$$\frac{z^{\text{neg}(\pi)} z_0^{\text{des}(\pi)} \prod_{i \in \text{DES}(\pi)} z_{\pi_1}^2 \cdots z_{\pi_i}^2 \prod_{\pi_i < 0} z_{\pi_i}}{\prod_{i=0}^m (1 - z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2)}.$$

We claim that

$$z_0^{\lceil \frac{\text{ch}(\pi)}{2} \rceil} \prod_{\substack{i \in \text{DES}(\pi) \\ \delta_i^\pi = 0}} z_0 z_{\pi_1}^2 \cdots z_{\pi_i}^2 \prod_{i=1}^m z_{\pi_1}^{\delta_i^\pi} \cdots z_{\pi_i}^{\delta_i^\pi} = z_0^{\text{des}(\pi)} \prod_{i \in \text{DES}(\pi)} z_{\pi_1}^2 \cdots z_{\pi_i}^2 \prod_{\pi_i < 0} z_{\pi_i}, \quad (3.26)$$

from which we see that $\sum_{\sigma \in B(\mathbf{m})} \text{wt}_B(\sigma)$ equals the left-hand side of (3.24). Clearly, the degrees of z_0 in both sides of (3.26) are equal follows from (3.19). To complete the proof of our claim, it remains to show that for each $1 \leq i \leq m$, the degrees of z_{π_i} in both sides of (3.26) are equal. By (3.21), the degree of z_{π_i} in left-hand side of (3.26) is

$$\begin{aligned} & 2|\{j \geq i : j \in \text{DES}(\pi), \delta_j^\pi = 0\}| + \sum_{j=i}^m \delta_j^\pi \\ &= 2(|\{j \geq i : j \in \text{DES}(\pi), \delta_j^\pi = 0\}| + |\{j \geq i : j \in \text{DES}(\pi), \delta_j^\pi = 1\}|) + \chi(\pi_i < 0) \\ &= 2|\{j \geq i : j \in \text{DES}(\pi)\}| + \chi(\pi_i < 0), \end{aligned}$$

which is exactly the degree of z_{π_i} in right-hand side of (3.26). This completes the proof of our claim.

Next, for a fixed integer $k \geq 0$, we can show that

$$\sum_{\sigma \in B_k(\mathbf{m})} \text{wt}_B(\sigma) = z_0^k \prod_{r=1}^n \left(\sum_{i=0}^{m_r} (z z_{\bar{r}})^i \begin{bmatrix} m_r - i + k \\ m_r - i \end{bmatrix}_{z_r^2} \begin{bmatrix} i + k - 1 \\ i \end{bmatrix}_{z_r^2} \right)$$

by similar discussions as in the proof of Theorem 3.3.2. Therefore, the enumerative polynomial $\sum_{\sigma \in B(\mathbf{m})} \text{wt}_B(\sigma)$ also equals the right-hand side of (3.24), which finishes the proof of the theorem. \square

Remark 3.7. For $z = 0$, identity (3.24) yields (3.16). Setting $m_1 = \cdots = m_n = 1$ in (3.25), we recover Theorem 6.10 in [7]. By (3.19) and (3.20), if we set $z_0 = t$ and $z_1 = z_{\bar{1}} = \cdots = z_n = z_{\bar{n}} = q$ in (3.24), then we go back to (3.6).

3.6 Some remarks

First of all, it would be interesting to find a bijective proof or a q -analog of the equidistribution (3.1). Probably, a combinatorial interpretation of recurrence (3.2) in terms of the inversion sequences would help us to understand this equidistribution better.

Recall that a polynomial $f(z_1, \dots, z_m) \in \mathbb{R}[z_1, \dots, z_m]$ is said to be *stable*, if whenever $\Im(z_i) > 0$ for all i then f does not vanish. It is well known that the stability of the multivariate generating functions implies that their univariate counterparts, obtained by diagonalization, have only real roots (see [49, Lemma 2.3]). For each $\mathbf{m} \in \mathbb{P}^n$, the three descent polynomials

$$\sum_{\pi \in P(\mathbf{m})} t^{\text{des}(\pi)}, \quad \sum_{\pi \in P^\pm(\mathbf{m})} t^{\text{des}(\pi)} \quad \text{and} \quad \sum_{\pi \in P^\pm(\mathbf{m})} t^{f\text{des}(\pi)}$$

appear in this chapter all have only real roots. It would be interesting to find some multivariate generalizations of the above three descent polynomials, similar to the results for generalized Stirling permutations discovered by Haglund and Visontai [49], which have the nice stable property.

Finally, the multivariate identities appearing in Section 3.5 can be generalized naturally from signed multipermutations to r -colored multipermutations for $r \geq 1$.

Chapter 4

Jacobi–Stirling polynomials and P -Partitions

4.1 Introduction and results

Let $\ell_{\alpha,\beta}[y](t)$ be the Jacobi differential operator:

$$\ell_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^\alpha(1+t)^\beta} \left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t) \right)'$$

It is well known [14, p. 143] that the Jacobi polynomial $y = P_n^{(\alpha,\beta)}(t)$ is an eigenvector for the differential operator $\ell_{\alpha,\beta}$ corresponding to $n(n + \alpha + \beta + 1)$, i.e.,

$$\ell_{\alpha,\beta}[y](t) = n(n + \alpha + \beta + 1)y(t).$$

For each $n \in \mathbb{N}$, the *Jacobi–Stirling numbers* $\text{JS}(n, k; z)$ of the second kind appeared originally as the coefficients in the expansion of the n -th composite power of $\ell_{\alpha,\beta}$ (see [27]):

$$(1-t)^\alpha(1+t)^\beta \ell_{\alpha,\beta}^n[y](t) = \sum_{k=0}^n (-1)^k \text{JS}(n, k; z) \left((1-t)^{\alpha+k}(1+t)^{\beta+k} y^{(k)}(t) \right)^{(k)},$$

where $z = \alpha + \beta + 1$, and can also be defined as the connection coefficients in

$$x^n = \sum_{k=0}^n \text{JS}(n, k; z) \prod_{i=0}^{k-1} (x - i(z + i)). \quad (4.1)$$

The Jacobi–Stirling numbers $\text{js}(n, k; z)$ of the first kind are defined by

$$\prod_{i=0}^{n-1} (x - i(z + i)) = \sum_{k=0}^n \text{js}(n, k; z) x^k. \quad (4.2)$$

When $z = 1$, the Jacobi–Stirling numbers become the *Legendre–Stirling numbers* [26] of the first and second kinds:

$$\text{ls}(n, k) = \text{js}(n, k; 1), \quad \text{LS}(n, k) = \text{JS}(n, k; 1). \quad (4.3)$$

Generalizing the work of Andrews and Littlejohn [4] on Legendre–Stirling numbers, Gelineau and Zeng [43] studied the combinatorial interpretations of the Jacobi–Stirling numbers and remarked on the connection with Stirling numbers and central factorial numbers. Further properties of the Jacobi–Stirling numbers have been given by Andrews, Egge, Gawronski, and Littlejohn [3].

The *Stirling numbers of the second and first kinds* $S(n, k)$ and $s(n, k)$ are defined by

$$x^n = \sum_{k=0}^n S(n, k) \prod_{i=0}^{k-1} (x - i), \quad \prod_{i=0}^{n-1} (x - i) = \sum_{k=0}^n s(n, k) x^k. \quad (4.4)$$

The lesser known *central factorial numbers* [69, p. 213–217] $T(n, k)$ and $t(n, k)$ are defined by

$$x^n = \sum_{k=0}^n T(n, k) x \prod_{i=1}^{k-1} \left(x + \frac{k}{2} - i \right), \quad (4.5)$$

and

$$x \prod_{i=1}^{n-1} \left(x + \frac{n}{2} - i \right) = \sum_{k=0}^n t(n, k) x^k.$$

Starting from the fact that for fixed k , the Stirling number $S(n + k, n)$ can be written as a polynomial in n of degree $2k$ and there exist nonnegative integers $c_{k,j}$, $1 \leq j \leq k$, such that

$$\sum_{n \geq 0} S(n + k, n) t^n = \frac{\sum_{j=1}^k c_{k,j} t^j}{(1 - t)^{2k+1}},$$

Gessel and Stanley [48] gave a similar combinatorial interpretation for the $c_{k,j}$ in terms of the descents in *Stirling permutations*. Recently, Egge [24] has given an analogous result for the Legendre–Stirling numbers, and Gelineau [42] has made a preliminary study of the analogous problem for Jacobi–Stirling numbers. In this chapter, we will prove some analogous results for the diagonal generating function for Jacobi–Stirling numbers. As noticed in [43], the leading coefficient of the polynomial $\text{JS}(n, k; z)$ is $S(n, k)$ and the constant term of $\text{JS}(n, k; z)$ is the

central factorial number of the second kind with even indices $T(2n, 2k)$. Similarly, the leading coefficient of the polynomial $\text{js}(n, k; z)$ is $s(n, k)$ and the constant term of $\text{js}(n, k; z)$ is the central factorial number of the first kind with even indices $t(2n, 2k)$.

Definition 4.1. The *Jacobi–Stirling polynomial* of the second kind is defined by

$$f_k(n; z) := \text{JS}(n + k, n; z). \quad (4.6)$$

The coefficient $p_{k,i}(n)$ of z^i in $f_k(n; z)$ is called the *Jacobi–Stirling coefficient* of the second kind for $0 \leq i \leq k$. Thus

$$f_k(n; z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k. \quad (4.7)$$

The main goal of this chapter is to prove Theorems 4.1.1 and 4.1.2 below.

Theorem 4.1.1. For each integer k and i such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j} t^j$ with positive integer coefficients such that

$$\sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \quad (4.8)$$

In order to give a combinatorial interpretation for $a_{k,i,j}$, we introduce the multiset

$$M_k := \{1, 1, \bar{1}, 2, 2, \bar{2}, \dots, k, k, \bar{k}\},$$

where the elements are ordered by

$$\bar{1} < 1 < \bar{2} < 2 \dots < \bar{k} < k.$$

Let $[\bar{k}] := \{\bar{1}, \bar{2}, \dots, \bar{k}\}$. For any subset $S \subseteq [\bar{k}]$, we set $M_{k,S} = M_k \setminus S$.

Definition 4.2. A permutation π of $M_{k,S}$ is a *Jacobi–Stirling permutation* if whenever $u < v < w$ and $\pi(u) = \pi(w)$, we have $\pi(u) < \pi(v)$. We denote by $\mathcal{JSP}_{k,S}$ the set of Jacobi–Stirling permutations of $M_{k,S}$ and

$$\mathcal{JSP}_{k,i} = \bigcup_{\substack{S \subseteq [\bar{k}] \\ |S|=i}} \mathcal{JSP}_{k,S}.$$

For example, the Jacobi–Stirling permutations of $\mathcal{JSP}_{2,1}$ are:

$$\begin{aligned} & 22\bar{2}11, \bar{2}2211, \bar{2}1221, \bar{2}1122, 221\bar{2}1, 122\bar{2}1, \bar{1}2221, \bar{1}\bar{2}122, 2211\bar{2}, 1221\bar{2}, \\ & 1122\bar{2}, 11\bar{2}22, 2211\bar{1}, 1221\bar{1}, 1122\bar{1}, 11\bar{1}22, 22\bar{1}11, \bar{1}2211, \bar{1}1221, \bar{1}1122. \end{aligned}$$

Let $\pi = \pi_1 \pi_2 \dots \pi_m$ be a word on a totally ordered alphabet. We say that π has a descent at l , where $1 \leq l \leq m-1$, if $\pi_l > \pi_{l+1}$. Let $\text{des}(\pi)$ be the number of descents of π . The following is our main interpretation for the coefficients $a_{k,i,j}$.

Theorem 4.1.2. For $k \geq 1$, $0 \leq i \leq k$, and $1 \leq j \leq 2k-i$, the coefficient $a_{k,i,j}$ is the number of Jacobi–Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j-1$ descents.

4.2 Jacobi–Stirling polynomials

Proposition 4.2.1. *For $0 \leq i \leq k$, the Jacobi–Stirling coefficient $p_{k,i}(n)$ is a polynomial in n of degree $3k - i$. Moreover, the leading coefficient of $p_{k,i}(n)$ is*

$$\frac{1}{3^{k-i} 2^i i! (k-i)!} \quad (4.9)$$

for all $0 \leq i \leq k$.

Proof. We proceed by induction on $k \geq 0$. For $k = 0$, we have $p_{0,0}(n) = 1$ since $f_0(n) = \text{JS}(n, n; z) = 1$. Let $k \geq 1$ and suppose that $p_{k-1,i}$ is a polynomial in n of degree $3(k-1) - i$ for $0 \leq i \leq k-1$. From (4.1) we deduce the recurrence relation:

$$\begin{cases} \text{JS}(0, 0; z) = 1, & \text{JS}(n, k; z) = 0, \text{ if } k \notin \{1, \dots, n\}, \\ \text{JS}(n, k; z) = \text{JS}(n-1, k-1; z) + k(k+z) \text{JS}(n-1, k; z), & \text{for } n, k \geq 1. \end{cases} \quad (4.10)$$

Substituting in (4.6) yields

$$f_k(n; z) - f_k(n-1; z) = n(n+z)f_{k-1}(n; z). \quad (4.11)$$

It follows from (4.7) that for $0 \leq i \leq k$,

$$p_{k,i}(n) - p_{k,i}(n-1) = n^2 p_{k-1,i}(n) + n p_{k-1,i-1}(n). \quad (4.12)$$

Applying the induction hypothesis, we see that $p_{k,i}(n) - p_{k,i}(n-1)$ is a polynomial in n of degree at most

$$\max(3(k-1) - i + 2, 3(k-1) - (i-1) + 1) = 3k - i - 1.$$

Hence $p_{k,i}(n)$ is a polynomial in n of degree at most $3k - i$. It remains to determine the coefficient of n^{3k-i} , say $\beta_{k,i}$. Extracting the coefficient of n^{3k-i-1} in (4.12) we have

$$\beta_{k,i} = \frac{1}{3k-i} (\beta_{k-1,i} + \beta_{k-1,i-1}).$$

Now it is fairly easy to see that (4.9) satisfies the above recurrence. \square

Proposition 4.2.2. *For all $k \geq 1$ and $0 \leq i \leq k$, we have*

$$p_{k,i}(0) = p_{k,i}(-1) = p_{k,i}(-2) = \dots = p_{k,i}(-k) = 0. \quad (4.13)$$

Proof. We proceed by induction on k . By definition, we have

$$f_1(n; z) = \text{JS}(n + 1, n; z) = p_{1,0}(n) + p_{1,1}(n)z.$$

As noticed in [43, Theorem 1], the leading coefficient of the polynomial $\text{JS}(n, k; z)$ is $S(n, k)$ and the constant term is $T(2n, 2k)$. We derive from (4.4) and (4.5) that

$$\begin{aligned} p_{1,1}(n) &= S(n + 1, n) = n(n + 1)/2, \\ p_{1,0}(n) &= T(2n + 2, 2n) = n(n + 1)(2n + 1)/6. \end{aligned}$$

Hence (4.13) is true for $k = 1$. Assume that (4.13) is true for some $k \geq 1$. By (4.12) we have

$$p_{k,i}(n) - p_{k,i}(n - 1) = n^2 p_{k-1,i}(n) + n p_{k-1,i-1}(n).$$

Since $\text{JS}(0, k; z) = 0$ if $k \geq 2$, we have $p_{k,i}(0) = 0$. The above equation and the induction hypothesis imply successively that

$$p_{k,i}(-1) = 0, \quad p_{k,i}(-2) = 0, \quad \dots, \quad p_{k,i}(-k + 1) = 0, \quad p_{k,i}(-k) = 0.$$

The proof is thus complete. \square

Lemma 4.2.3. *For each integer k and i such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j} t^j$ with integer coefficients such that*

$$\sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1 - t)^{3k-i+1}}. \quad (4.14)$$

Proof. By Proposition 4.2.1 and standard results concerning rational generating functions (cf. [83, Corollary 4.3.1]), for each integer k and i such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = a_{k,i,0} + a_{k,i,1}t + \dots + a_{k,i,3k-i}t^{3k-i}$ satisfying (4.14). Now, by [83, Proposition 4.2.3], we have

$$\sum_{n \geq 1} p_{k,i}(-n) t^n = -\frac{A_{k,i}(1/t)}{(1 - 1/t)^{2k-i+1}}. \quad (4.15)$$

Applying (4.13) we see that $a_{k,i,2k-i+1} = \dots = a_{k,i,3k-i} = 0$. \square

The first values of $A_{k,i}(t)$ are given in Table 1. The following result gives a recurrence for the coefficients $a_{k,i,j}$.

Proposition 4.2.4. *Let $a_{0,0,0} = 1$. For $k, i, j \geq 0$, we have the following recurrence for the integers $a_{k,i,j}$:*

$$\begin{aligned} a_{k,i,j} &= j^2 a_{k-1,i,j} + [2(j-1)(3k-i-j-1) + (3k-i-2)] a_{k-1,i,j-1} \\ &\quad + (3k-i-j)^2 a_{k-1,i,j-2} + j a_{k-1,i-1,j} + (3k-i-j) a_{k-1,i-1,j-1}, \end{aligned} \quad (4.16)$$

where $a_{k,i,j} = 0$ if any of the indices k, i, j is negative or if $j \notin \{1, \dots, 2k-i\}$.

Table 4.1: The first values of $A_{k,i}(t)$

$k \setminus i$	0	1	2	3
0	1			
1	$t + t^2$	t		
2	$t + 14t^2 + 21t^3 + 4t^4$	$2t + 12t^2 + 6t^3$	$t + 2t^2$	
3	$t + 75t^2 + 603t^3 + 1065t^4 + 460t^5 + 36t^6$	$3t + 114t^2 + 501t^3 + 436t^4 + 66t^5$	$3t + 55t^2 + 116t^3 + 36t^4$	$t + 8t^2 + 6t^3$

Proof. For $0 \leq i \leq k$, let

$$F_{k,i}(t) = \sum_{n \geq 0} p_{k,i}(n)t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \quad (4.17)$$

The recurrence relation (4.12) is equivalent to

$$F_{k,i}(t) = (1-t)^{-1}[t^2 F''_{k-1,i}(t) + tF'_{k-1,i}(t) + tF'_{k-1,i-1}(t)] \quad (4.18)$$

with $F_{0,0} = (1-t)^{-1}$. Substituting (4.17) into (4.18) we obtain

$$\begin{aligned} A_{k,i}(t) &= (1-t)^{3k-i} [t^2 (A_{k-1,i}(t)(1-t)^{-(3k-i-2)})'' \\ &\quad + t(A_{k-1,i}(t)(1-t)^{-(3k-i-2)})' + t(A_{k-1,i-1}(t)(1-t)^{-(3k-i-1)})'] \\ &= [t^2 A''_{k-1,i}(t)(1-t)^2 + 2(3k-i-2)t^2 A'_{k-1,i}(t)(1-t) \\ &\quad + (3k-i-2)(3k-i-1)t^2 A_{k-1,i}(t)] \\ &\quad + [tA'_{k-1,i}(t)(1-t)^2 + (3k-i-2)tA_{k-1,i}(t)(1-t)] \\ &\quad + [tA'_{k-1,i-1}(t)(1-t) + (3k-i-1)tA_{k-1,i-1}(t)]. \end{aligned}$$

Taking the coefficient of t^j in both sides of the above equation, we have

$$\begin{aligned} a_{k,i,j} &= j(j-1)a_{k-1,i,j} - 2(j-1)(j-2)a_{k-1,i,j-1} + (j-2)(j-3)a_{k-1,i,j-2} \\ &\quad + 2(3k-i-2)(j-1)a_{k-1,i,j-1} - 2(3k-i-2)(j-2)a_{k-1,i,j-2} \\ &\quad + (3k-i-2)(3k-i-1)a_{k-1,i,j-2} + ja_{k-1,i,j} - 2(j-1)a_{k-1,i,j-1} \\ &\quad + (j-2)a_{k-1,i,j-2} + (3k-i-2)a_{k-1,i,j-1} - (3k-i-2)a_{k-1,i,j-2} \\ &\quad + ja_{k-1,i-1,j} - (j-1)a_{k-1,i-1,j-1} + (3k-i-1)a_{k-1,i-1,j-1}, \end{aligned}$$

which gives (4.16) after simplification. \square

Corollary 4.2.5. For $k \geq 0$ and $0 \leq i \leq k$, the coefficients $a_{k,i,j}$ are positive integers for $1 \leq j \leq 2k - i$.

Proof. This follows from (4.16) by induction on k . Clearly, this is true for $k = 0$ and $k = 1$. Suppose that this is true for some $k \geq 1$. As each term in the right-hand side of (4.16) is nonnegative, we only need to show that at least one term on the right-hand side of (4.16) is strictly positive. Indeed, for $k \geq 2$, the induction hypothesis and (4.16) imply that

- if $j = 1$, then $a_{k,i,1} \geq a_{k-1,i-1,1} > 0$;
- if $2 \leq j \leq 2k - i$, then $a_{k,i,j} \geq (3k - i - j)a_{k-1,i-1,j-1} \geq ka_{k-1,i-1,j-1} > 0$.

These two cases cover all possibilities. \square

Theorem 4.1.1 follows then from Lemma 4.2.3, Proposition 4.2.4 and Corollary 4.2.5.

Now, define the *Jacobi-Stirling polynomial of the first kind* $g_k(n; z)$ by

$$g_k(n; z) = \text{js}(n, n - k; z). \quad (4.19)$$

Proposition 4.2.6. *For $k \geq 1$, we have*

$$g_k(n; z) = f_k(-n; -z). \quad (4.20)$$

If we write $g_k(n; z) = q_{k,0}(n) + q_{k,1}(n)z + \cdots + q_{k,k}(n)z^k$, then

$$\sum_{n \geq 1} q_{k,i}(n)t^n = (-1)^k \frac{\sum_{j=1}^{2k-i} a_{k,i,3k-i+1-j} t^j}{(1-t)^{3k-i+1}}. \quad (4.21)$$

Proof. From (4.2) we deduce

$$\begin{cases} \text{js}(0, 0; z) = 1, & \text{js}(n, k; z) = 0, & \text{if } k \notin \{1, \dots, n\}, \\ \text{js}(n, k; z) = \text{js}(n-1, k-1; z) - (n-1)(n-1+z) \text{js}(n-1, k; z), & n, k \geq 1. \end{cases}$$

It follows from the above recurrence and (4.19) that

$$g_k(n; z) - g_k(n-1; z) = -(n-1)(n-1+z)g_{k-1}(n-1; z).$$

Comparing with (4.11) we get (4.20), which implies that $q_{k,i}(n) = (-1)^i p_{k,i}(-n)$. Finally (4.21) follows from (4.15). \square

4.3 Jacobi–Stirling posets

We first recall some basic facts about Stanley’s theory of P -partitions [80, 82]. Let P be a poset, and let ω be a labeling of P , i.e., an injection from P to a totally ordered set (usually a set of integers). A (P, ω) -*partition* (or P -partition if ω is understood) is a function f from P to the positive integers satisfying

1. if $x <_P y$ then $f(x) \leq f(y)$
2. if $x <_P y$ and $\omega(x) > \omega(y)$ then $f(x) < f(y)$.

A *linear extension* of a poset P is an extension of P to a total order. We will identify a linear extension of P labeled by ω with the permutation obtained by taking the labels of P in increasing order with respect to the linear extension. For example, the linear extensions of the poset shown in Figure 4.1 are 213 and 231. We write $\mathcal{L}(P)$ for the set of linear extensions of P (which also depend on the labeling ω).

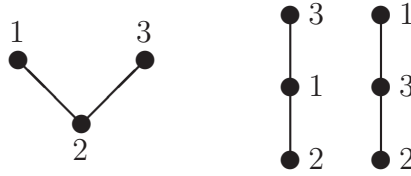


Figure 4.1: A poset (left) and its linear extensions

The *order polynomial* $\Omega_P(n)$ of P is the number of (P, ω) -partitions with parts in $[n] = \{1, 2, \dots, n\}$. It is known that $\Omega_P(n)$ is a polynomial in n whose degree is the number of elements of P . The following is a fundamental result in the P -partition theory [83, Theorem 4.5.14]:

$$\sum_{n \geq 1} \Omega_P(n) t^n = \frac{\sum_{\pi \in \mathcal{L}(P)} t^{\text{des } \pi + 1}}{(1-t)^{k+1}}, \quad (4.22)$$

where k is the number of elements of P and $\text{des } \pi$ is computed according to the natural order of the integers.

For example, the two linear extensions of the poset shown in Figure 4.1 each have one descent, and the order polynomial for this poset is $2 \binom{n+1}{3}$. So equation (4.22) reads

$$\sum_{n \geq 1} 2 \binom{n+1}{3} t^n = \frac{2t^2}{(1-t)^4}.$$

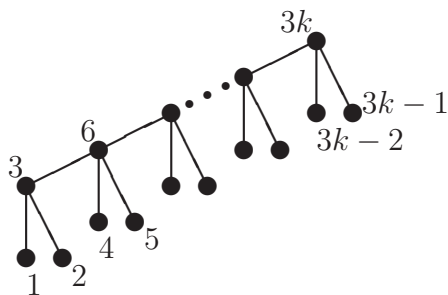


Figure 4.2: The labeled poset R_k .

By (4.10) the Jacobi-Stirling numbers have the generating function

$$\sum_{n \geq 0} \text{JS}(n, k; z) t^n = \frac{t^k}{(1 - (z + 1)t)(1 - 2(z + 2)t) \cdots (1 - k(z + k)t)}, \quad (4.23)$$

As $f_k(n; z) = \text{JS}(n + k, n; z)$, switching n and k in the last equation yields

$$\sum_{k \geq 0} f_k(n; z) t^k = \frac{1}{(1 - (z + 1)t)(1 - 2(z + 2)t) \cdots (1 - n(z + n)t)}.$$

Identifying the coefficients of t^k gives

$$f_k(n; z) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} j_1(z + j_1) \cdot j_2(z + j_2) \cdots j_k(z + j_k). \quad (4.24)$$

For any subset S of $[k]$, we define $\gamma_{S,m}(j)$ by

$$\gamma_{S,m}(j) = \begin{cases} j & \text{if } m \in S, \\ j^2 & \text{if } m \notin S, \end{cases}$$

and define $p_{k,S}(n)$ by

$$p_{k,S}(n) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} \gamma_{S,1}(j_1) \gamma_{S,2}(j_2) \cdots \gamma_{S,k}(j_k). \quad (4.25)$$

For example, if $k = 2$ and $S = \{1\}$ then

$$p_{k,S}(n) = \sum_{1 \leq j_1 \leq j_2 \leq n} j_1 j_2^2 = n(n + 1)(n + 2)(12n^2 + 9n - 1)/120.$$

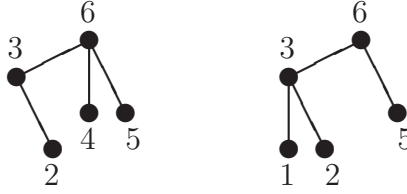


Figure 4.3: The labeled posets $R_{2,\{1\}}$ and $R_{2,\{2\}}$.

Definition 4.3. Let R_k be the labeled poset in Figure 4.2. Let S be a subset of $[k]$. The poset $R_{k,S}$ obtained from R_k by removing the points $3m - 2$ for $m \in S$ is called a *Jacobi–Stirling poset*.

For example, the posets $R_{2,\{1\}}$ and $R_{2,\{2\}}$ are shown in Figure 4.3.

Lemma 4.3.1. For any subset $S \subseteq [k]$, let $A_{k,S}(t)$ be the descent polynomial of $\mathcal{L}(R_{k,S})$, i.e., the coefficient of t^j in $A_{k,S}(t)$ is the number of linear extensions of $R_{k,S}$ with $j - 1$ descents, then

$$\sum_{n \geq 0} p_{k,S}(n)t^n = \frac{A_{k,S}(t)}{(1-t)^{3k-|S|+1}}. \tag{4.26}$$

Proof. It is easy to see that $\Omega_{R_{k,S}}(n) = p_{k,S}(n)$ and the result follows from (4.22). □

For $0 \leq i \leq k$, $R_{k,i}$ is defined as the set of $\binom{k}{i}$ posets

$$R_{k,i} = \{ R_{k,S} \mid S \subseteq [k] \text{ with cardinality } i \}.$$

The posets in $R_{2,1}$ are shown in Figure 4.3. We define $\mathcal{L}(R_{k,i})$ to be the (disjoint) union of $\mathcal{L}(P)$, over all $P \in R_{k,i}$; i.e.,

$$\mathcal{L}(R_{k,i}) = \bigcup_{\substack{S \subseteq [k] \\ |S|=i}} \mathcal{L}(R_{k,S}).$$

Now we are ready to give the first interpretation of the coefficients $a_{k,i,j}$ in the polynomial $A_{k,i}(t)$ defined in (4.14).

Theorem 4.3.2. We have

$$A_{k,i}(t) = \sum_{\substack{S \subseteq [k] \\ |S|=i}} A_{k,S}(t). \tag{4.27}$$

In other words, the integer $a_{k,i,j}$ is the number of elements of $\mathcal{L}(R_{k,i})$ with $j - 1$ descents.

Proof. Extracting the coefficient of z^i in both sides of (4.24), then applying (4.7) and (4.25), we obtain

$$p_{k,i}(n) = \sum_{\substack{S \subseteq [k] \\ |S|=i}} p_{k,S}(n),$$

so that

$$\sum_{n \geq 0} p_{k,i}(n)t^n = \sum_{n \geq 0} \sum_S p_{k,S}(n)t^n = \sum_S \sum_{n \geq 0} p_{k,S}(n)t^n,$$

where the summations on S are over all subsets of $[k]$ with cardinality i . The result follows then by comparing (4.14) and (4.26). \square

It is easy to compute $A_{k,S}(1)$ which is equal to $|\mathcal{L}(R_{k,S})|$ and is also $(3k - i)!$ times the leading coefficient of $p_{k,S}(n)$.

Proposition 4.3.3. *Let $S \subseteq [k]$, $|S| = i$ and let $l_j(S) = |\{s \in S \mid s \leq j\}|$ for $1 \leq j \leq k$. We have*

$$A_{k,S}(1) = \frac{(3k - i)!}{\prod_{j=1}^k (3j - l_j(S))}. \quad (4.28)$$

Proof. We construct a permutation in $\mathcal{L}(R_{k,S})$ by reading the elements of $R_{k,S}$ in increasing order of their labels and inserting each one into the permutation already constructed from the earlier elements. Each element of $R_{k,S}$ will have two natural numbers associated to it: the reading number and the insertion-position number. It is clear that the insertion-position number of $3j$ must be equal to its reading number, which is $3j - l_j(S)$, since it must be inserted to the right of all the previously inserted elements (those with labels less than $3j$). On the other hand, an element not divisible by 3 may be inserted anywhere, so its number of possible insertion positions is equal to its reading number. So the number of possible linear extensions of $R_{k,S}$ is equal to the product of the reading numbers of all elements with labels not divisible by 3. Since the product of all the reading numbers is $(3j - i)!$, we obtain the result by dividing this number by the product of the reading numbers of the elements with labels $3, 6, \dots, 3k$. \square

From (4.28) we can derive the formula for $A_{k,i}(1)$, which is equivalent to Proposition 4.2.1.

Proposition 4.3.4. *We have*

$$|\mathcal{L}(R_{k,i})| = A_{k,i}(1) = \frac{(3k - i)!}{3^{k-i} 2^i i! (k - i)!}.$$

Proof. By Proposition 4.3.3 it is sufficient to prove the identity

$$\sum_{1 \leq s_1 < \dots < s_i \leq k} \frac{(3k-i)!}{\prod_{j=1}^k (3j-l_j(S))} = \frac{(3k-i)!}{3^{k-i} 2^i i! (k-i)!}, \quad (4.29)$$

where $S = \{s_1, \dots, s_i\}$ and $l_j(S) = |\{s \in S : s \leq j\}|$.

The identity is obvious if $S = \emptyset$, i.e., $i = 0$. When $i = 1$, it is easy to see that (4.29) is equivalent to the $a = 2/3$ case of the indefinite summation

$$\sum_{s=0}^{k-1} \frac{(a)_s}{s!} = \frac{(a+1)_{k-1}}{(k-1)!}, \quad (4.30)$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. Since the left-hand side of (4.29) can be written as

$$\sum_{s_i=i}^k \frac{(3k-i)!}{\prod_{j=s_i}^k (3j-i)} \sum_{1 \leq s_1 < \dots < s_{i-1} \leq s_i-1} \frac{1}{\prod_{j=1}^{s_i-1} (3j-l_j(S))},$$

we derive (4.29) from the induction hypothesis and (4.30). \square

Remark 4.1. Alternatively, we may prove the formula for $A_{k,i}(1)$ as follows:

$$\begin{aligned} A_{k,i}(1) &= \sum_{\substack{S \subseteq [k] \\ |S|=i}} A_{k,S}(1) \\ &= \sum_{\substack{S \subseteq [k] \\ |S|=i, k \in S}} A_{k,S}(1) + \sum_{\substack{S \subseteq [k] \\ |S|=i, k \notin S}} A_{k,S}(1) \\ &= (3k-i-1)A_{k-1,i-1}(1) + (3k-i-1)(3k-i-2)A_{k-1,i}(1), \end{aligned}$$

from which we easily deduce that $A_{k,i}(1) = (3k-i)!/3^{k-i} 2^i i! (k-i)!$.

Since both of the above proofs of Proposition 4.3.4 use mathematical induction, it is desirable to have a more conceptual proof. Here we give such a proof based on the fact that Proposition 4.3.4 is equivalent to

$$|\mathcal{L}(R_{k,i})| = 2^{k-i} \cdot \frac{(3k-i)!}{2^i i! 3^{k-i} (k-i)!}. \quad (4.31)$$

A combinatorial proof of Proposition 4.3.4. We show that $|\mathcal{L}(R_{k,i})|$ is equal to 2^{k-i} times the number of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2.

Let S be an i -element subset of $[k]$ and let π be an element $\mathcal{L}(R_{k,S})$, viewed as a bijection from $[3k-i]$ to $R_{k,S}$. Let $\sigma = \pi^{-1}$. Then σ is a natural labeling of $R_{k,S}$, i.e., an order-preserving bijection from the poset $R_{k,S}$ to $[3k-i]$, and conversely, every natural labeling of $R_{k,S}$ is the inverse of an element of $\mathcal{L}(R_{k,S})$.

We will describe a map from the set of natural labelings of elements of $R_{k,i}$ to the set of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2, for which each such partition is the image of 2^{k-i} natural labelings. Given a natural labeling σ of $R_{k,S}$, the blocks of the corresponding partition are the sets $\{\sigma(3m-2), \sigma(3m-1), \sigma(3m)\}$ for $m \notin S$ and the sets $\{\sigma(3m-1), \sigma(3m)\}$ for $m \in S$. We note that since σ is a natural labeling, $\sigma(3m)$ is always the largest element of its block and $\sigma(3) < \sigma(6) < \dots < \sigma(3m)$.

Now let P be a partition of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2. We shall describe all natural labelings σ of posets $R_{k,S}$ that correspond to P under the map just defined. First, we list the blocks of P as B_1, B_2, \dots, B_k in increasing order of their largest elements. Then $\sigma(3m)$ must be the largest element of B_m . If B_m has two elements, then the smaller element must be $\sigma(3m-1)$, and m must be an element of S . If B_m has three elements then $m \notin S$, and $\sigma(3m-2)$ and $\sigma(3m-1)$ are the two smaller elements of B_m , but in either order. Thus S is uniquely determined by P , and there are exactly 2^{k-i} natural labelings of $R_{k,S}$ in the preimage of P . So $|\mathcal{L}(R_{k,i})|$ is 2^i times the number of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and i blocks of size 2, and is therefore equal to the right-hand side of (4.31). \square

4.4 Two proofs of Theorem 4.1.2

We shall give two proofs of Theorem 4.1.2. We first derive Theorem 4.1.2 from Theorem 4.3.2 by constructing a bijection from the linear extensions of Jacobi–Stirling posets to permutations. The second proof consists of verifying that the cardinality of Jacobi–Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j-1$ descents satisfies the recurrence relation (4.16). Given a word $w = w_1 w_2 \dots w_m$ of m letters, we define the j th slot of w by the pair (w_j, w_{j+1}) for $j = 0, \dots, m$. By convention $w_0 = w_{m+1} = 0$. A slot (w_j, w_{j+1}) is called a descent (resp. non-descent) slot if $w_j > w_{j+1}$ (resp. $w_j \leq w_{j+1}$).

4.4.1 First proof of Theorem 4.1.2

For any subset $S = \{s_1, \dots, s_i\}$ of $[k]$ we define $\bar{S} = \{\bar{s}_1, \dots, \bar{s}_i\}$, which is a subset of $[\bar{k}]$. Recall that $\mathcal{JSP}_{k,\bar{S}}$ is the set of Jacobi–Stirling permutations of $M_{k,\bar{S}}$. We construct a bijection $\phi : \mathcal{L}(R_{k,S}) \rightarrow \mathcal{JSP}_{k,\bar{S}}$ such that $\text{des } \phi(\pi) = \text{des } \pi$ for any $\pi \in \mathcal{L}(R_{k,S})$.

If $k = 1$, then $\mathcal{L}(R_{1,0}) = \{123, 213\}$ and $\mathcal{L}(R_{1,1}) = \{23\}$. We define ϕ by

$$\phi(123) = \bar{1}11, \quad \phi(213) = 11\bar{1}, \quad \phi(23) = 11.$$

Suppose that $k \geq 2$ and $\phi : \mathcal{L}(R_{k-1,S}) \rightarrow \mathcal{JSP}_{k-1,\bar{S}}$ is defined for any $S \subseteq [k-1]$. If $\pi \in \mathcal{L}(R_{k,S})$ with $S \subseteq [k]$, we consider the following two cases:

- (i) $k \notin S$, denote by π' the word obtained by deleting $3k$ and $3k-1$ from π , and π'' the word obtained by further deleting $3k-2$ from π' . As $\pi'' \in \mathcal{L}(R_{k-1,S})$, by induction hypothesis, the permutation $\phi(\pi'') \in \mathcal{JSP}_{k-1,\bar{S}}$ is well defined. Now,
 - a) if $3k-2$ is in the r th descent (or nondescent) slot of π'' , then we insert \bar{k} in the r th descent (or nondescent) slot of $\phi(\pi'')$ and obtain a word $\phi_1(\pi'')$;
 - b) if $3k-1$ is in the s th descent (or nondescent) slot of π' , we define $\phi(\pi)$ by inserting kk in the s th descent (or nondescent) slot of $\phi_1(\pi'')$.
- (ii) $k \in S$, denote by π' the word obtained from π by deleting $3k$ and $3k-1$. As $\pi' \in \mathcal{L}(R_{k-1,i-1})$, the permutation $\phi(\pi') \in \mathcal{JSP}_{k-1,\bar{S}}$ is well defined. If $3k-1$ is in the r th descent (or nondescent) slot of π' , we define $\phi(\pi)$ by inserting kk in the r th descent (or nondescent) slot of $\phi(\pi')$.

Clearly this mapping is a bijection and preserves the number of descents. For example, if $k = 3$ and $S = \{2\}$, then $\phi(25137869) = 112\bar{3}233\bar{1}$. This can be seen by applying the mapping ϕ as follows:

$$\begin{aligned} 213 &\rightarrow 25136 \rightarrow 251376 \rightarrow 25137869, \\ 11\bar{1} &\rightarrow 1122\bar{1} \rightarrow 112\bar{3}2\bar{1} \rightarrow 112\bar{3}233\bar{1}. \end{aligned}$$

Clearly we have $\text{des}(25137869) = 2$ and $\text{des}(112\bar{3}233\bar{1}) = 2$.

4.4.2 Second proof of Theorem 4.1.2

Let $\mathcal{JSP}_{k,i,j}$ be the set of Jacobi–Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j-1$ descents. Let $a'_{0,0,0} = 1$ and $a'_{k,i,j}$ be the cardinality of $\mathcal{JSP}_{k,i,j}$ for $k, i, j \geq 0$. By definition, $a'_{k,i,j} = 0$ if any of the indices $k, i, j < 0$ or $j \notin \{1, \dots, 2k-i\}$. We show that $a'_{k,i,j}$'s satisfy the same recurrence (4.16) and initial conditions as $a_{k,i,j}$'s.

Any Jacobi–Stirling permutation of $\mathcal{JSP}_{k,i,j}$ can be obtained from one of the following five cases:

- (i) Choose a Jacobi–Stirling permutation in $\mathcal{JSP}_{k-1,i,j}$, insert \bar{k} and then kk in one of the descent slots (an extra descent at the end of the permutation). Clearly, there are $a'_{k-1,i,j}$ ways to choose the initial permutation, j ways to insert \bar{k} , and j ways to insert kk .

- (ii) Choose a Jacobi–Stirling permutation of $\mathcal{JSP}_{k-1,i,j-1}$,
- 1) insert \bar{k} in a descent slot and then kk in a non-descent slot. In this case, there are $a'_{k-1,i,j-1}$ ways to choose the initial permutation, $j-1$ ways to insert \bar{k} , and $3k-i-j-1$ ways to insert kk .
 - 2) insert \bar{k} in a non-descent slot and then kk in a descent slot. In this case, there are $a'_{k-1,i,j-1}$ ways to choose the initial permutation, $3k-i-j-1$ ways to insert \bar{k} , and j ways to insert kk .
- (iii) Choose a Jacobi–Stirling permutation in $\mathcal{JSP}_{k-1,i,j-2}$, insert \bar{k} and then kk in one of the non-descent slots. In this case, there are $a'_{k-1,i,j-2}$ ways to choose the initial permutation, $3k-i-j$ ways to insert \bar{k} , and $3k-i-j$ ways to insert kk .
- (iv) Choose a Jacobi–Stirling permutation in $\mathcal{JSP}_{k-1,i-1,j}$ and insert kk in one of the descent slots. There are $a'_{k-1,i-1,j}$ ways to choose the initial permutation, and j ways to insert kk .
- (v) Choose a Jacobi–Stirling permutation in $\mathcal{JSP}_{k-1,i-1,j-1}$ and insert kk in one of the non-descent slots. There are $a'_{k-1,i-1,j-1}$ ways to choose the initial permutation, and $3k-i-j$ ways to insert kk .

Summarizing all the above five cases, we obtain

$$\begin{aligned} a'_{k,i,j} &= j^2 a'_{k-1,i,j} + [2(j-1)(3k-i-j-1) + (3k-i-2)] a'_{k-1,i,j-1} \\ &\quad + (3k-i-j)^2 a'_{k-1,i,j-2} + j a'_{k-1,i-1,j} + (3k-i-j) a'_{k-1,i-1,j-1}. \end{aligned}$$

Therefore, the numbers $a'_{k,i,j}$ satisfy the same recurrence and initial conditions as the $a_{k,i,j}$, so they are equal.

4.5 Legendre–Stirling posets

Let P_k be the poset shown in Figure 4.4, called the *Legendre–Stirling poset*. The order polynomial of P_k is given by

$$\begin{aligned} \omega_{P_k}(n) &= \sum_{2 \leq f(2) \leq \dots \leq f(3k-1) \leq n} \prod_{i=1}^k f(3i-1)(f(3i-1)-1) \\ &= [x^k] \frac{1}{(1-2x)(1-6x) \dots (1-(n-1)nx)}, \end{aligned}$$

Second proof. By (4.6), (4.7), and (4.8), we have

$$\sum_{n=0}^{\infty} \text{JS}(n+k, n; z)t^n = \sum_{i=0}^k z^i \frac{\sum_{j=1}^{2k-i} a_{k,i,j} t^j}{(1-t)^{3k-i+1}}.$$

Setting $z = 1$ and using (4.3) gives

$$\sum_{n=0}^{\infty} \text{LS}(n+k, n)t^n = \sum_{i=0}^k (1-t)^i \frac{\sum_{j=1}^{2k-i} a_{k,i,j} t^j}{(1-t)^{3k+1}}.$$

Multiplying both sides by $(1-t)^{3k+1}$ and applying (4.32) gives

$$\sum_{j=1}^{2k-1} b_{k,j} t^j = \sum_{i=0}^k (1-t)^i \sum_{j=1}^{2k-i} a_{k,i,j} t^j,$$

so

$$\sum_{i=0}^k \sum_{l=0}^i (-1)^l \binom{i}{l} a_{k,i,j-l} = b_{k,j}. \quad (4.33)$$

For any $S \subseteq [\bar{k}]$, let $\mathcal{JSP}_{k,S,j}$ be the set of all Jacobi–Stirling permutations of $M_{k,S}$ with $j-1$ descents. Let $B_{k,j} = \bigcup_{S \subseteq [\bar{k}]} \mathcal{JSP}_{k,S,j}$ be the set of Jacobi–Stirling permutations with $j-1$ descents. We show that the left-hand side of (4.33) is the number N_0 of permutations in $B_{k,j}$ with no pattern $u\bar{u}$.

For any $T \subseteq [\bar{k}]$, let $B_{k,j}(T, \geq)$ be the set of permutations in $B_{k,j}$ containing all the patterns $u\bar{u}$ for $\bar{u} \in T$. By the principle of inclusion-exclusion [83, Chapter 2],

$$N_0 = \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} |B_{k,j}(T, \geq)|. \quad (4.34)$$

Now, for any subsets $T, S \subseteq [\bar{k}]$ such that $T \subseteq [\bar{k}] \setminus S$, define the mapping

$$\varphi : \mathcal{JSP}_{k,S,j} \cap B_{k,j}(T, \geq) \rightarrow \mathcal{JSP}_{k,S \cup T, j-|T|}$$

by deleting the \bar{u} in every pattern $u\bar{u}$ of $\pi \in \mathcal{JSP}_{k,S,j} \cap B_{k,j}(T, \geq)$. Clearly, this is a bijection. Hence, we can rewrite (4.34) as

$$\begin{aligned} N_0 &= \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} \sum_{\substack{S, T \subseteq [\bar{k}] \\ T \cap S = \emptyset}} |\mathcal{JSP}_{k,S \cup T, j-|T|}| \\ &= \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} \sum_{\substack{S \subseteq [\bar{k}] \\ T \subseteq S}} |\mathcal{JSP}_{k,S, j-|T|}|. \end{aligned}$$

For any subset S of $[\bar{k}]$ with $|S| = i$, and any l with $0 \leq l \leq i$, there are $\binom{i}{l}$ subsets T of S such that $|T| = l$, and, by definition,

$$\sum_{\substack{S \subseteq [\bar{k}] \\ |S|=i}} |\mathcal{JSP}_{k,S,j-|T|}| = a_{k,i,j-l}.$$

This proves that N_0 is equal to the left-hand side of (4.33).

Let $\mathcal{LSP}_{k,j}$ be the set of all Legendre–Stirling permutations of M_k with $j - 1$ descents. It is easy to identify a permutation $\pi \in B_{k,j}$ with no pattern $u\bar{u}$ with a Legendre–Stirling permutation $\pi' \in \mathcal{LSP}_{k,j}$ by inserting each missing \bar{u} just to the right of the second u . This completes the proof. \square

4.6 A conjecture and a remark

The numerical experiments suggest the following conjecture, which has been verified for $0 \leq i \leq k \leq 9$.

Conjecture 4.6.1. *For $0 \leq i \leq k$, the polynomial $A_{k,i}(t)$ has only real roots.*

Note that by Lemma 2.4.6 and Corollary 4.2.5, Conjecture 4.6.1 would imply that $A_{k,i}(t)$ is unimodal. Let G_k be the multiset $\{1^{m_1}, 2^{m_2}, \dots, k^{m_k}\}$ with $m_i \in \mathbb{N}$. A permutation π of G_k is a *generalized Stirling permutation* (see [8, 53, 65]) if whenever $u < v < w$ and $\pi(u) = \pi(w)$, we have $\pi(v) > \pi(u)$. For any $S \subseteq [\bar{k}]$, the set of generalized Stirling permutations of $M_k \setminus S$ is equal to $\mathcal{JSP}_{k,S}$. By Lemma 4.3.1 and Theorem 4.1.2, the descent polynomial of $\mathcal{JSP}_{k,S}$ is $A_{k,S}(t)$. It follows from a result of Brenti [8, Theorem 6.6.3] that $A_{k,S}(t)$ has only real roots. By (4.27), this implies, in particular, that the above conjecture is true for $i = 0$ and $i = k$.

One can also use the methods of Haglund and Visontai [49] to show that $A_{k,S}(t)$ has only real roots, though it is not apparent how to use these methods to show that $A_{k,i}(t)$ has only real roots.

Remark 4.2. Very recently, some other positivity properties about Jacobi–Stirling numbers are studied in [57].

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