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Study of Localization for Disordered Systems on Quantum Graphs

Mostafa Sabri

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(ED 386)

Institut de mathématiques de Jussieu - PRG

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

Mostafa SABRI

Étude de la localisation pour des systèmes désordonnés sur un graphe quantique

dirigée par Anne BOUTET DE MONVEL et Victor CHULAEVSKY

Soutenue le 7 mai 2014 devant le jury composé de :

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*What ever happened to the crystal maze
Sweet like a candy, sour like a lime.
If all the year were playing holidays
Is midnight still considered dinner time?*

Pentametron

Remerciements

Les années sont passées, le chemin exploré
Il a plu et neigé, mais souvent la lumière
Traversant les nuages, résolvait les mystères
Et bravant les mirages je notais les idées

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Et c'est très sincèrement que j'aimerais remercier

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Il semble que ces vers
Demandent
Qu'on les libère
Je vais donc les laisser
Tranquilles
Et dispersés

J'ai eu un grand honneur d'avoir
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Jitomirskaya et Pankrashkin
Comme rapporteurs
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A demandé
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Pour venir assister

Marche
Pense
Note
Lance!

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Monsieur Pankrashkin, au milieu du trajet
Et monsieur Klopp, quand je finalisais
Et puis
Un grand merci
A Rojas-Molina

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Et passer un moment
Au plan
Plus personnel

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Les galères en TD
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Merci tout simplement

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Celle qui tendait la main
Celle qui encourageait
Dans les hauts et les bas
Dans le stress et la joie
Celle qui nous procurait tous ces
Moments parfaits

J'étais encore tout jeune quand j'ai trouvé la mienne
On a grandi ensemble, on s'est dit
« A jamais ! »

Les yeux pleins de lumières splendides
Électroniques
Délaissant le sentier de la
Réalité
Je dédie quelques lignes romantiques
Dans une étreinte sublime et robotique
A mes
Jeux vidéos
Et dessins
Animés

Résumé

Résumé

Ce travail est consacré à l'étude de certaines propriétés spectrales des opérateurs de Schrödinger aléatoires. Il est divisé en deux parties :

1. Une étude de la localisation d'Anderson pour des systèmes multi-particules sur un graphe quantique.
2. Une formulation abstraite de quelques estimées de Wegner, suivie par une liste d'applications pour des modèles concrets.

Au Chapitre 1 on essaie d'introduire les problèmes et les résultats de la thèse de façon élémentaire.

La première partie occupe les chapitres 2 et 3. Le Chapitre 2 consiste essentiellement en notre article "Anderson Localization for a multi-particle quantum graph" [97] sur le sujet. Au Chapitre 3 on discute quelques propriétés supplémentaires du modèle, et on donne surtout des démonstrations alternatives de certains résultats du Chapitre 2.

La deuxième partie occupe les chapitres 4 et 5. Le Chapitre 4 reproduit essentiellement notre article "Some abstract Wegner estimates with applications" [98]. Au Chapitre 5 on poursuit l'étude des estimées de Wegner, en donnant notamment quelques théorèmes abstraits supplémentaires dans la Section 5.2 et encore d'autres applications dans la Section 5.3.

On conclut avec deux annexes A et B. Dans la première on expose de manière très détaillée les développements en fonctions propres généralisées. Dans l'Annexe B, on démontre quelques résultats classiques utilisés dans le texte.

Mots-clefs

Opérateurs de Schrödinger aléatoires, graphes quantiques, localisation d'Anderson, estimées de Wegner.

Study of Localization for Disordered Systems on Quantum Graphs

Abstract

This work is devoted to the study of some spectral properties of random Schrödinger operators. It is divided into two parts:

1. A study of localization for multi-particle systems on quantum graphs.
2. An abstract formulation of some Wegner estimates, followed by a list of applications for concrete models.

In Chapter 1 we try to introduce the problems and the results of this thesis in an elementary way.

The first part occupies chapters 2 and 3. Chapter 2 essentially reproduces our article “Anderson Localization for a multi-particle quantum graph” [97] on this subject. In Chapter 3 we discuss some additional properties of our model, and we give alternative proofs to some results of Chapter 2.

The second part occupies chapters 4 and 5. Chapter 4 essentially reproduces our article “Some abstract Wegner estimates with applications” [98]. In Chapter 5 we continue the study of Wegner estimates by giving more abstract theorems in Section 5.2 and yet more applications in Section 5.3.

We conclude with two appendices A and B. In the first one we explain the theory of generalized eigenfunction expansions in great detail. In Appendix B, we prove some classical results used in the text.

Keywords

Random Schrödinger operators, quantum graphs, Anderson localization, Wegner estimates.

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Introduction (version Française)

0.1 Généralités

0.1.1 Opérateurs de Schrödinger aléatoires

On étudie dans cette thèse quelques aspects de la théorie spectrale des opérateurs de Schrödinger aléatoires. Un but de cette théorie est de comprendre la conductivité électronique d'un point de vue mathématique. Une des découvertes fondamentales en physique est que le désordre peut supprimer le transport électronique dans un semi-conducteur. Plus précisément, il y a deux situations dans lesquelles on s'attend à ce qu'une onde ne se propage plus à travers un matériau cristallin et devient piégée ou *localisée* :

- ou bien l'onde possède une faible énergie,
- ou bien le désordre ou les impuretés dans le solide ont dépassé une certaine valeur critique.

La première interprétation théorique de ce phénomène a été donnée en 1958 par le physicien P. W. Anderson [6], ce qui lui a valu le prix Nobel. D'autres physiciens ont aussi largement contribué à l'étude de problèmes analogues, parmi lesquels on cite N. F. Mott.

0.1.2 Localisation

D'un point de vue mathématique, on se donne un espace de Hilbert associé au modèle, et le problème est d'étudier les propriétés spectrales de certains opérateurs différentiels décrivant la propagation de l'onde dans le solide. Plus précisément, un Laplacien $(-\Delta)$ modélise l'énergie cinétique de la particule, un potentiel fixé V_0 représente le champ produit par les ions du solide, et le désordre dans le milieu peut être interprété comme une perturbation aléatoire $V(\omega)$. Ceci donne un *opérateur de Schrödinger aléatoire* $H(\omega) := -\Delta + V_0 + V(\omega)$.

Sous certaines conditions d'ergodicité, on peut montrer que presque tous les $H(\omega)$ possèdent le même spectre ; on parle alors d'un *spectre presque sûr*. Même sans ergodicité, on peut parfois montrer que le bas du spectre est le même pour presque tous les $H(\omega)$. Soit I un intervalle qui contient presque sûrement du spectre.

- Une première indication de la *localisation* est que $H(\omega)$ possède presque sûrement un spectre purement ponctuel dans I . On parle alors de *localisation spectrale* dans I .
- L'étape suivante est d'établir la *localisation exponentielle*, i.e. de montrer que presque sûrement, les fonctions propres qui correspondent à des énergies dans I décroissent de façon exponentielle.
- Enfin, pour comprendre l'évolution des états de $H(\omega)$, on peut considérer l'équation de Schrödinger dépendant du temps et établir la *localisation dynamique*. Grosso modo, cela veut dire que si un état initial possède une énergie dans I , et s'il est localisé dans un domaine borné, alors il ne quittera pas beaucoup ce domaine à mesure que le temps passe.

En supposant que les variables aléatoires modélisant le désordre sont i.i.d. avec une distribution commune Höldérienne, les différents aspects de la localisation ont été établis pour beaucoup de modèles sur $\ell^2(\mathbb{Z}^d)$ et $L^2(\mathbb{R}^d)$, comme prévu

- dans des intervalles proches du bas du spectre presque sûr de l'opérateur de Schrödinger aléatoire (qui est la zone dans laquelle les fonctions propres ont de faibles énergies)
- dans le cas où l'ampleur du désordre dans le potentiel aléatoire est très grande.

L'historique de ces preuves est long ; mentionnons simplement les méthodes applicables en toute dimension, à savoir l'analyse multi-échelle introduite par Fröhlich et Spencer dans [46] et améliorée plus tard dans une série de papiers, en particulier [112] et [47], et la méthode des moments fractionnaires introduite par Aizenmann et Molchanov dans [3] pour des modèles discrets, et adaptée pour des modèles continus dans [2].

Si les variables aléatoires sont discrètes (e.g. des variables de Bernoulli), la localisation est beaucoup plus difficile à établir. En 2005, Bourgain et Kenig [15] ont enfin réussi à démontrer la localisation exponentielle au bord du spectre pour des modèles de Bernoulli sur $L^2(\mathbb{R}^d)$. Ce résultat a plus tard été renforcé par Germinet et Klein, qui ont démontré entre autres la localisation dynamique pour ce modèle dans [48]. Il n'y a toujours pas de preuve de localisation pour des modèles de Bernoulli sur $\ell^2(\mathbb{Z}^d)$, $d \geq 2$.

0.1.3 Graphes Quantiques

Une question naturelle à présent, motivée entre autres par la nanotechnologie, est d'étudier différentes formes du semi-conducteur. Les graphes quantiques apparaissent lorsqu'on essaie de comprendre la propagation des ondes à travers un système quasi unidimensionnel qui ressemble à un graphe.

On appelle *graphe quantique* un triplet $(\mathcal{E}, \mathcal{V}, H)$, où $(\mathcal{E}, \mathcal{V})$ est un graphe, avec un ensemble de sommets \mathcal{V} et un ensemble d'arêtes \mathcal{E} . Une longueur l_e est attribuée à chaque arête e , et H est un opérateur de Schrödinger opérant sur $\oplus_{e \in \mathcal{E}} L^2(0, l_e)$. Bien que le terme "graphe quantique" semble d'introduction récente dans la littérature mathématique, l'étude des équations différentielles sur un graphe date des années 80 ; on trouvera un aperçu dans [90].

Pour vérifier la localisation sur de telles structures, on peut interpréter les impuretés comme étant des sources d'aléa dans le graphe. Pour des graphes quantiques mono-particules avec $\mathcal{V} = \mathbb{Z}^d$, la localisation au bas du spectre a été démontrée pour un modèle à potentiel aléatoire dans [42], pour un modèle à constantes de couplage aléatoires dans [71], et pour un modèle à longueurs d'arête aléatoires dans [72]. Quelques résultats ont aussi été obtenus dans [4] et [55] pour des arbres aléatoires.

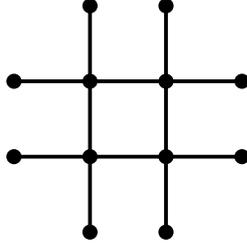
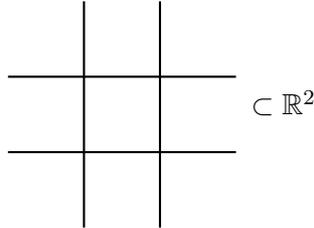
Décrivons un modèle simple. Soit $\mathcal{V} = \mathbb{Z}^d$, $(h_j)_{j=1}^d$ la base canonique de \mathbb{Z}^d et soit

$$\mathcal{E} = \{(m, j) : m \in \mathbb{Z}^d, j \in \{1, \dots, d\}\},$$

où (m, j) désigne l'arête entre m et $m + h_j$. Ainsi, on a une arête entre chaque paire de sommets voisins. On regarde à présent chaque arête comme un segment de longueur 1 : notons $[m, m + h_j] = \{(1-t)m + t(m + h_j) : t \in [0, 1]\} = \{m + th_j : t \in [0, 1]\}$. On définit

$$\begin{aligned} \Gamma^{(1)} &= \bigcup_{m \in \mathbb{Z}^d, j \in \{1, \dots, d\}} [m, m + h_j] \\ &= \{x \in \mathbb{R}^d : \exists m \in \mathbb{Z}^d, j \in \{1, \dots, d\}, 0 \leq t \leq 1, \text{ tels que } x = m + th_j\}. \end{aligned}$$

$\Gamma^{(1)}$ représente simplement la structure géométrique sous-jacente au graphe $(\mathcal{E}, \mathcal{V})$. Si $d = 1$, alors $\Gamma^{(1)} = \mathbb{R}$. Une partie de $(\mathcal{E}, \mathcal{V})$ et de $\Gamma^{(1)}$ est donnée dans les Figures 1 et 2, respectivement, lorsque $d = 2$.

FIGURE 1 – Une partie de $(\mathcal{E}, \mathcal{V})$ quand $d = 2$.FIGURE 2 – Une partie de $\Gamma^{(1)}$ quand $d = 2$.

En tant que partie de \mathbb{R}^d , $\Gamma^{(1)}$ hérite la métrique donnée par la norme sup de \mathbb{R}^d .

Soit $\mathcal{H} = \oplus_{e \in \mathcal{E}} L^2(0, 1)$. La mesure de Lebesgue sur $[0, 1]$ induit une mesure naturelle sur $\Gamma^{(1)}$ qu'on note $m^{(1)}$. On verra au Chapitre 2 que \mathcal{H} s'identifie facilement à $L^2(\Gamma^{(1)}, dm^{(1)})$. Soient $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$, et soit μ une mesure de probabilité sur \mathbb{R} de support $[q_-, q_+]$. Considérons l'espace de probabilité (Ω, \mathbb{P}) , où $\Omega := [q_-, q_+]^{\mathcal{E}}$ et $\mathbb{P} = \otimes_{e \in \mathcal{E}} \mu$. Alors étant donné $\omega = (\omega_e) \in \Omega$, on définit l'opérateur de Schrödinger $H(\omega)$ par

$$H(\omega) : (f_e) \mapsto (-f_e'' + \omega_e f_e),$$

de domaine

$$D(H(\omega)) = \left\{ f = (f_e) \in \oplus_{e \in \mathcal{E}} W^{2,2}((0, 1)) \left| \begin{array}{l} \text{pour tout } v \in \mathcal{V}, f \text{ est continue en } v, \\ \text{et } \sum_{e: \iota e = v} f_e'(0) - \sum_{e: \tau e = v} f_e'(1) = 0. \end{array} \right. \right\}.$$

Ici $\iota(m, j) = m$ et $\tau(m, j) = m + h_j$ représentent les sommets de départ et d'arrivée d'une arête (m, j) . Les conditions aux limites dans $D(H(\omega))$ sont connues sous le nom de *conditions aux limites de Kirchhoff*. Noter que ce sont ces conditions qui reflètent la connectivité du graphe. Ce modèle a été étudié en 2007 par Exner, Helm et Stollmann. Ils ont démontré dans [42] que $H(\omega)$ possède un spectre presque sûr Σ au bas duquel on a localisation exponentielle et localisation dynamique.

0.2 Graphe quantique multi-particule

Dans la première partie de cette thèse, on étudie l'analogie multi-particule du modèle décrit dans la Section 0.1.3. On parle d'un *graphe quantique multi-particule* pour souligner le fait que non seulement l'opérateur H change dans ce cas, mais que la structure géométrique Γ change elle aussi, à savoir, on n'a plus de collection de sommets et d'arêtes.

0.2.1 Le modèle

Pour étudier l'interaction entre deux particules, l'une sur une arête e_1 , l'autre sur une arête e_2 , on devrait intuitivement considérer le couple (e_1, e_2) . Ainsi, au lieu d'étudier une collection d'arêtes \mathcal{E} , on devrait à présent étudier une collection de couples d'arêtes, ou de rectangles par identification. On obtient ainsi une sorte de réseau de rectangles. Plus généralement, pour étudier l'interaction entre N particules, on devrait considérer un réseau de pavés droits de dimension N . Ceci est seulement l'image intuitive¹.

Plus formellement, si $\Gamma^{(1)} \subset \mathbb{R}^d$ est l'ensemble introduit dans la Section 0.1.3, alors l'espace de Hilbert à considérer pour l'étude d'un système mono-particule sur $\Gamma^{(1)}$ est $\mathcal{H}_1 = L^2(\Gamma^{(1)}, dm^{(1)})$. La mécanique quantique nous dit alors que l'espace de Hilbert correspondant à N particules distinguables, chacune vivant dans $\Gamma^{(1)}$, est le produit tensoriel $\mathcal{H}_N = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1$. En prenant le produit cartésien

$$\Gamma^{(N)} := \Gamma^{(1)} \times \dots \times \Gamma^{(1)} \subset (\mathbb{R}^d)^N$$

et la mesure produit $m := m^{(1)} \otimes \dots \otimes m^{(1)}$, on peut identifier² \mathcal{H}_N avec $L^2(\Gamma^{(N)}, dm)$.

Tout point $\mathbf{x} = (x_1, \dots, x_N) \in \Gamma^{(N)}$ prend la forme $x_k = m_k + t^k h_{j_k}$, avec $m_k \in \mathbb{Z}^d$, $t^k \in [0, 1]$ et $j_k \in \{1, \dots, d\}$. Ainsi, si pour $\mathbf{m} = (m_1, \dots, m_N) \in (\mathbb{Z}^d)^N$ et $\mathbf{j} = (j_1, \dots, j_N) \in \{1, \dots, d\}^N$ on pose

$$\kappa_{\mathbf{m}, \mathbf{j}} := [m_1, m_1 + h_{j_1}] \times \dots \times [m_N, m_N + h_{j_N}],$$

on a $\Gamma^{(N)} = \bigcup_{\mathbf{m} \in (\mathbb{Z}^d)^N, \mathbf{j} \in \{1, \dots, d\}^N} \kappa_{\mathbf{m}, \mathbf{j}}$. On peut donc voir $\Gamma^{(N)}$ comme un couple $(\mathcal{K}, \mathcal{S})$, où \mathcal{K} est une collection de cubes κ de dimension N de volume 1 et \mathcal{S} est l'ensemble des bords σ de ces cubes.

Pour $d = 1$ et $N = 2$, on a clairement $\Gamma^{(2)} = \mathbb{R}^2$. Le couple $(\mathcal{K}, \mathcal{S})$ correspondant à $\Gamma^{(2)}$ est décrit dans la Figure 3. Pour $d = 2$, $\Gamma^{(2)}$ est une collection infinie de plans qui s'entrecroisent dans \mathbb{R}^4 .

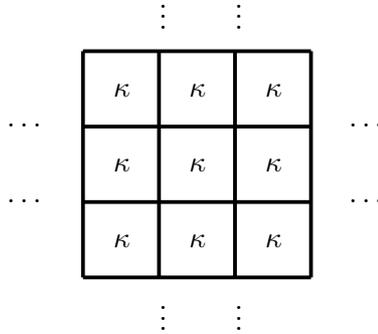


FIGURE 3 – Le couple $(\mathcal{K}, \mathcal{S})$ correspondant à $\Gamma^{(2)}$ pour $d = 1$. Il s'agit d'une collection de carrés κ dont les coins sont dans \mathbb{Z}^2 et qui recouvrent \mathbb{R}^2 .

Chaque σ est l'union fermée de $2N$ “faces ouvertes” σ^i , i.e. $\sigma = \cup_i \bar{\sigma}^i$. Par exemple, si σ est le bord de $\kappa_{\mathbf{m}, \mathbf{j}}$, alors $\sigma^1 = \{m_1\} \times (m_2, m_2 + h_{j_2}) \times \dots \times (m_N, m_N + h_{j_N})$ et sa face opposée est $\sigma^{o(1)} = \{m_1 + h_{j_1}\} \times (m_2, m_2 + h_{j_2}) \times \dots \times (m_N, m_N + h_{j_N})$.

Le cube $\kappa_{\mathbf{m}, \mathbf{j}}$ sera noté $\kappa = ((m_1, j_1), \dots, (m_N, j_N))$. Avec la notation des arêtes $e_k = (m_k, j_k)$, ceci devient $\kappa = (e_1, \dots, e_N)$.

1. Notons que Nicaise est probablement le premier à avoir étudié de façon systématique les problèmes aux limites sur des réseaux polygonaux, voir [85]. On note aussi un article récent [14] où des interactions singulières entre deux particules sur un graphe quantique compact ont été considérées.

2. Voir par exemple [93, Theorem II.10].

Introduisons à présent notre opérateur de Schrödinger à N -particules $H^{(N)}(\omega)$. Cela nous donnera un *graphe quantique N -particule* $(\mathcal{K}, \mathcal{S}, H^{(N)})$.

Fixons $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$ et prenons une mesure de probabilité μ sur \mathbb{R} de support $[q_-, q_+]$. Considérons l'espace de Hilbert $\mathcal{H} := \bigoplus_{\kappa \in \mathcal{K}} L^2((0, 1)^N)$. Il peut être facilement identifié à l'espace $L^2(\Gamma^{(N)}, dm)$ décrit précédemment, comme on verra au Chapitre 2. Soit (Ω, \mathbb{P}) l'espace de probabilité donné par $\Omega := [q_-, q_+]^{\mathcal{E}}$ et $\mathbb{P} := \bigotimes_{e \in \mathcal{E}} \mu$. Alors pour $\omega = (\omega_e) \in \Omega$, l'opérateur de Schrödinger est défini via la forme

$$\mathfrak{h}_\omega^{(N)}[f, g] = \sum_{\kappa \in \mathcal{K}} [\langle \nabla f_\kappa, \nabla g_\kappa \rangle + \langle V_\kappa^\omega f_\kappa, g_\kappa \rangle],$$

de domaine

$$D(\mathfrak{h}_\omega^{(N)}) = \left\{ f = (f_\kappa) \in \bigoplus_{\kappa \in \mathcal{K}} W^{1,2}((0, 1)^N) \mid \begin{array}{l} f \text{ est continue sur chaque } \sigma^i, \\ \sum_{\kappa \in \mathcal{K}} \|f_\kappa\|_{W^{1,2}}^2 < \infty \end{array} \right\}.$$

La continuité sur σ^i signifie que si σ^i est une face commune à κ_1 et κ_2 , alors $f_{\kappa_1}|_{\sigma^i} = f_{\kappa_2}|_{\sigma^i}$ au sens des traces.

Ici $V_\kappa^\omega := U_\kappa^{(N)} + W_\kappa^\omega$, où $U^{(N)} \geq 0$ est un potentiel d'interaction non-aléatoire. L'hypothèse importante sur $U^{(N)}$ est qu'il est de portée finie. Par exemple, si $N = 2$, cela signifie qu'il existe $r_0 > 0$ tel que pour tous $x_1, x_2 \in \Gamma^{(1)}$, $x_k = m_k + t^k h_{j_k}$ on ait

$$|x_1 - x_2| > r_0 \implies U^{(2)}(x_1, x_2) = U_{\kappa_{\mathbf{m}, \mathbf{j}}}^{(2)}(t^1, t^2) = 0,$$

où $|\cdot| := \|\cdot\|_\infty$ est la norme sup de \mathbb{R}^d . Cela signifie qu'il n'y a pas d'interaction entre deux particules en des positions respectives x_1 et x_2 éloignées.

W_κ^ω est un potentiel multi-particule aléatoire. Si $\kappa = (e_1, \dots, e_N)$, alors $W_\kappa^\omega := \omega_{e_1} + \dots + \omega_{e_N}$. Voyons le cas le plus simple $d = 1$ et $N = 2$ dans la Figure 4.



FIGURE 4 – Deux carrés κ^1 et κ^2 partageant une arête e_2 .

Ici f_{κ^1} est multipliée par $\omega_{e_1} + \omega_{e_2}$ et f_{κ^2} est multipliée par $\omega_{e_3} + \omega_{e_2}$. Ainsi, il n'y a pas d'indépendance à distance : si κ^1 et κ^2 partagent une arête e_2 , on aura toujours le terme ω_{e_2} , même si κ^1 et κ^2 sont très éloignés.

Il est assez facile de voir que $\mathfrak{h}_\omega^{(N)}$ correspond à l'unique opérateur auto-adjoint $H^{(N)}(\omega)$ donné par

$$H^{(N)}(\omega) : (f_\kappa) \mapsto ((-\Delta + V_\kappa^\omega)f_\kappa)$$

sur un certain domaine $D(H^{(N)}(\omega))$. La forme explicite de $D(H^{(N)}(\omega))$ ne semble pas être une question facile ; voir le Chapitre 2.

0.2.2 Résultats principaux

Voici les résultats principaux qu'on a obtenus dans [97] et qu'on reprend ici au Chapitre 2. Tout d'abord, le bas du spectre est déterministe :

Théorème 0.2.1. *Il existe $\Omega_0 \subseteq \Omega$ avec $\mathbb{P}(\Omega_0) = 1$ tel que pour tout $\omega \in \Omega_0$:*

$$[Nq_-, Nq_+] \subset \sigma(H^{(N)}(\omega)) \subseteq [Nq_-, +\infty).$$

En particulier, $\inf \sigma(H^{(N)}(\omega)) = Nq_-$ presque sûrement.

Ensuite, en supposant que μ est Höldérienne, le bas du spectre est localisé :

Théorème 0.2.2 (Localisation exponentielle). *Il existe $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ et $m > 0$ tels que pour presque tout ω le spectre de $H^{(N)}(\omega)$ dans $I = [Nq_-, Nq_- + \varepsilon_0]$ soit purement ponctuel et les fonctions propres qui correspondent à des valeurs propres dans I soient à décroissance exponentielle de taux m .*

Théorème 0.2.3 (Localisation dynamique HS-forte). *Il existe $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ tel que pour $I = [Nq_-, Nq_- + \varepsilon_0]$, on ait pour toute partie bornée $K \subset \Gamma^{(N)}$ et tout $s > 0$,*

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^{s/2} f(H^{(N)}(\omega)) E_\omega(I) \chi_K\|_2^2 \right\} < \infty,$$

où $(X\psi)(\mathbf{x}) := |\mathbf{x}| \cdot \psi(\mathbf{x})$ pour $\psi \in L^2(\Gamma^{(N)}, dm)$, E_ω est la projection spectrale de $H^{(N)}(\omega)$ et on prend le sup sur toutes les fonctions boréliennes, $\|f\| := \|f\|_\infty$.

Dans le cas particulier où $N = 1$, ces théorèmes sont plus forts que le résultat principal de [42], d'abord parce qu'on les établit sans la condition technique ($\exists \tau > \frac{d}{2} : \mu([q_-, q_- + h]) \leq h^\tau$ pour h petit), ensuite parce que la localisation dynamique est démontrée en norme Hilbert-Schmidt.

0.2.3 Idées de la preuve

La preuve de ces théorèmes est basée sur l'analyse multi-échelle multi-particule développée par A. Boutet de Monvel, Chulaevsky et Suhov dans [27] et [19]. L'analyse multi-échelle devient remarquablement plus compliquée dans le cas multi-particule car on perd l'indépendance à distance. Le passage des bornes fournies par l'analyse multi-échelle à la localisation s'appuie sur l'approche de Germinet et Klein [47].

Avant d'adapter ces méthodes aux graphes quantiques multi-particules, il y a une liste de choses à vérifier. Voici les difficultés principales qu'on rencontre en passant de $\Gamma^{(1)}$ à $\Gamma^{(N)}$.

Tout d'abord, pour l'estimée de Combes-Thomas, on a besoin de bien contrôler l'exposant de décroissance, car ici l'estimée de Combes-Thomas est utilisée non seulement pour obtenir le pas initial, mais aussi dans la récurrence multi-échelle elle-même. Grosso modo, l'exposant doit dépendre (comme d'habitude) de la distance de l'énergie au spectre, mais *pas* de la valeur absolue de l'énergie. On a réussi à obtenir une telle estimée en utilisant les semi-groupes, via une estimée améliorée de Davies-Gaffney.

Ensuite, on a dû démontrer quelques estimées de trace nécessaires pour obtenir des développements en fonctions propres généralisées.

Puis il y a l'inégalité de décroissance des fonctions propres (EDI) : dans la preuve usuelle, on a besoin de savoir que les fonctions propres généralisées sont régulières, disons localement de classe $W^{1,2}$. Cependant, pour utiliser l'approche de Germinet et Klein, on a besoin du développement en fonctions propres généralisées attribué d'habitude à Berezansky (voir [11]), dont les fonctions ne sont pas forcément régulières. Au lieu de démontrer qu'elles sont bien régulières³, on a réussi à prouver EDI en contournant ce

3. Plus tard on a démontré dans le Chapitre 3 que les fonctions propres généralisées sont régulières. Cependant, comme le lecteur verra, il est plus simple et plus naturel de raisonner directement avec des approximations, sans faire usage de la régularité.

problème à l'aide d'approximations dans l'espace de Hilbert négatif (i.e. celui dans lequel les fonctions propres généralisées vivent).

Enfin, on avait besoin d'un pas initial pour $H^{(1)}$ plus fort que celui démontré dans [42], c'est pourquoi on a démontré des asymptotiques de type Lifshitz. L'argument perturbatif qu'on utilise pour les obtenir dans $L^2(\mathbb{R}^d)$ marche aussi ici, mais à un moment donné, on a besoin de connaître la largeur du trou spectral entre les deux premières valeurs propres de $H^{(1)}$ restreint à un graphe fini. Il se trouve que ce problème a déjà été étudié en géométrie riemannienne, et une façon d'estimer ce trou spectral est via l'inégalité de Cheeger⁴. Cette inégalité a plus tard été démontrée par Nicaise pour les graphes quantiques finis [84], et elle est aussi apparue dans un article de Post [91].

Cela concerne les principales difficultés. Il y a aussi d'autres détails à régler pour pouvoir commencer l'analyse multi-échelle ; il faut notamment obtenir l'inégalité géométrique de la résolvante (GRI), ainsi que l'estimée de Wegner. Pour GRI, noter qu'en plus de l'inégalité habituelle dont on a besoin pour les mono-particules, on a aussi besoin d'une nouvelle inégalité dans le cadre multi-particule, qui compare les fonctions de Green associées à certains N -cubes à celles associées aux n -cubes pour $n < N$. Pour Wegner, la différence principale dans le cadre multi-particule, est qu'on a besoin d'être sûr que le potentiel aléatoire est bien distribué dans n'importe quelle direction mono-particulaire. Plus précisément, l'estimée de Wegner dont on a vraiment besoin n'est pas une espérance sur tout l'espace Ω , mais plutôt une espérance conditionnelle. Ceci est nécessaire pour pouvoir obtenir plus tard l'estimée de Wegner à deux volumes.

0.2.4 Variations

Au Chapitre 3, on propose quelques variations sur ces résultats. Notamment, on démontre la localisation exponentielle et dynamique sans faire usage de fonctions propres généralisées. Cependant, cette approche n'établit la localisation dynamique qu'en norme d'opérateur. On explique aussi à quel point on peut étendre nos résultats lorsque la distribution μ des variables aléatoires est seulement log-Höldérienne. Enfin, on donne une estimée de Combes-Thomas valable pour des énergies dans un trou spectral quelconque, mais le prix à payer est que la preuve est assez compliquée.

0.3 Estimées de Wegner abstraites

Dans la deuxième partie de cette thèse, on formule quelques estimées de Wegner abstraites, puis on les applique à des modèles concrets.

0.3.1 Estimées de Wegner en général

Le but des estimées de Wegner est d'obtenir de bonnes bornes sur le nombre moyen de valeurs propres d'un opérateur de Schrödinger aléatoire $H_\Lambda(\omega)$ restreint à un cube Λ dans un intervalle donné I . Plus précisément, si $\chi_I(H_\Lambda(\omega))$ est la projection spectrale de $H_\Lambda(\omega)$ sur I , alors on cherche des estimées du type

$$\mathbb{E}\{\mathrm{tr}[\chi_I(H_\Lambda(\omega))]\} \leq C_W \cdot |\Lambda|^\alpha \cdot |I|^\beta$$

pour certains $C_W \geq 0$, $\alpha \geq 1$ et $\beta > 0$. De telles estimées peuvent être utilisées dans une preuve de la localisation via l'analyse multi-échelle, ou dans l'étude de la continuité de la densité d'états intégrée (IDS) si $\alpha = 1$.

4. On donne une approche différente dans l'Annexe B, qui s'appuie sur l'inégalité de Faber-Krahn.

0.3.2 Motivation

La motivation pour ce travail est venue encore une fois des graphes quantiques. Une particularité très intéressante des graphes quantiques est qu'on peut considérer un aléa non seulement dans le potentiel, mais aussi dans le graphe lui-même. Cela donne naissance à deux nouveaux modèles : les graphes quantiques à constantes de couplage aléatoires (RCM, i.e. conditions aux limites aléatoires) et les graphes quantiques à longueurs d'arête aléatoires (RLM) ; voir [55], [71] et [72] pour des résultats de localisation pour ces modèles.

Je me suis intéressé dans un premier temps aux articles [72] et [78] qui étudient les RLM. Ces travaux établissent beaucoup de résultats y compris la localisation au bord du spectre et la continuité de l'IDS. Je me suis concentré sur l'estimée de Wegner, qui m'est parue très intéressante car ces modèles dépendent du paramètre aléatoire de manière assez insolite.

La description précise de ces modèles prendrait trop de place, donc je vais simplement mentionner deux astuces principales qui ont été appliquées dans ces travaux pour obtenir l'estimée de Wegner. Dans [72], l'idée était d'établir une relation entre l'opérateur de Schrödinger $H(l^\omega)$ sur le graphe quantique $\oplus_{e \in \mathcal{E}} L^2(0, l_e^\omega)$ et un homologue discret $M(l^\omega, I)$ qui agit sur l'ensemble des sommets, i.e. sur l'espace de Hilbert $\ell^2(\mathcal{V})$. Ici I est l'intervalle où on voudrait avoir une estimée de Wegner. Ceci a un peu simplifié la tâche, car pour des sous-graphes finis $(\mathcal{E}_\Lambda, \mathcal{V}_\Lambda)$, l'espace $\ell^2(\mathcal{V}_\Lambda)$ est de dimension finie. Cependant, la dépendance de $M(l^\omega, I)$ en $l^\omega = (l_e^\omega)$ devient difficile (des termes comme $(\sin l_e^\omega)^{-1}$ apparaissent), donc il faut concevoir de nouveaux arguments.

Dans [78], l'idée était de travailler avec les variables aléatoires $(\ln l_e^\omega)_{e \in \mathcal{E}}$ au lieu des variables $(l_e^\omega)_{e \in \mathcal{E}}$. La dépendance en le paramètre aléatoire devient alors plus simple, et quelques relations précises entre les valeurs propres peuvent être obtenues. L'espace de probabilité doit évidemment être modifié de façon analogue. Mais la conclusion n'est toujours pas immédiate, car la dépendance en le paramètre aléatoire reste inhabituelle.

Les deux travaux supposaient que les variables aléatoires (l_e^ω) étaient i.i.d. avec une distribution commune absolument continue. Je me suis alors demandé si on pouvait affaiblir cette condition et travailler avec une distribution Höldérienne.

Une autre motivation est venue des opérateurs de Schrödinger usuels sur $\ell^2(\mathbb{Z}^d)$ et $L^2(\mathbb{R}^d)$, mais dont le potentiel aléatoire ne recouvre pas l'espace. Pour ces opérateurs, les preuves disponibles sont compliquées si on ne suppose pas que la mesure de probabilité a une densité, voir e.g. [30].

0.3.3 Résultats

L'article [98] tente de donner un traitement simple et unifié pour les modèles précédents.

Espaces de Hilbert de dimension finie

On donne d'abord une estimée de Wegner pour certains opérateurs auto-adjoints aléatoires sur un espace de Hilbert de dimension finie. On ne suppose pas que la distribution a une densité.

Comme application, on donne des estimées de Wegner optimales pour le modèle d'Anderson discret multi-particule, et ceci pour trois cas différents :

- Si le potentiel aléatoire recouvre \mathbb{Z}^d , i.e. s'il existe $c > 0$ tel que $V^\omega = \sum_{\alpha \in \mathbb{Z}^d} \omega_\alpha u_\alpha$, $u_\alpha \geq c \cdot \delta_\alpha$, on obtient des estimées à un volume et à deux volumes valables pour un intervalle d'énergie quelconque.

- Sinon, si la perturbation est négative et portée par un demi-espace, on donne une estimée de Wegner au bas du spectre. On montre qu'elle est non-triviale, i.e. que l'intervalle considéré contient bien du spectre.
- Enfin, si la perturbation est positive, on a encore des bornes optimales, mais pour montrer qu'elles sont non-triviales, on utilise un résultat de [37], où les auteurs considèrent des perturbations positives portées par des ensembles de Delone (exemple de potentiel sans recouvrement).

Notons que les demi-espaces qu'on a considérés dans le cas de perturbations négatives ne sont pas des ensembles de Delone, et que les ensembles de Delone ne sont pas non plus des demi-espaces. Ainsi on peut dire que notre article [98] d'une part, et les articles [37] et [94] d'autre part, se complètent. Enfin, soulignons que toutes ces estimées supposent que le potentiel en chaque site est positif (il est important qu'il ne change pas de signe).

Plus tard on donne au Chapitre 5 plusieurs variations sur le théorème abstrait, y compris une adaptation du résultat de [30] aux espaces de dimension finie.

Espaces de Hilbert séparables

On passe ensuite plus généralement aux espaces de Hilbert séparables et on donne deux estimées. La première n'a presque pas de restriction sur l'intervalle d'énergie et sur la forme du paramètre aléatoire, mais elle suppose qu'on a une forme de monotonie et de recouvrement. La deuxième est un peu plus restrictive sur l'intervalle et sur la forme du paramètre aléatoire, mais elle n'a pas besoin de recouvrement.

On applique la première estimée aux RLM de [72]. On obtient alors une réponse positive à notre question initiale concernant ce modèle : l'estimée de [72] peut en effet être généralisée pour des distributions Höldériennes. En particulier, cela permet d'étendre les résultats de localisation de cet article à de telles distributions (si la constante de couplage est positive).

On applique notre deuxième estimée aux RCM, ainsi qu'aux Hamiltoniens continus multi-particules sans condition de recouvrement. Ici on a encore deux situations : pour une perturbation négative portée par un demi-espace, on démontre une estimée de Wegner au bord du spectre, et on montre qu'elle n'est pas triviale. Pour les perturbations positives, on doit supposer que le bord du spectre est fluctuant. Cependant, la situation ici est bien meilleure que dans le cas discret, car dans $L^2(\mathbb{R}^d)$, une grande classe d'opérateurs ont un bord fluctuant, notamment les potentiels de Delone et de surface ; voir [62, Theorem 2.2], [21, Sections 4,5] et [66, Lemma 4.2].

Notons que, contrairement au cas discret, les bornes qu'on obtient ici ne sont pas optimales, car la dépendance en le volume $|\Lambda|$ n'est pas linéaire. Ces estimées ne peuvent donc pas être utilisés pour étudier la densité d'états intégrée, mais elles sont néanmoins suffisantes pour établir la localisation.

Comparons rapidement nos résultats à la littérature. Sauf erreur, les potentiels à demi-espace n'ont pas été considérés avant⁵, ainsi notre résultat concernant les perturbations négatives semble nouveau. Pour les perturbations positives, notre résultat est très proche de celui de [20]. Cependant, cet article utilise les résultats de [30], qui sont assez compliqués. On prend une approche complètement différente pour démontrer notre théorème, et notre preuve est assez élémentaire. Néanmoins, l'article [20] a l'avantage de fournir une estimée de Wegner optimale, ce qui le rend utile pour l'étude de la densité d'états intégrée. Notre estimée pour les RCM semble nouvelle.

5. Notons que ces potentiels entrent dans le cadre de l'article [63], mais ici on montre qu'il y a bien du spectre dans l'intervalle d'énergie considéré, une question qui ne semble pas être abordée dans [63].

Enfin, pour les opérateurs de Delone sur $L^2(\mathbb{R}^d)$, les papiers récents [95] et [66] arrivent à étendre les estimées de Wegner à des intervalles arbitraires, pas seulement au bord du spectre. Ainsi, dans le cas de ces opérateurs, notre résultat donne un cas particulier de ces papiers, mais notre preuve est très différente (par exemple, [66] utilise les résultats de [30], ainsi qu'un résultat récent [16]).

Soulignons encore une fois que tous ces théorèmes supposent que le potentiel en chaque site est positif. Cette hypothèse est légèrement affaiblie dans [95], mais elle ne disparaît pas complètement.

Esquissons maintenant les preuves de nos théorèmes généraux. La première estimée suit d'une variation facile du lemme de Stollmann [103] ; il faut simplement faire attention à des questions de mesurabilité. La deuxième borne est basée sur deux idées : d'abord travailler avec les variables aléatoires $(\ln \omega_j)_j$ au lieu de $(\omega_j)_j$ comme dans [78]. Ensuite, modifier le lemme de Stollmann pour qu'il puisse traiter des dépendances diagonales plus générales.

Il est assez curieux qu'on n'ait pas réussi à appliquer notre théorème au modèle de [78] qui nous a pourtant inspiré une partie de la preuve. Le problème est que, pour ce modèle, le domaine de l'opérateur $H(\ell_\omega)$ dépend de ℓ_ω . On explique en détail cette difficulté dans la Section 5.4.

0.4 Perspectives

Il y a beaucoup de questions qui devraient encore être abordées dans les deux études.

0.4.1 Graphes quantiques

Ici on a seulement étudié le graphe quantique multi-particule le plus simple. Il y a deux questions naturelles :

- (i) d'abord la géométrie,
- (ii) ensuite les conditions aux limites.

Pouvons-nous facilement adapter notre preuve pour traiter de cas plus généraux ? Ou est ce qu'une situation vraiment difficile apparaît ? Il semblerait que ces deux questions aient été étudiées récemment dans le cas mono-particule dans le travail de Schubert [99], qui pourrait fournir un bon point de départ.

Nous pensons qu'une simple généralisation de la géométrie, e.g. des pavés droits au lieu de cubes, devrait être immédiate. Nous pensons aussi qu'on pourrait donner des volumes variés aux cubes élémentaires, voire leur donner des volumes croissants, peut-être qu'en contre-partie la localisation ne sera plus exponentielle mais polynomiale. Cependant, des géométries trop générales pourraient détruire l'ergodicité. Il sera donc important de déterminer des géométries qui nous donnent au moins un bas du spectre non-aléatoire.

Pour les conditions aux limites, on perdra peut-être les Lifshitz-tails ; il faut regarder de plus près. Il faudra aussi arranger quelques lemmes techniques.

Enfin, il y a le fait qu'on a utilisé dans plusieurs endroits le plongement dans l'espace euclidien, mais ceci pourrait probablement être évité sans trop de difficulté. En particulier, il nous semble plausible que, pour étudier des systèmes multi-particules sur un graphe quantique quelconque, on devrait étudier un réseau de pavés droits. En effet, on a un isomorphisme canonique entre $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$ et $L^2(M_1 \times M_2, d\mu_1 \otimes d\mu_2)$ pour des espaces mesurés quelconques (M_1, μ_1) et (M_2, μ_2) , donc en particulier pour des réseaux unidimensionnels munis de la mesure de Lebesgue.

Tout ceci concerne la localisation ; on pourrait aussi s'intéresser aux statistiques spectrales des valeurs propres.

0.4.2 Estimées de Wegner

Pour la deuxième étude, il y a au moins trois questions.

D'abord, pouvons-nous modifier les théorèmes abstraits, ou en prouver d'autres, qui soient capables de traiter les Hamiltoniens dont le potentiel aléatoire change de signe ? ou plus généralement, des opérateurs qui ne dépendent pas de façon monotone du désordre ? A notre connaissance, il n'existe pas à ce jour d'estimées de Wegner pour de tels modèles si on ne suppose pas que la distribution des variables aléatoires possède une densité, du moins pour $L^2(\mathbb{R}^d)$. Ceci nous semble donc un problème très intéressant à étudier. On *tente* de le résoudre dans la Section 5.4.1. Les articles [38, 69, 54, 111, 110, 75, 39] pourraient peut-être nous permettre d'aller plus loin.

La deuxième question concerne la non-linéarité des estimées de Wegner dans le cas d'espaces de Hilbert séparables. Ce problème apparaît car on utilise une version modifiée du lemme de Stollmann pour contrôler les valeurs propres. Pouvons-nous nous débarrasser de ce phénomène ? Faudrait-il d'abord manipuler la trace de la projection spectrale avant d'utiliser ce lemme, un peu comme on a fait pour les espaces de dimension finie ? Faudrait-il plutôt modifier le lemme de Stollmann lui-même ? Ce lemme est déjà optimal, mais si on a plus d'informations sur les valeurs propres, on pourrait peut-être ajouter quelques hypothèses à ce lemme qui nous donnerait alors une meilleure conclusion. Ou est-ce indispensable de prendre une approche complètement différente ? Cette question vaut la peine d'être résolue, car elle pourrait ouvrir une nouvelle porte à l'étude de la continuité de la densité d'états intégrée.

La troisième question est la suivante : nos estimées de Wegner dans le cas d'espaces de Hilbert séparables s'appliquent à des potentiels très généraux, mais elles ne sont valables que pour des intervalles au bas du spectre. La question est de savoir si on peut donner une preuve aussi élémentaire de l'estimée de Wegner, mais cette fois en toute énergie, si on considère une classe de potentiels sans recouvrement mais qui possède quelque périodicité comme dans [30]. L'éventuelle non-linéarité serait sans doute un prix raisonnable à payer si la preuve est élémentaire.

Chapter 1

Introduction

1.1 Background

1.1.1 Random Schrödinger operators

In this thesis we study some aspects of the spectral theory of random Schrödinger operators. One aim of this theory is to get a mathematical comprehension of conductivity in disordered solids. One of the fundamental findings of physics is that disorder can suppress the transport properties of a medium. More precisely, there are two situations in which one expects a wave to stop traveling through a crystalline material and become trapped or *localized*:

- either the wave has a low energy,
- or the disorder or impurities in the solid have passed a certain critical value.

The first theoretical interpretation of this phenomenon was given in 1958 by the physicist P. W. Anderson [6], who was later awarded the Nobel Prize for his contributions to this field. Other physicists have also largely contributed to related problems, in particular N.F. Mott.

1.1.2 Localization

From a mathematical point of view one considers a Hilbert space associated to the model, and the problem is to study the spectral properties of certain differential operators describing the propagation of the wave in the solid. More precisely, a Laplace operator $(-\Delta)$ models the kinetic energy of the particle, a fixed potential V_0 represents the field given by the ions in the solid, and the disorder in the medium can be interpreted as a random potential perturbation $V(\omega)$. This yields a *random Schrödinger operator* $H(\omega) := -\Delta + V_0 + V(\omega)$.

Under some assumptions of ergodicity, it can be shown that almost all $H(\omega)$ possess the same spectrum; one then speaks of an *almost sure spectrum*. Even without ergodicity, one can show sometimes that the lower part of the spectrum is the same for almost all $H(\omega)$. Let I be an interval that contains some spectrum almost surely.

- A first indication of *localization* is for $H(\omega)$ to exhibit pure point spectrum almost surely in I , in which case one speaks of *spectral localization* in I .
- A step further is to establish *exponential localization*, i.e., to show that almost surely, the eigenfunctions corresponding to energies in I decay exponentially fast.

- Finally, to understand the evolution of the states of $H(\omega)$, one may consider the time-dependent Schrödinger equation and establish *dynamical localization*. Roughly speaking, this means that if an initial state has an energy in I and if it is localized in a bounded domain, then it will not leave this domain too much as time goes on.

Assuming the random variables modeling the disorder are i.i.d. with a common distribution which is Hölder continuous, the different aspects of localization have been proven to hold for many models in $\ell^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, as expected

- in intervals near the bottom of the almost sure spectrum of the random Schrödinger operator (which is the zone in which the eigenfunctions have low energies)
- in the case where the disorder amplitude of the random potential is very large.

Such proofs have a long history; let us only mention the ones applicable in any dimension, namely the multiscale analysis proof introduced by Fröhlich and Spencer in [46] and later improved in a series of paper, in particular [112] and [47], and the fractional moments method introduced by Aizenman and Molchanov in [3] for models on $\ell^2(\mathbb{Z}^d)$ and later adapted to $L^2(\mathbb{R}^d)$ in [2].

If the random variables are discrete (e.g. Bernoulli random variables), a proof of localization is much more difficult. It was proven in 2005 by Bourgain and Kenig [15] that exponential localization holds near the bottom of the spectrum for Bernoulli models on $L^2(\mathbb{R}^d)$. This result was later extended by Germinet and Klein, who proved dynamical localization for this model, among other results in [48]. A proof of localization is still missing for Bernoulli models on $\ell^2(\mathbb{Z}^d)$ with $d \geq 2$.

1.1.3 Quantum graphs

A natural question now, motivated by nanotechnology among others, is to investigate different shapes of the semiconductor. Quantum graphs arise when one wants to understand the propagation of waves through a quasi-one-dimensional system that looks like a thin neighborhood of a graph.

A *quantum graph* is a triple $(\mathcal{E}, \mathcal{V}, H)$, where $(\mathcal{E}, \mathcal{V})$ is a graph, with a vertex set \mathcal{V} and an edge set \mathcal{E} . Each edge e is assigned a length l_e , and H is a Schrödinger operator acting on $\bigoplus_{e \in \mathcal{E}} L^2(0, l_e)$. While the term “quantum graph” seems to have been introduced quite recently in the mathematical literature, the study of differential equations on graphs actually goes back at least to the early 1980s, see [90] for a review.

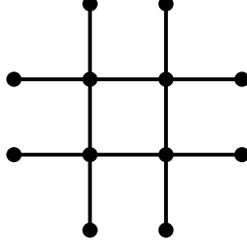
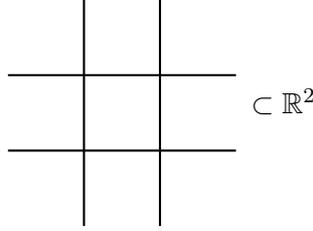
To verify localization on such structures, one may interpret the impurities as sources of randomness in the quantum graph. For models with a \mathbb{Z}^d structure, localization near the spectral edge has been established for a random potential model in [42], for a random vertex coupling model in [71], and for a random edge length model in [72]. Related results were obtained in [4] and [55] for random quantum tree graphs.

Let us describe a simple model. Let $\mathcal{V} = \mathbb{Z}^d$, $(h_j)_{j=1}^d$ be the canonical basis of \mathbb{Z}^d and let

$$\mathcal{E} = \{(m, j) : m \in \mathbb{Z}^d, j \in \{1, \dots, d\}\},$$

where (m, j) denotes the edge between m and $m + h_j$. We thus have one edge between each pair of neighbouring vertices. We now regard each edge as a segment of length 1: let $[m, m + h_j] := \{(1 - t)m + t(m + h_j) : t \in [0, 1]\} = \{m + th_j : t \in [0, 1]\}$. We define

$$\begin{aligned} \Gamma^{(1)} &= \bigcup_{m \in \mathbb{Z}^d, j \in \{1, \dots, d\}} [m, m + h_j] \\ &= \{x \in \mathbb{R}^d : \exists m \in \mathbb{Z}^d, j \in \{1, \dots, d\}, 0 \leq t \leq 1, \text{ such that } x = m + th_j\}. \end{aligned}$$

Figure 1.1: Part of $(\mathcal{E}, \mathcal{V})$ when $d = 2$.Figure 1.2: Part of $\Gamma^{(1)}$ when $d = 2$.

$\Gamma^{(1)}$ is simply the underlying geometric structure of the graph $(\mathcal{E}, \mathcal{V})$. If $d = 1$, then $\Gamma^{(1)} = \mathbb{R}$. A part of $(\mathcal{E}, \mathcal{V})$ and $\Gamma^{(1)}$ is given in Figures 1.1 and 1.2, respectively, when $d = 2$.

As a subset of \mathbb{R}^d , $\Gamma^{(1)}$ inherits the metric given by the supremum norm of \mathbb{R}^d .

Let $\mathcal{H} = \oplus_{e \in \mathcal{E}} L^2(0, 1)$. The Lebesgue measure on $[0, 1]$ induces a natural measure on $\Gamma^{(1)}$ which we denote by $m^{(1)}$. As shown in Chapter 2, it can easily be shown that \mathcal{H} can be identified with $L^2(\Gamma^{(1)}, dm^{(1)})$. Let $q_- < q_+ \in \mathbb{R}$ and let μ be a probability measure on \mathbb{R} with support $[q_-, q_+]$. Consider the probability space (Ω, \mathbb{P}) , where $\Omega := [q_-, q_+]^{\mathcal{E}}$, and $\mathbb{P} := \otimes_{e \in \mathcal{E}} \mu$. Then given $\omega = (\omega_e)_{e \in \mathcal{E}} \in \Omega$, the Schrödinger operator $H(\omega)$ is given by

$$H(\omega) : (f_e) \mapsto (-f_e'' + \omega_e f_e),$$

with domain

$$D(H(\omega)) = \left\{ f = (f_e) \in \bigoplus_{e \in \mathcal{E}} W^{2,2}((0, 1)) \left| \begin{array}{l} \text{for each } v \in \mathcal{V}, f \text{ is continuous at } v, \\ \text{and } \sum_{e: \iota e = v} f_e'(0) - \sum_{e: \tau e = v} f_e'(1) = 0. \end{array} \right. \right\}.$$

Here $\iota(m, j) = m$ and $\tau(m, j) = m + h_j$ represent the initial and terminal vertices of an edge (m, j) . The boundary conditions in $D(H(\omega))$ are known as *Kirchhoff boundary conditions*. Note that it is such boundary conditions that reflect the connectivity of the graph. The above model was studied in 2007 by Exner, Helm and Stollmann, and they proved in [42] that $H(\omega)$ has an almost sure spectrum Σ , and that both exponential and strong dynamical localization hold near the lower edge of Σ .

1.2 Multi-particle quantum graphs

In the first part of this thesis, we study the N -particle analog of the model described in Section 1.1.3. We call it a *multi-particle quantum graph* to emphasize the fact that not only the Schrödinger operator H changes in this case, but also the geometric structure Γ , namely it is no longer a collection of vertices and edges.

1.2.1 The model

To study the interaction between two particles, one lying on an edge e_1 and the other lying on an edge e_2 , one intuitively has to consider the couple (e_1, e_2) . So instead of studying a collection of edges \mathcal{E} , one now has to study a collection of edge couples or rectangles by identification. So one obtains a kind of two-dimensional network of rectangles. More generally, to study the interaction between N particles, one will have to consider an N -dimensional network of rectangular boxes. This is just the intuitive picture¹.

More formally, if $\Gamma^{(1)} \subset \mathbb{R}^d$ is the set introduced in Section 1.1.3, then the Hilbert space one considers for the study of 1-particle systems on $\Gamma^{(1)}$ is $\mathcal{H}_1 = L^2(\Gamma^{(1)}, dm^{(1)})$. Now quantum mechanics tells us that the Hilbert space corresponding to N distinguishable particles, each living in $\Gamma^{(1)}$ is the tensor product $\mathcal{H}_N = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1$. Taking the cartesian product

$$\Gamma^{(N)} := \Gamma^{(1)} \times \dots \times \Gamma^{(1)} \subset (\mathbb{R}^d)^N$$

and the product measure $m := m^{(1)} \otimes \dots \otimes m^{(1)}$, \mathcal{H}_N may be identified² with $L^2(\Gamma^{(N)}, dm)$.

Each $\mathbf{x} = (x_1, \dots, x_N) \in \Gamma^{(N)}$ takes the form $x_k = m_k + t^k h_{j_k}$, for some $m_k \in \mathbb{Z}^d$, $j_k \in \{1, \dots, d\}$ and $t^k \in [0, 1]$. Hence, if for $\mathbf{m} = (m_1, \dots, m_N) \in (\mathbb{Z}^d)^N$ and $\mathbf{j} = (j_1, \dots, j_N) \in \{1, \dots, d\}^N$ we put

$$\kappa_{\mathbf{m}, \mathbf{j}} := [m_1, m_1 + h_{j_1}] \times \dots \times [m_N, m_N + h_{j_N}],$$

we have $\Gamma^{(N)} = \bigcup_{\mathbf{m} \in (\mathbb{Z}^d)^N, \mathbf{j} \in \{1, \dots, d\}^N} \kappa_{\mathbf{m}, \mathbf{j}}$. We may thus regard $\Gamma^{(N)}$ as a couple $(\mathcal{K}, \mathcal{S})$, where \mathcal{K} is a collection of N -dimensional cubes κ of unit volume and \mathcal{S} is the set of the boundaries σ of these cubes.

For $d = 1$ and $N = 2$, we clearly have $\Gamma^{(2)} = \mathbb{R}^2$. The couple $(\mathcal{K}, \mathcal{S})$ corresponding to $\Gamma^{(2)}$ is described in Figure 1.3. For $d = 2$, $\Gamma^{(2)}$ consists of an infinite collection of sheets intersecting each other in \mathbb{R}^4 .

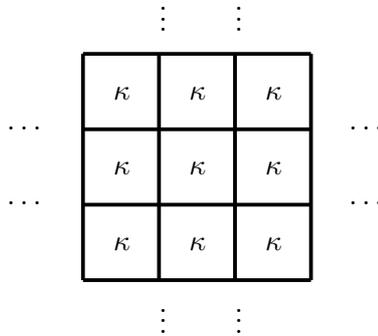


Figure 1.3: The couple $(\mathcal{K}, \mathcal{S})$ corresponding to $\Gamma^{(2)}$ for $d = 1$. It is an infinite collection of squares κ cornered in \mathbb{Z}^2 and covering \mathbb{R}^2 .

Each σ is a closed union of $2N$ “open faces” σ^i , i.e. $\sigma = \cup_i \sigma^i$. For example, if σ is the boundary of $\kappa_{\mathbf{m}, \mathbf{j}}$, then $\sigma^1 = \{m_1\} \times (m_2, m_2 + h_{j_2}) \times \dots \times (m_N, m_N + h_{j_N})$ and its opposite face is $\sigma^{o(1)} = \{m_1 + h_{j_1}\} \times (m_2, m_2 + h_{j_2}) \times \dots \times (m_N, m_N + h_{j_N})$.

The cube $\kappa_{\mathbf{m}, \mathbf{j}}$ will be denoted $\kappa = ((m_1, j_1), \dots, (m_N, j_N))$. With the edge notation $e_k = (m_k, j_k)$, this becomes $\kappa = (e_1, \dots, e_N)$.

1. Let us mention here that the first systematic study of boundary value problems on two-dimensional polygonal networks probably appeared in the works of Nicaise, see [85]. We also record the recent article [14] which considers compact quantum graphs with singular two-particle interactions.

2. See for example [93, Theorem II.10].

Let us now introduce our N -particle Schrödinger operator $H^{(N)}(\omega)$. This will give us a *multi-particle quantum graph* $(\mathcal{K}, \mathcal{S}, H^{(N)})$.

Fix $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$ and let μ be a probability measure on \mathbb{R} with support $[q_-, q_+]$. Now consider the Hilbert space $\mathcal{H} := \bigoplus_{\kappa \in \mathcal{K}} L^2((0, 1)^N)$. This space can easily be identified with $L^2(\Gamma^{(N)}, dm)$ as shown in Chapter 2. Let (Ω, \mathbb{P}) be the probability space given by $\Omega := [q_-, q_+]^{\mathcal{E}}$ and $\mathbb{P} := \bigotimes_{e \in \mathcal{E}} \mu$. Then for $\omega = (\omega_e) \in \Omega$, the Schrödinger operator is defined via the form

$$\mathfrak{h}_\omega^{(N)}[f, g] = \sum_{\kappa \in \mathcal{K}} [\langle \nabla f_\kappa, \nabla g_\kappa \rangle + \langle V_\kappa^\omega f_\kappa, g_\kappa \rangle],$$

with domain

$$D(\mathfrak{h}_\omega^{(N)}) = \left\{ f = (f_\kappa) \in \bigoplus_{\kappa \in \mathcal{K}} W^{1,2}((0, 1)^N) \mid \begin{array}{l} f \text{ is continuous on each } \sigma^i, \\ \sum_{\kappa \in \mathcal{K}} \|f_\kappa\|_{W^{1,2}}^2 < \infty \end{array} \right\}.$$

Continuity on σ^i means that if σ^i is a common face to κ_1 and κ_2 , then $f_{\kappa_1}|_{\sigma^i} = f_{\kappa_2}|_{\sigma^i}$ in the trace sense.

Here $V_\kappa^\omega := U_\kappa^{(N)} + W_\kappa^\omega$, where $U^{(N)} \geq 0$ is a non-random interaction potential. The important assumption on $U^{(N)}$ is that it has a finite range. For example, for $N = 2$, this means that there exists an $r_0 > 0$ such that for any $x_1, x_2 \in \Gamma^{(1)}$, $x_k = m_k + t^k h_{j_k}$ we have

$$|x_1 - x_2| > r_0 \implies U^{(2)}(x_1, x_2) = U_{\kappa_{\mathbf{m}, \mathbf{j}}}^{(2)}(t^1, t^2) = 0,$$

where $|\cdot| := \|\cdot\|_\infty$ is the sup norm of \mathbb{R}^d . Thus, there is no interaction between two particles at respective positions x_1 and x_2 which are far apart.

W_κ^ω is a multi-particle random potential. If $\kappa = (e_1, \dots, e_N)$, then $W_\kappa^\omega := \omega_{e_1} + \dots + \omega_{e_N}$. Again look at the simplest case $d = 1$ and $N = 2$ in Figure 1.4.



Figure 1.4: Two squares κ^1 and κ^2 sharing an edge e_2 .

Here f_{κ^1} is multiplied by $\omega_{e_1} + \omega_{e_2}$ while f_{κ^2} is multiplied by $\omega_{e_3} + \omega_{e_2}$. We thus see there is no independence at a distance: if κ^1 and κ^2 share an edge e_2 , then no matter how far κ^1 and κ^2 are, one always has the term ω_{e_2} .

It is quite easy to see that $\mathfrak{h}_\omega^{(N)}$ corresponds to a unique self-adjoint operator $H^{(N)}(\omega)$ given by

$$H^{(N)}(\omega) : (f_\kappa) \mapsto ((-\Delta + V_\kappa^\omega)f_\kappa)$$

on a certain domain $D(H^{(N)}(\omega))$. The explicit form of $D(H^{(N)}(\omega))$ seems to be a subtle question; see Chapter 2.

1.2.2 Main results

Here are the main results we give in Chapter 2, which were obtained in [97]. First of all, the lower part of the spectrum is deterministic:

Theorem 1.2.1. *There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$:*

$$[Nq_-, Nq_+] \subset \sigma(H^{(N)}(\omega)) \subseteq [Nq_-, +\infty).$$

In particular, $\inf \sigma(H^{(N)}(\omega)) = Nq_-$ almost surely.

Then, assuming μ is Hölder continuous, we have localization near the spectral bottom:

Theorem 1.2.2 (Exponential localization). *There exists $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ and $m > 0$ such that for a.e. ω the spectrum of $H^{(N)}(\omega)$ in $I = [Nq_-, Nq_- + \varepsilon_0]$ is pure point and the eigenfunctions corresponding to eigenvalues in I decay exponentially with mass m .*

Theorem 1.2.3 (Strong HS-dynamical localization). *There exists $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ such that for $I = [Nq_-, Nq_- + \varepsilon_0]$, we have for any bounded $K \subset \Gamma^{(N)}$ and all $s > 0$,*

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^{s/2} f(H^{(N)}(\omega)) E_\omega(I) \chi_K\|_2^2 \right\} < \infty,$$

where $(X\psi)(\mathbf{x}) := |\mathbf{x}| \cdot \psi(\mathbf{x})$ for $\psi \in L^2(\Gamma^{(N)}, dm)$, E_ω is the spectral projection of $H^{(N)}(\omega)$ and the supremum is taken over bounded Borel functions, $\|f\| := \|f\|_\infty$.

In the special case where $N = 1$, these theorems are stronger than the main result of [42], first because they hold without the technical disorder assumption ($\exists \tau > \frac{d}{2} : \mu([q_-, q_- + h]) \leq h^\tau$ for small h), next because strong dynamical localization holds in the Hilbert-Schmidt norm.

1.2.3 Ideas of the proof

The proof of these theorems is based on the multi-particle multi-scale analysis developed by A. Boutet de Monvel, Chulaevsky and Suhov in [27] and [19]. The multi-scale induction becomes significantly more involved in the multi-particle setting because one loses the independence at a distance. The passage from multiscale analysis bounds to localization is based on the approach of Germinet and Klein [47].

Before adapting these methods to multi-particle quantum graphs, there is a list of input to verify. Here are the main difficulties that lie in the passage from $\Gamma^{(1)}$ to $\Gamma^{(N)}$.

First of all in the Combes-Thomas estimate, we need to have a good control on the decay exponent, because here the Combes-Thomas estimate is used not only to derive the initial length scale estimate, but also in the multiscale induction itself. Roughly speaking, the exponent should depend (as usual) on the distance of the energy to the spectrum, but *not* on the absolute value of the energy. We derived such a Combes-Thomas estimate using semigroups, via an improved Davies-Gaffney estimate.

Next we had to derive some trace estimates which are necessary to obtain generalized eigenfunction expansions.

Next there is the eigenfunction decay inequality (EDI): in the usual derivation, one needs to be sure that the generalized eigenfunctions have some regularity, say they are locally of class $W^{1,2}$. However, to use the approach of Germinet and Klein, we needed the generalized eigenfunction expansion usually attributed to Berezansky (see [11]) whose functions are not a priori regular. Instead of proving they are indeed regular³, we were

3. We later proved in Chapter 3 that the generalized eigenfunctions are regular. But as the reader will see, it is simpler and more natural to reason directly with approximations, without relying on regularity.

able to derive EDI by means of approximations in the negative Hilbert space (that is, the Hilbert space in which the generalized eigenfunctions live).

Finally we needed a stronger initial length scale estimate for $H^{(1)}$ than the one proved in [42] and this is why we proved some Lifshitz-type asymptotics. The perturbation argument used to derive them in the continuum also works here, but at some point one needs to know the width of the gap between the first two eigenvalues of $H^{(1)}$ restricted to a finite graph. It turns out that this problem had been studied in Riemannian geometry, and one way to estimate this gap is by means of Cheeger Inequality⁴. This inequality was later proved for finite quantum graphs by Nicaise in [84] and it also appeared in an article by Post [91].

These are the main difficulties. There are also some details to settle to obtain the rest of the input, namely the geometric resolvent inequalities and the Wegner estimate. For the former, note that besides the usual resolvent inequality needed for single-particle models, one needs an additional one in the multi-particle setting, namely one that compares the Green functions of a certain type of N -cubes to those of n -cubes for $n < N$. A Combes-Thomas estimate with a good exponent enters again in the proof. For the latter, the main difference in the N -particle setting is that one needs to make sure that the random potential is well distributed in any 1-particle direction. More precisely, the Wegner bound one really needs for localization is not one that estimates the probability on the whole space Ω , but a stronger conditional one instead. This is needed to prove the two-volume bound later on.

1.2.4 Variations

In Chapter 3, we give a variation of these results. Namely, we are able to prove exponential localization and strong dynamical localization of any order without relying on generalized eigenfunction expansions. However, using this approach, we cannot achieve it in the Hilbert-Schmidt norm. We also explain how far our results can be generalized if the distribution of the random variables is only log-Hölder continuous. Finally, we give a Combes-Thomas estimate which is valid for energies in arbitrary spectral gaps, but the price to pay is that the proof becomes quite complicated.

1.3 Abstract Wegner estimates

The second part of this thesis is concerned with the formulation of some abstract Wegner estimates and their application to concrete models.

1.3.1 Wegner estimates in general

The objective of Wegner estimates is to derive good bounds on the average number of eigenvalues of a random Schrödinger operator $H_\Lambda(\omega)$ restricted to a cube Λ in a fixed interval I . More precisely, if $\chi_I(H_\Lambda(\omega))$ is the spectral projection of $H_\Lambda(\omega)$ onto the interval I , then one seeks estimates of the type

$$\mathbb{E}\{\mathrm{tr}[\chi_I(H_\Lambda(\omega))]\} \leq C_W \cdot |\Lambda|^\alpha \cdot |I|^\beta$$

for some $C_W \geq 0$, $\alpha \geq 1$ and $\beta > 0$. Such estimates can be used in a proof of localization via multiscale analysis, or in the study of continuity properties of the integrated density of states (IDS) if $\alpha = 1$.

4. We give a different approach in Appendix B, which goes by the name of Faber-Krahn inequality.

1.3.2 Motivation

The motivation for this work came again from quantum graphs. A very interesting feature of quantum graphs is that one can study randomness not only in the potential, but also in the graph itself. This gives rise to two new models: quantum graphs with random vertex couplings (RCM, i.e. random boundary conditions) and quantum graphs with random edge lengths (RLM); see [55], [71] and [72] for localization results for these models.

I was first interested in the RLM, so I studied the articles [72] and [78]. These works establish many results including localization near the spectral edge and continuity of the IDS. I focused on the Wegner bound, which I found to be particularly interesting as the dependence on the random parameter becomes quite unusual for such models.

The precise description of these models would be too long, so I will only discuss two major tricks that were applied in these works to obtain the Wegner bound. In [72], the idea was to establish a relationship between the Schrödinger operator $H(l^\omega)$ on the quantum graph $\oplus_{e \in \mathcal{E}} L^2(0, l_e^\omega)$ and a discrete counterpart $M(l^\omega, I)$ which acts on the set of vertices, i.e. in the Hilbert space $\ell^2(\mathcal{V})$. Here I is the interval where the Wegner bound is to be established. This simplified the task a little bit, because for finite subgraphs $(\mathcal{E}_\Lambda, \mathcal{V}_\Lambda)$, the space $\ell^2(\mathcal{V}_\Lambda)$ is finite-dimensional. However, the dependence of $M(l^\omega, I)$ on $l^\omega = (l_e^\omega)$ becomes difficult (terms like $(\sin l_e^\omega)^{-1}$ arise), so new arguments have to be conceived.

In [78], the first idea was to work with the random variables $(\ln l_e^\omega)_{e \in \mathcal{E}}$ instead of the random variables $(l_e^\omega)_{e \in \mathcal{E}}$. This makes the dependence on the random parameter simpler, and some precise relationships between the eigenvalues can be derived. Of course the probability space needs to be modified in a corresponding way. The conclusion is not immediate because even with this, the dependence on the random parameter is unusual.

Both works assumed the random variables (l_e^ω) are i.i.d. with an absolutely continuous distribution. So I asked myself how we could relax this condition and work with a Hölder continuous distribution.

A second motivation came from the usual Schrödinger operators on $\ell^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, but which have no covering condition (i.e. the single site potentials do not cover all points in \mathbb{Z}^d or all regions in \mathbb{R}^d respectively). For these operators the available proofs of the Wegner bound are complicated if one does not assume the probability measure has a density, see e.g. [30].

1.3.3 Results

We attempt to give a simple and unified treatment for these models in the paper [98].

Finite-dimensional Hilbert spaces

We first derive a Wegner bound for some random self-adjoint operators on a finite-dimensional Hilbert space. We do not assume the distribution of the random variables has a density.

As an application, we give optimal Wegner bounds for the discrete multi-particle Anderson model for three different cases:

- If there is a covering condition, i.e. if there exists $c > 0$ such that $V^\omega = \sum_{\alpha \in \mathbb{Z}^d} \omega_\alpha u_\alpha$, $u_\alpha \geq c \cdot \delta_\alpha$, we obtain both one-volume and two-volume bounds for arbitrary energy intervals.
- Otherwise, if the perturbation is negative and supported on a half-space, we give a Wegner bound near the spectral bottom. We show it is non-trivial, i.e. the interval

we consider does contain some spectrum.

- Finally, if the perturbation is positive, we still have an optimal bound, but to illustrate that it is non-trivial, we use a result from [37], where the authors consider positive perturbations supported on Delone sets (which are examples of random potentials with no covering condition).

Let us mention here that the half-space potentials we considered in the case of negative perturbations are not Delone potentials, and that Delone potentials are not half-space potentials either. So one can say that our paper [98] on one hand and the papers [37] and [94] on the other hand complement each other. Finally, let us mention that all these bounds assume the single-site potential has a fixed sign.

Later on we give in Chapter 5 many variations of the abstract theorem, including an adaptation of the result of [30] to finite-dimensional spaces.

Separable Hilbert space

Next we consider the more general framework of separable Hilbert spaces and give two bounds. The first one has almost no restriction on the energy interval or the form of the random parameter, but requires a form of monotonicity and covering. The second one is a bit more restrictive on the interval and on the form of the random parameter, but does not require covering.

We apply the first bound to the RLM of [72]. This gives us a positive answer to our initial question concerning this model: the bound of [72] can indeed be generalized to Hölder continuous distributions. In particular, this allows to extend the localization results of this paper to such distributions (if the coupling constant is positive).

We apply the second bound to RCMs, and also to continuum multi-particle Hamiltonians with no covering condition. Here we have again two situations: for negative perturbations supported on half-spaces, we prove a Wegner bound near the spectral bottom of the random operator, and we show it is non-trivial. For positive perturbations, our bound is only non-trivial for Schrödinger operators in the fluctuation boundary regime. However, here the situation is much better than in the lattice, because a large class of operators with no covering condition in the continuum satisfy this regime. Namely, potentials with some periodicity [62, Theorem 2.2], surface potentials and Delone potentials [21, Sections 4,5] and the general crooked potentials [66, Lemma 4.2].

Let us note that in contrast to the lattice case, the bounds that we obtain here are not optimal, namely the dependence on $|\Lambda|$ is not linear. But this is still sufficient for localization purposes.

Let us briefly compare our results with the literature. It seems that half-space and related potentials were not considered before⁵, so our result for negative perturbations seems to be new. For positive perturbations, our result is very close in spirit to [20]. However this paper builds on the results of [30], which are technically involved. We take a completely different approach to prove our theorem, and our proof is quite elementary. Still, the paper [20] has the advantage of providing a linear dependence on $|\Lambda|$, which makes it suitable for the study of the integrated density of states. Our estimate for the RCMs appears to be new.

Finally, let us mention that in the case of Delone operators in the continuum, the very recent papers [95] and [66] were able to extend Wegner estimates to arbitrary small intervals, not just intervals near the spectral bottom. Thus, in the case of continuum

5. Such potentials fall within the framework of [63], but here we show that there is indeed some spectrum in the considered energy interval, a question which does not appear to be addressed in [63].

Delone operators, our result is a special case of these papers, but our proof is very different (for example, [66] builds on the results of [30] and uses a recent result [16]).

Let us emphasize again that all these results assume the single-site potential has a fixed sign. This assumption is relaxed a little bit in [95], but does not disappear.

Let us discuss the proofs of our general theorems. The first bound follows from an easy variation of Stollmann's Lemma [103]; one should just pay attention to some measurability issues. The proof of the second bound is based on two tricks: first as in [78] work with the random variables $(\ln \omega_j)_j$ instead of $(\omega_j)_j$, next modify Stollmann's Lemma to make it work for more general diagonal dependencies.

It is quite curious that we could not apply this theorem to the model of [78] which inspired us part of the proof: the problem of this model is that the domain of the random operator $H(\ell_\omega)$ depends on ℓ_ω . We explain this issue in detail in Section 5.4.

1.4 Perspectives

There are many questions which are still waiting for an answer in both works.

1.4.1 Quantum graphs

Here we only studied the most simple multi-particle quantum graph. There are two natural questions:

- (i) first the geometry,
- (ii) second the boundary conditions.

Can the proof be easily modified if they are generalized? Or does a really difficult situation occur? It seems both questions are explored for single-particle models in the recent work of Schubert [99], which could provide a good starting point.

We think that simple generalizations of the geometry, e.g. rectangular boxes instead of cubes, should be straightforward. We also think one could assign many different volumes for the elementary cubes, perhaps even make them grow; maybe as a compromise the decay of the eigenfunctions would no longer be exponential but polynomial. Note however that too general geometries may destroy the ergodicity. So it will be important to search for geometries which give us at least a spectral bottom which is not random.

For the boundary conditions, we might lose the Lifshitz tails; one should take a closer look at the proof. A few technical lemmas will also need to be modified.

Finally there is the fact that we used the embedding in the Euclidean space in many situations, but this can probably be avoided without much difficulty. In particular, it seems plausible that, to study multi-particle systems on any quantum graph, one should work with a network of rectangular boxes. Indeed, we have a canonical isomorphism between $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$ and $L^2(M_1 \times M_2, d\mu_1 \otimes d\mu_2)$ for any measure spaces (M_1, μ_1) and (M_2, μ_2) , so in particular for networks with the 1-dimensional Lebesgue measure.

All this concerns localization; one could also investigate the spectral statistics of the eigenvalues.

1.4.2 Wegner estimates

For the second work there are at least three questions.

First, can we modify the abstract theorems, or prove new ones, which are capable of treating Hamiltonians with sign-changing potentials? or more generally, operators with non-monotone dependence on the random variables? To the best of our knowledge, the problem of deriving a Wegner bound for such models without assuming the distribution has a density is still open, at least in the continuum. So that's a question which we think will be very interesting. We *attempt* to solve it in Section 5.4.1. The papers [38, 69, 54, 111, 110, 75, 39] may allow us to go further.

The second question concerns the non-linearity issue in the Wegner bound for separable Hilbert spaces. This problem arose because we make use of a modified version of Stollmann's Lemma to control the eigenvalues. Can we get rid of this phenomenon? Should we first manipulate the trace of the spectral projection before using this lemma, like we did in the case of finite-dimensional spaces? Should we modify Stollmann's lemma itself? This lemma is already optimal, but maybe if we find better properties of the eigenvalues in question, we could add more hypotheses to the lemma and obtain a linear bound. Or should we think of a totally different approach? Solving this question is worth the effort because it can open a new road to the study of the continuity of the integrated density of states.

The third question is the following: our Wegner bound for separable Hilbert spaces is valid for very general potentials, but only holds for intervals near the spectral edge. So the question is whether we can give a similarly short proof of a Wegner bound, but this time at all energies, if we consider potentials that have some periodicity as in [30]. The eventual non-linearity in $|\Lambda|$ will certainly be a reasonable price to pay if the new proof is elementary.

Chapter 2

Localization for a Multi-Particle Quantum Graph

2.1 Multi-particle Quantum Graphs

In this chapter we study localization for $(\mathcal{K}, \mathcal{S}, H^{(N)})$, the multi-particle quantum graph introduced in Section 1.2. We shall construct it again in this section and recall the main results, because we omitted some details in Chapter 1 to simplify the exposition.

2.1.1 1-Graphs

Our building block is the quantum graph $(\mathcal{E}, \mathcal{V})$ of [42], given by the vertex set $\mathcal{V} = \mathbb{Z}^d$ and the edge set \mathcal{E} of all line segments of length 1 between two neighbouring vertices. This graph is naturally embedded in \mathbb{R}^d and we denote by $\Gamma^{(1)} \subset \mathbb{R}^d$ the image of the embedding. To describe $\Gamma^{(1)}$ explicitly, let $(h_j)_{j=1}^d$ be the standard basis of \mathbb{Z}^d . Then

$$\Gamma^{(1)} := \{x \in \mathbb{R}^d : x = m + th_j \text{ for some } m \in \mathbb{Z}^d, j \in \{1, \dots, d\} \text{ and } t \in [0, 1]\}.$$

We denote the edge between m and $m + h_j$ by $e = (m, j)$. Such an edge is identified with the interval $[0, 1]$ by sending $x = m + th_j \in e$ to the point t . The Lebesgue measure on $[0, 1]$ then induces a natural measure¹ on $\Gamma^{(1)}$ which we denote by $m^{(1)}$.

A function f on $\Gamma^{(1)}$ induces a sequence (f_e) , $f_e : (0, 1) \rightarrow \mathbb{C}$ by setting $f(x) =: f_{(m,j)}(t)$ when $x = m + th_j$, for some $m \in \mathbb{Z}^d$ and $t \in (0, 1)$. As equality in L^2 is a.e., this in turn identifies $L^2(\Gamma^{(1)}, dm^{(1)})$ with $\bigoplus_{e \in \mathcal{E}} L^2(0, 1)$.

Now fix $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$ and let μ be a probability measure on \mathbb{R} with support $[q_-, q_+]$. Consider the Hilbert space $\mathcal{H} := \bigoplus_{e \in \mathcal{E}} L^2(0, 1)$, the probability space (Ω, \mathbb{P}) , where $\Omega := [q_-, q_+]^{\mathcal{E}}$ and $\mathbb{P} = \bigotimes_{e \in \mathcal{E}} \mu$, and given $\omega = (\omega_e) \in \Omega$, define the form

$$\mathfrak{h}_\omega^{(1)}[f, g] = \sum_{e \in \mathcal{E}} [\langle f'_e, g'_e \rangle + \langle \omega_e f_e, g_e \rangle], \quad D(\mathfrak{h}_\omega^{(1)}) = W^{1,2}(\Gamma^{(1)}),$$

where

$$W^{1,2}(\Gamma^{(1)}) := \left\{ f \in \bigoplus_{e \in \mathcal{E}} W^{1,2}(0, 1) \left| \begin{array}{l} f \text{ is continuous at each } v \in \mathcal{V}, \\ \sum_{e \in \mathcal{E}} \|f_e\|_{W^{1,2}}^2 < \infty \end{array} \right. \right\}.$$

1. If $A \subset \Gamma^{(1)}$, then $A = \bigcup A_{e_j}$ for some disjoint $A_{e_j} \subset e_j$. Each $A_{e_j} \equiv B_{e_j} \subset [0, 1]$ and we set $m^{(1)}(A) := \sum |B_{e_j}|$.

This form corresponds to the self-adjoint operator $H^{(1)}(\omega) : (f_e) \mapsto (-f_e'' + \omega_e f_e)$ with Kirchhoff boundary conditions (i.e. if $f \in D(H^{(1)}(\omega))$ and $v \in \mathcal{V}$, then f is continuous at v and satisfies $\sum_{j=1}^d f'_{(v,j)}(0) - \sum_{j=1}^d f'_{(v-h_j,j)}(1) = 0$). It is shown in [42] that $H^{(1)}(\omega)$ has an almost sure spectrum $\Sigma = [q_-, +\infty)$ and that localization holds near q_- .

2.1.2 n -Graphs

Let us emphasize that throughout this chapter and the next one, the number of particles

$$N \text{ is fixed.}$$

We will need to consider Hamiltonians $H^{(n)}(\omega)$ for $1 \leq n \leq N$ because we will later deduce some spectral properties of $H^{(N)}(\omega)$ from those of $H^{(n)}(\omega)$.

So let us fix $1 \leq n \leq N$ and consider n -particle systems. Formally, quantum mechanics tells us that the Hilbert space corresponding to n distinguishable particles, each living in $\Gamma^{(1)}$, is the tensor product $L^2(\Gamma^{(1)}, dm^{(1)}) \otimes \dots \otimes L^2(\Gamma^{(1)}, dm^{(1)})$. Taking

$$\Gamma^{(n)} := \Gamma^{(1)} \times \dots \times \Gamma^{(1)} \quad \text{and} \quad m^{(n)} := m^{(1)} \otimes \dots \otimes m^{(1)},$$

this space may be identified with $L^2(\Gamma^{(n)}, dm^{(n)})$.

If $(h_j)_{j=1}^d$ is the canonical basis of \mathbb{Z}^d , then each point $\mathbf{x} = (x_1, \dots, x_n) \in \Gamma^{(n)}$ takes the form $x_k = m_k + t^k h_{j_k}$ for some $m_k \in \mathbb{Z}^d$, $t^k \in [0, 1]$ and $j_k \in \{1, \dots, d\}$. By varying t^k from 0 to 1, we thus obtain a cube κ which may be identified with $[0, 1]^n$ by sending such an \mathbf{x} to (t^1, \dots, t^n) . Thus, we may regard $\Gamma^{(n)}$ as a couple $(\mathcal{K}, \mathcal{S})$, where \mathcal{K} is a collection of n -dimensional cubes κ and \mathcal{S} is the collection of the boundaries σ of κ .

For $d = 1$, $\Gamma^{(2)} = \mathbb{R}^2$. If we regard it as a couple $(\mathcal{K}, \mathcal{S})$, then it consists of unit squares covering \mathbb{R}^2 and cornered in \mathbb{Z}^2 . For $d = 2$, let x, y, z, t be the coordinate axes of \mathbb{R}^4 . Then $\Gamma^{(2)}$ lives in the planes xz, xt, yz and yt , and all their \mathbb{Z}^4 -translates, and consists of unit squares cornered in \mathbb{Z}^4 . Squares in the planes xy and zt (and their \mathbb{Z}^4 -translates) are not allowed². More generally, $\Gamma^{(2)}$ lives in the translates of d^2 planes in \mathbb{R}^{2d} and each affine plane is an infinite collection of κ .

For $n = 3$, the only case that can be visualized is that of $d = 1$, in which case $\Gamma^{(3)} = \mathbb{R}^3$, and is regarded as the set of all cubes of unit volume cornered in the lattice \mathbb{Z}^3 .

If the points of κ take the form (x_1, \dots, x_n) with $x_k = m_k + t^k h_{j_k}$ for some $m_k \in \mathbb{Z}^d$, $t^k \in [0, 1]$ and $j_k \in \{1, \dots, d\}$, we will denote $\kappa = ((m_1, j_1), \dots, (m_n, j_n))$. Hence, any $\kappa \in \mathcal{K}$ may be written as $\kappa = (e_1, \dots, e_n)$ for some $e_j \in \mathcal{E}$.

A function f on $\Gamma^{(n)}$ induces a sequence (f_κ) , $f_\kappa : (0, 1)^n \rightarrow \mathbb{C}$ by setting $f(\mathbf{x}) := f_{((m_1, j_1), \dots, (m_n, j_n))}(t^1, \dots, t^n)$ when $x_k = m_k + t^k h_{j_k}$, for some $m_k \in \mathbb{Z}^d$ and $t^k \in (0, 1)$. As equality in L^p is a.e., this in turn identifies $L^p(\Gamma^{(n)}, dm^{(n)})$ with $\bigoplus_{\kappa \in \mathcal{K}} L^p(0, 1)^n$ for $1 \leq p < \infty$, where $\|(f_\kappa)\|_{L^p}^p := \sum_{\kappa \in \mathcal{K}} \|f_\kappa\|_{L^p(0, 1)^n}^p$.

Each σ is the closed union of $2n$ "open faces" σ^i which may be identified with $(0, 1)^{n-1}$.

Given $\mathbf{x} = (x_1, \dots, x_n) \in \Gamma^{(n)} \subset \mathbb{R}^{nd}$ and a partition $\{1, \dots, n\} = \mathcal{J} \cup \mathcal{J}^c$, we put $x_{\mathcal{J}} := (x_j)_{j \in \mathcal{J}}$, $x_{\mathcal{J}^c} := (x_j)_{j \in \mathcal{J}^c}$ and define

$$\text{dist}(x_{\mathcal{J}}, x_{\mathcal{J}^c}) := \min\{|x_i - x_j| : i \in \mathcal{J}, j \in \mathcal{J}^c\}, \quad \text{where } |y| := \|y\|_\infty \text{ for } y \in \mathbb{R}^d.$$

2. Here we identified $(\mathbb{R}^2)^2$ with \mathbb{R}^4 via $((a, b), (c, d)) \mapsto (a, b, c, d)$. If $(a, b, c, d) \in \Gamma^{(2)}$, then $(a, b) \in \Gamma^{(1)}$, so $(a, b) = (m_1, m_2) + s_1(1, 0)$ or $(a, b) = (m_1, m_2) + s_1(0, 1)$ for some $m_1, m_2 \in \mathbb{Z}$ and $s_1 \in [0, 1]$. Similarly, for (c, d) , so we may find $m_j \in \mathbb{Z}$ and $s_k \in [0, 1]$ such that $(a, b, c, d) = (m_1 + s_1, m_2, m_3 + s_2, m_4)$ or $(m_1, m_2 + s_1, m_3 + s_2, m_4)$ or $(m_1 + s_1, m_2, m_3, m_4 + s_2)$ or $(m_1, m_2 + s_1, m_3, m_4 + s_2)$. By varying s_k from 0 to 1, we obtain squares in the planes xz, yz, xt and yt respectively, but never in the planes xy or zt .

Now fix $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$, and let μ be a probability measure on \mathbb{R} with support $[q_-, q_+]$. Consider the probability space (Ω, \mathbb{P}) with $\Omega := [q_-, q_+]^{\mathcal{E}}$, $\mathbb{P} := \otimes_{e \in \mathcal{E}} \mu$, the Hilbert space $\mathcal{H} := \oplus_{\kappa \in \mathcal{K}} L^2(0, 1)^n$, and given $\omega = (\omega_e) \in \Omega$, define the form

$$\mathfrak{h}_\omega^{(n)}[f, g] = \sum_{\kappa \in \mathcal{K}} [\langle \nabla f_\kappa, \nabla g_\kappa \rangle + \langle V_\kappa^\omega f_\kappa, g_\kappa \rangle], \quad D(\mathfrak{h}_\omega^{(n)}) = W^{1,2}(\Gamma^{(n)}),$$

where

$$W^{1,2}(\Gamma^{(n)}) := \left\{ f \in \bigoplus_{\kappa \in \mathcal{K}} W^{1,2}((0, 1)^n) \mid \begin{array}{l} f \text{ is continuous on each } \sigma^i, \\ \sum_{\kappa \in \mathcal{K}} \|f_\kappa\|_{W^{1,2}}^2 < \infty \end{array} \right\}.$$

By continuity on σ^i we mean that whenever σ^i is a common face to κ_1 and κ_2 , then $f_{\kappa_1}|_{\sigma^i} = f_{\kappa_2}|_{\sigma^i}$ in the trace sense³. The potential is given by $V_\kappa^\omega := U_\kappa^{(n)} + W_\kappa^\omega$, where W_κ^ω is an n -particle random potential, $W_\kappa^\omega := \omega_{e_1} + \dots + \omega_{e_n}$ if $\kappa = (e_1, \dots, e_n)$. The sequence $(U_\kappa^{(n)})$ is induced from a non-random interaction potential $U^{(n)} : \Gamma^{(n)} \rightarrow \mathbb{R}$ with the following properties:

(1) $U^{(n)}$ is bounded and non-negative: there exists $u_0 > 0$ such that

$$0 \leq U^{(n)}(\mathbf{x}) \leq u_0 \quad \text{for } \mathbf{x} \in \Gamma^{(n)}.$$

(2) $U^{(n)}$ has finite range⁴: there exists $r_0 > 0$ such that

$$\text{dist}(x_{\mathcal{J}}, x_{\mathcal{J}^c}) \geq r_0 \implies U^{(n)}(\mathbf{x}) = U^{(n')}(\mathcal{J}) + U^{(n'')}(x_{\mathcal{J}^c})$$

for any partition $\{1, \dots, n\} = \mathcal{J} \cup \mathcal{J}^c$ with $|\mathcal{J}| = n'$ and $|\mathcal{J}^c| = n''$.

(3) There is no one-particle potential:

$$U^{(1)} \equiv 0.$$

For $n = 2$, $U^{(2)}$ is thus function satisfying for $(x_1, x_2) \in \Gamma^{(2)} \subset (\mathbb{R}^d)^2$

$$0 \leq U^{(2)}(x_1, x_2) \leq u_0 \quad \text{and} \quad |x_1 - x_2| \geq r_0 \implies U^{(2)}(x_1, x_2) = 0.$$

Notice that if $|x_i - x_j| \geq r_0$ for all $i \neq j$, then $U^{(n)}(\mathbf{x}) = U^{(1)}(x_1) + \dots + U^{(1)}(x_n) = 0$. Condition (2) says that more generally, if $x_{\mathcal{J}}$ and $x_{\mathcal{J}^c}$ are far apart, then $U^{(n)}$ decouples as prescribed.

We may assume that $r_0 \in \mathbb{N}$; if this is not the case, we just consider $\lfloor r_0 \rfloor + 1$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Theorem 2.1.1. *Given $\omega \in \Omega$, $\mathfrak{h}_\omega^{(n)}$ is closed, densely defined and bounded from below. The unique self-adjoint operator $H^{(n)}(\omega)$ associated with $\mathfrak{h}_\omega^{(n)}$ is given by*

$$H^{(n)}(\omega) : (f_\kappa) \mapsto (-\Delta f_\kappa + V_\kappa^\omega f_\kappa), \quad \text{for } (f_\kappa) \in D(H^{(n)}(\omega)).$$

Proof. See the Appendix, Theorem 2.11.1. □

3. This means that if $\gamma : W^{1,2}((0, 1)^n) \rightarrow L^2((0, 1)^{n-1})$ is the trace operator, then $\gamma(f_{\kappa_1})(x) = \gamma(f_{\kappa_2})(x)$ for a.e. $x \in \sigma^i$.

4. This includes the 2-body interaction potentials $U^{(n)}(\mathbf{x}) = \sum_{1 \leq i < j \leq n} F(x_i - x_j)$, where $F : \Gamma^{(1)} \rightarrow \mathbb{R}$ satisfies $F(y) = 0$ if $|y| \geq r_0$. Indeed, if $\text{dist}(x_{\mathcal{J}}, x_{\mathcal{J}^c}) \geq r_0$, then we will have $F(x_i - x_j) = 0$ whenever $i \in \mathcal{J}$ and $j \in \mathcal{J}^c$, so that $U^{(n)}(\mathbf{x})$ indeed decouples into $U^{(n')}(\mathcal{J}) + U^{(n'')}(x_{\mathcal{J}^c})$.

We did not provide the explicit domain of $H^{(n)}(\omega)$ as it is not needed in the sequel. It is a subtle question to know exactly how regular the functions $(f_\kappa) \in D(H^{(n)})$ are; in particular, it is not clear if the normal derivatives of f_κ have a trace on σ^i . For $n = 1$, it is easy to see that if $(f_e) \in D(H^{(1)})$, then $f_e \in W^{2,2}(0,1)$ for each e . This gives a meaning in particular to the Kirchhoff conditions. Once $n \geq 2$ however, corner singularities appear which, in general, destroy the regularity of the f_κ , see e.g. [51]. If we had asked each f_κ to satisfy Dirichlet or Neumann conditions, we would have $f_\kappa \in W^{2,2}((0,1)^n)$ (see [51, Section 3.2]). However, as we ask f_κ to be continuous on σ^i , this regularity result is no longer clear. See [85, Section 2.3.2] for some results when $n = 2$ and [14] for some boundary conditions ensuring regularity also when $n = 2$. For general n -dimensional polyhedral interface problems, we record the result of [7].

Given $\mathbf{x} \in \mathbb{Z}^{Nd}$ put $\mathbf{C}(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^{Nd} : |\mathbf{y} - \mathbf{x}| < 1\}$, where $|\mathbf{z}| := \|\mathbf{z}\|_\infty$ and let $\chi_{\mathbf{x}} := \chi_{\Gamma^{(N)} \cap \mathbf{C}(\mathbf{x})}$. We say that $\psi \in L^2(\Gamma^{(N)})$ decays exponentially with mass $m > 0$ if

$$\limsup_{|\mathbf{x}| \rightarrow \infty} \frac{\log \|\chi_{\mathbf{x}} \psi\|}{|\mathbf{x}|} \leq -m.$$

We now state our main results. First of all, the lower part of the spectrum is deterministic:

Theorem 2.1.2. *There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$:*

$$[nq_-, nq_+] \subset \sigma(H^{(n)}(\omega)) \subseteq [nq_-, +\infty).$$

In particular, $\inf \sigma(H^{(n)}(\omega)) = nq_-$ almost surely.

Proof. See the Appendix, Theorem 2.11.2. □

Next, if μ is Hölder continuous, we have localization near the spectral edge of $H^{(N)}(\omega)$.

Theorem 2.1.3 (Exponential localization). *There exist $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ and $m > 0$ such that for a.e. ω the spectrum of $H^{(N)}(\omega)$ in $I = [Nq_-, Nq_- + \varepsilon_0]$ is pure point and the eigenfunctions corresponding to eigenvalues in I decay exponentially with mass m .*

Theorem 2.1.4 (Strong HS-dynamical localization). *There exists $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ such that for $I = [Nq_-, Nq_- + \varepsilon_0]$, we have for any bounded $K \subset \Gamma^{(N)}$ and all $s > 0$,*

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^{s/2} f(H^{(N)}(\omega)) E_\omega(I) \chi_K\|_2^2 \right\} < \infty,$$

where $(X\psi)(\mathbf{x}) := |\mathbf{x}| \cdot \psi(\mathbf{x})$ for $\psi \in L^2(\Gamma^{(N)})$, E_ω is the spectral projection of $H^{(N)}(\omega)$ and the supremum is taken over bounded Borel functions, $\|f\| := \|f\|_\infty$.

Theorems 2.1.3 and 2.1.4 are proved in Sections 2.9 and 2.10 respectively, using the multi-particle multiscale analysis introduced by Chulaevsky and Suhov in [27], and adapted to the continuum by A. Boutet de Monvel *et al.* in [19]. The traditional single-particle multiscale analysis was introduced by Fröhlich and Spencer in [46].

Let us note that for $N = 1$ our theorems improve the main result of [42], first by removing the technical assumption $(\exists \tau > \frac{d}{2} : \mu([q_-, q_- + h]) \leq h^\tau$ for small h), next by establishing strong dynamical localization in the HS norm. We are able to remove the assumption on μ by proving Lifshitz-type asymptotics for this model in Section 2.6, which to the best of our knowledge, were not proven in earlier papers.

2.2 Finite-volume operators and geometry of cubes

2.2.1 Finite-volume operators

Fix $1 \leq n \leq N$. Throughout the chapter we use the sup norm of \mathbb{R}^{nd} :

$$|x| := \|x\|_\infty, \quad |\mathbf{x}| := \|\mathbf{x}\|_\infty$$

for $x \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^{nd}$. Given $L \in \mathbb{N}^*$, we define 1-cubes with center $u \in \mathbb{Z}^d$ by

$$\Lambda_L^{(1)}(u) = \{x \in \mathbb{R}^d : |x - u| < L\}, \quad |\Lambda_L^{(1)}(u)| = (2L)^d.$$

Given $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^{nd}$ and $\mathbb{L} = (L_1, \dots, L_n) \in \mathbb{N}^n$ with $L_j \geq 1$, we define n -rectangles and n -cubes by

$$\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) = \prod_{j=1}^n \Lambda_{L_j}^{(1)}(u_j), \quad \Lambda_L^{(n)}(\mathbf{u}) = \Lambda_{(L, \dots, L)}^{(n)}(\mathbf{u}) = \prod_{j=1}^n \Lambda_L^{(1)}(u_j).$$

Note that a cube is always open. We take $\mathbf{u} \in \mathbb{Z}^{nd}$ and $\mathbb{L} \in \mathbb{N}^n$ above to ensure that the closure⁵ of $\Gamma^{(n)} \cap \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ is a subgraph of $\Gamma^{(n)}$. Abusing notation, we also denote this closure by $\Gamma^{(n)} \cap \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$. For 1-graphs, taking the closure means that we add the vertices lying on $\partial\Lambda_L^{(1)}(u)$ that belong to inner edges. This should not be confused with the larger subgraph $\Gamma^{(1)} \cap \bar{\Lambda}_L^{(1)}(u)$.

Lemma 2.2.1. *The following estimates hold:*

$$(NB.1) \quad \#\{\mathcal{E}(\Gamma^{(1)} \cap \Lambda_L^{(1)})\} = d(2L)(2L-1)^{d-1} \leq d \cdot |\Lambda_L^{(1)}|,$$

$$(NB.n) \quad \#\{\mathcal{K}(\Gamma^{(n)} \cap \Lambda_{\mathbb{L}}^{(n)})\} = \prod_{j=1}^n (d(2L_j)(2L_j-1)^{d-1}) \leq d^n \cdot |\Lambda_{\mathbb{L}}^{(n)}|.$$

Proof. See the Appendix, Lemma 2.11.3. □

We define the *discrete cubes* $\mathbf{B}_L^{(n)}(\mathbf{u})$ and the *cells* $\mathbf{C}(\mathbf{u})$ by

$$\mathbf{B}_L^{(n)}(\mathbf{u}) = \Lambda_L^{(n)}(\mathbf{u}) \cap \mathbb{Z}^{nd}, \quad \mathbf{C}(\mathbf{u}) = \Lambda_1^{(n)}(\mathbf{u}) \subset \mathbb{R}^{nd}.$$

A finite union of cells will be called a *cellular set*. For $L \geq 7$, we denote

$$\Lambda_L^{\text{out}}(\mathbf{u}) = \Lambda_L^{(n)}(\mathbf{u}) \setminus \Lambda_{L-6}^{(n)}(\mathbf{u}), \quad \mathbf{B}_L^{\text{out}}(\mathbf{u}) = \Lambda_L^{\text{out}}(\mathbf{u}) \cap \mathbb{Z}^{nd}.$$

We define the restriction of $H^{(n)}(\omega)$ to a rectangle $\Lambda = \Lambda_{\mathbb{L}}^{(n)}$ to be the operator $H_{\Lambda}^{(n)}(\omega)$ associated with the form

$$\mathfrak{h}_{\omega, \Lambda}^{(n)}[f, g] = \sum_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda)} [\langle \nabla f_{\kappa}, \nabla g_{\kappa} \rangle + \langle V_{\kappa}^{\omega} f_{\kappa}, g_{\kappa} \rangle], \quad D(\mathfrak{h}_{\Lambda}^{\omega}) = W^{1,2}(\Gamma^{(n)} \cap \Lambda),$$

where $W^{1,2}(\Gamma^{(n)} \cap \Lambda)$ is the set of $f \in \oplus_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda)} W^{1,2}((0,1)^n)$ which are continuous on inner σ^i . For $n = 1$, the functions $f \in D(H_{\Lambda}^{(1)}(\omega))$ satisfy Kirchhoff conditions at each vertex in Λ . Note that for boundary vertices, Kirchhoff conditions are just Neumann conditions.

⁵ The closure here is with respect to the metric induced on $\Gamma^{(n)}$ from the sup norm of \mathbb{R}^{nd} . Taking the closure means that we add the σ^i lying on $\partial\Lambda_{\mathbb{L}}^{(n)}$ that belong to inner κ .

Lemma 2.2.2. $H_{\Lambda_{\mathbb{L}}}^{(n)}(\omega)$ has a compact resolvent. Its discrete set of eigenvalues denoted by $E_j(H_{\Lambda_{\mathbb{L}}}^{(n)}(\omega))$ counting multiplicity satisfies the following Weyl law:

$$(WEYL.n) \quad \forall S \in \mathbb{R} \exists C = C(n, d, S - nq_-) : j > C|\Lambda_{\mathbb{L}}^{(n)}| \implies E_j(H_{\Lambda_{\mathbb{L}}}^{(n)}(\omega)) > S.$$

Moreover, C is independent of ω , and if $S > S^*(n, q_-)$, then $C \leq \left\lfloor \frac{d^n (S - nq_-)^{n/2}}{(4\pi)^{n/2} \Gamma(n/2)} \right\rfloor + 1$.

Proof. See the Appendix, Lemma 2.11.4. □

In the rest of this chapter, for a bounded volume $K \subset \mathbb{R}^{nd}$ we put

$$\chi_K := \chi_{\Gamma^{(n)} \cap K}, \quad \chi_{\mathbf{x}} := \chi_{\Gamma^{(n)} \cap \mathbf{C}(\mathbf{x})}.$$

Given $\omega \in \Omega$, $E \notin \sigma(H_{\Lambda}^{(n)}(\omega))$ and $\mathbf{i}, \mathbf{j} \in \Lambda^{(n)} \cap \mathbb{Z}^{nd}$, we define⁶

$$G_{\Lambda^{(n)}}(E) := (H_{\Lambda}^{(n)}(\omega) - E)^{-1}, \quad G_{\Lambda^{(n)}}(\mathbf{i}, \mathbf{j}; E) := \chi_{\mathbf{i}} G_{\Lambda^{(n)}}(E) \chi_{\mathbf{j}}.$$

2.2.2 Geometry of cubes

This subsection gives a slight modification of some geometric arguments introduced in [24] and [19], which are relevant to us because $\Gamma^{(n)} \subset \mathbb{R}^{nd}$.

Definition 2.2.3. Given $n \geq 2$ and a partition $\{1, \dots, n\} = \mathcal{J} \cup \mathcal{J}^c$, we say that $\Lambda_L^{(n)}(\mathbf{u})$ is \mathcal{J} -decomposable if

$$\text{dist}(u_{\mathcal{J}}, u_{\mathcal{J}^c}) \geq 2L + r_0.$$

We say that $\Lambda_L^{(n)}(\mathbf{u})$ is *decomposable* if there exists a partition $\{1, \dots, n\} = \mathcal{J} \cup \mathcal{J}^c$ such that $\Lambda_L^{(n)}(\mathbf{u})$ is \mathcal{J} -decomposable.

A \mathcal{J} -decomposable cube $\Lambda_L^{(n)}(\mathbf{u})$ will henceforth be denoted by

$$\Lambda_L^{(n)}(\mathbf{u}) = \Lambda_L^{(n')}(\mathbf{u}_{\mathcal{J}}) \times \Lambda_L^{(n'')}(\mathbf{u}_{\mathcal{J}^c}), \quad \text{where } n' = |\mathcal{J}| \text{ and } n'' = |\mathcal{J}^c|.$$

Remark 2.2.4. Suppose $\Lambda_L^{(n)}(\mathbf{u})$ is \mathcal{J} -decomposable and identify $L^2(\Gamma^{(n)} \cap \Lambda_L^{(n)}(\mathbf{u})) \equiv L^2(\Gamma^{(n')} \cap \Lambda_L^{(n')}(\mathbf{u}_{\mathcal{J}})) \otimes L^2(\Gamma^{(n'')} \cap \Lambda_L^{(n'')}(\mathbf{u}_{\mathcal{J}^c}))$. Any $\mathbf{x} \in \Lambda_L^{(n)}(\mathbf{u})$ satisfies $\text{dist}(x_{\mathcal{J}}, x_{\mathcal{J}^c}) > r_0$, hence $U^{(n)}(\mathbf{x}) = U^{(n')}(\mathbf{x}_{\mathcal{J}}) + U^{(n'')}(\mathbf{x}_{\mathcal{J}^c})$. Consequently, $H_{\Lambda_L^{(n)}(\mathbf{u})}^{(n)} = H_{\Lambda_L^{(n')}(\mathbf{u}_{\mathcal{J}})}^{(n')} \otimes I + I \otimes H_{\Lambda_L^{(n'')}(\mathbf{u}_{\mathcal{J}^c})}^{(n'')}$. If now $\{(\varphi_a, \lambda_a)\}_a$ and $\{(\psi_b, \mu_b)\}_b$ are orthonormal bases of eigenfunctions of $H_{\Lambda_L^{(n')}(\mathbf{u}_{\mathcal{J}})}^{(n')}$ and $H_{\Lambda_L^{(n'')}(\mathbf{u}_{\mathcal{J}^c})}^{(n'')}$ respectively, then $\Psi_{a,b}(\mathbf{x}) := \varphi_a(x_{\mathcal{J}}) \otimes \psi_b(x_{\mathcal{J}^c})$ form an orthonormal basis of eigenfunctions for $H_{\Lambda_L^{(n)}(\mathbf{u})}^{(n)}$ with corresponding eigenvalues $E_{a,b} = \lambda_a + \mu_b$. Since

$$P_{a,b} := \langle \cdot, \Psi_{a,b} \rangle \Psi_{a,b} = \langle \cdot, \varphi_a \otimes \psi_b \rangle \varphi_a \otimes \psi_b = (\langle \cdot, \varphi_a \rangle \varphi_a) \otimes (\langle \cdot, \psi_b \rangle \psi_b) := P_a \otimes P_b,$$

by the functional calculus, we get for any Borel function $\eta : \sigma(H_{\Lambda_L^{(n)}(\mathbf{u})}^{(n)}) \rightarrow \mathbb{C}$

$$(2-1) \quad \eta(H_{\Lambda_L^{(n)}(\mathbf{u})}^{(n)}) = \sum_{a,b} \eta(E_{a,b}) P_{a,b} = \sum_a P_a \otimes \left(\sum_b \eta(E_{a,b}) P_b \right).$$

6. We should write $G_{\Lambda^{(n)}}^{\omega}$ to be very precise, but we follow the established custom of omitting ω .

Definition 2.2.5. Let $\mathbb{D} := \{\mathbf{x} = (x, \dots, x) : x \in \mathbb{Z}^d\} \subset \mathbb{Z}^{nd}$. A cube $\Lambda_L^{(n)}(\mathbf{u})$ is *partially interactive* (PI) if $\text{dist}(\mathbf{u}, \mathbb{D}) \geq (n-1)(2L+r_0)$, and *fully interactive* (FI) otherwise.

Lemma 2.2.6. *A partially interactive cube is decomposable.*

Proof. See the Appendix, Lemma 2.11.6. \square

For $n \geq 1$, $j = 1, \dots, n$, we define projections of n -rectangles on \mathbb{R}^d by

$$\Pi_j \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) = \Lambda_{L_j}^{(1)}(u_j), \quad \Pi \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) = \bigcup_{j=1}^n \Lambda_{L_j}^{(1)}(u_j).$$

We define $\Pi_{\emptyset} \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) := \emptyset$ and put for $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$,

$$\Pi_{\mathcal{J}} \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) = \bigcup_{j \in \mathcal{J}} \Pi_j \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) = \bigcup_{j \in \mathcal{J}} \Lambda_{L_j}^{(1)}(u_j).$$

Definition 2.2.7. We say $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ is \mathcal{J} -pre-separable from $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$ if

$$\Pi_{\mathcal{J}} \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) \cap (\Pi_{\mathcal{J}^c} \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}) \cup \Pi \Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})) = \emptyset.$$

$\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ and $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$ are said to be *pre-separable* if there exists $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$ such that $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ is \mathcal{J} -pre-separable from $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$ or $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$ is \mathcal{J} -pre-separable from $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$.

Two cubes $\Lambda_L^{(n)}(\mathbf{u})$ and $\Lambda_L^{(n)}(\mathbf{v})$ are said to be *separable* if they are pre-separable and if $|\mathbf{u} - \mathbf{v}| \geq r_{n,L}$, where

$$r_{n,L} := 4(n-1)(2L+r_0) + 2L.$$

Finally, they are said to be *completely separated* if they are separable with $\mathcal{J} = \{1, \dots, n\}$, i.e. if $\Pi \Lambda_L^{(n)}(\mathbf{u}) \cap \Pi \Lambda_L^{(n)}(\mathbf{v}) = \emptyset$ and $|\mathbf{u} - \mathbf{v}| \geq r_{n,L}$.

Notice that if two cubes are completely separated, the corresponding Hamiltonians $H_{\Lambda_L(\mathbf{u})}^{(n)}$ and $H_{\Lambda_L(\mathbf{v})}^{(n)}$ have independent spectra (because $\Pi \Lambda_L^{(n)}(\mathbf{u}) \cap \Pi \Lambda_L^{(n)}(\mathbf{v}) = \emptyset$).

Let us give some criteria for separability. Given $\mathbf{x} \in \mathbb{Z}^{nd}$, put $k_{\mathbf{x}} := \#\{x_1, \dots, x_n\}$. Then each $\mathbf{x} \in \mathbb{Z}^{nd}$ gives rise to $k_{\mathbf{x}}^n$ related points denoted by $\mathbf{x}^{(j)} = (x_1^{(j)}, \dots, x_n^{(j)})$, with $x_k^{(j)} \in \{x_1, \dots, x_n\}$ for all k . For example, for $d = 1$, the point $(1, 5) \in \mathbb{Z}^2$ gives rise to $(1, 1)$, $(1, 5)$, $(5, 1)$ and $(5, 5)$. Taking

$$K(n) := n^n,$$

we have $k_{\mathbf{x}}^n \leq K(n)$ and the following lemmas hold.

Lemma 2.2.8. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{nd}$, $L \in \mathbb{N}^*$ and take $r_{n,L}$ as in Definition 2.2.7. Then*

- 1) *If $\mathbf{y} \notin \bigcup_{j=1}^{K(n)} \Lambda_{2nL}^{(n)}(\mathbf{x}^{(j)})$, then $\Lambda_L^{(n)}(\mathbf{y})$ and $\Lambda_L^{(n)}(\mathbf{x})$ are pre-separable.*
- 2) *If $\mathbf{y} \notin \bigcup_{j=1}^{K(n)} \Lambda_{r_{n,L}}^{(n)}(\mathbf{x}^{(j)})$, then $\Lambda_L^{(n)}(\mathbf{y})$ and $\Lambda_L^{(n)}(\mathbf{x})$ are separable.*
- 3) *If $\mathbf{y} \notin \Lambda_{2r_{n,L}}^{(n)}(\mathbf{0})$, then $\Lambda_L^{(n)}(\mathbf{y})$ is separable from any $\Lambda_L^{(n)}(\mathbf{x})$ satisfying $\mathbf{x} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})$.*

Proof. See the Appendix, Lemma 2.11.7. \square

Lemma 2.2.9. *Separable FI cubes are completely separated.*

Proof. See the Appendix, Lemma 2.11.8. \square

2.2.3 MSA strategy

We summarize here the multiscale analysis (MSA) strategy which we follow to prove localization in an interval I . Let us start with 1-particle systems:

1. Find $L_0 > 0$ and $m_0 > 0$ such that the probability of having one “good” cube among any disjoint pair $\Lambda_{L_0}(u), \Lambda_{L_0}(v)$ is high. Here Λ_{L_0} is good if for any $E \in I$ and i, j far apart, $\|G_{\Lambda_{L_0}}(i, j; E)\| \leq e^{-m_0 L_0}$. This is the *initial length scale estimate* (ILS).
2. Find an increasing sequence of length scales $L_k \rightarrow \infty$ for which a similar decay property holds, with an increasingly good probability (typically $1 - L_k^{-2p}$ for some $p > 0$). This is done by induction on k and is the heart of multiscale analysis.
3. Use this sequence to show that the generalized eigenfunctions of $H^{(1)}$ corresponding to generalized eigenvalues in I exhibit an exponential decay.
4. Deduce exponential localization by proving that generalized eigenfunctions of $H^{(1)}$ exist spectrally almost everywhere.
5. Establish dynamical localization.

For step 1, one shows that if a cube Λ is “bad”, then $\text{dist}(\sigma(H_\Lambda^{(1)}), \inf \sigma(H^{(1)}))$ must be very small. This is done ad absurdum using a *Combes-Thomas estimate*. Then one proves this distance cannot be too small using *Lifshitz tails* (one can also prove step 1 without Lifshitz tails in some cases). For step 2, one first relates $G_{\Lambda'}(x, y; E)$ to $G_\Lambda(z, y; E)$ for $\Lambda' \supset \Lambda$ to deduce the decay of $G_{\Lambda_{L_k}}(x, y; E)$ from the decay of $G_{\Lambda_{L_{k-1}}}(z, y; E)$. This is done using the *Geometric resolvent inequality*. However, in this inequality the decay term from $G_{\Lambda_{L_{k-1}}}(z, y; E)$ gets multiplied by $\|G_{\Lambda_{L_k}}(x, w; E)\|$. So to make sure the product remains very small, it is necessary to show that $\|G_{\Lambda_{L_k}}(x, w; E)\|$ is not too big. This is done using *Wegner estimates*. The remaining steps will be explained in more detail later.

The main difficulty in adapting the previous scheme to multi-particle systems is the fact that Hamiltonians restricted to disjoint cubes are no longer independent. A natural idea is then to restrict the scheme to completely separated cubes, since the corresponding Hamiltonians will then be independent. Unfortunately this cannot work, as there is no analog of Lemma 2.2.8 for such cubes (e.g. $[0, 1] \times [1, 2]$ and $[0, 1] \times [r, r + 1]$ are not completely separated, no matter how big r is) and consequently no analog of Lemma 2.7.4 either. This is why one is forced to work with the larger class of separable cubes. As Hamiltonians restricted to such cubes are not independent, a new strategy must be conceived especially in the induction step; see Section 2.7.2.

2.3 Combes-Thomas estimate

We prove our Combes-Thomas estimate by deriving good bounds on the Schrödinger semigroup. This was done before in [43] using the Feynman-Kac formula and the explicit form of the heat kernel. We shall instead prove our bound via a Davies-Gaffney estimate. This method has several advantages: it does not presuppose a heat kernel estimate, it proves the Combes-Thomas estimate for any energy below the spectral bottom, not just below the infimum of the potential, and the resulting upper bound is easier to control.

Let us mention that the idea of proving Combes-Thomas estimates via semigroups appeared much earlier in [100, Lemma B.7.11]. Compared to our proof and the proof of [43], the method of [100] requires much more input, but it has the advantage of being valid for arbitrary energies outside the spectrum⁷.

⁷ We prove an estimate which is valid for energies in spectral gaps above the spectrum in Section 3.2 using the method of [9], which was adapted to quadratic forms in [104].

We start with a technical lemma.

Lemma 2.3.1. *Let $\Lambda^{(n)}$ be a cube or $\Lambda^{(n)} = \mathbb{R}^{nd}$. If $u \in W^{1,2}(\Gamma^{(n)} \cap \Lambda^{(n)})$ and φ is a bounded Lipschitz continuous function on $\Gamma^{(n)} \cap \Lambda^{(n)}$, then $\varphi u \in W^{1,2}(\Gamma^{(n)} \cap \Lambda^{(n)})$ and $\nabla(\varphi u) = u\nabla\varphi + \varphi\nabla u$.*

Here $\varphi u := (\varphi_\kappa u_\kappa)$, where (φ_κ) is obtained from φ as in Section 2.1.2.

Proof. By [104, Proposition 4.1.27] we have $\varphi_\kappa u_\kappa \in W^{1,2}((0,1)^n)$ and $\nabla(\varphi_\kappa u_\kappa) = u_\kappa \nabla\varphi_\kappa + \varphi_\kappa \nabla u_\kappa$ for all κ . So it remains to show φu is continuous on inner σ^i . By the density of $C^\infty([0,1]^n)$ in $W^{1,2}((0,1)^n)$ (see [82, Section 1.1.6]) and the continuity of the trace operator $\gamma : W^{1,2}((0,1)^n) \rightarrow L^2((0,1)^{n-1})$, we may assume all $u_\kappa \in C([0,1]^n)$. Since each φ_κ is bounded and uniformly continuous on $(0,1)^n$, it has a unique bounded continuous extension $\tilde{\varphi}_\kappa$ on $[0,1]^n$. Thus, $\tilde{\varphi}_\kappa u_\kappa \in C([0,1]^n)$ and $\gamma(\varphi_\kappa u_\kappa)$ is just the restriction of $\tilde{\varphi}_\kappa u_\kappa$ to $\partial\kappa$. Now if σ^i is a common face to κ_1 and κ_2 , the extensions $\tilde{\varphi}_{\kappa_1}$ and $\tilde{\varphi}_{\kappa_2}$ must coincide on σ^i since φ is Lipschitz continuous. Hence,

$$\gamma(\varphi_{\kappa_1} u_{\kappa_1}) = (\tilde{\varphi}_{\kappa_1} u_{\kappa_1})|_{\sigma^i} = (\tilde{\varphi}_{\kappa_2} u_{\kappa_2})|_{\sigma^i} = \gamma(\varphi_{\kappa_2} u_{\kappa_2}),$$

since u is continuous on σ^i . Hence φu is continuous on σ^i . \square

In the following $\text{dist}(\cdot, \cdot)$ refers to the distance induced by the sup norm of \mathbb{R}^{nd} .

Lemma 2.3.2 (Improved Davies-Gaffney estimate). *Let $\Lambda^{(n)}$ be a cube or $\Lambda^{(n)} = \mathbb{R}^{nd}$. Let $A_1, A_2 \subset \Lambda^{(n)}$ be cellular sets such that $\text{dist}(A_1, A_2) =: \delta \geq 1$ and suppose $f, g \in L^2(\Gamma^{(n)} \cap \Lambda^{(n)})$, $\text{supp } f \subset A_1$ and $\text{supp } g \subset A_2$. Then if $s_\omega := \inf \sigma(H_\Lambda^{(n)}(\omega))$, we have*

$$\forall t > 0 : |\langle e^{-tH_\Lambda^{(n)}(\omega)} f, g \rangle| \leq e^{-ts_\omega} e^{-\frac{\delta^2}{4t}} \|f\| \|g\|.$$

Proof. We first assume $\Lambda^{(n)}$ is a cube. Put $H := H_\Lambda^{(n)}(\omega) - s_\omega$. Given $\mathbf{x} \in \Lambda^{(n)}$, let $\tilde{w}(\mathbf{x}) := \text{dist}(\mathbf{x}, A_1)$. Then $|\tilde{w}(\mathbf{x}) - \tilde{w}(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|$, hence $\|\nabla \tilde{w}\|_\infty \leq 1$ and $e^{\zeta \tilde{w}(\cdot)}$ is bounded, Lipschitz continuous on $\Lambda^{(n)}$ for $\zeta > 0$. Let w be the restriction of \tilde{w} to $\Gamma^{(n)} \cap \Lambda^{(n)}$. Then by Lemma 2.3.1, if \mathfrak{h} is the form associated to H , then $e^{\zeta w} u \in D(\mathfrak{h})$ whenever $u \in D(\mathfrak{h})$. Now given $f \in D(H)$, $t > 0$ put $f_t := e^{-tH} f$ and note that $f_t \in D(H)$. Fix $\beta > 0$ and as in [33, Theorem 3.3] consider

$$E(t) = \langle f_t, f_t e^{\beta w} \rangle = \|f_t e^{\beta w/2}\|^2.$$

Then

$$E'(t) = -2 \text{Re} \langle H f_t, f_t e^{\beta w} \rangle = -2 \text{Re} \mathfrak{h}[f_t, f_t e^{\beta w}]$$

and thus

$$\begin{aligned} \frac{E'(t)}{2} &= -\text{Re} \langle \nabla f_t, \nabla(f_t e^{\beta w}) \rangle + \langle (V^\omega - s_\omega) f_t, f_t e^{\beta w} \rangle \\ &= -\text{Re} \langle \nabla f_t, \nabla(f_t e^{\beta w}) \rangle - \langle V^\omega f_t, f_t e^{\beta w} \rangle + s_\omega \|f_t e^{\beta w/2}\|^2. \end{aligned}$$

Now by min-max for forms we have

$$s_\omega = \inf_{f \in D(\mathfrak{h}_{\omega, \Lambda}^{(n)}), \|f\|=1} \mathfrak{h}_{\omega, \Lambda}^{(n)}[f, f] \leq \|f_t e^{\beta w/2}\|^{-2} \cdot \mathfrak{h}_{\omega, \Lambda}^{(n)}[f_t e^{\beta w/2}, f_t e^{\beta w/2}],$$

where $\mathfrak{h}_{\omega, \Lambda}^{(n)}$ is the form associated to $H_{\Lambda}^{(n)}(\omega)$. Thus,

$$\begin{aligned} s_{\omega} \|f_t e^{\beta w/2}\|^2 &\leq \langle \nabla(f_t e^{\beta w/2}), \nabla(f_t e^{\beta w/2}) \rangle + \langle V^{\omega} f_t e^{\beta w/2}, f_t e^{\beta w/2} \rangle \\ &= \langle (\nabla f_t) e^{\beta w/2}, (\nabla f_t) e^{\beta w/2} \rangle + 2 \operatorname{Re} \langle (\nabla f_t) e^{\beta w/2}, f_t (\frac{\beta}{2} \nabla w) e^{\beta w/2} \rangle \\ &\quad + \langle f_t (\frac{\beta}{2} \nabla w) e^{\beta w/2}, f_t (\frac{\beta}{2} \nabla w) e^{\beta w/2} \rangle + \langle V^{\omega} f_t, f_t e^{\beta w} \rangle \\ &= \langle \nabla f_t, (\nabla f_t) e^{\beta w} \rangle + \operatorname{Re} \langle \nabla f_t, f_t (\beta \nabla w) e^{\beta w} \rangle \\ &\quad + \frac{\beta^2}{4} \|f_t (\nabla w) e^{\beta w/2}\|^2 + \langle V^{\omega} f_t, f_t e^{\beta w} \rangle \\ &= \operatorname{Re} \langle \nabla f_t, \nabla(f_t e^{\beta w}) \rangle + \langle V^{\omega} f_t, f_t e^{\beta w} \rangle + \frac{\beta^2}{4} \|f_t (\nabla w) e^{\beta w/2}\|^2, \end{aligned}$$

where we used Lemma 2.3.1. We thus have

$$\frac{E'(t)}{2} \leq \frac{\beta^2}{4} \|f_t (\nabla w) e^{\beta w/2}\|^2 \leq \frac{\beta^2}{4} \|f_t e^{\beta w/2}\|^2 = \frac{\beta^2 E(t)}{4}.$$

Hence, $E(t) \leq e^{\beta^2 t/2} E(0)$. Moreover,

$$\|\chi_{A_2} f_t\|^2 \leq \|\chi_{A_2} e^{-\beta w/2}\|_{\infty}^2 \|e^{\beta w/2} f_t\|^2 \leq e^{-\beta \delta} E(t).$$

Since $\operatorname{supp} f \subset A_1$ and $w = 0$ on A_1 , we have $E(0) = \|e^{\beta w/2} f\|^2 = \|f\|^2$. Hence,

$$\|\chi_{A_2} f_t\|^2 \leq e^{-\beta \delta} E(t) \leq \exp\left(\frac{\beta^2 t}{2} - \beta \delta\right) E(0) = \exp\left(\frac{\beta^2 t}{2} - \beta \delta\right) \|f\|^2.$$

Choose $\beta = \delta/t$. Since $\operatorname{supp} g \subset A_2$ we finally get

$$|\langle e^{-tH} f, g \rangle|^2 = |\langle \chi_{A_2} f_t, g \rangle|^2 \leq \|\chi_{A_2} f_t\|^2 \cdot \|g\|^2 \leq e^{-\delta^2/2t} \|f\|^2 \|g\|^2.$$

The assertion follows (if $\Lambda^{(n)}$ is a cube) by noting that H is densely defined and that

$$e^{-tH} = \exp(-t(H_{\Lambda}^{(n)}(\omega) - s_{\omega})) = e^{ts_{\omega}} e^{-tH_{\Lambda}^{(n)}(\omega)}.$$

Finally, all the arguments remain valid if $\Lambda^{(n)} = \mathbb{R}^{nd}$, except that $e^{\zeta w}$ is no longer bounded. We thus consider a large cube Ξ containing A_1 and A_2 and replace \tilde{w} by a Lipschitz function ρ of compact support such that $\rho(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, A_1)$ if $\mathbf{x} \in \Xi$ and $\|\nabla \rho\|_{\infty} \leq 1$, then take w to be the restriction of ρ to $\Gamma^{(n)}$. \square

Theorem 2.3.3 (Combes-Thomas estimate). *Let $\Lambda^{(n)}$ be a cube or $\Lambda^{(n)} = \mathbb{R}^{nd}$ and let $A, B \subset \Lambda^{(n)}$ be cellular sets such that $\operatorname{dist}(A, B) =: \delta \geq 1$. Then for $E < s_{\omega} := \inf \sigma(H_{\Lambda}^{(n)}(\omega))$ and $\eta := s_{\omega} - E$ we have*

$$\|\chi_A (H_{\Lambda}^{(n)}(\omega) - E)^{-1} \chi_B\| \leq \sqrt{\frac{\pi}{2}} \left(\frac{\sqrt{\delta}}{\eta^{3/4}} + \frac{3}{8\sqrt{\delta}\eta^{5/4}} \right) e^{-\delta\sqrt{\eta}}.$$

Proof. Put $H = H_{\Lambda}^{(n)}(\omega)$. Given $f, g \in L^2(\Gamma^{(n)} \cap \Lambda^{(n)})$ with $\|f\| = \|g\| = 1$ we have

$$|\langle \chi_A e^{-tH} \chi_B f, g \rangle| = |\langle e^{-tH} \chi_B f, \chi_A g \rangle| \leq e^{-ts_{\omega}} e^{-\frac{\delta^2}{4t}} \|\chi_B f\| \|\chi_A g\| \leq e^{-ts_{\omega}} e^{-\frac{\delta^2}{4t}}$$

by Lemma 2.3.2. Thus

$$\|\chi_A e^{-tH} \chi_B\| \leq e^{-ts_{\omega}} e^{-\frac{\delta^2}{4t}}.$$

Now for $E < s_\omega$ we have $(H - E)^{-1} = \int_0^\infty e^{tE} e^{-tH} dt$. Hence

$$\|\chi_A(H - E)^{-1}\chi_B\| \leq \int_0^\infty e^{-t\eta} e^{-\frac{\delta^2}{4t}} dt = \frac{\delta}{\sqrt{\eta}} K_1(\delta\sqrt{\eta})$$

where K_1 is the modified Bessel function⁸ and we used [49, Formula 3.324] to evaluate the integral. Now by [1, Formula 9.7.2] and the remark after it, we have for real $z > 0$ the estimate $K_1(z) \leq \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{3}{8z}\right)$. This proves the assertion. \square

2.4 Geometric Resolvent Inequalities

In this section we follow [104] to prove Theorems 2.4.2 and 2.4.4 and use arguments from [17] to prove Theorem 2.4.5.

Throughout this section, $\Gamma := \Gamma^{nd}$. If $Q \subset \mathbb{R}^{nd}$ is a cellular set and $1 \leq k \leq \infty$, put

$$\tilde{C}_c^k(\Gamma \cap Q) := \{f|_\Gamma : f \in C_c^k(Q)\}, \quad W_0^{1,2}(\Gamma \cap Q) := \{f \in W^{1,2}(\Gamma \cap Q) : f|_{\partial Q} = 0\}$$

where $f|_{\partial Q}$ is understood in the trace sense. We start with a lemma which has to be justified in the context of multi-particle quantum graphs.

Lemma 2.4.1. *Let $\Lambda \subset \mathbb{R}^{nd}$ be a cube. Then for all $h \in (W_0^{1,2}(\Gamma \cap \Lambda))^n$ and $w \in W^{1,2}(\Gamma \cap \Lambda)$:*

$$\langle \nabla \cdot h, w \rangle = -\langle h, \nabla w \rangle.$$

Proof. Let $h = ((h_\kappa^{(1)}), \dots, (h_\kappa^{(n)}))$ and $w = (w_\kappa)$. Fix $\kappa \in \mathcal{K}(\Gamma \cap \Lambda)$ and let $\sigma = \partial\kappa$. Using the notation $\frac{\partial}{\partial x^i} \equiv \partial_i$, we have by Green's formula (see e.g. [51, Theorem 1.5.3.1])

$$(4-1) \quad \langle \partial_i h_\kappa^{(i)}, w_\kappa \rangle = -\langle h_\kappa^{(i)}, \partial_i w_\kappa \rangle + \int_{\sigma(\kappa)} h_\kappa^{(i)} \bar{w}_\kappa \nu^{(i)} d\sigma,$$

where the values of $h_\kappa^{(i)} \bar{w}_\kappa$ on $\sigma := \sigma(\kappa)$ are understood in the trace sense and $\nu := (\nu^{(1)}, \dots, \nu^{(n)})$ is the outward unit vector normal to σ , well defined on each σ^j . Identify $\kappa \equiv [0, 1]^n$ as in Section 2.1.2 and denote points in κ by (x^1, \dots, x^n) , with $x^i \in [0, 1]$. If σ^j is the face with points $(x^1, \dots, x^{j-1}, 0, x^{j+1}, \dots, x^n) := \hat{x}_0^j$ and if $\sigma^{o(j)}$ is the face opposite to it with points $(x^1, \dots, x^{j-1}, 1, x^{j+1}, \dots, x^n) := \hat{x}_1^j$, then $\nu|_{\sigma^j} = (0, \dots, 0, -1, 0, \dots, 0)$ and $\nu|_{\sigma^{o(j)}} = (0, \dots, 0, 1, 0, \dots, 0)$. Hence

$$\int_{\sigma(\kappa)} h_\kappa^{(i)} \bar{w}_\kappa \nu^{(i)} d\sigma = \int_{\sigma^{o(i)}(\kappa)} h_\kappa^{(i)}(\hat{x}_1^i) \bar{w}_\kappa(\hat{x}_1^i) d\hat{x}^i - \int_{\sigma^i(\kappa)} h_\kappa^{(i)}(\hat{x}_0^i) \bar{w}_\kappa(\hat{x}_0^i) d\hat{x}^i,$$

where $d\hat{x}^i := dx^1 \dots dx^{i-1} dx^{i+1} \dots dx^n$. Now consider

$$\sum_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda)} \left(\int_{\sigma^{o(i)}(\kappa)} h_\kappa^{(i)}(\hat{x}_1^i) \bar{w}_\kappa(\hat{x}_1^i) d\hat{x}^i - \int_{\sigma^i(\kappa)} h_\kappa^{(i)}(\hat{x}_0^i) \bar{w}_\kappa(\hat{x}_0^i) d\hat{x}^i \right).$$

Since $h|_{\partial\Lambda} = 0$, this sum may be re-arranged as

$$\sum_{\text{inner}} \sum_{\sigma^i} \sum_{j=1}^d \int_{\sigma^i} \left\{ h_{\kappa_j^-}^{(i)}(\hat{x}_1^i) \bar{w}_{\kappa_j^-}(\hat{x}_1^i) - h_{\kappa_j^+}^{(i)}(\hat{x}_0^i) \bar{w}_{\kappa_j^+}(\hat{x}_0^i) \right\} d\hat{x}^i,$$

⁸ The modified Bessel functions $I_{\pm\nu}(z)$ and $K_\nu(z)$ are the solutions of the differential equation $z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2)w = 0$.

where $\kappa_j^-(\sigma^i)$ and $\kappa_j^+(\sigma^i)$, $j = 1, \dots, d$ are the $2d$ cubes containing σ^i as a common face. But by hypothesis $h^{(i)}\bar{w}$ are continuous on σ^i , i.e. $h_{\kappa_j^-}^{(i)}(\hat{x}_1^i)\bar{w}_{\kappa_j^-}(\hat{x}_1^i) = h_{\kappa_j^+}^{(i)}(\hat{x}_0^i)\bar{w}_{\kappa_j^+}(\hat{x}_0^i)$ a.e.

Hence the sum vanishes and $\sum_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda)} \int_{\sigma(\kappa)} h_{\kappa}^{(i)} \bar{w}_{\kappa} \nu^{(i)} d\sigma = 0$. The assertion thus follows by summing in (4-1) over $\kappa \in \mathcal{K}(\Gamma \cap \Lambda)$ and $i = 1, \dots, n$. \square

Theorem 2.4.2 (Geometric Resolvent Equation). *Let $\Lambda_1^{(n)} \subseteq \Lambda_2^{(n)} \subset \mathbb{R}^{nd}$ be two cubes, $\psi \in \tilde{C}_c^\infty(\Gamma \cap \Lambda_1^{(n)})$ real-valued, and $E \in \rho(H_{\Lambda_1}^{(n)}) \cap \rho(H_{\Lambda_2}^{(n)})$. Then*

$$(GRE) \quad G_{\Lambda_1^{(n)}}(E)\psi = \psi G_{\Lambda_2^{(n)}}(E) + G_{\Lambda_1^{(n)}}(E)((\nabla\psi) \cdot \nabla + \nabla \cdot (\nabla\psi))G_{\Lambda_2^{(n)}}(E)$$

as operators on $L^2(\Gamma \cap \Lambda_2^{(n)})$.

Proof. Let $g \in L^2(\Gamma \cap \Lambda_2^{(n)})$, $u := [\psi G_{\Lambda_2^{(n)}} + G_{\Lambda_1^{(n)}}((\nabla\psi) \cdot \nabla + \nabla \cdot (\nabla\psi))G_{\Lambda_2^{(n)}}]g$, where $G_{\Lambda_i^{(n)}} := G_{\Lambda_i^{(n)}}(E)$ and put $\mathfrak{h}_{\Lambda_i^{(n)}} := \mathfrak{h}_{\omega, \Lambda_i}^{(n)}$. It suffices to show that $u \in D(\mathfrak{h}_{\Lambda_1^{(n)}})$ and

$$(\mathfrak{h}_{\Lambda_1^{(n)}} - E)[u, w] = \langle \psi g, w \rangle \quad \text{for all } w \in D(\mathfrak{h}_{\Lambda_1^{(n)}}).$$

Since $\psi \in \tilde{C}_c^\infty(\Gamma \cap \Lambda_1^{(n)})$ and $G_{\Lambda_2^{(n)}}g \in W^{1,2}(\Gamma \cap \Lambda_2^{(n)})$, we have $\psi G_{\Lambda_2^{(n)}}g \in D(\mathfrak{h}_{\Lambda_1^{(n)}})$ by Lemma 2.3.1. Similarly $(\nabla\psi)G_{\Lambda_2^{(n)}}g \in (W^{1,2}(\Gamma \cap \Lambda_1^{(n)}))^n$, so $\nabla \cdot (\nabla\psi)G_{\Lambda_2^{(n)}}g \in L^2(\Gamma \cap \Lambda_1^{(n)})$ and $G_{\Lambda_1^{(n)}}[\nabla \cdot (\nabla\psi)G_{\Lambda_2^{(n)}}g] \in D(H_{\Lambda_1}^{(n)})$. Finally $(\nabla\psi) \cdot \nabla G_{\Lambda_2^{(n)}}g \in L^2(\Gamma \cap \Lambda_1^{(n)})$, hence $G_{\Lambda_1^{(n)}}[(\nabla\psi) \cdot \nabla G_{\Lambda_2^{(n)}}g] \in D(H_{\Lambda_1}^{(n)})$. Thus, $u \in D(\mathfrak{h}_{\Lambda_1^{(n)}})$ and

$$\begin{aligned} (\mathfrak{h}_{\Lambda_1^{(n)}} - E)[u, w] &= (\mathfrak{h}_{\Lambda_1^{(n)}} - E)[\psi G_{\Lambda_2^{(n)}}g, w] + \langle ((\nabla\psi) \cdot \nabla + \nabla \cdot (\nabla\psi))G_{\Lambda_2^{(n)}}g, w \rangle \\ &= (\mathfrak{h}_{\Lambda_1^{(n)}} - E)[\psi G_{\Lambda_2^{(n)}}g, w] + \langle (\nabla\psi) \cdot \nabla(G_{\Lambda_2^{(n)}}g), w \rangle - \langle (\nabla\psi)G_{\Lambda_2^{(n)}}g, \nabla w \rangle \\ &= \langle \psi \nabla(G_{\Lambda_2^{(n)}}g), \nabla w \rangle + \langle (V^\omega - E)\psi G_{\Lambda_2^{(n)}}g, w \rangle + \langle \nabla(G_{\Lambda_2^{(n)}}g), (\nabla\psi)w \rangle \\ &= (\mathfrak{h}_{\Lambda_2^{(n)}} - E)[G_{\Lambda_2^{(n)}}g, \psi w] = \langle g, \psi w \rangle = \langle \psi g, w \rangle \end{aligned}$$

where we used Lemma 2.4.1 in the second equality. \square

Lemma 2.4.3. *Let $\Lambda^{(n)}$ be a cube or $\Lambda^{(n)} = \mathbb{R}^{nd}$, let $\tilde{Q} \subset Q \subset \Lambda^{(n)}$ be cellular sets with $\text{dist}(\partial Q, \partial \tilde{Q}) \geq 1$ and let $E_+ \in \mathbb{R}$. Then there exists $C = C(E_+, n, d, q_-) > 0$ such that for any $E \leq E_+$, if $f \in D(H_{\Lambda}^{(n)})$, then*

$$(SOL) \quad \|\chi_{\tilde{Q}} \nabla f\| \leq C \cdot (\|\chi_Q (H_{\Lambda}^{(n)} - E)f\| + \|\chi_Q f\|).$$

Proof. Since $\text{dist}(\partial Q, \partial \tilde{Q}) \geq 1$, we may choose a real $\psi \in \tilde{C}_c^\infty(\Gamma \cap Q)$, $0 \leq \psi \leq 1$ with $\psi \equiv 1$ on $\Gamma \cap \tilde{Q}$ and $\|\nabla\psi\|_\infty \leq C_1(nd)$. If $w := f\psi^2$, then $w \in D(\mathfrak{h}_\Lambda)$ by Lemma 2.3.1 and

$$\langle \nabla f, \nabla w \rangle = \langle \psi \nabla f, \psi \nabla f \rangle + 2\langle \psi \nabla f, f \nabla \psi \rangle.$$

Denoting $g := (H_{\Lambda}^{(n)} - E)f$ we thus get

$$\begin{aligned} \|\psi \nabla f\|^2 &= \langle \nabla f, \nabla w \rangle - 2\langle \psi \nabla f, f \nabla \psi \rangle \\ &= \langle g, w \rangle - \langle (V^\omega - E)f, w \rangle - 2\langle \psi \nabla f, f \nabla \psi \rangle \\ &= \langle g\psi, f\psi \rangle - \langle V^\omega f\psi, f\psi \rangle + E\|f\psi\|^2 - 2\langle \psi \nabla f, f \nabla \psi \rangle \\ &\leq \|g\|_Q \|f\|_Q + C_2 \|f\|_Q^2 + 2C_1 \|\psi \nabla f\| \|f\|_Q, \end{aligned}$$

where $\|\phi\|_Q := \|\chi_Q \phi\|$ and $C_2 := |E_+ - nq_-|$. Hence

$$(\|\psi \nabla f\| - C_1 \|f\|_Q)^2 \leq \|g\|_Q \|f\|_Q + (C_1^2 + C_2) \|f\|_Q^2 \leq \left(C_3 \|f\|_Q + \frac{1}{2C_3} \|g\|_Q \right)^2,$$

where $C_3 := \sqrt{C_1^2 + C_2}$. The assertion follows by taking square roots. \square

Theorem 2.4.4. *Let $\Lambda_l^{(n)} \subset \Lambda_L^{(n)}$ be cubes with $l \geq 7$, let $A \subseteq \Lambda_{l-6}^{(n)}$, $B \subseteq \Lambda_L^{(n)} \setminus \Lambda_l^{(n)}$ be cellular sets and let $E_+ \in \mathbb{R}$. Then there exists $C = C(E_+, n, d, q_-) > 0$ such that for all $E \in \rho(H_{\Lambda_L}^{(n)}) \cap \rho(H_{\Lambda_l}^{(n)}) \cap (-\infty, E_+]$:*

$$(GRI.1) \quad \|\chi_A G_{\Lambda_L^{(n)}}(E) \chi_B\| \leq C \cdot \|\chi_A G_{\Lambda_l^{(n)}}(E) \chi_{\Lambda_l^{\text{out}}}\| \cdot \|\chi_{\Lambda_l^{\text{out}}} G_{\Lambda_L^{(n)}}(E) \chi_B\|.$$

In particular, if $\mathbf{u} \in \Lambda_{l-7}^{(n)}$ and $\Lambda_l^{(n)} \subset \Lambda_{L-7}^{(n)}$, then given $\mathbf{y} \in \mathbf{B}_L^{\text{out}}$, we have

$$(GRI.2) \quad \|G_{\Lambda_L^{(n)}}(\mathbf{u}, \mathbf{y}; E)\| \leq C \cdot |\mathbf{B}_l^{\text{out}}|^2 \max_{\mathbf{w} \in \mathbf{B}_l^{\text{out}}} \|G_{\Lambda_l^{(n)}}(\mathbf{u}, \mathbf{w}; E)\| \max_{\mathbf{z} \in \mathbf{B}_l^{\text{out}}} \|G_{\Lambda_L^{(n)}}(\mathbf{z}, \mathbf{y}; E)\|.$$

Proof. Let $G_\Lambda := G_\Lambda(E)$, $Q = \text{int } \Lambda_l^{\text{out}}$ and choose a real $\psi \in \tilde{C}_c^\infty(\Gamma \cap \Lambda_l^{(n)})$ such that $\psi = 1$ on $\Gamma \cap \Lambda_{l-4}^{(n)}$, $\text{supp } \psi \subset \Lambda_{l-2}^{(n)}$ and $\|\nabla \psi\|_\infty$ is bounded independently of $\Lambda_l^{(n)}$. Then

$$\begin{aligned} \|\chi_A G_{\Lambda_L^{(n)}} \chi_B\| &= \|\chi_A (\psi G_{\Lambda_L^{(n)}} - G_{\Lambda_l^{(n)}} \psi) \chi_B\| \quad (\psi|_{\Gamma \cap A} = 1, \psi|_{\Gamma \cap B} = 0) \\ &= \|\chi_A (G_{\Lambda_l^{(n)}} ((\nabla \psi) \cdot \nabla + \nabla \cdot (\nabla \psi)) G_{\Lambda_L^{(n)}}) \chi_B\| \quad (\text{GRE}) \\ &\leq \|\chi_A G_{\Lambda_l^{(n)}} (\nabla \psi) \cdot \nabla G_{\Lambda_L^{(n)}} \chi_B\| + \|\chi_A G_{\Lambda_l^{(n)}} \nabla \cdot (\nabla \psi) G_{\Lambda_L^{(n)}} \chi_B\|. \end{aligned}$$

Now let $\tilde{Q} = \text{int}(\Lambda_{l-1}^{(n)} \setminus \Lambda_{l-5}^{(n)})$, so $\text{supp } \nabla \psi \subset \tilde{Q}$ and $\text{dist}(\partial Q, \partial \tilde{Q}) = 1$. Hence given $f_1, f_2 \in L^2(\Gamma \cap \Lambda_L^{(n)})$, $\|f_1\| = \|f_2\| = 1$, we have

$$\begin{aligned} |\langle \chi_A G_{\Lambda_l^{(n)}} (\nabla \psi) \cdot \nabla G_{\Lambda_L^{(n)}} \chi_B f_1, f_2 \rangle| &= |\langle \nabla G_{\Lambda_L^{(n)}} \chi_B f_1, (\nabla \psi) G_{\Lambda_l^{(n)}} \chi_A f_2 \rangle| \\ &\leq \|\nabla \psi\|_\infty \|\chi_{\tilde{Q}} \nabla G_{\Lambda_L^{(n)}} \chi_B\| \|\chi_{\tilde{Q}} G_{\Lambda_l^{(n)}} \chi_A\|. \end{aligned}$$

Furthermore, using Lemma 2.4.1 we have

$$\begin{aligned} |\langle \chi_A G_{\Lambda_l^{(n)}} \nabla \cdot (\nabla \psi) G_{\Lambda_L^{(n)}} \chi_B f_1, f_2 \rangle| &= |\langle f_1, \chi_B G_{\Lambda_L^{(n)}} (\nabla \psi) \cdot \nabla G_{\Lambda_l^{(n)}} \chi_A f_2 \rangle| \\ &\leq \|\nabla \psi\|_\infty \|\chi_B G_{\Lambda_L^{(n)}} \chi_{\tilde{Q}}\| \|\chi_{\tilde{Q}} \nabla G_{\Lambda_l^{(n)}} \chi_A\|. \end{aligned}$$

Noting that for a bounded operator T we have $\|T\| = \|T^*\|$, we thus get

$$\|\chi_A G_{\Lambda_L^{(n)}} \chi_B\| \leq \|\nabla \psi\|_\infty (\|\chi_A G_{\Lambda_l^{(n)}} \chi_{\tilde{Q}}\| \|\chi_{\tilde{Q}} \nabla G_{\Lambda_L^{(n)}} \chi_B\| + \|\chi_{\tilde{Q}} \nabla G_{\Lambda_l^{(n)}} \chi_A\| \|\chi_{\tilde{Q}} G_{\Lambda_L^{(n)}} \chi_B\|).$$

Now by Lemma 2.4.3, we can find C_1 such that

$$\|\chi_{\tilde{Q}} \nabla G_{\Lambda_L^{(n)}} \chi_B\| \leq C_1 \cdot \|\chi_Q G_{\Lambda_L^{(n)}} \chi_B\|.$$

Indeed, given $u \in L^2(\Gamma \cap \Lambda_L^{(n)})$, we apply (SOL) to $f = G_{\Lambda_L^{(n)}} \chi_B u$. Noting that $(H_{\Lambda_L}^{(n)} - E)f = \chi_B u = 0$ on Q we get $\|\chi_{\tilde{Q}} \nabla G_{\Lambda_L^{(n)}} \chi_B u\| \leq C_1 \cdot \|\chi_Q G_{\Lambda_L^{(n)}} \chi_B u\|$. As u is arbitrary, the assertion follows. In the same way we find C_2 such that

$$\|\chi_{\tilde{Q}} \nabla G_{\Lambda_l^{(n)}} \chi_A\| \leq C_2 \cdot \|\chi_Q G_{\Lambda_l^{(n)}} \chi_A\| = C_2 \cdot \|\chi_A G_{\Lambda_l^{(n)}} \chi_Q\|.$$

Noting that $\tilde{Q} \subset Q$, we finally get

$$\|\chi_A G_{\Lambda_L^{(n)}} \chi_B\| \leq C \cdot \|\chi_A G_{\Lambda_L^{(n)}} \chi_Q\| \cdot \|\chi_Q G_{\Lambda_L^{(n)}} \chi_B\|$$

for $C = \max(2C_1 \|\nabla \psi\|_\infty, 2C_2 \|\nabla \psi\|_\infty)$. We thus have (GRI.1).

For (GRI.2), note that $\Lambda_l^{\text{out}} \subseteq \bigcup_{\mathbf{w} \in \mathbf{B}_l^{\text{out}}} \mathbf{C}(\mathbf{w})$, so (GRI.1) gives us

$$\|\chi_{\mathbf{u}} G_{\Lambda_L^{(n)}}(E) \chi_{\mathbf{y}}\| \leq C \sum_{\mathbf{w}, \mathbf{z} \in \mathbf{B}_l^{\text{out}}} \|\chi_{\mathbf{u}} G_{\Lambda_l^{(n)}}(E) \chi_{\mathbf{w}}\| \|\chi_{\mathbf{z}} G_{\Lambda_L^{(n)}}(E) \chi_{\mathbf{y}}\|. \quad \square$$

We now give a resolvent inequality which is special to multi-particle systems⁹.

Theorem 2.4.5. *Let $\Lambda_L^{(n)}(\mathbf{u})$ be a \mathcal{J} -decomposable cube, let $\mathbf{x}, \mathbf{y} \in \mathbf{B}_L^{(n)}(\mathbf{u})$ and suppose that $E \in \rho(H_{\Lambda_L^{(n)}}(\mathbf{u}))$. There exists $S^* = S^*(n, q_-, E)$ such that for $S > S^*$, and under the notations of Remark 2.2.4, if $\delta_1 := |x_{\mathcal{J}^c} - y_{\mathcal{J}^c}| > 2$, then*

$$(GRI.3) \quad \|G_{\Lambda_L^{(n)}(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)\| \leq M_1 \cdot \max_{a \leq M_1} \|G_{\Lambda_L^{(n'')}(u_{\mathcal{J}^c})}(x_{\mathcal{J}^c}, y_{\mathcal{J}^c}; E - \lambda_a)\| + |\Lambda_L^{(n')}| e^{-\delta_1 S}$$

for $M_1 = \left(\left\lfloor \frac{d^{n'}((4S)^2 + E - nq_-)^{n'/2}}{(4\pi)^{n'/2} \Gamma(n'/2)} \right\rfloor + 1 \right) \cdot |\Lambda_L^{(n')}|$, and if $\delta_2 := |x_{\mathcal{J}} - y_{\mathcal{J}}| > 2$, then

$$(GRI.3') \quad \|G_{\Lambda_L^{(n)}(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)\| \leq M_2 \cdot \max_{b \leq M_2} \|G_{\Lambda_L^{(n')}(u_{\mathcal{J}})}(x_{\mathcal{J}}, y_{\mathcal{J}}; E - \mu_b)\| + |\Lambda_L^{(n'')}| e^{-\delta_2 S}$$

for $M_2 = \left(\left\lfloor \frac{d^{n''}((4S)^2 + E - nq_-)^{n''/2}}{(4\pi)^{n''/2} \Gamma(n''/2)} \right\rfloor + 1 \right) \cdot |\Lambda_L^{(n'')}|$.

Proof. We only prove the first bound; the second one is similar. Put $\Lambda^{(n)} := \Lambda_L^{(n)}(\mathbf{u})$, $\Lambda_1^{(n')} := \Lambda_L^{(n')}(u_{\mathcal{J}})$ and $\Lambda_2^{(n'')} := \Lambda_L^{(n'')}(u_{\mathcal{J}^c})$. Using (2-1) with $\eta(t) := (t - E)^{-1}$ we get

$$G_{\Lambda^{(n)}}(E) = \sum_a P_a \otimes \left(\sum_b \frac{1}{\mu_b - (E - \lambda_a)} P_b \right) = \sum_a P_a \otimes G_{\Lambda_2^{(n'')}}(E - \lambda_a).$$

Hence noting that $G_{\Lambda^{(n)}}(\mathbf{x}, \mathbf{y}; E) := \chi_{\mathbf{x}} G_{\Lambda^{(n)}}(E) \chi_{\mathbf{y}}$, we get

$$\begin{aligned} \|G_{\Lambda^{(n)}}(\mathbf{x}, \mathbf{y}; E)\| &\leq \sum_a \|\chi_{x_{\mathcal{J}}} P_a \chi_{y_{\mathcal{J}}} \otimes \chi_{x_{\mathcal{J}^c}} G_{\Lambda_2^{(n'')}}(E - \lambda_a) \chi_{y_{\mathcal{J}^c}}\| \\ &\leq \sum_a \|\chi_{x_{\mathcal{J}^c}} G_{\Lambda_2^{(n'')}}(E - \lambda_a) \chi_{y_{\mathcal{J}^c}}\|. \end{aligned}$$

Now given $S_j \gg 1$, by (WEYL. n') the constants $C_j = \left\lfloor \frac{d^{n'}(S_j + E - nq_-)^{n'/2}}{(4\pi)^{n'/2} \Gamma(n'/2)} \right\rfloor + 1$ satisfy

$$a > C_j |\Lambda_1^{(n')}| \implies \lambda_a > S_j + E - n''q_- \implies \eta_a > S_j,$$

where $\eta_a := n''q_- - (E - \lambda_a)$. Hence if $\delta_1 > 2$, taking $\delta := \text{dist}(C(x_{\mathcal{J}^c}), C(y_{\mathcal{J}^c})) = \delta_1 - 2$ and $S_j := (4S_j)^2$, we get by Combes-Thomas estimate,

$$\sum_{a=C_j |\Lambda_1^{(n')}| + 1}^{C_{j+1} |\Lambda_1^{(n')}|} \|G_{\Lambda_2^{(n'')}}(x_{\mathcal{J}^c}, y_{\mathcal{J}^c}; E - \lambda_a)\| \leq (C_{j+1} - C_j) |\Lambda_1^{(n')}| e^{-\delta \sqrt{\frac{S_j}{2}}} \leq |\Lambda_1^{(n')}| e^{-\delta \sqrt{\frac{S_j}{2}}}$$

9. Let us mention that, just as the preceding geometric resolvent inequalities allow one to perform an induction on k , the following inequality allows us to perform an induction on n . The error term $|\Lambda| \cdot e^{-\delta_j S}$ in (GRI.3) does not appear for discrete Anderson models on $\ell^2(\mathbb{Z}^{nd})$.

provided S is large enough. Hence

$$\sum_{a > C_1 |\Lambda_1^{(n')}|} \|G_{\Lambda_2^{(n')}}(x_{\mathcal{J}^c}, y_{\mathcal{J}^c}; E - \lambda_a)\| \leq |\Lambda_1^{(n')}| \sum_{j=1}^{\infty} e^{-\delta \frac{\sqrt{S_j}}{2}}.$$

But

$$\sum_{j=1}^{\infty} e^{-\delta \frac{\sqrt{S_j}}{2}} = \sum_{j=1}^{\infty} e^{-(2\delta S)j} = \frac{e^{-2\delta S}}{1 - e^{-2\delta S}} \leq 2e^{-2\delta S} \leq e^{-\delta_1 S}.$$

We thus obtain the first bound with $M_1 := C_1 |\Lambda_1^{(n')}|$. \square

2.5 Wegner Estimates

To establish Wegner estimates we use some ideas of [26], but we rely entirely on measure-theoretic arguments. For a probability measure μ on \mathbb{R} we put

$$s(\mu, \varepsilon) := \sup \{ \mu[a, b] : b - a \leq \varepsilon \}.$$

Given $J \subset \mathcal{E}(\Gamma^{(1)})$ and $\omega \in \Omega$, we denote $\omega = (\omega^J, \omega^{J^c})$, where $\omega^J := (\omega_e)_{e \in J}$. If $A \subseteq \Omega$ is measurable and ω^{J^c} is fixed, we define the section $A_{\omega^{J^c}} := \{ \omega^J : (\omega^J, \omega^{J^c}) \in A \}$ and put $\mathbb{P}_J := \otimes_{e \in J} \mu$. Then by definition of a product measure, we have $\mathbb{P}(A) = \mathbb{E}_{J^c} \{ \mathbb{P}_J(A_{\omega^{J^c}}) \}$, where \mathbb{E}_{J^c} denotes the integration over ω^{J^c} .

Theorem 2.5.1. *Let $E \in \mathbb{R}$ and $\varepsilon > 0$. There exists a non-random $C = C(n, d, E + \varepsilon - nq_-)$ such that for any $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ and any $1 \leq i \leq n$, if $J := \mathcal{E}(\Gamma^{(1)} \cap \Pi_i \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u}))$, then*

$$\mathbb{P}_J (\{ \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}^{(n)}}(\omega)), E) < \varepsilon \}_{\omega^{J^c}}) \leq C \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})| \cdot |\Pi_i \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})| \cdot s(\mu, 2\varepsilon)$$

for any ω^{J^c} .

Proof. Put $\Lambda := \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ and fix ω^{J^c} . By Lemma 2.2.2, we may find $C' = C'(n, d, E + \varepsilon - nq_-)$ such that $E_j(\omega) := E_j(H_{\Lambda}^{(n)}(\omega)) > E + \varepsilon$ if $j > C' \cdot |\Lambda|$. Hence,

$$(5-1) \quad \mathbb{P}_J (\{ \text{dist}(\sigma(H_{\Lambda}^{(n)}(\omega)), E) < \varepsilon \}_{\omega^{J^c}}) \leq \sum_{j \leq C' |\Lambda|} \mathbb{P}_J (\{ |E_j(\omega) - E| < \varepsilon \}_{\omega^{J^c}}).$$

Given $\kappa = (e_1, \dots, e_n) \in \mathcal{K}(\Gamma \cap \Lambda)$ we have

$$W_{\kappa}^{\omega} = \omega_{e_1} + \dots + \omega_{e_n} = \sum_{e \in \mathcal{E}} c_{\kappa}(e) \omega_e, \quad \text{where } c_{\kappa}(e) := \begin{cases} 1 & \text{if } e = e_j \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$W_{\kappa}^{\omega} = \sum_{e \in J} c_{\kappa}(e) \omega_e + \sum_{e \in J^c} c_{\kappa}(e) \omega_e = W_{\omega^J}(\kappa) + W_{\omega^{J^c}}(\kappa).$$

Now

$$H_{\Lambda}^{(n)}(\omega) = -\Delta + U + W_{\omega^{J^c}} + W_{\omega^J} = K_{\omega^{J^c}} + W_{\omega^J},$$

where the operator $K_{\omega^{J^c}}$ does not depend on ω^J . Let $(f_{\kappa}) \in L^2(\Gamma \cap \Lambda)$ with $\|(f_{\kappa})\| = 1$, let $t \geq 0$ and denote $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^J$. Then

$$H_{\Lambda}^{(n)}(\omega^J + t \cdot \mathbf{1}, \omega^{J^c})(f_{\kappa}) = (K_{\omega^{J^c}} + W_{\omega^J + t \cdot \mathbf{1}})(f_{\kappa}) = (K_{\omega^{J^c}} + W_{\omega^J})(f_{\kappa}) + t(n_{\kappa} f_{\kappa}),$$

where $n_\kappa := \sum_{e \in J} c_\kappa(e)$. Since every $\kappa \in \mathcal{K}(\Gamma \cap \Lambda)$ takes the form (e_1, \dots, e_n) with $e_i \in \mathcal{E}(\Gamma \cap \Pi_i \Lambda) = J$, we have $1 \leq n_\kappa \leq n$. Hence

$$\langle H_\Lambda^{(n)}(\omega^J + t \cdot \mathbf{1}, \omega^{J^c})(f_\kappa), (f_\kappa) \rangle \geq \langle H_\Lambda^{(n)}(\omega^J, \omega^{J^c})(f_\kappa), (f_\kappa) \rangle + t.$$

By the min-max principle, it follows that $E_j(\omega^J + t \cdot \mathbf{1}, \omega^{J^c}) \geq E_j(\omega^J, \omega^{J^c}) + t$. Finally, if $v_e \leq w_e$ for all $e \in J$, then $H(v^J, \omega^{J^c}) \leq H(w^J, \omega^{J^c})$ and thus $E_j(v^J, \omega^{J^c}) \leq E_j(w^J, \omega^{J^c})$. Hence the $E_j(\cdot, \omega^{J^c}) : \mathbb{R}^J \rightarrow \mathbb{R}$ satisfy the hypotheses of Stollmann's lemma (see [103] and [98]) for any ω^{J^c} , so we get

$$\mathbb{P}_J(\{\omega^J : |E_j(\omega^J, \omega^{J^c}) - E| < \varepsilon\}) \leq |J| \cdot s(\mu, 2\varepsilon) \leq d \cdot |\Pi_i \Lambda| \cdot s(\mu, 2\varepsilon)$$

by (NB.1). The theorem follows by (5-1). \square

Theorem 2.5.2. *Let $I = [a, b]$ be a bounded interval and let $\varepsilon > 0$. There exists $C = C(n, d, b + \varepsilon - nq_-)$ such that for any pre-separable $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ and $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$ we have*

$$\mathbb{P}\{\text{dist}(\sigma_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})}), \sigma_I(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})})) < \varepsilon\} \leq C \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})| \cdot |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})| \cdot |\Pi_0 \Lambda| \cdot s(\mu, 2\varepsilon),$$

where $\sigma_I(H_\Lambda^{(n)}) := \sigma(H_\Lambda^{(n)}(\omega)) \cap I$ and $|\Pi_0 \Lambda| := \max_{i,j} (|\Pi_i \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})|, |\Pi_j \Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})|)$.

Proof. Suppose $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$ is \mathcal{J} -pre-separable of $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ for some $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$, i.e. $\Pi_{\mathcal{J}} \Lambda_{\mathbb{K}}^{(n)}(\mathbf{v}) \cap (\Pi_{\mathcal{J}^c} \Lambda_{\mathbb{K}}^{(n)}(\mathbf{v}) \cup \Pi \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})) = \emptyset$. Fix $i \in \mathcal{J}$ and put $J := \mathcal{E}(\Gamma^{(1)} \cap \Pi_i \Lambda_{\mathbb{K}}^{(n)}(\mathbf{v}))$. Since the eigenvalues $E_j^{(\mathbf{u})}(\omega)$ of $H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})}(\omega)$ do not depend on ω^J , we may apply Theorem 2.5.1 with $E = E_j^{(\mathbf{u})} = E_j^{\mathbf{u}}(\omega^{J^c})$ to get

$$\begin{aligned} & \mathbb{P}\{\text{dist}(\sigma_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})}), \sigma_I(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})})) < \varepsilon\} \\ &= \mathbb{E}_{J^c} \left\{ \mathbb{P}_J \left\{ \text{dist}(\sigma_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})}), \sigma_I(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})})) < \varepsilon \right\}_{\omega^{J^c}} \right\} \\ &= \mathbb{E}_{J^c} \left\{ \mathbb{P}_J \left\{ \min_{a \leq E_j^{(\mathbf{u})} \leq b} \text{dist}(E_j^{(\mathbf{u})}, \sigma_I(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})})) < \varepsilon \right\}_{\omega^{J^c}} \right\} \\ &\leq \mathbb{E}_{J^c} \sum_{j \leq C_1 |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})|} \mathbb{P}_J(\{\text{dist}(E_j^{(\mathbf{u})}, \sigma_I(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})})) < \varepsilon\}_{\omega^{J^c}}) \\ &\leq C \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})| \cdot |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})| \cdot |\Pi_i \Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})| \cdot s(\mu, 2\varepsilon) \end{aligned}$$

where we used Lemma 2.2.2 to obtain $C_1 = C_1(n, d, b - nq_-)$. If however $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})$ was \mathcal{J} -pre-separable of $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})$, we would get for $i \in \mathcal{J}$,

$$\mathbb{P}\{\text{dist}(\sigma_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})}), \sigma_I(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})})) < \varepsilon\} \leq C \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})| \cdot |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{v})| \cdot |\Pi_i \Lambda_{\mathbb{L}}^{(n)}(\mathbf{u})| \cdot s(\mu, 2\varepsilon). \quad \square$$

2.6 Initial Length Scale Estimate

In this section we follow the ideas of [104] and use a Cheeger inequality from [91] to prove Lifshitz-type asymptotics for 1-particle systems. We then deduce the Initial Length Scale estimate (ILS) for our model. We speak of Lifshitz-type asymptotics because our result is not formulated in terms of the integrated density of states $N(E)$, as it is not needed here. Theorem 2.6.1 easily implies bounds of the form $N(E) \leq e^{-\gamma'(E - q_-)^{-1/2}}$ for

E near q_- if one knows that $N(E) \leq \frac{1}{|\Lambda|} \mathbb{E} \{ \text{tr}[\chi_{(-\infty, E)}(H_\Lambda^{(1)})] \}$; see [104, Theorem 2.1.4]. The existence of $N(E)$ was established in [53], see also [52].

In the following for $l \in \mathbb{N}^*$ we put

$$n_l := \#\mathcal{E}(\Gamma^{(1)} \cap \Lambda_l^{(1)}) = d(2l)(2l-1)^{d-1}.$$

Theorem 2.6.1. *There exist $b > 0$ and $\gamma > 0$ such that for any $u \in \mathbb{Z}^d$,*

$$\mathbb{P}\{E_1(H_{\Lambda_l(u)}^{(1)}(\omega)) \leq q_- + bn_l^{-2}\} \leq e^{-\gamma n_l}.$$

Proof. Put $\tilde{H}_{\Lambda_l}^{(1)}(\omega) = H_{\Lambda_l(u)}^{(1)}(\omega) - q_-$. Then

$$\mathbb{P}\{E_1(H_{\Lambda_l(u)}^{(1)}(\omega)) \leq q_- + bn_l^{-2}\} = \mathbb{P}\{E_1(\tilde{H}_{\Lambda_l}^{(1)}(\omega)) \leq bn_l^{-2}\}.$$

Now $\tilde{H}_{\Lambda_l}^{(1)}(\omega) = (-\Delta + \tilde{W}^\omega)_{\Lambda_l^{(1)}}$, where $\tilde{W}^\omega : (f_e) \mapsto ((\omega_e - q_-)f_e)$. We may assume $\tilde{W}^\omega \leq 1$ for all ω , since if \tilde{W}^ω is larger, $E_1(\tilde{H}_{\Lambda_l}^{(1)}(\omega))$ gets larger and the probability gets smaller. Define for $t \in [-1, 1]$,

$$H(\omega, t) := (-\Delta + t \cdot \tilde{W}^\omega)_{\Lambda_l^{(1)}}, \quad E_j(\omega, t) := E_j(H(\omega, t)).$$

Since the normalized ground state ϕ_0 of the Kirchhoff Laplacian $H(\omega, 0) = -\Delta_{\Lambda_l^{(1)}}$ is the constant function $(n_l^{-1/2})$, we have by the Feynman-Hellmann theorem

$$(6-1) \quad E_1(\omega, 0)' = \langle \tilde{W}^\omega \phi_0, \phi_0 \rangle = \frac{1}{n_l} \sum_{e \in \mathcal{E}(\Gamma^{(1)} \cap \Lambda_l^{(1)})} q_e(\omega) =: f_l(\omega),$$

where $q_e(\omega) = \omega_e - q_- \geq 0$. By [104, Lemma 2.1.1] we can find $s_0, \gamma > 0$ such that

$$\mathbb{P}\{f_l(\omega) \leq s_0\} \leq e^{-\gamma n_l}.$$

We now estimate the distance between $0 = E_1(\omega, 0)$ and the rest of the spectrum of $H(\omega, 0)$ using Cheeger inequality. Let $X := \Gamma^{(1)} \cap \Lambda_l^{(1)}$ and $\mathcal{O} := \{Y \subset X : Y \text{ open}, Y \neq X, Y \neq \emptyset\}$. For $Y \in \mathcal{O}$, let $|\partial Y|$ be the number of points on the boundary of Y , $\text{vol}_1 Y$ be the total length of Y and put $Y^c := X \setminus Y$. Then any $Y \in \mathcal{O}$ satisfies $\min(\text{vol}_1 Y, \text{vol}_1 Y^c) \leq \frac{1}{2}(\text{vol}_1 X) = \frac{n_l}{2}$, hence the Cheeger constant of X satisfies

$$h(X) := \inf_{Y \in \mathcal{O}} \frac{|\partial Y|}{\min(\text{vol}_1 Y, \text{vol}_1 Y^c)} \geq \frac{2}{n_l}.$$

By [91, Theorem 6.1], it follows that $E_2(\omega, 0) \geq \frac{1}{4}h(X)^2 \geq n_l^{-2}$. A similar estimate can also be obtained using the Faber-Krahn inequality; see [84]¹⁰.

We may now apply [104, Theorem 4.1.31], to find $c_1, c_2 > 0$ such that

$$|E_1(\omega, t) - t \cdot E_1(\omega, 0)'| \leq c_2 n_l^2 t^2 \quad \forall 0 \leq t \leq c_1 n_l^{-2}.$$

Since $E_1(\omega, t) \leq E_1(\omega, 1) = E_1(\tilde{H}_{\Lambda_l}^{(1)}(\omega))$, this gives by (6-1)

$$f_l(\omega) = E_1(\omega, 0)' \leq c_2 n_l^2 t + E_1(\tilde{H}_{\Lambda_l}^{(1)}(\omega)) t^{-1} \quad \forall 0 \leq t \leq c_1 n_l^{-2}.$$

10. or Appendix B.

Choose $0 < c_3 \leq c_1$ such that $c_2 c_3 \leq \frac{1}{2} s_0$. Then for $t = c_3 n_l^{-2}$ we get

$$f_l(\omega) \leq (s_0/2) + E_1(\tilde{H}_{\Lambda_l}^{(1)}(\omega)) c_3^{-1} n_l^2.$$

Hence, choosing $b > 0$ such that $b c_3^{-1} \leq \frac{1}{2} s_0$ we finally obtain

$$\mathbb{P}\{E_1(\tilde{H}_{\Lambda_l}^{(1)}(\omega)) \leq b n_l^{-2}\} \leq \mathbb{P}\{f_l(\omega) \leq s_0\} \leq e^{-\gamma n_l}. \quad \square$$

Theorem 2.6.2. *There exist $b > 0$ and $\gamma > 0$ such that for any $\mathbf{u} \in \mathbb{Z}^{nd}$,*

$$(6-2) \quad \mathbb{P}\{E_1(H_{\Lambda_l(\mathbf{u})}^{(n)}(\omega)) \leq n q_- + b n n_l^{-2}\} \leq e^{-\gamma n_l}.$$

Consequently, for all $\xi > 0$ and $\beta \in (0, 1)$, we may find $L_0 = L_0(N, d, \beta, \xi)$ as large as necessary such that for any $\mathbf{u} \in \mathbb{Z}^{nd}$,

$$\mathbb{P}\{\text{dist}(\sigma(H_{\Lambda_{L_0}(\mathbf{u})}^{(n)}(\omega)), n q_-) \leq L_0^{\beta-1}\} \leq L_0^{-\xi}.$$

Proof. Let $\hat{H}_{\Lambda_l(\mathbf{u})}^{(n)}(\omega) := H_{\Lambda_l(\mathbf{u})}^{(n)} - U_{\Lambda_l(\mathbf{u})}^{(n)} = (-\Delta + W^\omega)_{\Lambda_l(\mathbf{u})}$. Since $U^{(n)} \geq 0$, we have $H_{\Lambda_l(\mathbf{u})}^{(n)} \geq \hat{H}_{\Lambda_l(\mathbf{u})}^{(n)}$, hence $E_1(H_{\Lambda_l(\mathbf{u})}^{(n)}) \geq E_1(\hat{H}_{\Lambda_l(\mathbf{u})}^{(n)})$ and

$$\mathbb{P}\{E_1(H_{\Lambda_l(\mathbf{u})}^{(n)}(\omega)) \leq n q_- + b n n_l^{-2}\} \leq \mathbb{P}\{E_1(\hat{H}_{\Lambda_l(\mathbf{u})}^{(n)}(\omega)) \leq n q_- + b n n_l^{-2}\}.$$

But $\hat{H}_{\Lambda_l(\mathbf{u})}^{(n)} = H_{\Lambda_l(u_1)}^{(1)} \otimes I^{n-1} + \dots + I^{k-1} \otimes H_{\Lambda_l(u_k)}^{(1)} \otimes I^{n-k} + \dots + I^{n-1} \otimes H_{\Lambda_l(u_n)}^{(1)}$, where $H_{\Lambda_l}^{(1)} = \hat{H}_{\Lambda_l}^{(1)} : (f_e) \mapsto (-f_e'' + \omega_e f_e)$. Thus $E_1(\hat{H}_{\Lambda_l(\mathbf{u})}^{(n)}) = \sum_{j=1}^n E_1(H_{\Lambda_l(u_j)}^{(1)}) \geq n E_1(H_{\Lambda_l(u_{j_0})}^{(1)})$, where $E_1(H_{\Lambda_l(u_{j_0})}^{(1)}) := \min_{1 \leq j \leq n} E_1(H_{\Lambda_l(u_j)}^{(1)})$. Hence

$$\mathbb{P}\{E_1(H_{\Lambda_l(\mathbf{u})}^{(n)}(\omega)) \leq n q_- + b n n_l^{-2}\} \leq \mathbb{P}\{E_1(H_{\Lambda_l(u_{j_0})}^{(1)}(\omega)) \leq q_- + b n_l^{-2}\}.$$

The existence of b and γ now follows from Theorem 2.6.1. So take these b, γ , and given $\xi > 0, \beta \in (0, 1)$, choose $L^*(n, d, \beta, \xi)$ such that for $L \geq L^*$, we have

$$(6-3) \quad 6^{nd} b^{-n/2} d^n L^{nd + \frac{n(\beta-1)}{2}} e^{-\gamma 2^{-d} (bL^{1-\beta})^{1/2}} \leq (2L)^{-\xi}.$$

Let $L_* := \max_{1 \leq n \leq N} L^*(n, d, \beta, \xi)$. Given $L \geq L_*$, let $l := \lfloor \frac{1}{2} (\frac{bL^{1-\beta}}{d^2})^{1/2d} \rfloor$ and choose $L \leq L_0 \leq 2L$ such that $L_0 = r l$ for some $r \in \mathbb{N}$. Then $\Lambda_{L_0}^{(n)} := \Lambda_{L_0}^{(n)}(\mathbf{u})$ may be divided into $M = L_0^{nd} l^{-nd}$ disjoint cubes Λ_l^k . Since $H_{\Lambda_{L_0}}^{(n)} \geq \oplus_k H_{\Lambda_l^k}^{(n)}$, we get $E_1(H_{\Lambda_{L_0}}^{(n)}) \geq \min_k E_1(H_{\Lambda_l^k}^{(n)})$. But $n_l^2 \leq d^2 (2l)^{2d} \leq bL^{1-\beta}$ and thus $L^{\beta-1} \leq b n_l^{-2}$. So using (6-2) we get

$$\begin{aligned} \mathbb{P}\{\text{dist}(\sigma(H_{\Lambda_{L_0}}^{(n)}), n q_-) \leq L_0^{\beta-1}\} &\leq \mathbb{P}\{E_1(H_{\Lambda_{L_0}}^{(n)}) - n q_- \leq L^{\beta-1}\} \\ &\leq \mathbb{P}\{E_1(H_{\Lambda_l^k}^{(n)}) - n q_- \leq b n_l^{-2} \text{ for some } k\} \leq M e^{-\gamma n_l}. \end{aligned}$$

Noting that $M \leq (2L)^{nd} l^{-nd} \leq (2L)^{nd} (\frac{1}{3} (\frac{bL^{1-\beta}}{d^2})^{1/2d})^{-nd} = 6^{nd} b^{-n/2} d^n L^{nd + \frac{n(\beta-1)}{2}}$ and $n_l \geq d(2l-1)^d \geq d(l+1)^d \geq 2^{-d} (bL^{1-\beta})^{1/2}$, then using (6-3) we may bound the RHS by $(2L)^{-\xi} \leq L_0^{-\xi}$, which completes the proof. \square

Definition 2.6.3. Let $E \in \mathbb{R}$, $m > 0$ and $\omega \in \Omega$. A cube $\Lambda_L^{(n)}(\mathbf{u})$ is said to be (E, m) -Non Singular $((E, m)$ -NS) if $E \in \rho(H_{\Lambda_L(\mathbf{u})}^{(n)}(\omega))$ and

$$\max_{\mathbf{y} \in \mathbf{B}_L^{\text{out}}(\mathbf{u})} \|G_{\Lambda_L^{(n)}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)\| \leq e^{-mL},$$

otherwise it is said to be (E, m) -Singular $((E, m)$ -S).

Corollary 2.6.4 (ILS estimate). *For any $p > 0$ and $\beta \in (0, 1)$, we may find $L_0 = L_0(N, d, p, \beta)$ as large as necessary such that for $\varepsilon_0 = \frac{L_0^{\beta-1}}{2}$, $I_n = [nq_- - \frac{1}{2}, nq_- + \varepsilon_0]$, $m_{L_0} = \frac{L_0^{(\beta-1)/2}}{3}$ and any cube $\Lambda_{L_0}^{(n)}(\mathbf{u})$, we have*

$$(6-4) \quad \mathbb{P}\{\exists E \in I_n : \Lambda_{L_0}^{(n)}(\mathbf{u}) \text{ is } (E, m_{L_0})\text{-S}\} \leq L_0^{-2p}.$$

Proof. Given $2p \equiv \xi > 0$, $\beta \in (0, 1)$, we find L_0 as large as needed satisfying Theorem 2.6.2. Now let $\mathbf{y} \in \mathbf{B}_{L_0}^{\text{out}}(\mathbf{u})$, so $L_0 - 8 \leq \text{dist}(\mathbf{C}(\mathbf{u}), \mathbf{C}(\mathbf{y})) \leq L_0$. Let $\varepsilon_0 = \frac{L_0^{\beta-1}}{2}$ and suppose $s_\omega - nq_- > L_0^{\beta-1}$, where $s_\omega := \inf \sigma(H_{\Lambda_{L_0}(\mathbf{u})}^{(n)}(\omega))$. Then every $E \in I_n$ satisfies $E < s_\omega$ and $\eta := s_\omega - E \geq \frac{L_0^{\beta-1}}{2}$. So by Theorem 2.3.3,

$$\begin{aligned} \|G_{\Lambda_{L_0}^{(n)}(\mathbf{u})}(\mathbf{u}, \mathbf{y}; E)\| &\leq \sqrt{\frac{\pi}{2}} \left(\frac{L_0^{1/2}}{(L_0^{\beta-1}/2)^{3/4}} + \frac{3}{8(L_0 - 8)^{1/2}(L_0^{\beta-1}/2)^{5/4}} \right) e^{-(L_0-8)\sqrt{\frac{L_0^{\beta-1}}{2}}} \\ &\leq e^{-m_{L_0}L_0} \end{aligned}$$

for L_0 large enough. Hence $\Lambda_{L_0}^{(n)}(\mathbf{u})$ is (E, m_{L_0}) -NS. We thus showed that

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_0}^{(n)}(\mathbf{u}) \text{ is } (E, m_{L_0})\text{-S}\} \leq \mathbb{P}\{s_\omega - nq_- \leq L_0^{\beta-1}\}.$$

The claim follows by Theorem 2.6.2, since $s_\omega - nq_- = \text{dist}(\sigma(H_{\Lambda_{L_0}(\mathbf{u})}^{(n)}(\omega)), nq_-)$. \square

2.7 Multi-Particle Multiscale Analysis

We now introduce a multi-particle multiscale analysis following the main ideas of [19], providing modifications as necessary. Throughout this section we fix

$$\alpha = 3/2, \quad \beta = 1/2,$$

and given $1 \leq n \leq N$, we denote $K(n) := n^n$. We also assume that

μ is Hölder continuous.

Definition 2.7.1. We say that a cube $\Lambda_L^{(n)}(\mathbf{u})$ is E -Non Resonant (E -NR) if

$$\text{dist}(\sigma(H_{\Lambda_L(\mathbf{u})}^{(n)}), E) \geq e^{-L^\beta}.$$

We say it is E -Completely Non-Resonant (E -CNR) if any cube $\Lambda_\ell^{(n)} \subseteq \Lambda_L^{(n)}(\mathbf{u})$ with $\ell \in \mathbb{N}^*$, $L^{1/\alpha} \leq \ell \leq L$ is E -NR. In this case, $\Lambda_L^{(n)}(\mathbf{u})$ is thus E -NR in particular.

Definition 2.7.2. Let $l \in \mathbb{N}$, put

$$L = \lfloor l^\alpha \rfloor + 1,$$

and let $J \in \mathbb{N}$. We say that a cube $\Lambda_L^{(n)}(\mathbf{x})$ is (E, m_l, J) -good if it contains at most J pairwise separable cubes $\Lambda_l^{(n)}$ which are (E, m_l) -S. Otherwise, we say it is (E, m_l, J) -bad; in this case, there are at least $J + 1$ separable cubes $\Lambda_l^{(n)}$ which are (E, m_l) -S.

We start by adapting [112, Lemma 4.2] to n -graphs. For this, we first prove the following geometric argument: given a collection of cubes, either they are already pairwise disjoint, or we can construct larger cubes around each cluster, such that the larger cubes are disjoint. For technical reasons, we consider ϵ -enlargements of the cubes, with $\epsilon = 7$.

Lemma 2.7.3. *Given k cubes $\Lambda_L^{(n)}(\mathbf{u}_{(r)})$, $r = 1, \dots, k$, there exists $\tilde{k} \leq k$ disjoint cubes $\Lambda_{l_j}^{(n)}$, $j = 1, \dots, \tilde{k}$ such that $\bigcup_{j=1}^{\tilde{k}} \Lambda_{l_j-7}^{(n)} \supseteq \bigcup_{r=1}^k \Lambda_L^{(n)}(\mathbf{u}_{(r)})$, $l_j = n_j(L+7)$ for some $n_j \in \mathbb{N}^*$ and $\sum_{j=1}^{\tilde{k}} l_j = k(L+7)$.*

Proof. If the cubes $\Lambda_{L+7}^{(n)}(\mathbf{u}_{(r)})$ are disjoint, we put $\Lambda_{l_j}^{(n)} = \Lambda_{L+7}^{(n)}(\mathbf{u}_{(j)})$. Otherwise, divide $\bigcup_{r=1}^k \Lambda_{L+7}^{(n)}(\mathbf{u}_{(r)})$ into k' connected components with $1 \leq k' \leq k$ and order them. If the i -th component contains n_i cubes, find a cube Λ_{l_i}' containing it with $l_i := n_i(L+7)$. If these k' cubes are disjoint, then we are done. If not, divide them into k'' connected components and again find cubes Λ_{l_i}'' around each component with $l_i = n_i(L+7)$, where n_i is the number of the original cubes $\Lambda_{L+7}^{(n)}(\mathbf{u}_{(r)})$ which this component contains. Repeating this procedure we finally obtain the assertion. \square

Lemma 2.7.4. *Let $l \in \mathbb{N}^*$, $J \in \mathbb{N}$, $m_l > \frac{8NJK(N)}{l^{1-\beta}}$, $E_+ \in \mathbb{R}$ and $E \leq E_+$. Let $L = \lfloor l^\alpha \rfloor + 1$ and suppose that $\Lambda_L^{(n)}(\mathbf{x})$ is E -CNR and (E, m_l, J) -good. Then there exists $l^* = l^*(E_+, N, d, J, q_-, r_0)$ such that, if $l > l^*$, then $\Lambda_L^{(n)}(\mathbf{x})$ is $(E, m_L + L^{\beta-1})$ -NS, where*

$$m_L := m_l - \left(\frac{16NJK(N)}{l^{\alpha-1}} m_l + \frac{3}{l^{\alpha(1-\beta)}} \right) > \frac{8NJK(N)}{L^{1-\beta}}.$$

Proof. By hypothesis there are at most J pairwise separable cubes $\Lambda_l^{(n)}(\mathbf{u}_{(s)}) \subset \Lambda_L^{(n)}(\mathbf{x})$ which are (E, m_l) -S. Applying Lemma 2.2.8 to each of them, we may find $JK(n)$ cubes $\Lambda_{r_{n,l}}^{(n)}(\mathbf{z}^{(k)})$ such that if $\mathbf{v} \notin \bigcup_{k=1}^{JK(n)} \Lambda_{r_{n,l}}^{(n)}(\mathbf{z}^{(k)})$, then $\Lambda_l^{(n)}(\mathbf{v})$ is separable from all the $\Lambda_l^{(n)}(\mathbf{u}_{(s)})$. Now applying Lemma 2.7.3 to the $JK(n)$ cubes $\Lambda_{r_{n,l}}^{(n)}(\mathbf{z}^{(k)})$, we may construct disjoint cubes $\Lambda_{l_j}^{(n)}$ such that $\bigcup_j \Lambda_{l_j-7}^{(n)} \supseteq \bigcup_k \Lambda_{r_{n,l}}^{(n)}(\mathbf{z}^{(k)})$, $l_j = n_j(r_{n,l} + 7)$ for some $n_j \in \mathbb{N}^*$ and $\sum l_j \leq JK(n)(r_{n,l} + 7) \leq JK(N)(r_{N,l} + 7) =: l_{N,J}$. Thus, $\Lambda_l^{(n)}(\mathbf{v})$ is (E, m_l) -NS whenever $\mathbf{v} \in \Lambda_{L-l}^{(n)}(\mathbf{x}) \setminus \bigcup_j \Lambda_{l_j-7}^{(n)}$.

We first assume all the “bad cubes” $\Lambda_l^{(n)}$ are inside $\Lambda_{L-l-7}^{(n)}(\mathbf{x})$. Note that if $\mathbf{v} \in \mathbf{B}_{L-l}^{(n)}(\mathbf{x})$ satisfies $\mathbf{v} \in \mathbf{B}_{l_j}^{\text{out}} = \mathbf{B}_{l_j}^{(n)} \setminus \mathbf{B}_{l_j-6}^{(n)}$ for some j , then $\Lambda_l^{(n)}(\mathbf{v})$ is (E, m_l) -NS since $\mathbf{v} \notin \Lambda_{l_j-7}^{(n)}$ and $\mathbf{v} \notin \Lambda_{l_r-7}^{(n)}$ for $r \neq j$ (because $\mathbf{v} \in \Lambda_{l_j}^{(n)}$ and $\Lambda_{l_r}^{(n)}$ is disjoint from the other $\Lambda_{l_r}^{(n)}$).

Now fix $\mathbf{y} \in \mathbf{B}_{L-l}^{\text{out}}(\mathbf{x})$ and let $\mathbf{u} \in \mathbf{B}_{L-l-7}^{(n)}(\mathbf{x})$. We have 2 cases:

(a) $\Lambda_l^{(n)}(\mathbf{u})$ is (E, m_l) -NS. Then applying (GRI.2) to $\Lambda_l^{(n)} := \Lambda_l^{(n)}(\mathbf{u})$,

$$\begin{aligned} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{u}, \mathbf{y}; E)\| &\leq C \cdot |\mathbf{B}_l^{\text{out}}|^2 \max_{\mathbf{k} \in \mathbf{B}_l^{\text{out}}} \|G_{\Lambda_l^{(n)}}(\mathbf{u}, \mathbf{k}; E)\| \max_{\mathbf{k}' \in \mathbf{B}_l^{\text{out}}} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{k}', \mathbf{y}; E)\| \\ &\leq C_1 (2l-1)^{2(nd-1)} e^{-m_l l} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{w}_1, \mathbf{y}; E)\| \end{aligned}$$

for some $\mathbf{w}_1 \in \mathbf{B}_l^{\text{out}}(\mathbf{u})$.

(b) $\Lambda_l^{(n)}(\mathbf{u})$ is (E, m_l) -S. In this case, $\mathbf{u} \in \Lambda_{l_j}^{(n)}$ for some j , so applying (GRI.2) to $\Lambda_{l_j}^{(n)}$,

$$\begin{aligned} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{u}, \mathbf{y}; E)\| &\leq C \cdot |\mathbf{B}_{l_j}^{\text{out}}|^2 \max_{\mathbf{k} \in \mathbf{B}_{l_j}^{\text{out}}} \|G_{\Lambda_{l_j}^{(n)}}(\mathbf{u}, \mathbf{k}; E)\| \max_{\mathbf{k}' \in \mathbf{B}_{l_j}^{\text{out}}} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{k}', \mathbf{y}; E)\| \\ &\leq C_2(2l_{N,J} - 1)^{2(nd-1)} e^{l_{N,J}^\beta} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{w}, \mathbf{y}; E)\| \end{aligned}$$

for some $\mathbf{w} \in \mathbf{B}_{l_j}^{\text{out}}$ because $\Lambda_L^{(n)}(\mathbf{x})$ is E -CNR. But then $\Lambda_l^{(n)}(\mathbf{w})$ is (E, m_l) -NS, so applying (GRI.2) once more we get

$$\|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{u}, \mathbf{y}; E)\| \leq C_3((2l_{N,J} - 1)(2l - 1))^{2(nd-1)} e^{l_{N,J}^\beta} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{w}_1, \mathbf{y}; E)\|$$

for some $\mathbf{w}_1 \in \mathbf{B}_l^{\text{out}}(\mathbf{w})$. Hence

$$\|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{u}, \mathbf{y}; E)\| \leq e^{-m'_l l} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{w}_1, \mathbf{y}; E)\|,$$

where

$$m'_l = m_l - l^{-1} \{l_{N,J}^\beta + 2(nd - 1) \log((2l_{N,J} - 1)(2l - 1)) + \log C_3\} > 0$$

because for large l ,

$$l_{N,J} = JK(N)(2(4N - 3)l + 4(N - 1)r_0 + 7) \leq (8N - 5)JK(N)l$$

so that $m'_l \geq m_l - \frac{8NJK(N)}{l^{1-\beta}} > 0$ for l large enough.

Hence starting at $\mathbf{u} = \mathbf{w}_0 := \mathbf{x}$, we may iterate the procedure p times as long as $\mathbf{w}_{p-1} \in \mathbf{B}_{L-l-7}^{(n)}(\mathbf{x})$. If (a) occurs n_+ times and (b) occurs $n_0 = p - n_+$ times, we obtain

$$\|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E)\| \leq (C_1(2l - 1))^{2(nd-1)} e^{-m_l l} e^{-n_0 m'_l l} \|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{w}_p, \mathbf{y}; E)\|.$$

Now $\Lambda_L^{(n)}(\mathbf{x})$ is E -NR and $e^{-n_0 m'_l l} \leq 1$ since $m'_l > 0$. Hence

$$\|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E)\| \leq (C_1(2l - 1))^{2(nd-1)} e^{-m_l l} e^{L^\beta} \leq e^{-(\tilde{m}_L + L^{\beta-1})L},$$

where

$$(\tilde{m}_L + L^{\beta-1})L = -n_+(\log C_1 + 2(nd - 1) \log(2l - 1) - m_l l) - L^\beta.$$

In case (a), $\mathbf{w}_k \in \mathbf{B}_l^{\text{out}}(\mathbf{w}_{k-1})$, so each step cuts a length between $l - 6$ and $l - 1$. We thus have $\lfloor \frac{L - 2l_{N,J} - l - 7}{l - 1} \rfloor \leq n_+ \leq \lfloor \frac{L - l - 7}{l - 6} \rfloor$. Indeed, the lower bound represents the worst scenario in which the iteration met all the bad cubes in its way, a total length of $2l_{N,J}$. The upper bound occurs when it meets no bad cube. In particular, we have $\frac{L - 2l_{N,J} - l - 7}{l} - 1 \leq n_+ \leq \frac{L}{l - 6}$, so we get

$$\tilde{m}_L L \geq m_l(L - 2l_{N,J} - 2l - 7) - \frac{L}{l - 6} (\log C_1 + 2(nd - 1) \log(2l - 1)) - 2L^\beta.$$

But $2l_{N,J} + 2l + 7 = ((16N - 12)JK(N) + 2)l + C(r_0, N, J) \leq 16NJK(N)l$. Hence

$$\begin{aligned} \tilde{m}_L L &\geq m_l L - 16NJK(N)m_l l - \frac{2ndL \log(2l - 1)}{l - 6} - 2L^\beta \\ &\geq m_l L - 16NJK(N)m_l l - \frac{L}{l^{\alpha(1-\beta)}} - 2L^\beta \end{aligned}$$

for large l , because $\alpha(1 - \beta) = 3/4 < 1$. Noting that $L \geq l^\alpha$, we finally get

$$\tilde{m}_L \geq m_l - \frac{16NJK(N)}{l^{\alpha-1}}m_l - \frac{3}{l^{\alpha(1-\beta)}} = m_L.$$

Thus, $\|G_{\Lambda_L^{(n)}(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E)\| \leq e^{-(\tilde{m}_L + L^{\beta-1})L} \leq e^{-(m_L + L^{\beta-1})L}$ and $\Lambda_L^{(n)}(\mathbf{x})$ is $(E, m_L + L^{\beta-1})$ -NS. For the lower bound on m_L , note that for large l ,

$$(7-1) \quad \left(1 - \frac{16NJK(N)}{l^{\alpha-1}}\right)m_l \geq \frac{1}{2}m_l \geq \frac{4NJK(N)}{l^{1-\beta}} > \frac{3}{l^{\alpha(1-\beta)}} + \frac{8NJK(N)}{L^{1-\beta}}.$$

Finally, if a bad cube lies completely outside $F := \Lambda_{L-l-7}^{(n)}(\mathbf{x})$, the situation is obviously better. If a bad cube is not contained in F but intersects F , we stop the iteration if we reach this bad cube. Then again the situation is better (because here only part of the length $2l_j$ of this cube is counted as bad). \square

We define for $n \geq 2$,

$$p_n := \frac{p_{n-1}}{\alpha^2(1+\theta)} - \frac{(2n-1)d}{2\alpha} - nd - 1,$$

where $\theta := \frac{1}{2p_1}$. We then choose p_1 sufficiently large to make sure that

$$p_N \geq 3Nd + 1.$$

In particular, $0 < \theta < 1$.

Fix

$$E_+ := \max_{1 \leq n \leq N} (nq_- + 1), \quad J = 6,$$

and let l^* be as in Lemma 2.7.4. Then by Corollary 2.6.4, we may find $L_0 > l^*$ as large as necessary such that (6-4) is satisfied for all $1 \leq n \leq N$, with $\varepsilon_0 = \frac{L_0^{\beta-1}}{2}$, $I_n = [nq_- - \frac{1}{2}, nq_- + \varepsilon_0]$, $m_{L_0} = \frac{1}{3L_0^{(1-\beta)/2}}$ and $p := p_1$. We then define the sequences

$$L_{k+1} := \lfloor L_k^\alpha \rfloor + 1,$$

$$m_{L_{k+1}} := m_{L_k} - \left(\frac{96NK(N)}{L_k^{\alpha-1}} m_{L_k} + \frac{3}{L_k^{\alpha(1-\beta)}} \right).$$

Note that $m_{L_k} > \frac{48NK(N)}{L_k^{1-\beta}}$. Indeed, $m_{L_0} = \frac{1}{3L_0^{(1-\beta)/2}} > \frac{48NK(N)}{L_0^{1-\beta}}$ since L_0 is large, hence $m_{L_k} > \frac{48NK(N)}{L_k^{1-\beta}}$ by induction, using (7-1). We now introduce the property

(DS : n, k, m_{L_k}, I_n)

$$\begin{cases} \text{For all pairs of separable cubes } \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) : \\ \mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are } (E, m_{L_k})\text{-S}\} \leq L_k^{-2p_n(1+\theta)^k}. \end{cases}$$

The term $(1 + \theta)^k$ in the exponent was introduced in [19] and is new in comparison with the usual multiscale analysis. While it complicates a few estimates, it has a powerful advantage, namely it allows to prove dynamical localization of any order s in I_N , with ε_0 independent of s . This result (among others) was previously obtained for single-particle systems in the continuum using the bootstrap multiscale analysis of [47].

To prove this property, we shall need Lemma 2.7.4 and the following Wegner bound:

$$(W2 : n, k, I_n) \quad \begin{cases} \text{For all pairs of separable cubes } \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) : \\ \mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are not } E\text{-CNR}\} \leq \frac{1}{4}L_k^{-2p_1(1+\theta)^k}. \end{cases}$$

Lemma 2.7.5. *The property (W2: n, k, I_n) holds for all $k \geq 0$ and $1 \leq n \leq N$.*

Proof. Let $\Lambda_{L_k}^{(n)}(\mathbf{u})$ and $\Lambda_{L_k}^{(n)}(\mathbf{v})$ be separable. If $\Lambda_{\ell_1}^{(n)} \subseteq \Lambda_{L_k}^{(n)}(\mathbf{u})$ and $\Lambda_{\ell_2}^{(n)} \subseteq \Lambda_{L_k}^{(n)}(\mathbf{v})$, then $\Lambda_{\ell_1}^{(n)}$ and $\Lambda_{\ell_2}^{(n)}$ are pre-separable. Hence by Theorem 2.5.2,

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_n : \text{dist}(\sigma(H_{\Lambda_{\ell_1}}^{(n)}), E) < \varepsilon \text{ and } \text{dist}(\sigma(H_{\Lambda_{\ell_2}}^{(n)}), E) < \varepsilon\} \\ & \leq \mathbb{P}\{\text{dist}(\sigma_{J_n}(H_{\Lambda_{\ell_1}}^{(n)}), \sigma_{J_n}(H_{\Lambda_{\ell_2}}^{(n)})) < 2\varepsilon\} \leq C(2L_k)^{2nd+d} s(\mu, 4\varepsilon), \end{aligned}$$

where $J_n = [nq_- - \frac{1}{2} - \varepsilon, nq_- + \varepsilon_0 + \varepsilon]$. Bounding the number of cubes in $\Lambda_{L_k}^{(n)}$ by $|\mathbf{B}_{L_k}^{(n)}| \leq (2L_k)^{nd}$ and the number of $\ell \in \mathbb{N}^*$ satisfying $L_k^{1/\alpha} \leq \ell \leq L_k$ by L_k , we get for $\varepsilon := \max(e^{-\ell_1^\beta}, e^{-\ell_2^\beta}) \leq e^{-L_k^{\beta/\alpha}}$,

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are not } E\text{-CNR}\} \leq C(2L_k)^{4nd+d+2} s(\mu, 4e^{-L_k^{\beta/\alpha}}).$$

Since μ is Hölder continuous, there exist c_μ and $b > 0$ such that

$$C(2L_k)^{4nd+d+2} s(\mu, 4e^{-L_k^{\beta/\alpha}}) \leq Cc_\mu(2L_k)^{4nd+d+2} (4e^{-L_k^{\beta/\alpha}})^b \leq e^{-L_k^\zeta}$$

for some $\zeta > 0$, since L_0 is large. Now for any $k \geq 0$,

$$-\log(1/4) + 2p_1(1+\theta)^k \log L_k \leq \log(4) + 2p_1 2^k \log L_k \leq C_{N,d} 2^k \alpha^k \log L_0 \leq L_0^{\alpha^k \zeta}$$

since $\alpha^k \geq \frac{\log C}{\zeta \log L_0} + k \frac{\log 2\alpha}{\zeta \log L_0} + \frac{\log \log L_0}{\zeta \log L_0}$ for large L_0 , independently of k . But $L_0^{\alpha^k \zeta} \leq L_k^\zeta$. We thus showed that $e^{-L_k^\zeta} \leq \exp(\log(1/4) - 2p_1(1+\theta)^k \log L_k) = \frac{1}{4} L_k^{-2p_1(1+\theta)^k}$. \square

2.7.1 Single-particle case

For $n = 1$, separable cubes are just disjoint cubes; see Definition 2.2.7.

Theorem 2.7.6. *(DS: $1, k, m_{L_k}, I_1$) implies (DS: $1, k+1, m_{L_{k+1}}, I_1$).*

Proof. Put $L = L_{k+1}$, $l = L_k$ and let $\Lambda_L^{(1)}(u)$ and $\Lambda_L^{(1)}(v)$ be a pair of disjoint cubes. Since $\varepsilon_0 < 1$, any $E \in I_1$ satisfies $E \leq E_+ = \max_n(nq_- + 1)$, so applying Lemma 2.7.4 with $J = 6$, noting that $l > l^*$ because $L_0 > l^*$, we have

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)}(u) \text{ and } \Lambda_L^{(1)}(v) \text{ are } (E, m_L)\text{-S}\} \\ & \leq 3 \max_{x=u,v} \mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)}(x) \text{ is } (E, m_l, 6)\text{-bad}\} \\ & \quad + \mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)}(u) \text{ and } \Lambda_L^{(1)}(v) \text{ are not } E\text{-CNR}\}, \end{aligned}$$

since an (E, m_L) -S cube is a fortiori $(E, m_L + L^{\beta-1})$ -S. Now by (W2: $1, k+1, I_1$),

$$\mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)}(u) \text{ and } \Lambda_L^{(1)}(v) \text{ are not } E\text{-CNR}\} \leq \frac{1}{4} L^{-2p_1(1+\theta)^{k+1}}.$$

Moreover, given $J \in 2\mathbb{N}^*$, if $\Lambda_L^{(1)}$ is $(E, m_l, J-1)$ -bad then it contains at least J separable cubes which are (E, m_l) -S. Since $n = 1$, Hamiltonians on disjoint cubes are independent.

So by grouping these J cubes two by two, using $(\text{DS}: 1, k, m_{L_k}, I_1)$ and bounding the number of pairs of cubes in $\Lambda_L^{(1)}$ by $|\mathbf{B}_L^{(1)}|^2 \leq (2L)^{2d}$, we get

$$\begin{aligned}
(7-2) \quad \mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)} \text{ is } (E, m_l, J)\text{-bad}\} &\leq \mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)} \text{ is } (E, m_l, J-1)\text{-bad}\} \\
&\leq \mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)} \text{ is } (E, m_l, 1)\text{-bad}\}^{J/2} \\
&\leq ((2L)^{2d} l^{-2p_1(1+\theta)^k})^{J/2} \\
&\leq cL^{(d - \frac{p_1(1+\theta)^k}{\alpha})J} \leq cL^{(\frac{p_1-1}{3} - \frac{2p_1(1+\theta)^k}{3})J}
\end{aligned}$$

because $\alpha = 3/2$ and $d \leq \frac{p_N-1}{3} \leq \frac{p_1-1}{3}$. Now

$$cL^{\frac{Jp_1}{3}(1-2(1+\theta)^k - \frac{1}{p_1})} \leq \frac{1}{4}L^{\frac{Jp_1}{3}(1-2(1+\theta)^k - \theta)}$$

and since $(1-\theta) \leq (1-\theta)(1+\theta)^k = (1+\theta)^k - \theta(1+\theta)^k$, we have

$$(7-3) \quad 1 - 2(1+\theta)^k - \theta \leq -(1+\theta)^k - \theta(1+\theta)^k = -(1+\theta)^{k+1}.$$

Hence,

$$\mathbb{P}\{\exists E \in I_1 : \Lambda_L^{(1)} \text{ is } (E, m_l, J)\text{-bad}\} \leq \frac{1}{4}L^{-\frac{Jp_1}{3}(1+\theta)^{k+1}}.$$

The claim now follows by taking $J = 6^{11}$. \square

2.7.2 Multi-particle case: Strategy

The deduction of $(\text{DS}: 1, k+1, m_{L_{k+1}}, I_1)$ from $(\text{DS}: 1, k, m_{L_k}, I_1)$ was fairly simple. Once $n \geq 2$ however, we face a difficulty when trying to estimate the probability that a cube is (E, m_{L_k}, J) -bad. Indeed, Hamiltonians on separable sub-cubes are not independent, so we can no longer multiply the probabilities as in the previous subsection.

To overcome this, we reason as follows: if a cube $\Lambda_{L_{k+1}}^{(N)}$ is (E, m_{L_k}, J) -bad, then it contains at least $J+1$ pairwise separable cubes $\Lambda_{L_k}^{(N)}$ which are (E, m_{L_k}) -S. Hence, either it contains 2 separable (E, m_{L_k}) -S PI cubes, or it contains at least J separable (E, m_{L_k}) -S FI cubes. Now separable FI cubes are completely separated by Lemma 2.2.9, so taking $J = 6$, we can again multiply the probabilities. The main difficulty is in assessing the probability that a cube contains 2 separable (E, m_{L_k}) -S PI cubes. The idea is as follows: on PI cubes, the interaction potential decouples by Lemma 2.2.6, so the corresponding Hamiltonians take the form $H_{\Lambda_{L_k}(\mathbf{u})}^{(N)} = H_{\Lambda_{L_k}(u_{\mathcal{J}})}^{(n')} \otimes I + I \otimes H_{\Lambda_{L_k}(u_{\mathcal{J}^c})}^{(n'')}$, where $n', n'' < N$. Now using the new resolvent inequality (GRI.3), we may deduce that $\Lambda_{L_k}^{(N)}(\mathbf{u})$ is non-singular if both projections $\Lambda_{L_k}^{(n')}(u_{\mathcal{J}})$ and $\Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ are non-singular for an array of energies. To show both projections are indeed non-singular, we show that they cannot contain a lot of bad sub-cubes $\Lambda_{L_{k-1}}^{(n)}$, $n = n', n''$.

Notice that in the above scheme, we reduced the decay problem on PI N -cubes to that on n -cubes for $n < N$, which indicates that an induction on n will be performed. Also notice that unlike single-particle systems, here we will need good decay bounds on both orders $k-1$ and k to finally deduce the decay for $k+1$.

11. Obviously, we could have taken $J = 5$ by avoiding inequality (7-2), which gives an even better m_{L_k} than we need. We took $J = 6$ to be in accordance with the multi-particle situation later.

2.7.3 Pairs of PI cubes

We assume through this subsection that $2 \leq n \leq N$.

Recall that if $\Lambda_{L_k}^{(n)}(\mathbf{u})$ is a PI cube, it is \mathcal{J} -decomposable for some \mathcal{J} by Lemma 2.2.6. We then denote $\Lambda_{L_k}^{(n)}(\mathbf{u}) = \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$, where $n' = \#\mathcal{J}$ and $n'' = n - n'$. We also denote by Σ' and Σ'' the spectra of $H_{\Lambda_{L_k}(\mathcal{U}_{\mathcal{J}})}^{(n')}$ and $H_{\Lambda_{L_k}(u_{\mathcal{J}^c})}^{(n'')}$, respectively.

Definition 2.7.7. Let $\Lambda_{L_k}^{(n)}(\mathbf{u}) = \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ be a PI cube. We say that $\Lambda_{L_k}^{(n)}(\mathbf{u})$ is $(E, m_{L_{k-1}})$ -Non Tunneling (($E, m_{L_{k-1}}$)-NT) if

- (i) $\forall \mu_b \in \Sigma'' : \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}})$ is $(E - \mu_b, m_{L_{k-1}}, 1)$ -good.
- (ii) $\forall \lambda_a \in \Sigma' : \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ is $(E - \lambda_a, m_{L_{k-1}}, 1)$ -good.

Otherwise, we say it is $(E, m_{L_{k-1}})$ -Tunneling (($E, m_{L_{k-1}}$)-T).

The following definition is taken from [68, Definition 3.16].

Definition 2.7.8. Let $\Lambda_{L_k}^{(n)}(\mathbf{u}) = \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ be a PI cube. We say that $\Lambda_{L_k}^{(n)}(\mathbf{u})$ is E -Highly Non-Resonant (E -HNR) if

- (i) $\forall \mu_b \in \Sigma'' : \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}})$ is $(E - \mu_b)$ -CNR.
- (ii) $\forall \lambda_a \in \Sigma' : \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ is $(E - \lambda_a)$ -CNR.

Lemma 2.7.9. Let $\Lambda_{L_k}^{(n)}(\mathbf{u})$ be a PI cube, $k \geq 1$, and let $E \in I_n$. If $\Lambda_{L_k}^{(n)}(\mathbf{u})$ is E -HNR and $(E, m_{L_{k-1}})$ -NT, then $\Lambda_{L_k}^{(n)}(\mathbf{u})$ is (E, m_{L_k}) -NS.

Proof. Since $\mu_b \geq n''q_-$ for all $\mu_b \in \Sigma''$, given $E \in I_n$ and $\mu_b \in \Sigma''$ we have

$$(7-4) \quad E - \mu_b \leq E - n''q_- \leq (nq_- + \varepsilon_0) - n''q_- = n'q_- + \varepsilon_0.$$

As $\varepsilon_0 < 1$, $E - \mu_b \leq E_+ = \max_n(nq_- + 1)$. By hypothesis, $\Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}})$ is $(E - \mu_b)$ -CNR and $(E - \mu_b, m_{L_{k-1}}, 1)$ -good for all $\mu_b \in \Sigma''$, hence $\Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}})$ is $(E - \mu_b, m_{L_k} + L_k^{\beta-1})$ -NS by Lemma 2.7.4. Similarly, $\Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ is $(E - \lambda_a, m_{L_k} + L_k^{\beta-1})$ -NS for any $\lambda_a \in \Sigma'$.

Now let $\mathbf{v} \in \mathbf{B}_{L_k}^{\text{out}}(\mathbf{u})$. Then $|u_{\mathcal{J}^c} - v_{\mathcal{J}^c}| \geq L_k - 6$ or $|u_{\mathcal{J}} - v_{\mathcal{J}}| \geq L_k - 6$. In the first case, we take a large $S > 2m_{L_0} \geq 2m_{L_k}$ and apply (GRI.3) to obtain

$$\|G_{\Lambda_{L_k}^{(n)}(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)\| \leq cL_k^{n'd} e^{-(m_{L_k} + L_k^{\beta-1})L_k} + c'L_k^{n'd} e^{-(L_k-6)S} \leq e^{-m_{L_k}L_k}$$

since L_0 is large. The second case is similar, using (GRI.3'). \square

Lemma 2.7.10 (cf. [68], Lemma 3.18). Let $\Lambda_{L_k}^{(n)}(\mathbf{u}) = \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ be a PI cube. If $\Lambda_{L_k}^{(n)}(\mathbf{u})$ is not E -HNR, then

- a. either there exists a cube $\Lambda_{\ell}^{(n')} \subseteq \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}})$ with $\ell \in \mathbb{N}^*$, $L_k^{1/\alpha} \leq \ell \leq L_k$ such that for $\Lambda_{\mathbb{L}}^{(n)} := \Lambda_{\ell}^{(n')} \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ we have $\text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}}), E) < e^{-\ell^{\beta}}$,
- b. or there exists a cube $\Lambda_{\ell}^{(n'')} \subseteq \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ with $\ell \in \mathbb{N}^*$, $L_k^{1/\alpha} \leq \ell \leq L_k$ such that for $\Lambda_{\mathbb{L}}^{(n)} := \Lambda_{L_k}^{(n')}(\mathcal{U}_{\mathcal{J}}) \times \Lambda_{\ell}^{(n'')}$ we have $\text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}}), E) < e^{-\ell^{\beta}}$.

Proof. Suppose condition (i) of Definition 2.7.8 is not satisfied. Then there exist $\mu \in \Sigma''$ and $\Lambda_\ell^{(n')} \subseteq \Lambda_{L_k}^{(n')}(u_{\mathcal{J}})$, $L_k^{1/\alpha} \leq \ell \leq L_k$ such that $\text{dist}(\sigma(H_{\Lambda_\ell}^{(n')}), E - \mu) < e^{-\ell^\beta}$. Thus, there exists $\eta \in \sigma(H_{\Lambda_\ell}^{(n')})$ such that $|E - \mu - \eta| < e^{-\ell^\beta}$.

But $\Lambda_{L_k}^{(n')}(u_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ is PI and $\Lambda_\ell^{(n')} \subseteq \Lambda_{L_k}^{(n')}(u_{\mathcal{J}})$, so the interaction $U^{(n)}$ also decouples on $\Lambda_{\mathbb{L}}^{(n)} := \Lambda_\ell^{(n')} \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ and we get $H_{\Lambda_{\mathbb{L}}}^{(n)} = H_{\Lambda_\ell}^{(n')} \otimes I + I \otimes H_{\Lambda_{L_k}^{(n'')}}(u_{\mathcal{J}^c})$. In particular, the eigenvalues of $H_{\Lambda_{\mathbb{L}}}^{(n)}$ take the form $E_{a,b} = \eta_a + \mu_b$ for $\eta_a \in \sigma(H_{\Lambda_\ell}^{(n')})$ and $\mu_b \in \Sigma''$. We thus showed that $\text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}}^{(n)}), E) \leq |(\eta + \mu) - E| < e^{-\ell^\beta}$.

If instead (ii) of Definition 2.7.8 is not satisfied, we reason similarly and obtain b. \square

Lemma 2.7.11. *Let $\Lambda_{L_k}^{(n)}(\mathbf{u})$, $k \geq 1$ be a PI cube and suppose (DS: $n', k-1, m_{L_{k-1}}, I_{n'}$) holds for all $n' < n$. Then there exists $C_1 = C_1(n, d, q_-)$ such that*

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ is } (E, m_{L_{k-1}})\text{-T}\} \leq C_1 L_k^{(2n-1)d - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha}}.$$

Proof. Let $\Lambda_{L_k}^{(n)}(\mathbf{u}) = \Lambda_{L_k}^{(n')}(u_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ be PI and $\Sigma'' := \sigma(H_{\Lambda_{L_k}^{(n'')}}(u_{\mathcal{J}^c}))$. By (7-4), given $E \in I_n$ and $\mu_b \in \Sigma''$, either $E - \mu_b \in [n'q_- - \frac{1}{2}, n'q_- + \varepsilon_0] = I_{n'}$, or $E - \mu_b < n'q_- - \frac{1}{2}$. Suppose $E - \mu_b < n'q_- - \frac{1}{2}$, let $\Lambda_{L_{k-1}}^{(n')}(v_1), \Lambda_{L_{k-1}}^{(n')}(v_2) \subset \Lambda_{L_k}^{(n')}(u_{\mathcal{J}})$ be two separable cubes and let $\eta_b := n'q_- - (E - \mu_b) > \frac{1}{2}$. Then by Theorem 2.3.3 given $y_i \in B_{L_{k-1}}^{\text{out}}(v_i)$,

$$\begin{aligned} \|G_{\Lambda_{L_{k-1}}^{(n')}(v_i)}(v_i, y_i; E - \mu_b)\| &\leq \sqrt{\frac{\pi}{2}} \left(\frac{\sqrt{L_{k-1}}}{\eta_b^{3/4}} + \frac{3}{8\sqrt{L_{k-1} - 8\eta_b^{5/4}}} \right) e^{-(L_{k-1}-8)\sqrt{\eta_b}} \\ &\leq e^{-m_{L_{k-1}} L_{k-1}} \end{aligned}$$

because $\sqrt{\eta_b} > \frac{1}{\sqrt{2}} \geq 2m_{L_{k-1}}$ (in fact $\frac{1}{\sqrt{2}} \gg \frac{c}{L_0^{(1-\beta)/2}} = 2m_{L_0} \geq 2m_{L_{k-1}}$ for L_0 large enough). Thus both cubes are $(E, m_{L_{k-1}})$ -NS in this case. On the other hand,

$$\mathbb{P}\{\exists E - \mu_b \in I_{n'} : \Lambda_{L_{k-1}}^{(n')}(v_1) \text{ and } \Lambda_{L_{k-1}}^{(n')}(v_2) \text{ are } (E - \mu_b, m_{L_{k-1}})\text{-S}\} \leq L_{k-1}^{-2p_{n'}(1+\theta)^{k-1}}$$

by (DS: $n', k-1, m_{L_{k-1}}, I_{n'}$). But by Lemma 2.2.2 there exists $C > 0$ such that

$$b > C \cdot |\Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})| \implies \mu_b > E - n'q_- + \frac{1}{2} \implies E - \mu_b < n'q_- - \frac{1}{2}.$$

As the number of pairs of cubes in $\Lambda_{L_k}^{(n')}(u_{\mathcal{J}})$ is bounded by $|\mathbf{B}_{L_k}^{(n')}(u_{\mathcal{J}})|^2$, we finally obtain

$$\begin{aligned} &\mathbb{P}\{\exists E \in I_n, \exists \mu_b \in \Sigma'' \text{ such that } \Lambda_{L_k}^{(n')}(u_{\mathcal{J}}) \text{ is } (E - \mu_b, m_{L_{k-1}}, 1)\text{-bad}\} \\ &\leq |\mathbf{B}_{L_k}^{(n')}(u_{\mathcal{J}})|^2 \sum_{b \leq C|\Lambda^{(n'')}|} L_{k-1}^{-2p_{n'}(1+\theta)^{k-1}} \\ &\leq \tilde{C} L_k^{2n'd + n''d} L_k^{\frac{-2p_{n'}(1+\theta)^{k-1}}{\alpha}} = \tilde{C} L_k^{(n+n')d - \frac{2p_{n'}(1+\theta)^{k-1}}{\alpha}} \leq \frac{C_1}{2} L_k^{(2n-1)d - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha}} \end{aligned}$$

because $p_{n'} \geq p_{n-1}$ for $n' = 1, \dots, n-1$. The same reasoning with $\Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ and the spectrum Σ' of $H_{\Lambda_{L_k}^{(n')}}(u_{\mathcal{J}})$ yields the theorem. \square

From now on we declare that

$$(\text{DS} : n', -1, m_{L_{-1}}, I_{n'}) = \text{no assumption.}$$

Theorem 2.7.12. *Let $k \geq 0$. Suppose that (DS: $n', k-1, m_{L_{k-1}}, I_{n'}$) holds for all $n' < n$ and let $\Lambda_{L_k}^{(n)}(\mathbf{u})$ and $\Lambda_{L_k}^{(n)}(\mathbf{v})$ be separable PI cubes. Then there exist $C_2 = C_2(n, d, q_-)$ such that*

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are } (E, m_{L_k})\text{-S}\} \leq C_2 L_k^{(2n-1)d - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha}}.$$

Proof. If $k = 0$, recall that L_0 is chosen so that for $m_{L_0} = \frac{L_0^{(\beta-1)/2}}{3}$,

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_0}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_0}^{(n)}(\mathbf{v}) \text{ are } (E, m_{L_0})\text{-S}\} \leq L_0^{-2p_1} \leq C_2 L_0^{(2n-1)d - \frac{2p_{n-1}}{\alpha(1+\theta)}}.$$

So suppose $k \geq 1$. By Lemma 2.7.9,

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are } (E, m_{L_k})\text{-S}\} \\ & \leq 3 \max_{\mathbf{x}=\mathbf{u}, \mathbf{v}} \mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{x}) \text{ is } (E, m_{L_{k-1}})\text{-T}\} \\ & \quad + \mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are not } E\text{-HNR}\}. \end{aligned}$$

For $\mathbf{x} = \mathbf{u}, \mathbf{v}$, taking $C_2 := 4C_1$, we have by Lemma 2.7.11

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{x}) \text{ is } (E, m_{L_{k-1}})\text{-T}\} \leq \frac{C_2}{4} L_k^{(2n-1)d - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha}}.$$

Since both cubes are PI, they are decomposable, say $\Lambda_{L_k}^{(n)}(\mathbf{u}) = \Lambda_{L_k}^{(n')}(u_{\mathcal{J}}) \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ and $\Lambda_{L_k}^{(n)}(\mathbf{v}) = \Lambda_{L_k}^{(r')}(v_{\mathcal{I}}) \times \Lambda_{L_k}^{(r'')}(v_{\mathcal{I}^c})$, where $n' + n'' = r' + r'' = n$. If $\Lambda_{\ell_1}^{(n')} \subseteq \Lambda_{L_k}^{(n')}(u_{\mathcal{J}})$ and $\Lambda_{\ell_2}^{(r')} \subseteq \Lambda_{L_k}^{(r')}(v_{\mathcal{I}})$, where $L_k^{1/\alpha} \leq \ell_1, \ell_2 \leq L_k$, then the rectangles $\Lambda_{\mathbb{L}}^{(n)} := \Lambda_{\ell_1}^{(n')} \times \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ and $\Lambda_{\mathbb{K}}^{(n)} := \Lambda_{\ell_2}^{(r')} \times \Lambda_{L_k}^{(r'')}(v_{\mathcal{I}^c})$ are pre-separable. Let $J_n = [nq_- - \frac{1}{2} - \varepsilon, nq_- + \varepsilon_0 + \varepsilon]$ be an ε -enlargement of I_n . Then by Theorem 2.5.2, we may find $C = C(n, d, q_-)$ such that

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_n : \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}}^{(n)}), E) < \varepsilon \text{ and } \text{dist}(\sigma(H_{\Lambda_{\mathbb{K}}}^{(n)}), E) < \varepsilon\} \\ & \leq \mathbb{P}\{\text{dist}(\sigma_{J_n}(H_{\Lambda_{\mathbb{L}}}^{(n)}), \sigma_{J_n}(H_{\Lambda_{\mathbb{K}}}^{(n)})) < 2\varepsilon\} \leq C(2L_k)^{2nd+nd} s(\mu, 4\varepsilon). \end{aligned}$$

Reasoning similarly for $\Lambda_{\ell_3}^{(n'')} \subseteq \Lambda_{L_k}^{(n'')}(u_{\mathcal{J}^c})$ and $\Lambda_{\ell_4}^{(r'')} \subseteq \Lambda_{L_k}^{(r'')}(v_{\mathcal{I}^c})$, using Lemma 2.7.10, bounding the number cubes in $\Lambda_{L_k}^{(s)}$ by $|\mathbf{B}_{L_k}^{(s)}| \leq (2L_k)^{nd}$ for $s = n', n'', r', r''$, and the number of $\ell \in \mathbb{N}^*$ satisfying $L_k^{1/\alpha} \leq \ell \leq L_k$ by L_k , we get for $\varepsilon := \max_j e^{-\ell_j^\beta} \leq e^{-L_k^{\beta/\alpha}}$,

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_k}^{(n)}(\mathbf{u}) \text{ and } \Lambda_{L_k}^{(n)}(\mathbf{v}) \text{ are not } E\text{-HNR}\} \leq 4C(2L_k)^{4nd+d+2} s(\mu, 4e^{-L_k^{\beta/\alpha}}),$$

where $4C$ appear because we apply the above argument 4 times, since Lemma 2.7.10 provides 2 cases for $\Lambda_{L_k}^{(n)}(\mathbf{u})$ and 2 cases for $\Lambda_{L_k}^{(n)}(\mathbf{v})$. As estimated in Lemma 2.7.5,

$$4C(2L_k)^{4nd+d+2} s(\mu, 4e^{-L_k^{\beta/\alpha}}) \leq L_k^{-2p_1(1+\theta)^k} \leq \frac{C_2}{4} L_k^{(2n-1)d - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha}}.$$

We thus obtain the theorem for $k \geq 1$. \square

2.7.4 General pairs of cubes

We assume through this subsection that $2 \leq n \leq N$.

Lemma 2.7.13. *Let $k \geq 0$. Suppose $(\text{DS}: n, k, m_{L_k}, I_n)$ and $(\text{DS}: n', k-1, m_{L_{k-1}}, I_{n'})$ hold for $n' < n$. Then for any cube $\Lambda_{L_{k+1}}^{(n)}(\mathbf{z})$ and $J \in 2\mathbb{N}^*$,*

$$\mathbb{P}\{\exists E \in I_n : \Lambda_{L_{k+1}}^{(n)}(\mathbf{z}) \text{ is } (E, m_{L_k}, J)\text{-bad}\} \leq \frac{1}{8}(L_{k+1}^{-2p_n(1+\theta)^{k+1}} + L_{k+1}^{-Jp_n(1+\theta)^{k+1}/3}).$$

Proof. Put $L = L_{k+1}$, $l = L_k$. If $\Lambda_L^{(n)}(\mathbf{z})$ is (E, m_l, J) -bad, then it contains at least $J+1$ pairwise separable cubes which are (E, m_l) -S. Hence, either it contains 2 separable (E, m_l) -S PI cubes, or it contains at least J separable (E, m_l) -S FI cubes. By Theorem 2.7.12,

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ contains 2 separable } (E, m_l)\text{-S PI cubes}\} \\ & \leq C_2(2L)^{2nd} l^{(2n-1)d - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha}} \leq cL^{2nd + \frac{(2n-1)d}{\alpha} - \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha^2}}, \end{aligned}$$

where we bounded the number of pairs of cubes in $\Lambda_L^{(n)}$ by $|\mathbf{B}_L^{(n)}|^2 \leq (2L)^{2nd}$. Now

$$\begin{aligned} 2p_n(1+\theta)^{k+1} &= (2p_n + 2\theta p_n)(1+\theta)^k \\ &< (2p_n + 2)(1+\theta)^k \\ &= \left(\frac{2p_{n-1}}{\alpha^2(1+\theta)} - \frac{(2n-1)d}{\alpha} - 2nd \right) (1+\theta)^k \\ &\leq \frac{2p_{n-1}(1+\theta)^{k-1}}{\alpha^2} - \frac{(2n-1)d}{\alpha} - 2nd. \end{aligned}$$

Hence,

$$\mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ contains 2 separable } (E, m_l)\text{-S PI cubes}\} \leq \frac{1}{8}L^{-2p_n(1+\theta)^{k+1}}.$$

Next, by Lemma 2.2.9, pairs of separable FI cubes are completely separated, so the corresponding Hamiltonians $H_{\Lambda_l}^{(n)}$ are independent. Thus, bounding again the number of pairs of cubes in $\Lambda_L^{(n)}$ by $(2L)^{2nd}$, we get by $(\text{DS}: n, k, m_l, I_n)$,

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ contains at least } J \text{ separable } (E, m_l)\text{-S FI cubes}\} \\ & \leq \mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ contains at least 2 separable } (E, m_l)\text{-S FI cubes}\}^{J/2} \\ & \leq ((2L)^{2nd} l^{-2p_n(1+\theta)^k})^{J/2} \leq cL^{(nd - \frac{p_n(1+\theta)^k}{\alpha})J} \leq cL^{(\frac{p_n-1}{3} - \frac{2p_n(1+\theta)^k}{3})J} \end{aligned}$$

because $\alpha = 3/2$ and $nd \leq \frac{p_N-1}{3} \leq \frac{p_n-1}{3}$. Moreover,

$$cL^{\frac{Jp_n}{3}(1-2(1+\theta)^k - \frac{1}{p_n})} \leq \frac{1}{8}L^{\frac{Jp_n}{3}(1-2(1+\theta)^k - \theta)}.$$

We thus showed that

$$\mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ is } (E, m_l, J)\text{-bad}\} \leq \frac{1}{8}L^{-2p_n(1+\theta)^{k+1}} + \frac{1}{8}L^{\frac{Jp_n}{3}(1-2(1+\theta)^k - \theta)},$$

which completes the proof by (7-3). \square

Theorem 2.7.14. *Let $k \geq 0$. Then the properties $(\text{DS}: n', k-1, m_{L_{k-1}}, I_{n'})$ for $n' < n$ and $(\text{DS}: n, k, m_{L_k}, I_n)$ imply $(\text{DS}: n, k+1, m_{L_{k+1}}, I_n)$.*

Proof. Put $L = L_{k+1}$, $l = L_k$ and let $\Lambda_L^{(n)}(\mathbf{u})$ and $\Lambda_L^{(n)}(\mathbf{v})$ be a pair of separable cubes. Since $\varepsilon_0 < 1$, any $E \in I_n$ satisfies $E \leq E_+ = \max_n(nq_- + 1)$, so applying Lemma 2.7.4 with $J = 6$, noting that $l > l^*$ because $L_0 > l^*$, we have

$$\begin{aligned} & \mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{u}) \text{ and } \Lambda_L^{(n)}(\mathbf{v}) \text{ are } (E, m_L)\text{-S}\} \\ & \leq 3 \max_{\mathbf{z}=\mathbf{u}, \mathbf{v}} \mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ is } (E, m_l, 6)\text{-bad}\} \\ & \quad + \mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{u}) \text{ and } \Lambda_L^{(n)}(\mathbf{v}) \text{ are not } E\text{-CNR}\}, \end{aligned}$$

since an (E, m_L) -S cube is a fortiori $(E, m_L + L^{\beta-1})$ -S. Now by Lemma 2.7.13,

$$\mathbb{P}\{\exists E \in I_n : \Lambda_L^{(n)}(\mathbf{z}) \text{ is } (E, m_l, 6)\text{-bad}\} \leq \frac{1}{4} L^{-2p_n(1+\theta)^{k+1}}$$

for $\mathbf{z} = \mathbf{u}, \mathbf{v}$. The assertion follows, using $(\text{W2}: n, k+1, I_n)$. \square

2.7.5 Conclusion

Theorem 2.7.15. *There exists $m > 0$ such that $(\text{DS}: N, k, m, I_N)$ holds for all $k \geq 0$.*

Proof. By construction L_0 is a large integer such that $(\text{DS}: n, 0, m_{L_0}, I_n)$ holds for all $1 \leq n \leq N$, with $m_{L_0} = \frac{1}{3L_0^{(1-\beta)/2}} > \frac{48NK(N)}{L_0^{1-\beta}}$. We prove the theorem by induction on n .

For $n = 1$, we know that $(\text{DS}: 1, k, m_{L_k}, I_1)$ holds for all $k \geq 0$ by Theorem 2.7.6 and induction on k .

Now fix $n \geq 2$ and suppose that $(\text{DS}: n', k, m_{L_k}, I_{n'})$ holds for all $k \geq 0$ and all $n' < n$. We may then apply Theorem 2.7.14 to obtain $(\text{DS}: n, k, m_{L_k}, I_n)$ for all $k \geq 0$, by induction on k . (Recall that $(\text{DS}: n', -1, m_{L_{-1}}, I_{n'})$ means no assumption).

This completes the induction and we obtain $(\text{DS}: N, k, m_{L_k}, I_N)$ for all $k \geq 0$. Now

$$S := \sum_{j=0}^{\infty} (m_{L_j} - m_{L_{j+1}}) \leq 96NK(N) \sum_{j=0}^{\infty} \frac{m_{L_j}}{L_j^{\alpha-1}} + 3 \sum_{j=0}^{\infty} \frac{1}{L_j^{\alpha(1-\beta)}}.$$

Since $m_{L_j} \leq m_{L_0}$, we have

$$S \leq 96NK(N)m_{L_0} \sum_{j=0}^{\infty} \frac{1}{L_0^{(\alpha-1)\alpha^j}} + 3 \sum_{j=0}^{\infty} \frac{1}{L_0^{\alpha(1-\beta)\alpha^j}} \leq \frac{m_{L_0}}{2} \leq m_{L_0} - m$$

for any $0 < m \leq \frac{m_{L_0}}{2}$, assuming L_0 is large enough. Now given $k \geq 1$, put $S_k := \sum_{j=k}^{\infty} (m_{L_j} - m_{L_{j+1}})$. Again the m_{L_j} are decreasing, so $S_k \geq 0$ for all k . Since

$$m_{L_0} - m \geq S = m_{L_0} - m_{L_k} + S_k,$$

we get

$$m \leq m_{L_k} - S_k \leq m_{L_k},$$

so in particular, $(\text{DS}: N, k, m, I_N)$ holds for all $k \geq 0$. \square

2.8 Generalized Eigenfunctions

In this section we prove a generalized eigenfunction expansion for $H^{(n)}(\omega)$ which plays an important role in the proof of localization. For this we show that our model satisfies the hypotheses of [67, Theorem 3.1]¹².

Given a bounded potential $v = (v_\kappa) \geq 0$, we define H_v to be the operator associated with the form

$$\mathfrak{h}_v[f, g] = \sum_{\kappa \in \mathcal{K}} \mathfrak{a}_{v_\kappa}[f_\kappa, g_\kappa], \quad D(\mathfrak{h}_v) = W^{1,2}(\Gamma^{(n)}),$$

where

$$\mathfrak{a}_{v_\kappa}[\phi, \psi] := \langle \nabla \phi, \nabla \psi \rangle + \langle v_\kappa \phi, \psi \rangle, \quad D(\mathfrak{a}_{v_\kappa}) = W^{1,2}((0, 1)^n).$$

We first show that \mathfrak{h}_v is a Dirichlet form and that $(e^{-tH^{(n)}(\omega)})_{t \geq 0}$ is ultracontractive. For this we follow [87], as it covers the case where the Hilbert space is over \mathbb{C} .

Lemma 2.8.1. *\mathfrak{h}_v is a Dirichlet form.*

Proof. Combine [87, Corollary 4.3], [87, Corollary 4.10] and [87, Theorem 2.25] to see that $p(D(\mathfrak{a}_{v_\kappa})) \subseteq D(\mathfrak{a}_{v_\kappa})$ and $\mathfrak{a}_{v_\kappa}[p \circ f] \leq \mathfrak{a}_{v_\kappa}[f]$ for every $f \in D(\mathfrak{a}_{v_\kappa})$ and every normal contraction p . Now let $u = (u_\kappa) \in D(\mathfrak{h}_v)$ such that $u_\kappa \in C([0, 1]^n)$ for all κ and let p be a normal contraction. If $\sigma^i \equiv (0, 1)^{n-1}$ is a common face to κ_1 and κ_2 and if $\gamma : W^{1,2}(0, 1)^n \rightarrow L^2(0, 1)^{n-1}$ is the trace operator, then

$$\begin{aligned} \|\gamma(p(u_{\kappa_1})) - \gamma(p(u_{\kappa_2}))\|_{L^2(0,1)^{n-1}}^2 &= \|\gamma(p(u_{\kappa_1}) - p(u_{\kappa_2}))\|_{L^2(0,1)^{n-1}}^2 \\ &= \int_{(0,1)^{n-1}} |p(u_{\kappa_1}(x)) - p(u_{\kappa_2}(x))|^2 dx \\ &\leq \int_{(0,1)^{n-1}} |u_{\kappa_1}(x) - u_{\kappa_2}(x)|^2 dx \\ &= \|\gamma(u_{\kappa_1}) - \gamma(u_{\kappa_2})\|_{L^2(0,1)^{n-1}}^2 = 0, \end{aligned}$$

where the last equality holds since u is continuous on σ^i . By the density of $C^\infty([0, 1]^n)$ in $W^{1,2}((0, 1)^n)$ and the continuity of γ and p , the same is true for all $u \in D(\mathfrak{h}_v)$. Hence $p \circ u$ is continuous on σ^i for all $u \in D(\mathfrak{h}_v)$. Thus $p(D(\mathfrak{h}_v)) \subseteq D(\mathfrak{h}_v)$. Furthermore,

$$\mathfrak{h}_v[p \circ u] = \sum_{\kappa \in \mathcal{K}} \mathfrak{a}_{v_\kappa}[p \circ u_\kappa] \leq \sum_{\kappa \in \mathcal{K}} \mathfrak{a}_{v_\kappa}[u_\kappa] = \mathfrak{h}_v[u].$$

Hence by [87, Theorem 2.25], $(e^{-tH_v})_{t \geq 0}$ is sub-Markovian. Thus \mathfrak{h}_v is a Dirichlet form. \square

Lemma 2.8.2. *There exists $c = c(n) > 0$ such that for all $\omega \in \Omega$,*

$$\forall t > 0 : \|e^{-tH^{(n)}(\omega)}\|_{L^2(\Gamma) \rightarrow L^\infty(\Gamma)} \leq ct^{-n/4} e^{-(nq-1)t}.$$

Proof. Let $Q := (0, 1)^n$. By the Gagliardo-Nirenberg interpolation inequality¹³ (see [86]),

12. The reader is encouraged to check Appendix A, in which we collected all the facts we need here.

13. Alternatively, one could use Nash inequality: by [102, Theorem 5, p. 181], there exists an extension operator $E : W^{k,p}(Q) \rightarrow W^{k,p}(\mathbb{R}^n)$ such that $(Eu)|_Q = u$ and $\|Eu\|_{k,p,\mathbb{R}^n} \leq c_{k,p}\|u\|_{k,p,Q}$ for all $k \geq 0$ and $1 \leq p \leq \infty$. Here $\|g\|_{k,p,X} = \sum_{|\alpha| \leq k} \|D^\alpha g\|_{L^p(X)}$. Hence, using Nash inequality on \mathbb{R}^n [79, Theorem 8.13], we have for any $u \in W^{1,2}(Q)$,

$$\begin{aligned} \|u\|_{0,2,Q} &\leq \|Eu\|_{0,2,\mathbb{R}^n} \leq c_n \|\nabla Eu\|_{0,2,\mathbb{R}^n}^a \|Eu\|_{0,1,\mathbb{R}^n}^{1-a} \\ &\leq c_n \|Eu\|_{1,2,\mathbb{R}^n}^a \|Eu\|_{0,1,\mathbb{R}^n}^{1-a} \leq C \|u\|_{1,2,Q}^a \|u\|_{0,1,Q}^{1-a} \end{aligned}$$

for $C = c_n c_{1,2} c_{0,1}$. We used above that $Eu \in L^1(\mathbb{R}^n)$, since $u \in L^2(Q) \subset L^1(Q)$. Noting that $\|g\|_{1,2,Q} \leq (n+1)^{1/2} \left(\sum_{|\alpha| \leq 1} \|D^\alpha g\|_{L^2(X)}^2 \right)^{1/2}$, we get (8-1)

we have for any $u \in W^{1,2}(Q)$,

$$\|u\|_{L^2(Q)} \leq C(\|\nabla u\|_{L^2(Q)}^a + \|u\|_{L^1(Q)}^a)\|u\|_{L^1(Q)}^{1-a},$$

where $a = \frac{n}{n+2}$. By Hölder inequality, we have $\|u\|_{L^1(Q)} \leq \|u\|_{L^2(Q)}$. Using Hölder inequality again, with $p = \frac{2}{a}$ and $q = \frac{2}{2-a}$, we get $(x^a + y^a) \leq 2^{1/q}(x^2 + y^2)^{a/2}$. Thus,

$$(8-1) \quad \|u\|_{L^2(Q)} \leq \tilde{C}(\|\nabla u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2)^{a/2}\|u\|_{L^1(Q)}^{1-a} \leq \tilde{C}(\mathfrak{a}_{v_\kappa}[u])^{a/2}\|u\|_{L^1(Q)}^{1-a}$$

for any bounded potential $v_\kappa \geq 1$. Hence, for any $f \in D(\mathfrak{h}_v) \cap L^1(\Gamma)$ we have

$$\|f\|_{L^2(\Gamma)}^2 = \sum_{\kappa \in \mathcal{K}} \|f_\kappa\|_{L^2(Q)}^2 \leq \tilde{C}^2 \sum_{\kappa \in \mathcal{K}} (\mathfrak{a}_{v_\kappa}[f_\kappa])^a \|f_\kappa\|_{L^1(Q)}^{2(1-a)}.$$

Using Hölder inequality with $p = \frac{1}{a}$ and $q = \frac{1}{1-a}$ we get

$$\begin{aligned} \|f\|_{L^2(\Gamma)}^2 &\leq \tilde{C}^2 \left(\sum \mathfrak{a}_{v_\kappa}[f_\kappa] \right)^a \left(\sum \|f_\kappa\|_{L^1(Q)}^2 \right)^{(1-a)} \\ &\leq \tilde{C}^2 \left(\sum \mathfrak{a}_{v_\kappa}[f_\kappa] \right)^a \left(\sum \|f_\kappa\|_{L^1(Q)} \right)^{2(1-a)} = \tilde{C}^2 (\mathfrak{h}_v[f])^a \|f\|_{L^1(\Gamma)}^{2(1-a)}. \end{aligned}$$

Using Lemma 2.8.1 and applying [87, Theorem 6.3], it follows that

$$\forall t > 0 : \|e^{-tH_v}\|_{L^1(\Gamma) \rightarrow L^2(\Gamma)} \leq ct^{-n/4}.$$

But $\|e^{-tH_v}\|_{L^1 \rightarrow L^2} = \|e^{-tH_v}\|_{L^2 \rightarrow L^\infty}$ by duality. So the assertion follows by taking $v := V^\omega - (nq_- - 1) \geq 1$ and noting that

$$e^{-tH_v} = \exp(-t(H^{(n)}(\omega) - (nq_- - 1))) = e^{(nq_- - 1)t} e^{-tH^{(n)}(\omega)}. \quad \square$$

Let T be the self-adjoint operator on $L^2(\Gamma^{(n)})$ given by

$$Tf(\mathbf{x}) := w(\mathbf{x})f(\mathbf{x}), \quad \text{where } w(\mathbf{x}) = (1 + \|\mathbf{x}\|_2^2)^{\gamma/4}$$

for some fixed $\gamma > nd + 1$. We now establish

Lemma 2.8.3. *There exists $C = C(n)$ such that for all $\omega \in \Omega$ and $t > 0$:*

$$\text{tr}(T^{-1}e^{-2tH^{(n)}(\omega)}T^{-1}) \leq Ct^{-n/2}e^{-2(nq_- - 1)t}\|w^{-1}\|_{L^2}^2 < \infty.$$

Furthermore, if E_ω is the spectral projection of $H^{(n)}(\omega)$, then the set function ν_ω on \mathbb{R} given by¹⁴

$$\nu_\omega(J) := \text{tr}(T^{-1}E_\omega(J)T^{-1}) = \|E_\omega(J)T^{-1}\|_2^2$$

is a spectral measure for $H^{(n)}(\omega)$ which is finite on bounded Borel sets J .

Proof. Divide $\Gamma^{(n)}$ into annuli $\Gamma^{(n)} \cap (\Lambda_{k+1}^{(n)}(\mathbf{0}) \setminus \Lambda_k^{(n)}(\mathbf{0}))$. Then by (NB.n),

$$\int_{\Gamma^{(n)}} |w^{-1}|^2 dm \leq \sum_k \frac{m(\Gamma^{(n)} \cap \Lambda_{k+1}^{(n)}(\mathbf{0}))}{(1 + k^2)^{\gamma/2}} \leq C \sum_k \frac{(k+1)^{nd}}{(1 + k^2)^{\gamma/2}} < \infty.$$

14. Recall that the trace is well defined for any positive operator and has values in $[0, \infty]$.

Thus $w^{-1} \in L^2$ and T^{-1} is an operator from $L^\infty \rightarrow L^2$. Put $H := H^{(n)}(\omega)$. Then by Lemma 2.8.2, $T^{-1}e^{-tH} : L^2 \rightarrow L^2$ factorizes through L^∞ . It is thus Hilbert-Schmidt with

$$\|T^{-1}e^{-tH}\|_2 \leq \|w^{-1}\|_{L^2} \|e^{-tH}\|_{L^2 \rightarrow L^\infty} \leq ct^{-n/4} e^{-(nq_- - 1)t} \|w^{-1}\|_{L^2}$$

(see Appendix B). Thus $e^{-tH}T^{-1} = (T^{-1}e^{-tH})^*$ is also a Hilbert-Schmidt operator with the same norm. Hence,

$$\mathrm{tr}(T^{-1}e^{-2tH}T^{-1}) = \|e^{-tH}T^{-1}\|_2^2 \leq Ct^{-n/2} e^{-2(nq_- - 1)t} \|w^{-1}\|_{L^2}^2.$$

Now let J be a bounded Borel set and put $b := \sup\{\lambda \in J\}$. Then

$$0 \leq e^{-2b}E_\omega(J) \leq \int_J e^{-2\lambda} dE_\omega(\lambda) \leq \int_{\sigma(H)} e^{-2\lambda} dE_\omega(\lambda) = e^{-2H}.$$

Hence $\nu_\omega(J) \leq e^{2b} \mathrm{tr}(T^{-1}e^{-2H}T^{-1}) \leq C_J \|w^{-1}\|_{L^2}^2$ and ν_ω is finite on bounded Borel sets. It is easy to see that ν_ω is a Borel measure¹⁵. Finally, $\nu_\omega(J) = 0 \iff E_\omega(J) = 0$, so ν_ω is a spectral measure for H . \square

We note in passing that given a bounded interval I , the previous proof yields a constant $C = C(I, n, q_-) > 0$ independent of ω such that

$$(8-2) \quad \sup_\omega \nu_\omega(I) \leq C \|w^{-1}\|_{L^2}^2 := C_{\mathrm{tr}}.$$

Let \mathcal{H}_+ be the space $D(T)$ equipped with the norm $\|\phi\|_+ = \|T\phi\|$ and \mathcal{H}_- the completion of \mathcal{H} in the norm $\|\psi\|_- = \|T^{-1}\psi\|$. By construction $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is then a triple of Hilbert spaces¹⁶ with natural injections $\iota_+ : \mathcal{H}_+ \rightarrow \mathcal{H}$ and $\iota_- : \mathcal{H} \rightarrow \mathcal{H}_-$ continuous with dense range. The inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ extends to a sesquilinear form on $\mathcal{H}_+ \times \mathcal{H}_-$ which turns \mathcal{H}_+ and \mathcal{H}_- into conjugate duals (see [89, Lemma 1] and [11]). The adjoint of an operator O with respect to this duality is denoted by O^\dagger .

Lemma 2.8.4. *For all $\omega \in \Omega$, the space*

$$\mathcal{D}_+ = \{f \in D(H^{(n)}(\omega)) \cap \mathcal{H}_+ : H^{(n)}(\omega)f \in \mathcal{H}_+\}$$

is dense in \mathcal{H}_+ and is an operator core for $H^{(n)}(\omega)$ ¹⁷.

Proof. Set $H := H^{(n)}(\omega)$ and let $C_c^\infty(\Gamma) := (\oplus_{\kappa} C_c^\infty(0, 1)^n) \cap C_c(\Gamma)$. Clearly $\mathcal{D}_+ \supseteq C_c^\infty(\Gamma)$ (see the definition of $D(H)$ in the Appendix, Section 2.11). Moreover, $C_c^\infty(\Gamma)$ is dense in $L^2(\Gamma)$. Now let $f \in \mathcal{H}_+$, then $Tf \in L^2(\Gamma)$ may be approximated by $g_j \in C_c^\infty(\Gamma)$, hence $\|f - T^{-1}g_j\|_+ \rightarrow 0$ and clearly $T^{-1}g_j \in C_c^\infty(\Gamma)$. Hence \mathcal{D}_+ is dense in \mathcal{H}_+ .

To show \mathcal{D}_+ is a core we follow [22, Proposition 2.4]: let $E < nq_-$ and consider $D_0 = (H - E)^{-1}C_c(\Gamma)$. Since $C_c(\Gamma)$ is dense in $L^2(\Gamma)$, D_0 is a core for H . By Combes-Thomas estimate, each $f \in D_0$ is exponentially decreasing. Hence $f \in \mathcal{H}_+$ and

$$Hf = (H - E)f + Ef = \varphi + Ef \in \mathcal{H}_+$$

since $f = (H - E)^{-1}\varphi$ for some $\varphi \in C_c(\Gamma)$. This proves the claim. \square

15. If (J_i) are disjoint Borel sets, then $\nu_\omega(\bigcup J_i) = \mathrm{tr}[T^{-1}E_\omega(\bigcup J_i)T^{-1}] = \sum_j \langle T^{-1}E_\omega(\bigcup J_i)T^{-1}e_j, e_j \rangle = \sum_j \sum_i \langle T^{-1}E_\omega(J_i)T^{-1}e_j, e_j \rangle = \sum_i \sum_j \langle T^{-1}E_\omega(J_i)T^{-1}e_j, e_j \rangle = \sum_i \mathrm{tr}[T^{-1}E_\omega(J_i)T^{-1}] = \sum_i \nu_\omega(J_i)$. The previous interchange of summations is valid because all terms are positive (even if some series have an infinite value), see e.g. [96, Theorem 1.27].

16. By identifying $\mathcal{H}_+ \equiv \iota_+\mathcal{H}$ and $\mathcal{H} \equiv \iota_-\mathcal{H}$. We preferred to keep the distinction in Appendix A.

17. We prove a stronger statement in Lemma 3.4.3, namely functions of compact support in $D(H^{(n)}(\omega))$ form an operator core.

By [67, Lemma 3.1], $H^{(n)}(\omega)$ regarded as an operator on \mathcal{H}_- is thus closable and densely defined. We denote its closure by $H_-^{(n)}(\omega)$. We then say $\psi \in \mathcal{H}_-$ is a *generalized eigenfunction* of $H^{(n)}(\omega)$ with corresponding *generalized eigenvalue* $\lambda \in \mathbb{C}$ if ψ is an eigenfunction of $H_-^{(n)}(\omega)$ with eigenvalue λ , i.e. if $\psi \in D(H_-^{(n)}(\omega))$ and $H_-^{(n)}(\omega)\psi = \lambda\psi$.

By [67, Lemma 3.2], we have $H_-^{(n)}(\omega)\psi = H^{(n)}(\omega)\psi$ for any $\psi \in D(H_-^{(n)}) \cap \mathcal{H}$. In particular, if a generalized eigenfunction lies in \mathcal{H} , then it is an eigenfunction.

We may now state the main result of this section. Here $\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$ and $\mathcal{T}_{1,+}(\mathcal{H}_+, \mathcal{H}_-)$ are the spaces of trace class and positive trace class operators from \mathcal{H}_+ to \mathcal{H}_- respectively (see [67] for details).

Theorem 2.8.5. *Let ν_ω be the spectral measure of $H^{(n)}(\omega)$ introduced in Lemma 2.8.3. There exists a ν_ω -locally integrable function $P_\omega : \mathbb{R} \rightarrow \mathcal{T}_{1,+}(\mathcal{H}_+, \mathcal{H}_-)$ such that*

$$\iota_- f(H^{(n)}(\omega)) E_\omega(J) \iota_+ = \int_J f(\lambda) P_\omega(\lambda) d\nu_\omega(\lambda)$$

for all bounded Borel sets J and all bounded Borel functions f , where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$ -valued functions. Furthermore, for ν_ω -a.e. $\lambda \in \mathbb{R}$,

$$P_\omega(\lambda) = P_\omega(\lambda)^\dagger, \quad \text{tr } P_\omega(\lambda) = 1$$

and $P_\omega(\lambda)\phi \in \mathcal{H}_-$ is a generalized eigenfunction of $H^{(n)}(\omega)$ with generalized eigenvalue λ for any $\phi \in \mathcal{H}_+$.

Proof. Applying [67, Theorem 3.1] and [67, Corollary 3.1], it only remains to show that $P_\omega(\lambda) = P_\omega(\lambda)^\dagger$ ν_ω -a.e. This follows from [67, Eq.(46)] and the fact that $\iota_+^\dagger = \iota_-$ ¹⁸. \square

2.9 Exponential Localization

The fundamental link between multiscale analysis and localization is provided by the following eigenfunction decay inequality. Since we will not rely on regularity of generalized eigenfunctions, the proof is a bit longer than in [104].

Lemma 2.9.1. *Let $E_+ \in \mathbb{R}$. There exists $C = C(E_+, n, d, q_-)$ such that, if $\mathbf{x}_0 \in \mathbb{Z}^{nd}$ and $\mathbf{C}(\mathbf{x}) \subset \Lambda_{L-6}^{(n)}(\mathbf{x}_0)$, then every generalized eigenfunction ψ of $H^{(n)}(\omega)$ corresponding to $\lambda \in \rho(H_{\Lambda_L}^{(n)}(\mathbf{x}_0)) \cap (-\infty, E_+]$ satisfies*

$$\|\chi_{\mathbf{x}}\psi\| \leq C \cdot |\mathbf{B}_L^{\text{out}}(\mathbf{x}_0)| \max_{\mathbf{y} \in \mathbf{B}_L^{\text{out}}(\mathbf{x}_0)} \|G_{\Lambda_L^{(n)}(\mathbf{x}_0)}(\mathbf{x}, \mathbf{y}; \lambda)\| \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)}\psi\|.$$

Proof. Let $\Lambda := \Lambda_L^{(n)}(\mathbf{x}_0)$ and $\varphi \in \tilde{C}_c^1(\Gamma \cap \Lambda)$ such that $\varphi = 1$ on a neighborhood of $\Gamma^{(n)} \cap \mathbf{C}(\mathbf{x})$, $\text{supp } \nabla\varphi \subset \tilde{Q} := \text{int}(\Lambda_{L-2}^{(n)}(\mathbf{x}_0) \setminus \Lambda_{L-4}^{(n)}(\mathbf{x}_0))$, and $\|\nabla\varphi\|_\infty \leq C_1(nd)$. Then

$$\|\chi_{\mathbf{x}}\psi\|^2 = \langle \varphi\psi, \chi_{\mathbf{x}}\psi \rangle = \langle \varphi\psi, (H_\Lambda - \lambda)G_\Lambda(\lambda)\chi_{\mathbf{x}}\psi \rangle.$$

Put $H := H^{(n)}(\omega)$. Since $\psi \in D(H_-)$ and H_- is the closure of H , there exists (f_j) in $D(H)$ such that $\|f_j - \psi\|_- \rightarrow 0$ and $\|Hf_j - H_- \psi\|_- \rightarrow 0$ as $j \rightarrow \infty$. Now for any χ of compact support we have

$$(9-1) \quad \|\chi f_j - \chi\psi\| \leq \|\chi w\| \cdot \|f_j - \psi\|_- \rightarrow 0$$

18. See Section A.3 for details.

(recall that $Tg := wg$). Hence taking $v := G_\Lambda(\lambda)\chi_{\mathbf{x}}\psi$ we have

$$\|\chi_{\mathbf{x}}\psi\|^2 = \lim_{j \rightarrow \infty} \langle \varphi f_j, (H_\Lambda - \lambda)v \rangle = \lim_{j \rightarrow \infty} (\mathfrak{h}_\Lambda - \lambda)[\varphi f_j, v]$$

since $\varphi f_j \in D(\mathfrak{h}_\Lambda) = W^{1,2}(\Gamma^{(n)} \cap \Lambda)$ by Lemma 2.3.1. Now

$$\begin{aligned} (\mathfrak{h}_\Lambda - \lambda)[\varphi f_j, v] &= \langle \nabla(\varphi f_j), \nabla v \rangle + \langle (V^\omega - \lambda)\varphi f_j, v \rangle \\ &= [\langle \nabla f_j, \nabla(\varphi v) \rangle + \langle (V^\omega - \lambda)f_j, \varphi v \rangle] + \langle f_j \nabla \varphi, \nabla v \rangle - \langle \nabla f_j, v \nabla \varphi \rangle. \end{aligned}$$

Since $\varphi v \in W^{1,2}(\Gamma^{(n)} \cap \Lambda)$ has compact support in Λ , we may extend it by zero to a function g in $D(\mathfrak{h}) \cap C_c(\Gamma)$. Hence

$$(9-2) \quad (\mathfrak{h}_\Lambda - \lambda)[\varphi f_j, v] = \langle (H - \lambda)f_j, g \rangle + \langle f_j \nabla \varphi, \nabla v \rangle - \langle \nabla f_j, v \nabla \varphi \rangle.$$

Now $H_- \psi = \lambda \psi$, so by the choice of f_j

$$(9-3) \quad \|(H - \lambda)f_j\|_- \leq \|Hf_j - \lambda\psi\|_- + |\lambda| \cdot \|f_j - \psi\|_- \rightarrow 0.$$

Thus

$$|\langle (H - \lambda)f_j, g \rangle| \leq \|(H - \lambda)f_j\|_- \|g\|_+ \rightarrow 0.$$

The second term in (9-2) tends to $\langle \psi \nabla \varphi, \nabla v \rangle$ by (9-1). For the third term, note that by Lemma 2.4.3, taking $Q := \text{int } \Lambda_L^{\text{out}}(\mathbf{x}_0)$, we can find c_1 such that

$$\begin{aligned} \|\chi_{\tilde{Q}} \nabla f_j\| &\leq c_1 (\|\chi_Q (H - \lambda)f_j\| + \|\chi_Q f_j\|) \\ &\leq c_1 (\|\chi_Q w\| \cdot \|(H - \lambda)f_j\|_- + \|\chi_Q f_j\|) \rightarrow c_1 \|\chi_Q \psi\| \end{aligned}$$

using (9-1) and (9-3).

Recalling that $\text{supp } \nabla \varphi \subset \tilde{Q}$, the above derivation finally yields

$$\|\chi_{\mathbf{x}}\psi\|^2 \leq \|\nabla \varphi\|_\infty \|\chi_{\tilde{Q}} \psi\| \|\chi_{\tilde{Q}} \nabla v\| + c_1 \|\nabla \varphi\|_\infty \|\chi_Q \psi\| \|\chi_{\tilde{Q}} v\|$$

By Lemma 2.4.3, we can find c_2 such that

$$\|\chi_{\tilde{Q}} \nabla v\| \leq c_2 \|\chi_Q v\|$$

(note that $(H_\Lambda - \lambda)v = \chi_{\mathbf{x}}\psi = 0$ on Q). Taking $C = \max(2c_1 \|\nabla \varphi\|_\infty, 2c_2 \|\nabla \varphi\|_\infty)$ and noting that $\tilde{Q} \subset Q \subset \Lambda_L^{\text{out}}(\mathbf{x}_0)$ we thus get

$$\|\chi_{\mathbf{x}}\psi\|^2 \leq C \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} \psi\| \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} v\|.$$

Since $\|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} v\| \leq \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} G_\Lambda(\lambda)\chi_{\mathbf{x}}\| \|\chi_{\mathbf{x}}\psi\|$, we get

$$\|\chi_{\mathbf{x}}\psi\| \leq C \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} G_\Lambda(\lambda)\chi_{\mathbf{x}}\| \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} \psi\|.$$

The assertion now follows by the triangle inequality. \square

We now prove exponential localization by adapting [112, Theorem 2.3].

Proof of Theorem 2.1.3. Choose m and ε_0 such that (DS: N, k, m, I_N) holds for all $k \geq 0$ in $I_N = [Nq_- - \frac{1}{2}, Nq_- + \varepsilon_0]$, as guaranteed by Theorem 2.7.15. Let $\sigma_{\text{gen}}^\omega$ be the set of generalized eigenvalues of $H^{(N)}(\omega)$. By Theorem 2.8.5 there exists $A_0^\omega \subseteq \mathbb{R}$ of full ν_ω -measure such that $A_0^\omega \subseteq \sigma_{\text{gen}}^\omega$. If we show that every $\lambda \in \sigma_{\text{gen}}^\omega \cap I$ is an eigenvalue, $A_0^\omega \cap I$ will be countable (as $L^2(\Gamma^{(N)})$ is separable), so $\nu_\omega|_I$ will be concentrated on a

countable set and $\sigma(H^{(N)}(\omega)) \cap I$ will be pure point. It thus suffices to show that with probability one the generalized eigenfunctions of $H^{(N)}(\omega)$ corresponding to $\lambda \in \sigma_{\text{gen}}^\omega \cap I$ decay exponentially with mass m .

Let $b \in \mathbb{N}^*$ to be chosen later and define

$$A_{k+1} = \mathbf{B}_{2br_{k+1}}^{(N)}(\mathbf{0}) \setminus \mathbf{B}_{2r_k}^{(N)}(\mathbf{0}),$$

where $r_k := r_{N, L_k}$. Then by Lemma 2.2.8, any $\mathbf{x} \in A_{k+1}$ satisfies that $\Lambda_{L_k}^{(N)}(\mathbf{x})$ is separable from $\Lambda_{L_k}^{(N)}(\mathbf{y})$ for any $\mathbf{y} \in \mathbf{B}_{r_k}^{(N)}(\mathbf{0})$. Now define the event

$$E_k = \{\exists \lambda \in I, \mathbf{x} \in A_{k+1}, \mathbf{y} \in \mathbf{B}_{r_k}^{(N)}(\mathbf{0}) : \Lambda_{L_k}^{(N)}(\mathbf{x}) \text{ and } \Lambda_{L_k}^{(N)}(\mathbf{y}) \text{ are } (\lambda, m)\text{-S}\}.$$

Then by Theorem 2.7.15, we have

$$\mathbb{P}(E_k) \leq (4br_{k+1} - 1)^{Nd} (2r_k - 1)^{Nd} L_k^{-2p_N(1+\theta)^k} \leq cL_k^{2N\alpha d - 2p_N(1+\theta)^k}$$

Hence $\sum_{k=0}^{\infty} \mathbb{P}(E_k) < \infty$. So by the Borel-Cantelli Lemma, if we define the event

$$\Omega_1 = \{E_k \text{ occurs finitely often}\},$$

we have $\mathbb{P}(\Omega_1) = 1$. Now let $\omega \in \Omega_1$ and $\lambda \in \sigma_{\text{gen}}^\omega \cap I$ correspond to a generalized eigenfunction ψ . If $\|\chi_{\mathbf{x}}\psi\| = 0$ for all $\mathbf{x} \in \mathbb{Z}^{Nd}$, then $\psi = 0$ and the theorem holds. So suppose $\|\chi_{\mathbf{y}}\psi\| \neq 0$ for some $\mathbf{y} \in \mathbb{Z}^{Nd}$. Then by Lemma 2.9.1 we may find $C_1 = C_1(N, d, q_-, \gamma, \|\psi\|_-)$ such that

$$\|\chi_{\mathbf{y}}\psi\| \leq C_1 \cdot |\mathbf{B}_{L_k}^{\text{out}}(\mathbf{y})| \max_{\mathbf{z} \in \mathbf{B}_{L_k}^{\text{out}}(\mathbf{y})} \|G_{\Lambda_{L_k}^{(N)}(\mathbf{y})}(\mathbf{y}, \mathbf{z}; \lambda)\| \cdot (1 + (|\mathbf{y}| + L_k)^2)^{\gamma/4}.$$

Now if $\Lambda_{L_k}^{(N)}(\mathbf{y})$ is (λ, m) -NS, we get

$$\|\chi_{\mathbf{y}}\psi\| \leq C_1' L_k^{Nd-1} e^{-mL_k} (1 + (|\mathbf{y}| + L_k)^2)^{\gamma/4}.$$

Since $\|\chi_{\mathbf{y}}\psi\| \neq 0$, there exists k_0 such that $\Lambda_{L_k}^{(N)}(\mathbf{y})$ is (λ, m) -S for all $k \geq k_0$. But there exists k_1 such that $\mathbf{y} \in \mathbf{B}_{r_k}^{(N)}(\mathbf{0})$ for all $k \geq k_1$. Finally, since $\omega \in \Omega_1$, we may find k_2 such that E_k does not occur if $k \geq k_2$. Let $k_3 = \max(k_0, k_1, k_2)$. Then for $k \geq k_3$, we conclude that $\Lambda_{L_k}^{(N)}(\mathbf{x})$ is (λ, m) -NS for all $\mathbf{x} \in A_{k+1}$.

Now given $0 < \rho < 1$, we choose $b > \frac{1+\rho}{1-\rho}$ and define

$$\tilde{A}_{k+1} = \mathbf{B}_{\frac{2b}{1+\rho}r_{k+1}}^{(N)}(\mathbf{0}) \setminus \mathbf{B}_{\frac{2}{1-\rho}r_k}^{(N)}(\mathbf{0}).$$

Then $\tilde{A}_{k+1} \subset A_{k+1}$ and for any $\mathbf{x} \in \tilde{A}_{k+1}$, we have

$$\text{dist}(\mathbf{x}, \partial A_{k+1}) \geq \rho \cdot |\mathbf{x}|.$$

Indeed, if $\mathbf{x} \in \tilde{A}_{k+1}$, then

$$d(\mathbf{x}, \partial \mathbf{B}_{2br_{k+1}}^{(N)}(\mathbf{0})) \geq 2br_{k+1} - \frac{2b}{1+\rho}r_{k+1} = \rho \frac{2b}{1+\rho}r_{k+1} \geq \rho \cdot |\mathbf{x}|,$$

$$d(\mathbf{x}, \partial \mathbf{B}_{2r_k}^{(N)}(\mathbf{0})) = |\mathbf{x}| - 2r_k \geq |\mathbf{x}| - (1-\rho)|\mathbf{x}| = \rho \cdot |\mathbf{x}|.$$

Now for $\mathbf{x} \in \tilde{A}_{k+1}$ with $k \geq k_3$, $\Lambda_{L_k}^{(N)}(\mathbf{x})$ is (λ, m) -NS, so by Lemma 2.9.1,

$$\|\chi_{\mathbf{x}}\psi\| \leq C_2 L_k^{2(Nd-1)} e^{-mL_k} \|\chi_{\mathbf{w}_1}\psi\|$$

for some $\mathbf{w}_1 \in \mathbf{B}_{L_k}^{\text{out}}(\mathbf{x})$. We may iterate at least $\lfloor \frac{\rho|\mathbf{x}|}{L_k-1} \rfloor$ times and obtain

$$\begin{aligned} \|\chi_{\mathbf{x}}\psi\| &\leq (C_2 L_k^{2(Nd-1)} e^{-mL_k})^{\lfloor \frac{\rho|\mathbf{x}|}{L_k-1} \rfloor} C_3 \|\psi\| (1 + (2br_{k+1})^2)^{\gamma/4} \\ &\leq e^{-m\rho'\rho|\mathbf{x}|} \end{aligned}$$

for any $0 < \rho' < 1$, provided $k \geq k_4$ for some $k_4 \geq k_3$. But if $\mathbf{x} \notin \mathbf{B}_{\frac{2}{1-\rho}r_{k_4}}^{(N)}(\mathbf{0})$, then $\mathbf{x} \in \tilde{A}_{k+1}$ for some $k \geq k_4$ (since $\frac{2b}{1+\rho}r_{k+1} > \frac{2}{1-\rho}r_{k+1}$) and the bound is satisfied. Thus,

$$\log \|\chi_{\mathbf{x}}\psi\| \leq -m\rho'\rho \cdot |\mathbf{x}|$$

whenever $\mathbf{x} \notin \mathbf{B}_{\frac{2}{1-\rho}r_{k_4}}^{(N)}(\mathbf{0})$. Hence

$$\limsup_{|\mathbf{x}| \rightarrow \infty} \frac{\log \|\chi_{\mathbf{x}}\psi\|}{|\mathbf{x}|} \leq -m\rho'\rho$$

for all $\rho, \rho' \in (0, 1)$, which completes the proof of the theorem. \square

2.10 Dynamical Localization

We finally establish dynamical localization for $H^{(N)}(\omega)$ using the approach of [47]. In the following we consider the event

$$R(m, L, I, \mathbf{x}, \mathbf{y}) := \{ \forall \lambda \in I : \Lambda_L^{(N)}(\mathbf{x}) \text{ or } \Lambda_L^{(N)}(\mathbf{y}) \text{ is } (\lambda, m)\text{-NS} \}$$

for \mathbf{x}, \mathbf{y} such that the corresponding cubes are separable. We start with the following key lemma.

Lemma 2.10.1. *Let $m > 0$, $I \subset \mathbb{R}$ and assume $\omega \in R(m, L, I, \mathbf{x}, \mathbf{y})$. Then*

$$\|\chi_{\mathbf{x}}P_{\omega}(\lambda)\chi_{\mathbf{y}}\|_2 \leq C e^{-mL/2} (1 + |\mathbf{x}|)^{\gamma/2} (1 + |\mathbf{y}|)^{\gamma/2}$$

for ν_{ω} -a.e. $\lambda \in I$ and large L , with $C = C(I, m, N, d, \gamma, q_-) < \infty$.

Proof. Let A_0^{ω} be the set of full ν_{ω} -measure such that Theorem 2.8.5 holds for all $\lambda \in A_0^{\omega}$. Given $\lambda \in I \cap A_0^{\omega}$, either $\Lambda_L^{(N)}(\mathbf{x})$ or $\Lambda_L^{(N)}(\mathbf{y})$ is (λ, m) -NS. Since $P_{\omega}(\lambda) = P_{\omega}(\lambda)^{\dagger}$, we have $\|\chi_{\mathbf{x}}P_{\omega}(\lambda)\chi_{\mathbf{y}}\|_2 = \|\chi_{\mathbf{y}}P_{\omega}(\lambda)\chi_{\mathbf{x}}\|_2$, so we may assume that $\Lambda_L^{(N)}(\mathbf{x})$ is (λ, m) -NS. Now if $\phi \in \mathcal{H}$, then by Theorem 2.8.5, the vector $P_{\omega}(\lambda)\chi_{\mathbf{y}}\phi$ is a generalized eigenfunction of $H^{(N)}(\omega)$, hence by Lemma 2.9.1,

$$\|\chi_{\mathbf{x}}P_{\omega}(\lambda)\chi_{\mathbf{y}}\phi\| \leq C_1 (2L-1)^{Nd-1} e^{-mL} \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x})}P_{\omega}(\lambda)\chi_{\mathbf{y}}\phi\|.$$

Hence by definition of the HS norm,

$$\|\chi_{\mathbf{x}}P_{\omega}(\lambda)\chi_{\mathbf{y}}\|_2 \leq C_1 (2L-1)^{Nd-1} e^{-mL} \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x})}P_{\omega}(\lambda)\chi_{\mathbf{y}}\|_2.$$

But

$$\begin{aligned} \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x})}P_{\omega}(\lambda)\chi_{\mathbf{y}}\|_1 &\leq \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x})}\|_{\mathcal{H}_- \rightarrow \mathcal{H}} \|P_{\omega}(\lambda)\|_{\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)} \|\chi_{\mathbf{y}}\|_{\mathcal{H} \rightarrow \mathcal{H}_+} \\ &\leq c(1 + (|\mathbf{x}| + L)^2)^{\gamma/4} (1 + (|\mathbf{y}| + 1)^2)^{\gamma/4} \end{aligned}$$

since $\text{tr } P_{\omega}(\lambda) = 1$ and $P_{\omega}(\lambda) \geq 0$. The claim follows since $\|\cdot\|_2 \leq \|\cdot\|_1$. \square

We now establish the decay of the operator kernel. Given a bounded K as in the statement of Theorem 2.1.4, we find $k_0 > 0$ such that $K \subset \Gamma \cap \Lambda_{r_N, L_{k_0}}^{(N)}(\mathbf{0})$. For $j \geq k_0$ put

$$\begin{aligned} F_j &= \Lambda_{2r_N, L_j}^{(N)}(\mathbf{0}), & \tilde{F}_j &= \mathbf{B}_{2r_N, L_j}^{(N)}(\mathbf{0}), \\ M_j &= F_{j+1} \setminus F_j, & \tilde{M}_j &= \tilde{F}_{j+1} \setminus \tilde{F}_j. \end{aligned}$$

In the following, we choose m and ε_0 such that (DS: N, k, m, I_N) holds for all $k \geq 0$ in $I_N = [Nq_- - \frac{1}{2}, Nq_- + \varepsilon_0]$, as guaranteed by Theorem 2.7.15.

Lemma 2.10.2. *There exists $c = c(N, d, q_-, r_0, \gamma)$ such that for $\mathbf{x} \in \tilde{M}_j$ and $\mathbf{y} \in \mathbf{B}_{r_N, L_j}^{(N)}(\mathbf{0})$ with j large enough, we have for $I = [Nq_-, Nq_- + \varepsilon_0]$:*

$$\mathbb{E} \left(\sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}} f(H^{(N)}(\omega)) E_{\omega}(I) \chi_{\mathbf{y}}\|_2^2 \right) \leq c(e^{-mL_j/2} + L_j^{-2p_N(1+\theta)^j + \gamma}).$$

Proof. Given a bounded Borel function f put $f_I := f\chi_I$ and $H_{\omega} := H^{(N)}(\omega)$. By Theorem 2.8.5 and standard properties of the Bochner integral in the space of HS operators we have

$$\|\chi_{\mathbf{x}} f_I(H_{\omega}) \chi_{\mathbf{y}}\|_2 \leq \int_I |f(\lambda)| \|\chi_{\mathbf{x}} P_{\omega}(\lambda) \chi_{\mathbf{y}}\|_2 d\nu_{\omega}(\lambda).$$

Since $\mathbf{x} \in \tilde{M}_j$ and $\mathbf{y} \in \mathbf{B}_{r_N, L_j}^{(N)}(\mathbf{0})$, we know by Lemma 2.2.8 that $\Lambda_{L_j}^{(N)}(\mathbf{x})$ and $\Lambda_{L_j}^{(N)}(\mathbf{y})$ are separable. Hence if $\omega \in B_j := R(m, L_j, I, \mathbf{x}, \mathbf{y})$, we have by Lemma 2.10.1

$$\|\chi_{\mathbf{x}} P_{\omega}(\lambda) \chi_{\mathbf{y}}\|_2 \leq C_1 L_{j+1}^{\gamma/2} L_j^{\gamma/2} e^{-mL_j/2} \leq e^{-mL_j/4}$$

for ν_{ω} -a.e. $\lambda \in I$ and j large enough. Hence

$$\|\chi_{\mathbf{x}} f_I(H_{\omega}) \chi_{\mathbf{y}}\|_2 \leq \|f\|_{\infty} e^{-mL_j/4} \nu_{\omega}(I) \leq C_{\text{tr}} \|f\|_{\infty} e^{-mL_j/4}$$

where $C_{\text{tr}} = C_{\text{tr}}(N, d, q_-, \gamma)$ is given by (8-2). For $\omega \in B_j^c$ we have the bound

$$\begin{aligned} \|\chi_{\mathbf{x}} f_I(H_{\omega}) \chi_{\mathbf{y}}\|_2^2 &\leq \|f\|_{\infty}^2 \|E_{\omega}(I) \chi_{\mathbf{y}}\|_2^2 \\ &\leq \|f\|_{\infty}^2 \|\chi_{\mathbf{y}} T\|^2 \|E_{\omega}(I) T^{-1}\|_2^2 \leq C_2 \|f\|_{\infty}^2 L_j^{\gamma} \nu_{\omega}(I) \end{aligned}$$

for $C_2 = C_2(\gamma, N, d, r_0)$. Again $\nu_{\omega}(I) \leq C_{\text{tr}}$, so we finally get

$$\mathbb{E} \left(\sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}} f_I(H_{\omega}) \chi_{\mathbf{y}}\|_2^2 \right) \leq C_{\text{tr}}^2 e^{-mL_j/2} \mathbb{P}(B_j) + C_2 C_{\text{tr}} L_j^{\gamma} \mathbb{P}(B_j^c).$$

Using Theorem 2.7.15 to estimate $\mathbb{P}(B_j^c)$, we obtain the assertion. \square

We are finally ready to prove our main result. Note that if R is a Hilbert-Schmidt operator on $L^2(\Gamma)$ and if $A, B \subset \mathbb{R}^{Nd}$ are disjoint, then

$$(10-1) \quad \|\chi_{A \cup B} R\|_2^2 = \text{tr}[R^* \chi_{A \cup B} R] = \text{tr}[R^* \chi_A R] + \text{tr}[R^* \chi_B R] = \|\chi_A R\|_2^2 + \|\chi_B R\|_2^2,$$

$$(10-2) \quad \|R \chi_{A \cup B}\|_2^2 = \|\chi_{A \cup B} R^*\|_2^2 = \|\chi_A R^*\|_2^2 + \|\chi_B R^*\|_2^2 = \|R \chi_A\|_2^2 + \|R \chi_B\|_2^2.$$

Proof of Theorem 2.1.4. Let $k \geq k_0$ be sufficiently large so that Lemma 2.10.2 holds for $j \geq k$. Given $s > 0$ and a bounded Borel function f put $f_I := f\chi_I$ and $H_\omega := H^{(N)}(\omega)$. Since $F_k \cup (\cup_{j \geq k} M_j) = \mathbb{R}^{Nd}$, we have by (10-1)

$$\begin{aligned} \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^{s/2} f_I(H_\omega) \chi_K\|_2^2 \right\} &\leq \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_{F_k} X^{s/2} f_I(H_\omega) \chi_K\|_2^2 \right\} \\ &\quad + \mathbb{E} \left\{ \sum_{j \geq k} \sup_{\|f\| \leq 1} \|\chi_{M_j} X^{s/2} f_I(H_\omega) \chi_K\|_2^2 \right\}. \end{aligned}$$

Let us estimate the first term. We have

$$\begin{aligned} \|\chi_{F_k} X^{s/2} f_I(H_\omega) \chi_K\|_2^2 &\leq c_1 \|f\|_\infty^2 L_k^s \|E_\omega(I) \chi_K\|_2^2 \\ &\leq c_1 \|f\|_\infty^2 L_k^s \|\chi_K T\|^2 \|E_\omega(I) T^{-1}\|_2^2. \end{aligned}$$

Since $\|E_\omega(I) T^{-1}\|_2^2 = \nu_\omega(I) \leq C_{\text{tr}}$ by (8-2), we get

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_{F_k} X^{s/2} f_I(H_\omega) \chi_K\|_2^2 \right\} \leq c_2 L_k^{s+\gamma} < \infty.$$

For the second term, note that $\chi_{M_j} X^{s/2} g = X^{s/2} \chi_{M_j} g$ for $g \in D(X^{s/2})$, so using (10-1) and (10-2),

$$\begin{aligned} &\mathbb{E} \left\{ \sum_{j \geq k} \sup_{\|f\| \leq 1} \|\chi_{M_j} X^{s/2} f_I(H_\omega) \chi_K\|_2^2 \right\} \\ &\leq \sum_{j \geq k} c_3 L_{j+1}^s \sum_{\mathbf{x} \in \tilde{M}_j, \mathbf{y} \in \mathbf{B}_{r_{N, L_{k_0}}}^{(N)}(\mathbf{0})} \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}} f_I(H_\omega) \chi_{\mathbf{y}}\|_2^2 \right\}. \end{aligned}$$

Estimating $|\tilde{M}_j| \leq c L_{j+1}^{Nd}$, $|\mathbf{B}_{r_{N, L_{k_0}}}^{(N)}| \leq c' L_{k_0}^{Nd}$ and using Lemma 2.10.2, the series converges. This completes the proof of the theorem. \square

2.11 Appendix

In this section we prove various results used in the text. We shall repeat the statements of the theorems for the reader's convenience.

Theorem 2.11.1. *Given $\omega \in \Omega$, $\mathfrak{h}_\omega^{(n)}$ is closed, densely defined and bounded from below. The unique self-adjoint operator $H^{(n)}(\omega)$ associated with $\mathfrak{h}_\omega^{(n)}$ is given by*

$$H^{(n)}(\omega) : (f_\kappa) \mapsto (-\Delta f_\kappa + V_\kappa^\omega f_\kappa), \quad \text{for } (f_\kappa) \in D(H^{(n)}(\omega)).$$

Proof. As a direct sum of Hilbert spaces, the space

$$\left(\bigoplus_{\kappa \in \mathcal{K}} W^{1,2}((0,1)^n), \|\cdot\|_{W^{1,2}(\Gamma)} \right), \quad \|f\|_{W^{1,2}(\Gamma)}^2 := \sum_{\kappa \in \mathcal{K}} \|f_\kappa\|_{W^{1,2}((0,1)^n)}^2$$

is a Hilbert space. By the trace theorem for $W^{1,2}((0,1)^n)$ (see e.g. [83, Theorem 1.1.2]), $(W^{1,2}(\Gamma), \|\cdot\|_{W^{1,2}(\Gamma)})$ is a closed subspace of $(\bigoplus_{\kappa \in \mathcal{K}} W^{1,2}((0,1)^n), \|\cdot\|_{W^{1,2}(\Gamma)})$, hence a Hilbert space. Finally, $\mathfrak{h}_\omega^{(n)} \geq nq_-$. If for $f \in D(\mathfrak{h}_\omega^{(n)})$, we define $\|f\|_{\mathfrak{h}_\omega^{(n)}}^2 := \mathfrak{h}_\omega^{(n)}[f] + (-nq_- + 1)\|f\|_{L^2(\Gamma)}^2$, then $\|\cdot\|_{\mathfrak{h}_\omega^{(n)}}$ is equivalent to $\|\cdot\|_{W^{1,2}(\Gamma)}$. Hence, $\mathfrak{h}_\omega^{(n)}$ is closed.

Let $C_c^\infty(\Gamma) := (\oplus_\kappa C_c^\infty(0, 1)^n) \cap C_c(\Gamma)$. Since $D(\mathfrak{h}_\omega^{(n)}) \supset C_c^\infty(\Gamma)$, $\mathfrak{h}_\omega^{(n)}$ is densely defined. By [104, Theorem 4.1.5], the associated operator $H^{(n)}(\omega)$ is given by

$$D(H^{(n)}(\omega)) = \{f \in D(\mathfrak{h}_\omega^{(n)}) \mid \exists g \in \mathcal{H} : \forall v \in D(\mathfrak{h}_\omega^{(n)}), \mathfrak{h}_\omega^{(n)}[f, v] = \langle g, v \rangle\},$$

$$H^{(n)}(\omega)f := g.$$

So let $f \in D(H^{(n)}(\omega))$. Then in particular, given $v \in C_c^\infty(\Gamma)$, we have

$$\langle \nabla f, \nabla v \rangle = \langle g - V^\omega f, v \rangle.$$

Hence $-\Delta f = g - V^\omega f$ in the sense of distributions. As $g, V^\omega f \in L^2(\Gamma)$ and as $C_c^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, the equality holds in the L^2 sense¹⁹. Hence $H^{(n)}(\omega)f = g = -\Delta f + V^\omega f$. \square

Theorem 2.11.2. *There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$:*

$$[nq_-, nq_+] \subset \sigma(H^{(n)}(\omega)) \subseteq [nq_-, +\infty).$$

In particular, $\inf \sigma(H^{(n)}(\omega)) = nq_-$ almost surely.

Proof. Since $U_\kappa^{(n)} \geq 0$ and $W_\kappa^\omega \geq nq_-$, then $H^{(n)}(\omega) \geq nq_-$ and $\sigma(H^{(n)}(\omega)) \subseteq [nq_-, +\infty)$ for all $\omega \in \Omega$. To prove that $\sigma(H^{(n)}(\omega)) \supset [nq_-, nq_+]$ almost surely, let $E \in [nq_-, nq_+]$, put $I_m^E = [\frac{E}{n} - \frac{1}{nm}, \frac{E}{n} + \frac{1}{nm}]$ for $m \in \mathbb{N}^*$ and let

$$B_m := \{(x_1, \dots, x_n) \in (\mathbb{Z}^d)^n : \min_{i \neq j} |x_i - x_j| \geq 2m + r_0\},$$

where r_0 is the interaction range. Given $k \in \mathbb{N}^*$, consider the event

$$\Omega_m^E(k) := \{\omega \in \Omega : \omega_e \in I_m^E \quad \forall e \in \mathcal{E}(\Gamma^{(1)} \cap \Pi \Lambda_m^{(n)}(\mathbf{x}_{k,m}))\},$$

where $\mathbf{x}_{k,m} := 2^{kn}(2m + r_0)(1, 2, \dots, n)$. Then $\mathbf{x}_{k,m} \in B_m$ for each k , $\mathbb{P}(\Omega_m^E(k)) = \mu(I_m^E) \#\{\mathcal{E}(\Gamma^{(1)} \cap \Pi \Lambda_m^{(n)})\}$ is the same for all k and it is strictly positive since $\frac{E}{n} \in [q_-, q_+] = \text{supp } \mu$. Hence, $\sum_{k \geq 1} \mathbb{P}(\Omega_m^E(k)) = \infty$. Moreover, $\Pi \Lambda_m^{(n)}(\mathbf{x}_{k,m}) \cap \Pi \Lambda_m^{(n)}(\mathbf{x}_{k',m}) = \emptyset$ for $k \neq k'$, so the events $\{\Omega_m^E(k)\}_{k \in \mathbb{N}^*}$ are independent. Thus, by Borel-Cantelli lemma II, if $\Omega_m^E := \cap_{k' \geq 1} \cup_{k \geq k'} \Omega_m^E(k)$, then $\mathbb{P}(\Omega_m^E) = 1$. Let $\Omega^E := \cap_{m \in \mathbb{N}^*} \Omega_m^E$, then $\mathbb{P}(\Omega^E) = 1$.

Fix $\omega \in \Omega^E$ and let $m \in \mathbb{N}^*$. Then $\omega \in \Omega_m^E$, so we may find $k \in \mathbb{N}^*$ such that $\omega \in \Omega_m^E(k)$. We finally construct a Weyl sequence: choose $g_m \in D(H^{(n)})$ such that $0 \leq g_m \leq 1$, $g_m = 1$ on $\Gamma \cap \Lambda_{m-1}^{(n)}(\mathbf{x}_{k,m})$, $g_m = 0$ on $\Gamma \cap \Lambda_m^{(n)}(\mathbf{x}_{k,m})^c$ and $\|\Delta g_m\|_\infty \leq C$, for some $C = C(nd)$. Let $f_m := c_m g_m$, where $c_m := \|g_m\|^{-1}$. Then $\|f_m\| = 1$, $\|\Delta f_m\|_\infty \leq C c_m$ and

$$\|(H^{(n)}(\omega) - E)f_m\| = \|\chi_{\Lambda_m^{(n)}(\mathbf{x}_{k,m})}(-\Delta + U^{(n)} + W^\omega - E)f_m\|.$$

But $\mathbf{x}_{k,m} \in B_m$, so $U^{(n)} = 0$ on $\Lambda_m^{(n)}(\mathbf{x}_{k,m})$. Also $\omega \in \Omega_m^E(k)$, so $|W_\kappa^\omega - E| \leq \frac{1}{m}$ for all $\kappa \in \mathcal{K}(\Gamma \cap \Lambda_m^{(n)}(\mathbf{x}_{k,m}))$. Thus

$$\|(H^{(n)}(\omega) - E)f_m\| = \|\chi_{\Lambda_m^{(n)}(\mathbf{x}_{k,m})}(-\Delta + W^\omega - E)f_m\| \leq \|\Delta f_m\| + \frac{1}{m} \|f_m\| \rightarrow 0.$$

19. It is this part that distinguishes the difficulty of the domain for multi-particles: for $n = 1$, the fact that $-f'' \in L^2$ means that $f \in W^{2,2}$, but for $n > 1$, the fact that $-\Delta f \in L^2$ does not imply that $f \in W^{2,2}$. For f to be in $W^{2,2}$, we should have $\partial_i \partial_j f \in L^2$ for all $i, j = 1, \dots, n$.

Indeed, note that

$$1 = \|f_m\|^2 \geq \|\chi_{\Lambda_{m-1}^{(n)}} f_m\|^2 = c_m^2 (\#\{\mathcal{K}(\Gamma \cap \Lambda_{m-1}^{(n)})\}),$$

hence $c_m^2 \leq (\#\{\mathcal{K}(\Gamma \cap \Lambda_{m-1}^{(n)})\})^{-1}$ and using (NB.n),

$$\begin{aligned} \|\Delta f_m\|^2 &= \|\chi_{\Lambda_m^{(n)} \setminus \Lambda_{m-1}^{(n)}} \Delta f_m\|^2 \leq \|\Delta f_m\|_\infty^2 \left(\#\{\mathcal{K}(\Gamma \cap \Lambda_m^{(n)})\} - \#\{\mathcal{K}(\Gamma \cap \Lambda_{m-1}^{(n)})\} \right) \\ &\leq C^2 \frac{(2m)^n (2m-1)^{nd-n} - (2m-2)^n (2m-3)^{nd-n}}{(2m-2)^n (2m-3)^{nd-n}} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Thus, for any $\omega \in \Omega^E$ we have $E \in \sigma(H^{(n)}(\omega))$. Let $\Omega_0 := \bigcap_{E \in [nq_-, nq_+] \cap \mathbb{Q}} \Omega^E$. Then $\mathbb{P}(\Omega_0) = 1$ and for any $\omega \in \Omega_0$ we have $\sigma(H^{(n)}(\omega)) \supset [nq_-, nq_+] \cap \mathbb{Q}$. Since the spectrum is closed, the proof is complete. \square

Lemma 2.11.3. *The following estimates hold:*

$$(NB.1) \quad \#\{\mathcal{E}(\Gamma^{(1)} \cap \Lambda_L^{(1)})\} = d(2L)(2L-1)^{d-1} \leq d \cdot |\Lambda_L^{(1)}|,$$

$$(NB.n) \quad \#\{\mathcal{K}(\Gamma^{(n)} \cap \Lambda_{\mathbb{L}}^{(n)})\} = \prod_{j=1}^n \left(d(2L_j)(2L_j-1)^{d-1} \right) \leq d^n \cdot |\Lambda_{\mathbb{L}}^{(n)}|.$$

Proof. For $d = 1$, it is obvious that $\#\{\mathcal{E}(\Gamma^{(1)} \cap \Lambda_L^{(1)})\} = 2L$ since in this case $\Lambda_L^{(1)}$ is just an open segment of length $2L$ and each edge has length 1.

So let us suppose the estimate is true for $d = m$ and calculate the number of edges in a 1-cube in \mathbb{R}^{m+1} , with coordinate axes x_1, \dots, x_{m+1} . Since this number is invariant by translations, we may suppose the cube is $\Lambda_L^{(1)}(0)$. By hypothesis, the hyperplane $\{x_{m+1} = L-1\} \cap \Lambda_L^{(1)}(0)$ contains $m(2L)(2L-1)^{m-1}$ edges. The same holds for the hyperplane $\{x_{m+1} = L-2\} \cap \Lambda_L^{(1)}(0)$ and so on, by calculating the number of edges in the hyperplanes $x_{m+1} = L-1, L-2, \dots, -L+1$, we obtain $(2L-1)(m(2L)(2L-1)^{m-1}) = m(2L)(2L-1)^m$ edges. It remains to calculate the number of ‘‘vertical’’ edges, i.e. edges that lie in the translates of the axis x_{m+1} in $\Lambda_L^{(1)}(0)$. There are $(2L-1)^m$ such translates (since each x_j , $j = 1, \dots, m$ varies from $L-1$ to $-L+1$), and each axis contains $2L$ edges by the case $d = 1$. Hence we get $(2L-1)^m(2L)$ vertical edges. The total number of edges is thus $m(2L)(2L-1)^m + (2L-1)^m(2L) = (m+1)(2L)(2L-1)^m$. Thus (NB.1) holds $\forall d \geq 1$.

Since $\Gamma^{(n)} = \Gamma^{(1)} \times \dots \times \Gamma^{(1)}$, (NB.n) follows directly from (NB.1). \square

Lemma 2.11.4. $H_{\Lambda_{\mathbb{L}}}^{(n)}(\omega)$ has a compact resolvent. Its discrete set of eigenvalues denoted by $E_j(H_{\Lambda_{\mathbb{L}}}^{(n)}(\omega))$ counting multiplicity satisfies the following Weyl law:

$$(WEYL.n) \quad \forall S \in \mathbb{R} \exists C = C(n, d, S - nq_-) : j > C|\Lambda_{\mathbb{L}}^{(n)}| \implies E_j(H_{\Lambda_{\mathbb{L}}}^{(n)}(\omega)) > S.$$

Moreover, C is independent of ω , and if $S > S^*(n, q_-)$, then $C \leq \left\lfloor \frac{d^n (S - nq_-)^{n/2}}{(4\pi)^{n/2} \Gamma(n/2)} \right\rfloor + 1$.

Proof. Put $\Lambda = \Lambda_{\mathbb{L}}^{(n)}$ and define the Neumann-decoupled Laplacian $-\Delta_{\Lambda}^{N, \text{dec}}$ via the form $\mathfrak{h}_{\Lambda}^{\text{dec}}[f, g] = \sum_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda)} \langle \nabla f_{\kappa}, \nabla g_{\kappa} \rangle$, with $D(\mathfrak{h}_{\Lambda}^{\text{dec}}) = \bigoplus_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda)} W^{1,2}((0, 1)^n)$. Then $D(\mathfrak{h}_{\omega, \Lambda}^{(n)}) \subset D(\mathfrak{h}_{\Lambda}^{\text{dec}})$ and $\mathfrak{h}_{\omega, \Lambda}^{(n)}[f] \geq \mathfrak{h}_{\Lambda}^{\text{dec}}[f] + nq_- \|f\|^2$, hence

$$H_{\Lambda}^{(n)}(\omega) \geq -\Delta_{\Lambda}^{N, \text{dec}} + nq_-. \quad (\star)$$

Since $\Delta_{\Lambda}^{\mathbb{N}, \text{dec}} = \bigoplus_{\kappa \in \mathcal{K}(\Gamma \cap \Lambda^{(n)})} \Delta_{(0,1)^n}^{\mathbb{N}}$, the eigenvalues $E_j(-\Delta_{\Lambda}^{\mathbb{N}, \text{dec}})$ are just the eigenvalues $E_k(-\Delta_{(0,1)^n}^{\mathbb{N}})$ with multiplicity $\#\{\mathcal{K}(\Gamma \cap \Lambda^{(n)})\} \leq d^n |\Lambda^{(n)}|$. In particular, $E_j(-\Delta_{\Lambda}^{\mathbb{N}, \text{dec}}) \rightarrow \infty$ as $j \rightarrow \infty$, so by (\star) and [92, Theorem XIII.64], $H_{\Lambda}^{(n)}$ has a compact resolvent and thus a discrete spectrum. Now by Weyl law for $E_k(-\Delta_{(0,1)^n}^{\mathbb{N}})$ ([92, Section XIII.15]), there exists C_1 such that $k > C_1 \implies E_k(-\Delta_{(0,1)^n}^{\mathbb{N}}) > S - nq_-$, and if S is large, $C_1 \approx \frac{(S-nq_-)^{n/2}}{(4\pi)^{n/2}\Gamma(n/2)}$. Thus $j > C_1 d^n |\Lambda^{(n)}| \implies E_j(-\Delta_{\Lambda}^{\mathbb{N}, \text{dec}}) > S - nq_-$. But by (\star) , $E_j(H_{\Lambda}^{(n)}(\omega)) \geq E_j(-\Delta_{\Lambda}^{\mathbb{N}, \text{dec}}) + nq_-$. Thus $j > d^n C_1 |\Lambda^{(n)}| \implies E_j(H_{\Lambda}^{(n)}(\omega)) > S$. We get (WEYL. n) with $C = d^n C_1$ and $C \leq \left\lfloor \frac{d^n (S-nq_-)^{n/2}}{(4\pi)^{n/2}\Gamma(n/2)} \right\rfloor + 1$ if $S > S^*(n, q_-)$. \square

Before proceeding further, we need the following notion.

Definition 2.11.5. Given $\mathbf{y} \in \mathbb{Z}^{nd}$ and $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$, we say that $\mathcal{P} = \{y_j : j \in \mathcal{J}\}$ is *R-connected* if $\mathcal{Z} = \bigcup_{j \in \mathcal{J}} \Lambda_R^{(1)}(y_j) \subset \mathbb{R}^d$ is connected. In this case, if $\#\mathcal{J} \geq 2$, then²⁰

$$\forall i, j \in \mathcal{J} : |y_i - y_j| < (\#\mathcal{J} - 1)(2R) \leq 2(n-1)R.$$

Lemma 2.11.6. *A partially interactive cube is decomposable.*

Proof. Suppose $\Lambda_L^{(n)}(\mathbf{u})$ is not decomposable. Then $\exists i \neq 1$ such that $|u_1 - u_i| < 2L + r_0$ (otherwise $\mathcal{J} = \{1\}$ would give a possible partition). Let $\mathcal{J}_2 = \{1, i\}$. Since \mathcal{J}_2 is not a possible partition, $\exists i_2 \notin \mathcal{J}_2$ such that $|u_1 - u_{i_2}| < 2L + r_0$ or $|u_i - u_{i_2}| < 2L + r_0$. Taking $\mathcal{J}_3 = \{1, i, i_2\}$, the set $\{u_k : k \in \mathcal{J}_3\}$ is thus $(L + r_0/2)$ -connected. As \mathcal{J}_3 is not a possible partition, we may repeat the procedure and finally obtain $\mathcal{J}_n = \{1, \dots, n\}$ and $\{u_k : k \in \mathcal{J}_n\}$ is $(L + r_0/2)$ -connected. Consequently,

$$\forall 1 \leq j \leq n : |u_j - u_1| < (n-1)(2L + r_0)$$

Hence

$$\text{dist}(\mathbf{u}, \mathbb{D}) \leq |\mathbf{u} - (u_1, \dots, u_1)| = \max_{1 \leq j \leq n} |u_j - u_1| < (n-1)(2L + r_0)$$

The lemma now results by contraposition. \square

Lemma 2.11.7. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{nd}$, $L \geq 1$ and take $r_{n,L}$ as in Definition 2.2.7. Then*

- 1) *If $\mathbf{y} \notin \bigcup_{j=1}^{K(n)} \Lambda_{2nL}^{(n)}(\mathbf{x}^{(j)})$, then $\Lambda_L^{(n)}(\mathbf{y})$ and $\Lambda_L^{(n)}(\mathbf{x})$ are pre-separable.*
- 2) *If $\mathbf{y} \notin \bigcup_{j=1}^{K(n)} \Lambda_{r_{n,L}}^{(n)}(\mathbf{x}^{(j)})$, then $\Lambda_L^{(n)}(\mathbf{y})$ and $\Lambda_L^{(n)}(\mathbf{x})$ are separable.*
- 3) *If $\mathbf{y} \notin \Lambda_{2r_{n,L}}^{(n)}(\mathbf{0})$, then $\Lambda_L^{(n)}(\mathbf{y})$ is separable from any $\Lambda_L^{(n)}(\mathbf{x})$ satisfying $\mathbf{x} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})$.*

Proof. 1) Decompose $\{y_1, \dots, y_n\}$ into maximal L -connected subsets

$$\mathcal{P}_k = \{y_j : j \in \mathcal{J}_k\}, \quad k = 1, \dots, m,$$

and let $\mathcal{Z}_k = \bigcup_{j \in \mathcal{J}_k} \Lambda_L^{(1)}(y_j)$. Then $(\mathcal{Z}_k)_k$ forms a partition of $\Pi \Lambda_L^{(n)}(\mathbf{y})$. Suppose now that $\Lambda_L^{(n)}(\mathbf{x})$ and $\Lambda_L^{(n)}(\mathbf{y})$ are not pre-separable. Then

$$\forall \emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\} : \Pi_{\mathcal{J}} \Lambda_L^{(n)}(\mathbf{y}) \cap (\Pi_{\mathcal{J}^c} \Lambda_L^{(n)}(\mathbf{y}) \cup \Pi \Lambda_L^{(n)}(\mathbf{x})) \neq \emptyset$$

20. If $\#\mathcal{J} = 2$, i.e. $\mathcal{J} = \{i, j\}$, it is clear that $|y_i - y_j| < 2R$. Suppose the property holds for any \mathcal{J} with $\#\mathcal{J} = k$. Given \mathcal{J} with $\#\mathcal{J} = k+1$, let $i, j \in \mathcal{J}$. We may find $i_1 \in \mathcal{J}$, $i_1 \neq i$ such that $|y_i - y_{i_1}| < 2R$. But then $\#(\mathcal{J} \setminus \{i\}) = k$, so $|y_{i_1} - y_j| < (k-1)(2R)$ by the induction hypothesis. Thus, $|y_i - y_j| < k(2R)$.

Since $(\mathcal{Z}_k)_k$ forms a partition of $\Pi\Lambda_L^{(n)}(\mathbf{y})$, we have in particular

$$\forall 1 \leq k \leq m : \mathcal{Z}_k \cap \Pi\Lambda_L^{(n)}(\mathbf{x}) \neq \emptyset,$$

hence

$$\forall 1 \leq k \leq m, \exists y_j \in \mathcal{P}_k, \exists x_i : |y_j - x_i| < 2L.$$

But \mathcal{P}_k are L -connected, hence $\forall i, j \in \mathcal{J}_k : |y_i - y_j| < 2(n-1)L$. Thus,

$$\forall y_j \exists x_i : |y_j - x_i| < 2nL,$$

so that $\mathbf{y} \in \Lambda_{2nL}^{(n)}(\mathbf{x}^{(k)})$ for some k . The claim follows by contraposition.

2) This follows from 1) by noting that $r_{n,L} \geq 2nL$ and that $|\mathbf{y} - \mathbf{x}^{(j)}| \geq r_{n,L}$ for all j implies $|\mathbf{y} - \mathbf{x}| \geq r_{n,L}$ (since \mathbf{x} is one of the $\mathbf{x}^{(j)}$).

3) Let

$$F = \bigcup_{\mathbf{x} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})} \bigcup_{j=1}^{K(n)} \Lambda_{r_{n,L}}^{(n)}(\mathbf{x}^{(j)}).$$

Then by 2), if $\mathbf{y} \notin F$, then $\Lambda_L^{(n)}(\mathbf{y})$ is separable from any $\Lambda_L^{(n)}(\mathbf{x})$ with $\mathbf{x} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})$. Thus it suffices to show that $\Lambda_{2r_{n,L}}^{(n)}(\mathbf{0}) = F$. For this note that if $\mathbf{x} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})$, then $|x_k| < r_{n,L}$ for all k , so by definition of $\mathbf{x}^{(j)}$, $|\mathbf{x}^{(j)}| < r_{n,L}$ for all j and so $\mathbf{x}^{(j)} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})$ for all j . Thus $F = \bigcup_{\mathbf{x} \in \Lambda_{r_{n,L}}^{(n)}(\mathbf{0})} \Lambda_{r_{n,L}}^{(n)}(\mathbf{x}) = \Lambda_{2r_{n,L}}^{(n)}(\mathbf{0})$. □

Lemma 2.11.8. *Separable FI cubes are completely separated.*

Proof. Since $\Lambda_L^{(n)}(\mathbf{u})$ and $\Lambda_L^{(n)}(\mathbf{v})$ are FI, there exists $\mathbf{x}, \mathbf{y} \in \mathbb{D}$ such that $|\mathbf{u} - \mathbf{x}| < (n-1)(2L+r_0)$ and $|\mathbf{v} - \mathbf{y}| < (n-1)(2L+r_0)$. Hence for all $j, k = 1, \dots, n$:

$$(\star) \quad \Pi_j \Lambda_L^{(n)}(\mathbf{u}) \subseteq \Pi_j \Lambda_{(n-1)(2L+r_0)+L}^{(n)}(\mathbf{x}) \quad \text{and} \quad \Pi_k \Lambda_L^{(n)}(\mathbf{v}) \subseteq \Pi_k \Lambda_{(n-1)(2L+r_0)+L}^{(n)}(\mathbf{y}).$$

Now

$$|\mathbf{u} - \mathbf{v}| \leq |\mathbf{u} - \mathbf{x}| + |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{v}| < 2(n-1)(2L+r_0) + |\mathbf{x} - \mathbf{y}|.$$

Moreover, $\Lambda_L^{(n)}(\mathbf{u})$ and $\Lambda_L^{(n)}(\mathbf{v})$ are separable, so by definition $|\mathbf{u} - \mathbf{v}| \geq r_{n,L} = 4(n-1)(2L+r_0) + 2L$. We thus get

$$|\mathbf{x} - \mathbf{y}| > |\mathbf{u} - \mathbf{v}| - 2(n-1)(2L+r_0) \geq 2(n-1)(2L+r_0) + 2L.$$

Since $\mathbf{x}, \mathbf{y} \in \mathbb{D}$, this implies

$$(\star\star) \quad \Pi_j \Lambda_{(n-1)(2L+r_0)+L}^{(n)}(\mathbf{x}) \cap \Pi_k \Lambda_{(n-1)(2L+r_0)+L}^{(n)}(\mathbf{y}) = \emptyset$$

for all $j, k = 1, \dots, n$. By (\star) and $(\star\star)$, we see that $\Pi_j \Lambda_L^{(n)}(\mathbf{u}) \cap \Pi_k \Lambda_L^{(n)}(\mathbf{v}) = \emptyset$ for all j, k . Hence $\Pi\Lambda_L^{(n)}(\mathbf{u}) \cap \Pi\Lambda_L^{(n)}(\mathbf{v}) = \emptyset$, as asserted. □

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Chapter 3

Additional properties and alternative proofs

This chapter is a complement to Chapter 2 and discusses more properties of the operator $H^{(n)}(\omega)$, as well as some alternative proofs.

In Section 3.1, we show that if the interaction potential $U^{(n)}$ is invariant under diagonal translations, then $H^{(n)}(\omega)$ is ergodic and has an almost sure spectrum. However, this does not help us to locate the spectral edge of $H^{(n)}(\omega)$, so it is not a substitute to Theorem 2.1.2.

In Section 3.2, we give a Combes-Thomas estimate which is valid not only below the spectral edge, but also for arbitrary spectral gaps. The price to pay is that the proof becomes much more involved.

In Section 3.3 we show that the generalized eigenfunctions constructed in Section 2.8 are regular, i.e. locally in $W^{1,2}(\Gamma)$. This can simplify the proof of Lemma 2.9.1, but it takes more effort to prove regularity than to avoid it. In any case, regularity can be of independent interest.

In Sections 3.4 and 3.5, we establish localization without using generalized eigenfunctions. This makes the proof of localization conceptually more elementary. There are two facts to keep in mind however. First, this approach does not allow us to establish dynamical localization in the Hilbert-Schmidt norm, but only in the operator norm. And it seems to us that this *cannot* be improved. The second fact is that this approach needs essentially the same trace estimates needed to derive generalized eigenfunction expansions. So in conclusion, we believe that it is interesting in principle that one can derive localization without relying on generalized eigenfunctions, but for the time being, it seems better to just work in the usual way.

We conclude the chapter with Section 3.6, in which we explain to what extent our localization results can be extended to log-Hölder continuous distributions.

3.1 Ergodicity and almost sure spectrum

In this section we prove that if the interaction $U^{(n)}$ is invariant under diagonal translations, then the operator $H^{(n)}(\omega)$ is ergodic with respect to a certain family of translations. This will imply that $H^{(n)}(\omega)$ has an almost sure spectrum.

We shall follow the definitions of [104, Section 1.2]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $(T_i)_{i \in I}$ is called *measure preserving* if $\mathbb{P}(T_i^{-1}(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. It is called *ergodic* if, moreover, every set that is invariant under (T_i) is trivial; i.e. if any $A \in \mathcal{F}$ satisfying $T_i^{-1}(A) = A$ for all $i \in I$ must satisfy $\mathbb{P}(A) \in \{0, 1\}$.

In our setting, $\Omega := [q_-, q_+]^{\mathcal{E}}$. Recall that every edge e takes the form $m \rightarrow m + h_j$ for some $m \in \mathbb{Z}^d$, $j \in \{1, \dots, d\}$, where $(h_j)_{j=1}^d$ is the standard basis of \mathbb{Z}^d , and that we denote such e by $e = (m, j)$. Now consider the family $(T_i)_{i \in \mathbb{Z}^d}$, $T_i : \Omega \rightarrow \Omega$ defined by

$$T_i : (\omega_{(m,j)}) \mapsto (\omega_{(m-i,j)}).$$

Then we have the following lemma (which is quite classic, not yet dependent of the multi-particle context).

Lemma 3.1.1. *The family $T_i : \Omega \rightarrow \Omega$, $i \in \mathbb{Z}^d$ is ergodic.*

Proof. Let $A = \{\omega : \omega_{(m_1, j_1)} \in A_1, \dots, \omega_{(m_k, j_k)} \in A_k\}$ be a cylinder set. Then

$$T_i^{-1}A = \{\omega : T_i(\omega) \in A\} = \{\omega : \omega_{(m_1-i, j_1)} \in A_1, \dots, \omega_{(m_k-i, j_k)} \in A_k\}.$$

Hence $\mathbb{P}(T_i^{-1}A) = \mu(A_1) \dots \mu(A_k) = \mathbb{P}(A)$. Thus, $\mathcal{L} = \{A : \mathbb{P}(A) = \mathbb{P}(T_i^{-1}A) \forall i \in I\}$ is a λ -system that contains all cylinder sets. Since cylinder sets are a π -system, we get $\mathcal{L} = \mathcal{F}$ and the (T_i) are measure preserving (for a proof of the π - λ theorem, see [12]).

Now let \mathcal{R} be the set of cylinder sets and take

$$\mathcal{C} = \{A \mid \forall B \in \mathcal{R} : \mathbb{P}(T_i^{-1}A \cap B) \rightarrow \mathbb{P}(A)\mathbb{P}(B) \text{ as } \|i\|_{\infty} \rightarrow \infty\}.$$

Then \mathcal{C} is a λ -system¹ which clearly contains the cylinder sets, so $\mathcal{C} = \mathcal{F}$ by the π - λ theorem. Hence

$$\mathcal{Y} = \{B \mid \forall A \in \mathcal{F} : \mathbb{P}(T_i^{-1}A \cap B) \rightarrow \mathbb{P}(A)\mathbb{P}(B) \text{ as } \|i\|_{\infty} \rightarrow \infty\}$$

is a λ -system containing \mathcal{R} , so $\mathcal{Y} = \mathcal{F}$. Finally, if A is invariant under all T_i , then $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(T_i^{-1}A \cap A) \rightarrow \mathbb{P}(A)^2$, so that $\mathbb{P}(A) = 0$ or 1 . Thus, the (T_i) are ergodic². \square

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $H : \Omega \rightarrow \mathcal{S}(\mathcal{H})$ be measurable. H is called *ergodic* if there exists an ergodic family $(T_i)_{i \in I}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of unitary operators $(S_i)_{i \in I}$ on \mathcal{H} such that

$$H(T_i(\omega)) = S_i^* H(\omega) S_i \quad \text{for all } \omega \in \Omega, i \in I$$

Theorem 3.1.2. *Suppose that $U^{(n)}$ is invariant under diagonal translations, that is*

$$U^{(n)}(x_1 - i, \dots, x_n - i) = U^{(n)}(x_1, \dots, x_n)$$

for all $i \in \mathbb{Z}^d$ and $(x_1, \dots, x_n) \in \Gamma^{(n)}$. Then $H^{(n)}(\omega)$ is ergodic.

Proof. Since any e takes the form (m, j) , any $\kappa \equiv (\mathbf{m}, j_1, \dots, j_n)$ for some $\mathbf{m} \in \mathbb{Z}^{nd}$ and $j_k = 1, \dots, d$. In this case, we shall use the notation $f_{\kappa} \equiv f_{\mathbf{m}, j_1, \dots, j_n}$.

By Lemma 3.1.1, $(T_i)_{i \in \mathbb{Z}^d}$ is an ergodic family on Ω . Furthermore, for each T_i , there exists a unitary operator S_i on \mathcal{H} such that $H^{(n)}(T_i(\omega)) = S_i^* H^{(n)}(\omega) S_i$. Indeed, define $S_i : (f_{(m_1, \dots, m_n), j_1, \dots, j_n}) \mapsto (f_{(m_1+i, \dots, m_n+i), j_1, \dots, j_n})$, then

$$\begin{aligned} S_i^* H^{(n)}(\omega) S_i (f_{(m_1, \dots, m_n), j_1, \dots, j_n}) &= S_i^* H^{(n)}(\omega) (f_{(m_1+i, \dots, m_n+i), j_1, \dots, j_n}) \\ &= -\Delta f + (V_{(m_1-i, \dots, m_n-i), j_1, \dots, j_n}^{\omega} f_{(m_1, \dots, m_n), j_1, \dots, j_n}). \end{aligned}$$

1. To see that \mathcal{C} is closed under complements, first note that if $A \in \mathcal{C}$, then $\mathbb{P}(T_i^{-1}A \cap B) \rightarrow \mathbb{P}(A)\mathbb{P}(B)$ if B is a disjoint union of sets in \mathcal{R} . Now if $B \in \mathcal{R}$, then B^c is a disjoint union of sets in \mathcal{R} . Thus, $\mathbb{P}(T_i^{-1}A^c \cap B) = 1 - \mathbb{P}(T_i^{-1}A \cap B^c) = 1 - [\mathbb{P}(T_i^{-1}A) + \mathbb{P}(B^c) - \mathbb{P}(T_i^{-1}A \cap B^c)] = 1 - [\mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(T_i^{-1}A \cap B^c)] \rightarrow 1 - [\mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A)\mathbb{P}(B^c)] = \mathbb{P}(A^c)\mathbb{P}(B)$.

2. Actually we showed more: the (T_i) are *mixing*.

Now $V^\omega = U^{(n)} + W^\omega$, and by definition (cf. Section 2.1.2), if $\mathbf{x} \in \Gamma^{(n)}$ takes the form (x_1, \dots, x_n) with $x_k = m_k + t^k h_{j_k}$, then $U^{(n)}(\mathbf{x}) = U_{(m_1, \dots, m_n), j_1, \dots, j_n}^{(n)}(t^1, \dots, t^n)$ and if $\mathbf{i} = (i, \dots, i)$, then $U^{(n)}(\mathbf{x} - \mathbf{i}) = U_{(m_1 - i, \dots, m_n - i), j_1, \dots, j_n}^{(n)}(t^1, \dots, t^n)$. So using the hypothesis, we get $U_{(m_1 - i, \dots, m_n - i), j_1, \dots, j_n}^{(n)} = U_{(m_1, \dots, m_n), j_1, \dots, j_n}^{(n)}$. Next,

$$W_{(m_1 - i, \dots, m_n - i), j_1, \dots, j_n}^\omega = \omega_{(m_1 - i, j_1)} + \dots + \omega_{(m_n - i, j_n)} = W_{(m_1, \dots, m_n), j_1, \dots, j_n}^{T_i(\omega)}.$$

Hence,

$$\begin{aligned} S_i^* H^{(n)}(\omega) S_i(f_{(m_1, \dots, m_n), j_1, \dots, j_n}) &= -\Delta f + (V_{(m_1, \dots, m_n), j_1, \dots, j_n}^{T_i(\omega)} f_{(m_1, \dots, m_n), j_1, \dots, j_n}) \\ &= H^{(n)}(T_i(\omega))(f_{(m_1, \dots, m_n), j_1, \dots, j_n}). \end{aligned}$$

Thus, $H^{(n)}(\omega)$ is an ergodic operator. \square

Corollary 3.1.3 (Almost sure spectrum). *If the interaction $U^{(n)}$ is invariant under diagonal translations, then there exists a set $\Sigma \subset \mathbb{R}$ such that*

$$\sigma(H^{(n)}(\omega)) = \Sigma$$

almost surely. Similar statements hold for $\sigma_{ac}(H^{(n)}(\omega))$, $\sigma_{sc}(H^{(n)}(\omega))$ and $\sigma_{pp}(H^{(n)}(\omega))$.

Proof. This follows from Theorem 3.1.2 and [104, Theorem 1.2.5]. \square

3.2 Combes-Thomas estimate for general energies

The aim of this section is to prove the following estimate. We denote by $\text{dist}(\cdot, \cdot)$ the distance induced by the sup norm of \mathbb{R}^{nd} .

Theorem 3.2.1. *Let $\Lambda^{(n)}$ be an open cube or $\Lambda^{(n)} = \mathbb{R}^{nd}$ and $A, B \subseteq \Lambda^{(n)}$ be cellular sets such that $\text{dist}(A, B) := \delta \geq 1$. Let $E \in \rho(H_\Lambda^{(n)}(\omega))$, and put $v_0 = nq_-$.*

a. *If $E < v_0$, put $\eta := v_0 - E$. Then*

$$\|\chi_A (H_\Lambda^{(n)} - E)^{-1} \chi_B\| \leq \max\left(\frac{4}{\eta}, 1\right) \cdot e^{-\delta \sqrt{\frac{\eta}{2}}}.$$

b. *If there exists $r \geq v_0 - 1$ and $s > v_0$ such that $E \in (r, s) \subset \rho(H_\Lambda^{(n)}(\omega))$ with $\eta := \text{dist}(E, (r, s)^c) > 0$, then*

$$\|\chi_A (H_\Lambda^{(n)} - E)^{-1} \chi_B\| \leq \frac{1}{c\eta} \cdot e^{-c\delta \sqrt{\frac{\eta(s-r)}{2}}},$$

where $c = \frac{1}{4(s-v_0+2)}$.

Compared to the Combes-Thomas estimate we proved in Section 2.3, this one has the advantage of being valid for energies above the spectral bottom, which could be useful in the study of band-edge localization. The proof however is much more involved. Note that we formulated our bound for $H_\Lambda^{(n)}(\omega) = -\Delta + V^\omega$, but the same proof is valid for more general V^ω by replacing $v_0 = nq_-$ above by $v_\omega := \inf_{x \in \Lambda^{(n)}} V^\omega(x)$.

Our proof follows the strategy outlined in [104], which has its roots in [9]. Note however that our bounds contain explicit constants. This is important to us because, in contrast to single-particle theory, we need a Combes-Thomas estimate that remains efficient not only for energies close to the spectrum, but also for those lying far below the spectral edge. The constants that appear in [104] are not suited for the latter study.

We start with two lemmas that hold in any separable Hilbert space \mathcal{H} .

Lemma 3.2.2. *Let $\mathfrak{p} \geq 0$ be a closed densely defined symmetric form with associated operator P . Suppose \mathfrak{q} is a symmetric form relatively form bounded w.r.t. \mathfrak{p} satisfying*

$$|\mathfrak{q}[u]| \leq \gamma \cdot \mathfrak{p}[u]$$

for some $\gamma > 0$. Fix $\beta > 0$ and define $\mathfrak{r} := \mathfrak{p} + i\beta\mathfrak{q}$. Then the unique m -sectorial operator R associated with \mathfrak{r} takes the form

$$R = P^{1/2}(I + i\beta S)P^{1/2}$$

for some self-adjoint operator S with $\|S\| \leq \gamma$.

Note that the hypotheses of the lemma ensure that \mathfrak{r} is sectorial, so the existence and uniqueness of R are guaranteed by [57, Theorem VI.2.1].

Proof. The argument is contained in [57, Theorem VI.3.2]. By [57, Lemma VI.3.1], there exists a bounded operator S with $\|S\| \leq \gamma$ such that

$$\mathfrak{q}[u, v] = \langle SP^{1/2}u, P^{1/2}v \rangle$$

for all $u, v \in D(\mathfrak{p})$. Note that S is symmetric: given $f, g \in \mathcal{H}$, put $u := P^{-1/2}f$ and $v := P^{-1/2}g$, then $u, v \in D(\mathfrak{p})$ and

$$\langle Sf, g \rangle = \mathfrak{q}[u, v] = \bar{\mathfrak{q}}[v, u] = \langle f, Sg \rangle.$$

Thus S is self-adjoint. Finally, given $u, v \in D(\mathfrak{p})$ we have

$$\mathfrak{r}[u, v] = (\mathfrak{p} + i\beta\mathfrak{q})[u, v] = \langle (I + i\beta S)P^{1/2}u, P^{1/2}v \rangle.$$

But if $u \in D(R)$, we have $\mathfrak{r}[u, v] = \langle Ru, v \rangle$. Since $P^{1/2}$ is self-adjoint, it follows that $R \subset P^{1/2}(I + i\beta S)P^{1/2}$. But $P^{1/2}(I + i\beta S)P^{1/2}$ is easily seen to be accretive. Since R is m -accretive, we thus have $R = P^{1/2}(I + i\beta S)P^{1/2}$. \square

Lemma 3.2.3. *Let T be an invertible self-adjoint operator, S be bounded self-adjoint and put $d_{\pm} := \text{dist}(0, \sigma(T) \cap \mathbb{R}^{\pm})$*

- If $\sigma(T) \cap \mathbb{R}^+ = \emptyset$, then $T + i\beta S$ is invertible $\forall \beta \in \mathbb{R}$, and $\|(T + i\beta S)^{-1}\| \leq \frac{1}{d_-}$.*
- If $\sigma(T) \cap \mathbb{R}^- = \emptyset$, then $T + i\beta S$ is invertible $\forall \beta \in \mathbb{R}$, and $\|(T + i\beta S)^{-1}\| \leq \frac{1}{d_+}$.*
- In the general case, if $\|S\| \leq 1$, then $T + i\beta S$ is invertible $\forall |\beta| \leq \frac{1}{2}\sqrt{d_+d_-}$, and $\|(T + i\beta S)^{-1}\| \leq 2 \max(\frac{1}{d_+}, \frac{1}{d_-})$.*

Proof. Start like [9, Lemma 3.1]: let P_{\pm} be the spectral projections of T on $\sigma(T) \cap \mathbb{R}^{\pm}$, for $f \in D(T)$ put $P_{\pm}f := f_{\pm}$, so that $f = f_+ + f_-$ and $\|f\| = \|f_+ - f_-\|$ by orthogonality of f_+ to f_- . Hence $\|f\| \cdot \|(T + i\beta S)f\| \geq \text{Re}\langle f_+ - f_-, (T + i\beta S)(f_+ + f_-) \rangle$. Using orthogonality again with self-adjointness of T and S , this becomes

$$\|f\| \cdot \|(T + i\beta S)f\| \geq \langle f_+, Tf_+ \rangle - \langle f_-, Tf_- \rangle - 2\beta \text{Im}\langle f_+, Sf_- \rangle.$$

- In case a., $f_+ = 0$, so $f = f_-$ and the RHS becomes $-\langle f_-, Tf_- \rangle \geq d_- \|f_-\|^2$ because $f_- \in \text{Ran}(\chi_{(-\infty, -d_-)}(T))$. Thus $\|(T + i\beta S)f\| \geq d_- \|f\|$ and $T + i\beta S$ is injective. Since this is true $\forall \beta \in \mathbb{R}$, we also have $\text{Ran}(T + i\beta S)^{\perp} = \ker(T - i\beta S) = 0$, so $T + i\beta S$ is invertible with the given bound.

- b. In case b., $f_- = 0$, so $f = f_+$ and the RHS becomes $\langle f_+, Tf_+ \rangle \geq d_+ \|f_+\|^2$ because $f_+ \in \text{Ran}(\chi_{(d_+, \infty)}(T))$. Thus $\|(T + i\beta S)f\| \geq d_+ \|f\|$. Conclude as in (a.).
- c. In case c., since $\|S\| \leq 1$, the above reasoning gives us

$$\|f\| \cdot \|(T + i\beta S)f\| \geq d_+ \|f_+\|^2 + d_- \|f_-\|^2 - 2\beta \|f_+\| \|f_-\| \geq \frac{1}{2}(d_+ \|f_+\|^2 + d_- \|f_-\|^2)$$

because $|\beta| \leq \frac{1}{2}\sqrt{d_+ d_-}$ and $\frac{1}{2}(d_+ \|f_+\|^2 + d_- \|f_-\|^2 - 2\sqrt{d_+ d_-} \|f_+\| \|f_-\|) \geq 0$. Hence $\|(T + i\beta S)f\| \geq \frac{1}{2} \min(d_+, d_-) \|f\|$. Conclude as in (a.). \square

We now prove our main result.

Proof of Theorem 3.2.1. We first assume $\Lambda^{(n)}$ is a cube, put $H := H_\Lambda^{(n)}$, and for $\mathbf{x} \in \mathbb{R}^{nd}$, let $\tilde{w}(\mathbf{x}) := \text{dist}(\mathbf{x}, B)$. By the triangle inequality, $|\tilde{w}(\mathbf{x}) - \tilde{w}(\mathbf{y})| \leq \|\mathbf{x} - \mathbf{y}\|$, hence $\|\nabla \tilde{w}\|_\infty \leq 1$ and $e^{\tilde{w}(\mathbf{x})}$, $e^{-\tilde{w}(\mathbf{x})}$ are uniformly Lipschitz continuous in $\Lambda^{(n)}$. Let w be the restriction of \tilde{w} to Γ . Then by Lemma 2.3.1, $\varphi u := e^{w(\cdot)}u$ and $\psi u := e^{-w(\cdot)}u$ satisfy $\varphi u, \psi u \in D(\mathfrak{h})$ whenever $u \in D(\mathfrak{h})$. Hence for $u, v \in D(\mathfrak{h})$, we may define

$$\mathfrak{h}_\beta[u, v] := \mathfrak{h}[e^{-\beta w}u, e^{\beta w}v].$$

Now

$$\begin{aligned} \mathfrak{h}_\beta[u, v] &= \langle \nabla(e^{-\beta w}u), \nabla(e^{\beta w}v) \rangle + \langle V^\omega u, v \rangle \\ &= \langle e^{-\beta w} \nabla u - u\beta(\nabla w)e^{-\beta w}, e^{\beta w} \nabla v + v\beta(\nabla w)e^{\beta w} \rangle + \langle V^\omega u, v \rangle \\ &= \mathfrak{h}[u, v] - \beta \{ \langle u \nabla w, \nabla v \rangle - \langle \nabla u, v \nabla w \rangle \} - \beta^2 \langle m u, v \rangle, \end{aligned}$$

where $m(x) := \nabla w(x) \cdot \nabla w(x) \geq 0$. Note that $\|m\|_\infty \leq 1$ since $\|\nabla w\|_\infty \leq 1$. Taking

$$\mathfrak{k}[u, v] := i \{ \langle u \nabla w, \nabla v \rangle - \langle \nabla u, v \nabla w \rangle \},$$

we see that \mathfrak{k} is a symmetric form and that \mathfrak{h}_β takes the form

$$\mathfrak{h}_\beta[u, v] = \tilde{\mathfrak{h}}[u, v] + i\beta \mathfrak{k}[u, v], \quad \text{where } \tilde{\mathfrak{h}}[u, v] = \mathfrak{h}[u, v] - \beta^2 \langle m u, v \rangle.$$

Let us show that \mathfrak{h}_β is sectorial. First note that

$$|\mathfrak{k}[u]| \leq 2\|\nabla u\| \cdot \|u\| \leq \|\nabla u\|^2 + \|u\|^2.$$

Next, denoting by $\tilde{H} = H - \beta^2 m$ the operator associated with $\tilde{\mathfrak{h}}$,

$$\begin{aligned} \|(\tilde{H} + C)^{1/2} u\|^2 &= \langle \nabla u, \nabla u \rangle + \langle (V^\omega - \beta^2 m + C)u, u \rangle \\ (2-1) \quad &\geq \|\nabla u\|^2 + (v_0 - \beta^2 + C)\|u\|^2 \geq \|\nabla u\|^2 + \|u\|^2 \end{aligned}$$

for $C := \beta^2 - v_0 + 2$. Thus

$$|\mathfrak{k}[u]| \leq \|(\tilde{H} + C)^{1/2} u\|^2 = (\tilde{\mathfrak{h}} + C)[u]$$

and $\mathfrak{h}_\beta = \tilde{\mathfrak{h}} + i\beta \mathfrak{k}$ is indeed sectorial for all $\beta \in \mathbb{R}$, so there exists an associated m -sectorial operator H_β which is the formal realization of the expression $e^{\beta w} H e^{-\beta w}$. Moreover, taking $\mathfrak{p} = \tilde{\mathfrak{h}} + C$ and $\mathfrak{r} = \mathfrak{h}_\beta + C$, we have by Lemma 3.2.2

$$H_\beta + C = (\tilde{H} + C)^{1/2} (I + i\beta S) (\tilde{H} + C)^{1/2}$$

for some self-adjoint operator S with $\|S\| \leq 1$. So given $E \in \mathbb{R}$,

$$(2-2) \quad H_\beta - E = (\tilde{H} + C)^{1/2} (T + i\beta S) (\tilde{H} + C)^{1/2},$$

where $T := (\tilde{H} - E)(\tilde{H} + C)^{-1} = \phi(\tilde{H})$, for $\phi(t) := \frac{t-E}{t+C}$. Note that $\sigma(\tilde{H}) \subseteq [v_0 - \beta^2, +\infty)$. To apply Lemma 3.2.3, we need T to be invertible, i.e. we need $E \in \rho(\tilde{H})$.

- a. In case (a.), $E < v_0$. If we want $E \in \rho(\tilde{H})$, it suffices that $E < v_0 - \beta^2$, i.e. $\beta^2 < (v_0 - E) = \eta$. We thus take $\beta := \sqrt{\eta/2}$. Now by the spectral mapping theorem,

$$\sigma(T) = \overline{\{\phi(\lambda) : \lambda \in \sigma(\tilde{H})\}} = \overline{\left\{ \frac{\lambda - E}{\lambda + C} : \lambda \in \sigma(\tilde{H}) \right\}}.$$

If $\lambda \in \sigma(\tilde{H})$, then $\lambda - E \geq v_0 - \beta^2 - E = \eta - \beta^2 = \eta/2 > 0$. Also $\lambda + C \geq v_0 - \beta^2 + C = 2$. Thus, $\sigma(T) \subset \mathbb{R}^+$. Let us estimate $d_+ = \inf \sigma(T)$. Since $\phi'(\lambda) = \frac{C+E}{(\lambda+C)^2} = \frac{\beta^2 - v_0 + 2 + E}{(\lambda+C)^2} = \frac{2 - (\eta/2)}{(\lambda+C)^2}$, we have 2 cases:

- (i) If $E \geq v_0 - 4$, then $\eta \leq 4$ and $2 - (\eta/2) \geq 0$. Thus ϕ is increasing, so $\phi(\lambda) \geq \phi(v_0 - \beta^2)$ for any $\lambda \in \sigma(\tilde{H})$, since $\sigma(\tilde{H}) \subseteq [v_0 - \beta^2, +\infty)$. Thus

$$d_+ \geq \frac{v_0 - \beta^2 - E}{v_0 - \beta^2 + C} = \frac{\eta - \beta^2}{2} = \frac{\eta}{4}.$$

By Lemma 3.2.3, $T + i\beta S$ is thus invertible with $\|(T + i\beta S)^{-1}\| \leq \frac{1}{d_+} \leq \frac{4}{\eta}$.

- (ii) If $E < v_0 - 4$, then $\eta > 4$, so $2 - (\eta/2) < 0$. Thus ϕ is decreasing, so $\phi(\lambda) \geq \lim_{t \rightarrow \infty} \phi(t)$ for any $\lambda \in \sigma(\tilde{H})$. Hence, $d_+ \geq \lim_{t \rightarrow \infty} \frac{t-E}{t+C} = 1$. By Lemma 3.2.3, $T + i\beta S$ is invertible, $\|(T + i\beta S)^{-1}\| \leq \frac{1}{d_+} \leq 1$.

- b. In case (b.), $E \in (r, s) \subset \rho(H)$, and $\eta := \text{dist}(E, (r, s)^c) > 0$. Put $\beta_0 := \sqrt{\eta/2}$. Then for $|\beta| \leq \beta_0$, we have $\|\beta^2 m\|_\infty \leq \frac{\eta}{2}$, which ensures that $(r + \frac{\eta}{2}, s - \frac{\eta}{2}) \subset \rho(\tilde{H})$. Let $r' = r + \frac{\eta}{2}$ and $s' = s - \frac{\eta}{2}$. Then $E \in (r', s')$ and $\min\{E - r', s' - E\} = \frac{\eta}{2}$.

Since $C + r \geq C + v_0 - 1 = \beta^2 + 1 \geq 1$, we have $\rho(T) \supset \phi(r', s') = (\frac{r'-E}{r'+C}, \frac{s'-E}{s'+C})$. In particular, $0 \in \rho(T)$. Taking $d_\pm = \text{dist}(0, \sigma(T) \cap \mathbb{R}^\pm)$, we have

$$d_+ \geq \frac{s' - E}{s' + C} \quad \text{and} \quad d_- \geq \frac{E - r'}{r' + C}.$$

But $s' + C \leq s - \frac{\eta}{2} + \frac{\eta}{2} - v_0 + 2 = s - v_0 + 2$ and $r' + C \leq r + \frac{\eta}{2} + \frac{\eta}{2} - v_0 + 2 = r + \eta - v_0 + 2 \leq s - v_0 + 2$. Hence,

$$d_+ \geq \frac{s' - E}{s - v_0 + 2} \geq \frac{\eta}{2(s - v_0 + 2)} \quad \text{and} \quad d_- \geq \frac{E - r'}{s - v_0 + 2} \geq \frac{\eta}{2(s - v_0 + 2)}.$$

Furthermore,

$$d_+ d_- \geq \frac{(s' - E)(E - r')}{(s - v_0 + 2)^2} = \frac{\eta(s' - r' - \eta/2)}{2(s - v_0 + 2)^2}.$$

Indeed, if $\frac{\eta}{2} = s' - E$, then $(s' - E)(E - r') = \frac{\eta}{2}(E - r') = \frac{\eta}{2}(s' - r' - \frac{\eta}{2})$. The case $\frac{\eta}{2} = E - r'$ is similar. Now $(s' - r' - \frac{\eta}{2}) = (s - r - \frac{3\eta}{2})$. But we know that $\eta \leq \frac{1}{2}(s - r)$, with equality iff $E = \frac{1}{2}(s + r)$ is the midpoint of (r, s) . Hence $s - r - \frac{3\eta}{2} \geq s - r - \frac{3(s - r)}{4} = \frac{s - r}{4}$. Thus,

$$d_+ d_- \geq \frac{1}{8(s - v_0 + 2)^2} \eta(s - r).$$

Now take $\beta_1 := \frac{1}{4(s - v_0 + 2)} \sqrt{\frac{\eta(s - r)}{2}}$ and note that $|\beta_1| = \sqrt{\frac{\eta}{2} \frac{\sqrt{s - r}}{4(s - v_0 + 2)}} \leq \sqrt{\frac{\eta}{2}} = \beta_0$ (because $s - r \leq s - v_0 + 1$). Moreover, $\beta_1 \leq \frac{1}{2} \sqrt{d_+ d_-}$. $T + i\beta_1 S$ is thus invertible by Lemma 3.2.3, with $\|(T + i\beta_1 S)^{-1}\| \leq 2 \max(\frac{1}{d_+}, \frac{1}{d_-}) \leq \frac{4(s - v_0 + 2)}{\eta}$.

To conclude, for either (a.) or (b.), take the β that makes $T + i\beta S$ invertible. Then by (2-2), $H_\beta - E$ is invertible with

$$(H_\beta - E)^{-1} = (\tilde{H} + C)^{-1/2}(T + i\beta S)^{-1}(\tilde{H} + C)^{-1/2}.$$

By (2-1), $\|(\tilde{H} + C)^{1/2}v\| \geq \|v\|$ for $v \in D(\mathfrak{h})$, hence $\|(\tilde{H} + C)^{-1/2}\| \leq 1$ and

$$\|(H_\beta - E)^{-1}\| \leq \|(T + i\beta S)^{-1}\|.$$

Finally, $(H_\beta - E)^{-1} = e^{\beta w}(H - E)^{-1}e^{-\beta w}$, so

$$\|\chi_A(H - E)^{-1}\chi_B\| \leq \|\chi_A e^{-\beta w}\|_\infty \cdot \|(H_\beta - E)^{-1}\| \cdot \|e^{\beta w}\chi_B\|_\infty$$

Since $w = 0$ on B , $\|e^{\beta w}\chi_B\|_\infty \leq 1$. For $x \in A$, $w(x, B) = \text{dist}(x, B) \geq \text{dist}(A, B) = \delta$. Hence $\|\chi_A e^{-\beta w}\|_\infty \leq e^{-\delta\beta}$. Thus,

$$\|\chi_A(H - E)^{-1}\chi_B\| \leq \|(T + i\beta S)^{-1}\| \cdot e^{-\delta\beta}$$

The statement then follows directly from our estimates for cases (a.) and (b.) above.

Finally, all the arguments remain valid if $\Lambda^{(n)} = \mathbb{R}^{nd}$, except that $e^{\zeta w}$ is no longer bounded. We thus consider a large cube Ξ containing A_1 and A_2 and replace \tilde{w} by a uniformly Lipschitz function ρ of compact support such that $\rho(\mathbf{x}) = \text{dist}(\mathbf{x}, A_1)$ if $\mathbf{x} \in \Xi$ and $\|\nabla\rho\|_\infty \leq 1$, then take w to be the restriction of ρ to $\Gamma^{(n)}$. \square

3.3 Regularity of generalized eigenfunctions

We showed in Section 2.9 that one can avoid relying on the regularity of generalized eigenfunctions by making use of approximations in the negative Hilbert space \mathcal{H}_- . Actually the same approach shows that the generalized eigenfunctions are locally in $W^{1,2}(\Gamma)$.

We have the following result. Here $\Lambda_L^{(n)} \ni B$ means that $\Lambda_{L-2}^{(n)} \supset B$.

Lemma 3.3.1. *Let $\psi \in D(H_-)$ and $\varphi \in \tilde{C}_c^1(\Gamma)$. Then $\varphi\psi \in D(\mathfrak{h}_\Lambda)$ for any cube $\Lambda^{(n)} \ni \text{supp } \varphi$ (including $\Lambda^{(n)} = \mathbb{R}^{nd}$).*

Moreover, there exists a function $\varphi\nabla\psi \in \mathcal{H}^n$ such that

$$\mathfrak{h}_\Lambda[\varphi\psi, v] = \langle \varphi\nabla\psi + \psi\nabla\varphi, \nabla v \rangle + \langle V^\omega\varphi\psi, v \rangle \quad \text{for any } v \in D(\mathfrak{h}_\Lambda).$$

Proof. Put $H := H^{(n)}(\omega)$. Since $\psi \in D(H_-)$, we may find $(f_j) \subset D(H)$ such that $\|\iota_- f_j - \psi\|_- \rightarrow 0$ and $\|\iota_- H f_j - H_- \psi\|_- \rightarrow 0$. Hence, for any χ of compact support we have

$$(\star) \quad \|\chi f_j - \chi\psi\| \leq \|\chi\|_+ \cdot \|\iota_- f_j - \psi\|_- \rightarrow 0.$$

Hence, $\|\varphi f_j - \varphi\psi\| \rightarrow 0$ and $\varphi f_j \in D(\mathfrak{h}_\Lambda)$ by Lemma 2.3.1. Now assume $\Lambda^{(n)} = \Lambda_L^{(n)}$, we know that $\text{supp } \varphi \subset \tilde{Q} := \Lambda_{L-2}^{(n)}$. Hence

$$\begin{aligned} \mathfrak{h}_\Lambda[\varphi f_j - \varphi f_k] &= \|\nabla(\varphi(f_j - f_k))\|^2 + \langle V^\omega\varphi(f_j - f_k), \varphi(f_j - f_k) \rangle \\ &\leq (\|\nabla\varphi\|_\infty \cdot \|\chi_{\tilde{Q}}(f_j - f_k)\| + \|\varphi\|_\infty \cdot \|\chi_{\tilde{Q}}\nabla(f_j - f_k)\|)^2 \\ &\quad + \|V^\omega\|_\infty \cdot \|\varphi\|_\infty^2 \cdot \|\chi_{\tilde{Q}}(f_j - f_k)\|^2. \end{aligned}$$

The first and third term tend to zero by (\star) . To estimate the second, put $Q := \Lambda_{L-1}^{(n)}$. Then by Lemma 2.4.3, we may find $C > 0$ such that

$$\|\chi_{\tilde{Q}} \nabla(f_j - f_k)\| \leq C \cdot (\|\chi_Q H(f_j - f_k)\| + \|\chi_Q(f_j - f_k)\|).$$

But

$$\|\chi_Q H(f_j - f_k)\| \leq \|\chi_Q\|_+ \|H(f_j - f_k)\|_- \rightarrow 0$$

since $\|Hf_j - H_- \psi\|_- \rightarrow 0$. We thus showed that $\mathfrak{h}_\Lambda[\varphi f_j - \varphi f_k] \rightarrow 0$. Since \mathfrak{h}_Λ is closed, it follows that $\varphi \psi \in D(\mathfrak{h}_\Lambda)$ and $\mathfrak{h}_\Lambda[\varphi \psi, v] = \lim_{j \rightarrow \infty} \mathfrak{h}_\Lambda[\varphi f_j, v]$ for any $v \in D(\mathfrak{h}_\Lambda)$.

Finally, note that $\|\varphi \nabla(f_j - f_k)\| \leq \|\varphi\|_\infty \|\chi_{\tilde{Q}} \nabla(f_j - f_k)\| \rightarrow 0$ as we showed above, so the sequence $(\varphi \nabla f_j)$ is Cauchy in \mathcal{H} and converges to some element which we denote by $\varphi \nabla \psi$. Then for $v \in D(\mathfrak{h}_\Lambda)$ we have

$$\begin{aligned} \mathfrak{h}_\Lambda[\varphi \psi, v] &= \lim_{j \rightarrow \infty} \mathfrak{h}_\Lambda[\varphi f_j, v] = \lim_{j \rightarrow \infty} \langle \varphi \nabla f_j + f_j \nabla \varphi, \nabla v \rangle + \langle V^\omega \varphi f_j, v \rangle \\ &= \langle \varphi \nabla \psi + \psi \nabla \varphi, \nabla v \rangle + \langle V^\omega \varphi \psi, v \rangle, \end{aligned}$$

where we used (\star) again. □

This still doesn't really allow us to simplify the proof of Lemma 2.9.1, so we record the following result.

Lemma 3.3.2. *Let ψ be a generalized eigenfunction of $H^{(n)}$ corresponding to $\lambda \in I$. Given $\varphi \in \tilde{C}_c^1(\Gamma)$ and $\Lambda^{(n)} \ni \text{supp } \varphi$, we may find a function $\nabla \varphi \cdot \nabla \psi \in \mathcal{H}$ such that*

$$[\mathfrak{h}_\Lambda - \lambda][\varphi \psi, v] = \langle \psi \nabla \varphi, \nabla v \rangle - \langle \nabla \varphi \cdot \nabla \psi, v \rangle \quad \text{for any } v \in D(\mathfrak{h}_\Lambda).$$

Moreover, if $\text{supp } \nabla \varphi \subset \tilde{Q} \subset Q \subset \Lambda^{(n)}$ for some cellular sets \tilde{Q} and Q such that $\text{dist}(\partial Q, \partial \tilde{Q}) \geq 1$, then there exists $c = c(I, n, d, q_-)$ such that

$$\|\nabla \varphi \cdot \nabla \psi\| \leq c_1 \|\nabla \varphi\|_\infty \|\chi_Q \psi\|.$$

Sketch of proof. The proof is contained in Lemma 2.9.1. As we showed there, we may find $(f_j) \subset D(H)$ such that

$$[\mathfrak{h}_\Lambda - \lambda][\varphi \psi, v] = \lim_{j \rightarrow \infty} \{ \langle (H - \lambda)f_j, g \rangle + \langle f_j \nabla \varphi, \nabla v \rangle - \langle \nabla f_j, v \nabla \varphi \rangle \}.$$

Again the first term tends to zero, the second to $\langle \psi \nabla \varphi, \nabla v \rangle$. For the third, note that $\|\nabla \varphi \cdot \nabla(f_j - f_k)\| \leq \|\nabla \varphi\|_\infty \|\chi_{\tilde{Q}} \nabla(f_j - f_k)\| \rightarrow 0$ using the same arguments of this lemma, so $(\nabla \varphi \cdot \nabla f_j)$ is Cauchy and converges to some $\nabla \varphi \cdot \nabla \psi \in \mathcal{H}$. For the bound, note that $\|\nabla \varphi \cdot \nabla \psi\| = \lim \|\nabla \varphi \cdot \nabla f_j\| \leq \liminf \|\nabla \varphi\|_\infty \|\chi_{\tilde{Q}} \nabla f_j\| \leq c_1 \|\nabla \varphi\|_\infty \|\chi_Q \psi\|$ as we showed there. □

This result does allow us to simplify the proof, however it is certainly more natural to reason directly as in Lemma 2.9.1 if one doesn't already have it.

3.4 Localization without generalized eigenfunctions: prelude

In this section we want to establish three properties concerning our random operator $H^{(n)}(\omega)$. First, we would like to prove that $\chi_\Lambda(H^{(n)}(\omega) - z)^{-1}$ is compact for some $z \in \rho(H^{(n)}(\omega))$ and any bounded $\Lambda \subset \mathbb{R}^{nd}$. This will follow from Theorem 3.4.1 below.

Next we want to prove the eigenfunction decay inequality, Lemma 3.4.2, which relates the decay of the resolvent operator kernel to that of the eigenfunction. Finally we wish to prove that if $\Lambda_j \subset \mathbb{R}^{nd}$ is an increasing sequence of cubes to \mathbb{R}^{nd} , then $H_{\Lambda_j}^{(n)}(\omega)$ converge to $H^{(n)}(\omega)$ in the strong resolvent sense. All these properties will be used in the next section to deduce dynamical and exponential localization from multiscale analysis, without relying on generalized eigenfunction expansions. The price to pay is that strong dynamical localization is established in the operator norm, not in the Hilbert-Schmidt norm.

Theorem 3.4.1. *Let Ξ be a cube or $\Xi = \mathbb{R}^{nd}$ and T_f be the operator of multiplication by a function f . Then*

- 1) *If $f \in L^2(\Gamma \cap \Xi)$, the operators $T_f e^{-tH_{\Xi}^{(n)}(\omega)}$ and $T_f \chi_I(H_{\Xi}^{(n)}(\omega))$ are Hilbert-Schmidt operators for any $t > 0$ and any bounded interval $I \subset \mathbb{R}$. Furthermore,*

$$\|T_f \chi_I(H_{\Xi}^{(n)}(\omega))\|_2 \leq c \cdot \|f\|_{L^2}$$

for some $c = c(I, n, q_-)$ independent of ω and Ξ .

- 2) *If $f \in L^\infty(\Gamma \cap \Xi)$ and $g : \mathbb{R} \rightarrow \mathbb{C}$ are bounded Borel functions vanishing at infinity, then $T_f g(H_{\Xi}^{(n)}(\omega))$ is a compact operator.*

Proof. 1) It is easy to see that the proofs of Lemma 2.8.1 and Lemma 2.8.2 remain valid if the operator $H^{(n)}(\omega)$ is replaced by $H_{\Xi}^{(n)}(\omega)$. Hence we have the following bound

$$(SG) \quad \forall t > 0 : \|e^{-tH_{\Xi}^{(n)}(\omega)}\|_{L^2 \rightarrow L^\infty} \leq ct^{-n/4} e^{-(nq_- - 1)t}$$

for some $c = c(n)$ independent of ω and Ξ . Now put $H := H_{\Xi}^{(n)}(\omega)$. Since $f \in L^2$, T_f is an operator from $L^\infty \rightarrow L^2$, thus by (SG), $T_f e^{-tH} : L^2 \rightarrow L^2$ factorizes through L^∞ . It is thus Hilbert-Schmidt with

$$\|T_f e^{-tH}\|_2 \leq \|f\|_{L^2} \|e^{-tH}\|_{L^2 \rightarrow L^\infty} \leq c' t^{-n/4} e^{-(nq_- - 1)t} \cdot \|f\|_{L^2},$$

(see Appendix B). Hence $e^{-tH} T_{\bar{f}}$ is also Hilbert-Schmidt with the same norm. But given an orthonormal basis (e_j) of L^2 , if $\rho_{\bar{f}e_j}$ is the spectral measure of H in the state $\bar{f}e_j$, then for any bounded interval $I = [a, b]$ we have

$$\|\chi_I(H) T_{\bar{f}} e_j\|^2 = \int_a^b d\rho_{\bar{f}e_j}(\lambda) \leq e^{2b} \int_a^b e^{-2\lambda} d\rho_{\bar{f}e_j}(\lambda) \leq e^{2b} \cdot \|e^{-H} T_{\bar{f}} e_j\|^2$$

for all j , so $\chi_I(H) T_{\bar{f}}$ is Hilbert-Schmidt. Hence $T_f \chi_I(H)$ is also Hilbert-Schmidt. The estimate on the norm follows from the above estimates.

- 2) Let T_m be the multiplication by $f_m := \chi_{\Lambda_m^{(n)}(0)} f$ and let $g_m(H) := \chi_{[-m, m]}(H) g(H)$. Then by 1) the operator $T_m \chi_{[-m, m]}(H)$ is Hilbert-Schmidt, hence $T_m g_m(H)$ is a sequence of Hilbert-Schmidt operators. By the spectral theorem,

$$\|T_f g(H) - T_m g_m(H)\| \leq \|f\|_\infty \|g - g_m\|_\infty + \|f - f_m\|_\infty \|g_m\|_\infty \rightarrow 0$$

because f and g vanish at infinity. Hence $T_f g(H)$ is compact. \square

Lemma 3.4.2. *Let Ξ be a cube or $\Xi = \mathbb{R}^{nd}$. Let $E_+ \in \mathbb{R}$. There exists $C = C(E_+, n, d, q_-)$ such that, if $\mathbf{x}_0 \in \mathbb{Z}^{nd}$, $\mathbf{C}(\mathbf{x}) \subset \Lambda_{L-6}^{(n)}(\mathbf{x}_0)$ and $\Lambda_L^{(n)}(\mathbf{x}_0) \subset \Xi$, then every eigenfunction u of $H_{\Xi}^{(n)}$ corresponding to $E \in \rho(H_{\Lambda_L^{(n)}(\mathbf{x}_0)}) \cap (-\infty, E_+]$ satisfies*

$$\|\chi_{\mathbf{x}} u\| \leq C \cdot |\mathbf{B}_L^{\text{out}}(\mathbf{x}_0)| \max_{\mathbf{y} \in \mathbf{B}_L^{\text{out}}(\mathbf{x}_0)} \|G_{\Lambda_L^{(n)}(\mathbf{x}_0)}(\mathbf{x}, \mathbf{y}; E)\| \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} u\|$$

Strictly speaking this does not follow from Lemma 2.9.1 because of Ξ . In any case, the proof is a bit simpler now that we deal with innocent eigenfunctions.

Proof. Let $\Lambda := \Lambda_L^{(n)}(\mathbf{x}_0)$ and \mathfrak{h}_Λ be the form corresponding to $H_\Lambda^{(n)}(\omega)$. Let $\varphi \in \tilde{C}_c^1(\Gamma \cap \Lambda)$ such that $\varphi = 1$ on a neighborhood of $\Gamma \cap \mathbf{C}(\mathbf{x})$, $\text{supp } \nabla \varphi \subset \tilde{Q} := \text{int}(\Lambda_{L-2}^{(n)}(\mathbf{x}_0) \setminus \Lambda_{L-4}^{(n)}(\mathbf{x}_0))$ and $\|\nabla \varphi\|_\infty \leq C_1(nd)$. Then

$$\|\chi_{\mathbf{x}} u\|^2 = \langle \varphi u, \chi_{\mathbf{x}} u \rangle = \langle \varphi u, (H_\Lambda - E)G_\Lambda(E)\chi_{\mathbf{x}} u \rangle = (\mathfrak{h}_\Lambda - E)[\varphi u, v]$$

where $v := G_\Lambda(E)\chi_{\mathbf{x}} u$. Now

$$\begin{aligned} (\mathfrak{h}_\Lambda - E)[\varphi u, v] &= \langle \nabla(\varphi u), \nabla v \rangle + \langle (V^\omega - E)\varphi u, v \rangle \\ &= [\langle \nabla u, \nabla(\varphi v) \rangle + \langle (V^\omega - E)u, \varphi v \rangle] + \langle u \nabla \varphi, \nabla v \rangle - \langle \nabla u, v \nabla \varphi \rangle \end{aligned}$$

Since u is an eigenfunction of $H_\Xi^{(n)}(\omega)$ with energy E , the term in square brackets vanishes. We thus showed that

$$\|\chi_{\mathbf{x}} u\|^2 \leq \|\nabla \varphi\|_\infty \|\chi_{\tilde{Q}} u\| \|\chi_{\tilde{Q}} \nabla v\| + \|\nabla \varphi\|_\infty \|\chi_{\tilde{Q}} \nabla u\| \|\chi_{\tilde{Q}} v\|$$

Now by Lemma 2.4.3, taking $Q := \text{int } \Lambda_L^{\text{out}}(\mathbf{x}_0)$, we can find c_1 and c_2 such that

$$\|\chi_{\tilde{Q}} \nabla u\| \leq c_1 \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} u\| \quad \text{and} \quad \|\chi_{\tilde{Q}} \nabla v\| \leq c_2 \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} v\|$$

(for the first bound take $f = u$ and $H = H_\Xi^{(n)}$, for the second take $f = v$ and $H = H_\Lambda^{(n)}$ and note that $(H_\Lambda - E)v = \chi_{\mathbf{x}} u = 0$ on Q).

Taking $C = \max(2c_1 \|\nabla \varphi\|_\infty, 2c_2 \|\nabla \varphi\|_\infty)$ and noting that $\tilde{Q} \subseteq \Lambda_L^{\text{out}}$ we thus get

$$\|\chi_{\mathbf{x}} u\|^2 \leq C \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} u\| \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} v\|.$$

Since $\|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} v\| \leq \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} G_\Lambda(E)\chi_{\mathbf{x}}\| \|\chi_{\mathbf{x}} u\|$, we get

$$\|\chi_{\mathbf{x}} u\| \leq C \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} G_\Lambda(E)\chi_{\mathbf{x}}\| \cdot \|\chi_{\Lambda_L^{\text{out}}(\mathbf{x}_0)} u\|.$$

The assertion now follows by the triangle inequality. \square

Lemma 3.4.3. *If $\Lambda_j^{(n)} \subset \mathbb{R}^{nd}$ is an increasing sequence of cubes to \mathbb{R}^{nd} , then the operators $H_{\Lambda_j}^{(n)}(\omega)$ converge to $H^{(n)}(\omega)$ in the strong resolvent sense as $j \rightarrow \infty$.*

Proof. Let D_0 be the set of functions in $D(H^{(n)}(\omega))$ of compact support. We first show that D_0 is a core for $H^{(n)}(\omega)$. Let $f \in D(H^{(n)}(\omega))$ and $\psi_m = (\psi_{m,\kappa})$ such that $\psi_m = 1$ on $\Gamma \cap \Lambda_{m+\frac{1}{4}}^{(n)}(\mathbf{0})$ and $\psi_m = 0$ on $\Gamma \cap \Lambda_{m+\frac{3}{4}}^{(n)}(\mathbf{0})^c$. In the layer $\Lambda_{m+\frac{3}{4}}^{(n)}(\mathbf{0}) \setminus \Lambda_{m+\frac{1}{4}}^{(n)}(\mathbf{0})$, we take $\psi_{m,\kappa}$ a smooth continuation such that $\|\nabla \psi_m\|_\infty \leq C$ and $\|\Delta \psi_m\|_\infty \leq C$ for some $C = C(nd)$. Let $f_m = \psi_m f$. It is clear that $f_m \in D(\mathfrak{h}_\omega^{(n)})$ (eventually use Lemma 2.3.1). To see that $f_m \in D(H^{(n)}(\omega))$, note that for $v \in D(\mathfrak{h}_\omega^{(n)})$ we have

$$\mathfrak{h}_\omega^{(n)}[\psi_m f, v] = \mathfrak{h}_\omega^{(n)}[f, \psi_m v] + \langle f \nabla \psi_m, \nabla v \rangle - \langle \nabla f, v \nabla \psi_m \rangle$$

Since ψ_m is smooth and constant in the neighborhood of each σ^i , we have $f \nabla \psi_m \in W_0^{1,2}(\Gamma \cap \Lambda_{m+1}^{(n)}(\mathbf{0}))$, so we may apply Lemma 2.4.1 to the second term and obtain

$$\begin{aligned} \mathfrak{h}_\omega^{(n)}[\psi_m f, v] &= \langle H^{(n)}(\omega) f, \psi_m v \rangle - \langle \nabla \cdot (f \nabla \psi_m), v \rangle - \langle \nabla f \cdot \nabla \psi_m, v \rangle \\ &= \langle \psi_m H^{(n)}(\omega) f - 2 \nabla \psi_m \cdot \nabla f - (\Delta \psi_m) f, v \rangle \end{aligned}$$

Hence $f_m \in D(H^{(n)}(\omega))$ (see e.g. Section 2.11). Finally, we have $f_m \rightarrow f$ in $L^2(\Gamma)$ and

$$H^{(n)}(\omega)f_m = \psi_m H^{(n)}(\omega)f - 2\nabla\psi_m \cdot \nabla f - (\Delta\psi_m)f \rightarrow H^{(n)}(\omega)f$$

since $\nabla\psi_m$ and $\Delta\psi_m$ are uniformly bounded and supported in $\Lambda_{m+1}^{(n)}(\mathbf{0}) \setminus \Lambda_m^{(n)}(\mathbf{0})$. Hence D_0 is a core for $H^{(n)}(\omega)$. Finally, if $f \in D_0$ and if j is sufficiently large, we have $f \in D(H_{\Lambda_j}^{(n)}(\omega))$ and $H_{\Lambda_j}^{(n)}(\omega)f = H^{(n)}(\omega)f$. The claim now follows from [57, Corollary VIII.1.6]. \square

3.5 Localization

We first establish dynamical localization by adapting the approach of [47] to finite volumes and passing to the limit. Then we deduce spectral localization by showing that the RAGE theorem is valid in our setting. Finally, we prove exponential localization.

We start by fixing a large cube $\Lambda^{(N)} := \Lambda_{2r_N, L_{k^*+1}}^{(N)}(\mathbf{0})$ such that $K \subset \Gamma \cap \Lambda_{2r_N, L_{k^*}}^{(N)}(\mathbf{0})$. In the following we consider the event

$$R(m, L, I, \mathbf{x}, \mathbf{y}) := \{ \forall E \in I : \Lambda_L^{(n)}(\mathbf{x}) \text{ or } \Lambda_L^{(n)}(\mathbf{y}) \text{ is } (E, m)\text{-NS} \}$$

for \mathbf{x}, \mathbf{y} in $\Lambda^{(N)}$ such that the corresponding cubes are separable and contained in $\Lambda^{(N)}$.

We start with the following key lemma. Here $P_n(\omega)\psi := \langle \psi, \varphi_n(\omega) \rangle \varphi_n(\omega)$, where $\{\varphi_n(\omega)\}$ is an orthonormal basis of eigenvectors of $H_{\Lambda}^{(N)}(\omega)$.

Lemma 3.5.1. *Let $m > 0$, $I \subset \mathbb{R}$ and assume $\omega \in R(m, L, I, \mathbf{x}, \mathbf{y})$. Then*

$$\|\chi_{\mathbf{x}}P_n(\omega)\chi_{\mathbf{y}}\|_2 \leq CL^{Nd-1}e^{-mL}\|\chi_{\Lambda_L^{\text{out}}(\mathbf{x})}P_n(\omega)\chi_{\mathbf{y}}\|_2$$

for any n , where $C = C(I, N, d, q_-) < \infty$.

Proof. Given an eigenvalue $\lambda_n \in I$, either $\Lambda_L^{(N)}(\mathbf{x})$ or $\Lambda_L^{(N)}(\mathbf{y})$ is (λ_n, m) -NS for $H^{(N)}(\omega)$. Since $\|\chi_{\mathbf{x}}P_n(\omega)\chi_{\mathbf{y}}\|_2 = \|\chi_{\mathbf{y}}P_n(\omega)\chi_{\mathbf{x}}\|_2$, we may assume that $\Lambda_L^{(N)}(\mathbf{x})$ is (λ_n, m) -NS. Now given $\psi \in L^2(\Gamma \cap \Lambda^{(N)})$, $P_n(\omega)\chi_{\mathbf{y}}\psi$ is an eigenfunction of $H_{\Lambda}^{(N)}(\omega)$ with eigenvalue λ_n , hence by Lemma 3.4.2,

$$\|\chi_{\mathbf{x}}P_n(\omega)\chi_{\mathbf{y}}\psi\| \leq c(2L-1)^{Nd-1}e^{-mL}\|\chi_{\Lambda_L^{\text{out}}(\mathbf{x})}P_n(\omega)\chi_{\mathbf{y}}\psi\|.$$

The assertion follows by definition of the HS norm. \square

Before proving dynamical localization, we establish the following decay of the kernel. Let k_0 be the smallest integer such that $K \subset \Gamma \cap \Lambda_{r_N, L_{k_0}}^{(N)}(\mathbf{0})$ and for $j \geq k_0$ put

$$\begin{aligned} F_j &= \Lambda_{2r_N, L_j}^{(N)}(\mathbf{0}), & \tilde{F}_j &= \mathbf{B}_{2r_N, L_j}^{(N)}(\mathbf{0}) \\ M_j &= F_{j+1} \setminus F_j, & \tilde{M}_j &= \tilde{F}_{j+1} \setminus \tilde{F}_j \end{aligned}$$

In the following, we choose m and ε_0 such that (DS: N, k, m, I_N) holds for all $k \geq 0$ in $I_N = [Nq_- - \frac{1}{2}, Nq_- + \varepsilon_0]$, as guaranteed by Theorem 2.7.15.

Lemma 3.5.2. *There exists $c = c(m, N, d, q_-)$ such that for $\mathbf{x} \in \tilde{M}_j$ and $\mathbf{y} \in \mathbf{B}_{r_N, L_j}^{(N)}(\mathbf{0})$ with j large enough, we have for $I = [Nq_-, Nq_- + \varepsilon_0]$:*

$$\mathbb{E} \left(\sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}}f(H_{\Lambda}^{(N)}(\omega))\chi_I(H_{\Lambda}^{(N)}(\omega))\chi_{\mathbf{y}}\|_2 \right) \leq c(e^{-mL_j/2} + L_j^{-2p_N(1+\theta)^j}).$$

Proof. Given a bounded Borel f put $f_I := \chi_I f$. By the functional calculus we have $f_I(H_\Lambda^{(N)}(\omega)) = \sum_{\lambda_n \in I} f(\lambda_n) P_n(\omega)$. Hence

$$\|\chi_{\mathbf{x}} f_I(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\|_2 \leq \sum_{\lambda_n \in I} |f(\lambda_n)| \|\chi_{\mathbf{x}} P_n(\omega) \chi_{\mathbf{y}}\|_2$$

Since $\mathbf{x} \in \tilde{M}_j$ and $\mathbf{y} \in \mathbf{B}_{r_{N,L_j}}^{(N)}(\mathbf{0})$, we know by Lemma 2.2.8 that $\Lambda_{L_j}^{(N)}(\mathbf{x})$ and $\Lambda_{L_j}^{(N)}(\mathbf{y})$ are separable. Hence if $\omega \in B_j := R(m, L_j, I, \mathbf{x}, \mathbf{y})$ we have by Lemma 3.5.1

$$\|\chi_{\mathbf{x}} f_I(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\|_2 \leq C \|f\|_\infty L_j^{Nd-1} e^{-mL_j} \sum_{\lambda_n \in I} \|\chi_{\Lambda_{L_j}^{\text{out}}(\mathbf{x})} P_n(\omega) \chi_{\mathbf{y}}\|_2.$$

Since

$$\begin{aligned} \sum_{\lambda_n \in I} \|\chi_{\Lambda_{L_j}^{\text{out}}(\mathbf{x})} P_n(\omega) \chi_{\mathbf{y}}\|_2 &\leq \sum_{\lambda_n \in I} \|\chi_{\Lambda_{L_j}^{\text{out}}(\mathbf{x})} P_n(\omega)\|_2 \|P_n(\omega) \chi_{\mathbf{y}}\|_2 \\ &\leq \frac{1}{2} \sum_{\lambda_n \in I} (\|\chi_{\Lambda_{L_j}^{\text{out}}(\mathbf{x})} P_n(\omega)\|_2^2 + \|P_n(\omega) \chi_{\mathbf{y}}\|_2^2) \\ &= \frac{1}{2} (\|\chi_{\Lambda_{L_j}^{\text{out}}(\mathbf{x})} \chi_I(H_\Lambda^{(N)}(\omega))\|_2^2 + \|\chi_{\mathbf{y}} \chi_I(H_\Lambda^{(N)}(\omega))\|_2^2), \end{aligned}$$

then using Theorem 3.4.1 we obtain

$$\begin{aligned} \|\chi_{\mathbf{x}} f_I(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\|_2 &\leq C_1 \|f\|_\infty L_j^{Nd-1} e^{-mL_j} (\|\chi_{\Lambda_{L_j}^{\text{out}}(\mathbf{x})}\|_{L^2}^2 + \|\chi_{\mathbf{y}}\|_{L^2}^2) \\ &\leq C_2 \|f\|_\infty L_j^{3(Nd-1)} e^{-mL_j} \end{aligned}$$

for some $C_2 = C_2(N, d, q_-)$ independent of ω . For $\omega \in B_j^c$ we have

$$\|\chi_{\mathbf{x}} f_I(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\|_2^2 \leq \|f\|_\infty^2 \|\chi_I(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\|_2 \leq C_3 \|f\|_\infty$$

where we used again Theorem 3.4.1. We thus showed that

$$\mathbb{E} \left(\sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}} f(H_\Lambda^{(N)}(\omega)) \chi_I(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\| \right) \leq C_2 L_j^{3(Nd-1)} e^{-mL_j} \mathbb{P}(B_j) + C_3 \mathbb{P}(B_j^c).$$

Using Theorem 2.7.15 to estimate $\mathbb{P}(B_j^c)$, we obtain the assertion. \square

Corollary 3.5.3. *There exists $c = c(m, N, d, q_-)$ such that for $\mathbf{x} \in \tilde{M}_j$ and $\mathbf{y} \in \mathbf{B}_{r_{N,L_j}}^{(N)}(\mathbf{0})$ with j large enough, we have for $I = [Nq_-, Nq_- + \varepsilon_0]$:*

$$\mathbb{E} \left(\sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}} f(H^{(N)}(\omega)) E_\omega(I) \chi_{\mathbf{y}}\| \right) \leq c(e^{-mL_j/2} + L_j^{-2p_N(1+\theta)^j})$$

Proof. By Lemma 3.4.3, $H_\Lambda^{(N)}(\omega)$ converge to $H^{(N)}(\omega)$ in the strong resolvent sense, so by [93, Theorem VIII.20], $g(H_\Lambda^{(N)}(\omega))$ converge strongly to $g(H^{(N)}(\omega))$ for any bounded continuous function g . Thus

$$\|\chi_{\mathbf{x}} g(H^{(N)}(\omega)) \chi_{\mathbf{y}}\| \leq \liminf_{k^* \rightarrow \infty} \|\chi_{\mathbf{x}} g(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\|.$$

Now $\sup_g \liminf_{k^* \rightarrow \infty} \|\cdot\| \leq \liminf_{k^* \rightarrow \infty} \sup_g \|\cdot\|$ since $\|\cdot\| \leq \sup_g \|\cdot\|$ for any k^* . Thus, taking suprema and applying Fatou's lemma yields

$$\mathbb{E} \left\{ \sup_{\|g\| \leq 1} \|\chi_{\mathbf{x}} g(H^{(N)}(\omega)) \chi_{\mathbf{y}}\| \right\} \leq \liminf_{k^* \rightarrow \infty} \mathbb{E} \left\{ \sup_{\|g\| \leq 1} \|\chi_{\mathbf{x}} g(H_\Lambda^{(N)}(\omega)) \chi_{\mathbf{y}}\| \right\}$$

where the supremum is initially taken over continuous g supported in I , but may be extended to all Borel g without changing its value (see the proof of [2, Theorem 1.1.]). Since $\|A\| \leq \|A\|_2$ for an HS operator A , the claim follows from Lemma 3.5.2. \square

Remark 3.5.4. Unfortunately, although we have strong HS decay of the kernel for any $H_\Lambda^{(N)}(\omega)$, strong resolvent convergence alone is not sufficient to deduce the same is true for $H^{(N)}(\omega)$. The problem is that, given an orthonormal basis (e_j) of $L^2(\Gamma)$, while we know that for e_j , given $\varepsilon > 0$ we may find a large m_j such that

$$\|\chi_{\mathbf{x}}g(H^{(N)}(\omega))\chi_{\mathbf{y}}e_j\| \leq \varepsilon + \|\chi_{\mathbf{x}}g(H_\Lambda^{(N)}(\omega))\chi_{\mathbf{y}}e_j\|,$$

for $k^* \geq m_j$, we cannot deduce a similar bound on the HS norm of the operators because the m_j are not uniform, i.e. there is no largest m_j . We could have deduced such a bound if we had a *norm resolvent* convergence, i.e., if $G_\Lambda(z) \rightarrow G(z)$ for all non-real z . However, this is certainly not true for Schrödinger operators. Indeed, all H_Λ have compact resolvent, so if we had $G_\Lambda(z) \rightarrow G(z)$, then $G(z)$ would be compact, so H would have a compact resolvent, which is of course not true.

We are finally ready to prove strong dynamical localization of any order.

Theorem 3.5.5. *There exists $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ such that for $I = [Nq_-, Nq_- + \varepsilon_0]$, we have for any bounded $K \subset \Gamma^{(N)}$ and all $s > 0$,*

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^s f(H^{(N)}(\omega))E_\omega(I)\chi_K\| \right\} < \infty,$$

where $(X\psi)(\mathbf{x}) := |\mathbf{x}| \cdot \psi(\mathbf{x})$ for $\psi \in \mathcal{H}$, E_ω is the spectral projection of $H^{(N)}(\omega)$ and the supremum is taken over bounded Borel functions, $\|f\| := \|f\|_\infty$.

Proof. Let $k \geq k_0$ be sufficiently large so that Corollary 3.5.3 holds for $j \geq k$. Now

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^s \chi_{F_k} f(H^{(N)}(\omega))E_\omega(I)\chi_K\| \right\} \leq c_1(d, N)L_k^s$$

Furthermore, by the triangle inequality

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{j \geq k} \sup_{\|f\| \leq 1} \|X^s \chi_{M_j} f(H^{(N)}(\omega))E_\omega(I)\chi_K\| \right\} \\ & \leq \sum_{j \geq k} c_2 L_{j+1}^s \sum_{\mathbf{x} \in \tilde{M}_j, \mathbf{y} \in \mathbf{B}_{r_N, L_{k_0}}^{(N)}(\mathbf{0})} \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_{\mathbf{x}} f(H^{(N)}(\omega))E_\omega(I)\chi_{\mathbf{y}}\| \right\} \end{aligned}$$

Estimating $|\tilde{M}_j| \leq cL_{j+1}^{Nd}$, $|\mathbf{B}_{r_N, L_{k_0}}^{(N)}(\mathbf{0})| \leq c'L_{k_0}^{Nd}$ and using Corollary 3.5.3, the series converges. \square

We may now deduce spectral localization:

Theorem 3.5.6. *With probability one, the spectrum of $H^{(N)}(\omega)$ in $I = [Nq_-, Nq_- + \varepsilon_0]$ is pure point.*

Proof. Put $H_\omega := H^{(N)}(\omega)$. Let $\phi \in L^2(\Gamma)$ be compactly supported and let k_0 be the smallest integer with $K := \text{supp } \phi \subset \Lambda_{r_N, L_{k_0}}^{(N)}(\mathbf{0})$. Then for any $k \geq k_0$, by repeating the calculations in the proof of Theorem 3.5.5, we have

$$(\star) \quad \mathbb{E} \left\{ \|\chi_{F_k^c} e^{-itH_\omega} E_\omega(I)\chi_K\| \right\} \leq cL_{k_0}^{Nd} \sum_{j \geq k} L_j^{\alpha Nd - 2p_N(1+\theta)^j} \leq \sum_{j \geq k} L_j^{\alpha Nd - 2p_N}.$$

Recall that $\alpha = \frac{3}{2}$ and $p_N \geq 3Nd + 1$. Now for $j \geq k$ we have

$$\frac{L_j}{L_k} \geq \frac{L_k^{\alpha^{j-k}}}{L_k} \geq \frac{2^{\alpha^{j-k}}}{2}$$

since $(\frac{L_k}{2})^{\alpha^{j-k}} \geq \frac{L_k}{2}$. Hence for $c' := \frac{2^{2p_N - \alpha Nd}}{1 - 2^{\alpha(\alpha Nd - 2p_N)}}$, (\star) is bounded by

$$L_k^{\alpha Nd - 2p_N} \sum_{j \geq k} \left(\frac{L_j}{L_k}\right)^{\alpha Nd - 2p_N} \leq \frac{L_k^{\alpha Nd - 2p_N}}{2^{\alpha Nd - 2p_N}} \sum_{r \geq 0} (2^{\alpha(\alpha Nd - 2p_N)})^r = c' L_k^{\alpha Nd - 2p_N}.$$

For $k < k_0$, the expectation may of course be bounded by 1. Now let $P_c(H_\omega)$ be the projection onto the continuous spectrum of H_ω . By Theorem 3.4.1, the operators $\chi_{F_k}(H_\omega - i)^{-1}$ are compact for all k and clearly χ_{F_k} converge strongly to the identity. Hence it follows from the RAGE theorem (see [107, Corollary 5.9]) that

$$\begin{aligned} \|P_c(H_\omega)\chi_I(H_\omega)\phi\|^2 &= \liminf_{k \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\chi_{F_k^c} e^{-itH_\omega} \chi_I(H_\omega)\phi\|^2 dt \\ &\leq \liminf_{k \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\chi_{F_k^c} e^{-itH_\omega} \chi_I(H_\omega)\chi_K\|^2 \|\phi\|^2 dt \end{aligned}$$

Taking expectation, Fatou's lemma and Fubini theorem yield

$$\mathbb{E}\{\|P_c(H_\omega)\chi_I(H_\omega)\phi\|^2\} \leq \liminf_{k \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(k) \|\phi\|^2 dt = 0$$

where $g(k) = 1$ for $k < k_0$ and $g(k) = c' L_k^{\alpha Nd - 2p_N}$ for $k \geq k_0$. In particular, we have $P_c(H_\omega)\chi_I(H_\omega)\phi = 0$ almost surely. Since this is valid for arbitrary ϕ of compact support, the assertion follows. \square

We finally conclude with exponential localization.

Theorem 3.5.7. *There exists $m > 0$ such that for a.e. ω , any eigenfunction of $H^{(N)}(\omega)$ corresponding to an eigenvalue in $I = [Nq_-, Nq_- + \varepsilon_0]$ decays exponentially with mass m .*

Sketch of proof. The proof follows the same strategy outlined in Theorem 2.1.3 using Lemma 3.4.2 instead of Lemma 2.9.1. The paragraph on generalized eigenfunctions is of course no longer needed here since we already know that the spectrum is almost surely pure point. \square

3.6 Log-Hölder continuous distributions

In this section we briefly mention how our localization results change if the distribution μ is only log-Hölder continuous.

We say that μ is log-Hölder continuous with parameter ζ if there exists $c_\mu \geq 0$ and $\zeta > 0$ such that

$$s(\mu, \varepsilon) \leq c_\mu \cdot \frac{1}{|\log \varepsilon|^\zeta}$$

for all $0 < \varepsilon < 1$.

Recall that we only needed to speak of the regularity of μ when we needed to prove Lemma 2.7.5. Now as mentioned there, the event in (W2: n, k, I_n) may be estimated by

$$\mathbb{P}(A) \leq C(2L_k)^{4nd+d} s(\mu, 4e^{-L_k^{\beta/\alpha}}) \leq \tilde{C} \frac{L_k^{4nd+d}}{(L_k^{\beta/\alpha} - \log 4)^\zeta}.$$

Hence if ζ is sufficiently large, we may bound this event by $\frac{1}{4}L_k^{-2p_1}$. So what we lose when we pass from a Hölder continuous μ to a log-Hölder continuous one is the term $(1 + \theta)^k$ in the exponent. Consequently, we must remove the term $(1 + \theta)^k$ in all Section 2.7 and work in the more classical setting where the events in $(\text{DS}: n, k, m_{L_k}, I_n)$ are bounded by $L_k^{-2p_n}$, where we now choose

$$p_n = \frac{p_{n-1}}{\alpha^2} - \frac{(2n-1)d}{2\alpha} - nd - 1.$$

Such weaker bounds on the events in $(\text{DS}: n, k, m_{L_k}, I_n)$ still imply exponential localization. However, strong HS-dynamical localization becomes weaker, namely one first needs to fix a power $s > 0$ and then choose an $\varepsilon_0(s, N, d, q_-, r_0)$ accordingly. In other words, the interval of dynamical localization gets smaller as s gets larger.

Here are the statements.

Theorem 3.6.1. *There exists $\varepsilon_0 = \varepsilon_0(N, d, q_-, r_0) > 0$ and $m > 0$ such that for a.e. ω the spectrum of $H^{(N)}(\omega)$ in $I = [Nq_-, Nq_- + \varepsilon_0]$ is pure point and the eigenfunctions corresponding to eigenvalues in I decay exponentially with mass m .*

Theorem 3.6.2. *For any $s > 0$, there exists $\varepsilon_0 = \varepsilon_0(s, N, d, q_-, r_0) > 0$ such that for $I = [Nq_-, Nq_- + \varepsilon_0]$, we have for any bounded $K \subset \Gamma^{(N)}$*

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|X^{s/2} f(H^{(N)}(\omega)) E_\omega(I) \chi_K\|_2^2 \right\} < \infty,$$

where $(X\psi)(\mathbf{x}) := |\mathbf{x}| \cdot \psi(\mathbf{x})$ for $\psi \in \mathcal{H}$, E_ω is the spectral projection of $H^{(N)}(\omega)$ and the supremum is taken over bounded Borel functions, $\|f\| := \|f\|_\infty$.

To prove Theorem 3.6.2, given $s > 0$, we choose p_1 sufficiently large to ensure that $2p_N \geq \max(6Nd + 2, \alpha s + \alpha Nd + \gamma + 1)$ and follow the same strategy (recall that γ is the parameter from generalized eigenfunctions). Since p_1 now depends on s , ε_0 also depends on s (see Corollary 2.6.4).

Chapter 4

Abstract Wegner estimates with applications

4.1 Introduction

Wegner estimates for random Schrödinger operators have been the subject of active research for the last three decades. Given a random self-adjoint operator $A(\omega)$ with a discrete spectrum $\{E_j(\omega)\}$ and a fixed interval I , the aim is to obtain good bounds on the average number of eigenvalues $E_j(\omega)$ in I . These estimates are named after Wegner's work [113].

Let us give a typical Wegner bound. Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, let $D \subset \mathbb{R}^d$ be a discrete non-empty set, and consider a probability space (Ω, \mathbb{P}) , where $\Omega = [q_-, q_+]^D$ for some $q_-, q_+ \in \mathbb{R}$. Given $\omega = (\omega_\alpha)_{\alpha \in D} \in \Omega$, let

$$H(\omega) = -\Delta + V_0 + \sum_{\alpha \in D} \omega_\alpha u_\alpha.$$

Here V_0 and all u_α are bounded real potentials.

Let $I \subset \mathbb{R}$ be an interval. Then a Wegner estimate, when valid, asserts the existence of some $C > 0$, $a \geq 1$ and $0 < b \leq 1$ such that for any cube $\Lambda \subset \mathbb{R}^d$, we have

$$(1-1) \quad \mathbb{E}\{\text{tr}[\chi_I(H_\Lambda(\omega))]\} \leq C \cdot |\Lambda|^a \cdot |I|^b.$$

Such estimates can be used in a proof of Anderson localization via multiscale analysis. If $a = 1$, they can also be used to study the continuity of the integrated density of states.

A large body of literature has been devoted to the proof of (1-1) under weaker and weaker assumptions on the model, i.e. on D , u_α and ω_α . Let us mention [58, 103, 56, 30, 20, 95, 66] in case $u_\alpha \geq 0$ for all α , and [69, 54, 111, 110, 88, 75, 38, 39] in case the u_α may change sign. A common assumption for all these papers is that the ω_α have continuous distributions.

4.2 Outline

The aim of this chapter is to derive some abstract Wegner bounds for some random self-adjoint operators on a Hilbert space, and to apply them afterwards for specific models. This approach proves to be rewarding, if only because it considerably shortens the proof of a Wegner bound for the model at hand. This is not the first attempt to provide abstract bounds; see [31] for a previous one.

Results. The abstract Wegner estimates are stated in Section 4.3 and applied in Section 4.4. We first obtain optimal bounds on the lattice and non-optimal bounds in the continuum. We allow the potential to be sparse, i.e. make no covering assumption. This includes models with surface and Delone potentials. We then give Wegner bounds for quantum graphs with random edge lengths or random vertex couplings. In each application, we allow the coupling constants entering the randomness to be correlated and only assume that their distributions have no atoms. A comparison with previous results is provided for each application. We conclude the paper with an appendix describing the spectra of Anderson models with half-space potentials. This illustrates the non-triviality of some of our bounds.

Notations. We assume the probability space has the form $(\Omega, \mathfrak{F}, \mathbb{P})$, where $\Omega = \mathcal{B}^{\mathcal{I}}$ for some Borel set $\mathcal{B} \subseteq \mathbb{R}$ and some countable index set \mathcal{I} . Here \mathbb{P} is a probability measure on Ω and $\mathfrak{F} = \otimes_{\alpha \in \mathcal{I}} \mathfrak{B}$, where \mathfrak{B} is the Borel σ -algebra of \mathcal{B} . By definition, \mathfrak{F} is generated by cylinder sets of the form $\{\omega = (\omega_\alpha) : \omega_{\alpha_1} \in A_1, \dots, \omega_{\alpha_n} \in A_n\}$, with $\alpha_j \in \mathcal{I}$ and $A_j \in \mathfrak{B}$. Any product space $\mathcal{B}^{\mathcal{I}}$ is assumed to be endowed with the σ -algebra $\mathfrak{F} = \otimes_{\alpha \in \mathcal{I}} \mathfrak{B}$, which we shall often omit¹.

Fix $\alpha \in \mathcal{I}$, let $Y_\alpha := \mathcal{B}^{\mathcal{I} \setminus \{\alpha\}}$, $\mathcal{Y}_\alpha := \otimes_{\beta \neq \alpha} \mathfrak{B}$ and denote $\hat{\omega}_\alpha := (\omega_\beta)_{\beta \neq \alpha}$. Define $\tau_\alpha : \Omega \rightarrow \mathcal{B} \times Y_\alpha$ by $\tau_\alpha : \omega \mapsto (\omega_\alpha, \hat{\omega}_\alpha)$. Then applying [13, Corollary 10.4.15] to $(\mathcal{B} \times Y_\alpha, \mathfrak{B} \otimes \mathcal{Y}_\alpha, \mathbb{P} \circ \tau_\alpha^{-1})$, we may find for each $\hat{\omega}_\alpha \in Y_\alpha$ a probability measure $\mu_{\hat{\omega}_\alpha}$ on $(\mathcal{B}, \mathfrak{B})$ such that, if $A \in \mathfrak{F}$ and $A_{\hat{\omega}_\alpha} := \{\omega_\alpha : (\omega_\alpha, \hat{\omega}_\alpha) \in \tau_\alpha(A)\}$, then the map $\hat{\omega}_\alpha \mapsto \mu_{\hat{\omega}_\alpha}(A_{\hat{\omega}_\alpha})$ is \mathcal{Y}_α -measurable and $\mathbb{P}(A) = \int_{Y_\alpha} \mu_{\hat{\omega}_\alpha}(A_{\hat{\omega}_\alpha}) d\mathbb{P}_{Y_\alpha}(\hat{\omega}_\alpha)$. Here $\mathbb{P}_{Y_\alpha} := \mathbb{P} \circ \pi_{Y_\alpha}^{-1}$, where $\pi_{Y_\alpha} : \omega \mapsto \hat{\omega}_\alpha$. The measures $\mu_{\hat{\omega}_\alpha}(B)$, for $B \subset \mathbb{R}$, are essentially regular versions of the conditional probability $\mathbb{P}\{\omega_\alpha \in B | \hat{\omega}_\alpha\}$. We will usually omit the identification map τ_α and simply regard elements of Ω as ordered pairs $(\omega_\alpha, \hat{\omega}_\alpha)$, so that $A_{\hat{\omega}_\alpha}$ is a section of A , $A_{\hat{\omega}_\alpha} = \{\omega_\alpha : (\omega_\alpha, \hat{\omega}_\alpha) \in A\}$.

Now fix a finite set $\mathcal{I}_F \subseteq \mathcal{I}$ (e.g. $\mathcal{I} = \mathbb{Z}^d$ and $\mathcal{I}_F \subset \mathbb{Z}^d$ a cube). We express our Wegner bounds in terms of the following modulus of continuity

$$(2-1) \quad s_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \mathbb{E}_{Y_\alpha} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_\alpha}(E, E + \varepsilon) \right\}.$$

We show in Section 4.6.2 that for any probability measure μ on \mathbb{R} ,

$$(2-2) \quad \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) = \sup_{E \in \mathbb{Q}} \mu(E, E + \varepsilon).$$

In particular, $\hat{\omega}_\alpha \mapsto \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_\alpha}(E, E + \varepsilon)$ is \mathcal{Y}_α -measurable, so $s_F(\mathbb{P}, \varepsilon)$ is well defined. We also verify that in the special case where $\mathbb{P} = \otimes_{\alpha \in \mathcal{I}} \mu_\alpha$ for some probability measures μ_α on \mathbb{R} , we have $s_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \sup_{E \in \mathbb{R}} \mu_\alpha(E, E + \varepsilon)$.

Remark. Our bounds are useful if the probability measure \mathbb{P} is continuous. If $\mathbb{P} = \otimes_{\alpha \in \mathcal{I}} \mu_\alpha$, it is sufficient for localization to have μ Hölder (or even log-Hölder) continuous. This, of course, encompasses the case where μ has a bounded density $\mu = \rho(\lambda) d\lambda$.

We will not treat here random Schrödinger operators with sign-indefinite single-site potentials. The reader can find some Wegner estimates for such models in [69], [54], [111], [110], [88] and [75], assuming the distribution μ of the (ω_α) has a density. See also the recent survey [38]. For sign-indefinite models on the lattice, the density assumption on μ

1. Note that if $V_\omega(\alpha)$ is a real-valued random field on \mathcal{I} , it can always be realized on this probability space in such a way that $V_\omega(\alpha) = \omega_\alpha$, i.e. there is no loss of generality in considering the coordinate process (ω_α) , as we do here. See [34, Pages 165-166] and [23, Chapter 2].

can be relaxed if the disorder is large; see [39, Theorem 1.2] and [40, Proposition 5.1] for a related result. For sign-indefinite models in the continuum however, there are to the best of our knowledge no Wegner bounds without the hypothesis that μ has a density.

Further developments. More abstract theorems appear in Chapter 5. In particular, we give in Section 5.5 an adaptation of the result of [30] to finite-dimensional spaces (which thus applies to discrete models). We believe this is instructive as the proof greatly simplifies in this context and provides explicit constants. We also give a few more applications in Section 5.3 and explain some difficulties in Section 5.4.

4.3 Abstract Theorems

In the following we give three abstract Wegner estimates. Theorem 4.3.2 is optimal, but is only valid for finite-dimensional spaces. It can be applied for example to discrete Schrödinger operators on finite cubes Λ , acting on $\ell^2(\Lambda)$. Theorems 4.3.3 and 4.3.4 on the other hand are valid in an arbitrary separable Hilbert space, but they are not optimal.

4.3.1 Finite-dimensional Hilbert spaces

Hypotheses (A)

- 1) We fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with $\Omega = \mathcal{B}^{\mathcal{I}}$ for some Borel set $\mathcal{B} \subseteq \mathbb{R}$, some countable index set \mathcal{I} , and fix a finite-dimensional Hilbert space \mathcal{H} .
- 2) $H(\omega)$ is a self-adjoint operator on \mathcal{H} for each $\omega \in \Omega$.
- 3) Fix a bounded interval I . There exist a constant $\gamma > 0$ and a self-adjoint operator W such that \mathbb{P} -almost surely,

$$\chi_I(H(\omega))W\chi_I(H(\omega)) \geq \gamma\chi_I(H(\omega)).$$

- 4) The operator W takes the form

$$W = \sum_{\alpha \in \mathcal{I}_F} U_\alpha,$$

for some finite set $\mathcal{I}_F \subseteq \mathcal{I}$, where the U_α are self-adjoint operators.

- 5) Fix an orthonormal basis $\{e_j\}_{j \in J}$ for \mathcal{H} . We define $\mathcal{I}_j := \{\alpha \in \mathcal{I}_F : U_\alpha e_j \neq 0\}$, $C_{\text{fin}} := \max_{j \in J} |\mathcal{I}_j|$ and $J_{\text{eff}} := \{j \in J : U_\alpha e_j \neq 0 \text{ for some } \alpha \in \mathcal{I}_F\}$.

Note that one may take $\mathcal{I} = \mathcal{I}_F = J$ and $W = \sum_{j \in J} P_j = \text{Id}$, where $P_j f := \langle f, e_j \rangle e_j$, in which case conditions 3 and 4 hold trivially on any interval with $\gamma = 1$ and $C_{\text{fin}} = 1$. For random Schrödinger operators, the U_α can be the single-site potentials. Condition 3 is sometimes called an *uncertainty principle*, and an efficient criterion to check its validity was established in [20]. The constant γ often depends on I .

The following proposition is the key idea for obtaining optimal Wegner bounds without covering assumptions. It decomposes the trace into local contributions of the U_α . The proof is given in Section 4.5.1.

Proposition 4.3.1. *Suppose that $H(\omega)$ satisfies Hypotheses (A) in the interval I . Then \mathbb{P} -almost surely,*

$$\text{tr}[\chi_I(H(\omega))] \leq \gamma^{-2} C_{\text{fin}} \sum_{j \in J_{\text{eff}}} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle.$$

For our first Wegner bound, we need one more hypothesis:

Hypothesis (B) $H(\omega)$ satisfies Hypotheses (A). Moreover, given $\omega = (\omega_\alpha)_{\alpha \in \mathcal{I}} \in \Omega$, $H(\omega)$ has the form

$$H(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha,$$

where H_1 is self-adjoint, $U_\alpha \geq 0$ and $\|U_\alpha\| \leq C_U$ for all α .

Hence, randomness must appear as an additive perturbation and the U_α must be positive operators. The proof of the next theorem is given in Section 4.5.2.

Theorem 4.3.2. *Suppose that $H(\omega)$ satisfies Hypotheses (A) and (B) in the interval I . Then $\text{tr}[\chi_I(H(\omega))]$ is measurable and*

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\text{eff}}| \cdot s_F(\mathbb{P}, |I|),$$

where $C_W := 6\gamma^{-2}C_U^2C_{\text{fin}}^2$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (2-1).

The fact that uncertainty principles imply Wegner bounds was first realized in [29] and [30]. There however, the authors considered the spectral projectors $\chi_I(H_1)$. It was later noticed in [105] that the arguments become simpler if one considers $\chi_I(H(\omega))$, and this idea was used again in [66] and [37].

It is worthwhile to note that if H_1 has the special form $H_1 = \sum_{\alpha \in \mathcal{I}_F} c_\alpha U_\alpha$, that is, if $H(\omega) = \sum_{\alpha \in \mathcal{I}_F} (c_\alpha + \omega_\alpha) U_\alpha$ for some bounded non-random constants c_α , then analogs of Proposition 4.3.1 and Theorem 4.3.2 hold for intervals not containing 0, without the need for an uncertainty principle². Such models arise when studying discrete acoustic operators on $\ell^2(\mathbb{Z}^d)$. We refer the reader to [65] for details.

4.3.2 Separable Hilbert spaces

We now work in the general setting.

Given $\mathcal{C} \subseteq \mathbb{R}$, we say that $f : \mathcal{C}^{\mathcal{I}} \rightarrow \mathbb{R}$ is *monotone increasing* (resp. *monotone decreasing*) if $v_\alpha \leq w_\alpha$ for all $\alpha \in \mathcal{I}$ implies $f(v) \leq f(w)$ (resp. $f(v) \geq f(w)$).

Hypotheses (C)

- 1) We fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with $\Omega = \mathcal{C}^{\mathcal{I}}$ for some interval $\mathcal{C} \subseteq \mathbb{R}$ and some countable index set \mathcal{I} . We assume \mathbb{P} has no atoms, more precisely $s_F(\mathbb{P}, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We also fix a separable Hilbert space \mathcal{H} .
- 2) $H(\omega)$ is a self-adjoint operator on \mathcal{H} for each $\omega \in \Omega$. It is bounded from below and has an orthonormal basis of eigenvectors, with eigenvalues $\lambda_1(\omega) \leq \lambda_2(\omega) \leq \dots$.
- 3) Fix an open interval I . There exists a number K independent of ω such that

$$n > K \implies \lambda_n(\omega) \notin I.$$

- 4) Either $\mathcal{D} := D(H(\omega))$ or $\mathcal{D} := D(\mathfrak{h}^\omega)$ is independent of ω , where \mathfrak{h}^ω is the form associated with $H(\omega)$. In the first case we define $f_u(\omega) := \langle H(\omega)u, u \rangle$, in the second case we define $f_u(\omega) := \mathfrak{h}^\omega[u]$, for $u \in \mathcal{D}$.
- 5) There exists a finite set $\mathcal{I}_F \subseteq \mathcal{I}$ such that $f_u(\omega)$ only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$. We denote by $\mathbf{1}_F$ the element $\mathbf{1}_F := (x_\alpha) \in \mathbb{R}^{\mathcal{I}}$ with $x_\alpha = 1$ for $\alpha \in \mathcal{I}_F$ and $x_\alpha = 0$ otherwise.

We also assume that there exists $\gamma > 0$ such that for every $u \in \mathcal{D}$, $f_u(\omega)$ satisfies one of the following properties for every $\omega \in \Omega$ and $t \geq 0$ such that $\omega - t \cdot \mathbf{1}_F \in \Omega$:

2. We prove this and much more in Section 5.2.

- a. f_u is monotone increasing and $f_u(\omega) - f_u(\omega - t \cdot \mathbf{1}_F) \geq t\gamma\|u\|^2$.
- b. $f_u \in C^1(\Omega)$, $\frac{\partial f_u(\omega)}{\partial \omega_\alpha} \geq 0 \forall \alpha \in \mathcal{I}_F$ and $\sum_{\alpha \in \mathcal{I}_F} \frac{\partial f_u(\omega)}{\partial \omega_\alpha} \geq \gamma\|u\|^2$.
- c. f_u is monotone decreasing and $f_u(\omega) - f_u(\omega - t \cdot \mathbf{1}_F) \leq -t\gamma\|u\|^2$.
- d. $f_u \in C^1(\Omega)$, $\frac{\partial f_u(\omega)}{\partial \omega_\alpha} \leq 0 \forall \alpha \in \mathcal{I}_F$ and $\sum_{\alpha \in \mathcal{I}_F} \frac{\partial f_u(\omega)}{\partial \omega_\alpha} \leq -\gamma\|u\|^2$.

By $f_u \in C^1(\Omega)$, we mean that the map $(\omega_\alpha)_{\alpha \in \mathcal{I}_F} \mapsto f_u((\omega_\alpha)_{\alpha \in \mathcal{I}_F}, 0)$ is continuous on $\mathcal{C}^{\mathcal{I}_F}$ and continuously differentiable on $\dot{\mathcal{C}}^{\mathcal{I}_F}$ (recall that f_u only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$).

Note that Wegner bounds formulated in terms of $s_F(\mathbb{P}, \varepsilon)$ are useful precisely when \mathbb{P} has no atoms. We need this assumption for technical reasons in Section 4.6.2. Condition 3 typically holds for any bounded $I \subset \mathbb{R}$. If \mathcal{H} is finite-dimensional, it is satisfied with $K = \dim \mathcal{H}$ (since there is no eigenvalue with $n > K$). For infinite-dimensional spaces, it is satisfied if $H(\omega)$ is bounded from below by a non random operator H_0 with a compact resolvent. In this case, K usually depends on I . The only ‘‘real’’ conditions are 4 and 5. Condition 5 assumes monotonicity and ‘‘diagonal covering’’ for $H(\omega)$. We remove the latter restriction in Theorem 4.3.4.

In the applications, it will be convenient that $H(\omega)$ is not supposed to have the form $H(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}} \omega_\alpha U_\alpha$, and that Hypothesis (C.5.b) is still sufficient to conclude. Let us state the theorem, see Section 4.5.3 for a proof, which is based on ideas from [103].

Theorem 4.3.3. *Suppose that $H(\omega)$ satisfies Hypotheses (C) in the interval I . Then $\text{tr}[\chi_I(H(\omega))]$ is $\mathfrak{F}_{\mathbb{P}}$ -measurable, where $\mathfrak{F}_{\mathbb{P}}$ denotes the \mathbb{P} -completion of \mathfrak{F} , and*

$$\overline{\mathbb{E}}\{\text{tr}[\chi_I(H(\omega))]\} \leq 2K \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{|I|}{\gamma}\right),$$

where $\overline{\mathbb{E}}$ denotes the extension of \mathbb{E} to $\mathfrak{F}_{\mathbb{P}}$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (2-1).

Note that we have $s_F(\mathbb{P}, \cdot)$ in the RHS; the quantity $s_F(\overline{\mathbb{P}}, \cdot)$ has not been defined. For the applications, classic arguments from [61] show that $\text{tr}[\chi_I(H(\omega))]$ is actually \mathfrak{F} -measurable, so that $\overline{\mathbb{E}}$ reduces to \mathbb{E} in the LHS.

For a random Schrödinger operator restricted to a cube Λ , the constant K comes, e.g. from a Weyl law and takes the form $C \cdot |\Lambda|$. The term $|\mathcal{I}_F|$ measures the contribution of the random potential in Λ , and will be approximately $|\Lambda|$ for standard single-particle systems. Hence, the upper bound is not linear in $|\Lambda|$.

There are mainly two applications for Wegner estimates: the first to prove localization via multiscale analysis, the second to study the continuity of the integrated density of states (IDS) of $H(\omega)$. For the first purpose, Theorem 4.3.3 is satisfactory because the term $s_F(\mathbb{P}, \frac{|I|}{\gamma})$ will be very small assuming $\mathbb{P} = \otimes \mu$ with μ (log-)Hölder continuous, so it will completely outweigh the terms K and $|\mathcal{I}_F|$. For the study of the IDS however, this theorem is not satisfactory.

It seems the ‘‘bad’’ term here is K . Indeed, for discrete models with sparse potentials supported in a set G , one expects $|\Lambda \cap G|$ in the upper bound (see Section 4.4.1), and this is precisely the term $|\mathcal{I}_F|$ in this case, not K which arguably will be $|\Lambda|$.

Theorem 4.3.2 also has the advantage of avoiding the diagonal cover in Hypothesis (C.5) by means of the uncertainty principle. As we show in Section 4.5.3, there is a related counterpart of this idea for Theorem 4.3.3. Namely, if one can prove that the eigenvalues $\lambda_n(\omega)$ of $H(\omega)$ are monotone increasing and satisfy

$$(\lambda_n(\omega) - \lambda_n(\omega - t \cdot \mathbf{1}_F))\chi_I(\lambda_n(\omega)) \geq t\gamma \cdot \chi_I(\lambda_n(\omega)),$$

then Theorem 4.3.3 is still valid if we only assume Hypotheses (C.1) to (C.3).

We finally give our last abstract theorem, which is probably the most original result of this section.

Theorem 4.3.4. *Suppose that $H(\omega)$ satisfies Hypotheses (C.1) to (C.4) in the interval $I = (E_1, E_2)$, where $E_2 < 0$. Assume moreover that $\Omega = [q_-, q_+]^{\mathcal{I}}$, fix $q > q_+$, and suppose that there exists $\zeta \neq 0$ such that for any $u \in \mathcal{D}$,*

$$(3-1) \quad f_u(\omega) = a(u) - \sum_{\alpha \in \mathcal{I}_F} (q - \omega_\alpha)^\zeta b_\alpha(u)$$

for some finite set $\mathcal{I}_F \subseteq \mathcal{I}$ and some constants $a(u) \geq 0$ and $b_\alpha(u) \geq 0$. Then $\text{tr}[\chi_I(H(\omega))]$ is $\mathfrak{F}_{\mathbb{P}}$ -measurable and

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq 2K \cdot |\mathcal{I}_F| \cdot s_F \left(\mathbb{P}, (q - q_-) \left(\left(1 + \frac{|I|}{|E_2|}\right)^{\frac{1}{|\zeta|}} - 1 \right) \right).$$

In particular, if $\zeta = \pm 1$, we have

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq 2K \cdot |\mathcal{I}_F| \cdot s_F \left(\mathbb{P}, \frac{q - q_-}{|E_2|} |I| \right).$$

The proof is given in Section 4.5.4. It uses an idea from [78], who roughly considered the case $a(u) \equiv 0$ and $\mathbb{P} = \otimes \mu_\alpha$ with $\mu_\alpha = \rho_\alpha(\lambda) d\lambda$. Both hypotheses were important to their proof, and we overcome this difficulty by generalizing ideas from [103]. Note however that we need (C.4), the argument of [78] holds under a weaker assumption on $D(H(\omega))$.

Of course, the main advantage here in comparison with Theorem 4.3.3 is that we only suppose $b_\alpha(u) \geq 0$. Theorem 4.3.3 would need a condition like $\sum_{\alpha \in \mathcal{I}_F} b_\alpha(u) \geq \gamma \|u\|^2$ for all u . The price to pay is that the bound only holds for specific intervals.

In the applications we shall only need the case $\zeta = 1$. However the greater generality does not require additional effort, and we believe it could be useful for models not considered here. For example, the case $\zeta = -2$ appears in the model of [78]³.

4.4 Applications

4.4.1 Discrete multi-particle models

Consider the Hilbert space $\ell^2(\mathbb{Z}^{nd})$, where $n \in \mathbb{N}^*$ represents the number of particles living in \mathbb{Z}^d . Let $\mathcal{B} \subseteq \mathbb{R}$ be a Borel set and consider a probability space (Ω, \mathbb{P}) , where $\Omega = \mathcal{B}^{\mathbb{Z}^d}$. Given $\omega = (\omega_\alpha) \in \Omega$, let

$$H(\omega) := H_0 + V^\omega, \quad H_0 := -\Delta + V_0,$$

where $-\Delta$ is the discrete Laplace operator on $\ell^2(\mathbb{Z}^{nd})$, V_0 is a real non-random potential (possibly an interaction) and for $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{Z}^d)^n \equiv \mathbb{Z}^{nd}$,

$$V^\omega(\mathbf{x}) = \sum_{1 \leq i \leq n} v^\omega(x_i) = \sum_{1 \leq i \leq n} \sum_{\alpha \in \mathbb{Z}^d} \omega_\alpha u_\alpha(x_i).$$

Since \mathbb{P} is arbitrary, the ω_α are allowed to be correlated. We assume V_0 is bounded and the $u_\alpha : \mathbb{Z}^d \rightarrow \mathbb{R}$ satisfy $0 \leq u_\alpha \leq C_u$ for some uniform $C_u \geq 0$. We also assume the u_α are compactly supported, that is, if for $j \in \mathbb{Z}^d$, $\mathbf{j} \in \mathbb{Z}^{nd}$ and $L \in \mathbb{N}$ we define the cubes

$$\Lambda_L^{(1)}(j) := \{x \in \mathbb{Z}^d : \|x - j\|_\infty \leq L\}, \quad \text{and} \quad \Lambda_L^{(n)}(\mathbf{j}) := \{\mathbf{x} \in \mathbb{Z}^{nd} : \|\mathbf{x} - \mathbf{j}\|_\infty \leq L\},$$

then we assume there exists an $R \geq 0$ such that $u_\alpha(j) = 0$ for all $j \notin \Lambda_R^{(1)}(\alpha)$.

As $\text{supp } u_\alpha$ is compact, we may interchange the sums and write

$$V^\omega = \sum_{\alpha \in \mathbb{Z}^d} \omega_\alpha U_\alpha, \quad \text{with} \quad U_\alpha(x_1, \dots, x_n) = \sum_{1 \leq i \leq n} u_\alpha(x_i).$$

3. There is an additional difficulty in this model however, namely Hypothesis (C.4) is not satisfied. See Section 5.4 for details.

Example. A simple and interesting case is when $n = 1$, a non-empty set $G \subset \mathbb{Z}^d$ is given, and $u_\alpha = \delta_\alpha$ for α inside G and $u_\alpha \equiv 0$ for α outside G , where δ_α is the characteristic function of $\{\alpha\}$. In this case,

$$H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha.$$

For instance, we may take $G = \mathbb{Z}^d \setminus \{0\}$, which gives rise to a non-covering situation. More generally, G could be a *Delone set* (i.e. $\exists K \geq 0$ such that $\forall j \in \mathbb{Z}^d$, the cube $\Lambda_K^{(1)}(j)$ contains at least one point of G) or a subspace $\mathbb{Z}^{d_1} \times \{0\}$ of \mathbb{Z}^d , $0 < d_1 < d$, in which case one speaks of *surface potentials*.

Discussion of the results.

- In the case of covering, i.e. $u_\alpha \geq c \cdot \delta_\alpha$ for some $c > 0$ and for all α , we have an optimal Wegner bound in any interval I . This extends [60, Theorem 2.1] and [68, Theorem 2.3] (because we neither assume that $u_\alpha = \delta_\alpha$ nor that $\mathbb{P} = \otimes \mu$ with $\mu = \rho(t)dt$) and improves [26, Theorem 1] (because our bound is linear in $|\Lambda|$). Note that the arguments of [68] actually allow for \mathbb{P} as general as ours. The multiscale analysis also requires two-volume Wegner bounds (cf. [68, Corollary 2.4]); we prove these in Section 5.3.2.
- If we have no covering and $\Omega = [q_-, q_+]^{\mathbb{Z}^d}$ with $q_- < 0$, i.e. the perturbation can be negative, we obtain Wegner bounds below $E_0 := \inf \sigma(H_0)$. This extends [63, Theorems 8,13], first because we make no regularity assumption on \mathbb{P} , second because our bound is optimal and valid for multi-particles. Our result also extends the optimal bound [64, Theorem 4.1] because we allow for general u_α and n .

But is there any spectrum below E_0 ? We show in Section 4.6.1 that if $n = 1$, if $G \subseteq \mathbb{Z}^d$ contains a half-space, if V_0 is periodic and if $H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha$, then $H(\omega)$ has a spectral interval below E_0 almost surely, provided that $\mathbb{P} = \otimes \mu$ and $\text{supp } \mu \supseteq [a, b]$, $a < b \leq 0$. This illustrates that our bound is indeed non-trivial⁴. The advantage here compared to the first item is, of course, the fact that we allow $G \neq \mathbb{Z}^d$.

- If we have no covering and $\Omega = [q_-, q_+]^{\mathbb{Z}^d}$ with $0 \leq q_-$, i.e. the perturbation is positive, we obtain optimal Wegner bounds below $E_q := \inf \sigma(H_0 + qW)$ for any $q > q_-$, where $W := \sum_\alpha U_\alpha$. But again, is there any spectrum below E_q ?

The recent preprints [37] and [94] have the advantage of giving a complete Wegner bound for some operators in this situation. Namely, the paper [37] assumes that $n = 1$, $H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha$, where $G \subseteq \mathbb{Z}^d$ is a Delone set, $\mathbb{P} = \otimes \mu_\alpha$ and $\text{supp } \mu_\alpha \subset [0, M]$. Under some condition (cf. [37, Eq.(1.13)]), the authors establish Wegner bounds for intervals near E_0 , and in contrast to our result, they show that these intervals contain some spectrum of $H(\omega)$ almost surely. A different proof for this Delone Wegner bound can be found in [94], in the special case where $V_0 \equiv 0$.

To conclude, let us mention that we can actually use the results of [37] to illustrate that our Wegner bound for positive perturbations is indeed interesting. Namely, if we take $t := q$ sufficiently large, then [37, Theorem 1.3] combined with [37, Proposition 1.5] asserts that the above Delone operator has $E_q > E_0$ and some spectrum in $[E_0, E_q)$ almost surely. Our Wegner bound is thus nontrivial for $I \subseteq [E_0, E_q)$.

4. Note that a single nonzero $\omega_\alpha u_\alpha$ actually suffices to create a spectral point below E_0 if $q_+ < q_*$, $q_* = q_*(\|H_0\|, u_\alpha)$. So our Wegner bound is also useful when the perturbation is highly negative and the spectral bottom is not an isolated point. This is likely to be the case if the operator is ergodic, e.g. $n = 1$ and $H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha$, with $G = (M\mathbb{Z})^d$ and V_0 M -periodic.

Boundary conditions. Let $\Lambda := \Lambda_L^{(n)}(\mathbf{x}) \subset \mathbb{Z}^{nd}$ be a cube. The *simple boundary conditions* are obtained by restricting the matrix of $H(\omega)$ to Λ , i.e. if $(e_j)_{j \in \mathbb{Z}^{nd}}$ is the standard basis of $\ell^2(\mathbb{Z}^{nd})$, then $H_\Lambda^S(\omega)(\mathbf{i}, \mathbf{j}) = \langle H(\omega)e_{\mathbf{i}}, e_{\mathbf{j}} \rangle$ if both $\mathbf{i}, \mathbf{j} \in \Lambda$ and $H_\Lambda^S(\omega)(\mathbf{i}, \mathbf{j}) = 0$ otherwise. Next, there are the *Dirichlet* $H_\Lambda^D(\omega)$ and *Neumann* $H_\Lambda^N(\omega)$ realizations, which were introduced in [101] to provide analogs for the lattice of the Dirichlet-Neumann bracketing; see [59, Section 5.2] for details. The only identity we need is [59, Eq. (5.42)], which asserts that if $H = -\Delta + V$, then $H \leq H_\Lambda^D \oplus H_{\Lambda^c}^D$. In particular, if $E_0 := \inf \sigma(H)$, $E_0^{\Lambda, D} := \inf \sigma(H_\Lambda^D)$ and $g \in \ell^2(\Lambda^c)$ is identically zero, then

$$E_0^{\Lambda, D} = \inf_{f \in \ell^2(\Lambda), \|f\|=1} \langle H_\Lambda^D f, f \rangle + \langle H_{\Lambda^c}^D g, g \rangle \geq \inf_{\varphi \in \ell^2(\mathbb{Z}^{nd}), \|\varphi\|=1} \langle H\varphi, \varphi \rangle = E_0,$$

i.e. Dirichlet boundary conditions *shift the spectrum up*.

The result. Let $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{Z}^d)^n$ and $\Lambda_L^{(n)}(\mathbf{x}) \subset \mathbb{Z}^{nd}$. Consider the Hilbert space $\mathcal{H} := \ell^2(\Lambda_L^{(n)}(\mathbf{x}))$ with standard basis $(e_j)_{j \in \Lambda_L^{(n)}(\mathbf{x})}$ and the operator $H_{\Lambda_L^{(n)}(\mathbf{x})}^\bullet(\omega)$, where $\bullet = S, D$ or N . Let

$$W := \sum_{\alpha \in \mathbb{Z}^d} U_\alpha, \quad \text{and} \quad W_{\Lambda_L^{(n)}(\mathbf{x})} := \sum_{\alpha \in \mathcal{I}_F} U_\alpha, \quad \text{where} \quad \mathcal{I}_F := \bigcup_{i=1}^n \Lambda_{L+R}^{(1)}(x_i).$$

We first show in Theorem 4.4.1 that uncertainty principles imply Wegner bounds, then we give in Lemma 4.4.2 concrete cases in which the uncertainty principle holds.

Theorem 4.4.1. *Let $\Lambda := \Lambda_L^{(n)}(\mathbf{x})$ be a cube and suppose $H_\Lambda^\bullet(\omega)$ satisfies*

$$(4-1) \quad \chi_I(H_\Lambda^\bullet(\omega)) W_\Lambda \chi_I(H_\Lambda^\bullet(\omega)) \geq \gamma \chi_I(H_\Lambda^\bullet(\omega)) \quad \mathbb{P}\text{-a.s.}$$

in an interval I , for some $\gamma > 0$. Then

$$\mathbb{E}\{\text{tr}[\chi_I(H_\Lambda^\bullet(\omega))]\} \leq C_W \cdot |\tilde{\Lambda}_L^{(n)}| \cdot s_F(\mathbb{P}, |I|),$$

where $C_W = 6n^4 \gamma^{-2} C_u^2 (2R+1)^{2d}$ and $\tilde{\Lambda}_L^{(n)} := \{\mathbf{j} \in \Lambda_L^{(n)}(\mathbf{x}) : U_\alpha e_{\mathbf{j}} \neq 0 \text{ for some } \alpha \in \mathcal{I}_F\}$.

If $u_\alpha = c_\alpha \delta_\alpha$ with $c_\alpha \geq 0$, then $C_u = \sup_{\alpha \in \mathcal{I}} c_\alpha$ and $R = 0$. If, moreover $n = 1$ and $H(\omega) = H_0 + \sum_{\alpha \in G} \omega_\alpha \delta_\alpha$, then $\tilde{\Lambda}_L^{(1)} = \Lambda_L^{(1)}(x) \cap G$.

Proof. $H_\Lambda^\bullet(\omega)$ is a self-adjoint operator given by $H_\Lambda^\bullet(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$, with $H_1 = H_{0, \Lambda}^\bullet$ self-adjoint. Moreover, $U_\alpha \geq 0$, $\|U_\alpha\| \leq C_U := nC_u$ and $\mathcal{I}_j := \{\alpha : U_\alpha e_{\mathbf{j}} \neq 0\} \subseteq \bigcup_{k=1}^n \Lambda_R^{(1)}(j_k)$, hence $C_{\text{fin}} := \max |\mathcal{I}_j| \leq n(2R+1)^d$. The claim now follows from Theorem 4.3.2. \square

Lemma 4.4.2. *Fix $\eta > 0$. The uncertainty principle (4-1) holds in any interval*

- (1) $I \subset \mathbb{R}$, if $\exists c > 0$ such that $u_\alpha \geq c \cdot \delta_\alpha$ for all α , with $\gamma = nc$.
- (2) $I \subset (-\infty, E_q - \eta]$, if $\Omega = [q_-, q_+]^{\mathbb{Z}^d}$, $q > q_-$ and $E_q := \inf \sigma(H_0 + qW)$, for the Dirichlet restriction H_Λ^D , with $\gamma \geq \frac{\eta}{q - q_-}$.

Theorem 4.4.1 combined with Lemma 4.4.2 thus provides a Wegner bound in either situation. If $q_- < 0$, we may take $q = 0$ and obtain a Wegner bound below $E_0 := \inf \sigma(H_0)$. Otherwise, $0 \leq q_- < q$, and the bound is interesting if $E_q > E_0$, for $I \subseteq [E_0, E_q)$.

Proof. For (1), note that if $u_\alpha \geq c \cdot \delta_\alpha$, then for any $\mathbf{y} \in \Lambda_L^{(n)}(\mathbf{x})$,

$$W_\Lambda(\mathbf{y}) \geq c \sum_{1 \leq i \leq n} \sum_{\alpha \in \mathcal{I}_F} \delta_\alpha(y_i) = c \sum_{1 \leq i \leq n} 1 = nc,$$

so that $W_\Lambda \geq nc$ and (4-1) holds trivially in any interval with $\gamma = nc$.

For (2), let $H_q := H_0 + qW$ and given $\omega \in \Omega$, let $\lambda_\omega(t) := \inf \sigma(H_\Lambda^D(\omega) + tW_\Lambda)$. Then for any $t \geq q - q_-$ we have

$$\lambda_\omega(t) = \inf \sigma(H_{q,\Lambda}^D + V_\Lambda^\omega + (t - q)W_\Lambda) \geq \inf \sigma\left(H_q + \sum_{\alpha \in \mathbb{Z}^d} (\omega_\alpha + t - q)U_\alpha\right) \geq E_q$$

where we used the fact that Dirichlet boundary conditions shift the spectrum up⁵. Thus, if $I \subset (-\infty, E_q - \eta]$, we get $\lambda_\omega(q - q_-) - \max I \geq \eta$. By [20, Theorem 1.1], (4-1) thus holds in I with $\gamma \geq \frac{\eta}{q - q_-}$. \square

4.4.2 Continuum multi-particle models

Consider the Hilbert space $L^2(\mathbb{R}^{nd})$, where $n \in \mathbb{N}^*$ represents the number of particles living in \mathbb{R}^d . Let $G \subset \mathbb{R}^d$ be a discrete non-empty set such that $\#\{\Lambda \cap G\} < \infty$ for any bounded $\Lambda \subset \mathbb{R}^d$ and consider a probability space (Ω, \mathbb{P}) , where $\Omega := [q_-, q_+]^G$ and \mathbb{P} has no atoms. Given $\omega = (\omega_\alpha) \in \Omega$, let

$$H(\omega) = H_0 + V^\omega, \quad H_0 := -\Delta + V_0,$$

where V_0 is a bounded real non-random potential. We can consider more general H_0 ; we only need $H_{0,\Lambda}$ to satisfy a Weyl law, and this is true for $H_0 = (-i\nabla - A)^2 + V_0$ with weak conditions on A and V_0 ; see [56, Lemma 5]. Given $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \equiv \mathbb{R}^{nd}$,

$$V^\omega(\mathbf{x}) = \sum_{1 \leq i \leq n} v^\omega(x_i) = \sum_{1 \leq i \leq n} \sum_{\alpha \in G} \omega_\alpha u_\alpha(x_i).$$

Let $\Lambda_L^{(n)}(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^{nd} : \|\mathbf{y} - \mathbf{x}\|_\infty < L\}$. We assume the $u_\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy $0 \leq u_\alpha \leq C_u$ for some uniform $C_u > 0$ and $\text{supp } u_\alpha \subset \Lambda_R^{(1)}(\alpha)$ for some $R > 0$ independent of α . This model encompasses sparse potentials such as Delone and surface potentials. Now put

$$U_\alpha(y_1, \dots, y_n) := \sum_{i=1}^n u_\alpha(y_i) \quad \text{and} \quad W := \sum_{\alpha \in G} U_\alpha.$$

Given $\mathbf{z} = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ and a cube $\Lambda_L^{(n)}(\mathbf{z}) \subset \mathbb{R}^{nd}$, let $H_{\Lambda_L^{(n)}(\mathbf{z})}^\bullet(\omega)$ be a restriction of $H(\omega)$ acting on $\mathcal{H} := L^2(\Lambda_L^{(n)}(\mathbf{z}))$, with $\bullet = D, N, \text{per}$. Note that without a growth condition on G , $H(\omega)$ may not be self-adjoint (cf. [63]), but here we are only concerned with its restriction, which is self-adjoint.

Discussion of the results. Our bounds are not linear in $|\Lambda|$, but may be used for localization.

5. To see this in detail, let $f \in \ell^2(\Lambda)$, $\|f\| = 1$ and $g \in \ell^2(\Lambda^c)$, g identically zero. Then $\langle (V_\Lambda^\omega + tW_\Lambda)f, f \rangle = \langle (V^\omega + tW)f \oplus g, f \oplus g \rangle$, so by the bracketing, $\langle H_\Lambda^D(\omega)f, f \rangle + t\langle W_\Lambda f, f \rangle \geq \langle H_0 f \oplus g, f \oplus g \rangle + \langle (V^\omega + tW)f \oplus g, f \oplus g \rangle \geq \langle H_q f \oplus g, f \oplus g \rangle \geq E_q$.

- The covering situation, i.e. when $G = \mathbb{Z}^d$ and $u_\alpha \geq c\chi_\alpha$ for all α , where χ_α is the characteristic function of $[\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]^d$, has already been analyzed in [18] and [73]. There the authors proved Wegner bounds in any interval $I \subset \mathbb{R}$ and for arbitrary \mathbb{P} . We do the same here, simply to illustrate Theorem 4.3.3.
- For negative perturbations, we have a Wegner bound below $E_0 := \inf \sigma(H_0)$. This extends [63, Theorems 8,13] because we do not impose any regularity on \mathbb{P} , but [63] has some Wegner bounds which depend linearly on $|\Lambda|$. In the case of surface potentials, i.e. when $G = \mathbb{Z}^{d_1} \times \{0\}$, [64, Theorem 2.1] provides an optimal bound. As in the lattice, there is the issue of whether $H(\omega)$ has some spectrum below E_0 . We show in the Appendix (Section 4.6.1) that continuum single-particle operators with half-space potentials are good examples of operators which have no covering condition and to which we have a non-trivial Wegner bound.
- For positive perturbations, we obtain a Wegner bound below $E_q := \inf \sigma(H_0 + qW)$, for any $q > q_+$. This result is very close in spirit to [20, Theorem 2.1], because both are interesting when E_0 is a *weak fluctuation boundary*, i.e. when $E_0 < E_q$. Besides the fact that we allow for multi-particles⁶, note that our proof is quite elementary. On the other hand, [20] builds on the results of [30], which are technically involved, but they provide an optimal bound.

In contrast to the lattice, the question of whether there are interesting operators for which $E_0 < E_q$ is well established in the continuum when $n = 1$. Already in [62, Theorem 2.2], it is shown that if $E(t) := \inf \sigma(H_0 + tW)$, then $E(t) - E_0$ grows linearly in t , even if the u_α have small support, provided $G = \mathbb{Z}^d$ and V_0 is periodic. It was later shown in [21, Sections 4,5] that $E_q > E_0$ for more general operators with surface or Delone potentials, assuming V_0 is periodic. In the case of Delone potentials, this result was very recently improved in [66, Lemma 4.2], namely, it is shown that $E(t) - E_0$ grows linearly in t , and V_0 is no longer assumed to be periodic.

Much stronger results are known if $n = 1$, G is a Delone set, each $u_\alpha > 0$ in an open set and $\mathbb{P} = \otimes_{\alpha \in G} \mu_\alpha$. Namely, the Wegner bound of [95], which was improved in [66], is valid for any small interval, not just intervals near the spectral bottom. The result of [66] also extends the one of [30] who considered $G = \mathbb{Z}^d$, but relies on it.

Theorem 4.4.3. *For any $I = (E_1, E_2)$, there exists $C_W > 0$ such that for any cube $\Lambda_L^{(n)}(\mathbf{x})$,*

- (1) *If $G = \mathbb{Z}^d$ and $\exists c > 0$ with $u_\alpha \geq c \cdot \chi_\alpha$ for all α , where $\chi_\alpha := \chi_{[\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]^d}$, then*

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L^{(n)}(\mathbf{x})}^\bullet)]\} \leq C_W \cdot |\Lambda_L^{(n)}(\mathbf{x})| \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{|I|}{nc}\right),$$

where $\mathcal{I}_F := \left(\bigcup_{j=1}^n \Lambda_{L+R}^{(1)}(x_j)\right) \cap G$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (2-1).

- (2) *Otherwise, for any $q > q_+$, if $E_2 < E_q := \inf \sigma(H_0 + qW)$, then*

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L^{(n)}(\mathbf{x})}^D)]\} \leq C_W \cdot |\Lambda_L^{(n)}(\mathbf{x})| \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{q - q_-}{E_q - E_2} |I|\right).$$

Here $C_W = C_W(nd, E_2, v_0)$ if $q_- \geq 0$ and $C_W = C_W(nd, E_2, v_0, nq - C_u R)$ otherwise, where $v_0 = \inf V_0$.

If $q_+ < 0$, i.e. the perturbation is negative, we may take $q = 0$ and obtain a Wegner bound below $E_0 := \inf \sigma(H_0)$. Otherwise, $0 \leq q_+ < q$ and $E_q > E_0$, for many models.

⁶. Let us mention here that there is a work in progress by Hislop and Klopp in which an optimal Wegner estimate is derived for some non-covering multi-particle Hamiltonians.

Proof. Let $\Lambda := \Lambda_L^{(n)}(\mathbf{x})$. For (1), note that $H_\Lambda^\bullet(\omega)$ is a self-adjoint operator given by $H_\Lambda^\bullet(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$, where $H_1 := H_{0,\Lambda}^\bullet$. Given $u \in D(H_\Lambda^\bullet)$, if $f_u(\omega) = \langle H_\Lambda^\bullet(\omega)u, u \rangle$, then f_u is monotone increasing since $U_\alpha \geq 0$. Moreover, if $\mathbf{y} \in \Lambda_L^{(n)}(\mathbf{x})$, then $W_\Lambda(\mathbf{y}) := \sum_{\alpha \in \mathcal{I}_F} U_\alpha(\mathbf{y}) \geq c \sum_{1 \leq i \leq n} \sum_{\alpha \in \mathcal{I}_F} \chi_\alpha(y_i) = nc$. Hence, $f_u(\omega + t \cdot \mathbf{1}_F) - f_u(\omega) = t \langle W_\Lambda u, u \rangle \geq nct \|u\|$. Hypotheses (C) are thus satisfied with $\gamma = nc$, a Weyl constant $K = C|\Lambda|$, and the claim follows from Theorem 4.3.3.

For (2), let $A(\omega) := H_\Lambda^D(\omega) - E_q$ and $I' = (E_1 - E_q, E_2 - E_q)$. Then $\chi_{I'}(\lambda) = \chi_{I'}(\lambda - E_q)$, hence $\mathbb{E}\{\text{tr}[\chi_{I'}(H_\Lambda^D(\omega))]\} = \mathbb{E}\{\text{tr}[\chi_{I'}(A(\omega))]\}$.

Now $A(\omega)$ is self-adjoint and $A(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}_F} (\omega_\alpha - q)U_\alpha$, where $H_1 := H_{0,\Lambda}^D + qW_\Lambda - E_q$. Since Dirichlet boundary conditions shift the spectrum up, we have $H_1 \geq 0$. Thus, $A(\omega)$ satisfies the hypotheses of Theorem 4.3.4 in I' with $\zeta = 1$, a Weyl constant $K = C|\Lambda|$, and the claim follows since $|I'| = |I|$. \square

4.4.3 Quantum graphs with random edge length

Consider the metric graph $(\mathcal{E}, \mathcal{V})$ with vertex set $\mathcal{V} = \mathbb{Z}^d$ and edge set $\mathcal{E} = \{(m, m+h_j) : m \in \mathbb{Z}^d, j = 1, \dots, d\}$, where $(h_j)_{j=1}^d$ is the standard basis of \mathbb{Z}^d . Each edge $e = (v, v')$ has an initial vertex $\iota e = v$ and a terminal vertex $\tau e = v'$. Now fix $0 < l_{\min} < l_{\max} < \infty$ and let (Ω, \mathbb{P}) be a probability space, where $\Omega := [l_{\min}, l_{\max}]^\mathcal{E}$ and \mathbb{P} has no atoms. Given $l^\omega = (l_e^\omega) \in \Omega$, we identify each edge e with $[0, l_e^\omega]$, such that ιe and τe correspond to 0 and l_e^ω , respectively, and consider the Hilbert space $\mathcal{H} := \bigoplus_{e \in \mathcal{E}} L^2[0, l_e^\omega]$. Fix $\alpha \in \mathbb{R}$ and define the operator

$$H(l^\omega, \alpha) : (f_e) \mapsto (-f_e''),$$

$$D(H(l^\omega, \alpha)) := \left\{ f = (f_e) \in \bigoplus_{e \in \mathcal{E}} W^{2,2}(0, l_e^\omega) \left| \begin{array}{l} f \text{ is continuous at each} \\ v \in \mathcal{V} \text{ and } f'(v) = \alpha f(v). \end{array} \right. \right\}.$$

By continuity at v , we mean that if $\tau e = \iota b = v$, then $f_e(l_e^\omega) = f_b(0) =: f(v)$. Here $f'(v) := \sum_{e: \iota e=v} f_e'(0) - \sum_{e: \tau e=v} f_e'(l_e^\omega)$.

Given $L \in \mathbb{N}^*$, let $\Lambda_L := \{e \in \mathcal{E} : \|\iota e\|_\infty \leq L \text{ or } \|\tau e\|_\infty \leq L\}$ be a cube and put $\mathcal{V}_{\Lambda_L} := \{\iota e : e \in \Lambda_L\} \cup \{\tau e : e \in \Lambda_L\}$. This yields a graph $(\Lambda_L, \mathcal{V}_{\Lambda_L})$ and a corresponding operator $H_{\Lambda_L}(l^\omega, \alpha)$. We denote $H_{\Lambda_L}^\omega(\alpha) := H_{\Lambda_L}(l^\omega, \alpha)$.

Theorem 4.4.4. *Let $I \subset (0, \infty)$ be an interval such that $\bar{I} \cap D_0 = \emptyset$, where $D_0 := \bigcup_{k \in \mathbb{Z}} \left[\frac{\pi^2 k^2}{l_{\max}^2}, \frac{\pi^2 k^2}{l_{\min}^2} \right]$. Then there exists $c_1 = c_1(d)$ and $c_2 = c_2(I) > 0$ such that for any interval $J \subset I$ and any cube Λ we have*

$$\mathbb{P}\{\sigma(H_\Lambda^\omega(\alpha)) \cap J \neq \emptyset\} \leq c_1 \cdot |\Lambda|^2 \cdot s_F(\mathbb{P}, c_2|J|),$$

where $|\Lambda|$ is the number of edges in $\mathcal{I}_F := \Lambda$ and $s_F(\mathbb{P}, \varepsilon)$ is as in (2-1).

Previous estimates appeared in [78] and [72], both assumed that $\mathbb{P} = \otimes_{e \in \mathcal{E}} \mu_e$, with $\mu_e = h_e(\lambda) d\lambda$, but their bounds were linear in $|\Lambda|$. Our proof heavily relies on the analysis of [72]. Our point here is twofold: first, if one makes use of the black box Theorem 4.3.3, then a large part of the proof of [72] can be omitted, second this allows to extend their localization results in case $\alpha > 0$ to measures μ_e which are (log-)Hölder continuous.

Proof. It is proved in [72, Eq. (9)-(14)], by spectral analytic arguments and without any assumption on \mathbb{P} , that if E_J is the midpoint of J , then there exists a discrete random self-adjoint operator $M_\Lambda(l^\omega, E_J)$ acting on $\ell^2(\mathcal{V}_\Lambda)$ and $b > 0$ such that

$$\mathbb{P}\{\sigma(H_\Lambda^\omega(\alpha)) \cap J \neq \emptyset\} \leq \mathbb{P}\{\text{dist}(\sigma(M_\Lambda(l^\omega, E_J)), \alpha) \leq b|J|\}.$$

Moreover, given $u \in \ell^2(\mathcal{V}_\Lambda)$, the map $l^\omega \mapsto f_u(l^\omega) := \langle M_\Lambda(l^\omega, E_J)u, u \rangle$ is in $C^1(\Omega)$, only depends on $(l_e^\omega)_{e \in \Lambda}$ and there exists $\beta > 0$ such that

- (i) $\frac{\partial M_\Lambda(l^\omega, E_J)}{\partial l_e^\omega} \geq \beta \cdot I^e$ for all $e \in \Lambda$, where $I^e f(v) = f(v)$ if $v \in \{\iota e, \tau e\}$ and $I^e f(v) = 0$ otherwise,
- (ii) and $\sum_{e \in \Lambda} \frac{\partial M_\Lambda(l^\omega, E_J)}{\partial l_e^\omega} \geq \beta \cdot \sum_{e \in \Lambda} I^e \geq \beta \cdot \text{Id}_{\ell^2(\mathcal{V}_\Lambda)}$.

Thus, $\frac{\partial f_u(l^\omega)}{\partial l_e^\omega} \geq \beta(|u(\iota e)|^2 + |u(\tau e)|^2) \geq 0$ for $e \in \Lambda$ and $\sum_{e \in \Lambda} \frac{\partial f_u(l^\omega)}{\partial l_e^\omega} \geq \beta \cdot \|u\|^2$. Hence $M_\Lambda(l^\omega, E_J)$ satisfies Hypothesis (C.5.b). Since $\ell^2(\mathcal{V}_\Lambda)$ is finite-dimensional, the rest of Hypotheses (C) are clearly satisfied with $\mathcal{I}_F = \Lambda$ and $K = |\mathcal{V}_\Lambda| \leq c_d |\Lambda|$. We may thus apply Theorem 4.3.3 and Markov inequality to get

$$\begin{aligned} \mathbb{P} \{ \text{dist}(\sigma(M_\Lambda(l^\omega, E_J)), \alpha) \leq b|J| \} &= \mathbb{P} \{ \text{tr} \chi_{[\alpha-b|J|, \alpha+b|J|]}(M_\Lambda(l^\omega, E_J)) \geq 1 \} \\ &\leq 2c_d |\Lambda|^2 s_F \left(\mathbb{P}, \frac{2b}{\beta} |J| \right). \end{aligned} \quad \square$$

4.4.4 Quantum graphs with random vertex coupling

We finally show that Theorem 4.3.4 can tackle random vertex coupling models without any analytic effort. It seems there are no previous Wegner estimates for such models.

For simplicity consider the graph $(\mathcal{E}, \mathcal{V})$ given by $\mathcal{V} = \mathbb{Z}^d$ and \mathcal{E} the set of segments $e = (v, v')$ between neighbouring vertices, assigned lengths l_e with $l_{\min} \leq l_e \leq l_{\max}$. More general structures can be treated similarly. Given $e = (v, v')$, we put $\iota e = v$ and $\tau e = v'$.

Fix $\alpha_-, \alpha_+ \in \mathbb{R}$, $\alpha_- < \alpha_+$ and $\emptyset \neq G \subseteq \mathcal{V}$. Let (Ω, \mathbb{P}) be a probability space, where $\Omega = [\alpha_-, \alpha_+]^G$, \mathbb{P} has no atoms and let $\mathcal{H} = \bigoplus_{e \in \mathcal{E}} L^2[0, l_e]$. Let $V = (V_e)$ be a bounded real potential, $c_0 := \inf V$, and given $\alpha^\omega = (\alpha_v^\omega) \in \Omega$, consider the operator

$$H(\alpha^\omega) : (f_e) \mapsto (-f_e'' + V_e f_e),$$

acting on $(f_e) \in \bigoplus_{e \in \mathcal{E}} W^{2,2}(0, l_e)$ which are continuous at all vertices, i.e. $f_e(l_e) = f_b(0) =: f(v)$ if $\tau e = \iota b = v$, and which satisfy

$$f'(v) := \sum_{e: \iota e=v} f_e'(0) - \sum_{e: \tau e=v} f_e'(l_e) = \begin{cases} \alpha_v^\omega f(v) & \text{if } v \in G, \\ 0 & \text{otherwise.} \end{cases}$$

The authors in [71] studied the case $G = \mathcal{V}$ and established localization for high disorder and near spectral edges using the fractional moments method (which does not rely on Wegner bounds). Their idea was to reduce the problem to one on $\ell^2(\mathcal{V})$, for an associated discrete operator. Below we prove a direct Wegner bound instead.

Given $\Lambda \subseteq \mathcal{E}$, let $\mathcal{V}_\Lambda := \{\iota e : e \in \Lambda\} \cup \{\tau e : e \in \Lambda\}$ and $\partial\Lambda := \mathcal{V}_\Lambda \cap \mathcal{V}_{\Lambda^c}$. Consider the form

$$\mathfrak{h}_\Lambda^{\omega, \text{D}}[f] = \sum_{e \in \Lambda} (\|f_e'\|_{L^2(0, l_e)}^2 + \langle V_e f_e, f_e \rangle_{L^2(0, l_e)}) + \sum_{v \in G \cap \mathcal{V}_\Lambda} \alpha_v^\omega |f(v)|^2$$

acting on $(f_e) \in \bigoplus_{e \in \Lambda} W^{1,2}(0, l_e)$ which are continuous at $v \in \mathcal{V}_\Lambda \setminus \partial\Lambda$ and vanish at $v \in \partial\Lambda$. Note that $\partial\Lambda$ is empty if $\Lambda = \mathcal{E}$. It is known (see [76] or [78, Lemma 4.1]) that $\mathfrak{h}_\Lambda^{\omega, \text{D}}$ is closed and bounded from below, and thus corresponds to a self-adjoint operator $H_\Lambda^{\text{D}}(\alpha^\omega)$. Moreover, $H(\alpha^\omega) = H_\mathcal{E}^{\text{D}}(\alpha^\omega)$, so we denote $\mathfrak{h}^\omega := \mathfrak{h}_\mathcal{E}^{\omega, \text{D}}$.

Lemma 4.4.5. *For any $\Lambda \subseteq \mathcal{E}$, $H(\alpha^\omega) \leq H_\Lambda^{\text{D}}(\alpha^\omega) \oplus H_{\Lambda^c}^{\text{D}}(\alpha^\omega)$. If Λ is finite, $H_\Lambda^{\text{D}}(\alpha^\omega)$ has a compact resolvent. Its eigenvalues, denoted $E_1^{\Lambda, \text{D}} \leq E_2^{\Lambda, \text{D}} \leq \dots$ counting multiplicity, satisfy the following Weyl law: for any $S \in \mathbb{R}$, there exists a non-random $C = C(S, c_0, \alpha_-, l_{\min}, l_{\max})$ such that $E_j^{\Lambda, \text{D}} > S$ if $j > C \cdot |\Lambda|$, where $|\Lambda|$ is the number of edges in Λ .*

Proof. The bracketing result follows [78, Lemma 4.2], namely, $D(\mathfrak{h}_\Lambda^{\omega, \text{D}}) \oplus D(\mathfrak{h}_{\Lambda^c}^{\omega, \text{D}}) \subset D(\mathfrak{h}^\omega)$ since a function in $D(\mathfrak{h}_\Lambda^{\omega, \text{D}}) \oplus D(\mathfrak{h}_{\Lambda^c}^{\omega, \text{D}})$ is automatically continuous at all v . Moreover, if $f = f_1 \oplus f_2 \in D(\mathfrak{h}_\Lambda^{\omega, \text{D}}) \oplus D(\mathfrak{h}_{\Lambda^c}^{\omega, \text{D}})$, then $\mathfrak{h}_\Lambda^{\omega, \text{D}}[f_1] + \mathfrak{h}_{\Lambda^c}^{\omega, \text{D}}[f_2] = \mathfrak{h}^\omega[f]$ because $f(v) = 0$ on boundary vertices. Thus, $H \leq H_\Lambda^{\text{D}} \oplus H_{\Lambda^c}^{\text{D}}$.

Now suppose Λ is finite and as in [42], consider the Neumann-decoupled Laplacian $-\Delta_\Lambda^{\text{dec}, \text{N}}$ defined via the form $\mathfrak{k}[f] = \sum_{e \in \Lambda} \|f'_e\|_{L^2[0, l_e]}^2$ with $D(\mathfrak{k}) = \bigoplus_{e \in \Lambda} W^{1,2}(0, l_e)$. Then $D(\mathfrak{h}_\Lambda^{\omega, \text{D}}) \subset D(\mathfrak{k})$ and $\mathfrak{h}_\Lambda^{\omega, \text{D}}[f] \geq \mathfrak{k}[f] + c_0 \|f\|^2 + \alpha_- \sum_{v \in G \cap \mathcal{V}_\Lambda} |f(v)|^2 \geq \frac{1}{2}((\mathfrak{k} + C)[f])$ for some $C = C(l_{\min}, l_{\max}, \alpha_-, c_0)$ by standard trace estimates, see e.g. [76, Lemma 8]. Thus, $H_\Lambda^{\text{D}}(\alpha^\omega) \geq \frac{1}{2}(-\Delta_\Lambda^{\text{dec}, \text{N}} + C)$. But since $-\Delta_\Lambda^{\text{dec}, \text{N}} = \bigoplus_{e \in \Lambda} -\Delta_{(0, l_e)}^{\text{N}}$, its eigenvalues $E_j^{\text{dec}, \Lambda}$ are just the eigenvalues $E_k(-\Delta_{(0, l_e)}^{\text{N}}) = \frac{\pi^2 k^2}{4l_e^2}$ with multiplicity $|\Lambda|$. In particular, $E_j^{\text{dec}, \Lambda} \rightarrow \infty$ as $j \rightarrow \infty$, hence $E_j^{\Lambda, \text{D}} \rightarrow \infty$ as $j \rightarrow \infty$ and $H_\Lambda^{\text{D}}(\alpha^\omega)$ has a compact resolvent by [92, Theorem XIII.64]. Moreover, we have $E_j^{\Lambda, \text{D}} \geq \frac{1}{2}(E_j^{\text{dec}, \Lambda} + C)$. By the explicit form of $E_j^{\text{dec}, \Lambda}$, we know that $E_j^{\text{dec}, \Lambda} > 2S - C$ if $j > C_2 |\Lambda|$ for some $C_2 = C_2(l_{\max}, S, C)$. Thus, $E_j^{\Lambda, \text{D}} > S$ if $j > C_2 |\Lambda|$ and we are done. \square

We may now state our Wegner bound. Fix $q > \alpha_+$ and let H_0, H_q be the operators corresponding to $\mathfrak{h}_0[f] = \sum_{e \in \mathcal{E}} (\|f'_e\|_{L^2[0, l_e]}^2 + \langle V_e f_e, f_e \rangle)$ and $\mathfrak{h}_q[f] = \mathfrak{h}_0[f] + q \sum_{v \in G} |f(v)|^2$ respectively, with $D(\mathfrak{h}_0) = D(\mathfrak{h}_q) = D(\mathfrak{h}^\omega)$. Let $\mathcal{I}_F := G \cap \mathcal{V}_\Lambda$ and $s_F(\mathbb{P}, \varepsilon)$ as in (2-1).

Theorem 4.4.6. *Let $I = (E_1, E_2)$ be an open interval.*

There exists $C_W = C_W(E_2, c_0, \alpha_-, l_{\min}, l_{\max}) > 0$ such that for any finite $\Lambda \subset \mathcal{E}$ and any $q > \alpha_+$, if $E_2 < E_q := \inf \sigma(H_q)$, then

$$\mathbb{E}\{\text{tr}[\chi_I(H_\Lambda^{\text{D}}(\alpha^\omega))]\} \leq C_W \cdot |\Lambda| \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{q - \alpha_-}{E_q - E_2} |I|\right).$$

If $\alpha_+ < 0$, we may take $q = 0$ and obtain a Wegner bound below $E_0 := \inf(\sigma(H_0))$. This result is non-trivial at least when $G = \mathbb{Z}^d$ and the disorder is high, because $H(\alpha^\omega)$ will have some spectrum below E_0 in this case almost surely; see [71, Theorem 12] and the remark thereafter. If $\alpha_+ \geq 0$, the non-triviality will be ensured if $E_0 < E_q$.

Proof. Let $A(\omega) := H_\Lambda^{\text{D}}(\alpha^\omega) - E_q$ and $I' = (E_1 - E_q, E_2 - E_q)$. Then $\chi_I(\lambda) = \chi_{I'}(\lambda - E_q)$, hence $\mathbb{E}\{\text{tr}[\chi_I(H_\Lambda^{\text{D}}(\alpha^\omega))]\} = \mathbb{E}\{\text{tr}[\chi_{I'}(A(\omega))]\}$. Moreover, $A(\omega)$ corresponds to the form $\mathfrak{a}^\omega[f] = (\mathfrak{h}_\Lambda^{\omega, \text{D}} - E_q)[f]$ with $\mathcal{D} := D(\mathfrak{a}^\omega) = D(\mathfrak{h}_\Lambda^{\omega, \text{D}})$ non-random, and we have $\mathfrak{a}^\omega[f] = \mathfrak{h}_1[f] + \sum_{v \in G \cap \mathcal{V}_\Lambda} (\alpha_v^\omega - q) |f(v)|^2$, where $\mathfrak{h}_1 := \mathfrak{h}_{q, \Lambda}^{\text{D}} - E_q$. By the bracketing in Lemma 4.4.5, we have $\mathfrak{h}_1 \geq 0$. Thus, $A(\omega)$ satisfies Hypotheses (C.1) to (C.4) in I' , with a Weyl constant $K = C|\Lambda|$ from Lemma 4.4.5, and the claim follows from Theorem 4.3.4. \square

4.5 Proofs of the general theorems

4.5.1 Proof of Proposition 4.3.1

Proof. Put $\chi_I := \chi_I(H(\omega))$. By hypothesis, for a.e. ω ,

$$(5-1) \quad \text{tr}[\chi_I] \leq \gamma^{-1} \text{tr}[\chi_I W \chi_I] = \gamma^{-1} \text{tr}[\chi_I W].$$

Given $j \in J$ we have

$$\begin{aligned} \langle \chi_I W e_j, e_j \rangle &= \langle \chi_I W e_j, \chi_I e_j \rangle \leq \|\chi_I W e_j\| \cdot \|\chi_I e_j\| \\ &\leq \frac{c}{2} \|\chi_I W e_j\|^2 + \frac{1}{2c} \|\chi_I e_j\|^2 = \frac{c}{2} \langle W \chi_I W e_j, e_j \rangle + \frac{1}{2c} \langle \chi_I e_j, e_j \rangle \end{aligned}$$

for any $c > 0$. Summing over $j \in J$ and choosing $c = \gamma^{-1}$ we get by (5-1)

$$\mathrm{tr}[\chi_I] \leq \gamma^{-1} \left(\frac{\gamma^{-1}}{2} \mathrm{tr}[W\chi_I W] + \frac{1}{2\gamma^{-1}} \mathrm{tr}[\chi_I] \right).$$

Thus,

$$\begin{aligned} \mathrm{tr}[\chi_I] &\leq \gamma^{-2} \mathrm{tr}[W\chi_I W] \\ &= \gamma^{-2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_F} \langle U_\alpha \chi_I U_{\alpha'} e_j, e_j \rangle \\ &= \gamma^{-2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} \langle \chi_I U_{\alpha'} e_j, \chi_I U_\alpha e_j \rangle \\ &\leq \frac{\gamma^{-2}}{2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} (\|\chi_I U_{\alpha'} e_j\|^2 + \|\chi_I U_\alpha e_j\|^2) \\ &\leq \gamma^{-2} C_{\mathrm{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \|\chi_I U_\alpha e_j\|^2. \end{aligned}$$

This completes the proof, since $\|\chi_I U_\alpha e_j\|^2 = \langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle$, and the terms with $j \notin J_{\mathrm{eff}}$ are zero. \square

4.5.2 Proof of Theorem 4.3.2

We first recall [105, Theorem 3.2]:

Spectral Averaging. *Let μ be a probability measure on \mathbb{R} and \mathcal{H} a Hilbert space. If A is a self-adjoint operator and $0 \leq B$ is a bounded operator on \mathcal{H} , then for any interval I and any $\phi \in \mathcal{H}$ we have*

$$\int_{\mathbb{R}} \langle B^{1/2} \chi_I(A + tB) B^{1/2} \phi, \phi \rangle d\mu(t) \leq 6 \|B\| \|\phi\|^2 s(\mu, |I|),$$

where $s(\mu, \varepsilon) := \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon)$.

Note that we could use instead the spectral averaging of [30]; in this case the upper bound should be replaced by $4 \|B\| (1 + \|B\|) \|\phi\|^2 s(\mu, |I|)$.

The proof in [105] actually gives $s(\mu, \varepsilon) = \sup_{E \in \mathbb{R}} \mu[E, E + \varepsilon]$, but since

$$(5-2) \quad \sup_{E \in \mathbb{R}} \mu[E, E + \varepsilon] = \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon] = \sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon)$$

(see Section 4.6.2), the above bound holds.

Proof of Theorem 4.3.2. To show that $\chi_I(H(\omega))$ is weakly measurable, it suffices to show that $H(\omega)$ is weakly measurable; see [61]. Let $\varphi, \psi \in \mathcal{H}$ and let $g(\omega) = \langle H(\omega)\varphi, \psi \rangle = \langle H_0\varphi, \psi \rangle + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha \langle U_\alpha \varphi, \psi \rangle$. Then g only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$, i.e. $\{\omega : g(\omega) \geq a\} = A \times \mathcal{B}^{\mathcal{I} \setminus \mathcal{I}_F}$ for some $A \subseteq \mathcal{B}^{\mathcal{I}_F}$, so by definition of \mathfrak{F} , it suffices to show that $A \in \otimes_{\alpha \in \mathcal{I}_F} \mathfrak{B}$. In turn, it suffices to show that the map $g_0 : \mathcal{B}^{\mathcal{I}_F} \rightarrow \mathbb{R}$ given by $g_0 : (\omega_\alpha)_{\alpha \in \mathcal{I}_F} \mapsto \langle H_0\varphi, \psi \rangle + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha \langle U_\alpha \varphi, \psi \rangle$ is Borel measurable, but this is obvious since it is affine. Hence, $\chi_I(H(\omega))$ is weakly measurable and $\mathrm{tr}[\chi_I(H(\omega))]$ is measurable.

We may thus integrate in Proposition 4.3.1 to get

$$\mathbb{E}\{\mathrm{tr}[\chi_I(H(\omega))]\} \leq \gamma^{-2} C_{\mathrm{fin}} \sum_{j \in J_{\mathrm{eff}}} \sum_{\alpha \in \mathcal{I}_j} \mathbb{E}\{\langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle\}.$$

Fix $j \in J_{\text{eff}}$, $\alpha \in \mathcal{I}_j$ and put $\phi := U_\alpha^{1/2} e_j$. Then by [36, Theorem 10.2.1],

$$\mathbb{E}\{\langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle\} = \mathbb{E}_{Y_\alpha} \left\{ \int_B \langle U_\alpha^{1/2} \chi_I(H(\omega)) U_\alpha^{1/2} \phi, \phi \rangle d\mu_{\hat{\omega}_\alpha}(\omega_\alpha) \right\}.$$

Using the spectral averaging with $A = H_1 + \sum_{\beta \neq \alpha} \omega_\beta U_\beta$, $B = U_\alpha$ and $t = \omega_\alpha$, we get

$$\mathbb{E}\{\langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle\} \leq 6 \|U_\alpha\| \|U_\alpha^{1/2} e_j\|^2 \mathbb{E}_{Y_\alpha} \{s(\mu_{\hat{\omega}_\alpha}, |I|)\} \leq 6C_U^2 \mathbb{E}_{Y_\alpha} \{s(\mu_{\hat{\omega}_\alpha}, |I|)\}.$$

Since $\mathbb{E}_{Y_\alpha} \{s(\mu_{\hat{\omega}_\alpha}, |I|)\} \leq s_F(\mathbb{P}, |I|)$, the proof is complete. \square

4.5.3 Proof of Theorem 4.3.3

Throughout this subsection $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space with $\Omega := \mathcal{C}^{\mathcal{I}}$, where $\mathcal{C} \subseteq \mathbb{R}$ is an interval and \mathcal{I} is a countable index set. $\mathfrak{F}_{\mathbb{P}}$ denotes the \mathbb{P} -completion of \mathfrak{F} . We fix a finite set $\mathcal{I}_F \subseteq \mathcal{I}$ and denote by $\mathbf{1}_F$ the element $\mathbf{1}_F = (x_\alpha) \in \mathbb{R}^{\mathcal{I}}$ such that $x_\alpha = 1$ if $\alpha \in \mathcal{I}_F$ and $x_\alpha = 0$ otherwise.

We will use the fact that monotone functions $\varphi : \Omega \rightarrow \mathbb{R}$ which depend on finitely many ω_α are $\mathfrak{F}_{\mathbb{P}}$ -measurable; this is proved in Lemma 4.6.2. Note that for any fixed $x \in \mathbb{R}^{\mathcal{I}}$, the map $\varphi(\omega - x)$ is also monotone increasing, hence $\mathfrak{F}_{\mathbb{P}}$ -measurable. We may thus state the following lemma⁷, whose basic idea stems from [103], see also [25] and [18].

Lemma 4.5.1. *Suppose $\varphi : \Omega \rightarrow \mathbb{R}$ is monotone increasing and depends on finitely many ω_α . Given $c \in \mathbb{R}$ and $\eta > 0$, define $A := \{\omega : \varphi(\omega) \leq c\}$, $A^\eta := \{\omega : \omega - \eta \cdot \mathbf{1}_F \in A \text{ and } \varphi(\omega - \eta \cdot \mathbf{1}_F) \leq c\}$, $B := \{\omega : \varphi(\omega) \geq c\}$ and $B^\eta := \{\omega : \omega + \eta \cdot \mathbf{1}_F \in B \text{ and } \varphi(\omega + \eta \cdot \mathbf{1}_F) \geq c\}$. Then*

$$\overline{\mathbb{P}}(A^\eta \setminus A) \leq |\mathcal{I}_F| \cdot s_F(\mathbb{P}, \eta) \quad \text{and} \quad \overline{\mathbb{P}}(B^\eta \setminus B) \leq |\mathcal{I}_F| \cdot s_F(\mathbb{P}, \eta),$$

where $\overline{\mathbb{P}}$ denotes the extension of \mathbb{P} to $\mathfrak{F}_{\mathbb{P}}$ and $s_F(\mathbb{P}, \eta)$ is as in (2-1).

Proof. We prove the second bound; the first is similar. Let $\mathcal{I}_F = \{\alpha_1, \dots, \alpha_m\}$ and $\mathcal{I}_k = \{\alpha_1, \dots, \alpha_k\}$ for $1 \leq k \leq m$. Let $\mathbf{1}_j$ be the element $\mathbf{1}_j = (x_\alpha) \in \mathbb{R}^{\mathcal{I}}$ with $x_\alpha = 1$ if $\alpha \in \mathcal{I}_j$ and $x_\alpha = 0$ otherwise, so that $\mathbf{1}_m = \mathbf{1}_F$. Set

$$B_0^\eta := B \quad \text{and} \quad B_j^\eta := \{\omega : \omega + \eta \cdot \mathbf{1}_j \in B \text{ and } \varphi(\omega + \eta \cdot \mathbf{1}_j) \geq c\}$$

for $1 \leq j \leq m$. Note that if B_0, \dots, B_m is any collection of sets, then one checks by induction that $B_m \setminus B_0 \subseteq \bigcup_{j=1}^m (B_j \setminus B_{j-1})$ ⁸, so we have in particular

$$(5-3) \quad \overline{\mathbb{P}}(B_m^\eta \setminus B_0) \leq \sum_{j=1}^m \overline{\mathbb{P}}(B_j^\eta \setminus B_{j-1}^\eta).$$

Now fix $j \in \{1, \dots, m\}$, let $\hat{\omega}_j = (\omega_\beta)_{\beta \neq \alpha_j} \in \mathcal{C}^{\mathcal{I} \setminus \{\alpha_j\}}$ and denote by $(x, \hat{\omega}_j)$ the element $(x_\alpha) \in \mathbb{R}^{\mathcal{I}}$ with $x_{\alpha_j} = x$ and $x_\beta = \omega_\beta$ for $\beta \neq \alpha_j$. Define the section

$$C_{\hat{\omega}_j} := \{x \in \mathcal{C} : (x, \hat{\omega}_j) \in B_j^\eta \setminus B_{j-1}^\eta\} = (B_j^\eta \setminus B_{j-1}^\eta)_{\hat{\omega}_j}.$$

7. In contrast to [103], here we avoided to speak of sums of sets, because such sums are not measurable in general; see [41] and [28], so a justification would be needed.

8. If $m = 1$, then $B_1 \setminus B_0 = B_1 \setminus B_0$. Suppose that $B_m \setminus B_0 \subseteq \bigcup_{j=1}^m (B_j \setminus B_{j-1})$ and let $\omega \in B_{m+1} \setminus B_0$. If $\omega \notin B_m \setminus B_0$, then $\omega \notin B_m$, since $\omega \notin B_0$. But $\omega \in B_{m+1}$, hence $\omega \in B_{m+1} \setminus B_m$. Thus, $B_{m+1} \setminus B_0 \subseteq (B_m \setminus B_0) \cup (B_{m+1} \setminus B_m) \subseteq \bigcup_{j=1}^{m+1} (B_j \setminus B_{j-1})$ by the induction hypothesis.

We show that $C_{\hat{\omega}_j}$ is contained in an interval of length η . If $C_{\hat{\omega}_j} = \emptyset$, this is clear, so suppose $x \in C_{\hat{\omega}_j}$. Fix $\delta \geq \eta$. If $x - \delta \in C_{\hat{\omega}_j}$, then $(x - \delta, \hat{\omega}_j) \in B_j^\eta$ and thus $\varphi((x - \delta, \hat{\omega}_j) + \eta \cdot \mathbf{1}_j) \geq c$. But φ is monotone increasing, so $\varphi((x - \delta, \hat{\omega}_j) + \eta \cdot \mathbf{1}_j) = \varphi((x - \delta + \eta, \hat{\omega}_j) + \eta \cdot \mathbf{1}_{j-1}) \leq \varphi((x, \hat{\omega}_j) + \eta \cdot \mathbf{1}_{j-1}) < c$, since $(x, \hat{\omega}_j) \notin B_{j-1}^\eta$ ⁹. This contradiction shows that $x - \delta \notin C_{\hat{\omega}_j}$ for any $\delta \geq \eta$, i.e. $C_{\hat{\omega}_j}$ is contained in a semi-open interval¹⁰ $I_{\hat{\omega}_j}$ of length η . Let D_j^η be the set $B_j^\eta \setminus B_{j-1}^\eta$ with each section $C_{\hat{\omega}_j}$ replaced by $I_{\hat{\omega}_j}$. Then $B_j^\eta \setminus B_{j-1}^\eta \subseteq D_j^\eta$ and $I_{\hat{\omega}_j}$ is a Borel set for any $\hat{\omega}_j$. So applying [13, Corollary 10.4.15] to D_j^η , taking $Y_j := \mathcal{C}^{\mathcal{I} \setminus \{\alpha_j\}}$ and using (5-2), we may find $\bar{\mu}_{\hat{\omega}_j}$ such that

$$(5-4) \quad \bar{\mathbb{P}}(B_j^\eta \setminus B_{j-1}^\eta) \leq \bar{\mathbb{E}}_{Y_j} \{ \bar{\mu}_{\hat{\omega}_j}(I_{\hat{\omega}_j}) \} \leq \bar{\mathbb{E}}_{Y_j} \left\{ \sup_{E \in \mathbb{R}} \bar{\mu}_{\hat{\omega}_j}(E, E + \eta) \right\}.$$

But $\bar{\mathbb{E}}_{Y_j} \{ \mu_{\hat{\omega}_j}(E, E + \eta) \} = \mathbb{P}\{\omega_{\alpha_j} \in (E, E + \eta)\} = \bar{\mathbb{P}}\{\dots\} = \bar{\mathbb{E}}_{Y_j} \{ \bar{\mu}_{\hat{\omega}_j}(E, E + \eta) \}$, hence $\mu_{\hat{\omega}_j}(E, E + \eta) = \bar{\mu}_{\hat{\omega}_j}(E, E + \eta)$ outside a $\bar{\mathbb{P}}_{Y_j}$ -null set Ω_E . Let $\Omega_* = \cup_{E \in \mathbb{Q}} \Omega_E$. Then $\bar{\mathbb{P}}_{Y_j}(\Omega_*) = 0$ and $\sup_{E \in \mathbb{Q}} \mu_{\hat{\omega}_j}(E, E + \eta) = \sup_{E \in \mathbb{Q}} \bar{\mu}_{\hat{\omega}_j}(E, E + \eta)$ for any $\hat{\omega}_j \notin \Omega_*$. So using (2-2),

$$\bar{\mathbb{E}}_{Y_j} \left\{ \sup_{E \in \mathbb{R}} \bar{\mu}_{\hat{\omega}_j}(E, E + \eta) \right\} = \mathbb{E}_{Y_j} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_j}(E, E + \eta) \right\} \leq s_F(\mathbb{P}, \eta),$$

and the claim follows by (5-3) and (5-4). \square

We may now prove a first extension of Stollmann's Lemma from [103]. Namely, we allow intervals \mathcal{C} and relax the diagonal condition by adding cutoffs $\chi_I(\varphi(\omega))$. The inclusion of cutoffs is actually immediate and will not be used in the proof of Theorem 4.3.3. However, this idea plays a major role in the proof of Theorem 4.3.4.

Lemma 4.5.2. *Let $I \subset \mathbb{R}$ be an open interval. Suppose $\varphi : \Omega \rightarrow \mathbb{R}$ is monotone increasing, depends on finitely many ω_α and satisfies*

$$(5-5) \quad (\varphi(\omega) - \varphi(\omega - t \cdot \mathbf{1}_F)) \chi_I(\varphi(\omega)) \geq t\gamma \cdot \chi_I(\varphi(\omega))$$

for some $\gamma > 0$ and all $t \geq 0$ such that $\omega - t \cdot \mathbf{1}_F \in \Omega$. Then

$$\bar{\mathbb{P}}\{\varphi(\omega) \in I\} \leq \delta \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{|I|}{\gamma}\right), \quad \text{where } \delta = \begin{cases} 1 & \text{if } \inf \mathcal{C} = -\infty, \\ 2 & \text{otherwise.} \end{cases}$$

This bound is also true if φ is monotone decreasing and satisfies

$$(5-6) \quad (\varphi(\omega) - \varphi(\omega - t \cdot \mathbf{1}_F)) \chi_I(\varphi(\omega)) \leq -t\gamma \cdot \chi_I(\varphi(\omega)).$$

Proof. Let $I = (a, b)$, $\varepsilon := b - a$ and $\eta := \frac{\varepsilon}{\gamma}$. We have

$$\bar{\mathbb{P}}\{\varphi(\omega) \in I\} \leq \bar{\mathbb{P}}\{\varphi(\omega) \in I \text{ and } \omega - \eta \cdot \mathbf{1}_F \in \Omega\} + \mathbb{P}\{\omega - \eta \cdot \mathbf{1}_F \notin \Omega\}.$$

Put $A := \{\omega : \varphi(\omega) \leq a\}$, $A^\eta := \{\omega : \omega - \eta \cdot \mathbf{1}_F \in \Omega \text{ and } \varphi(\omega - \eta \cdot \mathbf{1}_F) \leq a\}$ and let $\omega \in \mathcal{A} := \{\omega : \varphi(\omega) \in I \text{ and } \omega - \eta \cdot \mathbf{1}_F \in \Omega\}$. Then by (5-5),

$$\varphi(\omega - \eta \cdot \mathbf{1}_F) \leq \varphi(\omega) - \gamma\eta = \varphi(\omega) - \varepsilon \leq b - \varepsilon = a.$$

9. Note that if $\omega + \eta \cdot \mathbf{1}_j \in \Omega$, then $\omega_\alpha + \eta \in \mathcal{C}$ for any $\alpha \in \mathcal{I}_j$, so in particular for any $\alpha \in \mathcal{I}_{j-1}$ and thus $\omega + \eta \cdot \mathbf{1}_{j-1} \in \Omega$.

10. Set $a_j := \inf C_{\hat{\omega}_j}$. If $a_j \in C_{\hat{\omega}_j}$, then we proved that $a_j + \delta \notin C_{\hat{\omega}_j}$ for any $\delta \geq \eta$, i.e. $C_{\hat{\omega}_j} \subseteq [a_j, a_j + \eta)$. If $a_j \notin C_{\hat{\omega}_j}$, then if $x > a_j + \eta$, we may find $y \in C_{\hat{\omega}_j}$ such that $y < a_j + (x - a_j - \eta) = x - \eta$, so we cannot have $x \in C_{\hat{\omega}_j}$. Thus, $C_{\hat{\omega}_j} \subseteq (a_j, a_j + \eta]$.

Hence $\omega \in A^\eta$. Furthermore, $\varphi(\omega) \in I$ implies $\varphi(\omega) > a$ and thus $\omega \notin A$. Hence $\mathcal{A} \subseteq A^\eta \setminus A$ and $\overline{\mathbb{P}}(\mathcal{A}) \leq |\mathcal{I}_F| \cdot s_F(\mathbb{P}, \eta)$ by Lemma 4.5.1.

If $\inf \mathcal{C} = -\infty$, then $\mathbb{P}\{\omega - \eta \cdot \mathbf{1}_F \notin \Omega\} = 0$, since \mathcal{C} is an interval. Otherwise, let $q_- := \inf \mathcal{C}$. If $q_- \in \mathcal{C}$, then using (5-2),

$$\mathbb{P}\{\omega - \eta \cdot \mathbf{1}_F \notin \Omega\} = \mathbb{P}\{\omega_\alpha \in [q_-, q_- + \eta] \text{ for some } \alpha \in \mathcal{I}_F\} \leq |\mathcal{I}_F| \cdot s_F(\mathbb{P}, \eta)$$

since $\mathbb{P}\{\omega_\alpha \in [q_-, q_- + \eta]\} = \mathbb{E}_{Y_\alpha}\{\mu_{\hat{\omega}_\alpha}[q_-, q_- + \eta]\}$. If $q_- \notin \mathcal{C}$, replace $[q_-, q_- + \eta]$ by $(q_-, q_- + \eta]$.

Finally, if φ is decreasing and satisfies (5-6), then $\psi := -\varphi$ is increasing and $\chi_I(\varphi(\omega)) = \chi_{I'}(\psi(\omega))$, where $I' := (-b, -a)$, hence ψ satisfies (5-5) in I' . Applying the first part we obtain $\overline{\mathbb{P}}\{\varphi(\omega) \in I\} = \overline{\mathbb{P}}\{\psi(\omega) \in I'\} \leq \delta \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{|I'|}{\gamma}\right)$. \square

Proof of Theorem 4.3.3. Let $\{\varphi_n(\omega)\}$ be an orthonormal basis of eigenvectors of $H(\omega)$ with eigenvalues $\lambda_n(\omega)$. Then $\langle \chi_I(H(\omega))\varphi_n(\omega), \varphi_n(\omega) \rangle = \chi_I(\lambda_n(\omega))$. So using (C.3), we get

$$\text{tr}[\chi_I(H(\omega))] = \sum_n \langle \chi_I(H(\omega))\varphi_n(\omega), \varphi_n(\omega) \rangle = \sum_{n \leq K} \chi_I(\lambda_n(\omega)).$$

By (C.4), (C.5) and min-max, each $\lambda_n : \Omega \rightarrow \mathbb{R}$ is monotone (see below for (C.5.b) and (C.5.d)) and only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$. So by Lemma 4.6.2, each λ_n is $\mathfrak{F}_{\mathbb{P}}$ -measurable, hence $\chi_I(\lambda_n(\omega)) = \chi_{\lambda_n^{-1}(I)}(\omega)$ is $\mathfrak{F}_{\mathbb{P}}$ -measurable, and we may integrate to get

$$\overline{\mathbb{E}}\{\text{tr}[\chi_I(H(\omega))]\} = \sum_{n \leq K} \overline{\mathbb{E}}\{\chi_I(\lambda_n(\omega))\} = \sum_{n \leq K} \overline{\mathbb{P}}\{\lambda_n(\omega) \in I\}.$$

Now assume (C.5.a) holds. Then by min-max, $\lambda_n(\omega)$ is monotone increasing and satisfies $\lambda_n(\omega) \geq \lambda_n(\omega - t \cdot \mathbf{1}_F) + t\gamma$ for all $t \geq 0$ such that $\omega - t \cdot \mathbf{1}_F \in \Omega$. So by Lemma 4.5.2,

$$\overline{\mathbb{P}}\{\lambda_n(\omega) \in I\} \leq 2 \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{|I|}{\gamma}\right),$$

as asserted. The case (C.5.c) is similar. Let us show that (C.5.b) implies (C.5.a) and (C.5.d) implies (C.5.c).

Let $f : \Omega \rightarrow \mathbb{R}$ be a function only depending on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$ and suppose $f \in C^1(\Omega)$. Given $v, w \in \Omega$, we have $v + t(w - v) \in \Omega$ for any $t \in [0, 1]$. Moreover, $t \mapsto f(v + t(w - v))$ is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$, hence

$$f(w) - f(v) = \int_0^1 \frac{d}{dt} f(v + t(w - v)) dt = \int_0^1 \sum_{\alpha \in \mathcal{I}_F} (w_\alpha - v_\alpha) \frac{\partial f}{\partial \omega_\alpha}(v + t(w - v)) dt.$$

If $\frac{\partial f}{\partial \omega_\alpha} \geq 0$ on $\Omega \forall \alpha \in \mathcal{I}_F$ and $w_\alpha \geq v_\alpha$, then $f(w) - f(v) \geq 0$, i.e. f is monotone increasing. Similarly, if $\forall \alpha \in \mathcal{I}_F, \frac{\partial f}{\partial \omega_\alpha} \leq 0$ on Ω , then f is monotone decreasing. Finally, for $w = \omega$ and $v = \omega - \nu \cdot \mathbf{1}_F$ we get

$$f(\omega) - f(\omega - \nu \cdot \mathbf{1}_F) = \nu \int_0^1 \sum_{\alpha \in \mathcal{I}_F} \frac{\partial f}{\partial \omega_\alpha}(\omega - \nu \cdot \mathbf{1}_F + t(\nu \cdot \mathbf{1}_F)) dt,$$

hence $\sum_{\alpha \in \mathcal{I}_F} \frac{\partial f}{\partial \omega_\alpha} \geq c$ on Ω implies $f(\omega) - f(\omega - \nu \cdot \mathbf{1}_F) \geq \nu c$ and $\sum_{\alpha \in \mathcal{I}_F} \frac{\partial f}{\partial \omega_\alpha} \leq -c$ on Ω implies $f(\omega) - f(\omega - \nu \cdot \mathbf{1}_F) \leq -\nu c$. \square

4.5.4 Proof of Theorem 4.3.4

The proof of Theorem 4.3.4 uses two ideas: the first one is roughly to consider the change of variables $v_\alpha = \ln \omega_\alpha$, so that $\mathbb{E}\{f(\omega)\} = \int f(\omega) \mathbb{P}(d\omega) = \int f((e^{v_\alpha})) \tilde{\mathbb{P}}(dv)$. This idea was used before in [78, Theorem 2.9]. The new measure $\tilde{\mathbb{P}}$ is easily described if \mathbb{P} is a product measure; the general case is given in Lemma 4.5.3. The second idea is to generalize Stollmann's lemma to include cutoffs $\chi_I(\varphi(\omega))$ (as we did in Lemma 4.5.2) and also extend the diagonal growth condition. This is done in Lemma 4.5.4.

Lemma 4.5.3. *Let $\Omega = [q_-, q_+]^{\mathcal{I}}$, fix $q > q_+$ and let $\tilde{\Omega} := [v_-, v_+]^{\mathcal{I}}$, where $v_- = \ln(q - q_+)$ and $v_+ = \ln(q - q_-)$. Define $T : \Omega \rightarrow \tilde{\Omega}$ by $T : (\omega_\alpha) \mapsto (\ln(q - \omega_\alpha))$ and let $\tilde{\mathbb{P}} := \mathbb{P} \circ T^{-1}$. Then*

$$s_F(\tilde{\mathbb{P}}, \varepsilon) \leq s_F(\mathbb{P}, (q - q_-)(e^\varepsilon - 1)).$$

Here $s_F(\tilde{\mathbb{P}}, \varepsilon)$ is defined as before, i.e. if $Z_\alpha := [v_-, v_+]^{\mathcal{I} \setminus \{\alpha\}}$, $\pi_{Z_\alpha} : \tilde{\Omega} \rightarrow Z_\alpha$ is defined by $\pi_{Z_\alpha} : v \mapsto \hat{v}_\alpha$ and if $\tilde{\mathbb{P}}_{Z_\alpha} = \tilde{\mathbb{P}} \circ \pi_{Z_\alpha}^{-1}$, then $s_F(\tilde{\mathbb{P}}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \tilde{\mathbb{E}}_{Z_\alpha} \{ \sup_{E \in \mathbb{R}} \tilde{\mu}_{\hat{v}_\alpha}(E, E + \varepsilon) \}$.

Proof. First recall that by [36, Theorem 4.1.11], if $\mathcal{T} : (X, \mathcal{X}, \mathbb{P}) \rightarrow (Y, \mathcal{Y})$ is any measurable map, and if $\mathbb{P}^{\mathcal{T}} = \mathbb{P} \circ \mathcal{T}^{-1}$, then for any measurable $g : Y \rightarrow \mathbb{R}$, we have

$$(5-7) \quad \mathbb{E}^{\mathcal{T}} \{g(y)\} = \mathbb{E} \{(g \circ \mathcal{T})(x)\},$$

whenever either side exists. Fix $\alpha \in \mathcal{I}_F$ and let $G := \{v_\alpha \in (E, E + \varepsilon)\}$. Then

$$\begin{aligned} \tilde{\mathbb{E}}_{Z_\alpha} \{ \tilde{\mu}_{\hat{v}_\alpha}(E, E + \varepsilon) \} &= \tilde{\mathbb{P}}(G) = \mathbb{P}(T(\omega) \in G) = \mathbb{P}\{\ln(q - \omega_\alpha) \in (E, E + \varepsilon)\} \\ &= \mathbb{P}\{\omega_\alpha \in (q - e^{E+\varepsilon}, q - e^E)\} = \mathbb{E}_{Y_\alpha} \{ \mu_{\hat{\omega}_\alpha}(q - e^{E+\varepsilon}, q - e^E) \}, \end{aligned}$$

where $Y_\alpha := [q_-, q_+]^{\mathcal{I} \setminus \{\alpha\}}$. Define $\hat{T}_2 : Z_\alpha \rightarrow Y_\alpha$ by $\hat{T}_2 : (v_\alpha) \mapsto (q - e^{v_\alpha})$. Then $\hat{T}_2 \circ \pi_{Z_\alpha} \circ T = \pi_{Y_\alpha}$, so $\tilde{\mathbb{P}}_{Z_\alpha}^{\hat{T}_2} = \mathbb{P}_{Y_\alpha}$ and using (5-7) we get $\mathbb{E}_{Y_\alpha} \{ \mu_{\hat{\omega}_\alpha}(q - e^{E+\varepsilon}, q - e^E) \} = \tilde{\mathbb{E}}_{Z_\alpha} \{ \mu_{\hat{T}_2(\hat{v}_\alpha)}(q - e^{E+\varepsilon}, q - e^E) \}$. Hence $\tilde{\mu}_{\hat{v}_\alpha}(E, E + \varepsilon) = \mu_{\hat{T}_2(\hat{v}_\alpha)}(q - e^{E+\varepsilon}, q - e^E)$ outside a $\tilde{\mathbb{P}}_{Z_\alpha}$ -null set Ω_E . Let $\Omega_* = \cup_{E \in \mathbb{Q}} \Omega_E$. Then $\tilde{\mathbb{P}}_{Z_\alpha}(\Omega_*) = 0$ and $\sup_{E \in \mathbb{Q}} \tilde{\mu}_{\hat{v}_\alpha}(E, E + \varepsilon) = \sup_{E \in \mathbb{Q}} \mu_{\hat{T}_2(\hat{v}_\alpha)}(q - e^{E+\varepsilon}, q - e^E)$ for any $\hat{v}_\alpha \notin \Omega_*$. So using (2-2) and (5-7),

$$\begin{aligned} \tilde{\mathbb{E}}_{Z_\alpha} \left\{ \sup_{E \in \mathbb{R}} \tilde{\mu}_{\hat{v}_\alpha}(E, E + \varepsilon) \right\} &= \tilde{\mathbb{E}}_{Z_\alpha} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{T}_2(\hat{v}_\alpha)}(q - e^{E+\varepsilon}, q - e^E) \right\} \\ &= \mathbb{E}_{Y_\alpha} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_\alpha}(q - e^{E+\varepsilon}, q - e^E) \right\}. \end{aligned}$$

If $q - e^E < q_-$, the RHS is zero, since $\mu_{\hat{\omega}_\alpha}$ is supported in $[q_-, q_+]$. So suppose $e^E \leq q - q_-$. Then $(q - e^E) - (q - e^{E+\varepsilon}) = e^E(e^\varepsilon - 1) \leq (q - q_-)(e^\varepsilon - 1)$. This completes the proof. \square

Lemma 4.5.4. *Let (Ω, \mathbb{P}) be a probability space, $\Omega = [c_-, c_+]^{\mathcal{I}}$ and $I \subset \mathbb{R}$ an open interval. Suppose $\varphi : \Omega \rightarrow \mathbb{R}$ is monotone increasing, depends on finitely many ω_α and satisfies*

$$(5-8) \quad (\varphi(\omega + t \cdot \mathbf{1}_F) - \varphi(\omega)) \chi_I(\varphi(\omega)) \geq \gamma(e^{\zeta t} - 1) \cdot \chi_I(\varphi(\omega))$$

for some $\zeta > 0$, $\gamma > 0$ and all $t \geq 0$ such that $\omega + t \cdot \mathbf{1}_F \in \Omega$. Then

$$\tilde{\mathbb{P}}\{\varphi(\omega) \in I\} \leq 2 \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, \frac{1}{\zeta} \ln\left(1 + \frac{|I|}{\gamma}\right)\right).$$

This bound is also true if φ is monotone decreasing and satisfies for all $t \geq 0$ such that $\omega - t \cdot \mathbf{1}_F \in \Omega$ the bound

$$(5-9) \quad (\varphi(\omega) - \varphi(\omega - t \cdot \mathbf{1}_F)) \chi_I(\varphi(\omega)) \leq \gamma(1 - e^{\zeta t}) \cdot \chi_I(\varphi(\omega)).$$

Proof. Let $I = (a, b)$, $\varepsilon := b - a$ and $\eta := \frac{1}{\zeta} \ln(1 + \frac{\varepsilon}{\gamma})$. Suppose first that φ is monotone increasing and satisfies (5-8). We have

$$\overline{\mathbb{P}}\{\varphi(\omega) \in I\} \leq \overline{\mathbb{P}}\{\varphi(\omega) \in I \text{ and } \omega + \eta \cdot \mathbf{1}_F \in \Omega\} + \mathbb{P}\{\omega + \eta \cdot \mathbf{1}_F \notin \Omega\}.$$

For the first term, let $\omega \in \mathcal{A} := \{\varphi(\omega) \in I \text{ and } \omega + \eta \cdot \mathbf{1}_F \in \Omega\}$. Then by (5-8),

$$\varphi(\omega + \eta \cdot \mathbf{1}_F) \geq \varphi(\omega) + \gamma(e^{\zeta\eta} - 1) = \varphi(\omega) + \varepsilon \geq a + \varepsilon = b,$$

hence if $B^\eta := \{\omega : \omega + \eta \cdot \mathbf{1}_F \in \Omega \text{ and } \varphi(\omega + \eta \cdot \mathbf{1}_F) \geq b\}$, we have $\omega \in B^\eta$. Moreover, $\varphi(\omega) \in I$ implies $\varphi(\omega) < b$ and thus $\omega \notin B := \{\omega : \varphi(\omega) \geq b\}$. Hence, $\mathcal{A} \subseteq B^\eta \setminus B$ and $\overline{\mathbb{P}}(\mathcal{A}) \leq |\mathcal{I}_F| \cdot s_F(\mathbb{P}, \eta)$ by Lemma 4.5.1.

For the second term, $\mathbb{P}\{\omega + \eta \cdot \mathbf{1}_F \notin \Omega\} = \mathbb{P}\{\omega_\alpha \in (c_+ - \eta, c_+]$ for some $\alpha \in \mathcal{I}_F\} \leq |\mathcal{I}_F| \cdot s_F(\mathbb{P}, \eta)$ by (5-2). This proves the first claim.

Now suppose φ is decreasing and satisfies (5-9). Again,

$$\overline{\mathbb{P}}\{\varphi(\omega) \in I\} \leq \overline{\mathbb{P}}\{\varphi(\omega) \in I \text{ and } \omega - \eta \cdot \mathbf{1}_F \in \Omega\} + \mathbb{P}\{\omega - \eta \cdot \mathbf{1}_F \notin \Omega\}.$$

The second term is assessed as before. For the first term, let $\psi(\omega) := -\varphi(\omega)$ and put $A := \{\omega : \psi(\omega) \leq -b\}$, $A^\eta := \{\omega : \omega - \eta \cdot \mathbf{1}_F \in \Omega \text{ and } \psi(\omega - \eta \cdot \mathbf{1}_F) \leq -b\}$ and let $\omega \in \mathcal{A}' := \{\varphi(\omega) \in I \text{ and } \omega - \eta \cdot \mathbf{1}_F \in \Omega\}$. Then by (5-9),

$$\varphi(\omega - \eta \cdot \mathbf{1}_F) \geq \varphi(\omega) - \gamma(1 - e^{\zeta\eta}) = \varphi(\omega) + \varepsilon \geq a + \varepsilon = b,$$

hence $\psi(\omega - \eta \cdot \mathbf{1}_F) \leq -b$ and $\omega \in A^\eta$. Moreover, $\varphi(\omega) \in I$ implies $\varphi(\omega) < b$, i.e. $\psi(\omega) > -b$ and thus $\omega \notin A$. Hence, $\mathcal{A}' \subseteq A^\eta \setminus A$ and the claim follows from Lemma 4.5.1. \square

Proof of Theorem 4.3.4. Let $A(\omega) = -H(\omega)$ and $I' = (-E_2, -E_1)$. Then $\text{tr}[\chi_I(H(\omega))] = \text{tr}[\chi_{I'}(A(\omega))]$. Moreover, if $r_u(\omega) = -f_u(\omega) = -a(u) + \sum_{\alpha \in \mathcal{I}_F} (q - \omega_\alpha)^\zeta b_\alpha(u)$, then using min-max for $H(\omega)$, we obtain the formula

$$(5-10) \quad \mu_n(\omega) = \inf_{\varphi_1, \dots, \varphi_{n-1}} \sup_{\substack{u \in \mathcal{D}, \|u\|=1, \\ u \in \{\varphi_1, \dots, \varphi_{n-1}\}^\perp}} r_u(\omega)$$

for the decreasing set $\mu_1(\omega) \geq \mu_2(\omega) \geq \dots$ of eigenvalues of $A(\omega)$ (here $\mu_j(\omega) = -\lambda_j(\omega)$). Since $b_\alpha(u) \geq 0$ for any u , each $\mu_n(\omega)$ is monotone and only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_F}$ by (5-10), hence each is $\mathfrak{F}_{\mathbb{P}}$ -measurable by Lemma 4.6.2. Thus, as in the proof of Theorem 4.3.3, $\text{tr}[\chi_{I'}(A(\omega))]$ is $\mathfrak{F}_{\mathbb{P}}$ -measurable and we may integrate to get

$$(5-11) \quad \overline{\mathbb{E}}\{\text{tr}[\chi_{I'}(A(\omega))]\} = \sum_{n \leq K} \overline{\mathbb{P}}\{\mu_n(\omega) \in I'\} = \sum_{n \leq K} \overline{\mathbb{P}}\{\mu_n(T_2(v)) \in I'\},$$

where, using the notations of Lemma 4.5.3, $T_2 : \tilde{\Omega} \rightarrow \Omega$ is given by $T_2 : (v_\alpha) \mapsto (q - e^{v_\alpha})$, and we applied (5-7) to $g(v) := \chi_{I'}(\mu_n(T_2(v)))$, noting that $(T_2 \circ T)(\omega) = \omega$.

Suppose now that (3-1) holds with $\zeta > 0$ and fix $u \in \mathcal{D}$. Since $r_u \circ T_2(v) = -a(u) + \sum_{\alpha \in \mathcal{I}_F} e^{\zeta v_\alpha} b_\alpha(u)$, given $v \in \tilde{\Omega}$ and $t \geq 0$ such that $v + t \cdot \mathbf{1}_F \in \tilde{\Omega}$, we have

$$\begin{aligned} (r_u \circ T_2)(v + t \cdot \mathbf{1}_F) &= -a(u) + \sum_{\alpha \in \mathcal{I}_F} e^{\zeta(v_\alpha + t)} b_\alpha(u) \\ &= -a(u) + e^{\zeta t} \sum_{\alpha \in \mathcal{I}_F} e^{\zeta v_\alpha} b_\alpha(u) \geq e^{\zeta t} (r_u \circ T_2)(v) \end{aligned}$$

since $-a(u) \geq -e^{\zeta t} a(u)$. Thus, if $\nu_n(v) := \mu_n(T_2(v))$, we get by (5-10)

$$\nu_n(v + t \cdot \mathbf{1}_F) \geq e^{\zeta t} \nu_n(v).$$

Now note that if $\nu_n(v) \in I'$, then $\nu_n(v) \geq -E_2 = |E_2| > 0$. Hence,

$$\begin{aligned} (\nu_n(v + t \cdot \mathbf{1}_F) - \nu_n(v)) \chi_{I'}(\nu_n(v)) &\geq (e^{\zeta t} \nu_n(v) - \nu_n(v)) \chi_{I'}(\nu_n(v)) \\ &\geq (e^{\zeta t} - 1) |E_2| \chi_{I'}(\nu_n(v)). \end{aligned}$$

As $\zeta > 0$, $\nu_n(v)$ is monotone increasing in v , so using Lemma 4.5.4 we get

$$\begin{aligned} \bar{\mathbb{P}}\{\nu_n(v) \in I'\} &\leq 2 \cdot |\mathcal{I}_F| \cdot s_F\left(\bar{\mathbb{P}}, \frac{1}{\zeta} \ln\left(1 + \frac{|I'|}{|E_2|}\right)\right) \\ &\leq 2 \cdot |\mathcal{I}_F| \cdot s_F\left(\mathbb{P}, (q - q_-)\left(\left(1 + \frac{|I|}{|E_2|}\right)^{\frac{1}{\zeta}} - 1\right)\right), \end{aligned}$$

where we applied Lemma 4.5.3 with $\varepsilon := \frac{1}{\zeta} \ln\left(1 + \frac{|I'|}{|E_2|}\right)$. Using (5-11), the proof is complete for $\zeta > 0$. Now suppose that $\zeta < 0$ and put $\theta := -\zeta > 0$. Then

$$\begin{aligned} (r_u \circ T_2)(v) &= -a(u) + \sum_{\alpha \in \mathcal{I}_F} e^{-\theta v_\alpha} b_\alpha(u) \\ &= -a(u) + e^{-\theta t} \sum_{\alpha \in \mathcal{I}_F} e^{-\theta(v_\alpha - t)} b_\alpha(u) \leq e^{-\theta t} (r_u \circ T_2)(v - t \cdot \mathbf{1}_F) \end{aligned}$$

for any $t \geq 0$ such that $v - t \cdot \mathbf{1}_F \in \tilde{\Omega}$, since $-a(u) \leq -e^{-\theta t} a(u)$. Hence,

$$\nu_n(v) \leq e^{-\theta t} \nu_n(v - t \cdot \mathbf{1}_F),$$

and thus, noting that $(1 - e^{\theta t}) \leq 0$ we get

$$\begin{aligned} (\nu_n(v) - \nu_n(v - t \cdot \mathbf{1}_F)) \chi_{I'}(\nu_n(v)) &\leq (\nu_n(v) - e^{\theta t} \nu_n(v)) \chi_{I'}(\nu_n(v)) \\ &\leq (1 - e^{\theta t}) |E_2| \chi_{I'}(\nu_n(v)). \end{aligned}$$

Furthermore, $\nu_n(v)$ is monotone decreasing. The claim of Theorem 4.3.4 for $\zeta < 0$ now follows as before using Lemma 4.5.4. \square

4.6 Appendix

4.6.1 Spectra of some Schrödinger operators

Let $G \subset \mathbb{Z}^d$ be non-empty, $\mathcal{B} \subseteq \mathbb{R}$ a Borel set and consider the probability space (Ω, \mathbb{P}) , where $\Omega = \mathcal{B}^G$ and $\mathbb{P} = \otimes_{\alpha \in G} \mu$, for some probability measure μ on \mathbb{R} with $\text{supp } \mu \subseteq \mathcal{B}$. Define

$$\begin{aligned} H^\omega &= H^0 + V^\omega \text{ on } \ell^2(\mathbb{Z}^d), \text{ where } H^0 = -\Delta + V^0, \quad V^\omega = \sum_{\alpha \in G} \omega_\alpha \delta_\alpha, \\ H_\omega &= H_0 + V_\omega \text{ on } L^2(\mathbb{R}^d), \text{ where } H_0 = -\Delta + V_0, \quad V_\omega = \sum_{\alpha \in G} \omega_\alpha \chi_\alpha. \end{aligned}$$

Here δ_α and χ_α are the characteristic functions of $\{\alpha\}$ and $[\alpha - \frac{1}{2}, \alpha + \frac{1}{2}]^d$ respectively and V^0, V_0 are \mathbb{Z}^d -periodic bounded real potentials. We denote points in \mathbb{R}^d by (x^1, \dots, x^d) .

We now suppose that G contains a half-space of \mathbb{Z}^d , i.e., there exists $r \in \mathbb{Z}$ and $i \in \{1, \dots, d\}$ such that $(x^1, \dots, x^d) \in G$ whenever $x^i > r$. Examples are half-spaces of \mathbb{Z}^d , and sets with a finite number of holes, i.e. with $\mathbb{Z}^d \setminus G$ finite. We can actually consider more general sets like quarter-spaces or rotated half-spaces. The only thing we need is that G should contain arbitrarily large cubes of \mathbb{Z}^d . This excludes $(2\mathbb{Z})^d$ and thus excludes Delone sets. On the other hand, half-spaces are not Delone sets either since we allow for arbitrarily large cubes with no points of G . So the sets we consider here are neither a special case nor a generalization of Delone sets.

Lemma 4.6.1. *If G contains a half-space of \mathbb{Z}^d , then $\sigma(H^\omega) \supseteq \sigma(H^0) + \text{supp } \mu$ and $\sigma(H_\omega) \supseteq \sigma(H_0) + \text{supp } \mu$ almost surely.*

Proof. We only prove the claim for H_ω ; the proof is similar for H^ω . All the arguments actually go back to [77], [62]; one simply needs to choose $\Omega_k^{\lambda,q}(n)$ carefully below.

Assume $(x^1, \dots, x^d) \in G$ whenever $x^i > r$. Let $E = \lambda + q \in \sigma(H_0) + \text{supp } \mu$. By Weyl's criterion [114, Theorem 7.22], we may find $f_k \in C_c^\infty(\mathbb{R}^d)$, $\|f_k\| = 1$ such that $\|(H_0 - \lambda)f_k\| \rightarrow 0$ as $k \rightarrow \infty$. Choose $l_k = l_k(\lambda) \in \mathbb{N}^*$ such that $\text{supp } f_k \subset \Lambda_{l_k}(0)$, put $I_k^q := [q - \frac{1}{k}, q + \frac{1}{k}]$ and consider the event

$$\Omega_k^{\lambda,q}(n) := \{\omega \in \Omega : \omega_\alpha \in I_k^q \quad \forall \alpha \in \Lambda_{l_k}(x_{n,k})\},$$

where $x_{n,k} := (3nl_k + r)e^i$ and $e^i \in \mathbb{Z}^d$ has 1 in the i th coordinate and 0 otherwise. First note that $\Lambda_{l_k}(x_{n,k}) \cap G = \Lambda_{l_k}(x_{n,k})$, so that the above event is well defined. Moreover, $\Lambda_{l_k}(x_{n,k}) \cap \Lambda_{l_k}(x_{m,k}) = \emptyset$ for $n \neq m$, so the events $\{\Omega_k^{\lambda,q}(n)\}_{n \in \mathbb{N}^*}$ are independent and $\mathbb{P}(\Omega_k^{\lambda,q}(n)) = \mu(I_k^q)^{|\Lambda_{l_k}|}$ is the same for all n and strictly positive since $q \in \text{supp } \mu$. It follows by Borel-Cantelli lemma II that if $\Omega_k^{\lambda,q} := \bigcap_{m \geq 1} \bigcup_{n \geq m} \Omega_k^{\lambda,q}(n)$, then $\mathbb{P}(\Omega_k^{\lambda,q}) = 1$. Let $\Omega^{\lambda,q} := \bigcap_{k \in \mathbb{N}^*} \Omega_k^{\lambda,q}$, then $\mathbb{P}(\Omega^{\lambda,q}) = 1$.

Now fix $\omega \in \Omega^{\lambda,q}$ and let $k \in \mathbb{N}^*$. Then $\omega \in \Omega_k^{\lambda,q}$, so we may find $n \in \mathbb{N}^*$ such that $\omega \in \Omega_k^{\lambda,q}(n)$. But

$$\|(H_\omega - E)f_k(\cdot - x_{n,k})\| \leq \|(H_0 - \lambda)f_k(\cdot - x_{n,k})\| + \|(V_\omega - q)f_k(\cdot - x_{n,k})\|.$$

Since V_0 is periodic, $\|(H_0 - \lambda)f_k(\cdot - x_{n,k})\| = \|(H_0 - \lambda)f_k\| \rightarrow 0$. Moreover $\omega \in \Omega_k^{\lambda,q}(n)$, so $\omega_\alpha \in I_k^q$ for all $\alpha \in \Lambda_{l_k}(x_{n,k})$. Recalling that $\Lambda_{l_k}(x_{n,k}) \cap G = \Lambda_{l_k}(x_{n,k})$, we get

$$\|(V_\omega - q)f_k(\cdot - x_{n,k})\|^2 = \sum_{\alpha \in \Lambda_{l_k}(x_{n,k})} (\omega_\alpha - q)^2 \|\chi_\alpha f_k\|^2 \leq \frac{1}{k^2} \|f_k\|^2 \rightarrow 0.$$

Hence (f_k) is a Weyl sequence for E . We thus showed that for any $\omega \in \Omega^{\lambda,q}$ we have $\lambda + q \in \sigma(H_\omega)$. Let $\Omega_0 := \bigcap_{\lambda \in \sigma(H_0) \cap \mathbb{Q}, q \in \text{supp } \mu \cap \mathbb{Q}} \Omega^{\lambda,q}$. Then $\mathbb{P}(\Omega_0) = 1$ and for any $\omega \in \Omega_0$ we have $\sigma(H_\omega) \supseteq \sigma(H_0) \cap \mathbb{Q} + \text{supp } \mu \cap \mathbb{Q}$. Since $\sigma(H_\omega)$ is closed, the proof is complete. \square

4.6.2 Technical details

We give here the details of some claims we made in Sections 4.2 and 4.5. Let μ be a probability measure on \mathbb{R} . To prove (2-2), let $E \in \mathbb{R}$ and $E_k := \frac{\lfloor 10^k E \rfloor}{10^k}$, where $\lfloor x \rfloor$ is the integer part of x . Then $E_k \nearrow E$ and $c < E + \varepsilon$ iff $c < E_k + \varepsilon$ for some k . Hence

$$\begin{aligned} \mu(E, E + \varepsilon) &= \mu(\cup_k (E, E_k + \varepsilon)) = \lim_{k \rightarrow \infty} \mu(E, E_k + \varepsilon) \\ &\leq \liminf_{k \rightarrow \infty} \mu(E_k, E_k + \varepsilon) \leq \sup_{F \in \mathbb{Q}} \mu(F, F + \varepsilon). \end{aligned}$$

Thus, $\sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) \leq \sup_{F \in \mathbb{Q}} \mu(F, F + \varepsilon)$. This proves (2-2).

Suppose $\mathbb{P} = \otimes \mu_\alpha$ for some probability measures μ_α on \mathbb{R} . Then given $A \in \mathfrak{F}$, we have $\mathbb{P}(A) = \int_{Y_\alpha} \mu_\alpha(A_{\hat{\omega}_\alpha}) d\mathbb{P}_{Y_\alpha}(\hat{\omega}_\alpha)$, so by [13, Corollary 10.4.15], $\mu_{\hat{\omega}_\alpha} = \mu_\alpha$ \mathbb{P}_{Y_α} -a.s., so $s_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \sup_{E \in \mathbb{R}} \mu_\alpha(E, E + \varepsilon)$ using (2-2). Next, note that

$$\begin{aligned} \mu(E, E + \varepsilon) &= \mu(\cup_k (E + \frac{1}{k}, E + \varepsilon)) = \lim_{k \rightarrow \infty} \mu(E + \frac{1}{k}, E + \varepsilon) \\ &\leq \liminf_{k \rightarrow \infty} \mu(E + \frac{1}{k}, E + \varepsilon + \frac{1}{k}) \leq \sup_{F \in \mathbb{R}} \mu(F, F + \varepsilon), \end{aligned}$$

so $\sup_{E \in \mathbb{R}} \mu(E, E + \varepsilon) \leq \sup_{F \in \mathbb{R}} \mu(F, F + \varepsilon)$ and this proves equality. Similarly, one checks that $\sup_{E \in \mathbb{R}} \mu[E, E + \varepsilon) \leq \sup_{F \in \mathbb{R}} \mu(F, F + \varepsilon)$, which proves (5-2).

We finally prove the following. Here $\Omega = \mathcal{B}^{\mathcal{I}}$ with $\mathcal{B} \subseteq \mathbb{R}$ a Borel set and \mathcal{I} is countable.

Lemma 4.6.2. *If \mathbb{P} has no atoms, then any monotone $\varphi : \Omega \rightarrow \mathbb{R}$ which depends on finitely many ω_α is $\mathfrak{F}_{\mathbb{P}}$ -measurable, where $\mathfrak{F}_{\mathbb{P}}$ is the \mathbb{P} -completion of \mathfrak{F} .*

Proof. Suppose φ is monotone increasing and only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_m}$. For notational simplicity, assume $\mathcal{I}_m = \{1, \dots, m\}$. Put $\mathcal{I}_k := \{1, \dots, k\}$ for $1 \leq k \leq m$ and let \mathfrak{F}_k be the σ -algebra generated by $(\otimes_{\alpha \in \mathcal{I}_k} \mathfrak{B}) \cup \mathcal{N}_k(\mathbb{P})$, where $\mathcal{N}_k(\mathbb{P}) := \{M \subseteq \mathcal{B}^{\mathcal{I}_k} : \mathbb{P}^*(M \times \mathcal{B}^{\mathcal{I}_k^c}) = 0\}$. Here \mathbb{P}^* is the outer measure defined by \mathbb{P} and $\mathcal{I}_k^c = \mathcal{I} \setminus \mathcal{I}_k$. Then $A \in \mathfrak{F}_k$ implies $A \times \mathcal{B}^{\mathcal{I}_k^c} \in \mathfrak{F}_{\mathbb{P}}$.

Since $\varphi : \mathcal{B}^{\mathcal{I}} \rightarrow \mathbb{R}$ only depends on $(\omega_\alpha)_{\alpha \in \mathcal{I}_m}$, then given $a \in \mathbb{R}$, $\{\omega : \varphi(\omega) \geq a\} = A' \times \mathcal{B}^{\mathcal{I}_m^c}$ for some $A' \subseteq \mathcal{B}^{\mathcal{I}_m}$. So to show that φ is measurable, it suffices to show that $A' \in \mathfrak{F}_m$. But if we define $\varphi_0 : \mathcal{B}^{\mathcal{I}_m} \rightarrow \mathbb{R}$ by $\varphi_0(\omega^m) := \varphi(\omega^m, 0)$ for $\omega^m = (\omega_\alpha)_{\alpha \in \mathcal{I}_m}$, then φ_0 is increasing and $\{\omega^m : \varphi_0(\omega^m) \geq a\} = A'$. Thus, it suffices to show that any monotone increasing $f : \mathcal{B}^{\mathcal{I}_m} \rightarrow \mathbb{R}$ is \mathfrak{F}_m -measurable. For this, we proceed by induction, adapting an argument of Nathaniel Eldredge showing that monotone functions on \mathbb{R}^m are Lebesgue-measurable, following [50, Theorem 4.4].

For $k = 1$ the assertion is clear: if $f : \mathcal{B} \rightarrow \mathbb{R}$ is increasing and $A = \{t : f(t) \geq a\}$, then $A = \emptyset$ or $A = I \cap \mathcal{B}$ for some interval I . Thus, $A \in \mathfrak{B} \subset \mathfrak{F}_1$.

Now suppose $f : \mathcal{B}^{\mathcal{I}_{k+1}} \rightarrow \mathbb{R}$ is increasing, fix $a \in \mathbb{R}$ and define $g : \mathcal{B}^{\mathcal{I}_k} \rightarrow \mathbb{R}$ by $g(\omega^k) = \inf\{t \in \mathcal{B} : f(\omega^k, t) \geq a\}$. Then g is monotone decreasing, hence \mathfrak{F}_k -measurable by the induction hypothesis. So by [13, Proposition 3.3.4], we have $E := \{(\omega^k, \omega_{k+1}) : g(\omega^k) < \omega_{k+1}\} \in \mathfrak{F}_k \otimes \mathfrak{B}$ and $G := \{(\omega^k, \omega_{k+1}) : g(\omega^k) = \omega_{k+1}\} \in \mathfrak{F}_k \otimes \mathfrak{B}$. Moreover, for any $\omega^k \in \mathcal{B}^{\mathcal{I}_k}$ and $y \in \mathcal{B}^{\mathcal{I}_{k+1}^c}$, we have $G_{\omega^k, y} := \{\omega_{k+1} : (\omega^k, \omega_{k+1}, y) \in G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}\} = \{\omega_{k+1} : \omega_{k+1} = g(\omega^k)\} = \{g(\omega^k)\}$. We may find $F \subseteq G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}$ such that $F \in \mathfrak{F}$ and $\overline{\mathbb{P}}(G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) = \mathbb{P}(F)$. The section $F_{\omega^k, y}$ of such F is either a singleton or empty. Thus,

$$\overline{\mathbb{P}}(G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) = \mathbb{P}(F) = \mathbb{E}_{Y_{k+1}} \{\mu_{\hat{\omega}_{k+1}}(F_{\omega^k, y})\} \leq \mathbb{E}_{Y_{k+1}} \left\{ \sup_{E \in \mathbb{R}} \mu_{\hat{\omega}_{k+1}}(E, E + \varepsilon) \right\}$$

for any $\varepsilon > 0$. Since $s_F(\mathbb{P}, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, it follows that $\overline{\mathbb{P}}(G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) = 0$.

Finally, if $M = M' \times B$ with $M' \in \mathcal{N}_k(\mathbb{P})$ and $B \subseteq \mathcal{B}$, then $\mathbb{P}^*(M \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) \leq \mathbb{P}^*(M' \times \mathcal{B}^{\mathcal{I}_k^c}) = 0$, hence $\mathfrak{F}_k \otimes \mathfrak{B} \subset \mathfrak{F}_{k+1}$ and $E, G \in \mathfrak{F}_{k+1}$. But if $A = \{(\omega^k, \omega_{k+1}) : f(\omega^k, \omega_{k+1}) \geq a\}$, then $E \subseteq A$ and $(A \setminus E) \subseteq G$. Since $E \in \mathfrak{F}_{k+1}$ and $\mathbb{P}^*((A \setminus E) \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) \leq \mathbb{P}^*(G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) = \overline{\mathbb{P}}(G \times \mathcal{B}^{\mathcal{I}_{k+1}^c}) = 0$, $A \in \mathfrak{F}_{k+1}$ and the proof is complete. \square

It is worthwhile to note that the completeness of $(\Omega, \mathfrak{F}_{\mathbb{P}}, \mathbb{P})$ is not only sufficient for the above argument to work, but also necessary. Indeed, following [50, Section 4], let us construct a monotone increasing map $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is not Borel-measurable. Suppose $M \subseteq \mathbb{R}$ is not Borel measurable. Define

$$N := \{(x, y) : x + y > 0\} \cup \{(x, -x) : x \in M\}$$

and consider the map $h : x \mapsto (x, -x)$. Then h is continuous, hence Borel measurable. If N were Borel-measurable, then so would be $N' = N \cap \{(x, -x) : x \in \mathbb{R}\} = \{(x, -x) : x \in M\}$. This would imply that $h^{-1}(N') = M$ is Borel-measurable, a contradiction.

Now define $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $\varphi = \mathbf{1}_N$. Then φ is monotone increasing: it suffices to show that if $(x, y) \in N$ and if $x' \geq x, y' \geq y$, then $(x', y') \in N$. So suppose $(x, y) \in N$:

- If $x + y > 0$, then $x' + y' \geq x + y > 0$, so $(x', y') \in N$.
- If $y = -x$ with $x \in M$, then $x' + y' \geq x + y = 0$. If $x' + y' > 0$ then $(x', y') \in N$.
Otherwise, $x' + y' = 0$, so $y' = -x'$. Moreover, $y' \geq y = -x \geq -x'$, so $x' = x \in M$.
Thus, $(x', y') \in N$.

Thus, φ is monotone increasing, but it is not Borel-measurable, since $\varphi^{-1}(\{1\}) = N$.

Chapter 5

More results on Wegner bounds

5.1 A refinement of the discrete bounds

In Section B.4 below we prove a refinement of the spectral averaging estimate. Using this estimate, the proofs of the previous chapter directly yield the following result.

Theorem 5.1.1. *Suppose $H(\omega)$ satisfies Hypotheses (A) and (B) in the interval I . Then $\text{tr}[\chi_I(H(\omega))]$ is measurable, and for any $0 < \lambda \leq \frac{1}{C_U}$ we have*

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\text{eff}}| \cdot s_F(\mathbb{P}, \lambda|I|),$$

where $C_W := \frac{4}{\lambda} \gamma^{-2} C_{\text{fin}}^2 C_U$ and $s_F(\mathbb{P}, \varepsilon)$ is defined in (4-2-1).

This refinement improves our result on multi-particle models, Theorem 4.4.1, a little bit. For definiteness, suppose we are in situation (1) of Lemma 4.4.2, so that the uncertainty principle holds with $\gamma = nc$. Then taking $\lambda = \frac{1}{nC_u}$, we obtain the bound

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L^{(n)}(\mathbf{x})}(\omega))]\} \leq C_W \cdot |\Lambda_L^{(n)}(\mathbf{x})| \cdot s_F\left(\mathbb{P}, \frac{|I|}{nC_u}\right),$$

where $C_W = 4n^2 C_u^2 (2R + 1)^{2d}$.

The advantage is that for the special case where $\mathbb{P} = \otimes \mu$, with $\mu = gdx$, the upper bound becomes linear in n , which is somehow more natural. We do not know if this has a theoretical importance however.

Another advantage of Theorem 5.1.1 is that it allows one to shrink the quantity $\lambda|I|$ by choosing an arbitrarily small λ . Of course this is accompanied by a growth in the constant C_W , but it seems to be useful in some situations; see e.g. [65, Eq.(2.11) and Eq.(3.9)].

5.2 Discrete bounds without UP

5.2.1 Special unperturbed operators

Here we prove the remark we gave in the previous chapter, namely that our discrete bounds hold without the need for an uncertainty principle if the unperturbed operator has a special form. We list our hypotheses.

Hypotheses (D)

- 1) We fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with $\Omega = [q_-, q_+]^{\mathcal{I}}$ for some $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$, some countable index set \mathcal{I} , and fix a finite-dimensional Hilbert space \mathcal{H} .

- 2) $H(\omega)$ is a self-adjoint operator on \mathcal{H} for each $\omega \in \Omega$.
 3) There exist some non-random constants c_α with $|c_\alpha| \leq M$ such that

$$H(\omega) = \sum_{\alpha \in \mathcal{I}_F} (c_\alpha + \omega_\alpha) U_\alpha,$$

for some finite set $\mathcal{I}_F \subseteq \mathcal{I}$, where $0 \leq U_\alpha \leq C_U$ are non-negative self-adjoint operators.

- 4) Fix an orthonormal basis $\{e_j\}_{j \in J}$ for \mathcal{H} . We define $\mathcal{I}_j := \{\alpha \in \mathcal{I}_F : U_\alpha e_j \neq 0\}$, $C_{\text{fin}} := \max_{j \in J} |\mathcal{I}_j|$ and $J_{\text{eff}} := \{j \in J : U_\alpha e_j \neq 0 \text{ for some } \alpha \in \mathcal{I}_F\}$.

Theorem 5.2.1. *Suppose $H(\omega)$ satisfies Hypotheses (D). Then $\text{tr}[\chi_I(H(\omega))]$ is measurable for any interval I . If $I = [a, b]$ with $a > 0$, then for any $0 < \lambda \leq \frac{1}{C_U}$ we have*

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\text{eff}}| \cdot s_F(\mathbb{P}, \lambda|I|),$$

where $C_W := \frac{4}{\lambda a^2} (M + q)^2 C_{\text{fin}}^2 C_U$ and $q := \max(|q_-|, |q_+|)$.

Again, the importance of this theorem lies in the fact that we do not assume that the U_α cover \mathcal{H} , that is, we do not assume that $\sum_\alpha U_\alpha \geq c > 0$. Operators of this form arise when studying discrete acoustic models on $\ell^2(\mathbb{Z}^d)$. In this case, $C_U = 2d$ and $C_{\text{fin}} = d + 1$.

As the proof will show, the estimate is also true if $I = [a, b]$ with $b < 0$, in which case the term a^{-2} in C_W is to be replaced by b^{-2} . However, the bound cannot be true for intervals around 0 without additional hypotheses. Indeed, if $H(\omega) = \omega_k \delta_k$ on $\ell^2(\Lambda)$ for some $k \in \Lambda$, then 0 is an eigenvalue of multiplicity $|\Lambda| - 1$ for any ω . Moreover, $J_{\text{eff}} = \{e_k\}$, and we have $\mathbb{E}\{\text{tr}[\chi_{(-\varepsilon, \varepsilon)}(H(\omega))]\} \geq |\Lambda| - 1 > C_W \cdot s_F(\mathbb{P}, 2\lambda\varepsilon)$, if Λ is large.

Proof. For the weak measurability of $\chi_I(H(\omega))$, see Section 4.5.2. For the Wegner bound, we follow [65, Lemma 2.1]. Let $I = [a, b]$ and put $\chi_I := \chi_I(H(\omega))$. We have $\text{tr}[\chi_I] = \sum_{j \in J} \langle \chi_I e_j, e_j \rangle$. Since

$$\langle \chi_I(H(\omega)) e_j, e_j \rangle = \int_I d\rho_{e_j}(\lambda) \leq \frac{1}{a^2} \int_a^b \lambda^2 d\rho_{e_j}(\lambda) = a^{-2} \langle H(\omega) \chi_I(H(\omega)) H(\omega) e_j, e_j \rangle,$$

where ρ is the spectral measure of $H(\omega)$ in the state e_j , it follows that

$$\begin{aligned} \text{tr}[\chi_I] &\leq a^{-2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_F} (c_\alpha + \omega_\alpha)(c_{\alpha'} + \omega_{\alpha'}) \langle U_\alpha \chi_I U_{\alpha'} e_j, e_j \rangle \\ &\leq a^{-2} (M + q)^2 \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} |\langle \chi_I U_{\alpha'} e_j, \chi_I U_\alpha e_j \rangle| \\ &\leq \frac{a^{-2} (M + q)^2}{2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} (\|\chi_I U_{\alpha'} e_j\|^2 + \|\chi_I U_\alpha e_j\|^2) \\ &\leq a^{-2} (M + q)^2 C_{\text{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \|\chi_I U_\alpha e_j\|^2 \\ &= a^{-2} (M + q)^2 C_{\text{fin}} \sum_{j \in J_{\text{eff}}} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle. \end{aligned}$$

Thus,

$$\mathbb{E}\{\text{tr}[\chi_I]\} \leq a^{-2} (M + q)^2 C_{\text{fin}} \sum_{j \in J_{\text{eff}}} \sum_{\alpha \in \mathcal{I}_j} \mathbb{E}\{\langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle\}.$$

Now fix $j \in J_{\text{eff}}$, $\alpha \in \mathcal{I}_j$ and put $\phi := U_\alpha^{1/2} e_j$. Then by [36, Theorem 10.2.1], we have

$$\mathbb{E}\{\langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle\} = \mathbb{E}_{Y_\alpha} \left\{ \int_{[q_-, q_+]} \langle U_\alpha^{1/2} \chi_I(H(\omega)) U_\alpha^{1/2} \phi, \phi \rangle d\mu_{\hat{\omega}_\alpha}(\omega_\alpha) \right\},$$

where $Y_\alpha = [q_-, q_+]^{\mathcal{I} \setminus \{\alpha\}}$. Using the spectral averaging of Section B.4 with $A = \sum_\beta c_\beta U_\beta + \sum_{\beta \neq \alpha} \omega_\beta U_\beta$, $B = U_\alpha$ and $t = \omega_\alpha$, we have for any $0 < \lambda \leq \frac{1}{C_U}$,

$$\mathbb{E}\{\langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle\} \leq \frac{4}{\lambda} \|U_\alpha^{1/2} e_j\|^2 \mathbb{E}_{Y_\alpha}\{s(\mu_{\hat{\omega}_\alpha}, \lambda |I|)\} \leq \frac{4}{\lambda} C_U \mathbb{E}_{Y_\alpha}\{s(\mu_{\hat{\omega}_\alpha}, \lambda |I|)\}.$$

Since $\mathbb{E}_{Y_\alpha}\{s(\mu_{\hat{\omega}_\alpha}, |I|)\} \leq s_F(\mathbb{P}, |I|)$, the proof is complete. \square

5.2.2 General unperturbed operators

We now prove a Wegner bound for general unperturbed operators, again without UP and without covering assumptions. Here are the hypotheses¹.

Hypotheses (E)

- 1) We fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with $\Omega = [q_-, q_+]^{\mathcal{I}}$ for some $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$, some countable index set \mathcal{I} , and fix a finite-dimensional Hilbert space \mathcal{H} .
- 2) $H(\omega)$ is a self-adjoint operator on \mathcal{H} for each $\omega \in \Omega$.
- 3) $H(\omega)$ has the form

$$H(\omega) = H_0 + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$$

for some finite set $\mathcal{I}_F \subseteq \mathcal{I}$, where H_0 and all U_α are self-adjoint operators.

- 4) Fix an orthonormal basis $\{e_j\}_{j \in J}$ for \mathcal{H} . We define $\mathcal{I}_j := \{\alpha \in \mathcal{I}_F : U_\alpha e_j \neq 0\}$, $C_{\text{fin}} := \max_{j \in J} |\mathcal{I}_j|$ and $J_{\text{eff}} := \{j \in J : U_\alpha e_j \neq 0 \text{ for some } \alpha \in \mathcal{I}_F\}$.

Theorem 5.2.2. *Suppose $H(\omega)$ satisfies Hypotheses (E). Then $\text{tr}[\chi_I(H(\omega))]$ is measurable for any interval I . If $I \subset \mathbb{R} \setminus \sigma(H_0)$ and $\delta := \text{dist}(I, \sigma(H_0))$, then*

$$\text{tr}[\chi_I(H(\omega))] \leq q^2 \delta^{-2} C_{\text{fin}} \sum_{j \in J_{\text{eff}}} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle.$$

If moreover the U_α satisfy $0 \leq U_\alpha \leq C_U$ for all α , then for any $0 < \lambda \leq \frac{1}{C_U}$ we have

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\text{eff}}| \cdot s_F(\mathbb{P}, \lambda |I|),$$

where $C_W := \frac{4}{\lambda \delta^2} q^2 C_{\text{fin}}^2 C_U$ and $q := \max(|q_-|, |q_+|)$.

The trivial choice $U_\alpha = 0$ for all α shows that this estimate cannot hold for intervals intersecting $\sigma(H_0)$ without additional hypotheses.

Proof. For the weak measurability of $\chi_I(H(\omega))$, see Section 4.5.2. Let $\{\varphi_n(\omega)\}$ be an orthonormal basis of eigenfunctions for $H(\omega)$ with corresponding eigenvalues $\{\lambda_n(\omega)\}$. Then

$$\text{tr}[\chi_I(H(\omega))] = \sum_{\lambda_n(\omega) \in I} \langle \varphi_n(\omega), \varphi_n(\omega) \rangle.$$

But if $\lambda_n(\omega) \in I$, then $\lambda_n(\omega) \notin \sigma(H_0)$ and $\|(H_0 - \lambda_n)^{-1}\| \leq \delta^{-1}$. Thus, taking $V_\omega := \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$ we have

$$\begin{aligned} \langle \varphi_n, \varphi_n \rangle &= \langle (H_0 - \lambda_n)^{-2} (H_0 - \lambda_n) \varphi_n, (H_0 - \lambda_n) \varphi_n \rangle \\ &= \langle (H_0 - \lambda_n)^{-2} V_\omega \varphi_n, V_\omega \varphi_n \rangle \\ &\leq \delta^{-2} \|V_\omega \varphi_n\|^2 = \delta^{-2} \langle V_\omega^2 \varphi_n, \varphi_n \rangle, \end{aligned}$$

1. The result of this subsection is generalized in Section 5.5, but the proof is harder.

where we used the fact that $(H(\omega) - \lambda_n(\omega))\varphi_n(\omega) = 0$ in the second equality. Hence, for $\chi_I := \chi_I(H(\omega))$ we have

$$\begin{aligned}
\mathrm{tr}[\chi_I] &\leq \delta^{-2} \sum_{\lambda_n(\omega) \in I} \langle V_\omega^2 \varphi_n(\omega), \varphi_n(\omega) \rangle \\
&= \delta^{-2} \sum_n \langle V_\omega^2 \chi_I \varphi_n(\omega), \varphi_n(\omega) \rangle \\
&= \delta^{-2} \mathrm{tr}[V_\omega^2 \chi_I] \\
&= \delta^{-2} \mathrm{tr}[V_\omega \chi_I V_\omega] \\
&= \delta^{-2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} \omega_\alpha \omega_{\alpha'} \langle U_\alpha \chi_I U_{\alpha'} e_j, e_j \rangle \\
&\leq \frac{q^2 \delta^{-2}}{2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} (\|\chi_I U_{\alpha'} e_j\|^2 + \|\chi_I U_\alpha e_j\|^2) \\
&\leq q^2 \delta^{-2} C_{\mathrm{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \|\chi_I U_\alpha e_j\|^2 \\
&= q^2 \delta^{-2} C_{\mathrm{fin}} \sum_{j \in J_{\mathrm{eff}}} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle.
\end{aligned}$$

This proves the first claim. Finally,

$$\mathbb{E}\{\mathrm{tr}[\chi_I]\} \leq q^2 \delta^{-2} C_{\mathrm{fin}} \sum_{j \in J_{\mathrm{eff}}} \sum_{\alpha \in \mathcal{I}_j} \mathbb{E}\{\langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle\},$$

so the second claim now follows using the spectral averaging as in the previous theorem, with $A = H_0 + \sum_{\beta \neq \alpha} \omega_\beta U_\beta$, $B = U_\alpha$, $t = \omega_\alpha$ and $\phi = U_\alpha^{1/2} e_j$. \square

5.2.3 Alternative proofs

In this subsection we give two alternative proofs to the theorem of the previous subsection. The statements are weaker, specially for the second proof. However, we chose to include them here as we think the methods used in the proofs can be interesting elsewhere.

Theorem 5.2.3. *Suppose $H(\omega)$ satisfies Hypotheses (E) and that $0 \leq U_\alpha \leq C_U$ for all α . Let $I \subset \mathbb{R} \setminus \sigma(H_0)$ be an interval, put $\delta := \mathrm{dist}(I, \sigma(H_0))$ and assume $|I| \leq \frac{\delta}{2}$. Then for any $0 < \lambda \leq \frac{1}{C_U}$ we have*

$$\mathbb{E}\{\mathrm{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\mathrm{eff}}| \cdot s_F(\mathbb{P}, \lambda|I|),$$

where $C_W := \frac{8}{\lambda \delta^2} q^2 C_{\mathrm{fin}}^2 C_U$ and $q := \max(|q_-|, |q_+|)$.

Proof. The first part of the proof goes like [64, Theorem 4.1]. Before we begin, let us mention that we will frequently use the inequality $\mathrm{tr}[A \chi_I B] \leq \|A\| \|B\| \|\chi_I\|_1 = \|A\| \|B\| \mathrm{tr}[\chi_I]$ for bounded operators A, B , which is true because $\chi_I \geq 0$.

Let $E \in I$ be the midpoint of I . Then $E \notin \sigma(H_0)$, so taking $R_0 := (H_0 - E)^{-1}$ and $V_\omega = \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$, we may write

$$\chi_I(H(\omega)) = R_0(-V_\omega) \chi_I(H(\omega)) + R_0(H(\omega) - E) \chi_I(H(\omega)),$$

so that

$$\mathrm{tr}[\chi_I(H(\omega))] = \mathrm{tr}[R_0(-V_\omega) \chi_I(H(\omega))] + \mathrm{tr}[R_0(H(\omega) - E) \chi_I(H(\omega))].$$

To estimate the second term, note that if $f \in \mathcal{H}$ then taking $\varepsilon := |I|$ we have

$$|\langle (H(\omega) - E)\chi_I(H(\omega))f, f \rangle| \leq \int_I |\lambda - E| d\rho_f \leq \frac{\varepsilon}{2} \int_I d\rho_f \leq \frac{\varepsilon}{2} \|f\|^2$$

since E is the mid-point of I . Hence $\|(H(\omega) - E)\chi_I(H(\omega))\| \leq \frac{\varepsilon}{2}$ and we get

$$\begin{aligned} \operatorname{tr}[R_0(H(\omega) - E)\chi_I(H(\omega))] &\leq \|R_0\| \|(H(\omega) - E)\chi_I(H(\omega))\| \operatorname{tr}[\chi_I(H(\omega))] \\ &\leq \frac{\varepsilon}{2\delta} \operatorname{tr}[\chi_I(H(\omega))] \leq \frac{1}{4} \operatorname{tr}[\chi_I(H(\omega))] \end{aligned}$$

since $\operatorname{dist}(E, \sigma(H_0)) \geq \delta$ and $\varepsilon \leq \frac{\delta}{2}$.

We now need a different argument than the one used in [64, Theorem 4.1]. Recall that for any two Hilbert-Schmidt operators A and B we have

$$\begin{aligned} \operatorname{tr}[B^*A] &= \langle A, B \rangle_{HS} \leq \|A\|_{HS} \|B\|_{HS} \\ &\leq \frac{c}{2} \|A\|_{HS}^2 + \frac{1}{2c} \|B\|_{HS}^2 = \frac{c}{2} \operatorname{tr}[A^*A] + \frac{1}{2c} \operatorname{tr}[B^*B] \end{aligned}$$

for any $c > 0$. Now $\operatorname{tr}[R_0(-V_\omega)\chi_I] = \operatorname{tr}[\chi_I R_0(-V_\omega)\chi_I]$, so applying this to $B^* = \chi_I R_0$ and $A = (-V_\omega)\chi_I$ we get

$$\begin{aligned} \operatorname{tr}[\chi_I R_0(-V_\omega)\chi_I] &\leq \frac{c}{2} \operatorname{tr}[\chi_I V_\omega^2 \chi_I] + \frac{1}{2c} \operatorname{tr}[\chi_I R_0^2 \chi_I] \\ &= \frac{c}{2} \operatorname{tr}[V_\omega \chi_I V_\omega] + \frac{1}{2c} \operatorname{tr}[\chi_I R_0^2] \\ &\leq \frac{c}{2} \operatorname{tr}[V_\omega \chi_I V_\omega] + \frac{\delta^{-2}}{2c} \operatorname{tr}[\chi_I] \end{aligned}$$

Choosing $c := 2\delta^{-2}$, we thus showed that

$$\operatorname{tr}[\chi_I] \leq \delta^{-2} \operatorname{tr}[V_\omega \chi_I V_\omega] + \frac{1}{4} \operatorname{tr}[\chi_I] + \frac{1}{4} \operatorname{tr}[\chi_I],$$

and thus

$$\operatorname{tr}[\chi_I] \leq 2\delta^{-2} \operatorname{tr}[V_\omega \chi_I V_\omega].$$

The rest of the proof now goes as in Theorem 5.2.2. \square

Theorem 5.2.4. *Suppose $H(\omega)$ satisfies Hypotheses (E), and assume moreover that $0 \leq U_\alpha \leq C_U$ for all α . Let $E \in (-\infty, \inf \sigma(H_0))$ and put $\delta := \operatorname{dist}(\sigma(H_0), E)$. Then for any $0 < \varepsilon < \delta$ and for any $0 < \lambda \leq \frac{2}{C_U}$ we have*

$$\mathbb{P}\{\operatorname{dist}(E, \sigma(H(\omega))) < \varepsilon\} \leq C_W \cdot |\mathcal{I}_F|^2 \cdot |J| \cdot s_F(\mathbb{P}, \lambda\varepsilon),$$

where $C_W := \frac{8}{\lambda(\delta-\varepsilon)^2} q^2 C_U$ and $q := \max(|q_-|, |q_+|)$.

Proof. We simply reduce the problem to an auxiliary diagonal operator. This idea appeared before in the paper [54], where a continuous model with sign indefinite potential was considered.

Let $V^\omega := \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$. Given $E < \inf \sigma(H_0)$ we have $(H_0 - E) > 0$, so taking $R_0 := (H_0 - E)^{-1}$, we may define the operator

$$D_\omega := R_0^{1/2} V^\omega R_0^{1/2} = \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha R_0^{1/2} U_\alpha R_0^{1/2}.$$

Now notice that $E \notin \sigma(H(\omega))$ iff $-1 \notin \sigma(D_\omega)$, in which case

$$R_\omega := (H(\omega) - E)^{-1} = R_0^{1/2}(1 + D_\omega)^{-1}R_0^{1/2}.$$

Thus,

$$\|R_\omega\| \leq \delta^{-1}\|(1 + D_\omega)^{-1}\|$$

and

$$\mathbb{P}\{\|R_\omega\| > \varepsilon^{-1}\} \leq \mathbb{P}\{\|(1 + D_\omega)^{-1}\| > \delta\varepsilon^{-1}\}.$$

We thus showed that

$$\begin{aligned} \mathbb{P}\{\text{dist}(E, \sigma(H(\omega))) < \varepsilon\} &= \mathbb{P}\{E \in \sigma(H(\omega))\} + \mathbb{P}\{\|R_\omega\| > \varepsilon^{-1}\} \\ &\leq \mathbb{P}\{-1 \in \sigma(D_\omega)\} + \mathbb{P}\{\|(1 + D_\omega)^{-1}\| > \delta\varepsilon^{-1}\} \\ &= \mathbb{P}\{\text{dist}(-1, \sigma(D_\omega)) < \delta^{-1}\varepsilon\} \end{aligned}$$

Let $I := [-1 - \delta^{-1}\varepsilon, -1 + \delta^{-1}\varepsilon] = [a, b]$. Since $\varepsilon < \delta$, we have $I \subset \mathbb{R}^-$. Now by Markov inequality,

$$\mathbb{P}\{\text{dist}(-1, \sigma(D_\omega)) < \delta^{-1}\varepsilon\} = \mathbb{P}\{\text{tr}[\chi_I(D_\omega)] \geq 1\} \leq \mathbb{E}\{\text{tr}[\chi_I(D_\omega)]\}.$$

But $D_\omega = \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha S_\alpha$, where $0 \leq S_\alpha \leq \delta^{-1}C_U$ satisfies Hypotheses (D). Hence, using Theorem 5.2.1, we have for any $0 < \zeta \leq \frac{1}{\delta^{-1}C_U}$,

$$\mathbb{E}\{\text{tr}[\chi_I(D_\omega)]\} \leq \tilde{C}_W \cdot |\tilde{J}_{\text{eff}}| \cdot s(\mathbb{P}, \zeta|I|),$$

where $\tilde{C}_W = \frac{4}{b^2\zeta}q^2\delta^{-1}C_U\tilde{C}_{\text{fin}}^2$, $\tilde{C}_{\text{fin}} \leq |\mathcal{I}_F|$ and $|\tilde{J}_{\text{eff}}| \leq |J|$. Take $\lambda = 2\delta^{-1}\zeta$, then $\tilde{C}_W \leq \frac{8}{\lambda\delta^2b^2}q^2C_U|\mathcal{I}_F|^2$. Recalling that $b = -1 + \delta^{-1}\varepsilon$ we obtain the claim. \square

5.3 More applications

5.3.1 Discrete multi-particle models

In this subsection we give a result complementing Theorem 4.4.1 and Lemma 4.4.2. We use the same notations of Section 4.4.1. We assume moreover that $\Omega = [q_-, q_+]^{\mathbb{Z}^d}$ for some $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$.

Theorem 5.3.1. *Let $I \subset \mathbb{R} \setminus \sigma(H_{0, \Lambda_L}^\bullet)$ be an interval and set $\eta = \text{dist}(I, \sigma(H_{0, \Lambda_L}^\bullet))$. Then for any $0 < \lambda \leq \frac{1}{nC_u}$,*

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L}^\bullet(\omega))]\} \leq C_W \cdot |\tilde{\Lambda}_L^{(n)}| \cdot s_F(\mathbb{P}, \lambda|I|),$$

where $C_W = \frac{4}{\lambda\eta^2}n^3q^2(2R+1)^{2d}C_u$, $q := \max(|q_-|, |q_+|)$ and $\tilde{\Lambda}_L^{(n)} := \{\mathbf{j} \in \Lambda_L^{(n)}(\mathbf{x}) : U_\alpha e_{\mathbf{j}} \neq 0 \text{ for some } \alpha \in \mathcal{I}_F\}$.

Proof. $H_{\Lambda}^\bullet(\omega)$ is a self-adjoint operator given by $H_{\Lambda}^\bullet(\omega) = H_1 + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$, with $H_1 = H_{0, \Lambda}^\bullet$ self-adjoint. Moreover, $U_\alpha \geq 0$, $\|U_\alpha\| \leq C_U := nC_u$ and $\mathcal{I}_{\mathbf{j}} := \{\alpha : U_\alpha e_{\mathbf{j}} \neq 0\} \subseteq \bigcup_{k=1}^n \Lambda_R^{(1)}(j_k)$, hence $C_{\text{fin}} := \max|\mathcal{I}_{\mathbf{j}}| \leq n(2R+1)^d$. The claim now follows from Theorem 5.2.2. \square

5.3.2 Two-volume bounds

In this section we prove a two-volume Wegner bound for the discrete multi-particle model of Section 4.4.1, which is needed for localization. If $n = 1$, such bounds are an easy consequence of the one-volume bound given in Theorem 4.4.1, see e.g. [59, Lemma 5.28]. This is no longer the case once $n > 1$, essentially because in the multi-particle multiscale analysis, it is not sufficient to consider pairs of *fully separated* cubes, i.e. satisfying $\Pi\Lambda_{L+R}^{(n)}(\mathbf{x}) \cap \Pi\Lambda_{L+R}^{(n)}(\mathbf{y}) = \emptyset$, where $\Pi\Lambda_S^{(n)}(\mathbf{u}) = \bigcup_{i=1}^n \Lambda_S^{(1)}(u_i)$, but one must consider more generally pairs of *pre-separable* cubes to be defined below. Consequently, one needs to consider a smaller W in Theorem 4.4.1 and integrate only on part of Ω .

The results of this section improve the upper bound of [26, Theorem 2] and extend the recent [68, Corollary 2.4] because we do not assume that $\mathbb{P} = \otimes \mu$ with $\mu = \rho(t)dt$, and we allow for more general single-site potentials u_α . Note however that the arguments of [68] allow for probabilities \mathbb{P} as general as the ones we consider here.

We use the notations of Section 4.4.1. We assume moreover that $\mathbb{P} := \otimes_{\alpha \in \mathbb{Z}^d} \mu_\alpha$ for some probability measures μ_α on \mathbb{R} , and that there is a constant $c > 0$ such that $u_\alpha \geq c\delta_\alpha$ for all α . This means we assume the $\{\omega_\alpha\}$ are independent and the single-site potentials cover all lattice points.

Given $\emptyset \neq D \subseteq \mathbb{Z}^d$, $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, we define

$$\mathbb{E}_D\{f(\omega)\} := \int_{\mathcal{B}^D} f(\omega) d\mathbb{P}_D((\omega_\alpha)_{\alpha \in D}), \quad \text{where } \mathbb{P}_D := \otimes_{\alpha \in D} \mu_\alpha$$

which depends on $(\omega_\beta)_{\beta \notin D}$.

Given $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{Z}^d)^n$, $\mathbb{L} = (L_1, \dots, L_n) \in \mathbb{N}^n$, define rectangles $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x}) := \prod_{i=1}^n \Lambda_{L_i}^{(1)}(x_i)$, and given $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$, put $\Pi_{\mathcal{J}}\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x}) := \bigcup_{i \in \mathcal{J}} \Lambda_{L_i}^{(1)}(x_i)$. Now define $\mathbb{L} + R := (L_1 + R, \dots, L_n + R)$ and put $H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}(\omega) = H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}^\bullet(\omega)$. Then we have the following

Theorem 5.3.2. *For any interval $I \subset \mathbb{R}$, any $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$ and any $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$, taking $G(\mathcal{J}) := \Pi_{\mathcal{J}}\Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x})$, we have*

$$\mathbb{E}_{G(\mathcal{J})} \{ \text{tr}[\chi_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}(\omega))] \} \leq C_W \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})| \cdot s_F\left(\mathbb{P}, \frac{|I|}{nC_u}\right)$$

for any $(\omega_\beta)_{\beta \notin G(\mathcal{J})}$, where $C_W := 4c^{-2}n^2C_u^2(2R+1)^{2d}$.

Proof. Let $W_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}^{\mathcal{J}} := \sum_{\alpha \in \Pi_{\mathcal{J}}\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})} U_\alpha$. We first notice that for any $\mathbf{y} \in \Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$,

$$W_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}^{\mathcal{J}}(\mathbf{y}) \geq c \sum_{1 \leq i \leq n} \sum_{\alpha \in \Pi_{\mathcal{J}}\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})} \delta_\alpha(y_i) \geq c \sum_{i \in \mathcal{J}} 1 = c \cdot |\mathcal{J}|,$$

so if (e_j) is the canonical basis of $\ell^2(\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x}))$, we have for any $\omega \in \Omega$,

$$\text{tr}[\chi_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}(\omega))] \leq c^{-2}|\mathcal{J}|^{-2}C_{\text{fin}} \sum_{\mathbf{j} \in \Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})} \sum_{\alpha \in \Pi_{\mathcal{J}}\Lambda_R^{(n)}(\mathbf{j})} \langle U_\alpha \chi_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}(\omega)) U_\alpha e_{\mathbf{j}}, e_{\mathbf{j}} \rangle$$

by Proposition 4.3.1, where $C_{\text{fin}} = \max_{\mathbf{j} \in \Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})} |\Pi_{\mathcal{J}}\Lambda_R^{(n)}(\mathbf{j})| \leq |\mathcal{J}| \cdot (2R+1)^d$. Now given $\mathbf{j} \in \Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$ and $\alpha \in \Pi_{\mathcal{J}}\Lambda_R^{(n)}(\mathbf{j})$ put $\phi := U_\alpha^{1/2} e_{\mathbf{j}}$. Then

$$\mathbb{E}_{G(\mathcal{J})} \{ \langle U_\alpha \chi_I(H_{\Lambda}(\omega)) U_\alpha e_{\mathbf{j}}, e_{\mathbf{j}} \rangle \} = \mathbb{E}_{G(\mathcal{J}) \setminus \{\alpha\}} \left\{ \int_{\mathcal{B}} \langle U_\alpha^{1/2} \chi_I(H_{\Lambda}(\omega)) U_\alpha^{1/2} \phi, \phi \rangle d\mu_\alpha(\omega_\alpha) \right\}.$$

Fix $(\omega_\beta)_{\beta \neq \alpha}$. Using the spectral averaging of Section B.4 with $A = H_{0,\Lambda} + \sum_{\beta \neq \alpha} \omega_\beta U_\beta$, $B = U_\alpha$, $t = \omega_\alpha$ and $\lambda = \frac{1}{nC_u}$ we get

$$\mathbb{E}_{G(\mathcal{J})} \{ \langle U_\alpha \chi_I(H_\Lambda(\omega)) U_\alpha e_j, e_j \rangle \} \leq 4 \|U_\alpha\| \|U_\alpha^{1/2} e_j\|^2 s(\mu_\alpha, \lambda|I|) \leq 4C_U^2 s(\mu_\alpha, \lambda|I|),$$

where $C_U \leq nC_u$. This completes the proof. \square

Given $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$, we say that $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$ is \mathcal{J} -pre-separable from $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})$ if

$$\Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x}) \cap (\Pi_{\mathcal{J}^c} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x}) \cup \Pi \Lambda_{\mathbb{K}+R}^{(n)}(\mathbf{y})) = \emptyset.$$

We say that $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$ and $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})$ are *pre-separable* if there exists $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$ such that $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$ is \mathcal{J} -pre-separable from $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})$ or $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})$ is \mathcal{J} -pre-separable from $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$. Note that if $\mathcal{J} = \{1, \dots, n\}$, this means that $\Pi \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x}) \cap \Pi \Lambda_{\mathbb{K}+R}^{(n)}(\mathbf{y}) = \emptyset$, i.e. the cubes are fully separated. We now have the following

Theorem 5.3.3. *If $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$ and $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})$ are pre-separable, then*

$$\mathbb{P} \left\{ \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}), \sigma(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})})) < \varepsilon \right\} \leq C_W \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})| \cdot |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})| \cdot s_F \left(\mathbb{P}, \frac{2\varepsilon}{nC_u} \right),$$

where $C_W := 4c^{-2}n^2C_u^2(2R+1)^{2d}$.

Note that for the model studied in [26], $u_\alpha = \delta_\alpha$ for all α , so $c = 1$, $C_u = 1$, $R = 0$ and C_W reduces to $C_W = 4n^2$.

Proof. Suppose $\Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x}) \cap (\Pi_{\mathcal{J}^c} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x}) \cup \Pi \Lambda_{\mathbb{K}+R}^{(n)}(\mathbf{y})) = \emptyset$ for some $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$. Then denoting the eigenvalues of $H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})}$ by $E_j^{\mathbf{y}}$ we have for $G(\mathcal{J}) := \Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x})$,

$$\begin{aligned} & \mathbb{P} \left\{ \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}), \sigma(H_{\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})})) < \varepsilon \right\} \\ &= \mathbb{E}_{G(\mathcal{J})^c} \left\{ \mathbb{P}_{G(\mathcal{J})} \left\{ \min_{1 \leq j \leq |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})|} \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}), E_j^{\mathbf{y}}) < \varepsilon \right\} \right\}. \end{aligned}$$

Now the $E_j^{\mathbf{y}}$ do not depend on $(\omega_\alpha)_{\alpha \in G(\mathcal{J})}$ since $G(\mathcal{J}) \cap \Pi \Lambda_{\mathbb{K}+R}^{(n)}(\mathbf{y}) = \emptyset$. Hence, using the conditional Markov inequality and Theorem 5.3.2 we have

$$\begin{aligned} & \mathbb{P}_{G(\mathcal{J})} \left\{ \min_{1 \leq j \leq |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})|} \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}), E_j^{\mathbf{y}}) < \varepsilon \right\} \\ & \leq |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})| \cdot \sup_{\lambda \in \mathbb{R}} \mathbb{P}_{G(\mathcal{J})} \left\{ \text{dist}(\sigma(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}), \lambda) < \varepsilon \right\} \\ & = |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})| \cdot \sup_{\lambda \in \mathbb{R}} \mathbb{P}_{G(\mathcal{J})} \left\{ \text{tr}[\chi_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})})] \geq 1 \right\} \\ & \leq |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})| \cdot \sup_{\lambda \in \mathbb{R}} \mathbb{E}_{G(\mathcal{J})} \left\{ \text{tr}[\chi_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})})] \right\} \\ & \leq C_W \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})| \cdot |\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})| \cdot s_F \left(\mathbb{P}, \frac{2\varepsilon}{nC_u} \right) \end{aligned}$$

for any $(\omega_\beta)_{\beta \notin G(\mathcal{J})}$, so the claim follows. If instead $\Lambda_{\mathbb{K}}^{(n)}(\mathbf{y})$ is \mathcal{J} -separable from $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$, then we repeat the same argument, taking $G(\mathcal{J}) := \Pi_{\mathcal{J}} \Lambda_{\mathbb{K}+R}^{(n)}(\mathbf{y})$ instead. \square

Remark 5.3.4. 1. Theorem 5.3.2 may be seen as a conditional Wegner estimate. More precisely, for our probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where $\Omega = \mathcal{B}^{\mathbb{Z}^d}$ and $\mathbb{P} = \otimes_{\alpha \in \mathbb{Z}^d} \mu_\alpha$, if $D \subseteq \mathbb{Z}^d$ is non-empty and if $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $f \geq 0$, then

$$\mathbb{E}\{f(\omega) | (\omega_\beta)_{\beta \notin D}\} = \mathbb{E}_D\{f(\omega)\} \quad \text{a.s.}$$

To see this, simply note that if \mathcal{G} is the σ -algebra generated by $(\omega_\beta)_{\beta \notin D}$, then the RHS is \mathcal{G} -measurable (see [96, Theorem 8.8]), and its integral on any $G \in \mathcal{G}$ yields $\int_G f(\omega) d\mathbb{P}(\omega)$ (since any $G \in \mathcal{G}$ takes the form $\mathcal{B}^D \times C$ for some $C \subseteq \mathcal{B}^{\mathbb{Z}^d \setminus D}$).

2. Theorem 5.3.2 actually holds for arbitrary \mathbb{P} , i.e. we can avoid the assumption that \mathbb{P} is a product measure. In this case, instead of estimating $\mathbb{E}_{G(\mathcal{J})}\{\cdot\}$, one has to estimate $\mathbb{E}\{\cdot | (\omega_\beta)_{\beta \notin G(\mathcal{J})}\}$, and this may be done using the same arguments, bypassing the independence by using regular conditional distributions as in Section 4.5.2. We only assumed in this section that $\mathbb{P} = \otimes_{\alpha \in \mathbb{Z}^d} \mu_\alpha$ to prove Theorem 5.3.3.

5.3.3 Discrete Delone operators

As a final application, we consider Wegner bounds near the spectral bottom of a discrete Delone operator. Such operators are already included in Section 4.4.1. However, here we are concerned with positive perturbations, which correspond to case (3) of Lemma 4.4.2, and there we had to rely on [37, Theorem 1.3] to illustrate that our bound is non-trivial. In the special case where the background potential vanishes, we describe here a different argument that appears in [94]. There the author proved a deterministic UP (cf. Section 5.5) and concluded using the results of [30]. We shall instead prove a random UP and conclude using Theorem 4.3.2. The advantage is that, on one hand the constant in the Wegner bound becomes explicit, and on the other hand the proof Theorem 4.3.2 is extremely simple compared to the one of [30].

Consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, let $D \subset \mathbb{Z}^d$ be a Delone set, i.e. there exists $K \geq 1$ such that, for any $n \in \mathbb{Z}^d$, the cube $\Lambda_K(n)$ contains at least one point of D . Now consider the probability space (Ω, \mathbb{P}) , where $\Omega = [0, M]^D$ and given $\omega \in \Omega$, define

$$H(\omega) = -\Delta + \sum_{\alpha \in D} \omega_\alpha \delta_\alpha.$$

Assume for simplicity that $\mathbb{P} = \otimes \mu$, for some probability measure μ on \mathbb{R} with $\text{supp } \mu \subseteq [0, M]$. Assuming moreover that $0 \in \text{supp } \mu$, we have that

$$[0, 4d] \subseteq \sigma(H(\omega)) \subseteq [0, 4d + M] \quad \text{almost surely.}$$

The second inclusion is easy, for the first one, the arguments of [104, Lemma 1.4.1] or [60, Theorem 3.9] essentially work. Namely,

- Given $E \in [0, 4d]$, take a Weyl sequence φ_n for $-\Delta$. This sequence may be chosen to have compact support, say $\text{supp } \varphi_n \subset \Lambda_{l_n}(0)$ for some $l_n \in \mathbb{N}$.
- With probability one, there exists a sequence $\Lambda_{l_n}(x_n) \cap D$ in which the potential $V_\omega = \sum_{\alpha} \omega_\alpha \delta_\alpha$ is very small.
- Translate the Weyl sequence φ_n around the cubes $\Lambda_{l_n}(x_n)$. The translated sequence is still a Weyl sequence for $-\Delta$, moreover V_ω is very small in $\Lambda_{l_n}(x_n)$, since it is very small in $\Lambda_{l_n}(x_n) \cap D$ and zero in $\Lambda_{l_n}(x_n) \cap D^c$. Hence the translated sequence is a Weyl sequence for $H(\omega)$.

Let $H_{\Lambda_L}(\omega)$ be the restriction of $H(\omega)$ to a cube $\Lambda_L = \Lambda_L(x)$, $x \in \mathbb{Z}^d$, with Neumann boundary conditions. Let $W(n) := \sum_{\alpha \in D \cap \Lambda_L} \delta_\alpha(n)$. Then

Lemma 5.3.5 (cf. [94]). *There exists $E_* > 0$ such that in $I = [0, E_*]$, we have for any cube $\Lambda_L \subset \mathbb{Z}^d$ with $L > K$, and for any $\omega \in \Omega$,*

$$\chi_I(H_{\Lambda_L}(\omega))W\chi_I(H_{\Lambda_L}(\omega)) \geq \gamma\chi_I(H_{\Lambda_L}(\omega)),$$

where $\gamma := \frac{1}{2}(5K)^{-d}$. Moreover, $E_* = c(d)K^{-2d-2}$.

Now instead of using the results of [30], we use our theorems and deduce

Corollary 5.3.6. *There exists $E_* > 0$ such that in $I = [0, E_*]$, we have for any cube $\Lambda_L \subset \mathbb{Z}^d$ with $L > K$,*

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L}(\omega))]\} \leq C_W \cdot |D \cap \Lambda_L| \cdot s(\mu, |I|),$$

where $C_W = 24(5K)^{2d}$. Moreover, $E_* = c(d)K^{-2d-2}$.

Let us mention that Elgart and Klein prove a stronger result in [37] using different methods. In particular, they can allow for background potentials.

Proof of Corollary 5.3.6. $H_{\Lambda_L}(\omega)$ is a self-adjoint operator given by $H_{\Lambda_L}(\omega) = -\Delta_{\Lambda_L} + \sum_{\alpha \in D \cap \Lambda_L} \omega_\alpha \delta_\alpha$. It satisfies Hypotheses (A) and (B) by Lemma 5.3.5, with $C_{\text{fin}} = C_U = 1$ and $\gamma = \frac{1}{2}(5K)^{-d}$. The claim thus follows from Theorem 4.3.2, since $J_{\text{eff}} = D \cap \Lambda_L$. \square

For completeness, we now prove Lemma 5.3.5.

Proof of Lemma 5.3.5. We repeat the arguments of [94, Lemma 2.1], which still work to establish the random UP. Define the average \bar{W} of W by

$$\bar{W}(n) = \frac{1}{|\Lambda_{2K}(0)|} \sum_{j \in \Lambda_{2K}(0)} W(n-j).$$

Let $n \in \Lambda_L$, then we may find $m \in \Lambda_{L-K}$ such that $\Lambda_K(m) \subset \Lambda_{2K}(n)$. But D is a Delone set, so we may find $m_D \in D \cap \Lambda_K(m) \subset D \cap \Lambda_L$. Thus,

$$\bar{W}(n) = |\Lambda_{2K}|^{-1} \sum_{j \in \Lambda_{2K}(n)} W(j) \geq |\Lambda_{2K}|^{-1} W(m_D) = |\Lambda_{2K}|^{-1}.$$

We thus showed that $\bar{W} \geq |\Lambda_{2K}|^{-1} \chi_{\Lambda_L}$.

Now let $I = [0, E_*]$ with E_* to be chosen later, fix $\omega \in \Omega$, and let $f \in \text{Ran } \chi_I(H_{\Lambda_L}(\omega))$ with $\|f\| = 1$. Then

$$\langle Wf, f \rangle = \langle \bar{W}f, f \rangle + \langle (W - \bar{W})f, f \rangle \geq |\Lambda_{2K}|^{-1} - \langle (\bar{W} - W)f, f \rangle.$$

But

$$\langle (\bar{W} - W)f, f \rangle = |\Lambda_{2K}|^{-1} \sum_{j \in \Lambda_{2K}(0)} \langle (W(\cdot - j) - W)f, f \rangle,$$

and

$$\begin{aligned} |\langle (W(\cdot - j) - W)f, f \rangle| &= |\langle Wf(\cdot + j), f(\cdot + j) \rangle - \langle Wf, f \rangle| \\ &= |\langle Wf(\cdot + j), (f(\cdot + j) - f) \rangle + \langle W(f(\cdot + j) - f), f \rangle| \\ &\leq 2\|W\|_\infty \|f(\cdot + j) - f\|, \end{aligned}$$

since $\|f(\cdot + j)\| = \|f\| = 1$. But $\|W\|_\infty = 1$, so it follows that

$$|\langle (\overline{W} - W)f, f \rangle| \leq 2 |\Lambda_{2K}|^{-1} \sum_{j \in \Lambda_{2K}(0)} \|f(\cdot + j) - f\|.$$

Now for $j \in \Lambda_{2K}(0)$, we have $\|j\|_1 \leq d \|j\|_\infty \leq 2Kd$, so there exists a path of length $2Kd$, say (a_0, \dots, a_{2Kd}) joining 0 to j , i.e. with $a_0 = 0$, $a_{2Kd} = j$ and $\|a_{r+1} - a_r\|_1 = 1$. Thus,

$$\|f(\cdot + j) - f\| \leq \sum_{r=0}^{2Kd-1} \|f(\cdot + a_{r+1}) - f(\cdot + a_r)\|.$$

But since $\|a_{r+1} - a_r\|_1 = 1$, we have

$$\begin{aligned} \|f(\cdot + a_{r+1}) - f(\cdot + a_r)\|^2 &= \sum_{k \in \Lambda_L} |f(k + a_{r+1}) - f(k + a_r)|^2 \\ &\leq \sum_{n \in \Lambda_L} \sum_{\|m-n\|_1=1} |f(m) - f(n)|^2 = 2 \langle (-\Delta)_{\Lambda}^N f, f \rangle \end{aligned}$$

(see [60, Section 5.2] for Neumann conditions). Since $V_\omega \geq 0$, it follows that

$$\|f(\cdot + j) - f\| \leq 2Kd \sqrt{2 \langle H_{\Lambda_L}(\omega) f, f \rangle}$$

for any $j \in \Lambda_{2K}(0)$, and consequently

$$|\langle (\overline{W} - W)f, f \rangle| \leq 4Kd \sqrt{2 \langle H_{\Lambda_L}(\omega) f, f \rangle}.$$

Summarizing, we showed that

$$\langle Wf, f \rangle \geq |\Lambda_{2K}|^{-1} - 4Kd \sqrt{2 \langle H_{\Lambda_L}(\omega) f, f \rangle}.$$

Since $f = \chi_{[0, E_*]}(H_{\Lambda_L}(\omega))f$, it follows from the functional calculus that $\langle H_{\Lambda_L}(\omega) f, f \rangle \leq E_* \|f\|^2 = E_*$. Since $|\Lambda_{2K}| = (4K + 1)^d \leq (5K)^d$, we obtain

$$\langle Wf, f \rangle \geq (5K)^{-d} - 4Kd \sqrt{2E_*}.$$

Taking $q \in (0, 1)$ and choosing $E_* = q^2 (4\sqrt{2}Kd)^{-2} (5K)^{-2d}$, we finally obtain

$$\langle Wf, f \rangle \geq (5K)^{-d} - (4\sqrt{2}Kd)q(4\sqrt{2}Kd)^{-1} (5K)^{-d} = (1 - q)(5K)^{-d}.$$

Choosing $q = \frac{1}{2}$ and noting that f was arbitrary, it follows that

$$\chi_I(H_{\Lambda_L}(\omega))W\chi_I(H_{\Lambda_L}(\omega)) \geq \gamma \chi_I(H_{\Lambda_L}(\omega))$$

with $\gamma := \frac{1}{2}(5K)^{-d}$. □

5.4 Some difficulties

5.4.1 Sign-indefinite potentials

If we *suppose* that our bounds in Theorems 4.3.3 and 4.3.4 are formulated in terms of the following modulus of continuity

$$(4-1) \quad \mathfrak{s}_F(\mathbb{P}, \varepsilon) = \max_{\alpha \in \mathcal{I}_F} \sup_{E \in \mathbb{R}} \mathbb{P}(\omega_\alpha \in [E, E + \varepsilon]),$$

then we can tackle sign-indefinite potentials and obtain quite important results. This modulus is just $s_F(\mathbb{P}, \varepsilon)$, but with the $\sup_{E \in \mathbb{R}}$ interchanged with \mathbb{E}_{Y_α} . Unfortunately, there is no justification for such an interchange; all we can say is that $\mathfrak{s}_F(\mathbb{P}, \varepsilon) \leq s_F(\mathbb{P}, \varepsilon)$, and that $\mathfrak{s}_F(\mathbb{P}, \varepsilon) = s_F(\mathbb{P}, \varepsilon)$ if \mathbb{P} is a product measure.

Consider the Hilbert space $L^2(\mathbb{R}^d)$. Let $G \subset \mathbb{R}^d$ be a discrete set such that $\#\{\Lambda \cap G\} < \infty$ for any bounded $\Lambda \subset \mathbb{R}^d$. Assume, moreover that $(G, +)$ is a group (e.g. $G = \mathbb{Z}^d$, $(M\mathbb{Z})^d$, $\mathbb{Z}^{d_1} \times \{0\}$, ...etc) and consider the probability space (Ω, \mathbb{P}) , where $\Omega := [q_-, q_+]^G$, $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$, and $\mathbb{P} = \otimes_{\alpha \in G} \mu_\alpha$, for some probability measures μ_α on \mathbb{R} supported in $[q_-, q_+]$. Given $\omega = (\omega_\alpha) \in \Omega$, let

$$H(\omega) = H_0 + V^\omega, \quad H_0 := -\Delta + V_0,$$

where $V_0 \geq v_0$ is a bounded real non-random potential. Again we can consider more general H_0 . Given $x \in \mathbb{R}^d$, we assume

$$V^\omega(x) = \sum_{\alpha \in G} \omega_\alpha u(x - \alpha), \quad \text{where} \quad u(x) = \sum_{\beta \in G} c_\beta w(x - \beta)$$

for some $c_\beta \in \mathbb{R}$. We assume $0 \leq w \leq C_w$. Since the c_β are not required to have a fixed sign, the potential u is not sign-definite. This model was introduced in [108] for $G = \mathbb{Z}^d$, $w \geq \kappa \chi_{[0,1]^d}$, and following that paper we will represent V^ω as $V^\omega(x) = \sum_{\zeta \in G} \eta_\zeta(\omega) w(x - \zeta)$ for some $\eta_\zeta(\omega)$. Unlike [108], [111] however, we will not need to invert any matrices, and this gives us more freedom on the choice of the c_β . In fact, we will only assume that $c_\beta = 0$ outside a finite $\Gamma \subset G$, and that $\|c\|_\infty := \max_{\beta \in \Gamma} |c_\beta| \neq 0$. We also assume $w : \mathbb{R}^d \rightarrow \mathbb{R}$ is compactly supported, with $\text{supp } w \subset \Lambda_R(0)$ for some $R > 0$.

Discussion of the results. Assuming that Theorems 4.3.3 and 4.3.4 hold with $\mathfrak{s}_F(\mathbb{P}, \varepsilon)$ instead of $s_F(\mathbb{P}, \varepsilon)$, which we cannot prove, we provide Wegner bounds for sign-indefinite potentials in the continuum and on the lattice, without any regularity assumption on \mathbb{P} . To the best of our knowledge, these would be the first results with such generality, and they would imply new localization results; see [70], [69], [54], [108]. For example, if $\mathbb{P} = \otimes \mu$, then our estimates would allow to extend [70, Theorem 0.4] to distributions μ which are merely log-Hölder continuous with a large exponent. The weakest regularity assumption so far is for lattice models in the large disorder limit, see [39], where the authors prove localization if μ is Hölder continuous. For continuous models, the only available results assume that μ admits a density $\mu = \rho(\lambda)d\lambda$, with ρ of bounded variation. Note that a Wegner bound for sign-indefinite potentials is also mentioned in [30], but since it relies on the method of [108], it has the same restrictions on c_β and \mathbb{P} .

On the other hand, our bounds are not linear in $|\Lambda|$ and thus have no direct application for the integrated density of states (IDS). This in contrast to the results of [54], [111], [110], [88], [75] and [39], all of which have applications for the IDS. See also the survey [38]. We said “direct” because one might hope to linearize the Wegner bound by combining it with multiscale analysis; see [75, Appendix B].

Our results would somehow unify the approaches of [108], [69] and [54], in the sense that, if we have a “generalized covering”, then our bound holds in any interval, otherwise it holds for intervals below the spectral bottom of the unperturbed operator.

The hypothesis $\mathbb{P} = \otimes \mu_\alpha$ is only used to prove the following simple lemma. This lemma is the key to pass from monotone correlated potentials to sign-indefinite ones, and it replaces the arguments of inversion of Toeplitz matrices in [108], [111]. This is the reason why we have almost no hypotheses on the c_β and do not need μ_α to have a density.

Lemma 5.4.1. *Let (Ω, \mathbb{P}) be a probability space, with $\Omega = \mathcal{C}^{\mathcal{I}}$, where $\mathcal{C} \subseteq \mathbb{R}$ is an interval, \mathcal{I} is a countable index set, and $\mathbb{P} = \otimes_{\alpha \in \mathcal{I}} \mu_\alpha$ for some probability measures μ_α on \mathbb{R} .*

Let $\Gamma \subseteq \mathcal{I}$ be finite and consider the map $\varphi : \Omega \rightarrow \mathbb{R}$ given by

$$\varphi(\omega) = r + \sum_{\alpha \in \Gamma} c_\alpha \omega_\alpha$$

for some $r, c_\alpha \in \mathbb{R}$. Then if $\|c\|_\infty := \max_{\alpha \in \Gamma} |c_\alpha| \neq 0$, we have for any interval I ,

$$\mathbb{P}\{\varphi(\omega) \in I\} \leq s_\Gamma(\mathbb{P}, \|c\|_\infty^{-1}|I|),$$

where $s_\Gamma(\mathbb{P}, \varepsilon) := \max_{\alpha \in \Gamma} \sup_{E \in \mathbb{R}} \mu_\alpha[E, E + \varepsilon]$.

Proof. Assume $I = [a, b]$. We have $\|c\|_\infty = |c_\alpha|$ for some $\alpha \in \Gamma$, which we fix. Given $x \in \mathcal{C}$ and $\hat{\omega}_\alpha := (\omega_\beta)_{\beta \neq \alpha} \in \mathcal{C}^{\mathcal{I} \setminus \{\alpha\}}$ denote by $(x, \hat{\omega}_\alpha)$ the element $(x_\beta) \in \Omega$ with $x_\alpha = x$ and $x_\beta = \omega_\beta$ for $\beta \neq \alpha$. Now let $Q := \{\omega : \varphi(\omega) \in I\}$, let $\hat{\omega}_\alpha \in \mathcal{C}^{\mathcal{I} \setminus \{\alpha\}}$ and consider the section $Q_{\hat{\omega}_\alpha} := \{x \in \mathcal{C} : (x, \hat{\omega}_\alpha) \in Q\}$. Then by definition of a product measure,

$$(4-2) \quad \mathbb{P}(Q) = \int_{\hat{\omega}_\alpha \in \mathcal{C}^{\mathcal{I} \setminus \{\alpha\}}} \mu_\alpha(Q_{\hat{\omega}_\alpha}) d\hat{\mathbb{P}}(\hat{\omega}_\alpha),$$

where $\hat{\mathbb{P}} := \otimes_{\beta \neq \alpha} \mu_\beta$. But for any fixed $\hat{\omega}_\alpha$,

$$Q_{\hat{\omega}_\alpha} = \{x \in \mathcal{C} : \varphi(x, \hat{\omega}_\alpha) \in I\} = \left\{x \in \mathcal{C} : a \leq r + \sum_{\beta \neq \alpha} c_\beta \omega_\beta + c_\alpha x \leq b\right\}.$$

Let $\Sigma := \sum_{\beta \neq \alpha} c_\beta \omega_\beta$. Then if $c_\alpha > 0$ we get $Q_{\hat{\omega}_\alpha} = [\frac{a-r-\Sigma}{c_\alpha}, \frac{b-r-\Sigma}{c_\alpha}]$, and if $c_\alpha < 0$ we get $Q_{\hat{\omega}_\alpha} = [\frac{b-r-\Sigma}{c_\alpha}, \frac{a-r-\Sigma}{c_\alpha}]$. In any case, $Q_{\hat{\omega}_\alpha}$ is an interval of length $\frac{b-a}{|c_\alpha|} = \frac{|I|}{\|c\|_\infty}$. Thus $\mu_\alpha(Q_{\hat{\omega}_\alpha}) \leq \sup_{E \in \mathbb{R}} \mu_\alpha[E, E + \frac{|I|}{\|c\|_\infty}]$ for any $\hat{\omega}_\alpha$, and the claim follows by (4-2). \square

Conjecture 5.4.2. For any $I = (E_1, E_2)$, there exists $C_W(d, E_2, v_0, q_\pm, C_w, \|c\|_\infty, |\Gamma|)$ such that for any cube $\Lambda_L(x)$,

(1) If $G = \mathbb{Z}^d$ and $\exists \kappa > 0$ with $w \geq \kappa \cdot \chi_0$, where $\chi_0 := \chi_{[-\frac{1}{2}, \frac{1}{2}]^d}$, then

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L(x)}^\bullet)]\} \leq C_W \cdot |\Lambda_L(x)| \cdot |\Lambda_{L+R}(x)| \cdot s(\mathbb{P}, \kappa^{-1}\|c\|_\infty^{-1}|I|),$$

for $\bullet = D, N, \text{per}$, where $s(\mathbb{P}, \varepsilon) = \sup_{\alpha \in G} \sup_{E \in \mathbb{R}} \mu_\alpha[E, E + \varepsilon]$.

(2) In the general case, for any $q > q^*(q_\pm, \|c\|_\infty, |\Gamma|)$, if $E_2 < E_q := \inf \sigma(H_0 + qW)$, where $W(x) := \sum_{\beta \in G} w(x - \beta)$, then there exists $c_W = c_W(q, q_\pm, \|c\|_\infty, |\Gamma|, E_q - E_2)$ such that

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L(x)}^D)]\} \leq C_W \cdot |\Lambda_L(x)| \cdot \#\{\Lambda_{L+R}(x) \cap G\} \cdot s(\mathbb{P}, c_W|I|).$$

Note that $E_0 \leq E_q$ for any $q \geq 0$, so if $E_2 < E_0$, then $E_2 < E_q$ and we have in particular a Wegner bound below E_0 .

Proof. Note that $V^\omega(x) = \sum_{\alpha \in G} \omega_\alpha u(x - \alpha) = \sum_{\alpha \in G} \omega_\alpha \sum_{\beta \in G} c_\beta w(x - \alpha - \beta)$. As β runs over G , $\zeta := \alpha + \beta$ also covers G because $(G, +)$ is a group. So we get

$$V^\omega(x) = \sum_{\zeta \in G} w(x - \zeta) \sum_{\alpha \in G} c_{\zeta - \alpha} \omega_\alpha = \sum_{\zeta \in G} \eta_\zeta(\omega) w(x - \zeta),$$

where $\eta_\zeta(\omega) := \sum_{\alpha \in G} c_{\zeta - \alpha} \omega_\alpha$. This representation of $V^\omega(x)$ appeared before in [108], [111]. Now the $(\eta_\zeta(\omega))$ live in some $\tilde{\Omega} = [v_-, v_+]^G$, for some $v_\pm = v_\pm(q_-, q_+, \|c\|_\infty, |\Gamma|)$. Define the random self-adjoint operator $A : \tilde{\Omega} \rightarrow S(L^2(\mathbb{R}^d))$ by $A(\eta) = H_0 + \sum_{\zeta \in G} \eta_\zeta w(x -$

ζ), for $\eta = (\eta_\zeta) \in \tilde{\Omega}$. Then if $T : \Omega \rightarrow \tilde{\Omega}$ is given by $T : \omega \mapsto (\eta_\zeta(\omega))$, we have $H(\omega) = A \circ T(\omega)$.

Now define $\tilde{\mathbb{P}}$ on $\tilde{\Omega}$ by $\tilde{\mathbb{P}}(B) := \mathbb{P}(T^{-1}(B))$. Then by [36, Theorem 4.1.11], $\tilde{\mathbb{P}}$ is a probability measure on $\tilde{\Omega}$, and for any measurable function $g : \tilde{\Omega} \rightarrow \mathbb{R}$, we have $\tilde{\mathbb{E}}\{g(\eta)\} = \mathbb{E}\{(g \circ T)(\omega)\}$ whenever either side exists. In particular, $\mathbb{E}\{\text{tr}[\chi_I(H_\Lambda(\omega))]\} = \mathbb{E}\{\text{tr}[\chi_I(A_\Lambda(T(\omega)))]\} = \tilde{\mathbb{E}}\{\text{tr}[\chi_I(A_\Lambda(\eta))]\}$.

Now note that $A : \tilde{\Omega} \rightarrow S(L^2(\mathbb{R}^d))$ is sign-definite. For case (1), if $G = \mathbb{Z}^d$ and $w \geq \kappa\chi_0$, then $w_\zeta(x) := w(x - \zeta) \geq \kappa\chi_\zeta$, so² by Theorem 4.4.3, for any interval I we have

$$\tilde{\mathbb{E}}\{\text{tr}[\chi_I(A_\Lambda(\eta))]\} \leq C_W \cdot |\Lambda_L(x)| \cdot |\mathcal{I}_F| \cdot \mathfrak{s}_F(\tilde{\mathbb{P}}, \kappa^{-1}|I|),$$

where $\mathcal{I}_F := \Lambda_{L+R}(x) \cap \mathbb{Z}^d$ and $\mathfrak{s}_F(\tilde{\mathbb{P}}, \varepsilon) = \sup_{\zeta \in \mathcal{I}_F} \sup_{E \in \mathbb{R}} \tilde{\mathbb{P}}\{\eta_\zeta \in [E, E + \varepsilon]\}$. Let $B := \{\eta_\zeta \in [E, E + \varepsilon]\}$. Then by definition of $\tilde{\mathbb{P}}$,

$$\tilde{\mathbb{P}}(B) = \mathbb{P}(T(\omega) \in B) = \mathbb{P}(\eta_\zeta(\omega) \in [E, E + \varepsilon]).$$

But $\eta_\zeta(\omega) = \sum_{\alpha \in G} c_{\zeta-\alpha} \omega_\alpha$, so by Lemma 5.4.1, $\mathbb{P}(\eta_\zeta(\omega) \in [E, E + \varepsilon]) \leq s_{\Gamma+\zeta}(\mathbb{P}, \varepsilon \|c\|_\infty^{-1})$. Thus, $\mathfrak{s}_F(\tilde{\mathbb{P}}, \varepsilon) \leq s(\mathbb{P}, \varepsilon \|c\|_\infty^{-1})$, which completes the proof of (1).

The same argument proves case (2), for $q > v_+ =: q^*$. \square

Now consider the lattice. Let $G \subseteq \mathbb{Z}^d$ be a subgroup and consider the probability space (Ω, \mathbb{P}) , with $\Omega = [q_-, q_+]^G$ and $\mathbb{P} = \otimes_{\alpha \in G} \mu_\alpha$ for some probability measures μ_α on \mathbb{R} . Given $\omega = (\omega_\alpha) \in \Omega$, let

$$H(\omega) = H_0 + V^\omega, \quad H_0 := -\Delta + V_0,$$

where $-\Delta$ is the discrete Laplace operator and V_0 is a bounded real non-random potential. Given $x \in \mathbb{Z}^d$, we assume

$$V^\omega(x) = \sum_{\alpha \in G} \omega_\alpha u(x - \alpha),$$

with $\text{supp } u \subseteq G \cap \Lambda_M(0)$ and u not identically zero. Now any such u may be written as $u = \sum_{\beta \in G} u(\beta) \delta_\beta = \sum_{\beta \in G} u(\beta) \delta_0(\cdot - \beta)$. In other words, in the lattice we automatically have a partial cover (the condition $w \geq \chi_0$ is satisfied), and it becomes a total cover if $G = \mathbb{Z}^d$. Let us denote $\|u\|_\infty = \max_{\beta \in G} |u(\beta)|$; it is a max since $u(\beta) = 0$ outside $\Lambda_M(0)$. Since Theorems 4.3.3 and 4.3.4 apply to the lattice with $K = \dim \ell^2(\Lambda) = |\Lambda|$, and since $W := \sum_{\beta \in G} \delta_\beta = \chi_G$, the above arguments would prove the following result.

Conjecture 5.4.3. *In the case of the lattice, for any $I = (E_1, E_2)$ and any cube $\Lambda_L(x)$,*

(1) *If $G = \mathbb{Z}^d$, then for $\bullet = \text{S, D, N}$,*

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L(x)}^\bullet)]\} \leq 2 \cdot |\Lambda_L(x)|^2 \cdot s(\mathbb{P}, \|u\|_\infty^{-1}|I|).$$

(2) *Otherwise, for any $q > q^*(q_\pm, \|u\|_\infty, M)$, if $E_2 < E_q := \inf \sigma(H_0 + q\chi_G)$, then there exists $c_W = c_W(q, q_\pm, \|u\|_\infty, M, E_q - E_2)$ such that*

$$\mathbb{E}\{\text{tr}[\chi_I(H_{\Lambda_L(x)}^D)]\} \leq 2 \cdot |\Lambda_L(x)| \cdot |\Lambda_L(x) \cap G| \cdot s(\mathbb{P}, c_W|I|).$$

2. This is the conjectural part.

5.4.2 A different approach to quantum graphs with random edge length

In this subsection we explain why we could not apply our theorems to the model considered in [78].

We will only sketch the model; the reader is referred to the article for details. One fixes a vertex set V (for example \mathbb{Z}^d), an edge set E (defined in terms of V , for example $e = (m, m')$ for $m, m' \in \mathbb{Z}^d$) then consider the probability space $\Omega = [\ell_{\min}, \ell_{\max}]^E$, where $0 < \ell_{\min} < \ell_{\max} < \infty$ and $\mathbb{P} = \otimes \mu_e$ for some probability measures μ_e on \mathbb{R} supported on $[\ell_{\min}, \ell_{\max}]$. For each $\ell_\omega \in \Omega$ one obtains a Hilbert space $\mathcal{H} = \oplus_{e \in E} L^2[0, \ell_\omega(e)]$ and defines the Laplacian $H(\omega) = -D_{\ell_\omega}^2$, where $(D_{\ell_\omega} f)_e(x) := f'_e(x)$ for $f \in \oplus_{e \in E} W^{1,2}(0, \ell_\omega(e))$ with Kirchhoff boundary conditions.

There are two ideas in the approach of [78]: the first one is to remove the randomness from the Hilbert spaces and put it in the operators, the second one is to rescale the probability space Ω by considering the random variables $\ln \ell_\omega(e)$ instead of $\ell_\omega(e)$. We already used the second idea when proving Theorem 4.3.4. The first idea is only sketched in [78], so let us describe it precisely to expose the difficulty.

Given $\ell_\omega \in \Omega$, define the operator $U_\omega : \oplus_{e \in E} L^2[0, \ell_\omega(e)] \rightarrow \oplus_{e \in E} L^2[0, 1]$ by $U_\omega f_e(x) := \sqrt{\ell_\omega(e)} f_e(x \ell_\omega(e))$ for $x \in [0, 1]$. Then

$$\langle U_\omega f, U_\omega g \rangle = \sum_{e \in E} \ell_\omega(e) \int_0^1 f(x \ell_\omega(e)) \bar{g}(x \ell_\omega(e)) dx = \sum_{e \in E} \int_0^{\ell_\omega(e)} f(y) \bar{g}(y) dy = \langle f, g \rangle.$$

Hence U_ω is an isometry, in particular it is injective, and it is obviously bijective since it has an inverse $U_\omega^{-1} f_e(x) = \frac{1}{\sqrt{\ell_\omega(e)}} f_e\left(\frac{x}{\ell_\omega(e)}\right)$. Thus U_ω is a unitary isomorphism, so it is equivalent to study the operator $G(\omega) = U_\omega H(\omega) U_\omega^{-1}$ on the Hilbert space $\oplus_{e \in E} L^2[0, 1]$. Let us first check how this operator acts then we will describe its domain. We have

$$(G(\omega) f)_e(x) = -\frac{1}{\sqrt{\ell_\omega(e)}} \cdot U_\omega \frac{d^2}{dx^2} f_e\left(\frac{x}{\ell_\omega(e)}\right) = -\frac{1}{\ell_\omega^2(e)} f_e''(x).$$

Notice that f_e'' is now independent of ω , i.e. if $\ell_0(e) := 1$ for all e , then $G(\omega) f = (-\ell_\omega(e)^{-2} D_{\ell_0}^2 f_e)$, so it looks like we could apply Theorem 4.3.4 with $\zeta = -2$. However, the real trouble is in the domain. Recall that for Kirchhoff conditions a function must be continuous at the vertices, and the sum of its derivatives must vanish there. For the first condition, suppose $\tau e = \iota b$, where ιe and τe denote the initial and terminal vertex of an edge. Then $(U_\omega^{-1} f)_e(\ell_\omega(e)) = (U_\omega^{-1} f)_b(0)$ means that

$$(i) \quad \frac{f_e(1)}{\sqrt{\ell_\omega(e)}} = \frac{f_b(0)}{\sqrt{\ell_\omega(b)}}.$$

Next,

$$(U_\omega^{-1} f)'_e(x) = \frac{d}{dx} \frac{1}{\sqrt{\ell_\omega(e)}} f_e\left(\frac{x}{\ell_\omega(e)}\right) = \frac{1}{\ell_\omega^{3/2}(e)} f'_e\left(\frac{x}{\ell_\omega(e)}\right),$$

so we must have at any v ,

$$(ii) \quad \sum_{\iota e=v} \frac{1}{\ell_\omega^{3/2}(e)} f'_e(0) - \sum_{\tau e=v} \frac{1}{\ell_\omega^{3/2}(e)} f'_e(1) = 0.$$

Summarizing, we showed that

$$D(G(\omega)) = \{f = (f_e) \in \oplus_{e \in E} W^{2,2}(0, 1) : f \text{ satisfies (i) when } \tau e = \iota b \text{ and (ii) at any } v\}.$$

One can easily check that for $f \in D(G(\omega))$ we have $\langle G(\omega)f, f \rangle = \sum_{e \in E} \ell_\omega(e)^{-2} \|D_{\ell_0} f_e\|^2$. Since $D(G(\omega))$ depends on ω via (i) and (ii), we cannot apply Theorem 4.3.4 to it.

The reason why [78] still succeed to work with this operator is that they only need $D(G(\omega))$ to be independent of ω in the diagonal direction, and this is indeed the case after rescaling Ω . Indeed, if $\tilde{\Omega} = [\omega_-, \omega_+]$, where $\omega_- = \ln \ell_{\min}$ and $\omega_+ = \ln \ell_{\max}$, then if $\mathbf{1} = (1, \dots, 1)$, we have $(\ell_{\omega+t.1}(e))_{e \in E} = (\exp(\omega_e + t))_{e \in E} = e^t (\ell_\omega(e))_{e \in E}$. Hence $D(G(\omega + t \cdot \mathbf{1})) = D(G(\omega))$, since one simply multiplies all $\ell_\omega(e)$ by a scalar e^t , which does not affect Kirchhoff conditions.

It is also visible from this that Kirchhoff conditions are really crucial to this approach. If one had the slightly more general condition $\sum_{\iota e=v} f'_e(0) - \sum_{\tau e=v} f'_e(\ell_e) = \alpha f(v)$, $\alpha \neq 0$, then the proof would break down because in condition (ii) above, the LHS and RHS would get multiplied by $e^{-3t/2}$ and $e^{-t/2}$, respectively, so the domain is no longer the same. The approach of [72] thus has an advantage here, as it still works in this situation.

5.4.3 Two volume bounds: the non-covering case

In this subsection we expose a difficulty concerning two-volume bounds in the non-covering case. We use the notations of Section 5.3.2. We assume moreover that $\Omega = [q_-, q_+]^{\mathbb{Z}^d}$ for some $q_- < q_+ \leq 0$.

To state our Wegner bound, fix $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x}) \subset \mathbb{Z}^{nd}$, let $\Pi_{\mathcal{J}} \Lambda_{\mathbb{L}}^{(n)}(\mathbf{x}) := \Lambda_{L_{\mathcal{J}}}^{(1)}(x_{\mathcal{J}})$, and for $n \geq 2$, define the operator

$$\tilde{W}_{\Lambda^{(n)}(\mathbf{x})}(\mathbf{y}) = \max_{1 \leq j \leq n} \sum_{\alpha \notin \Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x})} U_{\alpha}(\mathbf{y}) \quad \text{where} \quad U_{\alpha}(y_1, \dots, y_n) = \sum_{1 \leq i \leq n} u_{\alpha}(y_i).$$

For $n = 1$ we put $\tilde{W} = 0$. Now we have the following result

Theorem 5.4.4. *Let $E_0^{\Lambda} = \inf \sigma(H_{\Lambda,0} + q_- \tilde{W}_{\Lambda})$ and suppose $I \subset (-\infty, E_0^{\Lambda} - \eta)$ is an interval. Then for any $\emptyset \neq \mathcal{J} \subseteq \{1, \dots, n\}$ and any $\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})$, taking $G(\mathcal{J}) := \Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x})$, we have*

$$\mathbb{E}_{G(\mathcal{J})} \{ \text{tr}[\chi_I(H_{\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})}(\omega))] \} \leq C_W \cdot |\Lambda_{\mathbb{L}}^{(n)}(\mathbf{x})| \cdot s_F \left(\mathbb{P}, \frac{|I|}{nC_u} \right)$$

for any $(\omega_{\beta})_{\beta \notin G(\mathcal{J})}$, where $C_W := 4\eta^{-2} n^2 C_u^2 (2R+1)^{2d}$.

Proof. Fix any $\omega \in \Omega$ and let $H_1 = H_{\Lambda,0} + \sum_{\alpha \notin \Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x})} \omega_{\alpha} U_{\alpha}$. Then $H(\omega) = H_1 + \sum_{\alpha \in \Pi_{\mathcal{J}} \Lambda_{\mathbb{L}+R}^{(n)}(\mathbf{x})} \omega_{\alpha} U_{\alpha}$ and the claim follows from Theorem 5.2.2. \square

We do not know if this theorem is useful. The first question is whether there is any spectrum left below E_0^{Λ} . Note that there is no spectrum below $\inf \sigma(H_{\Lambda,0} + q_- W_{\Lambda})$, where $W_{\Lambda} = \sum_{\alpha \in \Pi \Lambda_{\mathbb{L}+R}^{(n)}} U_{\alpha}$, so is there a big difference between \tilde{W}_{Λ} and W_{Λ} ? Yes when $n = 1$ since $\tilde{W} = 0$, but for $n \geq 2$? If $n = 2$ and $u_{\alpha} = \delta_{\alpha}$ for all α , then one can see that $\tilde{W}_{\Lambda}(\mathbf{y}) = 0$ or 1 for $\mathbf{y} \in \Lambda$, while $W_{\Lambda}(\mathbf{y}) = 2$.

The second issue is that one studies localization near the lower edge of $\sigma(H)$, not $\sigma(H_{\Lambda})$, so it would be more convenient to have a bound formulated in terms of $\inf \sigma(H_0 + q_- \tilde{W})$ for example, instead of E_0^{Λ} . But how should we define \tilde{W} ?

Hence, although we could prove a (normal) Wegner bound below $\inf \sigma(H_0)$ in Section 4.4.1, we were not able to do the same for the conditional bound. The reason why we stumbled upon the condition in Theorem 5.4.4 is that we tried to prove the Wegner bound outside the spectrum of the unperturbed operator. It seems that this approach is not convenient here and that one needs more sophisticated arguments.

Note that we did not speak about the case of positive perturbations here, i.e. when $q_- \geq 0$. This is mainly because it is not clear how to define \tilde{W} .

Concerning the two-volume bound itself, one cannot hope to have an analog of Theorem 5.3.3 since the Wegner bound is only valid in low-energy intervals. However, this is not really problematic: it suffices to study instead $\text{dist}(\sigma_I(H_{\Lambda(\mathbf{x})}), \sigma_I(H_{\Lambda(\mathbf{y})}))$ for a low energy I , where $\sigma_I(A) := \sigma(A) \cap I$, just as we did in Theorem 2.5.2.

5.5 A deterministic variant of the UP

We conclude this chapter with a general theorem. Let us first give the hypotheses; we will then discuss its relation to the results established previously.

Hypotheses (F)

- 1) We fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with $\Omega = [q_-, q_+]^{\mathcal{I}}$ for some $q_-, q_+ \in \mathbb{R}$, $q_- < q_+$, some countable index set \mathcal{I} , and fix a finite-dimensional Hilbert space \mathcal{H} .
- 2) $H(\omega)$ is a self-adjoint operator on \mathcal{H} for each $\omega \in \Omega$.
- 3) $H(\omega)$ has the form

$$H(\omega) = H_0 + \sum_{\alpha \in \mathcal{I}_F} \omega_\alpha U_\alpha$$

for some finite set $\mathcal{I}_F \subseteq \mathcal{I}$, where H_0 and all U_α are self-adjoint operators.

- 4) Fix an orthonormal basis $\{e_j\}_{j \in J}$ for \mathcal{H} . We define $\mathcal{I}_j := \{\alpha \in \mathcal{I}_F : U_\alpha e_j \neq 0\}$, $C_{\text{fin}} := \max_{j \in J} |\mathcal{I}_j|$ and $J_{\text{eff}} := \{j \in J : U_\alpha e_j \neq 0 \text{ for some } \alpha \in \mathcal{I}_F\}$.
- 5) There exist an interval I_0 and a constant $\gamma > 0$ such that

$$\chi_{I_0}(H_0) W \chi_{I_0}(H_0) \geq \gamma \chi_{I_0}(H_0),$$

where $W := \sum_{\alpha \in \mathcal{I}_F} U_\alpha$ and $\chi_{I_0}(H_0)$ is the spectral projection of H_0 onto I_0 .

Notice that these are just Hypotheses (E) plus the fifth item, which is a deterministic version of the uncertainty principle discussed in Chapter 4. This version also goes by the name of a *quantitative unique continuation principle* (QUCP) in the context of Schrödinger operators. The constant γ often depends on I_0 . It is not clear if this property is stronger or weaker than the random one we used in Chapter 4, in fact they are probably incomparable. However, this version has a nice feature, namely it holds trivially in any interval $I_0 \subset \mathbb{R} \setminus \sigma(H_0)$ for any $\gamma > 0$, since in this case there is no condition at all.

Let us now state our theorem, which is an adaptation of the result of [30] to finite dimensions.

Theorem 5.5.1. *Suppose $H(\omega)$ satisfies Hypotheses (F). Then $\text{tr}[\chi_I(H(\omega))]$ is measurable for any interval I . If $I \subset I_0$ with $\text{dist}(I, I_0^c) =: \delta > 0$, then*

$$\text{tr}[\chi_I(H(\omega))] \leq C_1 \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I(H(\omega)) U_\alpha e_j, e_j \rangle,$$

where

$$C_1 := (4\gamma^{-2} + (1 + 2\gamma^{-2} \|W\|^2) \delta^{-2} q^2) C_{\text{fin}}, \quad q := \max(|q_-|, |q_+|).$$

If moreover the U_α satisfy $0 \leq U_\alpha \leq C_U$ for all α , then for any $0 < \lambda \leq \frac{1}{C_U}$ we have

$$\mathbb{E}\{\text{tr}[\chi_I(H(\omega))]\} \leq C_W \cdot |J_{\text{eff}}| \cdot s(\mathbb{P}, \lambda|I|),$$

where

$$C_W := \frac{4}{\lambda} C_{\text{fin}}^2 C_U \cdot \left(\frac{4}{\gamma^2} + \left(1 + \frac{2\|W\|^2}{\gamma^2}\right) \delta^{-2} q^2 \right).$$

The constant here is quite ugly; we have only displayed it to emphasize that it is explicit and to prove the following lemma.

Lemma 5.5.2. *Theorem 5.5.1 is stronger than Theorem 5.2.2. That is, if $H(\omega)$ satisfies the hypotheses of Theorem 5.2.2, then Theorem 5.5.1 yields the same conclusion.*

Proof. Given $I = [a, b]$, $I \subset \mathbb{R} \setminus \sigma(H_0)$ with $\delta := \text{dist}(I, \sigma(H_0))$, take $I_0 \supset I$ such that $I_0 \subset \mathbb{R} \setminus \sigma(H_0)$ and $\text{dist}(I, I_0^c) = \delta$. Then Hypotheses (F) are satisfied in the interval I_0 for any $\gamma > 0$. Applying Theorem 5.5.1 and taking $\gamma \rightarrow \infty$ yields the result. \square

It turns out that Theorem 5.5.1 is strictly better than Theorem 5.2.2, since it can be used to prove a Wegner bound for intervals intersecting $\sigma(H_0)$ for the model in [94], while Theorem 5.2.2 cannot. This is one motivation to give the proof. Another motivation comes from the fact that this theorem is a discrete version of the result of [30], but simplifies substantially³ due to the finite dimensionality. So we think it is also useful to better understand some key ideas in the proof of [30].

Before we prove the theorem, let us note that if the U_α are multiplication operators on an $\ell^2(\Gamma)$ space, then $\|W\|$ in C_1 and C_W can be estimated by $\|W\| \leq C_{\text{fin}} C_U$. Indeed, if (e_j) is the canonical basis one has for $f \in \ell^2(\Gamma)$

$$\begin{aligned} \|Wf\|^2 &= \sum_{j \in J} |\langle Wf, e_j \rangle|^2 \\ &= \sum_{j \in J} \left| \sum_{\alpha \in \mathcal{I}_j} \langle f, U_\alpha e_j \rangle \right|^2 \\ &= \sum_{j \in J} \left| \sum_{\alpha \in \mathcal{I}_j} U_\alpha(j) \langle f, e_j \rangle \right|^2 \\ &= \sum_{j \in J} |\langle f, e_j \rangle|^2 \left| \sum_{\alpha \in \mathcal{I}_j} U_\alpha(j) \right|^2 \\ &\leq C_{\text{fin}}^2 C_U^2 \sum_{j \in J} |\langle f, e_j \rangle|^2 = C_{\text{fin}}^2 C_U^2 \|f\|^2. \end{aligned}$$

For the discrete acoustic operator on $\ell^2(\mathbb{Z}^d)$ considered in [65], one has $\|W\| \leq 4d^4$.

Proof of Theorem 5.5.1. Let $I \subset I_0$ with $0 < \delta := \text{dist}(I, I_0^c)$. In the following we denote $\chi_I := \chi_I(H(\omega))$, $\chi_{I_0} := \chi_{I_0}(H_0)$ and $\chi_{I_0^c} := \chi_{I_0^c}(H_0)$. We have

$$(\dagger) \quad \text{tr}[\chi_I] = \text{tr}[\chi_I \chi_{I_0}] + \text{tr}[\chi_I \chi_{I_0^c}].$$

3. Compared to our previous results, our proof here may seem long. However, without counting spectral averaging, the continuum required a 7+3 pages proof, so the situation here is clearly simpler.

4. Indeed, following [65, P. 445], we have $W\delta_j = \sum_{\alpha} \nabla^* \Pi_{\alpha} \nabla \delta_j = (d\delta_j - \sum_{k=1}^d \delta_{j-e_k}) + \sum_{k=1}^d (\delta_j - \delta_{j+e_k})$, so that $\langle f, W\delta_j \rangle = 2df(j) - \sum_{k=1}^d f(j-e_k) - \sum_{k=1}^d f(j+e_k) = (-\Delta f)(j)$ and thus $\|Wf\|^2 = \sum_j |(-\Delta f)(j)|^2 = \|-\Delta f\|^2 \leq (4d)^2 \|f\|^2$.

Let us control the first term. Using the UP we have⁵

$$\begin{aligned}
(\dagger\dagger) \quad \operatorname{tr}[\chi_I \chi_{I_0}] &= \operatorname{tr}[\chi_I \chi_{I_0} \chi_I] \\
&\leq \gamma^{-1} \operatorname{tr}[\chi_I \chi_{I_0} W \chi_{I_0} \chi_I] \\
&= \gamma^{-1} \operatorname{tr}[\chi_I \chi_{I_0} W \chi_{I_0}] \\
&= \gamma^{-1} (\operatorname{tr}[\chi_I W \chi_{I_0}] - \operatorname{tr}[\chi_I \chi_{I_0^c} W \chi_{I_0}]) \\
&\leq \gamma^{-1} (|\operatorname{tr}[\chi_I W \chi_{I_0}]| + |\operatorname{tr}[\chi_I \chi_{I_0^c} W \chi_{I_0}]|).
\end{aligned}$$

We first control the second term. We have, using Hölder inequality for HS operators,

$$\begin{aligned}
|\operatorname{tr}[\chi_I \chi_{I_0^c} W \chi_{I_0}]| &= |\operatorname{tr}[\chi_I \chi_{I_0^c} W \chi_{I_0} \chi_I]| \\
&\leq \|\chi_I \chi_{I_0^c}\|_2 \|W \chi_{I_0} \chi_I\|_2 \\
&\leq \frac{c}{2} \operatorname{tr}[\chi_{I_0^c} \chi_I \chi_{I_0^c}] + \frac{1}{2c} \operatorname{tr}[\chi_I \chi_{I_0} W^2 \chi_{I_0} \chi_I]
\end{aligned}$$

for any $c > 0$. Recalling that $\operatorname{tr}[b^* a^* a b] \leq \|a^* a\| \operatorname{tr}[b^* b]$ for HS operators a and b , we have

$$\operatorname{tr}[\chi_I \chi_{I_0} W^2 \chi_{I_0} \chi_I] \leq \|W^2\| \operatorname{tr}[\chi_I \chi_{I_0} \chi_{I_0} \chi_I] = \|W^2\| \operatorname{tr}[\chi_I \chi_{I_0}],$$

so that

$$|\operatorname{tr}[\chi_I \chi_{I_0^c} W \chi_{I_0}]| \leq \frac{c}{2} \operatorname{tr}[\chi_I \chi_{I_0^c}] + \frac{\|W\|^2}{2c} \operatorname{tr}[\chi_I \chi_{I_0}].$$

Inserting this into $(\dagger\dagger)$, we thus see that

$$\left(1 - \frac{\gamma^{-1} \|W\|^2}{2c}\right) \operatorname{tr}[\chi_I \chi_{I_0}] \leq \gamma^{-1} \left(|\operatorname{tr}[\chi_I W \chi_{I_0}]| + \frac{c}{2} \operatorname{tr}[\chi_I \chi_{I_0^c}]\right).$$

Choosing $c := \gamma^{-1} \|W\|^2$ we thus get

$$(\dagger\dagger\dagger) \quad \operatorname{tr}[\chi_I \chi_{I_0}] \leq 2\gamma^{-1} |\operatorname{tr}[\chi_I W \chi_{I_0}]| + \gamma^{-2} \|W\|^2 \operatorname{tr}[\chi_I \chi_{I_0^c}].$$

The second term is already in (\dagger) , so let us focus on the first term. Using Hölder inequality for HS operators once again, we have

$$\begin{aligned}
|\operatorname{tr}[\chi_I W \chi_{I_0}]| &= |\operatorname{tr}[\chi_I W \chi_{I_0} \chi_I]| \\
&\leq \|\chi_I W \chi_{I_0}\|_2 \|\chi_{I_0} \chi_I\|_2 \\
&\leq \frac{c}{2} \operatorname{tr}[\chi_{I_0} W \chi_I W \chi_{I_0}] + \frac{1}{2c} \operatorname{tr}[\chi_I \chi_{I_0} \chi_I] \\
&= \frac{c}{2} \operatorname{tr}[W \chi_I W \chi_{I_0}] + \frac{1}{2c} \operatorname{tr}[\chi_I \chi_{I_0}]
\end{aligned}$$

for any $c > 0$. Choosing $c := 2\gamma^{-1}$ and noting that

$$\operatorname{tr}[W \chi_I W \chi_{I_0}] \leq \|W \chi_I W\|_1 \|\chi_{I_0}\| = \operatorname{tr}[W \chi_I W]$$

because $W \chi_I W \geq 0$, we thus get

$$|\operatorname{tr}[\chi_I W \chi_{I_0}]| \leq \gamma^{-1} \operatorname{tr}[W \chi_I W] + \frac{\gamma}{4} \operatorname{tr}[\chi_I \chi_{I_0}].$$

5. Compared to Proposition 4.3.1, one of the main difficulties of this proof is that there is no simple way to get rid of the χ_{I_0} that appear in the first inequality of $(\dagger\dagger)$.

Inserting this into (†††) and moving a term to the LHS we get

$$\frac{1}{2} \operatorname{tr}[\chi_I \chi_{I_0}] \leq 2\gamma^{-2} \operatorname{tr}[W \chi_I W] + \gamma^{-2} \|W\|^2 \operatorname{tr}[\chi_I \chi_{I_0^c}].$$

So by (†) we get

$$\operatorname{tr}[\chi_I] \leq 4\gamma^{-2} \operatorname{tr}[W \chi_I W] + (1 + 2\gamma^{-2} \|W\|^2) \operatorname{tr}[\chi_I \chi_{I_0^c}].$$

We now estimate the second term. Let $\{\varphi_n(\omega)\}$ be an orthonormal basis of eigenfunctions of $H(\omega)$ with eigenvalues $\{\lambda_n(\omega)\}$. Then

$$\operatorname{tr}[\chi_I \chi_{I_0^c}] = \sum_{\lambda_n(\omega) \in I} \langle \chi_{I_0^c} \varphi_n(\omega), \varphi_n(\omega) \rangle.$$

Now notice that the operator $f(H_0) = (H_0 - \lambda_n)^{-1} \chi_{I_0^c}(H_0)$ is well defined, with $\|f(H_0)\| \leq \delta^{-1}$ since $\lambda_n \in I$. Thus, taking $V_\omega := \sum_{\alpha \in \mathcal{I}} \omega_\alpha U_\alpha$ we have

$$\begin{aligned} \langle \chi_{I_0^c} \varphi_n, \varphi_n \rangle &= \langle (H_0 - \lambda_n)^{-2} \chi_{I_0^c}(H_0) (H_0 - \lambda_n) \varphi_n, (H_0 - \lambda_n) \varphi_n \rangle \\ &= \langle (H_0 - \lambda_n)^{-2} \chi_{I_0^c}(H_0) V_\omega \varphi_n, V_\omega \varphi_n \rangle \\ &\leq \delta^{-2} \|V_\omega \varphi_n\|^2 = \delta^{-2} \langle V_\omega^2 \varphi_n, \varphi_n \rangle, \end{aligned}$$

where we used the fact that $(H(\omega) - \lambda_n(\omega)) \varphi_n(\omega) = 0$ in the second equality. Hence,

$$\begin{aligned} \operatorname{tr}[\chi_I \chi_{I_0^c}] &\leq \delta^{-2} \sum_{\lambda_n(\omega) \in I} \langle V_\omega^2 \varphi_n(\omega), \varphi_n(\omega) \rangle \\ &= \delta^{-2} \sum_n \langle V_\omega^2 \chi_I \varphi_n(\omega), \varphi_n(\omega) \rangle \\ &= \delta^{-2} \operatorname{tr}[V_\omega^2 \chi_I] = \delta^{-2} \operatorname{tr}[V_\omega \chi_I V_\omega]. \end{aligned}$$

We thus finally get

$$\operatorname{tr}[\chi_I] \leq 4\gamma^{-2} \operatorname{tr}[W \chi_I W] + (1 + 2\gamma^{-2} \|W\|^2) \delta^{-2} \operatorname{tr}[V_\omega \chi_I V_\omega].$$

Now

$$\begin{aligned} \operatorname{tr}[W \chi_I W] &= \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} \langle U_\alpha \chi_I U_{\alpha'} e_j, e_j \rangle \\ &\leq \frac{1}{2} \sum_{j \in J} \sum_{\alpha, \alpha' \in \mathcal{I}_j} (\|\chi_I U_{\alpha'} e_j\|^2 + \|\chi_I U_\alpha e_j\|^2) \\ &\leq C_{\text{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \|\chi_I U_\alpha e_j\|^2 \\ &= C_{\text{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle, \end{aligned}$$

Similarly,

$$\operatorname{tr}[V_\omega \chi_I V_\omega] \leq q^2 C_{\text{fin}} \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle,$$

so we obtain the first claim. Finally,

$$\mathbb{E}\{\operatorname{tr}[\chi_I]\} \leq C_1 \sum_{j \in J} \sum_{\alpha \in \mathcal{I}_j} \mathbb{E}\{\langle U_\alpha \chi_I U_\alpha e_j, e_j \rangle\},$$

so the second claim now follows using the spectral averaging of Section B.4 with $A = H_0 + \sum_{\beta \neq \alpha} \omega_\beta U_\beta$, $B = U_\alpha$, $t = \omega_\alpha$ and $\phi = U_\alpha^{1/2} e_j$; see Section 4.5.2 for details. \square

Appendix A

Generalized Eigenfunction Expansions

A.1 Introduction

In this appendix we derive a generalized eigenfunction expansion for self-adjoint operators on abstract Hilbert spaces, a result that was needed in Sections 2.8 - 2.10. We follow the approach taken in [67, Section 3], but we provide much more details; we hope this will be helpful to (some) readers¹.

It is an elementary fact that if H is a self-adjoint operator with a finite trace, then it has an eigenfunction expansion; in fact this is true for any compact self-adjoint operator by the Hilbert-Schmidt theorem. It is well known that this result is not true for a general self-adjoint operator, even if it is bounded, since it may not have any eigenvalue.

Still, let us look at the operator p corresponding to the closure of $C_0^\infty(\mathbb{R}) \ni u \mapsto -iu'(t)$. It may be shown e.g. using the Fourier transform that $\sigma(p) = \mathbb{R}$. If p had an eigenvector ψ , this would mean that $-i\psi' = \lambda\psi$, i.e. $\psi(x) = ce^{i\lambda x}$. This is not an L^2 function, so p has no eigenvector. However, this gives us the impression that p does have eigenvectors that live in a larger Hilbert space, for instance $\mathcal{H}_- = L^2(\mathbb{R}, (1+x^2)^{-\gamma} dx)$, $\gamma > \frac{1}{4}$.

Suppose we have fixed a larger Hilbert space \mathcal{H}_- in which H has eigenvalues and let us denote them by $\sigma_{\text{gen}}(H)$, the *generalized spectrum of H* . The same example tells us that the larger Hilbert space must be chosen carefully if we would like to have $\sigma_{\text{gen}}(H) \approx \sigma(H)$. For instance, in one takes $\mathcal{H}_- = L^2(\mathbb{R}, e^{-x^2} dx)$, then for any $\lambda \in \mathbb{C}$, we have $e^{i\lambda x} \in \mathcal{H}_-$, in other words $\sigma_{\text{gen}}(p) = \mathbb{C}$ although $\sigma(p) = \mathbb{R}$. The problem is clear: the space \mathcal{H}_- was chosen too large. On the other hand, if one takes $\mathcal{H}_- = L^2(\mathbb{R}, dx)$, then $\sigma_{\text{gen}}(p) = \emptyset$ since p has no eigenvalues. So it is desirable to have somehow minimal sufficient conditions that guarantee that \mathcal{H}_- contains just enough eigenvectors.

The main result of this appendix is the following: let H be a self-adjoint operator and suppose there exists a “good” self-adjoint operator T with a bounded inverse and a bounded function f strictly positive on $\sigma(H)$ such that $\text{tr}(T^{-1}f(H)T^{-1}) < \infty$. Then H has an eigenfunction expansion in the completion of $(\mathcal{H}, \|\cdot\|_-)$, where $\|f\|_- := \|T^{-1}f\|$. We call this a *generalized eigenfunction expansion*.

We shall only need the results of Sections A.2 and A.3. We have included Sections A.4 and A.5 for completeness, and to show that generalized eigenfunctions can also be defined “by duality”, which is the approach taken in [89] and [11].

1. We have also modified some arguments; in particular, our proofs give as a byproduct a unitary equivalence between the generalized operator H_- and the operator $(THT^{-1})^*$.

We end this section by recalling some elementary properties of closable operators.

Lemma A.1.1. *Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and suppose $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded operator which is densely defined. Then A is closable and its closure \tilde{A} satisfies $D(\tilde{A}) = \mathcal{H}_1$. Moreover, $\|\tilde{A}\| = \|A\|$ and if $\|Af\|_{\mathcal{H}_2} \geq \gamma \cdot \|f\|_{\mathcal{H}_1}$ for some $\gamma > 0$ and all $f \in D(A)$, then \tilde{A} has a closed range.*

Proof. We denote the norms on \mathcal{H}_1 and \mathcal{H}_2 by $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. To see that A is closable, let $(f_j) \subset D(A)$ such that $\|f_j\|_1 \rightarrow 0$ and $\|Af_j - v\|_2 \rightarrow 0$. Then it suffices to show that $v = 0$. But $\|v\|_2 \leq \|Af_j - v\|_2 + \|Af_j\|_2 \leq \|Af_j - v\|_2 + \|A\| \cdot \|f_j\|_1 \rightarrow 0$. Thus $v = 0$ and A is closable. Now let $f \in \mathcal{H}_1$. As $D(A)$ is dense, we may find $(f_j) \subset D(A)$ such that $\|f_j - f\|_1 \rightarrow 0$. But then $\|A(f_j - f_k)\|_2 \leq \|A\| \|f_j - f_k\|_1$, so $(\tilde{A}f_j) = (Af_j)$ is Cauchy in \mathcal{H}_2 and thus converges to some $\psi \in \mathcal{H}_2$. Since \tilde{A} is closed, it follows that $f \in D(\tilde{A})$ and $\tilde{A}f = \psi$. Thus, $D(\tilde{A}) = \mathcal{H}_1$.

Let $f \in \mathcal{H}_1$, and take $(f_j) \subset D(A)$ such that $\|f - f_j\|_1 \rightarrow 0$. Then the above argument tells us that Af_j converges to $\tilde{A}f$, i.e. $\|\tilde{A}f - Af_j\|_2 \rightarrow 0$. Hence $\|\tilde{A}f\|_2 = \lim \|Af_j\|_2 \leq \liminf \|A\| \|f_j\|_1 = \|A\| \|f\|_1$. Hence $\|\tilde{A}\| \leq \|A\|$, so equality obviously holds.

Finally, let $f \in \mathcal{H}_1$ and $(f_j) \subset D(A)$ with $\|f - f_j\|_1 \rightarrow 0$. Then

$$\gamma \cdot \|f\|_1 = \gamma \lim \|f_j\|_1 \leq \liminf \|Af_j\|_2 = \|\tilde{A}f\|_2,$$

since $\|\tilde{A}f - Af_j\|_2 \rightarrow 0$. Hence the property extends to \tilde{A} . Now if $(\tilde{A}f_j)$ is a sequence in $R(\tilde{A})$ which converges to some $\psi \in \mathcal{H}_2$, then $\|f_j - f_k\|_1 \leq \gamma^{-1} \|\tilde{A}(f_j - f_k)\|_2$, so (f_j) is Cauchy and thus converges to some $f \in \mathcal{H}_1$. Noting that $\|\tilde{A}f - \psi\|_2 \leq \|\tilde{A}f - \tilde{A}f_j\|_2 + \|\tilde{A}f_j - \psi\|_2$, we see that $\tilde{A}f = \psi$. Thus $R(\tilde{A})$ is closed. \square

A.2 Main results

Hypothesis I. T is a self-adjoint operator on a Hilbert space \mathcal{H} with a bounded inverse.

Recall that by definition, a self-adjoint operator must be densely defined.

Let \mathcal{H}_+ be the space $D(T)$ equipped with the norm $\|\phi\|_+ = \|T\phi\|$ and \mathcal{H}_- the completion of \mathcal{H} in the norm $\|\psi\|_- = \|T^{-1}\psi\|$. By construction the natural injections $\iota_+ : \mathcal{H}_+ \rightarrow \mathcal{H}$ and $\iota_- : \mathcal{H} \rightarrow \mathcal{H}_-$ are continuous with dense range. Moreover, ι_+^{-1} is well defined on $D(T)$ and ι_-^{-1} is well defined on \mathcal{H}_- . We define $T_+ : \mathcal{H}_+ \rightarrow \mathcal{H}$ and $T_- : D(T) \rightarrow \mathcal{H}_-$ by $T_+ = T\iota_+$ and $T_- = \iota_-T$.

We shall denote the inner products on \mathcal{H}_+ , \mathcal{H} and \mathcal{H}_- by $(\cdot, \cdot)_+$, (\cdot, \cdot) and $(\cdot, \cdot)_-$ respectively². By construction, for $u, v \in \mathcal{H}_+$ and $f, g \in D(T)$, we have $(u, v)_+ = (T_+u, T_+v)$ and $(T_-f, T_-g)_- = (f, g)$.

We now define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_+ \times \mathcal{H}_-$ as follows: given $\phi \in \mathcal{H}_+$ and $\psi \in \mathcal{H}_-$, let $(f_j) \subset \mathcal{H}$ be a sequence such that $\|\iota_-f_j - \psi\|_- \rightarrow 0$ and put

$$\langle \phi, \psi \rangle := \lim_{j \rightarrow \infty} (\iota_+\phi, f_j)$$

Lemma A.2.1. (cf. [89, Lemma 1])

- (1) *The sesquilinear form $\langle \cdot, \cdot \rangle$ is well defined.*
- (2) *T_- is closable. Its closure \tilde{T}_- is an isomorphism of the Hilbert spaces \mathcal{H} and \mathcal{H}_- , i.e. $D(\tilde{T}_-) = \mathcal{H}$, $R(\tilde{T}_-) = \mathcal{H}_-$ and*

$$(f, g) = (\tilde{T}_-f, \tilde{T}_-g)_- \quad \text{for any } f, g \in \mathcal{H}.$$

² This differs from our convention in the previous chapters, where we denoted the inner product on \mathcal{H} by $\langle \cdot, \cdot \rangle$.

(3) Given $\phi \in \mathcal{H}_+$ and $\psi \in \mathcal{H}_-$, we have

$$\langle \phi, \psi \rangle = (\tilde{T}_- T_+ \phi, \psi)_-.$$

Proof. (1) Let $\phi \in \mathcal{H}_+$, $\psi \in \mathcal{H}_-$ and $(f_j) \subset \mathcal{H}$ with $\|\iota_- f_j - \psi\|_- \rightarrow 0$. First note that $(\iota_+ \phi, f_j)$ is a Cauchy sequence since

$$|(\iota_+ \phi, f_j - f_k)| = |(T_+ \phi, T_-^{-1} \iota_- (f_j - f_k))| \leq \|\phi\|_+ \|\iota_- (f_j - f_k)\|_-,$$

hence $(\iota_+ \phi, f_j)$ converges. To see that its value is independent of the choice of the sequence (f_j) , suppose (g_j) is another sequence in \mathcal{H} with $\|\iota_- g_j - \psi\|_- \rightarrow 0$. Then

$$|(\iota_+ \phi, f_j - g_j)| \leq \|\phi\|_+ \|\iota_- (f_j - g_j)\|_- \leq \|\phi\|_+ (\|\psi - \iota_- f_j\|_- + \|\psi - \iota_- g_j\|_-) \rightarrow 0$$

which completes the proof.

(2) Given $f \in D(T)$ we have $\|T_- f\|_- = \|f\|$, in particular T_- is bounded. As $D(T_-) = D(T)$ is dense in \mathcal{H} , it follows from Lemma A.1.1 that T_- is closable and $D(\tilde{T}_-) = \mathcal{H}$. Since $\|T_- f\|_- = \|f\|$ for $f \in D(T)$, this lemma tells us that $R(\tilde{T}_-)$ is closed in \mathcal{H}_- . But $R(\tilde{T}_-)$ is dense in \mathcal{H}_- since it contains $\iota_- \mathcal{H}$ (indeed, any $\psi \in \iota_- \mathcal{H} = D(T_-^{-1})$ may be written as $\psi = T_- T_-^{-1} \psi$). Hence $R(\tilde{T}_-) = \mathcal{H}_-$. Finally, given $f, g \in \mathcal{H} = D(\tilde{T}_-)$, let $(f_j), (g_j) \subset D(T)$ such that $\|f_j - f\| \rightarrow 0$ and $\|g_j - g\| \rightarrow 0$. Then the argument in Lemma A.1.1 tells us that $\|T_- f_j - \tilde{T}_- f\|_- \rightarrow 0$ and $\|T_- g_j - \tilde{T}_- g\|_- \rightarrow 0$, hence

$$(f, g) = \lim_{j \rightarrow \infty} (f_j, g_j) = \lim_{j \rightarrow \infty} (T_- f_j, T_- g_j)_- = (\tilde{T}_- f, \tilde{T}_- g)_-.$$

(3) Given $\phi \in \mathcal{H}_+$ and $\psi \in \mathcal{H}_-$, let $(f_j) \subset \mathcal{H}$ such that $\|\iota_- f_j - \psi\|_- \rightarrow 0$. Then

$$\langle \phi, \psi \rangle = \lim_{j \rightarrow \infty} (\iota_+ \phi, f_j) = \lim_{j \rightarrow \infty} (T_+ \phi, T^{-1} f_j) = \lim_{j \rightarrow \infty} (\tilde{T}_- T_+ \phi, \tilde{T}_- T^{-1} f_j)_-,$$

where we used (2). But $T^{-1} f_j \in D(T) = D(T_-)$, so $\tilde{T}_- T^{-1} f_j = T_- T^{-1} f_j = \iota_- f_j$. Thus,

$$\langle \phi, \psi \rangle = \lim_{j \rightarrow \infty} (\tilde{T}_- T_+ \phi, \iota_- f_j)_- = (\tilde{T}_- T_+ \phi, \psi)_-. \quad \square$$

We now make an important observation.

Lemma A.2.2. *The sesquilinear form $\langle \cdot, \cdot \rangle$ turns the spaces \mathcal{H}_+ and \mathcal{H}_- into conjugate duals, that is*

$$\mathcal{H}_- = (\mathcal{H}_+)'.$$

Proof. Any $\psi \in \mathcal{H}_-$ defines a linear functional on \mathcal{H}_+ via $\langle \cdot, \cdot \rangle$. Conversely, let $F \in (\mathcal{H}_+)'$. By the Riesz representation theorem, we may find a unique $v_F \in \mathcal{H}_+$ such that $F(\phi) = (\phi, v_F)_+$. Now put $\psi := \tilde{T}_- T_+ v_F \in \mathcal{H}_-$. Then by Lemma A.2.1,

$$\langle \phi, \psi \rangle = (\tilde{T}_- T_+ \phi, \psi)_- = (\tilde{T}_- T_+ \phi, \tilde{T}_- T_+ v_F)_- = (T_+ \phi, T_+ v_F) = (\phi, v_F)_+$$

which completes the proof. □

Hypothesis II. H is a self-adjoint operator on \mathcal{H} such that

$$\mathcal{D}_+ = \{\phi \in \mathcal{H}_+ : \iota_+\phi \in D(H) \text{ and } H\iota_+\phi \in D(T)\}$$

is dense in \mathcal{H}_+ .

This allows us to define an operator H_+ on \mathcal{H}_+ by $D(H_+) = \mathcal{D}_+$ and $H_+ = \iota_+^{-1}H\iota_+$. By hypothesis H_+ is then densely defined.

Lemma A.2.3. *The operator H_+ is closed.*

Proof. Let $(\phi_j) \subset \mathcal{H}_+$ be a sequence such that $\|\phi_j - \phi\|_+ \rightarrow 0$ and $\|H_+\phi_j - u\|_+ \rightarrow 0$ for some $\phi, u \in \mathcal{H}_+$, then

$$\|\iota_+\phi_j - \iota_+\phi\| \leq \|T^{-1}\| \cdot \|\phi_j - \phi\|_+ \rightarrow 0, \quad \|H\iota_+\phi_j - \iota_+u\| \leq \|T^{-1}\| \cdot \|H_+\phi_j - u\|_+ \rightarrow 0,$$

and since H is self-adjoint, it is closed, so $\iota_+\phi \in D(H)$ and $H\iota_+\phi = \iota_+u \in D(T)$. Hence $\phi \in \mathcal{D}_+$ and $H_+\phi = \iota_+^{-1}H\iota_+\phi = u$. Thus, H_+ is closed. \square

We may also try a similar construction on \mathcal{H}_- and we obtain the following

Lemma A.2.4. *The operator $\iota_-H\iota_-^{-1}$ on \mathcal{H}_- with domain $\iota_-D(H)$ is densely defined and closable.*

Proof. Since $D(H)$ is dense in \mathcal{H} and since ι_- is continuous with dense range, $\iota_-H\iota_-^{-1}$ is densely defined.

To see that $\iota_-H\iota_-^{-1}$ is closable, let $(f_j) \subset D(H)$ such that $\|\iota_-f_j\|_- \rightarrow 0$ and suppose there exists $v \in \mathcal{H}_-$ such that $\|\iota_-Hf_j - v\|_- \rightarrow 0$. Then $v = 0$. Indeed, if $\phi \in \mathcal{D}_+$, then

$$\langle \phi, v \rangle = \lim_{j \rightarrow \infty} (\iota_+\phi, Hf_j) = \lim_{j \rightarrow \infty} (H\iota_+\phi, f_j).$$

But $|(H\iota_+\phi, f_j)| \leq \|H\iota_+\phi\|_+ \|\iota_-f_j\|_- \rightarrow 0$, hence $\langle \phi, v \rangle = 0$ for every $\phi \in \mathcal{D}_+$. Since \mathcal{D}_+ is dense in \mathcal{H}_+ , the same holds for every $\phi \in \mathcal{H}_+$. Finally, by Lemma A.2.2, $\mathcal{H}_- = (\mathcal{H}_+)'$, so it follows that $v = 0$. \square

The closure of $\iota_-H\iota_-^{-1}$ in \mathcal{H}_- , which we denote by H_- , may be described as follows (see e.g. [11, Section 12.2.2])

$$D(H_-) = \{\psi \in \mathcal{H}_- \mid \exists (f_j) \subset D(H), \eta \in \mathcal{H}_- : \|\iota_-f_j - \psi\|_- \rightarrow 0 \text{ and } \|\iota_-Hf_j - \eta\|_- \rightarrow 0\},$$

$$H_-\psi := \eta.$$

Lemma A.2.5. *For any $\phi \in \mathcal{D}_+$ and $\psi \in D(H_-)$ we have*

$$\langle H_+\phi, \psi \rangle = \langle \phi, H_-\psi \rangle.$$

Proof. Let $(f_j) \subset D(H)$ with $\|\iota_-f_j - \psi\|_- \rightarrow 0$ and $\|\iota_-Hf_j - H_-\psi\|_- \rightarrow 0$. Then

$$\langle H_+\phi, \psi \rangle = \lim_{j \rightarrow \infty} (H\iota_+\phi, f_j) = \lim_{j \rightarrow \infty} (\iota_+\phi, Hf_j) = \langle \phi, H_-\psi \rangle$$

by definition of $\langle \cdot, \cdot \rangle$. \square

Definition. We say that $\psi \in \mathcal{H}_-$ is a *generalized eigenfunction* of H if it is an eigenfunction of H_- , i.e. if $\psi \in D(H_-)$ and there exists $\lambda \in \mathbb{C}$ such that $H_-\psi = \lambda\psi$. In this case, we say that λ is a *generalized eigenvalue* of H .

Hypothesis III. The set

$$D_+ = \{f \in D(H) \cap D(T) : Hf \in D(T)\}$$

is an operator core for H .

Note that if $f \in D_+$, then $\iota_+^{-1}f \in \mathcal{D}_+$.

Let $(H_-)_{\mathcal{H}}$ be the restriction of H_- to $\iota_- \mathcal{H}$, i.e.

$$D((H_-)_{\mathcal{H}}) = \{\psi \in D(H_-) \cap \iota_- \mathcal{H} : H_- \psi \in \iota_- \mathcal{H}\},$$

and $(H_-)_{\mathcal{H}}\psi = H_- \psi$. Then the following holds.

Lemma A.2.6. *We have $\iota_- H \iota_-^{-1} = (H_-)_{\mathcal{H}}$.*

Proof. Since H_- is the closure of $\iota_- H \iota_-^{-1}$, if $\psi \in D(\iota_- H \iota_-^{-1}) = \iota_- D(H)$, then $\psi \in D(H_-) \cap \iota_- \mathcal{H}$ and $H_- \psi = \iota_- H \iota_-^{-1} \psi \in \iota_- \mathcal{H}$. Hence $\psi \in D((H_-)_{\mathcal{H}})$ and $(H_-)_{\mathcal{H}}\psi = H_- \psi = \iota_- H \iota_-^{-1} \psi$. Thus, $\iota_- H \iota_-^{-1} \subset (H_-)_{\mathcal{H}}$.

Conversely, given $\psi \in D((H_-)_{\mathcal{H}})$ and $f \in D_+$, we have by Lemma A.2.5

$$(Hf, \iota_-^{-1} \psi) = (\iota_+ H \iota_+^{-1} f, \iota_-^{-1} \psi) = \langle H \iota_+^{-1} f, \psi \rangle = \langle \iota_+^{-1} f, H_- \psi \rangle = (f, \iota_-^{-1} H_- \psi)$$

by definition of $\langle \cdot, \cdot \rangle$, by approximating ψ and $H_- \psi$ with constant sequences. Since D_+ is a core for H , we conclude that $\iota_-^{-1} \psi \in D(H^*) = D(H)$ and $\iota_- H \iota_-^{-1} \psi = H_- \psi$. \square

This yields the following corollary. It roughly says that if a generalized eigenfunction lives in \mathcal{H} , then it is an eigenfunction.

Corollary A.2.7. *$\psi \in \iota_- \mathcal{H}$ is a generalized eigenfunction if and only if $\iota_-^{-1} \psi$ is an eigenfunction.*

Proof. If $\iota_-^{-1} \psi$ is an eigenfunction, then $\iota_-^{-1} \psi \in D(H)$ and $H \iota_-^{-1} \psi = \lambda \iota_-^{-1} \psi$ for some $\lambda \in \mathbb{C}$, hence $H_- \psi = \iota_- H \iota_-^{-1} \psi = \lambda \psi$.

Conversely, if $\psi \in \iota_- \mathcal{H}$ is a generalized eigenfunction, then $\psi \in D(H_-) \cap \iota_- \mathcal{H}$ and $H_- \psi = \lambda \psi$, so $H_- \psi \in \iota_- \mathcal{H}$, i.e. $\psi \in D((H_-)_{\mathcal{H}})$. So using Lemma A.2.6 we get $H \iota_-^{-1} \psi = \iota_-^{-1} (\iota_- H \iota_-^{-1}) \psi = \iota_-^{-1} (H_-)_{\mathcal{H}} \psi = \lambda \iota_-^{-1} \psi$. \square

In the following we denote by $E(J)$ the spectral projection of H onto a Borel subset J . The usual trace on \mathcal{H} is denoted by tr .

Hypothesis IV. There exists a Borel set O and a bounded continuous function f , strictly positive on the spectrum of H , such that

$$\text{tr}(T^{-1} E(O) f(H) T^{-1}) < \infty.$$

For our purposes, O will be \mathbb{R} , so that $T^{-1} E(O) f(H) T^{-1} = T^{-1} f(H) T^{-1}$. Allowing a general O requires no additional effort, and is useful when considering wave operators; see [67].

Lemma A.2.8. *The set function $\nu(J) = \text{tr}(T^{-1} E(J \cap O) T^{-1})$ is a spectral measure for the restriction of H to $E(O)\mathcal{H}$ which is finite for all bounded Borel sets J .*

Proof. Since $T^{-1}E(J \cap O)T^{-1} \geq 0$ for any Borel J , we have $\nu(J) \geq 0$. Moreover, $E(\emptyset) = 0$, so $\nu(\emptyset) = 0$. If $\{J_i\}$ are pairwise disjoint Borel sets, then $E(\cup_i J_i) = \sum_i E(J_i)$ strongly, so countable additivity follows from the linearity of the trace. Hence ν is a measure. Finally, $\nu(J) = 0 \iff E(J \cap O) = 0$, so ν is a spectral measure for the restriction of H to $E(O)\mathcal{H}$. To conclude, let J be a bounded Borel set. Then we may find $\varepsilon > 0$ such that $f(\lambda) \geq \varepsilon$ for $\lambda \in \sigma_{J,O}(H) := \sigma(H) \cap J \cap O$. Hence

$$0 \leq \varepsilon E(J \cap O) \leq \int_{\sigma_{J,O}(H)} f(\lambda) dE(\lambda) \leq \int_{\sigma(H) \cap O} f(\lambda) dE(\lambda) = E(O)f(H).$$

Thus, $\nu(J \cap O) \leq \varepsilon^{-1} \operatorname{tr}(T^{-1}E(O)f(H)T^{-1}) < \infty$. \square

Any bounded operator C on \mathcal{H} induces a bounded operator from $\mathcal{H}_+ \rightarrow \mathcal{H}_-$ given by \tilde{T}_-CT_+ . Inversely, given a bounded operator $D : \mathcal{H}_+ \rightarrow \mathcal{H}_-$, the operator $C = \tilde{T}_-^{-1}DT_+^{-1}$ defined on $D(T)$ is bounded, hence its closure is a bounded operator on \mathcal{H} (see Lemma A.1.1). Hence the map $\tau : B(\mathcal{H}) \rightarrow B(\mathcal{H}_+, \mathcal{H}_-)$ given by $\tau(C) = \tilde{T}_-CT_+$ is bijective. Moreover, it is a Banach space isomorphism since \tilde{T}_- and T_+ are unitary. This motivates us to define for $1 \leq q < \infty$

$$\mathcal{T}_q(\mathcal{H}_+, \mathcal{H}_-) := \tau(\mathcal{T}_q(\mathcal{H})), \quad \mathcal{T}_{q,+}(\mathcal{H}_+, \mathcal{H}_-) := \tau(\mathcal{T}_{q,+}(\mathcal{H})),$$

where $\mathcal{T}_q(\mathcal{H})$ is the Banach space of bounded operators Y on \mathcal{H} with $\|Y\|_q = |\operatorname{tr}|Y|^q|^{\frac{1}{q}} < \infty$ and $\mathcal{T}_{q,+}(\mathcal{H})$ is the subset of positive operators in $\mathcal{T}_q(\mathcal{H})$.

By construction, $\mathcal{T}_q(\mathcal{H}_+, \mathcal{H}_-)$ equipped with the norm $\|B\|_q = \|\tau^{-1}(B)\|_q$ is a Banach space isomorphic to $\mathcal{T}_q(\mathcal{H})$, a separable dual Banach space. Note that if $B = \tau(C)$ then

$$(A-1) \quad \langle \phi, B\phi \rangle = \langle \phi, \tilde{T}_-CT_+\phi \rangle = \langle T_+\phi, CT_+\phi \rangle.$$

In particular, $B \in \mathcal{T}_{q,+}(\mathcal{H}_+, \mathcal{H}_-)$ iff $B \in \mathcal{T}_q(\mathcal{H}_+, \mathcal{H}_-)$ and $\langle \phi, B\phi \rangle \geq 0$ for all $\phi \in \mathcal{H}_+$ (note that $\operatorname{Ran}(T_+) = D(T_+^{-1}) = \iota_+\mathcal{H}_+ = D(T)$ is dense in \mathcal{H}).

For $B \in \mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$ we set

$$\operatorname{tr} B := \operatorname{tr} \tau^{-1}B = \operatorname{tr} \overline{\tilde{T}_-^{-1}BT_+^{-1}}.$$

If ϕ_n is an orthonormal basis of \mathcal{H}_+ , then $T_+\phi_n$ is an orthonormal basis for \mathcal{H} , so using (A-1) with $C = \tau^{-1}B$ we have

$$\operatorname{tr} B = \operatorname{tr} \tau^{-1}B = \sum_n \langle T_+\phi_n, (\tau^{-1}B)T_+\phi_n \rangle = \sum_n \langle \phi_n, B\phi_n \rangle.$$

Moreover, for $B \in \mathcal{T}_2(\mathcal{H}_+, \mathcal{H}_-)$, $\|B\|_2$ is the usual Hilbert-Schmidt norm:

$$\|B\|_2^2 = \sum_n \|\tau^{-1}(B)T_+\phi_n\|^2 = \sum_n \|\tilde{T}_-^{-1}B\phi_n\|^2 = \sum_n \|B\phi_n\|_-^2.$$

Now note that if $C \in B(\mathcal{H})$ and $T^{-1}CT^{-1} \in \mathcal{T}_1(\mathcal{H})$, we have

$$\operatorname{tr} \iota_-C\iota_+ = \operatorname{tr} \tau^{-1}(\iota_-C\iota_+) = \operatorname{tr} \overline{(\tilde{T}_-^{-1}\iota_-C\iota_+T_+^{-1})} = \operatorname{tr} T^{-1}CT^{-1}.$$

Hence the spectral measure ν of Lemma A.2.8 may be written as $\nu(J) = \operatorname{tr} \iota_-E(J \cap O)\iota_+$. Noting that $J \rightarrow T^{-1}E(J \cap O)T^{-1}$ is a $\mathcal{T}_{1,+}(\mathcal{H})$ -valued measure, it also follows that $J \rightarrow \iota_-E(J \cap O)\iota_+$ is a $\mathcal{T}_{1,+}(\mathcal{H}_+, \mathcal{H}_-)$ -valued measure.

We are finally ready to state the main result of this section.

Theorem A.2.9. *There exists a ν -locally integrable function $P : \mathbb{R} \rightarrow \mathcal{T}_{1,+}(\mathcal{H}_+, \mathcal{H}_-)$ such that*

$$\iota_- f(H)E(J \cap O)\iota_+ = \int_J f(\lambda)P(\lambda)d\nu(\lambda)$$

for all bounded Borel sets J and all bounded Borel functions f , where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$ -valued functions. Furthermore, for ν -almost every $\lambda \in \mathbb{R}$, $\text{tr } P(\lambda) = 1$ and $P(\lambda)\phi \in \mathcal{H}_-$ is a generalized eigenfunction of H with generalized eigenvalue λ , for any $\phi \in \mathcal{H}_+$.

Proof. Since $J \rightarrow \iota_- E(J \cap O)\iota_+$ is a $\mathcal{T}_{1,+}(\mathcal{H}_+, \mathcal{H}_-)$ -valued Borel measure and $\nu(J) = \text{tr } \iota_- E(J \cap O)\iota_+$ is finite on bounded Borel sets, it follows from an operator-valued Radon-Nikodym theorem that there exists a ν -locally integrable function $P : \mathbb{R} \rightarrow \mathcal{T}_{1,+}(\mathcal{H}_+, \mathcal{H}_-)$, called the Radon-Nikodym derivative of $\iota_- E(J \cap O)\iota_+$, such that

$$\iota_- E(J \cap O)\iota_+ = \int_J P(\lambda)d\nu(\lambda)$$

for bounded Borel J . Moreover, $P(\lambda)$ satisfies $\text{tr } P(\lambda) = 1$ for ν -a.e. λ . The proof of these properties may be found in [10, Theorem V.1.1], see also [11, Theorem 15.1.1].

Using this form of $\iota_- E(J \cap O)\iota_+$ and taking limits of Borel step functions, we obtain the required expansion for $\iota_- f(H)E(J \cap O)\iota_+$.

Now fix $\lambda \in \mathbb{R}$ and assume $\lambda \in \text{supp } \nu$. Then if $I_{\lambda,\delta} := [\lambda - \delta, \lambda + \delta]$, we have $\nu(I_{\lambda,\delta}) \neq 0$. Now observe that if f is a bounded Borel function, then using the expansion we get

$$\iota_- f(H)E(I_{\lambda,\delta} \cap O)\iota_+ - f(\lambda)P(\lambda) = \int_{I_{\lambda,\delta}} \left\{ f(\lambda')P(\lambda') - \frac{f(\lambda)P(\lambda)}{\nu(I_{\lambda,\delta})} \right\} d\nu(\lambda').$$

Thus, by standard properties of the Bochner integral we get as $\delta \rightarrow 0$

$$\left\| \frac{\iota_- f(H)E(I_{\lambda,\delta} \cap O)\iota_+}{\nu(I_{\lambda,\delta})} - f(\lambda)P(\lambda) \right\|_1 \leq \frac{1}{\nu(I_{\lambda,\delta})} \int_{I_{\lambda,\delta}} \|f(\lambda')P(\lambda') - f(\lambda)P(\lambda)\|_1 d\nu(\lambda') \rightarrow 0$$

for ν -a.e. λ , by Lebesgue differentiation theorem (see [81, Corollary 2.14]). Note that ν is indeed a Radon measure on \mathbb{R} by [81, Corollary 1.11]. Applying this to $f(x) \equiv 1$ and $f(x) = x\chi_{[\lambda-1, \lambda+1]}(x)$, we thus get in particular

$$(A-2) \quad P(\lambda) = \lim_{\delta \downarrow 0} \frac{\iota_- E(I_{\lambda,\delta} \cap O)\iota_+}{\nu(I_{\lambda,\delta})} \quad \text{in } \mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-), \quad \text{for } \nu\text{-a.e. } \lambda,$$

$$(A-3) \quad \lambda P(\lambda) = \lim_{\delta \downarrow 0} \frac{\iota_- H E(I_{\lambda,\delta} \cap O)\iota_+}{\nu(I_{\lambda,\delta})} \quad \text{in } \mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-), \quad \text{for } \nu\text{-a.e. } \lambda.$$

Finally, let $\phi \in \mathcal{H}_+$ and fix $\lambda \in \mathbb{R}$ such that (A-2) and (A-3) hold. Let $I_j = [\lambda - j^{-1}, \lambda + j^{-1}]$. Then if $f_j = \frac{1}{\nu(I_j)} E(I_j \cap O)\iota_+ \phi$, we have $f_j \in D(H)$ since $\|H f_j\|^2 = \frac{1}{\nu(I_j)^2} \int_{I_j \cap O} x^2 d\rho_{\iota_+ \phi}(x) \leq (\max I_j)^2 \|f_j\|^2$. By (A-2) and (A-3) we have $\|\iota_- f_j - P(\lambda)\phi\|_- \rightarrow 0$ and $\|\iota_- H f_j - \lambda P(\lambda)\phi\|_- \rightarrow 0$. By definition of $D(H_-)$ we get $P(\lambda)\phi \in D(H_-)$ and $H_- P(\lambda)\phi = \lambda P(\lambda)\phi$. \square

A.3 A word on adjoints

Recall that if X and Y are Banach spaces with duals X' and Y' respectively, and if $B : X \rightarrow Y$ is a bounded operator, we define its Banach adjoint $B^\dagger : Y' \rightarrow X'$ by

$$(B^\dagger f)(u) = f(Bu) \quad \text{for all } u \in X \text{ and } f \in Y'.$$

Let us describe what this means for our Hilbert spaces. In the following all operators are assumed to be bounded.

1. If $R : \mathcal{H}_+ \rightarrow \mathcal{H}$, then by Lemma A.2.2 any $\psi \in \mathcal{H}'_+ \equiv \mathcal{H}_-$ takes the form $\psi(\phi) = \langle \phi, \psi \rangle$. Thus, $(R^\dagger f)(\phi) = \langle \phi, R^\dagger f \rangle$ for any $\phi \in \mathcal{H}_+$ and $f \in \mathcal{H}' \equiv \mathcal{H}$. On the other hand, any $f \in \mathcal{H}'$ takes the form $f(u) = (u, f)$ for $u \in \mathcal{H}$. Hence $R^\dagger : \mathcal{H} \rightarrow \mathcal{H}_-$ is defined by

$$\langle \phi, R^\dagger f \rangle = (R\phi, f) \quad \text{for all } \phi \in \mathcal{H}_+ \text{ and } f \in \mathcal{H}.$$

2. If $S : \mathcal{H} \rightarrow \mathcal{H}_-$, let us show that $\mathcal{H}'_- = \mathcal{H}_+$ and that any $\phi \in \mathcal{H}'_-$ takes the form $\phi(\psi) = \overline{\langle \phi, \psi \rangle}$. It is clear that this defines a linear functional on \mathcal{H}_- . Conversely, by the Riesz representation theorem, for any $F \in \mathcal{H}'_-$ there exists a unique $u_F \in \mathcal{H}_-$ such that $F(\psi) = (\psi, u_F)_-$. Now let $(u_j) \subset \iota_- \mathcal{H}$ such that $\|u_j - u_F\|_- \rightarrow 0$. Then $F(\psi) = \lim (\psi, u_j)_- = \lim \overline{\langle u_j, \psi \rangle_-} = \lim \langle T_+^{-1} T_-^{-1} u_j, \psi \rangle$. Now $\|T_+^{-1} T_-^{-1} (u_j - u_k)\|_+ = \|u_j - u_k\|_-$, hence $(T_+^{-1} T_-^{-1} u_j)$ is Cauchy and converges to some $\phi \in \mathcal{H}_+$, so that $F(\psi) = \overline{\langle \phi, \psi \rangle}$. This proves the claim.

Now for $\phi \in \mathcal{H}_+$, $S^\dagger \phi \in \mathcal{H}'$ takes the form $S^\dagger \phi(f) = (f, S^\dagger \phi)$. Moreover, by the above argument, $\phi(Sf) = \overline{\langle \phi, Sf \rangle}$. Thus, we have

$$(f, S^\dagger \phi) = \overline{\langle \phi, Sf \rangle} \quad \text{for all } f \in \mathcal{H} \text{ and } \phi \in \mathcal{H}_+.$$

In other words,

$$\langle \phi, Sf \rangle = (S^\dagger \phi, f) \quad \text{for all } f \in \mathcal{H} \text{ and } \phi \in \mathcal{H}_+.$$

3. If $U : \mathcal{H}_- \rightarrow \mathcal{H}$, then $U^\dagger : \mathcal{H} \rightarrow \mathcal{H}_+$ is given by

$$\overline{\langle U^\dagger f, \psi \rangle} = (U\psi, f) \quad \text{for all } \psi \in \mathcal{H}_- \text{ and } f \in \mathcal{H}.$$

In other words,

$$\langle U^\dagger f, \psi \rangle = (f, U\psi) \quad \text{for all } \psi \in \mathcal{H}_- \text{ and } f \in \mathcal{H}.$$

4. If $V : \mathcal{H} \rightarrow \mathcal{H}_+$, then $V^\dagger : \mathcal{H}_- \rightarrow \mathcal{H}$ is given by

$$(f, V^\dagger \psi) = \langle Vf, \psi \rangle \quad \text{for all } f \in \mathcal{H} \text{ and } \psi \in \mathcal{H}_-.$$

5. If $W : \mathcal{H}_+ \rightarrow \mathcal{H}_-$, then $W^\dagger : \mathcal{H}'_- \rightarrow \mathcal{H}'_+$. So for $\phi \in \mathcal{H}_+ \equiv \mathcal{H}'_-$, $W^\dagger \phi \in \mathcal{H}'_+$ and thus $W^\dagger \phi(u) = \langle u, W^\dagger \phi \rangle$. But for $\phi \in \mathcal{H}'_-$, $\phi(Wu) = \overline{\langle \phi, Wu \rangle}$. Thus $W^\dagger : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ is defined by

$$\langle u, W^\dagger \phi \rangle = \overline{\langle \phi, Wu \rangle} \quad \text{for all } \phi, u \in D(\mathcal{H}_+).$$

Since by definition we have for $\phi \in \mathcal{H}_+$ and $f \in \mathcal{H}$

$$(\iota_+ \phi, f) = \langle \phi, \iota_- f \rangle,$$

we see that $\iota_+^\dagger = \iota_-$ and $\iota_-^\dagger = \iota_+$. Moreover,

$$(T_+ \phi, f) = (\tilde{T}_- T_+ \phi, \tilde{T}_- f)_- = \langle \phi, \tilde{T}_- f \rangle$$

so that $T_+^\dagger = \tilde{T}_-$ and $\tilde{T}_-^\dagger = T_+$.

We now have the following result, which is used in Section 2.10.

Lemma A.3.1. *We have $P(\lambda)^\dagger = P(\lambda)$ for ν -a.e. $\lambda \in \mathbb{R}$.*

If $U : \mathcal{H}_- \rightarrow \mathcal{H}$ and $V : \mathcal{H} \rightarrow \mathcal{H}_+$ are bounded operators, then $UP(\lambda)V \in \mathcal{T}_1(\mathcal{H})$ and

$$\|UP(\lambda)V\|_1 \leq \|U\| \cdot \|V\|.$$

Moreover, we have $(UP(\lambda)V)^ = V^\dagger P(\lambda)U^\dagger$, where $(UP(\lambda)V)^* : \mathcal{H} \rightarrow \mathcal{H}$ is the usual Hilbert adjoint. In particular, $\|UP(\lambda)V\|_2 = \|V^\dagger P(\lambda)U^\dagger\|_2$.*

Proof. Let $\phi, u \in \mathcal{H}_+$ and fix a λ such that (A-2) is true. We have for any $\delta > 0$ and for $E := E(I_{\lambda, \delta} \cap O)$,

$$\langle \phi, \iota_- E \iota_+ u \rangle = (\iota_+ \phi, E \iota_+ u) = (E \iota_+ \phi, \iota_+ u) = \overline{(\iota_+ u, E \iota_+ \phi)} = \overline{\langle u, \iota_- E \iota_+ \phi \rangle}.$$

Thus,

$$\langle \phi, P(\lambda)u \rangle = \lim_{\delta \downarrow 0} \frac{1}{\nu(I_{\lambda, \delta})} \langle \phi, \iota_- E \iota_+ u \rangle = \lim_{\delta \downarrow 0} \frac{1}{\nu(I_{\lambda, \delta})} \overline{\langle u, \iota_- E \iota_+ \phi \rangle} = \overline{\langle u, P(\lambda)\phi \rangle}$$

for any $\phi, u \in \mathcal{H}_+$. This proves the first claim. Next, we have

$$\|UP(\lambda)V\|_1 = \|U\tilde{T}_-\tau^{-1}(P(\lambda))T_+V\|_1 \leq \|U\tilde{T}_-\| \cdot \|\tau^{-1}(P(\lambda))\|_1 \cdot \|T_+V\|$$

by the usual properties of $\|\cdot\|_1$ in $\mathcal{T}_1(\mathcal{H})$. Now $P(\lambda) \geq 0$, so $\|\tau^{-1}(P(\lambda))\|_1 = \text{tr } P(\lambda) = 1$ by Theorem A.2.9. Moreover, $\|\tilde{T}_-\| = \|T_+\| = 1$, so the second claim follows. For the third, let $f, g \in \mathcal{H}$. Then

$$(UP(\lambda)Vf, f) = \overline{\langle U^\dagger f, P(\lambda)Vf \rangle} = \langle Vf, P(\lambda)^\dagger U^\dagger f \rangle = \langle Vf, P(\lambda)U^\dagger f \rangle = (f, V^\dagger P(\lambda)U^\dagger f).$$

The proof is complete by noting that $\|UP(\lambda)V\|_2 = \|(UP(\lambda)V)^*\|_2$. \square

A.4 A characterization of generalized eigenfunctions

In this section we prove that the generalized eigenfunctions may equally well be defined in terms of the operator H_+ . This is the point of view taken in [89] and [11].

Let us continue the study of adjoints which we started in the previous section. Recall that if X and Y are Banach spaces and if $A : D(A) \subset X \rightarrow Y$ is densely defined, its Banach adjoint $A^\dagger : Y' \rightarrow X'$ is defined as follows

$$D(A^\dagger) = \{f \in Y' \mid \exists g \in X' : g(u) = f(Au) \text{ for all } u \in D(A)\},$$

$$A^\dagger f := g.$$

In particular, if $A : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ is densely defined, then its adjoint is an operator $A^\dagger : \mathcal{H}'_+ \rightarrow \mathcal{H}'_+$. So for $\psi \in D(A^\dagger)$, $\psi \in \mathcal{H}_- \equiv \mathcal{H}'_+$, we have $A^\dagger \psi \in \mathcal{H}'_+$ and thus $A^\dagger \psi(\phi) = \langle \phi, A^\dagger \psi \rangle$. Moreover, $\psi \in \mathcal{H}'_+$ so $\psi(A\phi) = \langle A\phi, \psi \rangle$. It follows that $A^\dagger : \mathcal{H}_- \rightarrow \mathcal{H}_-$ is defined by

$$D(A^\dagger) = \{\psi \in \mathcal{H}_- \mid \exists \theta \in \mathcal{H}_- : \langle A\phi, \psi \rangle = \langle \phi, \theta \rangle \text{ for all } \phi \in D(A)\},$$

$$A^\dagger \psi = \theta.$$

With this language, Lemma A.2.5 tells us that $H_- \subset H_+^\dagger$. To prove the desired characterization for generalized eigenfunctions, we prove in this section that actually $H_- = H_+^\dagger$. We start with the following observation.

Lemma A.4.1. *We have $H_+^\dagger = \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}$, where $(THT^{-1})^* : \mathcal{H} \rightarrow \mathcal{H}$ is the usual Hilbert adjoint.*

Proof. Let $\psi \in D(H_+^\dagger)$. If $f \in D(THT^{-1})$, then $T^{-1}f \in D(H) \cap D(T)$ and $HT^{-1}f \in D(T)$. Thus, $T^{-1}f \in D_+$ and $\iota_+^{-1}T^{-1}f \in \mathcal{D}_+$. Hence,

$$\begin{aligned} (THT^{-1}f, \tilde{T}_-^{-1}\psi) &= (T_+H_+\iota_+^{-1}T^{-1}f, \tilde{T}_-^{-1}\psi) \\ &= (\tilde{T}_-T_+H_+\iota_+^{-1}T^{-1}f, \psi)_- \\ &= \langle H_+\iota_+^{-1}T^{-1}f, \psi \rangle \\ &= \langle \iota_+^{-1}T^{-1}f, H_+^\dagger\psi \rangle \\ &= (\tilde{T}_-T_+\iota_+^{-1}T^{-1}f, H_+^\dagger\psi)_- \\ &= (\tilde{T}_-f, H_+^\dagger\psi)_- = (f, \tilde{T}_-^{-1}H_+^\dagger\psi) \end{aligned}$$

Thus, $\tilde{T}_-^{-1}\psi \in D((THT^{-1})^*)$ and $(THT^{-1})^*\tilde{T}_-^{-1}\psi = \tilde{T}_-^{-1}H_+^\dagger\psi$. We thus showed that $H_+^\dagger \subset \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}$.

Conversely, let $\psi \in D(\tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1})$. Note that any $\phi \in \mathcal{D}_+$ satisfies $\iota_+\phi \in D(T)$ and $T\iota_+\phi \in D(THT^{-1})$, hence

$$\begin{aligned} \langle H_+\phi, \psi \rangle &= (T_+H_+\phi, \tilde{T}_-^{-1}\psi) \\ &= (TH\iota_+\phi, \tilde{T}_-^{-1}\psi) \\ &= (THT^{-1}T\iota_+\phi, \tilde{T}_-^{-1}\psi) \\ &= (T_+\phi, (THT^{-1})^*\tilde{T}_-^{-1}\psi) \\ &= (\tilde{T}_-T_+\phi, \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi)_- = \langle \phi, \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi \rangle. \end{aligned}$$

Thus, $\psi \in D(H_+^\dagger)$ and $H_+^\dagger\psi = \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi$. Thus $\tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1} \subset H_+^\dagger$ and we are done. \square

Theorem A.4.2. *We have $H_- = H_+^\dagger$.*

Proof. We already know by Lemma A.2.5 that $H_- \subset H_+^\dagger$. To prove the converse, we show that $\tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1} \subset H_-$. The claim will then follow from Lemma A.4.1.

Let $\psi \in D(\tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1})$ and suppose that $\{\psi, \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi\}$ is orthogonal to the graph $G(\iota_-H\iota_-^{-1})$. Recalling that $D(\iota_-H\iota_-^{-1}) = \iota_-D(H)$ by definition, this means that for any $f \in D(H)$ we have

$$\begin{aligned} 0 &= (\iota_-f, \psi)_- + (\iota_-H\iota_-^{-1}(\iota_-f), \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi)_- \\ &= (T^{-1}f, \tilde{T}_-^{-1}\psi) + (T^{-1}Hf, (THT^{-1})^*\tilde{T}_-^{-1}\psi) \\ &= (f, T^{-1}\tilde{T}_-^{-1}\psi) + (Hf, T^{-1}(THT^{-1})^*\tilde{T}_-^{-1}\psi). \end{aligned}$$

Thus, $T^{-1}(THT^{-1})^*\tilde{T}_-^{-1}\psi \in D(H^*) = D(H)$ and $HT^{-1}(THT^{-1})^*\tilde{T}_-^{-1}\psi = -T^{-1}\tilde{T}_-^{-1}\psi \in D(T)$. Hence, $(THT^{-1})(THT^{-1})^*\tilde{T}_-^{-1}\psi = -\tilde{T}_-^{-1}\psi$. But $(THT^{-1})(THT^{-1})^* \geq 0$, so $\tilde{T}_-^{-1}\psi = 0$ and thus $\psi = 0$.

We thus showed that if $\{\psi, \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi\}$ is orthogonal to the graph $G(\iota_-H\iota_-^{-1})$, then $\{\psi, \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi\}$ must be zero. It follows that $\{\psi, \tilde{T}_-(THT^{-1})^*\tilde{T}_-^{-1}\psi\} \in \overline{G(\iota_-H\iota_-^{-1})} = G((\iota_-H\iota_-^{-1})) = G(H_-)$. This completes the proof of the theorem. \square

Corollary A.4.3. *$\psi \in \mathcal{H}_-$ is a generalized eigenfunction if and only if there exists $\lambda \in \mathbb{C}$ such that*

$$\langle H_+\phi, \psi \rangle = \lambda \langle \phi, \psi \rangle \quad \text{for any } \phi \in \mathcal{D}_+.$$

Proof. If $\psi \in \mathcal{H}_-$ is a generalized eigenfunction, then $\psi \in D(H_-)$ and the claim follows from Lemma A.2.5.

Conversely, if this property holds, then $\psi \in D(H_+^\dagger)$ and $H_+^\dagger \psi = \bar{\lambda} \psi$. The claim now follows from Theorem A.4.2. \square

Remark A.4.4. Note that as a byproduct of Lemma A.4.1 and Theorem A.4.2, we proved that H_- is unitarily equivalent to $(THT^{-1})^*$. Moreover, as a consequence of Theorem A.2.9, we have for ν -almost every $\lambda \in \mathbb{R}$, and every $\phi \in \mathcal{H}_+$,

$$(THT^{-1})^* \tilde{T}_-^{-1} P(\lambda) \phi = \tilde{T}_-^{-1} H_- P(\lambda) \phi = \lambda \tilde{T}_-^{-1} P(\lambda) \phi.$$

Hence, with respect to the spectral measure ν of H , almost every $\lambda \in \mathbb{R}$ is an eigenvalue of $(THT^{-1})^*$, with eigenvector $\tilde{T}_-^{-1} P(\lambda) \phi$.

A.5 A characterization of cores

In this section we give two results concerning the core $\iota_- H \iota_-^{-1}$ of H_- (recall that $D(\iota_- H \iota_-^{-1}) = \iota_- D(H)$).

Lemma A.5.1. *Suppose there exists $\psi \in \iota_- \mathcal{H}$ and $\lambda \in \mathbb{C}$ such that*

$$(A-1) \quad \langle H_+ \phi, \psi \rangle = \lambda \langle \phi, \psi \rangle \quad \text{for all } \phi \in \mathcal{D}_+.$$

Then $\psi \in \iota_- D(H)$ and $\iota_- H \iota_-^{-1} \psi = \bar{\lambda} \psi$.

To prove this, one may combine Corollary A.4.3 with Lemma A.2.6. However, the situation is really simpler as we show below; one may simply repeat the argument of Lemma A.2.6.

Proof. Given $\psi \in \iota_- \mathcal{H}$, let $f \in \mathcal{D}_+$. Then

$$(Hf, \iota_-^{-1} \psi) = (\iota_+ H_+ \iota_+^{-1} f, \iota_-^{-1} \psi) = \langle H_+ \iota_+^{-1} f, \psi \rangle = \lambda \langle \iota_+^{-1} f, \psi \rangle = (f, \bar{\lambda} \iota_-^{-1} \psi)$$

Since \mathcal{D}_+ is a core for H , we conclude that $\iota_-^{-1} \psi \in D(H^*) = D(H)$ and $\iota_- H \iota_-^{-1} \psi = \bar{\lambda} \psi$. \square

Next we have the following characterization.

Lemma A.5.2. *We have $\iota_- H \iota_-^{-1} = T_- T_+ H_+^* T_+^{-1} T_-^{-1}$, where $H_+^* : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ is the usual Hilbert adjoint of H_+ .*

Proof. Let $H_-^0 := \iota_- H \iota_-^{-1}$. First note that $D(T_-^{-1}) = \iota_- \mathcal{H}$ and $D(T_+^{-1}) = \iota_+ \mathcal{H}_+ = D(T)$, so that $D(T_+^{-1} T_-^{-1}) = \iota_- \mathcal{H}$. Thus, given $\psi \in \iota_- D(H)$ we have $\psi \in D(T_+^{-1} T_-^{-1})$, and for any $\phi \in \mathcal{D}_+$,

$$\begin{aligned} (H_+ \phi, T_+^{-1} T_-^{-1} \psi)_+ &= (T_+ H_+ \phi, T_-^{-1} \psi) \\ &= (TH \iota_+ \phi, T_-^{-1} \iota_-^{-1} \psi) \\ &= (\iota_+ \phi, H \iota_-^{-1} \psi) \\ &= (T_+ \phi, T_-^{-1} H \iota_-^{-1} \psi) \\ &= (T_+ \phi, T_-^{-1} H_-^0 \psi) = (\phi, T_+^{-1} T_-^{-1} H_-^0 \psi)_+. \end{aligned}$$

It follows that $T_+^{-1} T_-^{-1} \psi \in D(H_+^*)$ and $H_+^* T_+^{-1} T_-^{-1} \psi = T_+^{-1} T_-^{-1} H_-^0 \psi \in D(T_- T_+)$. Hence $\psi \in D(T_- T_+ H_+^* T_+^{-1} T_-^{-1})$ and $T_- T_+ H_+^* T_+^{-1} T_-^{-1} \psi = H_-^0 \psi$. Thus $H_-^0 \subset T_- T_+ H_+^* T_+^{-1} T_-^{-1}$.

Conversely, let $\psi \in D(T_-T_+H_+^*T_+^{-1}T_-^{-1})$. Then $\psi \in D(T_-^{-1}) = \iota_- \mathcal{H}$. Given $f \in D(H)$, since D_+ is a core for H , we may find $(f_j) \subset D_+$ such that $\|f_j - f\| \rightarrow 0$ and $\|Hf_j - Hf\| \rightarrow 0$. Thus,

$$\begin{aligned}
(Hf, \iota_-^{-1}\psi) &= \lim_{j \rightarrow \infty} (Hf_j, \iota_-^{-1}\psi) \\
&= \lim_{j \rightarrow \infty} (T_+H_+\iota_+^{-1}f_j, T_-^{-1}\psi) \\
&= \lim_{j \rightarrow \infty} (H_+\iota_+^{-1}f_j, T_+^{-1}T_-^{-1}\psi)_+ \\
&= \lim_{j \rightarrow \infty} (\iota_+^{-1}f_j, H_+^*T_+^{-1}T_-^{-1}\psi)_+ \\
&= \lim_{j \rightarrow \infty} (Tf_j, T_+H_+^*T_+^{-1}T_-^{-1}\psi) \\
&= (f, TT_+H_+^*T_+^{-1}T_-^{-1}\psi)
\end{aligned}$$

Thus, $\iota_-^{-1}\psi \in D(H^*) = D(H)$ and $H\iota_-^{-1}\psi = TT_+H_+^*T_+^{-1}T_-^{-1}\psi$. Thus, $\psi \in \iota_- D(H)$ and $H_-^0\psi = T_-T_+H_+^*T_+^{-1}T_-^{-1}\psi$. Hence $T_-T_+H_+^*T_+^{-1}T_-^{-1} \subset H_-^0$. \square

Appendix B

Miscellaneous results

B.1 Spectra of compact metric graphs

In Section 2.6 we needed to estimate the spectral gap between the ground eigenvalue of the Kirchhoff Laplacian on the compact graph $\Gamma^{(1)} \cap \Lambda$ and the rest of the spectrum. For this we relied on a Cheeger inequality and obtained the lower bound $E_2 \geq L^{-2}$, where L is the total length of $\Gamma^{(1)} \cap \Lambda$.

Actually much more is true. The following remarkable characterization holds.

Theorem B.1.1 (Nicaise-Friedlander). *Let $(\Gamma, -\Delta)$ be a connected metric graph of total length L , where $-\Delta$ is the Laplace operator with Kirchhoff boundary conditions. Then the eigenvalues $E_j(\Gamma)$ satisfy*

$$E_j(\Gamma) \geq \frac{\pi^2 j^2}{4L^2} \quad \text{for any } j \geq 2.$$

Moreover, equality occurs if and only if Γ is an interval of length L for $j = 2$ and Γ is a star graph with $j + 1$ vertices and j edges of length L/j for $j \geq 3$.

Friedlander proved this theorem in [44]. Actually Nicaise had proven the estimate on E_j much earlier in [84], for $j = 2$, and he also proved a slightly weaker estimate for any j in the case of trees (namely, $E_j \geq \frac{4\pi j^2}{e^2 L^2}$). As we show below, one can always assume the graph is a tree. Friedlander was surely unaware of this paper; in any case the second part concerning equality did not appear in the paper of Nicaise.

We already know that $E_1(\Gamma) = 0$ of course, since a constant function on Γ provides an eigenvector. So a byproduct of Theorem B.1.1 is that the ground eigenvalue is never degenerate; a result that was obtained using cohomology methods in [91] and positivity-improving techniques in [74]. A natural question now is whether the rest of the spectrum $E_j(\Gamma)$ is also simple. This cannot be true in general: as we show below, the spectrum of the Laplacian on the circle has all its eigenvalues E_j of multiplicity 2 for $j \geq 2$. As the Kirchhoff Laplacian on a polygon is equivalent to a Kirchhoff Laplacian on a circle of the same circumference (by removing all the vertices, since they all have degree 2), we see that the answer is negative. However, Friedlander showed the following result in another article.

Theorem B.1.2 (See [45]). *Let $(\Gamma, -\Delta)$ be a connected metric graph of total length L , where $-\Delta$ is the Laplace operator with Kirchhoff boundary conditions. If Γ is different from the circle, and if it has no vertex of degree 2, then the spectrum of Γ is generically simple.*

Finally one may ask if a complement of Theorem B.1.1 holds, that is, if there are upper bounds on the eigenvalues of a metric graph with a fixed total length. Here is the answer:

Theorem B.1.3. *There can be no upper bounds on the eigenvalues of a metric graph. More precisely, for any $L > 0$ and $M > 0$, one may construct a Kirchhoff metric graph $(\Gamma, -\Delta)$ of total length L such that $E_2(\Gamma) > M$.*

Sketch of proof. The example is given by a star graph with $k + 1$ vertices and k edges of length L/k . One may prove that $E_2(\Gamma) = \frac{\pi^2 k^2}{4L^2}$, so taking a sufficiently large k , we see that $E_2(\Gamma) > M$. For details, see [44, Example 3]. \square

We now give a complete proof of a special case of Theorem B.1.1, which is all that we needed for Section 2.6.

Theorem B.1.4 (Faber-Krahn inequality). *Let $(\Gamma, -\Delta)$ be a connected metric graph of total length L , where $-\Delta$ is the Laplace operator with Kirchhoff boundary conditions. Then*

$$E_2(\Gamma) \geq \frac{\pi^2}{L^2}.$$

Proof. Our argument is a variation of the proof of Nicaise [84]; we think this variation is necessary if we do not wish to use the fact that the ground energy is simple. We also provide more details concerning the reduction to a tree which we found in [44].

To see why we may assume Γ is a tree, suppose Γ is a metric graph and let Γ' be the graph that is obtained from Γ by cutting an edge e at some point x_0 . Then if $\mathfrak{h}_\Gamma, \mathfrak{h}_{\Gamma'}$ are the forms associated with $-\Delta_\Gamma, -\Delta_{\Gamma'}$, we have $D(\mathfrak{h}_{\Gamma'}) = W^{1,2}(\Gamma') \supset W^{1,2}(\Gamma) = D(\mathfrak{h}_\Gamma)$ and $\mathfrak{h}_{\Gamma'}[f] = \mathfrak{h}_\Gamma[f]$ for $f \in D(\mathfrak{h}_\Gamma)$, hence $E_j(\Gamma') \leq E_j(\Gamma)$. Thus, if Γ is not a tree, one can cut several edges of Γ to make a connected tree out of it, and the j -th eigenvalue of that tree will not exceed $E_j(\Gamma)$.

So suppose Γ is a connected tree, let C be a circle of circumference $2L$ and let $R : C \rightarrow \Gamma$ be the function that “explores” Γ , that is, R is a path in Γ that crosses each edge exactly once in the forward direction and once in the backward direction. Now since we have Kirchhoff conditions, the min-max principle tells us that

$$(\star) \quad E_2(\Gamma) = \sup_{\phi \in L^2(\Gamma)} \inf_{\substack{\psi \in W^{1,2}(\Gamma), \\ \|\psi\|=1, \psi \perp \phi}} \int_\Gamma |\psi'|^2 \geq \inf_{\substack{\psi \in W^{1,2}(\Gamma), \\ \|\psi\|=1, \psi \perp 1_\Gamma}} \int_\Gamma |\psi'|^2.$$

So let $\psi \in W^{1,2}(\Gamma)$ with $\|\psi\| = 1$ and $\psi \perp 1_\Gamma$. Then $(\psi \circ R) \perp 1_C$ since $\int_C (\psi \circ R) = 2 \int_\Gamma \psi = 0$. Now note that we know exactly the spectrum of the Laplacian on C ; it is given by

$$(\star\star) \quad E_1(C) = 0, \quad E_{2k}(C) = E_{2k+1}(C) = \frac{4\pi^2 k^2}{(2L)^2}, \quad k \geq 1.$$

Indeed, the exponentials $e_j(x) = e^{\frac{2\pi i j x}{2L}}$, $j \in \mathbb{Z}$ provide an orthogonal basis of $L^2(\mathbb{R}/2L\mathbb{Z})$, and they are also eigenvectors of the Laplacian with eigenvalues $\frac{4\pi^2 j^2}{(2L)^2}$; this is just the Parseval theorem. So arranging them in an increasing order we get $(\star\star)$. So let $(f_k)_{k=1}^\infty$ be the corresponding rearranged orthonormal basis of eigenvectors of the Laplacian on C , then $\psi \circ R = \sum \alpha_k f_k$, and since f_1 is a constant and $(\psi \circ R) \perp 1$, the sum is on $k \geq 2$. Thus,

$$\frac{\int_C |(\psi \circ R)'|^2}{\int_C |\psi \circ R|^2} = \frac{\sum_{k \geq 2} |\alpha_k|^2 E_k(C)}{\sum_{k \geq 2} |\alpha_k|^2} \geq \frac{\sum_{k \geq 2} |\alpha_k|^2 E_2(C)}{\sum_{k \geq 2} |\alpha_k|^2} = E_2(C) = \frac{\pi^2}{L^2}$$

by $(\star\star)$. The claim now follows from (\star) since $\frac{\int_\Gamma |\psi'|^2}{\int_\Gamma |\psi|^2} = \frac{\int_C |(\psi \circ R)'|^2}{\int_C |\psi \circ R|^2}$. \square

B.2 Factorization of Hilbert-Schmidt operators

This section collects some useful factorization results for Hilbert-Schmidt operators; we needed one of them in Section 2.8.

In the following we fix an arbitrary measure space (S, μ) and denote $L^p(\mu) := L^p(S, \mu)$ for $1 \leq p \leq \infty$. These spaces include $L^p(\mathbb{R}^d)$ and $\ell^p(\mathbb{Z}^d)$ by taking the Lebesgue measure and the counting measure respectively.

Given $1 \leq p < \infty$, the space of p -summing operators between two Banach spaces X and Y will be denoted by $\Pi_p(X, Y)$, and the p -summing norm of $u \in \Pi_p(X, Y)$ will be denoted by $\pi_p(u)$. We have the following facts.

- Theorem B.2.1.**
1. Let X_0, X, Y and Y_0 be Banach spaces. Suppose $u : Y \rightarrow Y_0$ and $w : X_0 \rightarrow X$ are bounded operators and let $v \in \Pi_p(X, Y)$. Then $uvw \in \Pi_p(X_0, Y_0)$ and $\pi_p(uvw) \leq \|u\| \cdot \pi_p(v) \cdot \|w\|$.
 2. Let H_1 and H_2 be Hilbert spaces. Then $u : H_1 \rightarrow H_2$ is Hilbert-Schmidt if and only if it is 2-summing. In this case, we have $\|u\|_2 = \pi_2(u)$.
 3. Given $1 \leq p < \infty$ and $\varphi \in L^p(\mu)$ denote by $M_\varphi : L^\infty(\mu) \rightarrow L^p(\mu)$ the induced multiplication operator $M_\varphi(f) := \varphi f$. Then M_φ is p -summing and $\pi_p(M_\varphi) = \|\varphi\|_{L^p}$.
 4. Suppose $B : L^2(\mu) \rightarrow L^\infty(\mu)$ is a bounded operator and $M_\varphi : L^\infty(\mu) \rightarrow L^2(\mu)$ is the operator of multiplication by $\varphi \in L^2(\mu)$. Then $M_\varphi B : L^2(\mu) \rightarrow L^2(\mu)$ is Hilbert-Schmidt and $\|M_\varphi B\|_2 \leq \|\varphi\|_{L^2} \|B\|_{L^2 \rightarrow L^\infty}$.

Proof. The first three items are given in [35, Theorem 2.4, Page 37], [35, Theorem 4.10, Page 84] and [35, Examples 2.9, Page 40] respectively. The fourth item follows by combining them. \square

An alternative proof of the fourth item can probably be extracted from the proof of [100, Proposition B.9.4].

The previous theorem is all that we needed for Section 2.8. Nevertheless, let us now extend this theorem to arbitrary operators $A : L^\infty \rightarrow L^2$, not just multiplications.

Given $1 \leq p \leq \infty$ and $\lambda > 1$, a Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ -space if every finite dimensional subspace E of X is contained in a finite dimensional subspace F of X for which there is an isomorphism $v : F \rightarrow \ell_p^{\dim F}$ with $\|v\| \cdot \|v^{-1}\| < \lambda$. We say that X is an \mathcal{L}_p -space if it is an $\mathcal{L}_{p,\lambda}$ -space for some $\lambda > 1$.

- Theorem B.2.2.**
1. If (S, μ) is any measure space and $1 \leq p \leq \infty$, then $L^p(\mu)$ is an $\mathcal{L}_{p,\lambda}$ -space for all $\lambda > 1$.
 2. Let $1 \leq p \leq 2$, let X be an $\mathcal{L}_{\infty,\lambda}$ -space and let Y be an $\mathcal{L}_{p,\lambda'}$ -space. Then any bounded operator $u : X \rightarrow Y$ is 2-summing with $\pi_2(u) \leq K_G \lambda \lambda' \|u\|$, where K_G is a universal constant called the Grothendieck constant.
 3. We have $K_G \leq 2$.
 4. Suppose $B : L^2(\mu) \rightarrow L^\infty(\mu)$ and $A : L^\infty(\mu) \rightarrow L^2(\mu)$ are bounded operators. Then $AB : L^2(\mu) \rightarrow L^2(\mu)$ is Hilbert-Schmidt and $\|AB\|_2 \leq K_G \cdot \|A\|_{L^\infty \rightarrow L^2} \|B\|_{L^2 \rightarrow L^\infty}$.
 5. Let H_1 and H_2 be Hilbert spaces. Then $u : H_1 \rightarrow H_2$ is Hilbert-Schmidt if and only if it factors through an \mathcal{L}_∞ -space.

The fourth item is stated in [104, Section 4.1.11], and this result is even more general than what we needed in Section 2.8. We only gave the fifth item for completeness.

Proof. The first three items are given in [35, Theorem 3.2, Page 61], [35, Theorem 3.7, Page 64] and [35, Page 29] respectively. For the fourth, note that by combining these items, A is 2-summing with $\pi_2(A) \leq K_G \lambda \lambda' \|A\|$ for any $\lambda, \lambda' > 1$. By the preceding theorem, AB is also 2-summing, hence Hilbert-Schmidt, with $\|AB\|_2 = \pi_2(AB) \leq \pi_2(A) \|B\| \leq K_G \lambda \lambda' \|A\| \|B\|$. Since this holds for any $\lambda, \lambda' > 1$, we get the bound. The last item is given in [35, Corollary 4.12, Page 85], see also [80, Theorem 6.3]. \square

B.3 Self-adjoint dilations of dissipative operators

In this section we derive some results concerning self-adjoint dilations of maximal dissipative operators in a complex separable Hilbert space. These results are applied in the following section to prove a spectral averaging estimate for general probability measures.

We shall use the notation

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \quad \text{and} \quad \mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

For two bounded operators B and \mathcal{B} on Hilbert spaces \mathcal{H} and \mathcal{G} , we say that

$$B = \operatorname{pr} \mathcal{B}$$

if \mathcal{H} is a subspace of \mathcal{G} , and

$$(Bf, g) = (\mathcal{B}f, g) \quad \text{for all } f, g \in \mathcal{H}.$$

An operator L , not necessarily bounded, on a Hilbert space \mathcal{H} is *dissipative* if

$$\operatorname{Im}(Lf, f) \geq 0 \quad \text{for all } f \in D(L).$$

It is *maximal dissipative* if it has no proper dissipative extension.

An operator Q on a Hilbert space \mathcal{H} is said to be *accretive*¹ if

$$\operatorname{Re}(Qf, f) \geq 0 \quad \text{for all } f \in D(Q).$$

It is said to be maximal accretive if it has no proper accretive extension.

We now state an important result. This result is basically known among specialists, but it took us some effort to find good references.

Theorem B.3.1. *Let L be a maximal dissipative operator on a Hilbert space \mathcal{H} . Then there exists a Hilbert space \mathcal{G} containing \mathcal{H} and a self-adjoint operator \mathcal{L} on \mathcal{G} such that*

$$(L - z)^{-1} = \operatorname{pr} (\mathcal{L} - z)^{-1} \quad \text{and} \quad e^{itL} = \operatorname{pr} e^{it\mathcal{L}}$$

provided $z \in \mathbb{C}_-$ and $t \geq 0$.

Proof. We fill the details of the sketch given in [5, Section 4]. Let $Q := -iL$, then Q is maximal accretive. Define T , the *Cayley transform* of Q by

$$T := (Q - 1)(Q + 1)^{-1}.$$

By [106, Section IV.4], it follows that T is a contraction which is defined everywhere on \mathcal{H} , that T does not have the eigenvalue 1 and that $Q = (1 + T)(1 - T)^{-1}$.

1. We caution the reader that we follow the definitions of [106], which differ from the definitions of [105]

Now by [106, Theorem I.4.2], T has a *unitary dilation* U , i.e. there exists a Hilbert space \mathcal{G} containing \mathcal{H} and a unitary operator U on \mathcal{G} such that

$$T^n = \text{pr } U^n \quad \text{for all } n \geq 1.$$

Moreover, since 1 is not an eigenvalue of T , 1 is not an eigenvalue of U (see [106, Proposition II.6.1]). Now by [106, Theorem III.2.3.(g)], we have more generally a functional calculus for T given by

$$g(T) = \text{pr } g(U) \quad \text{for all } g \in H_T^\infty,$$

in a certain class $H_T^\infty \subset H^\infty$, see [106, Section III.2] for details (recall that the Hardy space H^∞ is the set of bounded holomorphic functions in $D = \{\zeta : |\zeta| < 1\}$).

Now since 1 is not an eigenvalue of T , the one-point set $\{1\}$ is of measure 0 with respect to the spectral measure E_T corresponding to the unitary part of T . Therefore, by the same [106, Theorem III.2.3], the class H_T^∞ contains the functions $g \in H^\infty$ that are continuous on $\bar{D} \setminus \{1\}$. Now observe that $Q = \omega(T)$, where $\omega(\zeta) = \frac{1+\zeta}{1-\zeta}$. Since ω maps $\bar{D} \setminus \{1\}$ onto the closed half plane $\bar{\Delta} = \{\lambda : \text{Re } \lambda \geq 0\}$, the point $z = \infty$ not included, it follows that $f \circ \omega \in H_T^\infty$ for any bounded continuous f on $\bar{\Delta}$ which is holomorphic in $\Delta = \{\lambda : \text{Re } \lambda > 0\}$. In particular, if $\text{Im } z < 0$, then the relation holds for $g = f \circ \omega$, where $f(\lambda) := \frac{1}{\lambda + iz}$ and we get

$$(Q + iz)^{-1} = f \circ \omega(T) = \text{pr } f \circ \omega(U) = \text{pr } (Q + iz)^{-1},$$

where $Q := (1 + U)(1 - U)^{-1}$. Thus,

$$(L - z)^{-1} = \text{pr } (\mathcal{L} - z)^{-1},$$

where $\mathcal{L} := iQ$.

Finally, since U is unitary and 1 is not an eigenvalue of U , we have by [32, Corollary X.3.5] that \mathcal{L} is self-adjoint. This completes the proof of the first assertion.

For the second one, given $t \geq 0$, take $f(\lambda) = e^{-t\lambda}$, which is bounded and continuous on $\bar{\Delta}$ and holomorphic in Δ . □

We note in passing that the above proof yields more generally a functional calculus for maximal dissipative operators given by

$$f(L) = \text{pr } f(\mathcal{L}) \quad \text{for any } f \in H^\infty(\overline{\mathbb{C}_+}),$$

where $H^\infty(\overline{\mathbb{C}_+})$ is the set of bounded continuous functions on upper half plane $\overline{\mathbb{C}_+}$ which are holomorphic in \mathbb{C}_+ .

We now describe the maximal dissipative operators to which we want to apply the previous theorem. Note that if $B \geq \delta > 0$ is a bounded operator, then B is invertible: injectivity is obvious, for surjectivity, note that $\text{Ran}(B)^\perp = \ker(B^*) = \ker(B) = \{0\}$.

Lemma B.3.2. *Let R be a self-adjoint operator and let $S \geq 0$ be a bounded operator. Define $L_0 := R + iS$ and suppose $B \geq \delta > 0$ is a bounded operator. Then $(L_0 - zB)$ is invertible for any $z \in \mathbb{C}_-$. Moreover, if $L := B^{-1/2}L_0B^{-1/2}$, then L is maximal dissipative and*

$$(L - z)^{-1} = B^{1/2}(L_0 - zB)^{-1}B^{1/2} \quad \text{for any } z \in \mathbb{C}_-.$$

Proof. Let $z = x - iy \in \mathbb{C}_-$, where $y > 0$. Then $(L_0 - zB) = \tilde{R} + i\tilde{S}$, where $\tilde{R} = R - xB$ and $\tilde{S} = S + yB$ are self-adjoint. Hence, given $f \in D(L_0)$,

$$\|(L_0 - zB)f\| \|f\| \geq \operatorname{Im}((L_0 - zB)f, f) = \operatorname{Im}((\tilde{R}f, f) + i(\tilde{S}f, f)) = (\tilde{S}f, f) \geq y\delta \|f\|^2.$$

Hence, $(L_0 - zB)$ is injective. Moreover, $(\operatorname{Ran}(L_0 - zB))^\perp = \ker(\tilde{R} - i\tilde{S})$. But

$$\|(\tilde{R} - i\tilde{S})f\| \|f\| = \|(i\tilde{S} - \tilde{R})f\| \|f\| \geq \operatorname{Im}((i\tilde{S} - \tilde{R})f, f) \geq y\delta \|f\|^2,$$

so $\ker(\tilde{R} - i\tilde{S}) = \{0\}$. Thus, $(L_0 - zB)$ is invertible with $\|(L_0 - zB)^{-1}\| \leq (y\delta)^{-1}$. Finally,

$$\begin{aligned} B^{1/2}(L_0 - zB)^{-1}B^{1/2}(L - z) &= B^{1/2}(L_0 - zB)^{-1}B^{1/2}B^{-1/2}(L_0 - zB)B^{-1/2} \\ &= I = (L - z)B^{1/2}(L_0 - zB)^{-1}B^{1/2}, \end{aligned}$$

so $(L - z)$ is invertible for any $z \in \mathbb{C}_-$ with the given relation. In particular, for $z = -i$, we get that $\operatorname{Ran}(L + i) = \mathcal{H}$ so that L is maximal dissipative by [106, Theorem IV.4.1] (note that L is dissipative since $\operatorname{Im} L = B^{-1/2}SB^{-1/2} \geq 0$). \square

Note that if $B \geq 0$ is invertible, then $B \geq \delta > 0$ for $\delta = \|B^{-1/2}\|^{-2}$. Indeed,

$$(Bf, f) = (B^{1/2}f, B^{1/2}f) = \|B^{1/2}f\|^2 \geq \|B^{-1/2}\|^{-2} \|B^{-1/2}B^{1/2}f\|^2 = \delta \|f\|^2.$$

A variant of the previous lemma appears in [2, Appendix B]. Namely, one can assume instead that $S \geq \delta > 0$ and $B \geq 0$. The proof becomes a bit more complicated, and the conclusion only holds in $(\ker B)^\perp$.

B.4 Spectral averaging for general probability measures

We may now prove the spectral averaging estimate. We follow here the approach of [105], but we slightly improve the upper bound.

We start with the following lemma. Recall that for a probability measure μ on \mathbb{R} , we denote by $s(\mu, \varepsilon) = \sup_{E \in \mathbb{R}} \{\mu(E, E + \varepsilon)\}$.

Lemma B.4.1. *Let μ be a probability measure on \mathbb{R} , $\lambda > 0$ and $a \in \mathbb{R}$. Then for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}} \frac{\lambda\varepsilon^2}{(t-a)^2 + \lambda^2\varepsilon^2} d\mu(t) \leq \frac{\pi \tanh \pi}{\lambda} \cdot s(\mu, \lambda\varepsilon).$$

Note that $\tanh \pi < 1$.

Proof. We first note that $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} I_k$, where $I_k := [k\lambda\varepsilon + a - \frac{\lambda\varepsilon}{2}, k\lambda\varepsilon + a + \frac{\lambda\varepsilon}{2})$. Indeed, given $t \in \mathbb{R}$, we have for $k := \lfloor \frac{t-a}{\lambda\varepsilon} + \frac{1}{2} \rfloor$,

$$\begin{aligned} k\lambda\varepsilon + a - \frac{\lambda\varepsilon}{2} &\leq \left(\frac{t-a}{\lambda\varepsilon} + \frac{1}{2}\right) \cdot \lambda\varepsilon + a - \frac{\lambda\varepsilon}{2} = t \\ &< (k+1) \cdot \lambda\varepsilon + a - \frac{\lambda\varepsilon}{2} = k\lambda\varepsilon + a + \frac{\lambda\varepsilon}{2}. \end{aligned}$$

Hence, using (4-5-2),

$$\begin{aligned} \int_{\mathbb{R}} \frac{\lambda\varepsilon^2}{(t-a)^2 + \lambda^2\varepsilon^2} d\mu(t) &\leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \int_{I_k} \frac{\lambda^2\varepsilon^2}{(t-a)^2 + \lambda^2\varepsilon^2} d\mu(t) \\ &\leq \frac{1}{\lambda} s(\mu, \lambda\varepsilon) \sum_{k \in \mathbb{Z}} \frac{1}{(k - \frac{1}{2})^2 + 1} \end{aligned}$$

since $|I_k| = \lambda\varepsilon$ and $(t - a) \geq (k - \frac{1}{2})\lambda\varepsilon$ in I_k . Let $f(z) = \frac{1}{(z - \frac{1}{2})^2 + 1}$. Then f has no poles in \mathbb{Z} , so by the residue theorem (see e.g. [8, Section 11.2]), if z_j are the complex poles of f , we have

$$\sum_{k \in \mathbb{Z}} f(k) = - \sum_{z_j} \text{Res}(f(z)\pi \cot(\pi z); z_j).$$

For our f we have two poles, namely $z_1 = \frac{1}{2} + i$ and $z_2 = \frac{1}{2} - i$. Now

$$\text{Res}(f(z)\pi \cot(\pi z); z_1) = \lim_{z \rightarrow z_1} (z - z_1)\pi \cot(\pi z)f(z) = \lim_{z \rightarrow z_1} \frac{\pi \cot(\pi z)}{z - z_2} = \frac{\pi \cot \pi z_1}{2i}.$$

Similarly, $\text{Res}(f(z)\pi \cot(\pi z); z_2) = \frac{\pi \cot \pi z_2}{-2i}$. Hence,

$$- \sum_{z_j} \text{Res}(f(z)\pi \cot(\pi z); z_j) = \frac{\pi}{2i} (\cot \pi z_2 - \cot \pi z_1).$$

Now observe that for $z \in \mathbb{C}$, we have $e^{2i\bar{z}} = \overline{e^{-2iz}}$, so that

$$\cot \bar{z} = i \frac{e^{2i\bar{z}} + 1}{e^{2i\bar{z}} - 1} = \overline{-i \frac{e^{-2iz} + 1}{e^{-2iz} - 1}} = \overline{-i \frac{1 + e^{2iz}}{1 - e^{2iz}}} = \overline{\cot z}$$

so noting that $z_2 = \bar{z}_1$ we get

$$\sum_{k \in \mathbb{Z}} f(k) = \frac{\pi}{2i} (\overline{\cot \pi z_1} - \cot \pi z_1) = -\pi \text{Im} \cot \pi z_1.$$

Finally, if $z = x + iy$, $x, y \in \mathbb{R}$, then $\text{Re} e^{2iz} = \cos(2x)e^{-2y}$ and $\text{Im} e^{2iz} = \sin(2x)e^{-2y}$, so

$$\cot z = i \frac{e^{2iz} + 1}{e^{2iz} - 1} = i \frac{(e^{2iz} + 1)(e^{-2i\bar{z}} - 1)}{(e^{2iz} - 1)(e^{-2i\bar{z}} - 1)} = i \frac{e^{-4y} - 2i \sin(2x)e^{-2y} - 1}{e^{-4y} - 2 \cos(2x)e^{-2y} + 1},$$

so for $x = \frac{\pi}{2}$ and $y = \pi$ we get

$$\begin{aligned} \cot \pi z_1 &= i \frac{e^{-4\pi} - 1}{e^{-4\pi} + 2e^{-2\pi} + 1} = i \frac{(e^{-2\pi} - 1)(e^{-2\pi} + 1)}{(e^{-2\pi} + 1)^2} \\ &= i \frac{e^{-2\pi} - 1}{e^{-2\pi} + 1} = i \frac{e^{-\pi} - e^{\pi}}{e^{-\pi} + e^{\pi}} = -i \tanh \pi. \quad \square \end{aligned}$$

Theorem B.4.2 (Spectral Averaging). *Let μ be a probability measure on \mathbb{R} , A a self-adjoint operator and $B \geq 0$ a bounded operator on \mathcal{H} . Fix $\lambda > 0$ such that $\lambda B \leq 1$. Then for any bounded interval I and any $\phi \in \mathcal{H}$ we have*

$$\int_{\mathbb{R}} (B^{1/2} \chi_I(A + tB) B^{1/2} \phi, \phi) d\mu(t) \leq \frac{4}{\lambda} \|\phi\|^2 s(\mu, \lambda|I|).$$

By $\lambda B \leq 1$, we mean that $(\lambda B f, f) \leq \|f\|^2$ for all f . If $\|B\| \leq 1$, we may take $\lambda = 1$, otherwise we may take $\lambda = \frac{1}{\|B\|}$. Leaving the statement in terms of λ is more convenient for applications because in general we have no upper bound on $\frac{1}{\|B\|}$.

Note that λ may be taken arbitrarily small to shrink the quantity $\lambda|I|$, which could be useful. Moreover, our RHS is more or less independent of B , which is the estimate one expects; see e.g. [109, Lemma 5.3.2] for the special case where $d\mu = g(x)dx$.

Proof. First note that for $x \in I = (a, b)$ and $\varepsilon := |I| = b - a$ we have

$$f(x) := \int_I \operatorname{Im} \frac{1}{x - E - i\varepsilon} dE = \int_I \frac{\varepsilon}{(x - E)^2 + \varepsilon^2} dE = \tan^{-1}\left(\frac{b-x}{\varepsilon}\right) - \tan^{-1}\left(\frac{a-x}{\varepsilon}\right).$$

Writing $x = ta + (1-t)b$ for some $0 \leq t \leq 1$, this becomes $(\tan^{-1}(t) + \tan^{-1}(1-t)) := g(t)$. Since $g'(t) = \frac{1}{1+t^2} - \frac{1}{1+(1-t)^2} = \frac{-2t+1}{(1+t^2)(1+(1-t)^2)}$, we see that g increases from 0 to $\frac{1}{2}$ and decreases from $\frac{1}{2}$ to 1. Thus, $g(t) \geq \min(g(0), g(1))$ for all $t \in [0, 1]$. Since $g(0) = g(1) = \tan^{-1}(1) = \frac{\pi}{4}$, we finally get $f(x) \geq \frac{\pi}{4}$ for any $x \in I$. By the functional calculus, this implies that

$$\chi_I(A + tB) \leq \frac{4}{\pi} \int_I \operatorname{Im}(A + tB - E - i\varepsilon)^{-1} dE,$$

thus

$$(B^{1/2} \chi_I(A + tB) B^{1/2} \phi, \phi) \leq \frac{4}{\pi} \int_I \operatorname{Im}(B^{1/2}(A + tB - E - i\varepsilon)^{-1} B^{1/2} \phi, \phi) dE.$$

Fix $0 < \tilde{\lambda} < \lambda$ such that $\tanh \pi \leq \frac{\tilde{\lambda}}{\lambda}$, fix $\delta > 0$ and let $B_\delta := B + \delta$. Since $\lambda B \leq 1$, we may choose δ small enough to ensure $\tilde{\lambda} B_\delta \leq 1$. Now if $L_0^E = -(A - E - i\varepsilon(1 - \tilde{\lambda} B_\delta))$, then by Lemma B.3.2, the operator $L_0^E - (t - i\tilde{\lambda}\varepsilon)B_\delta$ is invertible and

$$\begin{aligned} (B_\delta^{1/2}(A + tB_\delta - E - i\varepsilon)^{-1} B_\delta^{1/2} \phi, \phi) &= -(B_\delta^{1/2}(L_0^E - (t - i\tilde{\lambda}\varepsilon)B_\delta)^{-1} B_\delta^{1/2} \phi, \phi) \\ &= -((L^E - t + i\tilde{\lambda}\varepsilon)^{-1} \phi, \phi), \end{aligned}$$

where $L^E = B_\delta^{-1/2} L_0^E B_\delta^{-1/2}$ is maximal dissipative. So we may apply Theorem B.3.1 to find a Hilbert space \mathcal{G} containing \mathcal{H} and a self-adjoint operator \mathcal{L}^E on \mathcal{G} such that

$$((L^E - t + i\tilde{\lambda}\varepsilon)^{-1} \phi, \phi) = ((\mathcal{L}^E - t + i\tilde{\lambda}\varepsilon)^{-1} \phi, \phi) = \int_{\mathbb{R}} \frac{1}{x - t + i\tilde{\lambda}\varepsilon} d\rho_\phi^E(x),$$

where ρ_ϕ^E is the spectral measure of \mathcal{L}^E in the state ϕ . Hence, using Fubini Theorem and Lemma B.4.1, we get

$$\begin{aligned} (\star) \quad & \frac{4}{\pi} \int_{\mathbb{R}} \int_I \operatorname{Im}(B_\delta^{1/2}(A + tB_\delta - E - i\varepsilon)^{-1} B_\delta^{1/2} \phi, \phi) dE d\mu(t) \\ & \leq \frac{4}{\pi} \int_{\mathbb{R}} \int_I \int_{\mathbb{R}} \frac{\tilde{\lambda}\varepsilon}{(x-t)^2 + \tilde{\lambda}^2\varepsilon^2} d\mu(t) d\rho_\phi^E(x) dE \\ & \leq \frac{4 \tanh \pi}{\tilde{\lambda}} s(\mu, \tilde{\lambda}\varepsilon) \frac{1}{\varepsilon} \int_I \int_{\mathbb{R}} d\rho_\phi^E(x) dE \leq \frac{4}{\tilde{\lambda}} \|\phi\|^2 s(\mu, \lambda\varepsilon) \end{aligned}$$

as $\frac{1}{\varepsilon}$ cancels $|I|$. Finally, as $\delta \rightarrow 0$ we have

$$\|B_\delta^{1/2} - B^{1/2}\| \leq \|B_\delta - B\| \| (B_\delta^{1/2} + B^{1/2})^{-1} \| \leq \delta \delta^{-1/2} \rightarrow 0,$$

since $(B_\delta^{1/2} + B^{1/2}) \geq \delta^{1/2}$. Moreover, by the second resolvent identity,

$$\|(A + tB_\delta - E - i\varepsilon)^{-1} - (A + tB - E - i\varepsilon)^{-1}\| = \|(A + tB_\delta - E - i\varepsilon)^{-1} (t\delta) (A + tB - E - i\varepsilon)^{-1}\| \rightarrow 0$$

for any $t \in \mathbb{R}$. So it follows from Lebesgue dominated convergence theorem that the estimate (\star) holds with $B_\delta^{1/2}$ replaced by $B^{1/2}$ and we are done. \square

B.5 A criterion to establish UP

We prove here the simple criterion that was derived in [20] to test uncertainty principles. We used this in Section 4.4.

Theorem B.5.1. *Let \mathcal{H} be a Hilbert space and H a self-adjoint operator on \mathcal{H} which is bounded below. Suppose W is a bounded self-adjoint operator and define*

$$\lambda(t) := \inf \sigma(H + tW), \quad t \geq 0.$$

Suppose that $\lambda(t_0) > \zeta$ for some $t_0 > 0$ and $\zeta \in \mathbb{R}$. Then for any interval $I \subset (-\infty, \zeta]$ we have

$$\chi_I(H)W\chi_I(H) \geq \gamma\chi_I(H), \quad \gamma := \frac{\lambda(t_0) - \zeta}{t_0}.$$

Proof. Let $g \in D(H)$, $I = [E_1, E_2]$ with $E_2 \leq \zeta$ and put $g_I := \chi_I(H)g$. Then

$$\begin{aligned} \langle \chi_I(H)W\chi_I(H)g, g \rangle &= \frac{1}{t_0} \{ \langle (H + t_0W)g_I, g_I \rangle - \langle Hg_I, g_I \rangle \} \\ &\geq \frac{1}{t_0} \{ \lambda(t_0) \cdot \|g_I\|^2 - \langle Hg_I, g_I \rangle \}. \end{aligned}$$

But if ρ_g is the spectral measure of H in the state g , then

$$\langle Hg_I, g_I \rangle = \int_{E_1}^{E_2} x \, d\rho_g(x) \leq E_2 \int_{E_1}^{E_2} d\rho_g(x) \leq \zeta \cdot \|g_I\|^2.$$

Hence

$$\langle \chi_I(H)W\chi_I(H)g, g \rangle \geq \gamma \cdot \|g_I\|^2 = \gamma \cdot \|\chi_I(H)g\|^2$$

for any $g \in D(H)$. As $D(H)$ is dense, the same holds for any $g \in \mathcal{H}$ and we are done. \square

For completeness we give the following converse which was also derived in [20], nevertheless we give a different argument for the proof.

Lemma B.5.2. *Let \mathcal{H} be a Hilbert space and H a self-adjoint operator on \mathcal{H} which is bounded below and has a discrete spectrum. Suppose $W \geq 0$ is bounded and define*

$$\lambda(t) := \inf \sigma(H + tW), \quad t \geq 0.$$

Suppose that in the interval $I = [\lambda(0), E_2]$ we have

$$\chi_I(H)W\chi_I(H) \geq \gamma\chi_I(H)$$

for some $\gamma > 0$. Then $\lambda(t) > \lambda(0)$ for all $t > 0$.

Proof. Let $f \in D(H)$, put $f_1 = \chi_I(H)f$ and $f_2 = \chi_{I^c}(H)f$. Note that $f_2 = \chi_{(E_2, +\infty)}(H)f$ since H has no spectrum below $\lambda(0)$. Hence, if ρ_f is a spectral measure for H in the state f , we have

$$\langle Hf_2, f_2 \rangle = \int_{E_2}^{\infty} x \, d\rho_f(x) \geq E_2 \cdot \|f_2\|^2,$$

2. The reader may notice that compared to the article [20], we imposed the additional condition that H has a discrete spectrum. Actually we also needed this condition to conclude the argument in [20], but perhaps this is not necessary. In any case, this assumption is always satisfied in applications since H is typically a Schrödinger operator restricted to a cube.

and thus,

$$\begin{aligned}
\langle (H + tW)f, f \rangle &\geq \langle Hf, f \rangle \\
&= \langle Hf_1, f_1 \rangle + \langle Hf_2, f_2 \rangle \\
&\geq \lambda(0)\|f_1\|^2 + E_2\|f_2\|^2 \\
&= \lambda(0)\|f\|^2 + (E_2 - \lambda(0))\|f_2\|^2 > \lambda(0)\|f\|^2
\end{aligned}$$

provided $f_2 \neq 0$. On the other hand, if $f_2 = 0$, then

$$\langle (H + tW)f, f \rangle = \langle Hf, f \rangle + t\langle Wf_1, f_1 \rangle \geq \lambda(0)\|f\|^2 + t\gamma\|f_1\|^2 = (\lambda(0) + t\gamma)\|f\|^2.$$

We thus showed that for any $t > 0$ and any $f \in D(H)$, we have $\langle (H + tW)f, f \rangle > \lambda(0)\|f\|^2$. In particular, if f is a normalized eigenvector of $H + tW$ corresponding to $E_1(H + tW)$ we get

$$\lambda(t) = E_1(H + tW) = \langle (H + tW)f, f \rangle > \lambda(0)\|f\|^2 = \lambda(0). \quad \square$$

Bibliography

- [1] M. Abramowitz and I. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover, 1965.
- [2] M. Aizenman, A. Elgart, S. Naboko, J. Schenker, and G. Stolz. Moment analysis for localization in random Schrödinger operators. *Invent. Math.*, 163(2):343–413, 2006.
- [3] M. Aizenman and S. Molchanov. Localization at large disorder and at extreme energies: an elementary derivation. *Comm. Math. Phys.*, 157(2):245–278, 1993.
- [4] M. Aizenman, R. Sims, and S. Warzel. Absolutely continuous spectra of quantum tree graphs with weak disorder. *Comm. Math. Phys.*, 264(2):371–389, 2006.
- [5] A. B. Aleksandrov and V. V. Peller. Functions of perturbed dissipative operators. *St. Petersburg Math. J.*, 23:209–238, 2012.
- [6] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492–1505, Mar 1958.
- [7] C. Băcuță, A. L. Mazzucato, V. Nistor, and L. Zikatanov. Interface and mixed boundary value problems on n -dimensional polyhedral domains. *Doc. Math.*, 15:687–745, 2010.
- [8] J. Bak and D. Newman. *Complex analysis*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1997.
- [9] J. M. Barbaroux, J. M. Combes, and P. D. Hislop. Localization near band edges for random Schrödinger operators. *Helv. Phys. Acta*, 70(1-2):16–43, 1997. Papers honouring the 60th birthday of Klaus Hepp and of Walter Hunziker, Part II (Zürich, 1995).
- [10] J. M. Berezanskii. *Expansions in Eigenfunctions of Selfadjoint Operators*, volume 17 of *Translations of Mathematical Monographs*. AMS, 1968.
- [11] Y. M. Berezansky, Z. G. Sheftel, and G. F. Us. *Functional analysis. Vol. II*, volume 86 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1996.
- [12] R. Bhattacharya and E. Waymire. *A Basic Course in Probability Theory*. Universitext. Springer, New York, 2007.
- [13] V. I. Bogachev. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007.
- [14] J. Bolte and J. Kerner. Quantum graphs with singular two-particle interactions. *J. Phys. A*, 46(4):045206, 25, 2013.
- [15] J. Bourgain and C. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2):389–426, 2005.
- [16] J. Bourgain and A. Klein. Bounds on the density of states for Schrödinger operators. *Invent. Math.*, pages 1–32, 2013.

- [17] A. Boutet de Monvel, V. Chulaevsky, P. Stollmann, and Y. Suhov. Anderson localization for a multi-particle model with alloy-type external potential. *preprint arXiv: 1004.1300*, 2010.
- [18] A. Boutet de Monvel, V. Chulaevsky, P. Stollmann, and Y. Suhov. Wegner-type bounds for a multi-particle continuous Anderson model with an alloy-type external potential. *J. Stat. Phys.*, 138(4-5):553–566, 2010.
- [19] A. Boutet de Monvel, V. Chulaevsky, and Y. Suhov. Dynamical localization for a multi-particle model with an alloy-type external random potential. *Nonlinearity*, 24:1451–1472, May 2011.
- [20] A. Boutet de Monvel, D. Lenz, and P. Stollmann. An uncertainty principle, Wegner estimates and localization near fluctuation boundaries. *Math. Z.*, 269(3-4):663–670, 2011.
- [21] A. Boutet de Monvel, S. Naboko, P. Stollmann, and G. Stolz. Localization near fluctuation boundaries via fractional moments and applications. *J. Anal. Math.*, 100:83–116, 2006.
- [22] A. Boutet de Monvel and P. Stollmann. Eigenfunction expansions for generators of Dirichlet forms. *Journal für die reine und angewandte Mathematik*, 561(561):131–144, 2003.
- [23] L. Breiman. *Probability*, volume 7 of *Classics in Applied Mathematics*. SIAM, Philadelphia, PA, 1992. Corrected reprint of the 1968 original.
- [24] V. Chulaevsky. Localization with less larmes: simply msa. *preprint arXiv: 0812.2634*, 2008.
- [25] V. Chulaevsky. A Wegner-type estimate for correlated potentials. *Math. Phys. Anal. Geom.*, 11(2):117–129, 2008.
- [26] V. Chulaevsky and Y. Suhov. Wegner Bounds for a Two-Particle Tight Binding Model. *Comm. Math. Phys.*, 283:479–489, 2008. 10.1007/s00220-008-0508-3.
- [27] V. Chulaevsky and Y. Suhov. Multi-particle Anderson Localisation: Induction on the Number of Particles. *Math. Phys. Anal. Geom.*, 12:117–139, 2009.
- [28] K. Ciesielski, H. Fejzić, and C. Freiling. Measure zero sets with non-measurable sum. *Real Anal. Exchange*, 27(2):783–793, 2001/02.
- [29] J-M. Combes, P. Hislop, and F. Klopp. Hölder continuity of the integrated density of states for some random operators at all energies. *Int. Math. Res. Not.*, (4):179–209, 2003.
- [30] J-M. Combes, P. Hislop, and F. Klopp. An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. *Duke Math. J.*, 140(3):469–498, 2007.
- [31] J. M. Combes, P. D. Hislop, F. Klopp, and S. Nakamura. The Wegner estimate and the integrated density of states for some random operators. *Proc. Indian Acad. Sci. Math. Sci.*, 112(1):31–53, 2002. Spectral and inverse spectral theory (Goa, 2000).
- [32] J. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [33] T. Coulhon and A. Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proc. Lond. Math. Soc. (3)*, 96(2):507–544, 2008.
- [34] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition, 1987.

- [35] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
- [36] R. M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.
- [37] A. Elgart and A. Klein. Ground state energy of trimmed discrete Schrödinger operators and localization for trimmed Anderson models. *preprint arXiv: 1301.5268v1*, 2013.
- [38] A. Elgart, H. Krüger, M. Tautenhahn, and I. Veselić. Discrete Schrödinger operators with random alloy-type potential. In *Spectral analysis of quantum Hamiltonians*, volume 224 of *Oper. Theory Adv. Appl.*, pages 107–131. Birkhäuser/Springer Basel AG, Basel, 2012.
- [39] A. Elgart, M. Shamis, and S. Sodin. Localisation for non-monotone Schrödinger operators. *preprint arXiv: 1201.2211*, 2012.
- [40] A. Elgart, M. Tautenhahn, and I. Veselić. Anderson localization for a class of models with a sign-indefinite single-site potential via fractional moment method. *Ann. Henri Poincaré*, 12(8):1571–1599, 2011.
- [41] P. Erdős and A. H. Stone. On the sum of two Borel sets. *Proc. Amer. Math. Soc.*, 25:304–306, 1970.
- [42] P. Exner, M. Helm, and P. Stollmann. Localization on a Quantum Graph with Random Potential on the Edges. *Rev. Math. Phys.*, 19:923–939, 2007.
- [43] W. Fischer, H. Leschke, and P. Müller. Spectral Localization by Gaussian Random Potentials in Multi-Dimensional Continuous space. *Journal Statist. Phys.*, 101:935–985, 2000. 10.1023/A:1026425621261.
- [44] L. Friedlander. Extremal properties of eigenvalues for a metric graph. *Ann. Inst. Fourier (Grenoble)*, 55(1):199–211, 2005.
- [45] L. Friedlander. Genericity of simple eigenvalues for a metric graph. *Israel J. Math.*, 146:149–156, 2005.
- [46] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Comm. Math. Phys.*, 88(2):151–184, 1983.
- [47] F. Germinet and A. Klein. Bootstrap multiscale analysis and localization in random media. *Comm. Math. Phys.*, 222(2):415–448, 2001.
- [48] F. Germinet and A. Klein. A comprehensive proof of localization for continuous Anderson models with singular random potentials. *J. Eur. Math. Soc.*, 15(1):53–143, 2013.
- [49] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.
- [50] B. T. Graham and G. R. Grimmett. Influence and sharp-threshold theorems for monotonic measures. *Ann. Probab.*, 34(5):1726–1745, 2006.
- [51] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [52] M. J. Gruber, D. Lenz, and I. Veselić. Uniform existence of the integrated density of states for random Schrödinger operators on metric graphs over \mathbb{Z} . *J. Funct. Anal.*, 253(2):515–533, 2007.

- [53] M. Helm and I. Veselic. Linear Wegner estimate for alloy-type Schrödinger operators on metric graphs. *J. Math. Phys.*, 48:092107, 2007.
- [54] P. Hislop and F. Klopp. The integrated density of states for some random operators with nonsign definite potentials. *J. Funct. Anal.*, 195(1):12–47, 2002.
- [55] P. Hislop and O. Post. Anderson localization for radial tree-like quantum graphs. *Waves Random Complex Media*, 19(2):216–261, 2009.
- [56] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, and I. Veselić. Bounds on the spectral shift function and the density of states. *Comm. Math. Phys.*, 262(2):489–503, 2006.
- [57] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [58] W. Kirsch. Wegner estimates and Anderson localization for alloy-type potentials. *Math. Z.*, 221(3):507–512, 1996.
- [59] W. Kirsch. An invitation to random Schrödinger operators. In *Random Schrödinger operators*, volume 25 of *Panor. Synthèses*, pages 1–119. Soc. Math. France, Paris, 2008. With an appendix by Frédéric Klopp.
- [60] W. Kirsch. A Wegner estimate for multi-particle random Hamiltonians. *Zh. Mat. Fiz. Anal. Geom.*, 4(1):121–127, 203, 2008.
- [61] W. Kirsch and F. Martinelli. On the ergodic properties of the spectrum of general random operators. *J. Reine Angew. Math.*, 334:141–156, 1982.
- [62] W. Kirsch, P. Stollmann, and G. Stolz. Localization for random perturbations of periodic Schrödinger operators. *Random Oper. Stoch. Eq.*, 6(3):241–268, 1998.
- [63] W. Kirsch and I. Veselić. Wegner estimate for sparse and other generalized alloy type potentials. *Proc. Indian Acad. Sci. Math. Sci.*, 112(1):131–146, 2002. Spectral and inverse spectral theory (Goa, 2000).
- [64] Y. Kitagaki. Wegner estimates for some random operators with Anderson-type surface potentials. *Math. Phys. Anal. Geom.*, 13(1):47–67, 2010.
- [65] Y. Kitagaki. Generalized eigenvalue-counting estimates for some random acoustic operators. *Kyoto J. Math.*, 51(2):439–465, 2011.
- [66] A. Klein. Unique continuation principle for spectral projections of Schrödinger operators and optimal wegner estimates for non-ergodic random Schrödinger operators. *Commun. Math. Phys.*, 323(2):1229–1246, 2013.
- [67] A. Klein, A. Koines, and M. Seifert. Generalized eigenfunctions for waves in inhomogeneous media. *J. Funct. Anal.*, 190(1):255–291, 2002. Special issue dedicated to the memory of I. E. Segal.
- [68] A. Klein and S.T. Nguyen. The Bootstrap Multiscale Analysis for the Multi-particle Anderson model. *J. Stat. Phys.*, 151(5):938–973, 2013.
- [69] F. Klopp. Localization for some continuous random Schrödinger operators. *Comm. Math. Phys.*, 167(3):553–569, 1995.
- [70] F. Klopp and S. Nakamura. Spectral extrema and Lifshitz tails for non-monotonous alloy type models. *Comm. Math. Phys.*, 287(3):1133–1143, 2009.
- [71] F. Klopp and K. Pankrashkin. Localization on Quantum Graphs with Random Vertex Couplings. *J. Stat. Phys.*, 131:651–673, 2008. 10.1007/s10955-008-9517-z.
- [72] F. Klopp and K. Pankrashkin. Localization on Quantum Graphs with Random Edge Lengths. *Letters in Mathematical Physics*, 87:99–114, 2009. 10.1007/s11005-009-0293-8.

- [73] F. Klopp and H. Zenk. The integrated density of states for an interacting multi-particle homogeneous model and applications to the Anderson model. *Adv. Math. Phys.*, pages Art. ID 679827, 15, 2009.
- [74] V. Kostykin and R. Schrader. Laplacians on metric graphs: eigenvalues, resolvents and semigroups. In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 201–225. Amer. Math. Soc., Providence, RI, 2006.
- [75] H. Krüger. Localization for random operators with non-monotone potentials with exponentially decaying correlations. *Ann. Henri Poincaré*, 13(3):543–598, 2012.
- [76] P. Kuchment. Quantum graphs I. Some basic structures. *Waves Random Media*, 14(1):S107–S128, 2004.
- [77] H. Kunz and B. Souillard. Sur le spectre des opérateurs aux différences finies aléatoires. *Comm. Math. Phys.*, 78(2):201–246, 1980.
- [78] D. Lenz, N. Peyerimhoff, O. Post, and I. Veselić. Continuity of the integrated density of states on random length metric graphs. *Math. Phys. Anal. Geom.*, 12(3):219–254, 2009.
- [79] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
- [80] J. Lindenstrauss and A. Pełczyński. Absolutely summing operators in L_p -spaces and their applications. *Studia Math.*, 29:275–326, 1968.
- [81] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
- [82] V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011.
- [83] J. Nečas. *Direct Methods in the Theory of Elliptic Equations*. Springer, 2012.
- [84] S. Nicaise. Spectre des réseaux topologiques finis. *Bull. Sci. Math. (2)*, 111(4):401–413, 1987.
- [85] S. Nicaise. *Polygonal interface problems*, volume 39 of *Methoden und Verfahren der Mathematischen Physik [Methods and Procedures in Mathematical Physics]*. Verlag Peter D. Lang, Frankfurt am Main, 1993.
- [86] L. Nirenberg. An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa (3)*, 20:733–737, 1966.
- [87] E.M. Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.
- [88] N. Peyerimhoff, M. Tautenhahn, and I. Veselić. Wegner estimates for alloy-type models with sign-changing exponentially decaying single-site potentials. *TU Chemnitz Preprint*, 2011.
- [89] T. Poerschke, G. Stolz, and J. Weidmann. Expansions in generalized eigenfunctions of selfadjoint operators. *Math. Z.*, 202(3):397–408, 1989.
- [90] Yu. V. Pokornyi and A. V. Borovskikh. Differential equations on networks (geometric graphs). *J. Math. Sci. (N. Y.)*, 119(6):691–718, 2004. Differential equations on networks.

- [91] O. Post. Spectral analysis of metric graphs and related spaces. In *Limits of graphs in group theory and computer science*, pages 109–140. EPFL Press, Lausanne, 2009.
- [92] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
- [93] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980.
- [94] C. Rojas-Molina. The Anderson model with missing sites. *preprint arXiv: 1302.3640*, 2013.
- [95] C. Rojas-Molina and I. Veselić. Scale-free unique continuation estimates and applications to random Schrödinger operators. *Comm. Math. Phys.*, 320(1):245–274, 2013.
- [96] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [97] M. Sabri. Anderson localization for a multi-particle quantum graph. *Rev. Math. Phys.*, 26:1350020, 2014.
- [98] M. Sabri. Some abstract Wegner estimates with applications. *Lett. Math. Phys.*, 104(3):311–339, 2014.
- [99] C. Schubert. Localization for quantum graphs with a random potential. *preprint arXiv:1208.6278*, 2012.
- [100] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)*, 7(3):447–526, 1982.
- [101] B. Simon. Lifschitz tails for the Anderson model. *J. Statist. Phys.*, 38(1-2):65–76, 1985.
- [102] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
- [103] P. Stollmann. Wegner estimates and localization for continuum Anderson models with some singular distributions. *Arch. Math. (Basel)*, 75(4):307–311, 2000.
- [104] P. Stollmann. *Caught by disorder: Bound states in random media*, volume 20 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2001.
- [105] P. Stollmann. From uncertainty principles to Wegner estimates. *Math. Phys. Anal. Geom.*, 13(2):145–157, 2010.
- [106] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy. *Harmonic analysis of operators on Hilbert space*. Universitext. Springer, New York, second edition, 2010.
- [107] G. Teschl. *Mathematical methods in quantum mechanics*, volume 99 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009. With applications to Schrödinger operators.
- [108] I. Veselić. Wegner estimate and the density of states of some indefinite alloy-type Schrödinger operators. *Lett. Math. Phys.*, 59(3):199–214, 2002.
- [109] I. Veselić. *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, volume 1917 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.
- [110] I. Veselić. Wegner estimate for discrete alloy-type models. *Ann. Henri Poincaré*, 11(5):991–1005, 2010.

-
- [111] I. Veselić. Wegner estimates for sign-changing single site potentials. *Math. Phys. Anal. Geom.*, 13(4):299–313, 2010.
 - [112] H. von Dreifus and A. Klein. A new proof of localization in the Anderson tight binding model. *Comm. Math. Phys.*, 124(2):285–299, 1989.
 - [113] F. Wegner. Bounds on the density of states in disordered systems. *Z. Phys. B*, 44(1-2):9–15, 1981.
 - [114] J. Weidmann. *Linear operators in Hilbert spaces*, volume 68 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980. Translated from the German by Joseph Szücs.