Random times, enlargement of filtration and arbitrages

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Contents

Introduction

Dans cette thèse, nous étudions quelques questions liées à la théorie de grossissement de filtration dans le but de les appliquer à des modèles financiers. Nous nous concentrons sur deux approches classiques pour grossir la filtration de référence: le grossissement initial et le grossissement progressif. Dans les deux cas, une nouvelle source d'information est ajoutée de manière correspondante: initialement ou progressivement. Plus précisement, si ξ est une variable aléatoire, la filtration grossie initialement, $\mathbb{F}^{\sigma(\xi)}$, est la plus petite filtration $\mathbb{G}$ continue à droite contenant $\mathbb{F}$ et telle que ξ est $\mathcal{G}_0$-mesurable. Si τ est un temps aléatoire (c'est à dire une variable aléatoire positive) alors la filtration grossie progressivement, $\mathbb{F}^{\tau}$, est la plus petite filtration $\mathbb{G}$ continue à droite contenant $\mathbb{F}$ et faisant de τ un $\mathbb{G}$-temps d'arrêt. En d'autres termes

$$\boxed{\mathcal{F}_t^{\sigma(\xi)} = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\xi) \quad \text{et} \quad \mathcal{F}_t^{\tau} = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s).}$$

La théorie de grossissement de filtration a été développée dans les années 70 et 80 dans les travaux de Itô, Barlow, Jacod, Jeulin et Yor, portant esentiellement sur le comportement de $\mathbb{F}$-(semi)martingales dans la filtration grossie. Trois ouvrages ont été consacrés à ce sujet: Jeulin [Jeu80], Jeulin et Yor [JY85], Mansuy et Yor [MY06]; on trouve également de nombreux résultats dans le livre de Jacod [Jac79]. Pour un panorama de la théorie, les lecteurs peuvent se référer aux chapitres correspondants dans les livres [DMM92, JYC09, Pro04, Yor92].

Une première question a porté sur la stabilité de la propriété de martingale. Brémaud and Yor [BY78] ont donné, dans le cas où $\mathbb{G}$ est une filtration (continue à droite) quelconque contenant $\mathbb{F}$, une condition nécessaire et suffisante (connue sous le nom d' hypothèse $(\mathcal{H})$) pour que toutes les $\mathbb{F}$-martingales soient des $\mathbb{G}$-martingales. Le cas du grossissement initial a été étudié en particulier dans Itô [Itô76], Jacod [Jac85] et Yor [Yor85b], puis dans les thèses de Amendinger [Ame99], Ankirchner [Ank05], Song [Son87] et dans les articles associés de Amendinger, Imkeller et Schweizer [AIS98], Ankirchner, Dereich et Imkeller [ADI06, ADI07], Ankirchner et Imkeller [AI07]. Le cas du grossissement progressif et les propriétés de temps aléatoires ont été examinés dans Azéma [Azé72], Jeulin et Yor [JY78], Williams [Wil02], Barlow [Bar78], Nikeghbali et Yor [NY06], Nikeghbali [Nik07], Jeanblanc et Song [JS11a, JS11b] et sont présentés dans le *survey paper* de Nikeghbali [Nik06] où l'auteur se restreint au cas où toutes les $\mathbb{F}$-martingales sont continues (condition (C)) et où τ évite les $\mathbb{F}$-temps d'arrêt (condition (A)). Plus récemment, des résultats importants pour un type de grossissement progressif avec une filtration, ont été obtenus par Kchia et Protter [KP14], Kchia, Larsson et Protter [KLP14] et dans la thèse de Kchia [Kch12].

La théorie de grossissement de filtration s'est avérée être un cadre adéquat pour modéliser plusieurs phénomènes financiers, puisque l'information supplémentaire conduit naturellement à ajouter de l'infomation à la filtration de référence. Ainsi, un certain type d'asymétrie d'information existant sur le marché financier peut être capturé en regardant diverses filtrations.

Des études de délit d'initié basées sur la théorie de grossissement de filtration ont été entreprises par Karatzas et Pikovsky [PK96] et Grorud et Pontier [GP98, GP01]. Dans le premier article, les auteurs s'intéressent au profit qui peut être réalisé par un investisseur; dans le second article, les auteurs généralisent la première étude et montrent comment détecter les délits d'initiés. Dans ces deux articles, la filtration de référence est une filtration continue. Dans certains cas, travailler avec des informations supplémentaires donne les opportunités d'arbitrage; autrement dit, l'information supplémentaire permet un plus grand profit. Le même problème est étudié dans un modèle présentant des sauts par Elliott et Jeanblanc [EJ98] et dans Hillairet et Jiao [HJ13] dans un cadre de risque de crédit. Des applications à la modélisation des événements de défaut ont commencé avec les articles de Kusuoka [Kus99] et Elliott, Jeanblanc et Yor [EJY00]. Dans le premier article, l'auteur étudie le rôle de l'hypothèse $(\mathcal{H})$; dans le deuxième article, les auteurs présentent un modèle où cette hypothèse n'est pas verifiée, et développent une méthode d'évaluation de produits soumis au défaut.

Cependant, dans la construction d'un modèle financier, on doit veiller à ce que ce modèle n'admette pas d'opportunités d'arbitrage. En particulier, il faut préciser quel type d'arbitrages, le cas échéant, l'information supplémentaire peut apporter. Des opportunités d'arbitrage obtenues au moyen d'information supplémentaire ont été étudiés dans Coculescu, Jeanblanc et Nikeghbali [CJN12], Imkeller [Imk02], Zwierz [Zwi07], Fontana, Jeanblanc et Song [FJS12]. Dans les trois derniers articles, les auteurs travaillent avec un grossissement progressif spécifique, plus précisément, un grossissement obtenu en travaillant avec des temps honnêtes, et ils limitent leur attention au cas où la filtration de référence $\mathbb{F}$ est continue et où le temps aléatoire τ évite les $\mathbb{F}$-temps d'arrêt. Dans Imkeller et Zwierz, les auteurs prêtent attention au *drift* d'information et à ses propriétés d'intégrabilité; les preuves dans Fontana et al. sont basées sur une décomposition multiplicative de la supermartingale d' Azéma, établie dans Nikeghbali et Yor [NY06] sous les conditions (CA). Après une première version des résultats de cette thèse, Acciao et al. [AFK14] ont produit une étude intéressante sur les problèmes d'arbitrage jusqu'à un temps aléatoire, en utilisant une méthode différente que celle présentée dans cette thèse, et ont obtenu la même caractérisation de non arbitrage pour tous les prix et une condition suffisante pour un prix donné (cette condition étant un cas particulier de la notre).

Cette thèse est divisée en deux parties: la première partie traite des problèmes généraux de la théorie de grossissement de filtration, tandis que la deuxième partie est consacrée à l'absence d'arbitrage en présence d'information supplémentaire, pour différents notions d'arbitrage. Dans les deux parties nous nous intéressons particulièrement aux cas discontinus, c'est-à-dire quand il existe une martingale discontinue dans la filtration de référence et au cas de temps aléatoires qui n'évitent pas les $\mathbb{F}$-temps d'arrêt.

Dans la première partie de cette thèse, nous étudions les propriétés de différentes classes de temps aléatoires. Nous commençons avec la notion classique de temps honnête et pseudo-temps d'arrêt qui généralisent la notion de temps d'arrêt. Par ailleurs, nous considérons les deux classes de temps aléatoires qui sont les plus connues dans les applications au risque de crédit, c'est-à-dire les temps de Cox et les temps initiaux. Les questions posées sur les temps aléatoires sont liées à leurs propriétés du point de vue du grossissement de filtration. La $\mathbb{F}$-projection duale optionnelle associée à un temps aléatoire représente un important information, est disponible dans la filtration de référence, sur ce temps aléatoire et s'avère être une notion trés utile pour traiter divers problèmes.

La deuxième partie de cette thèse est consacrée à l'étude de conditions de non-arbitrage dans les filtrations grossies progressivement et initialement. Nous commençons par fournir des exemples dans les cas progressif dans le cadre de filtration de référence engendrée par un mouvement brownien ou un processus de Poisson. Le résultat principal consiste en une analyse complète qui nous conduit à une condition suffisante et nécessaire pour non-arbitrage avant le temps alátoire apportant de l'information. Finalement, nous regardons l'absence d'arbitrage dans le cas de filtration grossie initialement.

Introduction

In this thesis, we study some questions settled in the enlargement of filtration theory with finance in view. We focus on the two classical ways to enlarge a reference filtration $\mathbb{F}$, namely the initial and progressive enlargements. In each of those cases, a new stream of information is added in the corresponding manner: initially or progressively. More precisely, letting ξ be a random variable, the initially enlarged filtration $\mathbb{F}^{\sigma(\xi)}$ is the smallest right-continuous filtration $\mathbb{G}$ containing $\mathbb{F}$ such that ξ is $\mathcal{G}_0$-measurable. Letting τ be a random time (i.e., a positive random variable), the progressively enlarged filtration $\mathbb{F}^{\tau}$ is the smallest right-continuous filtration $\mathbb{G}$ containing $\mathbb{F}$ and making τ a $\mathbb{G}$-stopping time. In other words,

$$\boxed{\mathcal{F}_t^{\sigma(\xi)} = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\xi) \quad \text{and} \quad \mathcal{F}_t^{\tau} = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s).}$$

The theory of enlargement of filtration was developed during the 70's and 80's, starting with the works of Itô, Barlow, Jacod, Jeulin and Yor. Main questions refer to the behaviour of $\mathbb{F}$-(semi)martingales in enlarged filtration. Three lecture notes volumes have been devoted to this subject; Jeulin [Jeu80], Jeulin and Yor [JY85], Mansuy and Yor [MY06], and many results are given in Jacod [Jac79]. For an overview of the theory, the reader can also refer to related chapters in the books [DMM92, JYC09, Pro04, Yor92].

The stability of the martingale property, called $(\mathcal{H})$ hypothesis (that is when all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales, for a general enlargement $\mathbb{G}$) is studied in details in Brémaud and Yor [BY78]. The initial enlargement case was studied in particular in Itô [Itô76], Jacod [Jac85] and Yor [Yor85b], then later in the theses of Amendinger [Ame99], Ankirchner [Ank05], Song [Son87] and in the related papers of Amendinger, Imkeller and Schweizer [AIS98], Ankirchner, Dereich and Imkeller [ADI06, ADI07], Ankirchner and Imkeller [AI07]. The case of progressive enlargement and properties of random times were investigated in Azéma [Azé72], Jeulin and Yor [JY78], Williams [Wil02], Barlow [Bar78], Nikeghbali and Yor [NY06], Nikeghbali [Nik07], Jeanblanc and Song [JS11a, JS11b] and presented in the survey paper Nikeghbali [Nik06] where the author restricts his attention to the case where all $\mathbb{F}$-martingales are continuous (condition (C)) and when τ avoids $\mathbb{F}$-stopping times (condition (A)). More recently, important results for another kind of enlargement, namely progressive enlargement with filtration, are obtained in Kchia and Protter [KP14], Kchia, Larsson and Protter [KLP14] and in Kchia thesis [Kch12].

The theory of enlargement of filtration turned out to be an adequate setting to model several financial phenomena, since additional information flow can be seen as an enlarged filtration. Thereby, the asymmetric information existing on the financial market can be captured by considering more than one filtration. Insider trading studies relying on enlargement of filtration theory started with Karatzas and Pikovsky [PK96], Grorud and Pontier [GP98, GP01]. In the first paper, the authors are interested in the profit that can be done by an investor, in the last paper, the authors generalize the study and answer the question to detect insider trading. In some cases, working with additional information leads to arbitrage opportunities; otherwise, the additional information leads to a larger profit. The same problem is studied in a model with jumps in Elliott and Jeanblanc [EJ98] where the presence of jumps leads to different conclusions and in Hillairet and Jiao [HJ13] in a credit risk setting. Applications to credit events modelling began with the papers of Kusuoka [Kus99] and Elliott, Jeanblanc and Yor [EJY00]. In the first paper, the author investigates the role of the $(\mathcal{H})$ hypothesis; in the second paper, the authors present some model where this hypothesis does not hold, developing a pricing methodology.

While building a financial model, an important question to ask is if it is arbitrage-free set-up. In particular, one has to make precise what kind of arbitrages, if any, the additional information can produce. Arbitrage opportunities arising from additional flow of information were investigated in Coculescu, Jeanblanc and Nikeghbali [CJN12], Imkeller [Imk02], Zwierz [Zwi07], Fontana, Jeanblanc and Song [FJS12]. In the three last papers, the authors are working with a specific progressive enlargement, namely with honest times, and they restrict their attention to the case where the reference filtration $\mathbb{F}$ is continuous and where the random time τ avoids the $\mathbb{F}$-stopping times. In Imkeller and Zwierz, the authors pay attention to the information drift and its integrability properties; the proofs in Fontana et al. paper are based on the multiplicative decomposition of the Azéma supermartingale, established in Nikeghbali and Yor [NY06], valid only under the (CA) conditions. After that a first version of the results of this thesis was posted on arkiv, Acciao et al. [AFK14] produced an interesting study of the arbitrage problems up to a random time, using a different method, with the same results.

This thesis consists of two parts: the first part treats pure enlargement of filtration problems, while the second part focuses on absence of arbitrages in different enlarged markets and for different notions of non-arbitrage. In both parts, we are particularly interested in discontinuous cases, i.e., when there exist discontinuous martingales in the reference filtration $\mathbb{F}$ and in the case of random times which do not avoid $\mathbb{F}$-stopping times.

In the first part of this thesis, we study the properties of several families of random times. We start with the classical notion of honest times and the notion of pseudo-stopping times which both generalize the notion of stopping times. Furthermore, we consider two other classes of random times which are mostly known from their applications in credit risk modelling, namely Cox's times and initial times. Questions on random times are related to their properties from enlargement of filtration point of view. The $\mathbb{F}$-dual optional projection associated with the random time represents an important information, available in the reference filtration, about this random time and proves to be a very useful notion in treating several problems.

The second part of the thesis is devoted to the study of non-arbitrage conditions in progressively and initially enlarged filtrations in a financial framework. We start with providing some examples in progressive setting for Brownian and Poisson filtrations. Then, as a main result, we carry out a complete analysis up to random time, providing necessary and sufficient conditions for no-arbitrage. Finally, we turn our attention to absence of arbitrage in initially enlarged filtration.

There are links between the chapters of this thesis, nevertheless each one can be read independently. Here we would like to emphasis, that only some notations are global, the remaining ones are introduced locally in each chapter. For this reason, each chapter begins with the introduction of an appropriate set-up. Below we summarize the content of each of the seven chapters of this thesis.

Chapter 1: General theory.

This chapter collects some known results which will be useful in further developments of the thesis. If the result can be found in the literature, we provide a reference. If not, i.e., if we did not find any reference, we provide a proof.

Chapter 2: Thin random times.

This chapter is based on joint work with Tahir Choulli and Monique Jeanblanc [ACJ14b]. We classify random times into thin and strict random times. Taking as a starting point the assumption on avoidance of all stopping times from the reference filtration, we define a class of thin random times. Then, we define a decomposition of a random time into thin and strict parts in analogous way to the stopping time decomposition into accessible and totally inaccessible parts. The notion of dual optional projection plays a crucial role. Furthermore, we develop properties of thin random times, namely relationship of thin honest times with a jumping filtration, and entropy of a thin random time. The importance of thin honest times is remarkable for non-arbitrage consideration presented in Section 7.4. In fact, the arbitrage problem was our motivation for the analysis of thin random times.

Chapter 3: Pseudo-stopping times and enlargement of filtration.

This chapter is based on joint work with Libo Li [AL14]. We study the properties of pseudo-stopping times in the context of enlargement of filtration theory. Based on the example given by Williams, the concept of a pseudo-stopping time was formally introduced by Nikeghbali and Yor [NY05]. As its name suggests, the class of pseudo-stopping times is a class of non-stopping times, which enjoy stopping time like properties. We examine the relationship between the hypothesis $(\mathcal{H})$ and pseudo-stopping times and we provide alternative characterization of the hypothesis $(\mathcal{H})$. We as well discuss several classification results for pseudo-stopping times, honest times and stopping times. Moreover we extend the construction of Nikeghbali and Yor of pseudo-stopping times by relaxing continuity assumptions. For that construction, we study the hypothesis $(\mathcal{H}')$ in the progressive enlargement of filtration setting, where we find useful viewing the problem as an excursion straddling on a random time. We finish this chapter with looking at pseudo-stopping times recovered from Jeanblanc-Song model [JS11b].

Chapter 4: On some classes of random times.

This chapter consists of three independent sections. In Section 4.2, which is based on a joint work with Monique Jeanblanc and Shiqi Song [AJS14], we consider a question how the pseudo-stopping time property is affected by equivalent change of measure. Contrary to stopping time, the definition of pseudo-stopping time does depend on probability measure and the pseudo-stopping time property is not in general stable under equivalent change of measure. In Section 4.3, we firstly focus on some basic properties of honest times, then we study an example of last passage time which is not honest. In Section 4.4, we look at the classical Cox's construction with discontinuous hazard processes.

Chapter 5: Arbitrages in a progressive enlargement setting.

This chapter is based on a joint paper with Tahir Choulli, Jun Deng and Monique Jeanblanc [ACDJ14a]. It completes the analysis of Chapter 6 and contains two principal contributions. The first contribution consists of providing and analysing many practical examples of market models, while the second contribution lies in providing simple proofs for the stability of the No Unbounded Profit with Bounded Risk (called NUPBR hereafter). In this paper, we treat the question whether the non-arbitrage conditions are stable with respect to progressive enlargement of filtration. We focus on two components of the No Free Lunch with Vanishing Risk concept, namely on No Arbitrage Opportunity and No Unbounded Profit with Bounded Risk. The problem is divided into stability before and after the random time containing extra information.
The question regarding the No Arbitrage Opportunity condition is answered in the case of Brownian filtration and Poisson filtration for the special case of an honest time. Particular examples of non-honest times are described, and explicit arbitrage strategies are given. Both Brownian and Poisson filtrations possess the important Predictable Representation Property, which implies that the financial market is complete, and will be crucial to obtain the arbitrages strategies. One may further investigate similar problem without assuming market completeness. One considers also some example/classes of non-honest random times. Afterwards, we deal with stability of NUPBR concept in very particular situations, namely when the reference filtration is a continuous filtration, or the filtration generated by a standard Poisson process or the filtration generated by a Lévy process. We provide results with simple proofs in those particular situations, giving the deflator in a closed form. Combining results on the NA and the NUPBR conditions we conclude that some local martingales in the enlarged filtration are in fact strict local martingales, which provides a way to construct strict local martingales in enlarged Brownian and Poisson filtrations.

Chapter 6: Non-Arbitrage up to Random Horizon and after Honest Times for Semimartingale Models.

This chapter is based on joint paper with Tahir Choulli, Jun Deng and Monique Jeanblanc [ACDJ14b]. It addresses the question of how an arbitrage-free semimartingale model is affected when stopped at a random horizon. We focus on No-Unbounded-Profit-with-Bounded-Risk concept, which is also known in the literature as the first kind of non-arbitrage. For this non-arbitrage notion, we obtain two principal results. The first result lies in describing the pairs of market model and random time for which the resulting stopped model fulfills NUPBR condition. The second main result characterises the random time models that preserve the NUPBR property after stopping for any market model. These results are elaborated in a general market model, and we pay attention to some particular and practical models. The analysis that drives these results is based on new stochastic developments in semimartingale theory with progressive enlargement. Furthermore, we construct explicit martingale densities (deflators) for some classes of local martingales when stopped at a random time.

Chapter 7: Optional semimartingale decomposition and NUPBR condition.

This chapter is based on joint paper with Tahir Choulli and Monique Jeanblanc [ACJ14a]. Our study addresses the same question as in Chapter 6 in two different settings, i.e., how an arbitrage-free semimartingale model is affected when stopped at a random horizon or when a random variable satisfying Jacod's hypothesis is incorporated. We recover some results from the previous chapter using a different approach. In the general semimartingale setting, we provide a necessary and sufficient condition on the random time for which the non-arbitrage is preserved for any process. Analogous result is formulated for initial enlargement with random variable satisfying Jacod's hypothesis. The crucial intermediate results in enlargement of filtration theory are obtained. For local martingales from the reference filtration we provide special optional semimartingale decomposition up to random time and in initially enlarged filtration under Jacod's hypothesis. We observe an interesting link with absolutely continuous change of measure problem.

Part I

Random times and enlargement of filtration

Chapter 1

General theory

In this chapter, we recall some well known facts from the general theory of stochastic processes with an emphasis on enlargement of filtration theory in Section 1.2 and properties of random times in Section 1.3. In Section 1.4 we summarize results needed in Part II on arbitrage problem of this thesis. We give the proofs of new results.

For further details and proofs we address the reader to Rogers and Williams [RW00b, RW00a], Chesney, Jeanblanc and Yor [JYC09], Dellacherie and Meyer [DM75, DM80], Dellacherie, Meyer and Maisonneuve [DMM92], He, Wang and Yan [HWY92], Revuz and Yor [RY99] and Protter [Pro04].

Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a **filtered probability space** where $\mathbb{F}$ is a filtration satisfying usual conditions of completeness and right continuity. Recall that **the optional σ-field** on $\Omega \times \mathbb{R}_+$, denoted by $\mathcal{O}$, is the σ-field generated by all càdlàg $\mathbb{F}$-adapted processes and **the predictable σ-field** on $\Omega \times \mathbb{R}_+$, denoted by $\mathcal{P}$, is the σ-field generated by all left-continuous $\mathbb{F}$-adapted processes. A stochastic set or process is called optional (respectively predictable) if it is $\mathcal{O}$-measurable (respectively $\mathcal{P}$-measurable).

For an $\mathbb{F}$-semimartingale Y, the set of $\mathbb{F}$-predictable processes integrable with respect to Y is denoted by $L(Y, \mathbb{F})$ and for $H \in L(Y, \mathbb{F})$, we denote by $H \centerdot Y$ **the stochastic integral** $\int_0^\cdot H_s dY_s$.

As usual, for a process X and a random time ϑ, we denote by X^ϑ **the stopped process**.

The set of martingales for the filtration $\mathbb{F}$ under $\mathbb{P}$ is denoted by $\mathcal{M}(\mathbb{F}, \mathbb{P})$. As usual, $\mathcal{A}^+(\mathbb{F})$ denotes **the set of increasing, right-continuous, $\mathbb{F}$-adapted and integrable processes**. If $\mathcal{C}(\mathbb{F})$ is a class of $\mathbb{F}$-adapted processes, we denote by $\mathcal{C}_0(\mathbb{F})$ the set of processes $X \in \mathcal{C}(\mathbb{F})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{F}$-stopping times that increases to ∞ and such that the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{F})$. We put $\mathcal{C}_{0,loc}(\mathbb{F}) = \mathcal{C}_0(\mathbb{F}) \cap \mathcal{C}_{loc}(\mathbb{F})$.

For a given semimartingale X, $\mathcal{E}(X)$ stands for **the stochastic exponential** of X.

The continuous local martingale part and **the jump process** of a semimartingale X are denoted respectively by X^c and ΔX.

1.1 Stochastic processes

1.1.1 Monotone class theorem

The Monotone Class Theorem allows us to work with a generating family instead of the whole family of functions or sets (see [HWY92, Theorems 1.2, 1.4], [JYC09, p.4], [DM75, Chapter 1]) and is an important tool in proving various results.

Theorem 1.1. *Let $\mathcal{C}$ be a class of sets on $\Omega \times \mathbb{R}_+$ such that $A \cap B \in \mathcal{C}$ if $A, B \in \mathcal{C}$, and $\mathcal{H}$ be a vector space of stochastic processes on $\Omega \times \mathbb{R}_+$. If the following conditions are satisfied:*
(a) $1 \in \mathcal{H}$
(b) if $X_n \in \mathcal{H}$, $X_n \uparrow X$, X is finite (respectively bounded) then $X \in \mathcal{H}$
(c) if $A \in \mathcal{C}$ then $\mathbb{1}_A \in \mathcal{H}$
then $\mathcal{H}$ contains all $\sigma(\mathcal{C})$-measurable real (respectively bounded) processes on $\Omega \times \mathbb{R}_+$.

1.1.2 Random times and stopping times

Let T be a random time, i.e., an $\mathbb{R}_+ \cup \{\infty\}$ valued random variable. Let us recall the notions of graph of a random time

$$[\![T]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : T(\omega) = t\}$$

and restriction of a random time to a given set

$$T_A(\omega) := \begin{cases} T(\omega) & \omega \in A \\ \infty & \omega \notin A \end{cases}. \tag{1.1}$$

For two random times T and S such that $S \leq T$, the following stochastic intervals are defined:

$$\begin{aligned}
[\![S, T]\!] &= \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) \leq t \leq T(\omega)\}, \\
[\![S, T[\![&= \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) \leq t < T(\omega)\}, \\
]\!]S, T]\!] &= \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) < t \leq T(\omega)\}, \\
]\!]S, T[\![&= \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) < t < T(\omega)\}.
\end{aligned}$$

Definition 1.2. For a random time T we define the σ-fields $\mathcal{F}_{T-}$ and $\mathcal{F}_T$ as

$$\begin{aligned}
\mathcal{F}_{T-} &:= \sigma\{H_T : H \text{ is an } \mathbb{F}\text{-predictable process on } [0, \infty]\}, \\
\mathcal{F}_T &:= \sigma\{H_T : H \text{ is an } \mathbb{F}\text{-optional process on } [0, \infty]\}.
\end{aligned}$$

Here, in each case, H_∞ can be an arbitrary $\mathcal{F}_\infty$-measurable random variable.

Proposition 1.3. *Let T be a random time.*
(a) For any $\mathcal{F}_{T-}$-measurable random variable κ, there exists an $\mathbb{F}$-predictable process k such that $k_T = \kappa$.
(b) For any $\mathcal{F}_T$-measurable random variable κ, there exists an $\mathbb{F}$-optional process k such that $k_T = \kappa$.

Proof. (a) By definition, $\mathcal{F}_{T-} = \sigma\{H_T : H \text{ is an } \mathbb{F}\text{-predictable process}\}$. We show that $\mathcal{H} = \{H_T : H \text{ is an } \mathbb{F}\text{-predictable process}\}$ is already a σ-field. Set A belongs to $\mathcal{H}$ if there exists an $\mathbb{F}$-predictable process H and a set $B \in \mathcal{B}(\mathbb{R})$ such that $A = H_T^{-1}(B)$. Then, $A^c \in \mathcal{H}$ as $A^c = H_T^{-1}(B^c)$. Let $A = \bigcup_{n=1}^{\infty} A_n$ with $A_n = (H_T^n)^{-1}(B_n)$ where H^n is a predictable process and $B_n \in \mathcal{B}(\mathbb{R})$. Then, $A_n = \{\omega : (\omega, T(\omega)) \in \Gamma_n\}$, where $\Gamma_n = \{(\omega, t) : H_t^n(\omega) \in B_n\}$. And it is easy to see that $A = H_T^{-1}(B)$, for $H_t(\omega) = \mathbb{1}_{\Gamma}(\omega, t)$ with $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ predictable set and $B =]0, 1]$. So indeed we have $\mathcal{F}_{T-} = \mathcal{H}$. This implies that for $\mathcal{F}_{T-}$-measurable κ there exists a predictable process k with $k_T = \kappa$. If moreover κ is bounded then k can be chosen with the same bounds.
(b) follows by the same argumentation. ■

We restrict our attention to the class of stopping times, providing the result on stopping time decomposition.

Definition 1.4. A random time T defined on $(\Omega, \mathcal{G})$ is called an $\mathbb{F}$-stopping time (or simply stopping time) if for each $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

Lemma 1.5. *For a stopping time T we have the following alternative characterization of σ-fields $\mathcal{F}_T$ and $\mathcal{F}_{T-}$*

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : \forall t,\ A \cap \{T \leq t\} \in \mathcal{F}_t\},$$
$$\mathcal{F}_{T-} = \mathcal{F}_0 \vee \sigma(\{T > t\} \cap A : A \in \mathcal{F}_t, t \geq 0).$$

In order to classify stopping times we introduce the following definition; see [JYC09, Definition 1.2.3.1], [HWY92, Definition 3.34].

Definition 1.6. (a) A stopping time T is predictable if there exists an increasing sequence $(T_n)_{n \geq 1}$ of stopping times such that a.s. $\lim_{n \to \infty} T_n = T$ and $T_n < T$ for every n on the set $\{T > 0\}$. The sequence (T_n) is then called foretelling sequence.
(b) A stopping time T is accessible if there exists a sequence (T_n) of predictable stopping times such that $[\![T]\!] \subset \bigcup_n [\![T_n]\!]$.
(c) A stopping time T is totally inaccessible if $\mathbb{P}(T = S < \infty) = 0$ for any predictable stopping time S.

Each stopping time can be decomposed into accessible and totally inaccessible parts as stated in [HWY92, Theorem 4.20].

Theorem 1.7. *For each stopping time T there exists $A \in \mathcal{F}_{T-}$ such that $A \subset \{T < \infty\}$, $T^a = T_A$ is accessible and $T^i = T_{A^c}$ is totally inaccessible. Such set A is a.s. unique.*
In particular, any stopping time T can be decomposed into accessible and totally inaccessible parts as $T = T^a \wedge T^i$ and $[\![T]\!] = [\![T^a]\!] \cup [\![T^i]\!]$.

1.1.3 Thin sets and processes

Let us recall the definition of a thin set, see [HWY92, 3.13], which generalizes the notion of graph of a stopping time.

Definition 1.8. A stochastic set Γ is called a thin set, if there exists a sequence of stopping times $(T_n)_n$ such that $\Gamma = \bigcup_n [\![T_n]\!]$.
(a) A thin set Γ is said to be a predictable thin set if each T_n is a predictable stopping time.
(b) A thin set Γ is said to be an accessible thin set if each T_n is an accessible stopping time.
(c) A thin set Γ is said to be a totally inaccessible thin set if each T_n is a totally inaccessible stopping time.

We define a thin process as follows (it corresponds to *summation process* from [HWY92, 7.39]).

Definition 1.9. A process X is thin if there exist a sequence of random variables ξ_n and a sequence of stopping times (T_n) with disjoint graphs such that $X_t = \sum_{n=1}^{\infty} \xi_n \mathbb{1}_{[\![T_n,\infty[\![}$. Its paths vary on a thin set only: $\mathbb{1}_{\cup_{n=1}^{\infty}[\![T_n]\!]} \centerdot X = \sum_{n=1}^{\infty} \mathbb{1}_{[\![T_n]\!]} \centerdot X = \sum_{n=1}^{\infty} \mathbb{1}_{[\![T_n,\infty[\![} \Delta X_{T_n}$.

1.1.4 Section theorem

Section theorem allows us to look at stochastic processes only at stopping times, see [HWY92, Theorems 4.7, 4.8], see [DM80, Theorems 84 and 85].

Theorem 1.10. *Let A be an optional (respectively predictable) set. For any given $\varepsilon > 0$ there exists a (respectively predictable) stopping time T such that*
(a) $[\![T]\!] \subset A$,
(b) $\mathbb{P}(T < \infty) \geq \mathbb{P}(\pi(A)) - \varepsilon$,
where $\pi(A)$ is the projection of A onto Ω, i.e., $\pi(A) = \{\omega : \exists\, t < \infty \quad (\omega, t) \in A\}$.

1.1.5 Projections and dual projections

Projections and dual projections onto the reference filtration $\mathbb{F}$ play an important role in the theory of enlargement of filtrations. First we recall the definition of optional and predictable projections, see [HWY92, Theorems 5.1 and 5.2] , [JYC09, p.264-265].

Definition 1.11. Let X be a measurable bounded (or positive) process. The optional projection of X is the unique optional process ${}^oX$ such that for every stopping time T we have

$$\mathbb{E}(X_T \mathbb{1}_{\{T<\infty\}} | \mathcal{F}_T) = {}^oX_T \mathbb{1}_{\{T<\infty\}} \quad \text{a.s..}$$

The predictable projection of X is the unique predictable process ${}^pX$ such that for every predictable stopping time T we have

$$\mathbb{E}(X_T \mathbb{1}_{\{T<\infty\}} | \mathcal{F}_{T-}) = {}^pX_T \mathbb{1}_{\{T<\infty\}} \quad \text{a.s..}$$

Projections depend on the filtration. In case of various filtration we shall denote ${}^{o,\mathbb{F}}X$ the $\mathbb{F}$-optional projection of X and ${}^{p,\mathbb{F}}X$ the $\mathbb{F}$-predictable projection of X.

For definition of dual optional projection and dual predictable projection see [JYC09, p.265], [Pro04, Chapter 3 Section 5], [DM80, Chapter 6 Paragraph 73 p.148], [HWY92, Sections 5.18, 5.19].

Definition 1.12. (a) Let A be a càdlàg pre-locally integrable variation process (not necessary adapted). The dual optional projection of A is the unique optional process A^o such that for every optional process H we have

$$\mathbb{E}\left(\int_{[0,\infty[} H_s dA_s\right) = \mathbb{E}\left(\int_{[0,\infty[} H_s dA_s^o\right).$$

(b) Let A be a càdlàg locally integrable variation process (not necessary adapted). The dual predictable projection of A is the unique predictable process A^p such that for every predictable process H we have

$$\mathbb{E}\left(\int_{[0,\infty[} H_s dA_s\right) = \mathbb{E}\left(\int_{[0,\infty[} H_s dA_s^p\right).$$

Dual projections depend on the filtration. In case of various filtration we shall denote $A^{o,\mathbb{F}}$ the $\mathbb{F}$-dual optional projection of A and $A^{p,\mathbb{F}}$ the $\mathbb{F}$-predictable projection of A.

The following result links the jump of the dual projection with the projection of the jump. See [DM80, Theorem 76 p.149–150] or [HWY92, Theorem 5.27 p.150].

Lemma 1.13. *Let A be an integrable increasing process. Then*

$$\Delta A^o = {}^o(\Delta A) \quad \textit{and} \quad \Delta A^p = {}^p(\Delta A).$$

Proof. We focus on the predictable case, the optional one follows by the same type of arguments. Let T be a predictable stopping time and $F \in \mathcal{F}_{T-}$. Then from the definition of the dual predictable projection of A applied to the predictable process $H = 1\!\!1_F 1\!\!1_{[\![T]\!]}$ we obtain

$$\mathbb{E}(1\!\!1_F \Delta A_T 1\!\!1_{\{T<\infty\}}) = \mathbb{E}(1\!\!1_F \Delta A_T^p 1\!\!1_{\{T<\infty\}}).$$

Thus for any predictable stopping time T we have

$$\mathbb{E}(\Delta A_T 1\!\!1_{\{T<\infty\}} | \mathcal{F}_{T-}) = \mathbb{E}(\Delta A_T^p 1\!\!1_{\{T<\infty\}} | \mathcal{F}_{T-})$$

and by Section theorem 1.10 we conclude that $\Delta A^p = {}^p(\Delta A)$. ∎

Let us now recall [RW00a, Theorem VI.21.4)].

Lemma 1.14. *Let A be an integrable increasing process. Then, A^p is the unique predictable integrable increasing process such that ${}^oA - A^p$ is a martingale.*

Example 1.15. A useful application of the previous lemma is the relation between square and predictable brackets. Namely, for two semimartingales X and Y, we have that $[X,Y]^p = \langle X,Y\rangle$ and $[X,Y] - \langle X,Y\rangle$ is a martingale.

1.1.6 Quasi-left continuity

Quasi-left continuity will play an important role in various proofs concerning semimartingale models and arbitrage problem, see [HWY92, 4.22].

Definition 1.16. Let X be an adapted càdlàg process. Let (T_n) be a sequence of stopping times exhausting the jumps of X, i.e., $\{\Delta X \neq 0\} = \bigcup_n [\![T_n]\!]$. Then,
(a) We say that X has only accessible jumps if each T_n is an accessible stopping time.
(b) We say that X has only totally inaccessible jumps if each T_n is a totally inaccessible stopping time. Then, X is said to be quasi-left continuous.

Alternative characterization is given in the next theorem, see [HWY92, 4.23].

Theorem 1.17. *Let X be an adapted càdlàg process. Then the following statements are equivalent:*
(a) X is quasi-left continuous;
(b) For every predictable time $T > 0$, $X_T = X_{T-}$ a.s. on $\{T < \infty\}$;
(c) If (T_n) is an increasing sequence of stopping times and $T = \lim_{n\to\infty} T_n$, then $\lim_{n\to\infty} X_{T_n} = X_T$ a.s. on $\{T < \infty\}$.

We recall the notion of quasi-left continuous filtration, see [HWY92, Definition 3.39, Theorem 3.40] [Pro04, p.189].

Definition 1.18. A filtration $\mathbb{F}$ is quasi-left continuous if for any predictable stopping time T, $\mathcal{F}_T = \mathcal{F}_{T-}$.

Theorem 1.19. *(a) A filtration $\mathbb{F}$ is quasi-left continuous if and only if each accessible stopping time is predictable.*
(b) Suppose that $\mathbb{F}$ is quasi-left continuous. Then for any sequence of stopping times (T_n) we have $\mathcal{F}_{\vee_n T_n} = \vee_n \mathcal{F}_{T_n}$.

Theorem 1.20. *If the filtration $\mathbb{F}$ is quasi-left continuous then each $\mathbb{F}$-martingale is a quasi left continuous process.*

Lemma 1.21. *Let X be a quasi-left continuous semimartingale. Then $\langle X \rangle$ is continuous.*

Proof. As $\langle X \rangle$ is a dual predictable projection of $[X] = \langle X^c \rangle + \sum (\Delta X)^2$ it is enough to prove the continuity of dual predictable projection of $(\Delta X)_T^2 \mathbb{1}_{]\!]T,\infty]\!]}$ for any totally inaccessible stopping time T. By Lemma 1.13 and since T is totally inaccessible we have

$$\Delta((\Delta X)_T^2 \mathbb{1}_{]\!]T,\infty]\!]})^p = {}^p((\Delta X)_T^2 \mathbb{1}_{[\![T]\!]}) = 0.$$

∎

1.1.7 Predictable Representation Property

The Predictable Representation Property presented in this section plays an important role in financial considerations, i.e., it corresponds to market completeness. See [HWY92, ch. XIII].

Definition 1.22. Let M be a local martingale with $M_0 = 0$ and

$$\mathcal{N}(M) := \{H \centerdot M : H \text{ is a predictable process, integrable with respect to } M\}.$$

We say that M has the Predictable Representation Property if $\mathcal{N}(M) = \mathcal{M}_{loc,0}$.

Example 1.23. We recall the following examples where the Predictable Representation Property holds.
(a) A Brownian motion W has the PRP in its natural filtration.
(b) The martingale $M_t = N_t - \eta t$ where N is a standard Poisson process with parameter η has the PRP in its natural filtration.

1.1.8 Optional stochastic integral

The notion of optional stochastic integral (or compensated stochastic integral) is defined in [Jac79, Chapter III.4.b p.106-109] and also studied in [DM80, Chapter VIII.2 sections 32-40 p.356-366]. The notion of optional stochastic integral is a crucial tool for developments presented in Chapter 6.

Definition 1.24. Let N be an $\mathbb{F}$-local martingale with continuous martingale part N^c, and let H be an $\mathbb{F}$-optional process.
(a) The process H is said to be integrable with respect to N if ${}^{p,\mathbb{F}}H$ is N^c integrable and the process

$$\left(\sum_{s \leq t} \left(H_s \Delta N_s - {}^{p,\mathbb{F}}(H\Delta N)_s \right)^2 \right)^{1/2}$$

is locally integrable. The set of integrable processes with respect to N is denoted by ${}^{o}L^1_{loc}(N, \mathbb{F})$.
(b) For $H \in {}^{o}L^1_{loc}(N, \mathbb{F})$, the optional stochastic integral of H with respect to N, denoted by $H \odot N$, is the unique local martingale M which satisfies

$$M^c = {}^{p,\mathbb{F}}H \centerdot N^c \quad \text{and} \quad \Delta M = H\Delta N - {}^{p,\mathbb{F}}(H\Delta N).$$

Among the most useful results of the literature involving this integral, there is the following (see [DM80]).

Proposition 1.25. *Let $H \in {}^{o}L^1_{loc}(N, \mathbb{F})$ and let $M = H \odot N$ be the optional stochastic integral of H with respect to N.*

(a) M is the unique $\mathbb{F}$-local martingale such that, for any $\mathbb{F}$-bounded martingale Y, the process $[M,Y] - H \centerdot [N,Y]$ is an $\mathbb{F}$-local martingale.

(b) For any $\mathbb{F}$-local martingale Y we have $[M,Y] \in \mathcal{A}_{loc}(\mathbb{F})$ if and only if $H \centerdot [N,Y] \in \mathcal{A}_{loc}(\mathbb{F})$ and in this case we get

$$\langle M, Y\rangle^{\mathbb{F}} = (H \centerdot [N,Y])^{p,\mathbb{F}}.$$

1.1.9 Random/Jump measures

In this section we recall the notion of jump measure associated to a semimartingale. Let S be an $\mathbb{F}$-semimartingale valued in $\mathbb{R}^d$. We denote

$$\widetilde{\mathcal{O}}(\mathbb{F}) := \mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d), \qquad \widetilde{\mathcal{P}}(\mathbb{F}) := \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. The jump measure of S is denoted by μ, and is given by

$$\mu(dt, dx) := \sum_{u>0} \mathbb{1}_{\{\Delta S_u \neq 0\}} \delta_{(u,\Delta S_u)}(dt, dx). \tag{1.2}$$

For a product-measurable functional $W \geq 0$ on $\Omega \times [0,\infty[\times \mathbb{R}^d$, we denote by $W \star \mu$ (or sometimes, with abuse of notation $W(x) \star \mu$) the process

$$W \star \mu_t := \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} W(u,x)\mu(du,dx) = \sum_{0<u\leq t} W(u, \Delta S_u)\mathbb{1}_{\{\Delta S_u \neq 0\}}. \tag{1.3}$$

Also on $\Omega \times [0,\infty[\times \mathbb{R}^d$, we define the measure $M^{\mathbb{P}}_\mu := \mathbb{P} \otimes \mu$ by $\int W dM^{\mathbb{P}}_\mu := \mathbb{E}\left(W \star \mu_\infty\right)$ (when the integrals are well defined).
The conditional "expectation" given $\widetilde{\mathcal{P}}(\mathbb{F})$ of a product-measurable functional W, is the unique $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $\widetilde{W} =: M^{\mathbb{P}}_\mu \left(W \middle| \widetilde{\mathcal{P}}\left(\mathbb{F}\right)\right)$ satisfying

$$\mathbb{E}\left(W \mathbb{1}_\Sigma \star \mu_\infty\right) = \mathbb{E}\left(\widetilde{W}\mathbb{1}_\Sigma \star \mu_\infty\right), \quad \text{for all } \Sigma \in \widetilde{\mathcal{P}}(\mathbb{F}).$$

The random measure ν is called the compensator of random measure μ if for each $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable W the process $W \star \nu$ is predictable and $\mathbb{E}(W \star \mu_\infty) = \mathbb{E}(W \star \nu_\infty)$.

1.1.10 Representation of Local Martingales

This section recalls an important result on representation of local martingales. This result relies on the continuous local martingale part and the jump random measure of a given semimartingale. Thus, throughout this section, we suppose given a d-dimensional semimartingale, $S = (S_t)_{0\leq t\leq T}$. To this semimartingale, we associate its predictable characteristics that we will present below (for more details about these and other related issues, we refer the reader to [JS03, Section II.2]). The random measure μ associated to the jumps of S is defined in 1.2. The compensator of the random measure μ is denoted by ν. For a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$, (i.e., h is bounded, with compact support and satisfies

$h(x) = x$ in a neighbourhood of 0) we denote by B the predictable part of the special semimartingale $S - \sum(\Delta S - h(\Delta S))$. The continuous local martingale part of S is denoted by S^c. The matrix C is given by $C^{ij} := \langle S^{c,i}, S^{c,j}\rangle$. Then, the triplet $(B,\ C,\ \nu)$ is called *predictable characteristics* of S. This leads to the following decomposition, called *the canonical representation* (see [JS03, Theorem 2.34, Section II.2]), namely the decomposition:

$$S = S_0 + S^c + h \star (\mu - \nu) + B + (x - h) \star \mu.$$

Furthermore, we can find a version of the characteristics triplet satisfying

$$B = b \centerdot \bar{A} \quad \text{and} \quad C = c \centerdot \bar{A} \quad \text{and} \quad \nu(\omega,\ dt,\ dx) = d\bar{A}_t(\omega)F_t(\omega,\ dx).$$

Here $\bar{A}$ is an increasing and predictable process which is continuous if and only if S is quasi-left continuous, b and c are predictable processes, $F_t(\omega,\ dx)$ is a predictable kernel, $b_t(\omega)$ is a vector in $\mathbb{R}^d$ and $c_t(\omega)$ is a symmetric $d \times d$-matrix , for all $(\omega,\ t) \in \Omega \times [0,\ T]$. In the sequel we will often drop ω and t and write, for instance, $F(dx)$ as a shorthand for $F_t(\omega, dx)$.

The characteristics, $B = b \centerdot \bar{A},\ C$, and ν, satisfy

$$F_t(\omega,\ \{0\}) = 0, \qquad \int (|x|^2 \wedge 1) F_t(\omega,\ dx) \le 1,$$

$$\Delta B_t = b_t \Delta \bar{A}_t = \int h(x)\nu(\{t\}, dx), \qquad \text{and} \qquad c = 0 \quad \text{on} \quad \{\Delta \bar{A} \neq 0\}.$$

We set

$$\nu_t(dx) := \nu(\{t\}, dx), \quad a_t := \nu_t(\mathbb{R}^d) = \Delta \bar{A}_t F_t(\mathbb{R}^d) \le 1.$$

For the following representation theorem, we refer to [Jac79, Theorem 3.75, page 103] and to [JS03, Lemma 4.24, Chap III].

Theorem 1.26. *Let $N \in \mathcal{M}_{0,loc}$. Then, there exist a predictable S^c-integrable process β, $N^\perp \in \mathcal{M}_{0,loc}$ with $N^\perp$ and S orthogonal (i.e., $[S, N^\perp] \in \mathcal{M}_{0,loc}$) and functionals $f \in \widetilde{\mathcal{P}}$ and $g \in \widetilde{\mathcal{O}}$ such that*

(a) $\left(\sum_{s \le t} f_s(\Delta S_s)^2 \mathbb{1}_{\{\Delta S_s \neq 0\}}\right)^{1/2}$ *and* $\left(\sum_{s \le t} g_s(\Delta S_s)^2 \mathbb{1}_{\{\Delta S_s \neq 0\}}\right)^{1/2}$ *belong to* $\mathcal{A}^+_{loc}$.

(b) $M^{\mathbb{P}}_\mu(g \mid \widetilde{\mathcal{P}}) = 0, \quad M^{\mathbb{P}}_\mu$*-a.s.*

(c) The process N satisfies

$$N = \beta \centerdot S^c + W \star (\mu - \nu) + g \star \mu + N^\perp, \quad \text{where } W = f + \frac{\widehat{f}}{1-a}\mathbb{1}_{\{a<1\}}.$$

Here $\widehat{f}_t = \int f_t(x)\nu(\{t\}, dx)$ and f has a version such that $\{a = 1\} \subset \{\widehat{f} = 0\}$. Moreover

$$\Delta N_t = \Big(f_t(\Delta S_t) + g_t(\Delta S_t)\Big)\mathbb{1}_{\{\Delta S_t \neq 0\}} - \frac{\widehat{f}_t}{1-a_t}\mathbb{1}_{\{\Delta S_t = 0\}} + \Delta N^\perp_t.$$

The quadruplet $(\beta, f, g, N^\perp)$ is called the Jacod's parameters of the local martingale N with respect to S.

1.2 Enlargement of filtration

Enlargement of filtration theory is settled in general theory of stochastic processes. We consider an enlarged filtration $\mathbb{G}$, i.e., a filtration satisfying $\mathbb{F} \subset \mathbb{G}$, i.e., $\mathcal{F}_t \subset \mathcal{G}_t$ for every $t \geq 0$. We may ask the question whether an $\mathbb{F}$-semimartingale is still a $\mathbb{G}$-semimartingale. In case it is, we are interested in how semimartingale decomposition writes in filtration $\mathbb{G}$. There are two classical hypotheses linked with those questions, namely:

Definition 1.27. (a) **Hypothesis ($\mathcal{H}$)** is satisfied for $\mathbb{F} \subset \mathbb{G}$ if every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. For ease of language, when the hypothesis ($\mathcal{H}$) is satisfied for $\mathbb{F} \subset \mathbb{G}$, we shall often say $\mathbb{F}$ is immersed in $\mathbb{G}$ and write $\mathbb{F} \hookrightarrow \mathbb{G}$.
(b) **Hypothesis ($\mathcal{H}'$)** is satisfied for $\mathbb{F} \subset \mathbb{G}$ if every $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale.

In some sense, the inverse question is answered in the following useful theorem, see [Str77].

Theorem 1.28 (Stricker's Theorem)**.** *Let X be a $\mathbb{G}$-semimartingale which is adapted to a right-continuous subfiltration $\mathbb{F}$. Then X is also a $\mathbb{F}$-semimartingale.*

It is worth noting that an $\mathbb{F}$-adapted $\mathbb{G}$-local martingale is not necessary an $\mathbb{F}$-local martingale as it may not be possible to choose a localizing sequence in the filtration $\mathbb{F}$ (see [FP11]).

Two particular cases of enlargement of filtration were widely studied in the literature, that is to say initial and progressive enlargements. Initial enlargement of filtration $\mathbb{F}$ with a random variable ξ is defined as

$$\mathcal{F}_t^{\sigma(\xi)} := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\xi)) . \tag{1.4}$$

Progressive enlargement of filtration $\mathbb{F}$ with a positive random variable τ is defined as

$$\mathcal{F}_t^{\tau} := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)) . \tag{1.5}$$

In the following subsections we recall classical results from these studies.

1.2.1 Initial enlargement with atomic σ-field

In this section, we recall results on initial enlargement with a random variable ξ which takes countably many values or equivalently initial enlargement with atomic σ-field $\sigma(\xi)$, i.e., there exists a partition $(C_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}(C_n) > 0$ for every $n \in \mathbb{N}$ generating $\sigma(\xi)$. For each set C_n, we denote by $(z_t^n)_{t\geq 0}$ the $\mathbb{F}$-martingale $z_t^n = \mathbb{P}(C_n|\mathcal{F}_t)$. We have $\{\inf_s z_s^n = 0\} \cap C_n = \emptyset$.

Let us define $\mathbb{F}^{\mathcal{C}}$ as an initial enlargement of the filtration $\mathbb{F}$ with an atomic σ-field $\mathcal{C} = \sigma((C_n)_n)$.

The next lemma expresses projections on the filtration $\mathbb{F}^{\mathcal{C}}$ in terms of projections on $\mathbb{F}$, see [Jeu80, Lemme (3,1)].

Lemma 1.29. *Let H be a measurable bounded process. Then,*
(a) the $\mathbb{F}^{\mathcal{C}}$-optional projection of H is given by

$$ {}^{o,\mathbb{F}^{\mathcal{C}}}(H) = \sum_{n\in\mathcal{I}} 1\!\!1_{C_n} \frac{1}{z^n} {}^{o,\mathbb{F}}(1\!\!1_{C_n} H); $$

(b) the $\mathbb{F}^{\mathcal{C}}$-predictable projection of H is given by

$$ {}^{p,\mathbb{F}^{\mathcal{C}}}(H) = \sum_{n\in\mathcal{I}} 1\!\!1_{C_n} \frac{1}{z^n_-} {}^{p,\mathbb{F}}(1\!\!1_{C_n} H). $$

In this case of enlargement the hypothesis $(\mathcal{H}')$ is satisfied and semimartingale decomposition is given in the next theorem, see [Jeu80, Theorem 3,2], [Mey78].

Theorem 1.30. *The filtration $\mathbb{F}^{\mathcal{C}}$ satisfies $(\mathcal{H}')$ hypothesis and each $\mathbb{F}$-martingale X can be decomposed in $\mathbb{F}^{\mathcal{C}}$ as*

$$ X_t = \widehat{X}_t + \sum_n 1\!\!1_{C_n} \int_0^t \frac{1}{z^n_{s-}} d\langle X, z^n\rangle_s, $$

where $\widehat{X}$ is an $\mathbb{F}^{\mathcal{C}}$-local martingale.

1.2.2 Initial enlargement under Jacod's hypothesis

Previous results can be generalized to any random variable ξ taking values in Lusin space $(\mathbb{U},\mathcal{U})$ and satisfying Jacod's hypothesis, see [Jac85]. We recall as well a stronger version of this hypothesis, namely the equivalence Jacod's hypothesis, studied and used in the literature e.g. [Ame99, EJJ10, EJJZ14, JL09].

Definition 1.31. (a) A random variable ξ satisfies Jacod's hypothesis if there exists a σ-finite positive measure $\widetilde{\eta}$ such that for every $t \geq 0$ $\mathbb{P}(\xi \in du|\mathcal{F}_t)(\omega) \ll \widetilde{\eta}(du)$ $\mathbb{P}$-a.s.
(b) A random variable ξ satisfies the equivalence Jacod's hypothesis, if for every $t \geq 0$ $\mathbb{P}(\xi \in du|\mathcal{F}_t)(\omega) \sim \widetilde{\eta}(du)$ $\mathbb{P}$-a.s.

As shown by Jacod, without loss of generality, $\widetilde{\eta}$ can be taken as the law of ξ in the above definition.

The following result is due to Jacod [Jac85, Lemme 1.8]. We recall here the formulation of Amendinger as it provides nice measurability property [Ame99, Remark 1 p.17].

Proposition 1.32. *For ξ satisfying Jacod's hypothesis, there exists a positive $\mathcal{O}\otimes\mathcal{U}$-measurable function $(t,\omega,u) \to q_t(\omega,u)$ càdlàg in t such that*
(a) for every u, the process $(q^u_t, t \geq 0)$ is an $\mathbb{F}$-martingale, and if

$$ R^u = \inf\{t : q^u_{t-} = 0\} \tag{1.6} $$

we have $q^u > 0$ and $q^u_- > 0$ on $]\!]0, R^u]\!]$ and $q^u = 0$ on $]\!]R^u, \infty]\!]$,
(b) for every $t \geq 0$, the measure $q^u_t(\omega)\widetilde{\eta}(du)$ is a version of $\mathbb{P}(\tau \in du|\mathcal{F}_t)(\omega)$.

In the following lemma the $\mathbb{F}$-predictable and $\mathbb{F}$-optional projections of particular $\mathbb{F}^{\sigma(\xi)}$-adapted processes are given. The first part in due to Jacod [Jac85, Lemme (1.10)], the second part can be found in Amendinger [Ame99, Lemma 1.3].

Lemma 1.33. *(a) Let the function $(t, \omega, u) \to Y_t^u(\omega)$ be $\mathcal{P} \otimes \mathcal{U}$-measurable, positive or bounded. Then, the $\mathbb{F}$-predictable projection of the process $(Y_t^\xi)_{t\geq 0}$ is given by*

$$^{p,\mathbb{F}}(Y^\xi)_t = \int_{\mathbb{U}} Y_t^u q_{t-}^u \widetilde{\eta}(du) \quad t \geq 0.$$

(b) Let the function $(t, \omega, u) \to Y_t^u(\omega)$ be $\mathcal{O} \otimes \mathcal{U}$-measurable, positive or bounded. Then, the $\mathbb{F}$-optional projection of the process $(Y_t^\xi)_{t\geq 0}$ is given by

$$^{o,\mathbb{F}}(Y^\xi)_t = \int_{\mathbb{U}} Y_t^u q_t^u \widetilde{\eta}(du) \quad t \geq 0. \tag{1.7}$$

As noticed in [Jac85, Corollary (1.11)], Lemma 1.33 implies in particular that

$$R^\xi = \infty \quad \mathbb{P}-\text{a.s.} \tag{1.8}$$

with R^u defined through (1.6), or equivalently $q_t^\xi > 0$ and $q_{t-}^\xi > 0$ for $t \geq 0$ $\mathbb{P}$-a.s. Then, the $\mathbb{F}^{\sigma(\xi)}$-optional process $\left(\frac{1}{q_t^\xi}\right)_{t\geq 0}$ is well-defined.

Proposition 1.34. *Assume that ξ satisfies Jacod's hypothesis. Let X be an $\mathbb{F}$-martingale. Then*

$$X_t = \widehat{X}_t + \int_0^t \frac{1}{q_{s-}^\xi} d\langle X, q^u\rangle_s|_{u=\xi} \tag{1.9}$$

where $\widehat{X}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale.

Example 1.35. Let ξ be a random variable which takes only countably many values $(c_n)_{n\in\mathbb{N}}$. So $\xi = \sum_n \mathbb{1}_{C_n} c_n$ where $(C_n)_{n\in\mathbb{N}}$ is a partition of Ω (for each n, $\mathbb{P}(C_n) > 0$). Then, the law of ξ and the conditional law of ξ with respect to the σ-field $\mathcal{F}_t$ can be written as

$$\mathbb{P}(\xi \in du) = \sum_n \mathbb{P}(C_n)\delta_{c_n}(u)du,$$
$$\mathbb{P}(\xi \in du|\mathcal{F}_t) = \sum_n \mathbb{P}(C_n|\mathcal{F}_t)\delta_{c_n}(u)du,$$

where δ_{c_n} denotes the Dirac measure with mass at c_n. In particular, the random variable T has a density q_t^u satisfying

$$\mathbb{P}(\xi \in du|\mathcal{F}_t) = q_t^u \mathbb{P}(\xi \in du) \quad \text{and} \quad q_t^u = \sum_n \frac{\mathbb{P}(C_n|\mathcal{F}_t)}{\mathbb{P}(C_n)} \mathbb{1}_{\{c_n=u\}}.$$

The next theorem introduces particular change of measure making the reference filtration $\mathbb{F}$ and the random variable ξ independent, see [Son87], [Ame99, Proposition 1.6], [GP98, GP01].

Theorem 1.36. *Assume the equivalence Jacod's hypothesis is satisfied. Then*
(a) the process $\frac{1}{q^\xi}$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-martingale.*
(b) the probability measure $\mathbb{P}^*$, *defined as*

$$\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}}\Big|_{\mathcal{F}_t^{\sigma(\xi)}} = \frac{1}{q_t^\xi}$$

has the following properties:
(i) under $\mathbb{P}^*$, τ *is independent from* $\mathcal{F}_t$ *for any* t, *(ii)* $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$, *(iii)* $\mathbb{P}^*|_{\sigma(\xi)} = \mathbb{P}|_{\sigma(\xi)}$.

1.2.3 Progressive enlargement

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ.

A central object for progressive enlargement of filtration is the process $A := 1\!\!1_{[\![\tau,\infty[\![}$, where τ is a random time. We define two supermartingales associated with a random time τ and the reference filtration $\mathbb{F}$, namely

$$Z_t := {}^{o,\mathbb{F}}(1\!\!1_{[\![0,\tau[\![})_t = \mathbb{P}(\tau > t|\mathcal{F}_t) \tag{1.10}$$

$$\widetilde{Z}_t := {}^{o,\mathbb{F}}(1\!\!1_{[\![0,\tau]\!]})_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t). \tag{1.11}$$

The supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $1\!\!1_{[\![0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $1\!\!1_{[\![0,\tau]\!]}$. Moreover we consider two increasing processes associated with the random time τ, namely

$$A_t^o := (1\!\!1_{[\![\tau,\infty[\![})_t^{o,\mathbb{F}} \quad \text{and} \quad A_t^p := (1\!\!1_{[\![\tau,\infty[\![})_t^{p,\mathbb{F}}.$$

Let us define the $\mathbb{F}$-martingales n and m as

$$n_t : = \mathbb{E}(A_\infty^p + Z_\infty|\mathcal{F}_t) \quad \text{and} \quad m_t := \mathbb{E}(A_\infty^o + Z_\infty|\mathcal{F}_t).$$

Then, the supermartingales Z and $\widetilde{Z}$ decompose as

$$Z_t = n_t - A_t^p, \quad Z_t = m_t - A_t^o, \quad \text{and} \quad \widetilde{Z} = m - A_-^o. \tag{1.12}$$

The supermartingales Z and $\widetilde{Z}$ are related through

$$\widetilde{Z} = Z + \Delta A^o \quad \text{and} \quad \widetilde{Z} = Z_- + \Delta m. \tag{1.13}$$

In the literature (e.g. [MY06]), there are two standard assumptions about (progressive) enlargement of filtration problem, namely

Definition 1.37. We say that:
assumption (C) is satisfied if all $\mathbb{F}$-martingales are continuous;
assumption (A) is satisfied if the random time τ avoids all $\mathbb{F}$-stopping times.

The first one concerns the reference filtration $\mathbb{F}$ and the second one the random time τ. If assumption (C) or (A) is satisfied, then $Z = \widetilde{Z}$. Under assumptions (C) and (A), the supermartingale $Z = \widetilde{Z}$ is a continuous process.

Conditional expectations with respect to progressively enlarged filtration can be rewritten, on appropriate intervals, as given in the following proposition, see [Del70, BR04].

Proposition 1.38. *Let $X \in \mathcal{G}$ be an integrable random variable. Then*

$$\mathbb{E}(X|\mathcal{F}_t^\tau)\mathbb{1}_{\{\tau>t\}} = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}(X\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)}{\mathbb{P}(\tau > t|\mathcal{F}_t)}$$

and

$$\mathbb{E}(X|\mathcal{F}_t^\tau)\mathbb{1}_{\{\tau\leq t\}} = \mathbb{1}_{\{\tau\leq t\}}\mathbb{E}(X|\mathcal{F}_t \vee \mathcal{A}_\infty),$$

where the filtration $\mathbb{A}$ is generated by the process $A = \mathbb{1}_{[\![\tau,\infty[\![}$.

The following result states that any $\mathbb{F}$-martingale stopped at τ remains an $\mathbb{F}^\tau$-semimartingale,, i.e., hypothesis $(\mathcal{H}')$ is satisfied up to random time τ. We recall a result from [JY78], [Jeu80, Proposition (4,16)].

Proposition 1.39. *Let M be an $\mathbb{F}$-local martingale. Then, M^τ is an $\mathbb{F}^\tau$-semimartingale which can be decomposed as*

$$M_t^\tau = \widehat{M}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle M, m\rangle_s^{\mathbb{F}}, \tag{1.14}$$

where $\widehat{M}$ is an $\mathbb{F}^\tau$-local martingale.

In the next proposition we present equivalent characterizations of hypothesis $(\mathcal{H})$ in the progressive enlargement case, see [JYC09, p.323].

Proposition 1.40. *In the progressive enlargement setting, $(\mathcal{H})$ holds between $\mathbb{F}$ and $\mathbb{F}^\tau$ if and only if one of the following equivalent conditions holds*
(a) $\forall (t,s), s \leq t \quad \mathbb{P}(\tau \leq s|\mathcal{F}_\infty) = \mathbb{P}(\tau \leq s|\mathcal{F}_t)$,
(b) $\forall t \quad \mathbb{P}(\tau \leq t|\mathcal{F}_\infty) = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$.

1.3 Random times

In this section four classes of random times are presented: honest times, pseudo-stopping times, initial times and Cox times. We recall definitions and the most useful properties in the context of enlargement of filtration. In the following chapters of this thesis we will rely on those properties in answering posed questions.

1.3.1 Honest times

An important class of random times is the class of honest times. It is a class which generalizes the notion of stopping time. We recall its definition and alternative characterization, see [Jeu80, Chapter 5, p.73].

Definition 1.41. A random time τ is an $\mathbb{F}$-honest time if for every $t > 0$ there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau < t\}$.

Note that each stopping time τ is an honest time. It is enough to take $\tau_t = \tau \wedge t$.

We will often use following lemma while working with honest times.

Lemma 1.42. *(a) A random time τ is an $\mathbb{F}$-honest time if for every $t > 0$ there exists an $\mathcal{F}_{t-}$-measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau < t\}$.*
(b) A random time τ is an $\mathbb{F}$-honest time if for every $t > 0$ there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau \leq t\}$.

Proof. Using notation from Definition 1.41 we define the process α^- (this is the process A in the proof of Theorem 1.43 in [Jeu80]; and the process R^0 in [Kar14a]) as $\alpha^-_t = \sup_{r\in Q, r<t} \tau_r$. This definition implies that α^- is an increasing, left-continuous, adapted process such that $\alpha^-_t = \tau$ on $\{\tau < t\}$ and (a) is proved.
Denote by α the right-continuous version of α^-, i.e., $\alpha_t = \alpha^-_{t+}$. Then, α is an increasing, càdlàg, adapted process such that $\alpha_t = \tau$ on $\{\tau \leq t\}$ and $\tau = \sup\{t : \alpha_t = t\}$ and (b) is proved. ■

Next theorem gives characterisations of an honest time, see [Jeu80, Proposition (5,1) p.73].

Theorem 1.43. *Let τ be a random time. Then, the following conditions are equivalent:*
(a) τ is an honest time;
(b) there exists an optional set Γ such that $\tau(\omega) = \sup\{t : (\omega, t) \in \Gamma\}$ on $\{\tau < \infty\}$;
(c) $\widetilde{Z}_\tau = 1$ a.s. on $\{\tau < \infty\}$;
(d) $\tau = \sup\{t : \widetilde{Z}_t = 1\}$ a.s. on $\{\tau < \infty\}$;
(e) $\mathcal{P}^\tau|_{]\!]0,\infty[\![}$ is generated by $\mathcal{P}|_{]\!]0,\infty[\![}$ and $]\!]0, \tau]\!]$, where $\mathcal{P}^\tau$ is the predictable σ-field linked to $\mathbb{F}^\tau$;
(f) $A^o_t = A^o_{t\wedge\tau}$.

Proof. The equivalence among conditions (a), (b), (c), (d) and (e) is stated in Theorem (5,1) from Jeulin [Jeu80]. Implication (a)$\Rightarrow$(f) comes from analogous arguments as in Azéma [Azé72]. To finish the proof, we show implication (f)$\Rightarrow$(b). Let Λ be the support of the measure dA^o, i.e.,

$$\Gamma = \{(\omega, t) : \ \forall \varepsilon > 0 \ A^o_t(\omega) > A^o_{t-\varepsilon}(\omega)\}.$$

The set Γ is optional since A^o is an optional process. Then, $[\![\tau]\!] \subset \Gamma$ and $A^o_t = A^o_{t\wedge\tau}$ imply that indeed τ is the end of Γ on $\{\tau < \infty\}$. ■

Analogous result to Proposition 1.38 (a) after honest time is given in the following Proposition.

Proposition 1.44. *Let τ be an honest time. Then, for $\mathcal{G}$-measurable integrable random variable X and $s \leq t$ we have*

$$\mathbb{E}(X|\mathcal{F}^\tau_t)1\!\!1_{\{\tau\leq s\}} = 1\!\!1_{\{\tau\leq s\}} \frac{\mathbb{E}(X 1\!\!1_{\{\tau\leq s\}}|\mathcal{F}_t)}{\mathbb{P}(\tau \leq s|\mathcal{F}_t)}.$$

Proof. Note that for each $G \in \mathcal{F}_t^\tau$ there exists $F \in \mathcal{F}_t$ such that $G \cap \{\tau \le s\} = F \cap \{\tau \le s\}$ as, by Monotone Class Theorem 1.1, it is enough to check it for $G \in \mathcal{F}_t$ for which it is obviously satisfied and for $G = \{\tau \in B\}$ where B is a Borel set for which, by honest time property, we have

$$\{\tau \in B\} \cap \{\tau \le s\} = \{\tau_s \in B\} \cap \{\tau \le s\}$$

with $\{\tau_s \in B\} \in \mathcal{F}_s \subset \mathcal{F}_t$.

Then, we have to show that

$$\mathbb{E}\left(X 1\!\!1_{\{\tau \le s\}} \mathbb{P}(\tau \le s|\mathcal{F}_t)|\mathcal{F}_t^\tau\right) = 1\!\!1_{\{\tau \le s\}} \mathbb{E}(X 1\!\!1_{\{\tau \le s\}}|\mathcal{F}_t).$$

For any $G \in \mathcal{F}_t^\tau$ we choose $F \in \mathcal{F}_t$ such that $G \cap \{\tau \le s\} = F \cap \{\tau \le s\}$, and we get

$$\begin{aligned}
\mathbb{E}\left(X 1\!\!1_{\{\tau \le s\}\cap G}\ P(\tau \le s|\mathcal{F}_t)\right) &= \mathbb{E}\left(X 1\!\!1_{\{\tau \le s\}\cap F}\ \mathbb{P}(\tau \le s|\mathcal{F}_t)\right) \\
&= \mathbb{E}\left(1\!\!1_{\{\tau \le s\}\cap F}\ \mathbb{E}(X 1\!\!1_{\{\tau \le s\}}|\mathcal{F}_t)\right) \\
&= \mathbb{E}\left(1\!\!1_{\{\tau \le s\}\cap G}\ \mathbb{E}(X 1\!\!1_{\{\tau \le s\}}|\mathcal{F}_t)\right).
\end{aligned}$$

which ends the proof. ∎

For the progressive enlargement with honest time hypothesis $(\mathcal{H}')$ is satisfied (not only for $\mathbb{F}$-local martingales stopped at τ), the decomposition is given in [Jeu80].

Proposition 1.45. *Let M be an $\mathbb{F}$-local martingale. Then, there exists an $\mathbb{F}^\tau$-local martingale $\widehat{M}$ such that:*

$$M_t = \widehat{M}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle M, m\rangle_s - \int_\tau^t \frac{1}{1 - Z_{s-}} d\langle M, m\rangle_s. \tag{1.15}$$

Example 1.46. Let W be a Brownian motion. Then we define an honest time σ as follows: $\sigma = \sup\{t \le T_1 : W_t = 0\}$, where $T_1 = \inf\{t : W_t = 1\}$. So σ is the last zero of the process W before it reaches 1. Moreover it is an finite honest time (in the definition of honest time we take $\sigma_t = \sup\{t \le t \wedge T_1 : W_t = 0\}$). The supermartingale Z associated with σ can be computed as:

$$Z_t = \mathbb{P}(\sigma > t|\mathcal{F}_t) = 1 - (W_t^{T_1})^+,$$

which implies that the biggest predictable set situated on the left of $[\![\sigma]\!]$ equals

$$\Gamma = \{Z_- = 1\}) = \{(t,\omega) : W_t^{T_1}(\omega) \le 0\} = [\![0, T_1]\!] \cap \{W_t \le 0\}.$$

We recall the following useful simple lemma.

Lemma 1.47. *Let τ be the last passage time of an $\mathbb{F}$-adapted process Λ below a deterministic level a, i.e., $\tau = \sup\{t : \Lambda_t \le a\}$. Then, τ is an honest time.*

Proof. On the set $(\tau < u)$, we have that $\tau = \tau_u$ for $\tau_u = \sup\{t \le u : \Lambda_t \le a\}$ and τ_u is $\mathcal{F}_u$-measurable. ∎

1.3.2 Pseudo-stopping times

Another generalization of the class of stopping times is the class of pseudo-stopping times. The class of pseudo-stopping times was introduced by Nikeghbali and Yor in [NY05] as an extension of Williams' example in [Wil02] and also as an example where the supermartingale Z is decreasing, but where the hypothesis $(\mathcal{H})$ is not satisfied for the progressive enlargement.

Let us recall its definition, see [Wil02], [NY05].

Definition 1.48. A random time τ is an $\mathbb{F}$-pseudo-stopping time if for every bounded $\mathbb{F}$-martingale M, we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty)$.

The name of pseudo-stopping time is connected to Knight-Maisonneuve characterization of $\mathbb{F}$-stopping times given in [KM94], i.e., a random time τ is an $\mathbb{F}$-stopping time if and only if for any bounded $\mathbb{F}$-martingale M one has $\mathbb{E}(M_\infty|\mathcal{F}_\tau) = M_\tau$.

Example 1.49 ([Wil02]). Let $\mathbb{F}$ be the filtration generated by a Brownian motion B. Let σ be the honest time from Example 1.46. Then, the random time $\rho = \sup\{t < \sigma : W_t = W_t^*\}$ with $W_t^* = \sup_{s\leq t} W_s$, is a pseudo-stopping time. This follows from the next proposition and the fact that the dual predictable projection of the process $A = 1\!\!1_{[\![\rho,\infty[\![}$ equals $(W^*)^{T_1}$.

In [NY05], the definition of pseudo-stopping times was limited to finite random times. Here we recall their result with slight extension (to random times which may take the value infinity).

To avoid confusion at infinity, for any process V, we set $V_\infty := \lim_{u\to\infty} V_u$, if the limit exists.

Proposition 1.50 ([NY05]). *Let τ be a random time and $A := 1\!\!1_{[\![\tau,\infty[\![}$. Then, the following six conditions are equivalent:*
(a) τ is an $\mathbb{F}$-pseudo-stopping time;
(b) $A^o_\infty = \mathbb{P}(\tau < \infty|\mathcal{F}_\infty)$;
(c) $m = 1$;
(d) ${}^oA = A^o$;
(e) for every $\mathbb{F}$- local martingale M, the process M^τ is an $\mathbb{F}^\tau$-local martingale.
(f) $\widetilde{Z}$ is a decreasing càglàd process.

Proof. Let us start with the equivalence $(a) \iff (b)$. Let M be a bounded $\mathbb{F}$-martingale. Integration by parts formula gives

$$\begin{aligned}
M_\infty A^o_\infty &= M_0 A^o_0 + \int_0^\infty A^o_{s-} dM_s + \int_0^\infty M_{s-} dA^o_s + [A^o, M]_\infty \\
&= M_0 A^o_0 + \int_0^\infty A^o_{s-} dM_s + \int_0^\infty M_{s-} dA^o_s + \int_0^\infty \Delta M_s dA^o_s \\
&= M_0 A^o_0 + \int_0^\infty A^o_{s-} dM_s + \int_0^\infty M_s dA^o_s \\
&= \int_0^\infty A^o_{s-} dM_s + \int_{[0,\infty[} M_s dA^o_s.
\end{aligned}$$

Then, by the definition of dual optional projection, we have

$$\mathbb{E}(M_\tau 1\!\!1_{\{\tau<\infty\}}) = \mathbb{E}(\int_{[0,\infty[} M_s dA^o_s) = \mathbb{E}(M_\infty A^o_\infty).$$

Therefore, the equality $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty)$ holds true for every bounded $\mathbb{F}$-martingale M if and only if $A^o_\infty = \mathbb{P}(\tau < \infty|\mathcal{F}_\infty)$, since

$$\begin{aligned}\mathbb{E}(M_\tau) &= \mathbb{E}(M_\tau 1\!\!1_{\{\tau<\infty\}}) + \mathbb{E}(M_\infty 1\!\!1_{\{\tau=\infty\}})\\ &= \mathbb{E}(M_\infty A^o_\infty) + \mathbb{E}(M_\infty \mathbb{P}(\tau = \infty|\mathcal{F}_\infty))\\ &= \mathbb{E}(M_\infty(A^o_\infty + \mathbb{P}(\tau = \infty|\mathcal{F}_\infty))).\end{aligned}$$

and one can look at martingales

$$M^1_t = \mathbb{E}(1\!\!1_{\{A^o_\infty>\mathbb{P}(\tau<\infty|\mathcal{F}_\infty)\}}|\mathcal{F}_t) \quad \text{and} \quad M^2_t = \mathbb{E}(1\!\!1_{\{A^o_\infty<\mathbb{P}(\tau<\infty|\mathcal{F}_\infty)\}}|\mathcal{F}_t).$$

To show the equivalence $(b) \iff (c)$, we work with an $\mathbb{F}$-predictable process $X_s := 1\!\!1_{\{s>t\}} 1\!\!1_{F_t}$ where t is fixed and $F_t \in \mathcal{F}_t$. Then,

$$\begin{aligned}\mathbb{E}(Z_t 1\!\!1_{F_t}) &= \mathbb{E}(1\!\!1_{\{\tau>t\}} 1\!\!1_{F_t} 1\!\!1_{\{\tau<\infty\}}) + \mathbb{E}(1\!\!1_{\{\tau=\infty\}} 1\!\!1_{F_t})\\ &= \mathbb{E}(1\!\!1_{F_t}(A^o_\infty - A^o_t)) + \mathbb{E}(1\!\!1_{\{\tau=\infty\}} 1\!\!1_{F_t})\end{aligned}$$

which implies that $Z_t = \mathbb{E}(A^o_\infty + 1\!\!1_{\{\tau=\infty\}}|\mathcal{F}_t) - A^o_t$ and indeed, $A^o_\infty = \mathbb{P}(\tau < \infty|\mathcal{F}_\infty)$ if and only if $m = 1$.

The equivalence $(c) \iff (d)$ and the implication $(e) \Rightarrow (a)$ are straightforward, while the implication $(c) \Rightarrow (e)$ comes from general decomposition result for stopped martingales. The implication $(c) \Rightarrow (f)$ comes from (1.13). To show $(f) \Rightarrow (c)$ we also use (1.13), i.e., $\widetilde{Z} = m - A^o_-$. (f) implies that m is a continuous finite variation martingale thus it is a constant, $m = 1$. ∎

1.3.3 Cox times

The Cox's construction of a random time is commonly used in credit risk modelling literature. Cox times are not $\mathcal{F}_\infty$-measurable.

Definition 1.51. A random time τ is an $\mathbb{F}$-Cox time if it is of the form $\tau := \inf\{s : X_s \geq U\}$, where X is an $\mathbb{F}$-adapted càdlàg non-decreasing process and U is a positive random variable independent from $\mathcal{F}_\infty$.

Moreover, for each Cox time τ, hypothesis $(\mathcal{H})$ is satisfied. Thus, in particular, τ is a pseudo-stopping time.

1.3.4 Initial times

The class of initial times is linked with Jacod's hypothesis, see Definition 1.31 (a). This class of random times appears in credit risk literature, see [EJJ10], [EJJZ14].

Definition 1.52. A random time τ is called an initial time if it satisfies Jacod's hypothesis.

For initial times, hypothesis $(\mathcal{H}')$ is satisfied, and the decomposition is given as follows, see [JL09]. Note that this is a mixture of the result up to random time and Jacod's decomposition after random time.

Proposition 1.53. *Let M be an $\mathbb{F}$ martingale. Then*

$$M_t = \widehat{M}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle M, m\rangle_s + \int_\tau^t \frac{1}{q^\tau_{s-}} d\langle M, q^u\rangle_s|_{u=\tau}$$

where $\widehat{M}$ is an $\mathbb{F}^\tau$ martingale

1.4 Arbitrages

1.4.1 Classical Arbitrages and No Free Lunch with vanishing Risk

In this section, we recall the basic definitions on arbitrages, and we give sufficient conditions for no arbitrages in a market with zero interest rate.

Let S be an $\mathbb{F}$-semimartingale. For $a \in \mathbb{R}_+$, an element $\theta \in L(S,\mathbb{F})$ is said to be an a-admissible $\mathbb{F}$-strategy if $(\theta \centerdot S)_\infty := \lim_{t\to\infty} (\theta \centerdot S)_t$ exists and $V_t(0,\theta) := (\theta \centerdot S)_t \geq -a$ $\mathbb{P}$-a.s. for all $t \geq 0$. We denote by $\mathcal{A}_a(\mathbb{F})$ the set of all a-admissible $\mathbb{F}$-strategies. A process $\theta \in L(S,\mathbb{F})$ is called an *admissible* $\mathbb{F}$*-strategy* if $\theta \in \mathcal{A}(\mathbb{F})$, where $\mathcal{A}(\mathbb{F}) := \bigcup_{a\in\mathbb{R}_+} \mathcal{A}_a(\mathbb{F})$.

An admissible strategy yields an **Arbitrage Opportunity** if $V(0,\theta)_\infty \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}\big(V(0,\theta)_\infty > 0\big) > 0$. In order to avoid confusions, we shall call these arbitrages *classical arbitrages*. If there exists no such $\theta \in \mathcal{A}(\mathbb{F})$ we say that the financial market $\mathcal{M}(\mathbb{F}) := (\Omega, \mathbb{F}, \mathbb{P}; S)$ satisfies the No Arbitrage (NA) condition.

No Free Lunch with Vanishing Risk (NFLVR) holds in the financial market $\mathcal{M}(\mathbb{F})$ if and only if there exists an Equivalent Martingale Measure in $\mathbb{F}$, i.e., a probability measure $\mathbb{Q}$, such that $\mathbb{Q} \sim \mathbb{P}$ and the process S is a $(\mathbb{Q}, \mathcal{G}, \mathbb{F})$-local martingale. If NFLVR holds, there are no classical arbitrages.

We recall that NFLVR holds if and only if both NA and NUPBR hold (see [DS94, Corollary 3.4] and [KK07, Proposition 3.6]. NUPBR condition is presented in the next section.

1.4.2 No Unbounded Profit with Bounded Risk

This section is contained in the paper [ACDJ14b]. We focus here on No Unbounded Profit with Bounded Risk condition of no arbitrage.

We introduce the non-arbitrage notion that will be addressed in this thesis, in particular in Chapter 6.

Definition 1.54. An $\mathbb{F}$-semimartingale X satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition[1] under $(\mathbb{F}, \mathbb{Q})$ (hereafter called NUPBR$(\mathbb{F}, \mathbb{Q})$) if for any $T \in]0, \infty[$ the set

$$\mathcal{K}_T(X, \mathbb{F}) := \Big\{ (H \centerdot X)_T \mid \quad H \in L(X, \mathbb{F}), \text{ and } H \centerdot X \geq -1 \Big\}$$

is bounded in $\mathbb{Q}$-probability, i.e.,

$$\lim_{c \to \infty} \sup_{(H \centerdot X)_T \in \mathcal{K}_T(X, \mathbb{F})} \mathbb{Q}(|(H \centerdot X)_T| > c) = 0.$$

When $\mathbb{Q} \sim \mathbb{P}$, we simply write, with an abuse of language, X satisfies NUPBR$(\mathbb{F})$.

Recall that a process X is said to be a σ-martingale if it is a semimartingale and if there exists a predictable process ϕ such that $0 < \phi \leq 1$ and $\phi \centerdot X$ is a local martingale.

Remark 1.55. (a) It is important to notice that this definition of NUPBR condition first appeared in [Kar14b] (up to our knowledge), and it differs when the time horizon is infinite from that of the literature given in Delbaen and Schachermayer [DS94], Kabanov [Kab97] and Karatzas and Kardaras [KK07]. It is obvious that, when the horizon is deterministic and finite, the current NUPBR condition coincides with that of the literature. We could name the current NUPBR as NUPBR$_{loc}$, but for the sake of simplifying notation, we opted for the usual terminology.
(b) In general, when the horizon is infinite, the NUPBR condition of the literature implies the NUPBR condition defined above. However, the reverse implication may not hold in general. In fact if we consider $S_t = \exp(W_t + t),\ t \geq 0$, then it is clear that S satisfies our NUPBR$(\mathbb{F})$, while the NUPBR$(\mathbb{F})$ of the literature is violated. To see this last claim, it is enough to remark that

$$\lim_{t \longrightarrow \infty} (S_t - 1) = \infty \quad \mathbb{P} - a.s. \quad S^t - 1 = H \centerdot S \geq -1 \quad H := 1\!\!1_{]\!]0,t]\!]}.$$

The following proposition slightly generalizes Takaoka's results obtained for a finite horizon (see Theorem 2.6 in [Tak13]) to our NUPBR context.

Proposition 1.56. *Let X be an $\mathbb{F}$-semimartingale. Then the following assertions are equivalent.*
(a) X satisfies NUPBR$(\mathbb{F})$.
(b) There exist a positive $\mathbb{F}$-local martingale, Y and an $\mathbb{F}$-predictable process ϕ satisfying $0 < \phi \leq 1$ such that $Y(\phi \centerdot X)$ is an $\mathbb{F}$-local martingale.

Proof. The proof of the implication (b)$\Rightarrow$ (a) is based on [Tak13] and is omitted. We now suppose that assertion (a) holds. A direct application of Theorem 2.6 in [Tak13] to each $(X_{t \wedge n})_{t \geq 0}$, leads to the existence of a positive $\mathbb{F}$-local martingale $Y^{(n)}$ and an $\mathbb{F}$-predictable

[1] This condition is also known in the literature as the first kind of non-arbitrage

process ϕ_n such that $0 < \phi_n \leq 1$ and $Y^{(n)}(\phi_n \centerdot X^n)$ is a local martingale. Then, it is obvious that the process

$$N := \sum_{n=1}^{\infty} 1\!\!1_{]\!]n-1,n]\!]} (Y_-^{(n)})^{-1} \centerdot Y^{(n)}$$

is a local martingale and $Y := \mathcal{E}(N) > 0$. On the other hand, the $\mathbb{F}$-predictable process $\phi := \sum_{n\geq 1} 1\!\!1_{]\!]n-1,n]\!]}\phi_n$ satisfies $0 < \phi \leq 1$ and $Y(\phi \centerdot X)$ is a local martingale. This ends the proof of the proposition. ∎

For any $\mathbb{F}$-semimartingale X, the local martingales fulfilling the assertion (b) of Proposition 1.56 are called σ-martingale densities for X. The set of these σ-martingale densities will be denoted by

$$\mathcal{L}(\mathbb{F}, X) := \left\{ Y \in \mathcal{M}_{loc}(\mathbb{F}) \middle| \; Y > 0, \; \exists \phi \in \mathcal{P}(\mathbb{F}), \; 0 < \phi \leq 1, \; Y(\phi \centerdot X) \in \mathcal{M}_{loc}(\mathbb{F}) \right\}$$

where, as usual, $\mathcal{P}(\mathbb{F})$ stands for the predictable σ-field on $\Omega \times [0, \infty[$ and by abuse of notation $\phi \in \mathcal{P}(\mathbb{F})$ means that ϕ is $\mathcal{P}(\mathbb{F})$-measurable. We state, without proof, an obvious lemma.

Lemma 1.57. *For any $\mathbb{F}$-semimartingale X and any $Y \in \mathcal{L}(\mathbb{F}, X)$, one has ${}^{p,\mathbb{F}}(Y|\Delta X|) < \infty$ and ${}^{p,\mathbb{F}}(Y \Delta X) = 0$.*

Remark 1.58. Proposition 1.56 implies that for any process X and any **finite** stopping time σ, the two concepts of NUPBR($\mathbb{F}$) (the current concept and the one of the literature) coincide for X^σ.

Below, we prove that, in the case of infinite horizon, the current NUPBR condition is stable under localization, while this is not the case for the NUPBR condition defined in the literature.

Proposition 1.59. *Let X be an $\mathbb{F}$-adapted process. Then, the following assertions are equivalent.*

(a) There exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{F}$-stopping times that increases to ∞, such that for each $n \geq 1$, there exists a probability $\mathbb{Q}_n$ on $(\Omega, \mathbb{F}_{T_n})$ such that $\mathbb{Q}_n \sim \mathbb{P}$ and X^{T_n} satisfies NUPBR($\mathbb{F}$) under $\mathbb{Q}_n$.

(b) X satisfies NUPBR($\mathbb{F}$).

(c) There exists an $\mathbb{F}$-predictable process ϕ, such that $0 < \phi \leq 1$ and $(\phi \centerdot X)$ satisfies NUPBR($\mathbb{F}$).

Proof. The proof for (a)$\Longleftrightarrow$(b) follows from the stability of NUPBR condition for a finite horizon under localization which is due to [Tak13] (see also [CS13] for further discussion about this issue), and the fact that the NUPBR condition is stable under any equivalent probability change.

The proof of (b)$\Rightarrow$(c) is trivial and is omitted. To prove the reverse, we assume that (c) holds. Then Proposition 1.56 implies the existence of an $\mathbb{F}$-predictable process ψ such that $0 < \psi \leq 1$ and a positive $\mathbb{F}$-local martingale Y such that $Y(\psi\phi \centerdot X)$ is a local martingale. Since $\psi\phi$ is predictable and $0 < \psi\phi \leq 1$, we deduce that X satisfies NUPBR($\mathbb{F}$). This ends the proof of the proposition. ∎

We end this section with a simple, but useful result for predictable process with finite variation.

Lemma 1.60. *Let X be an $\mathbb{F}$-predictable process with finite variation. Then X satisfies NUPBR$(\mathbb{F})$ if and only if $X \equiv X_0$ (i.e. the process X is constant).*

Proof. It is obvious that if $X \equiv X_0$, then X satisfies NUPBR($\mathbb{F}$). Suppose that X satisfies NUPBR($\mathbb{F}$). Consider a positive $\mathbb{F}$-local martingale Y, and an $\mathbb{F}$-predictable process ϕ such that $0 < \phi \leq 1$ and $Y(\phi \centerdot X)$ is a local martingale. Let $(T_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to ∞ such that Y^{T_n} and $Y^{T_n}(\theta \centerdot X)^{T_n}$ are true martingales. Then, for each $n \geq 1$, define $\mathbb{Q}_n := (Y_{T_n}/Y_0)\cdot\mathbb{P}$. Since X is predictable, then $(\phi\centerdot X)^{T_n}$ is also predictable with finite variation and is a $\mathbb{Q}_n$-martingale. Thus, we deduce that $(\phi \centerdot X)^{T_n} \equiv 0$ for each $n \geq 1$. Therefore, we deduce that X is constant (since $X^{T_n} - X_0 = \phi^{-1} \centerdot (\phi \centerdot X)^{T_n} \equiv 0$). This ends the proof of the lemma. ∎

Definition 1.61. Consider an $\mathbb{F}$-semimartingale X. Then, X is said to admit an $\mathbb{F}$-supermartingale deflator if there exists a strictly positive $\mathbb{F}$-supermartingale Y such that $Y(1 + H \centerdot X)$ is a supermartingale for any $H \in L(X, \mathbb{F})$ such that $H \centerdot X \geq -1$.

For supermartingale deflators, we refer the reader to Rokhlin [Rok10]. Again, the above definition differs from that of the literature when the horizon is infinite, while it is the same as the one of the literature when the horizon is finite (even random). Below, we slightly generalize [Rok10] to our context.

Lemma 1.62. *Let X be an $\mathbb{F}$-semimartingale. Then, the following assertions are equivalent.*
(a) X admits an $\mathbb{F}$-supermartingale deflator.
(b) X satisfies NUPBR$(\mathbb{F})$.

Proof. The proof of this lemma is straightforward, and is omitted. ∎

Chapter 2

Thin random times

2.1 Introduction

This chapter is based on a joint paper with Monique Jeanblanc and Tahir Choulli [ACJ14b].

Motivated by arbitrage questions from [CADJ14] we develop the decomposition of an arbitrary random time into thin and strict parts. This decomposition can be seen as analogous to stopping time decomposition. It is based on dual optional projection of a jump process associated with random time. Furthermore, we observe that thin random times appear naturally in various contexts linked to enlargement of filtration literature.

2.2 Decomposition of a random time

We work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ denotes a filtration satisfying the usual conditions such that $\mathcal{F}_\infty \subset \mathcal{G}$. We consider a random time τ and the associated progressively and initially enlarged filtrations $\mathbb{F}^\tau$ and $\mathbb{F}^{\sigma(\tau)}$ respectively. For the process $A = \mathbb{1}_{[\![\tau,\infty[\![}$, we denote by A^p its $\mathbb{F}$-dual predictable projection and by A^o its $\mathbb{F}$-dual optional projection. The supermartingales Z and $\widetilde{Z}$ are associated with τ through (1.10) and (1.11). By the abuse of language, A^o is also called the dual optional projection of τ. As the dual optional projection will play a crucial role in this chapter, we recall two equalities where it appears. By (1.12) and (1.13) we have

$$\boxed{A^o = m - Z \quad \text{and} \quad \Delta A^o = \widetilde{Z} - Z\,.}$$

2.2.1 Definition and first properties

Taking the assumption (A) (in Definition 1.37) as a starting point and motivation we define two classes of random times.

Definition 2.1. A random time τ is called
(a) a strict random time if $[\![\tau]\!] \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-stopping time T,
(b) a thin random time if its graph $[\![\tau]\!]$ is contained in a thin set, i.e., if there exists a sequence of $\mathbb{F}$-stopping times $(T_n)_{n=1}^{\infty}$ with disjoint graphs such that $[\![\tau]\!] \subset \bigcup_n [\![T_n]\!]$. We say that such a sequence $(T_n)_n$ exhausts the thin random time τ or that $(T_n)_n$ is an exhausting sequence of the thin random time τ.

A strict random time coincides with a kind of random time often considered in the literature, namely it is a random time which avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = T < \infty) = 0$ for any $\mathbb{F}$-stopping time T.
On the other hand, a thin random time τ is completely built with $\mathbb{F}$-stopping times, i.e., $\tau = \mathbb{1}_{C_0}\infty + \sum_n \mathbb{1}_{C_n} T_n$ where $(T_n)_n$ is an exhausting sequence for τ and

$$C_0 = \{\tau = \infty\} \quad \text{and} \quad C_n = \{\tau = T_n < \infty\} \quad \text{for} \quad n \geq 1. \tag{2.1}$$

We denote by z^n the $\mathbb{F}$-martingale with terminal value $\mathbb{P}(C_n|\mathcal{F}_\infty)$ namely

$$z_t^n = \mathbb{P}(C_n|\mathcal{F}_t). \tag{2.2}$$

Note that for a thin random time an exhausting sequence $(T_n)_n$ is not unique. The notion of thin random time is already mentioned in [DM78] under the name *variable aléatoire arlequine*.

As a simple observation, we state in the next lemma that those two classes of random times have trivial intersection.

Lemma 2.2. *A random time τ belongs to the class of strict random times and to the class of thin random times if and only if $\tau = \infty$.*

In the following definition, we introduce the main concept of this section. Note that τ is not necessarily $\mathcal{F}_\infty$-measurable.

Definition 2.3. Consider a random time τ. The pair of random times (τ_1, τ_2) is called the $(*)$-decomposition of τ if τ_1 is a strict random time, τ_2 is a thin random time, and

$$\tau = \tau_1 \wedge \tau_2 \qquad \tau_1 \vee \tau_2 = \infty.$$

In the next theorem we state that such a decomposition can be found for any random time. The decomposition of a random time into strict and thin parts is congruent with the decomposition of a stopping time into totally inaccessible and accessible parts.

Theorem 2.4. *Each random time τ has a $(*)$-decomposition (τ_1, τ_2).*

Proof. It is enough to take τ_1 and τ_2 of the following form

$$\tau_1 = \tau_{\{\Delta A^o_\tau = 0\}} \quad \text{and} \quad \tau_2 = \tau_{\{\Delta A^o_\tau > 0\}},$$

where τ_C is a restriction of a random time τ to the set C defined in (1.1). Properties of dual optional projection ensure that τ_1 and τ_2 satisfy the required conditions. Namely, the time τ_1 is a strict random time as

$$\begin{aligned}\mathbb{P}(\tau_1 = T < \infty) &= \mathbb{E}(\mathbb{1}_{\{\tau=T\}}\mathbb{1}_{\{\Delta A^o_\tau=0\}}\mathbb{1}_{(T<\infty)}) \\ &= \mathbb{E}(\int_0^\infty \mathbb{1}_{\{u=T\}}\mathbb{1}_{\{\Delta A^o_u=0\}}dA^o_u) = 0.\end{aligned}$$

and the time τ_2 is a thin random time as

$$[\![\tau_2]\!] = [\![\tau]\!] \cap \{\Delta A^o > 0\} = [\![\tau]\!] \cap \bigcup_n [\![T_n]\!] \subset \bigcup_n [\![T_n]\!],$$

where the sequence $(T_n)_n$ exhausts the jumps of the càdlàg process A^o, i.e., $\{\Delta A^o > 0\} = \bigcup_n [\![T_n]\!]$. ■

Corollary 2.5. (a) A random time is a strict random time if and only if its dual optional projection is a continuous process.
(b) A random time is a thin random time if and only if its dual optional projection is a pure jump process.

Proof. Let τ be a random time and A^o the dual optional projection of τ.
(a) Take an $\mathbb{F}$-stopping time T. Since $\mathbb{E}(\Delta A^o_T \mathbb{1}_{\{T<\infty\}}) = \mathbb{P}(\tau = T < \infty)$ and A^o is an increasing process we get that $\mathbb{P}(\tau = T < \infty) = 0$ if and only if $\Delta A^o_T = 0$. By Section theorem 1.10, we conclude that τ is a strict random time if and only if A^o is continuous.
(b) For $(T_n)_n$ a sequence of $\mathbb{F}$-stopping times with disjoint graphs, we have

$$\sum_n \mathbb{P}(\tau = T_n < \infty) = \sum_n \mathbb{E}(\Delta A^o_{T_n}\mathbb{1}_{\{T_n<\infty\}}).$$

Since $\mathbb{E}(A^o_\infty \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{P}(\tau < \infty)$ and A^o is an increasing process, we conclude that the sequence $(T_n)_n$ satisfies the condition $\sum_n \mathbb{P}(\tau = T_n < \infty) = \mathbb{P}(\tau < \infty)$ if and only if it satisfies the condition $\mathbb{E}(A^o_\infty \mathbb{1}_{\{\tau<\infty\}}) = \sum_n \mathbb{E}(\Delta A^o_{T_n}\mathbb{1}_{\{T_n<\infty\}})$. In other words, τ is a thin random time if and only if A^o is a pure jump process. ■

For $i \in \{1, 2\}$, corresponding to the two $(*)$-parts of a random time, i.e., τ_1 and τ_2, we define $A^i := \mathbb{1}_{[\![\tau_i,\infty[\![}$. Then $A^{i,p}$ and $A^{i,o}$ are respectively the $\mathbb{F}$-dual predictable projection and the $\mathbb{F}$-dual optional projection of A^i. Let us denote by Z^i and $\widetilde{Z}^i$ supermartingales associated with τ_i. Then, the following relations hold.

Lemma 2.6. *Let τ be a random time and (τ_1, τ_2) its $(*)$-decomposition. Then, the supermartingales Z and $\widetilde{Z}$ can be decomposed in terms of the supermartingales Z^1, Z^2 and $\widetilde{Z}^1$, $\widetilde{Z}^2$ as:*

$$Z = Z^1 + Z^2 - 1 \quad \textit{and} \quad \widetilde{Z} = \widetilde{Z}^1 + \widetilde{Z}^2 - 1.$$

Proof. The result follows from the property that $\tau_1 \vee \tau_2 = \infty$. ■

Lemma 2.7. *Let τ be a thin random time and $(z^n)_{n\geq 1}$ the family of $\mathbb{F}$-martingales associated with τ through* (2.2)*. Then*
(a) $z^n > 0$ and $z^n_- > 0$ a.s. on C_n for each n,
(b) $1 - Z_\tau > 0$ a.s. on $\{\tau < \infty\}$.

Proof. (a) Define the $\mathbb{F}$-stopping time $S_0^n = \inf\{t \geq 0 : z_t^n = 0\}$. As z^n is a positive càdlàg martingale, by [RY99, Proposition (3.4) p.70], it vanishes on $[\![S_0^n, \infty[\![$. Then:

$$\{S_0^n < \infty\} = \{\inf_t z_t^n = 0\} = \{z_\infty^n = 0\}.$$

Moreover, the equality

$$0 = \mathbb{E}(z_\infty^n 1\!\!1_{\{z_\infty^n = 0\}}) = \mathbb{E}(1\!\!1_{C_n} 1\!\!1_{\{z_\infty^n = 0\}})$$

implies that $C_n \cap \{z_\infty^n = 0\} = \emptyset$, so as well $C_n \cap \{\inf_t z_t^n = 0\} = \emptyset$. We obtain that $z^n > 0$ and $z_-^n > 0$ a.s. on C_n.
(b) We have $Z_\tau = \sum_n 1\!\!1_{C_n} Z_{T_n}$ and

$$1 - Z_{T_n} = \mathbb{P}(\tau \leq T_n | \mathcal{F}_{T_n}) \geq \mathbb{P}(\tau = T_n | \mathcal{F}_{T_n}) = z_{T_n}^n.$$

This implies that $1 - Z_\tau > 0$ a.s. ■

The next result provides the supermartingales Z and $\widetilde{Z}$ of a thin random time and its decompositions (1.12) in terms of its exhausting sequence and the associated martingales defined in (2.2). We omit the proof as it is straightforward.

Lemma 2.8. *Let τ be a thin random time with exhausting sequence $(T_n)_{n \geq 1}$. Then,*

$$\boxed{\widetilde{Z}_t = \sum_n 1\!\!1_{\{t \leq T_n\}} z_t^n, \quad Z_t = \sum_n 1\!\!1_{\{t < T_n\}} z_t^n, \quad A_t^o = \sum_n 1\!\!1_{\{t \geq T_n\}} z_{T_n}^n, \quad m_t = \sum_n z_{t \wedge T_n}^n.}$$

We finish this subsection with a remark on the $(*)$-decomposition of a random time τ as an $\mathbb{F}^\tau$-stopping time.

Remark 2.9. We can also decompose the random time τ_2 into two parts. Then, we consider a decomposition of τ onto three parts as:

$$\tau_1 = \tau_{\{\Delta A_\tau^o = 0\}}, \quad \tau_2^i = \tau_{\{\Delta A_\tau^o > 0,\ \Delta A_\tau^p = 0\}} \quad \text{and} \quad \tau_2^a = \tau_{\{\Delta A_\tau^o > 0,\ \Delta A_\tau^p > 0\}}.$$

Then $\tau_1 \wedge \tau_2^i$ is an $\mathbb{F}^\tau$-totally inaccessible part and τ_2^a is an $\mathbb{F}^\tau$-accessible part of the $\mathbb{F}^\tau$-stopping time τ. These types of results are already shown in [Jeu80, p.65] and [Coc09]. We note that τ is an $\mathbb{F}^\tau$-predictable stopping time if and only if τ is an $\mathbb{F}$-predictable stopping time. Moreover, a filtration $\mathbb{F}^\tau$ such that τ_2^a the accessible thin part of τ is not an $\mathbb{F}$-stopping time is not quasi-left continuous. The last observation provides a systemic way to construct examples of non quasi-left continuous filtrations.

2.2.2 The hypothesis $(\mathcal{H}')$

In this part we focus on the hypothesis $(\mathcal{H}')$ and the $(*)$-decomposition of a random time. First, in section 2.2.2.1, we examine the case of thin random times. Then, in section 2.2.2.2, we work with general random times.

2.2.2.1 Thin random time case

In establishing the following theorem on hypothesis $(\mathcal{H}')$ for thin random times, Theorem 1.30 plays a crucial role. Let $\mathbb{F}^{\mathcal{C}}$ denote the initial enlargement of filtration $\mathbb{F}$ with atomic σ-field

$$\mathcal{C} := \sigma(C_n, n \geq 0)$$

with C_n defined in (2.1), i.e.,

$$\mathcal{F}_t^{\mathcal{C}} := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(C_n, n \geq 0).$$

Moreover consider filtrations $\mathbb{F}^\tau$ and $\mathbb{F}^{\sigma(\tau)}$ defined as, respectively, progressive and initial enlargement of filtration $\mathbb{F}$ with random time τ, i.e.,

$$\mathcal{F}_t^\tau := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s) \quad \text{and} \quad \mathcal{F}_t^{\sigma(\tau)} := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau).$$

Then, we have the following sequence of filtrations inclusions:

$$\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{F}^{\mathcal{C}} \subset \mathbb{F}^{\sigma(\tau)}.$$

For the next result, particularly important is that $\mathbb{F}^\tau \subset \mathbb{F}^{\mathcal{C}}$.

Theorem 2.10. *Let τ be a thin random time. Then the hypothesis $(\mathcal{H}')$ is satisfied between $\mathbb{F}$ and $\mathbb{F}^\tau$. Moreover, each $\mathbb{F}$-local martingale X can be decomposed in $\mathbb{F}^\tau$ as*

$$\boxed{X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s + \sum_n \mathbb{1}_{C_n} \int_0^t \mathbb{1}_{\{s>T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n\rangle_s}$$

where $\widehat{X}$ is an $\mathbb{F}^\tau$-local martingale.

Proof. We give an analogical proof as in [Jeu80, Lemma (4,11)]. The first part remains the same (argument about the hypothesis $(\mathcal{H}')$). In order to find the decomposition, we consider an $\mathbb{F}^\tau$-predictable process H. Then, [Jeu80, Lemma (4,4)] implies that

$$H_t = \mathbb{1}_{\{t\leq\tau\}} J_t + \mathbb{1}_{\{\tau<t\}} K_t(\tau) \quad t > 0$$

where J is an $\mathbb{F}$-predictable process and $K : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable. As τ is thin, we can rewrite the $\mathbb{F}^\tau$-predictable process H as

$$H_t = J_t \mathbb{1}_{\{t\leq\tau\}} + \sum_n \mathbb{1}_{\{T_n<t\}} K_t(T_n) \mathbb{1}_{C_n}$$

with $C_n = \{\tau = T_n\}$. Note that each process $\mathbb{1}_{\{T_n<t\}} K_t(T_n)$ is $\mathbb{F}$-predictable. We continue as in the original proof. ∎

We may see that the progressive enlargement of filtration with a thin random time as an alternative generalization of Theorem 1.30 to generalization based on density hypothesis in progressive setting (initial times from [JL09]).

We finish this section with a result linking processes in $\mathbb{F}^\tau$ and $\mathbb{F}^{\mathcal{C}}$. It can be seen as an alternative aproach to show the decomposition in Theorem 2.10 using Theorem 1.30. It is as well related to the aproach in [CJZ13].

Proposition 2.11. *Let X be a process such that $X = \mathbb{1}_{]\!]\tau,\infty[\![} \centerdot X$. Then*
(a) The process X is an $\mathbb{F}^{\mathcal{C}}$-(super-, sub-) martingale if and only if the process X is an $\mathbb{F}^{\tau}$-(super-, sub-) martingale.
(b) Let ϑ be an $\mathbb{F}^{\mathcal{C}}$-stopping time. Then $\vartheta \vee \tau$ is an $\mathbb{F}^{\tau}$-stopping time.
(c) The process X is an $\mathbb{F}^{\mathcal{C}}$-local martingale if and only if the process X is an $\mathbb{F}^{\tau}$-local martingale.

Proof. (a) Note that the filtrations $\mathbb{F}^{\tau}$ and $\mathbb{F}^{\mathcal{C}}$ are equal after τ, i.e., for each t and for each set $G \in \mathcal{F}_t^{\mathcal{C}}$ there exists a set $F \in \mathcal{F}_t^{\tau}$ such that

$$\{\tau \leq t\} \cap G = \{\tau \leq t\} \cap F. \tag{2.3}$$

To show (2.3), by Monotone Class Theorem 1.1, it is enough to consider $G = C_n$ and to take $F = C_n \cap \{\tau \leq t\}$ which belongs to $\mathcal{F}_t^{\tau}$ as $C_n \in \mathcal{F}_{\tau}^{\tau}$ by [HWY92, Corollary 3.5]. That implies that the process $\mathbb{1}_{]\!]\tau,\infty[\![} \centerdot X$ is $\mathbb{F}^{\tau}$-adapted if and only if it is $\mathbb{F}^{\mathcal{C}}$-adapted. Equivalence of (super-, sub-) martingale property comes from (2.3).
(b) For each t we have $\{\vartheta \vee \tau \leq t\} = \{\vartheta \leq t\} \cap \{\tau \leq t\} \in \mathcal{F}_t^{\tau}$ by (2.3).
(c) We combine two previous points. ∎

2.2.2.2 General random time case

In this section we work with (τ_1, τ_2) the $(*)$-decomposition of random time τ. We define the following enlarged filtrations $\mathbb{F}^{\tau_1}$, $\mathbb{F}^{\tau_2}$ and $\mathbb{F}^{\tau_1,\tau_2}$ as

$$\mathcal{F}_t^{\tau_i} := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau_i \wedge s) \quad \text{for} \quad i = 1, 2$$
$$\mathcal{F}_t^{\tau_1,\tau_2} := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau_1 \wedge s) \vee \sigma(\tau_2 \wedge s).$$

Lemma 2.12. *Let τ be a random time and (τ_1, τ_2) its $(*)$-decomposition. Then, the hypothesis $(\mathcal{H}')$ is satisfied for $\mathbb{F} \subset \mathbb{F}^{\tau}$ if and only if the hypothesis $(\mathcal{H}')$ is satisfied for $\mathbb{F} \subset \mathbb{F}^{\tau_1}$.*

Proof. First we show the following inclusions of filtrations for $i = 1, 2$:

$$\mathbb{F} \subset \mathbb{F}^{\tau_i} \subset \mathbb{F}^{\tau_1,\tau_2} = \mathbb{F}^{\tau}.$$

1) To show $\mathbb{F}^{\tau_1} \subset \mathbb{F}^{\tau}$, take B in the set of generators of $\mathcal{F}_t^{\tau_1}$. If $B \in \mathcal{F}_t$, then obviously $B \in \mathcal{F}_t^{\tau}$. If $B = \{\tau_1 \leq s\}$ with $s \leq t$, then

$$B = \{\tau \leq s\} \cap \{\Delta A_{\tau}^{o} = 0\} = \bigcap_n \{\tau \leq s\} \cap \{\tau \neq T_n\} \in \mathcal{F}_s^{\tau}, \text{ and } B \in \mathcal{F}_t^{\tau} \text{ since } \mathcal{F}_s^{\tau} \subset \mathcal{F}_t^{\tau}.$$

2) To show $\mathbb{F}^{\tau_2} \subset \mathbb{F}^{\tau}$, take B in the set of generators of $\mathcal{F}_t^{\tau_2}$. If $B \in \mathcal{F}_t$, then obviously $B \in \mathcal{F}_t^{\tau}$. If $B = \{\tau_2 \leq s\}$ with $s \leq t$, then

$$B = \{\tau \leq s\} \cap \{\Delta A_{\tau}^{o} > 0\} = \bigcup_n \{\tau \leq s\} \cap \{\tau = T_n\} \in \mathcal{F}_s^{\tau}, \text{ and } B \in \mathcal{F}_t^{\tau} \text{ since } \mathcal{F}_s^{\tau} \subset \mathcal{F}_t^{\tau}.$$

3) $\mathbb{F}^{\tau_1,\tau_2} \subset \mathbb{F}^{\tau}$ is due to the two previous points.
4) To show $\mathbb{F}^{\tau} \subset \mathbb{F}^{\tau_1,\tau_2}$, take B in the set of generators of $\mathcal{F}_t^{\tau}$. If $B \in \mathcal{F}_t$, then obviously $B \in \mathcal{F}_t^{\tau_1,\tau_2}$. If $B = \{\tau \leq s\}$ with $s \leq t$, then

$$B = \{\tau_1 \leq s\} \cup \{\tau_2 \leq s\} \in \mathcal{F}_s^{\tau_1,\tau_2} \subset \mathcal{F}_t^{\tau_1,\tau_2}.$$

Then, the necessary condition comes from Stricker's Theorem 1.28 and the sufficient condition comes from Lemma 2.10 for the thin random time τ_2. ∎

In the next lemma we see that τ_1 and τ_2 are in some sense orthogonal (in terms of semimartingale decomposition and associated supermartingales, which is due to $\tau_1 \vee \tau_2 = \infty$).

Lemma 2.13. *The $\mathbb{F}$-supermartingale Z^2 of a thin random time τ_2 coincides with the $\mathbb{F}^{\tau_1}$-supermartingale $Z^{2,\mathbb{F}^{\tau_1}}$ of τ_2, i.e., $\mathbb{P}(\tau_2 > t|\mathcal{F}_t) = \mathbb{P}(\tau_2 > t|\mathcal{F}_t^{\tau_1})$.*

Proof. Let T be an $\mathbb{F}$-stopping time. For each $A \in \mathcal{F}_t^{\tau_1}$, there exists $B \in \mathcal{F}_t$ such that $A \cap \{\tau = T\} = B \cap \{\tau = T\}$, so $\mathbb{P}(\tau = T|\mathcal{F}_t) = \mathbb{P}(\tau = T|\mathcal{F}_t^{\tau_1})$ which ends the proof. ∎

2.3 Thin honest times

In this part we restrict our attention to a special class of random times, namely to honest times (see Definition 1.41).

2.3.1 Fundamental properties

Lemma 2.14. *Let τ be an honest time and denote by (τ_1, τ_2) its $(*)$-decomposition. Then, the times τ_1 and τ_2 are honest times.*

Proof. On the set $\{\tau < \infty\}$, τ is equal to γ, the end of the optional set Γ (Theorem 1.43). Then, as $\{\tau_1 < \infty\} \subset \{\tau < \infty\}$, on the set $\{\tau_1 < \infty\}$, one has $\tau_1 = \gamma$, so τ_1 is an honest time. Same argument for τ_2. ∎

We also give a simple characterisation of honest times avoiding $\mathbb{F}$- stopping times.

Lemma 2.15. *A random time τ is an honest time and avoids $\mathbb{F}$-stopping times if and only if $Z_\tau = 1$ a.s. on $\{\tau < \infty\}$.*

Proof. Assume that τ is an honest time avoiding $\mathbb{F}$-stopping times. Honesty, by Theorem 1.43, implies that $\widetilde{Z}_\tau = 1$ and the avoiding property implies the continuity of A^o since for each $\mathbb{F}$-stopping time T, $\mathbb{E}(\Delta A_T^o) = \mathbb{P}(\tau = T < \infty) = 0$. Then, the relation $\tilde{Z} = Z + \Delta A^o$ leads to the result.
Assume now that $Z_\tau = 1$ on the set $\{\tau < \infty\}$. Then, on $\{\tau < \infty\}$ we have $1 = Z_\tau \leq \widetilde{Z}_\tau \leq 1$, so $\widetilde{Z}_\tau = 1$ and τ is an honest time. Furthermore, as $\Delta A_\tau^o = \widetilde{Z}_\tau - Z_\tau = 0$, for each $\mathbb{F}$-stopping time T we have

$$\begin{aligned}
\mathbb{P}(\tau = T < \infty) &= \mathbb{E}(\mathbb{1}_{\{\tau = T\}}\mathbb{1}_{\{\Delta A_\tau^o = 0\}}\mathbb{1}_{(T<\infty)}) \\
&= \mathbb{E}(\int_0^\infty \mathbb{1}_{\{u=T\}}\mathbb{1}_{\{\Delta A_u^o = 0\}} dA_u^o) = 0.
\end{aligned}$$

So τ avoids $\mathbb{F}$-stopping times. ∎

Lemma 2.16. *Let τ be an honest time with $(*)$-decomposition (τ_1, τ_2). Then, $Z_\tau = 1$ on $\{\tau = \tau_1 < \infty\}$ and $Z_\tau < 1$ on $\{\tau = \tau_2 < \infty\}$.*

Proof. From the honest time property of τ and Lemma 2.6, on the set $\{\tau < \infty\}$

$$1 = \widetilde{Z}_\tau = \widetilde{Z}^1_\tau + \widetilde{Z}^2_\tau - 1.$$

On the set $\{\tau = \tau_1 < \infty\}$, we have

$$1 = \widetilde{Z}^1_{\tau_1} + \widetilde{Z}^2_{\tau_1} - 1 = \widetilde{Z}^2_{\tau_1},$$

where the last equality comes from Lemma 2.15. Now let us compute $Z^2_{\tau_1}$

$$Z^2_{\tau_1} = \widetilde{Z}^2_{\tau_1} - \Delta A^{2,o}_{\tau_1} = \widetilde{Z}^2_{\tau_1} = 1,$$

where we use the strict random time property of τ_1, i.e., $\{\Delta A^{2\,o} > 0\} = \bigcup_n [\![T_n]\!]$ (with (T_n) being an exhausting sequence of τ_2) and $\mathbb{P}(\tau_1 = T_n < \infty) = 0$. Finally, on $\{\tau = \tau_1 < \infty\}$

$$Z_\tau = Z^1_{\tau_1} + Z^2_{\tau_1} - 1 = 1.$$

On the set $\{\tau = \tau_2 < \infty\}$,

$$Z_\tau = Z^1_{\tau_2} + Z^2_{\tau_2} - 1 \le Z^2_{\tau_2} < 1,$$

where the last inequality is due to Lemma 2.7. ∎

Lemma 2.17. *Let τ be a thin honest time and τ_t be associated with τ as in Definition 1.41. Then, for each n:*
(a) on $\{T_n = \tau_t\} = \{T_n = \tau_t \le t\}$ we have $z^n_t = 1 - Z_t$, $A^o_t = z^n_{T_n}$ and $1 - m_t = z^n_t - z^n_{T_n}$;
(b) on $\{T_n < t\}$ we have $z^n_t = 1\!\!1_{\{\tau_t = T_n\}}(1 - \widetilde{Z}_t)$ and $z^n_{t-} = 1\!\!1_{\{\tau_t = T_n\}}(1 - Z_{t-})$; in particular

$$1 - \widetilde{Z}_t = \sum_n 1\!\!1_{\{\tau_t = T_n < t\}}(1 - \widetilde{Z}_t) \quad \textit{and} \quad 1 - Z_{t-} = \sum_n 1\!\!1_{\{\tau_t = T_n < t\}}(1 - Z_{t-}).$$

Proof. (a) Using properties of τ_t we have

$$\begin{aligned} 1\!\!1_{\{T_n = \tau_t\}} z^n_t &= \mathbb{P}(T_n = \tau_t \le t, \tau = T_n | \mathcal{F}_t) = \mathbb{P}(\tau \le t, T_n = \tau_t = \tau | \mathcal{F}_t) \\ &= \mathbb{P}(\tau \le t, T_n = \tau_t | \mathcal{F}_t) = 1\!\!1_{\{T_n = \tau_t\}}(1 - Z_t). \end{aligned}$$

The dual optional projection of a thin random time equals

$$1\!\!1_{\{T_n = \tau_t\}} A^o_t = \sum_k 1\!\!1_{\{T_n = \tau_t, T_k \le t\}} z^k_{T_k} = 1\!\!1_{\{T_n = \tau_t\}} z^n_{T_n},$$

where the second equality is due to the facts that $(T_n)_n$ have disjoint graphs and that $\tau_t \le t$ a.s. Combining the two previous points, we conclude that $1 - m_t = 1 - Z_t - A^o_t = z^n_t - z^n_{T_n}$ on the set $\{T_n = \tau_t\}$.
(b) Again using properties of random variable τ_t we get

$$\begin{aligned} 1\!\!1_{\{T_n < t\}} z^n_t &= \mathbb{P}(\tau = T_n = \tau_t < t | \mathcal{F}_t) = 1\!\!1_{\{T_n = \tau_t < t\}}(1 - \widetilde{Z}_t), \\ 1\!\!1_{\{T_n < t\}} z^n_{t-} &= \mathbb{P}(\tau = T_n = \tau_t < t | \mathcal{F}_{t-}) = 1\!\!1_{\{T_n = \tau_t < t\}}(1 - Z_{t-}). \end{aligned}$$

Then, Lemma 2.8 completes the proof. ∎

Remark 2.18. For a thin honest time τ, the two following decomposition formulas coincide

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}} + \sum_n 1\!\!1_{C_n} \int_0^t 1\!\!1_{\{s>T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n\rangle_s^{\mathbb{F}},$$

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}} + \int_0^t 1\!\!1_{\{s>\tau\}} \frac{1}{1-Z_{s-}} d\langle X, 1-m\rangle_s^{\mathbb{F}}$$

where the first one comes from Theorem 2.10 and the second one comes from Theorem 1.45.

Proof. It is enough to show that

$$\int_0^t 1\!\!1_{\{s>\tau\}} \frac{1}{1-Z_{s-}} d\langle X, 1-m\rangle_s^{\mathbb{F}} = \sum_n 1\!\!1_{C_n} \int_0^t 1\!\!1_{\{s>T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n\rangle_s^{\mathbb{F}}.$$

This is a simple consequence of the set inclusion $\{\tau < s\} \cap \{\tau = T_n\} \subset \{T_n = \tau_s \leq s\}$ and Lemma 2.17 (a):

$$\begin{aligned}
\int_0^t 1\!\!1_{\{s>\tau\}} \frac{1}{1-Z_{s-}} d\langle X, 1-m\rangle_s^{\mathbb{F}} &= \sum_n \int_0^t 1\!\!1_{\{s>\tau\}\cap\{\tau=T_n\}} \frac{1}{1-Z_{s-}} d\langle X, 1-m\rangle_s^{\mathbb{F}} \\
&= \sum_n \int_0^t 1\!\!1_{\{s>\tau\}\cap\{\tau=T_n\}} \frac{1}{z_{s-}^n} d\langle X, z_s^n - z_{T_n}^n\rangle_s^{\mathbb{F}} \\
&= \sum_n 1\!\!1_{C_n} \int_0^t 1\!\!1_{\{s>T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n\rangle_s^{\mathbb{F}}.
\end{aligned}$$

■

2.3.2 Relation with jumping filtration

In this section we develop relation between jumping filtration and thin honest times. Let us first recall the definition of a jumping filtration and the main result obtained in [JS94].

Definition 2.19. A filtration $\mathbb{F}$ is called a jumping filtration if there exists a localizing sequence $(\theta_n)_n$, i.e., a sequence of stopping times increasing a.s. to ∞, with $\theta_0 = 0$ and such that for all n and $t > 0$ the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{\theta_n}$ coincide up to null sets on $\{\theta_n \leq t < \theta_{n+1}\}$. The sequence $(\theta_n)_n$ is then called a jumping sequence.

There exists an important alternative characterization of jumping filtration in terms of martingale's variation ([JS94, Theorem 1]).

Theorem 2.20. *The two following conditions are equivalent:*
(a) a filtration $\mathbb{F}$ is a jumping filtration;
(b) all martingales in the filtration $\mathbb{F}$ are a.s. of locally finite variation.

In the remaining part of this section we investigate relation between jumping filtration and the condition stating that ***All honest times in the filtration*** $\mathbb{F}$ ***are thin honest times***.

We start with showing that there does not exist strict honest time in a jumping filtration.

Proposition 2.21. *If $\mathbb{F}$ is a jumping filtration then all $\mathbb{F}$-honest times are thin.*

Proof. Let τ be an honest time. Then, take the same process α as in the proof of Lemma 1.42, i.e., α is an increasing, càdlàg, adapted process such that $\alpha_t = \tau$ on $\{\tau \leq t\}$ and $\tau = \sup\{t : \alpha_t = t\}$. Let us define the partition $(C_n)_{n=0}^{\infty}$ such that

$$C_n = \{\theta_{n-1} \leq \tau < \theta_n\}$$

for $n \geq 1$ and $C_0 = \{\tau = \infty\}$ with $(\theta_n)_n$ being a jumping sequence for the jumping filtration $\mathbb{F}$. On each C_n with $n \geq 1$ we have

$$\tau = T_n := \inf\{t \geq \theta_{n-1} : t = \alpha_{\theta_n -}\}.$$

From the jumping filtration property, we know that $\alpha_{\theta_n -}$ is $\mathcal{F}_{\theta_{n-1}}$ measurable so each T_n is a stopping time and $[\![\tau]\!] \subset \bigcup_{n=1}^{\infty}[\![T_n]\!]$ which shows that the honest time τ is a thin random time. ∎

In the next proposition we focus on condition (b) in Theorem 2.20. As a first step we restrict our attention to continuous martingales which are not constant (i.e., which have infinite variation). The connection with the studied condition that *All honest times in the filtration $\mathbb{F}$ are thin honest times* is announced.

Proposition 2.22. *Let M be a continuous $\mathbb{F}$-local martingale with $M_0 = 0$. Define the $\mathbb{F}$-stopping time $S_1 = \inf\{t > 0 : \langle M\rangle_t = 1\}$. Then, the $\mathbb{F}$-honest time*

$$\tau := \sup\{t \leq S_1 : M_t = 0\}$$

is a strict honest time.

Proof. This follows from [RY99, Exercise (1.26) p.235].
Let us denote by $\mathcal{Z}(\omega) := \{t : M_t(\omega) = 0\}$. The set $\mathcal{Z}(\omega)$ is closed and $\mathcal{Z}^c(\omega)$ is the union of countably many open intervals. We call $G(\omega)$ the set of left ends of these open intervals. In what follows we show that for any $\mathbb{F}$-stopping time T we have $\mathbb{P}(T \in G) = 0$. Define the $\mathbb{F}$-stopping time

$$D_T := \inf\{t > T : M_t = 0\}$$

and note that

$$\{T \in G\} = \{M_T = 0\} \cap \{T < D_T\} \in \mathcal{F}_T.$$

Assume $\mathbb{P}(T \in G) = p > 0$. Consider the following process

$$Y_t = \mathbb{1}_{\{T\in G\}}|M_{T+t}|\mathbb{1}_{\{0\leq t\leq D_T - T\}}.$$

We check it is an $(\mathcal{F}_{T+t})_{t\geq 0}$-martingale, for $s \leq t$ we have

$$\begin{aligned}
&\mathbb{E}(Y_t|\mathcal{F}_{T+s}) = \mathbb{1}_{\{T\in G\}}\mathrm{sgn}(M_{T+t})\mathbb{E}(M_{T+t}\mathbb{1}_{\{t\leq D_T-T\}}|\mathcal{F}_{T+s})\\
&=\mathbb{1}_{\{T\in G\}}\mathrm{sgn}(M_{T+t})\left(M_{T+s}\mathbb{1}_{\{s\leq D_T-T\}} - \mathbb{E}(\mathbb{1}_{\{s\leq D_T-T\}}\mathbb{1}_{\{t>D_T-T\}}\mathbb{E}(M_{T+t}|\mathcal{F}_{D_T})|\mathcal{F}_{T+s})\right)\\
&=Y_s - \mathbb{1}_{\{T\in G\}}\mathrm{sgn}(M_{T+t})\mathbb{E}(\mathbb{1}_{\{s\leq D_T-T\}}\mathbb{1}_{\{t>D_T-T\}}M_{D_T}|\mathcal{F}_{T+s})\\
&=Y_s
\end{aligned}$$

where we used martingale property of M and $M_{D_T} = 0$. Moreover $Y_0 = 0$ and there exists $\varepsilon > 0$ such that
$$\mathbb{P}(M_T = 0, D_T - T > \varepsilon) \geq \frac{p}{2} > 0.$$
Since $Y_\varepsilon = 1\!\!1_{\{M_T=0\}} 1\!\!1_{\{D_T - T \geq \varepsilon\}} |M_{T+\varepsilon}| \geq 0$ and $\mathbb{P}(Y_\varepsilon > 0) > 0$, we have $\mathbb{E}(Y_\varepsilon) > 0 = Y_0$. So, $\mathbb{P}(T \in G) = 0$. Finally, as $\tau \in G$ a.s. we conclude that τ is a strict honest time. ■

The proof of the previous proposition cannot be extended to the case of any infinite variation martingale M as on the interval $[T, D_T[$ the sign of the martingale M may change.

Nevertheless we give two examples of strict honest times associated with purely discontinuous semimartingales of infinite variation. In the first Example 2.23 the case of Azéma martingale is studied (see [Pro04, IV.8 p.232-237]). In the second Example 2.24 we recall the example 2.1 from [Kar14a] on "Maximum of downwards drifting spectrally negative Lévy processes with paths of infinite variation".

Example 2.23. Let B be a Brownian motion and $\mathbb{F} := \mathbb{F}^B$. Define the process
$$g_t := \sup\{s \leq t : B_s = 0\}.$$
The process
$$\mu_t := \operatorname{sgn}(B_t)\sqrt{t - g_t}$$
is a martingale with respect to the filtration $\mathbb{G} := (\mathcal{F}_{g_t +})_{t \geq 0}$ and is called the Azéma martingale. Then, the random time
$$\tau := \sup\{t \leq 1 : \mu_t = 0\}$$
is a $\mathbb{G}$-strict honest time. This is due to the fact that $\tau = \tau^B := \sup\{t \leq 1 : B_t = 0\}$ and τ^B is an $\mathbb{F}$-strict honest thus $\mathbb{G}$-strict honest time as well.

Example 2.24. Let X be a Lévy process with characteristic triplet $(\alpha, \sigma^2 = 0, \nu)$ satisfying $\nu(]0, \infty[) = 0$, $\alpha + \int_{-\infty}^{-1} x\nu(dx) < 0$ and $\int_{-1}^{0} |x|\nu(dx) = \infty$. Then, $\rho = \sup\{t : X_{t-} = X^*_{t-}\}$ with $X^*_t = \sup_{s \leq t} X_s$ is a strict honest time.

Problems related to purely discontinuous martingale filtrations are treated in [Han03].

2.3.3 Examples of thin honest times

2.3.3.1 Compound Poisson process: last passage time at a barrier a

Let us consider the filtration $\mathbb{F}$ generated by a Compound Poisson Process (CCP) X, defined as
$$X_t = \sum_{n=1}^{N_t} Y_n,$$
where N is a Poisson process with parameter η and sequence of jump times $(\theta_n)_{n=1}^{\infty}$, and where $(Y_n)_{n=1}^{\infty}$ are i.i.d. positive random variables, independent from N, with cumulative distribution function F.

In this section, we study several honest times in the filtration of X which are thin honest times, and are not stopping times. In a progressive enlargement framework, in order to study the $\mathbb{F}^\tau$-semimartingale decomposition of $\mathbb{F}$-martingales before τ, one needs to compute the martingale $m = Z + A^o$. Therefore, we shall present the computations of A^o (hence m) in all these examples.

Define a random time τ as

$$\tau := \sup\{t : \mu t - X_t \leq a\} \tag{2.4}$$

with $a > 0$ and a constant μ.

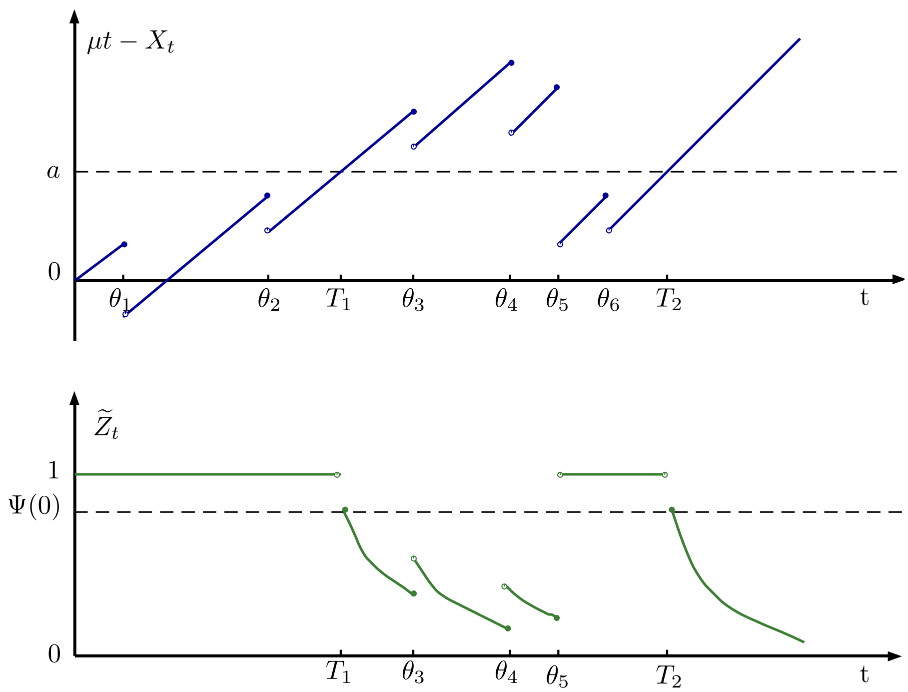

Figure 2.1: Higher line represents $\mu t - X_t$. $(T_n)_n$ is the exhausting sequence of τ, given in (2.5). Lower line represents the supermartingale $\widetilde{Z}$ associated to τ, given in (2.7). The supermartingale Z associated to τ, given in (2.6), is a right-limit of $\widetilde{Z}$.

From now on we assume that $\mu > \eta\,\mathbb{E}(Y_1)$. Under this condition, the random time τ is finite a.s.. Since τ is a last passage time as in Lemma 1.47, it is an honest time in the filtration $\mathbb{F}$. Furthermore, since the process $\mu t - X_t$ has only negative jumps, one has $\mu\tau - X_\tau = a$. This time τ does not avoid $\mathbb{F}$-stopping times as we shall see below.

Lemma 2.25. *The honest time τ is a thin random time with exhausting sequence $(T_n)_{n\geq 1}$ given by*

$$\begin{aligned} T_1 &:= \inf\{t > 0 : \mu t - X_t = a\} \quad \textit{and} \\ T_n &:= \inf\{t > T_{n-1} : \mu t - X_t = a\} \quad \textit{for} \quad n > 1. \end{aligned} \tag{2.5}$$

Each T_n is a predictable stopping time. Furthermore $Z_\tau < 1$.

Proof. The first part of the lemma is trivial. Sets $(C_n)_{n=0}^{\infty}$ with $C_n = \{\tau = T_n\}$ form a partition of Ω. Then, $\tau = \sum_{n=0}^{\infty} T_n \mathbb{1}_{C_n}$. Note that τ is not an $\mathbb{F}$-stopping time as $C_n \notin \mathcal{F}_{T_n}$ for any n.
To show that each T_n is predictable, let us define the stopping times J_d and J_u as

$$J_d = \inf\{t > 0 : \mu t - X_{t-} = a, \mu t - X_t < a\}$$
$$J_u = \inf\{t > 0 : \mu t - X_{t-} > a, \mu t - X_t = a\}.$$

First observe that $[\![J_d]\!] \subset \bigcup_n [\![\theta_n]\!]$ and $[\![J_u]\!] \subset \bigcup_n [\![\theta_n]\!]$. For each n, we have $\mathbb{P}(J_d = \theta_n) = 0$ as

$$\begin{aligned}\mathbb{P}(J_d = \theta_n) &= \mathbb{E}\left(\mathbb{P}\left(J_d = \theta_n | \mathcal{F}_{\theta_{n-1}}\right)\right) = \mathbb{E}\left(\mathbb{1}_{\{J_d > \theta_{n-1}\}} \mathbb{P}\left(\mu\theta_n - X_{\theta_{n-1}} = a | \mathcal{F}_{\theta_{n-1}}\right)\right) \\ &\leq \mathbb{E}\left(\mathbb{P}\left(\theta_n - \theta_{n-1} = \frac{a - \mu\theta_{n-1} + X_{\theta_{n-1}}}{\mu} | \mathcal{F}_{\theta_{n-1}}\right)\right) = 0,\end{aligned}$$

so we conclude that $J^d = \infty$ a.s. For each n, we have $\mathbb{P}(J_u = \theta_n) = 0$ as

$$\begin{aligned}\mathbb{P}(J_u = \theta_n) &= \mathbb{E}\left(\mathbb{P}\left(J_u = \theta_n | \mathcal{F}_{\theta_{n-1}} \vee \sigma(Y_n)\right)\right) = \mathbb{E}\left(\mathbb{1}_{\{J_u > \theta_{n-1}\}} \mathbb{P}\left(\mu\theta_n - X_{\theta_n} = a | \mathcal{F}_{\theta_{n-1}} \vee \sigma(Y_n)\right)\right) \\ &\leq \mathbb{E}\left(\mathbb{P}\left(\theta_n - \theta_{n-1} = \frac{a - \mu\theta_{n-1} + X_{\theta_{n-1}} + Y_n}{\mu} | \mathcal{F}_{\theta_{n-1}} \vee \sigma(Y_n)\right)\right) = 0,\end{aligned}$$

so we conclude that $J_u = \infty$ a.s. Now, for each $n \geq 1$, we simply define an announcing sequence $(T_{n,m})_{m \geq 1}$ for T_n as

$$T_{n,m} = \inf\{t > T_{n-1} : \mu t - X_t \geq a - \frac{1}{m}\}\}$$

with $T_0 = 0$. We see that $J_d = \infty$ and $J_u = \infty$ a.s. ensure that each sequence $(T_{n,m})_m$ is indeed an announcing sequence of T_n. ■

Let us remark that in fact the random time τ (defined in (2.4)) can be seen as the end of the optional set $\Gamma = \bigcup_n [\![T_n]\!]$ as

$$\tau(\omega) = \sup\{t : (\omega, t) \in \Gamma\}.$$

Proposition 2.26. *The supermartingales Z and $\widetilde{Z}$ associated with the honest time τ are given by*

$$Z_t = \Psi(\mu t - X_t - a)\mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}, \tag{2.6}$$

$$\widetilde{Z}_t = \Psi(\mu t - X_t - a)\mathbb{1}_{\{\mu t - X_t > a\}} + \mathbb{1}_{\{\mu t - X_t \leq a\}}, \tag{2.7}$$

where $\Psi(x)$ is the ruin probability associated with the process $\mu t - X_t$, i.e., for every $x \geq 0$

$$\Psi(x) := \mathbb{P}(t^x < \infty) \quad \text{with} \quad t^x := \inf\{t : x + \mu t - X_t < 0\}. \tag{2.8}$$

The function Ψ satisfies the following properties:
(i) for $x < 0$ we have $\Psi(x) = 1$;
(ii) the function Ψ is continuous and decreasing on $]0, \infty[$;
(iii) for $x = 0$, we have $\Psi(0) = \frac{\eta \mathbb{E}(Y_1)}{\mu} < 1$.

In particular, $Z_\tau = \frac{1}{1+\kappa}$ *where* $\kappa = \dfrac{\mu}{\eta \mathbb{E}(Y_1)} - 1.$
The supermartingale Z *admits the decomposition* $Z = m - A^o$ *where*

$$m_t = (1-\Psi(0)) \sum_n \mathbb{1}_{(t \geq T_n)} + \Psi(\mu t - X_t - a)\mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}$$

$$A^o_t = (1-\Psi(0)) \sum_n \mathbb{1}_{\{t \geq T_n\}}.$$

The $\mathbb{F}$*-dual optional projection and the* $\mathbb{F}$*-dual predictable projection of* $\mathbb{1}_{[\![\tau,\infty[\![}$ *are equal, i.e.* $A^o = A^p$.

Proof. The form of Z follows from the stationary and independent increments property of $\mu t - X_t$

$$\begin{aligned}
\mathbb{P}(\tau > t | \mathcal{F}_t) &= \mathbb{P}(\inf_{s \geq t}(\mu s - X_s) < a | \mathcal{F}_t) \\
&= \mathbb{P}(\inf_{s \geq t}(\mu(s-t) - (X_s - X_t)) < a - \mu t + X_t | \mathcal{F}_t) \\
&= \Psi(\mu t - X_t - a)\mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}.
\end{aligned}$$

Let us now compute the dual optional projection A^o of the process $\mathbb{1}_{[\![\tau,\infty[\![}$. For any bounded optional process X we have

$$\mathbb{E}(X_\tau) = \mathbb{E}(\sum_n \mathbb{1}_{\{\tau = T_n\}} X_{T_n}) = \sum_n \mathbb{E}(X_{T_n} \mathbb{E}(\mathbb{1}_{\{\tau = T_n\}} | \mathcal{F}_{T_n}))$$

which implies that $A^o = \sum_n \mathbb{E}(\mathbb{1}_{(\tau = T_n)} | \mathcal{F}_{T_n}) \mathbb{1}_{[T_n, \infty)}$. To compute $\mathbb{E}(\mathbb{1}_{\{\tau = T_n\}} | \mathcal{F}_{T_n})$ let us define the stopping time $S^T_x = \inf\{t > T : x + \mu t - X_t < 0\}$ and notice that

$$\mathbb{E}(\mathbb{1}_{\{\tau = T_n\}} | \mathcal{F}_{T_n}) = \mathbb{E}(S^{T_n}_{-a} = \infty | \mathcal{F}_{T_n}) = \mathbb{E}(S^0_0 = \infty) = 1 - \Psi(0).$$

Then, $A^o = (1-\Psi(0)) \sum_n \mathbb{1}_{[\![T_n,\infty[\![}$. This is also the dual predictable projection as, by previous lemma, T_n are predictable stopping times. The martingale $m = Z + A^o$ equals then

$$\begin{aligned}
m_t &= \mathbb{E}\Big((1-\Psi(0)) \sum_n \mathbb{1}_{\{T_n < \infty\}} | \mathcal{F}_t\Big) \\
&= (1-\Psi(0)) \sum_n \mathbb{1}_{\{t \geq T_n\}} + \Psi(\mu t - X_t - a)\mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}.
\end{aligned}$$

Finally, from the general relation $\widetilde{Z} = Z + \Delta A^o$, we conclude the form of $\widetilde{Z}$. ■

If F is an exponential distribution with parameter β, then (see [AA10, p.78-79])

$$\Psi(u) = \frac{1}{1+\kappa} \exp\Big(-\frac{\kappa\beta}{1+\kappa} u\Big) \quad \text{for} \quad u \geq 0.$$

In this case, the supermartingale Z is

$$\begin{aligned}
Z_t &= \frac{1}{1+\kappa} \exp\Big(-\frac{\kappa\beta}{1+\kappa}(\mu t - X_t - a)\Big)\mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}} \\
&= \frac{1}{1+\kappa} \exp\Big(-\frac{\kappa\beta}{1+\kappa}(\mu t - a)\Big) e^{\eta\kappa t} \exp\Big(\frac{\kappa\beta}{1+\kappa} X_t - \eta\kappa t\Big)\mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}.
\end{aligned}$$

Let us note that for c such that $\int_{\mathbb{R}} e^{cy}\, dF(dy) < \infty$, the process

$$\exp\Big(cX_t - \eta t \int_{\mathbb{R}} (e^{cy} - 1) dF(dy)\Big),$$

is a martingale ([JYC09, Proposition 8.6.3.4]). Using the fact that, for an exponential law with parameter β, $\int_{\mathbb{R}} (\exp(\frac{\kappa\beta}{1+\kappa} y) - 1) dF(dy) = \kappa$, and $\eta = \frac{\beta\mu}{1+\kappa}$, the supermartingale Z can be written as

$$Z_t = \frac{1}{1+\kappa} \exp\Big(-\frac{\kappa\beta}{1+\kappa}(\mu t - a)\Big) \exp(\eta\kappa t) \exp\Big(\frac{\kappa\beta}{1+\kappa} X_t - \eta\kappa t\Big) 1\!\!1_{\{\mu t - X_t \geq a\}} + 1\!\!1_{\{\mu t - X_t < a\}}.$$

Then,

$$Z_t = \frac{e^{\frac{\eta\kappa a}{\mu}}}{1+\kappa} V_t 1\!\!1_{\{\mu t - X_t \geq a\}} + 1\!\!1_{\{\mu t - X_t < a\}},$$

where $V_t := \exp\Big(\frac{\eta\kappa}{\mu}(X_t - \mu t)\Big)$ is a martingale.

2.3.3.2 Brownian motion: local time approximation

We give an example related to an approximation result for the local time. Let B be a Brownian motion. For $\varepsilon > 0$, define a double sequence of stopping times by

$$\begin{aligned}
&U_0^\varepsilon = 0, \quad V_0^\varepsilon = 0 \\
&U_n^\varepsilon = \inf\{t \geq V_{n-1}^\varepsilon : B_t = \varepsilon\}, \quad V_n^\varepsilon = \inf\{t \geq U_n^\varepsilon : B_t = 0\}.
\end{aligned}$$

We consider the random time

$$\boxed{\tau^\varepsilon := \sup\{V_n^\varepsilon : V_n^\varepsilon \leq T_1\}} \tag{2.9}$$

with $T_1 = \inf\{t : B_t = 1\}$. Let us introduce the processes X^ε, Y^ε and J^ε

$$\begin{aligned}
X_t^\varepsilon &:= \sup\{s \leq t \wedge T_1 : B_s = \varepsilon\} \\
Y_t^\varepsilon &:= \sup\{s \leq t \wedge T_1 : B_s = 0\} \\
J_t^\varepsilon &:= 1\!\!1_{\{X_t^\varepsilon > Y_t^\varepsilon\}}
\end{aligned}$$

and the function ζ

$$\zeta(x) := \mathbb{P}^x(T_0 < T_1) = 1 - x, \quad \text{for} \quad x \in [0,1].$$

The supermartingale Z^ε of τ^ε is equal to

$$\begin{aligned}
Z_t^\varepsilon &= J_t^\varepsilon \zeta(B_{t\wedge T_1}) + (1 - J_t^\varepsilon)\zeta(\varepsilon) \\
&= J_t^\varepsilon (1 - B_{t\wedge T_1}) + (1 - J_t^\varepsilon)(1 - \varepsilon) \\
&= 1 - J_t^\varepsilon B_{t\wedge T_1} - (1 - J_t^\varepsilon)\varepsilon
\end{aligned}$$

Let us define the process $D^\varepsilon_t = \max\{n : V^\varepsilon_n \le t\}$ which indicates the number of downcrossings of Brownian motion from level ε to level 0 before time t. By integration by parts we get

$$\begin{aligned} B_t J^\varepsilon_t + \varepsilon(1 - J^\varepsilon_t) &= \int_0^t J^\varepsilon_s \, dB_s + \int_0^t B_s \, dJ^\varepsilon_s + \varepsilon(1 - J^\varepsilon_t) \\ &= \int_0^t J^\varepsilon_s \, dB_s + \varepsilon D^\varepsilon_t + \varepsilon \end{aligned}$$

The dual optional projection of τ^ε equals

$$A^\varepsilon_t = (\mathbb{1}_{\{t \ge \tau^\varepsilon\}})^p = \varepsilon D^\varepsilon_{t \wedge T_1} + \varepsilon$$

and we easily see that it is a pure jump process with the property

$$\{\Delta A^\varepsilon > 0\} = [\![0, T_1]\!] \cap \bigcup_{n=0}^{\infty} [\![V^\varepsilon_n]\!].$$

We can interpret the sequence τ^ε with ε going to zero as an approximation of the strict honest time τ given by

$$\tau = \sup\{t < T_1 : B_t = 0\}, \tag{2.10}$$

as $\tau^\varepsilon \to \tau$ $\mathbb{P}$ a.s. (by time reversal at τ). The supermartingale Z of τ equals

$$Z_t = 1 - \int_0^{t \wedge T_1} \mathbb{1}_{\{B_s > 0\}} dB_s - \frac{1}{2} L^0_{t \wedge T_1}$$

and, by [RY99, Chapter VI Theorem (1.10)] and the fact that $\mathbb{E}(\sqrt{T_1}) < \infty$, we have the following convergence for dual optional projections

$$\lim_{\varepsilon \to 0} \mathbb{E}(\sup_t |\varepsilon D^\varepsilon_{t \wedge T_1} - \frac{1}{2} L_{t \wedge T_1}|) = 0.$$

Lemma 2.27. *Let $\mathbb{F}^n$ be a progressive enlargement of the filtration $\mathbb{F}$ with random time $\tau^{1/n}$ defined in* (2.9) *and $\mathbb{F}^\infty$ be a progressive enlargement of the filtration $\mathbb{F}$ with random time τ defined in* (2.10)*. Then, for each t, the sequence of σ-fields $(\mathcal{F}^n_t)_n$ converges weakly to $\mathcal{F}^\infty_t$.*

Proof. We have to check that for each $F \in \mathcal{F}^\infty_t$, $\mathbb{P}(F|\mathcal{F}^n_t)$ converges in probability to $\mathbb{1}_F$. We limit our attention to the sets F belonging to the generator of $\mathcal{F}^\infty_t$. If $F \in \mathcal{F}_t$, the condition is obviously satisfied. If $F = \{\tau \le s\}$ for $s \le t$, using Proposition 1.44 and the honesty of $\tau^{1/n}$, we have

$$\mathbb{E}(\mathbb{1}_{\{\tau \le s\}}|\mathcal{F}^n_t) = \mathbb{1}_{\{\tau^{1/n} \le s\}} \mathbb{E}(\mathbb{1}_{\{\tau \le s\}}|\mathcal{F}^n_t) = \mathbb{1}_{\{\tau^{1/n} \le s\}} \frac{\mathbb{E}(\mathbb{1}_{\{\tau \le s\}}|\mathcal{F}_t)}{\mathbb{P}(\tau^{1/n} \le s|\mathcal{F}_t)} \overset{n \to \infty}{\to} \mathbb{1}_{\{\tau \le s\}} \quad a.s.$$

where the convergence comes from $\tau^{1/n} \to \tau$ a.s. ■

2.4 Entropy of thin random time

Additional information carried by enlarged filtration and its measurement was studied by several authors. Already in [Mey78] and [Yor85a] the question on stability of martingale spaces with respect to initial enlargement with atomic σ-field was asked. From more recent studies, generalizing and applying previous results in different contexts, we would like to recall [AIS98], [ADI06], [ADI07], [AI07].

In this section we find a simple connection between progressive enlargement with thin random time and conditional entropy of a partition associated to this time. In [Mey78] the author asks the question about additional knowledge associated with thin random time:

Un problème voisin, mais plus intéressant peut-être, consiste à mesurer le bouleversement produit, sur un système probabiliste, non pas en forçant des connaissances à l'instant 0*, mais en les forçant progressivement dans le système.*

In the case of initial enlargement with partition $(C_n)_n$, the additional knowledge is measured by entropy, namely

$$\boxed{H(C) := -\textstyle\sum_n \mathbb{P}(C_n)\log\mathbb{P}(C_n).}$$

In the case of progressive enlargement with thin random time, we suggest the measurement of additional knowledge by

$$\boxed{H(\tau) := -\textstyle\sum_n \mathbb{E}\left(1\!\!1_{C_n}\log z^n_{T_n}\right).} \tag{2.11}$$

Remark 2.28. (a) If τ is an $\mathbb{F}$-stopping time then $H(\tau) = 0$.

(b) If for any n the set C_n is in $\mathcal{F}_{T_n}$ then we do not gain any additional information since $1\!\!1_{C_n}\log z^n_{T_n} = 1\!\!1_{C_n}\log 1\!\!1_{C_n} = 0$.

(c) $H(\tau)$ is invariant under different decompositions of τ since for $\mathbb{F}$-stopping times T, T_1 and T_2 such that $[\![T_1]\!]\cap[\![T_2]\!] = \emptyset$ and $\{\tau = T\} = \{\tau = T_1\}\cup\{\tau = T_2\}$ we have

$$\begin{aligned}1\!\!1_{\{\tau=T\}}\log\mathbb{P}(\tau = T|\mathcal{F}_T) &= 1\!\!1_{\{\tau=T_1\}}\log\mathbb{P}(\tau = T|\mathcal{F}_T) + 1\!\!1_{\{\tau=T_2\}}\log\mathbb{P}(\tau = T|\mathcal{F}_T)\\ &= 1\!\!1_{\{\tau=T_1\}}\log\mathbb{P}(\tau = T_1|\mathcal{F}_{T_1}) + 1\!\!1_{\{\tau=T_2\}}\log\mathbb{P}(\tau = T_2|\mathcal{F}_{T_2}).\end{aligned}$$

For any $p\in[1,\infty[$ we denote H^p_m and H^p_{sm} Banach spaces consisting respectively of local martingale and semimartingale equipped with the following norms:
– a continuous $\mathbb{F}$-local martingale X belongs to H^p_m if

$$\|X\|_{H^p_m} := \|\langle X\rangle_\infty\|_{L^p} < \infty;$$

– a continuous $\mathbb{F}$-semimartingale X belongs to H^p_{sm} if

$$\|X\|_{H^p_{sm}} := \|\langle M\rangle_\infty\|_{L^p} + \|\int_0^\infty |dV_t|\|_{L^p} < \infty.$$

The following proposition combines existing results from [Yor85a, Yor85c] and gives justification to (2.11) as an appropriate measure in a progressive setting.

Proposition 2.29. *Assume that all $\mathbb{F}$-local martingales are continuous and $H(\tau) < \infty$. Let an $\mathbb{F}$-local martingale X be an element of $H^2(\mathbb{F})$. Then, the $\mathbb{F}^\tau$-semimartingale X is an element of $H^1(\mathbb{F}^\tau)$.*

Proof. We consider separately $X - X^\tau$ and X^τ. For $X - X^\tau$ we refer the reader to [Yor85a, Theorem 2, p.47-50]. The $\mathbb{F}^\tau$-semimartingale $X - X^\tau$ has decomposition

$$X - X^\tau = \widehat{X} + \langle X, \sum_n \mathbb{1}_{C_n} \mathbb{1}_{\rrbracket T_n, \infty \llbracket} \frac{1}{z^n} \cdot \tilde{z}^n \rangle$$

where $\widehat{X}$ and $\tilde{z}^n$ for each n are $\mathbb{F}^\tau$ local martingales. Denote by $Y = \sum_n \mathbb{1}_{C_n} \mathbb{1}_{[T_n,\infty)} \frac{1}{z^n} \cdot \tilde{z}^n$ an $\mathbb{F}^\tau$ local martingale. Then,

$$\mathbb{E}(\langle Y \rangle_\infty) = \mathbb{E}\left(\sum_n \int_{T_n}^\infty \mathbb{1}_{C_n} \frac{1}{(z_t^n)^2} d\langle z^n \rangle_t \right) = \mathbb{E}\left(\sum_n \int_{T_n}^\infty \frac{1}{z_t^n} d\langle z^n \rangle_t \right).$$

Consider now the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $f(x) = x - x \log x$ for $x > 0$ and $f(0) = 0$. Then, Itô's formula implies

$$f(z_\infty^n) = f(z_{T_n}^n) - \int_{T_n}^\infty \log z_t^n dz_t^n - \frac{1}{2} \int_{T_n}^\infty \frac{1}{z_t^n} d\langle z^n \rangle_t.$$

Since $f(z_\infty^n) = \mathbb{1}_{C_n}$ and $f(z_{T_n}^n) = z_{T_n}^n - z_{T_n}^n \log z_{T_n}^n$ we get

$$z_{T_n}^n = z_{T_n}^n - z_{T_n}^n \log z_{T_n}^n - \frac{1}{2} \mathbb{E}\left(\int_{T_n}^\infty \frac{1}{z_t^n} d\langle z^n \rangle_t | \mathcal{F}_{T_n} \right)$$

so finally

$$\mathbb{E}(\langle Y \rangle_\infty) = \sum_n \mathbb{E}\left(\int_{T_n}^\infty \frac{1}{z_t^n} d\langle z^n \rangle_t \right) = -2 \sum_n \mathbb{E}\left(z_{T_n}^n \log z_{T_n}^n \right) = 2H(\tau) < \infty.$$

For X^τ, see [Yor85c, p.120-121]. ■

Remark 2.30. (a) The condition $H(\tau) = -\sum_n \mathbb{E}\left(\mathbb{1}_{C_n} \log z_{T_n}^n \right) < \infty$ is weaker than the condition $H(C) = -\sum_n \mathbb{E}\left(\mathbb{1}_{C_n} \log \mathbb{P}(C_n) \right) < \infty$. In this sense, the previous theorem describes a new result.
(b) In the example of local time approximation in Brownian motion case one has
$H(\tau) = -\log \varepsilon$.

Chapter 3

Pseudo stopping times and enlargement of filtration

3.1 Introduction

This chapter is based on the joint paper with Libo Li [AL14].

Starting with the Example 1.49 given in Williams [Wil02], the concept of a pseudo-stopping time was formally introduced by Nikeghabli and Yor in [NY05] (see Definition 1.48). As its name suggests, the class of pseudo-stopping times is larger that the one of stopping times, which enjoy stopping times like properties.

The aim of this chapter is to examine the relationship between hypothesis $(\mathcal{H})$ (see Definition 1.27) and pseudo-stopping times. In [BY78], several characterizations of the hypothesis $(\mathcal{H})$ have been given. In [NY05] the authors have observed that given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F}$ is immersed in $\mathbb{G}$, then every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo stopping time. Our main result, given in Theorem 3.1, shows that the converse is true and an alternative characterization of hypothesis $(\mathcal{H})$ can be given using the optional and the dual optional projections. Inspired by this new characterization, we characterize all $\mathbb{G}$-stopping times in Lemma 3.4, and we show, roughly speaking, that any $\mathbb{G}$-stopping time can be written as the minimum of a thin $\mathbb{F}$-pseudo stopping time and an $\mathbb{F}$-Cox time. The present chapter is devoted mostly to the study of pseudo-stopping times. However, we also explore the relationship between pseudo-stopping times and other classes of random times, namely honest times, thin times and Cox's times.

3.2 Pseudo-stopping times and hypothesis $(\mathcal{H})$

We work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ denotes a filtration satisfying the usual conditions.

The next theorem holds for a general enlargement of filtration, namely $\mathbb{F} \subset \mathbb{G}$ where $\mathbb{G}$ is not obtained in a specific way.

Theorem 3.1. *Given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$, the following conditions are equivalent:*
(a) every (bounded) $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale, i.e., the hypothesis ($\mathcal{H}$) is satisfied for $\mathbb{F} \subset \mathbb{G}$;
(b) every finite $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time;
(c) the $\mathbb{F}$-dual optional projection of any càdlàg $\mathbb{G}$-adapted process of pre-locally integrable variation is equal to its $\mathbb{F}$-optional projection.

Proof. To show $(a) \implies (b)$, let M be a bounded $\mathbb{F}$-martingale and γ a finite $\mathbb{G}$-stopping time. Then, from hypothesis $(\mathcal{H})$, M is a $\mathbb{G}$-martingale and $\mathbb{E}(M_\gamma) = \mathbb{E}(M_0)$, which implies that γ is an $\mathbb{F}$-pseudo-stopping time.

To show $(b) \implies (a)$, suppose that M is a bounded $\mathbb{F}$-martingale and γ a finite $\mathbb{G}$-stopping time. Since every finite $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo stopping time, we have $\mathbb{E}(M_\gamma) = \mathbb{E}(M_0)$. The last equality holds for each finite $\mathbb{G}$-stopping time, which, by [JYC09, Proposition 1.2.3.7], implies that M is a $\mathbb{G}$-martingale.

The implication $(c) \implies (b)$ follows directly from Proposition 1.50 (d). To complete the proof, we show the implication $(b) \implies (c)$. We will use monotone class type of arguments, which are presented in the following five steps.

Step 1. Define

$$\Pi^{\mathbb{G}} := \{[\![\gamma, \infty[\![: \gamma \text{ is a } \mathbb{G}\text{-stopping time}\}$$

$$\mathcal{H} := \{Y : \text{càdlàg } \mathbb{G}\text{-adapted of pre-locally integrable variation such that } {}^{o,\mathbb{F}}Y = Y^{o,\mathbb{F}}\}.$$

We note that $\Pi^{\mathbb{G}}$ is a π-class on $\Omega \times [0, \infty[$ and $\mathcal{H}$ is linear space such that for each $\Gamma \in \Pi^{\mathbb{G}}$ we have $\mathbb{1}_\Gamma \in \mathcal{H}$ by (b) and Proposition 1.50 (d).

Step 2. Let Y be a càdlàg $\mathbb{G}$-adapted process of pre-locally integrable variation. Then, Y can be decomposed as a difference of two increasing $\mathbb{G}$-optional processes of integrable variation, i.e.,
$Y = Y^+ - Y^-$. So, as $\mathcal{H}$ is a linear space, without loss of generality, we may assume that Y is an increasing $\mathbb{G}$-optional process of integrable variation. It can be approximated by

$$Y^n = \sum_{k=1}^{n2^n} \frac{1}{2^n} \mathbb{1}_{[\![S_k^n, \infty[\![} \quad \text{with} \quad S_k^n = \inf\left\{t : Y_t \geq \frac{k}{2^n}\right\}.$$

Then, for every ω we have $Y_t^n(\omega) \uparrow Y_t(\omega)$ for every $t \geq 0$. Since each S_k^n is a $\mathbb{G}$-stopping time and $\mathcal{H}$ is a linear space, we also have $Y^n \in \mathcal{H}$ for each n.

Step 3. Let T be an $\mathbb{F}$-stopping time. For $n \geq m$ we have $Y_T^n - Y_T^m \geq 0$ on $\{T < \infty\}$. From the definition of the optional projection we have that $\{Y_T^n = Y_T^m\} \subset \{{}^oY_T^n = {}^oY_T^m\}$

and $\{{}^oY^n_T < {}^oY^m_T\} \in \mathcal{F}_T$. Combining all the properties listed in this step, we obtain

$$\begin{aligned} 0 &\leq \mathbb{E}\left((Y^n_T - Y^m_T)\mathbb{1}_{\{{}^oY^n_T<{}^oY^m_T\}}\mathbb{1}_{\{T<\infty\}}\right) \\ &= \mathbb{E}\left(({}^oY^n_T - {}^oY^m_T)\mathbb{1}_{\{{}^oY^n_T<{}^oY^m_T\}}\mathbb{1}_{\{T<\infty\}}\right) \leq 0 \end{aligned}$$

and finally ${}^oY^n_T \geq {}^oY^m_T$. From Section theorem 1.10, we conclude that ${}^oY^n \geq {}^oY^m$ for $n \geq m$. Since $Y^n \in \mathcal{H}$ for each n, we have that ${}^oY^n = Y^{n,o}$, which implies that each ${}^oY^n$ is an increasing process. Therefore, $({}^oY^n)_n$ is an increasing sequence of increasing processes.

Step 4. Using the Lebesgue convergence theorem and step 3., we check the definition of the optional projection of Y:

$$\begin{aligned} \mathbb{E}\left({}^oY_T\mathbb{1}_{\{T<\infty\}}\right) &= \mathbb{E}\left(Y_T\mathbb{1}_{\{T<\infty\}}\right) = \mathbb{E}\left(\lim_{n\to\infty} Y^n_T\mathbb{1}_{\{T<\infty\}}\right) \\ &= \lim_{n\to\infty}\mathbb{E}\left(Y^n_T\mathbb{1}_{\{T<\infty\}}\right) = \lim_{n\to\infty}\mathbb{E}\left({}^oY^n_T\mathbb{1}_{\{T<\infty\}}\right) \\ &= \mathbb{E}\left(\lim_{n\to\infty} {}^oY^n_T\mathbb{1}_{\{T<\infty\}}\right). \end{aligned}$$

We conclude that ${}^oY^n \uparrow {}^oY$.

Step 5. Finally, to show that $Y^o = {}^oY$, it is equivalent to show that $\mathbb{E}(\int_{[0,\infty[} X_s dY_s) = \mathbb{E}(\int_{[0,\infty[} X_s d\,{}^oY_s)$ for any $\mathbb{F}$-optional process X. To this end, let us define

$$\Pi^{\mathbb{F}} := \{[\![0, T[\![: T \text{ is an } \mathbb{F}\text{-stopping time}\}$$

$$\mathcal{H}^Y := \{X : \mathbb{E}(\int_{[0,\infty[} X_s dY_s) = \mathbb{E}(\int_{[0,\infty[} X_s d\,{}^oY_s)\}.$$

First, we prove that $\mathbb{1}_\Gamma \in \mathcal{H}^Y$ for each $\Gamma \in \Pi^{\mathbb{F}}$. We start with

$$\mathbb{E}\left(\int_{[0,\infty[} \mathbb{1}_{[\![0,T[\![}(s)dY_s\right) = \mathbb{E}\left(\lim_{t\uparrow T} Y_t\right) = \mathbb{E}\left(\lim_{t\uparrow T}\lim_{n\to\infty} Y^n_t\right) = \mathbb{E}\left(\lim_{n\to\infty}\lim_{t\uparrow T} Y^n_t\right),$$

where the last equality comes from exchanging the limits in n and t, which can be done as Y^n_t is increasing in both n and in t. We continue using twice the Lebesgue convergence theorem and the fact that $\mathbb{E}\left(Y^n_{T-}\right) = \mathbb{E}\left({}^oY^n_{T-}\right)$ for $Y^n \in \mathcal{H}$. This gives

$$\mathbb{E}\left(\lim_{n\to\infty}\lim_{t\uparrow T} Y^n_t\right) = \lim_{n\to\infty}\mathbb{E}\left(\lim_{t\uparrow T} Y^n_t\right) = \lim_{n\to\infty}\mathbb{E}\left(\lim_{t\uparrow T} {}^oY^n_t\right) = \mathbb{E}\left(\lim_{n\to\infty}\lim_{t\uparrow T} {}^oY^n_t\right).$$

We end the proof with

$$\mathbb{E}\left(\lim_{n\to\infty}\lim_{t\uparrow T} {}^oY^n_t\right) = \mathbb{E}\left(\lim_{t\uparrow T}\lim_{n\to\infty} {}^oY^n_t\right) = \mathbb{E}\left(\lim_{t\uparrow T} {}^oY_t\right) = \mathbb{E}\left(\int_{[0,\infty[} \mathbb{1}_{[0,T[}(s)d\,{}^oY_s\right).$$

where the first equality is due to the exchange of limits, which can be done as ${}^oY^n_t$ is increasing both in n and in t (step 3) and the second equality is due to step 4. So indeed $\mathbb{1}_{[0,T[} \in \mathcal{H}^Y$.

We note that $\Pi^{\mathbb{F}}$ is a π-class on $\Omega \times [0,\infty[$ and $\mathcal{H}^Y$ is a linear space such that
1) $1 \in \mathcal{H}^Y$,
2) $X^n \in \mathcal{H}^Y$, $0 \le X^n \uparrow X$, X is finite, implies $X \in \mathcal{H}^Y$,
3) $1\!\!1_\Gamma \in \mathcal{H}^Y$ for each $\Gamma \in \Pi^{\mathbb{F}}$.
Then, by Monotone Class Theorem 1.1 we obtain that $\mathbb{E}(\int_{[0,\infty[} X_s dY_s) = \mathbb{E}(\int_{[0,\infty[} X_s d\,{}^oY_s)$ for any $\mathbb{F}$-optional process X. ■

Alternative proof of implication (a) $\implies$ (c) in Theorem 3.1. Without loss of generality we may assume that Y is an increasing $\mathbb{G}$-optional process of integrable variation. Since under the hypothesis $(\mathcal{H})$, we have $\mathbb{E}(Y_t \,|\, \mathcal{F}_t) = \mathbb{E}(Y_t \,|\, \mathcal{F}_\infty)$ (see [Jea11, Proposition 3.1.1] or [ÉS01]), the process ${}^oY$ is increasing and of integrable variation. Therefore ${}^oY - Y^o$ is an uniformly integrable martingale of finite variation and is of the form

$$({}^oY - Y^o)_t = \mathbb{E}\left(({}^oY)_\infty - (Y^o)_\infty \,|\, \mathcal{F}_t\right) \quad \text{for every} \quad t \ge 0. \tag{3.1}$$

For any $F \in \mathcal{F}_\infty$, set $Q_t := \mathbb{E}(1\!\!1_F \,|\, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence such that, after applying the integration by parts formula,

$$\mathbb{E}\left(Q^{T_n}_\infty ({}^oY - Y^o)^{T_n}_\infty\right) = \mathbb{E}\left(Q^{T_n}_0 ({}^oY - Y^o)^{T_n}_0\right) + \mathbb{E}\left(\sum_{t<\infty} \Delta Q^{T_n}_t \Delta({}^oY - Y^o)^{T_n}_t\right). \tag{3.2}$$

Since we know that ${}^o(\Delta Y) = \Delta(Y^o)$, to show that the sum of jumps in (3.2) is zero, it is therefore sufficient to show that ${}^o(\Delta Y) = \Delta\,{}^o(Y)$. For any finite stopping time T, we have

$${}^o(\Delta Y)_T = {}^o(Y)_T - {}^o(Y_-)_T \quad \text{and} \quad \Delta\,{}^oY_T = {}^o(Y)_T - {}^o(Y)_{T-}$$

therefore we have reduced the problem to show that the processes ${}^o(Y)_-$ and ${}^o(Y_-)$ are indistinguishable under the hypothesis $(\mathcal{H})$. To this end, we first notice that ${}^o(Y)_-$ is càglàd, therefore $\mathbb{F}$-predictable. On the other hand, from hypothesis $(\mathcal{H})$, we have for every finite stopping time T,

$${}^o(Y_-)_T = \mathbb{E}\left(Y_{T-} \,|\, \mathcal{F}_\infty\right).$$

Since Y is càdlàg, we can deduce from the above and dominated convergence theorem that ${}^o(Y_-)$ is also càglàd and therefore $\mathbb{F}$-predictable. Therefore, to show that the processes ${}^o(Y)_- = {}^p(Y_-)$ and ${}^o(Y_-)$ are indistinguishable, it is enough to show that for any finite $\mathbb{F}$-predictable stopping time v, the equality ${}^o(Y)_{v-} = {}^o(Y_-)_v$ holds.

Let v be any $\mathbb{F}$-predictable stopping time, then v is foretellable and suppose that $(v_n)_{n\in\mathbb{N}}$ is an announcing sequence for v, then by hypothesis $(\mathcal{H})$ and dominated convergence theorem we have

$${}^o(Y_-)_v = \mathbb{E}\left(Y_{v-} \,|\, \mathcal{F}_\infty\right) = \lim_{n\to\infty} \mathbb{E}\left(Y_{v_n} \,|\, \mathcal{F}_\infty\right)$$

and once again by hypothesis $(\mathcal{H})$, $\mathbb{E}(Y_{v_n} \,|\, \mathcal{F}_\infty) = \mathbb{E}(Y_{v_n} \,|\, \mathcal{F}_{v_n})$ and

$$\lim_{n\to\infty} \mathbb{E}\left(Y_{v_n} \,|\, \mathcal{F}_{v_n}\right) = \lim_{n\to\infty} {}^o(Y)_{v_n} = {}^o(Y)_{v-}.$$

Therefore, going back to (3.2), the jump terms disappear and in order to take the limit in (3.2) as $n \to \infty$, it is enough to notice that Q is bounded and

$$|({}^oY - Y^o)^{T_n}_\infty| \le |{}^oY|_\infty + |Y^o|_\infty$$

where the right-hand side is integrable since Y is increasing and of integrable variation. Then, by the dominated convergence theorem, we obtain

$$\mathbb{E}\left(\mathbb{1}_F({}^oY - Y^o)_\infty\right) = 0$$

for all $F \in \mathcal{F}_\infty$, and from (3.1) we see that ${}^oY = Y^o$. ∎

Example 3.2. For any stopping time T, one can shrink $\mathbb{F}$ to $\mathbb{F}_T := (\mathcal{F}_{T\wedge t})_{t\geq 0}$. Every $\mathbb{F}$-stopping time is an $\mathbb{F}_T$-pseudo stopping time since $\mathbb{F}_T$ is immersed in $\mathbb{F}$.

Inspired by Theorem 3.1, we suppose that the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$ and we characterize $\mathbb{G}$-stopping times. We first give an auxiliary lemma, which is to some extent known in the current literature (see [Gap14, Jea11]), however the assumption on the invertibility of the supermartingale Z is unclear. Therefore, we will give a concise proof and extend the result to the case of non-finite random times.

In the remaining part of this chapter, since we have to study several random times together, for a random time γ we shall denote by $A^{\gamma,o,\mathbb{F}}$ the $\mathbb{F}$-dual optional projection of $A^\gamma := \mathbb{1}_{[\![\gamma,\infty[\![}$. When there is no ambiguity about the filtrations we simply note $A^{\gamma,o}$. We denote by Z^γ and $\widetilde{Z}^\gamma$ the supermartingales associated with γ. We are not using stopped processes in this chapter so there is no ambiguity.

Proposition 3.3. *If the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$ and if γ is a $\mathbb{G}$-stopping time that avoids all finite $\mathbb{F}$-stopping times then*
(a) the $\mathcal{F}_\infty$-conditional distribution of $A^{\gamma,o}_\gamma$ is uniform on the interval $[0, A^{\gamma,o}_\infty[$, with an atom of size $1 - A^{\gamma,o}_\infty$ at $A^{\gamma,o}_\infty$;
(b) the $\mathbb{G}$-stopping time γ is an $\mathbb{F}$-barrier hitting time, meaning it satisfies

$$\gamma = \inf\{t > 0 : A^{\gamma,o}_t \geq A^{\gamma,o}_\gamma\}.$$

Proof. To show (a), we compute the $\mathcal{F}_\infty$-conditional distribution of $A^{\gamma,o}_\gamma$, that is

$$\mathbb{E}\left(\mathbb{1}_{\{A^{\gamma,o}_\gamma \leq u\}} \,\middle|\, \mathcal{F}_\infty\right) = \mathbb{E}\left(\mathbb{1}_{\{A^{\gamma,o}_\gamma \leq u\}} \,\middle|\, \mathcal{F}_\infty\right)\mathbb{1}_{\{u < A^{\gamma,o}_\infty\}} + \mathbb{1}_{\{u \geq A^{\gamma,o}_\infty\}}.$$

Let us set C to be the right inverse of $A^{\gamma,o}$, then the first term in the right-hand side in the above equality is

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}_{\{A^{\gamma,o}_\gamma \leq u\}}\mathbb{1}_{\{C_u<\infty\}} \,\middle|\, \mathcal{F}_\infty\right) &= \mathbb{E}\left(\mathbb{1}_{\{\gamma \leq C_u\}}\mathbb{1}_{\{C_u<\infty\}} \,\middle|\, \mathcal{F}_{C_u}\right)\\
&= ({}^oA^\gamma)_{C_u}\mathbb{1}_{\{C_u<\infty\}}\\
&= (A^{\gamma,o})_{C_u}\mathbb{1}_{\{C_u<\infty\}}\\
&= u\mathbb{1}_{\{u<A^{\gamma,o}_\infty\}}
\end{aligned}$$

where we have applied Theorem 3.1 in the third equality, while the last equality follows from the fact that $A^{\gamma,o}_{C_u} = u$, since $A^{\gamma,o}$ is continuous. This implies that the $\mathcal{F}_\infty$-conditional distribution of $A^{\gamma,o}_\gamma$ is uniform on $[0, A^{\gamma,o}_\infty[$, with an atom of size $1 - A^{\gamma,o}_\infty$ at $A^{\gamma,o}_\infty$.
To show (b), we first define another random time γ^* by setting

$$\gamma^* := \inf\{t > 0 : A^{\gamma,o}_t \geq A^{\gamma,o}_\gamma\}.$$

To see that $\gamma^* = \gamma$ (it is obvious that $\gamma^* \leq \gamma$), we use the fact that the support of A^γ is contained to the support of $A^{\gamma,o}$ (see Chapter IV, Lemma 4.2 of [Jeu80]). ■

As a special case, if we work under the same conditions as Proposition 3.3 and assume that γ is a finite $\mathbb{G}$-stopping time, then $A^{\gamma,o}_\infty = 1$ and $A^{\gamma,o}_\gamma$ is independent from $\mathcal{F}_\infty$ and uniformly distributed on the interval $[0,1]$. In this case, the stopping time γ is truly a $\mathbb{F}$-Cox time.

Corollary 3.4. If γ is a $\mathbb{G}$-stopping time, then it can be written as $\gamma^c \wedge \gamma^d$, where γ^c is a $\mathbb{G}$-stopping time that is an $\mathbb{F}$-barrier hitting time avoiding $\mathbb{F}$-stopping times and γ^d is a $\mathbb{G}$-stopping time whose graph is contained in the graphs of $\mathbb{F}$-stopping times.

Proof. By Theorem 2.4, γ has the representation $\gamma = \gamma^d \wedge \gamma^c$. The fact that γ^d and γ^c are $\mathbb{G}$-stopping times follows from the observation $\{\Delta A^{\gamma,o}_\gamma > 0\} \in \mathcal{F}_\gamma$ (from Corollary 3.23 in [HWY92] and Monotone Class Theorem 1.1). The $\mathbb{G}$-stopping time γ^c avoids all $\mathbb{F}$-stopping times and we conclude by applying Proposition 3.3. ■

Unlike stopping times, the minimum and maximum of two $\mathbb{F}$-pseudo-stopping times is in general not a $\mathbb{F}$-pseudo stopping time. The above corollary suggests that the minimum of a Cox time with a $\mathbb{F}$-pseudo stopping time is again a $\mathbb{F}$-pseudo stopping time. In the following, we explore extensions to Proposition 4. in [NY05], which states that the minimum of a pseudo-stopping time ρ with an $\mathbb{F}^\rho$-stopping time is an $\mathbb{F}$-pseudo stopping time.

Lemma 3.5. *Let ρ be an $\mathcal{F}_\infty$-measurable $\mathbb{F}$-pseudo-stopping time and τ be a random time such that $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$, then $\tau \wedge \rho$ is an $\mathbb{F}$-pseudo stopping time.*

Proof. Proposition 1.50 (f) and Theorem 3.1 imply that both $\widetilde{Z}^\rho$ and $\widetilde{Z}^\tau$ are decreasing càglàd processes. To compute the supermartingale $\widetilde{Z}^{\tau\wedge\rho}$ associated to $\tau \wedge \rho$, we write

$$\mathbb{P}\left(\tau \geq t, \rho \geq t \,|\, \mathcal{F}_\infty\right) = \mathbb{1}_{\{\rho \geq t\}} \mathbb{P}(\tau \geq t | \mathcal{F}_\infty) = \mathbb{1}_{\{\rho \geq t\}} \mathbb{P}\left(\tau \geq t \,|\, \mathcal{F}_t\right)$$

This implies that

$$\widetilde{Z}^{\tau\wedge\rho}_t = \mathbb{E}(\mathbb{P}\left(\tau \geq t, \rho \geq t \,|\, \mathcal{F}_\infty\right) | \mathcal{F}_t) = \mathbb{P}(\tau \geq t | \mathcal{F}_t) \mathbb{P}(\rho \geq t | \mathcal{F}_t) = \widetilde{Z}^\tau_t \widetilde{Z}^\rho_t,$$

so $\widetilde{Z}^{\tau\wedge\rho}$ is a decreasing càglàd process. Thus, by Proposition 1.50 (f), we conclude the assertion. ■

We observe that, under the same assumptions as in Lemma 3.5, since ρ is $\mathcal{F}_\infty$-measurable and $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$,

$$\mathbb{P}\left(\tau > t \,|\, \mathcal{F}^\rho_\infty\right) = \mathbb{P}\left(\tau > t \,|\, \mathcal{F}_\infty\right) = \mathbb{P}\left(\tau > t \,|\, \mathcal{F}_t\right)$$

and by taking the $\mathcal{F}^\rho_t$-conditional expectation, we obtain

$$\mathbb{P}\left(\tau > t \,|\, \mathcal{F}^\rho_t\right) = \mathbb{P}\left(\tau > t \,|\, \mathcal{F}_t\right).$$

Combined together it implies that $\mathbb{P}(\tau > t | \mathcal{F}^\rho_\infty) = \mathbb{P}(\tau > t | \mathcal{F}^\rho_t)$, i.e., $\mathbb{F}^\rho \hookrightarrow \mathbb{F}^{\rho,\tau}$. This gives us the following extension.

Lemma 3.6. *Suppose that ρ is an $\mathbb{F}$-pseudo-stopping time and τ is an $\mathbb{F}^\rho$-pseudo-stopping time, then $\tau \wedge \rho$ is again a $\mathbb{F}$-pseudo stopping time.*

Proof. We compute directly $\mathbb{E}(M_{\tau\wedge\rho})$ for M a bounded $\mathbb{F}$-martingale. Using the properties of the dual optional projection, we see that

$$\mathbb{E}(M_{\tau\wedge\rho}) = \mathbb{E}(\int_{[0,\infty]} M_{\rho\wedge u}\, dA_u^{\tau,o,\mathbb{F}^\rho}),$$

and by Proposition 1.50 (e), the process $\widehat{M}_u := M_{\rho\wedge u}$ is a bounded $\mathbb{F}^\rho$-martingale. Thus

$$\mathbb{E}(\int_{[0,\infty]} M_{\rho\wedge u}\, dA_u^{\tau,o,\mathbb{F}^\rho}) = \mathbb{E}(\widehat{M}_\infty \int_{[0,\infty]} dA_u^{\tau,o,\mathbb{F}^\rho}) = \mathbb{E}(\widehat{M}_\infty A_\infty^{\tau,o,\mathbb{F}^\rho}) = \mathbb{E}(M_\rho)$$

and therefore $\tau\wedge\rho$ is an $\mathbb{F}$-pseudo-stopping time. ∎

Finally, we relate pseudo-stopping times with honest times. Under assumption (C), a result of similar spirit was presented in [NY05, Proposition 6]. The authors gave a distributional argument. Here, we use path properties to show that the same kind of result holds in full generality.

Lemma 3.7. *Let τ be a random time. The following conditions are equivalent*
(a) τ is equal to an $\mathbb{F}$-stopping-time on $\{\tau<\infty\}$;
(b) τ is an $\mathbb{F}$-pseudo stopping time and an $\mathbb{F}$-honest time.

Proof. The implication $(a) \Longrightarrow (b)$ is trivial. To show $(b) \Longrightarrow (a)$ let us note that the honest time property of τ implies $\tau = \sup\{t : \widetilde{Z}_t^\tau = 1\}$ on $\{\tau<\infty\}$ (Proposition 1.43) and the pseudo-stopping time property of τ implies $Z_t^\tau = 1 - A_t^{\tau,o}$ (Proposition 1.50 and (1.12)). Moreover we use the general relation (stated in (1.13)) $\widetilde{Z}^\tau - Z^\tau = \Delta A^{\tau,o}$. Then, on $\{\tau<\infty\}$ we obtain

$$\begin{aligned}\tau &= \sup\{t : \widetilde{Z}_t^\tau = 1\} = \sup\{t : Z_t^\tau + \Delta A_t^{\tau,o} = 1\}\\ &= \sup\{t : 1 - A_t^{\tau,o} + \Delta A_t^{\tau,o} = 1\} = \sup\{t : A_{t-}^{\tau,o} = 0\}\\ &= \inf\{t : A_t^{\tau,o} > 0\}.\end{aligned}$$

So, τ equals an $\mathbb{F}$-stopping time on $\{\tau<\infty\}$. ∎

As a simple consequence of Lemma 3.7 and Theorem 3.1, we see that

Corollary 3.8. If τ is an $\mathbb{F}$-honest time which is not equal to an $\mathbb{F}$-stopping time on $\{\tau<\infty\}$, then the hypothesis $(\mathcal{H})$ is not satisfied for $\mathbb{F}\subset\mathbb{F}^\tau$.

3.3 Time change construction of pseudo-stopping times

In this subsection, we revisit and extend the construction of pseudo-stopping times suggested in [NY05]. In that paper, an honest time τ is given and under the assumptions (C) and (A), a pseudo-stopping time ρ is constructed by setting

$$\boxed{\rho := \sup\{t \le \tau : Z_t^\tau = \inf_{s\le t} Z_s^\tau\}.} \tag{3.3}$$

The goal of this section is to demonstrate that the honest time τ which appears in (3.3) can be replaced by an arbitrary finite random time and that assumptions (C) and (A) can be relaxed in certain cases, which allows us to construct an $\mathbb{F}$-pseudo-stopping time which is also an $\mathbb{F}$-thin time.

In the following, we assume that an arbitrary random time τ is given and we define the processes D and G:

$$D_t := \inf\{s > t : Z^\tau_s = \inf_{u \leq s} Z^\tau_u\},$$
$$G_t := \sup\{s \leq t : Z^\tau_s = \inf_{u \leq s} Z^\tau_u\}.$$

Lemma 3.9. *The processes D and G are increasing and càdlàg. The process D is a right-inverse of G.*

Proof. From [DMM92, Paragraph 1 Chapter XX] we know that D is a right-continuous process, and, since the set $\Gamma := \{(\omega, t) : Z^\tau_t(\omega) = \inf_{s\leq t} Z^\tau_s(\omega)\}$ is right closed (as Z^τ is càdlàg process), G is a right-continuous process as well. To show that the process D is the right-inverse of G, i.e., $D_t = \inf\{u : G_u > t\}$ let us fix t and ω. Then we have:
1) If $u \in [G_t, D_t[$ then, since $]G_t, D_t[\notin \Gamma$, we get $G_u = G_t \leq t$.
2) If $D_t = t$ and $u > D_t$ then, since there exists $s \in]D_t, u]$ such that $s \in \Gamma$, we get $G_u \geq D_s \geq s > t$.
3) If $D_t > t$ and $u > D_t$ then, since there exists $s \in [D_t, u]$ such that $s \in \Gamma$, we get $G_u \geq D_t > t$.
And the proof is completed. ∎

We denote by d the left limit of D, i.e., $d_t := D_{t-}$. Note that d is the left-inverse of G, i.e., $d_t = \inf\{s : G_s \geq t\}$. For each $\mathbb{F}$-stopping time T, D_T and d_T are $\mathbb{F}$-stopping times. Let ρ be defined as in (3.3), then we have

$$\{\rho \geq T\} = \{\tau \geq d_T\}. \tag{3.4}$$

Lemma 3.10. *Let ρ be defined by* (3.3). *Then*
(a) the $\mathbb{F}$-dual optional projection of A^ρ is given by $A^{\rho,o} = (A^{\tau,o}_D)^{\,o}$,
(b) the $\mathbb{F}$-supermartingale $\widetilde{Z}^\rho$ is given by $\widetilde{Z}^\rho = {}^o(\widetilde{Z}^\tau_d)$.

Proof. (a) For any $\mathbb{F}$-optional process X we have

$$\begin{aligned}\mathbb{E}(X_\rho) = \mathbb{E}(X_{G_\tau}) &= \mathbb{E}(\int_{[0,\infty[} X_{G_s} dA^{\tau,o}_s) \\ &= \mathbb{E}(\int_{[0,\infty[} X_{G_{C_s}} \mathbb{1}_{\{C_s<\infty\}}\, ds),\end{aligned}$$

where the second equality follows from the fact that X_{G_s} is $\mathbb{F}$-optional (see [Mey73]) and the third from a time change in the integrals with C being the right-inverse of $A^{\tau,o}$ (see [HWY92, Lemma 1.38]). Using once again time change in the integrals, as G_C is right-inverse of $A^{\tau,o}_D$, we obtain:

$$\mathbb{E}(X_\rho) = \mathbb{E}(\int_{[0,\infty[} X_s\, dA^{\tau,o}_{D_s}).$$

(b) From the definition of the optional projection and the identity (3.4), for every $\mathbb{F}$-stopping time T,

$$\begin{aligned}\widetilde{Z}^{\rho}_T 1\!\!1_{\{T<\infty\}} &= \mathbb{E}\left(1\!\!1_{\{\rho\geq T\}} 1\!\!1_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right)\\ &= \mathbb{E}\left(1\!\!1_{\{\tau\geq d_T\}} 1\!\!1_{\{d_T<\infty\}} 1\!\!1_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right) + \mathbb{E}\left(1\!\!1_{\{\tau\geq\infty\}} 1\!\!1_{\{d_T=\infty\}} 1\!\!1_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right)\end{aligned}$$

since τ is assumed to be finite

$$\begin{aligned}&= \mathbb{E}\left(\mathbb{E}\left(1\!\!1_{\{\tau\geq d_T\}} 1\!\!1_{\{d_T<\infty\}} \,\middle|\, \mathcal{F}_{d_T}\right) \,\middle|\, \mathcal{F}_T\right) 1\!\!1_{\{T<\infty\}}\\ &= \mathbb{E}\left(\widetilde{Z}^{\tau}_{d_T} 1\!\!1_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right),\end{aligned}$$

which shows the assertion.

■

Proposition 3.11. *Suppose that for every $\mathbb{F}$-stopping time T*

$$\widetilde{Z}^{\tau}_{d_T} = \inf_{s<T} Z^{\tau}_s. \tag{3.5}$$

Then, ρ (given in (3.3)) is an $\mathbb{F}$-pseudo-stopping time and $\widetilde{Z}^{\rho}_t = \inf_{s<t} Z^{\tau}_s$.

Proof. By the assumption (3.5) and Lemma 3.10, we get that

$$\widetilde{Z}^{\rho}_T 1\!\!1_{\{T<\infty\}} = {}^{o}(\widetilde{Z}^{\tau}_{d_T}) 1\!\!1_{\{T<\infty\}} = {}^{o}(\inf_{s<T} Z^{\tau}_s) 1\!\!1_{\{T<\infty\}} = \inf_{s<T} Z^{\tau}_s 1\!\!1_{\{T<\infty\}}.$$

Then, by Section theorem 1.10, we conclude that $\widetilde{Z}^{\rho}_t = \inf_{s<t} Z^{\tau}_s$ is a decreasing càglàd process, thus, by Proposition 1.50 (f), ρ is an $\mathbb{F}$-pseudo stopping time. ■

The equality (3.5) is rather technical and cannot be checked in general without knowing the exact structure of the processes Z^{τ} and $\widetilde{Z}^{\tau}$. In the following, we give examples of constructions where (3.5) is satisfied.

Example 3.12 (Poisson filtration example)**.** We present here an example without assuming neither (C) nor (A) nor the fact that the process $\inf_{s\leq t} Z^{\tau}_s$ is continuous.

We take as τ the random time defined in (2.4) and studied in Section 2.3.3.1. Supermartingales Z and $\widetilde{Z}$ computed in Proposition 2.26 satisfy condition (3.5) and ρ, given by (3.3), is an $\mathbb{F}$-pseudo-stopping time with $\widetilde{Z}^{\rho}_t = \inf_{s<t} Z^{\tau}$ and $Z^{\rho}_t = \inf_{s\leq t} Z^{\tau}$.

If one looks closer, the infimum process of Z^{τ} has only one negative jump at the predictable stopping time

$$T_1 := \inf\{t \geq 0 : \mu t - X_t - a = 0\},$$

where the process Z^{τ} jumps from 1 to $\psi(0)$. That implies that T_1 is the only $\mathbb{F}$-stopping time which ρ intersects.

Example 3.13 (Thin pseudo-stopping times)**.** In the following we work under the assumption that $\inf Z^{\tau}$ and $A^{\tau,o}$ are continuous. We show that one can systematically construct a thin pseudo-stopping times in the following manner.

Let $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ be a positive decreasing sequence of real numbers bounded by 1. We define the following sequence of $\mathbb{F}$-predictable stopping times

$$T_n = \inf\{t \geq 0 : \inf_{s\leq t} Z^\tau_s = \varepsilon_n\}, \qquad T_{N+1} = \infty$$

and the $\mathbb{F}$-adapted decreasing càglàd process $V = \varepsilon_0 \mathbb{1}_{[\![0,T_0]\!]} + \sum_{n=1}^N \varepsilon_n \mathbb{1}_{]\!]T_{n-1},T_n]\!]}$. Moreover we set

$$\begin{aligned} G'_t &:= \sup\{s \leq t : Z^\tau_s = V_s\} \\ d'_t &:= \inf\{s \geq t : Z^\tau_s = V_s\} \end{aligned}$$

and note that the process V is such that $V_T = V_{d'_T}$ for all stopping time T. The process d' is the left inverse of G' since the set $\{Z^\tau = V\}$ is a closed random set, (see [EW77], the reason is that for each fixed ω, it contains only a finite number of points).

Let us define $\rho := G'_\tau$. Clearly it is a thin random time as $[\![G'_\tau]\!] \subset \bigcup_n [\![T_n]\!]$. Similarly to the proof of Lemma 3.11, we compute $\widetilde{Z}^\rho_T \mathbb{1}_{\{T<\infty\}}$ for an $\mathbb{F}$-stopping time T:

$$\begin{aligned} \widetilde{Z}^\rho_T \mathbb{1}_{\{T<\infty\}} &= \mathbb{E}\left(\mathbb{1}_{\{\rho\geq T\}} \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right) \\ &= \mathbb{E}\left(\mathbb{1}_{\{\tau\geq d'_T\}} \mathbb{1}_{\{d'_T<\infty\}} \,\middle|\, \mathcal{F}_T\right) \mathbb{1}_{\{T<\infty\}} \\ &= \mathbb{E}\left(\widetilde{Z}^\tau_{d'_T} \mathbb{1}_{\{d'_T<\infty\}} \,\middle|\, \mathcal{F}_T\right) \mathbb{1}_{\{T<\infty\}} \\ &= \mathbb{E}\left(V_{d'_T} \mathbb{1}_{\{d'_T<\infty\}} \,\middle|\, \mathcal{F}_T\right) \mathbb{1}_{\{T<\infty\}} \\ &= V_T \mathbb{1}_{\{T<\infty\}}, \end{aligned}$$

where the fourth equality comes from continuity if $A^{\tau,o}$ (i.e., $\widetilde{Z}^\tau = Z^\tau$) and $Z^\tau_{d'_T} = V_{d'_T}$. One can conclude that ρ is an $\mathbb{F}$-pseudo-stopping time (Proposition 1.50 (f)). The random time ρ is an $\mathbb{F}$-thin time, since $A^{\rho,o} = 1 - V_+$ is a pure jump process.

Example 3.14 (Continuous infimum example). Let τ be a random time such that the running infimum of the supermartingale Z^τ is continuous. Then, the condition (3.5) is satisfied.

(a) Take the random time $\tau := \psi\Big(\int_0^\infty f(s)dB_s\Big)$, where $\psi : \mathbb{R} \to \mathbb{R}^+$ is a deterministic, strictly increasing, square integrable function and B is a Brownian motion. Its supermartingale Z^τ equals to

$$Z^\tau_t = 1 - \Phi\Big(\frac{\psi^{-1}(t) - \beta_t}{\sigma(t)}\Big),$$

where $\beta_t = \int_0^t f(s)dB_s$, $\sigma^2(t) = \int_t^\infty f^2(s)ds$, and Φ is the cumulative distribution function of standard normal law. Then

$$\rho := \sup\left\{t < \tau : \frac{\beta_t - \psi^{-1}(t)}{\sigma(t)} = \inf_{u\leq t} \frac{\beta_u - \psi^{-1}(u)}{\sigma(u)}\right\}$$

is a pseudo-stopping time.

(b) Take the random time $\tau := A_\infty^{(-\gamma)} = \int_0^\infty e^{2B_u^{(-\gamma)}} du$, where $B_t^{(-\gamma)} := B_t - \gamma t$ with B a Brownian motion. The supermartingale Z^τ equals to

$$Z_t^\tau = \Upsilon\Big(\frac{t - A_t^{(-\gamma)}}{e_t^{2B^{(-\gamma)}}}\Big),$$

where $\Upsilon(x) := \mathbb{P}\left(A_\infty^{(-\gamma)} > x\right)$. Then

$$\rho := \sup\left\{t < A_\infty^{(-\gamma)} : \frac{t - A_t^{(-\gamma)}}{e^{2B_t^{(-\gamma)}}} = \sup_{u \le t} \frac{u - A_u^{(-\gamma)}}{e^{2B_u^{(-\gamma)}}}\right\}$$

is a pseudo-stopping time.

3.4 Hypothesis ($\mathcal{H}'$) and semimartingale decomposition

In this section, we give some conditions so that hypothesis ($\mathcal{H}'$) is valid between $\mathbb{F}$ and $\mathbb{F}^\rho$, for random times constructed in (3.3) and, in that case, we give the semimartingale decomposition. We find it useful to change the point of view and consider the problem by examining the beginning and the end of the excursion of the process

$$Y_t^\tau := Z_t^\tau - \inf_{s \le t} Z^\tau \tag{3.6}$$

straddling the random time τ. That is

$$\rho(\tau) := \sup\{t \le \tau : Z_t^\tau - \inf_{s \le t} Z_s^\tau = 0\} = \sup\{s \le \tau : Y_t^\tau = 0\},$$
$$\delta(\tau) := \inf\{t \ge \tau : Z_t^\tau - \inf_{s \le t} Z_s^\tau = 0\} = \inf\{t \ge \tau : Y_t^\tau = 0\}.$$

When τ is an honest time, one is able to apply results from Jeulin [Jeu80] without referring to the specific structure of the excursion. However, once we replace τ by a general random time, to retrieve the decomposition, we exploit the structure of the excursion. We notice that, in view of enlargement of filtration, the properties of the beginning and the end of the excursion are symmetric.

In the following, we shall examine properties of the three random times τ, $\rho := \rho(\tau)$ and $\delta := \delta(\tau)$ in view of enlargement of filtration. For simplicity, we assume that the random time τ is finite.

Lemma 3.15. *We have the following relations between the random times τ, ρ and δ and associated enlarged filtrations:*
(a) the random time δ is an $\mathbb{F}^\rho$-stopping time;
(b) the random time ρ is an $\mathbb{F}^\delta$ and $\mathbb{F}^\tau$-honest time.

Proof. Assertion (a) follows as the random time

$$\delta = \inf\left\{t \ge \tau : Y_t^\tau = 0\right\} = \inf\left\{t \ge \rho : Y_t^\tau = 0\right\}$$

is a first passage time in $\mathbb{F}^\rho$, therefore is an $\mathbb{F}^\rho$-stopping time. To show (b), it is enough to see that the random time

$$\rho = \sup\left\{t \le \tau : Y_t^\tau = 0\right\} = \sup\left\{t \le \delta : Y_t^\tau = 0\right\}$$

is a last passage time in $\mathbb{F}^\delta$ and $\mathbb{F}^\tau$, therefore an $\mathbb{F}^\delta$-honest time and $\mathbb{F}^\tau$-honest time. ■

Proposition 3.16. *If the hypothesis ($\mathcal{H}'$) is satisfied between filtrations $\mathbb{F}$ and $\mathbb{F}^\tau$, then*
(a) the hypothesis ($\mathcal{H}'$) is also satisfied between $\mathbb{F}$ and $\mathbb{F}^\rho$;
(b) the hypothesis ($\mathcal{H}'$) is also satisfied between $\mathbb{F}$ and $\mathbb{F}^\delta$.

Proof. (a) Note that ρ is an honest time in the filtration $\mathbb{F}^\tau$. Let us introduce the progressive enlargement of $\mathbb{F}^\tau$ with ρ which is denoted by $\mathbb{F}^{\tau,\rho}$. Take any $\mathbb{F}$-martingale M, then, since the hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}$ and $\mathbb{F}^\tau$, the process M is an $\mathbb{F}^\tau$-semimartingale. On the other hand, the random time ρ is an honest time in $\mathbb{F}^\tau$. Then, by classic results (see [Jeu80]), the process M is an $\mathbb{F}^{\tau,\rho}$-semimartingale. Finally, from Stricker's Theorem 1.28, the process M is an $\mathbb{F}^\rho$-semimartingale.

(b) It is sufficient to notice that $\mathbb{F} \subset \mathbb{F}^\delta \subset \mathbb{F}^\tau$, as δ is an $\mathbb{F}^\tau$-stopping time. Then, by Stricker's Theorem 1.28, if hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}$ and $\mathbb{F}^\delta$, it is satisfied between $\mathbb{F}$ and $\mathbb{F}^\rho$. ■

Proposition 3.17. *The hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}$ and $\mathbb{F}^\rho$ if and only if the hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}$ and $\mathbb{F}^\delta$.*

Proof. Note that $\mathbb{F} \subset \mathbb{F}^\delta \subset \mathbb{F}^{\delta,\rho} = \mathbb{F}^\rho$, where the last equality follows from (a) of Lemma 3.15. Using (b) of Lemma 3.15, we see that ρ is an $\mathbb{F}^\delta$-honest time and this implies that the hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}^\delta$ and $\mathbb{F}^\rho$. If the hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}$ and $\mathbb{F}^\delta$ therefore the hypothesis ($\mathcal{H}'$) is satisfied between $\mathbb{F}$ and $\mathbb{F}^\rho$. The converse follows easily from Stricker's Theorem 1.28. ■

In the next proposition we give a $\mathbb{F}^\rho$-semimartingale decomposition for $\mathbb{F}$-local martingales stopped at δ (which, by Lemma 3.15, is an $\mathbb{F}^\rho$-stopping time). This result is in the same spirit as [Jeu80, Corollary 5.21]. Here we consider the particular case of the beginning and the end of an excursion. We do not use properties of the middle time τ. Moreover, the decomposition is given for stopped processes. It can be seen as extension of classical $\mathbb{F}^\rho$-semimartingale decomposition for $\mathbb{F}$-local martingales stopped at ρ up to the end of the excursion.

Proposition 3.18. *Let M be an $\mathbb{F}$-local martingale M. Then, the process*

$$\boxed{M_t - \int_0^{t\wedge\rho} \frac{1}{Z^\rho_{s-}} d\langle X, m^\rho\rangle^{\mathbb{F}}_s - \int_0^{t\wedge\delta} \mathbb{1}_{\{\rho<s\}} \frac{1}{Z^\delta_{s-} - Z^\rho_{s-}} d\langle X, m^\delta - m^\rho\rangle^{\mathbb{F}}_s}$$

is an $\mathbb{F}^\rho$-local martingale.

Proof. We use analogical arguments to [Jeu80, section V-3]. As each local martingale is locally in H^1_m it is enough to consider $M \in H^1_m(\mathbb{F})$. By Lemma 3.15, [Jeu80, Proposition

(4,16) 2)] and [Jeu80, Théorème (5,10)] we get that $M_{\cdot\wedge\delta} \in H^1_{sm}(\mathbb{F}^\delta)$ so $M_{\cdot\wedge\delta} \in H^1_{sm}(\mathbb{F}^\rho)$.
Let H be an $\mathbb{F}^\rho$-predictable bounded process. Then $H1\!\!1_{[\![0,\delta]\!]} = J^- 1\!\!1_{[\![0,\rho]\!]} + J^+ 1\!\!1_{]\!]\rho,\delta]\!]}$, where J^- and J^+ are $\mathbb{F}$-predictable processes. $\mathbb{F}$-predictability of J^- comes from [Jeu80, Lemme (4,4) b)]. Since ρ is an $\mathbb{F}^\delta$-honest time (Lemma 3.15 (b)), by [Jeu80, Proposition (5,3) a)], J^+ can be chosen to be $\mathbb{F}^\delta$-predictable. Next, using again [Jeu80, Lemme (4,4) b)], the $\mathbb{F}$-predictability of J^+ follows.
By the same arguments as in the proof of [Jeu80, Théorème (5,10)], based on properties of dual optional projections, we get

$$\begin{aligned}\mathbb{E}((H \centerdot M)_\infty) &= \mathbb{E}\left((1\!\!1_{[\![0,\rho]\!]} J^- \centerdot M)_\infty\right) + \mathbb{E}\left((1\!\!1_{]\!]\rho,\delta]\!]} J^+ \centerdot M)_\infty\right) \\ &= \mathbb{E}\left((J^- \centerdot M)_\rho\right) + \mathbb{E}\left((J^+ \centerdot M)_\delta\right) - \mathbb{E}\left((J^+ \centerdot M)_\rho\right) \\ &= \mathbb{E}\left(\int_0^\infty J^-_s d\langle M, m^\rho\rangle^{\mathbb{F}}_s\right) + \mathbb{E}\left(\int_0^\infty J^+_s d\langle M, m^\delta - m^\rho\rangle^{\mathbb{F}}_s\right) \\ &= \mathbb{E}\left(\int_0^\rho \frac{H_s}{Z^\rho_{s-}} d\langle M, m^\rho\rangle^{\mathbb{F}}_s\right) + \mathbb{E}\left(\int_\rho^\delta \frac{H_s}{Z^\delta_{s-} - Z^\rho_{s-}} d\langle M, m^\delta - m^\rho\rangle^{\mathbb{F}}_s\right).\end{aligned}$$

And the proof is completed. ■

Remark 3.19. Let us remark that all results from this section stay valid if we replace process Y^τ defined in (3.6) by any $\mathbb{F}$-optional process Y. We focused on Y^τ to illustrate the situation from previous section.

3.5 Construction from Jeanblanc-Song model

In this section, we present another technique to construct pseudo-stopping times which is based on Jeanblanc-Song model. Authors in [JS11b] present solutions to the following problem: construct random times on an extended space with a given supermartingale $Z = Ne^{-\Gamma}$, where N is a continuous local martingale and Γ is a continuous increasing process. In our case, it would be then enough to take $N \equiv 1$. The solution is expressed through increasing family of martingales $\{(M^u_t)_{t\geq u} : u \in [0,\infty[\}$ such that $M^u_t \geq M^v_t$ for $t \geq u \geq v$. Namely, given an $\mathbb{F}$-martingale Y and a Lipschitz function f with $f(0) = 0$, for each u the following SDE is considered:

$$\begin{cases} dM^u_t &= M^u_t f(M^u_t - (1 - e^{-\Gamma_t}))dY_t \quad \text{for} \quad t \geq u \\ M^u_u &= 1 - e^{-\Gamma_u} \end{cases}. \tag{3.7}$$

These martingales are proved to take values in $[0, 1]$.

Lemma 3.20. *Let $f(x) = x$, $Y = B$ be a Brownian motion and $\Gamma_\infty = \infty$. For M^u solution of SDE (3.7) we have $M^u_\infty = 0$ or $M^u_\infty = 1$.*

Proof. As M^u is a bounded martingale, we have

$$\begin{aligned}\infty > \mathbb{E}(M_t^u - M_\infty^u)^2 &= \mathbb{E}\Big(\int_t^\infty M_s^u(M_s^u - (1-e^{-\Gamma_s}))dB_s\Big)^2 \\ &= \mathbb{E}\Big(\int_t^\infty (M_s^u)^2(M_s^u - (1-e^{-\Gamma_s}))^2 ds\Big) \\ &= \int_t^\infty \mathbb{E}((M_s^u)^2(M_s^u - (1-e^{-\Gamma_s}))^2)ds.\end{aligned}$$

This implies that $\lim_{s\to\infty} \mathbb{E}((M_s^u)^2(M_s^u - (1-e^{-\Gamma_s}))^2) = 0$. From dominated convergence theorem and positivity of $(M_\infty^u)^2(M_\infty^u - 1)^2$ we finally receive the assertion. ∎

Lemma 3.20 and $M_t^u \geq M_t^v$ for $t \geq u \geq v$ allow us to define the random time τ in the following way.

Let us take the càdlàg version of the process $(M_\infty^u)_{u\geq 0}$ and define a random time τ as

$$\boxed{\tau := \inf\{u : M_\infty^u = 1\}.} \tag{3.8}$$

Lemma 3.21. *Let τ be defined in* (3.8). *Then*
(a) $M_\infty^u = \mathbb{1}_{\{\tau\leq u\}}$ and τ is $\mathcal{F}_\infty$-measurable;
(b) the supermartingale Z^τ equals $Z^\tau = e^{-\Gamma}$ and τ is an $\mathbb{F}$-pseudo-stopping time.

Proof. (a) comes from monotonicity of the family $\{M^u : u\}$ and Lemma 3.20.
(b) Using (a) and properties of the family $\{M^u : u\}$ we write that

$$Z^\tau = \mathbb{E}(1 - M_\infty^t|\mathcal{F}_t) = 1 - M_t^t = e^{-\Gamma_t}.$$

Thus, Z^τ is decreasing and continuous and by Proposition 1.50 we conclude that τ in a pseudo-stopping time. ∎

Chapter 4

On some classes of random times

4.1 Introduction

In this chapter we collect several results concerning different classes of random times. The first subsection, based on a joint work with Monique Jeanblanc and Shiqi Song [AJS14], is devoted to stability of pseudo-stopping time property with respect to equivalent change of probability measure. Unlike to stopping time property which depends only on the filtration, pseudo-stopping times are not in general invariant with respect to equivalent change of probability measure. In the second section we study basic properties of honest times like stability under maximum and minimum, and their intersection with initial times. Moreover an example of last passage time which is not honest is presented. This example relies on taking non-adapted barrier and is closely related to Brownian bridge construction. Last section is dedicated to Cox's construction where a hazard process is not necessary continuous.

4.2 Pseudo-stopping times and change of measure

In this section, we are concerned with change of measures. Hence, we shall make precise in our notation the probability measure under which we are working.

Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F}$ is a filtration satisfying usual conditions of completeness and right continuity. Let τ be a random time with associated $\mathbb{P}$-supermartingale $Z^{\mathbb{P}}$. Denote by $\mathbb{F}^{\tau}$ a progressive enlargement of filtration $\mathbb{F}$. Let $\mathbb{Q}$ be an equivalent probability measure defined on $\mathcal{G}$ given by Radon-Nikodym derivative ζ, i.e., ζ is a $\mathcal{G}$-measurable strictly positive random variable with $\mathbb{E}_{\mathbb{P}}(\zeta) = 1$. Define the $\mathbb{Q}$-supermartingale associated with τ by $Z^{\mathbb{Q}}$, namely $Z^{\mathbb{Q}}_t := \mathbb{Q}(\tau > t|\mathcal{F}_t)$. In this section, we look for ζ such that τ is a $(\mathbb{Q}, \mathbb{F})$-pseudo-stopping time. Note that we do not assume that τ is a $(\mathbb{P}, \mathbb{F})$-pseudo-stopping time.

Denote by ζ, ζ^a and ζ^b the processes related to the change of measure ζ (a is for "after τ" and b is for "before τ"):

$$\boxed{\zeta_t := \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_t), \quad \zeta_t^a := \mathbb{E}_{\mathbb{P}}(\zeta 1\!\!1_{(\tau\leq t)}|\mathcal{F}_t), \quad \zeta_t^b := \mathbb{E}_{\mathbb{P}}(\zeta 1\!\!1_{(\tau>t)}|\mathcal{F}_t).} \tag{4.1}$$

Obviously, we have the following decomposition $\zeta = \zeta^a + \zeta^b$. The Doob-Meyer decomposition of the $\mathbb{F}$-submartingale ζ^a is of the form $\zeta^a = N + B$, where N is an $\mathbb{F}$-martingale and B is an $\mathbb{F}$-predictable process of finite variation.

We start with analysing some specific situations. Lemma 3.7 implies that if τ is an honest time which is not a.s. equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$, then there is no equivalent probability measure $\mathbb{Q}$ such that τ is a $\mathbb{Q}$-pseudo-stopping time. The next example concerns hypothesis $(\mathcal{H})$ and exhibits a change of measure which preserves the pseudo-stopping time property of τ.

Example 4.1. Suppose that for a random time τ, the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{F}^\tau$ under $\mathbb{P}$. In particular, τ is a $\mathbb{P}$-pseudo-stopping time. Consider an equivalent probability measure $\mathbb{Q}$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = \zeta$ is $\mathcal{F}_\infty$-measurable. Then, for any bounded $(\mathbb{Q},\mathbb{F})$-martingale M, we have

$$\mathbb{E}_{\mathbb{Q}}(M_\tau) = \mathbb{E}_{\mathbb{P}}(M_\tau \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau^\tau)) = \mathbb{E}_{\mathbb{P}}(\zeta_\tau M_\tau) = \mathbb{E}_{\mathbb{P}}(\zeta_0 M_0) = \mathbb{E}_{\mathbb{Q}}(M_0).$$

The second equality comes from immersion, i.e., the process $(\zeta_t)_{t\geq 0}$ defined in (4.1) is $\mathbb{F}^\tau$-martingale and the fact that ζ is an $\mathcal{F}_\infty$-measurable random variable, which altogether imply that $\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_t^\tau)$. The third equality is due to the martingale property of $(\zeta M)_t$ under $\mathbb{P}$ and hypothesis $(\mathcal{H})$. We conclude that τ is a $\mathbb{Q}$-pseudo-stopping time.

The following example deals with the case of an initial time τ (see Definition 1.52).

Example 4.2. Let τ be an initial time satisfying the equivalence Jacod's hypothesis with density process $((q_t^u)_{t\geq 0}, u \in \mathbb{R}^+)$ with respect to its law $\widetilde{\eta}$. Let $\mathbb{Q}$ be given by $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t \vee \sigma(\tau)} = \frac{1}{q_t^\tau}$. Then, for any bounded $\mathbb{Q}$-martingale M we have

$$\mathbb{E}_{\mathbb{Q}}(M_{\tau\wedge t}) = \mathbb{E}_{\mathbb{P}}\left(\frac{M_{\tau\wedge t}}{q_t^\tau}\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty M_{u\wedge t}\widetilde{\eta}(du)\right) = \int_0^\infty \mathbb{E}_{\mathbb{P}}\left(M_{u\wedge t}\right)\widetilde{\eta}(du) = \mathbb{E}_{\mathbb{Q}}(M_0),$$

where the second equality comes from (1.7) and the last equality from $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ (see Theorem 1.36). Thus τ is a $\mathbb{Q}$-pseudo-stopping time.

In the following last particular case we focus on some changes of measure breaking the pseudo-stopping time property.

Example 4.3. Let τ be a finite $\mathbb{P}$-pseudo-stopping time. Suppose that $\mathcal{F}_0$ is trivial and consider a non trivial change of probability ζ such that $\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau^\tau) = 1$ and ζ_τ is not identically equal to 1 with process $(\zeta_t)_t$ defined in (4.1). Additionally suppose that ζ is bounded from below by a strictly positive constant. For the $(\mathbb{Q},\mathbb{F})$-bounded martingale $(M_t)_{t\geq 0} = (\zeta_t^{-1})_{t\geq 0}$ we have

$$\mathbb{E}_{\mathbb{Q}}(M_\tau) = \mathbb{E}_{\mathbb{P}}(\zeta M_\tau) = \mathbb{E}_{\mathbb{P}}(M_\tau) = \mathbb{E}_{\mathbb{P}}\left(\frac{1}{\zeta_\tau}\right) > \frac{1}{\mathbb{E}_{\mathbb{P}}(\zeta_\tau)} = 1 = \mathbb{E}_{\mathbb{Q}}(M_0)$$

where the inequality comes from Jensen's inequality with the convex function $\frac{1}{x}$ on $x > 0$, and the last equality comes from the $\mathbb{P}$-pseudo-stopping time property of τ. That proves that τ is not a $\mathbb{Q}$-pseudo-stopping time.

In the next proposition we investigate how the dual optional projection of the process $A := 1\!\!1_{[\![\tau,\infty[\![}$ changes under equivalent change of probability measure. We focus our attention on dual optional projection in view of Proposition 1.50 (d).

Proposition 4.4. *Denote by $A^{o,\mathbb{P}}$ the $\mathbb{P}$-dual optional projection of the process A. Let $\mathbb{Q}$ be an equivalent measure on $\mathcal{G}$ given by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \zeta$ and $\zeta_t = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_t)$. Then, the $\mathbb{Q}$-dual predictable projection of A under $\mathbb{Q}$ equals $A_t^{o,\mathbb{Q}} = \int_0^t \frac{k_u}{\zeta_u} dA_u^{o,\mathbb{P}}$, where k is the $\mathbb{F}$-optional process such that $k_\tau = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau)$.*
In particular, if τ is a $\mathbb{P}$-pseudo-stopping time and $\zeta_\tau = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau)$ then τ is a $\mathbb{Q}$-pseudo-stopping time.

Proof. For any $\mathbb{F}$-optional process H, we have

$$\mathbb{E}_{\mathbb{Q}}(H_\tau 1\!\!1_{\{\tau<\infty\}}) = \mathbb{E}_{\mathbb{P}}(\zeta H_\tau 1\!\!1_{\{\tau<\infty\}}) = \mathbb{E}_{\mathbb{P}}(H_\tau \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau) 1\!\!1_{\{\tau<\infty\}}).$$

By Proposition 1.3 (b) there exists an $\mathbb{F}$-optional process k such that $k_\tau = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau)$ and we get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(H_\tau\, k_\tau 1\!\!1_{\{\tau<\infty\}}\right) &= \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} H_u\, k_u\, dA_u^{o,\mathbb{P}}\right) = \mathbb{E}_{\mathbb{P}}\left(\zeta \int_{[0,\infty[} H_u\, \frac{k_u}{\zeta_u}\, dA_u^{o,\mathbb{P}}\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(\int_{[0,\infty[} H_u\, \frac{k_u}{\zeta_u}\, dA_u^{o,\mathbb{P}}\right).\end{aligned}$$

Above computations show that $A_t^{o,\mathbb{Q}} = \int_{[0,t[} \frac{k_u}{\zeta_u} dA_u^{o,\mathbb{P}}$. The particular case follows from Proposition 1.50 (d). ∎

Let us remark that the condition of the form $\zeta_\tau = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_\tau)$ was studied in the literature. We refer the reader for example to [NY05]. Authors therein give also an example when that condition is not satisfied.

In the remaining part of this section we work under assumptions (C) and (A) (see Definition 1.37). Note that under (A) the dual optional projection $A^{o,\mathbb{P}}$ and the dual predictable projection $A^{p,\mathbb{P}}$ are equal. Let us introduce two additional assumptions which will be used in the upcoming part of this section.

Assumption 4.5.
Assumption (P) is satisfied if $Z_t \in]0,1[$ for all $t \in]0,\infty[$.
Assumption (I) is satisfied if the process $A^{p,\mathbb{P}}$ is increasing at 0, i.e., for each $\varepsilon > 0$, we have $A_\varepsilon^{p,\mathbb{P}} > 0$.

In order to prove Proposition 4.8 we give some preliminary results in the two following lemmas.

Lemma 4.6. *The following statements hold.*
(a) Assume (P). Then for any $t \in]0,\infty[$, $\zeta^a_t > 0$ *and* $\zeta^b_t > 0$.
(b) Assume (I) and $\zeta \geq c > 0$. *Then the* $\mathbb{F}$*-dual predictable projection of* $\zeta 1\!\!1_{[\![\tau,\infty[\![}$ *is increasing at* 0.
(c) Assume (C) and (A). Then the processes ζ, N, B, $A^{p,\mathbb{P}}$, Z, ζ^a, ζ^b *and* $Z^{\mathbb{Q}}$ *are continuous.*

Proof. (a) Let T be a stopping time and introduce $F = \{\zeta^a_T = 0,\, T \in]0,\infty[\}$. Since $F \in \mathcal{F}_T$ we have

$$0 = 1\!\!1_F \mathbb{E}_{\mathbb{P}}(\zeta 1\!\!1_{(\tau \leq T)}|\mathcal{F}_T) = \mathbb{E}_{\mathbb{P}}(\zeta 1\!\!1_{(\tau \leq T)} 1\!\!1_F|\mathcal{F}_T),$$

which, combined with the strict positivity of ζ gives

$$0 = \mathbb{E}_{\mathbb{P}}(1\!\!1_{(\tau \leq T)} 1\!\!1_F|\mathcal{F}_T) = 1\!\!1_F \mathbb{P}(\tau \leq T|\mathcal{F}_T).$$

Therefore, $1-Z_T$ vanishes on the set F, i.e., for the $\mathbb{F}$-stopping time T_F (using the convention (1.1)), $1 - Z_{T_F} = 0$ on the set $\{T_F < \infty\}$. Assumption (P) implies $\mathbb{P}(F) = 0$. From the Section theorem 1.10 we get $\mathbb{P}(\forall t \in]0,\infty[\;\; \zeta^a_t > 0) = 1$. More precisely, let $C = \{(\omega,t) : \zeta^a_t(\omega) = 0\}$ and assume $\mathbb{P}(\pi(C)) > 0$. Then, there exists a stopping time T such that $[\![T]\!] \subset C$ and $\mathbb{P}(\pi([\![T]\!])) > 0$. Therefore,

$$0 < \mathbb{P}(\pi([\![T]\!])) = \mathbb{P}(\zeta^a_T = 0, T < \infty) = \mathbb{P}(F) = 0.$$

This contradiction leads to the conclusion that $\mathbb{P}(\pi(C)) = 0$. The proof for ζ^b follows by the same argumentation.
(b) Let H be a bounded $\mathbb{F}$-predictable process. Then, denoting by V the $\mathbb{F}$-dual predictable projection of $\zeta 1\!\!1_{[\![\tau,\infty[\![}$, we have

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} H_t \, dV_t\right) = \mathbb{E}_{\mathbb{P}}(\zeta H_\tau) = \mathbb{E}_{\mathbb{P}}(H_\tau \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_{\tau-})).$$

From Lemma 1.3, there exists an $\mathbb{F}$-predictable process k such that $k_\tau = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_{\tau-})$ and $k \geq c$, thus

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} H_t \, dV_t\right) = \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} k_t \, H_t \, dA^{p,\mathbb{P}}_t\right),$$

where $A^{p,\mathbb{P}}$ is the dual predictable projection of $1\!\!1_{[\![\tau,\infty[\![}$. So, $V_t = \int_{[0,t]} k_s \, dA^{p,\mathbb{P}}_s$. Assumption (I) and $k \geq c$ imply the assertion. ∎

Lemma 4.7. *Assume (C) and (A). Then, the random time* τ *is a* $\mathbb{Q}$*-pseudo-stopping time if and only if* $\frac{\zeta^a}{\zeta} = A^{o,\mathbb{Q}} = A^{p,\mathbb{Q}}$ *is a finite variation process.*

Proof. The process $Z^{\mathbb{Q}}$ can be expressed in terms of the processes ζ and ζ^a, as

$$Z^{\mathbb{Q}}_t = 1 - \frac{\mathbb{E}_{\mathbb{P}}(\zeta 1\!\!1_{(\tau \leq t)}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{F}_t)} = 1 - \frac{\zeta^a_t}{\zeta_t}.$$

Assumptions (C) and (A) are valid under $\mathbb{Q}$ and Proposition 1.50 implies that τ is a $\mathbb{Q}$-pseudo-stopping time if and only if $Z^{\mathbb{Q}}$ is a decreasing process. Since $Z^{\mathbb{Q}}$ is a supermartingale, we conclude that τ is a $\mathbb{Q}$-pseudo-stopping time if and only if $\frac{\zeta^a}{\zeta}$ is a finite variation process ■

The next proposition describes the $\mathbb{Q}$-pseudo-stopping time property in terms of the process ζ^a defined in (4.1).

Proposition 4.8. *Let assumptions (C), (A), (P) and (I) be satisfied. Assume that* $\zeta \geq c > 0$ *and* $\mathbb{E}(\zeta^2) < \infty$*. Then, the following two conditions are equivalent:*
(a) the random time τ *is a* $\mathbb{Q}$*-pseudo-stopping time;*
(b) the process ζ^a *satisfies*

(i) $\mathbb{E}\Big(\int_0^\infty \frac{1}{(\zeta^a_u)^2} d\langle \zeta^a \rangle_u \Big) < \infty,$

(ii) $\mathcal{E}(\frac{1}{\zeta^a} \centerdot N)$ *is a* $(\mathbb{P}, \mathbb{F})$*-uniformly integrable martingale,*

(iii) $\zeta^a_\infty = \mathcal{E}(\frac{1}{\zeta^a} \centerdot N)_\infty.$

Proof. (a) $\Rightarrow$ (b) Lemma (4.7) gives us that $\frac{\zeta^a}{\zeta}$ has finite variation, hence $\frac{\zeta^b}{\zeta}$ has finite variation as well. Then, Itô's formula applied to $\frac{\zeta^b}{\zeta}$ leads to

$$\begin{aligned}
\frac{\zeta^b_t}{\zeta_t} &= 1 - \int_0^t \frac{\zeta^b_u}{\zeta^2_u} d\zeta_u + \int_0^t \frac{\zeta^b_u}{\zeta^3_u} d\langle \zeta \rangle_u + \int_0^t \frac{1}{\zeta_u} d\zeta^b_u + \langle \zeta^b, \frac{1}{\zeta} \rangle_t \\
&= 1 + \Big(\int_0^t \frac{1}{\zeta_u} d(\zeta - N)_u - \int_0^t \frac{\zeta^b_u}{\zeta^2_u} d\zeta_u \Big) + \Big(\int_0^t \frac{\zeta^b_u}{\zeta^3_u} d\langle \zeta \rangle_u + \langle \zeta^b, \frac{1}{\zeta} \rangle_t - \int_0^t \frac{1}{\zeta_u} dB_u \Big).
\end{aligned}$$

The martingale part of $\frac{\zeta^b}{\zeta}$ must vanish, i.e.,

$$\frac{1}{\zeta_t} d(\zeta - N)_t = \frac{\zeta^b_t}{\zeta^2_t} d\zeta_t.$$

Since assumption (P) and Lemma 4.6 imply that $\zeta^b_t > 0$ for any $t \in]0, \infty[$, it is equivalent to

$$\frac{\zeta_t}{\zeta^b_t} d(\zeta - N)_t = d\zeta_t. \tag{4.2}$$

Using properties of the stochastic exponential, we can obtain another equivalent condition. Indeed, if the previous equality $\frac{\zeta}{\zeta^b} d(\zeta - N) = d\zeta$ holds, then

$$\zeta_t = \mathcal{E}\left(\frac{1}{\zeta^b} \centerdot (\zeta - N) \right)_t = \zeta^b_t \exp\Big(\int_0^t \frac{1}{\zeta^b_u} dB_u \Big),$$

where the second equality comes from Yor's exponential formula. So that $\zeta_t = \zeta^b_t e^{v_t}$ where $v_t := \int_0^t \frac{1}{\zeta^b_u} dB_u$. Since $\zeta = \zeta^b + \zeta^a$, this is equivalent to

$$\zeta(1 - e^{-v}) = \zeta^a.$$

Now, we transform the last condition into

$$\zeta_t^a = \zeta_t(1 - e^{-v_t}) = \mathcal{E}\left(\frac{1}{\zeta^b} \centerdot (\zeta - N)\right)_t (1 - e^{-v_t}).$$

Taking the logarithm of both sides for $t > 0$, we get

$$\ln \zeta_t^a = \ln \mathcal{E}\left(\frac{1}{\zeta^b} \centerdot (\zeta - N)\right)_t + \ln(1 - e^{-v_t}).$$

The left-hand side, for each $\varepsilon > 0$, can be written as

$$\ln \zeta_t^a = \ln \zeta_\epsilon^a + \int_\epsilon^t \frac{1}{\zeta_u^a} dN_u + \int_\epsilon^t \frac{1}{\zeta_u^a} dB_u - \frac{1}{2}\int_\epsilon^t \frac{1}{(\zeta^a)_u^2} d\langle \zeta^a \rangle_u$$

and the right-hand side as

$$\ln\left(\mathcal{E}\big(\frac{1}{\zeta^b} \centerdot (\zeta - N)\big)_t\right) + \ln(1 - e^{-v_t}) = \int_0^t \frac{1}{\zeta_u^b} d(\zeta - N)_u - \frac{1}{2}\int_0^t \frac{1}{(\zeta_u^b)^2} d\langle \zeta^b \rangle_u + \ln(1 - e^{-v_t}).$$

As martingale parts of both sides are equal, for each $t > \epsilon > 0$, we obtain that

$$\int_\epsilon^t \frac{1}{\zeta_u^a} dN_u = \int_\epsilon^t \frac{1}{\zeta_u^b} d(\zeta - N)_u,$$

and finally from (4.2) that

$$\int_\epsilon^t \frac{1}{\zeta_u^a} dN_u = \int_\epsilon^t \frac{1}{\zeta_u} d\zeta_u.$$

From the last integral equality, we conclude that
(i) the stochastic integral $\int_0^\infty \frac{1}{\zeta_u^a} dN_u$ exists as

$$\mathbb{E}\bigg(\int_0^\infty \frac{1}{(\zeta_u^a)^2} d\langle \zeta^a \rangle_u\bigg) = \mathbb{E}\bigg(\int_0^\infty \frac{1}{\zeta_u^2} d\langle \zeta \rangle_u\bigg) \leq \mathbb{E}\left(\frac{1}{c^2}\langle \zeta \rangle_\infty\right) < \infty,$$

where the last inequality comes from the fact that $\mathbb{E}(\zeta_\infty^2) \leq \mathbb{E}(\zeta^2) < \infty$ and we apply [RY99, Corollary (1.25) Chapter IV].
(ii) since

$$\zeta_t = \mathcal{E}\left(\frac{1}{\zeta^a} \centerdot N\right)_t,$$

we have that $\mathcal{E}(\frac{1}{\zeta^a} \centerdot N)$ is uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale.
(iii) moreover,

$$\zeta_\infty^a = \zeta_\infty = \mathcal{E}\left(\frac{1}{\zeta^a} \centerdot N\right)_\infty.$$

$(b) \Rightarrow (a)$ We will show that $\frac{\zeta^a}{\zeta}$ has finite variation and conclude that τ is a $\mathbb{Q}$-pseudo-stopping time from Lemma (4.7). We have

$$\frac{\zeta_t^a}{\zeta_t} = \frac{\zeta_t^a}{\zeta_\infty^a}\zeta_\infty^a \frac{1}{\zeta_t} = \exp(-\int_t^\infty \frac{1}{\zeta_u^a} dB_u)\frac{\mathcal{E}(\frac{1}{\zeta^a} \centerdot N)_t}{\zeta_t} = \exp(-\int_t^\infty \frac{1}{\zeta_u^a} dB_u),$$

where the last equality comes from the fact that $\mathcal{E}(\frac{1}{\zeta^a} \centerdot N)$ and ζ are uniformly integrable $\mathbb{P}$-martingales with the same terminal values $\zeta_\infty^a = \zeta_\infty$. We see that $\frac{\zeta^a}{\zeta}$ is an increasing process, so τ is a $\mathbb{Q}$-pseudo-stopping time. ∎

We would like to note that the stability of pseudo-stopping time property under equivalent change of measure was also studied in [Kre13, Kre14].

4.3 Last passage times

This section consists of several problems connected to honest times. In Section 4.3.1 we study stability of honest times under maximum and minimum. In Section 4.3.2 we look at the intersection of the sets of honest and initial times. Finally in Section 4.3.3 we consider a particular last passage time which is not honest.

4.3.1 Maximum and minimum of honest times

In Lemma 4.9 we show that maximum of two honest times is honest. In Lemma 4.10 we claim that for every honest time there exists a stopping time such that honest time property is broken for the minimum of the two. This asymmetry comes from characterization of honest time as last passage time, or, in other words, as the end of an optional set, which corresponds to taking the maximal element satisfying certain condition.

Lemma 4.9. *Let τ and $\widetilde{\tau}$ be two honest times. Then $\tau \vee \widetilde{\tau}$ is an honest time.*

Proof. First proof. As τ and $\widetilde{\tau}$ are honest, for every $t \geq 0$, there exist two random variables τ_t and $\widetilde{\tau}_t$ which are $\mathcal{F}_t$ measurable and satisfy

$$\tau \mathbb{1}_{(\tau<t)} = \tau_t \mathbb{1}_{(\tau<t)} \quad \text{and} \quad \widetilde{\tau} \mathbb{1}_{(\widetilde{\tau}<t)} = \widetilde{\tau}_t \mathbb{1}_{(\widetilde{\tau}<t)}.$$

Then, we have

$$(\tau \vee \widetilde{\tau}) \mathbb{1}_{\{\tau \vee \widetilde{\tau} < t\}} = (\tau \vee \widetilde{\tau}) \mathbb{1}_{\{\tau<t, \widetilde{\tau}<t\}} = (\tau_t \vee \widetilde{\tau}_t) \mathbb{1}_{\{\tau<t, \widetilde{\tau}<t\}} = (\tau_t \vee \widetilde{\tau}_t) \mathbb{1}_{\{\tau \vee \widetilde{\tau} < t\}},$$

which proves that it is in fact an honest time.
Second proof. Here, we give an alternative proof. From [Jeu80, Proposition 5.1] we know that τ (respectively $\widetilde{\tau}$) is an honest time if and only if τ (respectively $\widetilde{\tau}$) is the end of an optionnal set Γ (respectively $\widetilde{\Gamma}$) on the set $\{\tau < \infty\}$ (respectively $\{\widetilde{\tau} < \infty\}$). Then we can express the supremum $\tau \vee \widetilde{\tau}$ as the end of the optional set $\Gamma \cup \widetilde{\Gamma}$ on the set

$$\{\tau \vee \widetilde{\tau} < \infty\} = \{\tau < \infty\} \cap \{\widetilde{\tau} < \infty\}.$$

$$\begin{aligned}
(\tau \vee \widetilde{\tau}) \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}} &= \tau \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}} \vee \widetilde{\tau} \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}} \\
&= \left(\sup\{t : (\omega, t) \in \Gamma\} \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}}\right) \vee \left(\sup\{t : (\omega, t) \in \widetilde{\Gamma}\} \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}}\right) \\
&= \left(\sup\{t : (\omega, t) \in \Gamma\} \vee \sup\{t : (\omega, t) \in \widetilde{\Gamma}\}\right) \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}} \\
&= \sup\{t : (\omega, t) \in \Gamma \cup \widetilde{\Gamma}\} \mathbb{1}_{\{\tau \vee \widetilde{\tau} < \infty\}}.
\end{aligned}$$

∎

It is easy to see that the result can be obtained for countable set of honest times $(\tau^n)_{n=1}^{\infty}$, i.e., $\sup_n \tau^n$ is an honest time with associated optional set $\bigcup_n \Gamma^n$, where for each n, Γ_n is an optional set associated to τ^n.

Lemma 4.10. *Let τ be an honest time which is not a stopping time. There exists a stopping time T such that $T \wedge \tau$ is not an honest time.*

Proof. Let us assume that for all stopping times T the random time $\sigma := \tau \wedge T$ is an honest time. Then, $\widetilde{Z}^\sigma = 1\!\!1_{[\![0,T]\!]}\widetilde{Z}^\tau$ and by Proposition 1.45, σ is an honest time if and only if $\widetilde{Z}^\sigma_\sigma = 1$ on $\{\sigma < \infty\}$. On the set $\{\tau \leq T\} \cap \{\tau < \infty\}$, $\sigma = \tau = \sup\{t : \widetilde{Z}^\sigma_t = 1\}$ thus the honest time condition is satisfied. On the set $\{T \leq \tau\} \cap \{T < \infty\}$, $\sigma = T$ and the honest time condition is satisfied if and only if $\widetilde{Z}^\tau_T = 1$. Thus we get that for each stopping time T, $\widetilde{Z}^\tau_T = 1$ on $\{T \leq \tau\} \cap \{T < \infty\}$, and by Section theorem 1.10 we obtain that $\widetilde{Z}^\tau_t = 1$ on $\{t \leq \tau\}$. The last property implies that $\{\widetilde{Z}^\tau_\tau = 1\} = \{\tau \geq t\}$ thus $\{\tau < t\} \in \mathcal{F}_t$ for each t and we get a contradiction with the assumption that τ is not a stopping time. ∎

4.3.2 Honest – initial times

The next lemma gives a characterization of honest times satisfying Jacod's hypothesis. This class of honest times is limited to honest times taking countably many values.

Lemma 4.11. *Let τ be an honest time. Then τ satisfies Jacod's hypothesis if and only if τ takes countably many values.*

Proof. If τ takes countably many values then Jacod's hypothesis is satisfied (Example 1.35). For any bounded Borel function f

$$\begin{aligned}\mathbb{E}(f(\tau)|\mathcal{F}_t) &= \mathbb{E}(f(\tau)1\!\!1_{(\tau\leq t)}|\mathcal{F}_t) + \mathbb{E}(f(\tau)1\!\!1_{(\tau>t)}|\mathcal{F}_t)\\ &= f(\tau_t)(1-Z_t) + \mathbb{E}\left(\int_t^\infty f(s)dA^o_s|\mathcal{F}_t\right).\end{aligned}$$

In particular, for $f(s) = 1\!\!1_{\{s>u\}}$, we get

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) &= 1\!\!1_{\{\tau_t>u\}}(1-Z_t) + \mathbb{E}\left(A^o_\infty - A^o_{t\vee u}|\mathcal{F}_t\right)\\ &= \int_u^\infty (1-Z_t)\delta_{\tau_t}(ds) + \mathbb{E}\left(A^o_\infty - A^o_{t\vee u}|\mathcal{F}_t\right).\end{aligned}$$

Assume that η the law of τ is not purely atomic. Denote by D the set of atoms of η and take t such that $\eta([0,t[\setminus D) > 0$. Then, the first term of conditional law $\delta_{\tau_t}1\!\!1_{\{\tau<t\}} = \delta_\tau 1\!\!1_{\{\tau<t\}}$ is not absolutely continuous with respect to η $\mathbb{P}$-a.s. as it is enough to take set $\{\tau \in [0,t[\setminus D\}$ which has positive probability. ∎

4.3.3 Non honest last passage time

We study the enlargement of filtration formula for a particular case of last passage time which is not an honest time. Let W be a Brownian motion and $\mathbb{F}$ its natural filtration. Define the random time τ as

$$\boxed{\tau := \sup\{t \leq 1 : W_1 - 2W_t = 0\}.} \tag{4.3}$$

The aim is to study the hypothesis $(\mathcal{H}')$ and the semimartingale decomposition of $\mathbb{F}$-martingales in progressively enlarged filtration with time τ given in (4.3). We consider three enlarged filtrations $\mathbb{F}^{\sigma(W_1)}$, $\mathbb{F}^{\tau}$ and $\mathbb{F}^{\sigma(W_1),\tau}$ defined as

$$\begin{aligned}
\mathcal{F}_t^{\sigma(W_1)} &:= \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(W_1)) \qquad (4.4)\\
\mathcal{F}_t^{\tau} &:= \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau \wedge s))\\
\mathcal{F}_t^{\sigma(W_1),\tau} &:= \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(W_1) \vee \sigma(\tau \wedge s))
\end{aligned}$$

Both filtrations $\mathbb{F}^{\sigma(W_1)}$ and $\mathbb{F}^{\tau}$ are subfiltrations of $\mathbb{F}^{\sigma(W_1),\tau}$, but we cannot order $\mathbb{F}^{\sigma(W_1)}$ and $\mathbb{F}^{\tau}$. Let us note that τ is not an $\mathbb{F}$-honest time so we cannot apply standard decomposition formula for honest times. The random time τ is neither an $\mathbb{F}^{\sigma(W_1)}$-stopping time, but it is an $\mathbb{F}^{\sigma(W_1)}$-honest time.

Lemma 4.12. *(a) After τ, the filtration $\mathbb{F}^{\sigma(W_1)}$ is a subfiltration of the filtration $\mathbb{F}^{\tau}$, i.e., for all t and for all $F \in \mathcal{F}_t^{\sigma(W_1)}$ we have $F \cap \{t \geq \tau\} \in \mathcal{F}_t^{\tau}$.*
(b) After τ, the filtration $\mathbb{F}^{\sigma(W_1),\tau}$ coincides with filtration $\mathbb{F}^{\tau}$, i.e., for all t and for all $F \in \mathcal{F}_t^{\sigma(W_1),\tau}$ we have $F \cap \{t \geq \tau\} \in \mathcal{F}_t^{\tau}$.

Proof. (a) Take F from the generator of $\mathcal{F}_t^{\sigma(W_1)}$. If $F \in \mathcal{F}_t$, then the assertion is clear. If $F = \{W_1 \leq a\}$, then $F \cap \{t \geq \tau\} = \{2W_\tau \leq a\} \cap \{\tau \leq t\}$ and $\{2W_\tau \leq a\} \cap \{\tau \leq t\} \in \mathcal{F}_t^{\tau}$ as τ is an $\mathbb{F}^{\tau}$-stopping time.
Note that the opposite inclusion $\mathbb{F}^{\tau} \subset \mathbb{F}^{\sigma(W_1)}$ is not true as $\{\tau \leq t\} \notin \mathcal{F}_t^{\sigma(W_1)}$. For a general set F we conclude from Monotone Class Theorem 1.1.
(b) We use the same argument as in (a). ∎

Relying on [JY79, Theorem 3] we provide the following result.

Theorem 4.13. *Let X be an $\mathbb{F}$-local martingale with representation $X_t = X_0 + \int_0^t \varphi_s dW_s$ for an $\mathbb{F}$-predictable process φ such that $\int_0^t \varphi_s^2 ds < \infty$. Then, the following conditions are equivalent:*
(a) X is an $\mathbb{F}^{\tau}$-semimartingale;
(b) $\int_0^1 |\varphi_s| \frac{|W_1 - W_s|}{1-s} ds < \infty$ $\mathbb{P}$-a.s.;
(c) $\int_0^1 \frac{|\varphi_s|}{\sqrt{1-s}} ds < \infty$ $\mathbb{P}$-a.s.
Moreover, if those conditions are satisfied, the $\mathbb{F}^{\tau}$-semimartingale decomposition of X, for $t \leq 1$, is given by

$$\boxed{\begin{aligned}
X_t = \quad & X_0 + \int_0^t \varphi_s d\widehat{W}_s - \sqrt{\tfrac{2}{\pi}} \int_0^{t\wedge\tau} \varphi_s \frac{1}{Z_s} \frac{W_s^2}{(1-s)^{3/2}} \exp\left\{-\frac{W_s^2}{2(1-s)}\right\} sgn(W_s) ds \\
& + \int_\tau^t \varphi_s \frac{W_1 - W_s}{1-s} ds - \int_\tau^t \varphi_s \frac{1}{1-\exp\left\{\frac{2W_s W_1 - W_1^2}{2(1-s)}\right\}} \frac{W_1}{1-s} ds
\end{aligned}} \qquad (4.5)$$

where $\widehat{W}$ is an $\mathbb{F}^\tau$-Brownian motion and

$$Z_t = 1 - h\left(\frac{|W_t|}{\sqrt{1-t}}\right) \quad \textit{with} \quad h(y) = \sqrt{\frac{2}{\pi}}\int_0^y x^2\, e^{-x^2/2} dx.$$

Proof. 1) By [JY79, Theorem 3], W is an $\mathbb{F}^{\sigma(W_1)}$-semimartingale with decomposition given by

$$W_t = \widetilde{W}_t + \int_0^t \frac{W_1 - W_s}{1-s} ds,$$

where $\widetilde{W}$ is a $\mathbb{F}^{\sigma(W_1)}$-Brownian motion.
2) Since τ is an $\mathbb{F}^{\sigma(W_1)}$-honest time, $(\mathcal{H}')$ is satisfied between $\mathbb{F}^{\sigma(W_1)}$ and $\mathbb{F}^{\sigma(W_1),\tau}$ and $\widetilde{W}$ decomposes as

$$\widetilde{W}_t = \bar{W}_t + \int_0^{t\wedge\tau} \frac{1}{Y_{s-}} d\langle m^Y, \widetilde{W}\rangle_s^{\mathbb{F}^{\sigma(W_1)}} - \int_\tau^t \frac{1}{1-Y_{s-}} d\langle m^Y, \widetilde{W}\rangle_s^{\mathbb{F}^{\sigma(W_1)}},$$

where $\bar{W}$ is a $\mathbb{F}^{\sigma(W_1),\tau}$-Brownian motion and $Y_t := \mathbb{P}(\tau > t|\mathcal{F}_t^{\sigma(W_1)})$, m^Y is the martingale part from optional decomposition of Y and $\langle\ ,\ \rangle^{\mathbb{F}^{\sigma(W_1)}}$ means that the bracket is computed in $\mathbb{F}^{\sigma(W_1)}$.
3) Two previous points ensured that W is an $\mathbb{F}^{\sigma(W_1),\tau}$-semimartingale with decomposition given by

$$W_t = \bar{W}_t + \int_0^t \frac{W_1 - W_s}{1-s} ds + \int_0^{t\wedge\tau} \frac{1}{Y_{s-}} d\langle m^Y, \widetilde{W}\rangle_s^{\mathbb{F}^{\sigma(W_1)}} - \int_\tau^t \frac{1}{1-Y_{s-}} d\langle m^Y, \widetilde{W}\rangle_s^{\mathbb{F}^{\sigma(W_1)}}.$$

4) By Stricker's Theorem 1.28, W is an $\mathbb{F}^\tau$-semimartingale. Before τ, we have the standard decomposition for random time (see Proposition (1.39)). After τ, by Lemma 4.12 (b), we have the same decomposition as in $\mathbb{F}^{\sigma(W_1),\tau}$:

$$W_t = \widehat{W}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle m, W\rangle_s^{\mathbb{F}} + \int_\tau^t \frac{W_1 - W_s}{1-s} ds - \int_\tau^t \frac{1}{1-Y_{s-}} d\langle m^Y, \widetilde{W}\rangle_s^{\mathbb{F}^{\sigma(W_1)}},$$

where $\widehat{W}$ is an $\mathbb{F}^\tau$-Brownian motion.

By [JYC09, Proposition 4.3.5.3.] we have

$$\begin{aligned} Y_t := \mathbb{P}(\tau > t|\mathcal{F}_t^{\sigma(W_1)}) &= \mathbb{1}_{\{2W_t \le W_1, W_1 \ge 0\}} + \exp\left\{\frac{W_1^2 - 2W_tW_1}{2(1-t)}\right\}\mathbb{1}_{\{2W_t > W_1 \ge 0\}} \\ &+ \mathbb{1}_{\{2W_t \ge W_1, W_1 \le 0\}} + \exp\left\{\frac{W_1^2 - 2W_tW_1}{2(1-t)}\right\}\mathbb{1}_{\{2W_t < W_1 \le 0\}}. \end{aligned}$$

Note that after τ, $Y_t = \exp\left\{\frac{W_1^2 - 2W_tW_1}{2(1-t)}\right\}$. Since $\widetilde{W}$ is continuous, for $t > \tau$, we have

$$\begin{aligned} d\langle m^Y, \widetilde{W}\rangle_t^{\mathbb{F}^{\sigma(W_1)}} &= d\langle Y, \widetilde{W}\rangle_t^{\mathbb{F}^{\sigma(W_1)}} = d\langle \exp\left\{\frac{W_1^2 - 2WW_1}{2(1-t)}\right\}, W\rangle_t^{\mathbb{F}^{\sigma(W_1)}} \\ &= d\langle \exp\left\{\frac{u^2 - 2uW}{2(1-t)}\right\}, W\rangle_t^{\mathbb{F}}|_{u=W_1}, \end{aligned}$$

then, by Itô's lemma

$$d\langle m^Y, \widetilde{W}\rangle_t^{\mathbb{F}^{\sigma(W_1)}} = -Y_t \frac{W_1}{1-t} dt.$$

Summing up

$$\int_\tau^t \frac{1}{1-Y_{s-}} d\langle m^Y, \widetilde{W}\rangle_s^{\mathbb{F}^{\sigma(W_1)}} = \int_\tau^t \frac{1}{1-Y_s^{-1}} \frac{W_1}{1-s} ds.$$

The supermartingale Z is given in [JYC09, page 302]. Then, we compute $\langle m, W\rangle^{\mathbb{F}}$ as follows

$$\begin{aligned} d\langle m, W\rangle_t^{\mathbb{F}} &= -d\langle h\left(\frac{|W|}{\sqrt{1-t}}\right), W\rangle_t^{\mathbb{F}} = -h'\left(\frac{|W_t|}{\sqrt{1-t}}\right) d\langle |W|, W\rangle_t^{\mathbb{F}} \\ &= -\sqrt{\frac{2}{\pi}} \frac{W_t^2}{(1-t)} \exp\left\{-\frac{W_t^2}{2(1-t)}\right\} \operatorname{sgn}(W_t) dt, \end{aligned}$$

That ends the proof of the decomposition formula (4.5) for Brownian motion W.

For a general $\mathbb{F}$-local martingale $X = X_0 + \varphi \centerdot W$, we deduce equivalence between conditions (a), (b) and (c) from [JY79, Theorem 3] and the fact that $\tau < 1$ a.s. since the integrability problem appears close to 1. (For $t < 1$, the random variable W_1 satisfies Jacod's hypothesis.)
■

We want to remark here that above decomposition can be obtained (and is obtained) using methodology presented in [Son13].
Theorem 4.13 shows that the hypothesis $(\mathcal{H}')$ is not satisfied for the studied enlargement.

In the remaining part of this section, we study second example of last passage time which is not honest. Instead of Brownian motion we use geometric Brownian motion to define it. Let S be defined through $dS_t = \sigma S_t dW_t$, $S_0 = 1$, where W is a Brownian motion and σ a constant. Let

$$\boxed{\tau^S := \sup\{t \leq 1 \,:\, S_1 - 2S_t = 0\}} \tag{4.6}$$

with $\sup\{\emptyset\} = \infty$, that is the last time before 1 at which the geometric Brownian motion is equal to half of its terminal value at time 1. Define filtration $\mathbb{F}^{\tau^S}$ as a progressive enlargement of $\mathbb{F}$ with random time τ^S, i.e.,

$$\mathcal{F}_t^{\tau^S} := \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\tau^S \wedge s)\right).$$

Lemma 4.14. *(a) The random time τ^S is an $\mathbb{F}$-pseudo-stopping time and the process S^τ is a $\mathbb{F}^{\tau^S}$-martingale.*
(b) The process S is a $\mathbb{F}^{\tau^S}$-semimartingale.

Proof. (a) Note that

$$\{\tau^S \leq t\} = \left\{\inf_{t\leq s\leq 1} 2S_s \geq S_1\right\} = \left\{\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}\right\}$$

Since $\frac{S_s}{S_t}, s \geq t$ and $\frac{S_1}{S_t}$ are independent from $\mathcal{F}_t$,

$$\mathbb{P}\left(\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}|\mathcal{F}_t\right) = \mathbb{P}\left(\inf_{t\leq s\leq 1} 2S_{s-t} \geq S_{1-t}\right) = \Phi(1-t)$$

where $\Phi(u) = \mathbb{P}(\inf_{s\leq u} 2S_s \geq S_u)$. It follows that the supermartingale $Z^S_t := \mathbb{P}(\tau^S > t|\mathcal{F}_t)$ is a deterministic decreasing function, hence, by Proposition 1.50, τ is a pseudo-stopping time and S^τ is a $\mathbb{F}^\tau$-martingale.

(b) Let us consider the initial enlargement of $\mathbb{F}$ with W_1, namely the filtration $\mathbb{F}^{\sigma(W_1)}$ defined in (4.4). By [JY79, Theorem 3], S is a $\mathbb{F}^{\sigma(W_1)}$-semimartingale as

$$\int_0^1 \exp\left\{\sigma W_s - \frac{\sigma^2}{2}s\right\} \frac{1}{\sqrt{1-s}}ds < \infty.$$

The random time τ^S in an $\mathbb{F}^{\sigma(W_1)}$-honest time as

$$\tau^S = \sup\{t \leq 1 : W_t + \frac{1}{\sigma}\ln 2 + \frac{1}{2}\sigma(1-t) = W_1\}.$$

Thus, denoting by $\mathcal{F}_t^{\sigma(W_1),\tau^S} := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(W_1) \vee \sigma(\tau^S \wedge s)$, S is $\mathbb{F}^{\sigma(W_1),\tau^S}$-semimartingale (follows from (1.15)). Finally, by Stricker's theorem, as $\mathbb{F}^{\tau^S} \subset \mathbb{F}^{\sigma(W_1),\tau^S}$ and S is $\mathbb{F}^{\tau^S}$-adapted, we conclude that S is a $\mathbb{F}^{\tau^S}$-semimartingale. ∎

4.4 Cox's times

In this section, we are interested in a standard construction of a random time in credit risk modelling, namely Cox's construction. For a càdlàg increasing $\mathbb{F}$-adapted process Γ, we define a Cox's time as

$$\tau := \inf\{t : \Gamma_t \geq \Theta\}$$

where Θ is a random variable independent from $\mathcal{F}_\infty$ which has exponential law with parameter 1. The process Γ is called the hazard process of τ. We denote by $\mathbb{F}^\tau$ the progressive enlargement of $\mathbb{F}$ with random time τ. Since $\{\tau > t\} = \{\Theta > \Gamma_t\}$ and $\{\tau \geq t\} = \{\Theta \geq \Gamma_{t-}\}$ the supermartingales associated with τ can be written as

$$Z_t = e^{-\Gamma_t} \quad \text{and} \quad \widetilde{Z}_t = e^{-\Gamma_{t-}}. \tag{4.7}$$

Note that in this case $\widetilde{Z} = Z_-$.

We focus our attention on the $\mathbb{F}^\tau$-compensator of the default process $A_t = 1\!\!1_{\{\tau\leq t\}}$, i.e., the increasing $\mathbb{F}^\tau$-predictable process Λ such that $A - \Lambda$ is an $\mathbb{F}^\tau$-martingale. The compensator Λ is characterized in Proposition 3.1.5 in [BJR09] which says that the process M given by the formula

$$M_t := A_t - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} dA^p_s \tag{4.8}$$

is a $\mathbb{F}^\tau$-martingale. Thus, $\Lambda_t = \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} dA^p_s$. Moreover, if Λ is absolutely continuous with respect to Lebesgue's measure, it can be expressed as $\Lambda_t = \int_0^{t\wedge\tau} \lambda_s ds$. Then, the process λ is called the intensity of the default time τ.

Let us remark that if we assume that the hazard process Γ is continuous then $A_t^p = 1-e^{-\Gamma_t} = 1+\int_0^t e^{-\Gamma_s} d\Gamma_s$ and $\Lambda_t = -\Gamma_{t\wedge\tau}$. In the case of discontinuous Γ, this is no longer valid, see Section 4.4.3.

We would like to recall that compensators were studied for example in the papers [GZ08, JMP11].

4.4.1 Stability over supremum and infimum

Proposition 4.15. *If τ_1 and τ_2 are constructed as in the Cox model such that the associated random couple (Θ^1, Θ^2) is jointly independent from $\mathcal{F}_\infty$. Then $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ satisfy the immersion property.*

Proof. We use the characterization of hypothesis $(\mathcal{H})$ in a progressive enlargement of filtration setting given in Proposition 1.40. As τ_i for $i = 1, 2$ are Cox times we can express them as

$$\tau_1 = \inf\{t : \Gamma_t^1 \geq \Theta^1\} \quad \text{and} \quad \tau_2 = \inf\{t : \Gamma_t^2 \geq \Theta^2\}$$

where Γ^i for $i = 1, 2$ are increasing, càdlàg, $\mathbb{F}$-adapted processes and Θ^i for $i = 1, 2$ are two exponentially distributed random variables with mean 1, and jointly independent from $\mathcal{F}_\infty$. For the infimum we obtain

$$\mathbb{P}(\tau_1 \wedge \tau_2 > t|\mathcal{F}_\infty) = \mathbb{P}(\tau_1 > t, \tau_2 > t|\mathcal{F}_\infty) = \mathbb{P}(\Theta_1 > \Gamma_t^1, \Theta_2 > \Gamma_t^2|\mathcal{F}_\infty),$$
$$\mathbb{P}(\tau_1 \wedge \tau_2 > t|\mathcal{F}_t) = \mathbb{P}(\tau_1 > t, \tau_2 > t|\mathcal{F}_t) = \mathbb{P}(\Theta_1 > \Gamma_t^1, \Theta_2 > \Gamma_t^2|\mathcal{F}_t).$$

Then, as Θ_i for $i = 1, 2$ are independent from $\mathcal{F}_\infty$ (so from $\mathcal{F}_t$ as well) and Γ_t^i for $i = 1, 2$ are $\mathcal{F}_t$-measurable (so $\mathcal{F}_\infty$-measurable as well), we get that

$$\mathbb{P}(\tau_1 \wedge \tau_2 > t|\mathcal{F}_\infty) = \mathbb{P}(\tau_1 \wedge \tau_2 > t|\mathcal{F}_t) = \psi(\Gamma_t^1, \Gamma_t^2)$$

where $\psi(x, y) = \mathbb{P}(\Theta_1 > x, \Theta_2 > y)$. In a similar way, we obtain

$$\mathbb{P}(\tau_1 \vee \tau_2 \leq t|\mathcal{F}_\infty) = \mathbb{P}(\tau_1 \vee \tau_2 \leq t|\mathcal{F}_t) = \widetilde{\psi}(\Gamma_t^1, \Gamma_t^2)$$

where $\widetilde{\psi}(x, y) = \mathbb{P}(\Theta_1 \leq x, \Theta_2 \leq y)$. Thus the supremum and infimum of Cox's times fulfil the immersion property. ■

4.4.2 Dual optional projection and general hazard process

Lemma 4.16. *(a) The $\mathbb{F}$-dual optional projection of A equals $A^o = 1 - e^{-\Gamma}$.*
(b) An exhausting sequence for a thin part of τ coincides with exhausting sequence of the jumps of Γ.

Proof. (a) This is a direct application of (2.4) and (4.7).
(b) is a consequence of (a) and Lemma 2.8. ■

Lemma 4.17. *Let Y be a bounded $\mathbb{F}$-optional process. Then for any $t < T \leq \infty$*

$$\mathbb{E}(Y_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{F}_t^\tau) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\Big(- \int_t^T Y_s de^{-\Gamma_s} | \mathcal{F}_t \Big).$$

Proof. In the first step, we use the Lemma 1.38

$$\mathbb{E}(Y_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{F}_t^\tau) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}(Y_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{F}_t).$$

Now it is enough to show that for each set $B \in \mathcal{F}_t$ we have

$$\mathbb{E}(Y_\tau \mathbb{1}_{\{t<\tau\leq T\}} \mathbb{1}_B) = \mathbb{E}\Big(- \mathbb{1}_B \int_t^T Y_s de^{-\Gamma_s} \Big).$$

This follows from the property of the dual optional projection of the process $Y\mathbb{1}_{]t,T]}\mathbb{1}_B$. ■

4.4.3 Examples of pure jump process as hazard process

4.4.3.1 Compound Poisson process example

Let us take a compound Poisson process X with intensity η and jump size distribution F (with no atoms at 0 and with support in $\mathbb{R}^+$) as hazard process Γ. Equivalently $X_t = \sum_{n=1}^{N_t} Y_n$, where N is a Poisson process with intensity η and Y_n are i.i.d. random variables with distribution F independent from N. Let θ_n denote the time of the n-th jump of N with $\theta_0 = 0$. We also consider ν a positive finite measure on $\mathbb{R}$ such that $\nu(dy) = \eta F(dy)$ and μ a measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}))$ such that $\mu(dt, dy) = \sum_{n=1}^\infty \delta_{\theta_n, Y_n}(dt, dy)$. Then, the Doob-Meyer decomposition of the supermartingale Z is given by

$$\begin{aligned} e^{-X_t} = 1 + \int_0^t \int_{\mathbb{R}} (e^{-(X_{u-}+y)} - e^{-X_{u-}})(\eta(du, dy) - du\nu(dy)) \\ + \int_0^t \int_{\mathbb{R}} (e^{-(X_{u-}+y)} - e^{-X_{u-}})\nu(dy)du. \end{aligned}$$

Then, by (4.8), the compensator of $\mathbb{1}_{[\![\tau,\infty[\![}$ can be computed as

$$\begin{aligned} \Lambda_t &= \int_0^{t\wedge\tau} e^{X_{u-}} \int_{\mathbb{R}} (e^{-X_{u-}} - e^{-(X_{u-}+y)})\nu(dy)du \\ &= \int_0^{t\wedge\tau} \int_{\mathbb{R}} (1 - e^{-y})\nu(dy)du \end{aligned}$$

and the intensity equals to $\lambda \equiv \int_{\mathbb{R}} (1 - e^{-y})\nu(dy)$. In the case of Poisson process the intensity equals $\lambda \equiv \eta(1 - e^{-1})$.

4.4.3.2 Marked point process example

Let $\Phi = ((\theta_n, Y_n))_n$ be a marked point process (MPP), i.e.,

1. $(\theta_n)_n$ is a sequence of stopping times satisfying $0 \leq \theta_n \leq \theta_{n+1}$,

2. Y_n are random variables, linked with the stopping times $\theta_n$with values in $\mathbb{R}$, called marks.

We associate a pure jump process X to MPP Φ such that $X_t = \sum_{n:\ \theta_n \leq t} Y_n$. Moreover we define the counting measure μ of the MPP Φ which is a random measure on $\mathbb{R}_+ \times \mathbb{R}$ such that $\mu(dt, dy) = \sum_{n=1}^{\infty} \delta_{\theta_n, Y_n}(dt, dy)$ (i.e. $\mu(dt, dy) = \sum_{n=1}^{\infty} \delta_{\theta_n, \Delta X_n}(dt, dy)$). The notion of integral process with respect to random measure μ will be important for further consideration. In regards to that, let us first consider functions H on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ which are measurable with respect to the σ-field $\mathcal{O} \otimes \mathcal{B}(\mathbb{R})$. A function H is predictable if it is measurable with respect to the σ-field $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$.

We recall two definitions [Pri03, Definition 1.1.39] and [Pri03, Definition 1.1.40].

Definition 4.18. The integral process denoted by $H \star \mu$ is given by

$$(H \star \mu)_t := \int_0^t \int_{\mathbb{R}} H(u, y) \mu(du, dy),$$

if $\int_0^t \int_{\mathbb{R}} |H(u, y)| \mu(du, dy)$ is finite and is equal to ∞ otherwise.

Definition 4.19. The compensator of a random measure μ is the unique random measure ν such that

(a) ν is a predictable random measure, i.e., $H \star \nu$ is predictable for each predictable function H,

(b) $\mu - \nu$ is a local martingale measure, i.e. for every predictable function H such that $|H| \star \mu$ is increasing and locally integrable, the process $(H \star \mu - H \star \nu)$ is a local martingale.

Now it is possible to compute the Doob-Meyer decomposition of the supermartingale $Z = e^{-X}$. We have

$$\begin{aligned} e^{-X_t} = 1 &+ \int_0^t \int_{\mathbb{R}} (e^{-(X_{u-}+y)} - e^{-X_{u-}})(\mu(du, dy) - \nu(du, dy)) \\ &+ \int_0^t \int_{\mathbb{R}} (e^{-(X_{u-}+y)} - e^{-X_{u-}}) \nu(du, dy). \end{aligned}$$

Then, the compensator of $1\!\!1_{[\![\tau, \infty[\![}$ is equal to

$$\begin{aligned} \Lambda_t &= \int_0^{t \wedge \tau} e^{X_{u-}} \int_{\mathbb{R}} (e^{-X_{u-}} - e^{-(X_{u-}+y)}) \nu(du, dy) \\ &= \int_0^{t \wedge \tau} \int_{\mathbb{R}} (1 - e^{-y}) \nu(du, dy). \end{aligned}$$

The intensity exists if and only if the compensator measure ν is absolutely continuous in the first parameter with respect to Lebesgue measure, i.e., $\nu(dt, dy) = \widetilde{\nu}(t, dy)dt$. In this case intensity is equal to $\lambda_t = \int_{\mathbb{R}} (1 - e^{-y}) \widetilde{\nu}(t, dy)$.

Part II

Arbitrages in enlarged markets

Chapter 5

Arbitrages in a progressive enlargement setting

5.1 Introduction

This chapter is based on a joint paper with Tahir Choulli, Jun Deng and Monique Jeanblanc [ACDJ14a].

We study a financial market in which some assets, with prices adapted with respect to a reference filtration $\mathbb{F}$, are traded. One then assumes that an agent has some extra information, and may use strategies that are predictable with respect to a larger filtration $\mathbb{F}^\tau$. This extra information is modeled by the knowledge of some random time τ, when this time occurs. We restrict our study to progressive enlargement of filtration setting, and we pay a particular attention to honest times. Our goal is to detect if the knowledge of τ allows for some arbitrage, i.e., if using $\mathbb{F}^\tau$-predictable strategies, the agent can make profit.

In this chapter we consider two main notions of no-arbitrage, namely no classical arbitrage and No Unbounded Profit with Bounded Risk. To the best of our knowledge, there are no references for the case of classical arbitrages in a general setting. The goal of the present chapter is firstly to introduce the problem, to solve it in some specific cases and to give some explicit examples of classical arbitrages (with a proof different from the one in [FJS12]), and secondly to give, in some specific models, an easy proof of NUPBR condition.

In the case of honest times avoiding stopping times in a continuous filtration, the same problem was studied in [FJS12] where the authors have investigated several kinds of arbitrages. We refer the reader to that paper for an extensive list of related results in the literature.

This chapter is organized as follows: Section 2 presents the problem and recalls some definitions and results on arbitrages. In Section 3, we study two classical situations in enlargement of filtration theory, namely immersion and positive density hypothesis cases. Section 4 concerns honest times, and we show that, in case of a complete market, there exist classical arbitrages before and after the honest time, and we give a way to construct these arbitrages. This fact is illustrated by many examples, where we exhibit these arbitrages in a closed form.

In Section 5, we study some examples of non-honest times. In Section 5.6, we study NUPBR condition before a random time and after an honest time, in some specific examples. The general study will be done in Chapter 6 and Chapter 7.

5.2 General framework

We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F}$ satisfies the usual hypotheses and $\mathcal{F}_\infty \subset \mathcal{G}$, and a random time τ. We assume that the financial market where a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a riskless asset S^0 (assumed, for simplicity, to have a constant price so that the risk-free interest rate is null) are traded is arbitrage free. More precisely, without loss of generality we assume that S is a $(\mathbb{P}, \mathbb{F})$-(local) martingale. In this paper, the horizon is equal to ∞. We denote by $\mathbb{F}^\tau$ the progressively enlarged filtration of $\mathbb{F}$ by τ, i.e., $\mathbb{F}^\tau = (\mathcal{F}_t^\tau)_{t\geq 0}$ with

$$\mathcal{F}_t^\tau := \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\tau \wedge s)\right).$$

As in Section 1.2.3, we associate to τ two $\mathbb{F}$-supermartingales by

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) \quad \text{and} \quad \widetilde{Z}_t := \mathbb{P}(\tau \geq t | \mathcal{F}_t).$$

Let us denote by A^o the $\mathbb{F}$-dual optional projection of $A := 1\!\!1_{[\![\tau,\infty[\![}$ and define the $\mathbb{F}$-martingale

$$m_t := Z_t + A_t^o. \tag{5.1}$$

We start by an elementary remark: assume that there are no arbitrages using $\mathbb{F}^\tau$-predictable strategies and that $\mathbb{P}$ is the unique probability measure making S an $\mathbb{F}$-martingale. So, in particular, the $(S, \mathbb{F})$ market is complete (i.e., the market where (S, S^0) are traded). Then, roughly speaking, S would be a $(\mathbb{Q}, \mathbb{F}^\tau)$-martingale for some equivalent martingale measure $\mathbb{Q}$, hence would be also a $(\mathbb{Q}, \mathbb{F})$-martingale[1] and $\mathbb{Q}$ will coincide with $\mathbb{P}$ on $\mathbb{F}$. This implies that any $(\mathbb{Q}, \mathbb{F})$-martingale is a $(\mathbb{Q}, \mathbb{F}^\tau)$-martingale.

Another trivial remark is that, in the particular case where τ is an $\mathbb{F}$-stopping time, the enlarged filtration and the reference filtration are the same. Therefore, no-arbitrage conditions hold before and after τ.

5.2.1 Illustrative examples

We study here two basic examples, in order to show in a first step how arbitrages can occur in a Brownian filtration, and in a second step that discontinuous models present some difficulties.

[1]Note that if S is a $(\mathbb{Q}, \mathbb{F}^\tau)$-strict local martingale for some equivalent martingale measure $\mathbb{Q}$, one can not deduce that it is also a $(\mathbb{Q}, \mathbb{F})$-local martingale.

5.2.1.1 Brownian case

Let $dS_t = S_t \sigma dW_t$, where W is a Brownian motion and σ a constant, be the price of the risky asset. This martingale S goes to 0 a.s. when t goes to infinity, hence the random time $\tau := \sup\{t : S_t = S^*\}$ where $S^* = \sup_{s\geq 0} S_s$ is a finite honest time, and obviously leads to an arbitrage before τ: at time 0, buy one share of S (at price S_0), borrow S_0, then, at time τ, reimburse the loan S_0 and sell the share of the asset at price S_τ. The gain is $S_\tau - S_0 > 0$ with an initial wealth null. There are also arbitrages after τ: at time τ, take a short position on S, i.e., hold a self financing portfolio with value V such that $dV_t = -dS_t, V_\tau = 0$. Usually shortselling positions are not admissible, since $V_t = -S_t + S_\tau$ is not bounded below. Here $-S_t + S_\tau$ is positive, hence shortselling is an arbitrage opportunity.

5.2.1.2 Poisson case

Let N be a Poisson process with intensity η and M be its compensated martingale. We define the price process S as $dS_t = S_{t-}\psi dM_t, S_0 = 1$ with ψ is a constant satisfying $\psi > -1$ and $\psi \neq 0$, so that

$$S_t = \exp(-\lambda\psi t + \ln(1+\psi)N_t)\,.$$

Since $\frac{N_t}{t}$ goes to λ a.s. when t goes to infinity and $\ln(1+\psi) - \psi < 0$, S_t goes to 0 a.s. when t goes to infinity. The random time

$$\tau := \sup\{t : S_t = S^*\}$$

with $S^* = \sup_{s\geq 0} S_s$ is a finite honest time.

If $\psi > 0$, then $S_\tau \geq S_0$ and an arbitrage opportunity is realized at time τ, with a long position in the stock. If $\psi < 0$, then the arbitrage is not so obvious. We shall discuss that with more details in Section 5.4.2.

There are arbitrages after τ, selling at time τ a contingent claim with payoff 1, paid at the first time ϑ after τ when $S_t > \sup_{s\leq\tau} S_s$. For $\psi > 0$, it reduces to $S_\tau = \sup_{s\leq\tau} S_s$, and, for $\psi < 0$, one has $S_{\tau-} = \sup_{s\leq\tau} S_s$. At time $t_0 = \tau$, the non informed buyer will agree to pay a positive price, the informed seller knows that the exercise will be never done.

5.3 Some particular cases

5.3.1 Immersion assumption, equivalence Jacod's hypothesis

Firstly we look at the case where the filtration $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$ (see Definition 1.27).

Lemma 5.1. *If the immersion property is satisfied under a probability $\mathbb{Q}$ on $\mathbb{F}^\tau$, such that S is a $(\mathbb{Q},\mathbb{F})$-martingale, all the three concepts of NFLVR, NA and NUPBR hold.*

Proof. Let S be a $(\mathbb{Q},\mathbb{F})$-local martingale, then it is a $(\mathbb{Q},\mathbb{F}^\tau)$-local martingale as well. ∎

The second case considered in this section corresponds to a random time τ satisfying equivalence Jacod's hypothesis (see Definition 1.31 (b)).

Lemma 5.2. *If S is a $(\mathbb{P},\mathbb{F})$-martingale and if the random time τ satisfies the equivalence Jacod's hypothesis then NFLVR, NA and NUPBR hold for $\mathbb{F}^\tau$.*

Proof. By Theorem 1.36, there exists an equivalent measure $\mathbb{P}^*$ under which the immersion is satisfied. It is now obvious that, if S is a $(\mathbb{P},\mathbb{F})$-martingale, NFLVR holds in the enlarged filtration $\mathbb{F}\vee\sigma(\tau)$, hence in $\mathbb{F}^\tau$. Indeed, the $(\mathbb{P},\mathbb{F})$-martingale S is - using the independence property - a $(\mathbb{P}^*,\mathbb{F}\vee\sigma(\tau))$-martingale, so that S, being $\mathbb{F}^\tau$-adapted, is a $(\mathbb{P}^*,\mathbb{F}^\tau)$-martingale and $\mathbb{P}^*$ is an equivalent martingale measure. If S is only a $(\mathbb{P},\mathbb{F})$-local martingale, then one proceeds as follows. Let $\{T_n\}_{n\in\mathbb{N}}$ be an $\mathbb{F}$-localizing sequence for S, meaning that S^{T_n} is a $(\mathbb{P},\mathbb{F})$-martingale, for every $n\in\mathbb{N}$. Then, repeating previous resoning, it holds that S^{T_n} is a $(\mathbb{P}^*,\mathbb{F}^\tau)$-martingale. Thus S is a $(\mathbb{P}^*,\mathbb{F}^\tau)$-local martingale and $\mathbb{P}^*$ is an equivalent martingale measure. ∎

5.4 Classical arbitrages for a class of honest times

We start with the following obvious (but useful) result. Note that it is valid for any random time, not necessary an honest time.

Lemma 5.3. *Assume that the financial market $(S,\mathbb{F})$ is complete and let φ be a $\mathbb{F}$-predictable process satisfying $m=1+\varphi\centerdot S$, where m is defined in (5.1). If $m_\tau\geq 1$ and $\mathbb{P}(m_\tau>1)>0$, then, the $\mathbb{F}^\tau$-predictable process $\varphi 1\!\!1_{[\![0,\tau]\!]}$ is a classical arbitrage strategy in the market "before τ", i.e., in $(S^\tau,\mathbb{F}^\tau)$.*

Proof. The $\mathbb{F}$-predictable process φ exists due to the market completeness. Hence $1\!\!1_{[\![0,\tau]\!]}\varphi$ is an $\mathbb{F}^\tau$-predictable admissible self-financing strategy with initial value 1 and final value $m_\tau-1$ satisfying $m_\tau-1\geq 0$ a.s. and $\mathbb{P}(m_\tau-1>0)>0$, so it is a classical arbitrage strategy in $(S^\tau,\mathbb{F}^\tau)$. ∎

Herein, we generalize the results obtained in [FJS12] – which are established for honest times avoiding $\mathbb{F}$-stopping times in a complete market with continuous filtration – to any complete market and to a much broader class of honest times that will be defined below. Throughout this section, we denote by $\mathcal{T}_s$ the set of all $\mathbb{F}$-stopping times, $\mathcal{T}_h$ the subset of all $\mathbb{F}$-honest times, and $\mathcal{R}$ the set of random times given by

$$\mathcal{R}:=\Big\{\tau\ \text{ random time }\ \Big|\ \exists\ \Gamma\in\mathcal{G}\text{ and }T\in\mathcal{T}_s\text{ such that }\tau=T1\!\!1_\Gamma+\infty1\!\!1_{\Gamma^c}\Big\}. \tag{5.2}$$

Note also the connection between the class $\mathcal{R}$ and Lemma 3.7. $\mathcal{R}$ consists precisely of random times which are honest times and pseudo-stopping times.

Proposition 5.4. *The following inclusions hold*

$$\mathcal{T}_s\subset\mathcal{R}\subset\mathcal{T}_h.$$

Proof. The first inclusion is clear. For the inclusion $\mathcal{R}\subset\mathcal{T}_h$, we give, for ease of the reader two different proofs. Let us take $\tau\in\mathcal{R}$, i.e., $\tau=T1\!\!1_\Gamma+\infty1\!\!1_{\Gamma^c}$ for T an $\mathbb{F}$-stopping time and

$\Gamma \in \mathcal{G}$.
1) On $\{\tau < t\} = \{T < t\} \cap \Gamma$, we have $\tau = T \wedge t$ and $T \wedge t$ is $\mathcal{F}_t$-measurable. Thus, τ is an honest time.
2) By Theorem 1.43 (c) it is enough to show that on $\{\tau < \infty\}$, $\widetilde{Z}_\tau = 1$. Indeed,

$$\widetilde{Z}_t = \mathbb{1}_{\{T \geq t\}} \mathbb{P}(\Gamma|\mathcal{F}_t) + \mathbb{P}(\Gamma^c|\mathcal{F}_t),$$

so that

$$\begin{aligned}\mathbb{1}_{\{\tau<\infty\}} \widetilde{Z}_\tau &= \mathbb{1}_\Gamma \mathbb{1}_{\{T<\infty\}} \widetilde{Z}_T = \mathbb{1}_\Gamma \mathbb{1}_{\{T<\infty\}} (\mathbb{1}_{\{T \geq T\}} \mathbb{P}(\Gamma|\mathcal{F}_T) + \mathbb{P}(\Gamma^c|\mathcal{F}_T)) \\ &= \mathbb{1}_\Gamma \mathbb{1}_{\{T<\infty\}} = \mathbb{1}_{\{\tau<\infty\}}.\end{aligned}$$

This proves that τ is an honest time. ■

The following theorem represents our principal result in the general framework.

Theorem 5.5. *Assume that $(S, \mathbb{F})$ is a complete market and let φ be an $\mathbb{F}$-predictable process satisfying $m = 1 + \varphi \centerdot S$. Then the following assertions hold.*
(a) If τ is an honest time, and $\tau \notin \mathcal{R}$, then the $\mathbb{F}^\tau$-predictable process $\varphi^b = \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is a classical arbitrage strategy in the market "before τ", i.e., in $(S^\tau, \mathbb{F}^\tau)$.
(b) If τ is an honest time, which is not an $\mathbb{F}$-stopping time, and if $\{\tau = \infty\} \in \mathcal{F}_\infty$, then the $\mathbb{F}^\tau$-predictable process $\varphi^a = -\varphi \mathbb{1}_{]\!]\tau,\vartheta]\!]}$, with $\mathbb{F}^\tau$-stopping time defined as

$$\vartheta := \inf\{t > \tau : \widetilde{Z}_t \leq \frac{1 - \Delta A^o_\tau}{2}\}, \tag{5.3}$$

is a classical arbitrage strategy in the market "after τ", i.e., in $(S - S^\tau, \mathbb{F}^\tau)$.

Proof. (a) From $m = \widetilde{Z} + A^o_-$ and $\widetilde{Z}_\tau = 1$, we deduce that $m_\tau \geq 1$. Since $\tau \notin \mathcal{R}$, one has $\mathbb{P}(m_\tau > 1) = \mathbb{P}(A^o_{\tau-} > 0) > 0$. Then, by Lemma 5.3, the process $\varphi^b = \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage strategy in $(S^\tau, \mathbb{F}^\tau)$.
(b) From $m = Z + A^o$ and Theorem 1.43 (d), one obtains that, for $t > \tau$, $m_t - m_\tau = Z_t - Z_\tau \geq -1$. On the other hand, using $m = \widetilde{Z} + A^o_-$, one obtains that, for $t > \tau$, $m_t - m_\tau = \widetilde{Z}_t - 1 + \Delta A^o_\tau$. Assumption $\{\tau = \infty\} \in \mathcal{F}_\infty$ ensures that $\widetilde{Z}_\infty = \mathbb{1}_{\{\tau=\infty\}}$ and in particular $\{\tau < \infty\} \subset \{\widetilde{Z}_\infty = 0\}$. So, the $\mathbb{F}^\tau$-stopping time ϑ defined in (5.3) satisfies $\{\vartheta < \infty\} = \{\tau < \infty\}$. Then,

$$m_\vartheta - m_\tau = \widetilde{Z}_\vartheta - 1 + \Delta A^o_\tau \leq \frac{\Delta A^o_\tau - 1}{2} \leq 0,$$

and, as τ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\vartheta - m_\tau < 0) = \mathbb{P}(\Delta A^o_\tau < 1) > 0.$$

Hence $-\int_\tau^{t\wedge\vartheta} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\vartheta}$ is the value of an admissible self-financing strategy $\varphi^a = -\varphi \mathbb{1}_{]\!]\tau,\vartheta]\!]}$ with initial value 0 and terminal value $m_\tau - m_\vartheta \geq 0$ which satisfies $\mathbb{P}(m_\tau - m_\vartheta > 0) > 0$. This ends the proof of the theorem. ■

Remark 5.6. To make a link to Section 5.3.1 we recall that:

(a) if τ is a finite honest time and is not an $\mathbb{F}$-stopping time, then, by Corollary 3.8, immersion does not hold;

(b) if τ is an honest time which does not take only countably many values, then, by Lemma 4.11, the density hypothesis is not satisfied.

5.4.1 Classical arbitrage opportunities in a Brownian filtration

In this section, we develop practical market models S and honest times τ within the Brownian filtration for which one can compute explicitly the arbitrage opportunities for both before and after τ. For other examples of honest times, and associated classical arbitrages we refer the reader to [FJS12] (Note that the arbitrages constructed in that paper are different from our arbitrages). Throughout this section, we assume given a one-dimensional Brownian motion W and $\mathbb{F}$ is its augmented natural filtration. The market model is represented by the bank account process which is the constant one and one stock price process which is given by

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \qquad \sigma > 0 \text{ given.}$$

It is worth mentioning that in this context of Brownian filtration, for any process V with locally integrable variation, its $\mathbb{F}$-dual optional projection is equal to its $\mathbb{F}$-dual predictable projection, i.e., $V^{o,\mathbb{F}} = V^{p,\mathbb{F}}$. All the examples given below correspond to random times that are the end of optional sets, hence are honest times.

5.4.1.1 Last passage time at a given level

Proposition 5.7. *Consider the following random times*

$$\boxed{\tau := \sup\{t \,:\, S_t = a\} \qquad \textit{and} \qquad \vartheta := \inf\{t > \tau \;:\; S_t \leq \tfrac{a}{2}\}}$$

where $0 < a < 1$. Then, the following assertions hold.
(a) The model "before τ" $(S^\tau, \mathbb{F}^\tau)$ admits a classical arbitrage opportunity given by the $\mathbb{F}^\tau$-predictable process

$$\varphi^b = \frac{1}{a} \mathbb{1}_{\{S<a\}} \mathbb{1}_{]\!]0,\tau]\!]}.$$

(b) The model "after τ" $(S - S^\tau, \mathbb{F}^\tau)$ admits a classical arbitrage opportunity given by $\mathbb{F}^\tau$-predictable process

$$\varphi^a = -\frac{1}{a} \mathbb{1}_{\{S<a\}} \mathbb{1}_{]\!]\tau,\vartheta]\!]}.$$

Proof. It is clear that τ is a finite honest time [Theorem 1.43 (c)], and does not belong to the set $\mathcal{R}$ defined in (5.2). Thus τ fulfills the assumptions of assertions of Theorem 5.5. We now compute the predictable process φ such that $m = 1 + \varphi \centerdot S$. To this end, we calculate Z as follows. Using [JYC09, Exercise 1.2.3.10], we derive

$$1 - Z_t = \mathbb{P}\left(\sup_{t<u} S_u \le a|\mathcal{F}_t\right) = \mathbb{P}\left(\sup_u \widetilde{S}_u \le \frac{a}{S_t}|\mathcal{F}_t\right) = \Phi\left(\frac{a}{S_t}\right)$$

where $\widetilde{S}_u = \exp(\sigma\widetilde{W}_u - \frac{1}{2}\sigma^2 u)$, $\widetilde{W}$ independent of $\mathcal{F}_t$ and

$$\Phi(x) = \mathbb{P}\left(\sup_u \widetilde{S}_u \le x\right) = \mathbb{P}(\frac{1}{U} \le x) = \mathbb{P}(\frac{1}{x} \le U) = (1 - \frac{1}{x})^+,$$

where U is a random variable with uniform law. Thus we get $Z_t = 1-(1-\frac{S_t}{a})^+$ (in particular $Z_\tau = \widetilde{Z}_\tau = 1$), and

$$dZ_t = 1\!\!1_{\{S_t \le a\}}\frac{1}{a}dS_t - \frac{1}{2a}d\ell^a_t$$

where ℓ^a is the local time of the S at the level a (see [HWY92, p.252] for the definition of the local time). Therefore, we deduce that

$$m = 1 + \varphi \centerdot S.$$

Note that $\vartheta := \inf\{t > \tau \ : \ S_t \le \frac{a}{2}\} = \inf\{t > \tau \ : \ 1 - (1 - \frac{S_t}{a})^+ \le \frac{1}{2}\}$, so ϑ coincides with (5.3). Theorem 5.5 ends the proof of the proposition. ∎

5.4.1.2 Last passage time at a level before maturity

Our second example of random time in this section, takes into account finite horizon. In this example, we introduce the following notation

$$H(z, y, s) := e^{-zy}\mathcal{N}\left(\frac{zs - y}{\sqrt{s}}\right) + e^{zy}\mathcal{N}\left(\frac{-zs - y}{\sqrt{s}}\right), \tag{5.4}$$

where $\mathcal{N}(x)$ is the cumulative distribution function of the standard normal distribution.

Proposition 5.8. *Consider the following random time (an honest time)*

$$\boxed{\tau_1 := \sup\{t \le 1 \ : \ S_t = b\}}$$

where b is a positive real number, $0 < b < 1$. Let V and β be given by

$$V_t := \alpha - \gamma t - W_t \quad \textit{with } \alpha = \frac{\ln b}{\sigma} \textit{ and } \gamma = -\frac{\sigma}{2}$$

$$\beta_t := e^{\gamma V_t}\left(\gamma H(\gamma, |V_t|, 1 - t) + sgn(V_t)\, H'_x(\gamma, |V_t|, 1 - t)\right),$$

with H defined in (5.4), and let ϑ be as in (5.3). Then, the following assertions hold.
(a) The model "before τ_1" $(S^{\tau_1}, \mathbb{F}^\tau)$ admits a classical arbitrage opportunity given by the $\mathbb{F}^\tau$-predictable process

$$\varphi^b := \frac{1}{\sigma S_t}\beta_t 1\!\!1_{[\![0,\tau_1]\!]}.$$

(b) The model "after τ_1" $(S - S^{\tau_1}, \mathbb{F}^\tau)$ admits a classical arbitrage opportunity given by $\mathbb{F}^\tau$-predictable process

$$\varphi^a := -\frac{1}{\sigma S_t}\beta_t 1\!\!1_{]\!]\tau_1,\vartheta]\!]}.$$

Proof. The proof of this proposition follows from Theorem 5.5 as long as we can write explicitly the martingale m as an integral stochastic with respect to S. This is the main focus of the remaining part of this proof. By Theorem 1.43 (c), the time τ_1 is honest. It is a finite random time. Honest time τ_1 can be seen as

$$\tau_1 \quad = \quad \sup\{t \le 1 \,:\, \gamma t + W_t = \alpha\} = \sup\{t \le 1 \,:\, V_t = 0\}.$$

Setting $T_0(V) = \inf\{t \,:\, V_t = 0\}$, we obtain, using standard computations (see [JYC09, p.145-148])

$$1 - Z_t = (1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t))1\!\!1_{\{T_0(V) \le t \le 1\}} + 1\!\!1_{\{t>1\}},$$

where H is given in (5.4). In particular $Z_\tau = \widetilde{Z}_\tau = 1$. Using Itô's lemma, we obtain the decomposition of $1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t)$ as a semimartingale. The martingale part of Z is given by $dm_t = \beta_t dW_t = \frac{1}{\sigma S_t}\beta_t dS_t$, which ends the proof. ∎

5.4.2 Arbitrage opportunities in a Poisson filtration

Throughout this section, we suppose given a Poisson process N, with intensity rate $\eta > 0$, and natural filtration $\mathbb{F}$. For $n \in \mathbb{N}$, let θ_n denote the time of the n-th jump of N with $\theta_0 = 0$. The stock price process is given by

$$dS_t = S_{t-}\psi dM_t, \quad S_0 = 1, \qquad M_t := N_t - \eta t,$$

or equivalently $S_t = \exp(-\eta\psi t + \ln(1+\psi)N_t)$, where $\psi > -1$. In what follows, we introduce the notation

$$\alpha := \ln(1+\psi), \qquad \mu := \frac{\eta\psi}{\ln(1+\psi)} \quad \text{and} \quad Y_t := \mu t - N_t,$$

so that $S_t = \exp(-\ln(1+\psi)Y_t)$. We associate to the process Y its ruin probability, denoted by $\Psi(x)$, given in (2.8) by

$$\Psi(x) = \mathbb{P}(t^x < \infty) \quad \text{with} \quad t^x = \inf\{t : x + Y_t < 0\} \quad \text{for every} \quad x \ge 0.$$

5.4.2.1 Last passage time at a given level

The next proposition answers an arbitrage question in the case of the honest time defined by (2.4) and studied in Section 2.3.3.1.

Proposition 5.9. *Suppose that $\psi > 0$ and let $a := -\frac{1}{\alpha}\ln b$ and*

$$\varphi := \frac{\Psi(Y_- - a - 1)1\!\!1_{\{Y_- \ge a+1\}} - \Psi(Y_- - a)1\!\!1_{\{Y_- \ge a\}} + 1\!\!1_{\{Y_- < a+1\}} - 1\!\!1_{\{Y_- < a\}}}{\psi S_-}.$$

For $0 < b < 1$, consider the following random time

$$\boxed{\tau := \sup\{t : \ S_t \geq b\} = \sup\{t : Y_t \leq a\}}$$

Then the following assertions hold.
(a) The model "before τ" $(S^\tau, \mathbb{F}^\tau)$ admits a classical arbitrage opportunity given by the $\mathbb{F}^\tau$-predictable process $\varphi^b := \varphi 1\!\!1_{[\![0,\tau]\!]}$.
(b) The model "after τ" $(S - S^\tau, \mathbb{F}^\tau)$ admits a classical arbitrage opportunity given by the $\mathbb{F}^\tau$-predictable process $\varphi^a := -\varphi 1\!\!1_{]\!]\tau,\vartheta]\!]}$, with ϑ as in (5.3).

Proof. Since $\psi > 0$, one has $\mu > \eta$ so that Y goes to ∞ as t goes to infinity, and τ is finite. The supermartingale Z associated with the time τ is

$$Z_t = \Psi(Y_t - a) 1\!\!1_{\{Y_t \geq a\}} + 1\!\!1_{\{Y_t < a\}} = 1 + 1\!\!1_{\{Y_t \geq a\}} \left(\Psi(Y_t - a) - 1\right).$$

See Section 2.3.3.1 for more details on this example.

We set $\kappa = \dfrac{\mu}{\eta} - 1$, and deduce that $\Psi(0) = (1+\kappa)^{-1}$ (see [AA10]). Define

$$T_1 = \inf\{t > 0 : Y_t = a\}$$

and then, for each $n > 1$, $T_n = \inf\{t > T_{n-1} : Y_t = a\}$. It is proved in Proposition 2.25 that the times $(T_n)_n$ are $\mathbb{F}$-predictable stopping times. By Lemma 2.8, since

$$\mathbb{P}(\tau = T_n | \mathcal{F}_{T_n}) = 1 - \Psi(0) = \frac{\kappa}{1+\kappa},$$

the $\mathbb{F}$-dual optional projection A^o of the process $1\!\!1_{[\![\tau,\infty[\![}$ equals

$$A^o = \frac{\kappa}{1+\kappa} \sum_n 1\!\!1_{[\![T_n,\infty[\![}.$$

As a result the process A^o is predictable, and hence $Z = m - A^o$ is the Doob-Meyer decomposition of Z. Thus we can get

$$\Delta m = Z - {}^pZ$$

where ${}^pZ$ is the $\mathbb{F}$-predictable projection of Z. To calculate ${}^pZ$, we write the process Z in a more adequate form. To this end, we first remark that

$$\begin{aligned} 1\!\!1_{\{Y \geq a\}} &= 1\!\!1_{\{Y_- \geq a+1\}} \Delta N + (1 - \Delta N) 1\!\!1_{\{Y_- \geq a\}} \\ 1\!\!1_{\{Y < a\}} &= 1\!\!1_{\{Y_- < a+1\}} \Delta N + (1 - \Delta N) 1\!\!1_{\{Y_- < a\}}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \Delta m &= \left(\Psi(Y_- - a - 1) 1\!\!1_{\{Y_- \geq a+1\}} - \Psi(Y_- - a) 1\!\!1_{\{Y_- \geq a\}} + 1\!\!1_{\{Y_- < a+1\}} - 1\!\!1_{\{Y_- < a\}}\right) \Delta N \\ &= \psi S_- \varphi \Delta M = \varphi \Delta S. \end{aligned}$$

Since the two martingales m and S are purely discontinuous, we deduce that $m = 1 + \varphi \centerdot S$. Therefore, the proposition follows from Theorem 5.5. ∎

5.4.2.2 Time of supremum on fixed time horizon

The second example requires the following notations $S^*_t := \sup_{s\le t} S_s$, and

$$\bar\Phi(x,t) := \mathbb{P}(S^*_t > x), \quad \widehat\Phi(t) := \mathbb{P}(S^*_t \le 1), \quad \widetilde\Phi(x,t) := \mathbb{P}(S^*_t < x) \tag{5.5}$$

Proposition 5.10. *Consider the random time τ defined by*

$$\boxed{\tau := \sup\{t \le 1 : S_t = S^*_t\}}$$

where $S^*_t = \sup_{s\le t} S_s$. Then, the following assertions hold.
(a) The random time τ is an honest time.
(b) For $\psi > 0$, define the $\mathbb{F}^\tau$-predictable process φ as

$$\begin{aligned}\varphi_t := &\, 1\!\!1_{\{t<1\}}\left[\bar\Phi\left(\max(\tfrac{S^*_{t-}}{S_{t-}(1+\bar\Phi)},1),1-t\right)-\bar\Phi\left(\tfrac{S^*_{t-}}{S_{t-}},1-t\right)\right]\\ &+1\!\!1_{\{S^*_{t-}<S_{t-}(1+\bar\Phi)\}}\,\widehat\Phi(1-t)\\ &+\left[1\!\!1_{\{\max(S^*_{1-},S_{1-}(1+\bar\Phi))=S_0\}}-1\!\!1_{\{\max(S^*_{1-},S_{1-})=S_0\}}\right]1\!\!1_{\{t=1\}}.\end{aligned}$$

Then, $\varphi^b := \varphi 1\!\!1_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{F}^\tau)$, and $\varphi^a := -\varphi 1\!\!1_{]\!]\tau,\vartheta]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{F}^\tau)$. Here $\bar\Phi$ and $\widehat\Phi$ are defined in (5.5), and ϑ is defined similarly as in (5.3).
(c) For $-1 < \psi < 0$, define the $\mathbb{F}^\tau$-predictable process

$$\varphi_t := \frac{\bar\Phi 1\!\!1_{\{S^*_t=S_{t-}\}}\Phi(\frac{1}{1+\bar\Phi},1-t)+\bar\Phi(\widehat{\frac{S^*_t}{S_{t-}(1+\bar\Phi)}},1-t)-\bar\Phi(\frac{S^*_t}{S_{t-}},1-t)}{\bar\Phi S_{t-}}.$$

Then, $\varphi^b := \varphi 1\!\!1_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{F}^\tau)$, and $\varphi^a := -\varphi 1\!\!1_{]\!]\tau,\vartheta]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{F}^\tau)$.

Proof. Note that, if $-1 < \psi < 0$ the process S^* is continuous, $S_\tau < S^*_\tau = \sup_{t\in[0,1]} S_t$ on the set $\{\tau < 1\}$ and $S_{\tau-} = S^*_{\tau-} = \sup_{t\in[0,1]} S_t$. If $\psi > 0$, $S_{\tau-} < S^*_{\tau-} < \sup_{t\in[0,1]} S_t$ on the set $\{\tau < 1\}$.

Define the sets $(E_n)_{n=0}^\infty$ such that $E_0 = \{\tau = 1\}$ and $E_n = \{\tau = \theta_n\}$ with $n \ge 1$, with $(\theta_n)_n$ being the sequence of jumps of the Poisson process N. The sequence $(E_n)_{n=0}^\infty$ forms a partition of Ω. Then, $\tau = 1\!\!1_{E_0} + \sum_{n=1}^\infty \theta_n 1\!\!1_{E_n}$. Note that τ is not an $\mathbb{F}$ stopping time since $E_n \notin \mathcal{F}_{\theta_n}$ for any $n \ge 1$.

The supermartingale Z associated with the honest time τ is

$$\begin{aligned}Z_t &= \mathbb{P}(\sup_{s\in(t,1]} S_s > \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,1-t]} \widehat S_s > \frac{S^*_t}{S_t}|\mathcal{F}_t)\\ &= 1\!\!1_{(t<1)}\bar\Phi(\frac{S^*_t}{S_t},1-t),\end{aligned}$$

with $\widehat{S}$ an independent copy of S and $\bar{\Phi}(x,t)$ is given by (5.5).

As $\{\tau=\theta_n\}\subset\{\tau\le\theta_n\}\subset\{Z_{\theta_n}<1\}$, we have

$$Z_\tau = \mathbb{1}_{\{\tau=1\}}Z_1+\sum_{n=1}^{\infty}\mathbb{1}_{\{\tau=\theta_n\}}Z_{\theta_n}<1,\quad\text{and}\quad\{\widetilde{Z}=0<Z_-\}=\emptyset.$$

In the following, we will prove assertion (b). Thus, we suppose that $\psi>0$, and we calculate

$$\begin{aligned}
A^o_t &= \mathbb{P}(\tau=1|\mathcal{F}_1)\mathbb{1}_{\{t\ge1\}}+\sum_n\mathbb{P}(\tau=\theta_n|\mathcal{F}_{\theta_n})\mathbb{1}_{\{t\ge\theta_n\}}\\
&= \mathbb{1}_{\{S^*_1=S_0\,t\ge1\}}+\sum_n\mathbb{1}_{\{\theta_n<1\,S^*_{\theta_n-}<S_{\theta_n}\}}\mathbb{P}(\sup_{s\in[\theta_n,1[}S_s\le S_{\theta_n}|\mathcal{F}_{\theta_n})\mathbb{1}_{\{t\ge\theta_n\}}\\
&= \mathbb{1}_{\{S^*_1=S_0\,t\ge1\}}+\sum_n\mathbb{1}_{\{\theta_n<1,\,S^*_{\theta_n-}<S_{\theta_n-}(1+\bar{\Phi})\}}\,\widehat{\Phi}(1-\theta_n)\mathbb{1}_{\{t\ge\theta_n\}},
\end{aligned}$$

with $\widehat{\Phi}$ is given by (5.5). As before, we write

$$\begin{aligned}
A^o_t &= \mathbb{1}_{\{S^*_1=S_0\}}\mathbb{1}_{\{t\ge1\}}+\sum_{s\le t}\mathbb{1}_{\{s<1\}}\mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\bar{\Phi})\}}\,\widehat{\Phi}(1-s)\Delta N_s\\
&= \mathbb{1}_{\{S^*_1=S_0\}}\mathbb{1}_{\{t\ge1\}}+\int_0^{t\wedge1}\mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\bar{\Phi})\}}\,\widehat{\Phi}(1-s)\,dM_s\\
&\quad+\eta\int_0^{t\wedge1}\mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\bar{\Phi})\}}\,\hat{\Phi}(1-s)ds.
\end{aligned}$$

Remark that we have

$$\begin{aligned}
\mathbb{1}_{\{S^*_1=S_0\}} &= \left[\mathbb{1}_{\{\max(S^*_{1-},S_{1-}(1+\bar{\Phi}))=S_0\}}-\mathbb{1}_{\{\max(S^*_{1-},S_{1-})=S_0\}}\right]\Delta M_1\\
&\quad+\mathbb{1}_{\{\max(S^*_{1-},S_{1-})=S_0\}}.
\end{aligned}$$

and

$$\Delta m=\Delta Z+\Delta A^o = Z-\ {}^p(Z)+\Delta A^o-\ {}^p(\Delta A^o).$$

Then we re-write the process Z as follows

$$Z=\mathbb{1}_{[\![0,1[\![}\bar{\Phi}\left(\max(\frac{S^*_-}{S_-(1+\bar{\Phi})},1),1-t\right)\Delta M+(1-\Delta M)\mathbb{1}_{[\![0,1[\![}\bar{\Phi}\left(\frac{S^*_-}{S_-},1-t\right).$$

This implies that

$$Z-\ {}^p(Z)=\mathbb{1}_{[\![0,1[\![}\left[\bar{\Phi}\left(\max(\frac{S^*_-}{S_-(1+\bar{\Phi})},1),1-t\right)-\bar{\Phi}\left(\frac{S^*_-}{S_-},1-t\right)\right]\Delta M.$$

Thus by combining all these remarks, we deduce that

$$\Delta m = Z-\ {}^p(Z)+\Delta A^o-\ {}^p(\Delta A^o)=\varphi\Delta S,$$

where we used the fact that ${}^p(\widetilde{Z})=Z_-$ and $\widetilde{Z}=Z+\ \Delta A^o$. Then, the assertion (b) follows immediately from Theorem 5.5.

Next, we will prove assertion (c). Suppose that $-1 < \psi < 0$, and we calculate

$$
\begin{aligned}
A_t^o &= \mathbb{P}(\tau = 1|\mathcal{F}_1)1\!\!1_{\{t\geq 1\}} + \sum_n \mathbb{P}(\tau = \theta_n|\mathcal{F}_{\theta_n})1\!\!1_{\{t\geq\theta_n\}} \\
&= 1\!\!1_{\{S_1^*=S_1,\, t\geq 1\}} + \sum_n 1\!\!1_{\{\theta_n<1, S_{\theta_n}^*=S_{\theta_n -}\}}\mathbb{P}(\sup_{s\in[\theta_n,1[} S_s < S_{\theta_n -}|\mathcal{F}_{\theta_n})1\!\!1_{\{t\geq\theta_n\}} \\
&= 1\!\!1_{\{S_1^*=S_1, t\geq 1\}} + \sum_n 1\!\!1_{\{\theta_n<1, S_{\theta_n}^*=S_{\theta_n -}\}}\widetilde{\Phi}\left(\frac{S_{\theta_n -}}{S_{\theta_n}}, 1-\theta_n\right)1\!\!1_{\{t\geq\theta_n\}},
\end{aligned}
$$

with $\widetilde{\Phi}(x,t)$ is given by (5.5). In order to find the compensator of A^o, we write

$$
\begin{aligned}
A_t^o &= 1\!\!1_{\{S_1^*=S_1\}}1\!\!1_{\{t\geq 1\}} + \sum_{s\leq t} 1\!\!1_{\{s<1\}}1\!\!1_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}, 1-s\right)\Delta N_s \\
&= 1\!\!1_{\{S_1^*=S_1\}}1\!\!1_{\{t\geq 1\}} + \int_0^{t\wedge 1} 1\!\!1_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}, 1-s\right)dM_s \\
&\quad + \eta\int_0^{t\wedge 1} 1\!\!1_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}, 1-s\right)ds.
\end{aligned}
$$

As a result, due to the continuity of the process S^*, we get

$$
\begin{aligned}
A_t^o - {}^p(A^o)_t &= 1\!\!1_{\{S_t^*=S_{t-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}, 1-t\right)\Delta M_t, \\
Z_t - {}^pZ_t &= \left[\bar{\Phi}\left(\frac{S_t^*}{S_{t-}(1+\bar{\Phi})}, 1-t\right) - \bar{\Phi}\left(\frac{S_t^*}{S_{t-}}, 1-t\right)\right]\Delta N_t.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\Delta m_t &= Z_t - {}^pZ_t + A_t^o - {}^p(A^o)_t \\
&= \left\{\bar{\Phi}1\!\!1_{\{S_t^*=S_{t-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}, 1-t\right) + \bar{\Phi}\left(\frac{S_t^*}{S_{t-}(1+\bar{\Phi})}, 1-t\right)\right\}\Delta N_t \\
&\quad - \left\{\bar{\Phi}\left(\frac{S_t^*}{S_{t-}}, 1-t\right)\right\}\Delta N_t.
\end{aligned}
$$

Since m and S are pure discontinuous $\mathbb{F}$-local martingales, we conclude that m can be written in the form of

$$m = m_0 + \varphi \cdot S,$$

and the proof of the assertion (c) immediately follows from Theorem 5.5. This ends the proof of the proposition. ∎

5.4.2.3 Time of overall supremum

Below, we present our last example of this subsection. The analysis of this example is based on the following three functions. Here $S^* = \sup_s S_s$.

$$\bar{\Phi}(x) = \mathbb{P}(S^* > x), \quad \widehat{\Phi} = \mathbb{P}(S^* \leq 1), \text{ and } \widetilde{\Phi}(x) = \mathbb{P}(S^* < x). \tag{5.6}$$

Proposition 5.11. *Consider the random time τ given by*

$$\boxed{\tau := \sup\{t : S_t = S_t^*\}.}$$

Then, the following assertions hold.
(a) The random time τ is an honest time.
(b) For $\psi > 0$, define the $\mathbb{F}^\tau$-predictable process φ as

$$\varphi_t := \frac{\mathbb{1}_{\{S^*_{t-} < S_{t-}(1+\bar{\Phi})\}}\widehat{\Phi} + \bar{\Phi}\left(\max(\frac{S^*_{t-}}{S_{t-}(1+\bar{\Phi})}, 1\right) - \bar{\Phi}(\frac{S^*_{t-}}{S_{t-}})}{S_{t-}\bar{\Phi}}.$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{F}^\tau)$, and $\varphi^a := -\varphi \mathbb{1}_{]\!]\tau,\vartheta]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{F}^\tau)$. Here $\bar{\Phi}$ and $\widehat{\Phi}$ are defined in (5.6), and ϑ is defined in similar way as in (5.3).
(c) For $-1 < \psi < 0$, define the $\mathbb{F}^\tau$-predictable process φ as

$$\varphi := \frac{\bar{\Phi}(\frac{S^*}{S_-(1+\bar{\Phi})}) - \bar{\Phi}(\frac{S^*}{S_-}) + \mathbb{1}_{\{S^* = S_-\}}\widetilde{\Phi}(\frac{1}{1+\bar{\Phi}})\bar{\Phi}}{\bar{\Phi}S_-}.$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{F}^\tau)$, and $\varphi^a := -\varphi \mathbb{1}_{]\!]\tau,\vartheta]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{F}^\tau)$. Here again ϑ is defined as in (5.3).

Proof. It is clear that τ is an $\mathbb{F}$ honest time. Let us note that τ is finite and, as before, if $-1 < \psi < 0$, $S_\tau < S^*_\tau = \sup_t S_t$ and S^* is continuous and if $\psi > 0$, $S_\tau = S^*_\tau = \sup_t S_t$. The supermartingale Z associated with the honest time τ is

$$Z_t = \mathbb{P}(\sup_{s \in (t,\infty]} S_s > \sup_{s \in [0,t]} S_s | \mathcal{F}_t) = \mathbb{P}(\sup_{s \in [0,\infty]} \widehat{S}_s > \frac{S^*_t}{S_t} | \mathcal{F}_t) = \bar{\Phi}\left(\frac{S^*_t}{S_t}\right),$$

with $\widehat{S}$ an independent copy of S and $\bar{\Phi}$ is given by (5.6). As a result, we deduce that $Z_\tau < 1$. In the following, we will prove assertion (b). We suppose that $\psi > 0$, denoting by $(\theta_n)_n$ the sequence of jumps of the Poisson process N, we derive

$$\begin{aligned}
A^o_t &= \sum_n \mathbb{P}(\tau = \theta_n | \mathcal{F}_{\theta_n}) \mathbb{1}_{\{t \geq \theta_n\}} \\
&= \sum_n \mathbb{1}_{\{S^*_{\theta_n -} < S_{\theta_n}\}} \mathbb{P}(\sup_{s \geq \theta_n} S_s \leq S_{\theta_n} | \mathcal{F}_{\theta_n}) \mathbb{1}_{\{t \geq \theta_n\}} \\
&= \sum_n \mathbb{1}_{\{S^*_{\theta_n -} < S_{\theta_n -}(1+\bar{\Phi})\}} \widehat{\Phi} \mathbb{1}_{\{t \geq \theta_n\}},
\end{aligned}$$

with $\widehat{\Phi} = \mathbb{P}(\sup_s S_s \leq 1)$ given by (5.6).

We continue to compute the compensator of A^o

$$\begin{aligned}
A^o_t &= \sum_{s \leq t} \mathbb{1}_{\{S^*_{s-} < S_{s-}(1+\bar{\Phi})\}} \widehat{\Phi} \Delta N_s \\
&= \int_0^t \mathbb{1}_{\{S^*_{s-} < S_{s-}(1+\bar{\Phi})\}} \widehat{\Phi} dM_s + \eta \int_0^t \mathbb{1}_{\{S^*_{s-} < S_{s-}(1+\bar{\Phi})\}} \widehat{\Phi} ds.
\end{aligned}$$

Now as we did for the previous propositions, we calculate the jumps of m. To this end, we re-write Z as follows

$$Z=\left[\bar{\Phi}\left(\max(\frac{S_-^*}{S_-(1+\bar{\Phi})},1)\right)-\bar{\Phi}(\frac{S_-^*}{S_-})\right]\Delta M+\bar{\Phi}(\frac{S_-^*}{S_-}).$$

This implies that

$$Z-\ {}^pZ=\left[\bar{\Phi}\left(\max(\frac{S_-^*}{S_-(1+\bar{\Phi})},1)\right)-\bar{\Phi}(\frac{S_-^*}{S_-})\right]\Delta M.$$

Hence, we derive

$$\Delta m=\left[1\!\!1_{\{S_{s-}^*<S_{s-}(1+\bar{\Phi})\}}\widehat{\Phi}+\bar{\Phi}\left(\max(\frac{S_-^*}{S_-(1+\bar{\Phi})},1)\right)-\bar{\Phi}(\frac{S_-^*}{S_-})\right]\Delta M.$$

Since both martingales m and M are purely discontinuous, we deduce that $m=m_0+\varphi\centerdot S$. Then, the proposition immediately follows from Theorem 5.5.
In the following, we prove assertion (c). To this end, we suppose that $-1<\psi<0$, and we calculate

$$\begin{aligned}
A_t^o &= \sum_n \mathbb{P}(\tau=\theta_n|\mathcal{F}_{\theta_n})1\!\!1_{\{t\geq\theta_n\}}\\
&= \sum_n 1\!\!1_{\{S_{\theta_n}^*=S_{\theta_n-}\}}\mathbb{P}(\sup_{s\geq\theta_n}S_s<S_{\theta_n-}|\mathcal{F}_{\theta_n})1\!\!1_{\{t\geq\theta_n\}}\\
&= \sum_n 1\!\!1_{\{S_{\theta_n}^*=S_{\theta_n-}\}}\widetilde{\Phi}\left(\frac{S_{\theta_n-}}{S_{\theta_n}}\right)1\!\!1_{\{t\geq\theta_n\}},
\end{aligned}$$

with $\widetilde{\Phi}(x)=\mathbb{P}(\sup_s S_s<x)$. Therefore,

$$\begin{aligned}
A_t^o &= \sum_{s\leq t} 1\!\!1_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}\right)\Delta N_s\\
&= \int_0^t 1\!\!1_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}\right)dM_s+\eta\int_0^t 1\!\!1_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}\right)ds.
\end{aligned}$$

Since in the case of $\bar{\Phi}<0$, the process S^* is continuous, we obtain

$$\begin{aligned}
Z-\ {}^pZ &= \left[\bar{\Phi}\left(\frac{S^*}{S_-(1+\bar{\Phi})}\right)-\bar{\Phi}\left(\frac{S^*}{S_-}\right)\right]\Delta N,\\
A^o-\ {}^p(A^o) &= 1\!\!1_{\{S^*=S_-\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}\right)\Delta M.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
\Delta m &= Z-\ {}^pZ+A^o-\ {}^p(A^o)\\
&= \left\{\bar{\Phi}\left(\frac{S^*}{S_-(1+\bar{\Phi})}\right)-\bar{\Phi}\left(\frac{S^*}{S_-}\right)+1\!\!1_{\{S^*=S_-\}}\widetilde{\Phi}\left(\frac{1}{1+\bar{\Phi}}\right)\bar{\Phi}\right\}\Delta N.
\end{aligned}$$

This implies that the martingale m has the form of $m=1+\varphi\cdot S$, and assertion (c) immediately follows from Theorem 5.5, and the proof of the proposition is completed. ∎

5.5 Arbitrage opportunities for non-honest random times

This section is our second main part of this chapter. Herein, we develop a number of practical examples of market models and examples of random times that are not honest times, and we study the existence of classical arbitrages. This section contains two subsections that deal with two different situations.

5.5.1 In a Brownian filtration: example from section 4.3.3

We present here an example where τ is a pseudo stopping-time.

We take into consideration the random time studied in (4.6). We recall its definition once again here. Let S be defined through $dS_t = \sigma S_t dW_t$, where W is a Brownian motion and σ a constant. Let

$$\boxed{\tau^S := \sup\{t \leq 1 \,:\, S_1 - 2S_t = 0\}}$$

with $\sup\{\emptyset\} = \infty$, that is the last time before 1 at which the price is equal to half of its terminal value at time 1.

Proposition 5.12. *(a) NFLVR property holds in the model "before τ" $(S^\tau, \mathbb{F}^\tau)$.*
(b) NA and NUPBR properties fail in the model "after τ" $(S - S^\tau, \mathbb{F}^\tau)$.

Proof. (a) NFLVR property holds up to τ^S as S remains a $\mathbb{F}^\tau$-martingale up to time τ by Lemma 4.14 (a).

(b) There are obviously classical arbitrages after τ^S, since, at time τ^S, one knows the value of S_1 and $S_1 > S_\tau$. In fact, for $t > \tau^S$, one has $S_t > S_\tau$, and the arbitrage occurs at any time before 1. The arbitrage strategy is given by $\varphi_t = \mathbb{1}_{]\!]\tau^S,1]\!]}(t)$.
The NUPBR condition is not satisfied after τ^S. Indeed, NUPBR condition is equivalent to the following statement: there does not exist an arbitrage of the first kind, i.e., the random variable $\xi \geq 0$ with $\mathbb{P}(\xi > 0) > 0$ such that for every $x > 0$ there exits a strategy $\theta^x \in \mathcal{A}^x(\mathbb{F}^\tau)$ satisfying $V(x,\theta^x)_\infty > \xi$. Here it is enough to take the random variable $\xi := \frac{1}{2}S_1$ as an arbitrage of the first kind, since, for $t > \tau^S$ and $x > 0$, one has, for $\theta = \mathbb{1}_{]\!]\tau,1]\!]}$ $x + \int_{\tau^S}^1 \theta_s dS_s = x + S_1 - S_\tau = x + \xi > \xi$. ∎

5.5.2 In a Poisson filtration

This section develops similar examples of random times as in the Brownian filtration of the previous section.

In this section, we will work on a Poisson process N with intensity η and the compensated martingale $M_t = N_t - \eta t$. Denote

$$\theta_n = \inf\{t \geq 0 : N_t \geq n\}, \text{ and } H_t^n = \mathbb{1}_{\{\theta_n \leq t\}}, \quad n = 1, 2.$$

The stock price S is described by

$$dS_t = S_{t-}\psi dM_t, \quad \text{where}, \quad \psi > -1, \text{ and } \psi \neq 0,$$

or equivalently, $S_t = S_0 \exp(-\eta\psi t + \ln(1+\psi)N_t)$. Then,

$$\begin{aligned} M_t^1 &:= H_t^1 - \eta(t \wedge \theta_1) := H_t^1 - A_t^1, \\ M_t^2 &:= H_t^2 - (\eta(t \wedge \theta_2) - \eta(t \wedge \theta_1))) := H_t^2 - A_t^2 \end{aligned}$$

are two $\mathbb{F}$-martingales. Remark that if $\psi \in]-1, 0[$, between T_1 and T_2, the stock price increases; if $\psi > 0$, between T_1 and T_2, the stock process decreases. This would be the starting point of the existence of arbitrages.

5.5.2.1 Convex combination of two jump times

Below, we present an example of random time that avoids stopping times and the non-arbitrage property fails.

Proposition 5.13. *Consider the random time*

$$\boxed{\tau := k_1\theta_1 + k_2\theta_2}$$

that avoids $\mathbb{F}$ stopping times, where $k_1+k_2 = 1$ and $k_1, k_2 > 0$. Then the following properties hold.

(a) The random time τ is not an honest time.

(b) $\widetilde{Z}_\tau = Z_\tau = e^{-\eta k_1(\theta_2-\theta_1)} < 1$, and $\{\widetilde{Z} = 0 < Z_-\} = [\![\theta_2]\!]$.

(c) There is a classical arbitrage "before τ", given by

$$\varphi_t := -e^{-\eta\frac{k_2}{k_1}(t-\theta_1)}\left(\mathbb{1}_{\{N_{t-}\geq 1\}} - \mathbb{1}_{\{N_{t-}\geq 2\}}\right)\frac{1}{\psi S_{t-}}\mathbb{1}_{\{t\leq\tau\}}.$$

(d) There exist arbitrages "after τ": if $\psi \in (-1, 0)$, buy at τ and sell before θ_2; if $\psi > 0$, short sell at τ and buy back before θ_2.

Proof. First, we compute the supermartingale Z:

$$\mathbb{P}(\tau > t|\mathcal{F}_t) \quad = \quad \mathbb{1}_{\{\theta_1>t\}} + \mathbb{1}_{\{\theta_1\leq t\}}\mathbb{1}_{\{\theta_2>t\}}\mathbb{P}(k_1\theta_1 + k_2\theta_2 > t|\mathcal{F}_t).$$

On the set $E = \{\theta_1 \leq t\} \cap \{\theta_2 > t\}$, the quantity $\mathbb{P}(k_1\theta_1 + k_2\theta_2 > t|\mathcal{F}_t)$ is $\mathcal{F}_{\theta_1}$-measurable. It follows that, on E,

$$\mathbb{P}(k_1\theta_1 + k_2\theta_2 > t|\mathcal{F}_t) = \frac{\mathbb{P}(k_1\theta_1 + k_2\theta_2 > t, \theta_2 > t|\mathcal{F}_{\theta_1})}{\mathbb{P}(\theta_2 > t|\mathcal{F}_{\theta_1})} = e^{-\eta\frac{k_1}{k_2}(t-\theta_1)},$$

where we used the independence property of θ_1 and $\theta_2 - \theta_1$. Therefore, we deduce that,

$$Z_t := \mathbb{P}(\tau > t|\mathcal{F}_t) \quad = \quad \mathbb{1}_{\{\theta_1>t\}} + \mathbb{1}_{\{\theta_1\leq t\}}\mathbb{1}_{\{\theta_2>t\}}e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}.$$

Since $Z_t = (1 - H_t^1) + H_t^1(1 - H_t^2)e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}$, we deduce, using the fact that $e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dH_t^1 = dH_t^1$,

$$\begin{aligned}
dZ_t &= e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}(-H_t^2 dH_t^1 - H_t^1 dH_t^2) - \eta\frac{k_1}{k_2}H_t^1(1 - H_t^2)e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dt \\
&= -e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dH_t^2 - \eta\frac{k_1}{k_2}H_t^1(1 - H_t^2)e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dt \\
&= dm_t - e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dA_t^2 - \eta\frac{k_1}{k_2}H_t^1(1 - H_t^2)e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dt,
\end{aligned}$$

where

$$dm_t = -e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dM_t^2.$$

Hence

$$m_\tau = 1 - \int_0^\tau e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}dM_t^2 = 1 + \int_{\theta_1}^\tau e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}\eta dt > 1.$$

Now we will start proving the proposition.
i) Since τ avoids stopping times, $Z = \widetilde{Z}$. Note that $\widetilde{Z}_\tau = Z_\tau = e^{-\eta k_1(\theta_2-\theta_1)} < 1$. Hence, τ is not an honest time. Thus, we deduce that both assertions (a) and (b) hold.

ii) Now, we will prove assertion (c). We will describe explicitly the arbitrage strategy. Note that $\{\theta_2 \leq t\} = \{N_t \geq 2\}$. We deduce that

$$M_t^2 = 1\!\!1_{\{\theta_2 \leq t\}} - A_t^2 = 1\!\!1_{\{N_t \geq 2\}} - A_t^2 = 1\!\!1_{\{N_{t-} \geq 1\}}\Delta N_t + 1\!\!1_{\{N_{t-} \geq 2\}}(1 - \Delta N_t) - A_t^2.$$

Hence,

$$\begin{aligned}
\Delta M_t^2 &= M_t^2 - {}^p(M^2)_t = \left(1\!\!1_{\{N_{t-} \geq 1\}} - 1\!\!1_{\{N_{t-} \geq 2\}}\right)\Delta N_t \\
&= \left(1\!\!1_{\{N_{t-} \geq 1\}} - 1\!\!1_{\{N_{t-} \geq 2\}}\right)\Delta M_t.
\end{aligned}$$

Since M^2 and M are both purely discontinuous, we have $m_t = 1 + (\phi \centerdot M)_t = 1 + (\varphi \centerdot S)_t$, where

$$\phi_t = -e^{-\eta\frac{k_1}{k_2}(t-\theta_1)}\left(1\!\!1_{\{N_{t-} \geq 1\}} - 1\!\!1_{\{N_{t-} \geq 2\}}\right), \text{ and } \varphi_t = \phi_t \frac{1}{\psi S_{t-}}.$$

iii) Arbitrages after τ: at time τ, the value of θ_2 is known for the one who has $\mathbb{F}^\tau$-information. If $\psi > 0$, then the price process decreases before time θ_2, however, waiting up time θ_2 does not lead to an arbitrage Setting $\Delta = \theta_2 - \tau$ (which is known at time τ), there is an arbitrage selling short S at time τ for a delivery at time $\tau + \frac{1}{2}\Delta$. The strategy is admissible, since between θ_1 and θ_2, the quantity S_t is bounded by $S_0(1 + \varphi)$. This ends the proof of the proposition. ∎

5.5.2.2 Minimum of two scaled jump times

We give now an example of a non honest random time, which does not avoid $\mathbb{F}$ stopping time and induces classical arbitrage opportunities.

Proposition 5.14. *Consider the same market as before, and define*

$$\boxed{\tau := \theta_1 \wedge a\theta_2}$$

where $0 < a < 1$ *and* $\beta = \eta(1/a - 1)$*. Then, the following properties hold.*
(a) The random time τ *is not an honest time neither a pseudo-stopping time and does not avoid* $\mathbb{F}$*-stopping times.*
(b) $Z_\tau = \mathbb{1}_{\{\theta_1 > a\theta_2\}} e^{-\beta a\theta_2}(\beta a\theta_2 + 1) < 1$, $\widetilde{Z}_\tau = \mathbb{1}_{\{\theta_1 > a\theta_2\}} e^{-\beta a\theta_2}(\beta a\theta_2 + 1) + \mathbb{1}_{\{\theta_1 \leq a\theta_2\}} e^{-\beta\theta_1} < 1$, *and* $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$.
(c) There exist arbitrages "after τ*": if* $\psi \in (-1, 0)$*, buy at* τ *and sell before* τ/a*; if* $\psi > 0$*, short sell at* τ *and buy back before* τ/a*.*

Proof. First, let us compute the supermartingale Z,

$$\begin{aligned} Z_t &= \mathbb{1}_{\{\theta_1 > t\}} \mathbb{P}(a\theta_2 > t | \mathcal{F}_t) = \mathbb{1}_{\{\theta_1 > t\}} \frac{\mathbb{P}(a\theta_2 > t, \theta_1 > t)}{\mathbb{P}(\theta_1 > t)} \\ &= \mathbb{1}_{\{\theta_1 > t\}} e^{\eta t}\, \mathbb{E}(\mathbb{1}_{\{\theta_1 > t\}} e^{-\eta(\frac{t}{a} - \theta_1)^+}) \\ &= \mathbb{1}_{\{\theta_1 > t\}} e^{\eta t} \int_t^{t/a} e^{-\eta(\frac{t}{a} - x)} \eta e^{-\eta x} dx + \mathbb{1}_{\{\theta_1 > t\}} e^{\eta t} \int_{t/a}^{\infty} \eta e^{-\eta y} dy \\ &= \mathbb{1}_{\{\theta_1 > t\}} e^{-\beta t}(\beta t + 1), \end{aligned}$$

where $\beta = \eta(1/a - 1)$. In particular $Z_\tau = \mathbb{1}_{\{\theta_1 > a\theta_2\}} e^{-\beta a\theta_2}(\beta a\theta_2 + 1) < 1$.
Thus, $Z_{t-} = \mathbb{1}_{\{\theta_1 \geq t\}} e^{-\beta t}(\beta t + 1)$. As

$$\Delta A^o_{\theta_1} = \mathbb{P}(a\theta_2 > \theta_1 | \mathcal{F}_{\theta_1}) = e^{-\beta\theta_1}$$

we have that

$$\widetilde{Z}_t = \mathbb{1}_{\{\theta_1 > t\}} e^{-\beta t}(\beta t + 1) + \mathbb{1}_{[\![\theta_1]\!]}(t) e^{-\beta\theta_1}.$$

In particular $\widetilde{Z}_\tau = \mathbb{1}_{\{\theta_1 > a\theta_2\}} e^{-\beta a\theta_2}(\beta a\theta_2 + 1) + \mathbb{1}_{\{\theta_1 \leq a\theta_2\}} e^{-\beta\theta_1} < 1$.

Moreover, $\mathbb{F}$-martingale m has single jump at θ_1, i.e., $\Delta m_{\theta_1} = -\beta\theta_1 e^{-\beta\theta_1}$. Thus $m \neq 1$ and by Proposition 1.50 (c), it is not an $\mathbb{F}$-pseudo-stopping time. This proves assertions (a) and (b).
The proof of assertion (c) follows the same proof of assertion (d) of Proposition 5.13. This ends the proof of the proposition. ∎

5.6 NUPBR for particular models

In this section, we address some interesting practical models, for which we prove that the NUPBR condition remains valid up to τ. The originality of this part – as we mentioned in the introduction – lies in the simplicity of the proof. A general and complete analysis about the NUPBR condition is addressed in full generality in Chapter 6 of this thesis. Throughout this section, we will assume that $Z > 0$.

5.6.1 Before τ

Let $\widehat{m}$ be the $\mathbb{F}^\tau$-martingale stopped at time τ associated with m by (1.14), on $\{t \le \tau\}$

$$\widehat{m}_t := m_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle m, m\rangle_s^{\mathbb{F}}}{Z_s}\,.$$

5.6.1.1 Case of continuous filtration

We start with the particular case of continuous martingales and prove that, for any random time τ, NUPBR holds before τ.

We note that the continuity assumption implies that the martingale part of Z is continuous and that the optional and Doob-Meyer decompositions of Z are the same.

Proposition 5.15. *Assume that all $\mathbb{F}$-martingales are continuous. Then, for any random time τ, NUPBR holds before τ. An $\mathbb{F}^\tau$-local martingale deflator for S^τ is given by $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$, $L_0 = 1$.*

Proof. We make a use of Theorem 1.56 and we provide an $\mathbb{F}^\tau$-local martingale deflator for S^τ. Define the positive $\mathbb{F}^\tau$-local martingale L as $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$, $L_0 = 1$. Then, if SL is a $\mathbb{F}^\tau$-local martingale, NUPBR holds. Recall that, using (1.14) again,

$$\widehat{S}_t := S_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

is a $\mathbb{F}^\tau$-local martingale. From integration by parts, we obtain (using that the bracket of continuous martingales does not depend on the filtration)

$$\begin{aligned} d(LS^\tau)_t &= L_t dS_t^\tau + S_t dL_t + d\langle L, S^\tau\rangle_t^{\mathbb{F}^\tau} \\ &\overset{\mathbb{F}^\tau-mart}{=} L_t \frac{1}{Z_t} d\langle S, m\rangle_t^{\mathbb{F}} + \frac{1}{Z_t} L_t d\langle S, \widehat{m}\rangle_t^{\mathbb{F}^\tau} \\ &\overset{\mathbb{F}^\tau-mart}{=} L_t \frac{1}{Z_t} \left(d\langle S, m\rangle_t - d\langle S, m\rangle_t\right) = 0 \end{aligned}$$

where $X \overset{\mathbb{F}^\tau-mart}{=} Y$ is a notation for $X - Y$ is an $\mathbb{F}^\tau$-local martingale. ■

Remark 5.16. If τ is an honest time and Predictable Representation Property holds with respect to S then, as a consequence of Theorem 5.5, NA condition does not hold, hence NFLVR condition does not hold neither. That in turn implies that all the $\mathbb{F}^\tau$-local martingale deflators for S^τ are strict $\mathbb{F}^\tau$-local martingales.

5.6.1.2 Case of a Poisson filtration

We assume that S is an $\mathbb{F}$-martingale of the form $dS_t = S_{t-}\psi_t dM_t$, where ψ is a predictable process, satisfying $\psi > -1$ and $\psi \neq 0$, where M is the compensated martingale of a standard Poisson process.

In a Poisson setting, from Predictable Representation Property, $dm_t = j_t dM_t$ for some $\mathbb{F}$-predictable process j, so that, on $t \le \tau$,

$$d\widehat{m}_t = dm_t - \frac{1}{Z_{t-}} d\langle m, m\rangle_t = dm_t - \frac{1}{Z_{t-}} \lambda j_t^2 dt.$$

Proposition 5.17. *Let $\mathbb{F}$ be a Poisson filtration and τ be any random time satisfying $\int_0^\tau 1\!\!1_{\{Z_{t-}+j_t=0\}} dt = 0$ a.s. Then, NUPBR holds before τ since*

$$L = \mathcal{E}\left(-\frac{1}{Z_- + j} \centerdot \widehat{m}\right) = \mathcal{E}\left(-\frac{j}{Z_- + j} \centerdot \widehat{M}\right),$$

is a $\mathbb{F}^\tau$-local martingale deflator for S^τ.

Proof. We make a use of Theorem 1.56 and we are looking for an $\mathbb{F}^\tau$-local martingale deflator of the form $dL_t = L_{t-} \kappa_t \, d\widehat{m}_t$ (and $\psi_t \kappa_t > -1$) so that L is positive and $S^\tau L$ is an $\mathbb{F}^\tau$-local martingale. Integration by parts formula leads to (on $t \le \tau$)

$$\begin{aligned}
d(LS)_t &= L_{t-} dS_t + S_{t-} dL_t + d[L, S]_t \\
&\overset{\mathbb{F}^\tau - mart}{=} L_{t-}\, S_{t-}\, \psi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-}\, S_{t-}\, \kappa_t \psi_t j_t dN_t \\
&\overset{\mathbb{F}^\tau - mart}{=} L_{t-}\, S_{t-}\, \psi_t \frac{1}{Z_{t-}} j_t \lambda dt + L_{t-}\, S_{t-}\, \kappa_t\, \psi_t\, j_t\, \lambda (1 + \frac{1}{Z_{t-}} j_t) dt \\
&= L_{t-}\, S_{t-}\, \psi_t\, j_t\, \lambda \left(\frac{1}{Z_{t-}} + \kappa_t (1 + \frac{1}{Z_{t-}} j_t)\right) dt.
\end{aligned}$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-}+j_t}$, which is well-defined thanks to $\int_0^\tau 1\!\!1_{\{Z_{t-}+j_t=0\}} dt = 0$ a.s., one obtains a deflator. Note that

$$dL_t = L_{t-} \kappa_t d\widehat{m}_t = -L_{t-} \frac{1}{Z_{t-} + j_t} j_t d\widehat{M}_t$$

is indeed a positive $\mathbb{F}^\tau$-local martingale, since $\frac{1}{Z_{t-}+j_t} j_t < 1$. ∎

Remark 5.18. If τ is an honest time and Predictable Representation Property holds with respect to S, then all the $\mathbb{F}^\tau$-local martingale deflators for S^τ are strict $\mathbb{F}^\tau$-local martingales.

5.6.2 After τ

We now assume that τ is an honest time, which satisfies $Z_\tau < 1$ (for integrability reasons). Equivalently, by Lemma 2.16, we may assume that τ is a thin honest time. Note also that, in the case of continuous filtration, and $Z_\tau = 1$, NUPBR fails to hold after τ (see [FJS12]).

After (1.15), for any $\mathbb{F}$-martingale X (in particular for m and S)

$$\widehat{X}_t := X_t - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^{t} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{F}^\tau$-local martingale.

5.6.2.1 Case of continuous filtration

We start with the particular case of continuous martingales and prove that, for any honest time τ such that $Z_\tau < 1$, NUPBR holds after τ.

Proposition 5.19. *Assume that τ is an honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$-martingales are continuous. Then, for any honest time τ, NUPBR holds after τ. A $\mathbb{F}^\tau$-local martingale deflator for $S - S^\tau$ is given by $dL_t = -\frac{L_t}{1-Z_t} d\widehat{m}_t$.*

Proof. We use Theorem 1.56 as usual. The proof is based on Itô's calculus. Looking for an $\mathbb{F}^\tau$-local martingale deflator of the form $dL_t = L_t \kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1-Z)^{-1}$, the process $L(S-S^\tau)$ is an $\mathbb{F}^\tau$-local martingale. ■

Remark 5.20. If Predictable Representation Property holds with respect to S then, as a consequence of Theorem 5.5, the NA condition does not hold, hence NFLVR condition does not hold neither. That in turn implies that all the $\mathbb{F}^\tau$-local martingale deflators for $S - S^\tau$ are strict $\mathbb{F}^\tau$-local martingales.

5.6.2.2 Case of a Poisson filtration

We assume that S is an $\mathbb{F}$-martingale of the form $dS_t = S_{t-} \psi_t dM_t$, with ψ is a predictable process, satisfying $\psi > -1$.

The decomposition formula (1.15) reads after τ as

$$\widehat{S}_t = (1\!\!1_{]\!]\tau,\infty[\![} \cdot S)_t + \lambda \int_{t\vee\tau}^t \frac{1}{1-Z_{s-}} j_s \psi_s S_{s-} ds.$$

Proposition 5.21. *Let $\mathbb{F}$ be a Poisson filtration and τ be an honest time satisfying $Z_\tau < 1$ and $\int_\tau^\infty 1\!\!1_{\{1-Z_{t-}-j_t=0\}} dt = 0$ a.s. Then, NUPBR holds after τ since*

$$L = \mathcal{E}\left(\frac{1}{1 - Z_- - j} \centerdot \widehat{m} \right) = \mathcal{E}\left(\frac{j}{1 - Z_- - j} 1\!\!1_{]\tau,\infty[} \centerdot \widehat{M} \right),$$

is an $\mathbb{F}^\tau$-local martingale deflator for $S - S^\tau$.

Proof. We make a use of Theorem 1.56 and we are looking for an $\mathbb{F}^\tau$-local martingale deflator of the form $dL_t = L_{t-} \kappa_t d\widehat{m}_t$ (and $\psi_t \kappa_t > -1$) so that L is positive $\mathbb{F}^\tau$-local martingale and $(S - S^\tau)L$ is an $\mathbb{F}^\tau$-local martingale. Integration by parts formula leads to

$$\begin{aligned}
d(L(S-S^\tau))_t &= L_{t-} d(S - S^\tau)_t + (S_{t-} - S^\tau_{t-}) dL_t + d[L, S - S^\tau]_t \\
&\overset{\mathbb{F}^\tau-mart}{=} -\lambda L_{t-} S_{t-} j_t \psi_t \frac{1}{1 - Z_{t-}} 1\!\!1_{\{t>\tau\}} dt + L_{t-} S_{t-} \kappa_t \psi_t j_t 1\!\!1_{\{t>\tau\}} dN_t \\
&\overset{\mathbb{F}^\tau-mart}{=} \lambda L_{t-} S_{t-} \psi_t j_t 1\!\!1_{\{t>\tau\}} \left(-\frac{1}{1 - Z_{t-}} + \kappa_t \left(1 - \frac{1}{1 - Z_{t-}} j_t\right) \right) dt.
\end{aligned}$$

Therefore, for $\kappa_t = \frac{1}{1-Z_{t-}-j_t}$, which is well-defined thanks to $\int_\tau^\infty \mathbb{1}_{\{1-Z_{t-}-j_t=0\}} dt = 0$ a.s., one obtains an $\mathbb{F}^\tau$-local martingale deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = L_{t-}\frac{1}{1-Z_{t-}-j_t} j_t \mathbb{1}_{\{t>\tau\}} d\widehat{M}_t$$

is indeed a positive $\mathbb{F}^\tau$-local martingale, since $\frac{1}{1-Z_{t-}-j_t} j_t \Delta N_t > -1$. ∎

Remark 5.22. If Predictable Representation Property holds with respect to S then, all the $\mathbb{F}^\tau$-local martingale deflators for $S - S^\tau$ are strict $\mathbb{F}^\tau$-local martingales.

5.7 Conclusions

In this chapter we have treated the question whether the no-arbitrage conditions are stable with respect to progressive enlargement of filtration. We focused on two components of No Free Lunch with Vanishing Risk concept, namely on No Arbitrage Opportunity and No Unbounded Profit with Bounded Risk. The problem was divided into stability before and after random time containing extra information.

The question regarding No Arbitrage Opportunity condition was answered in the case of Brownian filtration and Poissonian filtration for special case of honest time, moreover particular examples of non-honest times were described. Both, Brownian and Poissonian filtrations possess an important, and crucial from our problem point of view, characteristic of Predictable Representation Property. One may further investigate a similar problem without assuming market completeness. One may as well consider other examples/classes of non-honest random times.

Afterwards, we handled the stability of NUPBR concept in some very particular situations, namely in continuous martingale case, standard Poisson process case and Lévy process case. We provided results with simple proofs in those particular situations. We emphasize again that in full generality the problem is solved in Chapter 6 revealing as well results within progressive enlargement of filtration theory.

Combining results on NA and NUPBR conditions, we concluded (in Remarks 5.16, 5.18, 5.20, 5.22) that some $\mathbb{F}^\tau$-local martingales are in fact $\mathbb{F}^\tau$-strict local martingales. That provides a way to construct strict local martingales in enlarged Brownian and Poisson filtrations.

Chapter 6

Non-Arbitrage up to Random Horizon for Semimartingale Models

6.1 Introduction

This chapter is based on a joint paper with Tahir Choulli, Jun Deng and Monique Jeanblanc [ACDJ14b].

In this chapter, we consider a general semimartingale model S satisfying the NUPBR property under the *public information* denoted by $\mathbb{F}$ and an arbitrary random time τ and we answer to the following questions:

for which pairs (S, τ), does the NUPBR property hold for S^τ? **(P1)**

for which τ, is NUPBR preserved for any S after stopping at τ? **(P2)**

To deepen our understanding of the precise interplay between the reference market model and the enlarged model, we address these two principal questions separately in the case of quasi-left-continuous models, and then in the case of thin processes with predictable jumps. Afterwards, we combine the two cases and state the results for the most general framework.

This chapter is organized as follows. The next section (Section 6.2) presents our main results in different contexts, and discusses their meaning and/or their economical interpretations and their consequences as well. Section 6.3 develops new stochastic results, which are the key mathematical ideas behind the answers to **(P1)**-**(P2)**. Section 6.4 gives an explicit form for the deflator in the case where S is quasi-left continuous. Section 6.5 contains the proofs of the main theorems announced, without proofs, in Section 6.2. The chapter concludes with an Appendix, where some classical results on the predictable characteristics of a semimartingale and other related results are recalled. Some technical proofs are also postponed to the Appendix, for the ease of the reader.

6.2 Main results and their interpretations

This section is devoted to the presentation of our main results and their immediate consequences. To this end, we start specifying our mathematical setting and the economical concepts that we will address.

6.2.1 Notations

We consider a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses (i.e., right continuity and completeness), and $\mathcal{F}_\infty \subseteq \mathcal{G}$. Financially speaking, the filtration $\mathbb{F}$ represents the flow of public information through time. On this basis, we consider an arbitrary but fixed d-dimensional càdlàg semimartingale S. This represents the discounted price processes of d-stocks, while the riskless asset's price is assumed to be constant.
Beside the reference model $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P}, S)$, we consider a random time τ, i.e., a non-negative $\mathcal{G}$-measurable random variable. To this random time, we associate the process A and the filtration $\mathbb{F}^\tau$ which is the progressive enlargement of $\mathbb{F}$ with τ, given by

$$A := \mathbb{1}_{[\![\tau,\infty[\![}, \quad \mathbb{F}^\tau := (\mathcal{F}_t^\tau)_{t\geq 0}\,, \quad \mathcal{F}_t^\tau := \bigcap_{s>t}\Big(\mathcal{F}_s \vee \sigma(\tau\wedge s)\Big).$$

As in Section 1.2.3, in addition to $\mathbb{F}^\tau$ and A, we associate to τ two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\left(\tau > t \;\middle|\; \mathcal{F}_t\right) \text{ and } \quad \widetilde{Z}_t := P\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right). \tag{6.1}$$

The decomposition of Z leads to an important $\mathbb{F}$-martingale m, given by

$$m := Z + A^{o,\mathbb{F}}, \tag{6.2}$$

where $A^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of A (see Section1.1.5 for more details).

6.2.2 The quasi-left-continuous processes

In this subsection, we present our two main results on the NUPBR condition under stopping at τ for quasi-left-continuous processes (see Section 1.1.6). The first result consists of characterizing the pairs (S, τ) of market and random time models, for which S^τ fulfills the NUPBR condition. The second result focuses on determining a necessary and sufficient condition on τ such that, for any semimartingale S enjoying NUPBR($\mathbb{F}$), the stopped process S^τ enjoys NUPBR($\mathbb{F}^\tau$) .

The following theorem gives a characterization of $\mathbb{F}$-quasi-left continuous processes that satisfy NUPBR($\mathbb{F}^\tau$) after stopping with τ. The proof of this theorem will be given in Subsection 6.5.1, while its statement is based on the following $\mathbb{F}$-semimartingale (we use the notation from Section 1.1.9)

$$S^{(0)} := x\mathbb{1}_{\{\psi=0<Z_-\}}\star\mu, \qquad \text{where} \qquad \psi := M^{\mathbb{P}}_\mu\left(\mathbb{1}_{\{\widetilde{Z}>0\}}\Big|\widetilde{\mathcal{P}}\left(\mathbb{F}\right)\right). \tag{6.3}$$

Theorem 6.1. *Suppose that S is $\mathbb{F}$-quasi-left-continuous. Then, the following assertions are equivalent.*
(a) S^τ satisfies NUPBR($\mathbb{F}^\tau$).
(b) For any $\delta > 0$, the process

$$\boxed{\mathbb{1}_{\{Z_- \geq \delta\}} \centerdot \left(S - S^{(0)}\right) \quad \textit{satisfies NUPBR}(\mathbb{F}).}$$

(c) For any $n \geq 1$, the process $(S - S^{(0)})^{\sigma_n}$ satisfies NUPBR($\mathbb{F}$), where

$$\sigma_n := \inf\{t \geq 0 \ : \ Z_t < 1/n\}.$$

Remark 6.2. (a) From assertion (c) one can understand that the NUPBR($\mathbb{F}^\tau$) property for S^τ can be verified by checking whether $S - S^{(0)}$ satisfies NUPBR($\mathbb{F}$) up to $\sigma_\infty := \sup_n \sigma_n$. This last fact means that $(S - S^{(0)})^T$ satisfies NUPBR($\mathbb{F}$) for any $\mathbb{F}$-stopping time T such that $[\![0, T]\!] \subset \{Z_- > 0\}$. It is important to mention that this property (i.e. the NUPBR($\mathbb{F}$) up to σ_∞) may not be equivalent to the NUPBR($\mathbb{F}$) of $Z_- \centerdot (S - S^{(0)})$, which is equivalent to the NUPBR($\mathbb{F}$) of $I_{\{Z_- > 0\}} \centerdot (S - S^{(0)})$ due to Proposition 1.59. We believe that a counter-example can be found in the same spirit of Remark 1.55–(b).
(b) The functionals ψ and f_m (defined as $Z_- + f_m := M^{\mathbb{P}}_\mu(\widetilde{Z} \big| \widetilde{\mathcal{P}}(\mathbb{F}))$ satisfy

$$\{\psi = 0\} = \{Z_- + f_m = 0\} \subset \{\widetilde{Z} = 0\}, \quad M^{\mathbb{P}}_\mu - a.e. \tag{6.4}$$

Indeed, due to $\widetilde{Z} \leq \mathbb{1}_{\{\widetilde{Z} > 0\}}$, we have

$$0 \leq Z_- + f_m = M^{\mathbb{P}}_\mu \left(\widetilde{Z} \big| \ \widetilde{\mathcal{P}}(\mathbb{F})\right) \leq \psi.$$

Thus, we get $\{\psi = 0\} \subset \{Z_- + f_m = 0\} \subset \{\widetilde{Z} = 0\}$ $\ M^{\mathbb{P}}_\mu - a.e.$ on the one hand. On the other hand, the reverse inclusion follows from

$$0 = M^{\mathbb{P}}_\mu \left(\mathbb{1}_{\{Z_- + f_m = 0\}} \mathbb{1}_{\{\widetilde{Z} = 0\}}\right) = M^{\mathbb{P}}_\mu \left(\mathbb{1}_{\{Z_- + f_m = 0\}} \psi\right).$$

(c) As a result of remark (b) above and $\{\widetilde{Z} = 0 < Z_-\} \subset [\![\sigma_\infty]\!]$, we deduce that $S^{(0)}$ is a càdlàg $\mathbb{F}$-adapted process with finite variation with $var(S^{(0)})_\infty \leq |\Delta S_{\sigma_\infty}| \mathbb{1}_{\{\sigma_\infty < \infty\}}$. Furthermore, it can be written as

$$S^{(0)} := \Delta S_{\sigma_\infty} \mathbb{1}_{\{\widetilde{Z}_{\sigma_\infty} = 0 = \psi(\sigma_\infty, \Delta S_{\sigma_\infty}), \ Z_{\sigma_\infty -} > 0\}} \mathbb{1}_{[\![\sigma_\infty, \infty[\![}.$$

The following corollary is useful for studying the problem (**P2**), and it describes examples of $\mathbb{F}$-quasi-left-continuous model S that fulfill the conditions of Theorem 6.1 as well.

Corollary 6.3. Suppose that S is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$). Then, the following assertions hold.

(a) If $\left(S, S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$), then S^τ satisfies NUPBR($\mathbb{F}^\tau$).

(b) If $S^{(0)} \equiv 0$, then the process S^τ satisfies NUPBR($\mathbb{F}^\tau$).

(c) If $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$, then S^τ satisfies NUPBR($\mathbb{F}^\tau$).

(d) If $\widetilde{Z} > 0$ (equivalently $Z > 0$ or $Z_- > 0$), then S^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. (a) Suppose that $\left(S, S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$). Then, $S - S^{(0)}$ satisfies NUPBR($\mathbb{F}$), and assertion (a) follows from Theorem 6.1.
(b) Since S satisfies NUPBR($\mathbb{F}$) and $S^{(0)} \equiv 0$, then $\left(S, S^{(0)}\right) \equiv (S, 0)$ satisfies NUPBR($\mathbb{F}$), and assertion (b) follows from assertion (a).
(c) It is easy to see that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$ implies that $S^{(0)} \equiv 0$ (due to (6.4)). The fact that $\widetilde{Z} > 0$ is equivalent to $Z > 0$ or to $Z_- > 0$ follows from the fact that the three sets have the same début (see the following Proposition 6.16) Hence, assertions (c) and (d) follow from assertion (b), and the proof of the corollary is completed. ∎

Remark 6.4. It is worth mentioning that $X - Y$ may satisfy NUPBR($\mathbb{H}$), while (X, Y) may not satisfy NUPBR($\mathbb{H}$). For a non trivial example, consider $X_t = \eta t$ and $Y_t = N_t$ where N is a Poisson process with intensity η.

We now give an answer to the second problem **(P2)** for quasi-left-continuous semimartingales. Later on (in Theorem 6.15) we will generalize this result.

Proposition 6.5. *The following assertions are equivalent.*
(a) The thin set $\{\widetilde{Z} = 0 < Z_-\}$ is accessible.
(b) For any (bounded) S that is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$), the process S^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. The implication (a)⇒(b) follows from Corollary 6.3(c), since we have

$$\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

We now focus on proving the reverse implication. To this end, we suppose that assertion (b) holds, and we consider an $\mathbb{F}$-stopping time σ such that $[\![\sigma]\!] \subset \{\widetilde{Z} = 0 < Z_-\}$. It is known that σ can be decomposed into a totally inaccessible part σ^i and an accessible part σ^a such that $\sigma = \sigma^i \wedge \sigma^a$ (see Theorem 1.7). Consider the quasi-left-continuous $\mathbb{F}$-martingale

$$M = V - \widetilde{V} \in \mathcal{M}_{0,loc}(\mathbb{F})$$

where $V := 1\!\!1_{[\![\sigma^i, \infty[\![}$ and $\widetilde{V} := V^{p,\mathbb{F}}$. It is known from [DMM92, paragraph 14, Chapter XX], that

$$\{\widetilde{Z} = 0\} \text{ and } \{Z_- = 0\} \text{ are disjoint from }]\!]0, \tau]\!]\,. \tag{6.5}$$

That implies that $\tau < \sigma \leq \sigma^i$ $\mathbb{P} - a.s.$. Hence, we get $M^\tau = -\widetilde{V}^\tau$ is $\mathbb{F}^\tau$-predictable. Since M^τ satisfies NUPBR($\mathbb{F}^\tau$), then we conclude that this process is null (i.e., $\widetilde{V}^\tau = 0$) due to Lemma 1.60. Thus, we obtain

$$0 = \mathbb{E}\left(\widetilde{V}_\tau\right) = \mathbb{E}\left(\int_0^\infty Z_{s-} d\widetilde{V}_s\right) = \mathbb{E}\left(Z_{\sigma^i -} 1\!\!1_{\{\sigma^i < \infty\}}\right),$$

or equivalently $Z_{\sigma^i-}\mathbb{1}_{\{\sigma^i<\infty\}} = 0$ $\mathbb{P}$-a.s. This is possible only if $\sigma^i = \infty$ $\mathbb{P}$-a.s. since on $\{\sigma^i < \infty\} \subset \{\sigma = \sigma^i < \infty\}$, one has $Z_{\sigma^i-} = Z_{\sigma-} > 0$. This proves that σ is an accessible stopping time. Since $\{\widetilde{Z} = 0 < Z_-\}$ is an optional thin set, assertion (a) immediately follows. This ends the proof of the proposition. ∎

6.2.3 Thin processes with predictable jump times

In this subsection, we outline the main results on the NUPBR condition for the stopped accessible parts of $\mathbb{F}$-semimartingales with a random time. This boils down to consider thin semimartingales with predictable jump times only. We start by addressing question **(P1)** in the case of single jump process with predictable jump time. (See Section 1.1.3 for the definition of a thin process.)

Theorem 6.6. *Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable random variable ξ such that $\mathbb{E}(|\xi|\big| \mathcal{F}_{T-}) < \infty$ $\mathbb{P}$-a.s. and define $S := \xi\mathbb{1}_{\{Z_{T-}>0\}}\mathbb{1}_{[\![T,\infty[\![}$. Then the following assertions are equivalent.*
(a) S^τ satisfies NUPBR$(\mathbb{F}^\tau)$.
(b) The process $\widetilde{S} := \xi\mathbb{1}_{\{\widetilde{Z}_T>0\}}\mathbb{1}_{[\![T,\infty[\![} = \mathbb{1}_{\{\widetilde{Z}>0\}} \cdot S$ satisfies NUPBR$(\mathbb{F})$.
(c) There exists a probability measure on $(\Omega, \mathcal{F}_T)$, denoted by $\mathbb{Q}_T$, such that $\mathbb{Q}_T$ is absolutely continuous with respect to $\mathbb{P}$, and S satisfies NUPBR$(\mathbb{F}, \mathbb{Q}_T)$.

The proof of this theorem is long and requires intermediary results that are interesting in themselves. Thus, this proof will be given later in Section 6.5.

Remark 6.7. (a) The importance of Theorem 6.6 goes beyond its vital role, as a building block for the more general result. This theorem provides two different characterizations for NUPBR$(\mathbb{F}^\tau)$ of S^τ. The first characterization is in terms of NUPBR$(\mathbb{F})$ of S under absolutely continuous change of measure, while the second characterization uses a transformation of S without any change of measure. Furthermore, Theorem 6.6 can be easily extended to the case of countably many ordered predictable jump times $T_0 = 0 \leq T_1 \leq T_2 \leq ...$ with $\sup_n T_n = \infty$ $\mathbb{P}$-a.s.
(b) In Theorem 6.6, the choice of S having the form $S := \xi\mathbb{1}_{\{Z_{T-}>0\}}\mathbb{1}_{[\![T,\infty[\![}$ is not restrictive. This can be understood from the fact that any single jump process S can be decomposed as follows

$$S := \xi\mathbb{1}_{[\![T,\infty[\![} = \xi\mathbb{1}_{\{Z_{T-}>0\}}\mathbb{1}_{[\![T,\infty[\![} + \xi\mathbb{1}_{\{Z_{T-}=0\}}\mathbb{1}_{[\![T,\infty[\![} =: \overline{S} + \widehat{S}.$$

Thanks to $\{T \leq \tau\} \subset \{Z_{T-} > 0\}$, we have $\widehat{S}^\tau = \xi\mathbb{1}_{\{Z_{T-}=0\}}\mathbb{1}_{\{T\leq\tau\}}\mathbb{1}_{[\![T,\infty[\![} \equiv 0$ is (obviously) an $\mathbb{F}^\tau$-martingale. Thus, the only part of S that requires careful attention is $\overline{S} := \xi\mathbb{1}_{\{Z_{T-}>0\}}\mathbb{1}_{[\![T,\infty[\![}$.

The following result is a complete answer to **(P2)** in the case of predictable single jump processes.

Proposition 6.8. *Let T be an $\mathbb{F}$-predictable stopping time. Then, the following assertions are equivalent.*
(a) On $\{T<\infty\}$, we have

$$\{\widetilde{Z}_T=0\}\subset\{Z_{T-}=0\}. \tag{6.6}$$

(b) For any $\mathbb{F}$-martingale $M:=\xi 1\!\!1_{[\![T,\infty[\![}$ where $\xi\in L^\infty(\mathcal{F}_T)$ such that $\mathbb{E}(\xi|\mathcal{F}_{T-})=0$, M^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. We start by proving $(a)\Rightarrow(b)$. Suppose that (6.6) holds; due to Remark 6.7(b), we can restrict our attention to the case $M:=\xi 1\!\!1_{\{Z_{T-}>0\}}1\!\!1_{[\![T,\infty[\![}$ where $\xi\in L^\infty(\mathcal{F}_T)$ such that $\mathbb{E}(\xi|\mathcal{F}_{T-})=0$. Since assertion (a) is equivalent to $[\![T]\!]\cap\{\widetilde{Z}=0<Z_-\}=\emptyset$, we deduce that

$$\widetilde{M}:=\xi 1\!\!1_{\{\widetilde{Z}_T>0\}}1\!\!1_{\{Z_{T-}>0\}}1\!\!1_{[\![T,\infty[\![}=M \qquad \text{is an } \mathbb{F}\text{-martingale.}$$

Therefore, a direct application of Theorem 6.6 (to M) allows us to conclude that M^τ satisfies the NUPBR($\mathbb{F}^\tau$). This ends the proof of (a)$\Rightarrow$ (b). To prove the reverse implication, we suppose that assertion (b) holds and consider

$$M:=\xi 1\!\!1_{[\![T,\infty[\![}, \qquad \text{where } \ \xi:=1\!\!1_{\{\widetilde{Z}_T=0\}}-\mathbb{P}(\widetilde{Z}_T=0|\mathcal{F}_{T-}).$$

From (6.5), we obtain $\{T\le\tau\}\subset\{\widetilde{Z}_T>0\}\subset\{Z_{T-}>0\}$ which implies that

$$M^\tau=-\mathbb{P}(\widetilde{Z}_T=0|\mathcal{F}_{T-})1\!\!1_{\{T\le\tau\}}1\!\!1_{[\![T,\infty[\![} \ \text{ is decreasing process.}$$

Therefore, M^τ satisfies NUPBR($\mathbb{F}^\tau$) if and only if it is a constant process equal to $M_0=0$. This is equivalent to

$$0=\mathbb{E}\Big(\mathbb{P}(\widetilde{Z}_T=0|\mathcal{F}_{T-})1\!\!1_{\{T\le\tau\}}1\!\!1_{\{T<\infty\}}\Big)=\mathbb{E}\left(Z_{T-}1\!\!1_{\{\widetilde{Z}_T=0,\ T<\infty\}}\right).$$

The last equality is equivalent to (6.6), and assertion (a) follows. This ends the proof of the theorem. ■

We now state the following version of Theorem 6.6, which provides, as already said, an answer to **(P1)** in the case where there are countable many arbitrary predictable jumps. The proof of this theorem will be given in Subsection 6.5.3.

Theorem 6.9. *Let S be a thin process with predictable jump times only and satisfying NUPBR($\mathbb{F}$). Then, the following assertions are equivalent.*
(a) The process S^τ satisfies NUPBR($\mathbb{F}^\tau$).
(b) For any $\delta>0$, there exists a positive $\mathbb{F}$-local martingale, Y, such that ${}^{p,\mathbb{F}}(Y|\Delta S|)<\infty$ and

$$\boxed{{}^{p,\mathbb{F}}\left(Y\Delta S 1\!\!1_{\{\widetilde{Z}>0,\ Z_-\ge\delta\}}\right)=0.}$$

Remark 6.10. (a) Suppose that S is a thin process with predictable jumps only, satisfying NUPBR($\mathbb{F}$), and that $\{\widetilde{Z}=0<Z_-\}\cap\{\Delta S\neq 0\}=\emptyset$ holds. Then, S$^\tau$ satisfies NUPBR($\mathbb{F}^\tau$).

This immediately follows from Theorem 6.9 by using $Y \in \mathcal{L}(S,\mathbb{F})$ and Lemma 1.57.
(b) Similarly to Proposition 6.5, we can easily prove that the thin set $\{\widetilde{Z} = 0 < Z_-\}$ is totally inaccessible if and only if X^τ satisfies NUPBR($\mathbb{F}^\tau$) for any thin process X with predictable jumps only satisfying NUPBR($\mathbb{F}$).

6.2.4 The general framework

Let $\mathbb{H} \in \{\mathbb{F}, \mathbb{F}^\tau\}$. Throughout the remaining part of this chapter, to any $\mathbb{H}$-semimartingale, X, we associate a sequence of $\mathbb{H}$-predictable stopping times $(T_n^X)_{n\geq 1}$ that exhaust the accessible jump times of X. Furthermore, we decompose X as follows.

$$X = X^{(qc)} + X^{(a)},\ X^{(a)} := I\!\!1_{\Gamma_X} \centerdot X,\ X^{(qc)} := X - X^{(a)},\ \Gamma_X := \bigcup_{n=1}^{\infty} [\![T_n^X]\!]. \tag{6.7}$$

The process $X^{(a)}$ (the accessible part of X) is a thin process with predictable jumps only, while $X^{(qc)}$ is an $\mathbb{H}$-quasi-left-continuous process (the quasi-left-continuous part of X).

Lemma 6.11. *Let X be an $\mathbb{H}$-semimartingale. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR($\mathbb{H}$).*

Proof. Thanks to Proposition 1.56, X satisfies NUPBR($\mathbb{H}$) if and only if there exist an $\mathbb{H}$-predictable real-valued process $\phi > 0$ and a positive $\mathbb{H}$-local martingale Y such that $Y(\phi \centerdot X)$ is an $\mathbb{H}$-local martingale. Then, it is obvious that $Y(\phi I\!\!1_{\Gamma_X} \centerdot X)$ and $Y(\phi I\!\!1_{{\Gamma_X}^c} \centerdot X)$ are both $\mathbb{H}$-local martingales. This proves that $X^{(a)}$ and $X^{(qc)}$ both satisfy NUPNR($\mathbb{H}$).
Conversely, if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPNR($\mathbb{H}$), then there exist two $\mathbb{H}$-predictable real-valued processes $\phi_1, \phi_2 > 0$ and two positive $\mathbb{H}$-local martingales $Y_1 = \mathcal{E}(L_1), Y_2 = \mathcal{E}(L_2)$ such that $Y_1(\phi_1 \centerdot (I\!\!1_{\Gamma_X} \centerdot S))$ and $Y_2(\phi_2 \centerdot (I\!\!1_{{\Gamma_X}^c} \centerdot X))$ are both $\mathbb{H}$-local martingales. Remark that there is no loss of generality in assuming $L_1 = I\!\!1_{\Gamma_X} \centerdot L_1$ and $L_2 = I\!\!1_{{\Gamma_X}^c} \centerdot L_2$. Setting

$$L_3 := I\!\!1_{\Gamma_X} \centerdot L_1 + I\!\!1_{{\Gamma_X}^c} \centerdot L_2 \quad \text{and} \quad \phi_3 := \phi_1 I\!\!1_{\Gamma_X} + \phi_2 I\!\!1_{{\Gamma_X}^c},$$

one obtains that $\mathcal{E}(L_3) > 0$, and the processes $\mathcal{E}(L_3)$ and $\mathcal{E}(L_3)(\phi_3 \centerdot S)$ are $\mathbb{H}$-local martingales, ϕ_3 is $\mathbb{H}$-predictable and $0 < \phi_3 \leq 1$. This ends the proof of the lemma. ∎

Below, we answer to question **(P1)** in this general framework, which, using Lemma 6.11 will be a consequence of Theorems 6.1 and 6.6.

Theorem 6.12. *Suppose that S satisfies NUPBR($\mathbb{F}$). Then, the following assertions are equivalent.*
(a) The process S^τ satisfies NUPBR($\mathbb{F}^\tau$).
(b) For any $\delta > 0$, the process

$$\boxed{I\!\!1_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - S^{(qc,0)}) := I\!\!1_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - I\!\!1_{\Gamma^c} \centerdot S^{(0)})}$$

satisfies NUPBR($\mathbb{F}$), and there exists a positive $\mathbb{F}$-local martingale, Y, such that ${}^{p,\mathbb{F}}(Y|\Delta S|) < \infty$ *and*

$$\boxed{{}^{p,\mathbb{F}}\left(Y\Delta S 1\!\!1_{\{\widetilde{Z}>0,\ Z_-\geq\delta\}}\right) = 0.}$$

Proof. Due to Lemma 6.11, it is obvious that S^τ satisfies NUPBR($\mathbb{F}^\tau$) if and only if both $(S^{(qc)})^\tau$ and $(S^{(a)})^\tau$ satisfy NUPBR($\mathbb{F}^\tau$). Thus, using both Theorems 6.1 and 6.9, we deduce that this last fact is true if and only if for any $\delta > 0$, the process $1\!\!1_{\{Z_-\geq\delta\}} \bullet (S^{(qc)} - 1\!\!1_{\Gamma^c} \bullet S^{(0)})$ satisfies NUPBR($\mathbb{F}$) and there exists a positive $\mathbb{F}$-local martingale Y such that

$$ {}^{p,\mathbb{F}}(Y|\Delta S|) = {}^{p,\mathbb{F}}\left(Y|\Delta S^{(a)}|\right) < \infty \quad \text{and}$$

$${}^{p,\mathbb{F}}\left(Y\Delta S 1\!\!1_{\{\widetilde{Z}>0,\ Z_-\geq\delta\}}\right) = {}^{p,\mathbb{F}}\left(Y\Delta S^{(a)} 1\!\!1_{\{\widetilde{Z}>0,\ Z_-\geq\delta\}}\right) = 0.$$

This ends the proof of the theorem. ∎

Corollary 6.13. The following assertions hold.
(a) If either m is continuous or Z is positive (equivalently $\widetilde{Z} > 0$ or $Z_- > 0$), S^τ satisfies NUPBR($\mathbb{F}^\tau$) whenever S satisfies NUPBR($\mathbb{F}$).
(b) If S satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$, then S^τ satisfies NUPBR($\mathbb{F}^\tau$).
(c) If S is continuous and satisfies NUPBR($\mathbb{F}$), then for any random time τ, S^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. 1) The proof of the assertion (a) of the corollary follows easily from Theorem 6.12. Indeed, in the two cases, one has $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$ which implies that $\{\widetilde{Z} = 0,\ Z_- \geq \delta\} = \emptyset$ and $S^{(qc,0)} \equiv 0$ (due to (6.4)). Then, due to Lemma 1.57, it suffices to take $Y \in \mathcal{L}(\mathbb{F}, S)$ – since this set is non-empty – and to apply Theorem 6.12.
2) It is obvious that assertion (c) follows from assertion (b). To prove this latter, it is enough to remark that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0,\ Z_- \geq \delta\} = \emptyset$ implies that

$$1\!\!1_{\{Z_-\geq\delta\}} \bullet S^{(qc,0)} \equiv 0 \quad \text{and} \quad \Delta S 1\!\!1_{\{\widetilde{Z}>0, Z_-\geq\delta\}} = \Delta S 1\!\!1_{\{Z_-\geq\delta\}}.$$

Thus, again, it is enough to take $Y \in \mathcal{L}(\mathbb{F}, S)$ and to apply Theorem 6.12. This ends the proof of the corollary. ∎

Remark 6.14. Any of the two assertions (a) or (c) of the above corollary generalizes the main result of [FJS12], obtained under some assumptions on the random time τ and the market model as well.

Below, we provide a general answer to question **(P2)**, as a consequence of Theorems 6.1, 6.5 and 6.9.

Theorem 6.15. *The following assertions are equivalent.*
(a) The thin set $\{\widetilde{Z} = 0 < Z_-\}$ is evanescent.
(b) For any (bounded) X satisfying NUPBR($\mathbb{F}$), X^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. Suppose that assertion (a) holds, and consider a process X satisfying NUPBR($\mathbb{F}$). Then, $X^{(qc,0)} := 1\!\!1_{\Gamma_X^c} \centerdot X^{(0)} \equiv 0$, where $X^{(0)}$ is defined as in (6.3). Hence $1\!\!1_{\{Z_-\geq\delta\}} \centerdot \left(X^{(qc)} - 1\!\!1_{\Gamma_X^c} \centerdot X^{(0)}\right) = 1\!\!1_{\{Z_-\geq\delta\}} \centerdot X^{(qc)}$ satisfies NUPBR($\mathbb{F}$) for any $\delta > 0$, and the NUPBR($\mathbb{F}^\tau$) property for $(X^{(qc)})^\tau$ immediately follows from Theorem 6.1 on the one hand. On the other hand, it is easy to see that $X^{(a)}$ fulfills assertion (b) of Theorem 6.9 with $Y \in \mathcal{L}(\mathbb{F}, X)$ due to Lemma 1.57. Thus, thanks to Theorem 6.9 (applied to the thin process $X^{(a)}$ satisfying NUPBR($\mathbb{F}$)), we conclude that $(X^{(a)})^\tau$ satisfies NUPBR($\mathbb{F}^\tau$). Thus, due to Lemma 6.11, the proof of (a)⇒(b) is completed.

We now suppose that assertion (b) holds. On the one hand, from Proposition 6.5, we deduce that $\{\widetilde{Z} = 0 < Z_-\}$ is accessible and can be covered with the graphs of $\mathbb{F}$-predictable stopping times $(T_n)_{n\geq 1}$. On the other hand, a direct application of Proposition 6.8 to all single predictable jump $\mathbb{F}$-martingales, we obtain $\{\widetilde{Z} = 0 < Z_-\} \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-predictable stopping time T. Therefore, we get

$$\{\widetilde{Z} = 0 < Z_-\} = \bigcup_{n=1}^{\infty} \left(\{\widetilde{Z} = 0 < Z_-\} \cap [\![T_n]\!]\right) = \emptyset.$$

This proves assertion (a), and the proof of the theorem is completed. ∎

6.3 Stochastics from–and–for informational non-arbitrage

In this section, we develop new stochastic results that will play a key role in the proofs and/or the statements of the main results outlined in the previous section. The first subsection compares the $\mathbb{F}^\tau$-compensators and the $\mathbb{F}$-compensators, while the second subsection studies an $\mathbb{F}^\tau$-martingale that is vital in the explicit construction of deflators.

Lemma 6.16. *Let Z and $\widetilde{Z}$ be the two supermartingales given by (6.1).*
(a) The three sets $\{\widetilde{Z} = 0\}$, $\{Z = 0\}$ and $\{Z_- = 0\}$ have the same début which is an $\mathbb{F}$-stopping time that we denote by

$$R := \inf\{t \geq 0 :\ Z_{t-} = 0\}. \tag{6.8}$$

Note that $\tau \leq R$.
(b) The $\mathbb{F}^\tau$-predictable process

$$H_t := (Z_{t-})^{-1}\, 1\!\!1_{[\![0,\tau]\!]}(t), \tag{6.9}$$

is $\mathbb{F}^\tau$-locally bounded.

Proof. From [DMM92, paragraph 14, Chapter XX], for any random time τ, the sets $\{\widetilde{Z} = 0\}$ and $\{Z_- = 0\}$ are disjoint from $]\!]0, \tau]\!]$ and have the same lower bound R, the smallest $\mathbb{F}$-stopping time greater than τ. Thus, we also conclude that $\{Z = 0\}$ is disjoint from $]\!]0, \tau[\![$. This leads to assertion (a). The process $X := Z^{-1} 1\!\!1_{]\!]0,\tau[\![}$ being a càdlàg $\mathbb{F}^\tau$-supermartingale [Yor78], its left limit is locally bounded. Then, due to

$$(Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} = X_-,$$

the local boundedness of H follows. This ends the proof of the lemma. ∎

6.3.1 Relationship between dual predictable projections under $\mathbb{F}^\tau$ and $\mathbb{F}$

The main results of this subsection are summarized in Lemmas 6.17 and 6.18, where we address the question of how to compute $\mathbb{F}^\tau$-dual predictable projections in term of $\mathbb{F}$-dual predictable projections and vice versa. These results are essentially based on the following standard result on progressive enlargement of filtration (see Proposition 1.39).

Let M be an $\mathbb{F}$-local martingale. Then, for any random time τ, the process

$$\widehat{M}_t \quad := \quad M_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle M, m\rangle_s^{\mathbb{F}}}{Z_{s-}} \tag{6.10}$$

is an $\mathbb{F}^\tau$-local martingale, where m is defined in (6.2).

In the following lemma, we express the $\mathbb{F}^\tau$-dual predictable projection of an $\mathbb{F}$-locally integrable variation process in terms of an $\mathbb{F}$-dual predictable projection, and $\mathbb{F}^\tau$-predictable projection in terms of $\mathbb{F}$-predictable projection.

Lemma 6.17. *The following assertions hold.*
(a) For any $\mathbb{F}$-adapted process V with locally integrable variation, we have

$$(V^\tau)^{p,\mathbb{F}^\tau} = (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \left(\widetilde{Z} \centerdot V\right)^{p,\mathbb{F}}. \tag{6.11}$$

(b) For any $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$

$$ {}^{p,\mathbb{F}^\tau}\!\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\!\left(\Delta M 1\!\!1_{\{\widetilde{Z}>0\}}\right)}{Z_-}, \quad \textit{and} \quad {}^{p,\mathbb{F}^\tau}\!\left(\frac{1}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\!\left(1\!\!1_{\{\widetilde{Z}>0\}}\right)}{Z_-}. \tag{6.12}$$

(c) For any quasi-left-continuous $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$

$$ {}^{p,\mathbb{F}^\tau}\!\left(\frac{\Delta M}{\widetilde{Z}}\right) = 0, \quad \textit{and} \quad {}^{p,\mathbb{F}^\tau}\!\left(\frac{1}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-},$$

where $m^{(qc)}$ is the quasi-left-continuous $\mathbb{F}$-martingale defined in (6.7).

Proof. (a) Using the process H introduced in (6.9), the equality (6.10) takes the form

$$M^\tau = \widehat{M} + H 1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}}.$$

By taking $M = V - V^{p,\mathbb{F}}$, we obtain

$$V^\tau = 1\!\!1_{]\!]0,\tau]\!]} \centerdot V^{p,\mathbb{F}} + \widehat{M} + H 1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle V, m\rangle^{\mathbb{F}} = \widehat{M} + 1\!\!1_{]\!]0,\tau]\!]} \centerdot V^{p,\mathbb{F}} + \frac{1}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \centerdot (\Delta m \centerdot V)^{p,\mathbb{F}},$$

which proves assertion (a).
(b) Let M be an $\mathbb{F}$-local martingale, then, for any positive integers (n,k) the process $V^{(n,k)} := \sum \frac{\Delta M}{\widetilde{Z}} 1\!\!1_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}$ has locally integrable variation. Then, by using the known equality ${}^{p,\mathbb{F}^\tau}(\Delta V) = \Delta(V^{p,\mathbb{F}^\tau})$ (see Lemma 1.13), and applying assertion (a) to the process $V^{(n,k)}$, we get, on $]\!]0,\tau]\!]$

$$ {}^{p,\mathbb{F}^\tau}\!\left(\frac{\Delta M}{\widetilde{Z}} 1\!\!1_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\right) \quad = \quad \frac{1}{Z_-}\, {}^{p,\mathbb{F}}\!\left(\Delta M 1\!\!1_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\right).$$

Since M is a local martingale, by stopping, we can exchange limits with projections in both sides. Then by letting n and k go to infinity, and using the fact that $\widetilde{Z} > 0$ on $]\!]0,\tau]\!]$, we deduce that

$$ {}^{p,\mathbb{F}^\tau}\left(\frac{\Delta M}{\widetilde{Z}}\right) \quad = \quad \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\left(\Delta M 1\!\!1_{\{\widetilde{Z}>0\}}\right). $$

This proves the first equality in (6.12), while the second equality follows from $\widetilde{Z} = \Delta m + Z_-$ (see [JY78]):

$$ Z_-\ {}^{p,\mathbb{F}^\tau}\left(\widetilde{Z}^{-1}\right) = {}^{p,\mathbb{F}^\tau}\left((\widetilde{Z}-\Delta m)/\widetilde{Z}\right) = 1 - {}^{p,\mathbb{F}^\tau}\left(\Delta m/\widetilde{Z}\right) $$
$$ = 1-(Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta m 1\!\!1_{\{\widetilde{Z}>0\}}\right) = 1 - {}^{p,\mathbb{F}}\left(1\!\!1_{\{\widetilde{Z}=0\}}\right) = {}^{p,\mathbb{F}}\left(1\!\!1_{\{\widetilde{Z}>0\}}\right). $$

In the above string of equalities, the third equality follows from the first equality in (6.12), while the fourth equality is due to ${}^{p,\mathbb{F}}(\Delta m) = 0$ and $\Delta m 1\!\!1_{\{\widetilde{Z}=0\}} = -Z_- 1\!\!1_{\{\widetilde{Z}=0\}}$. This ends the proof of assertion (b).

(c) If M is a quasi-left-continuous $\mathbb{F}$-local martingale, then ${}^{p,\mathbb{F}}\left(\Delta M 1\!\!1_{\{\widetilde{Z}>0\}}\right) = 0$, and the first property of the assertion (c) follows. Applying the first property to $M = m^{(qc)}$ and using that, on $]\!]0,\tau]\!]$, one has $\Delta m^{(qc)}\,(Z_- + \Delta m)^{-1} = \Delta m^{(qc)}\left(Z_- + \Delta m^{(qc)}\right)^{-1}$, we obtain

$$ \frac{1}{Z_-}\ {}^{p,\mathbb{F}^\tau}\left(\frac{Z_-}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}\left(1 - {}^{p,\mathbb{F}^\tau}\left(\frac{\Delta m^{(qc)}}{Z_- + \Delta m^{(qc)}}\right)\right) = \frac{1}{Z_-}. $$

This proves assertion (c), and the proof of the lemma is achieved. ∎

The next lemma proves that $\widetilde{Z}^{-1} 1\!\!1_{]\!]0,\tau]\!]}$ is Lebesgue-Stieljes-integrable with respect to any process that is $\mathbb{F}$-adapted with $\mathbb{F}$-locally integrable variation. Using this fact, the lemma addresses the question of how an $\mathbb{F}$-compensator stopped at τ can be written in terms of an $\mathbb{F}^\tau$-compensator, and constitutes a *sort of* converse result to Lemma 6.17(a).

Lemma 6.18. *Let V be an $\mathbb{F}$-adapted càdlàg process. Then the following properties hold.*
(a) If V belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$ (respectively $V \in \mathcal{A}^+(\mathbb{F})$), then the process

$$ U := \widetilde{Z}^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot V, \tag{6.13} $$

belongs to $\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$ (respectively to $\mathcal{A}^+(\mathbb{F}^\tau)$).
(b) If V has $\mathbb{F}$-locally integrable variation, then the process U is well defined, its variation is $\mathbb{F}^\tau$-locally integrable, and its $\mathbb{F}^\tau$-dual predictable projection is given by

$$ \boxed{ U^{p,\mathbb{F}^\tau} = \left(\tfrac{1}{\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot V\right)^{p,\mathbb{F}^\tau} = \tfrac{1}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \left(1\!\!1_{\{\widetilde{Z}>0\}} \centerdot V\right)^{p,\mathbb{F}}. } \tag{6.14} $$

In particular, if supp$V \subset \{\widetilde{Z} > 0\}$, then, on $]\!]0,\tau]\!]$, one has $V^{p,\mathbb{F}} = Z_- \centerdot U^{p,\mathbb{F}^\tau}$.

Proof. (a) Suppose that $V \in \mathcal{A}^+_{loc}(\mathbb{F})$. First, remark that, due to the fact that $\widetilde{Z}$ is positive on $]\!]0,\tau]\!]$, U is well defined. Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to ∞ such that $\mathbb{E}\left(V_{\vartheta_n}\right) < \infty$. Then, if $\mathbb{E}\left(U_{\vartheta_n}\right) \leq \mathbb{E}\left(V_{\vartheta_n}\right)$, assertion (a) follows. Thus, we calculate

$$\begin{aligned}
\mathbb{E}\left(U_{\vartheta_n}\right) &= \mathbb{E}\left(\int_0^{\vartheta_n} 1\!\!1_{\{0<t\leq\tau\}}\frac{1}{\widetilde{Z}_t}dV_t\right) = \mathbb{E}\left(\int_0^{\vartheta_n}\frac{\mathbb{P}(\tau\geq t|\mathcal{F}_t)}{\widetilde{Z}_t}1\!\!1_{\{\widetilde{Z}_t>0\}}dV_t\right)\\
&\leq \mathbb{E}\left(V_{\vartheta_n}\right).
\end{aligned}$$

The last inequality is obtained due to $\widetilde{Z}_t := \mathbb{P}(\tau \geq t|\mathcal{F}_t)$. This ends the proof of assertion (a) of the lemma.

(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$, and denote by $W := V^+ + V^-$ its variation. Then $W \in \mathcal{A}^+_{loc}(\mathbb{F})$, and a direct application of the first assertion implies that

$$\left(\widetilde{Z}\right)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot W \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau).$$

As a result, we deduce that U given by (6.13) for the case of $V = V^+ - V^-$ is well defined and its variation is equal to $\left(\widetilde{Z}\right)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot W$ which is $\mathbb{F}^\tau$-locally integrable. By setting $U_n := 1\!\!1_{]\!]0,\tau]\!]} \centerdot \left(\widetilde{Z}^{-1} 1\!\!1_{\{\widetilde{Z}\geq 1/n\}} \centerdot V\right)$, we derive, due to (6.11),

$$(U_n)^{p,\mathbb{F}^\tau} = \frac{1}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \left(1\!\!1_{\{\widetilde{Z}\geq 1/n\}} \centerdot V\right)^{p,\mathbb{F}}.$$

Hence, since $U^{p,\mathbb{F}^\tau} = \lim_{n\to\infty} (U_n)^{p,\mathbb{F}^\tau}$, by taking the limit in the above equality, (6.14) immediately follows, and the lemma is proved. ∎

6.3.2 An important $\mathbb{F}^\tau$-local martingale

In this subsection, we introduce an $\mathbb{F}^\tau$-local martingale that will be crucial for the construction of the deflator.

Lemma 6.19. *The following nondecreasing process*

$$V_t^{\mathbb{F}^\tau} := \sum_{0\leq u\leq t} {}^{p,\mathbb{F}}\left(1\!\!1_{\{\widetilde{Z}=0\}}\right)_u 1\!\!1_{\{u\leq\tau\}} \tag{6.15}$$

is $\mathbb{F}^\tau$-predictable, càdlàg, and locally bounded.

Proof. Let us define $\widetilde{R} = R_{\{\widetilde{Z}_R=0<Z_{R-}\}}$ using the convention introduced in (1.1) and where R was defined in Lemma 6.16. Then, one has

$$\begin{aligned}
V_t^{\mathbb{F}^\tau} &= \sum_{0\leq u\leq t} {}^{p,\mathbb{F}}\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}}\right)_u 1\!\!1_{\{u\leq\tau\}} = \sum_{0\leq u\leq t} {}^{p,\mathbb{F}}\left(1\!\!1_{[\![\widetilde{R}]\!]}\right)_u 1\!\!1_{\{u\leq\tau\}}\\
&= \sum_{0\leq u\leq t}\left(\Delta(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)_u^{p,\mathbb{F}} 1\!\!1_{\{u\leq\tau\}} = \left(1\!\!1_{[\![\widetilde{R}^a,\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau}.
\end{aligned}$$

Here $\widetilde{R}^a$ is the accessible part of $\widetilde{R}$, and there exists a sequence of $\mathbb{F}$-predictable times with disjoint graphs v_n such that

$$\left(1\!\!1_{[\![\widetilde{R}^a,\infty[\![}\right)^{p,\mathbb{F}} = \sum_n \mathbb{P}(\widetilde{R}^a = v_n|\mathcal{F}_{v_n-})1\!\!1_{[\![v_n,\infty[\![}.$$

The result follows. ∎

The important $\mathbb{F}^\tau$-local martingale will result from the optional integral recalled in this thesis in Section 1.1.8.

Now, we are ready to define the $\mathbb{F}^\tau$-local martingale which will play the role of deflator for a class of processes.

Proposition 6.20. *Consider the following $\mathbb{F}^\tau$-local martingale*

$$\widehat{m} := 1\!\!1_{]\!]0,\tau]\!]} \centerdot m - \frac{1}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle m\rangle^{\mathbb{F}},$$

and the process

$$K := \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} \frac{1}{\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]}.$$

Then, K belongs to the space ${}^oL^1_{loc}(\widehat{m},\mathbb{F}^\tau)$ defined in Definition 1.24. Furthermore, the $\mathbb{F}^\tau$-local martingale

$$L := -K \odot \widehat{m}, \tag{6.16}$$

satisfies the following:
(a) $\mathcal{E}(L) > 0$ (or equivalently $1 + \Delta L > 0$);
(b) for any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$, setting $\widehat{M} := M^\tau - Z_-^{-1} 1\!\!1_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}}$, we have

$$[L, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{F}^\tau) \quad \Big(i.e.,\ \langle L, \widehat{M}\rangle^{\mathbb{F}^\tau} \quad exists\Big). \tag{6.17}$$

Proof. We shall prove that $K \in {}^oL^1_{loc}(\widehat{m},\mathbb{F}^\tau)$ in the appendix 6.6.2.

We now prove assertions (a) and (b). Due to (6.45), we have, on $]\!]0,\tau]\!]$,

$$-\Delta L = K\Delta\widehat{m} - {}^{p,\mathbb{F}^\tau}(K\Delta\widehat{m}) = 1 - Z_-\left(\widetilde{Z}\right)^{-1} - {}^{p,\mathbb{F}}\Big(1\!\!1_{\{\widetilde{Z}=0\}}\Big).$$

Thus, we deduce that $1 + \Delta L > 0$, and assertion (a) is proved. In the rest of this proof, we will prove (6.17). To this end, let $M \in \mathcal{M}_{0,loc}(\mathbb{F})$. Thanks to Proposition 1.25, (6.17) is equivalent to

$$K \centerdot [\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{F}^\tau) \text{ or equivalently } \frac{1}{\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{F}^\tau)$$

for any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$. Then, it is easy to check that

$$\begin{aligned}
\frac{1}{\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [\widehat{m}, \widehat{M}] &= \frac{1}{\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [m, \widehat{M}] - \frac{1}{Z_-\,\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [\langle m\rangle^{\mathbb{F}}, \widehat{M}] \\
&= \frac{1}{\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [m, M] - \frac{1}{Z_-\,\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [m, \langle M, m\rangle^{\mathbb{F}}] \\
&\quad - \frac{1}{Z_-\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [\langle m\rangle^{\mathbb{F}}, M] + \frac{1}{Z_-^2\,\widetilde{Z}} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [\langle m\rangle^{\mathbb{F}}, \langle M, m\rangle^{\mathbb{F}}].
\end{aligned}$$

Since m is an $\mathbb{F}$-locally bounded local martingale, all the processes

$$[m, M],\ [m, \langle M, m\rangle^{\mathbb{F}}],\ [\langle m\rangle^{\mathbb{F}}, M],\ \text{and } [\langle m\rangle^{\mathbb{F}}, \langle M, m\rangle^{\mathbb{F}}]$$

belong to $\mathcal{A}_{loc}(\mathbb{F})$. Thus, by combining this fact with Lemma 6.18 and the $\mathbb{F}^\tau$-local boundedness of $Z_-^{-p} 1\!\!1_{]\!]0,\tau]\!]}$ for any $p > 0$, it follows that $A \in \mathcal{A}_{loc}(\mathbb{F}^\tau)$. This ends the proof of the proposition. ∎

6.4 Explicit deflators

This section describes some classes of $\mathbb{F}$-quasi-left-continuous local martingales for which the NUPBR is preserved after stopping with τ. For these stopped processes, we explicitly describe their local martingale densities in Theorems 6.21–6.25 with an increasing degree of generality. We recall that $m^{(qc)}$ was defined in (6.7) and L was defined in Proposition 6.20.

Theorem 6.21. *Suppose that S is a quasi-left-continuous $\mathbb{F}$-local martingale. If S and τ satisfy*

$$\{\Delta S \neq 0\} \cap \{Z_- > 0 = \widetilde{Z}\} = \emptyset, \tag{6.18}$$

then the following equivalent assertions hold.

(a) $\mathcal{E}(L)\, S^\tau$ is an $\mathbb{F}^\tau$-local martingale.

(b) $\mathcal{E}\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\right) S$ is an $\mathbb{F}$-local martingale.

Proof. We start by giving some useful observations. Since S is $\mathbb{F}$-quasi-left-continuous, on the one hand we deduce that (Γ_m is defined in (6.7))

$$\langle S, m\rangle^{\mathbb{F}} = \langle S, m^{(qc)}\rangle^{\mathbb{F}} = \langle S, 1\!\!1_{\Gamma_m^c} \centerdot m\rangle^{\mathbb{F}}. \tag{6.19}$$

On the other hand, we note that assertion (a) is equivalent to $\mathcal{E}(L^{(qc)})S^\tau$ is an $\mathbb{F}^\tau$-local martingale, where $L^{(qc)}$ is the quasi-left-continuous local martingale part of L given by $L^{(qc)} := 1\!\!1_{\Gamma_m^c} \centerdot L = -K \odot \widehat{m}^{(qc)}$. Here K is given in Proposition 6.20 and

$$\widehat{m}^{(qc)} := 1\!\!1_{]\!]0,\tau]\!]} \centerdot m^{(qc)} - (Z_-) 1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle m^{(qc)}\rangle^{\mathbb{F}}.$$

It is easy to check that (6.18) is equivalent to

$$1\!\!1_{\{Z_->0=\widetilde{Z}\}} \centerdot [S, m] = 0. \tag{6.20}$$

We now compute $-\langle L^{(qc)}, \widehat{S}\rangle^{\mathbb{F}^\tau}$, where $\widehat{S}$ is the $\mathbb{F}^\tau$-local martingale given by

$$\widehat{S} := S^\tau - (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle S, m\rangle^{\mathbb{F}}.$$

Due to the quasi-left continuity of S and that of $m^{(qc)}$, the two processes $\langle S, m\rangle^{\mathbb{F}}$ and $\langle m^{(qc)}\rangle^{\mathbb{F}}$ are continuous and $[S, m^{(qc)}] = [S, m]$. Hence, we obtain

$$\begin{aligned} K \centerdot [\widehat{S}, \widehat{m}^{(qc)}] &= K \centerdot [S, \widehat{m}^{(qc)}] - K \Delta \widehat{m}^{(qc)} (Z_-)^{-1} \centerdot \langle S, m\rangle^{\mathbb{F}} \\ &= (\widetilde{Z})^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [S, m^{(qc)}] = (\widetilde{Z})^{-1} 1\!\!1_{]\!]0,\tau]\!]} \centerdot [S, m]. \end{aligned}$$

It follows that

$$\begin{aligned}
&-\langle L^{(qc)}, \widehat{S}\rangle^{\mathbb{F}^\tau} = \left(K \cdot [\widehat{S}, \widehat{m}^{(qc)}]\right)^{p,\mathbb{F}^\tau} = \left((\widetilde{Z})^{-1} 1\!\!1_{]\!]0,\tau]\!]} \cdot [S,m]\right)^{p,\mathbb{F}^\tau} \\
= \;& (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \cdot \left(1\!\!1_{\{\widetilde{Z}>0\}} \cdot [S,m]\right)^{p,\mathbb{F}} \\
= \;& (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \cdot \langle S, m\rangle^{\mathbb{F}} - (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \cdot \left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \cdot [S,m]\right)^{p,\mathbb{F}} \\
= \;& (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \cdot \langle S, m\rangle^{\mathbb{F}} + (Z_-)^{-1} 1\!\!1_{]\!]0,\tau]\!]} \cdot \langle S, -1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\rangle^{\mathbb{F}}.
\end{aligned} \tag{6.21}$$

The first and the last equality follow from Proposition 1.25 applied respectively to $L^{(qc)}$ and $-1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}$. The second and the third equalities are respectively due to (6.19) and (6.14).

Now, we prove the theorem. Thanks to (6.21), it is obvious that assertion (a) is equivalent to $\langle S, -1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\rangle^{\mathbb{F}} \equiv 0$ which in turn is equivalent to assertion (b). This ends the proof of the equivalence between (a) and (b).

It is also clear that the condition (6.18) or equivalently (6.20) implies assertion (b), due to $[1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}, S] = 1\!\!1_{\{\widetilde{Z}=0<Z_-\}} \cdot [m,S] \equiv 0$. ∎

Remark 6.22. Suppose that S is a quasi-left-continuous $\mathbb{F}$-local martingale and let

$$R_0 := R_{\{\widetilde{Z}_R=0\}} \tag{6.22}$$

where R was defined in 6.8 and we use the convention (1.1). Then, $\mathcal{E}(L)\,\overline{S}^\tau$ is an $\mathbb{F}^\tau$-local martingale, where

$$\overline{S} := S^{R_0-} + \left(\Delta S_{R_0} 1\!\!1_{[\![R_0,\infty[\![}\right)^{p,\mathbb{F}}.$$

Indeed, writing

$$\overline{S} := S^{\;R_0} - \Delta S_{R_0} 1\!\!1_{[\![R_0,\infty[\![} + \left(\Delta S_{R_0} 1\!\!1_{[\![R_0,\infty[\![}\right)^{p,\mathbb{F}}$$

it is easy to see that the condition (6.18) is satisfied for $\overline{S}$.

Corollary 6.23. If S is quasi-left continuous and satisfies NUPBR($\mathbb{F}$) and

$$\{\Delta S \neq 0\} \cap \{Z_- > 0 = \widetilde{Z}\} = \emptyset,$$

then S satisfies NUPBR($\mathbb{F}^\tau$).

Proof. This follows from Proposition 1.59, Theorem 6.21 and the fact that, for any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, we have

$$\{Z_- > 0\} \cap \{\widetilde{Z} = 0\} = \{Z^{\mathbb{Q}}_- > 0\} \cap \{\widetilde{Z}^{\mathbb{Q}} = 0\}.$$

Here $Z^{\mathbb{Q}}_t = \mathbb{Q}(\tau > t|\mathcal{F}_t)$ and $\widetilde{Z}^{\mathbb{Q}}_t = \mathbb{Q}(\tau \geq t|\mathcal{F}_t)$. This last claim is a direct application of the optional and predictable Section theorem 1.10. ∎

In order to generalize the previous result, we need to introduce more notation and recall some results that are delegated in the Appendix. For the random measure μ defined in (1.3), we associate its predictable compensator random measure ν.

Definition 6.24. For a quasi-left-continuous process, $\mathcal{G}^1_{loc}(\mu,\mathbb{F})$ (respectively $\mathcal{H}^1_{loc}(\mu,\mathbb{F})$) is the set of all $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functions (respectively all $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functions) W such that

$$\sqrt{W^2\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{H}).$$

A direct application of Theorem 1.26, to the martingale m, leads to the existence of a local martingale $m^\perp$ as well as a process $\beta_m\in L(S^c,\mathbb{F})$, a $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional f_m and an $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functional g_m such that $f_m\in\mathcal{G}^1_{loc}(\mu,\mathbb{F})$ and $g_m\in\mathcal{H}^1_{loc}(\mu,\mathbb{F})$ such that

$$m=\beta_m\centerdot S^c+f_m\star(\mu-\nu)+g_m\star\mu+m^\perp. \tag{6.23}$$

We introduce $\mu^{\mathbb{F}^\tau}:=1\!\!1_{[\![0,\tau]\!]}\star\mu$ and its $\mathbb{F}^\tau$ compensated measure

$$\nu^{\mathbb{F}^\tau}(dt,dx):=(1+f_m(x)/Z_{t-})1\!\!1_{[\![0,\tau]\!]}(t)\nu(dt,dx). \tag{6.24}$$

We now state our general result that extends the previous Theorem 6.21.

Theorem 6.25. *Suppose that S is an $\mathbb{F}$-quasi-left-continuous local martingale. Consider $S^{(0)}$, ψ, and L defined in (6.3) and (6.16). If $\left(S,S^{(0)}\right)$ is an $\mathbb{F}$-local martingale, then $\mathcal{E}\left(L+L^{(1)}\right)S^\tau$ is an $\mathbb{F}^\tau$-local martingale, where*

$$L^{(1)}:=g_1\star(\mu^{\mathbb{F}^\tau}-\nu^{\mathbb{F}^\tau}),\quad \textit{and}\quad g_1:=\frac{1-\psi}{1+f_m/Z_-}1\!\!1_{\{\psi>0\}}.$$

Proof. We start by recalling from (6.4) that $\{\psi=0\}=\{Z_-+f_m=0\}$, $M^{\mathbb{P}}_\mu$-a.e. Thus the functional g_1 is a well defined non-negative $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional. The proof of the theorem will be completed in two steps. In the first step, we prove that the process $L^{(1)}$ is a well defined local martingale, while in the second step, we prove the main statement of the theorem.
1) Herein, we prove that the integral $g_1\star\left(\mu^{\mathbb{F}^\tau}-\nu^{\mathbb{F}^\tau}\right)$ is well-defined. To this end, it is enough to prove that $g_1\star\mu^{\mathbb{F}^\tau}\in\mathcal{A}^+(\mathbb{F}^\tau)$. Therefore, remark that

$$(1-\psi)1\!\!1_{\{0<Z_-\}}=M^{\mathbb{P}}_\mu\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right)=M^{\mathbb{P}}_\mu\left(1\!\!1_{[\![R_0]\!]}|\widetilde{\mathcal{P}}(\mathbb{F})\right)1\!\!1_{\{0<Z_-\}},$$

(where R_0 is defined in 6.22) and calculate

$$\begin{aligned}\mathbb{E}\left(g_1\star\mu^{\mathbb{F}^\tau}_\infty\right)&=\mathbb{E}\left(g_1\widetilde{Z}\star\mu_\infty\right)\\ &\leq\mathbb{E}\left(1\!\!1_{[\![R_0]\!]}\star\mu_\infty\right)=\mathbb{P}\left(\Delta S_{R_0}\neq0,\ R_0<\infty\right)\leq1.\end{aligned}$$

Thus, the process $L^{(1)}$ is a well defined $\mathbb{F}^\tau$-martingale.
2) In this part, we prove that $\mathcal{E}\left(L+L^{(1)}\right)S^\tau$ is an $\mathbb{F}^\tau$-local martingale. To this end, it is enough to prove that $\langle S^\tau,L+L^{(1)}\rangle^{\mathbb{F}^\tau}$ exists and

$$S^\tau+\left\langle S^\tau,L+g_1\star\left(\mu^{\mathbb{F}^\tau}-\nu^{\mathbb{F}^\tau}\right)\right\rangle^{\mathbb{F}^\tau}\quad\text{is an }\mathbb{F}^\tau\text{-local martingale.} \tag{6.25}$$

Recall that

$$L = -\frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}}\,\frac{1}{\widetilde{Z}}\,1\!\!1_{]\!]0,\tau]\!]} \odot \widehat{m},$$

and hence $\langle S^\tau, L\rangle^{\mathbb{F}^\tau}$ exists due to Proposition 6.20(b). By stopping, there is no loss of generality in assuming that S is a true martingale. Then, using similar calculation as in the first part 1), we can easily prove that

$$\mathbb{E}\left[|x|g_1 \star \mu_\infty^{\mathbb{F}^\tau}\right] \le \mathbb{E}\left(|\Delta S_{R_0}|1\!\!1_{\{R_0<\infty\}}\right) < \infty.$$

This proves that $\left\langle S^\tau, L + L^{(1)}\right\rangle^{\mathbb{F}^\tau}$ exists. Now, we calculate and simplify the expression in (6.25) as follows.

$$\begin{aligned}
S^\tau \quad + \quad & \left\langle S^\tau, L + g_1 \star \left(\mu^{\mathbb{F}^\tau} - \nu^{\mathbb{F}^\tau}\right)\right\rangle^{\mathbb{F}^\tau} = \widehat{S} + \frac{1}{Z_-}1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle S, m\rangle^{\mathbb{F}} + \langle S^\tau, L\rangle^{\mathbb{F}^\tau} + xg_1 \star \nu^{\mathbb{F}^\tau} \\
= \quad & \widehat{S} + \frac{1}{Z_-}1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle S, m\rangle - \frac{1}{Z_-}1\!\!1_{]\!]0,\tau]\!]} \cdot \left(1\!\!1_{\{\widetilde{Z}>0\}} \cdot [S, m]\right)^{p,\mathbb{F}} + xg_1 \star \nu^{\mathbb{F}^\tau} \\
= \quad & \widehat{S} + \frac{1}{Z_-}1\!\!1_{]\!]0,\tau]\!]} \centerdot \left(1\!\!1_{\{\widetilde{Z}=0\}} \centerdot [S, m]\right)^{p,\mathbb{F}} + xM_\mu^{\mathbb{P}}\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right)1\!\!1_{\{Z_-+f_m>0\}}1\!\!1_{]\!]0,\tau]\!]} \star \nu \\
= \quad & \widehat{S} - xM_\mu^{\mathbb{P}}\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right)1\!\!1_{\{\psi=0\}}1\!\!1_{]\!]0,\tau]\!]} \star \nu = \widehat{S} \in \mathcal{M}_{loc}(\mathbb{F}^\tau).
\end{aligned}$$

The second equality is due to (6.21), while the last equality follows directly from the fact that $S^{(0)}$ is an $\mathbb{F}$-local martingale (which is equivalent to $x1\!\!1_{\{\psi=0<Z_-\}} \star \nu \equiv 0$) and $M_\mu^{\mathbb{P}}\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right) = 1\!\!1_{\{0<Z_-\}}(1-\psi)$. This ends the proof of the theorem. ∎

Remark 6.26. (a) Both Theorems 6.21-6.25 provide methods that build-up explicitly σ-martingale density for S^τ, whenever S is an $\mathbb{F}$-quasi-left-continuous that is a local martingale under a locally equivalent probability measure, and fulfilling the assumptions of the theorems respectively.
(b) The extension of Theorem 6.21 to the general case where S is an $\mathbb{F}$-local martingale (not necessarily quasi-left-continuous) boils down to find a thin predictable process Φ such that $Y := \mathcal{E}(\Phi \centerdot L)$ will be the martingale density for S^τ. Finding the process Φ will be easy to guess when we will address the case of thin semimartingale. However the proof of Y is a local martingale density for S^τ is very technical. The extension of Theorem 6.25 to the case of arbitrary $\mathbb{F}$-local martingale S requires additional careful modification of the functional g_1 so that $1 + \Phi(\Delta L + \Delta L^{(1)})$ remains positive. While both extensions remain feasible, we opted to not overload the paper with technicalities.

6.5 Proofs of main theorems

This section is devoted to the proofs of Theorems 6.1, 6.6 and 6.9. They are quite long, since some integrability results have to be proved. We write the canonical decomposition of S (see Section 1.1.10)

$$S = S_0 + S^c + h \star (\mu - \nu) + b \centerdot \bar{A} + (x - h) \star \mu,$$

where h is defined as $h(x) := x 1\!\!1_{\{|x|\leq 1\}}$ is the truncation function and ν is the compensator of μ. The canonical decomposition of S^τ under $\mathbb{F}^\tau$ is given by

$$S^\tau = S_0 + \widehat{S^c} + h \star (\mu^{\mathbb{F}^\tau} - \nu^{\mathbb{F}^\tau}) + \tfrac{c\beta_m}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \bar{A} + h \tfrac{f_m}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \star \nu + b \centerdot \bar{A}^\tau + (x-h) \star \mu^{\mathbb{F}^\tau}$$

where $\mu^{\mathbb{F}^\tau}$ and $\nu^{\mathbb{F}^\tau}$ and (β_m, f_m) are given in (6.24) and (6.23) respectively and

$$\widehat{S^c} := 1\!\!1_{]\!]0,\tau]\!]} \centerdot S^c - \frac{1}{Z_-} 1\!\!1_{]\!]0,\tau]\!]} \centerdot \langle m, S^c \rangle^{\mathbb{F}}.$$

6.5.1 Proof of Theorem 6.1

The proof of Theorem 6.1 will be completed in four steps. The first step provides an equivalent formulation to assertion (a) using the filtration $\mathbb{F}$ instead. In the second step, we prove (a)$\Rightarrow$(b), while the reverse implication is proved in the third step. The proof of (b)$\Longleftrightarrow$ (c) is given in the last step.

Step 1. Formulation of assertion (a). Thanks to Proposition 1.56, S^τ satisfies NUPBR($\mathbb{F}^\tau$) if and only if there exist an $\mathbb{F}^\tau$-local martingale $L^{\mathbb{F}^\tau}$ with $1+\Delta L^{\mathbb{F}^\tau} > 0$ and an $\mathbb{F}^\tau$-predictable process $\phi^{\mathbb{F}^\tau}$ such that $0 < \phi^{\mathbb{F}^\tau} \leq 1$ and $\mathcal{E}\left(L^{\mathbb{F}^\tau}\right)\left(\phi^{\mathbb{F}^\tau} \centerdot S^\tau\right)$ is an $\mathbb{F}^\tau$-local martingale. We can reduce our attention to processes $L^{\mathbb{F}^\tau}$ such that

$$L^{\mathbb{F}^\tau} = \beta^{\mathbb{F}^\tau} \centerdot \widehat{S^c} + (f^{\mathbb{F}^\tau} - 1) \star (\mu^{\mathbb{F}^\tau} - \nu^{\mathbb{F}^\tau})$$

where $\beta^{\mathbb{F}^\tau} \in L(\widehat{S^c}, \mathbb{F}^\tau)$ and $f^{\mathbb{F}^\tau}$ is positive and such that $f^{\mathbb{F}^\tau} - 1 \in \mathcal{G}^1_{loc}(\mu^{\mathbb{F}^\tau}, \mathbb{F}^\tau)$.

Then, one notes that $\mathcal{E}\left(L^{\mathbb{F}^\tau}\right)\left(\phi^{\mathbb{F}^\tau} \centerdot S^\tau\right)$ is an $\mathbb{F}^\tau$-local martingale if and only if $\phi^{\mathbb{F}^\tau} \centerdot S^\tau + [\phi^{\mathbb{F}^\tau} \centerdot S^\tau, L^{\mathbb{F}^\tau}]$ is a $\mathbb{F}^\tau$-local martingale, which in turn, is equivalent to

$$\phi^{\mathbb{F}^\tau} |x f^{\mathbb{F}^\tau}(x) - h(x)| \left(1 + \frac{f_m(x)}{Z_-}\right) 1\!\!1_{[\![0,\tau]\!]} \star \nu \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau), \tag{6.26}$$

and $\mathbb{P} \otimes \bar{A} - a.e.$ on $[\![0,\tau]\!]$ (using the kernel F defined in Section 1.1.10 and studied in Appendix 6.6.1)

$$b + c(\frac{\beta_m}{Z_-} + \beta^{\mathbb{F}^\tau}) + \int \left[(x f^{\mathbb{F}^\tau}(x) - h(x)) \left(1 + \frac{f_m(x)}{Z_-}\right) + h(x) \frac{f_m(x)}{Z_-}\right] F(dx) = 0. \tag{6.27}$$

From Lemma 6.33, there exist $\phi^{\mathbb{F}}$ and $\beta^{\mathbb{F}}$ two $\mathbb{F}$-predictable processes and a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $f^{\mathbb{F}}$, such that $0 < \phi^{\mathbb{F}} \leq 1$,

$$\beta^{\mathbb{F}} = \beta^{\mathbb{F}^\tau},\ \phi^{\mathbb{F}} = \phi^{\mathbb{F}^\tau},\ f^{\mathbb{F}} = f^{\mathbb{F}^\tau} \text{ on } [\![0,\tau]\!]. \tag{6.28}$$

Then, from Proposition 6.34 we deduce that (6.26)–(6.27) imply that, on $\{Z_- \geq \delta\}$, we have

$$W^{\mathbb{F}} := \int |(x f^{\mathbb{F}}(x) - h(x))| \left(1 + \frac{f_m(x)}{Z_-}\right) F(dx) < \infty \quad \mathbb{P} \otimes \bar{A} - a.e, \tag{6.29}$$

and $\mathbb{P} \otimes \bar{A}$-a.e. on $\{Z_- \geq \delta\}$, we have

$$b + c\left(\beta^{\mathbb{F}} + \frac{\beta_m}{Z_-}\right) - \int h(x) 1\!\!1_{\{\psi=0\}} F(dx) + \int \left[x f^{\mathbb{F}}(x)(1 + \frac{f_m(x)}{Z_-}) - h(x)\right] 1\!\!1_{\{\psi>0\}} F(dx) = 0. \tag{6.30}$$

This latter equality follows as $\{\psi=0\}=\{Z_-+f_m=0\}$ (see 6.4).

Step 2. Proof of (a) $\Rightarrow$ (b). Suppose that S^τ satisfies NUPBR($\mathbb{F}^\tau$), hence (6.29)–(6.30) hold. To prove that $\mathbb{1}_{\{Z\geq\delta\}}\centerdot(S-S^{(0)})$ satisfies NUPBR($\mathbb{F}$), we consider

$$\beta:=\left(\frac{\beta_m}{Z_-}+\beta^{\mathbb{F}}\right)\mathbb{1}_{\{Z_-\geq\delta\}}\text{ and }\ f=f^{\mathbb{F}}\left(1+\frac{f_m}{Z_-}\right)\mathbb{1}_{\Sigma_0}+\mathbb{1}_{\Sigma_0^c},$$

where $\Sigma_0:=\{Z_-\geq\delta,\ \psi>0\}$. If $\beta\in L(S^c,\mathbb{F})$ and $(f-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{F})$, we conclude that

$$L':=\beta\centerdot S^c+(f-1)\star(\mu-\nu)$$

is a well defined $\mathbb{F}$-local martingale. Choosing $\phi=\left(1+W^{\mathbb{F}}\mathbb{1}_{\{Z_-\geq\delta\}}\right)^{-1}$ where $W^{\mathbb{F}}$ is defined in (6.29), using (6.30), and applying Itô's formula for $\mathcal{E}(L')\left(\phi\mathbb{1}_{\{Z_-\geq\delta\}}\cdot(S-S^{(0)})\right)$, we deduce that this process is an $\mathbb{F}$-local martingale. Hence, $\mathbb{1}_{\{Z_-\geq\delta\}}\cdot(S-S^{(0)})$ satisfies NUPBR($\mathbb{F}$), and the proof of (a)$\Rightarrow$(b) is completed.
Now, we focus on proving $\beta\in L(S^c,\mathbb{F})$ and $(f-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{F})$ (or equivalently $\sqrt{(f-1)^2\star\mu}$ belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$). Since $\beta_m\in L(S^c,\mathbb{F})$, then it is obvious that $\frac{\beta_m}{Z_-}\mathbb{1}_{\{Z_-\geq\delta\}}$ belongs to $L(S^c,\mathbb{F})$ on the one hand. On the other hand, $(\beta^{\mathbb{F}})^Tc\beta^{\mathbb{F}}\mathbb{1}_{\{Z_-\geq\delta\}}\centerdot\bar{A}\in\mathcal{A}^+_{loc}(\mathbb{F})$ due to $(\beta^{\mathbb{F}})^Tc\beta^{\mathbb{F}}\centerdot\bar{A}^\tau=(\beta^{\mathbb{F}^\tau})^Tc\beta^{\mathbb{F}^\tau}\centerdot\bar{A}^\tau\in\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$ and Appendix Proposition 6.34(c). This completes the proof of $\beta\in L(S^c,\mathbb{F})$.

Now, we focus on proving $(f-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{F})$. Since S is quasi-left-continuous, this is equivalent to prove $\sqrt{(f-1)^2\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{F})$. Thanks to Appendix Proposition 6.34 and $\sqrt{(\ f^{\mathbb{F}}-1)^2\star\mu^{\mathbb{F}^\tau}}=\sqrt{(\ f^{\mathbb{F}^\tau}-1)^2\star\mu^{\mathbb{F}^\tau}}$ belongs to $\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$, we deduce that

$$(\ f^{\mathbb{F}}-1)^2\mathbb{1}_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}\widetilde{Z}\mathbb{1}_{\{Z_-\geq\delta\}}\star\mu\text{ and }\ |f^{\mathbb{F}}-1|\mathbb{1}_{\{|f^{\mathbb{F}}-1|>\alpha\}}\widetilde{Z}\mathbb{1}_{\{Z_-\geq\delta\}}\star\mu\in\mathcal{A}^+_{loc}(\mathbb{F}).$$

By stopping, there is no loss of generality in assuming that these two processes and $[m,m]$ are integrable. Then,

$$f-1=\left(f^{\mathbb{F}}-1\right)\left(1+\frac{f_m}{Z_-}\right)\mathbb{1}_{\Sigma_0}+\frac{f_m}{Z_-}\mathbb{1}_{\Sigma_0}=:h_1+h_2.$$

Therefore, we derive that

$$\begin{aligned}\mathbb{E}\left[h_1^2\mathbb{1}_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}\star\mu_\infty\right]\ &\leq\ \delta^{-2}\mathbb{E}\left[\left(f^{\mathbb{F}}-1\right)^2(Z_-+f_m)^2\,\mathbb{1}_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}\mathbb{1}_{\{Z_-\geq\delta\}}\star\mu_\infty\right]\\ &\leq\ \delta^{-2}\mathbb{E}\left[\left(f^{\mathbb{F}}-1\right)^2\widetilde{Z}\mathbb{1}_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}\mathbb{1}_{\{Z_-\geq\delta\}}\star\mu_\infty\right]<\infty,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[|h_1|\mathbb{1}_{\{|f-1|>\alpha\}}\star\mu_\infty\right]\ &\leq\ \delta^{-1}\mathbb{E}\left[|f-1|\ |Z_-+f_m|\mathbb{1}_{\{|f-1|>\alpha\}}\mathbb{1}_{\{Z_-\geq\delta\}}\star\mu_\infty\right]\\ &=\ \delta^{-1}\mathbb{E}\left[|f-1|\widetilde{Z}\mathbb{1}_{\{|f-1|>\alpha\}}\mathbb{1}_{\{Z_-\geq\delta\}}\star\mu_\infty\right]<\infty.\end{aligned}$$

By combining the above two inequalities, we conclude that $\left(h_1^2\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{F})$. It is easy to see that $\left(h_2^2\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{F})$ follows from

$$\mathbb{E}\left[h_2^2\star\mu_\infty\right]\ \leq\ \delta^{-2}\mathbb{E}\left[f_m^2\star\mu_\infty\right]\leq\delta^2\mathbb{E}\left[(\Delta m)^2\star\mu_\infty\right]\leq\delta^{-2}\mathbb{E}\left[m,m\right]_\infty<\infty.$$

Step 3. Proof of (b) $\Rightarrow$ (a). Suppose that for any $\delta > 0$, the process $\mathbb{1}_{\{Z_- \geq \delta\}} \centerdot \left(S - S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$). Then, there exist an $\mathbb{F}$-local martingale $L^{\mathbb{F}}$ and an $\mathbb{F}$-predictable process ϕ such that $0 < \phi \leq 1$ and $\mathcal{E}\left(L^{\mathbb{F}}\right)\left[\phi \mathbb{1}_{\{Z_- \geq \delta\}} \centerdot \left(S - S^{(0)}\right)\right]$ is an $\mathbb{F}$-local martingale. We can restrict our attention to the case

$$L^{\mathbb{F}} := \beta^{\mathbb{F}} \centerdot S^c + (f^{\mathbb{F}} - 1) \star (\mu - \nu),$$

where $\beta^{\mathbb{F}} \in L(S^c, \mathbb{F})$ and $f^{\mathbb{F}}$ is positive such that $f^{\mathbb{F}} - 1 \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$.

Thanks to Itô's formula, the fact that $\mathcal{E}\left(L^{\mathbb{F}}\right)\left[\phi \mathbb{1}_{\{Z_- \geq \delta\}} \centerdot \left(S - S^{(0)}\right)\right]$ is an $\mathbb{F}$-local martingale implies that on $\{Z_- \geq \delta\}$

$$k^{\mathbb{F}} := \int |x f^{\mathbb{F}}(x) \mathbb{1}_{\{\psi(x) > 0\}} - h(x)| F(dx) < \infty \qquad \mathbb{P} \otimes \bar{A} - a.e.$$

and $\mathbb{P} \otimes \bar{A}$-a.e. on $\{Z_- \geq \delta\}$, we have

$$b - \int h(x) \mathbb{1}_{\{\psi = 0\}} F(dx) + c\beta^{\mathbb{F}} + \int \left[x f^{\mathbb{F}}(x) - h(x)\right] \mathbb{1}_{\{\psi(x) > 0\}} F(dx) = 0. \tag{6.31}$$

Recall that $\{\psi = 0\} = \{Z_- + f_m = 0\}$ and consider

$$\beta^{\mathbb{F}^\tau} := \left(\beta^{\mathbb{F}} - \frac{\beta_m}{Z_-}\right) \mathbb{1}_{]\!]0,\tau]\!]} \quad \text{and} \quad f^{\mathbb{F}^\tau} := \frac{f^{\mathbb{F}}}{1 + f_m/Z_-} \mathbb{1}_{\{\psi > 0\}} \mathbb{1}_{]\!]0,\tau]\!]} + \mathbb{1}_{\{\psi = 0\} \cup]\!]\tau, \infty[\![}.$$

If we assume that

$$\beta^{\mathbb{F}^\tau} \in L(\widehat{S^c}, \mathbb{F}) \quad \text{and} \quad (f^{\mathbb{F}^\tau} - 1) \in \mathcal{G}^1_{loc}(\mu^{\mathbb{F}^\tau}), \tag{6.32}$$

then, necessarily $L^{\mathbb{F}^\tau} := \beta^{\mathbb{F}^\tau} \centerdot \widehat{S^c} + (f^{\mathbb{F}^\tau} - 1) \star (\mu^{\mathbb{F}^\tau} - \nu^{\mathbb{F}^\tau})$ is a well defined $\mathbb{F}^\tau$-local martingale satisfying $\mathcal{E}(L^{\mathbb{F}^\tau}) > 0$. Furthermore, due to (6.31) and to $\{\psi = 0\} = \{Z_- + f_m = 0\}$ (see (6.4)), on $]\!]0, \tau]\!]$ we obtain

$$b + c\left(\beta^{\mathbb{F}^\tau} + \frac{\beta_m}{Z_-}\right) + \int \left(x f^{\mathbb{F}^\tau}\left(1 + \frac{f_m}{Z_-}\right) - h(x)\right) F(dx) = 0\,. \tag{6.33}$$

By taking $\phi^{\mathbb{F}^\tau} := \left(1 + k^{\mathbb{F}} \mathbb{1}_{\{Z_- \geq \delta\}}\right)^{-1}$, and applying Itô's formula to $(\phi^{\mathbb{F}^\tau} \mathbb{1}_{\{Z_- \geq \delta\}} \centerdot S^\tau)\mathcal{E}(L^{\mathbb{F}^\tau})$, we conclude that this process is an $\mathbb{F}^\tau$-local martingale due to (6.33). Thus, $\mathbb{1}_{\{Z_- \geq \delta\}} \centerdot S^\tau$ satisfies NUPBR($\mathbb{F}^\tau$) as long as (6.32) is fulfilled.

Since $Z^{-1} \mathbb{1}_{[\![0,\tau[\![}$ is $\mathbb{F}^\tau$-locally bounded, then there exists a family of $\mathbb{F}^\tau$-stopping times $(\tau_\delta)_{\delta > 0}$ such that $[\![0, \tau_\delta]\!] \subset \{Z_- \geq \delta\}$ (which implies that $\mathbb{1}_{\{Z_- \geq \delta\}} \centerdot S^{\tau \wedge \tau_\delta} = S^{\tau \wedge \tau_\delta}$) and τ_δ increases to infinity when δ goes to zero. Thus, using Proposition 1.59, we deduce that S^τ satisfies NUPBR($\mathbb{F}^\tau$). This achieves the proof of (b)$\Rightarrow$(a) under (6.32).

To prove that (6.32) holds true, we remark that $Z_-^{-1} \mathbb{1}_{]\!]0,\tau]\!]}$ is $\mathbb{F}^\tau$-locally bounded and both β_m and $\beta^{\mathbb{F}}$ belong to $L(S^c, \mathbb{F})$. This, easily, implies that $\beta^{\mathbb{F}^\tau} \in L(\widehat{S^c}, \mathbb{F}^\tau)$.
Now, we prove that $\sqrt{(f^{\mathbb{F}^\tau} - 1)^2 \star \mu^{\mathbb{F}^\tau}} \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau)$. Since $\sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$, Proposition 6.34 allows us to deduce that

$$(f^{\mathbb{F}} - 1)^2 \mathbb{1}_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}) \text{ and } |f^{\mathbb{F}} - 1| \mathbb{1}_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

Without loss of generality, we can assume that these two processes and $[m,m]$ are integrable. Put

$$f^{\mathbb{F}^\tau}-1=\mathbb{1}_{\{\psi>0\}}\mathbb{1}_{]\!]0,\tau]\!]}\frac{Z_-(f^{\mathbb{F}}-1)}{f_m+Z_-}-\mathbb{1}_{\{\psi>0\}}\mathbb{1}_{]\!]0,\tau]\!]}\frac{f_m}{f_m+Z_-}:=f_1+f_2.$$

Then, we calculate

$$\mathbb{E}\left(f_1^2\mathbb{1}_{\{f_m+Z_->\delta/2\}\cap\{|f^{\mathbb{F}}-1|\le\alpha\}}\star\mu^{\mathbb{F}^\tau}_\infty\right)\le\left(\tfrac{2}{\delta}\right)^2\mathbb{E}\left((f^{\mathbb{F}}-1)^2\mathbb{1}_{\{|f^{\mathbb{F}}-1|\le\alpha\}}\star\mu_\infty\right)<\infty$$

and

$$\begin{array}{rcl}\mathbb{E}\left(\sqrt{f_1^2\mathbb{1}_{\{f_m+Z_-\le\delta/2\}\cap\{|f^{\mathbb{F}}-1|\le\alpha\}}\star\mu^{\mathbb{F}^\tau}_\infty}\right)&\le&\alpha\mathbb{E}\big(\mathbb{1}_{\{f_m+Z_-\le\delta/2\}}(Z_-+f_m)^{-1}\star\mu^{\mathbb{F}^\tau}(\infty)\big)\\&\le&\mathbb{E}\big(\mathbb{1}_{\{|f_m|\ge\delta/2\}}\star\mu(\infty)\big)\\&\le&\dfrac{4\alpha}{\delta^2}\mathbb{E}([m,m]_\infty)\\&<&\infty.\end{array}$$

This proves that $\sqrt{f_1^2\mathbb{1}_{\{|f^{\mathbb{F}}-1|\le\alpha\}}\star\mu^{\mathbb{F}^\tau}}\in\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$. Similarly, we calculate

$$\begin{array}{rcl}\mathbb{E}\left(\sqrt{f_1^2\mathbb{1}_{\{|f^{\mathbb{F}}-1|>\alpha\}}\star\mu^{\mathbb{F}^\tau}_\infty}\right)&\le&\mathbb{E}\big(|f_1|\mathbb{1}_{\{|f^{\mathbb{F}}-1|>\alpha\}}\star\mu^{\mathbb{F}^\tau}_\infty\big)\\&\le&\mathbb{E}\left(\dfrac{|f^{\mathbb{F}}-1|}{1+f_m/Z_-}\mathbb{1}_{\{|f^{\mathbb{F}}-1|>\alpha\}}\star\mu^{\mathbb{F}^\tau}_\infty\right)\\&\le&\mathbb{E}\big(|f^{\mathbb{F}}-1|\mathbb{1}_{\{|f^{\mathbb{F}}-1|>\alpha\}}\star\mu_\infty\big)\\&<&\infty.\end{array}$$

Thus, by combining all the remarks obtained above, we conclude that $\sqrt{f_1^2\star\mu^{\mathbb{F}^\tau}}$ is $\mathbb{F}^\tau$-locally integrable. For the functional f_2, we proceed by calculating

$$\mathbb{E}\big(f_2^2\mathbb{1}_{\{f_m+Z_->\delta/2\}}\star\mu^{\mathbb{F}^\tau}_\infty\big)\le(2/\delta)^2\mathbb{E}\big(f_m^2\star\mu_\infty\big)\le(2/\delta)^2\mathbb{E}([m,m]_\infty)<\infty,$$

and

$$\begin{array}{rcl}\mathbb{E}\left(\sqrt{f_2^2\mathbb{1}_{\{f_m+Z_-\le\delta/2\}}\star\mu^{\mathbb{F}^\tau}_\infty}\right)&\le&\mathbb{E}\big(|f_m|\mathbb{1}_{\{|f_m|\ge\delta/2\}}\star\mu(\infty)\big)\\&\le&(2/\delta)\mathbb{E}\big(f_m^2\star\mu(\infty)\big)\\&\le&(2/\delta)\mathbb{E}([m,m]_\infty)\\&<&\infty.\end{array}$$

This proves that $\sqrt{f_2^2\star\mu^{\mathbb{F}^\tau}}$ is $\mathbb{F}^\tau$-locally integrable. Therefore, we conclude that (6.32) is valid, and the proof of (b)⇒(a) is completed.

Step 4. Proof of (b) $\Longleftrightarrow$ (c). For any $\delta>0$, we denote

$$\tau_\delta:=\sup\{t\,:\,Z_{t-}\ge\delta\}.$$

Then, due to $]\!]R,\infty[\![\subset\{Z_-=0\}\subset\{Z_-<\delta\}$, we deduce

$$\sigma_{1/\delta}\le\tau_\delta\le R\quad\text{ and }\quad Z_{\tau_\delta-}\ge\delta>0\qquad\mathbb{P}-a.s.\ \text{ on }\ \{\tau_\delta<\infty\}$$

where σ_n are defined in Theorem 6.1 and R is defined in Proposition 6.16. Furthermore, setting $\Sigma := \bigcap_{n\geq 1}(\sigma_n < R)$, we have

$$\text{on } \ \Sigma \cap \{R < \infty\} \quad Z_{R-} = 0, \quad \text{and} \quad \tau_\delta < R \quad \mathbb{P} - a.s.$$

We introduce the semimartingale $X := S - S^{(0)}$. For any $\delta > 0$, and any H predictable such that $H_\delta := H 1\!\!1_{\{Z_- \geq \delta\}} \in L(X, \mathbb{F})$ and $H_\delta \cdot X \geq -1$, due [DM80, Theorem 23 p.346],

$$(H_\delta \cdot X)_T = (H_\delta \cdot X)_{T\wedge\tau_\delta}, \quad \text{and on } \{\theta \geq \tau_\delta\} \ \ (H_\delta \cdot X)_T = (H_\delta \cdot X)_{T\wedge\theta}.$$

Then, for any $T \in (0, \infty)$, we calculate the following

$$\begin{aligned}
\mathbb{P}((H_\delta \cdot X)_T > c) &= \mathbb{P}((H_\delta \cdot X)_T > c,\ \sigma_n \geq \tau_\delta) + \mathbb{P}((H_\delta \cdot X)_T > c,\ \sigma_n < \tau_\delta) \\
&\leq 2 \sup_{\phi\in L(X^{\sigma_n}):\phi\centerdot X^{\sigma^n}\geq -1} \mathbb{P}((\phi \cdot X)_{\sigma_n\wedge T} > c) + \mathbb{P}(\sigma_n < \tau_\delta \wedge T).
\end{aligned} \tag{6.34}$$

It is easy to prove that $\mathbb{P}(\sigma_n < \tau_\delta \wedge T) \to 0$ as n goes to infinity. This can be seen due to the fact that on Σ, we have, on the one hand, $\tau_\delta \wedge T < R$ (distinguish the two cases whether R is finite or not). On the other hand, the event $(\sigma_n < R)$ increases to Σ with n. Thus, by combining these, we obtain the following

$$\begin{aligned}
\mathbb{P}(\sigma_n < \tau_\delta \wedge T) &= \mathbb{P}((\sigma_n < \tau_\delta \wedge T) \cap \Sigma) + \mathbb{P}((\sigma_n < \tau_\delta \wedge T) \cap \Sigma^c) \\
&\leq \mathbb{P}(\sigma_n < \tau_\delta \wedge T < R) + \mathbb{P}((\sigma_n < R) \cap \Sigma^c) \to 0.
\end{aligned} \tag{6.35}$$

Now suppose that for each $n \geq 1$, the process $(S - S^{(0)})^{\sigma_n}$ satisfies NUPBR($\mathbb{F}$). Then a combination of (6.34) and (6.35) implies that for any $\delta > 0$, the process $1\!\!1_{\{Z_-\geq\delta\}} \centerdot X := 1\!\!1_{\{Z_-\geq\delta\}} \centerdot (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), and the proof of (c)$\Rightarrow$ (b) is completed. The proof of the reverse implcation is obvious due to the fact that

$$[\![0, \sigma_n]\!] \subset \{Z_- \geq 1/n\} \subset \{Z_- \geq \delta\}, \quad \text{for} \quad n \leq \delta^{-1},$$

which implies that $(1\!\!1_{\{Z_-\geq\delta\}} \cdot X)^{\sigma_n} = X^{\sigma_n}$. This ends the proof of (b) $\Longleftrightarrow$(c), and the proof of the theorem is achieved.

6.5.2 Intermediate results

The proofs of Theorems 6.6 and 6.9 rely on the following intermediatory result about $\mathbb{F}$-martingales with a single jump, which is interesting in itself.

Proposition 6.27. *Let M be an $\mathbb{F}$-martingale with $M_0 = 0$ given by $M := \xi 1\!\!1_{[\![T,\infty[\![}$, where T is an $\mathbb{F}$-predictable stopping time, and ξ is an $\mathcal{F}_T$-measurable random variable. Then the following assertions are equivalent.*
(a) M is an $\mathbb{F}$-martingale under $\mathbb{Q}_T$ given by

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} := \frac{1\!\!1_{\{\widetilde{Z}_T>0\}\cap\Gamma(T)}}{\mathbb{P}(\widetilde{Z}_T > 0|\ \mathcal{F}_{T-})} + 1\!\!1_{\Gamma^c(T)}, \quad \Gamma(T) := \{\mathbb{P}(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\}. \tag{6.36}$$

(b) On the set $\{T<\infty\}$, we have

$$\mathbb{E}\left(M_T 1\!\!1_{\{\widetilde{Z}_T=0\}}\Big|\ \mathcal{F}_{T-}\right)=0, \quad \mathbb{P}-a.s. \tag{6.37}$$

(c) M^τ is an $\mathbb{F}^\tau$-martingale under $\mathbb{Q}_T^{\mathbb{F}^\tau}:=\left(U^{\mathbb{F}^\tau}(T)/\mathbb{E}(U^{\mathbb{F}^\tau}(T)\big|\ {\mathcal{F}^\tau}_{T-})\right)\cdot\mathbb{P}$ where

$$U^{\mathbb{F}^\tau}(T):=1\!\!1_{\{T>\tau\}}+1\!\!1_{\{T\le\tau\}}\frac{Z_{T-}}{\widetilde{Z}_T}>0. \tag{6.38}$$

Proof. The proof will be achieved in two steps.
Step 1. Here, we prove the equivalence between assertions (a) and (b). For simplicity we denote by $\mathbb{Q}:=\mathbb{Q}_T$, where $\mathbb{Q}_T$ is defined in (6.36), and remark that on $\{Z_{T-}=0\}$, $\mathbb{Q}$ coincides with $\mathbb{P}$ and (6.37) holds, due to $\{Z_{T-}=0\}\subset\{\widetilde{Z}_T=0\}$. Thus, it is enough to prove the equivalence between (a) and (b) on the set $\{T<\infty,\ Z_{T-}>0\}$. On this set, due to $\mathbb{E}(\xi|\mathcal{F}_{T-})=0$, we derive

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(\xi\big|\mathcal{F}_{T-}) &= \mathbb{E}(\xi 1\!\!1_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-})\Big(\mathbb{P}(\widetilde{Z}_T>0\big|\mathcal{F}_{T-})\Big)^{-1}\\ &= -\mathbb{E}(\xi 1\!\!1_{\{\widetilde{Z}_T=0\}}\big|\mathcal{F}_{T-})\left(\mathbb{P}(\widetilde{Z}_T>0\big|\mathcal{F}_{T-})\right)^{-1}.\end{aligned}$$

Therefore, we conclude that assertion (a) (or equivalently $\mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_{T-})=0$) is equivalent to (6.37). This ends the proof of (a) $\Longleftrightarrow$ (b).
Step 2. To prove (a)$\Longleftrightarrow$(c), we first notice that thanks to $(T\le\tau)\subset(\widetilde{Z}_T>0)\subset(Z_{T-}>0)$, on $\{T\le\tau\}$ we have

$$\begin{aligned}\mathbb{P}\left(\widetilde{Z}_T>0\big|\mathcal{F}_{T-}\right)\mathbb{E}^{\mathbb{Q}_T^{\mathbb{F}^\tau}}\left(\xi|\mathcal{F}^\tau_{T-}\right) &= \mathbb{E}\left(\frac{Z_{T-}}{\widetilde{Z}_T}\xi 1\!\!1_{\{T\le\tau\}}|\mathcal{F}^\tau_{T-}\right)=\mathbb{E}\left(\xi 1\!\!1_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right)\\ &= \mathbb{E}^{\mathbb{Q}}\left(\xi|\mathcal{F}_{T-}\right)\mathbb{P}\left(\widetilde{Z}_T>0\big|\mathcal{F}_{T-}\right).\end{aligned}$$

This equality proves that $M^\tau\in\mathcal{M}(\mathbb{Q}_T^{\mathbb{F}^\tau},\mathbb{F}^\tau)$ if and only if $M\in\mathcal{M}(\mathbb{Q},\mathbb{F})$, and the proof of (a)$\Longleftrightarrow$(c) is completed. This ends the proof of the theorem. ∎

6.5.3 Proof of Theorem 6.6

For the reader convenience, in order to prove Theorem 6.6, we state a more precise version of the theorem, in which we explicitly describe some possible choices for the probability measure $\mathbb{Q}_T$.

Theorem 6.28. *Suppose that the assumptions of Theorem 6.6 are in force. Then, the assertions (a) and (b) of Theorem 6.6 are equivalent to the following assertions.*
(d) S satisfies NUPBR($\mathbb{F},\widetilde{\mathbb{Q}}_T$), where $\widetilde{\mathbb{Q}}_T$ is

$$\widetilde{\mathbb{Q}}_T:=\left(\frac{\widetilde{Z}_T}{Z_{T-}}1\!\!1_{\{Z_{T-}>0\}}+1\!\!1_{\{Z_{T-}=0\}}\right)\cdot\mathbb{P},$$

(e) S satisfies NUPBR($\mathbb{F},\mathbb{Q}_T$), where $\mathbb{Q}_T$ is defined in (6.36).

Proof. The proof of this theorem will be achieved by proving $(d) \Longleftrightarrow (e) \Longleftrightarrow (b)$ and $(b) \Rightarrow (a) \Rightarrow (d)$. These will be carried out in four steps.

Step 1. In this step, we prove $(d) \Longleftrightarrow (e)$. Since S is a single jump process with predictable jump time T, it is easy to see that the fact that S satisfies NUPBR under some probability $\mathbb{Q}^*$ is equivalent to the fact that $1\!\!1_A S$ and $1\!\!1_{A^c} S$ satisfy NUPBR($\mathbb{Q}^*$) for any $\mathcal{F}_{T-}$-measurable event A. Hence, it is enough to prove the equivalence between the assertions (d) and (e) separately on the events $\{Z_{T-} = 0\}$ and $\{Z_{T-} > 0\}$. Since $\{Z_{T-} = 0\} \subset \{\widetilde{Z}_T = 0\}$ and $\mathbb{E}(\widetilde{Z}_T|\mathcal{F}_{T-}) = Z_{T-}$ on $\{T < \infty\}$, by putting $\Gamma_0 := \Big\{\mathbb{P}(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-}) = 0\Big\}$, we derive

$$\mathbb{E}\left(Z_{T-}1\!\!1_{\Gamma_0\cap\{T<\infty\}}\right) = \mathbb{E}\left(\widetilde{Z}_T 1\!\!1_{\Gamma_0\cap\{T<\infty\}}\right) = 0,$$

and

$$\begin{aligned} 0 &= \mathbb{P}\Big(\{Z_{T-} = 0\} \cap \{\widetilde{Z}_T > 0\} \cap \{T < \infty\}\Big) \\ &= \mathbb{E}\left(1\!\!1_{\{Z_{T-}=0\}\cap\{T<\infty\}}\mathbb{P}\left(\widetilde{Z}_T > 0|\mathcal{F}_{T-}\right)\right). \end{aligned}$$

These equalities imply that on $\{T < \infty\}$, $\mathbb{P} - a.s.$, we have

$$\{Z_{T-} = 0\} = \Gamma_0 \subset \{\widetilde{Z}_T = 0\}. \tag{6.39}$$

Thus, on the set $\{T < \infty\}\cap\Gamma_0$, the three probabilities $\mathbb{P}$, $\mathbb{Q}_T$ and $\widetilde{\mathbb{Q}}_T$ coincide, and the equivalence between assertions (d) and (e) is obvious. On the set $\{T < \infty,\ \mathbb{P}(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\}$, one has $\widetilde{\mathbb{Q}}_T \sim \mathbb{Q}_T$, and the equivalence between (d) and (e) is also obvious. This achieves this first step.

Step 2. This step proves (e)$\Longleftrightarrow$(b). Again thanks to (6.39), we deduce that on $\{Z_{T-} = 0\}$, one has $\widetilde{S} \equiv S \equiv 0$ and $\mathbb{Q}_T$ coincides with $\mathbb{P}$ as well. Hence, the equivalence between assertions (e) and (b) is obvious for this case. Thus, it is enough to prove the equivalence between these assertions on $\{T < \infty,\ \mathbb{P}(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\}$.
Assume that (e) holds. Then, there exists an $\mathcal{F}_T$-measurable random variable, Y, such that $Y > 0$ $\mathbb{Q}_T - a.s.$ and on $\{T < \infty\}$, we have

$$\mathbb{E}^{\mathbb{Q}_T}(Y|\mathcal{F}_{T-}) = 1,\quad \mathbb{E}^{\mathbb{Q}_T}(Y|\xi||\mathcal{F}_{T-}) < \infty,\quad \text{and}\quad \mathbb{E}^{\mathbb{Q}_T}(Y\xi 1\!\!1_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}) = 0.$$

Since $Y > 0$ on $\{\widetilde{Z}_T > 0\}$, by putting

$$Y_1 := Y 1\!\!1_{\{\widetilde{Z}_T>0\}} + 1\!\!1_{\{\widetilde{Z}_T=0\}}\quad \text{and}\quad \widetilde{Y}_1 := \frac{Y_1}{\mathbb{E}[Y_1|\mathcal{F}_{T-}]},$$

it is easy to check that $Y_1 > 0$, $\widetilde{Y}_1 > 0$,

$$\mathbb{E}\left[\widetilde{Y}_1|\mathcal{F}_{T-}\right] = 1 \text{ and } \mathbb{E}\left[\widetilde{Y}_1\xi 1\!\!1_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right] = \frac{\mathbb{E}\left[Y\xi 1\!\!1_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{\mathbb{E}[Y_1|\mathcal{F}_{T-}]} = 0.$$

Therefore, $\widetilde{S}$ is a martingale under $\mathbb{P}^* := \widetilde{Y}_1 \cdot \mathbb{P} \sim \mathbb{P}$, and hence $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). This ends the proof of (e)$\Rightarrow$(b).

To prove the reverse sense, we suppose that assertion (b) holds. Then, there exists $Y \in L^0(\mathcal{F}_T)$, $Y > 0$, such that $\mathbb{E}[Y|\xi|\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < \infty$, $\mathbb{E}[Y|\mathcal{F}_{T-}] = 1$ and $\mathbb{E}[Y\xi\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0$ on $\{Z_{T-} > 0\}$. Then, consider

$$Y_2 := \frac{Y\mathbb{1}_{\{\widetilde{Z}_T>0\}}\mathbb{P}(\widetilde{Z}_T > 0|\mathcal{F}_{T-})}{\mathbb{E}[Y\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} + \mathbb{1}_{\{\widetilde{Z}_T=0\}}$$

Then it is easy to verify that $Y_2 > 0 \ \ \mathbb{Q}_T - a.s.$,

$$\mathbb{E}^{\mathbb{Q}_T}\left(Y_2|\mathcal{F}_{T-}\right) = 1, \qquad \text{and} \qquad \mathbb{E}^{\mathbb{Q}_T}\left(Y_2\xi\mathbb{1}_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}\right) = \frac{\mathbb{E}\left[Y\xi\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{\mathbb{E}[Y\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} = 0.$$

This proves assertion (e), and the proof of (e)$\Longleftrightarrow$(b) is achieved.

Step 3. Herein, we prove (a) $\Rightarrow$ (d). Suppose that S^τ satisfies NUPBR($\mathbb{F}^\tau$). Then there exists a positive $\mathcal{F}^\tau_t$-measurable random variable $Y^{\mathbb{F}^\tau}$ such that $\mathbb{E}[\xi Y^{\mathbb{F}^\tau}\mathbb{1}_{\{T\leq\tau\}}|\mathcal{F}^\tau_{T-}] = 0$ on $\{T < \infty\}$. Due to Lemma 6.33(a), we deduce the existence of a positive $\mathcal{F}_T$-measurable variable $Y^{\mathbb{F}}$ such that $Y^{\mathbb{F}^\tau}\mathbb{1}_{\{T\leq\tau\}} = Y^{\mathbb{F}}\mathbb{1}_{\{T\leq\tau\}}$. Then, on $\{T < \infty\}$, we obtain

$$0 \ \ = \ \ \mathbb{E}[\xi Y^{\mathbb{F}}\mathbb{1}_{\{T\leq\tau\}}|\mathcal{F}^\tau_{T-}] = \mathbb{E}[\xi Y^{\mathbb{F}}\widetilde{Z}_T|\mathcal{F}_{T-}]\frac{\mathbb{1}_{\{T\leq\tau\}}}{Z_{T-}}.$$

Therefore, by taking conditional expectation in the above equality, we get

$$0 = \mathbb{E}[\xi Y^{\mathbb{F}}\frac{\widetilde{Z}_T}{Z_{T-}}\mathbb{1}_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}] = \mathbb{E}^{\widetilde{\mathbb{Q}}_T}[\xi Y^{\mathbb{F}}|\mathcal{F}_{T-}]\mathbb{1}_{\{Z_{T-}>0\}} = \mathbb{E}^{\widetilde{\mathbb{Q}}_T}[S_T Y^{\mathbb{F}}|\mathcal{F}_{T-}].$$

This proves that assertion (d) holds and the proof of (a)$\Rightarrow$(d) is achieved.

Step 4. This last step proves (b)$\Rightarrow$(a). Suppose that $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). Then, there exists an $\mathcal{F}_T$-measurable, integrable r.v. Y such that on $\{T < \infty\}$ we have

$$\mathbb{E}[Y|\mathcal{F}_{T-}] = 1, \quad Y > 0, \qquad \mathbb{E}[Y|\xi|\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < \infty, \quad \mathbb{P} - a.s.$$

and

$$\mathbb{E}[Y\xi\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0.$$

Then by considering $\mathbb{Q}^* := Y \cdot \mathbb{P} \sim \mathbb{P}$, we get

$$\mathbb{E}^{\mathbb{Q}^*}\left[\widetilde{S}_T|\mathcal{F}_{T-}\right] = \mathbb{E}^{\mathbb{Q}^*}\left[\xi\mathbb{1}_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right] = 0.$$

Therefore, assertion (a) directly follows from Proposition 6.27 applied to $M = \widetilde{S}$ under $\mathbb{Q}^* \sim \mathbb{P}$ (it is easy to see that (6.37) holds for $(\widetilde{S}, \mathbb{Q}^*)$, i.e. $\mathbb{E}^{\mathbb{Q}^*}(\widetilde{S}_T\mathbb{1}_{\{\widetilde{Z}_T=0\}}|\mathcal{F}_{T-}) = 0$). This ends the fourth step and the proof of the theorem is completed. ∎

6.5.4 Proof of Theorem 6.9

To highlight the precise difficulty in proving Theorem 6.9, we remark that on $\{T<\infty\}$,

$$\frac{U^{\mathbb{F}^\tau}(T)}{\mathbb{E}(U^{\mathbb{F}^\tau}(T)\big|\,\mathcal{F}^\tau_{T-})}=\frac{1+\Delta L_T-\Delta V^{\mathbb{F}^\tau}_T}{1-\Delta V^{\mathbb{F}^\tau}_T}\neq 1+\Delta L_T=\frac{\mathcal{E}(L)_T}{\mathcal{E}(L)_{T-}},$$

where $U^{\mathbb{F}^\tau}(T)$ is defined in (6.38) and $V^{\mathbb{F}^\tau}$ and L are defined in (6.15) and (6.16). This highlights one of the main difficulties that we will face when we will formulate the results for possible many predictable jumps that might not be ordered. It might not be possible to piece up

$$U^{\mathbb{F}^\tau}(T_n)=1-\frac{\Delta m_{T_n}}{\widetilde{Z}_{T_n}}1\!\!1_{\{T_n\leq\tau\}},\quad n\geq 1,$$

to form a positive $\mathbb{F}^\tau$-local martingale density for the process $(1\!\!1_{\cup[\![T_n]\!]}\centerdot S)^\tau$.

Thus, in virtue of the above, the key idea behind the proof of Theorem 6.9 lies in connecting NUPBR with the existence of a positive supermartingale (instead) that is a deflator for the market model under consideration.

Now, we start giving the proof of Theorem 6.9.

Proof of Theorem 6.9. The proof of the theorem will be given in two steps, where we prove (b)⇒(a) and the reverse implication respectively. For the sake of simplifying the overall proof of the theorem, we remark that

$$\{\widetilde{Z}^{\mathbb{Q}}_T=0\}=\{\widetilde{Z}_T=0\},\quad\text{for any }\mathbb{Q}\sim\mathbb{P}\text{ and any }\mathbb{F}\text{-stopping time }T,\tag{6.40}$$

where $\widetilde{Z}^{\mathbb{Q}}_t:=\mathbb{Q}[\tau\geq t|\mathcal{F}_t]$. This equality follows from

$$\mathbb{E}\left(\widetilde{Z}_T1\!\!1_{\{\widetilde{Z}^{\mathbb{Q}}_T=0\}}\right)=\mathbb{E}\left(1\!\!1_{\{\tau\geq T\}}1\!\!1_{\{\widetilde{Z}^{\mathbb{Q}}_T=0\}}\right)=0,$$

(which implies $\{\widetilde{Z}^{\mathbb{Q}}_T=0\}\subset\{\widetilde{Z}_T=0\}$) and the symmetric role of $\mathbb{Q}$ and $\mathbb{P}$.

Step 1. Here, we prove (b)⇒ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that Y^{τ_n} is an $\mathbb{F}$-martingale, where Y is given in Proposition 1.56. Then, setting $\mathbb{Q}_n:=Y_{\tau_n}/Y_0\cdot\mathbb{P}$, and using (6.40) and Proposition 1.59, we deduce that there is no loss of generality in assuming $Y\equiv 1$. Condition (6.37) in Theorem 6.27 holds for $\Delta S_{T_n}1\!\!1_{\{\widetilde{Z}_{T_n}>0\}}$ and $\Delta S_{T_n}1\!\!1_{\{\widetilde{Z}_{T_n}>0\}}1\!\!1_{[\![T_n,\infty[\![}$. Therefore, using the notation $V^{\mathbb{F}^\tau}$ and L defined in (6.15) and (6.16), for each n, $(1+\Delta L_{T_n}-\Delta V^{\mathbb{F}^\tau}_{T_n})\Delta S_{T_n}1\!\!1_{\{T_n\leq\tau\}}1\!\!1_{[\![T_n,\infty[\![}$ is an $\mathbb{F}^\tau$-martingale. Then, by a direct application of Yor's exponential formula, we get that, for any $\theta\in L(S^\tau,\mathbb{F}^\tau)$

$$\mathcal{E}\left(1\!\!1_\Gamma\centerdot L-1\!\!1_\Gamma\centerdot V^{\mathbb{F}^\tau}\right)\mathcal{E}\left(\theta1\!\!1_\Gamma\centerdot S^\tau\right)=\mathcal{E}\left(X\right)$$

where

$$X:=1\!\!1_\Gamma\centerdot L-1\!\!1_\Gamma\centerdot V^{\mathbb{F}^\tau}+\sum_{n\geq1}\theta_{T_n}\left(1+\Delta L_{T_n}-\Delta V^{\mathbb{F}^\tau}_{T_n}\right)\Delta S_{T_n}1\!\!1_{\{T_n\leq\tau\}}1\!\!1_{[\![T_n,\infty[\![}.$$

Consider now the $\mathbb{F}^\tau$-predictable process

$$\begin{aligned}\phi &= \sum_{n\geq 1}\xi_n 1\!\!1_{[\![T_n]\!]\cap[\![0,\tau]\!]} + 1\!\!1_{\Gamma^c\cup]\!]\tau\infty[\![}, \qquad \text{where}\\ \xi_n &:= \frac{2^{-n}\left(1+\mathcal{E}(X)_{T_n-}\right)^{-1}}{\left(1+\mathbb{E}\left[|\Delta L_{T_n}|\,\Big|\mathcal{F}^\tau_{T_n-}\right]+\Delta V^{\mathbb{F}^\tau}_{T_n-}+\mathbb{E}\left[|\theta_{T_n}\frac{Z_{T_n-}}{\widetilde{Z}_{T_n}}1\!\!1_{\{T_n\leq\tau\}}\Delta S_{T_n}|\Big|\mathcal{F}^\tau_{T_n-}\right]\right)}.\end{aligned}$$

Then, it is easy to verify that $0<\phi\leq 1$ and $\mathbb{E}\left(|\phi\centerdot\mathcal{E}(X)|_{var}(\infty)\right)\leq\sum_{n\geq 1}2^{-n}=1$. Hence, $\phi\centerdot\mathcal{E}(X)\in\mathcal{A}(\mathbb{F}^\tau)$. Since $\Delta L_{T_n}1\!\!1_{[\![T_n,\infty[\![}$ and $(1+\Delta L_{T_n}-\Delta V^{\mathbb{F}^\tau}_{T_n})\Delta S_{T_n}1\!\!1_{\{T_n\leq\tau\}}1\!\!1_{[\![T_n,\infty[\![}$ are $\mathbb{F}^\tau$-martingales, we derive

$$(\phi\centerdot\mathcal{E}(X))^{p,\mathbb{F}^\tau}=\sum_n\phi_{T_n}\mathcal{E}_{T_n-}(X)E(\Delta X_{T_n}|\mathcal{G}_{T_n-})1\!\!1_{[\![T_n,\infty[\![}=-\phi\mathcal{E}_-(X)\centerdot V^{\mathbb{F}^\tau}\leq 0.$$

This proves that $\mathcal{E}(X)$ is a positive σ-supermartingale. Thus, thanks to Kallsen [Kal04], we conclude that it is a supermartingale and $\left(1\!\!1_{\{Z_-\geq\delta\}}\centerdot S\right)^\tau$ admits a $\mathbb{F}^\tau$-deflator. Then, thanks to Lemma 1.62, we deduce that $\left(1\!\!1_{\{Z_-\geq\delta\}}\centerdot S\right)^\tau$ satisfies NUPBR($\mathbb{F}^\tau$). Remark that, due to the $\mathbb{F}^\tau$-local boundedness of $(Z_-)^{-1}1\!\!1_{[\![0,\tau]\!]}$, there exists a family of $\mathbb{F}^\tau$-stopping times $\tau_\delta,\ \delta>0$ such that τ_δ almost surely converges to infinity when δ goes zero and

$$[\![0,\tau\wedge\tau_\delta]\!]\subset\{Z_-\geq\delta\}.$$

This implies that $S^{\tau\wedge\tau_\delta}$ satisfies NUPBR($\mathbb{F}^\tau$), and the assertion (a) follows from Proposition 1.59 (by taking $\mathbb{Q}_n=\mathbb{P}$ for all $n\geq 1$). This ends the proof of (b)$\Rightarrow$(a).

Step 2. In this step, we focus on (a)$\Rightarrow$(b). Suppose that S^τ satisfies NUPBR($\mathbb{F}^\tau$). Then, there exists a σ-martingale density under $\mathbb{F}^\tau$, for $1\!\!1_{\{Z_-\geq\delta\}}\centerdot S^\tau$, $(\delta>0)$, that we denote by $D^{\mathbb{F}^\tau}$. Then, from a direct application of Theorem 1.26 and Theorem 6.31, we deduce the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{F}^\tau)$-measurable functional, $f^{\mathbb{F}^\tau}$, such that $D^{\mathbb{F}^\tau}:=\mathcal{E}(N^{\mathbb{F}^\tau})>0$, with

$$N^{\mathbb{F}^\tau}:=W^{\mathbb{F}^\tau}\star(\mu^{\mathbb{F}^\tau}-\nu^{\mathbb{F}^\tau}),\qquad W^{\mathbb{F}^\tau}:=f^{\mathbb{F}^\tau}-1+\frac{\widehat{f}^{\mathbb{F}^\tau}-a^{\mathbb{F}^\tau}}{1-a^{\mathbb{F}^\tau}}1\!\!1_{\{a^{\mathbb{F}^\tau}<1\}},$$

where $\nu^{\mathbb{F}^\tau}$ was defined in (6.24),and

$$xf^{\mathbb{F}^\tau}1\!\!1_{\{Z_-\geq\delta\}}\star\nu^{\mathbb{F}^\tau}=xf^{\mathbb{F}^\tau}\left(1+\frac{f_m}{Z_-}\right)1\!\!1_{]\!]0,\tau]\!]}1\!\!1_{\{Z_-\geq\delta\}}\star\nu\equiv 0. \tag{6.41}$$

Thanks to Lemma 6.33, we conclude to the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, f, such that $f^{\mathbb{F}^\tau}1\!\!1_{]\!]0,\tau]\!]}=f1\!\!1_{]\!]0,\tau]\!]}$. Thus (6.41) becomes

$$U^{\mathbb{F}^\tau}:=xf\left(1+\frac{f_m}{Z_-}\right)1\!\!1_{]\!]0,\tau]\!]}1\!\!1_{\{Z_->0\}}\star\nu\equiv 0. \tag{6.42}$$

Introduce the following notations

$$\begin{aligned}\mu_0 &:= 1\!\!1_{\{\widetilde{Z}>0\ \&\ Z_-\geq\delta\}}\cdot\mu,\quad \nu_0:=h_0 1\!\!1_{\{Z_-\geq\delta\}}\cdot\nu,\ h_0:=M^{\mathbb{P}}_\mu\left(1\!\!1_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right),\\ g &:= \frac{f(1+\frac{f_m}{Z_-})}{h_0}1\!\!1_{\{h_0>0\}}+1\!\!1_{\{h_0=0\}},\quad a_0(t):=\nu_0(\{t\},\mathbb{R}^d),\end{aligned} \tag{6.43}$$

and assume that

$$\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F}). \tag{6.44}$$

Then, thanks to Lemma 6.32, we deduce that $W := (g-1)/(1-a^0+\widehat{g}) \in \mathcal{G}^1_{loc}(\mu_0, \mathbb{F})$, and the local martingales

$$N^0 := \frac{g-1}{1-a^0+\widehat{g}} \star (\mu_0 - \nu_0), \quad Y^0 := \mathcal{E}(N^0),$$

are well defined satisfying $1 + \Delta N^0 > 0$, $[N^0, S] \in \mathcal{A}(\mathbb{F})$, and on $\{Z_- > 0\}$ we have

$$\begin{aligned}
\frac{{}^{p,\mathbb{F}}\left(Y^0 \Delta S 1\!\!1_{\{\widetilde{Z}>0\}}\right)}{Y^0_-} &= {}^{p,\mathbb{F}}\left((1+\Delta N^0)\Delta S 1\!\!1_{\{\widetilde{Z}>0\}}\right) = {}^{p,\mathbb{F}}\left(\frac{g}{1-a^0+\widehat{g}} \Delta S 1\!\!1_{\{\widetilde{Z}>0\}}\right) \\
&= \Delta \frac{g x h_0}{1-a^0+\widehat{g}} \star \nu = \Delta \frac{x f (1 + f_m/Z_-)}{1-a^0+\widehat{g}} \star \nu = Z_-^{-1} \frac{{}^{p,\mathbb{F}}(\Delta U)}{1-a^0+\widehat{g}} \equiv 0
\end{aligned}$$

where U is defined in (6.42). This proves that assertion (b) holds under the assumption (6.44).

The remaining part of the proof will show that this assumption always holds. To this end, we start by noticing that on the set $\{h_0 > 0\}$,

$$\begin{aligned}
g - 1 &= \frac{f(1+\frac{f_m}{Z_-})}{h_0} - 1 = \frac{(f-1)(1+\frac{f_m}{Z_-})}{h_0} + \frac{f_m}{Z_- h_0} + \frac{M^{\mathbb{P}}_\mu\left(1\!\!1_{\{\widetilde{Z}=0\}} | \widetilde{\mathcal{P}}\right)}{h_0} \\
&:= g_1 + g_2 + g_3.
\end{aligned}$$

Since $\left((f-1)^2 1\!\!1_{]\!]0,\tau]\!]} \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau)$, then due to Proposition 6.34(e)

$$\sqrt{(f-1)^2 1\!\!1_{\{Z_- \geq \delta\}} \star (\widetilde{Z} \cdot \mu)} \in \mathcal{A}^+_{loc}(\mathbb{F}), \quad \text{for any} \quad \delta > 0.$$

Then, by a direct application of Proposition 6.34(a), for any $\delta > 0$, we have

$$(f-1)^2 1\!\!1_{\{|f-1|\leq\alpha \ \& \ Z_-\geq\delta\}} \star (\widetilde{Z} \cdot \mu), \quad |f-1| 1\!\!1_{\{|f-1|>\alpha \ \& \ Z_-\geq\delta\}} \star (\widetilde{Z} \cdot \mu) \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

By stopping, without loss of generality, we assume these two processes and $[m, m]$ belong to $\mathcal{A}^+(\mathbb{F})$. Remark that $Z_- + f_m = M^{\mathbb{P}}_\mu\left(\widetilde{Z} | \widetilde{\mathcal{P}}\right) \leq M^{\mathbb{P}}_\mu\left(1\!\!1_{\{\widetilde{Z}>0\}} | \widetilde{\mathcal{P}}\right) = h_0$ that follows from $\widetilde{Z} \leq 1\!\!1_{\{\widetilde{Z}>0\}}$. Therefore, we derive

$$\begin{aligned}
\mathbb{E}\left[g_1^2 1\!\!1_{\{|f-1|\leq\alpha\}} \star \mu_0(\infty)\right] &= \mathbb{E}\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2} 1\!\!1_{\{|f-1|\leq\alpha\}} \star \mu_0(\infty)\right] \\
&\leq \delta^{-2} \mathbb{E}\left[(f-1)^2 (Z_- + f_m) 1\!\!1_{\{|f-1|\leq\alpha \ \& \ Z_-\geq\delta\}} \star \nu_\infty\right] \\
&= \delta^{-2} \mathbb{E}\left[(f-1)^2 1\!\!1_{\{|f-1|\leq\alpha\}} \star (\widetilde{Z} 1\!\!1_{\{Z_-\geq\delta\}} \cdot \mu)_\infty\right] \\
&< \infty,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[g_1 1\!\!1_{\{|f-1|>\alpha\}} \star \mu_0(\infty)\right] &= \mathbb{E}\left[\frac{|f-1|(1+\frac{f_m}{Z_-})}{h_0} 1\!\!1_{\{|f-1|>\alpha\}} \star \mu_0(\infty)\right] \\
&= \mathbb{E}\left[|f-1|(1+\frac{f_m}{Z_-}) 1\!\!1_{\{|f-1|>\alpha\}} 1\!\!1_{\{Z_-\geq\delta\}} \star \nu_0(\infty)\right] \\
&\leq \delta^{-1}\mathbb{E}\left[|f-1| 1\!\!1_{\{|f-1|>\alpha\}} \star (\widetilde{Z} 1\!\!1_{\{Z_-\geq\delta\}} \cdot \mu)_\infty\right] \\
&< \infty.
\end{aligned}$$

Here μ_0 and ν_0 are defined in (6.43). Therefore, again by Proposition 6.34–(a), we conclude that $\sqrt{g_1^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$.

Notice that $g_2 + g_3 = \frac{M^{\mathbb{P}}_\mu\left(\Delta m 1\!\!1_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_- h_0}$. Thanks to Lemma 6.29, we derive

$$\begin{aligned}
\mathbb{E}\left[(g_2+g_3)^2 \star \mu_0(\infty)\right] &= \mathbb{E}\left[\frac{M^{\mathbb{P}}_\mu\left(\Delta m 1\!\!1_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)^2}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\
&\leq \mathbb{E}\left[\frac{M^{\mathbb{P}}_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M^{\mathbb{P}}_\mu\left(1\!\!1_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\
&= \mathbb{E}\left[\frac{M^{\mathbb{P}}_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{Z_-^2} 1\!\!1_{\{Z_-\geq\delta\}} \star \mu(\infty)\right] \\
&\leq \delta^{-2}\mathbb{E}\left[[m,m]_\infty\right] < \infty.
\end{aligned}$$

Hence, we conclude that $\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$. This ends the proof of (6.44), and the proof of the theorem is completed. ∎

6.6 Appendix

6.6.1 Representation of local martingales

The following is a simple but useful result on the conditional expectation with respect to $M^{\mathbb{P}}_\mu$.

Lemma 6.29. *Consider a filtration $\mathbb{H}$ satisfying the usual conditions. Let f and g two nonnegative $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable functionals. Then we have*

$$M^{\mathbb{P}}_\mu\left(fg\middle|\, \widetilde{\mathcal{P}}\right)^2 \leq M^{\mathbb{P}}_\mu\left(f^2\middle|\, \widetilde{\mathcal{P}}\right) M^{\mathbb{P}}_\mu\left(g^2\middle|\, \widetilde{\mathcal{P}}\right), \qquad M^{\mathbb{P}}_\mu\text{–}a.e.$$

Proof. The proof is the same as the one of the regular Cauchy-Schwarz formula, by putting $\bar{f} := f/\left(M^{\mathbb{P}}_\mu\left(f^2\middle|\, \widetilde{\mathcal{P}}\right)\right)^{1/2}$ and $\bar{g} := g/\left(M^{\mathbb{P}}_\mu\left(g^2\middle|\, \widetilde{\mathcal{P}}\right)\right)^{1/2}$ and using the simple inequality $xy \leq (x^2+y^2)/2$. This ends the proof of the lemma. ∎

The following lemma is borrowed from [Jac79, Theorem 3.75] (see also [CS13, Proposition 2.2]).

Lemma 6.30. *Let $\mathcal{E}(N)$ be a positive local martingale and $\left(\beta, f, g, N^{\perp}\right)$ be the Jacod's parameters of N. Then $\mathcal{E}(N) > 0$ (or equivalently $1 + \Delta N > 0$) implies that*

$$f > 0, \quad M_{\mu}^{\mathbb{P}} - a.e.$$

Theorem 6.31. *Let S be a semimartingale with predictable characteristic triplet $(b, c, \nu = \bar{A} \otimes F)$, N be a local martingale such that $\mathcal{E}(N) > 0$, and (β, f, g, N') be its Jacod's parameters. Then the following assertions hold.*
1) $\mathcal{E}(N)$ is a σ-martingale density of S if and only if the following two properties hold:

$$\begin{aligned} \int |x - h(x) + xf(x)| F(dx) &< \infty, \quad \mathbb{P} \otimes \bar{A} - a.e. \\ b + c\beta + \int \Big(x - h(x) + xf(x)\Big) F(dx) &= 0, \quad \mathbb{P} \otimes \bar{A} - a.e. \end{aligned}$$

2) In particular, we have

$$\int x(1 + f_t(x)) \nu(\{t\}, dx) = \int x(1 + f_t(x)) F_t(dx) \Delta \bar{A}_t = 0, \qquad \mathbb{P} - a.e.$$

Proof. The proof can be found in Choulli et al. [CSL07, Lemma 2.4], and also Choulli and Schweizer [CS13]. ∎

Lemma 6.32. ***([CS13]).*** *Consider a filtration $\mathbb{H}$ satisfying the usual conditions. Let f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional such that $f > 0$ and*

$$\Big[(f-1)^2 \star \mu\Big]^{1/2} \in \mathcal{A}_{loc}^{+}(\mathbb{H}).$$

Then, the $\mathbb{H}$-predictable process $\left(1 - a^{\mathbb{H}} + \widehat{f}^{\mathbb{H}}\right)^{-1}$ is locally bounded, and hence

$$W_t(x) := \frac{f_t(x) - 1}{1 - a_t^{\mathbb{H}} + \widehat{f}_t^{\mathbb{H}}} \in \mathcal{G}_{loc}^{1}(\mu, \mathbb{H}).$$

Here, $a_t^{\mathbb{H}} := \nu^{\mathbb{H}}(\{t\}, \mathbb{R}^d)$, $\widehat{f}_t^{\mathbb{H}} := \int f_t(x) \nu^{\mathbb{H}}(\{t\}, dx)$ and $\nu^{\mathbb{H}}$ is the $\mathbb{H}$-predictable random measure compensator of μ under $\mathbb{H}$.

6.6.2 Proof of $K \in {}^{o}L_{loc}^{1}(\widehat{m}, \mathbb{F}^{\tau})$

We start by calculating on $]\!]0, \tau]\!]$, making use of Lemma 6.17. We denote $\kappa := Z_{-}^2 + \Delta \langle m \rangle^{\mathbb{F}}$.

$$\begin{aligned}
K\Delta\widehat{m} - {}^{p,\mathbb{F}^\tau}(K\Delta\widehat{m}) &= \frac{\mathbb{1}_{]\!]0,\tau]\!]} Z_-^2 \Delta\widehat{m}}{\kappa\widetilde{Z}} - {}^{p,\mathbb{F}^\tau}\left(\frac{\mathbb{1}_{]\!]0,\tau]\!]}\, Z_-^2}{\kappa\widetilde{Z}}\,\Delta\widehat{m}\right) \\
&= \frac{(Z_-^2\Delta m - Z_-\Delta\langle m\rangle^{\mathbb{F}})}{\kappa\,\widetilde{Z}} + \frac{{}^{p,\mathbb{F}}(\mathbb{1}_{\{\widetilde{Z}>0\}}\,\Delta\langle m\rangle^{\mathbb{F}})}{\kappa} - \frac{{}^{p,\mathbb{F}}(\Delta m\mathbb{1}_{\{\widetilde{Z}>0\}})\,Z_-}{\kappa} \\
&= \frac{\Delta m}{\widetilde{Z}}\mathbb{1}_{]\!]0,\tau]\!]} - {}^{p,\mathbb{F}}\left(\mathbb{1}_{\{\widetilde{Z}=0\}}\right)\mathbb{1}_{]\!]0,\tau]\!]} =: \Delta V - \Delta V^{\mathbb{F}^\tau}.
\end{aligned} \tag{6.45}$$

Here, $V^{\mathbb{F}^\tau}$, defined in (6.15), is nondecreasing, càdlàg and $\mathbb{F}^\tau$-locally bounded (see Proposition 6.19). Hence, we immediately deduce that $\sum(\Delta V^{\mathbb{F}^\tau})^2 = \Delta V^{\mathbb{F}^\tau} \centerdot V^{\mathbb{F}^\tau}$ is locally bounded, and in the rest of this part we focus on proving $\sqrt{\sum(\Delta V)^2} \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau)$. To this end, we consider $\delta \in (0,1)$, and define $C := \{\Delta m < -\delta Z_-\}$ and C^c its complement in $\Omega \otimes [0,\infty[$. Then we obtain

$$\begin{aligned}
\sqrt{\sum(\Delta V)^2} &\leq \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2}\mathbb{1}_C\mathbb{1}_{]\!]0,\tau]\!]}\right)^{1/2} + \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2}\mathbb{1}_{C^c}\mathbb{1}_{]\!]0,\tau]\!]}\right)^{1/2} \\
&\leq \sum \frac{|\Delta m|}{\widetilde{Z}}\mathbb{1}_C\mathbb{1}_{]\!]0,\tau]\!]} + \frac{1}{1-\delta}\left(\mathbb{1}_{]\!]0,\tau]\!]}\frac{1}{Z_-^2}\centerdot [m]\right)^{1/2} =: V_1 + V_2.
\end{aligned}$$

The last inequality above is due to $\sqrt{\sum(\Delta X)^2} \leq \sum|\Delta X|$ and $\widetilde{Z} \geq Z_-(1-\delta)$ on C^c. Using the fact that $(Z_-)^{-1}\mathbb{1}_{]\!]0,\tau]\!]}$ is $\mathbb{F}^\tau$-locally bounded and that m is an $\mathbb{F}$-locally bounded martingale, it follows that V_2 is $\mathbb{F}^\tau$-locally bounded. Hence, we focus on proving the $\mathbb{F}^\tau$-local integrability of V_1.

Consider a sequence of $\mathbb{F}^\tau$-stopping times $(\vartheta_n)_n$ that increases to ∞ and

$$\left((Z_-)^{-1}\mathbb{1}_{]\!]0,\tau]\!]}\right)^{\vartheta_n} \leq n.$$

For the process $V_3 := \sum\frac{(\Delta m)^2}{1+|\Delta m|}$ consider an $\mathbb{F}$-localizing sequence of stopping times $(\tau_n)_n$. Then, it is easy to prove

$$U_n := \sum|\Delta m|\mathbb{1}_{\{\Delta m < -\delta/n\}} \leq \frac{n+\delta}{\delta}V_3,$$

and to conclude that $(U_n)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$. Therefore, due to

$$\begin{aligned}
C \cap]\!]0,\tau]\!] \cap [\![0,\vartheta_n]\!] &= \{\Delta m < -\delta Z_-\} \cap]\!]0,\vartheta_n]\!] \cap]\!]0,\tau]\!] \\
&\subset]\!]0,\tau]\!] \cap]\!]0,\vartheta_n]\!] \cap \{\Delta m < -\tfrac{\delta}{n}\},
\end{aligned}$$

we derive

$$(V_1)^{\vartheta_n\wedge\tau_n} \leq \left(\widetilde{Z}\right)^{-1}\mathbb{1}_{]\!]0,\tau]\!]} \centerdot (U_n)^{\tau_n}.$$

Since $(U_n)^{\tau_n}$ is $\mathbb{F}$-adapted, nondecreasing and integrable, thanks to Lemma 6.18, we deduce that the process $V_1^{\vartheta_n\wedge\tau_n}$ is nondecreasing, $\mathbb{F}^\tau$-adapted and integrable. Since $\vartheta_n\wedge\tau_n$ increases to ∞, we conclude that the process V_1 is $\mathbb{F}^\tau$-locally integrable. This completes the proof of $K \in {}^oL^1_{loc}(\widehat{m},\mathbb{F}^\tau)$, and the process L (given via (6.16) and Definition 1.24) is an $\mathbb{F}^\tau$-local martingale.

6.6.3 $\mathbb{F}^\tau$-localization versus $\mathbb{F}$-localization

We now present results which are important for the proofs of Subsection 6.5.1.

Lemma 6.33. *Let $H^{\mathbb{F}^\tau}$ be a $\widetilde{\mathcal{P}}(\mathbb{F}^\tau)$-measurable functional. The following assertions hold.*
(a) There exist an $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$ and a $\mathcal{B}(\mathbb{R}_+)\otimes\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functionals $K^{\mathbb{F}}:\mathbb{R}_+\times\mathbb{R}_+\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ such that

$$H^{\mathbb{F}^\tau}(\omega,t,x)=H^{\mathbb{F}}(\omega,t,x)1\!\!1_{]\!]0,\tau]\!]}+K^{\mathbb{F}}(\tau(\omega),t,\omega,x)1\!\!1_{]\!]\tau,\infty]\!]}. \tag{6.46}$$

(b) If furthermore $H^{\mathbb{F}^\tau}>0$ (respectively $H^{\mathbb{F}^\tau}\leq 1$), then we can choose $H^{\mathbb{F}}>0$ (respectively $H^{\mathbb{F}}\leq 1$) such that

$$H^{\mathbb{F}^\tau}(\omega,t,x)1\!\!1_{]\!]0,\tau]\!]}=H^{\mathbb{F}}(\omega,t,x)1\!\!1_{]\!]0,\tau]\!]}.$$

Proof. The proof of assertion (a) exactly mimics the approach of Jeulin[Jeu80], and will be omitted. To prove the positivity of $H^{\mathbb{F}}$ when $H^{\mathbb{F}^\tau}>0$ holds, we consider

$$\overline{H}^{\mathbb{F}}:=(H^{\mathbb{F}})^+ + 1\!\!1_{\{H^{\mathbb{F}}=0\}}>0,$$

and we remark that due to (6.46), we have $]\!]0,\tau]\!]\subset\{H^{\mathbb{F}^\tau}=H^{\mathbb{F}}\}\subset\{H^{\mathbb{F}}>0\}$. Thus, we get $H^{\mathbb{F}^\tau}1\!\!1_{]\!]0,\tau]\!]}=\overline{H}^{\mathbb{F}}1\!\!1_{]\!]0,\tau]\!]}$. Similarly, we consider $H^{\mathbb{F}}\wedge 1$, and we deduce that if $H^{\mathbb{F}^\tau}$ is upper-bounded by one, the process $H^{\mathbb{F}}$ can also be chosen not to exceed one. This ends the proof of the proposition. ∎

Proposition 6.34. *For any $\alpha>0$, the following assertions hold.*
(a) Let h be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional. Then, $\sqrt{(h-1)^2\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{H})$ if and only if

$$(h-1)^2 1\!\!1_{\{|h-1|\leq\alpha\}}\star\mu \ \text{ and } \ |h-1|1\!\!1_{\{|h-1|>\alpha\}}\star\mu \ \text{ belong to } \ \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) Let $(\sigma_n^{\mathbb{F}^\tau})_n$ be a sequence of $\mathbb{F}^\tau$-stopping times that increases to infinity. Then, there exists a nondecreasing sequence of $\mathbb{F}$-stopping times, $(\sigma_n^{\mathbb{F}})_{n\geq1}$, satisfying the following properties

$$\sigma_n^{\mathbb{F}^\tau}\wedge\tau=\sigma_n^{\mathbb{F}}\wedge\tau,\quad \sigma_\infty:=\sup_n\sigma_n^{\mathbb{F}}\geq R\ \ \mathbb{P}-a.s., \tag{6.47}$$

$$\text{and}\qquad Z_{\sigma_\infty -}=0\quad \mathbb{P}-a.s.\quad \text{on}\quad \Sigma\cap(\sigma_\infty<\infty), \tag{6.48}$$

where $\Sigma:=\bigcap_{n\geq1}(\sigma_n^{\mathbb{F}}<\sigma_\infty)$.
(c) Let V be an $\mathbb{F}$-predictable and non-decreasing process. Then, $V^\tau\in\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$ if and only if $1\!\!1_{\{Z_-\geq\delta\}}\centerdot V\in\mathcal{A}^+_{loc}(\mathbb{F})$ for any $\delta>0$.
(d) Let h be a nonnegative and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional. Then, $h1\!\!1_{]\!]0,\tau]\!]}\star\mu\in\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$ if and only if for all $\delta>0$, $h1\!\!1_{\{Z_-\geq\delta\}}\star\mu^1\in\mathcal{A}^+_{loc}(\mathbb{F})$, where $\mu^1:=\widetilde{Z}\centerdot\mu$.
(e) Let f be positive and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable, and $\mu^1:=\widetilde{Z}\centerdot\mu$. Then

$$\sqrt{(f-1)^2 1\!\!1_{]\!]0,\tau]\!]}\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{F}^\tau)$$

if and only if $\sqrt{(f-1)^2 1\!\!1_{\{Z_-\geq\delta\}}\star\mu^1}\in\mathcal{A}^+_{loc}(\mathbb{F})$, for all $\delta>0$.

Proof. (a) Put $W := (h-1)^2 \star \mu = W_1 + W_2$, where $W_1 := (h-1)^2 1\!\!1_{\{|h-1|\le\alpha\}} \star \mu$, $W_2 := (h-1)^2 1\!\!1_{\{|h-1|>\alpha\}} \star \mu$. Let us introduce $W_2' := |h-1| 1\!\!1_{\{|h-1|>\alpha\}} \star \mu$. Note that

$$\sqrt{W} = \sqrt{W_1 + W_2} \le \sqrt{W_1} + \sqrt{W_2} \le \sqrt{W_1} + W_2'.$$

Therefore $\sqrt{W_1} \in \mathcal{A}^+_{loc}$ and $W_2' \in \mathcal{A}^+_{loc}$ imply that $\sqrt{W}$ is locally integrable.
Conversely, if $\sqrt{W} \in \mathcal{A}^+_{loc}$, then $\sqrt{W_1}$ and $\sqrt{W_2}$ are both locally integrable. Since W_1 is locally bounded and has finite variation, W_1 is locally integrable. In the following, we focus on the proof of the local integrability of W_2'. Denote

$$\tau_n := \inf\{t \ge 0 : \ V_t > n\}, \quad \text{where} \ \ V := W_2.$$

It is easy to see that τ_n increases to infinity and $V_- \le n$ on the set $]\!]0, \tau_n]\!]$. On the set $\{\Delta V > 0\}$, we have $\Delta V \ge \alpha^2$.
By using the elementary inequality $\sqrt{1+\frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}} \le \sqrt{1+x} - \sqrt{x} \le 1$, when $0 \le x \le \frac{n}{\alpha^2}$, we have

$$\sqrt{V_- + \Delta V} - \sqrt{V_-} \ge \beta_n \sqrt{\Delta V} \ \ \text{on }]\!]0, \tau_n]\!], \text{ where } \ \beta_n := \sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}},$$

and

$$\left(W_2'\right)^{\tau_n} = \left(\sum \sqrt{\Delta V}\right)^{\tau_n} \le \frac{1}{\beta_n}\left(\sum \Delta\sqrt{V}\right)^{\tau_n} = \frac{1}{\beta_n}\left(\sqrt{W_2}\right)^{\tau_n} \text{ and } \frac{1}{\beta_n}\left(\sqrt{W_2}\right)^{\tau_n} \in \mathcal{A}^+_{loc}(\mathbb{H}).$$

Therefore $W_2' \in (\mathcal{A}^+_{loc}(\mathbb{H}))_{loc} = \mathcal{A}^+_{loc}(\mathbb{H})$.

(b) Thanks to Jeulin [Jeu80], there exists a sequence of $\mathbb{F}$-stopping times $(\sigma_n^{\mathbb{F}})_n$ such that

$$\sigma_n^{\mathbb{F}^\tau} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau. \tag{6.49}$$

By putting $\sigma_n := \sup_{k\le n} \sigma_k^{\mathbb{F}}$, we shall prove that

$$\sigma_n^{\mathbb{F}^\tau} \wedge \tau = \sigma_n \wedge \tau, \tag{6.50}$$

or equivalently $\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\}$ is negligible. Due to (6.49) and to the fact that $\sigma_n^{\mathbb{F}^\tau}$ is nondecreasing, we derive

$$\{\sigma_n^{\mathbb{F}} < \tau\} = \{\sigma_n^{\mathbb{F}^\tau} < \tau\} \subset \bigcap_{i=1}^{n} \{\sigma_i^{\mathbb{F}^\tau} = \sigma_i^{\mathbb{F}}\} \subset \{\sigma_n^{\mathbb{F}} = \sigma_n\}.$$

This implies that

$$\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\} = \{\sigma_n^{\mathbb{F}} < \tau \ \& \ \sigma_n^{\mathbb{F}} < \sigma_n\} = \emptyset,$$

and the proof of (6.50) is completed. Without loss of generality we assume that the sequence $\sigma_n^{\mathbb{F}}$ is nondecreasing. By taking limit in (6.49), we obtain $\tau = \sigma_\infty \wedge \tau, \mathbb{P}$-a.s., which is equivalent to $\sigma_\infty \ge \tau, \mathbb{P}$-a.s. Since R is the smallest $\mathbb{F}$-stopping time greater or equal than τ almost surely, we obtain, $\sigma_\infty \ge R \ge \tau$ $\mathbb{P}$-a.s. This achieves the proof of (6.47).
On the set Σ, it is easy to show that

$$1\!\!1_{[\![0,\sigma_n^{\mathbb{F}}]\!]} \to 1\!\!1_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \quad \text{when } n \text{ goes to } \infty.$$

Then, thanks again to (6.49) (by taking $\mathbb{F}$-predictable projection and let n go to infinity afterwards), we obtain

$$Z_- = Z_- 1\!\!1_{[\![0,\sigma_\infty^{\mathbb{F}}[\![} \qquad \text{on } \Sigma.$$

Hence, (6.48) immediately follows, and the proof of assertion (b) is completed.

(d) Suppose that $h1\!\!1_{[\![0,\tau]\!]} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau)$. Then, there exists a sequence of $\mathbb{F}^\tau$-stopping times $(\sigma_n^{\mathbb{F}^\tau})$ increasing to infinity such that $h1\!\!1_{[\![0,\tau]\!]} \star \mu^{\sigma_n^{\mathbb{F}^\tau}}$ is integrable. Consider (σ_n) a sequence of $\mathbb{F}$-stopping times satisfying (6.47)–(6.48) (its existence is guaranteed by assertion (b)). Therefore, for any fixed $\delta > 0$

$$W^n := M^{\mathbb{P}}_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) 1\!\!1_{\{Z_- \geq \delta\}} h \star \nu^{\sigma_n} \in \mathcal{A}^+(\mathbb{F}),$$

or equivalently, this process is càdlàg predictable with finite values. Thus, it is obvious that the proof of assertion (d) will immediately follow if we prove that the $\mathbb{F}$-predictable and nondecreasing process

$$W := M^{\mathbb{P}}_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) 1\!\!1_{\{Z_- \geq \delta\}} h \star \nu \quad \text{is càdlàg with finite values.} \tag{6.51}$$

To prove this last fact, we consider the random time τ^δ defined by

$$\tau^\delta := \sup\{t \geq 0 : Z_{t-} \geq \delta\}.$$

Then, it is clear that $1\!\!1_{]\!]\tau^\delta,\infty[\![} \centerdot W \equiv 0$ and

$$\tau^\delta \leq R \leq \sigma_\infty \quad \text{and} \quad Z_{\tau^\delta -} \geq \delta \quad \mathbb{P}\text{–}a.s. \qquad \text{on } \{\tau^\delta < \infty\}.$$

The proof of (6.51) will be achieved by considering three sets, namely $\{\sigma_\infty = \infty\}$, $\Sigma \cap \{\sigma_\infty < \infty\}$, and $\Sigma^c \cap \{\sigma_\infty < \infty\}$. It is obvious that (6.51) holds on $\{\sigma_\infty = \infty\}$. Due to (6.48), we deduce that $\tau^\delta < \sigma_\infty, \mathbb{P}$-a.s. on $\Sigma \cap \{\sigma_\infty < \infty\}$. Since W is supported on $[\![0,\tau^\delta]\!]$, then (6.51) immediately follows on the set $\Sigma \cap \{\sigma_\infty < \infty\}$. Finally, on the set

$$\Sigma^c \cap \{\sigma_\infty < \infty\} = \left(\bigcup_{n\geq 1}\{\sigma_n = \sigma_\infty\}\right) \cap \{\sigma_\infty < \infty\},$$

the sequence σ_n stationary increases to σ_∞, and thus (6.51) holds on this set. This completes the proof of (6.51), and hence $h1\!\!1_{\{Z_- \geq \delta\}} \star (\widetilde{Z} \centerdot \mu)$ is locally integrable, for any $\delta > 0$.

Conversely, if $h1\!\!1_{\{Z_- \geq \delta\}} \widetilde{Z} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$, there exists a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq 1}$ that increases to infinity and $\left(h1\!\!1_{\{Z_- \geq \delta\}} \widetilde{Z} \star \mu\right)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$. Then, we have

$$\mathbb{E}\left[h1\!\!1_{\{Z_- \geq \delta\}} 1\!\!1_{[\![0,\tau]\!]} \star \mu(\tau_n)\right] = \mathbb{E}\left[h1\!\!1_{\{Z_- \geq \delta\}} \widetilde{Z} \star \mu(\tau_n)\right] < \infty.$$

This proves that $h1\!\!1_{\{Z_- \geq \delta\}} 1\!\!1_{[\![0,\tau]\!]} \star \mu$ is $\mathbb{F}^\tau$-locally integrable, for any $\delta > 0$. Since $(Z_-)^{-1} 1\!\!1_{[\![0,\tau]\!]}$ is $\mathbb{F}^\tau$-locally bounded, then there exists a family of $\mathbb{F}^\tau$-stopping times $(\tau_\delta)_{\delta>0}$ that increases to infinity when δ decreases to zero, and

$$[\![0,\tau \wedge \tau_\delta]\!] \subset \{Z_- \geq \delta\}.$$

This implies that the process $\left(h\mathbb{1}_{[\![0,\tau]\!]} \star \mu\right)^{\tau_\delta}$ is $\mathbb{F}^\tau$-locally integrable, and hence the assertion (d) immediately follows.

(e) The proof of assertion (e) follows from combining assertions (a) and (d). This ends the proof of the proposition. ∎

Chapter 7

Optional semimartingale deomposition and NUBPR condition in enlarged filtration

7.1 Introduction

This chapter is based on a joint paper with Tahir Choulli and Monique Jeanblanc [ACJ14a].

We study two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$. We consider the cases where the enlarged filtration $\mathbb{G}$ is constructed in two different ways: progressively and initially. In this chapter we focus on two specific situations where hypothesis $(\mathcal{H}')$ is satisfied. In both situations, under suitable conditions, we develop the $\mathbb{G}$-optional semimartingale decomposition for adequately changed $\mathbb{F}$-local martingales (Sections 7.2.1 and 7.2.1). An interesting link with absolutely continuous change of measure problem is observed in Section 7.5. Our study addresses as well the question of how an arbitrage-free semimartingale model is affected when stopped at a random horizon or when a random variable satisfying Jacod's hypothesis is incorporated. Precisely, we focus on the No-Unbounded-Profit-with-Bounded-Risk. In Section 7.2.2 we provide alternative proofs of Theorem 6.15, Corollary 6.3 (c) and Corollary 6.13 (b), using a new optional semimartingale decomposition. In Section 7.3.2 analogous results are formulated for initial enlargement with a random variable satisfying Jacod's hypothesis. Finally, in Section 7.4 we study the stability of NUPBR after thin random times.

7.2 Progressive enlargement up to random time

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space and $\mathbb{F}$ be a filtration satisfying the usual conditions. Consider a random time τ. Then, as in Section 1.2.3, we define several processes associated with τ. Two $\mathbb{F}$-supermartingales are given by

$$Z_t := \mathbb{P}(\tau > t|\mathcal{F}_t) \quad \text{and} \quad \widetilde{Z}_t := \mathbb{P}(\tau \geq t|\mathcal{F}_t).$$

Let us denote by A^o the $\mathbb{F}$-dual optional projection of $A := \mathbb{1}_{[\![\tau,\infty[\![}$ and define the $\mathbb{F}$-martingale

$$m_t := Z_t + A^o_t.$$

Then, by (1.13), $\widetilde{Z} = Z_- + \Delta m$.

Since we are here dealing with several enlarged filtrations, we have to change a little bit the notation used in the previous chapter. We denote by $\mathbb{F}^\tau = (\mathcal{F}^\tau_t)_{t\geq 0}$ the progressively enlarged filtration with random time τ, as defined in (1.5),

$$\mathcal{F}^\tau_t := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)).$$

In what follows we will call $\mathbb{F}^\tau$-predictable semimartingale decomposition up to the random time τ the equality introduced in Proposition 1.39, i.e., for an $\mathbb{F}$-local martingale X, one has

$$X^\tau_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle^{\mathbb{F}}_s, \tag{7.1}$$

where $\widehat{X}$ is an $\mathbb{F}^\tau$-local martingale. Using the $\mathbb{F}$-local martingale N which appears in Kardaras multiplicative decomposition [Kar10], i.e.,

$$N := \mathcal{E}\left(\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}} \cdot m\right) \tag{7.2}$$

the $\mathbb{F}^\tau$-predictable decomposition may be written as

$$X^\tau_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\langle X, N\rangle^{\mathbb{F}}_s. \tag{7.3}$$

7.2.1 Optional semimartingale decomposition for progressive enlargement

In this section, using the $\mathbb{F}^\tau$-predictable semimartingale decomposition and the results on various kinds of projections presented in Lemma 6.17 and Lemma 6.18, we derive another $\mathbb{F}^\tau$-decomposition of the $\mathbb{F}$-local martingales stopped at τ. Let us start by recalling the definition of the $\mathbb{F}$-stopping time R introduced in (6.8)

$$R := \inf\{t \geq 0 \ : \ Z_t = 0\}.$$

Then, the $\mathbb{F}$-stopping times defined as (using the convention given in (1.1))

$$\bar{R} := R_{\{Z_{R-}=0\}} \qquad \widetilde{R} := R_{\{\widetilde{Z}_R=0<Z_{R-}\}} \qquad \widehat{R} := R_{\{\widetilde{Z}_R>0\}}$$

have disjoint graphs and $R = \widehat{R}\wedge\widetilde{R}\wedge\bar{R}$. We note that $\bar{R}$ is an $\mathbb{F}$-predictable stopping time.

We establish an optional decomposition in the following theorem. By abuse of language, we shall refer to this decomposition as "the" optional decomposition even if there is no uniqueness for optional decomposition of a semimartingale.

Theorem 7.1. *Let X be an $\mathbb{F}$-local martingale. Then the process*

$$\boxed{\bar{X}_t := X^\tau_t - \int_0^{t\wedge\tau} \frac{1}{\widetilde{Z}_s} d[X,m]_s + \left(\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau}} \tag{7.4}$$

is an $\mathbb{F}^\tau$-local martingale.

Proof. By predictable decomposition (7.1) and Lemma 6.18 we obtain

$$\begin{aligned} X^\tau &= \widehat{X} + \frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]} \cdot \left(\mathbb{1}_{\{\widetilde{Z}>0\}} \cdot [X,m]\right)^{p,\mathbb{F}} + \frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]} \cdot \left(\mathbb{1}_{\{\widetilde{Z}=0\}} \cdot [X,m]\right)^{p,\mathbb{F}} \\ &= \widehat{X} + \left(\frac{1}{\widetilde{Z}}\mathbb{1}_{[\![0,\tau]\!]} \cdot [X,m]\right)^{p,\mathbb{F}^\tau} + \mathbb{1}_{[\![0,\tau]\!]} \cdot \left(\frac{1}{Z_-}\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}} \cdot [X,m]\right)^{p,\mathbb{F}}. \end{aligned}$$

Note that

$$\frac{1}{Z_-}\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}} \cdot [X,m] = \frac{1}{Z_{\widetilde{R}-}}\Delta X_{\widetilde{R}}\Delta m_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![} = -\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![},$$

where the first equality comes from $\{\widetilde{Z} = 0 < Z_-\} = [\![\widetilde{R}]\!]$ and the second equality from $\Delta m_{\widetilde{R}} = -Z_{\widetilde{R}-}$. Thus we obtain

$$X^\tau = \bar{X} + \frac{1}{\widetilde{Z}}\mathbb{1}_{[\![0,\tau]\!]} \cdot [X,m] - \mathbb{1}_{[\![0,\tau]\!]} \cdot \left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}$$

where $\bar{X}$ is an $\mathbb{F}^\tau$-local martingale. That ends the proof. ∎

Remark 7.2. In [DMM92, Paragraph 77, Chapter XX] an optional semimartingale decomposition is mentioned (without any proof) in the form: given an $\mathbb{F}$-local martingale X, the process

$$\bar{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{\widetilde{Z}_s} d[X,m]_s$$

is an $\mathbb{F}^\tau$-local martingale. Note that this decomposition is valid for any $\mathbb{F}$-local martingale if and only if $\widetilde{R} = \infty$.

Remark 7.3. The $\mathbb{F}^\tau$-local martingale introduced in (7.4) can be expressed in terms of the $\mathbb{F}$-local martingale N defined in (7.2).

$$\bar{X}_t = X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{N_s} d[X,N]_s + \left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau}.$$

From equalities $N = N_-\left(\mathbb{1}_{\{Z_->0\}}\frac{\widetilde{Z}}{Z_-} + \mathbb{1}_{\{Z_-=0\}}\right)$ and $N = 1 + N_-\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}} \cdot m$ it follows that

$$\frac{1}{N} \cdot [X,N] = \frac{1}{\widetilde{Z}}\mathbb{1}_{\{Z_->0\}} \cdot [X,m]$$

and the fact that $Z_- > 0$ on $[\![0,\tau]\!]$.

The next lemma presents another form of $\mathbb{F}^\tau$-local martingale introduced in (6.16). Here we denote it by L^{pr} where "pr" stands for progressive.

Lemma 7.4. *(a) The $\mathbb{F}^\tau$-predictable process $\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}$ is integrable with respect to $\bar{m}$ – the $\mathbb{F}^\tau$-martingale part from the optional decomposition of m.*
(b) The $\mathbb{F}^\tau$-local martingale L^{pr} (defined in (6.16)) can be expressed as

$$L^{\mathrm{pr}} := \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}}\ \frac{1}{\widetilde{Z}}\ \mathbb{1}_{[\![0,\tau]\!]} \odot \widehat{m} = \frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]} \cdot \bar{m},$$

where $\widehat{m}$ is $\mathbb{F}^\tau$-local martingale part from predictable decomposition of m and $\odot$ stands for the optional integral (see Section 1.1.8).

Proof. In the proof, we set $L = L^{\mathrm{pr}}$ for simplicity.
(a) The process $\frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}$ is càglàd so it is locally bounded.
(b) The $\mathbb{F}^\tau$-continuous martingale part and the jump part of $\frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot \bar{m}$ are given by

$$\left(\frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot \bar{m}\right)^c = \frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot\left(m^c - \frac{1}{Z_-}\centerdot\langle m^c\rangle^{\mathbb{F}}\right)$$
$$\Delta\left(\frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot \bar{m}\right) = \frac{\Delta m}{\widetilde{Z}}1\!\!1_{[\![0,\tau]\!]} - {}^{p,\mathbb{F}}\left(1\!\!1_{[\![\widetilde{R}]\!]}\right)1\!\!1_{[\![0,\tau]\!]}.$$

where m^c is $\mathbb{F}$-continuous martingale part of m. Let us compute now the $\mathbb{F}^\tau$-continuous martingale part and the jump part of L. By Definition 1.24 of optional stochastic integral and Lemma 6.17 (b), we have:

$$\begin{aligned}L^c &= \frac{Z_-^2}{Z_-^2+\Delta\langle m\rangle^{\mathbb{F}}}\,{}^{p,\mathbb{F}^\tau}\left(\frac{1}{\widetilde{Z}}\right)1\!\!1_{[\![0,\tau]\!]}\centerdot\widehat{m}^c\\ &= \frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot\widehat{m}^c - \frac{{}^{p,\mathbb{F}}(1\!\!1_{\{\widetilde{Z}=0<Z_-\}})}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot\widehat{m}^c.\end{aligned}$$

As $\{\widetilde{Z}=0<Z_-\}$ is a thin set, so it is $\{{}^{p,\mathbb{F}}(1\!\!1_{\{\widetilde{Z}=0<Z_-\}})\neq 0\}$, and from continuity of $\widehat{m}^c$ we conclude that

$$L^c = \frac{1}{Z_-}1\!\!1_{[\![0,\tau]\!]}\centerdot\widehat{m}^c.$$

From the proof of Proposition 6.20, we have that the jump of L equals

$$\Delta L = \frac{\Delta m}{\widetilde{Z}}1\!\!1_{[\![0,\tau]\!]} - {}^{p,\mathbb{F}}\left(1\!\!1_{\{\widetilde{Z}=0<Z_-\}}\right)1\!\!1_{[\![0,\tau]\!]}. \tag{7.5}$$

That completes the proof. ∎

The connection between the $\mathbb{F}^\tau$-local martingale L^{pr} and the $\mathbb{F}^\tau$-adapted process $\frac{1}{N^\tau}$ is exploited in the next lemma.

Proposition 7.5. *Let N be defined in (7.2).*
(a) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-supermartingale of the form

$$\boxed{\frac{1}{N^\tau} = \mathcal{E}\left(-(L^{\mathrm{pr}})^\tau - \left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\centerdot\wedge\tau}\right).}$$

(b) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-local martingale if and only if $\widetilde{R}=\infty$. Then, $\frac{1}{N^\tau} = \mathcal{E}\left(-(L^{\mathrm{pr}})^\tau\right)$.

Proof. (a) By Itô's formula and the obvious equality $dN = N_-\frac{1}{Z_-}1\!\!1_{\{Z_->0\}}dm$

$$\begin{aligned}\frac{1}{N_t^\tau} &= 1 - \int_0^{t\wedge\tau}\frac{1}{N_{s-}^2}dN_s + \int_0^{t\wedge\tau}\frac{1}{N_{s-}^3}d\langle N^c\rangle_s + \sum_{s\le t\wedge\tau}\left(\frac{1}{N_s} - \frac{1}{N_{s-}} + \frac{1}{N_{s-}^2}\Delta N_s\right)\\ &= 1 - \int_0^{t\wedge\tau}\frac{1}{N_{s-}Z_{s-}}dm_s + \int_0^{t\wedge\tau}\frac{1}{N_{s-}Z_{s-}^2}d\langle m^c\rangle_s + \sum_{s\le t\wedge\tau}\frac{(\Delta m_s)^2}{N_{s-}Z_{s-}\widetilde{Z}_s},\end{aligned}$$

where we have used the fact that $\widetilde{Z} = Z_- + \Delta m$. We continue with

$$\begin{aligned}\frac{1}{N_t^\tau} &= 1 - \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\left(\frac{1}{Z_-} \centerdot m - \frac{1}{Z_-^2} \centerdot \langle m^c\rangle - \sum \frac{(\Delta m)^2}{Z_-\widetilde{Z}}\right)_s \\ &= 1 - \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\left(\frac{1}{Z_-} \centerdot \bar{m} + \left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}\right)_s\end{aligned}$$

where the second equality comes from Theorem 7.1 applied to the $\mathbb{F}$-martingale m. Finally we conclude that

$$\frac{1}{N^\tau} = \mathcal{E}\left(-\frac{1}{Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot \bar{m} - \left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\centerdot\wedge\tau}\right).$$

(b) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-local martingale if and only if $\left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau} = 0$. The last equality is equivalent to

$$0 = \mathbb{E}\left(\left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_\tau\right) = \mathbb{E}\left(\int_0^\infty Z_{s-} d\left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_s\right) = \mathbb{E}\left(Z_{\widetilde{R}-} 1\!\!1_{\{\widetilde{R}<\infty\}}\right),$$

which in turn is equivalent to $\widetilde{R} = \infty$ $\mathbb{P}$-a.s. since $Z_{\widetilde{R}-} > 0$ on $\{\widetilde{R} < \infty\}$. ∎

7.2.2 NUPBR for progressive enlargement before τ

In this section, we give alternative proofs, based on the optional semimartingale decomposition, to some results from Chapter 6. In Proposition 7.6 we examine the $\mathbb{F}^\tau$-local martingale deflator for $\mathbb{F}$-local martingales. In Proposition 7.7, an $\mathbb{F}^\tau$-supermartingale deflator for $\mathbb{F}$-local martingales is studied.

Proposition 7.6. *Let $Y^{\rm pr} := \mathcal{E}\left(-L^{\rm pr}\right)$. Then, for any $\mathbb{F}$-local martingale X, the two processes*

$$Y^{\rm pr}X^\tau - \sum Y^{\rm pr}_- 1\!\!1_{[\![0,\tau]\!]}\, {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]})\Delta X + Y^{\rm pr}_- 1\!\!1_{[\![0,\tau]\!]} \centerdot \left(\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}$$

$$Y^{\rm pr}X^\tau + Y^{\rm pr}_- 1\!\!1_{[\![0,\tau]\!]} \centerdot \left((\Delta X_{\widetilde{R}} - \frac{\Delta\langle m, X\rangle^{\mathbb{F}}_{\widetilde{R}}}{Z_{\widetilde{R}-}}) 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}$$

are $\mathbb{F}^\tau$-local martingales.
In particular, if X is quasi-left continuous and $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$, then $Y^{\rm pr}$ is an $\mathbb{F}^\tau$-local martingale deflator for X^τ.

Proof. In the proof, we set $Y = Y^{\rm pr}$ and $L = L^{\rm pr}$ for simplicity. Using integration by parts and the optional decomposition (7.4) given in Theorem 7.1 for X and then for m, we obtain

$$\begin{aligned} &YX^\tau = X^\tau_- \centerdot Y + Y_- \centerdot X^\tau + [Y, X^\tau] \\ &= X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + Y_- \frac{1}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \centerdot [m, X] - Y_- 1\!\!1_{[\![0,\tau]\!]} \centerdot (\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}} - Y_- 1\!\!1_{[\![0,\tau]\!]} \centerdot [L, X] \\ &= X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + Y_- \frac{1}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \centerdot [\bar{m}, X] + Y_- \frac{1}{\widetilde{Z}^2} 1\!\!1_{[\![0,\tau]\!]} \centerdot [[m], X] \\ &\quad - Y_- \frac{1}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \centerdot [(\Delta m_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}}, X] - Y_- 1\!\!1_{[\![0,\tau]\!]} \centerdot (\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}} - \frac{Y_-}{Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot [\bar{m}, X] \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned}$$

We look closer at the sum of third and seventh term of the last expression

$$\begin{aligned}I_3 + I_7 =& Y_- \left(\frac{1}{\widetilde{Z}} - \frac{1}{Z_-}\right) 1\!\!1_{[\![0,\tau]\!]} \centerdot [\bar{m}, X] = -Y_- \frac{\Delta m}{\widetilde{Z} Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot [\bar{m}, X] \\ =& -\sum Y_- \frac{\Delta m}{\widetilde{Z} Z_-} 1\!\!1_{[\![0,\tau]\!]} \Delta \bar{m} \Delta X,\end{aligned}$$

where the third equality comes from the fact that $\{\Delta m \neq 0\}$ is a thin set. We add the term I_4 to the previous two

$$\begin{aligned}I_4 + (I_3 + I_7) =& \sum Y_- \frac{1}{\widetilde{Z}^2} 1\!\!1_{[\![0,\tau]\!]} (\Delta m)^2 \Delta X - \sum Y_- \frac{\Delta m}{\widetilde{Z} Z_-} 1\!\!1_{[\![0,\tau]\!]} \Delta \bar{m} \Delta X \\ =& -\sum Y_- \frac{\Delta m}{\widetilde{Z}} \Delta X 1\!\!1_{[\![0,\tau]\!]} \left(\frac{1}{Z_-} \Delta \bar{m} - \frac{1}{\widetilde{Z}} \Delta m\right) \\ =& \sum Y_- \frac{\Delta m}{\widetilde{Z}} \Delta X 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}]\!]}\right),\end{aligned}$$

where the last equality comes from (7.5). Note that, by Yoeurp's lemma and properties of dual predictable projection, the fifth term in the expression for YX^τ is equal to

$$\begin{aligned}I_5 =& -Y_- \frac{1}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \centerdot [(\Delta m_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}}, X] = -Y_- \frac{1}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}}(\Delta m_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R}]\!]}) \centerdot X \\ =& \sum Y_- \frac{Z_-}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]}) \Delta X.\end{aligned}$$

Finally, by $Z_- + \Delta m = \widetilde{Z}$, we get

$$\begin{aligned}I_5 + (I_4 + I_3 + I_7) =& \sum Y_- \frac{Z_-}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]}) \Delta X + \sum Y_- \frac{\Delta m}{\widetilde{Z}} \Delta X 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}]\!]}\right) \\ =& \sum Y_- 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]}) \Delta X.\end{aligned}$$

Summing up we have that

$$\begin{aligned}YX^\tau =& X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + \sum Y_- 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]}) \Delta X - Y_- 1\!\!1_{[\![0,\tau]\!]} \centerdot \left(\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}} \\ =& X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + \sum Y_- 1\!\!1_{[\![0,\tau]\!]} \, {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]}) \Delta X + \frac{Y_-}{Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot \left(1\!\!1_{[\![\widetilde{R}]\!]} \centerdot [X, m]\right)^{p,\mathbb{F}}\end{aligned}$$

Finally, thanks to the predictable semimartingale decomposition of X, we get

$$\begin{aligned}YX^\tau =& X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + Y_- \, {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}]\!]}\right) \centerdot \widehat{X} + \frac{Y_-}{Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot \left(1\!\!1_{[\![\widetilde{R}]\!]} \centerdot ([m, X] + \langle m, X \rangle^{\mathbb{F}})\right)^{p,\mathbb{F}} \\ =& X^\tau_- \centerdot Y + Y_- \centerdot \left(\bar{X} + {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}]\!]}\right) \centerdot \widehat{X}\right) - Y_- 1\!\!1_{[\![0,\tau]\!]} \centerdot \left((\Delta X_{\widetilde{R}} - \frac{\Delta \langle m, X \rangle^{\mathbb{F}}_{\widetilde{R}}}{Z_{\widetilde{R}-}}) 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}\end{aligned}$$

which ends the proof of the first assertion. Since for any $\mathbb{F}$-quasi-left continuous martingale X, the process $\langle m, X \rangle^{\mathbb{F}}$ is continuous, the particular case follows. ■

We shall now use the $\mathbb{F}^\tau$-predictable process introduced in Lemma 6.19 which was crucial for proofs therein. Denoting by $\widetilde{R}^a$ the accessible part of the $\mathbb{F}$-stopping time $\widetilde{R}$, we set

$$V^{\mathrm{pr}}_t := \left(1\!\!1_{[\![\widetilde{R}^a,\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau}.$$

Using the process V^{pr} we study, in the next lemma, the behaviour of a particular $\mathbb{F}^\tau$-supermartingale.

Proposition 7.7. *Let $\widetilde{Y}^{\mathrm{pr}} := \mathcal{E}(-L^{\mathrm{pr}} - V^{\mathrm{pr}})$. Let X be an $\mathbb{F}$-local martingale and H be an $\mathbb{F}^\tau$-predictable process such that $H \centerdot X \geq -1$. Then, the process*

$$(1 + H \centerdot X^\tau)\ \widetilde{Y}^{\mathrm{pr}} + H\ \widetilde{Y}^{\mathrm{pr}}_- \centerdot \left(\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}$$

is an $\mathbb{F}^\tau$-supermartingale.
In particular, if $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$ then $\widetilde{Y}^{\mathrm{pr}}$ is an $\mathbb{F}^\tau$-supermartingale deflator for X^τ and X^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. In the proof, we set $\widetilde{Y} = \widetilde{Y}^{\mathrm{pr}}$, $L = L^{\mathrm{pr}}$ and $V = V^{\mathrm{pr}}$ for simplicity. By integration by parts, we get

$$(1 + H \centerdot X^\tau)\ \widetilde{Y} = (1 + H \centerdot X^\tau)_- \centerdot \widetilde{Y} + H\ \widetilde{Y}_- \centerdot X^\tau - H\ \widetilde{Y}_- \centerdot [X^\tau, L] - H\ \widetilde{Y}_- \centerdot [X^\tau, V].$$

Note that

$$H\widetilde{Y}_- \centerdot [X^\tau, V] = \sum H\ \widetilde{Y}_-\ 1\!\!1_{[\![0,\tau]\!]}\ {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}]\!]})\Delta X.$$

Then, using the same arguments as in the proof of Theorem 7.6, we get

$$(1 + H \centerdot X^\tau)\widetilde{Y} = (1 + H \centerdot X^\tau)_- \centerdot \widetilde{Y} + H\widetilde{Y}_- \centerdot \bar{X} - H\widetilde{Y}_- 1\!\!1_{[\![0,\tau]\!]} \centerdot \left(\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}$$

and the first assertion is proved.
The second assertion comes from Definition 1.61 of the supermartingale deflator and Theorem 1.62. ∎

The next result recovers Theorem 6.15 providing explicit local martingale deflators for $\mathbb{F}$-local martingales. The proof differs from the proof of Theorem 6.15 and is based on the optional semimartingale decomposition and direct computations.

Theorem 7.8. *The following conditions are equivalent.*
(a) The thin set $\left\{\widetilde{Z} = 0 < Z_-\right\}$ is evanescent.
(b) The $\mathbb{F}$-stopping time $\widetilde{R}$ is infinite ($\widetilde{R} = \infty$).
(c) For any $\mathbb{F}$-local martingale X, the process X^τ admits an $\mathbb{F}^\tau$-local martingale deflator Y^{pr} (and satisfies NUPBR($\mathbb{F}^\tau$)).
(d) For any (bounded) process X satisfying NUPBR($\mathbb{F}$), the process X^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof. The equivalence between (a) and (b) is obvious from definition of $\widetilde{R}$.
The implication (b)⇒(c) follows from Proposition 7.6. To prove (c)⇒(b) (and (d)⇒(b)), we consider the $\mathbb{F}$-martingale

$$X = 1\!\!1_{[\![\widetilde{R},\infty[\![} - \left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}.$$

Note that $\mathbb{P}(\tau = \widetilde{R}) = \mathbb{E}(\Delta A^o_{\widetilde{R}}) = \mathbb{E}(\widetilde{Z}_{\widetilde{R}} - Z_{\widetilde{R}}) = 0$ (since $0 = \widetilde{Z}_{\widetilde{R}} \geq Z_{\widetilde{R}} \geq 0$). This implies that $\tau < \widetilde{R}$ and

$$X^\tau = -\left(1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}$$

is an $\mathbb{F}^\tau$-predictable decreasing process. Thus, X^τ satisfies NUPBR($\mathbb{F}^\tau$) if and only if it is a null process, due to Lemma 1.60. Then, we conclude that $\widetilde{R}$ is infinite using the same argument as in the proof of Lemma 7.5 (b).
(c) $\Rightarrow$ (d) Let X be an $\mathbb{F}$-semimartingale satisfying the NUPBR($\mathbb{F}$). Thanks to Proposition 1.56, we deduce the existence of a real-valued $\mathbb{F}$-predictable process ϕ and a positive $\mathbb{F}$-local martingale l such that

$$0 < \phi \leq 1 \quad \text{ and } \quad l\,(\phi \centerdot X) \quad \text{ is an } \mathbb{F}\text{-local martingale.}$$

Then there exists a sequence of $\mathbb{F}$-stopping times $(v_n)_n$ that increases to infinity such that the stopped process l^{v_n} is an $\mathbb{F}$-martingale. Put $\mathbb{Q}_n := l_{v_n} \centerdot \mathbb{P} \sim \mathbb{P}$. Then, by applying (c) to $(\phi \centerdot X)^{v_n}$ under $\mathbb{Q}_n$, we conclude that $(\phi \centerdot X)^{v_n \wedge \tau}$ satisfies NUPBR($\mathbb{F}^\tau$) under $\mathbb{Q}_n$. Thanks to Proposition 1.59, NUPBR($\mathbb{F}^\tau$) under $\mathbb{P}$ of X^τ follows immediately. ■

7.3 Initial enlargement under Jacod's hypothesis

The results presented in this section were obtained in parallel and independently to [AFK14]. Let ξ be a random variable with values in a Lusin space $(\mathbb{U}, \mathcal{U})$. We assume that the random variable ξ satisfies Jacod's hypothesis (Definition 1.31 (a)) with the density process $(q^u, u \in \mathbb{U})$ with respect to the law $\widetilde{\eta}$ (Proposition 1.32). We do not impose any conditions on $\widetilde{\eta}$, in particular it is not necessary atomless. Then, let $\mathbb{F}^{\sigma(\xi)} = (\mathcal{F}^{\sigma(\xi)})_{t\geq 0}$ be the initial enlargement of the filtration $\mathbb{F}$ with the random variable ξ, defined as in (1.4),

$$\mathcal{F}_t^{\sigma(\xi)} := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\xi)) \,.$$

Consider a mapping $X : (t, \omega, u) \to X_t^u(\omega)$ on $\mathbb{R}_+ \times \Omega \times \mathbb{U}$ with values in $\mathbb{R}$. Let $\mathcal{J}$ be a class of $\mathbb{F}$-optional processes, for example the class of $\mathbb{F}$-(local) martingales or the class of $\mathbb{F}$-locally integrable variation processes. Then, $(X^u, u \in \mathbb{U})$ is called a parametrized $\mathcal{J}$-process if for each $u \in \mathbb{U}$ the process X^u belongs to $\mathcal{J}$ and if it is measurable with respect to $\mathcal{O}(\mathbb{F}) \otimes \mathcal{U}$. The second condition, by [SY78, Proposition 3], can be obtained by taking appropriate versions of processes X^u.

To ensure the existence of well measurable versions of dual projections of parametrized process we assume from now on that the space $L^1(\Omega, \mathcal{G}, \mathbb{P})$ is separable. Then we apply [SY78, Proposition 4].

In the next proposition we state an analogical result to Proposition 1.34 for a bigger class of processes: parametrized $\mathbb{F}$-local martingales.

Proposition 7.9. *Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale. Then*

$$X_t^\xi = \widehat{X}_t^\xi + \int_0^t \frac{1}{q_{s-}^\xi} d\langle X^u, q^u \rangle_s|_{u=\xi}$$

where $\widehat{X}^\xi$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale.

Proof. Let X be of the form $X_t^u(\omega) = g(u)x_t(\omega)$ where x is an $\mathbb{F}$-martingale. Then, $X^\xi = g(\xi)x$ and for $t \geq s$ using Jacod's decomposition (1.9), we have

$$\begin{aligned}\mathbb{E}(X_t^\xi - X_s^\xi | \mathcal{F}_s^{\sigma(\xi)}) &= g(\xi)\mathbb{E}(x_t - x_s | \mathcal{F}_s^{\sigma(\xi)}) \\ &= g(\xi)\mathbb{E}\left(\int_s^t \frac{1}{q_{v-}^\xi} d\langle x, q^u\rangle_v|_{u=\xi} \middle| \mathcal{F}_s^{\sigma(\xi)}\right) \\ &= \mathbb{E}\left(\int_s^t \frac{1}{q_{v-}^\xi} d\langle X^u, q^u\rangle_v|_{u=\xi} \middle| \mathcal{F}_s^{\sigma(\xi)}\right).\end{aligned}$$

For a general X, we proceed by Monotone Class Theorem 1.1. ∎

7.3.1 Optional semimartingale decomposition for initial enlargement

In this section we develop the $\mathbb{F}^{\sigma(\xi)}$-optional semimartingale decomposition of parametrized $\mathbb{F}$-local martingales. In order to find it, we decompose the $\mathbb{F}$-stopping time R^u, introduced in (1.6), as $R^u = \widetilde{R}^u \wedge \bar{R}^u$ with

$$\widetilde{R}^u = R^u_{\{q^u_{R^u-}>0\}} \quad \text{and} \quad \bar{R}^u = R^u_{\{q^u_{R^u-}=0\}}. \tag{7.6}$$

Clearly $\bar{R}^u$ is an $\mathbb{F}$-predictable stopping time and $\{R^u = \infty\} \subset \{\bar{R}^u = \infty\}$ so

$$\left(\mathbb{1}_{[\![R^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi} = \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}.$$

In the following Lemma, we express the $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection in terms of the $\mathbb{F}$-dual predictable projection. This is a result analogous to Lemma 6.17 (a) and Lemma 6.18 for initial enlargement case.

Lemma 7.10. *Let $(V^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-adapted càdlàg process with locally integrable variation ($V \in \mathcal{A}_{loc}(\mathbb{F})$). Then the following properties hold.*
(a) The $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection of V^ξ is

$$(V^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q_-^\xi} \cdot \left(q^u \cdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}. \tag{7.7}$$

(b) If $(V^u, u \in \mathbb{U})$ belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$ (respectively $V \in \mathcal{A}^+(\mathbb{F})$), then the process $(U^u, u \in \mathbb{U})$ with

$$U^u := \frac{1}{q^\xi} \cdot V^u, \tag{7.8}$$

belongs to $\mathcal{A}^+_{loc}(\mathbb{F}^{\sigma(\xi)})$ (respectively to $\mathcal{A}^+(\mathbb{F}^{\sigma(\xi)})$).
(c) The process $(U^u, u \in \mathbb{U})$ is well defined, its variation is $\mathbb{F}^{\sigma(\xi)}$-locally integrable, and the $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection of U^ξ is given by

$$\boxed{(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q_-^\xi} \cdot \left(\mathbb{1}_{\{q^u>0\}} \cdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.}$$

Proof. (a) We apply the predictable semimartingale decomposition given in Lemma 7.9 to the parametrized $\mathbb{F}$-local martingale $(X^u, u \in \mathbb{U}) = (V^u - (V^u)^{p,\mathbb{F}}, u \in \mathbb{U})$, obtaining

$$\begin{aligned} V^\xi &= \widehat{X}^\xi + (V^u)^{p,\mathbb{F}}|_{u=\xi} + \frac{1}{q^\xi_-} \centerdot \langle V^u, q^u \rangle^{\mathbb{F}}|_{u=\xi} \\ &= \widehat{X}^\xi + \left(\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi} + \left(\frac{\Delta q^u}{q^u_-}\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}} \\ &= \widehat{X}^\xi + \frac{1}{q^\xi_-} \centerdot (q^u \centerdot V^u)^{p,\mathbb{F}}|_{u=\xi}, \end{aligned}$$

which proves assertion (a).
(b) Suppose that $(V^u, u \in \mathbb{U}) \in \mathcal{A}^+_{loc}(\mathbb{F})$. For fixed u, let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to infinity such that $\mathbb{E}\left(V^u_{\vartheta_n}\right) < \infty$. Then, $\mathbb{E}\left(U^u_{\vartheta_n}\right) < \infty$ since

$$\mathbb{E}\left(U^u_{\vartheta_n}\right) = \mathbb{E}\left(\int_0^{\vartheta_n} \frac{1}{q^\xi_t} dV^u_t\right) = \mathbb{E}\left(\int_0^{\vartheta_n}\int_{\mathbb{R}} \mathbb{1}_{\{q^y_t>0\}}\widetilde{\eta}(dy)\, dV^u_t\right) \leq \mathbb{E}\left(V^u_{\vartheta_n}\right) < \infty,$$

where the last equality comes from (1.7) applied to $\frac{1}{q^u_t}\mathbb{1}_{\{q^u_t>0\}}$.
(c) Suppose that $(V^u, u \in \mathbb{U}) \in \mathcal{A}_{loc}(\mathbb{F})$, and denote by $W := V^+ + V^-$ its variation. Then $(W^u, u \in \mathbb{U}) \in \mathcal{A}^+_{loc}(\mathbb{F})$, and a direct application of (b) implies that

$$\left(\frac{1}{q^\xi} \centerdot W^u, u \in \mathbb{U}\right) \in \mathcal{A}^+_{loc}(\mathbb{F}^\tau).$$

As a result, we deduce that U given by (7.8) for the case of $V = V^+ - V^-$ is well defined and has variation equal to $\frac{1}{q^\xi} \centerdot W$ which is $\mathbb{F}^{\sigma(\xi)}$-locally integrable. For each $n \geq 1$, let us consider the parametrized process $(U^u_n, u \in \mathbb{U})$ with

$$U^u_n := \frac{1}{q^u}\mathbb{1}_{\{q^u \geq \frac{1}{n}\}} \centerdot V^u.$$

Due to (7.7), we derive

$$\left(U^\xi_n\right)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-} \centerdot \left(\mathbb{1}_{\{q^u\geq 1/n\}} \centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

Hence, since $(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \lim_{n\to\infty}\left(U^\xi_n\right)^{p,\mathbb{F}^{\sigma(\xi)}}$ by taking the limit in the above equality, we get

$$(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-} \centerdot \left(\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

∎

We are now ready to state, in the next theorem, the main result of this section which is based on Lemma 7.10.

Theorem 7.11. *Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale. Then,*

$$\boxed{\bar{X}^\xi_t := X^\xi_t - \int_0^t \frac{1}{q^\xi_s} d[X^\xi, q^\xi]_s + \left(\Delta X^u_{\widetilde{R}^u}\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}_t|_{u=\xi}} \tag{7.9}$$

is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. Here, $\widetilde{R}^u$ was defined in (7.6).

Proof. From the predictable decomposition given in Proposition 7.9 and Lemma 7.10 (c) we develop the decomposition of X^ξ as

$$\begin{aligned} X^\xi &= \widehat{X}^\xi + \frac{1}{q^\xi_-} \centerdot \left(1\!\!1_{\{q^u>0\}} \centerdot [X^u, q^u]\right)^{p,\mathbb{F}} |_{u=\xi} + \frac{1}{q^\xi_-} \centerdot \left(1\!\!1_{\{q^u=0\}} \centerdot [X^u, q^u]\right)^{p,\mathbb{F}} |_{u=\xi} \\ &= \widehat{X}^\xi + \left(\frac{1}{q^\xi} \centerdot [X^\xi, q^\xi]\right)^{p,\mathbb{F}^{\sigma(\xi)}} + \frac{1}{q^\xi_-} \centerdot \left(\Delta X^u_{\widetilde{R}^u} \Delta q_{\widetilde{R}^u}(x) 1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi} \\ &= \bar{X}^\xi + \frac{1}{q^\xi} \centerdot [X^\xi, q^\xi] - \left(\Delta X^u_{\widetilde{R}^u} 1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi} \end{aligned}$$

where

$$\bar{X}^\xi = \widehat{X}^\xi - \frac{1}{q^\xi} \centerdot [X^\xi, q^\xi] + \left(\frac{1}{q^\xi} \centerdot [X^\xi, q^\xi]\right)^{p,\mathbb{F}^{\sigma(\xi)}}$$

is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. ∎

In the next two lemmas we study the properties of the process q^ξ. In Lemma 7.12 we define particular $\mathbb{F}^{\sigma(\xi)}$-local martingales based on q^ξ. Then in Lemma 7.13, we focus on the process $\frac{1}{q^\xi}$. We explicitly give the $\mathbb{F}^{\sigma(\xi)}$-supermartingales decomposition and necessary and sufficient condition such that $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. In [Ame99] the process $\frac{1}{q^\xi}$ was studied in the case of a random variable ξ satisfying equivalence Jacod's hypothesis. Here we work under less restrictive assumption.

Lemma 7.12. *Let $\bar{q}^\xi$ be the $\mathbb{F}^{\sigma(\xi)}$-local martingale part of q^ξ given by* (7.9)*, i.e.,*

$$\bar{q}^\xi := q^\xi - \frac{1}{q^\xi} \centerdot [q^\xi] - q^\xi_- \centerdot \left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}.$$

(a) The $\mathbb{F}^{\sigma(\xi)}$-predictable process $\frac{1}{q^\xi_-}$ is integrable with respect to $\bar{q}^\xi$.

(b) The $\mathbb{F}^{\sigma(\xi)}$-local martingale

$$L^i := \frac{1}{q^\xi_-} \centerdot \bar{q}^\xi \tag{7.10}$$

is such that $1 - \Delta L^i > 0$. Here, the superscript "i" stands for initial.

Proof. (a) The process $\frac{1}{q^\xi_-}$ is càglàd so it is locally bounded.
(b) The definition of $\bar{q}^\xi$ implies

$$\begin{aligned} 1 - \Delta L^i &= 1 - \frac{1}{q^\xi_-}\left(\Delta q^\xi - \frac{1}{q^\xi}(\Delta q^\xi)^2\right) + \Delta\left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi} \\ &= 1 - \frac{\Delta q^\xi}{q^\xi} + {}^{p,\mathbb{F}}\left(1\!\!1_{[\![\widetilde{R}^u]\!]}\right) |_{u=\xi} \\ &= \frac{q^\xi_-}{q^\xi} + {}^{p,\mathbb{F}}\left(1\!\!1_{[\![\widetilde{R}^u]\!]}\right) |_{u=\xi} > 0 \end{aligned}$$

which completes the proof. ∎

Lemma 7.13. *(a) The process* $\frac{1}{q^\xi}$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-supermartingale with decomposition*

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-} \centerdot \bar{q}^\xi - \frac{1}{q^\xi_-} \centerdot \left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}. \tag{7.11}$$

Moreover, it can be written as a stochastic exponential of the form

$$\boxed{\frac{1}{q^\xi} = \mathcal{E}\left(-L^i - \left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}\right).}$$

(b) The process $\frac{1}{q^\xi}$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-local martingale if and only if* $\widetilde{R}^u = \infty$ $\mathbb{P}\otimes\widetilde{\eta}$*-a.s. Then* $\frac{1}{q^\xi} = \mathcal{E}(-L^i)$.

Proof. (a) $(q^u, u \in \mathbb{U})$ is a parametrized $\mathbb{F}$-martingale, then by Proposition 7.9, q^ξ is a $\mathbb{F}^{\sigma(\xi)}$-semimartingale. By (1.8), q^ξ is strictly positive. Itô's lemma implies that $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale which can be written as

$$\begin{aligned}
\frac{1}{q^\xi} =& 1 - \frac{1}{(q^\xi)^2_-} \centerdot q^\xi + \frac{1}{(q^\xi)^3_-} \centerdot \langle (q^\xi)^c\rangle^{\mathbb{F}^{\sigma(\xi)}} + \sum\left(\frac{1}{q^\xi} - \frac{1}{q^\xi_-} + \frac{1}{(q^\xi)^2_-}\Delta q^\xi\right)\\
& 1 - \frac{1}{(q^\xi)^2_-} \centerdot q^\xi + \frac{1}{(q^\xi)^3_-} \centerdot \langle (q^\xi)^c\rangle^{\mathbb{F}^{\sigma(\xi)}} + \sum\left(\frac{1}{(q^\xi)^2_- q^\xi}(\Delta q^\xi)^2\right)\\
=& 1 - \frac{1}{(q^\xi)^2_-} \centerdot q^\xi + \frac{1}{(q^\xi)^2_- q^\xi} \centerdot [q^\xi].
\end{aligned} \tag{7.12}$$

Applying (7.9), we finally get that

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-} \centerdot \bar{q}^\xi - \frac{1}{q^\xi_-} \centerdot \left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi},$$

where $\frac{1}{q^\xi_-} \centerdot \left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-predictable increasing process. The exponential form immediately follows.

(b) The process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi} = 0$. The last is equivalent to have that, for each t

$$\begin{aligned}
0 = \mathbb{E}\left((1\!\!1_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}_t|_{u=\xi}\right) &= \mathbb{E}\left({}^{o,\mathbb{F}}\left((1\!\!1_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi}\right)_t\right)\\
&= \mathbb{E}\left(\int_{\mathbb{R}} (1\!\!1_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}_t q^u_t \widetilde{\eta}(du)\right)\\
&= \int_{\mathbb{R}} \mathbb{E}\left((1\!\!1_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}_t q^u_t\right)\widetilde{\eta}(du),
\end{aligned}$$

where the second equality comes from (1.7). Next, by Yoeurp's lemma we conclude that, for each t

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \mathbb{E}\left(\int_0^t q^u_{s-} d\left((1\!\!1_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}\right)_s\right)\widetilde{\eta}(du)\\
&= \int_{\mathbb{R}} \mathbb{E}\left(q^u_{\widetilde{R}^u-} 1\!\!1_{\{\widetilde{R}^u\le t\}}\right)\widetilde{\eta}(du)
\end{aligned}$$

which in turn is equivalent to $\widetilde{R}^u > t$ $\mathbb{P} \otimes \widetilde{\eta}$-a.s. for each t since $q^u_{\widetilde{R}^u -} > 0$. Thus, $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\widetilde{R}^u$ is infinite $\mathbb{P} \otimes \widetilde{\eta}$-a.s. ∎

In the following proposition, we investigate the $\mathbb{F}^{\sigma(\xi)}$-semimartingale decomposition of a parametrized $\mathbb{F}$-local martingale X when ξ is plugged in and when multiplied by $\frac{1}{q^\xi}$ from previous lemma.

Proposition 7.14. *Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale. Then $\frac{X^\xi}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale with $\mathbb{F}^{\sigma(\xi)}$-local martingale part equal to*

$$X^\xi_0 - \frac{X^\xi_-}{(q^\xi)^2_-} \centerdot \bar{q}^\xi + \frac{1}{q^\xi_-} \centerdot \bar{X}^\xi,$$

and $\mathbb{F}^{\sigma(\xi)}$-predictable finite variation part equal to

$$-\frac{1}{q^\xi_-} \centerdot \left(X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}.$$

Proof. We compute, applying integration by parts formula:

$$\begin{aligned}
\frac{X^\xi}{q^\xi} &= X^\xi_0 + X^\xi_- \centerdot \frac{1}{q^\xi} + \frac{1}{q^\xi_-} \centerdot X^\xi + \left[X^\xi, \frac{1}{q^\xi} \right] \\
&= X^\xi_0 - \frac{X^\xi_-}{(q^\xi)^2_-} \centerdot \bar{q}^\xi - \frac{X^\xi_-}{q^\xi_-} \centerdot \left(\mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \\
&\quad + \frac{1}{q^\xi_-} \centerdot \bar{X}^\xi + \frac{1}{q^\xi_- q^\xi} \centerdot [X^\xi, q^\xi] - \frac{1}{q^\xi_-} \centerdot \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \\
&\quad - \frac{1}{(q^\xi)^2_-} \centerdot [X^\xi, q^\xi] + \frac{1}{(q^\xi)^2_- q^\xi} \centerdot \left[X^\xi, [q^\xi] \right],
\end{aligned}$$

where the second equality comes from (7.11), (7.12). It follows that

$$\begin{aligned}
\frac{X^\xi}{q^\xi} &= X^\xi_0 - \frac{X^\xi_-}{(q^\xi)^2_-} \centerdot \bar{q}^\xi + \frac{1}{q^\xi_-} \centerdot \bar{X}^\xi \\
&\quad - \frac{1}{q^\xi_-} \centerdot \left(X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} - \frac{\Delta q^\xi}{p_(\xi)^2 q^\xi} \centerdot [X^\xi, q^\xi] + \frac{\Delta X^\xi}{(q^\xi)^2_- q^\xi} \centerdot [q^\xi] \\
&= X^\xi_0 - \frac{X^\xi_-}{(q^\xi)^2_-} \centerdot \bar{q}^\xi + \frac{1}{q^\xi_-} \centerdot \bar{X}^\xi - \frac{1}{q^\xi_-} \centerdot \left(X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}.
\end{aligned}$$

∎

As a corollary, from Proposition 7.14, we recover [IP11, Proposition 5.2] on universal supermartingale density.

Corollary 7.15. If X is a positive $\mathbb{F}$-supermartingale, then, $\frac{X}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale.

Proof. Let X be decomposed as $X = M^X - V^X$ where M^X is a positive $\mathbb{F}$-local martingale and V^X is an increasing $\mathbb{F}$-predictable process. Then, $\frac{M^X}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale since from the positiveness of M^X, by Proposition 7.14, we get that $\frac{1}{q^\xi_-} \cdot \left(M^X_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-predictable increasing process. Moreover, as $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale and V^X is increasing, the process $-\frac{V^X}{q^\xi}$ is as well an $\mathbb{F}^{\sigma(\xi)}$-supermartingale which ends the proof. ■

7.3.2 NUPBR for initial enlargement

In this section, we focus on the NUPBR condition. Using simple arguments based on the optional semimartingale decomposition, we prove the NUPBR stability with respect to a initial enlargement of filtration under Jacod's hypothesis. In Proposition 7.16 we study the $\mathbb{F}^{\sigma(\xi)}$-local martingale deflators for parametrized $\mathbb{F}$-local martingales. Then in Proposition 7.17 we focus on $\mathbb{F}^{\sigma(\xi)}$-supermartingale deflators for parametrized $\mathbb{F}$-local martingales. In Theorem 7.18 we formulate a necessary and sufficient condition such that each parametrized $\mathbb{F}$-local martingale satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$). We close this section giving two examples of very particular initial enlargements under Jacod's hypothesis. We also address the reader to [AFK14] for similar study to the one contained in this section.

Proposition 7.16. *Let $Y^i := \mathcal{E}\left(-L^i\right)$. Then, for any parametrized $\mathbb{F}$-local martingale $(X^u, u \in \mathbb{R})$, the two processes*

$$Y^i X^\xi - \sum Y^i_- \,{}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi + Y^i_- \cdot \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}$$

$$Y^i X^\xi + Y^i_- \cdot \left(\left(\Delta X^u_{\widetilde{R}^u} - \frac{\Delta\langle q^u, X^u\rangle_{\widetilde{R}^u}}{q^u_{\widetilde{R}^u-}}\right) \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |u = \xi$$

are $\mathbb{F}^{\sigma(\xi)}$-local martingales.
In particular, if X^u is quasi-left continuous and $\Delta X^u_{\widetilde{R}^u} = 0$ on $\{\widetilde{R}^u < \infty\}$ $\mathbb{P} \otimes \widetilde{\eta}$-a.s., then Y^i is an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator for X^ξ.

Proof. Using the optional decomposition (7.9) (Theorem 7.11) of X^ξ and q^ξ we obtain

$$\begin{aligned}
Y^i X^\xi =& X^\xi_- \cdot Y^i + Y^i_- \cdot X^\xi + [Y^i, X^\xi] \\
=& X^\xi_- \cdot Y^i + Y^i_- \cdot \bar{X}^\xi + Y^i_- \frac{1}{q^\xi} \cdot [X^\xi, q^\xi] - Y^i_- \cdot (\Delta X^u_{\widetilde{R}^u})^{p,\mathbb{F}}|_{u=\xi} - Y^i_- \cdot [L^i, X^\xi] \\
=& X^\xi_- \cdot Y^i + Y^i_- \cdot \bar{X}^\xi + Y^i_- \frac{1}{q^\xi} \cdot [X^\xi, \bar{q}^\xi] + Y^i_- \frac{1}{(q^\xi)^2} \cdot [[q^\xi], X^\xi] \\
&- Y^i_- \frac{1}{q^\xi} \cdot [(\Delta q^u_{\widetilde{R}^u-} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi}, X^\xi] - Y^i_- \cdot (\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi} - \frac{Y^i_-}{q^\xi_-} \cdot [\bar{q}^\xi, X^\xi].
\end{aligned}$$

We continue as

$$\begin{aligned}
Y^i X^\xi =& X^\xi_- \centerdot Y^i + Y^i_- \centerdot \bar{X}^\xi - \sum \frac{Y^i_- \Delta q^\xi}{q^\xi q^\xi_-} \Delta X^\xi \Delta \bar{q}^\xi + \sum \frac{Y^i_-}{(q^\xi)^2} (\Delta q^\xi)^2 \Delta X^\xi \\
&+ \sum \frac{Y^i_- q^\xi_-}{q^\xi} {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi - Y^i_- \centerdot \left(\Delta X^u_{\widetilde{R}^u} 1\!\!1_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \\
=& X^\xi_- \centerdot Y^i + Y^i_- \centerdot \bar{X}^\xi + \sum Y^i_- {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi - Y^i_- \centerdot \left(\Delta X^u_{\widetilde{R}^u} 1\!\!1_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \\
=& X^\xi_- \centerdot Y^i + Y^i_- \centerdot \bar{X}^\xi + \sum Y^i_- {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi + \frac{Y^i_-}{q^\xi_-} \centerdot \left(1\!\!1_{[\![\widetilde{R}^u]\!]} \centerdot [X^u, q^u] \right)^{p,\mathbb{F}} |_{u=\xi}.
\end{aligned}$$

Finally, due to the predictable semimartingale decomposition of X^ξ, we get

$$\begin{aligned}
Y^i X^\xi =& X^\xi_- \centerdot Y^i + Y^i_- \centerdot \bar{X}^\xi + Y^i_- {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}^u]\!]} \right) \centerdot \widehat{X}^\xi + \frac{Y^i_-}{q^\xi_-} \centerdot \left(1\!\!1_{[\![\widetilde{R}^u]\!]} \centerdot ([q^u, X^u] + \langle q^u, X^u \rangle) \right)^{p,\mathbb{F}} |_{u=\xi} \\
=& X^\xi_- \centerdot Y^i + Y^i_- \centerdot \left(\bar{X}^\xi + {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}^u]\!]} \right) |_{u=\xi} \centerdot \widehat{X}^\xi \right) - Y^i_- \centerdot \left((\Delta X^u_{\widetilde{R}^u} - \frac{\Delta \langle q^u, X^u \rangle_{\widetilde{R}^u}}{q^u_{\widetilde{R}^u -}}) 1\!\!1_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}
\end{aligned}$$

which ends the proof. ∎

Let us denote by $\widetilde{R}^{u,a}$ the accessible part of the $\mathbb{F}$-stopping time $\widetilde{R}^u$. Define the process V^i as

$$V^i_t := \sum_{0 \le s \le t} {}^{p,\mathbb{F}} \left(1\!\!1_{[\![\widetilde{R}^u]\!]} \right)_s |_{u=\xi} = \left(1\!\!1_{[\![\widetilde{R}^{u,a},\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}. \tag{7.13}$$

Proposition 7.17. *Let $\widetilde{Y}^i = \mathcal{E}(-L^i - V^i)$. Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale and H be an $\mathbb{F}^{\sigma(\xi)}$-predictable process such that $H \centerdot X^\xi \ge -1$. Then, the process*

$$(1 + H \centerdot X^\xi)\ \widetilde{Y}^i + H\ \widetilde{Y}^i_- \centerdot \left(\Delta X^u_{\widetilde{R}^u} 1\!\!1_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}$$

is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale.
In particular, if $\Delta X^u_{\widetilde{R}^u} = 0$ on $\{\widetilde{R}^u < \infty\}$ $\mathbb{P} \otimes \widetilde{\eta}$-a.s., then $\widetilde{Y}^i$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale deflator for X^ξ.

Proof. By integration by parts, we get

$$(1 + H \centerdot X^\xi)\ \widetilde{Y}^i = (1 + H \centerdot X^\xi)_- \centerdot \widetilde{Y}^i + H\ \widetilde{Y}^i_- \centerdot X^\xi - H\ \widetilde{Y}^i_- \centerdot [X^\xi, L^i] - H\ \widetilde{Y}^i_- \centerdot [X^\xi, V^i].$$

Note that

$$H\widetilde{Y}^i_- \centerdot [X^\xi, V^i] = \sum H\widetilde{Y}^i_- {}^{p,\mathbb{F}}(1\!\!1_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi.$$

Then, using the same arguments as in the proof of Theorem 7.16 we get

$$(1 + H \centerdot X^\xi)\widetilde{Y}^i = (1 + H \centerdot X^\xi)_- \centerdot \widetilde{Y}^i + H\ \widetilde{Y}^i_- \centerdot \bar{X}^\xi - H\ \widetilde{Y}^i_- \left(\Delta X^u_{\widetilde{R}^u} 1\!\!1_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}$$

and the assertion is proved. ∎

Theorem 7.18. *The following conditions are equivalent.*
(a) The thin set $\{q^u = 0 < q^u_-\}$ *is evanescent for* $\widetilde{\eta}$*-a.e.*
(b) The $\mathbb{F}$*-stopping time* $\widetilde{R}^u$ *is infinite* $\mathbb{P} \otimes \widetilde{\eta}$*-a.s.*
(c) For any parametrized $\mathbb{F}$*-local martingale* $(X^u, u \in \mathbb{U})$*, the process* X^ξ *admits an* $\mathbb{F}^{\sigma(\xi)}$*-local martingale deflator* $\frac{1}{q^\xi}$ *(and satisfies NUPBR*$(\mathbb{F}^{\sigma(\xi)})$*).*

Proof. The equivalence between (a) and (b) is obvious from the definition of $\widetilde{R}^u$.
The implication (b) $\Rightarrow$(c) follows from Theorem 7.16. To prove (c)$\Rightarrow$(b), we consider a parametrized $\mathbb{F}$-martingale $(M^u, u \in \mathbb{U})$ with

$$M^u := 1\!\!1_{[\![\widetilde{R}^u,\infty[\![} - (1\!\!1_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}.$$

Then, due to $R^\xi = \infty$ (1.8) it is clear that

$$M^\xi = -\left(1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}$$

is $\mathbb{F}^{\sigma(\xi)}$-predictable. Thus, M^ξ satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$) if and only if it is a null process, due to Lemma 1.60. Then, we conclude that $\widetilde{R}$ is infinite using the same argument as in the proof of Lemma 7.13 (b). ∎

In the two following examples we look at two extreme situations.

Example 7.19. Let $\mathbb{F}$ be a filtration such that each $\mathbb{F}$-martingale is continuous. Then, NUPBR is preserved in an initially enlarged filtration for any parametrized $\mathbb{F}$-local martingale from the reference filtration.

Example 7.20. Let B be a $\mathcal{G}$-measurable set such that $\mathbb{P}(B) = \frac{1}{2}$ and consider the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ defined as

$$\mathcal{F}_t = \{\emptyset, \Omega\} \text{ for } t \in [0,1[\quad \text{and} \quad \mathcal{F}_t := \{\emptyset, B, B^c, \Omega\} \text{ for } t \in [1,\infty[.$$

Define a random variable ξ as $\xi := 1\!\!1_B + 2 \cdot 1\!\!1_{B^c}$. The random variable ξ satisfies Jacod's hypothesis with density $(q^u, u \in \mathbb{U})$ equal to

$$q^1 = 1\!\!1_{[\![0,1[\![} + 2 \cdot 1\!\!1_{\{\xi=1\}} 1\!\!1_{[\![1,\infty[\![},$$
$$q^2 = 1\!\!1_{[\![0,1[\![} + 2 \cdot 1\!\!1_{\{\xi=2\}} 1\!\!1_{[\![1,\infty[\![}.$$

Let the filtration $\mathbb{F}^{\sigma(\xi)} = (\mathcal{F}^{\sigma(\xi)}_t)_{t\geq 0}$ be an initial enlargement of the filtration $\mathbb{F}$ with ξ, i.e.,

$$\mathcal{F}^{\sigma(\xi)}_t := \{\emptyset, B, B^c, \Omega\} \quad \text{for} \quad t \in [0,\infty[.$$

Let X be an $\mathbb{F}$-martingale defined as

$$X := \left(1\!\!1_{\{\xi=1\}} - \frac{1}{2}\right) 1\!\!1_{[\![1,\infty[\![}.$$

Then, X is an $\mathbb{F}^{\sigma(\xi)}$-predictable process. Thus, by Proposition 1.60 it does not satisfy NUPBR($\mathbb{F}^{\sigma(\xi)}$).

7.4 Thin random times

In this section, we focus our attention again on the progressively enlarged filtration $\mathbb{F}^\tau$. We assume additionally that τ is a thin random time with exhausting sequence $(T_n)_n$ (Definition 2.1). We associate to τ the partition $(C_n)_n$ and the family of $\mathbb{F}$-martingales $(z^n)_n$ as in (2.1) and (2.2). Then we study stability of NUPBR after the thin random time τ. We use results from Section 7.3.2 for the particular random variable $\xi := \sum_n n \mathbb{1}_{C_n}$ which satisfies Jacod's hypothesis (Example 1.35). Note that the following filtrations inclusion $\mathbb{F}^\tau \subset \mathbb{F}^{\sigma(\xi)}$ holds as $\mathbb{F}^{\sigma(\xi)}$ is the initial enlargement of $\mathbb{F}$ with an atomic σ-field $\mathcal{C} := \sigma(C_n, n \geq 0)$.

In Proposition 7.21 we give an $\mathbb{F}^\tau$-local martingale deflator and an $\mathbb{F}^\tau$-supermartingale deflator for $\mathbb{F}$-local martingales after τ. Next in Theorem 7.22 we provide a necessary and sufficient condition such that NUPBR($\mathbb{F}^\tau$) holds after τ for any process satisfying NUPBR($\mathbb{F}$).

Proposition 7.21. *Let τ be a thin random time. Assume that*

$$\text{for each } n, \text{ the set } \quad \{\Delta X \neq 0\} \cap \{z^n = 0 < z^n_-\} \cap \,]\!]T_n, \infty[\![\quad \text{is evanescent.} \tag{7.14}$$

Then
(a) for each $\mathbb{F}$-quasi-left continuous local martingale X, the process

$$Y^{\text{th}} := \prod_n \mathcal{E}\left(-\mathbb{1}_{C_n} \mathbb{1}_{]\!]T_n,\infty[\![} \frac{1}{z^n_-} \cdot \bar{z}^n \right)$$

is an $\mathbb{F}^\tau$-local martingale deflator for $\mathbb{1}_{]\!]\tau,\infty[\![} \cdot X$; here "th" stands for thin;
(b) for each $\mathbb{F}$-local martingale X, the process

$$\widetilde{Y}^{\text{th}} := \prod_n \mathcal{E}\left(-\mathbb{1}_{C_n} \mathbb{1}_{]\!]T_n,\infty[\![} \cdot \left(\frac{1}{z^n_-} \cdot \bar{z}^n + \left(\mathbb{1}_{[\![\widetilde{R}^{n,a},\infty[\![} \right)^{p,\mathbb{F}} \right) \right)$$

is an $\mathbb{F}^\tau$-supermartingale deflator for $\mathbb{1}_{]\!]\tau,\infty[\![} \cdot X$.

Proof. (a) The $\mathbb{F}^{\sigma(\xi)}$-local martingale L^i defined in (7.10) takes the form

$$L^i = \sum_n \mathbb{1}_{C_n} \frac{1}{z^n_-} \cdot \bar{z}^n.$$

Take a parametrized $\mathbb{F}$-quasi-left continuous local martingale $(X^n, n \in \mathbb{N})$ with $X^n = \mathbb{1}_{]\!]T_n,\infty[\![} \cdot X$. Then, by Proposition 7.16 and assumption (7.14), $Y^i = \mathcal{E}(-L^i)$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator for $X^\xi = \mathbb{1}_{]\!]\tau,\infty[\![} \cdot X$. The $\mathbb{F}^{\sigma(\xi)}$-local martingale $Y^i X^\xi$ satisfies $Y^i X^\xi = \mathbb{1}_{]\!]\tau,\infty[\![} \cdot (Y^i X^\xi)$ thus, by Proposition 2.11, it is an $\mathbb{F}^\tau$-local martingale. Similarly, by Proposition 2.11, the process

$$Y^{\text{th}} := \mathcal{E}(-\mathbb{1}_{]\!]\tau,\infty[\![} \cdot L^i) = \prod_n \mathcal{E}\left(-\mathbb{1}_{C_n} \mathbb{1}_{]\!]T_n,\infty[\![} \frac{1}{z^n_-} \cdot \bar{z}^n \right)$$

is an $\mathbb{F}^\tau$-local martingale. Note that $Y^{\text{th}} X^\xi = (\mathcal{E}(-\mathbb{1}_{[\![0,\tau]\!]} \cdot L^i))^{-1} Y^i X^\xi$ and $\mathcal{E}(-\mathbb{1}_{[\![0,\tau]\!]} \cdot L^i)_\tau > 0$ and $\mathcal{E}(-\mathbb{1}_{[\![0,\tau]\!]} \cdot L^i)_\tau \in \mathcal{F}^\tau_\tau$ so $Y^{\text{th}} X^\xi$ is an $\mathbb{F}^\tau$-local martingale. Finally, we conclude that Y^{th} is an $\mathbb{F}^\tau$-local martingale deflator for $\mathbb{1}_{]\!]\tau,\infty[\![} \cdot X$.

(b) Note that $\widetilde{Y}^{\text{th}} = \mathcal{E}(-1\!\!1_{]\!]\tau,\infty[\![} \centerdot (L^i + V^i))$ with V^i defined in (7.13), thus, by Proposition 2.11, it is an $\mathbb{F}^\tau$-supermartingale. Then, by the same type of computations as in the proof of Proposition 7.17 we derive that for each $\mathbb{F}^\tau$-predictable process H such that $H 1\!\!1_{]\!]\tau,\infty[\![} \centerdot X \geq -1$ the process $\widetilde{Y}^{\text{th}}(1 + H 1\!\!1_{]\!]\tau,\infty[\![} \centerdot X)$ is an $\mathbb{F}^\tau$-supermartingale and the assertion follows. ∎

Theorem 7.22. *The following conditions are equivalent.*
(a) For each n, the thin set $\left\{z^n = 0 < z^n_-\right\} \cap]\!]T_n, \infty[\![$ is evanescent.
(b) For each n, the $\mathbb{F}$-stopping time $\widetilde{R}^n = \infty$ on the set $\{\widetilde{R}^n > T_n\}$.
(c) For any $\mathbb{F}$-local martingale X, the process $1\!\!1_{]\!]\tau,\infty[\![} \centerdot X$ admits an $\mathbb{F}^\tau$-local martingale deflator Y^t (and satisfies NUPBR$(\mathbb{F}^\tau)$).
(d) For any (bounded) process X satisfying NUPBR$(\mathbb{F})$, the process $1\!\!1_{]\!]\tau,\infty[\![} \centerdot X$ satisfies NUPBR$(\mathbb{G})$.

Proof. The equivalence between (a) and (b) is obvious from the definition of $\widetilde{R}^n$.
The implication (b)⇒(c) follows from Proposition 7.21. To prove (c)⇒(b) (and (d)⇒(b)), let us fix n and consider the $\mathbb{F}$-martingale

$$X = 1\!\!1_{[\![\widetilde{R}^n,\infty[\![} - \left(1\!\!1_{[\![\widetilde{R}^n,\infty[\![}\right)^{p,\mathbb{F}}.$$

Then, since $\widetilde{R}^n = \infty$ on C_n we have that

$$1\!\!1_{]\!]\tau,\infty[\![} \centerdot X = \sum_{k\neq n} 1\!\!1_{C_k} 1\!\!1_{]\!]T_k,\infty[\![} \centerdot X - 1\!\!1_{C_n} 1\!\!1_{]\!]T_n,\infty[\![} \centerdot \left(1\!\!1_{[\![\widetilde{R}^n,\infty[\![}\right)^{p,\mathbb{F}}$$

and from the assumption it satisfies NUPBR$(\mathbb{F}^\tau)$. Take the $\mathbb{F}^\tau$-stopping time $\nu := (T_n)_{C_n}$ (using the convention given in (1.1)). Then, clearly, the process $1\!\!1_{]\!]\tau\vee\nu,\infty[\![} \centerdot X$ satisfies NUPBR$(\mathbb{F}^\tau)$. Note that

$$1\!\!1_{]\!]\tau\vee\nu,\infty[\![} \centerdot X = -1\!\!1_{C_n} 1\!\!1_{]\!]T_n,\infty[\![} \centerdot \left(1\!\!1_{[\![\widetilde{R}^n,\infty[\![}\right)^{p,\mathbb{F}}$$

is an $\mathbb{F}^\tau$-predictable decreasing process, thus it satisfies NUPBR$(\mathbb{F}^\tau)$ if and only if it is a null process, due to Lemma 1.60. The last assertion is equivalent to

$$\begin{aligned}
0 &= \mathbb{E}\left(1\!\!1_{C_n} 1\!\!1_{]\!]T_n,\infty[\![} \centerdot \left(1\!\!1_{[\![\widetilde{R}^n,\infty[\![}\right)^{p,\mathbb{F}}\right)\\
&= \mathbb{E}\left(\int_0^\infty z^n_{s-} 1\!\!1_{\{s>T_n\}} d\left(1\!\!1_{[\![\widetilde{R}^n,\infty[\![}\right)^{p,\mathbb{F}}_s\right)\\
&= \mathbb{E}\left(z^n_{\widetilde{R}^n-} 1\!\!1_{\{T_n<\widetilde{R}<\infty\}}\right),
\end{aligned}$$

which in turn is equivalent to $\widetilde{R}^n = \infty$ on $\{\widetilde{R}^n > T_n\}$ since $z^n_{\widetilde{R}^n-} > 0$ on $\{\widetilde{R}^n < \infty\}$.
(c) ⇒ (d) Let X be an $\mathbb{F}$-semimartingale satisfying NUPBR$(\mathbb{F})$. Thanks to Proposition 1.56, we deduce the existence of a real-valued $\mathbb{F}$-predictable process ϕ and a positive $\mathbb{F}$-local martingale l such that

$$0 < \phi \leq 1 \quad \text{and} \quad l\,(\phi \centerdot X) \quad \text{is a } \mathbb{F}\text{-local martingale.}$$

Then there exists a sequence of $\mathbb{F}$-stopping times $(v_n)_n$ that increases to infinity such that the stopped process l^{v_n} is a martingale. Put $\mathbb{Q}_n := l_{v_n} \centerdot \mathbb{P} \sim \mathbb{P}$. Then, by applying (c) to $(\phi \centerdot X)^{v_n}$ under $\mathbb{Q}_n$, we conclude that $1\!\!1_{]\!]\tau,\infty[\![} \centerdot (\phi \centerdot X)^{v_n}$ satisfies the NUPBR($\mathbb{F}^\tau$) under $\mathbb{Q}_n$. Thanks to Proposition 1.59, the NUPBR($\mathbb{F}^\tau$) under $\mathbb{P}$ of $1\!\!1_{]\!]\tau,\infty[\![} \centerdot X$ immediately follows. ■

In the next lemma we translate the condition (7.14) and the condition (a) in Theorem 7.22 into another form for the case of thin honest time. To this end we use Lemma 2.17. Then we easily see that Proposition 7.21 and Theorem 7.22 are one of the results from [CADJ14] after thin honest times. Note that honest times studied in [CADJ14], i.e., honest times satisfying the condition $Z_\tau < 1$ a.s, are exactly thin honest times (see Lemma 2.16).

Lemma 7.23. *Let τ be a thin honest time. Then,*
(a) for each n, the set $]\!]T_n,\infty[\![\, \cap \, \{z^n = 0 < z^n_-\}$ is evanescent if and only if the set $\{\widetilde{Z} = 1 > Z_-\}$ is evanescent;
(b) for each n, the set $\{\Delta X \neq 0\} \cap \,]\!]T_n,\infty[\![\, \cap \, \{z^n = 0 < z^n_-\}$ is evanescent if and only if the set $\{\Delta X \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\}$ is evanescent.

Proof. (a) The set $\{\widetilde{Z} = 1 > Z_-\}$ is evanescent if and only if

$$\mathbb{P}\left(\exists\, t \;\text{ such that }\; 1 - \widetilde{Z}_t = 0 < 1 - Z_{t-}\right) = 0.$$

By Lemma 2.17 (b) and the fact that $(T_n)_{n\geq 1}$ have disjoint graphs, it is equivalent to

$$\mathbb{P}\left(\exists\, t\; \exists\, n \;\text{ such that }\; \tau_t = T_n < t \;\text{ and }\; 1 - \widetilde{Z}_t = 0 < 1 - Z_{t-}\right) = 0.$$

The last condition is satisfied if and only if for each n we have

$$\begin{aligned} 0 &= \mathbb{P}\left(\exists\, t \;\text{ such that }\; \tau_t = T_n < t \;\text{ and }\; 1 - \widetilde{Z}_t = 0 < 1 - Z_{t-}\right) \\ &= \mathbb{P}\left(\exists\, t \;\text{ such that }\; T_n < t \;\text{ and }\; z^n_t = 0 < z^n_{t-}\right), \end{aligned}$$

where the second equality comes from Lemma 2.17 (b). That is exactly the condition that for each n the set $]\!]T_n,\infty[\![\, \cap \, \{z^n = 0 < z^n_-\}$ is evanescent.
Assertion (b) follows by the same type of argumentation. ■

7.5 Connection to absolutely continuous change of measure

In this section we look at the relationship between new optional semimartingale decompositions in progressive and initial enlargement of filtration cases and the optional semimartingale decomposition in absolutely continuous change of measure set-up. First let us recall [Pro04, Theorem 42, Chapter III].

Theorem 7.24. *Let X be a $\mathbb{P}$-local martingale with $X_0 = 0$. Let $\mathbb{Q}$ be a probability measure absolutely continuous with respect to $\mathbb{P}$, and let $\zeta_t := \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t)$. Let $r := \inf\{t > 0 : \zeta_t = 0\}$ and $\widetilde{r} := r_{\{\zeta_{r-}>0\}}$. Then*

$$\bar{X} := X - \frac{1}{\zeta} \centerdot [X,\zeta] + \left(\Delta X_{\widetilde{r}} 1\!\!1_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}} \tag{7.15}$$

is a $\mathbb{Q}$-local martingale.

It is clear that Theorem 7.24 implies the same type of decompositions as the two decompositions stated in Sections 7.2.1 and 7.3.1.

Up to random time τ:	$X^\tau = \bar{X} + \frac{1}{N^\tau} \centerdot [X^\tau, N^\tau] - \left(\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\centerdot\wedge\tau}$
Under Jacod's hypothesis:	$X^\xi = \bar{X} + \frac{1}{q^\xi} \centerdot [X^\xi, q^\xi] - \left(\Delta X^u_{\widetilde{R}^u} 1\!\!1_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} \|_{u=\xi}$
Under measure $\mathbb{Q}$:	$X^{\mathbb{Q}} = \bar{X} + \frac{1}{\zeta^{\mathbb{Q}}} \centerdot [X^{\mathbb{Q}}, \zeta^{\mathbb{Q}}] - \left(\left(\Delta X_{\widetilde{r}} 1\!\!1_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}\right)^{\mathbb{Q}}$

In the table above, the superscript $^{\mathbb{Q}}$ means that we consider the given process under the measure $\mathbb{Q}$. In each of the three cases, there is a different mechanism to ensure the strict positivity of N^τ, q^ξ and $\zeta^{\mathbb{Q}}$. In the case of progressive enlargement up to a random time, we stop at τ. In the case of initial enlargement with random variable satisfying Jacod's hypothesis, we plug ξ. In the case of absolutely continuous change of measure, we look through measure $\mathbb{Q}$.

The optional decomposition in the change of measure case can be used in the same way to obtain similar result on stability of NUPBR condition with respect to absolutely continuous change of measure.

Let us define $\mathbb{Q}$-local martingale L^a, where $\bar{\zeta}$ given by (7.15), by

$$L^a := \frac{1}{\zeta} \centerdot \bar{\zeta}.$$

Proposition 7.25. *Let $Y^a := \mathcal{E}\left(-L^a\right)$. Then, for any $\mathbb{P}$-local martingale X, the process*

$$Y^a X - \sum Y^a_- \, {}^{p,\mathbb{P}}(1\!\!1_{[\![\widetilde{r}]\!]}) \Delta X + Y^a_- \left(\Delta X_{\widetilde{r}} 1\!\!1_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}$$

is $\mathbb{Q}$-local martingale.

In particular, if X is quasi-left continuous and $\Delta X_{\widetilde{r}} = 0$ on $\{\widetilde{r} < \infty\}$ $\mathbb{P}$-a.s., then Y^a is a $\mathbb{Q}$-local martingale deflator for X.

Proof. Using integration by parts and the optional decomposition (7.15) given in Theorem 7.24 for X and then for ζ, we obtain

$$\begin{aligned}
Y^a X =& X_- \centerdot Y^a + Y^a_- \centerdot X + [Y^a, X] \\
=& X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + Y^a_- \frac{1}{\zeta} \centerdot [\zeta, X] - Y^a_- \centerdot (\Delta X_{\widetilde{r}} 1\!\!1_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}} - Y^a_- \centerdot [L^a, X] \\
=& X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + Y^a_- \frac{1}{\zeta} \centerdot [\bar{\zeta}, X] + Y^a_- \frac{1}{\zeta^2} [[\zeta], X] \\
&- Y^a_- \frac{1}{\zeta} \centerdot [(\Delta \zeta_{\widetilde{r}} 1\!\!1_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}}, X] - Y^a_- \centerdot (\Delta X_{\widetilde{r}} 1\!\!1_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}} - \frac{Y^a_-}{\zeta_-} [\bar{\zeta}, X].
\end{aligned}$$

We continue as

$$\begin{aligned}
Y^a X =& X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} - \sum \frac{Y^a_- \Delta\zeta}{\zeta\zeta_-}\Delta X \Delta\bar{\zeta} + \sum \frac{Y^a_-}{\zeta^2}(\Delta\zeta)^2 \Delta X \\
&+ \sum \frac{Y^a_- \zeta_-}{\zeta}\, {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]})\Delta X - Y^a_- \centerdot (\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}} \\
=& X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + \sum Y^a_- \, {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]})\Delta X - Y^a_- \centerdot \left(\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}} \qquad (7.16)\\
=& X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + \sum Y^a_- \, {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]})\Delta X + \frac{Y^a_-}{\zeta_-} \centerdot \left(\mathbb{1}_{[\![\widetilde{r}]\!]} \centerdot [X,\zeta]\right)^{p,\mathbb{P}}
\end{aligned}$$

which ends the proof of the first assertion. Since for any $\mathbb{P}$-quasi-left continuous martingale X, the process ${}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]})\Delta X$ is null, the particular case follows. ∎

Let us denote by $\widetilde{r}^a$ the accessible part of the stopping time $\widetilde{r}$, we set

$$V^a_t := \left(\mathbb{1}_{[\![\widetilde{r}^a,\infty[\![}\right)^{p,\mathbb{P}}$$

Using the process V^a we study, in the next lemma, the behaviour of a particular $\mathbb{Q}$-supermartingale.

Proposition 7.26. *Let $\widetilde{Y}^a := \mathcal{E}(-L^a - V^a)$. Let X be a $\mathbb{P}$-local martingale and H be a predictable process such that $H \centerdot X \geq -1$. Then, the process*

$$(1 + H \centerdot X)\ \widetilde{Y}^a + H\ \widetilde{Y}^a_- \centerdot \left(\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}$$

is a $\mathbb{Q}$-supermartingale.
In particular, if $\Delta X_{\widetilde{r}} = 0$ on $\{\widetilde{r} < \infty\}$ $\mathbb{P}$-a.s., then $\widetilde{Y}^a$ is a $\mathbb{Q}$-supermartingale deflator for X.

Proof. By integration by parts, we get

$$(1 + H \centerdot X)\ \widetilde{Y}^a = (1 + H \centerdot X)_- \centerdot \widetilde{Y}^a + H\ \widetilde{Y}^a_- \centerdot X - H\ \widetilde{Y}^a_- \centerdot [X, L^a] - H\ \widetilde{Y}^a_- \centerdot [X, V^a].$$

Note that

$$H\widetilde{Y}^a_- \centerdot [X, V^a] = \sum H\ \widetilde{Y}^a_- \ {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]})\Delta X.$$

Then, using the same arguments as in the proof of Theorem 7.25 to derive (7.16) we get

$$(1 + H \centerdot X)\widetilde{Y}^a = (1 + H \centerdot X)_- \centerdot \widetilde{Y}^a + H\widetilde{Y}^a_- \centerdot \bar{X} - H\widetilde{Y}^a_- \centerdot \left(\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{S},\infty[\![}\right)^{p,\mathbb{P}}$$

and the assertion is proved. ∎

[Fon13, Theorem 5.3] and [Fon13, Proposition 5.7] are recovered with alternative proof in the next result.

Theorem 7.27. *The following conditions are equivalent.*
(a) The thin set $\{\zeta = 0 < \zeta_-\}$ is $\mathbb{P}$-evanescent.
(b) The stopping time $\widetilde{r}$ is infinite $\mathbb{P}$-a.s.
(c) Any $\mathbb{P}$-local martingale X admits Y^a as a $\mathbb{Q}$-local martingale deflator, so X satisfies NUPBR($\mathbb{Q}$).
(d) Any (bounded) process X satisfying NUPBR($\mathbb{P}$) satisfies NUPBR($\mathbb{Q}$).

Proof. The equivalence between (a) and (b) is obvious from the definition of $\widetilde{r}$.
The implication (b)⇒(c) follows from Proposition 7.25. To prove (c)⇒(b) (and (d)⇒(b)), we consider the $\mathbb{P}$-martingale

$$X = 1\!\!1_{[\![\widetilde{r},\infty[\![} - \left(1\!\!1_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}.$$

Then, due to $\widetilde{r} = \infty$ $\mathbb{Q}$-a.s. we have that, under $\mathbb{Q}$,

$$X = -\left(1\!\!1_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}$$

is a predictable decreasing process. Thus, X satisfies NUPBR($\mathbb{Q}$) if and only if it is a null process, due to Lemma 1.60. Then, we conclude that $\widetilde{S}$ is infinite using the same argument as in the proof of Lemma 7.5 (b).
(c) ⇒ (d) Let X be an $\mathbb{P}$-semimartingale satisfying NUPBR($\mathbb{P}$). Thanks to Proposition 1.56, we deduce the existence of a real-valued predictable process ϕ and a positive $\mathbb{P}$-local martingale l such that

$$0 < \phi \leq 1 \quad \text{ and } \quad l\,(\phi \centerdot X) \quad \text{ is a } \mathbb{P}\text{-local martingale.}$$

Then there exists a sequence of stopping times $(v_n)_n$ that increases to infinity and such that the stopped process l^{v_n} is a $\mathbb{P}$-martingale. Put $\mathbb{P}_n := l_{v_n} \centerdot \mathbb{P} \sim \mathbb{P}$ and

$$\mathbb{Q}_n := \frac{l_{v_n}}{\mathbb{E}_{\mathbb{P}}(\zeta_{v_n} l_{v_n})} \centerdot \mathbb{Q} = \frac{\zeta_{v_n}}{\mathbb{E}_{\mathbb{P}}(\zeta_{v_n} l_{v_n})} \centerdot \mathbb{P}_n \ll \mathbb{P}_n.$$

Define $\zeta^n_t := \mathbb{E}_{\mathbb{P}_n}\left(\frac{\zeta_{v_n}}{\mathbb{E}_{\mathbb{P}_n}(\zeta_{v_n})} | \mathcal{F}_t\right)$ and note that the condition that $\{\zeta = 0 < \zeta_-\}$ is $\mathbb{P}$-evanescent implies that $\left\{\zeta^n = 0 < \zeta^n_-\right\}$ is $\mathbb{P}_n$-evanescent. Then, by applying (c) to $(\phi \centerdot X)^{v_n}$ under $\mathbb{P}_n$, we conclude that $(\phi \centerdot X)^{v_n}$ satisfies NUPBR($\mathbb{Q}_n$). Thanks to Proposition 1.59 since $\mathbb{Q}_n \sim \mathbb{Q}$, the NUPBR($\mathbb{Q}$) property of X immediately follows. ∎

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