



**HAL**  
open science

# Variational Design of Rational Bezier Curves and Surfaces

Georges-Pierre Bonneau

► **To cite this version:**

Georges-Pierre Bonneau. Variational Design of Rational Bezier Curves and Surfaces. Graphics [cs.GR]. University of Kaiserslautern, Germany, 1993. English. NNT: . tel-01064604

**HAL Id: tel-01064604**

**<https://theses.hal.science/tel-01064604v1>**

Submitted on 16 Sep 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Variational Design of Rational Bézier Curves and Surfaces

Vom Fachbereich Informatik der Universität Kaiserslautern  
zur Verleihung des akademischen Grades  
Doktor der Naturwissenschaften (Dr. rer. nat.)  
genehmigte  
Dissertation  
von  
Dipl. Math. Georges-Pierre Bonneau

Gutachter: Prof. Dr. Hans Hagen  
Prof. Dr. Gerald Farin  
Vorsitzender der Promotionskommission: Prof. Dr. Otto Mayer  
Dekan des Fachbereichs Informatik: Prof. Dr. Otto Mayer

Tag der wissenschaftlichen Aussprache: 2. Juli 1993

# Variational Design of Rational Bézier Curves and Surfaces

Vom Fachbereich Informatik der Universität Kaiserslautern  
zur Verleihung des akademischen Grades  
Doktor der Naturwissenschaften (Dr. rer. nat.)  
genehmigte  
Dissertation  
von  
Dipl. Math. Georges-Pierre Bonneau

Gutachter: Prof. Dr. Hans Hagen  
Prof. Dr. Gerald Farin  
Vorsitzender der Promotionskommission: Prof. Dr. Otto Mayer  
Dekan des Fachbereichs Informatik: Prof. Dr. Otto Mayer

Tag der wissenschaftlichen Aussprache: 2. Juli 1993

A mes parents

## Abstract

The design of curves and surfaces in C.A.D. systems has many applications in car, plane or ship industry. Because they offer more flexibility, rational functions are often preferred to polynomial functions to modelize curves and surfaces.

In this work, several methods to generate rational Bézier curves and surfaces which minimize some functionals are proposed. The functionals measure a technical smoothness of the curves and surfaces, and are related to the energy of beams and plates in the sense of the elasticity theory.

# Contents

<b>0 Introduction</b> .....	1
<b>1 Fundamentals</b> .....	5
1.1 Notations .....	5
1.2 Differential analysis .....	6
1.3 Differential geometry of curves .....	6
1.4 Differential geometry of surfaces .....	10
1.5 Bézier curves and surfaces .....	12
1.6 Strain energy of a surface .....	18
<b>2 Variational Design of Rational Bézier Curves based upon Local Minimization of the Bending Energy</b> .....	22
2.1 Restriction of the problem .....	22
2.2 Calculus of the local bending energy .....	24
2.3 Minimization of the criterion function in the cubic case .....	27
<b>3 Stiffness Degree Concept</b> .....	32
3.1 A local criterion .....	32
3.1.1 Derivatives of a rational Bézier curve .....	32
3.1.2 The criterion .....	33
3.2 Minimization in the endpoints .....	34
3.2.1 The resolving system of equations .....	34
3.2.2 Examples .....	41
3.3 Minimization in any two parameter values .....	45
3.3.1 Reduction of the degree of the resolving system .....	45
3.3.2 Implementation .....	49
3.3.3 Examples .....	50
<b>4 Rational Tensor Product Bézier Patches with Twist of Minimum Energy</b> 54	
4.1 An optimal normal component of the twist vector .....	54
4.2 Minimum energy twist weights .....	55
4.2.1 The first and second fundamental form in the corners of a rational tensor product Bézier patch .....	56
4.2.2 The weights for the optimal normal component of the twist vectors ..	57
4.2.3 Example .....	58

<b>5 Rational Tensor Product Bézier Patches with Locally Minimum Norm of Derivatives</b> .....	60
5.1 Standard reparametrization for rational tensor product Bézier surfaces .....	61
5.2 The resolving system .....	64
5.2.1 The derivatives of a rational tensor product Bézier patch in the corners .....	64
5.2.2 Degree of the resolving system .....	65
5.3 Other criterions .....	66
<b>6 Stiffness Degree for Rational Triangular Bézier Patches</b> .....	69
6.1 Standard reparametrization for rational triangular Bézier patches .....	69
6.2 The resolving system .....	73
6.2.1 The derivatives of a rational triangular Bézier patch in the corners ...	73
6.2.2 The degree of the resolving system .....	75
<b>7 References</b> .....	77

## 0 Introduction

CAD systems deal not only with simple geometric forms like cubes, cylinders or conics. They also have to modelize complex curves or surfaces, such as a car hood, or plane's wings.

The purpose of Computer Aided Geometric Design (CAGD) is to define some mathematical modelization of such curves and surfaces, to study their properties, and to improve their quality.

In this introduction, an overview of the main mathematical models is given, and the curves and surfaces used in this work are introduced. In a second part, the historical background is presented. This enables to point out the innovation of this work. In a third part, the content of the chapters is described. Eventually, an essential result used in several chapters ends this introduction.

In the middle 60's, S. Coons develop at the M.I.T. the so-called Coons patches [COO67]. These are rectangular patches, which interpolate a mesh of curves of any kind. They were included in the first CAD system : SKETCHPAD, written by A. Sutherland also at the M.I.T.

Slightly later, Paul Bézier develop the Bézier curves and surfaces, for the CAD system UNISURF at Renault. These curves and surfaces have the advantage for the designer to be controlled by a small number of points.

The B-splines appear in the 70's in CAGD. with the works of Cox [COX71] and de Boor [DEB72]. They contain the Bézier schemes as a special case. But while moving a control point in the Bézier case modify the entire curve or surface, it only induce a local modification in the B-spline case.

Coons, Bézier and B-spline surfaces consist of rectangular patches. The rectangular topology is naturally related to the cartesian system of coordinates, and is sufficient in almost all cases. Yet it is not the best appropriate in some critical cases. Although



triangular patches are less natural than rectangular one, they are sometimes necessary. Triangular Bézier patches are introduced in their actual form in the field of CAGD by G. Farin.

The wish to give always more freedom to the designer has resulted in the generalization of the polynomial schemes with the use of rational functions.

Although these functions are first introduced by S. Coons in CAGD, their practical use begins later in the 70's, with the success of the rational B-splines, (also called NURBS, for non uniform rational B-splines), which are becoming a standard in CAD systems.

The flexibility of these curves and surfaces is achieved through the adjunction of a scalar - called weight - to each control point. If a weight increases while the others remain constant, the curve or surface is pulled in the direction of the corresponding control point.

The win of flexibility (in particular the ability to describe exactly the conics) has resulted in an increasing complexity of already known algorithms, (for ex. the evaluation of derivatives), but has also brought new algorithms (for instance the reparametrization of rational curves introduced by G. Farin [FAR88]).

The present work deals with the rational counterpart of the Bézier schemes : the rational Bézier curves (chapter 2 and 3), the rational rectangular Bézier patches (chapter 4 and 5), and the rational triangular Bézier patches (chapter 6).

In the amount of works that aim to improve the quality of the mathematical modelization of curves and surfaces, two classes may be seen : algorithms for continuity and algorithms based on the minimization of a functional.

The continuity between curves and between rectangular patches is an area of intensive research since the beginning of CAGD. Fewer articles deal with the continuity between triangular patches ([HAG86]).

The algorithms based on the minimization of a functional appear in the middle 50's. Holladay minimizes the integral  $\int \|f''(t)\|^2 dt$  to produce a  $C^1$  curve that interpolates a set of given points [HOL57]. He uses this integral as an approximation of the bending energy of a curve. The resulting curves are called spline curves.

Nielson in [NIE74] introduces the polynomial spline curves in tension. These are  $C^1$  polynomial curves that interpolate a set of points  $(x_i, y_i)$ , and minimize the integral  $\int \|f''(t)\|^2 dt + \sum_{x_i} \nu_i \|f''(t)\|^2$  ( $y = f(x)$  is the equation of the curve). The tension parameters  $\nu_i$  allow a control of the shape of the curve. This result is extended by Hagen, who minimizes the integral  $\int \|f''(t)\|^2 dt + \sum_{x_i} \tau_i \|f''(t)\|^2 + \sum_{x_i} \nu_i \|f''(t)\|^2$ . The resulting curves are called  $\tau$ -splines and are both curvature and torsion continuous.

A least square condition is sometimes more suitable than the interpolation condition. Hagen and Santarelli use the minimization of the integral  $\int \alpha \|X''(t)\|^2 + \beta \|X'''(t)\|^2 dt$  together with a least square constraint, to obtain Bézier and B-spline polynomial curves ( $X(t) = [x(t), y(t), z(t)]^T$  is a parametric equation of the curve) [HAG92a].

They extend this result to the surface case [HAG92b], and apply it in practical cases for the company HELLA.

Nowacki [NOW83] uses the minimization of the functional  $\int \kappa_1^2 + \kappa_2^2 ds$  to construct ship hull surfaces, that interpolate a mesh of curves. None of these minimization algorithms are dedicated to rational functions.

In the present work, new functionals are introduced, that can be used as minimization criteria to produce rational Bézier curves, rational rectangular patches or rational triangular patches.

The keypoint is that the only allowed variable parameters in the variational process of minimization are the weights. Doing this, the control points can be given directly by a user, and are not affected by the variational process.

The presentation is organized as follows :

In chapter 1, after the mathematical notations, a brief overview of the elasticity theory is given. This results in a physical interpretation of the functional  $\int \kappa_1^2 + \kappa_2^2 ds$ , the so called strain energy of a surface.

In all the following chapters, new results are presented.

In Chapter 2, a functional related to the bending energy of a curve is introduced, and is minimized in the case of cubic rational Bézier curves.

Two functionals derived from the integral  $\int \alpha \|X''\|^2 + \beta \|X'''\|^2$  used by Hagen and Santarelli are presented in chapter 3. Their minimization is achieved in the case of rational Bézier curves of any degrees.

Chapter 4, 5 and 6 are dedicated to rational surfaces.

In chapter 4, the strain energy is used to find rational rectangular Bézier patches with twists of minimum energy.

Another minimization criterion for the design of the same patches is the subject of Chapter 5. This criterion tends to minimize the norm of the second derivatives in all directions.

Eventually, a functional related to the second and third derivatives in the three directions of a rational triangular Bézier patch serves as a minimization criterion in chapter 6.

All the results of this work involve the evaluation of the derivatives of rational curves and surfaces. It was already said in the introduction that this evaluation is much more complicate than in the polynomial case. An important idea, used in chapter 3,5 and 6, is to find an appropriate reparametrization for which the curve (or the surface) and all its derivatives become a polynomial function in the weights, at a particular parameter point. This result is quite independent of the results of the present work, and could be used in other problems involving derivatives of rational curves and surfaces.

Mein besonderer Dank gilt meinem Lehrer Herrn Prof. Dr. Hans Hagen. Er hat es mir nicht nur ermöglicht, an seinem Institut meine Doktorarbeit anfertigen zu können, sondern stand mir stets mit vielen fruchtbaren, wissenschaftlichen Aussprachen zur Seite.

Auch bei allen Mitarbeitern der Arbeitsgruppe Hagen, Mady, Philip, Rolf, Thomas, Heiko, Frank und vor allem Stefanie, möchte ich mich bedanken für ihre Unterstützung während meiner drei Jahre Aufenthalt in Deutschland.

# 1 Fundamentals

## 1.1 Notations

- The real  $n$ -tuples are denoted by bold characters :

$$\mathbf{a} = (a_1, \dots, a_n)$$

$$\mathbf{b} = (b_1, \dots, b_n).$$

- $\mathbf{R}^n$  is the  $n$ -dimensional euclidian space of all real  $n$ -tuples, with the scalar product

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^n a_i b_i.$$

- $\| \cdot \|$  denotes the norm induced by the scalar product :  $\| \mathbf{a} \| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$ .
- $|v_1, \dots, v_n|$  denotes the determinant of the  $n$  vectors  $v_1, \dots, v_n$  of  $\mathbf{R}^n$ .
- $[\cdot, \cdot]$  denotes the cross product in  $\mathbf{R}^3$  :

$$[\mathbf{a}, \mathbf{b}] = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

- $L(\mathbf{R}^n, \mathbf{R}^m)$  denotes the set of all linear mappings from  $\mathbf{R}^n$  into  $\mathbf{R}^m$ .
- A *domain* of  $\mathbf{R}^n$  is an open connected subset of  $\mathbf{R}^n$ .
- $K_d[x_1, \dots, x_n]$  is the vector space of the polynomials in the unknowns  $x_1, \dots, x_n$ , with the monom's coefficient in the body  $K$ , and total degree lower or equal than  $d$ .

$$K_d[x_1, \dots, x_n] = \left\{ \sum_{i=0}^k \lambda_i x_1^{p_1^i} \dots x_n^{p_n^i} / k \in \mathbf{N}; (p_1^i + \dots + p_n^i \leq d; ip_j^i \geq 0; \forall i \forall j) \right\}$$

## 1.2 Differential Analysis

### Definition 1.1

Let  $U$  be an open subset of  $\mathbb{R}^n$ ,  $f$  be a continuous mapping from  $U$  into  $\mathbb{R}^m$ , and  $x_0$  be an element of  $U$ .  $f$  is called *differentiable* in  $x_0$ , if there exists a linear mapping  $df_{x_0} \in L(\mathbb{R}^n, \mathbb{R}^m)$  such that  $\lim_{\|h\| \rightarrow 0} \frac{f(x_0+h) - f(x_0) - df_{x_0}(h)}{\|h\|} = 0$ .

If  $f$  is differentiable in all  $x_0 \in U$ ,  $f$  is said to be *differentiable in  $U$* , and  $df$  is the following mapping:

$$\begin{aligned} df : U &\rightarrow L(\mathbb{R}^n, \mathbb{R}^m) \\ x_0 &\mapsto df_{x_0} \end{aligned}$$

$f$  is called a mapping of class  $C^0$  if  $f$  is continuous.

$f$  is called an *mapping of class  $C^r$*  ( $r \geq 1$ ) if and only if  $f$  is differentiable and  $df$  is a mapping of class  $C^{r-1}$ .

### Definition 1.2

Let  $U, V$  be two open subsets of  $\mathbb{R}^n$ . A mapping  $\tau : U \rightarrow V$  is called a *diffeomorphism* of class  $C^r$  ( $r \geq 1$ ) if and only if

- (i)  $\tau$  is a one-to-one mapping
- (ii)  $\tau$  and  $\tau^{-1}$  are mappings of class  $C^r$ .

### Proposition 1.3

Let  $f$  be a  $C^2$  mapping from an open subset  $U$  of  $\mathbb{R}^n$  into  $\mathbb{R}$ . Let  $x_0 \in U$ .

- (i) If  $x_0$  is a local extremum of  $f$ , then  $df_{x_0} = 0$
- (ii) If  $df_{x_0} = 0$  and  $d^2f_{x_0}$  is a positive definite matrix, then  $f$  has a local minimum in  $x_0$ .

### Remark 1.4

The equation  $df_x = 0$  represent a system of  $n$  equations in  $n$  real unknowns. In the following we call it *the resolving system* of the problem  $f \rightarrow \min$ .

## 1.3 Differential Geometry of Curves

### Definition 1.5

Let  $I$  be an open interval of  $\mathbb{R}$ . A *parametric curve* in  $\mathbb{R}^n$  is a mapping  $X$  from  $I$  into  $\mathbb{R}^n$  of class  $C^r$  ( $r \geq 1$ ).

$X$  is called *regular*, if  $\frac{dX}{dt}(t) \neq 0$  for all  $t \in I$ .

$t$  is called the *parameter value* of the point  $X(t)$ .

$I$  is called the *parameter interval* of the curve  $X$ .

**Remark 1.6**

In the following, we only deal with Bézier curves. These curves are defined on bounded intervals. Therefore, the parameter interval can be extended to its closure, and become a closed interval of  $\mathbf{R}$ .

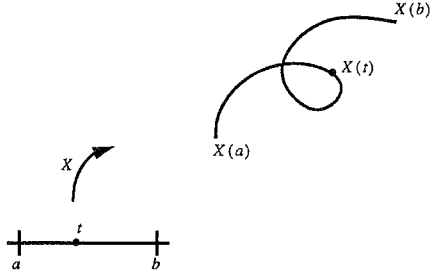


Fig. 1

**Definition 1.7**

If  $J$  and  $I$  are two open intervals of  $\mathbf{R}$ , then the diffeomorphisms  $\Phi$  from  $J$  into  $I$ , of class  $C^r$ , are called *parameter transformations of curves* of class  $C^r$ .

If  $X : I \rightarrow \mathbf{R}^n$  is a parametric curve of class  $C^r$  and  $\Phi : J \rightarrow I$  a parameter transformation of class  $C^r$ , then  $\bar{X} := X \circ \Phi : J \rightarrow \mathbf{R}^n$  is also a parametric curve of class  $C^r$ . Moreover,  $\bar{X}$  is regular if and only if  $X$  is also regular.

A set or property related to the curve, which is invariant under parameter transformation of the curve is called a *geometric invariant*. The image  $X(I)$  of the parametric curve  $X$  is an example of geometric invariant.

**Definition 1.8**

An arc  $X([a, b])$  of a parametric curve is called *rectifiable* if and only if the set of lengths of all interpolating polygons to  $X([a, b])$  has an upper bound. In this case, the length of the arc  $X([a, b])$  is defined to be the least upper bound of this set.

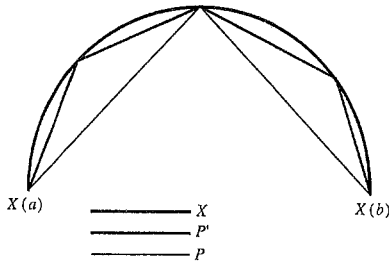


Fig. 2

**Theorem 1.9**

Let  $X : [a, b] \rightarrow \mathbb{R}^n$  be a regular parametric curve. The following holds :

- (i)  $X$  is rectifiable.
- (ii) if  $L$  is the length of  $X([a, b])$ , there exists a unique parameter transformation  $s$  from  $I$  into  $[0, L]$  such that for all  $t_0, t_1 \in [0, L]$  the length of the arc  $X([t_0, t_1])$  is equal to  $s(t_1) - s(t_0)$ .
- (iii)  $\forall t \in [a, b], s(t) = \int_a^t \|\frac{dX}{dt}\| dt$ .

$s$  is called the *arc length parametrization*.

**Remark 1.10**

The arc length parametrization is a geometric invariant of a curve and is therefore also called the *natural parametrization*.

**Definition 1.11**

A parametric curve is said to be *naturally parametrized* if and only if  $\|X'(s)\| = 1, \forall s \in ]0, L[$ .

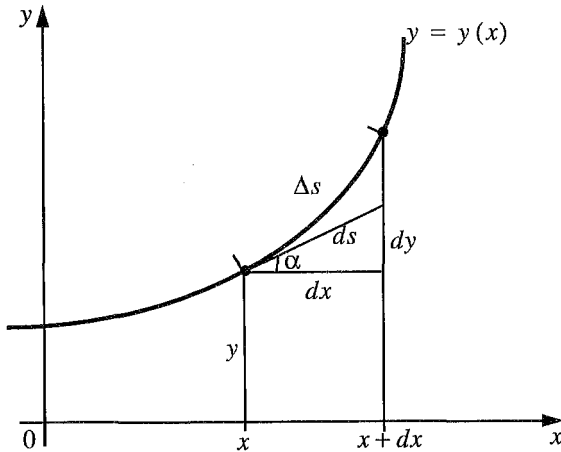


Fig. 3

In the future we will denote  $s$  the natural parameter and  $t$  the general parameter. The derivative along  $s$  will be marked with  $'$  :  $\frac{dX}{ds} = X'$  and the derivative along  $t$  with :  $\frac{dX}{dt} = \dot{X}$ .

**Definition and Theorem 1.12**

Let  $X$  be a regular and naturally parametrized curve of class  $C^3$ , in  $\mathbf{R}^3$ .

$$\begin{aligned} X &: [0, L] \rightarrow \mathbf{R}^3 \\ s &\mapsto X(s) \end{aligned}$$

such that  $\|X''(s)\| \neq 0 \forall s \in ]0, L[$ .

- $v_1(s) := X'(s)$  is called *tangent vector* of  $X$  in  $s$
- $v_2(s) := \frac{X''(s)}{\|X''(s)\|}$  is called *principal normal vector* of  $X$  in  $s$
- $v_3(s) := [v_1(s), v_2(s)]$  is called *binormal vector* of  $X$  in  $s$ .

$\{v_1(s), v_2(s), v_3(s)\}$  form an orthonormal basis of  $\mathbf{R}^3$  called *the frenet frame* of  $X$  in  $s$ .

And the following holds :

- (a)  $\{v_1, v_2, v_3\}$  are mappings of class  $C^1$
- (b) 
$$\begin{aligned} v_1' &= \kappa v_2 \\ v_2' &= -\kappa v_1 + \tau v_3 \\ v_3' &= -\tau v_2 \end{aligned}$$

where

$$\kappa(s) = \|X''(s)\| \quad \tau(s) = \frac{|X'(s), X''(s), X'''(s)|}{\|X''(s)\|} \tag{1.1}$$

$\kappa, \tau$  are mappings of class  $C^1$  and  $C^0$  respectively called *curvature* and *torsion* of  $X$ .

- (c) If  $X(t)$  is a general parametrization of  $X$  then

$$\kappa(t) = \frac{\|[\dot{X}, \ddot{X}]\|}{\|\dot{X}\|^3} \quad \tau(t) = \frac{|\dot{X}, \ddot{X}, X|}{\|[\dot{X}, \ddot{X}]\|^2} \tag{1.2}$$

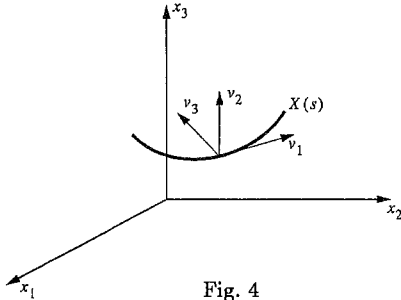


Fig. 4

**Fundamental Theorem 1.13**

Let  $\kappa, \tau$  be two mappings from  $[0, L]$  into  $\mathbf{R}$  of class  $C^1$  and  $C^0$  resp., such that  $\kappa(s) > 0$  for all  $s$  in  $[0, L]$ . There exists only one naturally parametrized curve  $X : [0, L] \rightarrow \mathbf{R}^3$  such that  $\kappa$  and  $\tau$  are the curvature and the torsion of  $X$ .



### 1.4 Differential Geometry of Surfaces

**Definition 1.14**

Let  $U$  be a domain of  $\mathbb{R}^2$ . A *parametric surface* is a mapping  $X$  from  $U$  into  $\mathbb{R}^3$  of class  $C^r$  ( $r \geq 1$ ).  $X$  is called *regular* if for all  $\mathbf{u} \in U$ ,  $dX_{\mathbf{u}}$  is an invertible linear mapping. The elements  $\mathbf{u} = (u, v)$  of  $U$  are called *parameter values* of the surface.  $U$  is called the *parameter domain* of the surface. The two partial derivatives of  $X$  in  $\mathbf{u}$  are denoted by  $X_u(\mathbf{u})$  and  $X_v(\mathbf{u})$ .

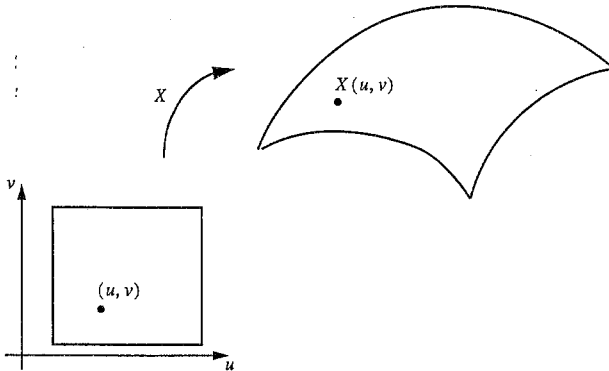


Fig. 5

**Definition 1.15**

Let  $U, V$  be two domains of  $\mathbb{R}^2$ . The diffeomorphisms from  $V$  into  $U$  of class  $C^r$  are called *parameter transformations* of surfaces of class  $C^r$ .

If  $X : U \rightarrow \mathbb{R}^3$  is a parametric surface of class  $C^r$  and  $\Phi : V \rightarrow U$  a parameter transformation of class  $C^r$ , then  $\tilde{X} := X \circ \Phi : V \rightarrow \mathbb{R}^3$  is also a parametric surface of class  $C^r$ . Moreover,  $\tilde{X}$  is regular if and only if  $X$  is also regular.

A set or property related to the surface, which is invariant under parameter transformation of the surface is called a *geometric invariant*. The image  $X(U)$  of the parametric surface  $X$  is an example of geometric invariant.

**Definition 1.16**

Let  $X$  be a regular parametric surface, with parameter domain  $U$ . Let  $\mathbf{u} \in U$ . The affine subspace  $T_{\mathbf{u}}X := \{X(\mathbf{u}) + \lambda X_u(\mathbf{u}) + \mu X_v(\mathbf{u}) \mid (\lambda, \mu) \in \mathbb{R}^2\}$  is called *tangential plane* of  $X$  in  $\mathbf{u}$ .

$N\mathbf{u} := \frac{[\mathbf{X}_u, \mathbf{X}_v]}{\|[\mathbf{X}_u, \mathbf{X}_v]\|}$  is called the *unit normal vector* of  $X$  in the point  $\mathbf{X}_u$ .  
The mapping  $N$

$$\begin{aligned} N : U &\rightarrow \mathbb{R}^3 \\ u &\mapsto N\mathbf{u} \end{aligned}$$

is called the *unit normal vector field* of the surface  $X$ .

**Definition 1.17**

The bilinear symmetric form  $I_u$ ,

$$\begin{aligned} I_u : T_u X \times T_u X &\rightarrow \mathbb{R} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

is called the *first fundamental form* of the surface  $X$ .

Its matrix in the basis  $(X_u(u), X_v(u))$  of  $T_u X$  is denoted  $G = (g_{ij})_{(i,j) \in \{1,2\}^2}$ .

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{bmatrix}$$

**Definition 1.18**

Let  $X$  be a  $C^2$  regular surface with parameter domain  $U$ . Let  $\mathbf{u} \in U$ .

The linear mapping  $L_u$ ,

$$\begin{aligned} L_u : T_u X &\rightarrow T_u X \\ \mathbf{x} &\mapsto -dN_u \circ dX_u^{-1}(\mathbf{x}) \end{aligned}$$

is called the *Weingarten map*.

The bilinear symmetric form  $II_u$  defined on  $T_u X$  by :

$$II_u(\mathbf{x}, \mathbf{y}) = \langle L_u(\mathbf{x}), \mathbf{y} \rangle,$$

is called the *second fundamental form* of the surface  $X$ .

Its matrix in the basis  $(X_u(u), X_v(u))$  of  $T_u X$  is denoted  $H = (h_{ij})_{(i,j) \in \{1,2\}^2}$ .

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \langle -N_u, \mathbf{X}_u \rangle & \langle -N_u, \mathbf{X}_v \rangle \\ \langle -N_v, \mathbf{X}_u \rangle & \langle -N_v, \mathbf{X}_v \rangle \end{bmatrix} = \begin{bmatrix} \langle N, \mathbf{X}_{uu} \rangle & \langle N, \mathbf{X}_{uv} \rangle \\ \langle N, \mathbf{X}_{vu} \rangle & \langle N, \mathbf{X}_{vv} \rangle \end{bmatrix}$$

**Remark 1.19**

The matrix  $HG^{-1}$  of the Weingarten map  $L_u$  is symmetric and real, and therefore has two real eigenvalues, with corresponding orthogonal eigenvectors.

**Definition 1.20**

The two real eigenvalues of the Weingarten map are called *principle curvatures* of the surface  $X$  in  $u$ , and are denoted  $\kappa_1, \kappa_2$ . Their corresponding directions are called *principle directions*.

**Definition 1.21**

The product of the principle curvatures  $K = \kappa_1 \cdot \kappa_2$  is called the *Gaussian curvature*, the mean sum  $M = \frac{1}{2}(\kappa_1 + \kappa_2)$  is called *the mean curvature*.

**Remark 1.22**

- $K$  and  $M$  can be calculated in term of the fundamental forms :

$$K = \frac{h}{g}$$

$$M = \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{2g},$$

with  $g = \det G$  and  $h = \det H$ .

- The quantity  $\kappa_1^2 + \kappa_2^2$ , which will be of interest later in this work, is given in term of the fundamental forms by :

$$\kappa_1^2 + \kappa_2^2 = \left( \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g} \right)^2 - \frac{2h}{g} \quad (1.3)$$

## 1.5 Bézier Curves and Surfaces

The aim of this chapter is to give the basic definitions and notations about Bézier curves and surfaces needed in this work, and not to present the whole theory of this topic. A complete presentation is given in [FAR88] and [HOS89].

**Definition 1.23**

Let  $n, i \in \mathbf{N}$ , with  $i \leq n$ .

The polynomial  $B_i^n$  defined by

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

is called  $t^{\text{th}}$  *Bernstein polynomial* of degree  $n$ .

**Remark 1.24**

- $\sum_{i=0}^n B_i^n(t) = 1$
- $B_i^n(t) \geq 0, t \in [0, 1]$ .

**Definition 1.25**

Let  $n \in \mathbf{N}$ ,  $(b_0, \dots, b_n)$  be  $n + 1$  points in  $\mathbb{R}^3$ , and  $[u_0, u_1]$  be an interval of  $\mathbf{R}$ . The parametric curve  $X$  defined by

$$X : [u_0, u_1] \rightarrow \mathbb{R}^3$$

$$u \mapsto X(u) = \sum_{i=0}^n b_i B_i^n \left( \frac{u - u_0}{u_1 - u_0} \right)$$

is called *Bézier curve* of degree  $n$ , with *control points*  $(b_0, \dots, b_n)$ .

The polygon with vertices  $(b_0, \dots, b_n)$  is called the *control polygon* of the curve  $X$ .

**Proposition 1.26**

Let  $n \in \mathbf{N}$ , and  $(b_0, \dots, b_n)$  be the control points of the Bézier curve  $X$  parametrized over  $[u_0, u_1]$ .

Let  $u \in [u_0, u_1]$ .

If  $b_i^r(u)$  is recursively defined by :

$$b_i^0(u) = b_i, \quad i = 0, \dots, n$$

$$b_i^r(u) = \left( \frac{u_1 - u}{u_1 - u_0} \right) b_i^{r-1}(u) + \left( \frac{u - u_0}{u_1 - u_0} \right) b_{i+1}^{r-1}(u), \quad 1 \leq r \leq n, \quad i = 0, \dots, n - r,$$

then  $b_0^n(u)$  is the point with parameter value  $u$  on the curve  $X$ .

**Remark 1.27**

$b_i^r(u)$  is a convex combination of the points  $b_i^{r-1}(u)$  and  $b_{i+1}^{r-1}(u)$ .

This result, known as the *de Casteljau algorithm*, is illustrated in Fig. 6.

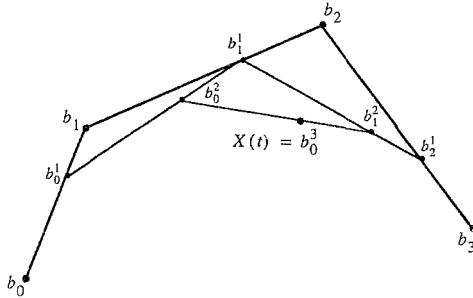


Fig. 6

**Definition 1.28**

Let  $n \in \mathbb{N}$ ,  $(b_0, \dots, b_n)$  be  $n + 1$  points in  $\mathbb{R}^3$ , and  $(\omega_0, \dots, \omega_n)$  be  $n + 1$  scalars. Let  $[u_0, u_1]$  be an interval of  $\mathbb{R}$ . The parametric curve  $X$  defined by

$$X : [u_0, u_1] \rightarrow \mathbb{R}^3$$

$$u \mapsto X(u) = \frac{\sum_{i=0}^n \omega_i b_i B_i^n \left( \frac{u-u_0}{u_1-u_0} \right)}{\sum_{i=0}^n \omega_i B_i^n \left( \frac{u-u_0}{u_1-u_0} \right)}$$

is called *rational Bézier curve* of degree  $n$ , with *control points*  $(b_0, \dots, b_n)$  and *control weights*  $(\omega_0, \dots, \omega_n)$ .

**Remark 1.29**

- For clarity purpose, the Bézier curves defined in 1.25 are often called non-rational Bézier curves.
- An advantage of the rational curves is that they can describe exactly the conic curves. These curves are used by almost all CAD-systems. A second advantage is that by changing the weights of a rational curve, one is allowed to control the shape of the curve, without having to move the control points. The geometric effect of such a change is easy to expect : if all other weights are fixed, then an increasing value of the  $i^{\text{th}}$  weight pulls the curve in the direction of the  $i$ .th control point.

**Definition 1.30**

Let  $m, n \in \mathbb{N}$ , and  $(b_{ij})_{\substack{i=0, \dots, m \\ j=0, \dots, n}}$  be  $(n+1)(m+1)$  points in  $\mathbb{R}^3$ .

let  $[u_0, u_1] \times [v_0, v_1]$  be a square in  $\mathbb{R}^2$ .

The parametric surface  $X$  defined by

$$X : [u_0, u_1] \times [v_0, v_1] \rightarrow \mathbb{R}^3$$

$$(u, v) \mapsto X(u, v) = \sum_{i=0}^m \sum_{j=0}^n b_{ij} B_i^m \left( \frac{u - u_0}{u_1 - u_0} \right) B_j^n \left( \frac{v - v_0}{v_1 - v_0} \right),$$

is called *tensor product Bézier surface* of degree  $(m, n)$ , with control points  $(b_{ij})$ .

**Remark 1.31**

These surfaces are called “tensor product”, because the basis functions  $B_i^m(s)B_j^n(t)$  are the product of the curve’s basis functions  $B_i^m(s)$  and  $B_j^n(t)$ .

More precisely, the parameter lines  $u \mapsto X(u, v)$  (resp.  $v \mapsto X(u, v)$ ) are the Bézier curves of degree  $m$  (resp.  $n$ ), with the control points  $\left( \sum_{j=0}^n b_{ij} B_j^n \left( \frac{v - v_0}{v_1 - v_0} \right) \right)_{i=0, \dots, m}$ ,

(resp.  $\left( \sum_{i=0}^m b_{ij} B_i^m \left( \frac{u - u_0}{u_1 - u_0} \right) \right)_{j=0, \dots, n}$ ), parametrized over  $[u_0, u_1]$  (resp.  $[v_0, v_1]$ ).

Thus, the de Casteljau algorithm for curves can be applied in each direction to evaluate the points of the surface, as illustrated in Fig. 7.

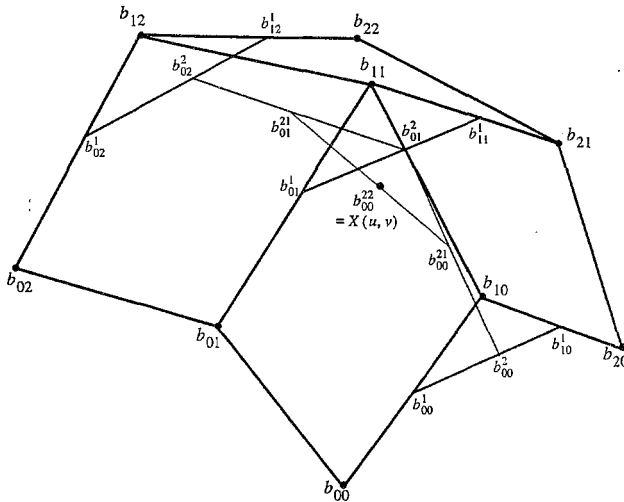


Fig. 7

**Definition 1.32**

Let  $m, n \in \mathbf{N}$ ,  $(b_{ij})_{\substack{i=0, \dots, m \\ j=0, \dots, n}}$  be  $(m+1)(n+1)$  points in  $\mathbf{R}^3$ , and  $(\omega_{ij})_{\substack{i=0, \dots, m \\ j=0, \dots, n}}$  be  $(m+1)(n+1)$  scalars.

let  $[u_0, u_1] \times [v_0, v_1]$  be a square in  $\mathbf{R}^2$ .

The parametric surface  $X$  defined by

$$X : [u_0, u_1] \times [v_0, v_1] \rightarrow \mathbf{R}^3$$

$$(u, v) \mapsto X(u, v) = \frac{\sum_{i=0}^m \sum_{j=0}^n \omega_{ij} b_{ij} B_i^m \left( \frac{u-u_0}{u_1-u_0} \right) B_j^n \left( \frac{v-v_0}{v_1-v_0} \right)}{\sum_{i=0}^m \sum_{j=0}^n \omega_{ij} B_i^m \left( \frac{u-u_0}{u_1-u_0} \right) B_j^n \left( \frac{v-v_0}{v_1-v_0} \right)},$$

is called *rational tensor product Bézier surface* of degree  $(m, n)$ , with control points  $(b_{ij})$  and control weights  $(\omega_{ij})$ .

**Remark 1.33**

Although the basis functions

$$\left( \frac{\omega_{ij} B_i^m(s) B_j^n(t)}{\sum_{k=0}^m \sum_{l=0}^n \omega_{kl} B_k^m(s) B_l^n(t)} \right)_{\substack{i=0, \dots, m \\ j=0, \dots, n}}$$

are not product of the rational Bézier curve's basis functions, the parameter lines  $(u \mapsto X(u, v))$  and  $(v \mapsto X(u, v))$  are still rational Bézier curves. This is why these rational surfaces are called "tensor product" surfaces, which is not quite correct.

**Definition 1.34**

Let  $n, i, j, k \in \mathbf{N}$ , such that  $i + j + k = n$ .

The polynoms  $B_{ijk}^n$  in the three unknowns  $u, v, w$  are defined by :

$$B_{ijk}^n(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k, \quad i + j + k = n$$

**Definition 1.35**

Let  $n \in \mathbf{N}$ ,  $(b_{ijk})_{\substack{i, j, k \geq 0 \\ i+j+k=n}}$  be  $(n+1)(n+2)/2$  points in  $\mathbf{R}^3$ , and  $\tau$  be the triangle with vertices  $d_0, d_1, d_2$  in  $\mathbf{R}^2$ .

Let  $u, v, w$  be the barycentric coordinates in the triangle  $\tau$ .

The parametric surface  $X$  defined by :

$$X : \tau \rightarrow \mathbf{R}^3$$

$$u d_0 + v d_1 + w d_2 \mapsto \sum_{\substack{i+j+k=n \\ i, j, k \geq 0}} b_{ijk} B_{ijk}^n(u, v, w)$$

is called *triangular Bézier patch* of degree  $n$ , with control points  $(b_{ijk})$ .

**Definition 1.36**

Let  $n \in \mathbb{N}$ ,  $(b_{ijk})_{\substack{i,j,k \geq 0 \\ i+j+k=n}}$  be the control points of the triangular Bézier patch  $X$ , parametrized over the triangle  $\tau$  with vertices  $d_0, d_1, d_2$ .

let  $ud_0 + vd_1 + wd_2$  be a point in  $\tau$ .

If  $b_{ijk}^r(u, v, w)$  is recursively defined by :

- $b_{ijk}^0 = b_{ijk}$  ,  $i + j + k = n$  ,  $i, j, k \geq 0$
- $b_{ijk}^r(u, v, w) = ub_{i+1,j,k}^{r-1} + vb_{i,j+1,k}^{r-1} + wb_{i,j,k+1}^{r-1}$ , where  $r = 1, \dots, n$ ;  $i + j + k = n - r$  and  $i, j, k \geq 0$ ,

then  $b_{000}^n(u, v, w)$  is the point with parameter value  $ud_0 + vd_1 + wd_2$  on the triangular Bézier patch, and is illustrated in Fig. 8.

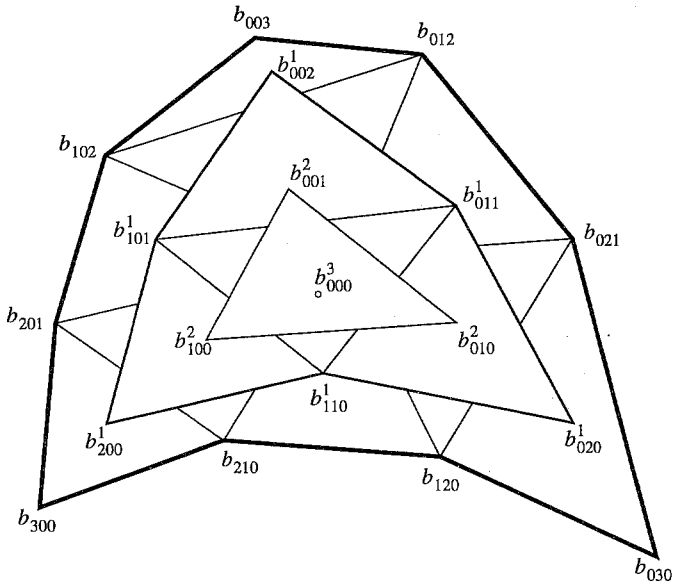


Fig. 8



**Definition 1.37**

Let  $n \in \mathbb{N}$ ,  $(b_{ijk})_{\substack{i,j,k \geq 0 \\ i+j+k=n}}$  be  $(n+1)(n+2)/2$  points in  $\mathbb{R}^3$ ,  $(\omega_{ijk})_{\substack{i,j,k \geq 0 \\ i+j+k=n}}$  be  $(n+1)(n+2)/2$  scalars and  $\tau$  be the triangle with vertices  $d_0, d_1, d_2$  in  $\mathbb{R}^2$ .

Let  $u, v, w$  be the barycentric coordinates in the triangle  $\tau$ .

The parametric surface  $X$  defined by :

$$X : \tau \rightarrow \mathbb{R}^3$$

$$ud_0 + vd_1 + wd_2 \mapsto \frac{\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \omega_{ijk} b_{ijk} B_{ijk}^n(u, v, w)}{\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \omega_{ijk} B_{ijk}^n(u, v, w)}$$

is called *rational triangular Bézier patch* of degree  $n$ , with control points  $(b_{ijk})$  and control weights  $(\omega_{ijk})$ .

## 1.6 Strain Energy of a Surface

A complete presentation of the theory of elasticity is given in [TIM34]. This chapter is based on the results of this book, and use the same notations.

Consider a thin rectangular elastic plate in equilibrium under the action of external forces.

Let  $\sigma_x$  and  $\sigma_y$  denote the *normal stresses* in the direction  $x$  and  $y$  respectively, and  $\tau_{xy}$  denotes the *shearing stress*.

Let  $u, v$  be the *displacement* components of the plate.

The *strains* in the directions  $x$  and  $y$  are respectively denoted by  $\epsilon_x$  and  $\epsilon_y$ , and the *shearing strain* by  $\gamma_{xy}$ .

The components of strain are related to the displacement coordinates by :

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Fig. 9 illustrates the meaning of the strain components.

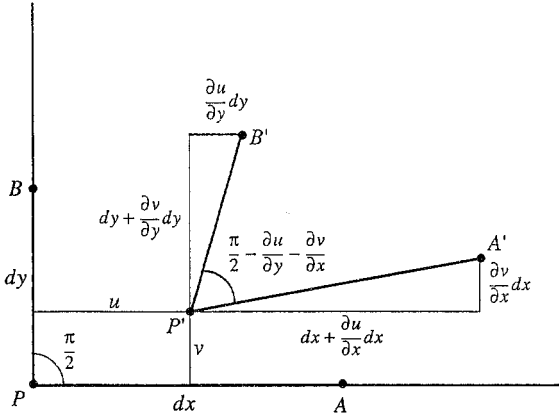


Fig. 9

The Hook's law gives the components of strain in term of the stress components :

**Proposition 1.38**      **Hook's Law**

$$\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \gamma_{xy} &= \frac{\tau_{xy}}{G}, \end{aligned}$$

where

- $E$  is the *modulus of elasticity in tension*
- $\nu$  is the *Poisson's ratio*
- $G = \frac{E}{2(1+\nu)}$  is the *modulus of elasticity in shear, or modulus of rigidity*.

**Proposition 1.39**

The amount of work per unit area, done by the forces during loading is equal to :

$$V = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy})$$

In term of the strain components,  $V$  is given by:

$$V = \frac{E}{2(1+\nu)}(\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y) + \frac{G}{2}\gamma_{xy}^2$$

$W = \iint V dx dy$  is called the *strain energy* of the plate.

**Proposition 1.40**

Consider a small rectangular plate of dimensions  $dx$ ,  $dy$  under a deformation  $\Phi(x, y)$ . This deformation induces the following displacement and strain components (see Fig. 10):

$$\begin{aligned} u &= \frac{\partial \Phi}{\partial x} \\ v &= \frac{\partial \Phi}{\partial y} \\ \epsilon_x &= \frac{\partial^2 \Phi}{\partial x^2} \\ \epsilon_y &= \frac{\partial^2 \Phi}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial^2 \Phi}{\partial x \partial y} \end{aligned}$$

And the strain energy per unit area due to the deformation  $\Phi$  is equal to :

$$V = \frac{E}{2(1-\nu^2)} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + 2\nu \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} \right) + \frac{G}{2} \frac{\partial^2 \Phi}{\partial x \partial y}$$

$V$  can be rewritten if the modulus of rigidity  $G$  is replaced by its value  $\frac{E}{2(1+\nu)}$  :

$$V = \frac{E}{2(1-\nu^2)} \left[ \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right)^2 - 2(1-\nu) \left( \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 \Phi}{\partial y^2} - \left( \frac{\partial^2 \Phi}{\partial x \partial y} \right)^2 \right) \right]$$

If the Poisson's ratio vanishes, then

$$V = \frac{E}{2} \left[ \left( \frac{\partial^2 \Phi}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 \Phi}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \Phi}{\partial y^2} \right)^2 \right]$$

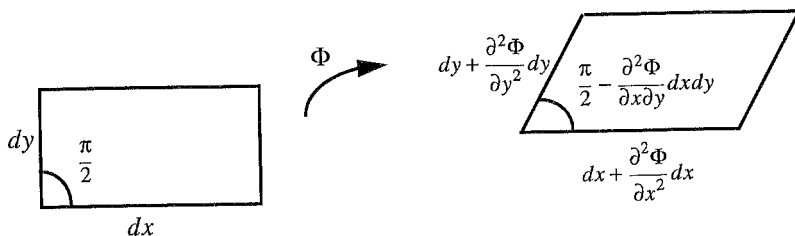


Fig. 10

**Proposition and Definition 1.41**

Let  $X$  be a parametric surface of class  $C^2$ :

$$\begin{aligned} X : [u_0, u_1] \times [v_0, v_1] &\rightarrow \mathbf{R}^3 \\ (u, v) &\mapsto X(u, v) \end{aligned}$$

Let  $\mathbf{u}_p = (u_p, v_p)$  be a parameter point.

In the affine basis with origin  $X(\mathbf{u}_p)$  and base vectors  $(X_u(\mathbf{u}_p), X_v(\mathbf{u}_p), N(\mathbf{u}_p))$ , the surface has locally the following parametric equation :

$$Y(\tilde{\mathbf{u}}) = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ f(\tilde{u}, \tilde{v}) \end{pmatrix},$$

where  $f(0, 0) = \frac{\partial f}{\partial \tilde{u}}(0, 0) = \frac{\partial f}{\partial \tilde{v}}(0, 0) = 0$ .

And the following holds :

$$(\kappa_1^2 + \kappa_2^2)(\mathbf{u}_p) = \frac{\partial^2 f}{\partial \tilde{u}^2}(0, 0) + 2 \frac{\partial^2 f}{\partial \tilde{u} \partial \tilde{v}}(0, 0) + \frac{\partial^2 f}{\partial \tilde{v}^2}(0, 0)$$

Thus  $\kappa_1^2 + \kappa_2^2$  represents the work done on the area  $d\tilde{u}d\tilde{v}$  by the deformation  $f(\tilde{u}, \tilde{v})$ , if the Poisson's ratio vanishes. Therefore, by analogy with the theory of elasticity, the quantity

$$W = \int_{u_0}^{u_1} \int_{v_0}^{v_1} (\kappa_1^2 + \kappa_2^2) \sqrt{g} du dv$$

is called *strain energy* of the surface  $X$ .

## 2 Variational Design of Rational Bézier Curves based upon Local Minimization of the Bending Energy

### 2.1 Restriction of the Problem

Given a control polygon of a rational Bézier curve of degree  $n$ , we want to find values of the weights for which the sum of the local bending energy in some parameter values is minimized. The vanishing of the first partial derivatives of this criterion function gives a system of equations for the weights. The degree of this system depends on the parameter values in which the local bending energy is minimized. In this chapter, we will find out the parameter values of the curve which ensure a low degree polynomial system of equations for our problem.

The local bending energy in any parameter value  $u$  of a parametric curve ( $u \mapsto X(u)$ ) is equivalent to

$$\kappa^2(u) \|X'(u)\| = \frac{\|[X'(u), X''(u)]\|^2}{\|X'(u)\|^5} \quad (2.1)$$

If  $X$  is a segment of a rational Bézier curve of degree  $n$ , (2.1) is a function of the weights  $\omega_0, \dots, \omega_n$ . The vector functions  $X'(u)$ ,  $X''(u)$  are rational functions of these weights. Thus the local bending energy in  $u$  is a rational function of these weights if and only if  $\|X'(u)\|$  is also rational. If not,  $\|X'(u)\|$  contains a square root term and the degree of the resolving system is multiplied by two. The next proposition gives the parameter values in which  $\|X'(u)\|$  is rational.

**Proposition 2.1**

Let  $X$  be a segment of a rational Bézier curve of degree  $n$ , parametrized over  $[u_0, u_1]$ , with control points  $b_0, \dots, b_n$  and control weights  $\omega_0, \dots, \omega_n$ .

$\|X'(u)\|$  is a rational function of the weights  $\omega_0, \dots, \omega_n$  if and only if ( $u = u_0$  or  $u = u_1$  or  $b_0, \dots, b_n$  are lying on a straight line).

Proof :

We write  $X(u) = \frac{p(u)}{w(u)}$  where

$$p(u) = \sum_{i=0}^n \omega_i b_i B_i^n(t),$$

$$\omega(u) = \sum_{i=0}^n \omega_i B_i^n(t),$$

$$t = \frac{u - u_0}{u_1 - u_0}.$$

The first derivative of  $X$  is equal to

$$X'(u) = \frac{p'(u)w(u) - p(u)w'(u)}{w(u)^2}.$$

This equation shows that  $\|X'\|$  is a rational function of  $\omega_0, \dots, \omega_n$  if and only if  $\|p'w - \omega'p\|$  is a polynomial function of  $\omega_0, \dots, \omega_n$ .

Calculating  $p'w - \omega'p$  we get

$$p'w - \omega'p = \frac{1}{u_1 - u_0} \sum_{i=0}^n \sum_{j=0}^n \omega_i \omega_j b_i B_i^{n'}(t) B_j^n(t) - \omega_i \omega_j b_j B_i^{n'}(t) B_j^n(t)$$

$$\Rightarrow p'w - \omega'p = \frac{1}{u_1 - u_0} \sum_{i=0}^n \sum_{j=0}^n \omega_i \omega_j (b_i - b_j) B_i^{n'}(t) B_j^n(t)$$

If  $p'w - \omega'p$  is a polynomial function of  $\omega_0, \dots, \omega_n$  then

$$\exists (\alpha_{ij})_{\substack{i=0, \dots, n \\ j=0, \dots, n}} / \|p'w - \omega'p\| = \sum_{i=0}^n \sum_{j=0}^n \alpha_{ij} \omega_i \omega_j$$

The last two equations gives us two different equations of  $\|p'w - w'p\|^2$ . Therefore we get necessary conditions on the coefficients  $(\alpha_{ij})_{\substack{i=0,\dots,n \\ j=0,\dots,n}}$ :

$$\forall i, j \quad \forall k, l \quad \alpha_{ij}\alpha_{kl} = \langle \mathbf{b}_i - \mathbf{b}_j, \mathbf{b}_k - \mathbf{b}_l \rangle > B_i^{n'}(t)B_k^{n'}(t)B_j^n(t)B_l^n(t) \quad (2.2)$$

For  $k = i$  and  $l = j$ , the last equation yields

$$\begin{aligned} \alpha_{ij}^2 &= \|\mathbf{b}_i - \mathbf{b}_j\|^2 \left[ B_i^{n'}(t) \right]^2 \left[ B_j^n(t) \right]^2 \\ \Rightarrow \alpha_{ij} &= \pm \|\mathbf{b}_i - \mathbf{b}_j\| B_i^{n'}(t) B_j^n(t) \end{aligned}$$

Inserting these values in the condition (2.2) for  $i = 0$  and  $k = 0$ , the following necessary conditions on the control points is obtained :

$$\langle \mathbf{b}_0 - \mathbf{b}_j, \mathbf{b}_0 - \mathbf{b}_l \rangle > B_0^{n'}(t)^2 B_j^n(t) B_l^n(t) = \|\mathbf{b}_0 - \mathbf{b}_j\| * \|\mathbf{b}_0 - \mathbf{b}_l\| B_0^{n'}(t)^2 B_j^n(t) B_l^n(t)$$

Now, the common coefficient  $B_0^{n'}(t)^2 B_j^n(t) B_l^n(t)$  on both sides of this equation vanishes if and only if  $u = u_0$  or  $u = u_1$ .

Therefore, if  $u \neq u_0$  and  $u \neq u_1$ , then

$$\langle \mathbf{b}_0 - \mathbf{b}_j, \mathbf{b}_0 - \mathbf{b}_l \rangle = \|\mathbf{b}_0 - \mathbf{b}_j\| \|\mathbf{b}_0 - \mathbf{b}_l\| \quad \forall j, l = 0, \dots, n$$

i.e.

$$(\mathbf{b}_j - \mathbf{b}_0) \text{ is colinear to } (\mathbf{b}_l - \mathbf{b}_0) \quad \forall j, l = 1, \dots, n$$

□

**Conclusion 2.2:**

To reduce the degree of the resulting system of equations of our problem, we must take  $u = u_0$  or  $u = u_1$ .

Our aim is now the following one : given a control polygon of a rational Bézier curve, find the values of the weights, for which the sum of the local bending energies in the endpoints of each segment of the curve is minimized. This sum is the new criterion function.

**2.2 Calculus of the Local Bending Energy**

We will first calculate the local bending energies in the endpoints of one segment of a rational Bézier curve.

**Proposition 2.3**

Let  $X$  be a segment of a rational Bézier curve of degree  $n$ , parametrized over  $[u_0, u_1]$ , with control points  $b_0, \dots, b_n$  and control weights  $\omega_0, \dots, \omega_n$ .

$$\kappa^2(u_0)\|X'(u_0)\| = \frac{(n-1)^2}{n(u_1-u_0)} \frac{\|[(b_1-b_0), (b_2-b_0)]\|^2}{\|(b_1-b_0)\|^5} \cdot \frac{\omega_2^2\omega_0}{\omega_1^3}$$

$$\kappa^2(u_1)\|X'(u_1)\| = \frac{(n-1)^2}{n(u_1-u_0)} \frac{\|[(b_{n-1}-b_n), (b_{n-2}-b_n)]\|^2}{\|(b_{n-1}-b_n)\|^5} \cdot \frac{\omega_{n-2}^2\omega_n}{\omega_{n-1}^3}$$

Proof :

The first and the second order derivatives of  $X$  in  $u_0$  are given in Prop 3.1:

$$X'(u_0) = \frac{n}{(u_1-u_0)} \cdot \frac{\omega_1}{\omega_0} (b_1-b_0)$$

$$X''(u_0) = \frac{n}{(u_1-u_0)^2} \cdot \frac{1}{\omega_0^2} \left[ 2(\omega_0\omega_1 - n\omega_1^2)(b_1-b_0) + (n-1)\omega_0\omega_2(b_2-b_0) \right]$$

The vector product of these two vectors is equal to:

$$[X'(u_0), X''(u_0)] = \frac{n^2(n-1)}{(u_1-u_0)^3} \frac{\omega_1\omega_2}{\omega_0^2} \cdot [(b_1-b_0), (b_2-b_0)]$$

Taking the square norm of this last vector, and dividing it by  $\|X'(u_0)\|^5$ , we get the first equation of our proposition.

To prove the second equation, we change the variable  $u$  into  $\bar{u} = u_0 + (u_1 - u)$ . The reparametrized curve  $Y(u) := X(\bar{u})$  is the rational Bézier curve with control points  $b_n, \dots, b_0$  and control weights  $\omega_n, \dots, \omega_0$ . Its curvature in  $u_0$  is equal to the curvature of  $X$  in  $u_1$  and is given by the first equation of our proposition.

□



**Proposition 2.4**

Let  $X$  be a  $C^0$  rational Bézier curve of degree  $n$ , with  $p$  segments. Let  $\{b_{ni+k}, 0 \leq i \leq p-1, 0 \leq k \leq n\}$  be the control points of  $X$ , and  $\{\omega_{ni+k}, 0 \leq i \leq p-1, 0 \leq k \leq n\}$  be the control weights of  $X$ . Let  $u_0 < u_1 < \dots < u_p$  be  $(p+1)$  scalars such that the  $i$ -th segment of  $X$  is parametrized over  $]u_i, u_{i+1}[$ .

The sum of the local bending energies in the endpoints of each segment of  $X$  is equal to :

$$\sum_{i=0}^p \frac{(n-1)^2}{n(u_{i+1} - u_i)} \left( D_i^+ \frac{\omega_{ni+2}^2 \omega_{ni}}{\omega_{ni+1}^3} + D_{i+1}^- \frac{\omega_{n(i+1)-2}^2 \omega_{n(i+1)}}{\omega_{n(i+1)-1}^3} \right) \quad (2.3)$$

$$\text{where } D_i^+ = \frac{\|[(b_{ni+1} - b_{ni}), (b_{ni+2} - b_{ni})]\|^2}{\|(b_{ni+1} - b_{ni})\|^5} \quad 0 \leq i \leq p-1$$

$$D_i^- = \frac{\|[(b_{ni-1} - b_{ni}), (b_{ni-2} - b_{ni})]\|^2}{\|(b_{ni-1} - b_{ni})\|^5} \quad 1 \leq i \leq p$$

**Proof :**

Because  $X$  is not supposed to be  $C^2$ , the local bending energy has a different limit in the right and the left of  $u_i$  :

$$\lim_{\substack{u \rightarrow u_i \\ u > u_i}} \kappa^2(u) \|X'(u)\| := \kappa^2(u_i^+) \|X'(u_i^+)\| \neq \kappa^2(u_i^-) \|X'(u_i^-)\| := \lim_{\substack{u \rightarrow u_i \\ u < u_i}} \kappa^2(u) \|X'(u)\|$$

Proposition 2.3 applied to the  $i$ -th segment of  $X$  yields :

$$1 \leq i \leq p-1 \quad \Leftrightarrow \quad \kappa^2(u_i^+) \|X'(u_i^+)\| = \frac{(n-1)^2}{n(u_{i+1} - u_i)} D_i^+ \frac{\omega_{ni+2}^2 \omega_{ni}}{\omega_{ni+1}^3}$$

$$\kappa^2(u_{i+1}^-) \|X'(u_{i+1}^-)\| = \frac{(n-1)^2}{n(u_{i+1} - u_i)} D_{i+1}^+ \frac{\omega_{n(i+1)-2}^2 \omega_{n(i+1)}}{\omega_{n(i+1)-1}^3}$$

□

### 2.3 Minimization of the Criterion Function in the Cubic Case

The function (2.3) for a rational Bézier curve of degree 3 is equal to :

$$\sum_{i=0}^{p-1} \frac{4}{3(u_{i+1} - u_i)} \left( D_i^+ \frac{\omega_{3i+2}^2 \omega_{3i}}{\omega_{3i+1}^3} + D_{i+1}^- \frac{\omega_{3i+1}^2 \omega_{3(i+1)}}{\omega_{3i+2}^3} \right) \quad (2.4)$$

If the weights  $(\omega_j)_{j \in \{0, \dots, 3p+1\}}$  minimize (2.4), then the partial derivatives of (2.4) must vanish in these weights. In particular, the partial derivative of (2.4) along  $\omega_{3i}$  should vanish :

$$\frac{4}{3(u_{i+1} - u_i)} D_i^+ \frac{\omega_{3i+2}^2}{\omega_{3i+1}^3} + \frac{4}{3(u_i - u_{i-1})} D_i^- \frac{\omega_{3i-2}^2}{\omega_{3i-1}^3} = 0, \quad 1 \leq i \leq p-1 \quad (2.5)$$

But the scalars  $D_i^+$  and  $D_i^-$  are positive, and  $\omega_{3i-1}, \omega_{3i+1}$  must be positive. Therefore, the condition (2.5) implies

$$\omega_{3i+2} = \omega_{3i-2} = 0 \quad 1 \leq i \leq p-1$$

For these values of the weights, the curve  $X$  is the polygon line with vertices  $(b_{3i})_{1 \leq i \leq p}$ , for which in fact, the local bending energy vanishes everywhere.

To find a non-trivial solution, we impose the following conditions on the weights :

$$\begin{aligned} \omega_{3i+1} &= \omega_{3i} & \forall i \in \{0, \dots, p-1\} \\ \omega_{3i-1} &= \omega_{3i} & \forall i \in \{1, \dots, p\} \end{aligned} \quad (2.6)$$

These conditions are compatible with the  $C^1$ -continuity in the sense that if the non rational Bézier curve with control points  $(b_{3i+k})_{i=0, \dots, p-1}$  is  $C^1$ , so is the rational Bézier curve with the same control points and weights fulfilling the conditions (2.6). But the main advantage of the conditions (2.6) is that they transform the minimization of the function (2.4) into a linear problem, with a unique solution.

To see this, we introduce the new unknowns

$$\alpha_i = \frac{\omega_{3(i+1)}^2}{\omega_{3i}^2} \quad \forall i \in \{0, \dots, p-1\}.$$

If the first weight  $\omega_0$  is given, the  $p$  scalars  $\alpha_0, \dots, \alpha_{p-1}$  and the conditions (2.6) determines uniquely the  $3p$  weights  $\omega_1, \dots, \omega_{3p+1}$ .

With these new notations, the function (2.4) is equal to :

$$F = \sum_{i=0}^{p-1} \frac{4}{3(u_{i+1} - u_i)} \left( D_i^+ \alpha_i + D_{i+1}^- \frac{1}{\alpha_i} \right) \quad (2.7)$$

Minimizing (2.7) is equivalent to minimize the  $p$  functions  $F_i$ ,

$$F_i = D_i^+ \alpha_i + D_{i+1}^- \frac{1}{\alpha_i} \quad i \in \{0, \dots, p-1\}.$$

The first order derivative of  $F_i$  gives the variations of the function  $F_i$

$$\frac{dF_i}{d\alpha_i} = D_i^+ - \frac{D_{i+1}^-}{\alpha_i^2} \tag{2.8}$$

If the control polygon of the curve is non-degenerate, then  $D_i^+ \neq 0$  and (2.8) has the unique root  $\sqrt{\frac{D_{i+1}^-}{D_i^+}}$ . In this case, the variations of the function  $F_i$  can be represented as follows:

$\alpha_i$	0	$\sqrt{\frac{D_{i+1}^-}{D_i^+}}$	$+\infty$
$\frac{dF_i}{d\alpha_i}$	-	0	+
$F_i$	$+\infty$ ↘	$2\sqrt{D_i^+ D_{i+1}^-}$	$+\infty$ ↗

We see that  $F_i$  reaches a global minimum for  $\alpha_i = \sqrt{\frac{D_{i+1}^-}{D_i^+}}$ .

Proposition 2.5 resumes the results of chapter 2.3 :

**Proposition 2.5**

Let  $X$  be a rational Bézier curve of degree 3 with  $p$  segments, control weights  $\{\omega_{3i+k}, 0 \leq i \leq p-1, 0 \leq k \leq 3\}$ , and a non-degenerate control polygon  $\{b_{3i+k}, 0 \leq i \leq p-1, 0 \leq k \leq 3\}$ . The sum of the local bending energies of  $X$  in the endpoints of each segment, under the conditions

$$\begin{aligned} \omega_{3i+1} &= \omega_{3i} & \forall i \in \{0, \dots, p-1\} \\ \omega_{3i-1} &= \omega_{3i} & \forall i \in \{1, \dots, p\}, \end{aligned}$$

is minimal if and only if

$$\frac{\omega_{3(i+1)}}{\omega_{3i}} = \left( \frac{\|[(b_{3i+1} - b_{3i+3}), (b_{3i+2} - b_{3i+3})]\|}{\|[(b_{3i+2} - b_{3i}), (b_{3i+1} - b_{3i})]\|} \right)^{1/2} \left( \frac{\|b_{3i+1} - b_{3i}\|}{\|b_{3i+2} - b_{3i+3}\|} \right)^{5/4}$$

**Remarks 2.6:**

- (a) To calculate all the weights, one weight must be chosen arbitrarily. Anyway, the curve doesn't depend on this choice.
- (b) The global minimum is reached if and only if the local bending energy is the same in both end points for each segment :

$$\kappa^2(u_i^+) \|X'(u_i^+)\| = \kappa^2(u_i^-) \|X'(u_i^-)\| \quad \forall i \in \{0, \dots, p-1\}$$

**Examples :**

In the following examples, each curve ( $u \mapsto X(u)$ ) is drawn together with the curve ( $u \mapsto X(u) + f(u)N(u)$ ), where  $N$  is the normal of  $X$  at the point  $X(u)$  and  $f$  is proportional to the local bending energy  $\kappa^2(u) \|X'(u)\|$ . The sum of the distances between the two curves at the endpoints of each segment is proportional to the function that we have to minimize. Remark (b) of proposition 2.5 implies that the distances between the two curves must be the same in both endpoints of each segment, for our solution.

- In Fig.11, we compare a cubic segment with all weights equal to one (at the top), with the curve given by Prop. 2.5 (at the bottom).
- Fig.12 shows (at the top) a cubic curve with four segments. The curve given by Prop. 2.5 is drawn at the bottom.

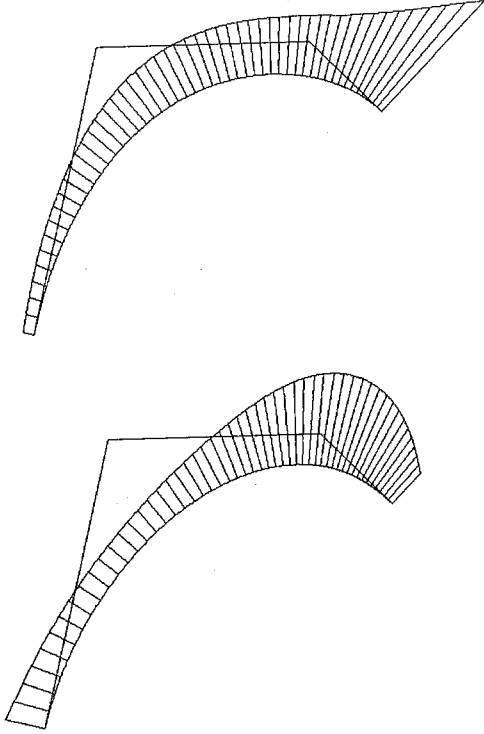


Fig. 11

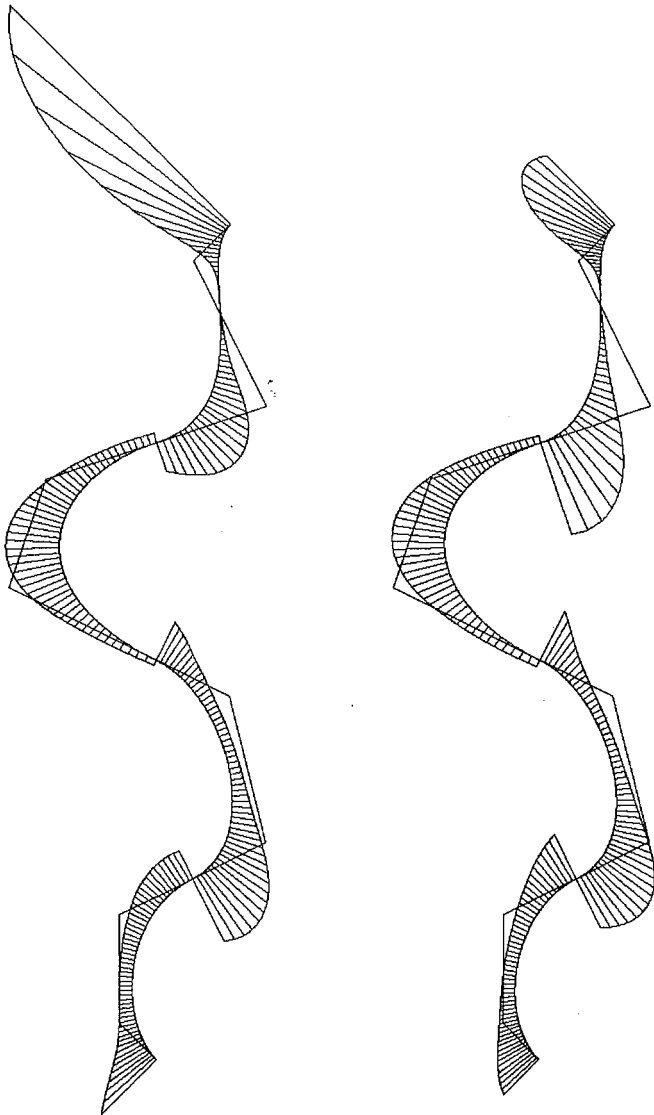


Fig. 12

### 3 Stiffness Degree Concept

In the previous chapter we try to minimize quantities related to the bending energy of a curve :

$$\int \kappa^2(t) \|X'(t)\| dt$$

As mentioned in the introduction, other integral criterions were successfully used in the literature of CADG ([HOL57], [CLI74], [NIE74], [HAG84]).

Recently, Hagen and Santarelli [H-S92] minimized the functional

$$\alpha \cdot \int \|X''(t)\|^2 dt + \beta \cdot \int \|X'''(t)\|^2 dt \quad (3.1)$$

$$\alpha + \beta = 1$$

over the set of non-rational quintic  $C^2$ -Bézier curves or non-rational B-Spline curves, together with a least square constraint. In this chapter, we will adapt this criterion for our problem : find weights for a rational Bézier curve whose control polygon is given by a user. We will first explain why we use a local criterion instead of an integral one. This new local criterion will be applied in the parameter values of the Bézier curve corresponding to the endpoints of each segment. In the last part we will show that it is possible to apply the criterion in any parameter value.

#### 3.1 A Local Criterion

##### 3.1.1 Derivatives of a rational Bézier curve

Let

$$X(t) = \frac{\mathbf{p}(t)}{\omega(t)}$$

be a parametric equation of a segment of a rational Bézier curve of degree  $n$ , with  $p \in \mathbf{R}_n^3[t]$ ;  $\omega \in \mathbf{R}_n[t]$ .

Rewriting the parametric equation of  $X$  as

$$p(t) = X(t) \omega(t),$$

and deriving it  $d$  times, we get

$$p^{(d)}(t) = \sum_{k=0}^d \binom{d}{k} X^{(k)}(t) \omega^{(d-k)}(t)$$

This equation can be solved for  $X^{(n)}$  :

$$X^{(d)}(t) = \frac{1}{\omega(t)} \left[ p^{(d)}(t) - \sum_{k=0}^{d-1} \binom{d}{k} X^{(k)}(t) \omega^{(d-k)}(t) \right] \quad (3.2)$$

We will use (3.2) as a recurrence relation to compute the derivatives of the curve.

Equation (3.2) shows that the derivatives of  $X$  are of the form

$$X^{(d)}(\omega_j, t) = \frac{q_d(\omega_j, t)}{[\omega(\omega_j, t)]^d} \quad (3.3)$$

where  $q_d(\omega_j, t) \in \mathbf{R}_{2d-2}^3[t]$  for any fixed  $\omega_0, \dots, \omega_n$  and  $q_d(\omega_i, t) \in \mathbf{R}_{d+1}^3[\omega_j]$  for any fixed  $t$ .

### 3.1.2 The criterion

The function (3.1) introduced by Hagen and Santarelli [H-S92] is, for a rational Bézier curve, the integral of a rational function. To minimize this function, we would have to calculate it as a function of the weights. This requires first to find the roots of  $\omega$  as functions of the weights (which is impossible if the degree of the curve is greater or equal than 5), and then to calculate the simple form of the fractional function equal to the integrand of (3.1), integrate it in the variable  $t$ , and derivate it in term of the weights. This is theoretically possible, but leads, even for degree 2 curves, to a non polynomial system of equations.

Therefore, we replace the integral (3.1) by a quadrature formula :

$$\sum_i a_i \left( \alpha \|X''(t_i)\|^2 + \beta \|X'''(t_i)\|^2 \right), \quad (3.4)$$

where the  $(t_i)$  are parameter values, and the  $(a_i)$  are scalars.



Our aim is to minimize the functional (3.4) over the set of all rational Bézier curves with a given control polygon  $(b_{ni+k})_{\substack{i=0,\dots,p-1 \\ k=0,\dots,n}}$ .

Because the derivatives of  $X$  are rational, minimizing (3.4) leads to a non polynomial system of equations for the weights. But, the degree of the resolving system is in general very high. (degree  $9 \times m$  if  $m$  is the maximal number of parameter values in a same segment).

However, as we will see in chapter 3.2, if the  $(t_i)$  are the parameter values of the endpoints of each segment, the degree of the system is reduced to 3 if  $\beta = 0$ , and 5 if  $\beta \neq 0$ . This result will be generalized in chapter 3.3 : If at most two parameter values  $(t_i)$  per segment are choosen, the degree of the system is reduced to 5 if  $\beta = 0$ , and 7 if  $\beta \neq 0$ .

## 3.2 Minimization in the Endpoints

### 3.2.1 The resolving system of equations

Let us first recall that the  $G^1$ -continuity at a common point of two segments just depends on the control points, so that moving the weights of one segment independently of those of the second segment doesn't disturb the  $G^1$ -continuity. Therefore, the sum (3.4) will be minimized independently for each segment.

Our aim is yet to minimize the functional

$$a_0 \left( \alpha \|X''(u_0)\|^2 + \beta \|X'''(u_0)\|^2 \right) + a_1 \left( \alpha \|X''(u_1)\|^2 + \beta \|X'''(u_1)\|^2 \right) \quad (3.5)$$

where

- $X$  is a segment of a rational Bézier curve parametrized over  $]u_0, u_1[$ ,
- $a_0, a_1$  are scalars,
- $\alpha, \beta$  are scalar values with  $\alpha + \beta = 1$ .

**Proposition 3.1**

Let  $X : [u_0, u_1] \rightarrow \mathbb{R}^3$  be a segment of a rational Bézier curve of degree  $n$ , with control points  $\mathbf{b}_0, \dots, \mathbf{b}_n$  and control weights  $\omega_0, \dots, \omega_n$ .

The first three derivatives of  $X$  in  $u_0$  are given by :

$$X'(u_0) = \frac{n}{(u_1 - u_0)} \cdot \frac{\omega_1}{\omega_0} (\mathbf{b}_1 - \mathbf{b}_0) \quad (3.6a)$$

$$X''(u_0) = \frac{n}{(u_1 - u_0)^2} \cdot \frac{1}{\omega_0^2} \cdot \left[ 2(\omega_0\omega_1 - n\omega_1^2)(\mathbf{b}_1 - \mathbf{b}_0) + (n-1)\omega_0\omega_2(\mathbf{b}_2 - \mathbf{b}_0) \right] \quad (3.6b)$$

$$X'''(u_0) = \frac{n}{(u_1 - u_0)^3} \cdot \frac{1}{\omega_0^3} \cdot \left[ 3(2\omega_0^2\omega_1 - 4n\omega_0\omega_1^2 - n(n-1)\omega_0\omega_1\omega_2 + 2n^2\omega_1^3)(\mathbf{b}_1 - \mathbf{b}_0) + 3(2(n-1)\omega_0^2\omega_2 - n(n-1)\omega_0\omega_1\omega_2)(\mathbf{b}_2 - \mathbf{b}_0) + (n-1)(n-2)\omega_0^2\omega_3(\mathbf{b}_3 - \mathbf{b}_0) \right] \quad (3.6c)$$

Proof :

To calculate the derivatives of  $X$ , we use the recurrence formula (3.2). First of all we have to calculate the derivatives of the denominator  $\omega$  and the nominator  $\mathbf{p}$  of  $X$ .  $\mathbf{p}$  and  $\omega$  are non-rational Bézier curves; their derivatives are calculated with the de Casteljau algorithm:

$$\begin{aligned} \mathbf{p}(u_0) &= \omega_0 \mathbf{b}_0 & \omega(u_0) &= \omega_0 \\ \mathbf{p}'(u_0) &= \frac{n}{(u_1 - u_0)} (\omega_1 \mathbf{b}_1 - \omega_0 \mathbf{b}_0) & \omega'(u_0) &= \frac{n}{(u_1 - u_0)} (\omega_1 - \omega_0) \\ \mathbf{p}''(u_0) &= \frac{n(n-1)}{(u_1 - u_0)^2} (\omega_2 \mathbf{b}_2 - 2\omega_1 \mathbf{b}_1 + \omega_0 \mathbf{b}_0) & \omega''(u_0) &= \frac{n(n-1)}{(u_1 - u_0)^2} (\omega_2 - 2\omega_1 + \omega_0) \\ \mathbf{p}'''(u_0) &= \frac{n(n-1)(n-2)}{(u_1 - u_0)^3} (\omega_3 \mathbf{b}_3 - 3\omega_2 \mathbf{b}_2 + 3\omega_1 \mathbf{b}_1 - \omega_0 \mathbf{b}_0) & \omega'''(u_0) &= \frac{n(n-1)(n-2)}{(u_1 - u_0)^3} (\omega_3 - 3\omega_2 + 3\omega_1 - \omega_0) \end{aligned}$$

Inserting these values in the recurrence formula (3.2) for  $d = 1$ , we get the first derivatives of  $X$  in  $u_0$  :

$$\begin{aligned} X'(u_0) &= \frac{1}{\omega(u_0)} \left[ p'(u_0) - X(u_0)\omega'(u_0) \right] \\ \Leftrightarrow X'(u_0) &= \frac{n}{(u_1 - u_0)} \frac{1}{\omega_0} \left[ (\omega_1 b_1 - \omega_0 b_0) - b_0(\omega_1 - \omega_0) \right] \\ \Leftrightarrow X'(u_0) &= \frac{n}{(u_1 - u_0)} \frac{\omega_1}{\omega_0} (b_1 - b_0) \end{aligned}$$

This value is inserted into the same recursive formula, for  $d = 2$  gives :

$$\begin{aligned} X''(u_0) &= \frac{1}{\omega(u_0)} \left[ p''(u_0) - 2X'(u_0)\omega'(u_0) - X(u_0)\omega''(u_0) \right] \\ \Leftrightarrow X''(u_0) &= \frac{n}{(u_1 - u_0)^2} \frac{1}{\omega_0^2} \left[ (n-1)(\omega_2 b_2 - 2\omega_1 b_1 + \omega_0 b_0)\omega_0 \right. \\ &\quad \left. - 2n\omega_1(b_1 - b_0)(\omega_1 - \omega_0) - \omega_0 b_0(n-1)(\omega_2 - 2\omega_1 + \omega_0) \right] \end{aligned}$$

We rewrite  $X''(u_0)$  as a linear combination of the vectors  $(b_1 - b_0)$ ,  $(b_2 - b_0)$  :

$$\begin{aligned} X''(u_0) &= \frac{n}{(u_1 - u_0)^2} \frac{1}{\omega_0^2} \left[ (n-1)\omega_0\omega_2(b_2 - b_0) - 2(n-1)\omega_0\omega_1(b_1 - b_0) \right. \\ &\quad \left. + 2n(\omega_0\omega_1 - \omega_1^2)(b_1 - b_0) \right] \end{aligned}$$

Adding the coefficients of  $(b_1 - b_0)$ , we get (3.6b)

Eventually, the third order derivative is equal to :

$$\begin{aligned} X'''(u_0) &= \frac{1}{\omega_0} \left[ p'''(u_0) - 3X''(u_0)\omega'(u_0) - 3X'(u_0)\omega''(u_0) - X(u_0)\omega'''(u_0) \right] \\ \Leftrightarrow X'''(u_0) &= \frac{n}{(u_1 - u_0)^3} \frac{1}{\omega_0^3} \left[ (n-1)(n-2)\omega_0^2(\omega_2 b_3 - 3\omega_2 b_2 + 3\omega_1 b_1 + \omega_0 b_0) \right. \\ &\quad \left. - 3n \left( 2(\omega_0\omega_1 - n\omega_1^2)(b_1 - b_0) \right. \right. \\ &\quad \left. \left. + (n-1)\omega_0\omega_2(b_2 - b_0) \right) (\omega_1 - \omega_0) \right. \\ &\quad \left. - 3n(n-1)\omega_0\omega_1(b_1 - b_0)(\omega_2 - 2\omega_1 + \omega_0) \right. \\ &\quad \left. (n-1)(n-2)\omega_0^2 b_0(\omega_3 - 3\omega_2 + 3\omega_1 - \omega_0) \right] \end{aligned}$$

To simplify this expression, we rewrite it as a linear combination of the vectors  $\mathbf{b}_1 - \mathbf{b}_0$ ,  $\mathbf{b}_2 - \mathbf{b}_0$  and  $\mathbf{b}_3 - \mathbf{b}_0$  :

$$\begin{aligned}
 X'''(u_0) = \frac{n}{(u_1 - u_0)^3} \frac{1}{\omega_0^3} & \left[ (n-1)(n-2)\omega_0^2\omega_3(\mathbf{b}_3 - \mathbf{b}_0) \right. \\
 & - 3(n-1)(n-2)\omega_0^2\omega_2(\mathbf{b}_2 - \mathbf{b}_0) \\
 & + 3(n-1)(n-2)\omega_0^2\omega_2(\mathbf{b}_1 - \mathbf{b}_0) \\
 & - 6n(\omega_0\omega_1 - n\omega_1^2)(\mathbf{b}_1 - \mathbf{b}_0) \\
 & - 3n(n-1)\omega_0\omega_2(\omega_1\omega_0)(\mathbf{b}_1 - \mathbf{b}_0) \\
 & \left. - 3n(n-1)\omega_0\omega_1(\omega_2 - 2\omega_1 + \omega_0)(\mathbf{b}_1 - \mathbf{b}_0) \right]
 \end{aligned}$$

Adding the two coefficients of  $(\mathbf{b}_2 - \mathbf{b}_0)$  and the three coefficients of  $(\mathbf{b}_1 - \mathbf{b}_0)$  we get the final formula for the derivatives of  $X$  in  $u_0$ .

To calculate the derivatives of  $X$  in  $u_1$ , we use the reparametrized curve  $Y(\bar{u}) = X(u_1 + u_0 - \bar{u})$ .  $Y$  is the rational Bézier curve with the control polygon  $(\mathbf{b}_n, \dots, \mathbf{b}_0)$  and the weights  $(\omega_n, \dots, \omega_0)$ , and

$$\begin{aligned}
 Y'(u_0) &= -X'(u_1) \\
 Y''(u_0) &= X''(u_1) \\
 Y'''(u_0) &= -X'''(u_1)
 \end{aligned}$$

Thus by changing the indices  $i$  of the weights and control points by  $(n-i)$  in (3.6), and take the opposite for the first and third derivatives, we get the derivatives of  $X$  in  $u_1$ .

□

The derivatives of  $X$  in the endpoints are still rational. To avoid this, we will use the standard form of the rational Bézier curves :

**Proposition 3.2**

Let  $X$  be a rational Bézier curve of degree  $n$  parametrized over  $[u_0, u_1]$ , with control points  $(b_0, \dots, b_n)$ , and positive control weights  $(\omega_0, \dots, \omega_n)$ . There exists a unique rational linear reparametrization  $\varphi$  such that

(i)

$$\begin{aligned} \varphi(u_0) &= u_0 \\ \varphi(u_1) &= u_1 \\ \varphi([u_0, u_1]) &= [u_0, u_1] \end{aligned} \tag{3.7}$$

(ii) The reparametrized curve  $X(\varphi)$  is a rational Bézier curve with control points  $(b_0, \dots, b_n)$  and control weights  $(\bar{\omega}_0, \dots, \bar{\omega}_n)$ , and

$$\begin{aligned} \bar{\omega}_0 &= 1 \\ \bar{\omega}_i &> 0 \quad 1 \leq i \leq n \\ \bar{\omega}_n &= 1. \end{aligned}$$

Proof : The conditions (3.7) imply that  $\varphi$  is of the form :

$$\varphi(u) = \frac{\rho u_1(u - u_0) + \hat{\rho} u_0(u_1 - u)}{\rho(u - u_0) + \hat{\rho}(u_1 - u)}$$

where  $\rho$  and  $\hat{\rho}$  are two positive scalar values.

The reparametrized Bernstein polynomials are equal to :

$$B_i^n \left( \frac{\varphi(u) - u_0}{u_1 - u_0} \right) = \left( \frac{u_1 - u_0}{\rho(u - u_0) + \hat{\rho}(u_1 - u)} \right)^n \rho^i \hat{\rho}^{n-i} B_i^n \left( \frac{u - u_0}{u_1 - u_0} \right),$$

so that the reparametrized curve  $\bar{X}$  has the following parametric equation :

$$\bar{X}(u) = X(\varphi(u)) = \frac{\sum_{i=0}^n \bar{\omega}_i b_i B_i^n \left( \frac{u - u_0}{u_1 - u_0} \right)}{\sum_{i=0}^n \bar{\omega}_i B_i^n \left( \frac{u - u_0}{u_1 - u_0} \right)}$$

with  $\bar{\omega}_i = \rho^i \hat{\rho}^{n-i} \omega_i, i = 0, \dots, n$ .

(ii) is fulfilled if and only if

$$\hat{\rho}^n \bar{\omega}_0 = 1$$

$$\rho^n \bar{\omega}_n = 1$$

$$\rho, \hat{\rho} > 0$$

This system has the unique solution

$$\hat{\rho} = \bar{\omega}_0^{-\frac{1}{n}}$$

$$\rho = \bar{\omega}_n^{-\frac{1}{n}}$$

□

The effect of such reparametrization is shown in figure 13.

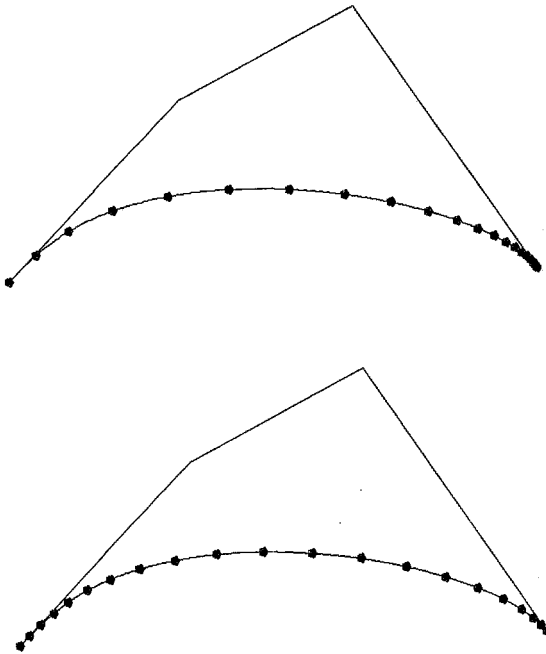


Fig. 13

The curve at the top has the weights  $(1, 1, 1, 20)$  and is parametrized over  $]0, 1[$ . The new weights are  $(1, 0.37, 0.14, 1)$ . The points on the curve correspond to uniformly spaced parameter values.

**Remark 3.3:**

Our criterion function (3.5) depends on the parametrization. Two different parametrizations of the same segment give two different values of this function. Assuming that the segments are reparametrized as it is done in Proposition (3.2) is an arbitrary but natural choice, because it is compatible with the non-rational case. Indeed, other rational linear reparametrizations would lead, for a non-rational curve, to a rational curve with weights not equal to one.

**Proposition 3.4**

Let  $X : [u_0, u_1] \rightarrow \mathbb{R}^3$  be a segment of a rational Bézier curve of degree  $n$  with control points  $(b_0, \dots, b_n)$  and control weights  $(1, \omega_1, \dots, \omega_{n-1}, 1)$ .

- $\|X''(u_0)\|^2$  ( resp.  $\|X''(u_1)\|^2$  ) is a polynomial of total degree 4 in  $\omega_1, \omega_2$  ( resp.  $\omega_{n-1}, \omega_{n-2}$  ), partial degree 4 in  $\omega_1$  ( resp.  $\omega_{n-1}$  ), partial degree 2 in  $\omega_2$  ( resp.  $\omega_{n-2}$  ).
- $\|X'''(u_0)\|^2$  ( resp.  $\|X'''(u_1)\|^2$  ) is a polynomial of total degree 6 in  $\omega_1, \omega_2, \omega_3$  ( resp.  $\omega_{n-3}, \omega_{n-2}, \omega_{n-1}$  ), partial degree 6 in  $\omega_1$  ( resp.  $\omega_{n-1}$  ), partial degree 2 in  $\omega_2$ , ( resp.  $\omega_{n-2}$  ) and partial degree 2 in  $\omega_3$ , ( resp.  $\omega_{n-3}$  ).
- The extremal values of the functional (3.5) with respect to the weights  $\omega_1, \omega_2, \omega_3, \omega_{n-3}, \omega_{n-2}, \omega_{n-1}$  are solutions of a polynomial system of degree 3 if  $\beta = 0$ , and degree 5 if  $\beta \neq 0$ .

Proof: Obvious with the formulas of Prop. 3.1

□

**Remarks 3.5:**

- (a) If the degree of the rational curve is greater or equal than 4, the partial derivatives of (3.5) along  $\omega_2$  and  $\omega_{n-2}$  are linear in  $\omega_2$  and  $\omega_{n-2}$  (this is no longer true for cubic rational curves, because in this case we have  $\omega_1 = \omega_{n-2}$  and  $\omega_2 = \omega_{n-1}$ ). Suppose now that the degree is greater or equal than 4, and that the weights  $\omega_1, \omega_{n-1}$  are fixed (with  $C^1$ -continuity conditions for example), then the minimization of the function (3.5) for  $\beta = 0$ , with respect to the weights  $\omega_2, \omega_{n-2}$  leads to a linear system of equations.

Moreover, if the degree of the curve is greater than 5, then minimizing (3.5) for any  $\beta$  with respect to the weights  $\omega_2, \omega_3, \omega_{n-3}, \omega_{n-2}$  also leads to a linear system of equations.

- (b) If an extremal value  $\omega_1, \dots, \omega_{n-1}$  of the function (3.5) is found, the jacobian matrix in this value is calculated. This extremum is a minimum if and only if this matrix is positive.
- (c) The functional (3.5) has a unique global minimum reached in  $\omega_1 = \dots = \omega_{n-1} = 0$ .

### 3.2.2 Examples

In the first example,  $X$  is a rational Bézier curve of degree 4, with control points

$$\begin{aligned} b_0 &= (0, 0, 0) \\ b_1 &= (0, 1, 1.2) \\ b_2 &= (0, 2, 0.8) \\ b_3 &= (0, 3, 1) \\ b_4 &= (0, 4, 0) \end{aligned}$$

parametrized over  $]0, 1[$ .

The second and third order derivatives of  $X$  in 0 and 1 are the following polynomials in the unknowns  $\omega_1, \omega_2, \omega_3$  :

$$\begin{aligned} \frac{d^2 X}{dt^2}(0.0) &= + (0.00, -32.00, -38.40)\omega_1^2 + (0.00, 8.00, 9.60)\omega_1 + (0.00, 0.00, 0.00) \\ &\quad + (0.00, 24.00, 9.60)\omega_2 \end{aligned}$$

$$\begin{aligned} \frac{d^3 X}{dt^3}(0.0) &= + (0.00, 384.00, 460.80)\omega_1^3 + (0.00, -192.00, -230.40)\omega_1^2 \\ &\quad + (0.00, 24.00, 28.80)\omega_1 + (0.00, -432.00, -288.00)\omega_1\omega_2 \\ &\quad + (0.00, 0.00, 0.00) + (0.00, 144.00, 57.60)\omega_2 \\ &\quad + (0.00, 72.00, 24.00)\omega_3 \end{aligned}$$

$$\begin{aligned} \frac{d^2 X}{dt^2}(1.0) &= + (0.00, 0.00, 0.00) + (0.00, -8.00, 8.00)\omega_3 + (0.00, 32.00, -32.00)\omega_3^2 \\ &\quad + (0.00, -24.00, 9.60)\omega_2 \end{aligned}$$

$$\begin{aligned} \frac{d^3 X}{dt^3}(1.0) &= + (0.00, 0.00, 0.00) + (0.00, 24.00, -24.00)\omega_3 + (0.00, -192.00, 192.00)\omega_3^2 \\ &\quad + (0.00, 144.00, -57.60)\omega_2 + (0.00, 384.00, -384.00)\omega_3^3 \\ &\quad + (0.00, -432.00, 259.20)\omega_2\omega_3 + (0.00, 72.00, -28.80)\omega_1 \end{aligned}$$



The functional (3.5) for  $a_0 = a_1 = \frac{1}{2}$ ,  $\alpha = 0.8$ ,  $\beta = 0.2$  is equal to :

$$\begin{aligned}
& 0.8/2 \left( \left\| \frac{d^2 X}{dt^2}(0) \right\|^2 + \left\| \frac{d^2 X}{dt^2}(1) \right\|^2 \right) + 0.2/2 \left( \left\| \frac{d^3 X}{dt^3}(0) \right\|^2 + \left\| \frac{d^3 X}{dt^3}(1) \right\|^2 \right) = \\
& + (29491.20)\omega_3^6 + (-29491.20)\omega_3^5 + (-53084.16)\omega_2\omega_3^4 + (11878.40)\omega_3^4 \\
& + (42024.96)\omega_2\omega_3^3 + (7741.44)\omega_1\omega_3^3 + (-2252.80)\omega_3^3 + (25380.86)\omega_2^2\omega_3^2 \\
& + (-11919.36)\omega_2\omega_3^2 + (-3870.72)\omega_1\omega_3^2 + (742.40)\omega_3^2 + (-15427.58)\omega_2^2\omega_3 \\
& + (-15316.99)\omega_1\omega_2\omega_3 + (3532.80)\omega_2\omega_3 + (7741.44)\omega_1^3\omega_3 + (-3870.72)\omega_1^2\omega_3 \\
& + (967.68)\omega_1\omega_3 + (26956.80)\omega_1^2\omega_2^2 + (-15759.36)\omega_1\omega_2^2 + (5345.28)\omega_2^2 \\
& + (-59719.68)\omega_1^4\omega_2 + (46227.45)\omega_1^3\omega_2 + (-12825.60)\omega_1^2\omega_2 + (3655.68)\omega_1\omega_2 \\
& + (35979.26)\omega_1^6 + (-35979.26)\omega_1^5 + (14491.65)\omega_1^4 + (-2748.42)\omega_1^3 \\
& + (804.35)\omega_1^2
\end{aligned}$$

For the non-rational curve with control points  $b_0, \dots, b_4$  ( $\omega_i = 1, i = 0, \dots, 4$ ), this functional take the value **695.8**

A partial minimum is reached in

$$\begin{aligned}
\omega_1 &= 0.73 \\
\omega_2 &= 0.65 \\
\omega_3 &= 0.72
\end{aligned}$$

In these weights the functional (3.5) is equal to **53.27**

If the weights  $\omega_1$  and  $\omega_3$  are fixed to one, the minimization of the functional leads to a linear equation in  $\omega_2$ , as it is said in the remark 3.3(a). The solution of this equation is  $\omega_2 = 1.08$ . In the weights  $(1, 1, 1.08, 1, 1)$  the functional is equal to **510.4**.

Figure 14 shows the 3 curves with the control points  $b_0, \dots, b_4$  and control weights respectively  $(1, 1, 1, 1, 1)$ ;  $(1, 0.73, 0.65, 0.72, 1)$ ;  $(1, 1, 1.08, 1, 1)$  (at the bottom). Each curve ( $u \mapsto X(u)$ ) is drawn together with the curve ( $u \mapsto X(u) + f(u) \cdot \vec{N}(u)$ ), where  $\vec{N}(u)$  is the normal of the curve at the point  $X(u)$ , and  $f(u)$  is proportional to the functional  $\alpha \|X''(u)\|^2 + \beta \|X'''(u)\|^2$ .

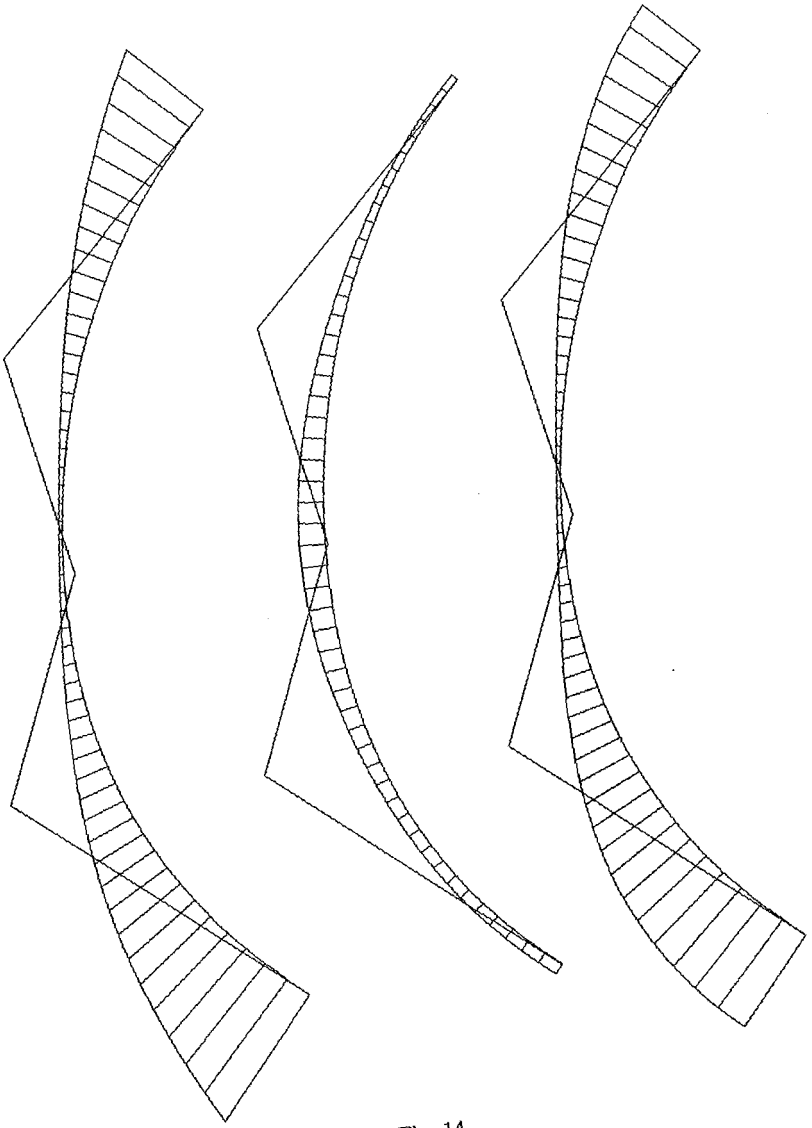


Fig. 14

Figure 15 shows two closed rational Bézier curves with four segments each. The first segment (at the top of each curve) has degree five, the second and the third (to the right and at the bottom) have degree 4, the fourth segment (to the left) has degree 5.

The first curve (at the top) has all weights equal to one.

We use the functional (3.5) with  $a_0 = a_1 = \frac{1}{2}$ ;  $\alpha = 0.8$ ;  $\beta = 0.2$  for each segment and find the new weights :

- (1, 0.69, 0.53, 0.52, 0.61, 1) for the first segment
- (1, 0.62, 0.57, 0.62, 1) for the second segment
- (1, 0.60, 0.58, 0.60, 1) for the third segment
- (1, 0.64, 0.51, 0.51, 0.64, 1) for the fourth segment

The value of the functional (3.5) is

- 15667.2 for the first curve
- 626.0 for the second one.

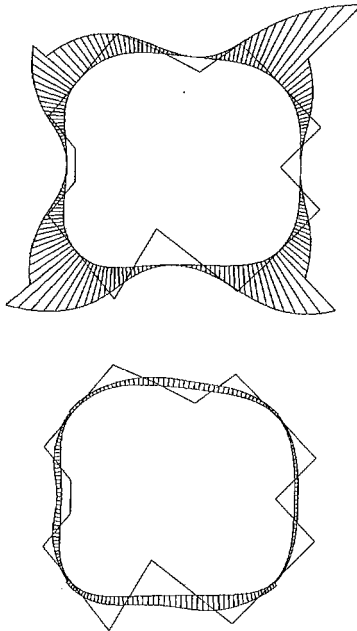


Fig. 15

### 3.3 Minimization in any two Parameter Values

#### 3.3.1 Reduction of the degree of the resolving system

We find in the previous chapter that the degree of the resolving system of our problem (see 3.1.2 for the statement of this problem) can be reduced if our criterion function (3.4) is minimized in the endpoints of each segment. This has been done using a reparametrization of the rational curves, which sets the first and the last weights to one, thus transforming the derivatives of the curve in the endpoints into polynomial functions of the weights.

More precisely, if the rational Bézier curve  $X(u)$  is equal to  $\frac{P(u)}{\omega(u)}$ , then the denominator of  $X^{(d)}(u)$  is equal to  $\omega^{d+1}(u)$  (see (3.3)). In  $u = u_0$  ( resp.  $u = u_1$  ) this denominator is equal to  $\omega_0^{d+1}$  ( resp.  $\omega_n^{d+1}$  ), and the reparametrization of the proposition 3.2 sets these denominators to one. To generalize this result for any two parameter values  $a, b$  in  $[u_0, u_1]$ , we shall find a reparametrized curve  $\bar{X} = \bar{p}/\bar{\omega}$ , such that

$$\bar{\omega}(a) = \bar{\omega}(b) = 1.$$

This is the object of the next proposition.

#### Proposition 3.6

Let  $(b_0, \dots, b_n)$  and  $(\omega_0, \dots, \omega_n) \in \mathbb{R}_+^n$  be the control points and control weights of a rational Bézier curve  $X$  of degree  $n$  parametrized over  $]u_0, u_1[$ . Let  $a, b$  be two parameter values in  $[u_0, u_1]$  with  $a < b$ . There exists a rational linear function  $\varphi$  such that

$$\begin{aligned} \varphi(u_0) &= u_0 \\ \varphi(u_1) &= u_1 \\ \varphi([u_0, u_1]) &= [u_0, u_1], \end{aligned} \tag{3.8}$$

that the control points of the reparametrized curve  $u \mapsto \bar{X}(u) = X(\varphi(u))$ . are the  $(b_i)$ , and the control weights  $(\bar{\omega}_0, \dots, \bar{\omega}_n)$  of  $\bar{X}$  verify

$$\begin{aligned} \sum_{i=0}^n \bar{\omega}_i B_i^n \left( \frac{a - u_0}{u_1 - u_0} \right) &= 1 \\ \sum_{i=0}^n \bar{\omega}_i B_i^n \left( \frac{b - u_0}{u_1 - u_0} \right) &= 1 \end{aligned} \tag{3.9}$$

Proof :

The conditions (3.8) imply that  $\varphi$  is of the form :

$$\varphi(u) = \frac{\rho u_1(u - u_0) + \hat{\rho} u_0(u_1 - u)}{\rho(u - u_0) + \hat{\rho}(u_1 - u)}$$

where  $\rho$  and  $\hat{\rho}$  are two non zero scalar values with the same sign, so that the denominator doesn't vanish.

We already see in the proof of Prop. 3.2 that the reparametrized curve  $\bar{X}$  has the following parametric equation :

$$\bar{X}(u) = X(\varphi(u)) = \frac{\sum_{i=0}^n \bar{\omega}_i b_i B_i^n \left( \frac{u - u_0}{u_1 - u_0} \right)}{\sum_{i=0}^n \bar{\omega}_i B_i^n \left( \frac{u - u_0}{u_1 - u_0} \right)}$$

with  $\bar{\omega}_i = \rho^i \hat{\rho}^{n-i} \omega_i$ ,  $i = 0, \dots, n$ .

Dividing the two equations (3.9) by  $\hat{\rho}^n$  and writing  $\alpha = \rho/\hat{\rho}$ , we find the following equivalent conditions to (3.9).

$$\begin{aligned} \exists \alpha > 0 / \sum_{i=0}^n \alpha^i \omega_i \left[ B_i^n \left( \frac{a - u_0}{u_1 - u_0} \right) - B_i^n \left( \frac{b - u_0}{u_1 - u_0} \right) \right] &= 0 \\ \frac{1}{\rho^n} &= \sum_{i=0}^n \alpha^i \omega_i B_i^n \left( \frac{a - u_0}{u_1 - u_0} \right) \\ \hat{\rho} &= \frac{\rho}{\alpha} \end{aligned}$$

Let us call  $f$  the left member of the first of these three equations.  $f(\alpha)$  is a polynomial of degree  $n$ , with constant and degree- $n$  coefficients of opposite sign (because  $\omega_0, \omega_1$  are positive,  $B_0^n$  is a strictly decreasing function,  $B_n^n$  a strictly increasing function, and  $a < b$ ). So  $f$  must have at least one positive root. □

Figure 16 shows the effect of such reparametrization. The curve at the top (same as in Fig. 13) has the weights (1,1,1,20) and is parametrized over  $[0, 1]$ .

We reparametrize this curve with

$$a = 0.8$$

$$b = 1.0$$

The new weights are (16.20, 2.36, 0.34, 1.00). The points on the curves correspond to uniformly spaced parameter values.

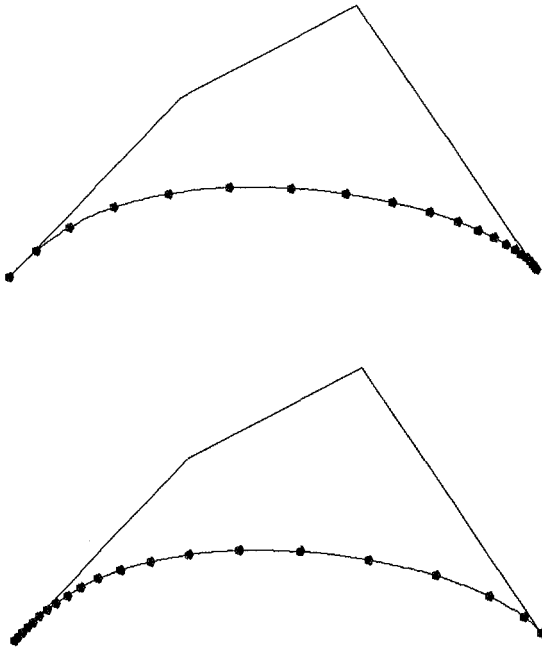


Fig. 16

The consequences of such a reparametrization on the degree of the resulting system of our problem are the object of the next proposition.

**Proposition 3.7**

Let  $(b_0, \dots, b_n)$  and  $(\omega_0, \dots, \omega_n) \in (\mathbb{R}_+)^n$  be the control points and -weights of a rational Bézier curve  $X$  of degree  $n$ , parametrized over  $]u_0, u_1[$ .

Let  $a, b$  be two parameter values,  $u_0 < a < b < u_1$ .

We suppose that

$$\sum_{i=0}^n \omega_i B_i^n \left( \frac{a - u_0}{u_1 - u_0} \right) = 1$$

$$\sum_{i=0}^n \omega_i B_i^n \left( \frac{b - u_0}{u_1 - u_0} \right) = 1$$
(3.10)

The following hold:

- i) For all  $0 \leq i_0 < i_1 \leq n$ , the system of equations (3.10) has a unique solution in  $\omega_{i_0}, \omega_{i_1}$ .
- ii) if  $\omega_{i_0}, \omega_{i_1}$  are replaced by the solution given in i), then

$$a_0 \left( \alpha \|X''(a)\|^2 + \beta \|X'''(a)\|^2 \right) + a_1 \left( \alpha \|X''(b)\|^2 + \beta \|X'''(b)\|^2 \right)$$

is a polynomial function in the unknowns  $\omega_0, \dots, \omega_{i_0-1}, \omega_{i_0+1}, \dots, \omega_{i_1-1}, \omega_{i_1+1}, \dots, \omega_n$ ; with total degree 6 if  $\beta = 0$ , 8 if  $\beta \neq 0$ .

**Proof :**

Let  $\bar{a} = \frac{a - u_0}{u_1 - u_0}$  and  $\bar{b} = \frac{b - u_0}{u_1 - u_0}$ .

To prove i) we must prove that the following determinant doesn't vanish :

$$\begin{vmatrix} B_{i_0}^n(\bar{a}) & B_{i_1}^n(\bar{a}) \\ B_{i_0}^n(\bar{b}) & B_{i_1}^n(\bar{b}) \end{vmatrix}.$$

This determinant vanishes if and only if

$$\frac{B_{i_1}^n(\bar{a})}{B_{i_0}^n(\bar{a})} = \frac{B_{i_1}^n(\bar{b})}{B_{i_0}^n(\bar{b})}.$$

But the function  $\left( \frac{B_{i_0}^n(t)}{B_{i_0}^n(t)} = \frac{\binom{n}{i_1}}{\binom{n}{i_0}} \left( \frac{1-t}{t} \right)^{i_1-i_0} \right)$  is a strictly increasing function in  $]0, 1[$ , and  $0 < \bar{a} < \bar{b} < 1$ .

To prove ii), we use the recursive formula (3.2) giving the derivatives of  $X$ . Let  $X(t) = \frac{P(t)}{\omega(t)}$ , then

$$X^{(d)} = \frac{1}{\omega(t)} \cdot \left[ p^{(d)}(t) - \sum_{i=0}^{d-1} \binom{d}{i} X^{(i)}(t) \omega^{(d-i)}(t) \right].$$

The conditions (3.10) on the weights are equivalent to  $\omega(a) = \omega(b) = 1$ . Replacing  $\omega_{i_0}$  and  $\omega_{i_1}$  by the solution found in i), we see that  $p(a)$  and  $p(b)$  are two affine functions of the weights  $\omega_0, \dots, \omega_{i_0-1}, \omega_{i_0+1}, \dots, \omega_{i_1-1}, \omega_{i_1+1}, \dots, \omega_n$ . The recursive formula applied in  $a$  and  $b$  shows that  $X^{(d)}(a)$  and  $X^{(d)}(b)$  are polynomial functions of degree total  $d + 1$  in the same unknowns.

□

**Remark:**

A natural choice for  $i_0$  (resp.  $i_1$ ) is to take the closest bezier ordinate  $\frac{i_0}{n}$  to  $\bar{a}$  (resp. the closest bezier ordinate  $\frac{i_1}{n}$  to  $\bar{b}$ ).

**3.3.2 Implementation**

In chapter 3.2.1 we calculate explicitly the derivatives of a rational Bézier curve  $X = \frac{P}{\omega}$  of degree  $b$ , in the endpoints of a segment. We get (after the reparametrization) a polynomial function with a low number of monomials. But the calculus of the derivatives of  $X$  in any two parameter values  $a$  and  $b$  of a segment, after the reparametrization of proposition 3.6 leads to a polynomial with much more monomials. To do this calculus, we use polynomial's routines (addition, scalar product, multiplication) for polynomials with  $(n - 2)$  unknowns and coefficients in  $\mathbb{R}^3$  or  $\mathbb{R}$ , and we program the de Casteljau algorithm for non-rational Bézier curves with control points in  $\mathbb{R}^4[\omega_0, \dots, \omega_{i_0-1}, \omega_{i_0+1}, \dots, \omega_{i_1-1}, \omega_{i_1+1}, \dots, \omega_n]$ .

We apply the deCasteljau algorithm in  $a$  and  $b$  with the control points

$$\omega_0 \begin{pmatrix} b_0 \\ 1 \end{pmatrix}, \dots, \omega_{i_0} \begin{pmatrix} b_{i_0} \\ 1 \end{pmatrix}, \dots, \omega_{i_1} \begin{pmatrix} b_{i_1} \\ 1 \end{pmatrix}, \dots, \omega_n \begin{pmatrix} b_n \\ 1 \end{pmatrix}$$

where  $\omega_0, \dots, \omega_{i_0-1}, \omega_{i_0+1}, \dots, \omega_{i_1-1}, \omega_{i_1+1}, \dots, \omega_n$  are unknowns, and  $\omega_{i_0}$  and  $\omega_{i_1}$  are the affine functions in these unknowns, defined in i) of proposition (3.7). This gives us the derivatives of  $p$  and  $\omega$  in  $a$  and  $b$ . Eventually, we apply  $d$ -times the recursive formula (3.2) to find the  $d$ -th derivatives of  $X$  in  $a$  and  $b$ .



## 3.3.3 Examples

In the next example,  $X$  is a rational Bézier curve of degree 3, with control points

$$\mathbf{b}_0 = (0, 0, 0)$$

$$\mathbf{b}_1 = (0, 1, 1)$$

$$\mathbf{b}_2 = (0, 2, 5)$$

$$\mathbf{b}_3 = (0, 3, 0)$$

parametrized over  $]0, 1[$ .

We compare, for these control points, the result given by chapter 3.2 and chapter 3.3. In both cases we minimize the functional (3.5) for two parameter values, with  $a_0 = a_1 = \frac{1}{2}$ ;  $\alpha = 0.8$  and  $\beta = 0.2$ .

For the first method, the two parameter values are 0 and 1, and the polynomials are :

$$\begin{aligned} \frac{d^2 X}{dt^2}(0.0) = &+ (0.00, -18.00, -18.00)\omega_1^2 + (0.00, 6.00, 6.00)\omega_1 + (0.00, 0.00, 0.00) \\ &+ (0.00, 12.00, 30.00)\omega_2 \end{aligned}$$

$$\begin{aligned} \frac{d^3 X}{dt^3}(0.0) = &+ (0.00, 162.00, 162.00)\omega_1^3 + (0.00, -108.00, -108.00)\omega_1^2 \\ &+ (0.00, 18.00, 18.00)\omega_1 + (0.00, -162.00, -324.00)\omega_1\omega_2 \\ &+ (0.00, 18.00, 0.00) + (0.00, 72.00, 180.00)\omega_2 \end{aligned}$$

$$\begin{aligned} \frac{d^2 X}{dt^2}(1.0) = &+ (0.00, 0.00, 0.00) + (0.00, -6.00, 30.00)\omega_2 \\ &+ (0.00, 18.00, -90.00)\omega_2^2 + (0.00, -12.00, 6.00)\omega_1 \end{aligned}$$

$$\begin{aligned} \frac{d^3 X}{dt^3}(1.0) = &+ (0.00, 18.00, 0.00) + (0.00, 18.00, -90.00)\omega_2 \\ &+ (0.00, -108.00, 540.00)\omega_2^2 + (0.00, 72.00, -36.00)\omega_1 \\ &+ (0.00, 162.00, -810.00)\omega_2^3 + (0.00, -162.00, 324.00)\omega_1\omega_2 \end{aligned}$$

$$\begin{aligned} 0.8/2 \left( \left\| \frac{d^2 X}{dt^2}(0) \right\|^2 + \left\| \frac{d^2 X}{dt^2}(1) \right\|^2 \right) + 0.2/2 \left( \left\| \frac{d^3 X}{dt^3}(0) \right\|^2 + \left\| \frac{d^3 X}{dt^3}(1) \right\|^2 \right) = \\ + (68234.40)\omega_2^6 + (-90979.20)\omega_2^5 + (-57736.80)\omega_1\omega_2^4 + (48859.20)\omega_2^4 \\ + (46656.00)\omega_1\omega_2^3 + (-11772.00)\omega_2^3 + (26244.00)\omega_1^2\omega_2^2 \\ + (-26460.00)\omega_1\omega_2^2 + (5004.00)\omega_2^2 + (-15746.40)\omega_1^4\omega_2 \\ + (18662.40)\omega_1^3\omega_2 + (-12463.20)\omega_1^2\omega_2 + (1051.20)\omega_1\omega_2 \\ + (324.00)\omega_2 + (5248.80)\omega_1^5 + (-6998.40)\omega_1^5 \\ + (3758.40)\omega_1^4 + (-367.20)\omega_1^3 + (424.80)\omega_1^2 \\ + (324.00)\omega_1 + (64.80) \end{aligned}$$

For  $\omega_1 = \omega_2 = 1$ , the value of this last polynomial is equal to **2332.80**.

We find a partial minimum for the weights

$$w_1 = 1.000000$$

$$w_2 = 1.000000$$

In these weights the polynomial is equal to **415,90**.

With the second method, we minimize the functional (3.4) for the two parameter values  $a = 0.2$  and  $b = 0.64$ . (0.64 is the parameter value in which the non-rational Bézier curve with control points  $(b_0, \dots, b_4)$  takes its highest curvature value). The polynomials involved are the following :

$$\begin{aligned} \frac{d^2 X}{dt^2}(0.2) = &+ (0.00, -0.14, -0.62)\omega_3^3 + (0.00, -0.45, -2.97)\omega_3^2 \\ &+ (0.00, -1.62, -5.40)\omega_0\omega_3^2 + (0.00, 2.62, 4.88)\omega_3 \\ &+ (0.00, 9.90, 19.38)\omega_0\omega_3 + (0.00, -6.48, -13.39)\omega_0^2\omega_3 \\ &+ (0.00, 0.00, 0.00) + (0.00, -42.00, -48.00)\omega_0 \\ &+ (0.00, 46.80, 56.64)\omega_0^2 + (0.00, -8.64, -6.91)\omega_0^3 \end{aligned}$$

$$\begin{aligned} \frac{d^3 X}{dt^3}(0.2) = &+ (0.00, -0.30, -1.40)\omega_3^4 + (0.00, -0.84, -5.91)\omega_3^3 \\ &+ (0.00, -4.86, -17.74)\omega_0\omega_3^3 + (0.00, 6.84, 16.41)\omega_3^2 \\ &+ (0.00, 30.38, 70.20)\omega_0\omega_3^2 + (0.00, -29.16, -78.73)\omega_0^2\omega_3^2 \\ &+ (0.00, 2.34, -11.72)\omega_3 + (0.00, -108.38, -142.12)\omega_0\omega_3 \\ &+ (0.00, 283.50, 537.30)\omega_0^2\omega_3 + (0.00, -77.76, -136.08)\omega_0^3\omega_3 \\ &+ (0.00, 0.00, 0.00) + (0.00, 600.00, 600.00)\omega_0 \\ &+ (0.00, -1218.00, -1488.00)\omega_0^2 + (0.00, 594.00, 648.00)\omega_0^3 \\ &+ (0.00, -77.76, -62.21)\omega_0^4 \end{aligned}$$

$$\begin{aligned} \frac{d^2 X}{dt^2}(0.8) = &+ (0.00, 8.64, -48.38)\omega_3^3 + (0.00, -46.80, 204.48)\omega_3^2 \\ &+ (0.00, 6.48, -15.12)\omega_0\omega_3^2 + (0.00, 42.00, -192.00)\omega_3 \\ &+ (0.00, -9.90, 15.78)\omega_0\omega_3 + (0.00, 1.62, 1.51)\omega_0^2\omega_3 \\ &+ (0.00, 0.00, 0.00) + (0.00, -2.62, -8.62)\omega_0 \\ &+ (0.00, 0.45, 2.19)\omega_0^2 + (0.00, 0.13, 0.57)\omega_0^3 \end{aligned}$$

$$\begin{aligned}
\frac{d^3 X}{dt^3}(0.8) = & + (0.00, -77.76, 435.46)\omega_3^4 + (0.00, 594.00, -2808.00)\omega_3^5 \\
& + (0.00, -77.76, 244.94)\omega_0\omega_3^3 + (0.00, -1218.00, 5280.00)\omega_3^2 \\
& + (0.00, 283.50, -677.70)\omega_0\omega_3^2 + (0.00, -29.16, 20.41)\omega_0^2\omega_3^2 \\
& + (0.00, 600.00, -3000.00)\omega_3 + (0.00, -108.38, 431.62)\omega_0\omega_3 \\
& + (0.00, 30.37, -10.80)\omega_0^2\omega_3 + (0.00, -4.86, -8.50)\omega_0^3\omega_3 \\
& + (0.00, 0.00, 0.00) + (0.00, 2.34, 2.34)\omega_0 + (0.00, 6.84, 23.72)\omega_0^2 \\
& + (0.00, -0.84, -4.22)\omega_0^3 + (0.00, -0.30, -1.28)\omega_0^4 \\
0.8/2 \left( \left\| \frac{d^2 X}{dt^2}(0.2) \right\|^2 + \left\| \frac{d^2 X}{dt^2}(0.8) \right\|^2 \right) + 0.2/2 \left( \left\| \frac{d^3 X}{dt^3}(0.2) \right\|^2 + \left\| \frac{d^3 X}{dt^3}(0.8) \right\|^2 \right) = \\
& + (19567.05)\omega_3^8 + (22547.03)\omega_0\omega_3^7 + (-253788.23)\omega_3^7 + (8893.22)\omega_0^2\omega_3^6 \\
& + (-210228.77)\omega_0\omega_3^6 + (1303518.75)\omega_3^6 + (1138.75)\omega_0^5\omega_3^5 \\
& + (-54297.27)\omega_0^2\omega_3^5 + (731681.56)\omega_0\omega_3^5 + (-3388805.25)\omega_3^5 \\
& + (964.63)\omega_0^4\omega_3^4 + (-5084.38)\omega_0^3\omega_3^4 + (117871.34)\omega_0^2\omega_3^4 \\
& + (-1199216.25)\omega_0\omega_3^4 + (4717641.50)\omega_3^4 + (2828.40)\omega_0^5\omega_3^3 \\
& + (-14143.81)\omega_0^4\omega_3^3 + (18327.41)\omega_0^3\omega_3^3 + (-112940.15)\omega_0^2\omega_3^3 \\
& + (925985.31)\omega_0\omega_3^3 + (-3347188.25)\omega_3^3 + (3895.65)\omega_0^8\omega_3^2 \\
& + (-33913.58)\omega_0^5\omega_3^2 + (84297.98)\omega_0^4\omega_3^2 + (-64059.95)\omega_0^3\omega_3^2 \\
& + (61339.09)\omega_0^2\omega_3^2 + (-270847.19)\omega_0\omega_3^2 + (951477.88)\omega_3^2 \\
& + (2904.84)\omega_0^7\omega_3 + (-37958.81)\omega_0^6\omega_3 + (166180.48)\omega_0^5\omega_3 \\
& + (-286455.41)\omega_0^4\omega_3 + (172153.98)\omega_0^3\omega_3 + (-41568.92)\omega_0^2\omega_3 \\
& + (-1288.78)\omega_0\omega_3 + (-0.12)\omega_3 + (991.82)\omega_0^8 + (-17298.91)\omega_0^7 \\
& + (114773.91)\omega_0^6 + (-354996.84)\omega_0^5 + (521578.19)\omega_0^4 \\
& + (-328469.25)\omega_0^3 + (73660.86)\omega_0^2 + (-0.01)\omega_0
\end{aligned}$$

For  $\omega_0 = \omega_3 = 1$  the last polynomial is equal to **1669.48**, and we find a partial minimum in the weights

$$w_0 = 0.924456$$

$$w_3 = 1.132421$$

In these weights the polynomial to minimize is equal to **647.52**.

Eventually, the condition i) of proposition (3.7) gives us the value of the other weights

$$w_1 = 1.15$$

$$w_2 = 1.13$$

Figure 17 shows, at the top, the non-rational curve, in the middle of the page, the curve found by the first method, and at the bottom, the curve found with the second method.

Each curve  $(u \mapsto X(u))$  is represented twice, to the left together with the curve  $(u \mapsto X(u) + f(u)N(u))$  where  $f(u) = \kappa^2 \|X''(u)\|$ , and to the right taking  $f(u) = \frac{0.8}{2} \|X''(u)\|^2 + \frac{0.2}{2} \|X'''(u)\|^2$ .

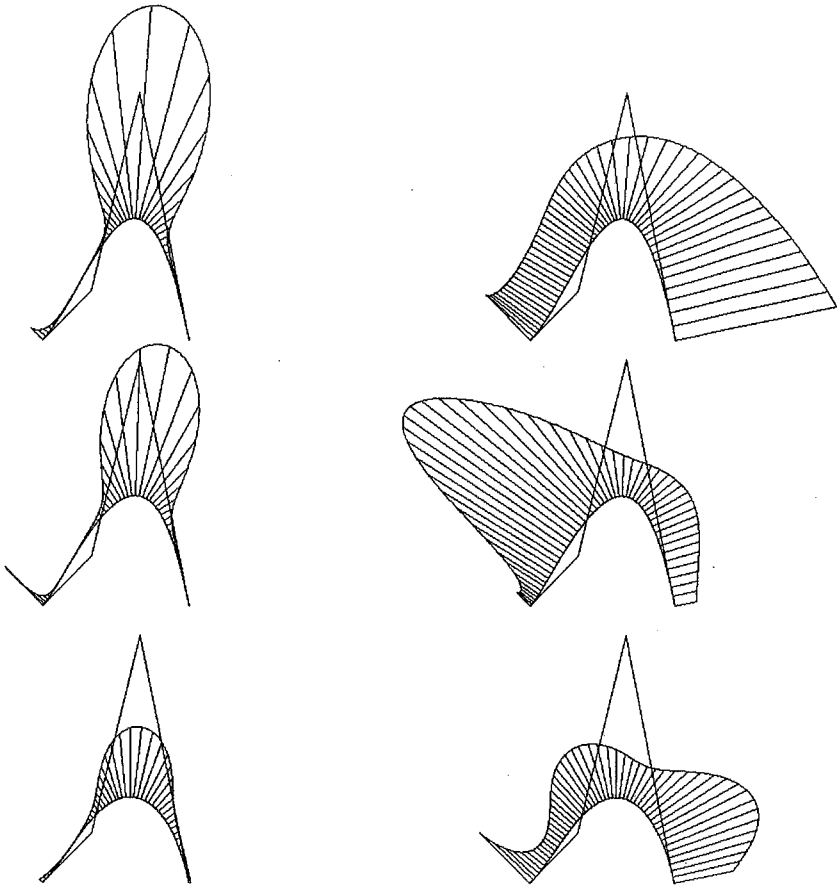


Fig. 17

## 4 Rational Tensor Product Bézier Patches with Twist of Minimum Energy

In chapter 2, we locally minimized the bending energy of a curve. The equivalent of the bending energy of a curve for a surface  $S$ , is the strain energy  $\int_S \kappa_1^2 + \kappa_2^2 ds$  (see 1.6).

Hagen and Farin find an optimal value of the normal component of the twist vector, for which the quantity  $(\kappa_1^2 + \kappa_2^2)ds$  is minimized. They apply this result to non-rational tensor product surfaces. The object of this chapter is to use this optimal component of the twist vector, to find values for the inner-weights of a rational tensor product Bézier patch.

In paragraph 4.1, we recall the result of Hagen and Farin [FAR90], while in paragraph 4.2 we apply this result to rational tensor product patches.

### 4.1 An Optimal Normal Component of the Twist Vector

**Theorem 4.1** (Farin, Hagen)

- Let  $X : U \rightarrow \mathbf{R}^3$  be a regular parametric surface of class  $C^3$ .
  - Let  $(g_{ij}), (h_{ij})$  be the first and second fundamental forms of  $X$ .
- (i) There exists a function  $f : U \times \mathbf{R} \rightarrow \mathbf{R}$  of class  $C^1$  such that

$$\iint_U (\kappa_1^2 + \kappa_2^2) \sqrt{g} du dv = \iint_U f(u, v, h_{12}(u, v)) du dv.$$

- (ii) The partial derivative of  $f$  along  $h_{12}$  vanishes if and only if

$$h_{12} = \frac{g_{12}(g_{11}h_{22} + g_{22}h_{11})}{g_{11}g_{22} - g_{12}^2}. \quad (4.1)$$

Proof :

(i) As a consequence of (1.3), it follows :

$$\iint_U (\kappa_1^2 + \kappa_2^2) \sqrt{g} dudv = \iint_U \left[ \left( \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g} \right)^2 - 2 \frac{h}{g} \right] \sqrt{g} dudv ,$$

and the function  $f$

$$f : U \times \mathbf{R} \rightarrow \mathbf{R}$$

$$(u, v, x) \mapsto \left[ \left( \frac{g_{11}h_{22} - 2g_{12}x + g_{22}h_{11}}{g} \right)^2 - 2 \left( \frac{h_{11}h_{22} - x^2}{g} \right) \right] \sqrt{g}$$

is of class  $C^1$  because the surface  $X$  is of class  $C^3$ .

(ii) is proved by calculating the partial derivative  $\partial f / \partial x$

$$\frac{\partial f}{\partial x}(u, v, x) = \frac{-4\sqrt{g}}{g^2} [(-2g_{12}^2 - g)x + g_{12}(g_{11}h_{22} + g_{22}h_{11})] .$$

Now,  $X$  is regular implies that  $g$  is positiv, and all the more so  $g + 2g_{12}^2$ . Therefore, dividing by  $\frac{-4\sqrt{g}}{g^2}$  and  $2g_{12}^2 + g$  leads to the necessary and sufficient condition. □

## 4.2 Minimum Energy Twist Weights

Farin and Hagen applied this theorem to non-rational tensor product surfaces in the following way : They supposed that all the data (i.e. Bézier control points for a Bézier patch, partial derivatives in the corners for a generalized Coons patch) except the normal component of the twist vectors in the corners (i.e. normal component of  $b_{11}, b_{m-1,1}, b_{1,m-1}, b_{m-1,m-1}$  for a Bézier patch and normal component of  $\frac{\partial X}{\partial u \partial v}$  in the corners for a generalized Coons patch) are fixed. Then they applied the condition (4.1) in the four corners of the patch to find the rest of the data.

Therefore they find the unique patch, which interpolates the fixed data, and minimizes the local strain energy in each corner.

In paragraph 4.2 we apply theorem 4.1 in an analogous sense, to rational tensor product Bézier patches. We supposed that all the data, except the four weights  $\omega_{11}, \omega_{1,m-1}, \omega_{m-1,1}, \omega_{m-1,m-1}$ , are fixed, and we find the unique values of these weights, for which the strain energy of flexure and torsion is locally, in each corner of the patch, minimized.

We call these weights the "minimum energy twist weights".

4.2.1 The first and second fundamental form in the corners of a rational tensor product Bézier patch

**Proposition 4.2**

Let  $X$  be a rational tensor product Bézier patch of degree  $(m, n)$  parametrized over  $[u_0, u_1] \times [v_0, v_1]$ .

Let  $(b_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  and  $(\omega_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  be the control points and -weights of  $X$ .

The first fundamental form  $(g_{ij})_{\substack{i=1,2 \\ j=1,2}}$  in  $(u_0, v_0)$  is given by :

$$g_{11}(u_0, v_0) = \frac{m^2}{(u_1 - u_0)^2} \frac{\omega_{10}^2}{\omega_{00}^2} \|b_{10} - b_{00}\|^2$$

$$g_{22}(u_0, v_0) = \frac{n^2}{(v_1 - v_0)^2} \frac{\omega_{01}^2}{\omega_{00}^2} \|b_{01} - b_{00}\|^2$$

$$g_{12}(u_0, v_0) = \frac{mn}{(u_1 - u_0)(v_1 - v_0)} \frac{\omega_{10}\omega_{01}}{\omega_{00}^2} \langle b_{10} - b_{00}, b_{01} - b_{00} \rangle,$$

$$g_{21}(u_0, v_0) = g_{12}(u_0, v_0)$$

and the second fundamental form  $(h_{ij})_{\substack{i=1,2 \\ j=1,2}}$  in  $(u_0, v_0)$  is given as follows :

$$h_{11}(u_0, v_0) = \frac{m(m-1)}{(u_1 - u_0)^2} \frac{\omega_{20}}{\omega_{00}} \langle b_{20} - b_{00}, N \rangle$$

$$h_{22}(u_0, v_0) = \frac{n(n-1)}{(v_1 - v_0)^2} \frac{\omega_{02}}{\omega_{00}} \langle b_{02} - b_{00}, N \rangle$$

$$h_{12}(u_0, v_0) = \frac{mn}{(u_1 - u_0)(v_1 - v_0)} \frac{\omega_{11}}{\omega_{00}} \langle b_{11} - b_{00}, N \rangle$$

$$h_{21}(u_0, v_0) = h_{12}(u_0, v_0)$$

with  $N = \frac{[b_{10} - b_{00}, b_{01} - b_{00}]}{\| [b_{10} - b_{00}, b_{01} - b_{00}] \|}$ .

Proof :

The first and second order derivatives in  $(u_0, v_0)$  are given in Prop. 5.2. Prop. 5.2, together with the definitions (1.17) and (1.18) of the first and second fundamental forms yields Prop. 4.2. □

**Remark 4.3:**

- For symmetry reasons, the first and second fundamental forms in  $(u_1, v_0)$  ( resp.  $(u_0, v_1)$ ,  $(u_1, v_1)$  ) are given by Prop. 4.2, after changing the indices  $(i, j)$  of the weights and control points by  $(m - i, j)$  ( resp.  $(i, n - j)$ ,  $(m - i, n - j)$ ) and taking the opposite sign for  $N(u_1, v_0)$  ( resp. opposite sign for  $N(u_0, v_1)$ , same sign for  $N(u_1, v_1)$ ).

**4.2.2 The weights for the optimal normal component of the twist vectors**

**Proposition 4.4**

Let  $X$  be a rational tensor product Bézier patch of degree  $(m, n)$ .

Let  $(b_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  and  $(\omega_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  be the control points and -weights of  $X$ .

If  $m \geq 3$ ,  $n \geq 3$  and if  $b_{00}, b_{10}, b_{01}, b_{11}$  are not coplanar then

- $(\kappa_1^2 + \kappa_2^2)\sqrt{g}(u_0, v_0)$  is a polynomial of degree 2 in  $\omega_{11}$  (degree 0 in  $\omega_{1, n-1}$ ,  $\omega_{m-1, 1}$ ,  $\omega_{m-1, n-1}$ )
- $(\kappa_1^2 + \kappa_2^2)\sqrt{g}(u_0, v_0)$  is minimum if and only if

$$\omega_{11} = \frac{\langle b_{10} - b_{00}, b_{01} - b_{00} \rangle}{\langle b_{11} - b_{00}, N \rangle} \tag{4.2}$$

$$\frac{m(n-1)\omega_{10}^2\omega_{20}\|b_{10} - b_{00}\|^2\langle b_{02} - b_{00}, N \rangle + n(m-1)\omega_{01}^2\omega_{02}\|b_{01} - b_{00}\|^2\langle b_{20} - b_{00}, N \rangle}{\|b_{10} - b_{00}\|^2\|b_{01} - b_{00}\|^2 + \langle b_{10} - b_{00}, b_{01} - b_{00} \rangle^2}$$

where  $N = \frac{[b_{10} - b_{00}, b_{01} - b_{00}]}{\|[b_{10} - b_{00}, b_{01} - b_{00}]\|}$ .

i) and ii) are also true in  $(u_1, v_0)$  ( resp.  $(u_0, v_1)$ ,  $(u_1, v_1)$  ) after changing the indices  $(i, j)$  of the weights and control points in  $(m - i, j)$  ( resp.  $(i, n - j)$ ,  $(m - i, n - j)$ ).

**Proof :**

The conditions  $m \geq 3$ ,  $n \geq 3$  imply that the patch has actually four "twist" weights  $\omega_{1,1}$ ,  $\omega_{m-1,1}$ ,  $\omega_{1, n-1}$ ,  $\omega_{m-1, n-1}$ . Equation (1.3) together with Prop. 4.2 proves that  $(\kappa_1^2 + \kappa_2^2)\sqrt{g}(u_0, v_0)$  is a polynomial of degree 2 in  $\omega_{1,1}$  if and only if the leading coefficient (i.e.  $\langle b_{11} - b_{00}, N \rangle$ ) doesn't vanish. This is ensured by the condition  $b_{00}, b_{10}, b_{01}, b_{11}$  not coplanar.

Replacing the values given by Prop. 4.2 in the equation (4.1) leads to the solution (4.2). □



**Remark 4.5:**

- One has to be careful with the fact that the weights given by Prop. 4.4 only ensure the minimization of  $(\kappa_1^2 + \kappa_2^2)\sqrt{g}$  in the corners of the patch. This choice of weights can be inconvenient for the rest of the patch.

If, for example,  $\mathbf{b}_{11} - \mathbf{b}_{00}$  is almost orthogonal to the normal  $N$  ( $\langle \mathbf{b}_{11} - \mathbf{b}_{00}, N \rangle \simeq 0$ ), (4.2) can lead to a negative value of  $\omega_{1,1}$ , for which the surface  $X$  is not defined over the whole parameter domain  $[u_0, u_1] \times [v_0, v_1]$ . However, in this case, there still exists a neighbourhood of  $(u_0, v_0)$  in which  $X$  is defined, and the strain energy is still minimized in  $(u_0, v_0)$ .

**4.2.3 Example**

We choose a rational Bézier patch with high curvatures at the four corners to visualize the results of Prop. 4.4. Fig. 18 shows at the top the original patch, and at the bottom the rational patch with the minimum energy "twist" weights resulting from Prop. 4.4. Each surface  $X$  ( $X \rightarrow X(u, v)$ ) is represented together with the surface  $Y$  ( $Y(u, v) \rightarrow X(u, v) + f(u, v) \cdot N(u, v)$ ), where  $f(u, v)$  is proportional to the local bending energy  $(\kappa_1^2 + \kappa_2^2)\sqrt{g}$  in the point  $X(u, v)$  and  $N(u, v)$  is the normal to the surface  $X$  at this point. Thus, the distances between the two surfaces  $X$  and  $Y$  at the corners, are a measure of the function that we want to minimize.

Although Prop. 4.4 only ensured in general a minimization of the local energy in the corners (s. Remark 4.5), Fig. 18 shows that, the strain energy is minimized over the entire patch in this example.

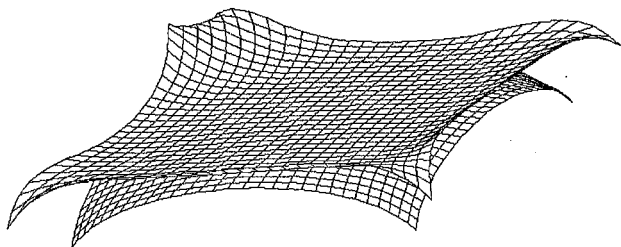
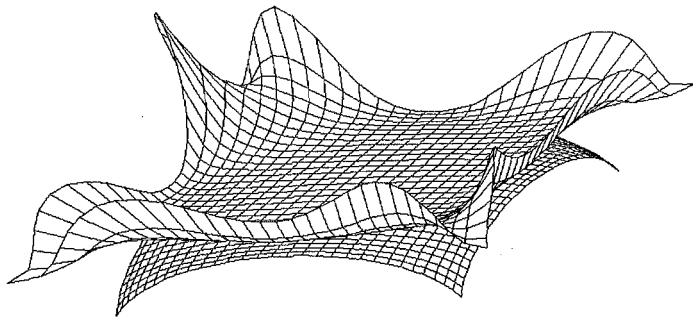


Fig. 18

## 5 Rational Tensor Product Bézier Patches with Locally Minimum Norm of Derivatives

The generalization of classical spline functions for functions of two variables was first introduced by Atteia in 1966 [ATT66]. He minimized the integral

$$\iint \|X_{uu}\|^2 + 2\|X_{uv}\|^2 + \|X_{vv}\|^2 \quad (5.1)$$

over all the surfaces  $X$  interpolating a given set of points in  $\mathbb{R}^3$ . Other analogous criteria are successfully used by Harder and Desmaray [HAR72], Duchon [DUC77] and Franke [FRA85].

For the same reasons as in chapter 3, we replace the integral by a quadrature formula : we minimize the sum of the scalar values  $\|X_{uu}\|^2 + 2\|X_{uv}\|^2 + \|X_{vv}\|^2$  in the corners of a rational tensor product Bézier patch, which control points are given by a user.

In chapter 3 we used the standard reparametrization of the Bézier curves (Prop. 3.2) to reduce the degree of the resolving system. In paragraph 5.1, we will see that such a rational linear reparametrization, setting the corner weights to one while remaining the control polygon, doesn't always exist in the case of rational tensor product Bézier patches. Anyway, the condition that the four corner weights are equal to one can be imposed. In this case, the degree of the resolving system is reduced, with the drawback that the criterion function is minimized over a smaller set : The set of rational tensor product Bézier patches with corner weights equal to one (and the users given control structure).

In paragraph 5.2, we use this condition to resolve our problem and present an example. In paragraph 5.3, we find out other stronger but useful conditions on the weights, which allow to minimize the function  $\|X_{uu}\|^2 + 2\|X_{uv}\|^2 + \|X_{vv}\|^2$  in other parameter points of the boundary curves.

### 5.1 Standard Reparametrization for Rational Tensor Product Bézier Surfaces

**Proposition 5.1**

Let  $X$  be a rational tensor product Bézier patch of degree  $m, n$  parametrized over  $[u_0, u_1] \times [v_0, v_1]$ , with control points  $(b_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  and positive control weights  $(\omega_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$ .

There exists a rational linear reparametrization  $\varphi$  such that

- ii)  $\varphi(u_i, v_i) = (u_i, v_i) \quad \varphi([u_0, u_1] \times [v_0, v_1]) = [u_0, u_1] \times [v_0, v_1] \quad (5.2)$
- i) the reparametrized patch  $X(\varphi)$  is a rational tensor product Bézier patch with the same control points as  $X$  and with control weights  $(\bar{\omega}_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  such that

$$\bar{\omega}_{ij} > 0 \tag{5.3}$$

$$\bar{\omega}_{00} = \bar{\omega}_{m0} = \bar{\omega}_{0n} = \bar{\omega}_{mn} = 1$$

if and only if the following condition holds :

- iii) 
$$\frac{\omega_{m0}}{\omega_{00}} = \frac{\omega_{mn}}{\omega_{0n}} \tag{5.4}$$

**Proof :**

The proof is analogous to the proof of Prop. 3.2. The conditions (5.2) imply that  $\varphi$  is of the following form :

$$\varphi = \left( \frac{\rho_u u_1(u - u_0) + \hat{\rho}_u u_0(u_1 - u)}{\rho_u(u - u_0) + \hat{\rho}_u(u_1 - u)}, \frac{\rho_v v_1(v - v_0) + \hat{\rho}_v v_0(v_1 - v)}{\rho_v(v - v_0) + \hat{\rho}_v(v_1 - v)} \right)$$

where  $\rho_u, \hat{\rho}_u, \rho_v, \hat{\rho}_v$  are four positive scalars.

The reparametrized surface  $X(\varphi)$  is equal to :

$$X(\varphi(u, v)) = \frac{\sum_{i=0}^m \sum_{j=0}^n \rho_u^i \hat{\rho}_u^{m-i} \rho_v^j \hat{\rho}_v^{n-j} \omega_{ij} b_{ij} B_i^m(s) B_j^n(t)}{\sum_{i=0}^m \sum_{j=0}^n \rho_u^i \hat{\rho}_u^{m-i} \rho_v^j \hat{\rho}_v^{n-j} \omega_{ij} B_i^m(s) B_j^n(t)}$$

$X(\varphi)$  is the rational tensor product Bézier surface with the same control points as  $X$  and with the control weights  $(\bar{\omega}_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$

$$\bar{\omega}_{ij} = \rho_u^i \hat{\rho}_u^{m-i} \rho_v^j \hat{\rho}_v^{n-j} \omega_{ij}$$

The conditions (5.3) are equivalent to :

$$\hat{\rho}_u^m \hat{\rho}_v^n = \frac{1}{\omega_{00}} \quad (5.5.1)$$

$$\rho_u^m \hat{\rho}_v^n = \frac{1}{\omega_{m0}} \quad (5.5.2)$$

$$\hat{\rho}_u^m \rho_v^n = \frac{1}{\omega_{0n}} \quad (5.5.3)$$

$$\rho_u^m \rho_v^n = \frac{1}{\omega_{mn}} \quad (5.5.4)$$

Dividing (5.5.1) by (5.5.2) and (5.5.3) by (5.5.4) leads to the necessary condition (5.4).

Now if (5.4) is fulfilled, then taking

$$\begin{aligned} \hat{\rho}_u^n &= \left( \frac{1}{\omega_{00}} \right)^{\frac{1}{m}}, \\ \rho_u &= \left( \frac{1}{\omega_{m0}} \right)^{\frac{1}{m}}, \\ \hat{\rho}_v &= 1, \\ \rho_v &= \left( \frac{\omega_{00}}{\omega_{0n}} \right)^{\frac{1}{n}} \end{aligned}$$

yields a reparametrization  $\varphi$  such that i) and ii) are true. □

### Remark 5.2:

- In the curve case we saw (Prop. 3.2) that all rational Bézier segments with a given control polygon can be reparametrized with a rational linear function such that the endpoint weights are changed into one. Prop. 5.1 shows that the equivalent for the surfaces is no longer true: For any control polygon, there exist some rational tensor product Bézier surfaces for which no rational linear reparametrization change the corner weights into one.

This will have consequences in paragraph 5.2.

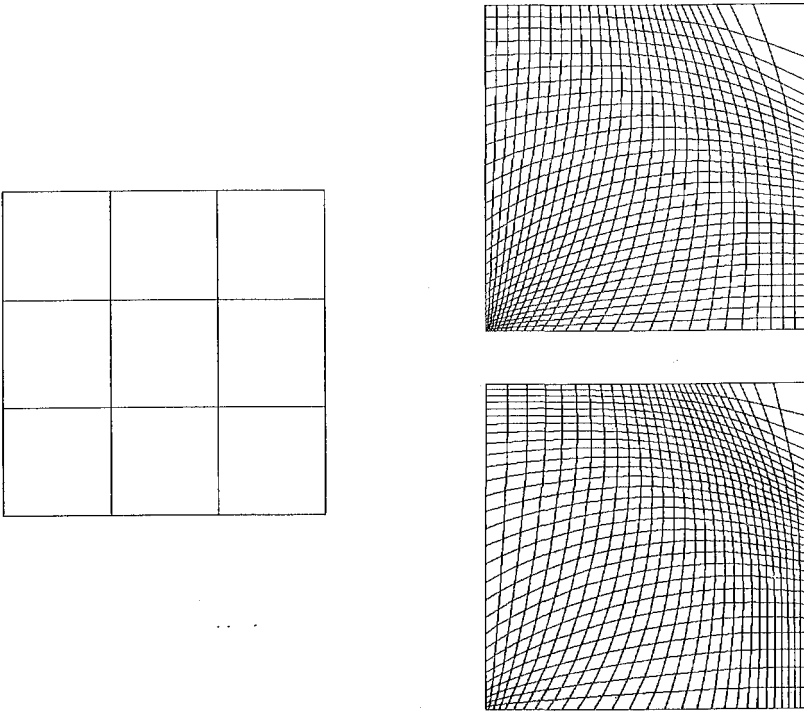


Fig. 19

**Example :**

We choose a plane bicubic rational Bézier patch to focus the attention on the parameter lines, and not on the surface, which is not modified by the reparametrization. The control points are  $(i, j)_{\substack{i=0,\dots,3 \\ j=0,\dots,3}}$ . The control polygon is represented to the left of Fig. 19.

The weights of the upper right patch are equal to one, except the bottom left and upper right corner weights, which are respectively equal to 8 and  $1/8$ . Thus the condition iii) of Prop. 5.1 is verified, and the upper right patch can be reparametrized. The result of this reparametrization is the bottom right patch.

## 5.2 The Resolving System

### 5.2.1 The derivatives of a rational tensor product Bézier patch in the corners

#### Proposition 5.2

Let  $X$  be a rational tensor product Bézier patch of degree  $m, n$  parametrized over  $[u_0, u_1] \times [v_0, v_1]$ , with control points  $(b_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$  and control weights  $(\omega_{ij})_{\substack{i \in \{0, \dots, m\} \\ j \in \{0, \dots, n\}}}$ .

The second partial derivatives of  $X$  in  $(u_0, v_0)$  are given by :

$$\frac{\partial^2 X}{\partial u^2}(u_0, v_0) = \frac{m}{(u_1 - u_0)^2} \frac{1}{\omega_{00}^2} \left[ 2(\omega_{00}\omega_{10} - m\omega_{10}^2)(b_{10} - b_{00}) + (m+1)\omega_{00}\omega_{20}(b_{20} - b_{00}) \right] \quad (5.6.a)$$

$$\frac{\partial^2 X}{\partial u \partial v}(u_0, v_0) = \frac{mn}{(u_1 - u_0)(v_0 - v_1)} \frac{1}{\omega_{00}^2} \left[ \omega_{00}\omega_{11}(b_{11} - b_{00}) - \omega_{10}\omega_{01} \left( (b_{10} - b_{00}) + (b_{01} - b_{00}) \right) \right] \quad (5.6.b)$$

$$\frac{\partial^2 X}{\partial v^2}(u_0, v_0) = \frac{n}{(v_1 - v_0)^2} \frac{1}{\omega_{00}^2} \left[ 2(\omega_{00}\omega_{01} - n\omega_{01}^2)(b_{01} - b_{00}) + (n+1)\omega_{00}\omega_{02}(b_{02} - b_{00}) \right] \quad (5.6.c)$$

Proof :

- $(u \rightarrow X(u, v_0))$  (resp.  $(v \rightarrow X(u_0, v))$ ) is the rational Bézier curve with control points  $(b_{00}, \dots, b_{m0})$  (resp.  $(b_{00}, \dots, b_{0n})$ ) and control weights  $(\omega_{00}, \dots, \omega_{m0})$  (resp.  $(\omega_{00}, \dots, \omega_{0n})$ ). Therefore, the partial derivatives  $\frac{\partial^2 X}{\partial u^2}(u_0, v_0)$  and  $\frac{\partial^2 X}{\partial v^2}(u_0, v_0)$  are given by Prop. 3.1.
- Let us write as usual  $X = \frac{p}{\omega}$ .

The equivalent of the recurrence formula (3.2) in the surface case gives for  $\frac{\partial^2 X}{\partial u \partial v}(u_0, v_0)$  :

$$\frac{\partial^2 X}{\partial u \partial v}(u_0, v_0) = \frac{1}{\omega(u_0, v_0)} \left[ \frac{\partial^2 p}{\partial u \partial v}(u_0, v_0) - \frac{\partial X}{\partial u}(u_0, v_0) \frac{\partial \omega}{\partial v}(u_0, v_0) - \frac{\partial X}{\partial v}(u_0, v_0) \frac{\partial \omega}{\partial u}(u_0, v_0) - X(u_0, v_0) \frac{\partial^2 \omega}{\partial u \partial v}(u_0, v_0) \right] \quad (5.7)$$

The partial derivatives of the right member are given by the de Casteljau algorithm and by Prop. 3.1 :

$$\begin{aligned} \frac{\partial^2 \mathbf{p}}{\partial u \partial v}(u_0, v_0) &= \frac{mn}{(u_1 - u_0)(v_1 - v_0)} \cdot (\omega_{11} \mathbf{b}_{11} - \omega_{10} \mathbf{b}_{10} - \omega_{01} \mathbf{b}_{01} + \omega_{00} \mathbf{b}_{00}) \\ \frac{\partial^2 \omega}{\partial u \partial v}(u_0, v_0) &= \frac{mn}{(u_1 - u_0)(v_1 - v_0)} \cdot (\omega_{11} - \omega_{10} - \omega_{01} + \omega_{00}) \\ \frac{\partial \omega}{\partial u}(u_0, v_0) &= \frac{n}{(u_1 - u_0)} \cdot (\omega_{10} - \omega_{00}) \\ \frac{\partial \omega}{\partial v}(u_0, v_0) &= \frac{m}{(v_1 - v_0)} \cdot (\omega_{01} - \omega_{00}) \\ \frac{\partial X}{\partial u}(u_0, v_0) &= \frac{m}{(u_1 - u_0)} \frac{\omega_{10}}{\omega_{00}} \cdot (\mathbf{b}_{10} - \mathbf{b}_{00}) \\ \frac{\partial X}{\partial v}(u_0, v_0) &= \frac{n}{(v_1 - v_0)} \frac{\omega_{01}}{\omega_{00}} \cdot (\mathbf{b}_{01} - \mathbf{b}_{00}) \end{aligned}$$

Inserting these formulas in (5.7), we get :

$$\begin{aligned} \frac{\partial^2 X}{\partial u \partial v}(u_0, v_0) &= \frac{mn}{(u_1 - u_0)(v_1 - v_0)} \cdot \frac{1}{\omega_{00}^2} \left[ \omega_{00}(\omega_{11} \mathbf{b}_{11} - \omega_{10} \mathbf{b}_{10} - \omega_{01} \mathbf{b}_{01} + \omega_{00} \mathbf{b}_{00}) \right. \\ &\quad - \omega_{10}(\omega_{01} - \omega_{00})(\mathbf{b}_{10} - \mathbf{b}_{00}) \\ &\quad - \omega_{01}(\omega_{10} - \omega_{00})(\mathbf{b}_{01} - \mathbf{b}_{00}) \\ &\quad \left. - \omega_{00}(\omega_{11} - \omega_{10} - \omega_{01} + \omega_{00}) \mathbf{b}_{00} \right] \end{aligned}$$

We rewrite  $\frac{\partial^2 X}{\partial u \partial v}(u_0, v_0)$  as a linear combination of the vectors  $(\mathbf{b}_{11} - \mathbf{b}_{00})$ ,  $(\mathbf{b}_{10} - \mathbf{b}_{00})$ ,  $(\mathbf{b}_{01} - \mathbf{b}_{00})$  :

$$\begin{aligned} \frac{\partial^2 X}{\partial u \partial v}(u_0, v_0) &= \frac{mn}{(u_1 - u_0)(v_1 - v_0)} \cdot \frac{1}{\omega_{00}^2} \left[ \omega_{00} \omega_{11} (\mathbf{b}_{11} - \mathbf{b}_{00}) \right. \\ &\quad - \omega_{00} \omega_{10} (\mathbf{b}_{10} - \mathbf{b}_{00}) \\ &\quad - \omega_{00} \omega_{01} (\mathbf{b}_{01} - \mathbf{b}_{00}) \\ &\quad - \omega_{10} (\omega_{01} - \omega_{00}) (\mathbf{b}_{10} - \mathbf{b}_{00}) \\ &\quad \left. - \omega_{01} (\omega_{10} - \omega_{00}) (\mathbf{b}_{01} - \mathbf{b}_{00}) \right] \end{aligned}$$

Adding the two coefficients of  $(\mathbf{b}_{10} - \mathbf{b}_{00})$  and  $(\mathbf{b}_{01} - \mathbf{b}_{00})$  we get (5.6.b). □

### 5.2.2 Degree of the resolving system

Our aim in chapter 5 is to minimize the sum of the scalar functions  $\|X_{uu}\|^2 + 2\|X_{uv}\|^2 + \|X_{vv}\|^2$  in the corners of a rational tensor product Bézier patch which control points are



given by a user. Prop. 5.3 shows that the second partial derivatives are rational functions of the weights of total degree 2, with denominator equal to  $\frac{1}{\omega_{00}^2}, \frac{1}{\omega_{m0}^2}, \frac{1}{\omega_{0n}^2}, \frac{1}{\omega_{mn}^2}$ .

In the curve case, we could set without lost of generality the endpoint weights to one, so that the degree of the resolving system was reduced.

In the surface case, we still have to fix the value of the corner weights (if not, the degree of the resolving system would be 16 !), but this implies now a restriction of the set of patches in which we minimize the criterion function : We only can minimize this function over patches fullfilling condition iii) of Prop. 5.1. This proposition says that if the condition iii) is verified, then there exists a reparametrization transforming the corner weights in one. With this restriction, the degree of the resolving system is reduced to 3, because the second partial derivatives become polynomial functions of the weights with total degree 2.

### 5.3 Other Criteria

In paragraph 5.2.2, we make a restriction to solve our problem : we consider only patches such that the corner weights can be fix to one. Now, it is possible to set arbitrary other weights to one. The set of patches in which the criterion function is minimized become smaller, both further applications are possible.

In the following, we dicuss two possible applications :

- If the boundary weights are equal to one, the denominator of the parametric equation is identically equal to one along the boundaries. In other words, the boundary curves are non rational curves. Therefore, the second partial derivatives in the points of the boundary curves are polynomial functions of the weights, with low degree.

This allows to use the following criterion function :

$$\sum_{i \in I} \left\| \frac{\partial^2 X}{\partial u^2}(u_i, v_i) \right\|^2 + 2 \left\| \frac{\partial^2 X}{\partial u \partial v}(u_i, v_i) \right\|^2 + \left\| \frac{\partial^2 X}{\partial v^2}(u_i, v_i) \right\|^2 \tag{5.8}$$

where  $\{(u_i, v_i), i \in I\}$  is a set of parameter points of the boundary curves.

This criterion function is a polynomial function of the inner weights.

- For surfaces with adjacent patches, the  $G^1$ -continuity between two adjacent patches depends on the weights (this is not the case for the  $G^1$ -continuity between two consecutive curve segments). The necessary and sufficient  $G^1$ -continuity conditions between two adjacent patches with same control-points and -weights along a common boundary curve of degree  $n$  consists of  $16 \cdot n$  linear homogeneous equations with  $12n + 1$  arbitrary constants (see [LUI90]). In our configuration, the points are not unknowns, they are given by a user and may not move. Therefore, the number of unknowns is  $3 \times (n + 1)$  : the  $2 \times (n + 1)$  weights on each side of the boundary curve, and the  $(n + 1)$  weights on this curve. This

means that the degrees of freedom (the  $(12n + 1)$  arbitrary constants and the  $3(n + 1)$  weights) are less than the number of equations. Moving these  $3(n + 1)$  weights doesn't give enough degrees of freedom to reach the  $G^1$ -continuity. Therefore we decide to leave these  $3(n + 1)$  weights unchanged during the variational process, and move only the weights, which do not affect the  $G^1$ -continuity. This implies that to have a  $G^1$ -continuous solution, the control points given by a user must define a  $G^1$ -continuous non rational surface.

The colorplate 20 illustrates the results of chapter 5 on a biquintic Bézier surface with four patches.

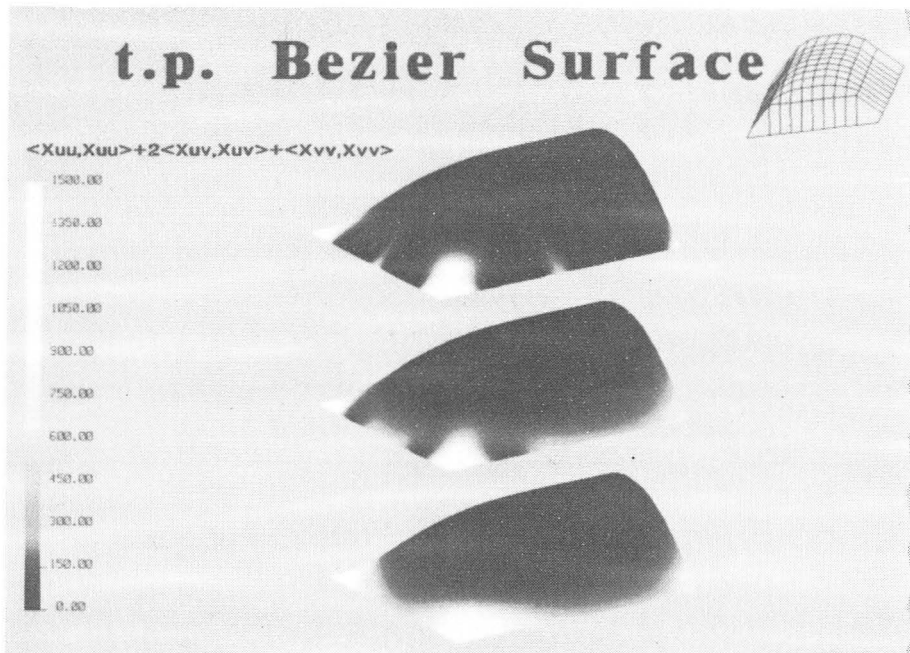


Fig. 20

At the upper right corner is the control polygon of the surface. The corresponding non-rational Bézier surface is represented at the bottom of the colorplate. The surface drawn in the middle of plate 20 is the result of the minimization of the sum of  $\left\| \frac{\partial^2 X}{\partial u^2} \right\|^2 + 2 \left\| \frac{\partial^2 X}{\partial u \partial v} \right\|^2 + \left\| \frac{\partial^2 X}{\partial v^2} \right\|^2$  in the corners of the four patches. At the top is represented the surface with the same control points after minimization of the sum of  $\left\| \frac{\partial^2 X}{\partial u^2} \right\|^2 + 2 \left\| \frac{\partial^2 X}{\partial u \partial v} \right\|^2 + \left\| \frac{\partial^2 X}{\partial v^2} \right\|^2$  in the middle parameter points of the 10 boundary curves.

During the variational process we only allowed the weights  $\omega_{22}$ ,  $\omega_{23}$ ,  $\omega_{32}$ ,  $\omega_{33}$  of the four patches to move. Thus the three surfaces of plate 20 are  $C^1$ -continuous.

## 6 Stiffness Degree for Rational Triangular Bézier Patches

Triangular Bézier patches are a more natural generalization of Bézier curves than are tensor product surfaces. Therefore, some results, true for Bézier curves, can be generalized for triangular Bézier patches, but do not find direct equivalent for tensor product Bézier surfaces. Among such results is the so called standard reparametrization.

While in paragraph 5.1, we saw that, for some tensor product Bézier patches, no rational linear reparametrization transform the four corner weights in one, we will prove in paragraph 6.1, that for any triangular Bézier patches with positive weights, there exists a unique rational linear reparametrization changing the three corner weights in one, and remaining the positive sign of the other weights.

This will enable us to generalize the results of chapter 3 to triangular Bézier patches : we will minimize the following criterion function :

$$\sum_{\text{corner}} \alpha (\|X_{uu}\|^2 + \|X_{vv}\|^2 + \|X_{ww}\|^2) + \beta (\|X_{uuu}\|^2 + \|X_{vvv}\|^2 + \|X_{www}\|^2) \quad (6.1)$$

In paragraph 6.2.1 the second and third derivatives are calculated, and in paragraph 6.2.3 two examples are presented.

### 6.1 Standard Reparametrization for Rational Triangular Bézier Patches

The next proposition is the direct equivalent of Prop. 3.2 for rational triangular Bézier patches.

**Proposition 6.1**

Let  $X$  be a rational triangular Bézier patch of degree  $n$ , parametrized over the domain  $T(a, b, c)$  with the control points  $(b_{ijk})_{\substack{i+j+k=n \\ i,j,k \geq 0}}$ , and positive control weights  $(\omega_{ijk})_{\substack{i+j+k=n \\ i,j,k \geq 0}}$ .

There exists a unique rational linear reparametrization  $\varphi$  such that

- i)  $\varphi(a) = \varphi(b) = \varphi(c)$  ,  $\varphi(T(a, b, c)) = T(a, b, c)$
- ii) The parametrized Bézier triangle  $X(\varphi)$  has the same control points, but new control weights  $(\bar{\omega}_{ijk})_{\substack{i+j+k=n \\ i,j,k \geq 0}}$  with

$$\bar{\omega}_{n00} = \bar{\omega}_{0n0} = \bar{\omega}_{00n} = 1$$

$$\bar{\omega}_{ijk} > 0$$

**Proof :**

Let  $(u, v, w)$  be the affine coordinates of the parameter point  $au + bv + cw$  in  $T(a, b, c)$ .

A rational linear reparametrization  $\varphi$  fulfilling the condition i) must transform  $u, v$  and  $w$  in :

$$\begin{aligned} \bar{u} &= \frac{\rho_a u}{\rho_a u + \rho_b v + \rho_c w} \\ \bar{v} &= \frac{\rho_b v}{\rho_a u + \rho_b v + \rho_c w} \\ \bar{w} &= \frac{\rho_c w}{\rho_a u + \rho_b v + \rho_c w} \end{aligned}$$

where  $\rho_a, \rho_b, \rho_c$  are three positive scalars.

The reparametrized generalized Bernstein polynomials  $B_{ijk}^n$  are equal to :

$$B_{ijk}^n(\bar{u}, \bar{v}, \bar{w}) = \frac{\rho_a^i \rho_b^j \rho_c^k}{(\rho_a u + \rho_b v + \rho_c w)^n} \cdot B_{ijk}^n(u, v, w) \quad i + j + k = n$$

And the parametric equation of the reparametrized surface  $X(\varphi)$  is now :

$$X(\varphi(u, v, w)) = \frac{\sum_{i+j+k=n} \rho_a^i \rho_b^j \rho_c^k \omega_{ijk} b_{ijk} B_{ijk}^n(u, v, w)}{\sum_{i+j+k=n} \rho_a^i \rho_b^j \rho_c^k \omega_{ijk} B_{ijk}^n(u, v, w)}$$

We see that  $X(\varphi)$  has the same control points as  $X$ , but new control weights  $(\bar{\omega}_{ijk})_{\substack{i+j+k=n \\ i,j,k \geq 0}}$

$$\bar{\omega}_{ijk} = \rho_a^i \rho_b^j \rho_c^k \omega_{ijk} , \quad i + j + k = n$$

The condition ii) is equivalent to :

$$\rho_a = \left( \frac{1}{\omega_{n00}} \right)^{\frac{1}{n}}$$

$$\rho_b = \left( \frac{1}{\omega_{0n0}} \right)^{\frac{1}{n}}$$

$$\rho_c = \left( \frac{1}{\omega_{00n}} \right)^{\frac{1}{n}}$$

□

The effect of such a reparametrization is shown on Fig. 21

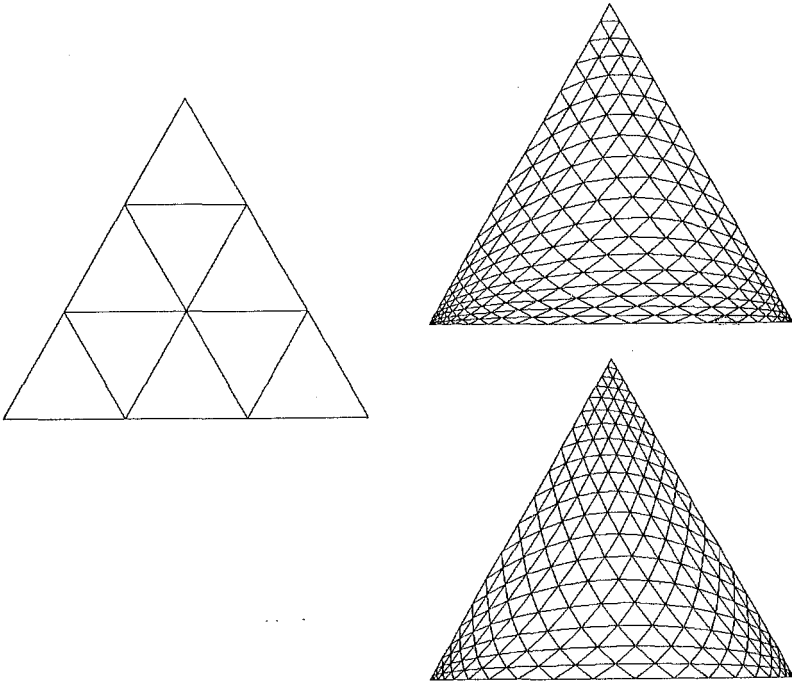


Fig. 21

The control polygon of a plane cubic triangular Bézier patch is represented to the left of Fig. 21.

The control points are :  $(i, j, k)_{\substack{i+j+k=3 \\ i,j,k \geq 0}}$ . The weights of the upper right patch are equal to one, except the bottom left and bottom right corner weights, which are respectively equal to 10 and 5.

The bottom right patch is the same surface, after the reparametrization.

## 6.2 The Resolving System

### 6.2.1 The derivatives of a rational triangular Bézier patch in the corners

#### Proposition 6.2

Let  $X$  be a rational triangular Bézier patch of degree  $n$ , parametrized over the domain triangle  $T(a, b, c)$  with control points  $(b_{ijk})_{\substack{i+j+k=n \\ i,j,k \geq 0}}$ , and control weights  $(\omega_{ijk})_{\substack{i+j+k=n \\ i,j,k \geq 0}}$ .

The derivatives of  $X$  in  $b_{n00}$  in the direction  $u = c - b$  are the following rational functions in the weights :

$$X_u = \frac{n}{\omega_{n00}} \sum_{i=0}^1 (-1)^i \omega_{n-1,1-i,i} (b_{n-1,1-i,i} - b_{n00}) \quad (6.2.a)$$

$$X_{uu} = \frac{n}{\omega_{n00}^2} \left[ -2n(\omega_{n-1,1,0} - \omega_{n-1,0,1}) \sum_{i=0}^1 (-1)^i \omega_{n-1,1-i,i} (b_{n-1,1-i,i} - b_{n00}) \right. \\ \left. + (n-1)\omega_{n00} \sum_{i=0}^2 (-1)^i \binom{2}{i} \omega_{n-2,2-i,i} (b_{n-2,2-i,i} - b_{n00}) \right] \quad (6.2.b)$$

$$X_{uuu} = \frac{n}{\omega_{n00}^3} \left[ 3n(2n(\omega_{n-1,1,0} - \omega_{n-1,0,1})^2 \right. \\ \left. - (n-1)\omega_{n00}(\omega_{n-2,2,0} - 2\omega_{n-2,1,1} + \omega_{n-2,0,2})) \right. \\ \left. \cdot \sum_{i=0}^1 (-1)^i \omega_{n-1,1-i,i} (b_{n-1,1-i,i} - b_{n00}) \right. \\ \left. - 3n(n-1)\omega_{n00}(\omega_{n-1,1,0} - \omega_{n-1,0,1}) \sum_{i=0}^2 (-1)^i \binom{2}{i} \omega_{n-2,2-i,i} (b_{n-2,2-i,i} - b_{n00}) \right. \\ \left. + (n-1)(n-2)\omega_{n00}^2 \sum_{i=0}^3 (-1)^i \binom{3}{i} \omega_{n-3,3-i,i} (b_{n-3,3-i,i} - b_{n00}) \right] \quad (6.2.c)$$

The derivatives of  $X$  in the direction  $v$  in  $b_{0n0}$ , and in the direction  $w$  in  $b_{00n}$  follow from symmetry.

Proof :

The proof is based on the de Casteljau algorithm and on the recurrence formula (3.2). Although (3.2) is written for a function of one variable, it remains true if the  $d$ -th derivative is replaced by the  $d$ -th directional derivative of a function of several variables.



The directional derivatives of the numerator  $p$  and the denominator  $\omega$  of  $X$  in  $\mathbf{b}_{n00}$  is given by the triangular de Casteljau algorithm :

$$\begin{aligned}
 p &= \omega_{n00} \mathbf{b}_{n00} \\
 p_u &= n(\omega_{n-1,1,0} \mathbf{b}_{n-1,1,0} - \omega_{n-1,0,1} \mathbf{b}_{n-1,0,1}) \\
 p_{uu} &= n(n-1)(\omega_{n-2,2,0} \mathbf{b}_{n-2,2,0} - 2\omega_{n-2,1,1} \mathbf{b}_{n-2,1,1} + \omega_{n-2,0,2} \mathbf{b}_{n-2,0,2}) \\
 p_{uuu} &= n(n-1)(n-2)(\omega_{n-3,3,0} \mathbf{b}_{n-3,3,0} - 3\omega_{n-3,2,1} \mathbf{b}_{n-3,2,1} + 3\omega_{n-3,1,2} \mathbf{b}_{n-3,1,2} \\
 &\quad - \omega_{n-3,0,3} \mathbf{b}_{n-3,0,3}) \\
 \omega &= \omega_{n00} \\
 \omega_u &= n(\omega_{n-1,1,0} - \omega_{n-1,0,1}) \\
 \omega_{uu} &= n(n-1)(\omega_{n-2,2,0} - 2\omega_{n-2,1,1} + \omega_{n-2,0,2}) \\
 \omega_{uuu} &= n(n-1)(n-2)(\omega_{n-3,3,0} - 3\omega_{n-3,2,1} + 3\omega_{n-3,1,2} - \omega_{n-3,0,3})
 \end{aligned}$$

These derivatives depend not only on the boundary weights, but also on the interior weights  $\omega_{n-2,1,1}$ ,  $\omega_{n-3,2,1}$ ,  $\omega_{n-3,1,2}$ .

For the first derivative, (3.2) yields :

$$\begin{aligned}
 X_u &= \frac{1}{\omega} [p_u - \omega_u X] \\
 \Rightarrow X_u &= \frac{n}{\omega_{n00}} \left[ (\omega_{n-1,1,0} \mathbf{b}_{n-1,1,0} - \omega_{n-1,0,1} \mathbf{b}_{n-1,0,1}) - (\omega_{n-1,1,0} - \omega_{n-1,0,1}) \mathbf{b}_{n00} \right]
 \end{aligned}$$

Rewriting  $X_u$  as a linear combination of the vectors  $(\mathbf{b}_{n-1,1,0} - \mathbf{b}_{n,0,0})$  and  $(\mathbf{b}_{n-1,0,1} - \mathbf{b}_{n,0,0})$ , we get (6.2.a).

The second derivative is equal to :

$$\begin{aligned}
 X_{uu} &= \frac{1}{\omega} [p_{uu} - 2\omega_u X_u - \omega_{uu} X] \\
 \Rightarrow X_{uu} &= \frac{n}{\omega_{n00}^2} \left[ (n-1)\omega_{n00} \sum_{i=0}^2 (-1)^i \binom{2}{i} \omega_{n-2,2-i,i} \mathbf{b}_{n-2,2-i,i} \right. \\
 &\quad - 2n(\omega_{n-1,1,0} - \omega_{n-1,0,1}) \sum_{i=0}^1 (-1)^i \omega_{n-1,1-i,i} (\mathbf{b}_{n-1,1-i,i} - \mathbf{b}_{n00}) \\
 &\quad \left. - (n-1)\omega_{n00} \sum_{i=0}^2 (-1)^i \binom{2}{i} \omega_{n-2,2-i,i} \mathbf{b}_{n00} \right]
 \end{aligned}$$

Adding the monoms corresponding to  $p_{uu}$  and  $-\omega_{uu} X$ , we get (6.2.b).

The calculus of the third derivative is similar :

$$\begin{aligned}
X_{uuu} &= \frac{1}{\omega} [p_{uuu} - 3\omega_u X_{uu} + 3\omega_{uu} X_u - \omega_{uuu} X] \\
\Rightarrow X_{uuu} &= \frac{n}{\omega^3} \left[ (n-1)(n-2)\omega_{n00}^2 \sum_{i=0}^3 (-1)^i \binom{3}{i} \omega_{n-3,3-i,i} \mathbf{b}_{n-3,3-i,i} \right. \\
&+ 6n^2 (\omega_{n-1,1,0} - \omega_{n-1,0,1})^2 \sum_{i=0}^1 (-1)^i \omega_{n-1,1-i,i} (\mathbf{b}_{n-1,1-i,i} - \mathbf{b}_{n00}) \\
&- 3n(n-1)\omega_{n00} (\omega_{n-1,1,0} - \omega_{n-1,0,1}) \sum_{i=0}^2 (-1)^i \binom{2}{i} \omega_{n-2,2-i,i} (\mathbf{b}_{n-2,2-i,i} - \mathbf{b}_{n00}) \\
&- 3n(n-1)\omega_{n00} (\omega_{n-2,2,0} - 2\omega_{n-2,1,1} + \omega_{n-2,0,2}) \sum_{i=0}^1 (-1)^i \omega_{n-1,1-i,i} (\mathbf{b}_{n-1,1-i,i} - \mathbf{b}_{n00}) \\
&\left. - (n-1)(n-2)\omega_{n00}^2 \sum_{i=0}^3 (-1)^i \binom{3}{i} \omega_{n-3,3-i,i} \mathbf{b}_{n00} \right].
\end{aligned}$$

We get (6.2.c) after adding the coefficients of  $(\mathbf{b}_{n-1,1,0} - \mathbf{b}_{n00})$  and  $(\mathbf{b}_{n-1,0,1} - \mathbf{b}_{n00})$ , and the monoms corresponding to  $p_{uuu}$  and  $-\omega_{uuu}X$ .

□

### Remark 6.3

Prop. 6.2 only gives the cross derivatives in each corner. The derivatives parallel to the edges of the triangle follow directly from Prop. 3.1.

### 6.2.2 The degree of the resolving system

The criterion function (6.1) depends on the parametrization of the triangular patch. We make now the assumption that the rational triangular Bézier patches are reparametrized such as in Prop. 6.2. This has two consequences. First, the same corner weights are set to one, and the criterion function (6.1) became a polynomial function in the weights, with total degree 4 if  $\beta = 0$ , degree 6 if  $\beta \neq 0$ . Second, a unique value of the function (6.1) is associated to each rational triangular Bézier patch.

Figure 22 shows the result of the minimization of (6.1) on a cubic triangular patch for  $\alpha = 0.8$ . The values of the criterion function (6.1) and of the energy function  $(\kappa_1^2 + \kappa_2^2)\sqrt{g}$  are represented with the help of a color map. To the left of the picture is the original triangle (with all weights equal to one), and to the right the triangle with the weights minimizing the criterion function (6.1). The value of (6.1) for the original triangle is 814.45.

We find a minimum for the weights

$$\omega_{300} = 1.0$$

$$\omega_{210} = 0.727$$

$$\omega_{201} = 0.732$$

$$\omega_{120} = 0.728$$

$$\omega_{111} = 0.650$$

$$\omega_{102} = 0.731$$

$$\omega_{030} = 1.0$$

$$\omega_{021} = 0.731$$

$$\omega_{012} = 0.730$$

$$\omega_{003} = 1.0$$

In these weights, (6.1) takes the value 494.17.

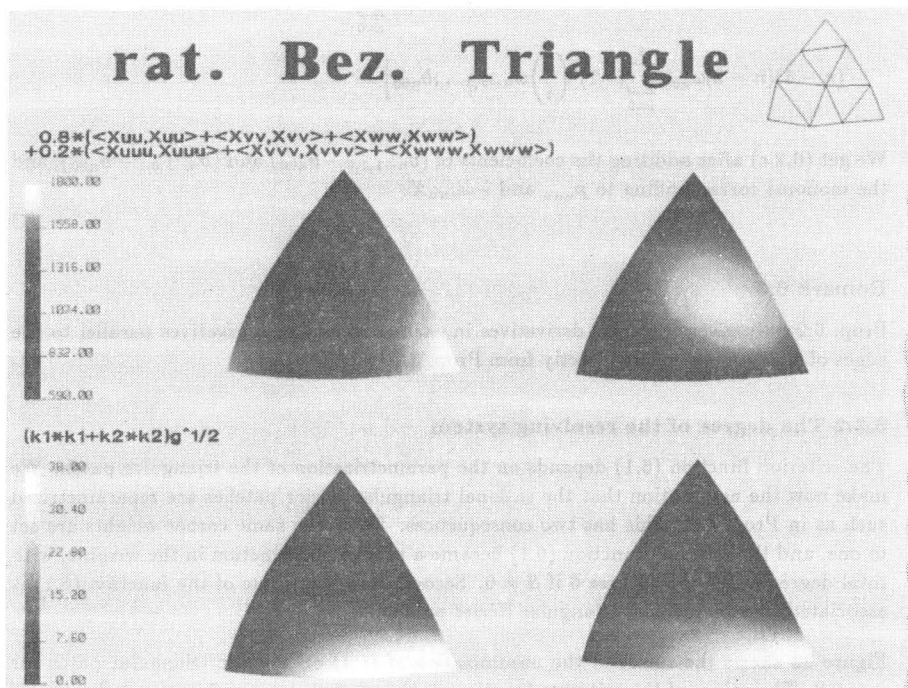


Fig. 22

## 7 References

- [CLI74] A. Cline, *Scalar- and Planar-Valued Curve Fitting Using Splines under Tensions*, Comm. ACM 17, 218-220, (1974)
- [COO67] S.A. Coons, *Surfaces for Computer-Aided Design of Space Forms*, Technical Report, MAC-TR-41, MIT, (1967)
- [COX71] M. Cox, *The numerical evaluation of B-splines*, DNAC 4, National Physical Laboratory, (1971)
- [DEB72] C. de Boor, *On calculating with B-splines*, J. Approx. Theory, 6, 50-62 (1972)
- [DUC76] J. Duchon, *Fonctions-Spline à Energie invariante par Rotation*, Report #27, Université de Grenoble, (1976)
- [DUC77] J. Duchon, *Splines Minimizing Rotation-invariant Semi-norms in Sobolev Spaces*, Lecture Notes in Math. 571, Springer Berlin, 85-100, (1977)
- [DUC79] J. Duchon, *Fonctions-Spline du Type Plaque mince en Dimension 2*, Report #231, Université de Grenoble, (1979)
- [FAR83] G. Farin, *Algorithms for rational Bézier Curves*, CAD, 15, 73-77, (1983)
- [FAR88] G. Farin: *Curves and Surfaces for Computer Aided Geometric Design, A Practical Guide*, Academic Press, (1988)
- [FAR90] G. Farin, H. Hagen, *A Local Twist Estimator*, Curve and Surface Design, Hagen (ed.), (1990)
- [FRA85] R. Franke *Thin Plate Splines with Tension*, CAGD 2, 87-95, (1985)
- [HAG85] H. Hagen, *Geometric Spline Curves*, CAGD 2, 223-227, (1985)
- [HAG86] H. Hagen, *Geometric Surfaces Patches without Twist Constraints*, CAGD 3, 179-184, (1986)
- [HAG92a] H. Hagen, P. Santarelli, *Variational Design of Smooth Bézier Curves*, to be published in CAD.
- [HAG92b] H. Hagen, P. Santarelli, *Variational Design of Smooth B-spline Surfaces*, to be published in SIAM Proceedings.
- [HAR72] R.L. Harder, R.N. Desmarais, *Interpolation using Surface Splines*, J. Aircraft 9, 189-191, (1972)

- [HOL57] J.C. Holladay, *Smoothest Curve Approximation*,  
Math. Table Aids Comput. 11, 233-243, (1957)
- [HOS89] J. Hoschek, D. Lasser: Grundlagen der geometrischen Datenverarbeitung,  
Teubner Stuttgart, (1989)
- [LIU90] D. Liu,  *$GC^1$  Continuity Conditions between two Adjacent Rational Bézier Surface patches*,  
CAGD 7, 151-163, (1990)
- [NIE74] G. Nielson, *Some Piecewise Polynomial Alternatives to Splines under Tension*,  
in R.E. Barnhill, R.F. Riesenfeld (ed): Computer Aided Geometric Design,  
209-235, (1974)
- [NOW83] H. Nowacki, D. Reese, *Design and Faring of Ship Surfaces*,  
Barnhill, R.E.; W. Böhm (ed): Surfaces in CAGD, 121-134, (1983)
- [SCH66] D.G. Schweickert, *An Interpolation Curve Using a Spline in Tension*,  
J. Math.&Phys. 45, 312-317, (1966)
- [TIM34] S.P. Timoshenko, J.N. Goodier: Theory of Elasticity,  
McGraw-Hill (1934)

# Lebenslauf

## Persönliche Daten :

Name : Bonneau  
Vorname : Georges-Pierre  
Geburtsdatum : 23.07.1967  
Geburtsort : Pithiviers, Frankreich  
Nationalität : französisch  
Eltern : Georges Bonneau, Mathematiklehrer  
Marie-Therese Bonneau, Mathematik- und Musiklehrerin  
Geschwister : 3 Schwestern, 2 Brüder

## Schulausbildung :

9/1973 - 6/1977 Grundschole, Denis-Poisson, Pithiviers  
3. Klasse übersprungen  
9/1977 - 6/1981 Collège, Denis-Poisson, Pithiviers  
9/1981 - 6/1984 Gymnasium, Denis-Poisson, Pithiviers  
Juli 1984 Abitur

## Studium :

2 Jahre Classes Préparatoires für den Eintritt in die Grandes Ecoles :  
9/1984 - 6/1985 Maths Sup  
9/1985 - 6/1986 Maths Spé M'  
9/1986 - 6/1990 Studium an der Ecole Normale Supérieure de Cachan

## Abschlüsse :

Juni 1987 Licence in Mathematik an der Université d'Orsay (PARIS XI)  
Juni 1988 Maitrise in Mathematik mit Nebenfach Informatik an der  
Université d'Orsay (PARIS XI)  
Juni 1989 Agrégation in Mathematik mit Nebenfach Informatik  
Juni 1990 Diplôme d'Etudes Approfondies (D.E.A.) in Numerischer  
Mathematik an der Université Pierre et Marie Curie (PARIS VI)  
  
seit 9/1990 Assistant Moniteur Normalien (A.M.N.) an der Université  
Louis Pasteur in Straßburg im Fachbereich Mathematik  
Doktorand an der Universität Kaiserslautern im Fachbereich  
Informatik

druck + verarbeitung :  
sofort druck darmstadt  
schaubeck kg  
telefon 0 61 51 / 24 24 7