Rank n swapping algebra and its applications

Zhe Sun

N° d'ordre :

Université Paris-Sud

Ecole doctorale de Mathématiques

Laboratoire de Mathématiques

Spécialité : Mathématiques

THÈSE DE DOCTORAT

soutenue le 03/07/2014

par

Zhe SUN

Rank n swapping algebra and its applications

Directeur de thèse : M. François LABOURIE

Composition du jury:

Président de jury:	M.	Pierre	PANSU
Directeur de thèse :	M.	François	LABOURIE
Rapporteurs :	M.	Vladimir	FOCK
	M.	Valentin	OVSIENKO
Examinateur :	M.	Louis	FUNAR
	M.	Olivier	GUICHARD

The research leading to these results has received funding from the European Research Council under the *European Community*'s seventh Framework Programme (FP7/2007-2013)/ERC *grant agreement*.

https://sites.google.com/site/zhesunmath/

Thèse préparée au
Département de Mathématiques d'Orsay
Laboratoire de Mathématiques d'Orsay (UMR 8628), Bât. 425
Université Paris-Sud
91405 Orsay Cedex
France

ABSTRACT

Inspired by the swapping algebra [L4] and the rank n cross-ratio [L3] introduced by F. Labourie, we construct a ring equipped with the swapping Poisson structure—the rank n swapping algebra $\mathcal{Z}_n(\mathcal{P})$ to characterize the moduli spaces of cross ratios in rank n. We prove that $\mathcal{Z}_n(\mathcal{P})$ inherits a Poisson structure form the swapping bracket. To consider the "cross-ratios" in the fraction ring, by interpreting $\mathcal{Z}_n(\mathcal{P})$ by a geometric model, we prove that $\mathcal{Z}_n(\mathcal{P})$ is an integral domain. Then we consider the ring $\mathcal{B}_n(\mathcal{P})$ generated by the cross ratios in the fraction ring of $\mathcal{Z}_n(\mathcal{P})$. As an application, for $n = 2, 3$, we embed in a Poisson way the ring generated by Fock-Goncharov coordinates for configuration space of flags in $\mathbb{R}^n$ into $\mathcal{B}_n(\mathcal{P})$. By studying the discrete integrable system for the configuration space $\mathcal{M}_{N,1}$ of N-twisted polygons in $\mathbb{RP}^1$, we asymptotically identify the rank 2 swapping algebra with the dual of Virasoro algebra on a hypersurface of $\mathcal{M}_{N,1}$. And we find a discrete bihamiltonian system of $\mathcal{M}_{N,1}$.

Keywords: Swapping algebra, cross ratio, Poisson structure, Fock-Goncharov algebra, Virasoro algebra.

ACKNOWLEDGEMENTS

First of all, I express my profoundest gratitude to my advisor: Prof. François Labourie. First, I thank him for giving me very inspiring stuff about path integral as essay in Master Two, providing such interesting stuff as my PhD thesis. Second, I thank him and his ERC grant for offering me the chances to participate many conferences and visit some places which I learned new things around my subject, made new friends and found potential cooperations. Third, I thank him for sharing his time to answer my questions patiently, share his inspiring ideas to understand deep mathematics with simple facts, review very carefully my thesis.

I thank Vladimir Fock and Valentin Ovsienko sincerely for accepting the task of reviewing my thesis. I also thank Louis Funar, Olivier Guichard, Pierre Pansu sincerely for their participation in my thesis committee.

I would like to thank Vladimir Fock, first, for discussing with me in many occasions which inspires me to carry much deeper researches around my thesis in the future; second, for reading my thesis carefully and providing remarks and references for many corrections.

I thank our secretaries Mme Valérie Blandin-Lavigne, Mme Fabienne Jacquemin, Mme Marie-Christine Myoupo, Mme Isabelle Souriou, who make our life much more easier.

This thesis was carried out at the department of Mathematics of Orsay who provided me wonderful working conditions, I thank all its members. I express my gratitude to Yongqi Liang who introduced me this department and encouraged me to study here.

I thank Anna Wienhard for her invitation for the conference in Oberwolfach and her hospitality during my stay in Heidelberg for the lectures of Andrew Neitzke.

I also express my gratitude to Xiaonan Ma for his help in my life and mathematics in France.

I would like to thank Jørgen Ellegaard Andersen, Norbert A'Campo, Nariya Kawazumi, Qionglin Li, Feng Luo, Andrew Neitzke, Robert Penner, Ser Peow Tan, Tian Yang for mathematical discussions and their interest in my thesis.

With my friends: Norbert A'Campo, Daniele Alessandrini, Junye Cao, Ke Chen, Li Chen, Miaofen Chen, Zongbin Chen, Xin Fang, Chengxin Feng, Lie Fu, Sourav Ghosh, Hengnan Hu, Yong Hu, Yongquan Hu, Zhaoyu Hu, Wei Huang, Xun Jiang, Gye-Seon Lee, Le Li, Qionglin Li, Shuxian Li, Yongqi Liang, Ling-Min Liao, Xiao Liu, Xin Nie, Zhe Peng, Fan Qin, Andres Sambarino, Shu Shen, Xu Shen, Zhenya Song, Fei Sun, Linhao Sun, Shenghao Sun, Ser Peow Tan, Lu Tang, Shun Tang, Shun Tang, Zhiyu Tian, Chunhui Wang, Haoran Wang, Hao Wu, Shengnan Wu, Binbin Xu, Xiaojie Xu, Lei Yang, Tian Yang, Chi Zhang, Wei Zhang etc, my life becomes more colorful.

Finally, I thank Lin Cao for her love, I thank my parents for supporting me throughout my life.

CONTENTS

1. Introduction 7

1.1 Background 8

1.1.1 Swapping algebra 8

1.1.2 Swapping algebra and the Atiyah-Bott-Goldman symplectic structure 9

1.1.3 Fock-Goncharov algebra on $\mathcal{X}_{\mathrm{PSL}(n,\mathbb{R}),S}$ 10

1.1.4 Motivation of our study 11

1.2 Our Results 12

1.2.1 Rank n swapping algebra 12

1.2.2 The relations between the rank n swapping algebra and the Fock-Goncharov algebra 13

1.2.3 The relation between the rank 2 swapping algebra and the Virasoro algebra 13

1.3 Summary 15

2. Rank n swapping algebra 16

2.1 Swapping algebra 17

2.1.1 Linking number of 4 points in the circle 17

2.1.2 Swapping algebra 18

2.1.3 Swapping fraction algebra, swapping multifraction algebra 18

2.2 Rank n swapping algebra 19

2.2.1 The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ 20

2.2.2 Swapping bracket over $\mathcal{Z}_n(\mathcal{P})$ 20

2.3 $\mathcal{Z}_n(\mathcal{P})$ is an integral domain 29

2.3.1 A geometry model for $\mathcal{Z}_n(\mathcal{P})$ 29

2.3.2 Proof of the main theorem 30

2.3.3 Rank n swapping fraction algebra of $\mathcal{P}$ 36

2.3.4 Rank n swapping multifraction algebra of $\mathcal{P}$ 36

2.4 The rank n swapping algebra and the weak cross ratios . . . 37
2.4.1 The weak cross ratios . . . 37
2.4.2 Rank n cross ratio . . . 38
2.4.3 Injectivity . . . 40

3. Fock-Goncharov algebra and swapping algebra . . . 41
3.1 Configuration space of flags and its coordinates . . . 42
3.1.1 Configuration space of flags . . . 42
3.1.2 Triangulation . . . 42
3.1.3 Fock–Goncharov coordinates . . . 44
3.2 Fock–Goncharov and swapping algebras for $\mathrm{Conf}_{k,2}$ . . . 46
3.2.1 Rank n Fock–Goncharov algebra on $\mathrm{Conf}_{k,n}$ . . . 46
3.2.2 Relations between Fock–Goncharov and swapping algebras for $\mathrm{Conf}_{k,2}$ 46
3.2.3 Main theorem . . . 47
3.3 Fock–Goncharov and swapping algebras for $\mathrm{Conf}_{k,3}$ . . . 56
3.3.1 Fock–Goncharov algebra for $\mathrm{Conf}_{k,3}$ . . . 56
3.3.2 Relations between Fock–Goncharov and swapping for $\mathrm{Conf}_{k,3}$ . . . 57
3.3.3 Main theorem . . . 61
3.4 Poisson embedding . . . 83
3.4.1 Poisson embedding τ_2 . . . 83
3.4.2 Poisson embedding τ_3 . . . 85

4. Swapping algebra and the discrete version of the space of opers . . . 87
4.1 Virasoro algebra and Hill's operators . . . 88
4.1.1 Central extension . . . 88
4.1.2 Virasoro algebra . . . 89
4.1.3 Hill's operators . . . 90
4.2 The configuration space of N-twisted polygons in $\mathbb{RP}^n$ . . . 91
4.2.1 Configuration space of N-twisted polygons . . . 91
4.2.2 Construction of a coordinate system . . . 92
4.3 Large N asymptotic relation between the swapping algebra and the Virasoro algebra . . . 94
4.3.1 The discrete Hill's operator and the cross-ratios . . . 94
4.3.2 Relate discrete version with continuous version . . . 96

4.3.3 Swapping algebra on S^1 98
4.3.4 Main result 104
4.4 Schwartz algebra on $\mathbb{R}(B_1, ..., B_N)$ and its relation with swapping algebra . 107
4.4.1 Schwartz algebra and its asymptotic phenomenon 107
4.4.2 Bihamiltonian system 109
4.4.3 Casimir elements 112

1. INTRODUCTION

In [L4], F. Labourie introduces the swapping algebra by pair of points on a circle. He relates, through the cross ratios, the swapping algebra to the Atiyah-Bott-Goldman symplectic structure on the universal(in genus) Hitchin space and to the Drinfeld-Sokolov reduction on the space of real opers for all possible ranks. From his point of view, the swapping algebra is the natural Poisson algebra for all the moduli spaces of cross ratios. Here, the moduli space of cross ratios refers to the moduli space equipped with a cross ratio like coordinate system and a cyclic order on points. For example, moduli space of limit curves for Hitchin component, configuration space of flags, etc. In my thesis, to characterize the moduli spaces of cross ratios in each rank and the relations between special functions of the moduli spaces, inspired by the swapping algebra and the rank n cross ratio, I introduce the rank n swapping algebra, then:

1. for $n = 2, 3$, we embed in a Poisson way the ring generated by Fock-Goncharov coordinates, for configuration space of flags(which correspond to $\mathcal{X}$ variety) in $\mathbb{R}^n$ into $\mathcal{B}_n(\mathcal{P})$.
2. By studying the discrete integrable system for the configuration space $\mathcal{M}_{N,1}$ of N-twisted polygons in $\mathbb{RP}^1$, we asymptotically identify the rank 2 swapping algebra with the dual of Virasoro algebra on a hypersurface of $\mathcal{M}_{N,1}$. And we find a discrete bihamiltonian system of $\mathcal{M}_{N,1}$.

I will treat the general cases of Case 1 in my upcoming paper. I hope in the future to obtain results in higher dimensions for Case 2 and find possible discrete bihamiltonian system of $\mathcal{M}_{N,n}$.

1.1 Background

1.1.1 Swapping algebra

The swapping algebra is constructed through ordered pairs of points on a circle. Let $\mathcal{P}$ be a cyclic subset of S^1, we represent an ordered pair (r, x) of $\mathcal{P}$ by the expression rx. Then we consider the associative commutative ring $\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{\forall x,y\in\mathcal{P}}]/\{xx|\forall x \in \mathcal{P}\}$ over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$), where $\{xy\}_{\forall x,y\in\mathcal{P}}$ are variables. Notably, $rx = 0$ in $\mathcal{Z}(\mathcal{P})$ if $r = x$. Then we equip $\mathcal{Z}(\mathcal{P})$ with a Poisson bracket defined by F. Labourie in [L4].

Definition 1.1.1. [Swapping bracket] The swapping bracket over $\mathcal{Z}(\mathcal{P})$ *is defined by extending the following formula on generators to* $\mathcal{Z}(\mathcal{P})$ *by using Leibniz's rule:*

$$\{rx, sy\} = \mathcal{J}(r, x, s, y) \cdot ry \cdot sx, \tag{1.1}$$

where $\mathcal{J}(r, x, s, y)$ *is* the linking number *of* ry *and* sx*.(See Section 2.1. This is the case for* $\alpha = 0$ *in [L4].)*
[Leibniz's rule]

$$\{rx \cdot sy, tz\} = rx\{sy, tz\} + sy\{rx, tz\}. \tag{1.2}$$

In Section 2.1, we give more details on configurations of points with different linking numbers.

In [L4], F. Labourie proved that

Theorem 1.1.2. [F. Labourie [L4]] *The swapping bracket is a Poisson bracket.*

Definition 1.1.3. [swapping fraction algebra of $\mathcal{P}$] *The* swapping fraction algebra of $\mathcal{P}$ *is the total fraction $\mathcal{Q}(\mathcal{P})$ of $\mathcal{Z}(\mathcal{P})$ equipped with the induced swapping bracket.*

Definition 1.1.4. [multifraction algebra of $\mathcal{P}$] *Let x, y, z, t belong to $\mathcal{P}$ so that $x \neq t$ and $y \neq z$. The* cross fraction *determined by (x, y, z, t) is the element of $\mathcal{Q}(\mathcal{P})$:*

$$[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{1.3}$$

Let $\mathcal{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P},\ x \neq t, y \neq z\}$ be the set of all the cross-fractions in $\mathcal{Q}(\mathcal{P})$. Let $\mathcal{B}(\mathcal{P})$ be the subring of $\mathcal{Q}(\mathcal{P})$ generated by $\mathcal{CR}(\mathcal{P})$. Then, the multifraction algebra of $\mathcal{P}$ *is the ring $\mathcal{B}(\mathcal{P})$ equipped with the swapping bracket.*

1.1.2 Swapping algebra and the Atiyah-Bott-Goldman symplectic structure

A theorem of F. Labourie [L1] and O. Guichard[Gu1] allows us to consider Hitchin component as the space of the limit double curves. Let us recall some definitions.

Definition 1.1.5. [Hitchin component [H1]] *An* n-Fuchsian homomorphism from $\pi_1(S)$ to $\mathrm{PSL}(n, \mathbb{R})$ *is a homomorphism $\rho = i \circ \rho_0$, where ρ_0 is a discrete faithful homomorphism with values in $\mathrm{PSL}(2, \mathbb{R})$ and i is the irreducible homomorphism from $PSL(2, \mathbb{R})$ to $PSL(n, \mathbb{R})$. A homomorphism is* Hitchin *if it may be deformed into an n-Fuchsian homomorphism. The* Hitchin component *$H_n(S)$ is the space of Hitchin homomorphisms up to conjugation by $\mathrm{PSL}(n, \mathbb{R})$. The Hitchin component and the representations of surface groups can be defined for surfaces with boundary [L2].*

Definition 1.1.6. [Hyperconvex map] *A continuous map ξ from a set $\mathcal{P}$ to $\mathbb{RP}^{n-1}$ is* hyperconvex *if for any pairwise distinct points $(x_1, ..., x_p)$ with $p \leq n$, the following sum is direct*

$$\xi(x_1) + ... + \xi(x_p)$$

Let $\partial_\infty \pi_1(S)$ be the boundary at infinity of $\pi_1(S)$. When we fix an uniformisation of the universal cover of the surface S equipped with a complex structure, $\partial_\infty \pi_1(S)$ can be identified with the real projective line $\mathbb{RP}^1$ as the boundary of $\mathbb{H}^2$.

Definition 1.1.7. [n-hyperconvex] *A homomorphism ρ from $\pi_1(S)$ to $\mathrm{PSL}(n, \mathbb{R})$ is n-*hyperconvex*, if there exists a ρ-equivariant hyperconvex map ξ from $\partial_\infty \pi_1(S)$ to $\mathbb{RP}^{n-1}$, namely $\xi(\gamma x) = \rho(\gamma)\xi(x)$. Such a map is called the* limit curve *of the homomorphism.*

Theorem 1.1.8. [F. Labourie [L1], O. Guichard[Gu1]] *A homomorphism ρ from $\pi_1(S)$ to $PSL(n, \mathbb{R})$ is Hitchin if and only if ρ is n-hyperconvex.*

Moreover, by F. Labourie [L1] and O. Guichard[Gu1], If ρ is n-hyperconvex, there is an unique ρ-equivariant hyperconvex map $\xi(\xi^*$ resp.) from $\partial_\infty \pi_1(S)$ to $\mathbb{P}(\mathbb{R}^n)(\mathbb{P}(\mathbb{R}^n)^*$ resp.) such that $\xi(x) \in ker(\xi^*(y)) \Leftrightarrow x = y$. Hence $\rho \in H_n(S)$ is associated with a limit double curve (ξ, ξ^*) by Theorem 1.1.8.

Definition 1.1.9. [Weak cross ratio] *The* weak cross ratio $\mathbb{B}_{\xi,\xi^*}$ *of four different points* x, y, z, t *in* $\partial_\infty \pi_1(S)$ *is defined to be*

$$\mathbb{B}_{\xi,\xi^*}(x,y,z,t) = \frac{\left\langle \tilde{\xi}(x)|\tilde{\xi^*}(z)\right\rangle}{\left\langle \tilde{\xi}(x)|\tilde{\xi^*}(t)\right\rangle}\frac{\left\langle \tilde{\xi}(y)|\tilde{\xi^*}(t)\right\rangle}{\left\langle \tilde{\xi}(y)|\tilde{\xi^*}(z)\right\rangle}, \tag{1.4}$$

which is independent of the lifts $\tilde{\xi}$ *of* ξ *with values in* $\mathbb{R}^n$ *and the lifts* $\tilde{\xi^*}$ *of* ξ^* *with values in* $\mathbb{R}^{n*}$.

Let $\mathcal{P}$ be a subset of $\partial_\infty \pi_1(S)$, F. Labourie [L4] define a ring homomorphism $I : \mathcal{B}(\mathcal{P}) \to \mathcal{C}^\infty(H_n(S))$ by extending the following formula on generators to the whole ring $\mathcal{B}(\mathcal{P})$:

$$I\left(\frac{xz}{xt}\cdot\frac{yt}{yz}\right)(\rho) = \mathbb{B}_{\xi,\xi^*}(x,y,z,t). \tag{1.5}$$

Then the swapping bracket induces a Poisson bracket on the space of weak cross ratios. In [L4], F. Labourie chooses a vanishing sequence $\{\Gamma_n\}_{n=1}^\infty$ of finite index subgroup of $\pi_1(S)$. Vanishing means: When n is sufficiently large, every geodesic of $\mathbb{H}/\Gamma_n$ becomes simple and the intersection of two geodesics becomes either zero or one. He proved that

Theorem 1.1.10. [F. Labourie [L4]] *Let* $\{\Gamma_n\}_{n=1}^\infty$ *be a vanishing sequence of* $\pi_1(S)$, $S_n = \mathbb{H}/\Gamma_n$. *Let* $\mathcal{P} \subset \partial_\infty \pi(S)$ *be the set of end points of geodesics. Let* b_0 *and* b_1 *be two elements in* $\mathcal{B}(\mathcal{P})$. *Let* $\{.,.\}_{S_n}$ *be the Atiyah-Bott-Goldman bracket [AB] [G1] for* $H(S_n)$. *Let* $\{.,.\}$ *be the swapping bracket. Then we have,*

$$\lim_{n\to\infty}\{I(b_0), I(b_0)\}_{S_n} = I \circ \{b_0, b_1\}. \tag{1.6}$$

1.1.3 Fock-Goncharov algebra on $\mathcal{X}_{\mathrm{PSL}(n,\mathbb{R}),S}$

V. V. Fock and A. B. Goncharov introduce $\mathcal{X}$-higher Teichmuller space ($\mathcal{X}_{G,S}$ in [FG1] where G is real split reductive Lie group, S is a compact oriented surface with boundaries. We suppose S has negative Euler number. Certain finite cover of $H_n(S)$ can be identified with an open subset of $\mathcal{X}_{\mathrm{PSL}(n,\mathbb{R}),S}$ (see section 2 of [FG1]). Moreover, they give a coordinate system of $\mathcal{X}_{\mathrm{PSL}(n,\mathbb{R}),S}$.

Definition 1.1.11. [$\mathcal{X}$-higher Teichmuller space $\mathcal{X}_{G,S}$, Section 2 of [FG1]] *Let* G *be a reductive group. Let* B *be the Borel group of* G. *Let* $\mathcal{F} = G/B$ *be the flag variety of* G. *Let* T *be the finite subset of* ∂S, $\hat{S} = S\backslash T$. *An element in* $\mathcal{X}$-higher Teichmuller space $\mathcal{X}_{G,S}$ *is a pair* (∇, f) , *where* ∇ *is a flat connection on the* G*-principal bundle* $P \to S$ *and* $f : \partial S\backslash T \to G/B$ *is* B*-reduction flat on* $\partial S\backslash T$, *that is to say, for any* $x \in \partial S\backslash T$, *we choose a flag* $f(x)$ *in* $P|_x$, *such that* ∇ *translates* $f(x_1)$ *to* $f(x_2)$ *when* x_1 *and* x_2 *are in the same component of* $x \in \partial S\backslash T$.

For the disc D , a finite subset $\mathcal{P} \subset \partial D$, $\mathcal{X}_{\mathrm{PSL}(n,\mathbb{R}),\hat{D}} = \mathrm{Conf}_{k,n}$, where $\mathrm{Conf}_{k,n}$ is the *configuration space of flags* (see Definition 3.1.2), it is considered as the building block of $\mathcal{X}_{\mathbb{PSL}(n,\mathbb{R}),S}$ (See section 7 of [FG1]).

Given an ideal triangulation T and its n-triangulation T_n(see Chapter 3), V. V. Fock and A. B. Goncharov associate some vertices $I_n \cup J_n$ of T_n with rational functions which we call the *Fock-Goncharov coordinates* $X_i : \mathcal{X}_{\mathbb{PSL}(n,\mathbb{R}),S} \to \mathbb{R}$ for any i belongs to $I_n \cup J_n$. They define a Poisson bracket which we call the *Fock-Goncharov bracket* on the function ring generated by these functions by extending the following formulas to the whole ring.

$$\{X_u,\ X_w\}_n = \epsilon_{uw} \cdot X_u \cdot X_w \tag{1.7}$$

For any $u, w \in I_n \cup J_n$, $\epsilon_{uw} = 0, 1, -1$. The value of ϵ_{uw} depends on the orientation of the surface. Moreover, they show that

Proposition 1.1.12. [V. V. Fock and A. B. Goncharov [FG1]] *The Poisson bracket* $\{.,.\}_n$ *does not depend on the triangulation that we choose.*

1.1.4 Motivation of our study

F. Labourie [L4] have constructed a universal Poisson algebra—swapping algebra for weak cross ratio functions of Hitchin component. There are rank n weak cross ratio relations [L3] between these weak cross ratios. So we restrict to rank n swapping algebra to characterize weak cross ratio function ring pure algebraically. The free $\mathbb{K}$-module $\mathcal{Z}(\mathcal{P})$ becomes into a quotient ring $\mathcal{Z}_n(\mathcal{P})$ with some relations(see next section), which is not free any more. To construct rank n swapping algebra, we have to answer two fundamental questions at the beginning:

1. Is that $\mathcal{Z}_n(\mathcal{P})$ compatible with the swapping bracket?
2. Is that cross ratios well-defined in the fraction ring of $\mathcal{Z}_n(\mathcal{P})$, in other words, is that $\mathcal{Z}_n(\mathcal{P})$ an integral domain?

Fortunately, the answers to these two fundamental questions are positive by Chapter 2.

For the configuration space of flags which is the building block of the universal covering(in genus) of Hitchin component, there are the Fock-Goncharov algebra considered by V. V. Fock and A. B. Goncharov [FG1] and the rank n swapping algebra by considering the cyclic set $\mathcal{P}$ to be cyclically ordered points in each configuration. Since Fock-Goncharov algebra is a natural Poisson algebra invariant under the cluster transformations, we hope these two natural Poisson algebras in some sense coincide with each other. In fact, we embed in a Poisson way the Fock-Goncharov algebra into the rank n swapping algebra in general, we will show the cases for $n = 2, 3$ in my thesis. The general case will be shown in the upcoming paper by the same method.

Discrete integrable system of the configuration space $\mathcal{M}_{N,n}$ of RP^n is considered by L. Faddeev, A. Yu. Volkov [FV] for $n = 1$, R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT] for $n = 2$ and B. Khsein, F. Soloviev [KS] for n in general. We plan to construct a Poisson structure on coordinates defined by weak cross ratios on $\mathcal{M}_{N,n}$ through the swapping algebra. Then compare this Poisson structure with the Poisson structure considered by R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT] for $n = 2$ to show that they are compatible, where their continuous limit are the natural Lie-Poisson structure and the

freezing structure [SOT]. This plan is to reply the conjecture mentioned by [SOT]. But we do not success in finding such a nice swapping Poisson structure. We have only result for $n = 1$ shown in the last section. More general case will be considered later on.

1.2 Our Results

1.2.1 Rank n swapping algebra

Inspired by the swapping algebra and the rank n cross-ratio, we introduce the rank n swapping algebra $\mathcal{Z}_n(\mathcal{P})$.

Definition 1.2.1. [The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$] *For $n \geq 2$, let $R_n(\mathcal{P})$ be the subring of $\mathcal{Z}(\mathcal{P})$ generated by*

$$\left\{ D \in \mathcal{Z}_n(\mathcal{P}) \mid D = \det \begin{pmatrix} x_1y_1 & \dots & x_1y_{n+1} \\ \dots & \dots & \dots \\ x_{n+1}y_1 & \dots & x_{n+1}y_{n+1} \end{pmatrix}, \forall x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1} \in \mathcal{P} \right\}.$$

Let $\mathcal{Z}_n(\mathcal{P})$ be the quotient ring $\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$.

We prove two fundamental Theorems 1.2.2 and 1.2.4 for the rank n swapping algebra below.

Theorem 1.2.2. *For $n \geq 2$, $R_n(\mathcal{P})$ is an ideal for the swapping bracket, thus $\mathcal{Z}_n(\mathcal{P})$ inherits a Poisson bracket from the swapping bracket.*

By the above theorem, we have

Definition 1.2.3. [rank n swapping algebra of $\mathcal{P}$] *The* rank n swapping algebra of $\mathcal{P}$ *is the ring $\mathcal{Z}_n(\mathcal{P})$ equipped with the swapping bracket.*

Theorem 1.2.4. *For $n \geq 2$, $\mathcal{Z}_n(\mathcal{P})$ is an integral domain.*

Definition 1.2.5. [Rank n swapping fraction algebra of $\mathcal{P}$] *The* rank n swapping fraction algebra of $\mathcal{P}$ *is the total fraction ring $\mathcal{Q}_n(\mathcal{P})$ of $\mathcal{Z}_n(\mathcal{P})$ equipped with the swapping bracket.*

Definition 1.2.6. [rank n multifraction algebra of $\mathcal{P}$] *Let x, y, z, t belong to $\mathcal{P}$ so that $x \neq t$ and $y \neq z$. The* cross fraction *determined by (x, y, z, t) is the element of $\mathcal{Q}_n(\mathcal{P})$:*

$$[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{1.8}$$

Let $\mathcal{CR}_n(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}_n(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P},\ x \neq t, y \neq z\}$ be the set of all the cross-fractions in $\mathcal{Q}_n(\mathcal{P})$. Let $\mathcal{B}_n(\mathcal{P})$ be the subring of $\mathcal{Q}_n(\mathcal{P})$ generated by $\mathcal{CR}_n(\mathcal{P})$.

Then, the rank n multifraction algebra of $\mathcal{P}$ *is the ring $\mathcal{B}_n(\mathcal{P})$ equipped with the swapping bracket.*

1.2.2 The relations between the rank n swapping algebra and the Fock-Goncharov algebra

Let D_k be a disc with k marked points on the boundary. The configuration space of flags $\mathrm{Conf}_{k,n} = \mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),D_k}$ is the building block of the $\mathcal{X}$-higher Teichmüller space $\mathcal{X}_{\mathrm{PSL}(n,\mathbb{R}),S}$ (see [FG1] Section 7). We want to study the relation between Fock-Goncharov algebra and the swapping algebra on $\mathrm{Conf}_{k,n}$ at the beginning. In fact, the swapping algebra is very complicated on $\mathrm{Conf}_{k,n}$. But when we restrict to the rank n swapping algebra, the Poisson bracket suddenly becomes canonical. We have our main result, which says that we embed in a Poisson way the rank n Fock-Goncharov algebra $\mathcal{A}(T_n)$ into the rank n swapping multifraction algebra $\mathcal{B}_n(\mathcal{P})$.

Definition 1.2.7. *The* rank n Fock-Goncharov ring $\mathcal{A}(T_n)$ *is a ring generated by the triple ratios and the edge functions(see Definition 3.1.8 and Definition 3.1.9).*

Let $\mathcal{P} = \{x_1, ..., x_k\}$ be the set of marked points of D_k. We construct the swapping multifraction $\mathcal{B}_n(\mathcal{P}')$ on $\mathrm{Conf}_{k,n}$ by adding some points with geometric meaning into $\mathcal{P}$. Then we construct a ring homomorphism $\theta_{T_n} : \mathcal{A}(T_n) \to \mathcal{B}_n(\mathcal{P}')$ by interpreting the volumes in Fock-Goncharov coordinates by the determinants in $\mathcal{Z}_n(\mathcal{P}')$. In my thesis, we only consider the case for $n = 2, 3$, where $\mathcal{P}' = \mathcal{P}$. Then we prove that

Theorem 1.2.8. *For $n = 2, 3$, the ring homomorphism θ_{T_n} is injective and Poisson.*

The general case will be shown in the upcoming paper.

1.2.3 The relation between the rank 2 swapping algebra and the Virasoro algebra

Definition 1.2.9. [Configuration space of N-twisted polygons in $\mathbb{RP}^n$] *A N-*twisted polygon in $\mathbb{RP}^n$ *is a map f from $\mathbb{Z}$ to $\mathbb{RP}^n$ such that for any $k \in \mathbb{Z}$, we have $f(k+N) = M_f \cdot f(k)$ where M_f belongs to $\mathrm{PSL}_{n+1}(R)$. We call M_f the* monodromy *of the N-twisted polygon in $\mathbb{RP}^n$. We say that f is* in general position *if for any $k \in \mathbb{N}$, the points $\{f(k+i-1)\}_{i=1}^{n+1}$ are in general position in $\mathbb{RP}^n$.*

The configuration space of N-twisted polygons in $\mathbb{RP}^n$, *denoted by $\mathcal{M}_{N,n}$, is the space of the N-twisted polygons in general position in $\mathbb{RP}^n$ up to projective transformations.*

We construct a cross ratio coordinate system $\{B_k\}_{k=1}^N$ of $\mathcal{M}_{N,1}$. There is a natural Poisson structure induced from the swapping algebra on the cross ratios. Then we relates the swapping algebra to the Virasoro algebra on $\mathcal{M}_{N,1}$ as follows.

Definition 1.2.10. [(t_1, t_2, N)-Virasoro bracket] *Let $t_1, t_2 \in \mathbb{R}$ and $N \in \mathbb{N}$, the* (t_1, t_2, N)-Virasoro bracket *on $\{I_k\}_{k=-N}^N$ is defined to be:*

For $p, q = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$,

1. *when $p \neq -q$, we have*
$$\{I_p, I_q\}_{N,t_1,t_2} = (p-q) \cdot I_{p+q}$$
with the convention $I_{k+N} = I_k$;

2. when $p = -q$, we have

$$\{I_p, I_{-p}\}_{N,t_1,t_2} = 2p \cdot I_0 + t_1 \cdot p^3 + t_2 \cdot p.$$

Definition 1.2.11. [Discrete Fourier transformation] *Let $\{B_k\}_{k=1}^N$ be the cross ratio coordinates of $\mathcal{M}_{N,1}$. Let $\mathbb{B} = \{B_1, ..., B_N\}$. The* discrete Fourier transformation *$\mathcal{F}$ of $\mathbb{B}$ is defined to be*

$$\mathcal{F}_p\mathbb{B} = \sum_{k=1}^{N} B_k e^{-\frac{2pk\pi i}{N}}. \tag{1.9}$$

Our main result is

Theorem 1.2.12. *Let $\{B_k\}_{k=1}^N$ be the cross ratio coordinates of $\mathcal{M}_{N,1}$ such that*

$$\sum_{k=1}^{N} B_k = 0.$$

For $k = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$,

$$V_k = \frac{\mathcal{F}_k\mathbb{B} \cdot N}{8\pi i}.$$

We have

$$\{V_p, V_q\}_{B2} = \{V_p, V_q\}_{N,\frac{8\pi^2}{N},8N} + o\left(\frac{1}{N^3}\right). \tag{1.10}$$

We have a similar theorem for lower rank version of the Poisson algebra defined in [SOT] $\{\cdot,\cdot\}_{S1}$ on (see Section 4.4) on $\mathcal{M}_{N,1}$.

Theorem 1.2.13. *Let $\{B_k\}_{k=1}^N$ be the cross ratio coordinates of $\mathcal{M}_{N,1}$ such that*

$$\sum_{k=1}^{N} B_k = 0.$$

For $k = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$,

$$W_k = \frac{\mathcal{F}_k\mathbb{B} \cdot N}{16\pi i}.$$

We have

$$\{W_p, W_q\}_{S1} = \{W_p, W_q\}_{N,\frac{8\pi^2}{3N},4-8N} + o\left(\frac{1}{N^3}\right). \tag{1.11}$$

Then we observe that

Theorem 1.2.14. *For $N \geq 5$, $\{\cdot,\cdot\}_{B2}$ and $\{\cdot,\cdot\}_{S1}$ are compatible on $\mathcal{M}_{N,1}$.*

1.3 Summary

1. In chapter 2, we construct and study the rank n swapping multifraction algebra —a quotient ring equipped with the swapping Poisson structure. In §2.1, we recall the swapping algebra introduced by F. Labourie. In §2.2, we define the rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ and prove that it is compatible with the swapping bracket. In §2.3, we prove that $\mathcal{Z}_n(\mathcal{P})$ is a integral domain. After two fundamental Theorems 2.2.4 and 2.3.7, we construct the rank n swapping multifraction algebra. In §2.4, we study the relations between the rank n swapping algebra and the weak cross ratios in the Hitchin component and the configuration space of the flags.

2. In chapter 3, for $n = 2, 3$, we relate the rank n swapping multifraction algebra to the rank n Fock–Goncharov algebra on the configuration space of flags $\mathrm{Conf}_{k,n}$. In §3.1, we recall the configuration space of flags which is the building block of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$, and its coordinates coined by Fock and Goncharov. In §3.2, we embed in a Poisson way the rank 2 Fock–Goncharov algebra into the rank 2 swapping multifraction algebra. In §3.3, we embed in a Poisson way the rank 3 Fock–Goncharov algebra into the rank 3 swapping multifraction algebra. In §3.4, we relate θ_{T_2} to θ_{T_3} by a Poisson embedding τ_2 and we generalize to τ_3.

3. In chapter 4, we relate the rank 2 swapping multifraction algebra to the Virasoro algebra on a hypersurface of the configuration space $\mathcal{M}_{N,1}$ of N twisted polygons in $\mathbb{RP}^1$. In §4.1, we recall the central extension, Virasoro algebra, Hill's operators and some of their properties. In §4.2, we recall the configuration space $\mathcal{M}_{N,n}$ of N-twisted polygons in $\mathbb{RP}^n$. When $(N, n+1) = 1$, we recall a coordinate system of $\mathcal{M}_{N,n}$ [SOT] [KS]. In §4.3, we asymptotically identify the rank 2 swapping algebra with the dual of Virasoro algebra on a hypersurface of $\mathcal{M}_{N,1}$.. In §4.4, we asymptotically identify the Schwartz algebra with the dual of Virasoro algebra on a hypersurface of $\mathcal{M}_{N,1}$.. We prove that swapping bracket is compatible with Schwartz bracket on $\mathcal{M}_{N,1}$.

2. RANK N SWAPPING ALGEBRA

In this chapter, we construct and study the rank n swapping multifraction algebra —a ring equipped with the swapping Poisson structure. In §2.1, we recall the swapping algebra introduced by F. Labourie. In §2.2, we define the rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ and prove that it is compatible with the swapping bracket. In §2.3, to ensure that "cross ratios" are well-defined in the fraction ring of $\mathcal{Z}_n(\mathcal{P})$, we prove that $\mathcal{Z}_n(\mathcal{P})$ is a integral domain. After two fundamental Theorems 2.2.4 and 2.3.7, we construct the rank n swapping multifraction algebra. In §2.4, we study the relations between the rank n swapping algebra and the weak cross ratios in the Hitchin component and the configuration space of flags.

2.1 Swapping algebra

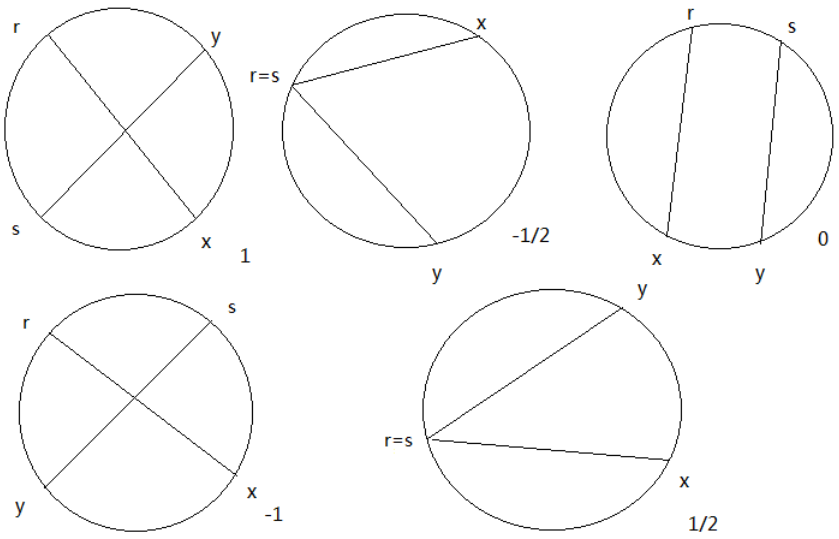
In this section, we recall some definitions about the swapping algebra created by F. Labourie. Our definitions here are based on Section 2 of [L4].

2.1.1 Linking number of 4 points in the circle

Definition 2.1.1. [LINKING NUMBER] *Let (r,x,s,y) be a quadruple of 4 different points in the interval $]0,1[$. Let $\sigma(\triangle)=-1,0,1$ whenever $\triangle<0,\triangle=0,\triangle>0$ respectively. We call $\mathcal{J}(r,x,s,y)$ the* linking number *of (r,x,s,y), where*

$$\mathcal{J}(r,x,s,y)=\frac{1}{2}\cdot(\sigma(r-x)\cdot\sigma(r-y)\cdot\sigma(y-x)-\sigma(r-x)\cdot\sigma(r-s)\cdot\sigma(s-x))\,. \quad (2.1)$$

Fig. 2.1:



If (r,x,s,y) is a quadruple of 4 points in the oriented circle S^1, the linking number of 4 points in the interval $S^1\backslash o$ for $o\notin\{r,x,s,y\}$ does not depend on the choice of o. So, $\mathcal{J}(r,x,s,y)$ is defined to be the linking number of 4 points in the circle S^1. We describe four cases in Figure 2.1.

2.1.2 Swapping algebra

Let $\mathcal{P}$ be a finite subset of the circle S^1 provided with cyclic order. $\mathbb{K}$ is a field ($\mathbb{C}$ or $\mathbb{R}$) We represent an ordered pair (r, x) of $\mathcal{P}$ by the expression rx.

Definition 2.1.2. [SWAPPING RING OF $\mathcal{P}$] The swapping ring of $\mathcal{P}$ *is the ring* $\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{\forall x,y\in\mathcal{P}}]/\{xx|\forall x \in \mathcal{P}\}$ *over* $\mathbb{K}$*, where* $\{xy\}_{\forall x,y\in\mathcal{P}}$ *are variables with values in* $\mathbb{K}$.

Notably, $rx = 0$ if $r = x$ in $\mathcal{Z}(\mathcal{P})$. Then we equip $\mathcal{Z}(\mathcal{P})$ with a Poisson bracket defined by F. Labourie in Section 2 of [L4].

Definition 2.1.3. [SWAPPING BRACKET] The swapping bracket *over* $\mathcal{Z}(\mathcal{P})$ *is defined by extending the following formula to* $\mathcal{Z}(\mathcal{P})$ *by using* Leibniz's rule *and* additive rule*:*

$$\{rx, sy\} = \mathcal{J}(r, x, s, y) \cdot ry \cdot sx. \tag{2.2}$$

(Here is the case for $\alpha = 0$ *in Section 2 of [L4].)*
Leibniz's rule*:*

$$\{rx \cdot sy, tz\} = rx\{sy, tz\} + sy\{rx, tz\} \tag{2.3}$$

for any rx*,* xy*,* tz *in* $\mathcal{P}$.
Additive rule*:*

$$\{a + b, c\} = \{a, c\} + \{b, c\} \tag{2.4}$$

For any $a, b, c \in \mathcal{Z}(\mathcal{P})$.

Theorem 2.1.4. [F. LABOURIE [L4]] *The swapping bracket as above verifies the Jacobi identity. So the swapping bracket defines a Poisson structure on* $\mathcal{Z}(\mathcal{P})$.

Definition 2.1.5. [SWAPPING ALGEBRA OF $\mathcal{P}$] The swapping algebra of $\mathcal{P}$ *is* $\mathcal{Z}(\mathcal{P})$ *equipped with the swapping bracket.*

2.1.3 Swapping fraction algebra, swapping multifraction algebra

Definition 2.1.6. [CLOSED UNDER SWAPPING BRACKET] *For a ring* R*, if* $\forall a, b \in R$*, we have* $\{a, b\} \in R$*, then we say that* R *is* closed under swapping bracket*. Moreover, we say that* R *is* equipped with the closed swapping bracket.

$\mathcal{Z}(\mathcal{P})$ is a integral domain, let $\mathcal{Q}(\mathcal{P})$ be the total fraction of $\mathcal{Z}(\mathcal{P})$. By Leibniz's rule, since $\{a, \frac{1}{b}\} = -\frac{\{a,b\}}{b^2}$, the swapping bracket is well defined on $\mathcal{Q}(\mathcal{P})$. So we have

Definition 2.1.7. [SWAPPING FRACTION ALGEBRA OF $\mathcal{P}$] *The* swapping fraction algebra of $\mathcal{P}$ *is* $\mathcal{Q}(\mathcal{P})$ *equipped with the induced swapping bracket.*

Definition 2.1.8. [CROSS FRACTION] *Let* x, y, z, t *belong to* $\mathcal{P}$ *so that* $x \neq t$ *and* $y \neq z$*. The* cross fraction *determined by* (x, y, z, t) *is the element of* $\mathcal{Q}(\mathcal{P})$*:*

$$[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{2.5}$$

Let $\mathcal{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P},\ x \neq t, y \neq z\}$ *be the set of all the cross-fractions in* $\mathcal{Q}(\mathcal{P})$.

Remark 2.1.9. *Notice that the cross fractions verify* the cross-ratio conditions *[L3] :*

Symmetry: $[a,b,c,d] = [b,a,d,c]$,

Normalisation: $[a,b,c,d] = 0$ *if and only if* $a = c$ *or* $b = d$,

Normalisation: $[a,b,c,d] = 1$ *if and only if* $a = b$ *or* $c = d$,

Cocycle identity: $[a,b,c,d] \cdot [a,b,d,e] = [a,b,c,e]$,

Cocycle identity: $[a,b,d,e] \cdot [b,c,d,e] = [a,c,e,f]$.

Let $\mathcal{B}(\mathcal{P})$ be the subring of $\mathcal{Q}(\mathcal{P})$ generated by $\mathcal{CR}(\mathcal{P})$.

Proposition 2.1.10. *$\mathcal{B}(\mathcal{P})$ is closed under swapping bracket.*

Proof. By Leibniz's rule, $\forall c_1, ..., c_n, d_1, ..., d_m \in \mathcal{Z}(\mathcal{P})$

$$\frac{\{c_1 \cdots c_n, d_1 \cdots d_m\}}{c_1 \cdots c_n \cdot d_1 \cdots d_m} = \sum_{i,j=1}^{n,m} \frac{\{c_i, d_j\}}{c_i \cdot d_j}, \tag{2.6}$$

we have only to show that for any two elements $\frac{xz}{xt} \cdot \frac{yt}{yz}$ and $\frac{uw}{us} \cdot \frac{vs}{vw}$ in $\mathcal{CR}(\mathcal{P})$, where $x \neq t$, $y \neq z$, $u \neq s$, $v \neq w$, then $\{\frac{xz}{xt} \cdot \frac{yt}{yz}, \frac{uw}{us} \cdot \frac{vs}{vw}\} \in \mathcal{B}(\mathcal{P})$. Let $e_1 = xz$, $e_2 = \frac{1}{xt}$, $e_3 = yt$, $e_4 = \frac{1}{yz}$, $h_1 = uw$, $h_2 = \frac{1}{us}$, $h_3 = vs$, $h_4 = \frac{1}{vw}$. By the definition of the swapping bracket $\frac{\{e_1,h_1\}}{e_1 \cdot h_1} \in \mathcal{CR}(\mathcal{P})$. Then by the Leibniz's rule, we deduce that for any $e, h \in \mathcal{Z}(\mathcal{P})$,

$$\frac{\{e, \frac{1}{h}\}}{e/h} = -\frac{\{e,h\}}{e \cdot h}.$$

So for any $i,j = 1,2,3,4$, we have $\frac{\{e_i,h_j\}}{e_i \cdot h_j} \in \mathcal{CR}(\mathcal{P})$. $e_1e_2e_3e_4$ and $h_1h_2h_3h_4$ are also in $\mathcal{CR}(\mathcal{P})$, so

$$\{e_1e_2e_3e_4, h_1h_2h_3h_4\} = \sum_{i,j=1}^{4} \frac{\{e_i,h_j\}}{e_i \cdot h_j} \cdot (e_1e_2e_3e_4h_1h_2h_3h_4) \in \mathcal{B}(\mathcal{P}).$$

Finally, we conclude that $\mathcal{B}(\mathcal{P})$ is closed under swapping. □

Definition 2.1.11. [SWAPPING MULTIFRACTION ALGEBRA OF $\mathcal{P}$] The swapping multifraction algebra of $\mathcal{P}$ *is* $\mathcal{B}(\mathcal{P})$ *equipped with the swapping bracket.*

2.2 Rank n swapping algebra

In this section, we define the rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$, then we prove one of our fundamental theorems, says that $\mathcal{Z}_n(\mathcal{P})$ is compatible with the swapping bracket.

2.2.1 The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$

In spirit of rank n cross-ratio in [L3] defined by F. Labourie, combining with the definition of the swapping ring above, we define the rank n swapping ring as follows.

Notation 2.2.1. *Let*

$$\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1}))=\det\begin{pmatrix} x_1y_1 & ... & x_1y_{n+1}\\ ... & ... & ...\\ x_{n+1}y_1 & ... & x_{n+1}y_{n+1}\end{pmatrix}.$$

Definition 2.2.2. [The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$] *For $n\geq 2$, let $R_n(\mathcal{P})$ be the ideal of $\mathcal{Z}(\mathcal{P})$ generated by*
$\{D\in\mathcal{Z}(\mathcal{P})\mid D=\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1})),\forall x_1,...,x_{n+1},y_1,...,y_{n+1}\in\mathcal{P}\}$.

The rank n swapping ring $\mathcal{Z}_n(\mathcal{P})$ *is the quotient ring $\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$.*

Remark 2.2.3. *Decomposing the determinant D in first row, we have by induction that*

$$R_2(\mathcal{P})\supseteq R_3(\mathcal{P})\supseteq ...\supseteq R_n(\mathcal{P}). \tag{2.7}$$

2.2.2 Swapping bracket over $\mathcal{Z}_n(\mathcal{P})$

Our main result is

Theorem 2.2.4. *For $n\geq 2$, $R_n(\mathcal{P})$ is an ideal for the swapping bracket, thus $\mathcal{Z}_n(\mathcal{P})$ inherits a Poisson bracket from the swapping bracket.*

Proof. The above theorem is equivalent to say $\forall h\in R_n(\mathcal{P})$, $\forall f\in\mathcal{Z}(\mathcal{P})$, we have $\{f,h\}\in R_n(\mathcal{P})$ where $n\geq 2$. By Leibniz's rule of the swapping bracket, it suffices to consider $f=ab\in\mathcal{Z}(\mathcal{P})$, $h=\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1}))$.

Lemma 2.2.5. *Let $n\geq 2$. Let $x_1,...,x_{n+1}$ ($y_1,...,y_{n+1}$ resp.) different from each other in $\mathcal{P}$, a,b belong to $\mathcal{P}$. Then $\{ab,\Delta((x_1,...,x_{n+1}),(y_1,...,y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$.*

In particular, $x_1,...,x_l,y_1,...,y_k$ are (strictly) on the right side of $\overrightarrow{ab}$ and $x_{l+1},...,x_m,y_{k+1},...,y_p$ are on the left side of $\overrightarrow{ab}$ as in Figure 2.2. Let

$$\begin{aligned}\Delta^R=&\sum_{d=1}^{l}x_db\cdot\Delta((x_1,...,x_{d-1},a,x_{d+1},...,x_{n+1}),(y_1,...,y_{n+1}))\\ &-\sum_{d=1}^{k}ay_d\cdot\Delta((x_1,...,x_{n+1}),(y_1,...,y_{d-1},b,y_{d+1},...,y_{n+1})),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\Delta^L=&\sum_{d=l+1}^{m}x_db\cdot\Delta((x_1,...,x_{d-1},a,x_{d+1},...,x_{n+1}),(y_1,...,y_{n+1}))\\ &-\sum_{d=k+1}^{p}ay_d\cdot\Delta((x_1,...,x_{n+1}),(y_1,...,y_{d-1},b,y_{d+1},...,y_{n+1})).\end{aligned} \tag{2.9}$$

we obtain that

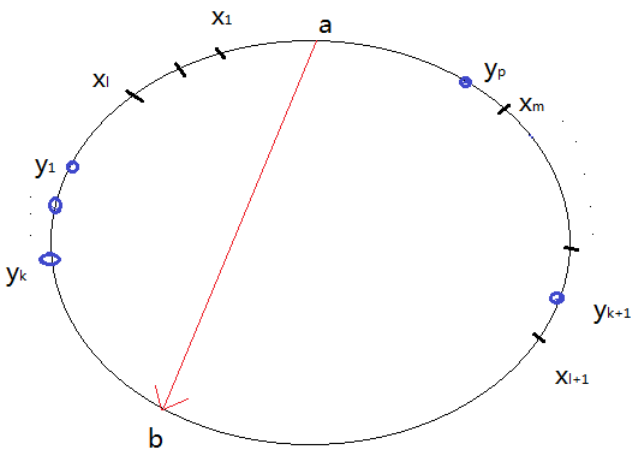


Fig. 2.2:

1. *when a does not belong to $\{x_1, ..., x_{n+1}\}$ and b does not belong to $\{y_1, ..., y_{n+1}\}$, $m = p = n + 1$, we have*

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R = -\Delta^L; \tag{2.10}$$

2. *when a coincides with x_{n+1}, b does not belong to $\{y_1, ..., y_{n+1}\}$, $m = n$, $p = n + 1$, we have*

$$\begin{aligned} &\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \\ &= \frac{1}{2} \cdot ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) + \Delta^R \\ &= -\frac{1}{2} \cdot ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) - \Delta^L; \end{aligned} \tag{2.11}$$

3. *when a does not belong to $\{x_1, ..., x_{n+1}\}$, b coincides with y_{n+1}, $m = n + 1$, $p = n$, we have*

$$\begin{aligned} &\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \\ &= -\frac{1}{2} \cdot ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) + \Delta^R \\ &= \frac{1}{2} \cdot ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) - \Delta^L. \end{aligned} \tag{2.12}$$

4. *when a coincides with some x_{n+1} and b coincides with some y_{n+1}, $m = p = n$, we have*

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R = -\Delta^L; \tag{2.13}$$

Proof. We suppose that $x_1, x_2, ..., x_{n+1}$ (resp. $y_1, y_2, ..., y_{n+1}$) cyclically ordered on the circle with anticlockwise orientation.

We assoicate to $(x_1, ..., x_{n+1}, y_1, ..., y_{n+1})$ the oriented complete bipartite graph $K_{n+1,n+1}$ with the vertex sets $X = (x_1, x_2, ..., x_{n+1})$ and $Y = (y_1, y_2, ..., y_{n+1})$ in $\mathcal{P}$, so that every

vertex in X is adjacent to every vertex in Y by an arrow from x_i to y_j, but there are no arrows within X or Y. Let Γ be the set of all the subgraphs of $K_{n+1,n+1}$ so that every vertex appear on exactly one edge.

Notation 2.2.6. *Let γ be a oriented graph, let $E(\gamma)$ be the set of all the arrows in γ, for any e in $E(\gamma)$, $e = \overrightarrow{e_+e_-}$ where $e_+(e_-)$ is said to be the* origin *(end respectively) of e. If $e = \overrightarrow{x_iy_j}$, we denote x_iy_j in $\mathcal{Z}(\mathcal{P})$ by e without loss of ambiguity.*

For any γ in Γ, we have exactly one σ in the permutation group of $n+1$ elements S_{n+1}, such that, for any e in γ, we have $x_{i_l} = e_+$, $y_{j_l} = e_-$, $\sigma(i_l) = j_l$; for any σ in S_{n+1}, we have exactly one γ in Γ, such that for any e in γ, we have $x_{i_l} = e_+$, $y_{j_l} = e_-$, $\sigma(i_l) = j_l$. For example, the subgraph γ_0 with the arrows $\{\overrightarrow{x_iy_i}\}_{i=1}^{n+1}$ corresponds to $I \in S_n$. We identify the set Γ with the group S_{n+1} of permutations of $\{1, ..., n+1\}$. We have

$$\Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) = \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} \prod_{i=1}^{n+1} x_i y_{\sigma(i)}. \tag{2.14}$$

Then $\Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))$ can be written as the sum over Γ:

$$\Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) = \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \prod_{e \in \sigma} e_+e_-, \tag{2.15}$$

where $sign(\sigma)$ is the sign of the permutation in S_{n+1}. By Leibniz's rule, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \sum_{e \in \sigma} \left(\left(\prod_{e' \in \sigma} e' \right) \cdot \frac{\{ab, e\}}{e} \right). \tag{2.16}$$

Notation 2.2.7. *The arrow $\overrightarrow{ab}$ seperates S^1 into disjoint unions*

$$\{a\} \cup \{b\} \cup I \cup J,$$

where I is on the left side *of $\overrightarrow{ab}$. Let*

$$d(a, b) := \sharp I \cap \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}.$$

If a point u in S^1 does not coincide with any one of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$, we say that u is in general position.

We say that a point u(different from v) is next to v *if $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\} \backslash \{u, v\}$ is included in the component of $S^1 \backslash \{u, v\}$ which is on the left side of $\overrightarrow{vu}$.*

Notice that every element of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ appear only once in $\prod_{i=1}^{n+1} x_i y_{\sigma(i)}$. We prove the lemma by induction on $d(a, b)$. The following calculations are all based on the combinatorial model described above and the formula 2.16.

We start our induction with $d(a, b) = 0$, then there is no vertex of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ on the left side of ab.

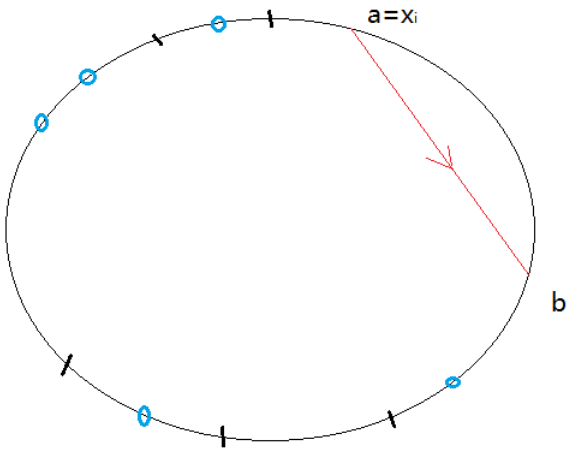


Fig. 2.3:

1. If a does not coincide with any x_i and b does not coincide with any y_j, then for any x_s , y_t, we have $\{ab, x_s y_t\} = 0$. By the formula 2.16, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = 0 \in R_n(\mathcal{P}).$$

2. If a coincides with some x_i and b does not coincide with any y_j as in Figure 2.3, then for each $\sigma \in \Gamma$, each $e \in E(\sigma)$, we have $\{ab, e\} = 0$ except for the arrow e in σ so that $e_+ = x_i$, $\mathcal{J}(a, b, e_+, e_-) = -\frac{1}{2}$. So $\{ab, e\} = -\frac{1}{2} \cdot ab \cdot e$. Hence in this case, By the formula 2.16, we deduce that

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = -\frac{1}{2} ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) \in R_n(\mathcal{P}).$$

3. If a does not coincide with any x_i, b coincides with some y_j. Similarly as Case 2, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \frac{1}{2} ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) \in R_n(\mathcal{P}).$$

4. If a coincides with some x_i, b coincides with some y_j. By the formula 2.16, we have

$$\begin{aligned} &\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \\ &= \sum_{\sigma \in \Gamma \text{ such that } ab \in \sigma} (-1)^{sign(\sigma)} \left(\sum_{e \in \sigma} \left(\prod_{e' \in \sigma} e' \right) \frac{\{ab, e\}}{e} \right) \\ &+ \sum_{\sigma \in \Gamma \text{ such that } ab \notin \sigma} (-1)^{sign(\sigma)} \left(\sum_{e \in \sigma} \left(\prod_{e' \in \sigma} e' \right) \frac{\{ab, e\}}{e} \right). \end{aligned} \tag{2.17}$$

For $\sigma \in \Gamma$ such that $ab \in \sigma$, for any $e \in \sigma$, we have $\mathcal{J}(a, b, e_+, e_-) = 0$. So the first part of the right hand side of the equation is zero.

For the second part of the right hand side of the equation, for each σ such that $ab \notin \sigma$, we have

$$\sum_{e\in\sigma}\left(\prod_{e'\in\sigma} e'\right)\frac{\{ab,e\}}{e} = \sum_{e\in\sigma}\left(\prod_{e'\in\sigma} e'\right)(\mathcal{J}(a,b,x_i,y_{\sigma(i)}) + \mathcal{J}(a,b,x_{\sigma^{-1}(j)},y_j))\cdot ab$$
$$= \sum_{e\in\sigma}\left(\prod_{e'\in\sigma} e'\right)\left(-\frac{1}{2}+\frac{1}{2}\right)\cdot ab = 0 \in R_n(\mathcal{P}). \tag{2.18}$$

So the lemma is valid for $d(a,b) = 0$.

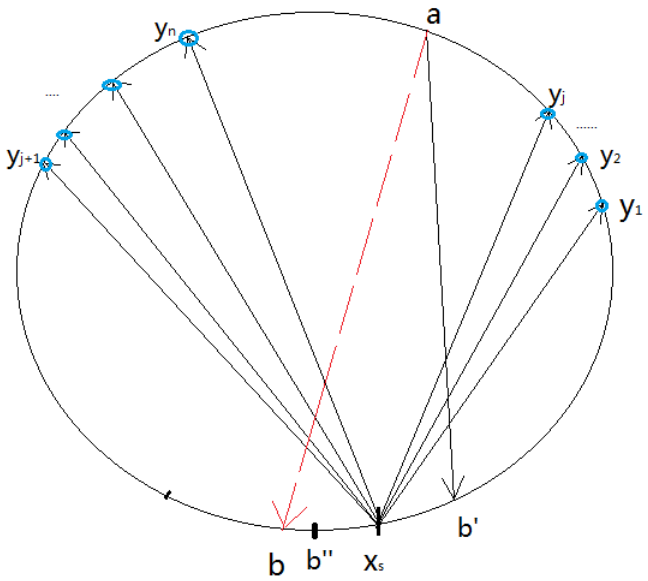


Fig. 2.4:

We suppose that the lemma is true for $d(a,b) = k \geq 0$. When $\overrightarrow{ab}$ is the dotted arrow as in Figure 2.4 with $d(a,b) = k+1$, we prove that $\{ab, \Delta(x_1,...,x_{n+1},y_1,...,y_{n+1})\}$ belongs to $R_n(\mathcal{P})$ as follows.

1. If x_s is the next to the point b as in Figure 2.4.

   (a) If b is in general position. Let b' be a point in general position next to x_s on the other side of $\overrightarrow{ax_s}$ as in Figure 2.4. Since there are d vertices on the left side

of $\overrightarrow{ab'}$, by hypothesis and the formula 2.16, we have

$$\begin{aligned}\{ab', \Delta(x_1, ..., x_{n+1}, y_1, ..., y_{n+1})\} &= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\sum_{e\in\sigma}\left(\prod_{e'\in\sigma}e'\right)\frac{\{ab', e\}}{e}\right)\\ &= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\sum_{e\in\sigma}\left(\prod_{e'\in\sigma}e'\right)\frac{\mathcal{J}(a, b', e_+, e_-)\cdot ae_-\cdot e_+b'}{e}\right)\in R_n(\mathcal{P}).\end{aligned} \tag{2.19}$$

Since the right hand side of the above formula is a polynomial in $ab', x_1b', ..., x_{n+1}b'$, we denote the right hand side of the formula by $P(ab', x_1b', ..., x_{n+1}b')$. When we replace $ab', x_1b', ..., x_{n+1}b'$ by $ab, x_1b, ..., x_{n+1}b$ in $P(ab', x_1b', ..., x_{n+1}b')$, we obtain that

$$P(ab, x_1b, ..., x_{n+1}b) = \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\sum_{e\in\sigma}\left(\prod_{e'\in\sigma}e'\right)\frac{\mathcal{J}(a, b', e_+, e_-).ae_-.e_+b}{e}\right).$$

Still, we have $P(ab, x_1b, ..., x_{n+1}b)$ belongs to $R_n(\mathcal{P})$. Then, by the formula 2.19, we have

$$\begin{aligned}&\{ab, \Delta(x_1, ..., x_{n+1}, y_1, ..., y_{n+1})\} - P(ab, x_1b, ..., x_{n+1}b)\\ &= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\sum_{e\in\sigma}\left(\prod_{e'\in\sigma}e'\right)\frac{(\mathcal{J}(a, b, e_+, e_-) - \mathcal{J}(a, b', e_+, e_-)).ae_-.e_+b}{e}\right)\\ &= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\sum_{e\in\sigma}\left(\prod_{e'\in\sigma}e'\right)\frac{(\mathcal{J}(b', b, e_+, e_-)).ae_-.e_+b}{e}\right).\end{aligned} \tag{2.20}$$

For a is in any position, we always have:
For any σ in Γ, e in σ such that $e_+ \neq x_s$,

$$\mathcal{J}(b', b, e_+, e_-) = 0;$$

For any σ in Γ, e in σ such that $e_+ = x_s$,

$$\mathcal{J}(b', b, e_+, e_-) = -1.$$

Then we have

$$\begin{aligned}&\sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\sum_{e\in\sigma}\left(\prod_{e'\in\sigma}e'\right)\frac{(\mathcal{J}(b', b, e_+, e_-)).ae_-.e_+b}{e}\right)\\ &= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)}\left(\prod_{e'\in\sigma}e'\right)\frac{(-1).ay_{\sigma(s)}\cdot x_sb}{x_sy_{\sigma(s)}}\\ &= -x_sb\cdot\Delta((x_1, ..., x_{s-1}, a, x_{s+1}, ..., x_{n+1}), (y_1, ..., y_{n+1}))\in R_n(\mathcal{P}).\end{aligned} \tag{2.21}$$

So $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$ in this case.

(b) If b coincides with x_i for some i. Let b'' be a point in general position next to b as in Figure 2.4. By the result of the case (1a), we have

$$\{ab'', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \in R_n(\mathcal{P}).$$

Since $\{ab'', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ is a polynomial in $ab'', x_1b'', ..., x_{n+1}b''$, we denote it by $C(ab'', x_1b'', ..., x_{n+1}b'')$. When $ab'', x_1b'', ..., x_{n+1}b''$ is replaced by $ab, x_1b, ..., x_{n+1}b$ in $C(ab'', x_1b'', ..., x_{n+1}b'')$, we still have $C(ab, x_1b, ..., x_{n+1}b)$ belongs to $R_n(\mathcal{P})$. By the similar argument, we have

$$\begin{aligned}
&\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} - C(ab, x_1b, ..., x_{n+1}b) \\
&= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)} \left(\sum_{e\in\sigma} \left(\prod_{e'\in\sigma} e' \right) \frac{(\mathcal{J}(a, b, e_+, e_-) - \mathcal{J}(a, b'', e_+, e_-)).ae_-.e_+b}{e} \right) \\
&= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)} \left(\sum_{e\in\sigma} \left(\prod_{e'\in\sigma} e' \right) \frac{\mathcal{J}(b'', b, e_+, e_-).ae_-.e_+b}{e} \right).
\end{aligned} \tag{2.22}$$

For a is in any position, we always have:
For any σ in Γ, e in σ such that $e_+ \neq x_i$,

$$\mathcal{J}(b'', b, e_+, e_-) = 0;$$

For any σ in Γ, e in σ such that $e_+ = x_i$,

$$\mathcal{J}(b'', b, e_+, e_-) = -\frac{1}{2}.$$

By the above formula, we have

$$\begin{aligned}
&\sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)} \left(\sum_{e\in\sigma} \left(\prod_{e'\in\sigma} e' \right) \frac{(\mathcal{J}(b'', b, e_+, e_-)).ae_-.e_+b}{e} \right) \\
&= \sum_{\sigma\in\Gamma}(-1)^{sign(\sigma)} \left(\prod_{e'\in\sigma} e' \right) \frac{(-\frac{1}{2}).ay_{\sigma(s)}.x_ib}{x_iy_{\sigma(s)}} \\
&= 0.
\end{aligned} \tag{2.23}$$

So $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$ in this case.

(c) If b coincides with y_i for some i. Let b'' be a point in general position next to b as in Figure 2.4. Without loss of ambiguity, we mention that this b'' is not the same as the case (b). By the result of the case (1a), we have

$$\{ab'', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \in R_n(\mathcal{P}).$$

Since $\{ab'', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ is a polynomial in $ab'', x_1b'', ..., x_{n+1}b''$, we denote it by $D(ab'', x_1b'', ..., x_{n+1}b'')$. When $ab'', x_1b'', ..., x_{n+1}b''$ is replaced by $ab, x_1b, ..., x_{n+1}b$ in $D(ab'', x_1b'', ..., x_{n+1}b'')$, we still have $D(ab, x_1b, ..., x_{n+1}b)$

belongs to $R_n(\mathcal{P})$. By the similar argument, we have

$$\begin{aligned}
&\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} - D(ab, x_1 b, ..., x_{n+1} b) \\
&= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left(\sum_{e \in \sigma} \left(\prod_{e' \in \sigma} e' \right) \frac{(\mathcal{J}(a, b, e_+, e_-) - \mathcal{J}(a, b'', e_+, e_-)).ae_-.e_+ b}{e} \right) \\
&= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left(\sum_{e \in \sigma} \left(\prod_{e' \in \sigma} e' \right) \frac{\mathcal{J}(b'', b, e_+, e_-).ae_-.e_+ b}{e} \right).
\end{aligned} \tag{2.24}$$

For a is in any position, we always have :
For any σ in Γ, e in σ such that $e_- \neq y_i$,

$$\mathcal{J}(b'', b, e_+, e_-) = 0;$$

For any σ in Γ, e in σ such that $e_- = y_i$,

$$\mathcal{J}(b'', b, e_+, e_-) = \frac{1}{2}.$$

Then we have

$$\begin{aligned}
&\sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left(\sum_{e \in \sigma} \left(\prod_{e' \in \sigma} e' \right) \frac{(\mathcal{J}(b'', b, e_+, e_-)).ae_-.e_+ b}{e} \right) \\
&= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left(\prod_{e' \in \sigma} e' \right) \frac{(\frac{1}{2}).ay_i.x_{\sigma^{-1}(i)} b}{x_{\sigma^{-1}(i)} y_i} \\
&= \frac{1}{2} ay_i \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1})).
\end{aligned} \tag{2.25}$$

So $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$ in this case.

(d) For b coincides with certain point x such that $x = x_i = y_j$, combining the cases (b) and (c), we have $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$.

2. If x_s is replaced by certain y_t as in Figure 2.4. We have $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$ by the similar arguments.

3. If x_s is replaced by certain x such that $x = x_s = y_t$ as in Figure 2.4. We have $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(\mathcal{P})$ by combining the above two cases.

We conclude that $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ in $R_n(\mathcal{P})$ for any $\Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))$ in $R_n(\mathcal{P})$ and any ab in $\mathcal{Z}(\mathcal{P})$.

Lemma 2.2.8. *For any $a, b, x_1, ..., x_{n+1}, y_1, ..., y_{n+1} \in \mathcal{P}$, we have*

$$\begin{aligned}
&\sum_{i=1}^{n+1} ay_i \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1})) \\
&= \sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_{n+1}), (y_1, ..., y_{n+1})).
\end{aligned} \tag{2.26}$$

Proof. We develop the left hand side of Equation 2.26 in the column of b, we have

$$\begin{aligned}&\sum_{i=1}^{n+1} ay_i \cdot \Delta((x_1,...,x_{n+1}),(y_1,...,y_{i-1},b,y_{i+1},...,y_{n+1}))\\&=\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} ay_i \cdot x_j b \cdot \Delta((x_1,...,x_{j-1},x_{j+1},...,x_{n+1}),(y_1,...,y_{i-1},y_{i+1},...,y_{n+1})).\end{aligned} \tag{2.27}$$

We develop the right hand side of Equation 2.26 in the row of a, we have

$$\begin{aligned}&\sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1,...,x_{i-1},a,x_{i+1},...,x_{n+1}),(y_1,...,y_{n+1})\\&=\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} x_i b \cdot ay_j \cdot \Delta((x_1,...,x_{i-1},x_{i+1},...,x_{n+1}),(y_1,...,y_{j-1},y_{j+1},...,y_{n+1})).\end{aligned} \tag{2.28}$$

Compare Equation 2.27 with Equation 2.28, we conclude that

$$\begin{aligned}&\sum_{i=1}^{n+1} ay_i \cdot \Delta((x_1,...,x_{n+1}),(y_1,...,y_{i-1},b,y_{i+1},...,y_{n+1}))\\&=\sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1,...,x_{i-1},a,x_{i+1},...,x_{n+1}),(y_1,...,y_{n+1})).\end{aligned} \tag{2.29}$$

□

Formulas 2.10 2.11 2.12 are proved by our induction procedures and the lemma above. □

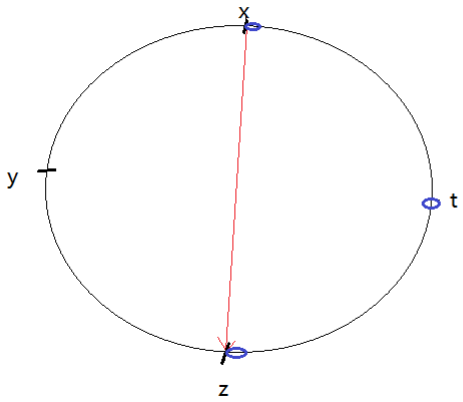


Fig. 2.5:

Remark 2.2.9. *1. For example, as in Figure 2.5, we have*

$$\{xz, \Delta((x,z,y),(z,x,t))\} = yz \cdot \Delta((x,z,x),(z,x,t)) = 0. \tag{2.30}$$

2. *It is easy to notice that a coincides with some y_j and b coincides with some x_i can be combined into generic case. But the triviality of these cases are not due to the linking number, but due to the definition that $xx = 0$ in $\mathcal{Z}(\mathcal{P})$.*

Finally, we conclude that the swapping bracket over $\mathcal{Z}_n(\mathcal{P})$ is well defined for $n \geq 2$. □

Definition 2.2.10. [RANK n SWAPPING ALGEBRA OF $\mathcal{P}$] *The* rank n swapping algebra of $\mathcal{P}$ *is the ring $\mathcal{Z}_n(\mathcal{P})$ equipped with the swapping bracket.*

2.3 $\mathcal{Z}_n(\mathcal{P})$ is an integral domain

In this section, the field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We want to know that if the cross fractions are well-defined in the fraction ring of $\mathcal{Z}_n(\mathcal{P})$. To this end, we prove that $\mathcal{Z}_n(\mathcal{P})$ is an integral domain through a geometry model considered by H. Weyl [W1] and C. De Concini and C. Procesi [CP].

2.3.1 A geometry model for $\mathcal{Z}_n(\mathcal{P})$

Let us introduce a geometry model essentially associated to the weak cross ratios. Let M_p be the configuration space of p vectors in $\mathbb{K}^n$ and p co-vectors in $\mathbb{K}^{n*}$. Here we denote $a_i = (a_{i,l})_{l=1}^n$, $b_i = (b_{i,l})_{l=1}^n$. Let $\mathbb{K}[M_p]$ be the polynomial ring generated by coordinates functions on $M_p = (\mathbb{K}^n \times \mathbb{K}^{n*})^p$.

Notation 2.3.1. *We define the product between a vector a_i in $\mathbb{K}^n$ and a co-vector b_j in $\mathbb{K}^{n*}$ by*

$$\langle a_i | b_j \rangle := b_j(a_i) = \sum_{k=1}^{n} a_{i,l} \cdot b_{j,l}. \tag{2.31}$$

Let $\mathrm{GL}(n,\mathbb{K})$ acts naturally on the vectors and the covectors by $g \circ a_i := g \cdot a_i$, $g \circ b_j := (g^{-1})^T \cdot b_j$ where T is the transpose of the matrix. It induces a $\mathrm{GL}(n,\mathbb{K})$ action on $\mathbb{K}[M_p]$, defined by:

- *For any $g \in \mathrm{GL}(n,\mathbb{K})$, $a, b \in \mathbb{K}[M_p]$, we have $g \circ (a+b) = g \circ a + g \circ b$;*
- *For any $g \in \mathrm{GL}(n,\mathbb{K})$, $a, b \in \mathbb{K}[M_p]$, we have $g \circ (a \cdot b) = (g \circ a) \cdot (g \circ b)$.*

Then $\mathbb{K}[M_p]$ is a $\mathrm{GL}(n,\mathbb{K})$-module, since for any $g, g_1, g_2 \in \mathrm{GL}(n,\mathbb{K})$, $a, b \in \mathbb{K}[M_p]$, we have:

$$g \circ (a+b) = g \circ a + g \circ b,$$
$$(g_1 + g_2) \circ a = g_1 \circ a + g_2 \circ a,$$
$$(g_1 \cdot g_2) \circ a = g_1 \circ (g_2 \circ a).$$

We denote the $\mathrm{GL}(n,\mathbb{K})$ invariant ring of $\mathbb{K}[M_p]$ by $\mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})}$.

Let $B_{n\mathbb{K}}$ be the subring of $\mathbb{K}[M_p]$ generated by $\{\langle a_i | b_j \rangle\}_{i=1,j=1}^p$.

Since $\langle a_i|b_j\rangle \in \mathbb{K}[M_p]$ is invariant under $\mathrm{GL}(n,\mathbb{K})$, we have $B_{n\mathbb{K}} \subseteq \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})}$. Moreover

Theorem 2.3.2. [C. De Concini and C. Procesi [CP] [1]] $B_{n\mathbb{K}} = \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})}$.

Since $\mathbb{K}[M_p]$ is an integral domain, we have

Corollary 2.3.3. [C. De Concini and C. Procesi [CP]] *The subring $B_{n\mathbb{K}}$ is an integral domain.*

H. Weyl describe $B_{n\mathbb{K}}$ as a quotient ring.

Theorem 2.3.4. [H. Weyl [W1]] *The relations in $B_{n\mathbb{K}}$ are generated by* $R = \{f \in B_{n\mathbb{K}} \mid f = \det \begin{pmatrix} \langle a_{i_1}|b_{j_1}\rangle & \dots & \langle a_{i_1}|b_{j_{n+1}}\rangle \\ \dots & \dots & \dots \\ \langle a_{i_{n+1}}|b_{j_1}\rangle & \dots & \langle a_{i_{n+1}}|a_{j_{n+1}}\rangle \end{pmatrix}, \forall i_k, j_l = 1, \dots, p\}$.

Remark 2.3.5. *In other words, let W be the polynomial ring $\mathbb{K}[\{x_{i,j}\}_{i,j=1}^{p}]$,* $r = \{f \in W \mid f = \det \begin{pmatrix} x_{i_1,j_1} & \dots & x_{i_1,j_{n+1}} \\ \dots & \dots & \dots \\ x_{i_{n+1},j_1} & \dots & x_{i_{n+1},j_{n+1}} \end{pmatrix}, \forall i_k, j_l = 1, \dots, p\}$, *let T be the ideal of W generated by r, then we have $B_{n\mathbb{K}} \cong W/T$.*

Let us recall that $\mathcal{Z}_n(\mathcal{P}) = \mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$ is the rank n swapping ring where $\mathcal{P} = \{x_1, \dots, x_p\}$. When we identify a_i with x_i on the left and b_i with x_i on the right of the pairs of points in $\mathcal{Z}_n(\mathcal{P})$, we obtain the main result of this section

Theorem 2.3.6. *Let $\mathcal{Z}_n(\mathcal{P})$ be the rank n swapping ring. Let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{\langle a_i|b_i\rangle\}_{i=1}^{p}$, then $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$.*

2.3.2 Proof of the main theorem

Our main result is

Theorem 2.3.7. *For $n > 1$, $\mathcal{Z}_n(\mathcal{P})$ is an integral domain.*

Proof. Consider these $\mathrm{GL}(n,\mathbb{K})$-modules: Let L be the ideal of $\mathbb{K}[M_p]$ generated by $(\{\langle a_i|b_i\rangle\}_{i=1}^{p})$, let K_p be the quotient ring $\mathbb{K}[\{a_{i,l}, b_{i,l}\}_{i,l=1}^{p,n}]/L$, let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{\langle a_i|b_i\rangle\}_{i=1}^{p}$.

There is an exact sequence of $\mathrm{GL}(n,\mathbb{K})$-modules:

$$0 \to L \to \mathbb{K}[M_p] \to K_p \to 0. \tag{2.32}$$

By Lie group cohomology[CE], the exact sequence above induces the long exact sequence :

$$0 \to L^{\mathrm{GL}(n,\mathbb{K})} \to \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})} \to K_p^{\mathrm{GL}(n,\mathbb{K})} \to H^1(\mathrm{GL}(n,\mathbb{K}), L) \to .... \tag{2.33}$$

[1] Thanks for the reference provided by J. B. Bost.

Proposition 2.3.8. *Let S be the finite subset $\{\langle a_i|b_i\rangle\}_{i=1}^{p}$. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Then*

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{K})} = \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})}\cdot S.$$

Remark 2.3.9. *For $K=\mathbb{C}$, this lemma can be proved by Weyl's unitary trick. But for $K=\mathbb{R}$, this lemma depends on Corollary 2.3.12.*

Proof. We prove the proposition in three steps. Firstly, we prove the proposition when we substitute $\mathrm{U}(n)$ for $\mathrm{GL}(n,\mathbb{K})$; secondly, by Corollary 2.3.12, we prove the proposition for $\mathrm{GL}(n,\mathbb{C})$; finally, by Corollary 2.3.12, we prove the proposition for $\mathrm{GL}(n,\mathbb{R})$.

1. Let
$$\mathrm{U}(n) = \{g\in \mathrm{GL}(n,\mathbb{C}) \mid g\cdot \bar{g}^T = I\}.$$
Let us prove that
$$(\mathbb{C}[M_p]\cdot S)^{\mathrm{U}(n)} = \mathbb{C}[M_p]^{\mathrm{U}(n)}\cdot S.$$
Of course, we have
$$(\mathbb{C}[M_p]\cdot S)^{\mathrm{U}(n)} \supseteq \mathbb{C}[M_p]^{\mathrm{U}(n)}\cdot S.$$
We now prove that $(\mathbb{C}[M_p]\cdot S)^{\mathrm{U}(n)} \subseteq \mathbb{C}[M_p]^{\mathrm{U}(n)}\cdot S$. Let $\mathrm{d}g$ be a Haar measure on $\mathrm{U}(n)$. Let x belongs to $(\mathbb{C}[M_p]\cdot S)^{\mathrm{U}(n)}$. we represent x by $\sum_{l=1}^{k} t_l\cdot s_l$ where $t_l\in\mathbb{C}[M_p]$ and $s_l\in S$. Since $S\subseteq \mathbb{C}[M_p]^{\mathrm{GL}(n,\mathbb{C})}\subseteq \mathbb{C}[M_p]^{\mathrm{U}(n)}$, for any $g\in\mathrm{U}(n)$, $g\circ s_l = s_l$. Thus we have
$$x = g\circ x = \sum_{l=1}^{k}(g\circ t_l)\cdot(g\circ s_l) = \sum_{l=1}^{k}(g\circ t_l)\cdot s_l.$$
So
$$g\circ x = \int_{\mathrm{U}(n)}\sum_{l=1}^{k}(g\circ t_l)\cdot s_l\,\mathrm{d}g = \sum_{l=1}^{k}\left(\int_{\mathrm{U}(n)} g\circ t_l\,\mathrm{d}g\right)\cdot s_l. \tag{2.34}$$
Let
$$b_l = \int_{\mathrm{U}(n)} g\circ t_l\,\mathrm{d}g.$$
For any g_1 in $\mathrm{U}(n)$, we have
$$\begin{aligned} g_1\circ b_l &= \int_{\mathrm{U}(n)} g_1\circ(g\circ t_l)\ \mathrm{d}g = \int_{\mathrm{U}(n)}((g_1\cdot g)\circ t_l)\ \mathrm{d}g \\ &= \int_{\mathrm{U}(n)}((g_1\cdot g)\circ t_l)\ \mathrm{d}\,(g_1\cdot g) = b_l. \end{aligned} \tag{2.35}$$
So b_l belongs to $\mathbb{C}[M_p]^{\mathrm{U}(n)}$, hence x belongs to $\mathbb{C}[M_p]^{\mathrm{U}(n)}\cdot S$. Therefore, we have
$$(\mathbb{C}[M_p]\cdot S)^{\mathrm{U}(n)} \subseteq \mathbb{C}[M_p]^{\mathrm{U}(n)}\cdot S.$$
We conclude that
$$(\mathbb{C}[M_p]\cdot S)^{\mathrm{U}(n)} = \mathbb{C}[M_p]^{\mathrm{U}(n)}\cdot S.$$

2. Secondly, let us prove that

$$(\mathbb{C}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{C})} = \mathbb{C}[M_p]^{\mathrm{GL}(n,\mathbb{C})}\cdot S.$$

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. Let $\mathfrak{s}$ be a subset of $\mathfrak{g}$. Let V be a subset of $\mathbb{C}[M_p]$. Let

$$V^{\mathfrak{s}} = \{a \in V \mid \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\cdot h)\circ a = 0\ \forall h\ \in\ \mathfrak{s}\ \}.$$

Since for the groups $\mathrm{U}(n)$ and $\mathrm{GL}(n,\mathbb{C})$, the exponential map is surjective, for any subset V of $\mathbb{C}[M_p]$, we have

$$V^{\mathrm{U}(n)} = V^{\mathfrak{u}(n)}$$

and

$$V^{\mathrm{GL}(n,\mathbb{C})} = V^{\mathfrak{gl}(n,\mathbb{C})}.$$

To prove that

$$V^{\mathfrak{gl}(n,\mathbb{C})} = V^{\mathfrak{u}(n)},$$

since

$$V^{\mathfrak{gl}(n,\mathbb{C})} = V^{\mathfrak{u}(n)+i\cdot\mathfrak{u}(n)} = V^{\mathfrak{u}(n)}\cap V^{i\cdot\mathfrak{u}(n)},$$

we only have to prove that

$$V^{\mathfrak{u}(n)} = V^{i\cdot\mathfrak{u}(n)}.$$

Lemma 2.3.10. *For any $u\in\mathfrak{gl}(n,\mathbb{C})$, any $v\in\mathbb{C}[M_p]$, we have*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\exp\left(t\cdot i\cdot u\right)\circ(v)\right) = i\cdot\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\exp\left(t\cdot u\right)\circ(v)\right). \tag{2.36}$$

Proof. Let $a_{i_1,j_1}\cdots a_{i_s,j_s}b_{k_1,l_1}\cdots b_{k_t,l_t}$ be a monomial in $\mathbb{C}[M_p]$, where $i_1,...,i_s,k_1,...,k_t\in\{1,...,p\}$, $j_1,...,j_s,l_1,...,l_t\in\{1,...,n\}$. Then, for any $u\in\mathfrak{gl}(n,\mathbb{C})$, we have

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\exp\left(t\cdot u\right)\circ\left(a_{i_1,j_1}\cdots a_{i_s,j_s}\cdot b_{k_1,l_1}\cdots b_{k_t,l_t}\right)\right)\\
&=\sum_{m=1}^{s}\left(\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\exp\left(t\cdot u\right)\circ a_{i_m,j_m}\right)\cdot a_{i_1,j_1}\cdots a_{i_{m-1},j_{m-1}}\cdot a_{i_{m+1},j_{m+1}}\cdots a_{i_s,j_s}\cdot b_{k_1,l_1}\cdots b_{k_t,l_t}\right)\\
&+\sum_{d=1}^{t}\left(\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\exp\left(-t\cdot u^T\right)\circ b_{i_d,j_d}\right)\cdot a_{i_1,j_1}\cdots a_{i_s,j_s}\cdot b_{k_1,l_1}\cdots b_{k_{d-1},l_{d-1}}\cdot b_{k_{d+1},l_{d+1}}\cdots b_{k_t,l_t}\right)\\
&=\sum_{m=1}^{s}\left(\left(u\circ a_{i_m,j_m}\right)\cdot a_{i_1,j_1}\cdots a_{i_{m-1},j_{m-1}}\cdot a_{i_{m+1},j_{m+1}}\cdots a_{i_s,j_s}\cdot b_{k_1,l_1}\cdots b_{k_t,l_t}\right)\\
&-\sum_{d=1}^{t}\left(\left(u^T\circ b_{i_d,j_d}\right)\cdot a_{i_1,j_1}\cdots a_{i_s,j_s}\cdot b_{k_1,l_1}\cdots b_{k_{d-1},l_{d-1}}\cdot b_{k_{d+1},l_{d+1}}\cdots b_{k_t,l_t}\right).
\end{aligned} \tag{2.37}$$

Replacing u by $i \cdot u$ in the above formula, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot i \cdot u) \circ (a_{i_1,j_1} \cdots a_{i_s,j_s} \cdot b_{k_1,l_1} \cdots b_{k_t,l_t})) \\ &= i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot u) \circ (a_{i_1,j_1} \cdots a_{i_s,j_s} \cdot b_{k_1,l_1} \cdots b_{k_t,l_t})) . \end{aligned} \tag{2.38}$$

Since for any $v_1, ..., v_r \in \mathbb{C}[M_p]$, any $\lambda_1, ..., \lambda_r \in \mathbb{C}$ and any $h \in \mathfrak{gl}(n, \mathbb{C})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot h) \circ (\lambda_1 \cdot v_1 + ... + \lambda_r \cdot v_r)) = \sum_{k=1}^{r} \lambda_r \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot h) \circ (v_k)) ,$$

we obtain that for any $v \in \mathbb{C}[M_p]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot i \cdot u) \circ (v)) = i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot u) \circ (v)) . \tag{2.39}$$

□

Remark 2.3.11. *This lemma largely depends on the group action of* $\mathrm{GL}(n, \mathbb{K})$ *on* $\mathbb{K}[M_p]$*, see Notation 2.3.1.*

Corollary 2.3.12. *Let* V *be a subset of* $\mathbb{C}[M_p]$*. Let* $\mathfrak{g}$ *be a linear Lie algebra. Then*

$$V^{\mathfrak{g}} = V^{i \cdot \mathfrak{g}}.$$

Proof. If $v \in V^{\mathfrak{g}}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot u) \circ (v)) = 0$$

for any $u \in \mathfrak{g}$. By Lemma 2.3.10, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t \cdot i \cdot u) \circ (v)) = 0$$

for any $u \in \mathfrak{g}$. So $v \in V^{i \cdot \mathfrak{g}}$. Hence, we have $V^{\mathfrak{g}} \subseteq V^{i \cdot \mathfrak{g}}$. Similarly, we have $V^{\mathfrak{g}} \supseteq V^{i\mathfrak{g}}$. We conclude that

$$V^{\mathfrak{g}} = V^{i \cdot \mathfrak{g}}.$$

□

By Corollary 2.3.12, we obtain that

$$V^{\mathfrak{u}(n)} = V^{i \cdot \mathfrak{u}(n)},$$

thus we have

$$V^{\mathfrak{gl}(n,\mathbb{C})} = V^{\mathfrak{u}(n)}.$$

Hence

$$\begin{aligned} &(\mathbb{C}[M_p] \cdot S)^{\mathrm{GL}(n,\mathbb{C})} = (\mathbb{C}[M_p] \cdot S)^{\mathfrak{gl}(n,\mathbb{C})} = (\mathbb{C}[M_p] \cdot S)^{\mathfrak{u}(n)} = (\mathbb{C}[M_p] \cdot S)^{\mathrm{U}(n)} \\ &= \mathbb{C}[M_p]^{\mathrm{U}(n)} \cdot S = \mathbb{C}[M_p]^{\mathfrak{u}(n)} \cdot S = \mathbb{C}[M_p]^{\mathfrak{gl}(n,\mathbb{C})} \cdot S = \mathbb{C}[M_p]^{\mathrm{GL}(n,\mathbb{C})} \cdot S. \end{aligned} \tag{2.40}$$

We conclude that

$$(\mathbb{C}[M_p] \cdot S)^{\mathrm{GL}(n,\mathbb{C})} = \mathbb{C}[M_p]^{\mathrm{GL}(n,\mathbb{C})} \cdot S.$$

3. Finally, let us prove that

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{R})} = \mathbb{R}[M_p]^{\mathrm{GL}(n,\mathbb{R})}\cdot S.$$

Of course, we have

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{R})} \supseteq \mathbb{R}[M_p]^{\mathrm{GL}(n,\mathbb{R})}\cdot S.$$

To prove that

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{R})} \subseteq \mathbb{R}[M_p]^{\mathrm{GL}(n,\mathbb{R})}\cdot S,$$

firstly, we have

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{R})} \subseteq (\mathbb{R}[M_p]\cdot S)^{\mathfrak{gl}(n,\mathbb{R})}. \tag{2.41}$$

By Corollary 2.3.12, we have

$$(\mathbb{R}[M_p]\cdot S)^{\mathfrak{gl}(n,\mathbb{R})} = (\mathbb{R}[M_p]\cdot S)^{\mathfrak{gl}(n,\mathbb{R})+i\cdot\mathfrak{gl}(n,\mathbb{R})} = (\mathbb{R}[M_p]\cdot S)^{\mathfrak{gl}(n,\mathbb{C})} = (\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{C})}. \tag{2.42}$$

On the other hand, by case 2, we have

$$(\mathbb{C}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{C})} = \mathbb{C}[M_p]^{\mathrm{GL}(n,\mathbb{C})}\cdot S = B_{\mathbb{C}}\cdot S.$$

When we restrict to the polynomials with real coefficients of the above equation, we have

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{C})} = B_{\mathbb{R}}\cdot S = \mathbb{R}[M_p]^{\mathrm{GL}(n,\mathbb{R})}\cdot S. \tag{2.43}$$

By Equations 2.41,2.43, 2.42, we obtain that

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{R})} \subseteq \mathbb{R}[M_p]^{\mathrm{GL}(n,\mathbb{R})}\cdot S.$$

We conclude that

$$(\mathbb{R}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{R})} = \mathbb{R}[M_p]^{\mathrm{GL}(n,\mathbb{R})}\cdot S.$$

Finally, we conclude that

$$(\mathbb{K}[M_p]\cdot S)^{\mathrm{GL}(n,\mathbb{K})} = \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})}\cdot S$$

for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

□

Lemma 2.3.13. *There is a ring homomorphism $\theta : B_{n\mathbb{K}}/S_{n\mathbb{K}} \to K_p^{\mathrm{GL}(n,\mathbb{K})}$ induced from the exact sequence:*

$$0 \to L^{\mathrm{GL}(n,\mathbb{K})} \to \mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})} \to K_p^{\mathrm{GL}(n,\mathbb{K})} \to H^1(\mathrm{GL}(n,\mathbb{K}),L) \to .... \tag{2.44}$$

is injective.

Proof. We have $\mathbb{K}[M_p]^{\mathrm{GL}(n,\mathbb{K})} = B_{n\mathbb{K}}$. By Proposition 2.3.8, we have $L^{\mathrm{GL}(n,\mathbb{K})} = (\mathbb{K}[M_p](\{\langle a_i|b_i\rangle\}_{i=1}^p))^{\mathrm{GL}(n,\mathbb{K})} = B_{n\mathbb{K}}(\{\langle a_i|b_i\rangle\}_{i=1}^p)) = S_{n\mathbb{K}}$. So the exact sequence becomes into:

$$0 \to S_{n\mathbb{K}} \to B_{n\mathbb{K}} \to K_p^{\mathrm{GL}(n,\mathbb{K})} \to H^1(\mathrm{GL}(n,\mathbb{K}),L) \to ... \tag{2.45}$$

So there is an injective ring homomorphism θ from $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ to $K_p^{\mathrm{GL}(n,\mathbb{K})}$. □

Proposition 2.3.14. *K_p is an integral domain.*

Proof. We proof the theorem by induction on the number of the vectors p. When $p = 1$, $K_1 = \mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]/(\{\sum_{k=1}^n a_{1,k} \cdot b_{1,k}\})$.

Lemma 2.3.15. *For $n > 1$, $\sum_{k=1}^n a_{1,k}.b_{1,k}$ is an irreducible polynomial in the integral domain $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$.*

Proof. Let us define the degree of a monomial in $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$ to be the sum of the degrees in all the variables. Let the degree of a polynomial f in $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$ be the maximal degree of the monomials in f, denoted by $\deg(f)$. Suppose that $\sum_{k=1}^n a_{1,k}.b_{1,k}$ is a reducible polynomial in $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$, we have

$$\sum_{k=1}^{n} a_{1,k}.b_{1,k} = g \cdot h,$$

where $g, h \in \mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$, $\deg(g) > 0$ and $\deg(h) > 0$. Since $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$ is a integral domain, $2 = \deg(gh) = \deg(g) + \deg(h)$, so we have $\deg(g) = \deg(h) = 1$. Suppose that

$$g = c + \lambda_1 \cdot x_1 + ... + \lambda_r \cdot x_r,$$

$$h = d + \mu_1 \cdot y_1 + ... + \mu_s \cdot y_s,$$

where $\lambda_1, ..., \lambda_r, \mu_1, ..., \mu_s$ are non zero elements in $\mathbb{K}$, $x_1, ..., x_r$ ($y_1, ..., y_s$ resp.) are different elements in $\{a_{1,k}, b_{1,k}\}_{k=1}^n$. Since there is no square in $g \cdot h$, we have

$$\{x_1, ..., x_r\} \cap \{y_1, ..., y_s\} = \emptyset$$

and

$$r \cdot s = n.$$

Since there are $2n$ variables in $g \cdot h$, we have

$$r + s = 2n.$$

Then $r \cdot s \geq 2n - 1$. Since $n > 1$, we obtain that

$$r \cdot s \geq 2n - 1 > n = r \cdot s,$$

which is a contradiction. We conclude that $\sum_{k=1}^n a_{1,k}.b_{1,k}$ is an irreducible polynomial in $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$. □

Since $\mathbb{K}[\{a_{1,k}, b_{1,k}\}_{k=1}^n]$ is an integral domain and $\sum_{k=1}^n a_{1,k}.b_{1,k}$ is an irreducible polynomial by Lemma 2.3.15, we obtain that K_1 is an integral domain. Suppose that the theorem is true for $p = m \geq 1$. When $p = m + 1$,

$$K_{m+1} = \mathbb{K}[\{a_{i,k}, b_{i,k}\}_{i,k=1}^{m+1,n}]/(\{\sum_{k=1}^{n} a_{i,k}.b_{i,k}\}_{i=1}^{m+1}) = K_m[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^n]/(\{\sum_{k=1}^{n} a_{m+1,k} \cdot b_{m+1,k}\}), \tag{2.46}$$

we have $K_m[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^n]$ is an integral domain. $\sum_{k=1}^n a_{m+1,k} \cdot b_{m+1,k}$ is an irreducible polynomial over $K_m[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^n]$. Since $a_{m+1,k}, b_{m+1,k} (k = 1, ..., n)$ are not variable that appear in K_m, so $\sum_{k=1}^n a_{m+1,k} \cdot b_{m+1,k}$ is an irreducible polynomial over $K_m[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^n]$, so K_{m+1} is an integral domain.

We conclude that K_p is an integral domain for any $p \geq 1$. □

By Proposition 2.3.14, K_p is an integral domain, we deduce that $K_p^{\mathrm{GL}(n,\mathbb{K})}$ is an integral domain. By Lemma 2.3.13, there is an injective ring homomorphism θ from $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ to $K_p^{\mathrm{GL}(n,\mathbb{K})}$, $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ is an integral domain. Moreover, by Theorem 2.3.6 $\mathcal{Z}_n(\mathcal{P}) \cong B_{n\mathbb{K}}/S_{n\mathbb{K}}$, finally, we conclude that $\mathcal{Z}_n(\mathcal{P})$ is an integral domain. □

Remark 2.3.16. *$\mathcal{Z}_1(\mathcal{P})$ is not a integral domain, since*

$$D = xy.yz = \det \begin{pmatrix} xy & xz \\ yy & yz \end{pmatrix}$$

is zero in $\mathcal{Z}_1(\mathcal{P})$, but we have xy and yz are not zero in $\mathcal{Z}_1(\mathcal{P})$ whenever $x \neq y$, $y \neq z$.

Remark 2.3.17. *We observe that this geometry model has other important usage than the theorem above. It is easy to see that the stability condition of the images of enough finite points of the limit double curve is related to the hyper-convexity property of the representation of surface group. We shall interpret some algebraic geometry property by some dynamic property related to $\mathcal{Z}_n(\mathcal{P})$ in the future.*

2.3.3 Rank n swapping fraction algebra of $\mathcal{P}$

Definition 2.3.18. *The* rank n swapping fraction ring $\mathcal{Q}_n(\mathcal{P})$ *is the total fraction ring of* $\mathcal{Z}_n(\mathcal{P})$.

Similar to $\mathcal{Q}(\mathcal{P})$, we have the swapping bracket is well defined on $\mathcal{Q}_n(\mathcal{P})$.

Definition 2.3.19. *The* rank n swapping fraction algebra *of $\mathcal{P}$ is the ring $\mathcal{Q}_n(\mathcal{P})$ equipped with the closed swapping bracket.*

2.3.4 Rank n swapping multifraction algebra of $\mathcal{P}$

Let $\mathcal{CR}_n(\mathcal{P}) = \{ [x, y, z, t] = \frac{xz}{xt} \cdot \frac{yt}{yz} \in \mathcal{Q}_n(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P},\ x \neq t,\ y \neq z \}$ be the set of all the cross-ratios in $\mathcal{Q}_n(\mathcal{P})$.

Definition 2.3.20. *The* rank n swapping multifraction ring $\mathcal{B}_n(\mathcal{P})$ *is the subring of $\mathcal{Q}_n(\mathcal{P})$ generated by $\mathcal{CR}_n(\mathcal{P})$.*

Similar to Proposition 2.1.10, we have

Proposition 2.3.21. *$\mathcal{B}_n(\mathcal{P})$ is closed under swapping bracket.*

Definition 2.3.22. *The* rank n swapping multifraction algebra of $\mathcal{P}$ *is the ring $\mathcal{B}_n(\mathcal{P})$ equipped with the closed swapping bracket.*

2.4 The rank n swapping algebra and the weak cross ratios

In this section, let $\mathcal{P}$ be a subset of $\partial_\infty\pi_1(S)$ where S is a surface of negative Euler class with boundaries. We associate the rank n swapping algebra with the rank n cross ratios, similar to the ring homomorphism in [L4], but up to $R_n(\mathcal{P})$.

2.4.1 The weak cross ratios

A theorem of F. Labourie [L1] and O. Guichard[Gu1] help us to consider Hitchin component as the space of the double limit curves $\{(\xi,\xi^*)\}$. Let us recall some definitions.

Definition 2.4.1. [Hitchin component] *An* n-Fuchsian homomorphism from $\pi_1(S)$ to $\mathrm{PSL}(n,\mathbb{R})$ *is a homomorphism* $\rho = i\circ\rho_0$*, where* ρ_0 *is a discrete faithful homomorphism with values in* $\mathrm{PSL}(2,\mathbb{R})$ *and* i *is the irreducible homomorphism from* $\mathrm{PSL}(2,\mathbb{R})$ *to* $\mathrm{PSL}(n,\mathbb{R})$*. A homomorphism is* Hitchin *if it may be deformed into an n-Fuchsian homomorphism.* Hitchin component $H_n(S)$ *is the space of Hitchin homomorphisms up to adjoint action of* $\mathrm{PSL}(n,\mathbb{R})$*.*

Definition 2.4.2. [Hyperconvex map] *A continuous map* ξ *from a set* $\mathcal{P}$ *to* $\mathbb{RP}^{n-1}$ *is* hyperconvex *if for any pairwise distinct points* $(x_1,...,x_p)$ *with* $p\leq n$*, the following sum is direct*

$$\xi(x_1)+...+\xi(x_p).$$

Let $\partial_\infty\pi_1(S)$ be the boundary at infinity of $\pi_1(S)$. When we fix an uniformisation of the universal cover of the surface S equipped with a complex structure, $\partial_\infty\pi_1(S)$ can be identified with the real projective line $\mathbb{RP}^1$ as the boundary of $\mathbb{H}^2$.

Definition 2.4.3. [n-hyperconvex] *A homomorphism* ρ *from* $\pi_1(S)$ *to* $PSL(n,\mathbb{R})$ *is* n-hyperconvex*, if there exists a* ρ*-equivariant hyperconvex map* ξ *from* $\partial_\infty\pi_1(S)$ *to* $\mathbb{RP}^{n-1}$*, namely* $\xi(\gamma x)=\rho(\gamma)\xi(x)$*. Such a map is called the* limit curve *of the homomorphism.*

Theorem 2.4.4. [F. Labourie [L1], O. Guichard[Gu1]] *Every homomorphism* ρ *from* $\pi_1(S)$ *to* $PSL(n,\mathbb{R})$ *is Hitchin if and only if* ρ *is n-hyperconvex.*

If ρ is n-hyperconvex, there is uniquely another ρ-equivariant hyperconvex map ξ^* from $\partial_\infty\pi_1(S)$ to $\mathbb{P}(\mathbb{R}^n)^*$ such that $\xi(x)\in ker(\xi^*(y))\Leftrightarrow x=y$. Hence $\rho\in H_n(S)$ is associated with a double limit curve (ξ,ξ^*) by Theorem 2.4.4.

Definition 2.4.5. [Weak cross ratio] *Let* (ξ,ξ^*) *be a double limit curve. Let* $\tilde{\xi}$*(*$\tilde{\xi^*}$ *resp.) be the lifts of* ξ*(*ξ^* *resp.) with values in* $\mathbb{R}^n$*(*$\mathbb{R}^{n*}$ *resp.). The* weak cross ratio $\mathbb{B}_{\xi,\xi^*}$ *of 4 different points* x,y,z,t *in* $\partial_\infty\pi_1(S)$ *is defined to be*

$$\mathbb{B}_{\xi,\xi^*}(x,y,z,t)=\frac{\left\langle\tilde{\xi}(x)\middle|\tilde{\xi^*}(z)\right\rangle}{\left\langle\tilde{\xi}(x)\middle|\tilde{\xi^*}(t)\right\rangle}\cdot\frac{\left\langle\tilde{\xi}(y)\middle|\tilde{\xi^*}(t)\right\rangle}{\left\langle\tilde{\xi}(y)\middle|\tilde{\xi^*}(z)\right\rangle}, \tag{2.47}$$

which is independent of the lifts $\tilde{\xi}$ *with values in* $\mathbb{R}^n$ *and* $\tilde{\xi^*}$ *with values in* $\mathbb{R}^{n*}$*.*

F. Labourie [L4] define a ring homomorphism $I : \mathcal{B}(\mathcal{P}) \to \mathcal{C}^\infty(H_n(S))$ such that

$$I\left(\frac{xz}{xt}\cdot\frac{yt}{yz}\right)(\rho) = \mathbb{B}_{\xi,\xi^*}(x,y,z,t) \tag{2.48}$$

for all $\frac{xz}{xt}\cdot\frac{yt}{yz} \in \mathcal{CR}(\mathcal{P})$. Since $\mathcal{CR}(\mathcal{P})$ generates $\mathcal{B}(\mathcal{P})$ over $\mathbb{R}$, the map above shall be extended to the whole ring $\mathcal{B}(\mathcal{P})$.

Since there is the swapping bracket on $\mathcal{B}(\mathcal{P})$, the ring homomorphism I induces a Poisson bracket $\{\cdot,\cdot\}_I$ on $I(\mathcal{B}(\mathcal{P}))$.

Definition 2.4.6. *For any $\alpha,\beta \in \mathcal{B}(\mathcal{P})$, the Poisson bracket $\{\cdot,\cdot\}_I$ on $I(\mathcal{B}(\mathcal{P}))$ is*

$$\{I(\alpha), I(\beta)\}_I := I(\{\alpha,\beta\}). \tag{2.49}$$

Definition 2.4.7. *For $n>1$, let $i'_n : \mathcal{CR}_n(\mathcal{P}) \to \mathcal{C}^\infty(H_n(S))$ be the map such that for any $\frac{xz}{xt}\cdot\frac{yt}{yz} \in \mathcal{CR}_n(\mathcal{P})$:*

$$i'_n\left(\frac{xz}{xt}\cdot\frac{yt}{yz}\right)(\rho) = \mathbb{B}_{\xi,\xi^*}(x,y,z,t), \tag{2.50}$$

where ρ is associated with the double limit curve (ξ,ξ^*).

Proposition 2.4.8. *For $n>1$, the map i'_n extends to a ring homomorphism $i_n : \mathcal{B}_n(\mathcal{P}) \to \mathcal{C}^\infty(H_n(S))$ where $i_n|_{\mathcal{CR}_n(\mathcal{P})} = i'_n$.*

Proof. Let j be the natural surjective ring projection from $\mathcal{B}(\mathcal{P})$ to $\mathcal{B}_n(\mathcal{P})$. We define a map i_n as follows: for any Q_n in $\mathcal{B}_n(\mathcal{P})$, let $Q(\alpha_1,...,\alpha_l)$ be one of its representative in $\mathcal{B}(\mathcal{P})$, where $\alpha_1,...,\alpha_l$ belong to $\mathcal{CR}(\mathcal{P})$, Q is a rational function, such that

$$Q_n = j(Q(\alpha_1,...,\alpha_l)) = Q(j(\alpha_1),...,j(\alpha_l)),$$

let

$$i_n(Q_n) := I(Q(\alpha_1,...,\alpha_l)).$$

We have to prove that i_n is a well defined ring homomorphism. By our definition, we only have to prove that if Q_n equals to zero in $\mathcal{B}_n(\mathcal{P})$, then $I(Q(\alpha_1,...,\alpha_l))$ equals to zero in $\mathcal{C}^\infty(H_n(S))$ which does not depend on the choice of the representative in $\mathcal{B}(\mathcal{P})$. To this end, we identify the rank n cross ratio conditions with the generators in $R_n(\mathcal{P})$ after we introduce some notations in the next section.

2.4.2 Rank n cross ratio

Definition 2.4.9. [Cross ratio [L3]] *A cross ratio on $\partial_\infty\pi_1(S)$ is a Hölder function $\mathbb{B}$ defined on $\partial_\infty\pi_1(S)^{4*} = \{(a,b,c,d) \in \partial_\infty\pi_1(S)^4 \,|\, x\neq t, y\neq z\}$ invariant under the diagonal action of $\pi_1(S)$ and which satisfies :*

Symmetry: $\mathbb{B}(a,b,c,d) = \mathbb{B}(b,a,d,c)$,

Normalisation: $\mathbb{B}(a,b,c,d) = 0$ *if and only if* $a=c$ *or* $b=d$,

Normalisation: $\mathbb{B}((a,b,c,d) = 1$ *if and only if* $a=b$ *or* $c=d$,

Cocycle identity: $\mathbb{B}(a,b,c,d)\cdot\mathbb{B}(a,b,d,e) = \mathbb{B}(a,b,c,e)$,

Cocycle identity: $\mathbb{B}(a,b,d,e)\cdot\mathbb{B}(b,c,d,e) = \mathbb{B}(a,c,d,e)$.

Definition 2.4.10. [Rank n cross ratio [L3]] *For every integer p, let $\mathcal{K}^p$ be the set of*

$$(u,v)=((v_0,...,v_p),(u_0,...,u_p)) \tag{2.51}$$

where $u_i,v_i\in\partial_\infty\pi_1(S)$, such that $u_j\neq u_i\neq u_0$ and $v_j\neq v_i\neq v_0$ whenever $j>i>0$. Let $\mathbb{B}$ be a cross ratio and let $\chi^p_{\mathbb{B}}$ be the map from $\mathcal{K}^p$ to $\mathbb{R}$ defined by

$$\chi^p_{\mathbb{B}}=\det_{i,j>0}^{p}(\mathbb{B}(u_i,v_j,u_0,v_0)). \tag{2.52}$$

The cross ratio $\mathbb{B}$ has rank n if

1. *$\chi^n_{\mathbb{B}}(u,v)\neq 0$ for all $(u,v)\in\mathcal{K}^n$,*
2. *$\chi^{n+1}_{\mathbb{B}}(u,v)=0$ for all $(u,v)\in\mathcal{K}^{n+1}$.*

Of course, the weak cross ratio associate with the double limit curve (ξ,ξ^*) is a rank n cross ratio.

Lemma 2.4.11. *Let*

$$(u,v)=((v_0,...,v_{n+1}),(u_0,...,u_{n+1})) \tag{2.53}$$

where $u_i,v_i\in\partial_\infty\pi_1(S)$, such that $u_0\neq v_0$ are different from all the $\{u_i,v_i\}_{i=1}^{n+1}$. Let $\mathbb{B}$ be a rank n cross ratio. Let (ξ,ξ^*) be the double limit curve associate to $\mathbb{B}$. Let $\tilde{\xi}$ ($\tilde{\xi}^*$ resp.) be the lifts of ξ (ξ^* resp.) with values in $\mathbb{R}^n$ ($\mathbb{R}^{n*}$ resp.), then $\chi^{n+1}_{\mathbb{B}}=\det^{n+1}_{i,j=1}(\mathbb{B}(u_i,v_j,u_0,v_0))$ is zero is equivalent to say $\det^{n+1}_{i,j=1}\left(\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_j)\right\rangle\right)$ is zero.

Proof. We multiple $\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_0)\right\rangle$ in rows and $\frac{1}{\left\langle\tilde{\xi}(u_0)\middle|\tilde{\xi}^*(v_j)\right\rangle}$ in columns in $\chi^{n+1}_{\mathbb{B}}$, we obtain

$$\begin{aligned}\chi^{n+1}_{\mathbb{B}}=&\frac{1}{\left\langle\tilde{\xi}(u_0)\middle|\tilde{\xi}^*(v_0)\right\rangle^{n+1}}\cdot\prod_{j=1}^{n+1}\left(\left\langle\tilde{\xi}(u_0)\middle|\tilde{\xi}^*(v_j)\right\rangle\right)\cdot\prod_{i=1}^{n+1}\frac{1}{\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_0)\right\rangle}\cdot\\&\det_{i,j=1}^{n+1}\left(\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_j)\right\rangle\right).\end{aligned} \tag{2.54}$$

Since $u_0\neq v_0$ are different from all the $\{u_i,v_i\}_{i=1}^{n+1}$, $\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_j)\right\rangle\neq 0$, $\left\langle\tilde{\xi}(u_0)\middle|\tilde{\xi}^*(v_j)\right\rangle\neq 0$, $\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_0)\right\rangle\neq 0$. By the formula above, we conclude that $\chi^{n+1}_{\mathbb{B}}=0$ is equivalent to say $\det^{n+1}_{i,j=1}\left(\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_j)\right\rangle\right)=0$. □

We continue to prove Proposition 2.4.8.

Since Q_n equals to zero in $\mathcal{B}_n(\mathcal{P})$, for any representative $Q(\alpha_1,...,\alpha_l)$, we suppose $Q(\alpha_1,...,\alpha_l)$ equals to P'/Q' in $\mathcal{B}(\mathcal{P})$ where P' and Q' belong to $\mathcal{Z}(\mathcal{P})$, we have P' belongs to $R_n(\mathcal{P})$. Since I identifies x on the left of the ordered pairs in $\mathcal{Z}(\mathcal{P})$ with $\tilde{\xi}(x)$ and x on the right with $\tilde{\xi}^*(x)$, then $I(P'/Q')$ equals to the sum of products with a factor of the type $\det^{n+1}_{i,j=1}\left(\left\langle\tilde{\xi}(u_i)\middle|\tilde{\xi}^*(v_j)\right\rangle\right)$. By Lemma 2.4.11, $I(Q(\alpha_1,...,\alpha_l))$ equals to zero.

We conclude that for $n>1$, i_n is well defined and $i_n|_{\mathcal{CR}_n(\mathcal{P})}=i'_n$. □

The ring homomorphism i_n is not injective, F. Labourie conjectures that

Conjecture 2.4.12. *For $n > 2$, $\ker(i_n)$ equals to the ideal generated by $\{\frac{xz}{xt}\frac{yt}{yz} - \gamma \circ \frac{xz}{xt}\frac{yt}{yz} \mid \forall \gamma \in \pi_1(S)\}$.*

2.4.3 Injectivity

The ring homomorphism i_n is not injective at least due to the fact that $\frac{xz}{xt} \cdot \frac{yt}{yz} - \gamma \circ \left(\frac{xz}{xt} \cdot \frac{yt}{yz}\right) = 0$, now we consider a discrete version of i_n where the corresponding $\pi_1(S)$ is identity.

Let us recall that $\mathcal{Z}_n(\mathcal{P}) = \mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$ is the rank n swapping ring where $\mathcal{P} = \{x_1, ..., x_p\}$. Recall that M_p is the configuration space of p vectors a_i in $\mathbb{K}^n$ and p covectors b_i in $\mathbb{K}^{n*}$. When we identify x_i on the left with a_i and x_i on the right with b_i, we induce a map j_n from $\mathcal{CR}_n(\mathcal{P})$ to $\mathcal{C}^\infty(M_p)$:

$$j_n\left(\frac{x_i x_k}{x_i x_l} \cdot \frac{x_j x_l}{x_j x_k}\right)(f) = \frac{\langle a_i | b_k\rangle}{\langle a_i | b_l\rangle} \cdot \frac{\langle a_j | b_l\rangle}{\langle a_j | b_k\rangle} \tag{2.55}$$

where $f = (a_1, ..., a_p, b_1, ..., b_p) \in M_p$.

Similar as Proposition 2.4.8, we have

Proposition 2.4.13. *For $n > 1$, the map j_n extends to a ring homomorphism $k_n : \mathcal{B}_n(\mathcal{P}) \to \mathcal{C}^\infty(M_p)$. where $k_n|_{\mathcal{CR}_n(\mathcal{P})} = j_n$.*

Moreover, the isomorphism between $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ and $\mathcal{Z}_n(\mathcal{P})$ induce the isomorphism between $k_n(\mathcal{B}_n(\mathcal{P}))$ and $\mathcal{B}_n(\mathcal{P})$.

Proposition 2.4.14. *For $n > 1$, there exists a ring homomorphism l_n from $k_n(\mathcal{B}_n(\mathcal{P}))$ to $\mathcal{B}_n(\mathcal{P})$ such that*

$$l_n \circ k_n = Id_{\mathcal{B}_n(\mathcal{P})}.$$

Corollary 2.4.15. *For $n > 1$, k_n is injective.*

3. FOCK-GONCHAROV ALGEBRA AND SWAPPING ALGEBRA

In this chapter, we relate the rank n swapping multifraction algebra to the rank n Fock–Goncharov algebra on the configuration space of flags $\mathrm{Conf}_{k,n}$ for $n = 2, 3$. This is one of the reason that we consider the quotient ring $\mathcal{Z}_n(\mathcal{P})$ and the rank in the last chapter. In §3.1, we recall the configuration space of flags which is the building block of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$, and its coordinates coined by Fock and Goncharov [FG1]. In §3.2, we embed in a Poisson way the rank 2 Fock–Goncharov algebra into the rank 2 swapping multifraction algebra. In §3.3, we embed in a Poisson way the rank 3 Fock–Goncharov algebra into the rank 3 swapping multifraction algebra. In §3.4, we relate θ_{T_2} to θ_{T_3} by a Poisson embedding τ_2 and we generalize to τ_3.

3.1 Configuration space of flags and its coordinates

3.1.1 Configuration space of flags

Definition 3.1.1. [Flag] *A* flag X *in* $\mathbb{R}^n$ *is a family* $(X_1, ..., X_{n-1})$ *so that* X_k *is a* k*-dimensional vector space in* $\mathbb{R}^n$*,* $X_k \subset X_{k+1}$*. A* base of a flag X *is a basis* $x_1, , , x_n$ *of* $\mathbb{R}^n$ *so that* $\hat{x}_k$ *is generated by* $\{x_1, ..., x_k\}$*.*
We denote the space of all the flags *in* $\mathbb{R}$ *by* $\mathcal{F}_n$*.*

Definition 3.1.2. [Configuration space of flags] *Let* (F, G, H) *be a triple of flags in* $\mathbb{R}^n$*. We call* (F, G, H) *are* in general position *if for all triples of the positive numbers* (m, l, p) *with* $m + l + p = n$*, the sum* $F_m + G_l + H_p = \mathbb{R}^n$ *is direct.*

A configuration of flags $(F_1, ..., F_k)(k \geq 3)$ *in* $\mathbb{R}^n$ *is in* general position *if any three of them is in general position.*

Let the configuration space of flags $\mathrm{Conf}_{k,n}$ *to be the ordered sets of flags* $(X_1, ..., X_k)$ *in general position up to projective transformations.*

3.1.2 Triangulation

Definition 3.1.3. [Ideal triangulation] *Let* S *be a connected oriented topological surface of finite type and genus* g *with boundaries or punctures and marked points. We assume that* S *has negative Euler number. We denote by* $\overline{S}$ *the surface obtained from* S *by filling in the punctures. Let* B *be the union of the punctures and the marked points on the boundaries.*

An arc *in* S *is the image in* $\overline{S}$ *of a closed interval, whose interior is homeomorphically embedded in* $S \setminus B$ *and whose endpoints are on* B*. An arc in* S *is said to be* essential *if it is not homotopic (relative to* B*) to a point in* S*.*

An ideal triangulation of S *is a maximal collection of disjoint essential arcs that are pairwise non-homotopic.*

Let $D_k(k \geq 3)$ be a disk with k marked points $B = \{x_1, ..., x_k\}$ on the boundary S^1. Let B be the marked points ordered cyclically on S^1. Let $\overline{x_1...x_k}$ be the convex k-gon with the edges $x_1x_2, x_2x_3..., x_kx_1$. Then the ideal triangulation T of (D_k, B) is same as a triangulation of $\overline{x_1...x_k}$. Later on, we draw a convex polygon to represent D_k.

Definition 3.1.4. [N-TRIANGULATION] *Given a triangulation T, we define the* n-triangulation *T_n of T to be: for each interior triangle(no edge of the triangle is on the boundary of T) of T, we subdivide this triangle into n^2 triangles as in Figure 3.1. Let*

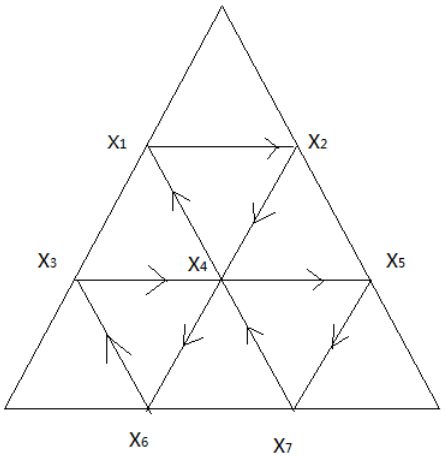


Fig. 3.1:

$$I_n = \{n-1 \text{ vertices of } T_n \text{ on each edge of } T \text{ which is not the edge of the } k-gon\}, \tag{3.1}$$

as A, B, C, E, F, G in Figure 3.1. Let

$$J_n = \left\{ \frac{(n-2)(n-1)}{2} \text{ vertices of } T_n \text{ in the interior of each triangle of } T \right\}, \tag{3.2}$$

as D in Figure 3.1.

If $\overline{rs}$ is an edge of a triangle $\overline{rst}$ of T, the inner vertex u of $\overline{rs}$ in I_n is described by a pair of positive numbers (x, y) where $x + y = n$ and

$$x : y = \frac{dist(u, \overline{st})}{dist(r, \overline{st})} : \frac{dist(u, \overline{rt})}{dist(s, \overline{rt})}.$$

As in Figure 3.1, A is described by $(2, 1)$ on the edge $\overline{x_5x_1}$.

If $\overline{abc}$ is a triangle of T, the inner vertex v of $\overline{abc}$ in J_n is described by a triple of positive numbers (m, l, p) where $m + l + p = n$ and

$$m : l : p = \frac{dist(v, \overline{bc})}{dist(a, \overline{bc})} : \frac{dist(v, \overline{ac})}{dist(b, \overline{ac})} : \frac{dist(v, \overline{ab})}{dist(c, \overline{ab})}.$$

As in Figure 3.1, D is described by $(1, 1, 1)$ in the triangle $\overline{x_1x_3x_5}$.

we will assign to each vertex of I_n an edge function and each vertex of J_n an triple ratio in the next section.

3.1.3 Fock–Goncharov coordinates

In [FG1],V. V. Fock and A. B. Goncharov introduce the higher Teichmuller space $(\mathcal{X}_{G,S}, \mathcal{A}_{G',S})$ where G is real semi-simple Lie group, G' is Langlands dual of G, S is a compact oriented surface with boundaries. We suppose S has negative Euler number. They relate $(\mathcal{X}_{G,S}, \mathcal{A}_{G',S})$ to the cluster orbi-ensemble(a generalization of cluster algebra) $(\mathcal{X}_{h(G,S)}, \mathcal{A}_{h(G',S)})$. These algebraic structure has geometric meaning, since certain finite cover of the Penner's decorated Teichmüller space can be identified with a open subset of $\mathcal{A}_{\mathrm{SL}(2,\mathbb{R}),S}$ [P1], certain finite cover of the Hitchin component $H_n(S)$ can be identified with a open set of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$. More precisely, they give a special coordinate system for the spaces $\mathcal{A}_{\mathrm{SL}(n,\mathbb{K}),S}$ and $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{K}),S}$ for any field $\mathbb{K}$ and the Hitchin component corresponds to the positive part of the space $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$.

Definition 3.1.5. [$\mathcal{X}$-HIGHER TEICHMULLER SPACE $\mathcal{X}_{G,S}$] *Let G be a reductive group. Let B be the Borel group of G. Let $\mathcal{F} = G/B$ be the flag variety of G. Let T be the finite subset of ∂S, $\hat{S} = S \backslash T$. An element in $\mathcal{X}_{G,S}$ is a pair (∇, f) , where ∇ is a flat connection on the G-principal bundle $P \to S$ and $f : \partial S \backslash T \to G/B$ is B-reduction flat on $\partial S \backslash T$, that is to say, for any $x \in \partial S \backslash T$, we choose a flag $f(x)$ in $P|_x$, such that ∇ translates $f(x_1)$ to $f(x_2)$ when x_1 and x_2 are in the same component of $x \in \partial S \backslash T$.*

Remark 3.1.6. *For the disc D with a finite subset $\mathcal{P} \subset \partial D$, $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),\hat{D}} = \mathrm{Conf}_{k,n}$, where $\mathrm{Conf}_{k,n}$ is the configuration space of flags, it is the building block of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$(see Section 7.8 of [FG1]).*

The edge functions and the triple ratios are functions on $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ defined by Fock and Goncharov in [FG1]. We give some notations on the flags before we define the edge functions and the triple ratios.

Notation 3.1.7. *Let $\{f^i\}_{i=1}^n$ be a base of a flag F in $\mathbb{R}^n$. For $k = 1, , , n$, let*

$$\hat{f^k} := f^1 \wedge ... \wedge f^k, \tag{3.3}$$

and by convention $\hat{f^0} = 1$.

Definition 3.1.8. [(m,l,p)-TRIPLE RATIO] *For $m, l, p > 0$, the (m,l,p)-*triple ratio *of the triple of flags (F, G, H) in $\mathbb{RP}^{n-1}$ is*

$$T^{m,l,p}(G,F,H) := \frac{\Omega\left(\hat{f}^{m+1} \wedge \hat{g}^{l} \wedge \hat{h}^{p-1}\right) \cdot \Omega\left(\hat{f}^{m-1} \wedge \hat{g}^{l+1} \wedge \hat{h}^{p}\right) \cdot \Omega\left(\hat{f}^{m} \wedge \hat{g}^{l-1} \wedge \hat{h}^{p+1}\right)}{\Omega\left(\hat{f}^{m+1} \wedge \hat{g}^{l-1} \wedge \hat{h}^{p}\right) \cdot \Omega\left(\hat{f}^{m} \wedge \hat{g}^{l+1} \wedge \hat{h}^{p-1}\right) \cdot \Omega\left(\hat{f}^{m-1} \wedge \hat{g}^{l} \wedge \hat{h}^{p+1}\right)}, \tag{3.4}$$

where Ω is the volume form on $\mathbb{R}^n$, $\{f^i\}_{i=1}^n$, $\{g^i\}_{i=1}^n$, $\{h^i\}_{i=1}^n$ are bases of F, G, H respectively.

This triple ratio does not depend on the bases that we choose for the flags since each of $\{\hat{f}^{m-1}, \hat{f}^{m}, \hat{f}^{m+1}, \hat{g}^{l-1}\hat{g}^{l}, \hat{g}^{l+1}, \hat{h}^{p-1}, \hat{h}^{p}, \hat{h}^{p+1}\}$ appears once in the numerator , once in the denominator of the right hand side of the definition.

Definition 3.1.9. [$(i,n-i)$-EDGE FUNCTION] *For $0<i<n$ and a quadruple $(Y,T,Z,X)\in\mathcal{F}_n^4$ in general position, The $(i,n-i)$-*edge function *of the edge $\overline{zx}$ is*

$$\mathbb{B}_i(Y,T,Z,X)=\frac{\Omega\left(\hat{x}^i\wedge\hat{z}^{n-i-1}\wedge t_1\right)\cdot\Omega\left(\hat{x}^{i-1}\wedge\hat{z}^{n-i}\wedge y_1\right)}{\Omega\left(\hat{x}^i\wedge\hat{z}^{n-i-1}\wedge y_1\right)\cdot\Omega\left(\hat{x}^{i-1}\wedge\hat{z}^{n-i}\wedge t_1\right)}. \tag{3.5}$$

By the same reason, this edge function does not depend on the bases that we choose for the flags.

Remark 3.1.10. *Attention, we define the edge function as the minus of the original definition of the edge function in [FG1].*

By definition, we have

Proposition 3.1.11.

$$\mathbb{B}_i(T,Y,Z,X)=\frac{1}{\mathbb{B}_i(Y,T,Z,X)} \tag{3.6}$$

$$\mathbb{B}_i(Y,T,X,Z)=\frac{1}{\mathbb{B}_{n-i}(Y,T,Z,X)} \tag{3.7}$$

Since

$$\Omega(g\cdot v_1,...,g\cdot v_n)=\det g\cdot\Omega(v_1,...,v_n),$$

we have

Proposition 3.1.12. *The triple ratios and the edge functions are invariant by projective translations.*

Fock–Goncharov coordinate for the space $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$ is constructed as follows. Given an ideal triangulation T and its n-triangulation T_n of the topological surface S of negative Euler class with boundaries. Let $\overline{rs}$ be an edge of T, let $\overline{rsf}$ and $\overline{rsg}$ be the triangles of T, with the flags (F,G,R,S)(corresponding to (f,g,r,s)) in general position, for each element t of I_n on $\overline{rs}$ associated to $(i,n-i)$, we assign to t an $(i,n-i)$-edge function; Let $\overline{abc}$ be an triangle of T, with the flags (A,B,C)(corresponding to (a,b,c)) in general position, for each element v of J_n in the interior of $\overline{abc}$ associated to (m,l,p), we assign to v a (m,l,p)-triple ratio.

Theorem 3.1.13. [V. V . FOCK, A. B. GONCHAROV [FG1]] *Given an ideal triangulation T and its n-triangulation T_n of the topological surface S of negative Euler class with boundaries, the triple ratios and the edge functions are algebraic independent. These functions give us a coordinate system of $\mathcal{X}_{\mathrm{PGL}(n,\mathbb{R}),S}$.*

Using Theorem 3.1.13 for the case $S=D_k$, we have

Corollary 3.1.14. *All the edge functions and triple ratios for a triangulation T of the k-gon $\overline{x_1...x_k}$ and its n-triangulation T_n are algebraic independent. These functions give us a coordinate system of* $\mathrm{Conf}_{k,n}$.

3.2 Fock–Goncharov and swapping algebras for $\mathrm{Conf}_{k,2}$

3.2.1 Rank n Fock–Goncharov algebra on $\mathrm{Conf}_{k,n}$

Definition 3.2.1. [Rank n Fock–Goncharov ring] *For a disk with k marked points D_k, let $\mathcal{P} = \{x_1, ..., x_k\}$ be the set of marked points, let T be an ideal triangulation of D_k, let T_n be its n-triangulation. Let X_v be the edge function corresponding to v in I_n, let X_u be the triple ratio corresponding to u in J_n. The* rank n Fock–Goncharov ring *is the ring generated by the edge functions and the triple ratios over $\mathbb{K}$($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).*

We denote the rank n Fock–Goncharov ring by $\mathcal{A}(T_n)$ or $\mathbb{K}[\{X_v\}_{v \in I_n \cup J_n}]$.

Definition 3.2.2. [Rank n Fock–Goncharov bracket *[FG1], see [FG2] for a simple version]*

The Rank n Fock–Goncharov bracket *is defined by extending the following formula for the generators to $\mathcal{A}(T_n)$.*

$$\{X_u,\ X_w\}_n = \epsilon_{uw} \cdot X_u \cdot X_w \tag{3.8}$$

Where ϵ_{uw} = (arrows from u to w) - (arrows form w to u).

For each triangle of T and its n-triangulation, we have the arrows described as in Figure 3.1.

Theorem 3.2.3. [V. V . Fock, A. B. Goncharov [FG1]] *The rank n Fock–Goncharov bracket verifies Jacobi identity, so the rank n Fock–Goncharov ring equipped with a rank n Fock–Goncharov bracket is a Poisson algebra.*

Definition 3.2.4. [Rank n Fock–Goncharov algebra] *The* rank n Fock–Goncharov algebra *of T_n is the ring $\mathcal{A}(T_n)$ equipped with the rank n Fock–Goncharov bracket.*

3.2.2 Relations between Fock–Goncharov and swapping algebras for $\mathrm{Conf}_{k,2}$

For a disk with k marked points on the boundary D_k, let $\mathcal{P} = \{x_1, ..., x_k\}$ be the set of marked points, we construct the rank 2 swapping multifraction algebra $\mathcal{B}_2(\mathcal{P})$ as in the chapter 2. Let T be a triangulation of $\overline{x_1...x_k}$ and T_2 its 2-triangulation, let $\mathcal{A}(T_2)$ be the rank 2 Fock–Goncharov algebra. We show that $\mathcal{A}(T_2)$ embed in a Poisson way into $\mathcal{B}_2(\mathcal{P})$ by a natural ring monomorphism which relates volume forms to ordered pairs of marked points.

Definition 3.2.5. *We define a ring homomorphism θ_{T_2} by extending the following map defined on the generators to $\mathcal{A}(T_2)$:*

$$\begin{gathered}\theta_{T_2} : \mathcal{A}(T_2) \to \mathcal{B}_2(\mathcal{P}) \\ X = \frac{\Omega\left(\hat{y}^1 \wedge \hat{z}^1\right)}{\Omega\left(\hat{t}^1 \wedge \hat{z}^1\right)} \cdot \frac{\Omega\left(\hat{t}^1 \wedge \hat{x}^1\right)}{\Omega\left(\hat{y}^1 \wedge \hat{x}^1\right)} \\ \theta_{T_2}(X) := \frac{yz}{tz} \cdot \frac{tx}{yx}\end{gathered} \tag{3.9}$$

where x, y, z, t are in S^1 as in Figure 3.2.

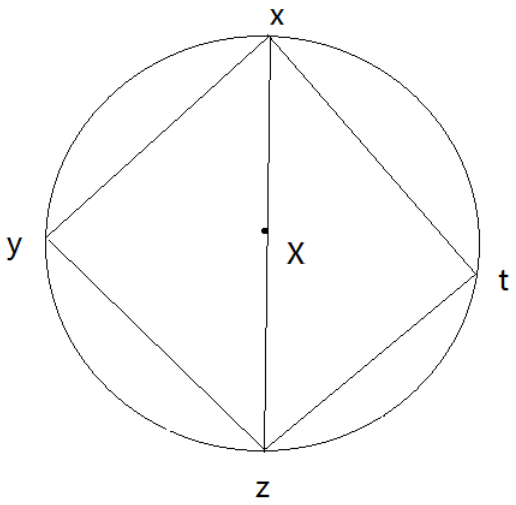


Fig. 3.2:

Remark 3.2.6. *By Theorem 3.1.14, there is no algebraic relation between these edge functions, hence θ_{T_2} is well defined.*

Proposition 3.2.7. *The ring homomorphism θ_{T_2} is injective.*

Proof. Let k_2 be the ring homomorphism from $\mathcal{B}_2(\mathcal{P})$ to $\mathcal{C}^\infty(\mathrm{Conf}_{k,n})$ defined in Proposition 2.4.13. By the definition, we have $k_2 \circ \theta_{T_2}$ is an identity map on the generators of $\mathcal{A}(T_2)$. Since k_2 and θ_{T_2} are ring homomorphism, $k_2 \circ \theta_{T_2}$ is an identity map on $\mathcal{A}(T_2)$. We conclude that θ_{T_2} is injective. □

Remark 3.2.8. *The ring homomorphism θ_{T_2} is not surjective, for instance, in Figure 3.2, $\frac{zy}{xy} \cdot \frac{xt}{zt} \notin \theta_{T_2}(\mathcal{A}(T_2))$. But any generator of $\mathcal{B}_2(\mathcal{P})$ belongs to $Im(\theta_{T_2'})$ for some ideal triangulation T_2'.*

3.2.3 Main theorem

We embed in a Poisson way the rank 2 Fock–Goncharov algebra into the rank 2 multifraction algebra for $\mathrm{Conf}_{k,2}$ by the following theorem.

Theorem 3.2.9. *The ring monomorphism θ_{T_2} is Poisson.*

Proof. We associate each edge function with a lozenge and an edge. By symmetry, we have the following possible cases for two graphs associated to edge functions as in Figure 3.3:

1. Trivial cases: Two graphs associate with two edge functions can be separated by a line in the disk.

2. Two graphs associated to two edge functions have one common point.

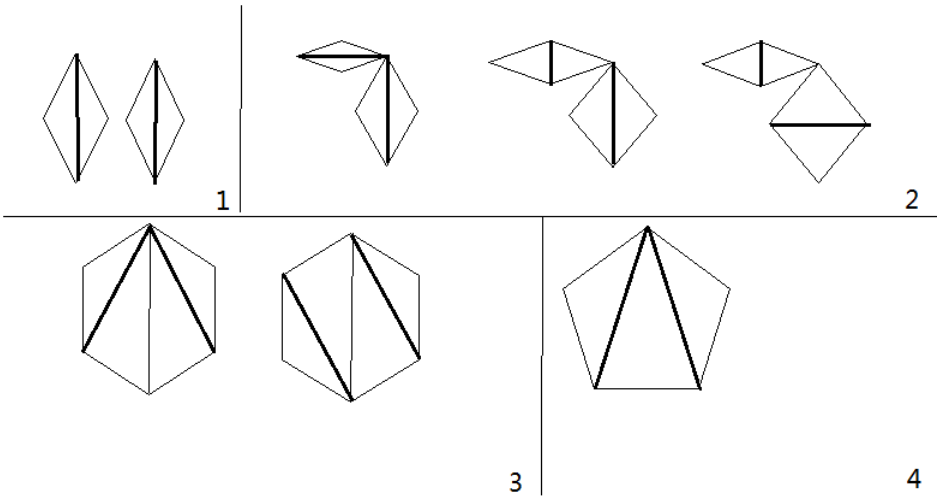


Fig. 3.3:

3. Two graphs associated to two edge functions have two common points.
4. Two graphs associated to two edge functions have three common points.

We prove the theorem for all the cases as follows.

1. **Trivial case**

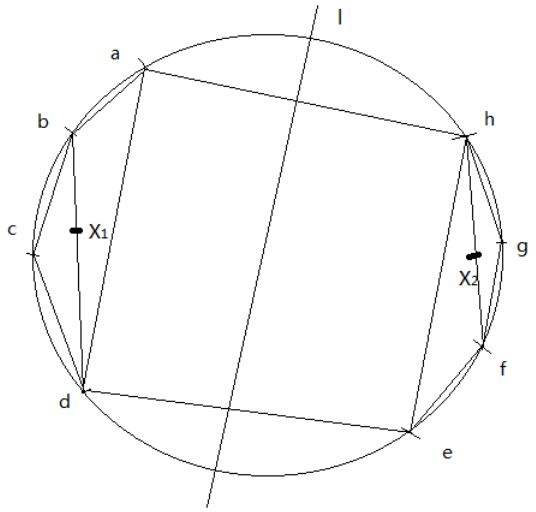


Fig. 3.4:

As in Figure 3.3.1, two graphs associate with two edge functions can be separated by a line in the disk.

Let $\alpha, \beta, \gamma, \theta, y, z, t, x$ be cyclically ordered $[\alpha, \beta, \gamma, \theta, y, z, t, x]$ as in Figure 3.4. Let

$$\theta_{T_2}(\epsilon) = \frac{\beta\gamma}{\theta\gamma} \cdot \frac{\theta\alpha}{\beta\alpha}$$

and

$$\theta_{T_2}(e) = \frac{yz}{tz} \cdot \frac{tx}{yx},$$

we have to verify that

Proposition 3.2.10.

$$\{\theta_{T_2}(\epsilon), \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

Proof. The linking number between the ordered pair of points of $\{\alpha, \beta, \gamma, \theta\}$ and the ordered pair of points of $\{x, y, z, t\}$ is always zero, thus we have

$$\left\{ \frac{\beta\gamma}{\theta\gamma} \cdot \frac{\theta\alpha}{\beta\alpha},\ \frac{yz}{tz} \cdot \frac{tx}{yx} \right\} = 0.$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2) = 0,$$

we conclude that

$$\{\theta_{T_2}(\epsilon), \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

□

2. **One common point**

As in Figure 3.3.2, the cases that two graphs associated to two edge functions have one common point. By symmetry, we have to check the following three cases:

(a) Let $x, \beta, \gamma, \theta, y, z, t$ be cyclically ordered $[x, \beta, \gamma, \theta, y, z, t]$ as in Figure 3.5. Let

$$\theta_{T_2}(\epsilon) = \frac{\beta\gamma}{\theta\gamma} \cdot \frac{\theta x}{\beta x}$$

and

$$\theta_{T_2}(e) = \frac{yz}{tz} \cdot \frac{tx}{yx},$$

we have to verify that

Proposition 3.2.11.

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

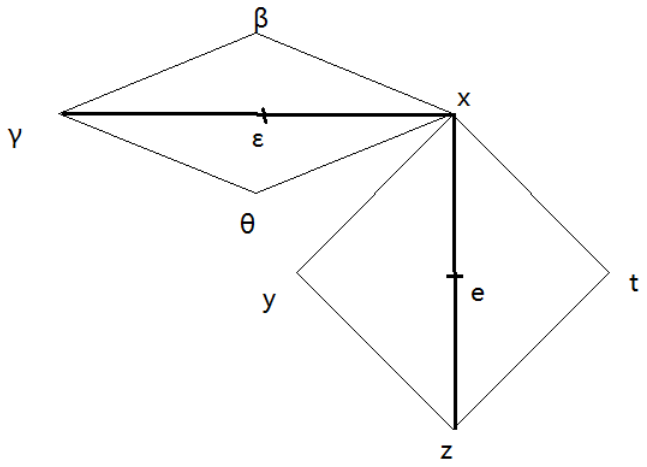


Fig. 3.5:

Proof. When the swapping bracket between the factors of $\theta_{T_2}(\epsilon)$ and $\theta_{T_2}(e)$ is not zero, two of the factors have common point x on left side(right side respectively). We have

$$\begin{aligned}
&\frac{\left\{\frac{\beta\gamma}{\theta\gamma}\cdot\frac{\theta x}{\beta x},\ \frac{yz}{tz}\cdot\frac{tx}{yx}\right\}}{\frac{\beta\gamma}{\theta\gamma}\cdot\frac{\theta x}{\beta x}\cdot\frac{yz}{tz}\cdot\frac{tx}{yx}}\\
&=\frac{\{\theta x,tx\}}{\theta x\cdot tx}-\frac{\{\theta x,yx\}}{\theta x\cdot yx}-\frac{\{\beta x,tx\}}{\beta x\cdot tx}+\frac{\{\beta x,yx\}}{\beta x\cdot yx}\\
&=\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{2}\\
&=0.
\end{aligned}\tag{3.10}$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2)=0,$$

we conclude that

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\}=\theta_{T_2}(\{\epsilon,\ e\}_2)=0.$$

□

(b) Let $x,\beta,\gamma,\theta,y,z,t$ be cyclically ordered $[x,\beta,\gamma,\theta,y,z,t]$ as in Figure 3.6. Let

$$\theta_{T_2}(\epsilon)=\frac{\gamma\theta}{x\theta}\cdot\frac{x\beta}{\gamma\beta}$$

and

$$\theta_{T_2}(e)=\frac{yz}{tz}\cdot\frac{tx}{yx},$$

we have to verify that

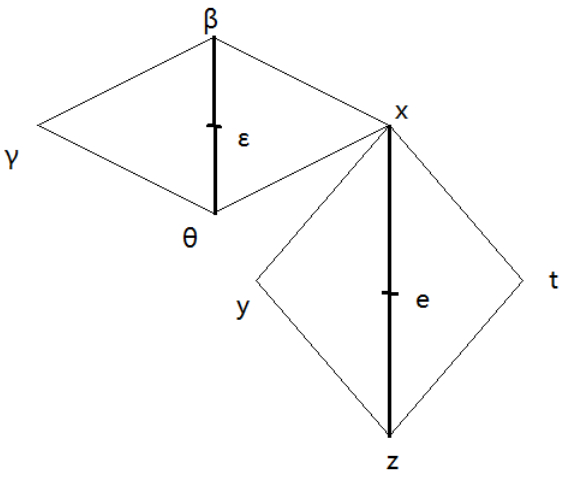


Fig. 3.6:

Proposition 3.2.12.

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

Proof. When the swapping bracket between the factors of $\theta_{T_2}(\epsilon)$ and $\theta_{T_2}(e)$ is not zero, two of the factors have common point x on left side(right side respectively). We have

$$\frac{\left\{\frac{\gamma\theta}{x\theta}\cdot\frac{x\beta}{\gamma\beta},\ \frac{yz}{tz}\cdot\frac{tx}{yx}\right\}}{\frac{\gamma\theta}{x\theta}\cdot\frac{x\beta}{\gamma\beta}\cdot\frac{yz}{tz}\cdot\frac{tx}{yx}} = 0. \tag{3.11}$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2) = 0,$$

we conclude that

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

□

(c) Let $x, \beta, \gamma, \theta, y, z, t$ be cyclically ordered $[x, \beta, \gamma, \theta, y, z, t]$ as in Figure 3.7. Let

$$\theta_{T_2}(\epsilon) = \frac{\gamma\theta}{x\theta}\cdot\frac{x\beta}{\gamma\beta}$$

and

$$\theta_{T_2}(e) = \frac{xy}{zy}\cdot\frac{zt}{xt},$$

we have to verify that

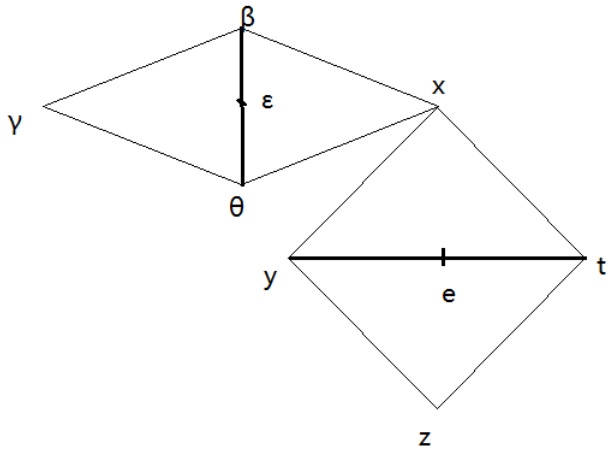


Fig. 3.7:

Proposition 3.2.13.

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

Proof. When the swapping bracket between the factors of $\theta_{T_2}(\epsilon)$ and $\theta_{T_2}(e)$ is not zero, two of the factors have common point x on left side(right side respectively). We have

$$\begin{aligned}
&\frac{\left\{\frac{\gamma\theta}{x\theta}\cdot\frac{x\beta}{\gamma\beta},\ \frac{xy}{zy}\cdot\frac{zt}{xt}\right\}}{\frac{\gamma\theta}{x\theta}\cdot\frac{x\beta}{\gamma\beta}\cdot\frac{xy}{zy}\cdot\frac{zt}{xt}} \\
&= \frac{\{x\beta, xy\}}{x\beta\cdot xy} - \frac{\{x\beta, xt\}}{x\beta\cdot xt} - \frac{\{x\theta, xy\}}{x\theta\cdot xy} + \frac{\{x\beta, xt\}}{x\beta\cdot xt} \qquad (3.12)\\
&= \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \\
&= 0.
\end{aligned}$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2) = 0,$$

we conclude that

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

□

3. **Two common points**

As in Figure 3.3.3, the cases that two graphs associated to two edge functions have two common points. By symmetry, we have to check the following two cases:

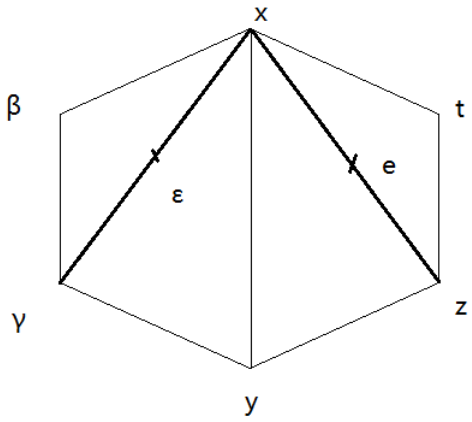


Fig. 3.8:

(a) Let $x, \beta, \gamma, y, z, t$ be cyclically ordered $[x, \beta, \gamma, y, z, t]$ as in Figure 3.8. Let

$$\theta_{T_2}(\epsilon) = \frac{\beta\gamma}{y\gamma} \cdot \frac{yx}{\beta x}$$

and

$$\theta_{T_2}(e) = \frac{yz}{tz} \cdot \frac{tx}{yx},$$

we have to verify that

Proposition 3.2.14.

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

Proof. When the swapping bracket between the factors of $\theta_{T_2}(\epsilon)$ and $\theta_{T_2}(e)$ is not zero, two of the factors have common point x or y on left side(right side respectively). We have

$$\begin{aligned}
&\frac{\left\{\frac{\beta\gamma}{y\gamma} \cdot \frac{yx}{\beta x},\ \frac{yz}{tz} \cdot \frac{tx}{yx}\right\}}{\frac{\beta\gamma}{y\gamma} \cdot \frac{yx}{\beta x} \cdot \frac{yz}{tz} \cdot \frac{tx}{yx}} \\
&= -\frac{\{y\gamma, yz\}}{y\gamma \cdot yz} + \frac{\{y\gamma, yx\}}{y\gamma \cdot yx} + \frac{\{yx, yz\}}{yx \cdot yz} + \frac{\{yx, tx\}}{yx \cdot tx} - \frac{\{\beta x, tx\}}{\beta x \cdot tx} + \frac{\{\beta x, yx\}}{\beta x \cdot yx} \\
&= -\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \\
&= 0.
\end{aligned} \tag{3.13}$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2) = 0,$$

we conclude that

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

□

(b) Let $x, \beta, \gamma, y, z, t$ be cyclically ordered $[x, \beta, \gamma, y, z, t]$ as in Figure 3.9. Let

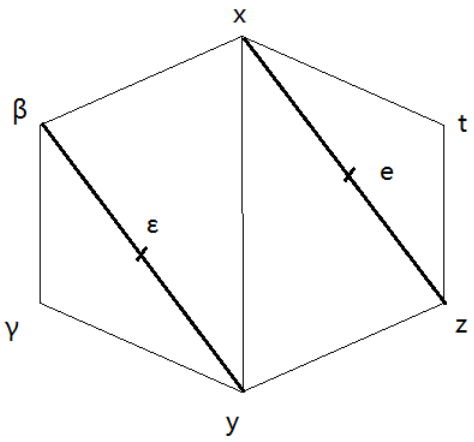


Fig. 3.9:

$$\theta_{T_2}(\epsilon) = \frac{\gamma y}{xy} \cdot \frac{x\beta}{\gamma\beta}$$

and

$$\theta_{T_2}(e) = \frac{yz}{tz} \cdot \frac{tx}{yx},$$

we have to verify that

Proposition 3.2.15.

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

Proof. When the swapping bracket between the factors of $\theta_{T_2}(\epsilon)$ and $\theta_{T_2}(e)$ is not zero, two of the factors have common point x or y on left side(right side respectively). We have

$$\begin{aligned}
&\frac{\left\{\frac{\gamma y}{xy} \cdot \frac{x\beta}{\gamma\beta},\ \frac{yz}{tz} \cdot \frac{tx}{yx}\right\}}{\frac{\gamma y}{xy} \cdot \frac{x\beta}{\gamma\beta} \cdot \frac{yz}{tz} \cdot \frac{tx}{yx}} \\
&= -\frac{\{y\gamma, yz\}}{y\gamma \cdot yz} + \frac{\{y\gamma, yx\}}{y\gamma \cdot yx} + \frac{\{yx, yz\}}{yx \cdot yz} + \frac{\{yx, tx\}}{yx \cdot tx} - \frac{\{\beta x, tx\}}{\beta x \cdot tx} + \frac{\{\beta x, yx\}}{\beta x \cdot yx} \\
&= 0.
\end{aligned} \tag{3.14}$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2) = 0,$$

we conclude that

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2) = 0.$$

□

4. **Three common points**

As in Figure 3.3.4, the cases that two graphs associated to two edge functions have three common points. By symmetry, we have to check the following case.

Let x, β, y, z, t be cyclically ordered $[x, \beta, y, z, t]$ as in Figure 3.10. Let

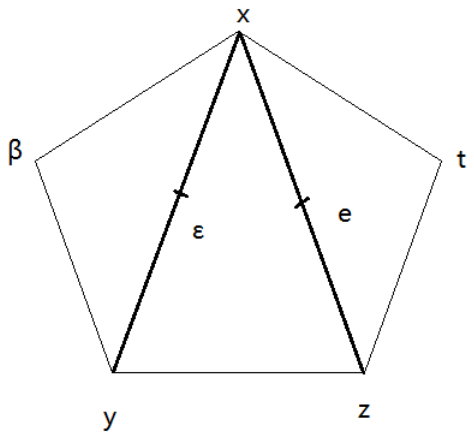


Fig. 3.10:

$$\theta_{T_2}(\epsilon) = \frac{\beta y}{zy} \cdot \frac{zx}{\beta x}$$

and

$$\theta_{T_2}(e) = \frac{yz}{tz} \cdot \frac{tx}{yx},$$

we have to verify that

Proposition 3.2.16.

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\} = \theta_{T_2}(\{\epsilon,\ e\}_2).$$

Proof. When the swapping bracket between the factors of $\theta_{T_2}(\epsilon)$ and $\theta_{T_2}(e)$ is not zero, two of the factors have common point x or y or z on left side(right side respec-

tively). We have

$$\begin{aligned}
&\frac{\left\{\frac{\beta y}{zy}\cdot\frac{zx}{\beta x},\ \frac{yz}{tz}\cdot\frac{tx}{yx}\right\}}{\frac{\beta y}{zy}\cdot\frac{zx}{\beta x}\cdot\frac{yz}{tz}\cdot\frac{tx}{yx}}\\
&=\frac{\{zx,tx\}}{zx\cdot tx}-\frac{\{zx,yx\}}{zx\cdot yx}-\frac{\{\beta x,tx\}}{\beta x\cdot tx}+\frac{\{\beta x,yx\}}{\beta x\cdot yx}\\
&=\frac{1}{2}-\left(-\frac{1}{2}\right)-\frac{1}{2}+\frac{1}{2}\\
&=1.
\end{aligned}\tag{3.15}$$

Since

$$\theta_{T_2}(\{\epsilon,\ e\}_2)=\theta_{T_2}(\epsilon\cdot e),$$

we conclude that

$$\{\theta_{T_2}(\epsilon),\ \theta_{T_2}(e)\}=\theta_{T_2}(\{\epsilon,\ e\}_2).$$

□

Finally, we conclude that θ is a Poisson homomorphism.

□

3.3 Fock–Goncharov and swapping algebras for $\mathrm{Conf}_{k,3}$

For a disk with k marked points D_k, let $\mathcal{P}=\{x_1,...,x_k\}$ be the set of marked points, we have the rank 3 swapping multifraction algebra $\mathcal{B}_3(\mathcal{P})$. Given D_k an ideal triangulation T and its 3-triangulation T_3, we have the rank 3 Fock–Goncharov ring $\mathcal{A}(T_3)$. In fact, $\mathcal{A}(T_3)$ is related to $\mathcal{B}_3(\mathcal{P})$ by a natural Poisson ring monomorphism which relates volume forms to ordered pairs of points.

3.3.1 Fock–Goncharov algebra for $\mathrm{Conf}_{k,3}$

Let us recall the edge functions and the triple ratios of $\mathcal{F}_3$. By Definitions 3.1.8 3.1.9, we have

Definition 3.3.1. [The edge function and the triple ratio of $\mathcal{F}_3$] *Let Ω be the volume form on $\mathbb{R}^3$. Let the* triple ratio *of the three flags F,G,H in $\mathbb{R}^3$ be*

$$\mathbb{T}(G,F,H)=\frac{\Omega\left(\hat{g}^1\wedge\hat{f}^2\right)\Omega\left(\hat{f}^1\wedge\hat{h}^2\right)\Omega\left(\hat{h}^1\wedge\hat{g}^2\right)}{\Omega\left(\hat{f}^1\wedge\hat{g}^2\right)\Omega\left(\hat{h}^1\wedge\hat{f}^2\right)\Omega\left(\hat{g}^1\wedge\hat{h}^2\right)}.\tag{3.16}$$

Let the edge functions *of the flags Y,T,Z,X in $\mathbb{R}_3$ be*

$$\mathbb{B}_1(Y,T,Z,X)=\frac{\Omega\left(\hat{y}^1\wedge\hat{z}^2\right)}{\Omega\left(\hat{t}^1\wedge\hat{z}^2\right)}\cdot\frac{\Omega\left(\hat{t}^1\wedge\hat{x}^1\wedge\hat{z}^1\right)}{\Omega\left(\hat{y}^1\wedge\hat{x}^1\wedge\hat{z}^1\right)},\tag{3.17}$$

$$\mathbb{B}_2(Y,T,Z,X)=\frac{1}{\mathbb{B}_1(Y,T,X,Z)}=\frac{\Omega\left(\hat{t}^1\wedge\hat{x}^2\right)}{\Omega\left(\hat{y}^1\wedge\hat{x}^2\right)}\cdot\frac{\Omega\left(\hat{y}^1\wedge\hat{x}^1\wedge\hat{z}^1\right)}{\Omega\left(\hat{t}^1\wedge\hat{x}^1\wedge\hat{z}^1\right)}\tag{3.18}$$

3.3.2 Relations between Fock–Goncharov and swapping for $\mathrm{Conf}_{k,3}$

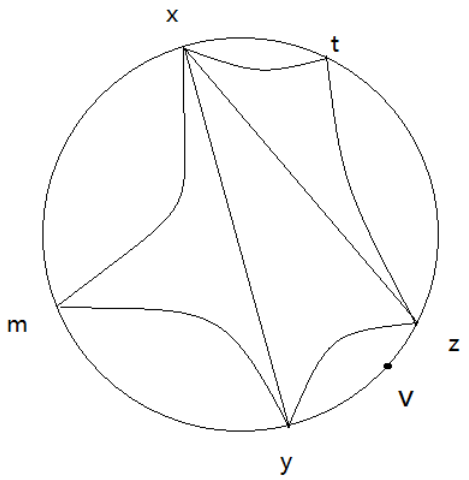


Fig. 3.11:

Lemma 3.3.2. *Let $x, z, y, u, s, v \in \mathcal{P}$, $x, z, y(u, s, v)$ are different from each other, we have*

$$\Delta\left((x, z, y), (u, s, v)\right) \neq 0. \tag{3.19}$$

in $\mathcal{Z}_3(\mathcal{P})$.

Proof. Let us recall the geometric model for $\mathcal{Z}_3(\mathcal{P})$. Each point x corresponds to vector x_1 in $\mathbb{K}^3$ and covector x_2 in $\mathbb{K}^{3*}$. By Theorem 2.3.6, we have $B_{3\mathbb{K}}/S_{3\mathbb{K}} \cong \mathcal{Z}_3(\mathcal{P})$. This isomorphism is realized by identifying x on the left hand side with vector x_1 and x on the right hand side with covector x_2. Thus $\Delta\left((x, z, y), (u, s, v)\right) \neq 0 \in \mathcal{Z}_3(\mathcal{P})$ corresponds to say that

$$\det \begin{pmatrix} \langle x_1|u_2\rangle & \langle x_1|s_2\rangle & \langle x_1|v_2\rangle \\ \langle z_1|u_2\rangle & \langle z_1|s_2\rangle & \langle z_1|v_2\rangle \\ \langle y_1|u_2\rangle & \langle y_1|s_2\rangle & \langle y_1|v_2\rangle \end{pmatrix} \in B_{3\mathbb{K}}/S_{3\mathbb{K}}.$$

is not always zero. Actually,

$$\det \begin{pmatrix} \langle x_1|u_2\rangle & \langle x_1|s_2\rangle & \langle x_1|v_2\rangle \\ \langle z_1|u_2\rangle & \langle z_1|s_2\rangle & \langle z_1|v_2\rangle \\ \langle y_1|u_2\rangle & \langle y_1|s_2\rangle & \langle y_1|v_2\rangle \end{pmatrix}$$

is interpreted as volume form of x_1, z_1, y_1 with the dual base of u_2, s_2, v_2. Let x_1, z_1, y_1 and u_2, s_2, v_2 are both in general position in $\mathbb{R}^3$. We obtain that

$$\det \begin{pmatrix} \langle x_1|u_2\rangle & \langle x_1|s_2\rangle & \langle x_1|v_2\rangle \\ \langle z_1|u_2\rangle & \langle z_1|s_2\rangle & \langle z_1|v_2\rangle \\ \langle y_1|u_2\rangle & \langle y_1|s_2\rangle & \langle y_1|v_2\rangle \end{pmatrix} \in B_{3\mathbb{K}}/S_{3\mathbb{K}}.$$

is not always zero. We conclude that

$$\Delta\left((x,z,y),(u,s,v)\right)\neq 0.$$

in $\mathcal{Z}_3(\mathcal{P})$. □

Lemma 3.3.3. *Let $x,y,z,t,u,s,v,w\in\mathcal{P}$, $x,z,y(u,s,v,w)$ are different from each other, we have*

$$\frac{\Delta\left((x,z,t),(u,s,v)\right)}{\Delta\left((x,z,y),(u,s,v)\right)}=\frac{\Delta\left((x,z,t),(u,s,w)\right)}{\Delta\left((x,z,y),(u,s,w)\right)}. \tag{3.20}$$

Proof. Consider the matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}=\begin{pmatrix} xu & xs & xv & xw\\ zu & zs & zv & zw\\ yu & ys & yv & yw\\ tu & ts & tv & tw\end{pmatrix}. \tag{3.21}$$

The adjugate of A is

$$A^{\star}=\begin{pmatrix} A_{11} & -A_{21} & A_{31} & -A_{41}\\ -A_{12} & A_{22} & -A_{32} & A_{42}\\ A_{13} & -A_{23} & A_{33} & -A_{43}\\ -A_{14} & A_{24} & -A_{34} & A_{44}\end{pmatrix}. \tag{3.22}$$

Where A_{ij} equals the determinant of the matrix without the ith row and the jth colomn of A.
We have already

$$\det A=0,$$

then

$$A^{\star}\cdot A=0_{4\times 4}. \tag{3.23}$$

A, $A^{\star}$ and $A^{\star}\cdot A$ correspond to linear endomorphisms of $\mathbb{K}^4$ f,g and $g\circ f$ respectively. By Equation 3.23, the rank of $g\circ f$(the dimension of the image of $g\circ f$) is 0. By the above lemma, we have

$$\Delta\left((x,z,y),(u,s,v)\right)\neq 0. \tag{3.24}$$

Thus, the rank of f is at least 3. Therefore, we have the rank of g is at most 1 (If not so, we will get the rank of $g\circ f$ is not 0). We conclude that

$$A_{34}\cdot A_{43}-A_{33}\cdot A_{44}=0. \tag{3.25}$$

This is same to say

$$\frac{\Delta\left((x,z,t),(u,s,v)\right)}{\Delta\left((x,z,y),(u,s,v)\right)}=\frac{\Delta\left((x,z,t),(u,s,w)\right)}{\Delta\left((x,z,y),(u,s,w)\right)}.$$

□

Remark 3.3.4. *This is the essential step that we use the rank n condition.*

Moreover, by applying Lemma 3.3.3 three times, we have

Corollary 3.3.5. *Let $x,z,t,y,u,w,v,u_1,w_1,v_1 \in \mathcal{P}$, u,w,v (u_1,w_1,v_1, or x,z,y resp.) are different from each other. We have*

$$\frac{\Delta\left((x,z,t),(u,w,v)\right)}{\Delta\left((x,z,y),(u,w,v)\right)} = \frac{\Delta\left((x,z,t),(u_1,w_1,v_1)\right)}{\Delta\left((x,z,y),(u_1,w_1,v_1)\right)} \tag{3.26}$$

in $\mathcal{B}_3(\mathcal{P})$.

Notation 3.3.6. *Let*

$$T(x,z,y) := \frac{xz\cdot zy\cdot yx}{zx\cdot yz\cdot xy}.$$

By Corollary 3.3.5, let

$$E(x,z,t,y) := \frac{\Delta\left((x,z,t),(u,w,v)\right)}{\Delta\left((x,z,y),(u,w,v)\right)}.$$

By Corollary 3.3.5 and Theorem 2.2.4,

Corollary 3.3.7. *Let $a,b,x,z,t,y,u,w,v \in \mathcal{P}$, u,w,v are different from each other. We have the value of*

$$\{ab, E(x,z,t,y)\}$$

in $\mathcal{Q}_3(\mathcal{P})$ does not depend on the choice of three different points $u,w,v \in \mathcal{P}$.

Definition 3.3.8. *We define a ring homomorphism θ_{T_3} by extending the following map defined on the generators to $\mathcal{A}(T_3)$:*

$$\theta_{T_3} : \mathcal{A}(T_3) \to \mathcal{B}_3(\mathcal{P})$$

For a triangle $\overline{xzy}$ of T, the corresponding flags are X,Z,Y, let

$$\theta_{T_3}(T(X,Z,Y)) = T(x,z,y). \tag{3.27}$$

For two triangles $\overline{zyx}$ and $\overline{zxt}$ of T, the corresponding flags of x,y,z,t are X,Y,Z,T respectively. Let

$$\theta_{T_3}(\mathbb{B}_1(Y,T,Z,X)) = \frac{yz}{tz}\cdot E(x,z,t,y), \tag{3.28}$$

$$\theta_{T_3}(\mathbb{B}_2(Y,T,Z,X)) = \frac{tx}{yx}\cdot E(x,z,y,t). \tag{3.29}$$

Remark 3.3.9. *Sometimes, to simplify our calculations of certain swapping bracket, we consider $E(x,z,y,t) \in \mathcal{Q}_3(\mathcal{P}')$ by adding some points to $\mathcal{P}$.*

Let $k_3 : \mathcal{B}_3(\mathcal{P}) \to \mathcal{C}^\infty(\mathrm{Conf}_{k,3})$ be the ring homomorphism defined in Proposition 2.4.13.

Proposition 3.3.10.

$$k_3 \circ \theta_{T_3} = Id_{\mathcal{A}(T_3)}.$$

Proof.

Notation 3.3.11. *For each flag F, we choose a base $\{f_i\}_{i=1}^n$ of a flag F in $\mathbb{R}^n$. For $k = 1, , , n$, let*

$$\hat{f^k} := f^1 \wedge ... \wedge f^k. \tag{3.30}$$

We denote $\Omega\left(\hat{x}^1 \wedge \hat{y}^2\right)$ by $\widehat{XY}$ for any X, Y in the flag variety $\mathcal{F}_3$.

For a triple ratio $T(X, Z, Y)$, we have

$$\begin{aligned} & k_3 \circ \theta_{T_3}(T(X, Z, Y)) \\ &= k_3\left(\frac{xz \cdot zy \cdot yx}{zx \cdot yz \cdot xy}\right) \\ &= \frac{\widehat{XZ} \cdot \widehat{ZY} \cdot \widehat{YX}}{\widehat{ZX} \cdot \widehat{YZ} \cdot \widehat{XY}} \\ &= T(X, Z, Y). \end{aligned} \tag{3.31}$$

Let

$$E(x, z, t, y) = \frac{\Delta\left((x, z, t), (z, x, v)\right)}{\Delta\left((x, z, y), (z, x, v)\right)}.$$

For an edge function $\mathbb{B}_1(Y, T, Z, X)$, we have

$$\begin{aligned} & k_3 \circ \theta_{T_3}(\mathbb{B}_1(Y, T, Z, X)) \\ &= k_3\left(\frac{yz}{tz} \cdot \frac{\Delta\left((x, z, t), (z, x, v)\right)}{\Delta\left((x, z, y), (z, x, v)\right)}\right) \\ &= \frac{\widehat{YZ}}{\widehat{TZ}} \cdot \left(\frac{\widehat{XZ}.\widehat{ZX}.\widehat{TV} - \widehat{XZ}.\widehat{ZV}.\widehat{TX} - \widehat{XV}.\widehat{ZX}.\widehat{TZ}}{\widehat{XZ}.\widehat{ZX}.\widehat{YV} - \widehat{XZ}.\widehat{ZV}.\widehat{YX} - \widehat{XV}.\widehat{ZX}.\widehat{YZ}}\right). \end{aligned} \tag{3.32}$$

Let $U = (U_1, U_2)$ be a flag in $\mathbb{R}^3$ where $U_2 = X_1 \wedge Z_1$. In $\mathbb{R}^3$, we have

$$\widehat{t_1} = s_0 \cdot \widehat{x}_1 + s_1 \cdot \widehat{z}_1 + s_2 \cdot \widehat{y}_1 \tag{3.33}$$

where $s_0, s_1, s_2 \in \mathbb{R}$. Thus we obtain that

$$\det\begin{pmatrix} \widehat{XU} & \widehat{XX} & \widehat{XV} & \widehat{XZ} \\ \widehat{ZU} & \widehat{ZX} & \widehat{ZV} & \widehat{ZZ} \\ \widehat{YU} & \widehat{YX} & \widehat{YV} & \widehat{YZ} \\ \widehat{TU} & \widehat{TX} & \widehat{TV} & \widehat{TZ} \end{pmatrix} = 0. \tag{3.34}$$

And we have $\widehat{XU} = \widehat{ZU} = 0$. Then, we decompose Determinant 3.34 in the first column and we obtain that

$$\frac{\widehat{TU}}{\widehat{YU}} = \frac{\det\begin{pmatrix} \widehat{XX} & \widehat{XV} & \widehat{XZ} \\ \widehat{ZX} & \widehat{ZV} & \widehat{ZZ} \\ \widehat{TX} & \widehat{TV} & \widehat{TZ} \end{pmatrix}}{\det\begin{pmatrix} \widehat{XX} & \widehat{XV} & \widehat{XZ} \\ \widehat{ZX} & \widehat{ZV} & \widehat{ZZ} \\ \widehat{YX} & \widehat{YV} & \widehat{YZ} \end{pmatrix}} = \frac{\widehat{XZ}.\widehat{ZX}.\widehat{TV} - \widehat{XZ}.\widehat{ZV}.\widehat{TX} - \widehat{XV}.\widehat{ZX}.\widehat{TZ}}{\widehat{XZ}.\widehat{ZX}.\widehat{YV} - \widehat{XZ}.\widehat{ZV}.\widehat{YX} - \widehat{XV}.\widehat{ZX}.\widehat{YZ}}. \tag{3.35}$$

So

$$k_3 \circ \theta_{T_3}(\mathbb{B}_1(Y,T,Z,X)) = \frac{\widehat{YZ}}{\widehat{TZ}} \cdot \frac{\widehat{TU}}{\widehat{YU}} = \mathbb{B}_1(Y,T,Z,X). \tag{3.36}$$

Similarly, we have

$$k_3 \circ \theta_{T_3}(\mathbb{B}_2(Y,T,Z,X)) = \mathbb{B}_2(Y,T,Z,X).$$

So $k_3 \circ \theta_{T_3}$ is an identity map on all the generators. Since $k_3 \circ \theta_{T_3}$ is a ring homomorphism, we have $k_3 \circ \theta_{T_3} = Id_{\mathcal{A}(T_3)}$. □

In particular, we have

Corollary 3.3.12. *θ_{T_3} is injective.*

3.3.3 Main theorem

We embed in a Poisson way the rank 3 Fock–Goncharov algebra into the rank 3 multifraction algebra for $\mathrm{Conf}_{k,3}$ by Theorem 3.3.16.

Let us prove some lemmas before we start to prove the main theorem.

Lemma 3.3.13. *Let $\alpha_1, \alpha_2, x_1, x_2, x_3 \in \mathcal{P}$. Assume that $|\mathcal{J}(\alpha_1, \alpha_2, x_i, x_j)| \neq 1$ for any $i, j = 1, 2, 3$, and α_1, α_2 are in the arc $\overset{\frown}{x_1x_2}$. We calculate $\frac{\{\alpha_1\alpha_2, x_1x_2 \cdot x_2x_3 \cdot x_3x_1\}}{\alpha_1\alpha_2 \cdot x_1x_2 \cdot x_2x_3 \cdot x_3x_1}$ in $\mathcal{Q}_n(\mathcal{P})$. Up to symmetry, we have seven cases as in Figure 3.12. We have*

$$\frac{\{\alpha_1\alpha_2, x_1x_2 \cdot x_2x_3 \cdot x_3x_1\}}{\alpha_1\alpha_2 \cdot x_1x_2 \cdot x_2x_3 \cdot x_3x_1} = \begin{cases} \frac{1}{2} & \textit{case 1 and case 2} \\ -\frac{1}{2} & \textit{case 3 and case 4} \\ 0 & \textit{others} \end{cases} \tag{3.37}$$

in $\mathcal{Q}_n(\mathcal{P})$.

Proof. By Leibniz's rule, we have

$$\frac{\{\alpha_1\alpha_2, x_1x_2 \cdot x_2x_3 \cdot x_3x_1\}}{\alpha_1\alpha_2 \cdot x_1x_2 \cdot x_2x_3 \cdot x_3x_1} = \frac{\{\alpha_1\alpha_2, x_1x_2\}}{\alpha_1\alpha_2 \cdot x_1x_2} + \frac{\{\alpha_1\alpha_2, x_2x_3\}}{\alpha_1\alpha_2 \cdot x_2x_3} + \frac{\{\alpha_1\alpha_2, x_3x_1\}}{\alpha_1\alpha_2 \cdot x_3x_1}. \tag{3.38}$$

In case 1, we have

$$\frac{\{\alpha_1\alpha_2, x_1x_2\}}{\alpha_1\alpha_2 \cdot x_1x_2} = \frac{1}{2},$$
$$\frac{\{\alpha_1\alpha_2, x_2x_3\}}{\alpha_1\alpha_2 \cdot x_2x_3} = 0,$$
$$\frac{\{\alpha_1\alpha_2, x_3x_1\}}{\alpha_1\alpha_2 \cdot x_3x_1} = 0.$$

We obtain that

$$\frac{\{\alpha_1\alpha_2, x_1x_2 \cdot x_2x_3 \cdot x_3x_1\}}{\alpha_1\alpha_2 \cdot x_1x_2 \cdot x_2x_3 \cdot x_3x_1} = \frac{1}{2}.$$

We obtain the results for the other cases by the same way.

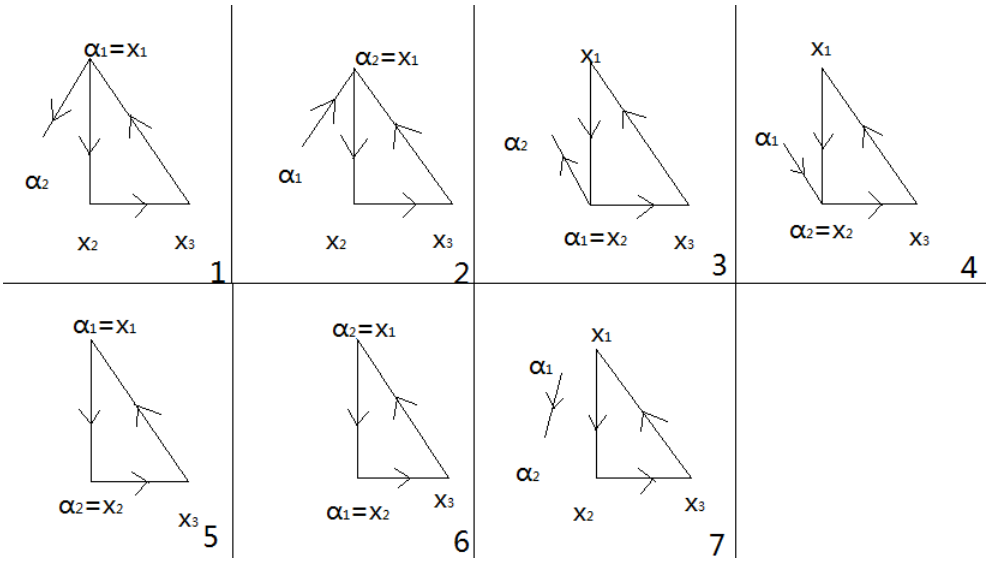


Fig. 3.12:

Finally, we conclude that

$$\frac{\{\alpha_1\alpha_2, x_1x_2\cdot x_2x_3\cdot x_3x_1\}}{\alpha_1\alpha_2\cdot x_1x_2\cdot x_2x_3\cdot x_3x_1}=\begin{cases}\frac{1}{2} & case\ 1\ and\ case\ 2\\ -\frac{1}{2} & case\ 3\ and\ case\ 4\\ 0 & others\end{cases} \tag{3.39}$$

in $\mathcal{Q}_n(\mathcal{P})$. □

As a corollary, we have

Corollary 3.3.14. *Let $\alpha_1,\alpha_2,\alpha_3,u_1,...,u_s,v_1,...,v_s\in\mathcal{P}$. Let P be a polynomial in variables $u_1v_1,...,u_sv_s$. Recall that $T(\alpha_1,\alpha_2,\alpha_3)=\frac{\alpha_1\alpha_2\cdot\alpha_2\alpha_3\cdot\alpha_3\alpha_1}{\alpha_2\alpha_1\cdot\alpha_3\alpha_2\cdot\alpha_1\alpha_3}$. Assume that $|\mathcal{J}(\alpha_i,\alpha_j,u_l,v_l)|\neq 1$ for any $i,j=1,2,3$, $l=1,...,s$, we have*

$$\{T(\alpha_1,\alpha_2,\alpha_3),P(u_1v_1,...,u_sv_s)\}=0$$

in $\mathcal{Q}_n(\mathcal{P})$.

Proof. In fact, the corollary follows from an observation by Lemma 3.3.13 that

$$\frac{\{u_lv_l,\alpha_1\alpha_2\cdot\alpha_2\alpha_3\cdot\alpha_3\alpha_1\}}{u_lv_l\cdot\alpha_1\alpha_2\cdot\alpha_2\alpha_3\cdot\alpha_3\alpha_1}=\frac{\{u_lv_l,\alpha_2\alpha_1\cdot\alpha_3\alpha_2\cdot\alpha_1\alpha_3\}}{u_lv_l\cdot\alpha_2\alpha_1\cdot\alpha_3\alpha_2\cdot\alpha_1\alpha_3}. \tag{3.40}$$

□

Lemma 3.3.15. *Let $\alpha_1,\alpha_2,x_1,x_2,x_3,x_4,u,v,w\in\mathcal{P}$. Assume that $|\mathcal{J}(\alpha_1,\alpha_2,x_i,x_j)|\neq 1$ for any $i,j=1,2,3,4$. Let $\delta_{a;b,c}=1$ if $a=b$ or $a=c$, $\delta_{a,b}=0$ otherwise. Recall that $E(x_1,x_3,x_2,x_4)=\frac{\Delta((x_1,x_3,x_2),(u,v,w))}{\Delta((x_1,x_3,x_4),(u,v,w))}$. Then we have*

$$\frac{\{\alpha_1\alpha_2,E(x_1,x_3,x_2,x_4)\}}{\alpha_1\alpha_2\cdot E(x_1,x_3,x_2,x_4)}=\mathcal{J}(\alpha_1,\alpha_2,x_2,x_4)\cdot\delta_{\alpha_1;x_2,x_4}$$

in $\mathcal{Q}_n(\mathcal{P})$.

Proof. We prove the lemma by considering all possible positions of α_1, α_2 in S^1:

1. α_1, α_2 are in the arc $\overset{\frown}{x_1x_2}$,
2. α_1, α_2 are in the arc $\overset{\frown}{x_2x_3}$,
3. α_1, α_2 are in the arc $\overset{\frown}{x_3x_4}$,
4. α_1, α_2 are in the arc $\overset{\frown}{x_4x_1}$.

We prove the case 1, the others are similar. When α_1, α_2 are different from each other in the arc $\overset{\frown}{x_1x_2}$. By Lemma 3.3.3, the value of $\frac{\Delta((x_1,x_3,x_2),(u,v,w))}{\Delta((x_1,x_3,x_4),(u,v,w))}$ does not depend on the different position of u, v, w. We assume that u, v, w are in the arc $\overset{\frown}{x_3x_4}$ as in Figure.

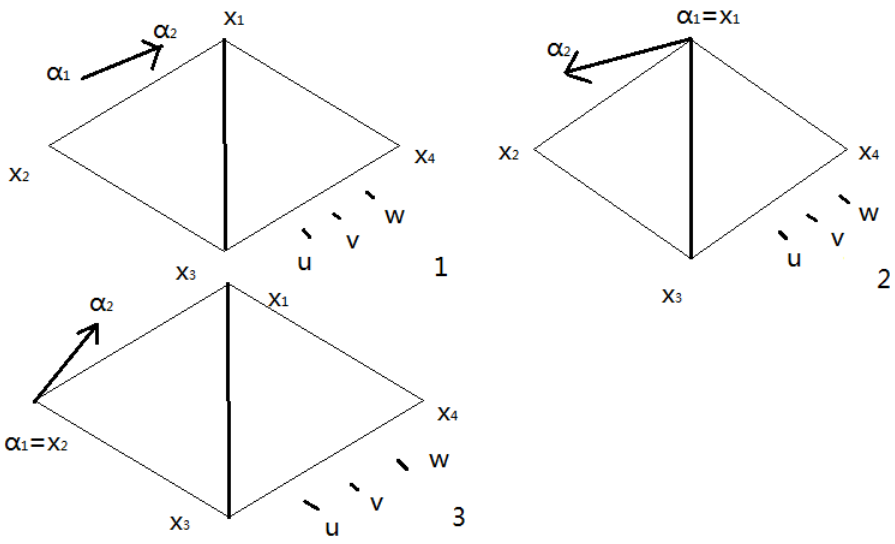


Fig. 3.13:

1. when α_1 is different from x_1 and x_2, by Equation 2.10, we have

$$\{\alpha_1\alpha_2, \Delta((x_1, x_3, x_2), (u, v, w))\} = 0$$

and

$$\{\alpha_1\alpha_2, \Delta((x_1, x_3, x_4), (u, v, w))\} = 0.$$

So by Leibniz's rule, we have

$$\frac{\{\alpha_1\alpha_2, E(x_1, x_3, x_2, x_4)\}}{\alpha_1\alpha_2 \cdot E(x_1, x_3, x_2, x_4)} = 0. \tag{3.41}$$

2. when α_1 coincides with x_1, by Equation 2.10, we have

$$\{\alpha_1\alpha_2, \Delta((x_1, x_3, x_2), (u, v, w))\} = -\frac{1}{2} \cdot \alpha_1\alpha_2 \cdot \Delta((x_1, x_3, x_2), (u, v, w))$$

and

$$\{\alpha_1\alpha_2, \Delta((x_1, x_3, x_4), (u, v, w))\} = -\frac{1}{2} \cdot \alpha_1\alpha_2 \cdot \Delta((x_1, x_3, x_4), (u, v, w)).$$

So by Leibniz's rule, we have

$$\frac{\{\alpha_1\alpha_2, E(x_1, x_3, x_2, x_4)\}}{\alpha_1\alpha_2 \cdot E(x_1, x_3, x_2, x_4)} = 0. \tag{3.42}$$

3. when α_1 coincides with x_2, by Equation 2.11, we have

$$\{\alpha_1\alpha_2, \Delta((x_1, x_3, x_2), (u, v, w))\} = -\frac{1}{2} \cdot \alpha_1\alpha_2 \cdot \Delta((x_1, x_3, x_2), (u, v, w)),$$

$$\{\alpha_1\alpha_2, \Delta((x_1, x_3, x_4), (u, v, w))\} = 0.$$

So by Leibniz's rule, we have

$$\frac{\{\alpha_1\alpha_2, E(x_1, x_3, x_2, x_4)\}}{\alpha_1\alpha_2 \cdot E(x_1, x_3, x_2, x_4)} = -\frac{1}{2}. \tag{3.43}$$

We conclude that when α_1, α_2 are in the arc $\overset{\frown}{x_1x_2}$,

$$\frac{\{\alpha_1\alpha_2, E(x_1, x_3, x_2, x_4)\}}{\alpha_1\alpha_2 \cdot E(x_1, x_3, x_2, x_4)} = \mathcal{J}(\alpha_1, \alpha_2, x_2, x_4) \cdot \delta_{\alpha_1; x_2, x_4}.$$

in $\mathcal{Q}_n(\mathcal{P})$.

We obtain the results for the other three cases by the same way.

Finally, we conclude that

$$\frac{\{\alpha_1\alpha_2, E(x_1, x_3, x_2, x_4)\}}{\alpha_1\alpha_2 \cdot E(x_1, x_3, x_2, x_4)} = \mathcal{J}(\alpha_1, \alpha_2, x_2, x_4) \cdot \delta_{\alpha_1; x_2, x_4}.$$

in $\mathcal{Q}_n(\mathcal{P})$. □

Our main result is

Theorem 3.3.16. *The ring homomorphism θ_{T_3} is Poisson.*

Proof. We associate each edge function with a 4-gons and an edge, each triple ratio with a triangle. For example, we associate $\mathbb{B}_1(Y, T, Z, X)$(or $\mathbb{B}_2(Y, T, Z, X)$) with $\overline{yztx}$ and $\overline{zx}$, we associate $T(X, Z, Y)$ with $\overline{xzy}$. By symmetry, we have the following possible cases as in Figure 3.14:

1. Trivial cases.
2. Brackets of triple ratios.
3. Brackets of triple ratio and edge function.

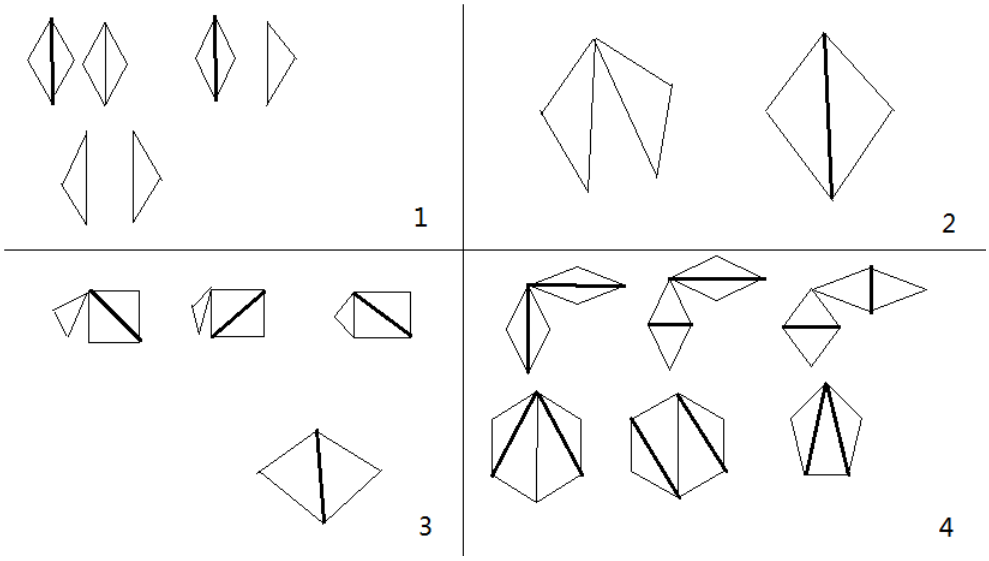


Fig. 3.14:

4. Brackets of edge functions.

We prove the theorem for all the cases as follows.

1. **Trivial case**

   Two graphs associate with two Fock–Goncharov coordinates α, b are separated by a line in the disk. As in Figure 3.15, there are three cases, in each case, α is the Fock–Goncharov coordinate on the left and a is the Fock–Goncharov coordinate on the right.

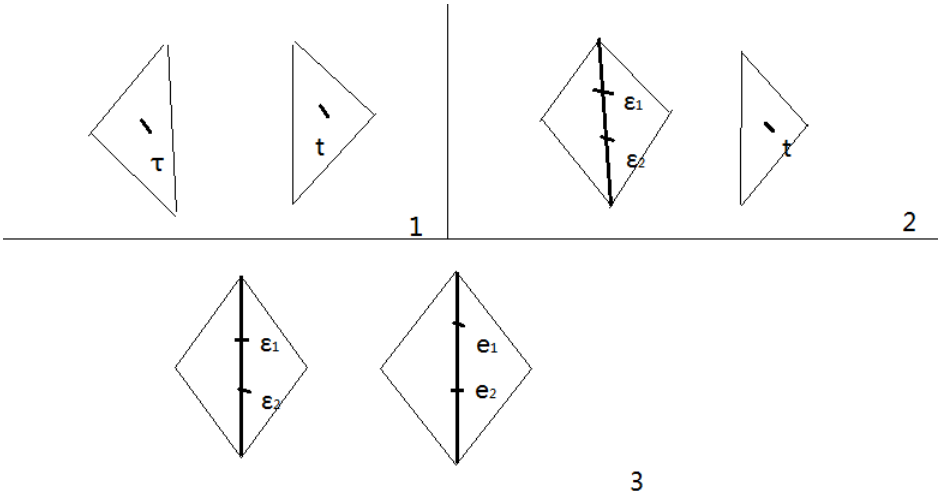


Fig. 3.15:

Proposition 3.3.17. *For all these cases, we have*

$$\{\theta_{T_3}(\alpha),\ \theta_{T_3}(b)\} = \theta_{T_3}\left(\{\alpha,\ b\}_3\right).$$

Proof. For all these cases, we have

$$\{\theta_{T_3}(\alpha),\ \theta_{T_3}(b)\} = 0.$$

For the Fock–Goncharov bracket, we have

$$\{\alpha,\ b\}_3 = 0.$$

We conclude that

$$\{\theta_{T_3}(\alpha),\ \theta_{T_3}(b)\} = \theta_{T_3}\left(\{\alpha,\ b\}_3\right).$$

□

2. **Brackets of triple ratios** Two graphs associate with two triple ratios have at least

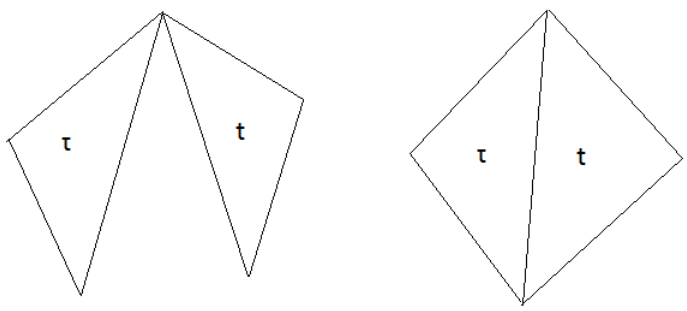


Fig. 3.16:

one common point. By symmetry, we have the following two cases as in Figure 3.16. By Corollary 3.3.14, we obtain that

Corollary 3.3.18. *For these two cases, we have*

$$\{\theta_{T_3}(\tau), \theta_{T_3}(t)\} = \theta_{T_3}(\{\tau,\ t\}_3) = 0.$$

3. **Brackets of triple ratio and edge function**

   Two graphs associate with a triple ratio and an edge function have at least one common point. By symmetry, we have the following four cases as in Figure 3.17.

   (a) By Corollary 3.3.14, we have

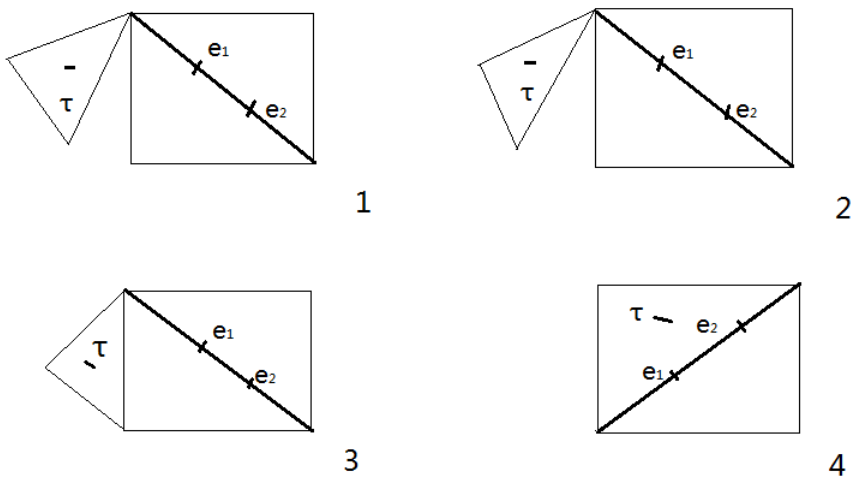


Fig. 3.17:

Proposition 3.3.19. *For case 1,2,3 in Figure 3.17, we have*

$$\{\theta_{T_3}(\tau), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\tau,\ e_1\}_3) = 0 \tag{3.44}$$

and

$$\{\theta_{T_3}(\tau), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\tau,\ e_2\}_3) = 0. \tag{3.45}$$

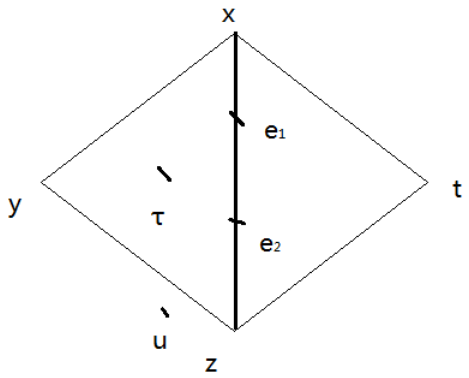


Fig. 3.18:

(b) As in Figure 3.18, let $u \in S^1$. Let x, y, u, z, t be cyclically ordered $[x, y, u, z, t]$ as in Figure 3.18,

$$\theta_{T_3}(\tau) = \theta_{T_3}(T(X, Z, Y)) = T(y, x, z) = \frac{yx \cdot xz \cdot zy}{xy \cdot zx \cdot yz}, \tag{3.46}$$

$$\theta_{T_3}(e_2) = \theta_{T_3}(B_1(Y,T,Z,X)) = \frac{yz}{tz} \cdot E(x,z,t,y), \tag{3.47}$$

To simplify our calculations, let

$$E(x,z,t,y) = \frac{\Delta\left((x,z,t),(z,x,u)\right)}{\Delta\left((x,z,y),(z,x,u)\right)}.$$

By symmetry, we only have to verify that

Proposition 3.3.20.

$$\{\theta_{T_3}(\tau), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\tau,\ e_2\}_3). \tag{3.48}$$

Proof. By Equation 2.10, we have

$$\{xz,\ \Delta\left((x,z,t),(z,x,u)\right)\} = -xu \cdot \Delta\left((x,z,t),(z,x,z)\right) = 0$$

and

$$\{xz,\ \Delta\left((x,z,y),(z,x,u)\right)\} = yz \cdot \Delta\left((x,z,x),(z,x,u)\right) - xu \cdot \Delta\left((x,z,y),(z,x,z)\right) = 0.$$

We obtain that

$$\{xz,\ E(x,z,t,y)\} = 0. \tag{3.49}$$

Similarly, we obtain that

$$\{zx,\ E(x,z,t,y)\} = 0. \tag{3.50}$$

By Leibniz's rule, we have

$$\begin{aligned}
&\frac{\left\{T(y,x,z),\ \frac{yz}{tz} \cdot E(x,z,t,y)\right\}}{T(y,x,z) \cdot \frac{yz}{tz} \cdot E(x,z,t,y)} \\
&= \frac{\left\{T(y,x,z),\ \frac{yz}{tz}\right\}}{T(y,x,z) \cdot \frac{yz}{tz}} \\
&+ \frac{\{yx,\ E(x,z,t,y)\}}{yx \cdot E(x,z,t,y)} - \frac{\{xy,\ E(x,z,t,y)\}}{xy \cdot E(x,z,t,y)} \\
&+ \frac{\{xz,\ E(x,z,t,y)\}}{xz \cdot E(x,z,t,y)} - \frac{\{zx,\ E(x,z,t,y)\}}{zx \cdot E(x,z,t,y)} \\
&+ \frac{\{zy,\ E(x,z,t,y)\}}{zy \cdot E(x,z,t,y)} - \frac{\{yz,\ E(x,z,t,y)\}}{yz \cdot E(x,z,t,y)}.
\end{aligned} \tag{3.51}$$

By Corollary 3.3.14, we have

$$\left\{T(y,x,z),\ \frac{yz}{tz}\right\} = 0.$$

By Lemma 3.3.15, we have

$$\frac{\{yx,\ E(x,z,t,y)\}}{yx \cdot E(x,z,t,y)} = \frac{1}{2}, \tag{3.52}$$

$$\frac{\{xy,\ E(x,z,t,y)\}}{xy \cdot E(x,z,t,y)} = 0, \tag{3.53}$$

$$\frac{\{zy,\ E(x,z,t,y)\}}{zy \cdot E(x,z,t,y)} = 0, \tag{3.54}$$

$$\frac{\{yz,\ E(x,z,t,y)\}}{yz \cdot E(x,z,t,y)} = -\frac{1}{2}. \tag{3.55}$$

Thus the right hand side of Equation 3.51 equals to

$$\begin{aligned} &= 0 + \frac{1}{2} - 0 + 0 - 0 + 0 - \left(-\frac{1}{2}\right) \\ &= 1. \end{aligned} \tag{3.56}$$

For the Fock-Goncharov algebra, we have

$$\{\tau,\ e_2\}_3 = \tau \cdot e_2,$$

we conclude that

$$\{\theta_{T_3}(\tau), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\tau,\ e_2\}_3).$$

□

4. **Brackets of edge functions**

Two graphs associate with two edge functions have at least one common point. By symmetry, we have the following cases as in Figure 3.14.4.

(a) Let $u, \nu \in S^1$. Let $x, \alpha, \nu, \beta, \gamma, y, u, z, s$ be cyclically ordered $[x, \alpha, \nu, \beta, \gamma, y, u, z, s]$ as in Figure 3.19,

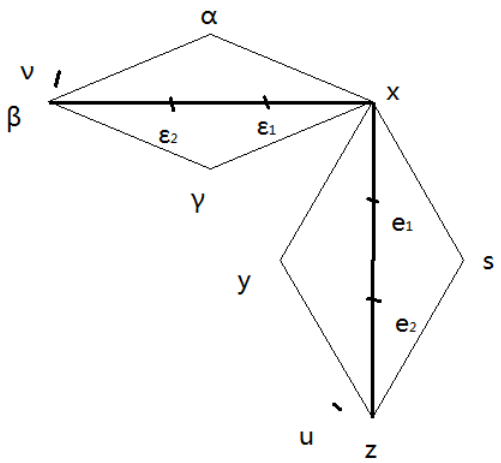


Fig. 3.19:

$$\theta_{T_3}(\epsilon_1) = \theta_{T_3}(B_2(A,\Gamma,B,X)) = \frac{\gamma x}{\alpha x}\cdot E(x,\beta,\alpha,\gamma), \tag{3.57}$$

$$\theta_{T_3}(\epsilon_2) = \theta_{T_3}(B_1(A,\Gamma,B,X)) = \frac{\alpha\beta}{\gamma\beta}\cdot E(x,\beta,\gamma,\alpha), \tag{3.58}$$

$$\theta_{T_3}(e_1) = \theta_{T_3}(B_2(Y,S,Z,X)) = \frac{sx}{yx}\cdot E(x,z,y,s), \tag{3.59}$$

$$\theta_{T_3}(e_2) = \theta_{T_3}(B_1(Y,S,Z,X)) = \frac{yz}{sz}\cdot E(x,z,s,y). \tag{3.60}$$

To simplify our calculations, let

$$E(x,\beta,\alpha,\gamma) = \frac{\Delta\left((x,\beta,\alpha),(\beta,x,\nu)\right)}{\Delta\left((x,\beta,\gamma),(\beta,x,\nu)\right)},$$

$$E(x,z,y,s) = \frac{\Delta\left((x,z,y),(z,x,u)\right)}{\Delta\left((x,z,s),(z,x,u)\right)}.$$

By symmetry, we only have to verify that

i.
$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0.$$

ii.
$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_1,\ e_2\}_3) = 0.$$

iii.
$$\{\theta_{T_3}(\epsilon_2),\theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_2,\ e_2\}_3) = 0.$$

Proposition 3.3.21.

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0.$$

Proof. By Lemma 3.3.15, for any factor $\mathcal{S}$ of $\frac{sx}{yx}\cdot\frac{\Delta((x,z,y),(z,x,u))}{\Delta((x,z,s),(z,x,u))}$ ($\mathcal{S}\in\{sx,yx,xz,xu,zx,zu,yz,yx,yu\}$), we have

$$\{\mathcal{S},\ E(x,\beta,\alpha,\gamma)\} = 0.$$

Thus we have

$$\left\{E(x,\beta,\alpha,\gamma),\ \frac{sx}{yx}\cdot E(x,z,y,s)\right\} = 0 \tag{3.61}$$

Then by Lemma 3.3.15, we have

$$\{\gamma x,\ E(x,z,y,s)\} = 0, \tag{3.62}$$

and

$$\{\alpha x,\ E(x,z,y,s)\} = 0. \tag{3.63}$$

Thus, by Leibniz's rule, we have

$$\begin{aligned}
&\frac{\left\{\frac{\gamma x}{\alpha x}\cdot E(x,\beta,\alpha,\gamma),\ \frac{sx}{yx}\cdot E(x,z,y,s)\right\}}{\frac{\gamma x}{\alpha x}\cdot E(x,\beta,\alpha,\gamma)\cdot\frac{sx}{yx}\cdot E(x,z,y,s)}\\
&=\frac{\{\gamma x,\ sx\}}{\gamma x\cdot sx}-\frac{\{\gamma x,\ yx\}}{\gamma x\cdot yx}-\frac{\{\alpha x,\ sx\}}{\alpha x\cdot sx}+\frac{\{\alpha x,\ yx\}}{\alpha x\cdot yx}\\
&+\frac{\{\gamma x,\ E(x,z,y,s)\}}{\gamma x\cdot E(x,z,y,s)}-\frac{\{\alpha x,\ E(x,z,y,s)\}}{\alpha x\cdot E(x,z,y,s)}\\
&+\frac{\left\{E(x,\beta,\alpha,\gamma),\ \frac{sx}{yx}\cdot E(x,z,y,s)\right\}}{E(x,\beta,\alpha,\gamma)\cdot\frac{sx}{yx}\cdot E(x,z,y,s)}\\
&=\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{2}+0-0+0\\
&=0
\end{aligned}\tag{3.64}$$

Since

$$\theta_{T_3}(\{\epsilon_1,\ e_1\}_3)=0,$$

we conclude that

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_1)\}=\theta_{T_3}(\{\epsilon_1,\ e_1\}_3)=0.$$

□

Similarly, we have

Proposition 3.3.22.

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_2)\}=\theta_{T_3}(\{\epsilon_1,\ e_2\}_3)=0.$$

Proposition 3.3.23.

$$\{\theta_{T_3}(\epsilon_2),\theta_{T_3}(e_2)\}=\theta_{T_3}(\{\epsilon_2,\ e_2\}_3)=0.$$

(b) Let $u,\nu\in S^1$. Let $x,\alpha,\beta,\nu,\gamma,y,u,z,s$ be cyclically ordered $[x,\alpha,\beta,\nu,\gamma,y,u,z,s]$ as in Figure 3.20,

$$\theta_{T_3}(\epsilon_1)=\theta_{T_3}(B_2(B,X,\Gamma,A))=\frac{x\alpha}{\beta\alpha}\cdot E(\alpha,\gamma,\beta,x),\tag{3.65}$$

$$\theta_{T_3'}(\epsilon_2)=\theta_{T_3}(B_1(B,X,\Gamma,A))=\frac{\beta\gamma}{x\gamma}\cdot E(\alpha,\gamma,x,\beta),\tag{3.66}$$

$$\theta_{T_3}(e_1)=\theta_{T_3}(B_2(Y,S,Z,X))=\frac{sx}{yx}\cdot E(x,z,y,s),\tag{3.67}$$

$$\theta_{T_3}(e_2)=\theta_{T_3}(B_1(Y,S,Z,X))=\frac{yz}{sz}\cdot E(x,z,s,y).\tag{3.68}$$

Similarly as Proposition 3.3.21, we have

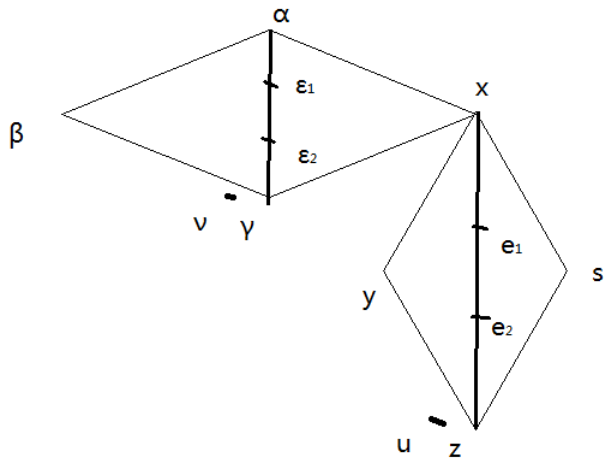


Fig. 3.20:

i.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0,$$

ii.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_1,\ e_2\}_3) = 0,$$

iii.
$$\{\theta_{T_3}(\epsilon_2), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_2,\ e_1\}_3) = 0,$$

iv.
$$\{\theta_{T_3}(\epsilon_2), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_2,\ e_2\}_3) = 0.$$

(c) Let $u, \nu \in S^1$. Let $x, \alpha, \beta, \nu, \gamma, u, y, z, s$ be cyclically ordered $[x, \alpha, \beta, \nu, \gamma, u, y, z, s]$ as in Figure 3.21, let

$$\theta_{T_3}(\epsilon_1) = \theta_{T_3}(B_2(B, X, \Gamma, A)) = \frac{x\alpha}{\beta\alpha} \cdot E(\alpha, \gamma, \beta, x), \tag{3.69}$$

$$\theta_{T_3'}(\epsilon_2) = \theta_{T_3}(B_1(B, X, \Gamma, A)) = \frac{\beta\gamma}{x\gamma} \cdot E(\alpha, \gamma, x, \beta), \tag{3.70}$$

$$\theta_{T_3}(e_1) = \theta_{T_3}(B_2(X, Z, Y, S)) = \frac{zs}{xs} \cdot E(s, y, x, z), \tag{3.71}$$

$$\theta_{T_3}(e_2) = \theta_{T_3}(B_1(X, Z, Y, S)) = \frac{xy}{zy} \cdot E(s, y, z, x). \tag{3.72}$$

Similarly as Proposition 3.3.21, we have

i.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0,$$

ii.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_1,\ e_2\}_3) = 0,$$

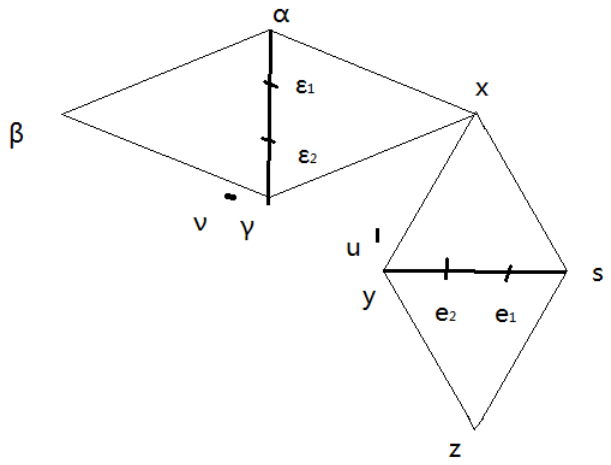


Fig. 3.21:

iii.

$$\{\theta_{T_3}(\epsilon_2), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_2,\ e_2\}_3) = 0.$$

(d) Let $u, \nu \in S^1$. Let $x, \alpha, \nu, \beta, y, u, z, s$ be cyclically ordered $[x, \alpha, \nu, \beta, y, u, z, s]$ as in Figure 3.22, let

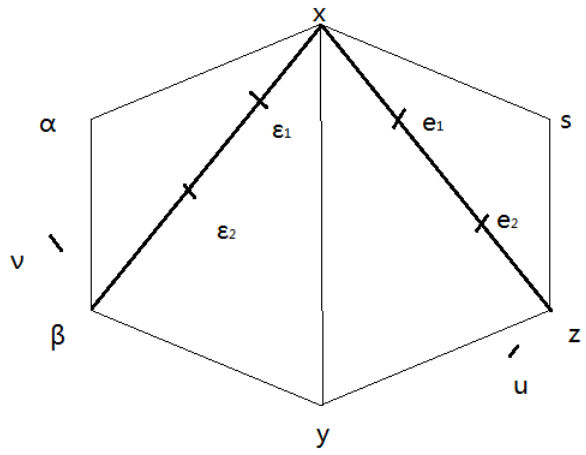


Fig. 3.22:

$$\theta_{T_3}(\epsilon_1) = \theta_{T_3}(B_2(A, Y, B, X)) = \frac{yx}{\alpha x} \cdot E(x, \beta, \alpha, y), \tag{3.73}$$

$$\theta_{T_3}(\epsilon_2) = \theta_{T_3}(B_1(A, Y, B, X)) = \frac{\alpha\beta}{y\beta} \cdot E(x, \beta, y, \alpha), \tag{3.74}$$

$$\theta_{T_3}(e_1) = \theta_{T_3}(B_2(Y,S,Z,X)) = \frac{sx}{yx} \cdot E(x,z,y,s), \tag{3.75}$$

$$\theta_{T_3}(e_2) = \theta_{T_3}(B_1(Y,S,Z,X)) = \frac{yz}{sz} \cdot E(x,z,s,y). \tag{3.76}$$

By symmetry, we only have to verify that

i.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0.$$

ii.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_1,\ e_2\}_3) = 0.$$

iii.
$$\{\theta_{T_3}(\epsilon_2), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_2,\ e_2\}_3) = 0.$$

Proposition 3.3.24.

$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0.$$

Proof. To simplify our calculations, let

$$E(x,\beta,\alpha,y) = \frac{\Delta\left((x,\beta,\alpha),(\beta,x,\nu)\right)}{\Delta\left((x,\beta,y),(\beta,x,\nu)\right)}.$$

By Lemma 3.3.15, for any factor $\mathcal{S}$ of

$$\Delta\left((x,\beta,\alpha),(\beta,x,\nu)\right) = x\beta\cdot\beta x\cdot\alpha\nu - x\nu\cdot\beta x\cdot\alpha\beta - x\beta\cdot\alpha x\cdot\beta\nu,$$

we have

$$\{\mathcal{S},\ E(x,z,y,s)\} = 0.$$

Thus we have

$$\{\Delta\left((x,\beta,\alpha),(\beta,x,\nu)\right),\ E(x,z,y,s)\} = 0.$$

Moreover, by Lemma 3.3.15, for $r = x,\nu,\beta$, we have

$$\frac{\{yr,\ E(x,z,y,s)\}}{yr\cdot E(x,z,y,s)} = -\frac{1}{2}.$$

Since $\Delta\left(x,\beta,y),(\beta,x,\nu)\right) = x\beta\cdot\beta x\cdot y\nu - x\nu\cdot\beta x\cdot y\beta - x\beta\cdot yx\cdot\beta\nu$, we have

$$\frac{\{\Delta\left((x,\beta,y),(\beta,x,\nu)\right),\ E(x,z,y,s)\}}{\Delta\left((x,\beta,y),(\beta,x,\nu)\right)\cdot E(x,z,y,s)} = -\frac{1}{2}. \tag{3.77}$$

By Lemma 3.3.15, we have

$$\{E(x,\beta,\alpha,y),\ sx\} = 0, \tag{3.78}$$

$$\frac{\{E(x,\beta,\alpha,y),\ yx\}}{E(x,\beta,\alpha,y)\cdot yx} = \frac{1}{2}. \tag{3.79}$$

Thus we have

$$\begin{aligned}
&\frac{\left\{\frac{yx}{\alpha x}\cdot E(x,\beta,\alpha,y),\ \frac{sx}{yx}\cdot E(x,z,y,s)\right\}}{\frac{yx}{\alpha x}\cdot E(x,\beta,\alpha,y)\cdot\frac{sx}{yx}\cdot E(x,z,y,s)}\\
&=\frac{\{yx,\ sx\}}{yx\cdot sx}-\frac{\{yx,\ yx\}}{yx\cdot yx}-\frac{\{\alpha x,\ sx\}}{\alpha x\cdot sx}+\frac{\{\alpha x,\ yx\}}{\alpha x\cdot yx}\\
&+\frac{\{yx,\ E(x,z,y,s)\}}{yx\cdot E(x,z,y,s)}-\frac{\{\alpha x,\ E(x,z,y,s)\}}{\alpha x\cdot E(x,z,y,s)}\\
&+\frac{\{E(x,\beta,\alpha,y),\ sx\}}{E(x,\beta,\alpha,y)\cdot sx}-\frac{\{E(x,\beta,\alpha,y),\ yx\}}{E(x,\beta,\alpha,y)\cdot yx}\\
&+\frac{\{\Delta\left((x,\beta,\alpha),(\beta,x,\nu)\right),\ E(x,z,y,s)\}}{\Delta\left((x,\beta,\alpha),(\beta,x,\nu)\right)\cdot E(x,z,y,s)}-\frac{\{\Delta\left((x,\beta,y),(\beta,x,\nu)\right),\ E(x,z,y,s)\}}{\Delta\left((x,\beta,y),(\beta,x,\nu)\right)\cdot E(x,z,y,s)}\\
&=\frac{1}{2}-0-\frac{1}{2}+\frac{1}{2}+\left(-\frac{1}{2}\right)-0+0-\frac{1}{2}+0-\left(-\frac{1}{2}\right)\\
&=0
\end{aligned}\tag{3.80}$$

Since

$$\theta_{T_3}(\{\epsilon_1,\ e_1\}_3)=0,$$

we conclude that

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_1)\}=\theta_{T_3}(\{\epsilon_1,\ e_1\}_3)=0.$$

□

Similarly, we have

Proposition 3.3.25.

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_2)\}=\theta_{T_3}(\{\epsilon_1,\ e_2\}_3)=0.$$

Proposition 3.3.26.

$$\{\theta_{T_3}(\epsilon_2),\theta_{T_3}(e_2)\}=\theta_{T_3}(\{\epsilon_2,\ e_2\}_3)=0.$$

(e) Let $u,\nu\in S^1$. Let $x,\alpha,\beta,\nu,y,u,z,s$ be cyclically ordered $[x,\alpha,\beta,\nu,y,u,z,s]$ as in Figure 3.23, let

$$\theta_{T_3}(\epsilon_1)=\theta_{T_3}(B_2(B,X,Y,A))=\frac{x\alpha}{\beta\alpha}\cdot E(\alpha,y,\beta,x),\tag{3.81}$$

$$\theta_{T_3'}(e_2)=\theta_{T_3}(B_1(B,X,Y,A))=\frac{\beta y}{xy}\cdot E(\alpha,y,x,\beta),\tag{3.82}$$

$$\theta_{T_3}(e_1)=\theta_{T_3}(B_2(Y,S,Z,X))=\frac{sx}{yx}\cdot E(x,z,y,s),\tag{3.83}$$

$$\theta_{T_3}(e_2)=\theta_{T_3}(B_1(Y,S,Z,X))=\frac{yz}{sz}\cdot E(x,z,s,y).\tag{3.84}$$

By symmetry, we only have to verify that

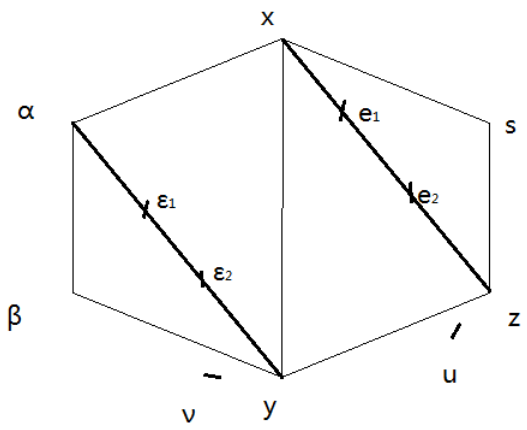


Fig. 3.23:

i.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0,$$

ii.
$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_1,\ e_2\}_3) = 0.$$

Proposition 3.3.27.

$$\{\theta_{T_3}(\epsilon_1), \theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0.$$

Proof. To simplify our calculations, let

$$E(\alpha, y, \beta, x) = \frac{\Delta\left((\alpha, y, \beta), (y, \alpha, \nu)\right)}{\Delta\left((\alpha, y, x), (y, \alpha, \nu)\right)}.$$

By Equation 3.77, we obtain that

$$\frac{\{\Delta\left((\alpha, y, \beta), (y, \alpha, \nu)\right),\ E(x, z, y, s)\}}{\left(\Delta\left((\alpha, y, \beta), (y, \alpha, \nu)\right)\right) \cdot E(x, z, y, s)} = -\frac{1}{2}, \tag{3.85}$$

$$\frac{\{\Delta\left((\alpha, y, x), (y, \alpha, \nu)\right),\ E(x, z, y, s)\}}{\left(\Delta\left((\alpha, y, x), (y, \alpha, \nu)\right)\right) \cdot E(x, z, y, s)} = -\frac{1}{2}. \tag{3.86}$$

Thus we have

$$\begin{aligned} \frac{\{E(\alpha, y, \beta, x),\ E(x, z, y, s)\}}{E(\alpha, y, \beta, x) \cdot E(x, z, y, s)} &= \frac{\left\{\frac{\Delta((\alpha,y,\beta),(y,\alpha,\nu))}{\Delta((\alpha,y,x),(y,\alpha,\nu))},\ E(x, z, y, s)\right\}}{\frac{\Delta((\alpha,y,\beta),(y,\alpha,\nu))}{\Delta((\alpha,y,x),(y,\alpha,\nu))} \cdot E(x, z, y, s)} \\ &= \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) \\ &= 0. \end{aligned} \tag{3.87}$$

By Lemma 3.3.15, we obtain that

$$\{x\alpha,\ E(x,z,y,s)\} = 0 \tag{3.88}$$

$$\{\beta\alpha,\ E(x,z,y,s)\} = 0 \tag{3.89}$$

$$\{E(\alpha,y,\beta,x),\ sx\} = 0 \tag{3.90}$$

$$\{E(\alpha,y,\beta,x),\ yx\} = 0 \tag{3.91}$$

Thus we have

$$\begin{aligned}
&\frac{\left\{\frac{x\alpha}{\beta\alpha}\cdot E(\alpha,y,\beta,x),\ \frac{sx}{yx}\cdot E(x,z,y,s)\right\}}{\frac{x\alpha}{\beta\alpha}\cdot E(\alpha,y,\beta,x)\cdot\frac{sx}{yx}\cdot E(x,z,y,s)}\\
&=\frac{\{x\alpha,\ sx\}}{x\alpha\cdot sx}-\frac{\{x\alpha,\ yx\}}{x\alpha\cdot yx}-\frac{\{\beta\alpha,\ sx\}}{\beta\alpha\cdot sx}+\frac{\{\beta\alpha,\ yx\}}{\beta\alpha\cdot yx}+\frac{\{x\alpha,\ E(x,z,y,s)\}}{x\alpha\cdot E(x,z,y,s)}\\
&-\frac{\{\beta\alpha,\ E(x,z,y,s)\}}{\beta\alpha\cdot E(x,z,y,s)}+\frac{\{E(\alpha,y,\beta,x),\ sx\}}{\alpha,y,\beta,x)\cdot sx}-\frac{\{E(\alpha,y,\beta,x),\ yx\}}{\alpha,y,\beta,x)\cdot yx}\\
&+\frac{\{E(\alpha,y,\beta,x),\ E(x,z,y,s)\}}{E(\alpha,y,\beta,x)\cdot E(x,z,y,s)}\\
&=0-0-0+0+0-0+0-0+0\\
&=0.
\end{aligned} \tag{3.92}$$

Since

$$\theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0,$$

we conclude that

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_1)\} = \theta_{T_3}(\{\epsilon_1,\ e_1\}_3) = 0.$$

□

Similarly, we have

Proposition 3.3.28.

$$\{\theta_{T_3}(\epsilon_1),\theta_{T_3}(e_2)\} = \theta_{T_3}(\{\epsilon_1,\ e_2\}_3) = 0.$$

(f) Let $w, v \in S^1$. Let x, m, w, y, v, z, t be cyclically ordered $[x, m, w, y, v, z, t]$ as in Figure 3.24. We have

$$\theta_{T_3}(K) = \theta_{T_3}(B_2(Y,T,Z,X)) = \frac{tx}{yx}\cdot E(x,z,y,t), \tag{3.93}$$

$$\theta_{T_3}(L) = \theta_{T_3}(B_1(Y,T,Z,X)) = \frac{yz}{tz}\cdot E(x,z,t,y), \tag{3.94}$$

$$\theta_{T_3}(A) = \theta_{T_3}(B_2(M,Z,Y,X)) = \frac{zx}{mx}\cdot E(x,y,m,z), \tag{3.95}$$

$$\theta_{T_3}(B) = \theta_{T_3}(B_1(M,Z,Y,X)) = \frac{my}{zy}\cdot E(x,y,z,m). \tag{3.96}$$

By symmetry, we only have to verify that

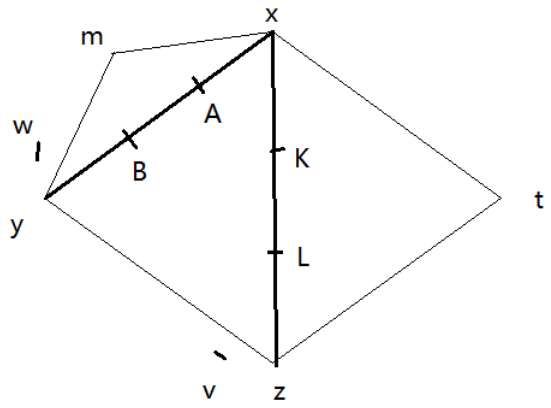


Fig. 3.24:

i.
$$\{\theta_{T_3}(K), \theta_{T_3}(A)\} = \theta_{T_3}(\{K,\ A\}_3) = -\theta_{T_3}(K) \cdot \theta_{T_3}(A),$$

ii.
$$\{\theta_{T_3}(K), \theta_{T_3}(B)\} = \theta_{T_3}(\{K,\ B\}_3) = 0,$$

iii.
$$\{\theta_{T_3}(L), \theta_{T_3}(B)\} = \theta_{T_3}(\{L,\ B\}_3) = 0,$$

We start with a lemma.

Lemma 3.3.29.
$$\{E(x,y,m,z),\ E(x,z,y,t)\} = 0. \tag{3.97}$$

Proof. To simplify our calculations, let

$$E(x,y,m,z) = \frac{\Delta\left((x,y,m),(y,x,w)\right)}{\Delta\left((x,y,z),(y,x,w)\right)}.$$

By Equation 3.77, we have

$$\frac{\{\Delta\left((x,y,m),(y,x,w)\right),\ E(x,z,y,t)\}}{\Delta\left((x,y,m),(y,x,w)\right) \cdot E(x,z,y,t)} = -\frac{1}{2}.$$

By Equation 3.50, we have

$$\{zx,\ E(x,z,y,t)\} = 0. \tag{3.98}$$

By Lemma 3.3.3, let

$$E(x,z,y,t) = \frac{\Delta\left((x,z,y),(v_1,v_2,v_3)\right)}{\Delta\left((x,z,t),(v_1,v_2,v_3)\right)},$$

where v_1, v_2, v_3 are in the interior of the arc $\widehat{zt}$. By Equation 2.11, we have

$$\begin{aligned}&\{zw,\ \Delta((x,z,y),(v_1,v_2,v_3))\}\\&=\frac{1}{2}\cdot zw\cdot\Delta((x,z,y),(v_1,v_2,v_3))-yw\cdot\Delta((x,z,z),(v_1,v_2,v_3))\\&=\frac{1}{2}\cdot zw\cdot\Delta((x,z,y),(v_1,v_2,v_3))\end{aligned}\tag{3.99}$$

and

$$\{zw,\ \Delta((x,z,t),(v_1,v_2,v_3))\}=\frac{1}{2}\cdot zw\cdot\Delta((x,z,y),(v_1,v_2,v_3)),\tag{3.100}$$

so

$$\frac{\{zw,\ E(x,z,y,t)\}}{zw\cdot E(x,z,y,t)}=\frac{1}{2}-\frac{1}{2}=0.\tag{3.101}$$

We calculate five terms of type $\frac{\{a,b\}}{a\cdot b}$ by using Lemma 3.3.15 as in table below:

$a\setminus b$	$E(x,z,y,t)$
xy	0
yx	$-1/2$
zw	
xw	0
zy	0
zx	
yw	$-1/2$

.

Since $\Delta((x,z,y),(y,x,w))=xy\cdot zx\cdot yw+xw\cdot zy\cdot yx-xy\cdot zw\cdot yx$, we have

$$\frac{\{\Delta((x,y,z),(y,x,w)),\ E(x,z,y,t)\}}{\Delta((x,y,z),(y,x,w))\cdot E(x,z,y,t)}=-\frac{1}{2}\tag{3.102}$$

Thus we have

$$\begin{aligned}&\frac{\{E(x,y,m,z),\ E(x,z,y,t)\}}{E(x,y,m,z)\cdot E(x,z,y,t)}\\&=\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)=0.\end{aligned}\tag{3.103}$$

We conclude that

$$\{E(x,y,m,z),\ E(x,z,y,t)\}=0.$$

□

Proposition 3.3.30.

$$\frac{\{\theta_{T_3}(K),\theta_{T_3}(A)\}}{\theta_{T_3}(K)\cdot\theta_{T_3}(A)}=-1.$$

Proof. We have

$$\begin{aligned}
&\frac{\left\{\frac{tx}{yx}\cdot E(x,z,y,t),\ \frac{zx}{mx}\cdot E(x,y,m,z)\right\}}{\frac{tx}{yx}\cdot E(x,z,y,t)\cdot\frac{zx}{mx}\cdot E(x,y,m,z)}\\
&=\frac{\{tx,\ zx\}}{tx\cdot zx}-\frac{\{tx,\ mx\}}{tx\cdot mx}-\frac{\{yx,\ zx\}}{yx\cdot zx}+\frac{\{yx,\ mx\}}{yx\cdot mx}\\
&+\frac{\{tx,\ E(x,y,m,z)\}}{tx\cdot E(x,y,m,z)}-\frac{\{yx,\ E(x,y,m,z)\}}{yx\cdot E(x,y,m,z)}\\
&+\frac{\{E(x,z,y,t),\ zx\}}{E(x,z,y,t)\cdot zx}-\frac{\{E(x,z,y,t),\ mx\}}{E(x,z,y,t)\cdot mx}\\
&+\frac{\{E(x,z,y,t),\ E(x,y,m,z)\}}{E(x,z,y,t)\cdot E(x,y,m,z)}.
\end{aligned}\tag{3.104}$$

By Lemma 3.3.15, we have

$$\{tx,\ E(x,y,m,z)\}=0,\tag{3.105}$$

$$\{E(x,z,y,t),\ mx\}=0.\tag{3.106}$$

By Equation 3.50, we have

$$\{yx,\ E(x,y,m,z)\}=0,\tag{3.107}$$

$$\{E(x,z,y,t),\ zx\}=0.\tag{3.108}$$

Together with Lemma 3.3.29, the right hand side of Equation 3.104 equals to

$$\begin{aligned}
&=\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)-\frac{1}{2}+\left(-\frac{1}{2}\right)+0-0+0-0+0\\
&=-1.
\end{aligned}\tag{3.109}$$

We conclude that

$$\frac{\{\theta_{T_3}(K),\theta_{T_3}(A)\}}{\theta_{T_3}(K)\cdot\theta_{T_3}(A)}=-1.$$

□

Proposition 3.3.31.

$$\{\theta_{T_3}(K),\theta_{T_3}(B)\}=0.$$

Proof. We have

$$\begin{aligned}
&\frac{\left\{\frac{tx}{yx}\cdot E(x,z,y,t),\ \frac{my}{zy}\cdot E(x,y,z,m)\right\}}{\frac{tx}{yx}\cdot E(x,z,y,t)\cdot\frac{my}{zy}\cdot E(x,y,z,m)}\\
&=\frac{\{tx,\ my\}}{tx\cdot my}-\frac{\{tx,\ zy\}}{tx\cdot zy}-\frac{\{yx,\ my\}}{yx\cdot my}+\frac{\{yx,\ zy\}}{yx\cdot zy}\\
&+\frac{\{tx,\ E(x,y,z,m)\}}{tx\cdot E(x,y,z,m)}-\frac{\{yx,\ E(x,y,z,m)\}}{yx\cdot E(x,y,z,m)}\\
&+\frac{\{E(x,z,y,t),\ my\}}{E(x,z,y,t)\cdot my}-\frac{\{E(x,z,y,t),\ zy\}}{E(x,z,y,t)\cdot zy}\\
&+\frac{\{E(x,z,y,t),\ E(x,y,z,m)\}}{E(x,z,y,t)\cdot E(x,y,z,m)}.
\end{aligned}\tag{3.110}$$

By Lemma 3.3.15, we have

$$\{tx,\ E(x,y,z,m)\}=0,\tag{3.111}$$

$$\{E(x,z,y,t),\ my\}=0,\tag{3.112}$$

$$\{E(x,z,y,t),\ zy\}=0.\tag{3.113}$$

By Equation 3.50, we have

$$\{yx,\ E(x,y,z,m)\}=0.\tag{3.114}$$

With Lemma 3.3.29, the right hand side of Equation 3.110 equals to

$$\begin{aligned}
&=0-0-0+0+0-0+0-0+0\\
&=0
\end{aligned}\tag{3.115}$$

We conclude that

$$\{\theta_{T_3}(K),\theta_{T_3}(B)\}=0.$$

□

Proposition 3.3.32.

$$\{\theta_{T_3}(L),\theta_{T_3}(B)\}=0.$$

Proof. We have

$$\begin{aligned}
&\frac{\left\{\frac{yz}{tz}\cdot E(x,z,t,y),\ \frac{my}{zy}\cdot E(x,y,z,m)\right\}}{\frac{yz}{tz}\cdot E(x,z,t,y)\cdot\frac{my}{zy}\cdot E(x,y,z,m)}\\
&=\frac{\{yz,\ my\}}{yz\cdot my}-\frac{\{yz,\ zy\}}{yz\cdot zy}-\frac{\{tz,\ my\}}{tz\cdot my}+\frac{\{tz,\ zy\}}{tz\cdot zy}\\
&+\frac{\{yz,\ E(x,y,z,m)\}}{yz\cdot E(x,y,z,m)}-\frac{\{tz,\ E(x,y,z,m)\}}{tz\cdot E(x,y,z,m)}\\
&+\frac{\{E(x,z,t,y),\ my\}}{E(x,z,t,y)\cdot my}-\frac{\{E(x,z,t,y),\ zy\}}{E(x,z,t,y)\cdot zy}\\
&+\frac{\{E(x,z,t,y),\ E(x,y,z,m)\}}{E(x,z,t,y)\cdot E(x,y,z,m)}.
\end{aligned}\tag{3.116}$$

By Lemma 3.3.15, we have

$$\{yz,\ E(x,y,z,m)\} = 0, \tag{3.117}$$

$$\{tz,\ E(x,y,z,m)\} = 0, \tag{3.118}$$

$$\{E(x,z,t,y),\ my\} = 0, \tag{3.119}$$

$$\{E(x,z,t,y),\ zy\} = 0. \tag{3.120}$$

Together with Lemma 3.3.29, the right hand side of Equation 3.116 equals to

$$\begin{aligned} &= 0-0-0+0+0-0+0-0+0 \\ &= 0 \end{aligned} \tag{3.121}$$

We conclude that

$$\{\theta_{T_3}(L), \theta_{T_3}(B)\} = 0.$$

□

(g) Let $w, v \in S^1$. Let x, m, w, y, v, z, t be cyclically ordered $[x, m, w, y, v, z, t]$ as in Figure 3.24. We have

$$\theta_{T_3}(K) = \theta_{T_3}(B_2(Y,T,Z,X)) = \frac{tx}{yx} \cdot E(x,z,y,t), \tag{3.122}$$

$$\theta_{T_3}(L) = \theta_{T_3}(B_1(Y,T,Z,X)) = \frac{yz}{tz} \cdot E(x,z,t,y), \tag{3.123}$$

Proposition 3.3.33.

$$\{\theta_{T_3}(K), \theta_{T_3}(L)\} = 0.$$

Proof. To simplify our calculations, let

$$E(x,z,y,t) = \frac{1}{E(x,z,t,y)} = \frac{\Delta\left((x,z,y),(z,x,v)\right)}{\Delta\left((x,z,y),(z,x,v)\right)}.$$

We have

$$\begin{aligned} &\frac{\left\{\frac{tx}{yx} \cdot E(x,z,y,t),\ \frac{yz}{tz} \cdot E(x,z,t,y)\right\}}{\frac{tx}{yx} \cdot E(x,z,y,t) \cdot \frac{yz}{tz} \cdot E(x,z,t,y)} \\ &= \frac{\{tx,\ yz\}}{tx \cdot yz} - \frac{\{tx,\ tz\}}{tx \cdot tz} - \frac{\{yx,\ yz\}}{yx \cdot yz} + \frac{\{yx,\ tz\}}{yx \cdot tz} \\ &+ \frac{\{tx,\ E(x,z,t,y)\}}{tx \cdot E(x,z,t,y)} + \frac{\{yx,\ E(x,z,t,y)\}}{yx \cdot E(x,z,t,y)} \\ &+ \frac{\{E(x,z,y,t),\ yz\}}{E(x,z,y,t) \cdot yz} - \frac{\{E(x,z,y,t),\ tz\}}{E(x,z,y,t) \cdot tz}. \end{aligned} \tag{3.124}$$

We calculate four terms of type $\frac{\{a,\ b\}}{a \cdot b}$ by using Lemma 3.3.15 as in the table below:

$a \setminus b$	yz	tz	$\frac{\Delta((x,z,t),(z,x,v))}{\Delta((x,z,y),(z,x,v))}$
tx			$1/2$
yx			$1/2$
$\frac{\Delta((x,z,y),(z,x,v))}{\Delta((x,z,t),(z,x,v))}$	$-1/2$	$-1/2$	

.

Thus the right hand side of Equation 3.124 equals to

$$\begin{aligned} &= 0 - \left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right) + 0 + \frac{1}{2} - \frac{1}{2} + \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) + 0 \\ &= 0. \end{aligned} \tag{3.125}$$

Since

$$\theta_{T_3}(\{K,\ L\}_3) = 0,$$

we conclude that

$$\frac{\{\theta_{T_3}(K), \theta_{T_3}(L)\}}{\theta_{T_3}(K) \cdot \theta_{T_3}(L)} = 0.$$

□

We conclude that the monomorphism θ_{T_3} is Poisson. □

3.4 Poisson embedding

For a disk with $k(k \geq 3)$ marked points on the boundary, given an ideal triangulation T and its n-triangulation T_n for $n \geq 2$. We have the Fock–Goncharov rings $\{\mathcal{A}(T_n)\}_{n=2}$, We consider the relations among $\{\mathcal{A}(T_2), \mathcal{A}(T_3), ...\}$. In fact, $\mathcal{A}(T_k)$ is naturally Poisson embedded in $\mathcal{A}(T_{k+1})$ for $k = 2, 3$. Hence, $\mathcal{A}(T_2)$ is Poisson embedded in $\mathcal{A}(T_k)$ for all $k = 3, 4$.

3.4.1 Poisson embedding τ_2

Definition 3.4.1. *We consider a ring homomorphism τ_2 from $\mathcal{A}(T_2)$ to $\mathcal{A}(T_3)$ by extending the following map on generators for each triangle of T to $\mathcal{A}(T_2)$:*

$$\tau_2(A) = A'B',$$
$$\tau_2(B) = C'D',$$
$$\tau_2(C) = E'F'.$$

Proposition 3.4.2. *The ring homomorphism τ_2 is injective.*

Proof. If $\tau_2(P(\alpha_1, ..., \alpha_m)) = P(\tau_2(\alpha_1), ..., \tau_2(\alpha_m)) = 0$, where $P(\alpha_1, ..., \alpha_m)$ is a polynomial of the edge functions $(\alpha_1, ..., \alpha_m)$ in $\mathcal{A}(T_2)$. Let $\tau_2(\alpha_i) = \beta_i\gamma_i$ for $i = 1, ..., m$. By observing that $\beta_1, \gamma_1, ..., \beta_m, \gamma_m$ are different from each other. Since there is no algebraic relation among $\{\beta_1, \gamma_1, ..., \beta_m, \gamma_m\}$. So, there is no algebraic relation among $\{\tau_2(\alpha_1), ..., \tau_2(\alpha_m)\}$. We have $P(\alpha_1, ..., \alpha_m) = 0$. We conclude that τ_2 is injective. □

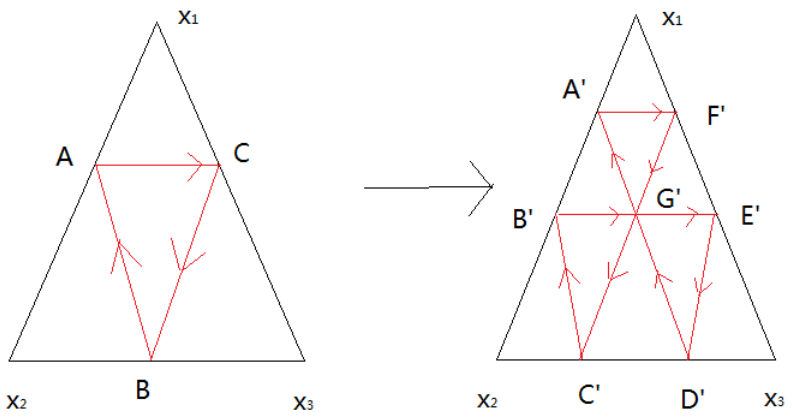


Fig. 3.25:

Proposition 3.4.3. *The ring homomorphism τ_2 is Poisson.*

Proof. By symmetry, we only have to verify that if

$$\{\tau_2(A), \tau_2(C)\}_3 = \tau_2 \circ \{A, C\}_2.$$

The notations are shown in Figure 3.25, We have

$$\{\tau_2(A), \tau_2(C)\}_3 = \{A'B', E'F'\}_3 = A'B'E'F' = \tau_2(A)\tau_2(C) = \tau_2(AC) = \tau_2 \circ \{A, C\}_2.$$

□

Let i be the natural embedding from $\mathcal{B}_2(\mathcal{P})$ to $\mathcal{B}_3(\mathcal{P})$.

Proposition 3.4.4. *The following diagram of ring homomorphisms commutes:*

$$\begin{array}{ccc} \mathcal{A}(T_2) & \xrightarrow{\tau_2} & \mathcal{A}(T_3) \\ \downarrow{\scriptstyle \theta_{T_2}} & & \downarrow{\scriptstyle \theta_{T_3}} \\ \mathcal{B}_2(\mathcal{P}) & \xrightarrow{i} & \mathcal{B}_3(\mathcal{P}) \end{array}$$

Which means that $\theta_{T_3} \circ \tau_2 = i \circ \theta_{T_2}$.

Proof. By definitions of these ring homomorphisms, we only have to check one generator of $\mathcal{A}(T_2)$. By Formula 3.28, 3.29, we have

$$\theta_{T_3} \circ \tau_2 \left(\frac{\Omega\left(\hat{y}^1 \wedge \hat{z}^1\right)}{\Omega\left(\hat{t}^1 \wedge \hat{z}^1\right)} \cdot \frac{\Omega\left(\hat{t}^1 \wedge \hat{x}^1\right)}{\Omega\left(\hat{y}^1 \wedge \hat{x}^1\right)} \right) = \frac{yz}{tz} \cdot \frac{tx}{yx},$$

$$i \circ \theta_{T_2} \left(\frac{\Omega\left(\hat{y}^1 \wedge \hat{z}^1\right)}{\Omega\left(\hat{t}^1 \wedge \hat{z}^1\right)} \cdot \frac{\Omega\left(\hat{t}^1 \wedge \hat{x}^1\right)}{\Omega\left(\hat{y}^1 \wedge \hat{x}^1\right)} \right) = i \left(\frac{yz}{tz} \cdot \frac{tx}{yx} \right) = \frac{yz}{tz} \cdot \frac{tx}{yx}.$$

We conclude that $\theta_{T_3} \circ \tau_2 = i \circ \theta_{T_2}$. □

3.4.2 Poisson embedding τ_3

Definition 3.4.5. *We consider a ring homomorphism τ_3 from $\mathcal{A}(T_3)$ to $\mathcal{A}(T_4)$ by extending the following map on generators for each $\overline{xyz}$ and $\overline{xzt}$ of T to $\mathcal{A}(T_n)$:*

$$\tau_3(B_{i,3-i}(Y,T,Z,X))) = B_{i+1,3-i}(Y,T,Z,X))B_{i,4-i}(Y,T,Z,X)),$$
$$\tau_3(T_{1,1,1}(X,Z,Y)) = T_{2,1,1}(X,Z,Y) \cdot T_{1,2,1}(X,Z,Y) \cdot T_{1,1,2}(X,Z,Y).$$

Proposition 3.4.6. *The ring homomorphism τ_3 is injective.*

Proof. As in Proposition 3.4.1, to proof τ_3 is injective, we have to proof the algebraic independence of $\{\tau_3(B_{1,2}(Y,T,Z,X))), \tau_3(B_{2,1}(Y,T,Z,X))), \tau_3(T_{1,1,1}(X,Z,Y)), ...\}$. Since the Fock–Goncharov coordinates $\{B_{1,3}(Y,T,Z,X)), B_{2,2}(Y,T,Z,X)), B_{3,1}(Y,T,Z,X)), T_{2,1,1}(X,Z,Y), T_{1,2,1}(X,Z,Y), T_{1,1,2}(X,Z,Y)...\}$ are algebraic independent. With

$$\tau_3(B_{1,2}(Y,T,Z,X))) = B_{2,2}(Y,T,Z,X))B_{1,3}(Y,T,Z,X)),$$
$$\tau_3(B_{2,1}(Y,T,Z,X))) = B_{3,1}(Y,T,Z,X))B_{2,2}(Y,T,Z,X)),$$
$$\tau_3(T_{1,1,1}(X,Z,Y)) = T_{2,1,1}(X,Z,Y) \cdot T_{1,2,1}(X,Z,Y) \cdot T_{1,1,2}(X,Z,Y),$$

we have that $\{\tau_3(B_{1,2}(Y,T,Z,X))), \tau_3(B_{2,1}(Y,T,Z,X))), \tau_3(T_{1,1,1}(X,Z,Y)), ...\}$ are algebraic independent. We conclude that τ_3 is injective. □

Proposition 3.4.7. *The ring homomorphism τ_3 is Poisson.*

Proof. By symmetry, as in Figure 3.26, we only have to verify

$$\{\tau_3(A), \tau_3(F)\}_4 = \tau_3 \circ \{A, F\}_3,$$
$$\{\tau_3(G), \tau_3(A)\}_4 = \tau_3 \circ \{G, A\}_3,$$
$$\{\tau_3(B), \tau_3(G)\}_4 = \tau_3 \circ \{B, G\}_3,$$
$$\{\tau_3(A), \tau_3(E)\}_4 = \tau_3 \circ \{A, E\}_3 = 0,$$
$$\{\tau_3(A), \tau_3(D)\}_4 = \tau_3 \circ \{A, D\}_3 = 0.$$

By calculations, we have

$$\{\tau_3(A), \tau_3(F)\}_4 = \{A'B', I'H'\}_4 = A'B'I'H' = \tau_3 \circ \{A, F\}_3,$$
$$\{\tau_3(G), \tau_3(A)\}_4 = \{J'K'L', A'B'\}_4 = J'K'L'A'B' = \tau_3 \circ \{G, A\}_3,$$
$$\{\tau_3(B), \tau_3(G)\}_4 = \{B'C', J'K'L'\}_4 = B'C'J'K'L' = \tau_3 \circ \{B, G\}_3,$$
$$\{\tau_3(A), \tau_3(E)\}_4 = 0,$$
$$\{\tau_3(A), \tau_3(D)\}_4 = 0.$$

We conclude that τ_3 is Poisson. □

Corollary 3.4.8. *The ring homomorphism $\tau_2 \circ \tau_3$ is a Poisson embedding.*

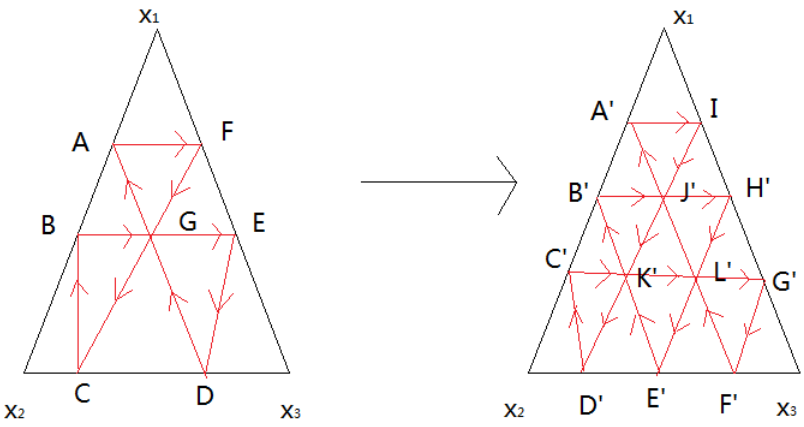


Fig. 3.26:

4. SWAPPING ALGEBRA AND THE DISCRETE VERSION OF THE SPACE OF OPERS

In this chapter, we relate the rank 2 swapping multifraction algebra to the Virasoro algebra on a hypersurface of the configuration space $\mathcal{M}_{N,1}$ of N twisted polygons in $\mathbb{RP}^1$. In §4.1, we recall the central extension, Virasoro algebra, Hill's operators and some of their properties. In §4.2, we recall the configuration space $\mathcal{M}_{N,n}$ of N-twisted polygons in $\mathbb{RP}^n$. When $(N, n+1) = 1$, we recall a coordinate system of $\mathcal{M}_{N,n}$ [SOT] [KS]. In §4.3, we asymptotically identify the rank 2 swapping algebra with the dual of Virasoro algebra on a hypersurface of $\mathcal{M}_{N,1}$. In §4.4, we asymptotically identify the Schwartz algebra with the dual of Virasoro algebra on a hypersurface of $\mathcal{M}_{N,1}$. We prove that swapping bracket is compatible with Schwartz bracket on $\mathcal{M}_{N,1}$.

4.1 Virasoro algebra and Hill's operators

This section is for the purpose of self-contained, which is not original. we recall the definition of central extension, Virasoro algebra, Hill's operators and some well known facts. We refer to [KW] , [OT] and [S1] for further details.

4.1.1 Central extension

Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{n}$ be a vector space.

Definition 4.1.1. [2-COCYCLE] *The map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$ is a* 2-cocycle *if ω is bilinear, antisymmetric and satisfies the cocycle identity*

$$\omega([X,Y],Z) + \omega([Z,X],Y) + \omega([Y,Z],X) = 0. \tag{4.1}$$

Definition 4.1.2. [2-COBOUNDARY] *A 2-cocycle ω on $\mathfrak{g}$ with values in $\mathfrak{n}$ is called a* 2-coboundary *if there exists a linear map $\alpha : \mathfrak{g} \to \mathfrak{n}$ such that*

$$\omega(X,Y) = \alpha([X,Y])$$

for all $X, Y \in \mathfrak{g}$.

Proposition 4.1.3. [CHAPTER 1 SECTION 3.1 IN [KW]] *Let $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$ be a bilinear 2-form. Let*

$$\widehat{\mathfrak{g}} := \mathfrak{g} \bigoplus \mathfrak{n}$$

be a vector space equipped with the following bracket:

$$[(X,u),(Y,v)] = ([X,Y], \omega(X,Y)). \tag{4.2}$$

Then $\widehat{\mathfrak{g}}$ is a Lie algebra if and only if ω is a 2-cocycle.

Definition 4.1.4. [CENTRAL EXTENSION] *A* central extension *defined by a 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ by a vector space $\mathfrak{n}$ is a Lie algebra $\widehat{\mathfrak{g}}$ whose underlying vector space $\widehat{\mathfrak{g}} = \mathfrak{g} \bigoplus \mathfrak{n}$ is equipped with the following Lie bracket:*

$$[(X,u),(Y,v)] = ([X,Y], \omega(X,Y)). \tag{4.3}$$

One can easily see that the central extension defined by a 2-coboundary becomes the trivial extension by the zero cocycle after the change of coordinates $(X, u) \to (X, u-\alpha(X))$. Moreover, we have the following proposition.

Proposition 4.1.5. [Chapter 1 Proposition 3.4 in [KW]] *There is a one-to-one correspondence between equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{n}$ and the elements of $H^2(\mathfrak{g}, \mathfrak{n})$.*

4.1.2 Virasoro algebra

We denote $\frac{\mathrm{d}}{\mathrm{d}\theta}$ by ∂_θ.

Definition 4.1.6. [Lie algebra $\mathrm{Vect}(S^1)$] *Let $\mathrm{Vect}(S^1)$ be the space of all the smooth vector fields on S^1. After fixing a coordinate θ on the circle S^1, any smooth vector field on S^1 can be written as $f(\theta)\partial_\theta$, where f is a smooth function on S^1. Under this identification, the Lie bracket of two elements $f(\theta)\partial_\theta, g(\theta)\partial_\theta$ in $\mathrm{Vect}(S^1)$ is given by*

$$[f(\theta)\partial_\theta, g(\theta)\partial_\theta] = (f'(\theta)g(\theta) - g'(\theta)f(\theta))\partial_\theta. \tag{4.4}$$

where $f'(\theta)$ denotes the derivative in θ of the function $f(\theta)$. The Lie algebra $\mathrm{Vect}(S^1)$ *is the vector space $\mathrm{Vect}(S^1)$ equipped with the Lie bracket defined above.*

Since $f(\theta)$ is smooth in S^1, we have the Fourier coefficient decomposition

$$f(\theta) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta},$$

where $f_j \in \mathbb{C}$.

Definition 4.1.7. [Witt algebra] *Let $L_k := ie^{ik\theta}\partial_\theta$. The* Witt algebra $\mathfrak{Witt}$ *is the Lie subalgebra of $\mathrm{Vect}(S^1) \bigotimes \mathbb{C}$ generated by $\{L_k\}_{k=-\infty}^{\infty}$. The restriction of the Lie bracket is the* Witt bracket *given by*

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{4.5}$$

for any $m, n \in \mathbb{Z}$.

Definition 4.1.8. [Virasoro algebra] *The map $\omega_{GF} : \mathfrak{Witt} \times \mathfrak{Witt} \to \mathbb{R}$ given by*

$$\omega\left(f(\theta)\partial_\theta, g(\theta)\partial_\theta\right) = \int_{S^1} f'(\theta)g''(\theta)\mathrm{d}\theta \tag{4.6}$$

is a nontrivial 2-cocycle on $\mathfrak{Witt}$, called the Gelfand–Fuchs cocycle. *The corresponding central extension of $\mathfrak{Witt}$ is called the* Virasoro algebra *and is denoted by $\mathfrak{vir}$.*

Proposition 4.1.9. [Chapter 2 Proposition 2.3 in [KW]] *Let ω be a 2-cocycle of $\mathfrak{Witt}$. Let $\delta_{a,b} = 1$ if $a = b$, $\delta_{a,b} = 0$ if $a \neq b$. Then there are two constants c_1, c_2 such that*

$$\omega(L_n, L_m) = (c_1 \cdot n^3 + c_2 \cdot n) \cdot \delta_{n,-m}. \tag{4.7}$$

When $c_1 = 0$, ω is a 2-coboundary.

Remark 4.1.10. *For Gelfand-Fuchs 2-cocycle, we have*

$$\omega_{GF}(L_n, L_m) = i \cdot n^3 \cdot \delta_{n,-m}. \tag{4.8}$$

4.1.3 Hill's operators

Definition 4.1.11. [Hill's operator and Hill's equation] *Let $a \in \mathbb{R}$. Let $H(t) \in C^2(\mathbb{R},\mathbb{R})$ with $H(s) = H(s+1)$ for any $s \in \mathbb{R}$. Then the* Hill's operator *defined by $(H(S), a)$ is a map $a\frac{\mathrm{d}^2}{\mathrm{d}s^2} + H(s)$ from $C^2(\mathbb{R},\mathbb{R})$ to $C^0(\mathbb{R},\mathbb{R})$. When*

$$a \cdot \frac{\mathrm{d}^2 X(s)}{\mathrm{d}s^2} + H(s) \cdot X(s) = 0$$

for any $s \in \mathbb{R}$, we say $X(s) \in C^2(\mathbb{R},\mathbb{R})$ is a solution of the Hill's equation $a\frac{\mathrm{d}^2X(s)}{\mathrm{d}s^2} + H(s) \cdot X(s) = 0$.

We can also consider H as a function belongs to $C^2(S^1,\mathbb{R})$. By definition, the *dual space* of $\mathrm{Vect}(S^1)$ is the space of the quadratic differential

$$\Omega^{\otimes 2} := \left\{ H(\theta)(\mathrm{d}\theta)^2 \mid H(\theta) \in C^2(S^1,\mathbb{R}) \right\}.$$

The dual space of $\mathfrak{vir}$ is

$$\mathfrak{vir}^* := \left\{ (H(\theta)(\mathrm{d}\theta)^2, a) \mid H(\theta) \in C^2(S^1,\mathbb{R}), a \in \mathbb{R} \right\}$$

We identify $\mathfrak{vir}^*$ with the space of Hill's operators

$$\left\{ a\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + H(\theta) \mid \ H(\theta) \in C^2(S^1,\mathbb{R}), a \in \mathbb{R} \right\}.$$

Later on, we consider the hyperplane $\left\{ -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + H(\theta) \mid \ H(\theta) \in C^2(S^1,\mathbb{R}) \right\}$.

Definition 4.1.12. [Schwarzian derivative] *Let $u \in C^2(\mathbb{R},\mathbb{R})$. Then the* Schwarzian derivative *of u is*

$$\mathrm{S}(u) := \frac{u'u''' - \frac{3}{2}(u'')^2}{(u')^2} \tag{4.9}$$

Proposition 4.1.13. [Chapter 1 Section 1.2 in [OT]] *Let us fix $x, x_1, x_2, x_3, x_4 \in \mathbb{R}$. Let $[a:b:c:d] := \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}$. The* Schwarzian derivative $\mathrm{S}(u)(x)$ *satisfies for all small ϵ*

$$[u(x+x_1\epsilon) : u(x+x_2\epsilon) : u(x+x_3\epsilon) : u(x+x_4\epsilon)] = [x_1 : x_2 : x_3 : x_4] - 2\,\mathrm{S}(u)(x)\epsilon^2 + O(\epsilon^3). \tag{4.10}$$

There is a well known result which relates the Hill's operator to the Schwarzian derivative.

Proposition 4.1.14. [Chapter 2 Proposition 2.9 in [KW]] *If f, g are two linear independent solutions of the Hill's equation for $(H(s), -1)$. Let $u = \frac{f}{g}$. Let $\mathrm{S}(u)$ be the* Schwarzian derivative *of u. Then, we have*

$$H(t) = -\frac{1}{2} \cdot \mathrm{S}(u)(t) \tag{4.11}$$

4.2 The configuration space of N-twisted polygons in $\mathbb{RP}^n$

Discrete integrable system of the configuration space $\mathcal{M}_{N,n}$ of RP^n is considered by L. Faddeev, A. Yu. Volkov [FV] for $n = 1$, R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT] for $n = 2$ and B. Khsein, F. Soloviev [KS] for n in general. When $(N, n+1) = 1$, a coordinate system of $\mathcal{M}_{N,n}$ is given in [SOT] and [KS], we recall in our own way.

4.2.1 Configuration space of N-twisted polygons

Definition 4.2.1. [Configuration space of N-twisted polygons in $\mathbb{RP}^n$] *A N-twisted polygon in $\mathbb{RP}^n$ is a map f from $\mathbb{Z}$ to $\mathbb{RP}^n$ such that for any $k \in \mathbb{Z}$, we have $f(k+N) = M_f \cdot f(k)$ where M_f belongs to $\mathrm{PSL}_{n+1}(R)$. We call M_f the* monodromy *of the N-twisted polygon in $\mathbb{RP}^n$. We say that f is* in general position *if for any $k \in \mathbb{N}$, the points $\{f(k+i-1)\}_{i=1}^{n+1}$ are in general position in $\mathbb{RP}^n$.*

The configuration space of N-twisted polygons in $\mathbb{RP}^n$, *denoted by $\mathcal{M}_{N,n}$, is the space of the N-twisted polygons in general position in $\mathbb{RP}^n$ up to projective transformations.*

Remark 4.2.2. *If f is in general position, the monodromy is determined by f.*

We a similar definition for $\mathbb{R}^{n+1} \setminus \{0\}$.

Definition 4.2.3. [Configuration space of N-twisted polygons in $\mathbb{R}^{n+1} \setminus \{0\}$] *A N-twisted polygon in $\mathbb{R}^{n+1} \setminus \{0\}$ is a map $\widehat{f}$ from $\mathbb{Z}$ to $\mathbb{R}^{n+1} \setminus \{0\}$ such that for any $k \in \mathbb{Z}$, we have $\widehat{f}(k+N) = G_{\widehat{f}} \cdot \widehat{f}(k)$ where $G_{\widehat{f}}$ belongs to $\mathrm{GL}_{n+1}(R)$. We call $G_{\widehat{f}}$ the* monodromy *of the N-twisted polygon in $\mathbb{R}^{n+1} \setminus \{0\}$. We say that $\widehat{f}$ is* in general position *if for any $k \in \mathbb{N}$, the points $\{\widehat{f}(k+i-1)\}_{i=1}^{n+1}$ are in general position in $\mathbb{R}^{n+1} \setminus \{0\}$.*

The configuration space of N-twisted polygons in $\mathbb{R}^{n+1} \setminus \{0\}$, *denoted by $\widehat{\mathcal{M}_{N,n}}$, is the space of the N-twisted polygons in general position in $\mathbb{R}^{n+1} \setminus \{0\}$ up to linear transformations.*

Remark 4.2.4. *1. There is a natural projection π_N from $\widehat{\mathcal{M}_{N,n}}$ to $\mathcal{M}_{N,n}$ given by:*

$$\pi_N(\widehat{f}) = f, \tag{4.12}$$

where $\widehat{f} = (..., \widehat{f}(1), \widehat{f}(2), ...)$ and $f = (..., f(1), f(2), ...)$ with for any $j \in \mathbb{Z}$,

$$\widehat{f}(j) = (X_{1,j}, ..., X_{n+1,j}) \in \mathbb{R}^{n+1} \setminus \{0\}$$

and

$$f(j) = [X_{1,j}, ..., X_{n+1,j}] \in \mathbb{RP}^n.$$

2. For $n = 2$, $\mathcal{M}_{N,2}$ is the configuration space considered by Richard Schwartz, Valentin Ovsienko and Sergei Tabachnikovin in [SOT]. They constructed a Poisson structure(discrete version of Gelfand–Dickey second Poisson structure) such that the pentagram map defined in Introduction of [SOT] relative to this Poisson structure is completely integrable.

3. For $n = 1$, $\mathcal{M}_{N,1}$ should be thought as a discrete version of the space of the Hill's operators. We will construct a Poisson structure on $\mathcal{M}_{N,1}$ from the swapping algebra in the next section.

4.2.2 Construction of a coordinate system

The configuration space of N-twisted polygons in $\mathbb{RP}^n$ is a discrete version of quasi periodic curves in $\mathbb{RP}^n$. In the continuous case, there is a one to one map between quasi periodic curves without inflection points in $\mathbb{RP}^n$ up to projective transformations and generic $(n+1)$-th order linear differential operators up to some relations, since a curve without inflection points in $\mathbb{RP}^n$ up to projective translations is interpreted as $n+1$ linearly independent solutions of one generic $(n+1)$-th order linear differential operator up to some relations.

In [SOT], they give the (a,b)-coordinates for $\mathcal{M}_{N,2}$ when $(N,3)=1$. In [KS], they generalize the (a,b)-coordinates for $\mathcal{M}_{N,n}$ without much detail. We recall this coordinate system in our own way.

For any element

$$\widehat{f}=(...,\widehat{f}(1),\widehat{f}(2),...)\in\widehat{\mathcal{M}_{N,n}},$$

where

$$\widehat{f}(j)=(X_{1,j},...,X_{n+1,j})\in\mathbb{R}^{n+1},$$

for any $k\in\mathbb{Z}$, since $\{\widehat{f}(k-i)\}_{i=0}^{n}$ are in general position, the $n+1$ vectors $\widehat{f}(k-n),...,\widehat{f}(k)$ are linearly independent, but the $n+2$ vectors $\widehat{f}(k-n),...,\widehat{f}(k+1)$ are not linearly independent, so there are $(a_{k,k-n},...,a_{n+1,k-n})\in\mathbb{R}^{n+1}$ such that

$$X_{i,k+1}=a_{1,k-n}X_{i,k}+a_{2,k-n}X_{i,k-1}+...a_{n,k-n}X_{i,k-n+1}+a_{n+1,k-n}X_{i,k-n}\tag{4.13}$$

for $i=1,...,n+1$, where

$$a_{j,k-n}=\frac{\det\left(\widehat{f}(k-n),...,\widehat{f}(k-j-1),\widehat{f}(k-j),\widehat{f}(k+1),\widehat{f}(k+2-j),...,\widehat{f}(k)\right)}{\det\left(\widehat{f}(k-n),...,\widehat{f}(k)\right)}.$$

For $i=1,...,n+1$, $k\in\mathbb{Z}$, $a_{i,k-n}$ is considered as a map from $\widehat{\mathcal{M}_{N,n}}$ to $\mathbb{R}$.

As we vary k, we get a family of functions $\{a_{1,k},...,a_{n+1,k}\}_{k=-\infty}^{\infty}$. Since $\widehat{f}(k+N)=G_f\cdot\widehat{f}(k)$, we have $a_{i,k}=a_{i,k+N}$. Thus we have

Proposition 4.2.5. *The functions $\{a_{1,k},...,a_{n+1,k}\}_{k=1}^{N}$ defines a coordinate system of $\widehat{\mathcal{M}_{N,n}}$.*

Proof. We have already shown that given an element in $\widehat{\mathcal{M}_{N,n}}$, we obtain a set $\{a_{1,k},...,a_{n+1,k}\}_{k=1}^{N}$. Given a set $\{a_{1,k},...,a_{n+1,k}\}_{k=1}^{N}$, we want to get an element in $\widehat{\mathcal{M}_{N,n}}$. Since the elements in $\widehat{\mathcal{M}_{N,n}}$ is equivalent under $\mathrm{GL}(n,\mathbb{R})$, for any element

$$\widehat{f}=(...,\widehat{f}(1),\widehat{f}(2),...)\in\widehat{\mathcal{M}_{N,n}},$$

where

$$\widehat{f(j)}=(X_{1,j},...,X_{n+1,j})\in\mathbb{R}^{n+1},$$

we suppose that $\left(\widehat{f}(0),...,\widehat{f}(n)\right) = I_{n+1,n+1}$.

Then, given $\{a_{1,k},...,a_{n+1,k}\}_{k=1}^{N}$, we get $\{a_{1,k},...,a_{n+1,k}\}_{k=-\infty}^{\infty}$ with $a_{i,k} = a_{i,k+N}$. We obtain an element $\{\widehat{f(i)}\}_{i=-\infty}^{\infty}$ belongs to $\widehat{\mathcal{M}_{N,n}}$ through Equations 4.13. □

Corollary 4.2.6. $\dim \widehat{\mathcal{M}_{N,n}} = (n+1)\cdot N$.

Proof. By counting the cardinal of variables in $\{a_{1,k},...,a_{n+1,k}\}_{k=1}^{N}$, we have $\dim \widehat{\mathcal{M}_{N,n}} = n\cdot N$. □

We generalize Proposition 4.1 in [SOT] by using the same method to give a coordinate system of $\mathcal{M}_{N,n}$ when $(N,n+1)=1$.

Definition 4.2.7. *Let $c\neq 0$, let π be the natural projection from $\widehat{\mathcal{M}_{N,n}}$ to $\mathcal{M}_{N,n}$. Let $\widehat{\mathcal{M}_{N,n}}^{c}$ be the subset of $\widehat{\mathcal{M}_{N,n}}$ such that for any $g\in\widehat{\mathcal{M}_{N,n}}^{c}$, we have*

$$\det\left(g(k),...,g(k+n)\right) = c \neq 0 \tag{4.14}$$

for any $k\in\mathbb{Z}$.

Lemma 4.2.8. *For any $f\in\mathcal{M}_{N,n}$, we have a unique $g\in\widehat{\mathcal{M}_{N,n}}^{c}$ such that $\pi(g)=f$.*

Proof. For any element

$$f = (...,f(0),f(1),...)\in\mathcal{M}_{N,n},$$

we lift f to

$$\widehat{f} = (...,\widehat{f}(0),\widehat{f}(1),...)\in\widehat{\mathcal{M}_{N,n}}$$

where

$$\widehat{f}(j) = (X_{1,j},...,X_{n+1,j})\in\mathbb{R}^{n+1}$$

for any $j\in\mathbb{Z}$. The general position property implies that $\det\left(\widehat{f}(k),...,\widehat{f}(k+n)\right)\neq 0$ for any $k\in\mathbb{Z}$. We rescale $g(j) = t_j\cdot\widehat{f}(j)$ (where $g(j)=(Y_{1,j},...,Y_{n+1,j})$ for any $j\in\mathbb{Z}$) such that

$$\det\left(g(k),...,g(k+n)\right) = c\neq 0 \tag{4.15}$$

for any $k\in\mathbb{Z}$. We obtain that

$$t_k\cdots t_{k+n} = \frac{c}{\det\left(\widehat{f}(k),...,\widehat{f}(k+n)\right)} \tag{4.16}$$

for $k=1,...,N$ with the convention $j+N=j$. Since $(N,n+1)=1$, Equations 4.16 have a unique solution. We conclude that there is a unique g belongs to $\widehat{\mathcal{M}_{N,n}}^{c}$ such that $\pi(g)=f$. □

When we restrict $\widehat{\mathcal{M}_{N,n}}$ to the subspace $\widehat{\mathcal{M}_{N,n}}^{c}$, Equations 4.13 becomes into

$$Y_{i,k+1} = b_{1,k-n}Y_{i,k} + b_{2,k-n}Y_{i,k-1} + ...b_{n,k-n}Y_{i,k-n+1} + (-1)^n Y_{i,k-n}. \tag{4.17}$$

Similarly to Proposition 4.2.5, we have

Lemma 4.2.9. *The functions* $\{b_{1,k},...,b_{n,k}\}_{k=1}^{N}$ *defines a coordinate system of* $\widehat{\mathcal{M}_{N,n}}^{c}$.

Combining Lemma 4.2.8 and Lemma 4.2.9, we have

Proposition 4.2.10. *When* $(N,n+1)=1$, *the functions* $\{b_{1,k},...,b_{n,k}\}_{k=1}^{N}$ *defines a coordinate system of* $\mathcal{M}_{N,n}$.

Corollary 4.2.11. $\dim \mathcal{M}_{N,n} = n\cdot N$.

Proof. By counting the cardinal of variables in $\{b_{1,k},...,b_{n,k}\}_{k=1}^{N}$, we have $\dim \mathcal{M}_{N,n} = n\cdot N$. □

Later on, we only consider the case $n=1$.

4.3 Large N asymptotic relation between the swapping algebra and the Virasoro algebra

By considering a cross ratio coordinate system $\{B_k\}_{k=1}^{N}$ of $\mathcal{M}_{N,1}$, we define a Poisson structure on $\mathbb{R}(B_1,...,B_N)$ through the swapping algebra, which is asymptotic to the dual of some Poisson algebra generated by Virasoro algebra.

4.3.1 The discrete Hill's operator and the cross-ratios

Definition 4.3.1. [Discrete Hill's equation] *Let* $N\geq 1$ *be an integer. Given a periodic sequence* $\{H_k\}_{k=-\infty}^{\infty}$ *in* $\mathbb{R}$ *where* $H_{N+k}=H_k$ *for any* $k\in\mathbb{Z}$. *The* discrete Hill's equation *is the difference equation in* $\{C_k\}_{k=-\infty}^{\infty}$:

$$C_{k+1}=\left(\frac{H_k}{N^2}+2\right)\cdot C_k - C_{k-1}, \tag{4.18}$$

for any k *belongs to* $\mathbb{Z}$.

We define $\{H_k\}_{k=-\infty}^{\infty}$ *to be a* discrete Hill's operator, *and* $\{C_k\}_{k=-\infty}^{\infty}$ *as in Equation 4.18 is the* solution *to the discrete Hill's equation.*

By Proposition 4.2.10, we have

Corollary 4.3.2. *Let* N *be an odd integer. For any* $f\in\mathcal{M}_{N,1}$, *let* $b_k=b_k(f)$ *be the coordinate system of* $\mathcal{M}_{N,1}$. *Let*

$$H_k=(b_k-2)\cdot N^2.$$

Let $\{X_i\}_{i=-\infty}^{\infty}$ *and* $\{Y_i\}_{i=-\infty}^{\infty}$ *are two solutions to the Hill's equation 4.18 with the initial conditions*

$$X_0=0, X_1=1, Y_0=1, Y_1=0.$$

Let

$$f_i=\frac{X_i}{Y_i},$$

let $\widehat{f}=\{[f_i:1]\}_{i=-\infty}^{\infty}\in\mathcal{M}_{N,1}$, *then* $b_k(\widehat{f})=b_k(f)$.

Remark 4.3.3. *Let N be an odd integer. Since H_k is linear to b_k, $\{H_k\}_{k=1}^N$ is also a coordinate system of $\mathcal{M}_{N,1}$.*

Notation 4.3.4. *Let $c_1, ..., c_n \in \mathbb{R}$, the* continued fraction

$$[c_1; c_2, ..., c_n] := c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}.$$

Proposition 4.3.5. *Let $N \geq 4$,*

1. *for $n \geq 2$, we have*
$$f_n = [-b_1; b_2, -b_3, ..., (-1)^{n-1} b_{n-1}]; \tag{4.19}$$
2. *for $k \geq 2$, we have*
$$b_{2k+1} = (f_3 - f_2) \cdot \frac{f_3 - f_2}{f_3 - f_4} \cdot \frac{f_4 - f_5}{f_4 - f_3} \cdot \cdots \cdot \frac{f_{2k} - f_{2k+1}}{f_{2k} - f_{2k-1}} \cdot \frac{f_{2k+2} - f_{2k}}{f_{2k+2} - f_{2k+1}}, \tag{4.20}$$
and for any $k \geq 1$, we have
$$b_k \cdot b_{k+1} = [f_{k-1}, f_{k+2}, f_{k+1}, f_k]. \tag{4.21}$$

Proof. 1. We prove Formula 4.19 by induction on n. By Formula 4.18, we have $X_{k+1} = b_k X_k - X_{k-1}$ and $Y_{k+1} = b_k Y_k - Y_{k-1}$ for any $k \in \mathbb{Z}$. Since $X_0 = 0, X_1 = 1, Y_0 = 1, Y_1 = 0$, we have

$$X_2 = b_1,\ X_3 = b_1 b_2 + 1,\ X_4 = b_1 b_2 b_3 - b_1 - b_3,\ Y_2 = -1,\ Y_3 = -b_2,\ Y_4 = -b_2 b_3 + 1.$$

Then, we obtain that

$$f_2 = -b_1,\ f_3 = -b_1 + \frac{1}{b_2},\ f_4 = [-b_1; b_2, -b_3, b_4].$$

So Formula 4.19 is true for $n = 2, 3, 4$. Suppose that for $n = k$,

$$f_k = \frac{X_k}{Y_k} = \frac{b_{k-1} X_{k-1} - X_{k-2}}{b_{k-1} Y_{k-1} - Y_{k-2}} = [-b_1; b_2, -b_3, ..., (-1)^{k-1} b_{k-1}]. \tag{4.22}$$

For $n = k + 1$

$$\begin{aligned} f_{k+1} = \frac{X_{k+1}}{Y_{k+1}} = \frac{b_k X_k - X_{k-1}}{b_k Y_k - Y_{k-1}} &= \frac{b_{k-1} X_{k-1} - X_{k-2} - \frac{X_{k-1}}{b_k}}{b_{k-1} Y_{k-1} - Y_{k-2} - \frac{Y_{k-1}}{b_k}} \\ &= \frac{\left(b_{k-1} - \frac{1}{b_k}\right) \cdot X_{k-1} - X_{k-2}}{\left(b_{k-1} - \frac{1}{b_k}\right) \cdot Y_{k-1} - Y_{k-2}} \end{aligned} \tag{4.23}$$

By substitute $b_{k-1} - \frac{1}{b_k}$ for b_{k-1} in the continued fraction f_k, we have

$$\begin{aligned} f_{k+1} &= \left[-b_1; b_2, -b_3, ..., (-1)^{k-1} (b_{k-1} - \frac{1}{b_k})\right] \\ &= [-b_1; b_2, -b_3, ..., (-1)^{k-1} b_{k-1}, (-1)^k b_k)]. \end{aligned} \tag{4.24}$$

We conclude that the formula $f_n = [-b_1; b_2, -b_3, ..., (-1)^{n-1} b_{n-1}]$ for $n \geq 2$.

2. For any $k \geq 0$, we have

$$X_{k+1}Y_k - X_kY_{k+1} = (b_kX_k - X_{k-1}) \cdot Y_k - X_k \cdot (b_kY_k - Y_{k-1}) = X_kY_{k-1} - X_{k-1}Y_k$$
$$= ... = X_1Y_0 - X_0Y_1 = 1, \tag{4.25}$$

$$f_{k+1} - f_k = \frac{X_{k+1}}{Y_{k+1}} - \frac{X_k}{Y_k} = \frac{1}{Y_kY_{k+1}}, \tag{4.26}$$

$$\begin{aligned} f_{k+2} - f_k &= \frac{X_{k+2}}{Y_{k+2}} - \frac{X_k}{Y_k} = \frac{X_{k+2}Y_k - X_kY_{k+2}}{Y_kY_{k+2}} \\ &= \frac{(b_{k+1}X_{k+1} - X_k) \cdot Y_k - X_k \cdot (b_{k+1}Y_{k+1} - Y_k)}{Y_kY_{k+2}} = \frac{b_{k+1}}{Y_kY_{k+1}}. \end{aligned} \tag{4.27}$$

By the above formulas, we conclude that for any $k \geq 2$, we have

$$\begin{aligned} &(f_3 - f_2) \cdot \frac{f_3 - f_2}{f_3 - f_4} \cdot \frac{f_4 - f_5}{f_4 - f_3} \cdot \dots \cdot \frac{f_{2k} - f_{2k+1}}{f_{2k} - f_{2k-1}} \cdot \frac{f_{2k+2} - f_{2k}}{f_{2k+2} - f_{2k+1}} \\ &= \frac{1}{Y_2Y_3} \cdot \frac{\frac{1}{Y_2Y_3}}{\frac{1}{Y_3Y_4}} \cdot \frac{\frac{1}{Y_4Y_5}}{\frac{1}{Y_4Y_3}} \cdot \dots \cdot \frac{\frac{1}{Y_{2k}Y_{2k+1}}}{\frac{1}{Y_{2k}Y_{2k-1}}} \cdot \frac{\frac{b_{2k+1}}{Y_{2k+2}Y_{2k}}}{\frac{1}{Y_{2k+2}Y_{2k+1}}} \\ &= \frac{b_{2k+1}}{Y_2^2} = b_{2k+1}. \end{aligned} \tag{4.28}$$

For any $k \geq 1$, we have

$$\begin{aligned} [f_{k-1}, f_{k+2}, f_{k+1}, f_k] &= \frac{f_{k-1} - f_{k+1}}{f_{k-1} - f_k} \cdot \frac{f_{k+2} - f_k}{f_{k+2} - f_{k+1}} \\ &= \frac{\frac{-b_k}{Y_{k-1}Y_{k+1}}}{\frac{1}{Y_{k-1}Y_k}} \cdot \frac{\frac{b_{k+1}}{Y_{k+2}Y_k}}{\frac{1}{Y_{k+2}Y_{k+1}}} \\ &= b_kb_{k+1}. \end{aligned} \tag{4.29}$$

□

Definition 4.3.6. *Since we relate b_kb_{k+1} to a cross ratio $[f_{k-1}, f_{k+2}, f_{k+1}, f_k]$, let*

$$B_k := b_kb_{k+1}.$$

By the same argument in [SOT], we have

Corollary 4.3.7. *The functions $\{B_k\}_{k=1}^N$ defines a coordinate system of $\mathcal{M}_{N,1}$. We call $\{B_k\}_{k=1}^N$ the* cross ratio coordinate system.

4.3.2 Relate discrete version with continuous version

By Proposition 4.3.5, we relate the Schwartzian with B_k as follows.

Proposition 4.3.8. *Let $H(t) \in C^2(\mathbb{R},\mathbb{R})$ with $H(s) = H(s+1)$ for any $s \in \mathbb{R}$ be a Hill's operator, let $X(t)$, $Y(t)$ be two linear independent solutions of $H(t)$.*

Let $N > 4$, let $H_k = H(k/N)$, $X_k = X(k/N)$, $Y_k = Y(k/N)$, suppose that

$$\frac{X_1}{X_0} \neq \frac{Y_1}{Y_0},$$

then we have

$$H_k = -\frac{1}{2} \cdot \mathrm{S}\left(\frac{X}{Y}\left(\frac{k}{N}\right)\right) + o\left(\frac{1}{N}\right).$$

Proof. Let

$$h := X_1Y_0 - X_0Y_1 \neq 0.$$

For any $k \geq 0$, we have

$$\begin{aligned} &X_{k+1}Y_k - X_kY_{k+1} = (b_kX_k - X_{k-1}) \cdot Y_k - X_k \cdot (b_kY_k - Y_{k-1}) = X_kY_{k-1} - X_{k-1}Y_k \\ &= ... = X_1Y_0 - X_0Y_1 = h, \end{aligned} \tag{4.30}$$

$$f_{k+1} - f_k = \frac{X_{k+1}}{Y_{k+1}} - \frac{X_k}{Y_k} = \frac{h}{Y_kY_{k+1}}, \tag{4.31}$$

$$\begin{aligned} f_{k+2} - f_k &= \frac{X_{k+2}}{Y_{k+2}} - \frac{X_k}{Y_k} = \frac{X_{k+2}Y_k - X_kY_{k+2}}{Y_kY_{k+2}} \\ &= \frac{(b_{k+1}X_{k+1} - X_k) \cdot Y_k - X_k \cdot (b_{k+1}Y_{k+1} - Y_k)}{Y_kY_{k+2}} = \frac{hb_{k+1}}{Y_kY_{k+1}}. \end{aligned} \tag{4.32}$$

For any $k \geq 1$, we have

$$\frac{f_{k-1} - f_{k+1}}{f_{k-1} - f_k}\frac{f_{k+2} - f_k}{f_{k+2} - f_{k+1}} = \frac{\frac{-hb_k}{Y_{k-1}Y_{k+1}}}{\frac{h}{Y_{k-1}Y_k}}\frac{\frac{hb_{k+1}}{Y_{k+2}Y_k}}{\frac{h}{Y_{k+2}Y_{k+1}}} = b_kb_{k+1}. \tag{4.33}$$

Then we have

$$\begin{aligned} &[f_{k-1} : f_{k+2} : f_{k+1} : f_k] = b_kb_{k+1} \\ &= (2 + H_{k+1}/N^2)(2 + H_k/N^2) \\ &= [k-1 : k+2 : k+1 : k] + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_kH_{k+1}}{N^4}. \end{aligned} \tag{4.34}$$

When N converge to infinity, since $H(t)$ is continuous, we have

$$H_{k+1} = H_k + o\left(\frac{1}{N}\right).$$

Comparing with the Schwarzian derivative in Proposition 4.1.13, we have

$$4H_k = -2 \cdot \mathrm{S}\left(\frac{X}{Y}\left(\frac{k}{N}\right)\right) + o\left(\frac{1}{N}\right).$$

We conclude that

$$H_k = -\frac{1}{2} \cdot \mathrm{S}\left(\frac{X}{Y}\left(\frac{k}{N}\right)\right) + o\left(\frac{1}{N}\right).$$

□

4.3.3 Swapping algebra on S^1

Definition 4.3.9. *Let $N \geq 5$ be odd. Let us consider a cyclic ordered set in S^1*

$$\mathcal{P} := \{r_k, 1 \leq k \leq N \mid r_1 < r_2 \cdot \cdots \cdot r_N < r_1 ...\}.$$

Recall that $B_k = [f_{k-1}, f_{k+2}, f_{k+1}, f_k]$. Let θ' be a map from $\{B_1, ..., B_N\}$ to $\mathcal{B}_2(\mathcal{P})$ such that:

$$\theta'(B_k) := \frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \cdot \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}} \tag{4.35}$$

for $k = 1, ..., N$ with the convention $r_k = r_{k+N}$.

Proposition 4.3.10. *Let $N \geq 5$ be odd. Then the map θ' extends to a ring homomorphism $\theta : \mathbb{R}(B_1, ..., B_N) \to \mathcal{B}_2(\mathcal{P})$.*

Proof. For a polynomial $P(B_1, ..., B_N)$, let

$$\theta(P(B_1, ..., B_N)) := P(\theta'(B_1), ..., \theta'(B_N)).$$

To prove that θ is a well defined ring homomorphism, we have only to prove that for any $P(B_1, ..., B_N) = 0$, we have $P(\theta'(B_1), ..., \theta'(B_N)) = 0$.

Since $\{b_i\}_{i=1}^N$ give a coordinate system of $\mathcal{M}_{N,1}$, we have $b_1, ..., b_N$ are algebraic independent. We have $b_2 = \frac{B_1}{b_1}, ..., b_N = \frac{B_{N-1}}{b_1}$, so $b_1, B_1, ..., B_{N-1}$ are algebraic independent. Since N is odd, let $N = 2m + 1$. We have

$$b_1^2 = \frac{B_1 B_3 \cdots B_{2m+1}}{B_2 B_4 \cdots B_{2m}}.$$

So $B_1, ..., B_{2m+1}$ are algebraic independent. If

$$P(B_1, ..., B_{2m+1}) = 0,$$

then

$$P(X_1, ..., X_{2m+1}) = 0$$

for any variables $X_1, ..., X_{2m+1}$, so

$$P(\theta'(B_1), ..., \theta'(B_{2m+1})) = 0.$$

□

Proposition 4.3.11. *Let $N \geq 5$ be odd, let us consider the finite family of elements of $\mathcal{B}_2(\mathcal{P})$*

$$\mathcal{C} := \left\{ \frac{r_{k-1}r_{k+2} \cdot r_k r_{k+1}}{r_{k-1}r_{k+1} \cdot r_k r_{k+2}}, \frac{r_{k+2}r_{k+1} \cdot r_{k+3}r_k}{r_{k+2}r_k \cdot r_{k+3}r_{k+1}}, \frac{r_{k+2}r_{k+3} \cdot r_{k+1}r_k}{r_{k+2}r_k \cdot r_{k+1}r_{k+3}} \right\}_{k=1}^N$$

with the convention $r_k = r_{k+N}$. Let $\langle \mathcal{C} \rangle$ be the subring of $\mathcal{B}_2(\mathcal{P})$ generated by $\mathcal{C}$. Then for any $i, j = 1, ..., N$, we have $\frac{\{\theta(B_i), \theta(B_j)\}}{\theta(B_i) \cdot \theta(B_j)}$ belongs to $\langle \mathcal{C} \rangle$.

Proof. Since we have the convention $r_k = r_{k+N}$, the index of B_k have the convention $k = k + N$. By direct calculation, we obtain that

1. For $i = k$, $j = k + 1$, we have
$$\begin{aligned}&\frac{\{\theta(B_k), \theta(B_{k+1})\}}{\theta(B_k) \cdot \theta(B_{k+1})}\\ &= -1 + \frac{r_{k-1}r_{k+2} \cdot r_k r_{k+1}}{r_{k-1}r_{k+1} \cdot r_k r_{k+2}} + \frac{r_{k+2}r_{k+1} \cdot r_{k+3}r_k}{r_{k+2}r_k \cdot r_{k+3}r_{k+1}},\end{aligned} \tag{4.36}$$

2. For $i = k$, $j = k + 1$, we have
$$\begin{aligned}&\frac{\{\theta(B_{k+1}), \theta(B_k)\}}{\theta(B_{k+1}) \cdot \theta(B_k)}\\ &= 1 - \frac{r_{k-1}r_{k+2} \cdot r_k r_{k+1}}{r_{k-1}r_{k+1} \cdot r_k r_{k+2}} - \frac{r_{k+2}r_{k+1} \cdot r_{k+3}r_k}{r_{k+2}r_k \cdot r_{k+3}r_{k+1}},\end{aligned} \tag{4.37}$$

3. For $i = k$, $j = k + 2$, we have
$$\begin{aligned}\frac{\{\theta(B_k), \theta(B_{k+2})\}}{\theta(B_k) \cdot \theta(B_{k+2})} &= \left\{ \frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}}, \frac{r_{k+1}r_{k+3}}{r_{k+1}r_{k+2}} \frac{r_{k+4}r_{k+2}}{r_{k+4}r_{k+3}} \right\}\\ &= -\frac{r_{k+2}r_{k+3} \cdot r_{k+1}r_k}{r_{k+2}r_k \cdot r_{k+1}r_{k+3}}.\end{aligned} \tag{4.38}$$

4. For $i = k + 2$, $j = k$, we have
$$\begin{aligned}&\frac{\{\theta(B_{k+2}), \theta(B_k)\}}{\theta(B_{k+2}) \cdot \theta(B_k)}\\ &= \frac{r_{k+2}r_{k+3} \cdot r_{k+1}r_k}{r_{k+2}r_k \cdot r_{k+1}r_{k+3}}.\end{aligned} \tag{4.39}$$

5. For all the other cases, we have
$$\frac{\{\theta(B_i), \theta(B_j)\}}{\theta(B_i) \cdot \theta(B_j)} = 0.$$

□

Definition 4.3.12. *Let $N \geq 5$ be odd, let $\mathcal{D} = \mathcal{C} \cup \left\{ \frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}} \right\}_{k=1}^{N}$ with the convention $r_k = r_{k+N}$. Let $\mathbb{R}(B_1, ..., B_N)$ be a free fraction ring generated by $\{B_1, ..., B_N\}$. We define a map $\eta' : \mathcal{D} \to \mathbb{R}(B_1, ..., B_N)$ by:*

$$\eta' \left(\frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}} \right) = B_k, \tag{4.40}$$

$$\eta' \left(\frac{r_{k-1}r_{k+2} \cdot r_k r_{k+1}}{r_{k-1}r_{k+1} \cdot r_k r_{k+2}} \right) = 1 - \frac{1}{B_k}, \tag{4.41}$$

$$\eta' \left(\frac{r_{k+2}r_{k+1} \cdot r_{k+3}r_k}{r_{k+2}r_k \cdot r_{k+3}r_{k+1}} \right) = 1 - \frac{1}{B_{k+1}}, \tag{4.42}$$

$$\eta' \left(\frac{r_{k+2}r_{k+3} \cdot r_{k+1}r_k}{r_{k+2}r_k \cdot r_{k+1}r_{k+3}} \right) = \frac{1}{B_{k+1}}, \tag{4.43}$$

for $k = 1, ..., N$ with the convention $r_k = r_{k+N}$.

Proposition 4.3.13. *The map η' extends to a well defined ring homomorphism η from $\langle\mathcal{D}\rangle$ to $\mathbb{R}(B_1, ..., B_N)$.*

Proof. Since

$$[f_{k-1} : f_{k+2} : f_{k+1} : f_k] = B_k,$$

$$[f_{k-1} : f_k : f_{k+2} : f_{k+1}] = 1 - \frac{1}{B_k},$$

$$[f_{k+2} : f_{k+3} : f_{k+1} : f_k] = 1 - \frac{1}{B_{k+1}},$$

$$[f_{k+2} : f_{k+1} : f_{k+3} : f_k] = \frac{1}{B_{k+1}}.$$

Let P belongs to $\langle\mathcal{D}\rangle$, then a represent of P in $\mathcal{Q}(\mathcal{P})$ with a factor of $T_2(\mathcal{P})$ in the numerator of P maps to zero in $\mathbb{R}(B_1, ..., B_N)$. □

Proposition 4.3.14. *The ring homomorphism θ is injective.*

Proof. Of course, we have

$$\eta \circ \theta = I_{\mathbb{R}(B_1,...,B_N)}.$$

We conclude that θ is injective. □

Thus allows us to define a swapping bracket on $\mathbb{R}(B_1, ..., B_N)$.

Definition 4.3.15. *Let $N \geq 5$ be odd, we define the* swapping bracket $\{\cdot, \cdot\}_{B2}$ *on $\mathbb{R}(B_1, ..., B_N)$ by:*

$$\{P, Q\}_{B2} := \eta\left(\{\theta(P), \theta(Q)\}\right) \tag{4.44}$$

for any $P, Q \in \mathbb{R}(B_1, ..., B_N)$.

Proposition 4.3.16. *Let $N \geq 5$ be odd, with the convention $k + N = k$, for $k = 1, ..., N$, we have*

$$\{B_k, B_{k+1}\}_{B2} = \left(1 - \frac{1}{B_k} - \frac{1}{B_{k+1}}\right) B_k \cdot B_{k+1}, \tag{4.45}$$

$$\{B_k, B_{k-1}\}_{B2} = -\left(1 - \frac{1}{B_k} - \frac{1}{B_{k-1}}\right) B_k \cdot B_{k-1}, \tag{4.46}$$

$$\{B_k, B_{k+2}\}_{B2} = -\frac{1}{B_{k+1}} \cdot B_k \cdot B_{k+2}, \tag{4.47}$$

$$\{B_k, B_{k-2}\}_{B2} = \frac{1}{B_{k-1}} \cdot B_k \cdot B_{k-2}, \tag{4.48}$$

for the other cases

$$\{B_i, B_j\}_{B2} = 0. \tag{4.49}$$

Proof. The results follow from Proposition 4.3.11 and Definition 4.3.15.

□

Without the restriction for N to be odd, we define the swapping bracket on $\mathbb{R}(B_1, ..., B_N)$ as follows.

Definition 4.3.17. [Swapping bracket on $\mathbb{R}(B_1, ..., B_N)$] *Let $N \geq 5$, we define the* swapping bracket $\{\cdot, \cdot\}_{B2}$ *on $\mathbb{R}(B_1, ..., B_N)$ by extending the following formula on generators to $\mathbb{R}(B_1, ..., B_N)$:*

with the convention $k + N = k$, for $k = 1, ..., N$, we have

$$\{B_k, B_{k+1}\}_{B2} = \left(1 - \frac{1}{B_k} - \frac{1}{B_{k+1}}\right) B_k \cdot B_{k+1}, \tag{4.50}$$

$$\{B_k, B_{k-1}\}_{B2} = -\left(1 - \frac{1}{B_k} - \frac{1}{B_{k-1}}\right) B_k \cdot B_{k-1}, \tag{4.51}$$

$$\{B_k, B_{k+2}\}_{B2} = -\frac{1}{B_{k+1}} \cdot B_k \cdot B_{k+2}, \tag{4.52}$$

$$\{B_k, B_{k-2}\}_{B2} = \frac{1}{B_{k-1}} \cdot B_k \cdot B_{k-2}, \tag{4.53}$$

for the other cases

$$\{B_i, B_j\}_{B2} = 0. \tag{4.54}$$

Proposition 4.3.18. *For $N \geq 5$(odd or even), the swapping bracket $\{\cdot, \cdot\}_{B2}$ on $\mathbb{R}(B_1, ..., B_N)$ is a Poisson bracket.*

Proof. Let

$$J(B_i, B_j, B_k) := \{\{B_i, B_j\}_{B2}, B_k\}_{B2} + \{\{B_j, B_k\}_{B2}, B_i\}_{B2} + \{\{B_k, B_i\}_{B2}, B_j\}_{B2}. \tag{4.55}$$

By definition, we have to check that

$$J(B_i, B_j, B_k) = 0$$

for any $i, j, k = 1, ..., N$. Since

$$J(B_i, B_j, B_k) = -J(B_j, B_i, B_k),$$

when some indexes coincide, for example $i = j$, we have

$$J(B_i, B_i, B_k) = 0.$$

Let σ_s be the permutation of the N indexes such that $\sigma(l) = l + s$. The permutation σ_s induce a ring automorphism χ_s of $\mathbb{R}(B_1, ..., B_N)$ such that

$$\chi_s(B_l) = B_{l+s}$$

for $l = 1, ..., N$. Moreover, we have

$$\{\chi_s(B_i), \chi_s(B_j)\}_{B2} = \chi_s(\{B_i,\ B_j\}_{B2}).$$

Let τ be the permutation of the N indexes such that

$$\tau(l) = N + 1 - l$$

for $l = 1, ..., N$. The permutation τ induce a ring automorphism ν of $\mathbb{R}(B_1, ..., B_N)$ such that

$$\nu(B_l) = B_{N+1-l}.$$

Moreover, we have

$$\{\nu(B_i), \nu(B_j)\}_{B2} = -\nu(\{B_i,\ B_j\}_{B2}).$$

By the above symmetry, we suppose that

$$i = 1$$

and

$$1 < j < k \leq N.$$

Let

$$l := \min\{|j - i|,\ |j - i - N|,\ |k - j|,\ |k - j - N|,\ |i - k|,\ |i - k - N|\}.$$

We suppose that $l = |j - 1|$, we have to verify the following cases:

1. When $1 < j - 2 < k - 4 < N - 1$, we have
$$J(B_1, B_j, B_k) = 0. \tag{4.56}$$

2. When $j = 2$, $k = 3$, we have
$$\begin{aligned} &J(B_1, B_2, B_3) \\ &= \{\{B_1,\ B_2\}_{B2},\ B_3\}_{B2} + \{\{B_2,\ B_3\}_{B2},\ B_1\}_{B2} + \{\{B_3,\ B_1\}_{B2},\ B_2\}_{B2} \\ &= \{B_1B_2 - B_1 - B_2,\ B_3\}_{B2} + \{B_2B_3 - B_2 - B_3,\ B_1\}_{B2} + \left\{\frac{B_1B_3}{B_2},\ B_2\right\}_{B2} \\ &= \left(B_1 - 1 - \frac{B_1}{B_2}\right) \cdot \{B_2,\ B_3\}_{B2} + \left(B_3 - 1 - \frac{B_3}{B_2}\right) \cdot \{B_2,\ B_1\}_{B2} \\ &= \frac{\{B_2,\ B_1\}_{B2} \cdot \{B_2,\ B_3\}_{B2} + \{B_3,\ B_2\}_{B2} \cdot \{B_2,\ B_1\}_{B2}}{B_2} = 0. \end{aligned} \tag{4.57}$$

3. When $j = 2$, $k = 4$ and $N > 5$, we have
$$\begin{aligned} &J(B_1, B_2, B_4) \\ &= \{\{B_1,\ B_2\}_{B2},\ B_4\}_{B2} + \{\{B_2,\ B_4\}_{B2},\ B_1\}_{B2} + \{\{B_4,\ B_1\}_{B2},\ B_2\}_{B2} \\ &= \{B_1B_2 - B_1 - B_2,\ B_4\}_{B2} + \left\{-\frac{B_2B_4}{B_3},\ B_1\right\}_{B2} \\ &= (B_1 - 1) \cdot \{B_2,\ B_4\}_{B2} - \frac{B_4}{B_3} \cdot \{B_2,\ B_1\}_{B2} + \frac{B_2B_4}{B_3^2} \cdot \{B_3,\ B_1\}_{B2} \\ &= -(B_1 - 1) \cdot \frac{B_2B_4}{B_3} - \frac{B_4}{B_3} \cdot (B_1 + B_2 - B_1B_2) + \frac{B_2B_4}{B_3^2} \cdot \frac{B_3B_1}{B_2} \\ &= 0. \end{aligned} \tag{4.58}$$

4. When $j=2$, $k=4$ and $N=5$, we have

$$\begin{aligned}
&J(B_1,B_2,B_4)\\
&=\{\{B_1,\ B_2\}_{B2},\ B_4\}_{B2}+\{\{B_2,\ B_4\}_{B2},\ B_1\}_{B2}+\{\{B_4,\ B_1\}_{B2},\ B_2\}_{B2}\\
&=\{B_1B_2-B_1-B_2,\ B_4\}_{B2}+\left\{-\frac{B_2B_4}{B_3},\ B_1\right\}_{B2}+\left\{-\frac{B_4B_1}{B_5},\ B_2\right\}_{B2}\\
&=(B_1-1)\cdot\{B_2,\ B_4\}_{B2}+(B_2-1)\cdot\{B_1,\ B_4\}_{B2}-\frac{B_4}{B_3}\cdot\{B_2,\ B_1\}_{B2}+\frac{B_2B_4}{B_3^2}\cdot\{B_3,\ B_1\}_{B2}\\
&-\frac{B_2}{B_3}\cdot\{B_4,\ B_1\}_{B2}-\frac{B_4}{B_5}\cdot\{B_1,\ B_2\}_{B2}-\frac{B_1}{B_5}\cdot\{B_4,\ B_2\}_{B2}+\frac{B_4B_1}{B_5^2}\cdot\{B_5,\ B_2\}_{B2}\\
&=(B_2-1)\cdot\{B_1,\ B_4\}_{B2}-\frac{B_2}{B_3}\cdot\{B_4,\ B_1\}_{B2}-\frac{B_4}{B_5}\cdot\{B_1,\ B_2\}_{B2}-\frac{B_1}{B_5}\cdot\{B_4,\ B_2\}_{B2}\\
&+\frac{B_4B_1}{B_5^2}\cdot\{B_5,\ B_2\}_{B2}\\
&=(B_2-1)\cdot\frac{B_1B_4}{B_5}+\frac{B_2}{B_3}\cdot\frac{B_1B_4}{B_5}-\frac{B_4}{B_5}\cdot(B_1B_2-B_1-B_2)-\frac{B_1}{B_5}\cdot\frac{B_4B_2}{B_3}-\frac{B_4B_1}{B_5^2}\cdot\frac{B_5B_2}{B_1}\\
&=0
\end{aligned}\tag{4.59}$$

5. When $j=2$, $4<k<N-1$, we have

$$\begin{aligned}
&J(B_1,B_2,B_k)\\
&=\{\{B_1,\ B_2\}_{B2},\ B_k\}_{B2}+\{\{B_2,\ B_k\}_{B2},\ B_1\}_{B2}+\{\{B_k,\ B_1\}_{B2},\ B_2\}_{B2}\\
&=\{B_1B_2-B_1-B_2,\ B_k\}_{B2}\\
&=0
\end{aligned}\tag{4.60}$$

6. When $j=2$, $k=N$, by the cyclic permutation and the case 2, we have

$$J(B_1,B_2,B_N)=0\tag{4.61}$$

7. When $j=3$, $k=5$ and $N>6$, we have

$$\begin{aligned}
&J(B_1,B_3,B_5)\\
&=\{\{B_1,\ B_3\}_{B2},\ B_5\}_{B2}+\{\{B_3,\ B_5\}_{B2},\ B_1\}_{B2}+\{\{B_5,\ B_1\}_{B2},\ B_3\}_{B2}\\
&=\left\{-\frac{B_1B_3}{B_2},\ B_5\right\}_{B2}+\left\{-\frac{B_3B_5}{B_4},\ B_1\right\}_{B2}\\
&=\frac{B_1B_3B_5}{B_2B_4}-\frac{B_1B_3B_5}{B_2B_4}\\
&=0
\end{aligned}\tag{4.62}$$

8. When $j=3$, $k=5$ and $N=6$, we have

$$\begin{aligned}
&J(B_1,B_3,B_5)\\
&=\{\{B_1,\ B_3\}_{B2},\ B_5\}_{B2}+\{\{B_3,\ B_5\}_{B2},\ B_1\}_{B2}+\{\{B_5,\ B_1\}_{B2},\ B_3\}_{B2}\\
&=\left\{-\frac{B_1B_3}{B_2},\ B_5\right\}_{B2}+\left\{-\frac{B_3B_5}{B_4},\ B_1\right\}_{B2}+\left\{-\frac{B_5B_1}{B_6},\ B_3\right\}_{B2}\\
&=-\frac{B_3}{B_2}\cdot\{B_1,\ B_5\}_{B2}-\frac{B_1}{B_2}\cdot\{B_3,\ B_5\}_{B2}-\frac{B_5}{B_4}\cdot\{B_3,\ B_1\}_{B2}\\
&-\frac{B_3}{B_4}\cdot\{B_5,\ B_1\}_{B2}-\frac{B_1}{B_6}\cdot\{B_5,\ B_3\}_{B2}-\frac{B_5}{B_6}\cdot\{B_1,\ B_3\}_{B2}\\
&=\frac{B_1B_3B_5}{B_2B_6}+\frac{B_1B_3B_5}{B_2B_4}-\frac{B_1B_3B_5}{B_2B_4}+\frac{B_1B_3B_5}{B_4B_6}-\frac{B_1B_3B_5}{B_4B_6}-\frac{B_1B_3B_5}{B_2B_6}\\
&=0
\end{aligned}\tag{4.63}$$

9. When $j=3$, $5<k<N-1$, we have

$$\begin{aligned}
&J(B_1,B_3,B_k)\\
&=\{\{B_1,\ B_3\}_{B2},\ B_k\}_{B2}+\{\{B_3,\ B_k\}_{B2},\ B_1\}_{B2}+\{\{B_k,\ B_1\}_{B2},\ B_3\}_{B2}\\
&=\left\{-\frac{B_1B_3}{B_2},\ B_k\right\}_{B2}\\
&=0
\end{aligned}\tag{4.64}$$

10. When $j=3$, $k=N-1>5$, by cyclic permutation, we have

$$\chi_2(J(B_1,B_3,B_{N-1}))=J(B_3,B_5,B_1)=J(B_1,B_3,B_5).\tag{4.65}$$

By Case 7, we have

$$J(B_1,B_3,B_5)=0.\tag{4.66}$$

Thus we have

$$J(B_1,B_3,B_{N-1})=0.\tag{4.67}$$

We conclude that for any i,j,k, we have

$$J(B_i,B_j,B_k)=0$$

□

4.3.4 Main result

Definition 4.3.19. [(t_1,t_2,N)-Virasoro bracket] *Let $t_1,t_2\in\mathbb{R}$ and $N\in\mathbb{N}$, the* (t_1,t_2,N)-Virasoro bracket *on $\{I_k\}_{k=-N}^{N}$ is defined to be:*

For $p,q=-\left[\frac{N-1}{2}\right],...,\left[\frac{N}{2}\right]$,

1. *when* $p \neq -q$, *we have*
$$\{I_p, I_q\}_{N,t_1,t_2} = (p-q)\cdot I_{p+q}$$
with the convention $I_{k+N} = I_k$;

2. *when* $p = -q$, *we have*
$$\{I_p, I_{-p}\}_{N,t_1,t_2} = 2p\cdot I_0 + t_1\cdot p^3 + t_2\cdot p.$$

Remark 4.3.20. *Very specifical values of* t_1 *and* t_2 *correspond to Virasoro algebra. When* t_1 *is fixed,* t_2 *varies, they correspond to same cohomology class. Different* t_1 *corresponds to different cohomology class. Virasoro algebra generates all the possible central extension by Proposition 4.1.9.* (t_1, t_2, N)*-Virasoro bracket is asymptotic to the Poisson bracket associated to the 2-cocycle with* $c_1 = t_1$, $c_2 = t_2$ *in Proposition 4.1.9, but it is not a Poisson bracket.*

Definition 4.3.21. [Discrete Fourier transformation] *Let* $\{B_k\}_{k=1}^N$ *be the cross ratio coordinates of* $\mathcal{M}_{N,1}$. *Let* $\mathbb{B} = \{B_1, ..., B_N\}$. *The* discrete Fourier transformation $\mathcal{F}$ *of* $\mathbb{B}$ *is defined to be*
$$\mathcal{F}_p\mathbb{B} = \sum_{k=1}^{N} B_k e^{-\frac{2pk\pi i}{N}}. \tag{4.68}$$

Our main result of this chapter is

Theorem 4.3.22. *Let* $\{B_k\}_{k=1}^N$ *be the cross ratio coordinates of* $\mathcal{M}_{N,1}$ *such that*
$$\sum_{k=1}^{N} B_k = 0.$$
For $k = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$,
$$V_k = \frac{\mathcal{F}_k\mathbb{B}\cdot N}{8\pi i}.$$
We have
$$\{V_p, V_q\}_{B2} = \{V_p, V_q\}_{N,\frac{8\pi^2}{N},8N} + o\left(\frac{1}{N^3}\right). \tag{4.69}$$

Proof. For $p, q = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$, we have
$$\begin{aligned}\{\mathcal{F}_p\mathbb{B}, \mathcal{F}_q\mathbb{B}\}_{B2} = &\sum_{k=1}^{N}\left(e^{\frac{-2pk\pi i}{N}}\cdot e^{\frac{-2q(k+1)\pi i}{N}} - e^{\frac{-2p(k+1)\pi i}{N}}\cdot e^{\frac{-2qk\pi i}{N}}\right)\cdot(B_kB_{k+1} - B_k - B_{k+1})\\ &-\left(e^{\frac{-2pk\pi i}{N}}\cdot e^{\frac{-2q(k+2)\pi i}{N}} - e^{\frac{-2p(k+2)\pi i}{N}}\cdot e^{\frac{-2qk\pi i}{N}}\right)\cdot\frac{B_kB_{k+2}}{B_{k+1}}\end{aligned} \tag{4.70}$$
By
$$B_k = b_kb_{k+1} = 4 + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_kH_{k+1}}{N^4},$$

we have

$$B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right).$$

We have the above formula equals to

$$\begin{aligned}
&\sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} \cdot \left[\left(\left(1 + \frac{-2q\pi i}{N} + \frac{-2\pi^2 q^2}{N^2} + \frac{4\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right)\right) - \left(1 + \frac{-2p\pi i}{N} + \frac{-2\pi^2 \cdot p^2}{N^2} + \right.\right.\right. \\
&\left.\left.\left. \frac{4\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right)\right)\right) \cdot \left(8 + \frac{24H_k}{N^2} + o\left(\frac{1}{N^2}\right)\right) - \left(\left(1 + \frac{-4q\pi i}{N} + \frac{-8\pi^2 q^2}{N^2} + \frac{32\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right)\right)\right. \\
&\left. - \left(1 + \frac{-4p\pi i}{N} + \frac{-8\pi^2 p^2}{N^2} + \frac{32\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right)\right)\right) \cdot \left(4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right)\right)\Bigg] \\
&= \sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} \cdot \left[\frac{32H_k(p-q)\pi i}{N^3} - \frac{16\pi^2(p^2-q^2)}{N^2} + \frac{32\pi^3(p^3-q^3)i}{N^3} + o\left(\frac{1}{N^3}\right)\right].
\end{aligned} \tag{4.71}$$

When $p \neq -q$, since $\sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} = 0$, by $B_k = 4 + \frac{4H_k}{N^2} + o(\frac{1}{N^2})$, the above formula equals to

$$\sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} \cdot \frac{8B_k(p-q)\pi i}{N} = \frac{8(p-q)\pi i}{N} \cdot \mathcal{F}_{p+q}\mathbb{B} + o\left(\frac{1}{N^3}\right); \tag{4.72}$$

When $p = -q$, the above formula equals to

$$\begin{aligned}
&\frac{64p\pi i}{N^3}\sum_{k=1}^{N} H_k + \frac{64p^3\pi^3 i}{N^2} + o\left(\frac{1}{N^3}\right) \\
&= \frac{16p\pi i}{N}\sum_{k=1}^{N}(B_k - 4) + \frac{64p^3\pi^3 i}{N^2} + o\left(\frac{1}{N^3}\right) \\
&= -64p\pi i + \frac{64p^3\pi^3 i}{N^2} + o\left(\frac{1}{N^3}\right).
\end{aligned} \tag{4.73}$$

Since $\sum_{k=1}^{N} B_k = 0$, we have $V_0 = 0$.

We obtain that:

for $p \neq -q$,

$$\{V_p, V_q\}_{B2} = (p-q) \cdot V_{p+q} + o\left(\frac{1}{N^3}\right);$$

for $p = -q$

$$\{V_p, V_{-p}\}_{B2} = 2p \cdot V_0 + \left(\frac{8\pi^2}{N}\right) \cdot p^3 - 8N \cdot p + o\left(\frac{1}{N^3}\right).$$

We conclude that

$$\{V_p, V_q\}_{B2} = \{V_p, V_q\}_{N, \frac{8\pi^2}{N}, 8N} + o\left(\frac{1}{N^3}\right). \tag{4.74}$$

□

4.4 Schwartz algebra on $\mathbb{R}(B_1, ..., B_N)$ and its relation with swapping algebra

We plan to construct a Poisson structure on coordinates defined by weak cross ratios on $\mathcal{M}_{N,n}$ through the swapping algebra. Then compare this swapping Poisson structure with the Poisson structure considered by R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT] for $n = 2$ to show that they are compatible, where their continuous limit are the natural Lie-Poisson structure and the freezing structure [SOT]. This plan is to reply the conjecture mentioned in [SOT]. But we do not success in finding such a nice swapping Poisson structure. We have only result for $n = 1$ as shown in this section. More general case will be considered later on.

4.4.1 Schwartz algebra and its asymptotic phenomenon

The Schwartz algebra appears in [SOT] as a Poisson algebra on the cross ratio coordinate system of $\mathcal{M}_{N,2}$ which is a discrete version of the second Gelfand–Dickey Poisson structure. Here, we consider the case for $\mathbb{RP}^1$ where the bracket is referring to [FV]. We have the similar result as in Section 4.3.

Definition 4.4.1. [Schwartz bracket on $\mathbb{R}(B_1, ..., B_N)$] *The Schwartz bracket $\{\cdot,\cdot\}_1$ on $\mathbb{R}(B_1, ..., B_N)$ is defined by extending the following formula on generators to the whole ring:*

$$\{B_i, B_{i\pm1}\}_{S1} = \pm B_i \cdot B_{i\pm1} \tag{4.75}$$

For the other cases

$$\{B_i, B_j\}_{S1} = 0 \tag{4.76}$$

We have similar result as Theorem 4.3.22 for the Schwartz algebra on $\mathbb{R}(B_1, ..., B_N)$.

Theorem 4.4.2. *Let $\{B_k\}_{k=1}^N$ be the cross ratio coordinates of $\mathcal{M}_{N,1}$ such that*

$$\sum_{k=1}^{N} B_k = 0.$$

For $k = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$,

$$W_k = \frac{\mathcal{F}_k\mathbb{B} \cdot N}{16\pi i}.$$

We have

$$\{W_p, W_q\}_{S1} = \{W_p, W_q\}_{N,\frac{8\pi^2}{3N},4-8N} + o\left(\frac{1}{N^3}\right). \tag{4.77}$$

Proof. For $p, q = -\left[\frac{N-1}{2}\right], ..., \left[\frac{N}{2}\right]$, we have

$$\{\mathcal{F}_p\mathbb{B}, \mathcal{F}_q\mathbb{B}\}_{S1} = \sum_{k=1}^{N} \left(e^{\frac{-2pk\pi i}{N}} \cdot e^{\frac{-2q(k+1)\pi i}{N}} - e^{\frac{-2p(k+1)\pi i}{N}} \cdot e^{\frac{-2qk\pi i}{N}} \right) \cdot (B_k B_{k+1}) \tag{4.78}$$

By

$$B_k = a_k a_{k+1} = 4 + \frac{2\left(H_k + H_{k+1}\right)}{N^2} + \frac{H_k H_{k+1}}{N^4},$$

we have

$$B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right).$$

We have the above formula equals to

$$\begin{aligned}
&\sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} \cdot \left(\left(1 + \frac{-2q\pi i}{N} + \frac{-2\pi^2 q^2}{N^2} + \frac{4\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right)\right) - \left(1 + \frac{-2p\pi i}{N} + \frac{-2\pi^2 \cdot p^2}{N^2} + \right.\right. \\
&\left.\left. \frac{4\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right)\right)\right) \cdot \left(16 + \frac{32H_k}{N^2} + o\left(\frac{1}{N^2}\right)\right) \\
&= \sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} \cdot \left[\frac{32\pi i(p-q)}{N} + \frac{32\pi^2(p^2-q^2)}{N^2} - \frac{64\pi^3(p^3-q^3)i}{3N^3} + \frac{64H_k(p-q)\pi i}{N^3} + o\left(\frac{1}{N^3}\right)\right].
\end{aligned} \tag{4.79}$$

When $p \neq -q$, since $\sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} = 0$, by $B_k = 4 + \frac{4H_k}{N^2} + o(\frac{1}{N^2})$, the above formula equals to

$$\sum_{k=1}^{N} e^{\frac{-2(p+q)k\pi i}{N}} \cdot \frac{16B_k(p-q)\pi i}{N} = \frac{16(p-q)\pi i}{N} \cdot \mathcal{F}_{p+q}\mathbb{B} + o(\frac{1}{N^3}); \tag{4.80}$$

When $p = -q$, since $\sum_{k=1}^{N} B_k = 0$, the above formula equals to

$$\begin{aligned}
&\frac{64p\pi i}{N} + \frac{128p\pi i}{N^3}\sum_{k=1}^{N} H_k - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right) \\
&= \frac{64p\pi i}{N} + \frac{32p\pi i}{N}\sum_{k=1}^{N}(B_k - 4) - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right) \\
&= \frac{64p\pi i}{N} - 128p\pi i - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right)
\end{aligned} \tag{4.81}$$

and

$$W_0 = 0.$$

Thus we have:

1. for $p \neq -q$
$$\{W_p, W_q\}_{S1} = (p-q) \cdot W_{p+q} + o\left(\frac{1}{N^3}\right);$$
2. for $p = -q$
$$\{W_p, W_{-p}\}_{S1} = 2p \cdot W_0 + (4 - 8N) \cdot p - \frac{8\pi^2}{3N} \cdot p^3 + o\left(\frac{1}{N^3}\right).$$

We conclude that

$$\{W_p, W_q\}_{S1} = \{W_p, W_q\}_{N, \frac{8\pi^2}{3N}, 4-8N} + o\left(\frac{1}{N^3}\right). \tag{4.82}$$

□

4.4.2 Bihamiltonian system

Let us recall the traditional definition of the bihamiltonian system.

Definition 4.4.3. *Two Poisson structures $\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$ on a manifold M are said to be compatible if and only if for any λ, $\{\cdot,\cdot\}_a + \lambda\{\cdot,\cdot\}_b$ is Poisson on M.*
A dynamic system $\frac{d}{dt}m = \xi(m)$ over M is bihamiltonian if its vector field ξ is Hamiltonian with respect to these two Poisson structure $\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$.

Then we define the compatibility on a ring R with two Poisson structures.

Definition 4.4.4. *Two Poisson brackets $\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$ on a ring R are said to be compatible if and only if for any λ, $\{\cdot,\cdot\}_a + \lambda\{\cdot,\cdot\}_b$ is Poisson on R.*

Proposition 4.4.5. *$\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$ are compatible if and only if for any $x, y, z \in R$, we have*

$$\begin{aligned}&\{\{x,y\}_a,z\}_b + \{\{y,z\}_a,x\}_b + \{\{z,x\}_a,y\}_b + \\ &+ \{\{x,y\}_b,z\}_a + \{\{y,z\}_b,x\}_a + \{\{z,x\}_b,y\}_a = 0\end{aligned}$$

Proof. Let the bracket

$$\{\cdot,\ \cdot\}_{a\lambda b} := \{\cdot,\cdot\}_a + \lambda\{\cdot,\cdot\}_b.$$

The bracket $\{\cdot,\ \cdot\}_{a\lambda b}$ is Poisson if and only if $\{\cdot,\ \cdot\}_{a\lambda b}$ satisfies the Jacobi identity. For any $x, y, z \in R$, Let $\sum$ runs over the triplets (x,y,z), (y,z,x), (z,x,y), the Jacobi identity of $\{\cdot,\ \cdot\}_{a\lambda b}$ equals

$$\begin{aligned}&\sum \{\{x,\ y\}_{a\lambda b},\ z\}_{a\lambda b} \\ &= \sum \{\{x,\ y\}_a + \lambda\{x,\ y\}_b,\ z\}_{a\lambda b} \\ &= \sum \left(\{\{x,\ y\}_a,\ z\}_a + \lambda\left(\{x,\ y\}_b,\ z\}_a + \{x,\ y\}_a,\ z\}_b\right) + \lambda^2\{x,\ y\}_b,\ z\}_b\right) \\ &= \lambda \sum \left(\{x,\ y\}_b,\ z\}_a + \{x,\ y\}_a,\ z\}_b\right)\end{aligned} \tag{4.83}$$

The last equation uses the fact that $\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$ are Poisson. We conclude that $\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$ are compatible if and only if

$$\sum \left(\{x,\ y\}_b,\ z\}_a + \{x,\ y\}_a,\ z\}_b\right) = 0.$$

□

Our main result of this section is

Theorem 4.4.6. *For $N \geq 5$, $\{\cdot,\cdot\}_{B2}$ and $\{\cdot,\cdot\}_{S1}$ are compatible on $\mathcal{M}_{N,1}$.*

Remark 4.4.7. *For $N = 2$, we have also constructed a swapping algebra on $\mathcal{M}_{N,2}$, but we found that it is not compatible with the Schwartz algebra on $\mathcal{M}_{N,2}$.*

Proof. Let

$$K(B_i,B_j,B_k) := \{\{B_i,B_j\}_{B2},B_k\}_{S1}+\{\{B_j,B_k\}_{B2},B_i\}_{S1}+\{\{B_k,B_i\}_{B2},B_j\}_{S1} \\ +\{\{B_i,B_j\}_{S1},B_k\}_{B2}+\{\{B_j,B_k\}_{S1},B_i\}_{B2}+\{\{B_k,B_i\}_{S1},B_j\}_{B2}. \tag{4.84}$$

By definition, we have to check that

$$K(B_i,B_j,B_k)=0$$

for any $i,j,k=1,...,N$. Since

$$K(B_i,B_j,B_k)=-K(B_j,B_i,B_k),$$

when some indexes coincide, for example $i=j$, we have

$$K(B_i,B_i,B_k)=0.$$

As in the proof of Proposition 4.3.18, let σ_s be the permutation of the N indexes such that $\sigma(l)=l+s$. The permutation σ_s induce a ring automorphism χ_s of $\mathbb{R}(B_1,...,B_N)$ such that

$$\chi_s(B_l)=B_{l+s}$$

for $l=1,...,N$. Moreover, we have

$$\{\chi_s(B_i),\chi_s(B_j)\}_{S1}=\chi_s(\{B_i,\ B_j\}_{S1})$$

and

$$\{\chi_s(B_i),\chi_s(B_j)\}_{B2}=\chi_s(\{B_i,\ B_j\}_{B2}).$$

Let τ be the permutation of the N indexes such that

$$\tau(l)=N+1-l$$

for $l=1,...,N$. The permutation τ induce a ring automorphism ν of $\mathbb{R}(B_1,...,B_N)$ such that

$$\nu(B_l)=B_{N+1-l}.$$

Moreover, we have

$$\{\nu(B_i),\nu(B_j)\}_{S1}=-\nu(\{B_i,\ B_j\}_{S1}),$$
$$\{\nu(B_i),\nu(B_j)\}_{B2}=-\nu(\{B_i,\ B_j\}_{B2}).$$

By the above symmetry, we suppose that

$$i=1$$

and

$$1<j<k\leq N.$$

Let

$$l:=\min\{|j-i|,\ |j-i-N|,\ |k-j|,\ |k-j-N|,\ |i-k|,\ |i-k-N|\}.$$

We suppose that $l=|j-1|$, we have to verify the following cases:

1. When $1 < j-1 < k-2 < N-1$, we have

$$\begin{aligned} &K(B_1, B_i, B_k) \\ &= \{\{B_1, B_j\}_{B2}, B_k\}_{S1} + \{\{B_j, B_k\}_{B2}, B_1\}_{S1} + \{\{B_k, B_1\}_{B2}, B_j\}_{S1} \\ &+ \{\{B_1, B_j\}_{S1}, B_k\}_{B2} + \{\{B_j, B_k\}_{S1}, B_1\}_{B2} + \{\{B_k, B_1\}_{S1}, B_j\}_{B2} \\ &= \{\{B_1, B_j\}_{B2}, B_k\}_{S1} + \{\{B_j, B_k\}_{B2}, B_1\}_{S1} + \{\{B_k, B_1\}_{B2}, B_j\}_{S1}. \end{aligned} \tag{4.85}$$

   Since

$$\{B_1, B_j\}_{B2}$$

   is a polynomial of $B_1, ..., B_j$, we have

$$\{\{B_1, B_j\}_{B2}, B_k\}_{S1} = 0.$$

   Similarly, we have

$$\{\{B_j, B_k\}_{B2}, B_1\}_{S1} = 0$$

   and

$$\{\{B_k, B_1\}_{B2}, B_j\}_{S1} = 0.$$

   We conclude that

$$K(B_1, B_i, B_k) = 0.$$

2. When $j = 2$, $k = 3$, we have

$$\begin{aligned} &K(B_1, B_2, B_3) \\ &= \{\{B_1, B_2\}_{B2}, B_3\}_{S1} + \{\{B_2, B_3\}_{B2}, B_1\}_{S1} + \{\{B_3, B_1\}_{B2}, B_2\}_{S1} \\ &+ \{\{B_1, B_2\}_{S1}, B_3\}_{B2} + \{\{B_2, B_3\}_{S1}, B_1\}_{B2} + \{\{B_3, B_1\}_{S1}, B_2\}_{B2} \\ &= \{B_1B_2 - B_1 - B_2, B_3\}_{S1} + \{B_2B_3 - B_2 - B_3, B_1\}_{S1} + \{\frac{B_3B_1}{B_2}, B_2\}_{S1} \\ &+ \{B_1B_2, B_3\}_{B2} + \{B_2B_3, B_1\}_{B2} \\ &= (B_1 - 1)B_2B_3 - (B_3 - 1)B_1B_2 + B_1(B_2B_3 - B_2 - B_3) \\ &- \frac{B_1B_3}{B_2}B_2 - (B_1B_2 - B_1 - B_2)B_3 + \frac{B_3B_1}{B_2}B_2 \\ &= 0. \end{aligned} \tag{4.86}$$

3. When $j = 2$, $k = 4$ and $N > 5$, we have

$$\begin{aligned} &K(B_1, B_2, B_4) \\ &= \{\{B_1, B_2\}_{B2}, B_4\}_{S1} + \{\{B_2, B_4\}_{B2}, B_1\}_{S1} + \{\{B_4, B_1\}_{B2}, B_2\}_{S1} \\ &+ \{\{B_1, B_2\}_{S1}, B_4\}_{B2} + \{\{B_2, B_4\}_{S1}, B_1\}_{B2} + \{\{B_4, B_1\}_{S1}, B_2\}_{B2} \\ &= \{B_1B_2 - B_1 - B_2, B_4\}_{S1} + \{-\frac{B_2B_4}{B_3}, B_1\}_{S1} + \{B_1B_2, B_4\}_{B2} \\ &= \frac{B_1B_2B_4}{B_3} - \frac{B_1B_2B_4}{B_3} \\ &= 0. \end{aligned} \tag{4.87}$$

4. When $j = 2$, $k = 4$ and $N = 5$, we have

$$\begin{aligned}
&K(B_1, B_2, B_4) \\
&= \{\{B_1, B_2\}_{B2}, B_4\}_{S1} + \{\{B_2, B_4\}_{B2}, B_1\}_{S1} + \{\{B_4, B_1\}_{B2}, B_2\}_{S1} \\
&+ \{\{B_1, B_2\}_{S1}, B_4\}_{B2} + \{\{B_2, B_4\}_{S1}, B_1\}_{B2} + \{\{B_4, B_1\}_{S1}, B_2\}_{B2} \\
&= \{B_1B_2 - B_1 - B_2, B_4\}_{S1} + \{-\frac{B_2B_4}{B_3}, B_1\}_{S1} + \{-\frac{B_4B_1}{B_5}, B_2\}_{S1} \\
&+ \{B_1B_2, B_4\}_{B2} \\
&= \frac{B_1B_2B_4}{B_3} - \frac{B_1B_4B_2}{B_5} - \frac{B_1B_2B_4}{B_3} + \frac{B_1B_4B_2}{B_5} \\
&= 0.
\end{aligned} \tag{4.88}$$

□

4.4.3 Casimir elements

Definition 4.4.8. [Casimir element] *An element f of a ring R is called a* Casimir element *for the Poisson bracket on the ring R if f Poisson commutes with all other elements in R.*

Proposition 4.4.9. *Let $N \geq 5$. For $N = 2l$ even, $B_1B_2.....B_{2l}$ and $B_1B_3.....B_{2l-1}$ are two Casmir elements with respect to the Schwartz bracket on $\mathbb{R}(B_1, ..., B_N)$; for $N = 2l+1$ odd, $B_1B_2.....B_{2l+1}$ is a Casmir with respect to the Schwartz bracket on $\mathbb{R}(B_1, ..., B_N)$.*

Proof. It is sufficient to prove that the elements Poisson commute with the generators $B_1, ..., B_N$ of $\mathbb{R}(B_1, ..., B_N)$.

1. When $N \geq 5$, for any $i = 1, ..., N$ we have

$$\begin{aligned}
&\{B_i, B_1B_2.....B_N\}_1 \\
&= \frac{B_1B_2.....B_N}{B_{i-1}}\{B_i, B_{i-1}\}_1 + \frac{B_1B_2.....B_N}{B_{i+1}}\{B_i, B_{i+1}\}_1 \\
&= -B_i \cdot B_1B_2.....B_N + B_i \cdot B_1B_2.....B_N = 0
\end{aligned} \tag{4.89}$$

   With the convention that $k + N = k$. We conclude that $B_1B_2.....B_N$ is a Casmir element with respect to the Schwartz bracket on $\mathbb{R}(B_1, ..., B_N)$.

2. When $N \geq 5$ even, $N = 2l$, for any $i = 1, ..., l$ we have

$$\{B_{2i-1}, B_1B_3.....B_{2l-1}\}_1 = 0. \tag{4.90}$$

$$\begin{aligned}
&\{B_{2i}, B_1B_3.....B_{2l-1}\}_1 \\
&= \frac{B_1B_3.....B_{2l-1}}{B_{2i-1}}\{B_{2i}, B_{2i-1}\}_1 + \frac{B_1B_3.....B_{2l-1}}{B_{2i+1}}\{B_{2i}, B_{2i+1}\}_1 \\
&= -B_{2i} \cdot B_1B_3.....B_{2l-1} + B_{2i} \cdot B_1B_3.....B_{2l-1} = 0.
\end{aligned} \tag{4.91}$$

   With the convention that $k + N = k$. We conclude that $B_1B_3.....B_{2l-1}$ is a Casmir element with respect to the Schwartz bracket on $\mathbb{R}(B_1, ..., B_N)$.

□

Proposition 4.4.10. *Let $N \geq 5$. For $N = 2l$ even,*

$$\frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l-1}}$$

is a Casmir element with respect to the swapping bracket on $\mathbb{R}(B_1, ..., B_N)$; for $N = 2l+1$ odd,

$$\frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l+1}}$$

is a Casmir with respect to the swapping bracket on $\mathbb{R}(B_1, ..., B_N)$.

Proof. It is sufficient to prove that the elements Poisson commute with the generators $B_1, ..., B_N$ of $\mathbb{R}(B_1, ..., B_N)$.

1. When $N = 2l$ even, for any $i = 1, ..., l$ we have

$$\begin{aligned}
&\frac{\left\{B_{2i}, \frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l-1}}\right\}_{B2}}{B_{2i} \cdot \frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l-1}}} \\
&= \frac{\{B_{2i}, B_{2i-2}\}_{B2}}{B_{2i}B_{2i-2}} - \frac{\{B_{2i}, B_{2i-1}\}_{B2}}{B_{2i}B_{2i-1}} - \frac{\{B_{2i}, B_{2i+1}\}_{B2}}{B_{2i}B_{2i+1}} + \frac{\{B_{2i}, B_{2i+2}\}_{B2}}{B_{2i}B_{2i+2}} \qquad (4.92) \\
&= \frac{1}{B_{2i-1}} - (-1 + \frac{1}{B_{2i}} + \frac{1}{B_{2i-1}}) - (1 - \frac{1}{B_{2i}} - \frac{1}{B_{2i+1}}) + \left(-\frac{1}{B_{2i+1}}\right) \\
&= 0.
\end{aligned}$$

   With the convention that $k + N = k$. Thus we have

$$\left\{B_{2i}, \frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l-1}}\right\}_{B2} = 0. \qquad (4.93)$$

   Similarly, we have

$$\left\{B_{2i+1}, \frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l-1}}\right\}_{B2} = 0. \qquad (4.94)$$

   We conclude that

$$\frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l-1}}$$

   is a Casmir element with respect to the swapping bracket on $\mathbb{R}(B_1, ..., B_N)$.

2. When $N = 2l + 1$ odd, Similarly, we have

$$\frac{B_2B_4.....B_{2l}}{B_1B_3.....B_{2l+1}}$$

   is a Casimir element with respect to the swapping bracket on $\mathbb{R}(B_1, ..., B_N)$.

□

BIBLIOGRAPHY

[AB] Micheal. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A, 308(1505):523-615, 1983.

[CE] Claude Chevalley and Samuel Eilenberg, *Cohomology Theory Of Lie Groups And Lie Algebras* , Transactions of the American Mathematical Society, Vol. 63, No. 1 (1948), 85-124.

[CP] C. De Concini, C. Procesi, *A Characteristic Free Approach to Invariant Theory*, Advances in Mathematics 21, 330-354(1976).

[L1] François Labourie, *Anosov Flows, Surface Groups and Curves in Projective Space*, Inventiones Mathematicae 165 no. 1, 51–114 (2006).

[L2] François Labourie, *Cross Ratios and Identities for Higher Thurston Theory*, Duke Math Journal 148 n. 9, 279–345 (2009).

[L3] François Labourie, *Cross Ratios, Surface Groups, SL(n,R) and Diffeomorphisms of the Circle*, Publications de l'IHES, n. 106, 139-213 (2007).

[L4] François Labourie, *Goldman algebra, opers and the swapping algebra*, available in his homepage.

[FG1] Vladimir. V. Fock and Alexander. B. Goncharov, *Moduli spaces of local systems and higher Teichm uller theory*, Inst. Hautes Études Sci. Publ. Math. (2006), no. 103, 1-211.

[FG2] Vladimir. V. Fock and Alexander. B. Goncharov, *Moduli spaces of convex projective structures on surfaces*, Adv. Math (2004) Volume: 208, Issue: 1, Pages: 249-273.

[FG3] Vladimir. V. Fock and Alexander. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Annales scientifiques de l'Ecole normale supérieure (2009) Volume: 42, no.6, pages: 865-930.

[FV] L. Faddeev, A. Yu. Volkov, *Abelian Current Algebra and the Virasoro Algebra on the Lattice*, Physics Letters B, Volume 315, Issue 3-4, p. 311-318.

[G1] William. M. Goldman, *The Symplectic Nature of Fundamental Groups of Surfaces* , Adv. Math. 54 (1984), no. 2, 200-225.

[GSV] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. Volume 127, Number 2 (2005), 193-393.

[Gu1] Olivier Guichard, *Composantes de Hitchin et représentations hyperconvexes de groupes de surface*, J. Differential Geom. Volume 80, Number 3 (2008), 391-431.

[H1] Nigel. J. Hitchin, *Lie groups and Teichmüller space* , Topology 31 (1992), no. 3, 449-473.

[KS] B. Khsein, F. Soloviev, *Integrability of higher pentagram maps*, Mathematische Annalen November 2013, Volume 357, Issue 3, pp 1005-1047.

[KW] B. Khsein, R. Wendt, *The Geometry of Infinite-Dimensional Groups*, A Series of Modern Surveys in Mathematics, Vol. 51 (2009).

[OT] V. Ovsienko, S. Tabachnikov, *Projective differential geometry old and new: from Schwarzian derivative to cohomology of diffeomorphism groups* , Cambridge Univ. Press, 2005.

[P1] R. C. Penner, *The decorated Teichmüller space of punctured surfaces* , Comm. Math. Phys, Volume 113, Number 2 (1987), 299-339.

[GSV] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. Volume 127, Number 2 (2005), 193-393.

[W1] Hermann Weyl, *The Classical Groups: Their Invariants and Representations* , Princeton university press, (1939), 320 pages.

[Wi] Edward Witten, *Surprises with topological field theories* , 4th International Superstring Workshop, Texas A and M Univ., College Station, TX, USA, 12 - 17 Mar 1990, pp.50-61.

[S1] R. C. Graeme Segal, *The geometry of the kdv equation*, Internat. J. Modern Phys. A 6 (1991), no. 16, 2859-2869.

[SOT] Richard Schwartz, Valentin Ovsienko and Sergei Tabachnikov, *The Pentagram map: A discrete integrable system* , Communications in Mathematical Physics (2010), arxiv:0810.5605.