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Yuehong Feng

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Master en Chine en 2008

**Stabilités de solutions régulières pour des systèmes  
d'Euler-Maxwell et de Navier-Stokes-Maxwell  
compressibles**

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# Stabilités de solutions régulières pour des systèmes d'Euler-Maxwell et de Navier-Stokes-Maxwell compressibles

## RÉSUMÉ

Cette thèse est essentiellement composée de deux parties traitant des problèmes de Cauchy ou des problèmes périodiques. Dans la première partie, on étudie la stabilité de solutions régulières au voisinage d'états d'équilibre non constants pour un système d'Euler-Maxwell isentropique compressible bipolaire. Par des estimations d'énergie classiques et un argument de récurrence sur l'ordre des dérivées des solutions, on montre l'existence globale et l'unicité des solutions régulières du système lorsque les données initiales sont proches des états d'équilibre. On obtient aussi le comportement asymptotique des solutions quand le temps tend vers l'infini.

Dans la deuxième partie, on considère la stabilité en temps long des solutions régulières de systèmes d'Euler-Maxwell et de Navier-Stokes-Maxwell compressibles dans le cas non isentropique lorsque les états d'équilibre sont constants. Grâce à des choix convenables de symétriseurs des systèmes et à des estimations d'énergie, on montre l'existence globale et l'unicité des solutions régulières des systèmes avec données initiales petites. De plus, par le principe de Duhamel et l'outil d'analyse de Fourier, on obtient des taux de décroissance des solutions quand le temps tend vers l'infini.

**Mots clés :** Système d'Euler-Maxwell, système de Navier-Stokes-Maxwell, états d'équilibre stationnaires, existence globale de solutions régulières, comportement en temps long, taux de décroissance en temps, estimations d'énergie, argument de récurrence, choix de symétriseur, analyse de Fourier



# Stabilities of smooth solutions for compressible Euler-Maxwell and Navier-Stokes-Maxwell systems

## ABSTRACT

This thesis is essentially composed of two parts dealing with Cauchy problems and periodic problems. In the first part, we study the stability of smooth solutions near non constant equilibrium states for a two-fluid isentropic compressible Euler-Maxwell system. By classical energy estimates together with an induction argument on the order of the derivatives of solutions, we prove the existence and uniqueness of global solutions to the system when the given initial data are near the equilibrium states. We also obtain the asymptotic behavior of solutions when the time goes to infinity.

In the second part, we consider the long time stability of the global smooth solutions for compressible Euler-Maxwell and Navier-Stokes-Maxwell systems in non isentropic case when the equilibrium solutions are constants. With the help of suitable choices of symmetrizers and energy estimates, we prove the existence and uniqueness of global solutions to the systems with given small initial data. Furthermore, using the Duhamel principle and the Fourier analysis tool, we obtain the decay rates of smooth solutions as the time goes to infinity.

**Keywords :** Euler-Maxwell system, Navier-Stokes-Maxwell system, stationary equilibrium states, global existence of smooth solutions, long time behavior, time decay rates, energy estimates, induction argument, choice of symmetrizer, Fourier analysis





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# Chapitre 1

## Introduction

On considère les équations d'Euler et de Navier-Stokes compressibles couplées aux équations de Maxwell. Les équations obtenues sont appelées systèmes d'Euler-Maxwell et de Navier-Stokes-Maxwell, respectivement. On les trouve notamment dans la modélisation de plasmas ionisés, voir par exemple [8, 62]. On considère ces systèmes selon les cas isentropique ou non isentropique et unipolaire ou bipolaire. Dans le cas isentropique, les équations d'Euler ou de Navier-Stokes sont constituées des lois de conservation de la masse et de la quantité de mouvement. On y ajoute une équation de la conservation de l'énergie totale dans le cas non isentropique. Le système unipolaire correspond au cas habituel d'un fluide et le système bipolaire concerne deux fluides des particules, qui sont typiquement électrons et ions pour un plasma.

On étudie le problème de Cauchy dans l'espace tout entier  $\mathbb{R}^3$  ou le problème périodique dans un tore  $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$ . Plus précisément, on étudie la stabilité de solutions régulières autour d'états d'équilibre qui sont solutions stationnaires des systèmes. Il s'agit de problèmes d'existence globale et de comportement en temps long avec taux de décroissance des solutions. On note que si les états d'équilibre ne sont pas constants, l'existence globale de des solutions n'est obtenue que dans le cas périodique. Ces problèmes et résultats sont présentés dans les quatre sections suivantes.

### 1.1 Le système d'Euler-Maxwell isentropique

#### 1.1.1 Présentation générale

Le système d'Euler-Maxwell bipolaire décrit des plasmas magnétisés d'électrons et d'ions, représentés respectivement par les indices  $\nu = e$  et  $\nu = i$ . Dans le cas isentropique,

il est composé de l'équation de la conservation de la masse

$$(1.1) \quad \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \quad \nu = e, i,$$

et des équations de la conservation de la quantité de mouvement

$$(1.2) \quad m_\nu \partial_t (n^\nu u^\nu) + m_\nu \nabla \cdot (n^\nu u^\nu \otimes u^\nu) + \nabla p_\nu (n^\nu) = q_\nu n^\nu (E + \gamma u^\nu \times B) - \frac{m_\nu n^\nu u^\nu}{\tau_\nu}, \nu = e, i,$$

couplées avec les équations de Maxwell

$$(1.3) \quad \begin{cases} \gamma \lambda^2 \partial_t E - \nabla \times B = -\gamma (q_e n^e u^e + q_i n^i u^i), & \lambda^2 \nabla \cdot E = n^i - n^e + b(x), \\ \gamma \partial_t B + \nabla \times E = 0, & \nabla \cdot B = 0. \end{cases}$$

En vue des équations de Maxwell, le système (1.1)-(1.3) est valable en dimension 3, et donc pour la variable de temps  $t > 0$  et la variable d'espace  $x \in \mathbb{R}^3$ . Ici,  $n^e$  et  $u^e$  (respectivement,  $n^i$  et  $u^i$ ) sont la densité et la vitesse des électrons (respectivement, des ions),  $E$  et  $B$  sont le champ électrique et le champ magnétique. Les paramètres physiques sont

$$m_\nu > 0, \quad \lambda > 0, \quad \tau_\nu > 0, \quad q_e = -1, \quad q_i = 1.$$

Ils représentent la masse du particule  $\nu$ , la longueur de Debye, le temps de relaxation, la charge des électrons et la charge des ions, respectivement. On a aussi la relation  $\gamma = c^{-1}$  où  $c$  est la vitesse de la lumière. La fonction de pression  $p_\nu = p_\nu(n)$  est supposée régulière et strictement croissante pour  $n > 0$ . Dans ce modèle,  $b$  est une fonction donnée et on suppose toujours que  $b$  est suffisamment régulière et  $b \geq \text{const.} > 0$ .

On note que le dernier terme dans (1.2) est un terme de relaxation. Lorsque  $(n^\nu, u^\nu, E, B)$  est suffisamment régulière et  $n^\nu > 0$ , l'équation (1.2) est équivalente à

$$(1.4) \quad m_\nu \partial_t u^\nu + m_\nu (u^\nu \cdot \nabla) u^\nu + \nabla h_\nu (n^\nu) = q_\nu (E + \gamma u^\nu \times B) - \frac{m_\nu u^\nu}{\tau_\nu}, \quad \nu = e, i,$$

où  $h_\nu$  est la fonction d'enthalpie définie par

$$(1.5) \quad h'_\nu (n^\nu) = \frac{1}{n^\nu} p'_\nu (n^\nu).$$

En effet,

$$(1.6) \quad \nabla \cdot (n^\nu u^\nu \otimes u^\nu) = u^\nu \nabla \cdot (n^\nu u^\nu) + n^\nu (u^\nu \cdot \nabla) u^\nu,$$

on obtient donc (1.4) à l'aide de (1.1).

Le système (1.1)-(1.3) est un modèle bipolaire. Dans certains cas, on considère aussi le modèle simplifié où les ions sont immobiles ( $u^i = 0$ ) et de densité stationnaire ( $n^i = 0$ ).

On obtient alors un modèle unipolaire du système d'Euler-Maxwell pour les électrons. En notant

$$n = n^e, \quad u = u^e, \quad p = p_e, \quad m = m_e, \quad \tau = \tau_e \text{ etc.}$$

le modèle unipolaire s'écrit :

$$(1.7) \quad \begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ m\partial_t u + m(u \cdot \nabla)u + \nabla h(n) = -(E + \gamma u \times B) - \frac{mu}{\tau}, \\ \gamma\lambda^2\partial_t E - \nabla \times B = \gamma nu, \quad \lambda^2\nabla \cdot E = b - n, \\ \gamma\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \end{cases}$$

Pour un modèle simplifié de (1.7) en une dimension d'espace, Chen-Jerome-Wang [9] ont montré l'existence globale de solutions faibles par la méthode de compacité par compensation. Pour la solution régulière du modèle complet (1.7) en dimension 3, Peng-Wang ont établi une série de résultats sur des limites singulières lorsque des petits paramètres tendent vers zéro. Dans [57, 58], ils étudient la limite non relativiste  $\gamma \rightarrow 0$  (i.e.  $c \rightarrow \infty$ ) et la limite de quasi-neutralité  $\lambda \rightarrow 0$ , respectivement. Les limites de (1.7) sont le système d'Euler-Poisson compressible quand  $c \rightarrow \infty$  ou le système d'e-MHD quand  $\lambda \rightarrow 0$ . Une limite combinée  $\gamma = \lambda^2 \rightarrow 0$  et la limite de relaxation  $\tau \rightarrow 0$  sont considérées dans [60, 61]. Le système d'Euler-Maxwell bipolaire (1.1)-(1.3) est visiblement plus compliqué dû aux termes de couplage. Dans [60], Peng-Wang ont établi des limites formelles de (1.1)-(1.3) lorsque ces petits paramètres tendent vers zéro. Une justification a été faite par Yang-Wang [74] sur la limite non relativiste  $\gamma \rightarrow 0$  pour des solutions régulières.

Dans cette thèse, on s'intéresse à l'existence et au comportement en temps long des solutions régulières du système d'Euler-Maxwell unipolaire (1.7) ou bipolaire (1.1)-(1.3). Cette question est liée à la stabilité des solutions qui est étudiée dans un voisinage d'états d'équilibre stationnaires avec vitesse nulle. La taille du voisinage dépend en général des petits paramètres. Pour simplifier la présentation, les petits paramètres physiques sont égales à 1, de sorte que

$$m_\nu = \gamma = \lambda = \tau_\nu = 1, \quad \nu = e, i.$$



Ainsi, le système d'Euler-Maxwell isentropique bipolaire devient :

$$(1.8) \quad \left\{ \begin{array}{l} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla h_\nu(n^\nu) = q_\nu (E + u^\nu \times B) - u^\nu, \\ \partial_t E - \nabla \times B = n^e u^e - n^i u^i, \quad \nabla \cdot E = n^i - n^e + b(x), \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \end{array} \right.$$

Il est complété par une condition initiale :

$$(1.9) \quad (n^\nu, u^\nu, E, B)|_{t=0} = (n^{\nu 0}, u^{\nu 0}, E^0, B^0), \quad \nu = e, i, \quad \text{dans } \mathbb{R}^3, \quad \text{ou dans } \mathbb{T}.$$

Compte tenu des contraintes différentielles des équations de Maxwell, la donnée initiale doit vérifier la condition de compatibilité :

$$\nabla \cdot E^0 = n^{i0} - n^{e0} + b(x), \quad \nabla \cdot B^0 = 0.$$

De même, le système d'Euler-Maxwell isentropique unipolaire devient :

$$(1.10) \quad \left\{ \begin{array}{l} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \nabla h(n) = -(E + u \times B) - u, \\ \partial_t E - \nabla \times B = nu, \quad \nabla \cdot E = b - n, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \end{array} \right.$$

Il est complété par une condition initiale :

$$(1.11) \quad (n, u, E, B)|_{t=0} = (n^0, u^0, E^0, B^0), \quad \text{dans } \mathbb{R}^3, \quad \text{ou dans } \mathbb{T},$$

qui satisfait la condition de compatibilité :

$$\nabla \cdot E^0 = b - n^0, \quad \nabla \cdot B^0 = 0.$$

Pour le système unipolaire (1.10), les états d'équilibre stationnaires de vitesse nulle  $(\bar{n}, 0, \bar{E}, \bar{B})$  sont des solutions particulières du système. Ils vérifient

$$(1.12) \quad \left\{ \begin{array}{l} \nabla h(\bar{n}) = -\bar{E}, \\ \nabla \times \bar{B} = 0, \quad \nabla \cdot \bar{E} = b - \bar{n}, \\ \nabla \times \bar{E} = 0, \quad \nabla \cdot \bar{B} = 0. \end{array} \right.$$

Ceci implique que  $\bar{B}$  est une constante de  $\mathbb{R}^3$  et  $\bar{n}$  satisfait une équation elliptique semi-linéaire :

$$(1.13) \quad -\Delta h(\bar{n}) = b - \bar{n}.$$

Lorsque  $\bar{n}$  est résolue,  $\bar{E}$  est donné par

$$\bar{E} = -\nabla h(\bar{n}).$$

Par conséquent,  $(\bar{n}, 0, \bar{E})$  est une solution particulière d'un flot potentiel stationnaire pour des semi-conducteurs, voir [56].

On note  $\bar{\phi} = h(\bar{n})$  et  $h^{-1}$  la fonction réciproque de  $h$ . Alors, (1.13) est équivalent à une équation elliptique semi-linéaire monotone :

$$-\Delta \bar{\phi} = b - h^{-1}(\bar{\phi}).$$

Dans un domaine borné, l'existence de solutions régulières à cette équation peut être obtenue facilement par une méthode de minimisation ou par un théorème de point fixe du Schauder. L'unicité découle de la monotonie stricte de fonction  $h$ . Pour ces résultats, nous renvoyons à [30] avec une condition homogène de Neumann ou à [15] avec une condition de Dirichlet, respectivement. Comme  $b \geq \text{const.} > 0$ , les solutions satisfont  $\bar{n} \geq \text{const.} > 0$ . Dans le cas périodique dans  $\mathbb{T}$ , un tel résultat sur l'existence et l'unicité des solutions périodiques pour (1.13) est évidente.

**Proposition 1.1.** *Soit  $b$  une fonction régulière périodique telle que  $b \geq \text{const.} > 0$  dans  $\mathbb{T}$ . Alors le problème périodique (1.12) admet une unique solution régulière stationnaire  $(\bar{n}, \bar{E}, \bar{B})$  satisfaisant  $\bar{n} \geq \text{const.} > 0$  dans  $\mathbb{T}$ .*

Ainsi on obtient l'existence et l'unicité d'états d'équilibre périodiques du système (1.10). En particulier, si  $b = 1$ , on a  $\bar{n} = 1$  et  $\bar{E} = 0$ . Donc, l'unique état d'équilibre du système (1.10) est donnée par  $(\bar{n}, \bar{u}, \bar{E}, \bar{B}) = (1, 0, 0, \bar{B})$ .

De même, pour le système d'Euler-Maxwell isentropique bipolaire (1.8), l'état d'équilibre stationnaire de vitesse nulle  $(\bar{n}^e, \bar{n}^i, 0, 0, \bar{E}, \bar{B})$  vérifie

$$(1.14) \quad \begin{cases} -\nabla h_e(\bar{n}^e) = \bar{E} = \nabla h_i(\bar{n}^i), \\ \nabla \times \bar{B} = 0, \quad \nabla \cdot \bar{E} = \bar{n}^i - \bar{n}^e + b(x), \\ \nabla \times \bar{E} = 0, \quad \nabla \cdot \bar{B} = 0. \end{cases}$$

Ceci implique que  $\bar{B}$  est encore une constante. De plus, si on note  $\bar{\phi} = h_e(\bar{n}^e)$ , alors

$$-\nabla h_i(\bar{n}^i) = \nabla \bar{\phi},$$

de sorte que

$$\bar{n}^e = h_e^{-1}(\bar{\phi}), \quad \bar{n}^i = h_i^{-1}(C_1 - \bar{\phi}),$$

où  $h_\nu^{-1}$  est la fonction réciproque de  $h_\nu$  et  $C_1$  est une constante quelconque. Par la contrainte différentielle sur  $\bar{E}$ , on obtient l'équation satisfaite par  $\bar{\phi}$  :

$$(1.15) \quad \Delta \bar{\phi} = h_e^{-1}(\bar{\phi}) - h_i^{-1}(-\bar{\phi} + C_1) - b(x).$$

Puisque  $f : \phi \mapsto h_e^{-1}(\phi) - h_i^{-1}(C_1 - \phi)$  est également une fonction strictement croissante, un résultat similaire à Proposition 1.1 est encore valable. On a donc l'existence et l'unicité d'une solution régulière périodique pour le problème stationnaire (1.14).

Pour établir  $\bar{n}^i > 0$  et  $\bar{n}^e > 0$ , en notant  $b_2 \geq b_1 > 0$  être deux constantes de sorte que  $b_1 \leq b(x) \leq b_2$  dans  $\mathbb{T}$ . Par le principe du maximum, le solution  $\bar{\phi}$  de (1.15) satisfait

$$f^{-1}(b_1) \leq \bar{\phi} \leq f^{-1}(b_2).$$

Alors,

$$\bar{n}^e = h_e^{-1}(\bar{\phi}) \geq h_e^{-1}(f^{-1}(b_1)), \quad \bar{n}^i = h_i^{-1}(C_1 - \bar{\phi}) \geq h_i^{-1}(C_1 - f^{-1}(b_2)).$$

Donc,  $\bar{n}^i > 0$  et  $\bar{n}^e > 0$  sont vrai si

$$h_e^{-1}(f^{-1}(b_1)) > 0, \quad h_i^{-1}(C_1 - f^{-1}(b_2)) > 0,$$

ou équivalente

$$b_1 > -h_i^{-1}(C_1 - h_e(0)), \quad b_2 < h_e^{-1}(C_1 - h_i(0)).$$

Il est facile de voir que ces conditions sont toujours satisfaites si  $C_1$  est assez grande. En particulier, si  $b = 1$ ,  $(n^e, n^i, u^e, u^i, E, B) = (2, 1, 0, 0, 0, \bar{B})$  est un état d'équilibre constant du système.

### 1.1.2 Résultats existants

Les deux systèmes d'Euler-Maxwell (1.8) et (1.10) sont hyperboliques symétrisables au sens de Friedrichs et l'existence locale en temps de solutions régulières est bien connue grâce à Kato [41]. Par ailleurs, les systèmes sont partiellement dissipatifs dûs aux termes de relaxation. Cependant, ils ne vérifient pas la condition de stabilité de Shizuta-Kawashima [64]. Donc les résultats sur l'existence globale dans [31, 76] et sur le comportement en temps long des solutions dans [4, 2] ne s'appliquent pas à ces deux systèmes.

D'autre part, il est aussi connu que la condition de stabilité de Shizuta-Kawashima n'est pas toujours nécessaire pour l'existence globale comme des exemples l'ont montré dans [77, 7, 61]. La structure des systèmes d'Euler-Maxwell joue donc un rôle important

dans notre étude. Lorsque les données initiales sont proches des états d'équilibre constants, l'existence globale de solutions régulières est obtenue pour le problème de Cauchy dans  $\mathbb{R}^3$  ou pour le problème périodique dans  $\mathbb{T}$ . Ces résultats sont démontrés d'abord pour le système unipolaire puis pour le système bipolaire.

**Théorème 1.1.** (*Peng-Wang-Gu, 2011 [61]*) Soient  $s \geq 3$  un entier,  $b = 1$  et  $\bar{B} \in \mathbb{R}^3$  un vecteur donné. Si  $\|(n^0 - 1, u^0, E^0, B^0 - \bar{B})\|_{H^s(\mathbb{T})}$  est suffisamment petit, le problème périodique du système d'Euler-Maxwell unipolaire (1.10) avec (1.11) admet une solution globale unique  $(n, u, E, B)$  satisfaisant

$$(n - 1, u, E, B - \bar{B}) \in C^1(\mathbb{R}^+, H^{s-1}(\mathbb{T})) \cap C(\mathbb{R}^+, H^s(\mathbb{T})).$$

**Théorème 1.2.** (*Ueda-Kawashima, 2011 [67]*) Soient  $s \geq 6$  un entier et  $b = 1$ . Si  $\|(n^0 - 1, u^0, E^0, B^0)\|_{H^s(\mathbb{R}^3)}$  est suffisamment petit, le problème de Cauchy du système (1.10) avec (1.11) admet une solution globale unique  $(n, u, E, B)$  satisfaisant, pour  $t > 0$  suffisamment grand

$$\|(n - 1, u, E, B)(t)\|_{H^{s-2k}(\mathbb{R}^3)} \leq C \|(n^0 - 1, u^0, E^0, B^0)\|_{H^s(\mathbb{R}^3)} (1+t)^{-k/2}, \quad \forall 0 \leq k \leq [s/2],$$

où  $C > 0$  est une constante indépendante du temps.

**Théorème 1.3.** (*Peng, 2012 [55]*) Soient  $s \geq 3$  un entier et  $\bar{B} \in \mathbb{R}^3$  un vecteur donné. Pour  $\Omega = \mathbb{R}^3$  ou  $\Omega = \mathbb{T}$ , si  $\|(n^{\nu 0} - 1, u^{\nu 0}, E^0, B^0 - \bar{B})\|_{H^s(\Omega)}$  est suffisamment petit pour  $\nu = e, i$ , le problème de Cauchy ou le problème périodique du système bipolaire (1.8) avec (1.9) admet une solution globale unique  $(n^\nu, u^\nu, E, B)$ ,  $\nu = e, i$ . Cette solution satisfait

$$(n^\nu - 1, u^\nu, E, B) \in C^1(\mathbb{R}^+, H^{s-1}(\Omega)) \cap C(\mathbb{R}^+, H^s(\Omega)), \quad \nu = e, i,$$

$$\lim_{t \rightarrow +\infty} \|(n^e(t) - n^i(t), u^\nu(t), E(t))\|_{H^{s-1}(\Omega)} = 0, \quad \nu = e, i.$$

De plus, pour  $\nu = e, i$ ,

$$\lim_{t \rightarrow +\infty} \|(n^\nu(t) - 1, B(t) - \bar{B})\|_{W^{s-2,6}(\mathbb{R}^3)} = 0, \quad \text{si } \Omega = \mathbb{R}^3,$$

et si  $\Omega = \mathbb{T}$  et

$$\int_{\mathbb{T}} n^{\nu 0}(x) dx = 1, \quad \int_{\mathbb{T}} B^0(x) dx = \bar{B}, \quad \nu = e, i,$$

alors

$$\lim_{t \rightarrow +\infty} \|(n^\nu(t) - 1, B(t) - \bar{B})\|_{H^{s-1}(\mathbb{T})} = 0.$$

Les états d'équilibre stationnaires du système unipolaire (1.10) sont déterminés par Proposition 1.1. Lorsqu'ils ne sont pas constants, la stabilité de solutions a été étudiée récemment par Peng [56]. L'existence globale de solutions régulières périodiques au voisinage des états d'équilibre est obtenue par l'utilisation des estimations fines de l'énergie et d'un argument de récurrence sur l'ordre des dérivées des solutions par rapport à  $t$  et  $x$ .

**Théorème 1.4.** (Peng, 2013 [56]) Soient  $s \geq 3$  un entier et  $\bar{B} \in \mathbb{R}^3$  un vecteur donné. Si  $\|(n^0 - \bar{n}, u^0, E^0 - \bar{E}, B^0 - \bar{B})\|_{H^s(\mathbb{T})}$  est suffisamment petit, le problème périodique du système d'Euler-Maxwell isentropique unipolaire (1.10) avec (1.11) admet une solution globale unique  $(n, u, E, B)$  satisfaisant

$$(n - \bar{n}, u, E - \bar{E}, B - \bar{B}) \in \bigcap_{k=0}^s C^k(\mathbb{R}^+; H^{s-k}(\mathbb{T})).$$

De plus, si

$$\int_{\mathbb{T}} B^0(x) dx = \bar{B},$$

alors

$$\lim_{t \rightarrow +\infty} \|(n(t) - \bar{n}, u(t), E(t) - \bar{E}, B(t) - \bar{B})\|_{H^{s-1}(\mathbb{T})} = 0,$$

où  $\|\cdot\|_s$  est une norme plus forte que  $\|\cdot\|_{H^s(\mathbb{T})}$ , qui sera définie dans le chapitre suivant.

### 1.1.3 Résultats obtenus

Dans le chapitre 2, on étudie la stabilité de solutions du problème périodique du système d'Euler-Maxwell isentropique bipolaire (1.8). Les états d'équilibre stationnaires sont définis dans (1.14) et sont déterminés grâce à Proposition 1.1. On établit l'existence globale de solutions régulières pour des données initiales proches des états d'équilibre.

**Théorème 1.5.** Soient  $s \geq 3$  et  $\bar{B} \in \mathbb{R}^3$  un vecteur donné. Alors il existe une constante  $\delta_0 > 0$  telle que si

$$\|(n^{\nu 0} - \bar{n}^{\nu}, u^{\nu 0}, E^0 - \bar{E}, B^0 - \bar{B})\|_{H^s(\mathbb{T})} \leq \delta_0, \quad \nu = e, i,$$

le problème périodique du système bipolaire (1.8) avec (1.9) admet une solution globale unique

$$(n^{\nu} - \bar{n}^{\nu}, u^{\nu}, E - \bar{E}, B - \bar{B}) \in \bigcap_{k=0}^s C^k(\mathbb{R}^+, H^{s-k}(\mathbb{T})), \quad \nu = e, i.$$

De plus, si

$$\int_{\mathbb{T}} n^{\nu 0}(x) dx = \int_{\mathbb{T}} \bar{n}^{\nu}(x) dx, \quad \int_{\mathbb{T}} B^0(x) dx = \bar{B}, \quad \nu = e, i,$$

alors on a

$$(1.16) \quad \lim_{t \rightarrow +\infty} \|(n^\nu(t) - \bar{n}^\nu, u^\nu, E(t) - \bar{E}, B(t) - \bar{B})\|_{s-1} = 0, \quad \nu = e, i,$$

La preuve du Théorème 1.5 repose sur des estimations d'énergie et sur un argument de récurrence sur l'ordre des dérivés des solutions par rapport à  $t$  et  $x$ . Elle suit principalement des techniques dans Peng [55] pour le système bipolaire et dans Peng [56] traitant la stabilité de solutions au voisinage d'un état d'équilibre non constant du système unipolaire. On souligne que la stabilité dans le cas d'équilibre non constant est beaucoup plus compliquée que dans le cas d'équilibre constant. Dans le cas d'équilibre constant des équations d'Euler-Maxwell isentropique bipolaire (1.8), des estimations d'énergie classiques dans l'espace de Sobolev usuel  $H^s(\mathbb{R}^3)$  suffisent pour obtenir l'existence globale. Cela n'est plus possible dans le cas d'équilibre non constant. Par exemple, si  $\bar{n}^\nu$  dépend de  $x$ , alors  $\partial_x n^\nu \neq \partial_x(n^\nu - \bar{n}^\nu)$  et  $\partial_x n^\nu$  n'est pas un terme petit d'ordre 1, ce qui crée des difficultés dans des estimations d'énergie d'ordre supérieur. Par ailleurs, à cause d'interaction des variables, le système bipolaire pose aussi des problèmes dans des estimations par rapport au cas du système unipolaire. Pour surmonter ces difficultés, des nouvelles techniques sont nécessaires (voir Proposition 2.8).

## 1.2 Les systèmes d'Euler-Maxwell et de Navier-Stokes-Maxwell non isentropiques

### 1.2.1 Présentation générale

Le système d'Euler-Maxwell non isentropique est une version généralisée du système isentropique incluant les conservations des énergies totales. Dans le cas bipolaire pour les fluides des électrons et des ions, il s'écrit :

$$(1.17) \quad \begin{cases} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t (n^\nu u^\nu) + \nabla \cdot (n^\nu u^\nu \otimes u^\nu) + \nabla p_\nu = q_\nu n^\nu (E + u^\nu \times B) - n^\nu u^\nu, \\ \partial_t \mathcal{E}_\nu + \nabla \cdot (\mathcal{E}_\nu u^\nu + p_\nu u^\nu) = q_\nu n^\nu u^\nu E - n^\nu |u^\nu|^2 - (\mathcal{E}_\nu - e_* n^\nu), \\ \partial_t E - \nabla \times B = n^e u^e - n^i u^i, \quad \nabla \cdot E = n^i - n^e, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \end{cases}$$

où les inconnues sont, pour  $\nu = e, i$ ,  $(n^\nu, u^\nu, E, B)$ , la température absolue  $\theta^\nu > 0$ , l'énergie interne  $e^\nu$  définie par

$$e^\nu = \frac{3}{2} K_B \theta^\nu,$$

et l'énergie totale  $\mathcal{E}_\nu$  donnée par

$$\mathcal{E}_\nu = n^\nu \left( \frac{1}{2} |u^\nu|^2 + C_\nu e^\nu \right).$$

La fonction de pression est définie par

$$p_\nu = R_\nu n^\nu e^\nu, \quad \nu = e, i.$$

L'énergie interne de référence est  $e_* = \frac{3}{2} K_B \theta_*$  où  $\theta_* > 0$  est la température de référence. Ici, les constantes  $K_B > 0$ ,  $C_\nu > 0$  et  $R_\nu > 0$  sont la constante de Boltzmann, la capacité thermique à volume constant et le coefficient de conductivité thermique, respectivement. Comme dans la section précédente, on prend

$$C_\nu = K_B = 1, \quad R_\nu = \frac{2}{3}, \quad \nu = e, i.$$

Lorsque  $n^\nu > 0$  pour  $\nu = e, i$ , le système (1.17) devient :

$$(1.18) \quad \left\{ \begin{array}{l} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \frac{1}{n^\nu} \nabla (n^\nu \theta^\nu) = q_\nu (E + u^\nu \times B) - u^\nu, \\ \partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu + \frac{2}{3} \theta^\nu \nabla \cdot u^\nu + \frac{1}{3} |u^\nu|^2 + (\theta^\nu - \theta_*) = 0, \\ \partial_t E - \nabla \times B = n^e u^e - n^i u^i, \quad \nabla \cdot E = n^i - n^e, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \end{array} \right.$$

On considère le problème de Cauchy du système (1.18) dans  $\mathbb{R}^3$  associé à une condition initiale :

$$(1.19) \quad (n^\nu, u^\nu, \theta^\nu, E, B)|_{t=0} = (n^{\nu 0}, u^{\nu 0}, \theta^{\nu 0}, E^0, B^0), \quad \nu = e, i, \quad \text{dans } \mathbb{R}^3,$$

qui satisfait la condition de compatibilité :

$$(1.20) \quad \nabla \cdot E^0 = n^{i0} - n^{e0}, \quad \nabla \cdot B^0 = 0.$$

Les états d'équilibre considérés sont toujours des constants qui sont de la forme :

$$(n^e, n^i, u^e, u^i, \theta^e, \theta^i, E, B) = (1, 1, 0, 0, \theta_*, \theta_*, 0, 0).$$

Parallèlement, on considère aussi le système d'Euler-Maxwell unipolaire non isentro-

rique. En notant  $n = n^e$ ,  $u = u^e$  et  $\theta = \theta^e$  etc., il s'écrit :

$$(1.21) \quad \left\{ \begin{array}{l} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \frac{1}{n} \nabla (n\theta) = -(E + u \times B) - u, \\ \partial_t \theta + u \cdot \nabla \theta + \frac{2}{3} \theta \nabla \cdot u + \frac{1}{3} |u|^2 + (\theta - \theta_*) = 0, \\ \partial_t E - \nabla \times B = nu, \quad \nabla \cdot E = 1 - n, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \end{array} \right.$$

Le système de Navier-Stokes-Maxwell compressible non isentropique est obtenu en remplaçant le terme de relaxation  $-nu$  par un terme de viscosité  $\Delta u$  dans le système d'Euler-Maxwell non isentropique unipolaire (1.21). Il est de la forme :

$$(1.22) \quad \left\{ \begin{array}{l} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \frac{1}{n} \nabla (n\theta) = -(E + u \times B) + \frac{1}{n} \Delta u, \\ \partial_t \theta + \frac{2}{3} \theta \nabla \cdot u + u \cdot \nabla \theta = -\frac{1}{3} |u|^2 - (\theta - \theta_*), \\ \partial_t E - \nabla \times B = nu, \quad \nabla \cdot E = 1 - n, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \end{array} \right.$$

On considère le problème de Cauchy pour (1.21) et pour (1.22) dans  $\mathbb{R}^3$  associé à des conditions initiales :

$$(1.23) \quad (n, u, \theta, E, B)_{t=0} = (n^0, u^0, \theta^0, E^0, B^0), \quad \text{dans } \mathbb{R}^3,$$

qui satisfont la condition de compatibilité :

$$(1.24) \quad \nabla \cdot E^0 = 1 - n^0, \quad \nabla \cdot B^0 = 0.$$

Il est facile de voir que les systèmes (1.21) et (1.22) possèdent les mêmes états d'équilibre constants :

$$(n, u, \theta, E, B) = (1, 0, \theta_*, 0, 0).$$

Si le changement de température n'est pas pris en compte dans le système (1.22), on obtient le système de Navier-Stokes-Maxwell compressible isentropique :

$$(1.25) \quad \left\{ \begin{array}{l} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \frac{1}{n} \nabla p(n) = -(E + u \times B) + \frac{1}{n} \Delta u, \\ \partial_t E - \nabla \times B = nu, \quad \nabla \cdot E = 1 - n, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \end{array} \right.$$



avec la condition initiale :

$$(1.26) \quad (n, u, E, B)|_{t=0} = (n^0, u^0, E^0, B^0), \quad \text{dans } \mathbb{R}^3,$$

qui satisfait la condition de compatibilité :

$$(1.27) \quad \nabla \cdot E^0 = 1 - n^0, \quad \nabla \cdot B^0 = 0.$$

Il est facile aussi de voir que le système (1.25) possède des états d'équilibre constants :

$$(n, u, E, B) = (1, 0, 0, 0).$$

Dans [18], Duan a considéré le problème de Cauchy du système (1.25) dans  $\mathbb{R}^3$ . Grâce à l'utilisation de la fonction de Green, il a obtenu l'existence globale et le comportement en temps long des solutions régulières au voisinage d'états d'équilibre constants  $(1, 0, 0, 0)$ .

**Théorème 1.6.** *(Duan, 2012 [18]) Soit  $s \geq 4$  un entier. Si  $\|(n^0 - 1, u^0, E^0, B^0)\|_{H^s(\mathbb{R}^3)}$  est suffisamment petit, le problème de Cauchy (1.25)-(1.26) admet une solution globale unique*

$$(n - 1, u, E, B) \in C(\mathbb{R}^+; H^s(\mathbb{R}^3)), \quad (n - 1, \nabla u) \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^3)),$$

$$\nabla E \in L^2(\mathbb{R}^+; H^{s-2}(\mathbb{R}^3)), \quad \nabla B \in L^2(\mathbb{R}^+; H^{s-3}(\mathbb{R}^3)).$$

De plus, si  $\|(n^0 - 1, u^0, E^0, B^0)\|_{H^{s+2}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)}$  est suffisamment petit, pour tout  $t > 0$  suffisamment grand, on a

$$\|n(t) - 1\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-1}, \quad \|u(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{5}{8}},$$

$$\|E(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}} \ln(3+t), \quad \|B(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{8}}.$$

### 1.2.2 Résultats obtenus

Dans le chapitre 3, on monte l'existence globale et le comportement en temps long de solutions régulières du problème de Cauchy du système unipolaire (1.21) lorsque les données initiales proches des états d'équilibre  $(1, 0, \theta_*, 0, 0)$  avec  $\theta_* = 1$ . Le résultat obtenu est annoncé comme suit et la preuve dépend fortement du choix des symétriseurs et de l'outil d'analyse de Fourier.

**Théorème 1.7.** *Soit  $s \geq 4$  un entier. Si  $\|(n^0 - 1, u^0, \theta^0 - 1, E^0, B^0)\|_{H^s(\mathbb{R}^3)}$  est suffisamment petit, le problème de Cauchy (1.21) et (1.23) admet une solution globale unique  $(n, u, \theta, E, B)$  qui satisfait*

$$(n - 1, u, \theta - 1, E, B) \in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{R}^3)).$$

De plus, si  $\|(n^0 - 1, u^0, \theta^0 - 1, E^0, B^0)\|_{H^{13}(\mathbb{R}^3)}$  et  $\|(u^0, E^0, B^0)\|_{L^1(\mathbb{R}^3)}$  sont suffisamment petits, pour tout  $2 \leq q \leq \infty$  et tout  $t > 0$  suffisamment grand, on a

$$\|(n(t) - 1, \theta(t) - 1)\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{11}{4}},$$

$$\|(u(t), E(t))\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-2+\frac{3}{2q}},$$

$$\|B(t)\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}+\frac{3}{2q}},$$

où  $C > 0$  est une constante indépendante du temps.

Ensuite, dans le chapitre 4, on étend ce résultat au système bipolaire (1.18). La preuve du résultat est basée sur des estimations d'énergie classiques et des combinaisons des variables d'inconnues du système.

**Théorème 1.8.** *Soit  $s \geq 4$  un entier. Alors il existe des constantes  $\delta_0 > 0$  et  $\delta_1 > 0$  telles que si  $\|(n^{\nu 0} - 1, u^{\nu 0}, \theta^{\nu 0} - 1, E^0, B^0)\|_{H^s(\mathbb{R}^3)} \leq \delta_0$  pour  $\nu = e, i$ , le problème de Cauchy (1.18)-(1.19) admet une solution globale unique  $(n^\nu, u^\nu, \theta^\nu, E, B)$  qui satisfait*

$$(n^\nu - 1, u^\nu, \theta^\nu - 1, E, B) \in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{R}^3)), \quad \nu = e, i.$$

De plus, si

$$\|(n^{\nu 0} - 1, u^{\nu 0}, \theta^{\nu 0} - 1, E^0, B^0)\|_{L^1(\mathbb{R}^3) \cap H^{13}(\mathbb{R}^3)} \leq \delta_1, \quad \nu = e, i$$

alors pour tout  $2 \leq q \leq \infty$  et tout  $t > 0$  suffisamment grand, on a

$$(1.28) \quad \|(n^e(t) - n^i(t), \theta^e(t) - \theta^i(t))\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-2-\frac{1}{q}},$$

$$(1.29) \quad \|(n^e(t) + n^i(t) - 2, \theta^e(t) + \theta^i(t) - 2)\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}+\frac{3}{2q}},$$

$$(1.30) \quad \|(u^e(t) \pm u^i(t), E(t))\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}+\frac{1}{2q}},$$

$$(1.31) \quad \|B(t)\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}+\frac{3}{2q}},$$

où  $C > 0$  est une constante indépendante du temps.

Enfin, dans le chapitre 5, on obtient le résultat suivant pour le système de Navier-Stokes-Maxwell (1.22).

**Théorème 1.9.** Soient  $s \geq 4$  un entier,  $\theta_* > 0$  une constante donnée et  $\bar{B} \in \mathbb{R}^3$  un vecteur donné. Alors il existe une constante  $\delta_0 > 0$  telle que si

$$\left\| (n^0 - 1, u^0, \theta^0 - \theta_*, E^0, B^0 - \bar{B}) \right\|_{H^s(\mathbb{R}^3)} \leq \delta_0,$$

le problème de Cauchy (1.22)-(1.23) admet une solution globale unique  $(n, u, \theta, E, B)$  qui satisfait

$$(1.32) \quad \begin{aligned} u &\in C^1(\mathbb{R}^+; H^{s-2}(\mathbb{R}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{R}^3)), \\ (n - 1, \theta - \theta_*, E, B - \bar{B}) &\in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{R}^3)). \end{aligned}$$

De plus, on a

$$(1.33) \quad \lim_{t \rightarrow +\infty} \|(n - 1, \theta - \theta_*)(t)\|_{H^{s-1}(\mathbb{R}^3)} = 0, \quad \lim_{t \rightarrow +\infty} \|\nabla u(t)\|_{H^{s-3}(\mathbb{R}^3)} = 0,$$

$$(1.34) \quad \lim_{t \rightarrow +\infty} \|\nabla E(t)\|_{H^{s-2}(\mathbb{R}^3)} = 0,$$

et

$$(1.35) \quad \lim_{t \rightarrow +\infty} \|\nabla^2 B(t)\|_{H^{s-4}(\mathbb{R}^3)} = 0.$$

Le système (1.22) ne diffère du système d'Euler-Maxwell que par la présence du terme de viscosité  $\Delta u$  au lieu du terme de relaxation  $-nu$ . La preuve du théorème 1.9 suit donc des techniques des preuves des théorèmes 1.6-1.7. Néanmoins, il faut traiter le terme de viscosité  $\Delta u$  pour l'estimation de  $u$ . Ceci est présenté dans Lemme 5.2.

## Chapitre 2

# Stability of non constant equilibrium solutions for two-fluid Euler-Maxwell systems

### 2.1 Introduction and main results

In this chapter, we consider the periodic problem for the two-fluid Euler-Maxwell system :

$$(2.1) \quad \begin{cases} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t (n^\nu u^\nu) + \nabla \cdot (n^\nu u^\nu \otimes u^\nu) + \nabla p_\nu (n^\nu) = q_\nu n^\nu (E + u^\nu \times B) - n^\nu u^\nu, \\ \partial_t E - \nabla \times B = - (q_e n^e u^e + q_i n^i u^i), \quad \nabla \cdot E = n^i - n^e + b(x), \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \nu = e, i, \quad \text{in } \mathbb{R}^+ \times \mathbb{T}. \end{cases}$$

For smooth solutions with  $n^\nu > 0$ , the second equation of (2.1) is equivalent to

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla h_\nu (n^\nu) = q_\nu (E + u^\nu \times B) - u^\nu,$$

where  $h_\nu$  is the enthalpy function defined by  $n^\nu h'_\nu (n^\nu) = p'_\nu (n^\nu)$ . Since  $p_\nu$  is strictly increasing on  $\mathbb{R}^+$ , so is  $h_\nu$ . Then system (2.1) becomes :

$$(2.2) \quad \begin{cases} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla h_\nu (n^\nu) = q_\nu (E + u^\nu \times B) - u^\nu, \\ \partial_t E - \nabla \times B = n^e u^e - n^i u^i, \quad \nabla \cdot E = n^i - n^e + b(x), \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{T}. \end{cases}$$

Initial values are given as :

$$(2.3) \quad (n^\nu, u^\nu, E, B)|_{t=0} = (n^{\nu 0}, u^{\nu 0}, E^0, B^0), \quad \text{in } \mathbb{T},$$

with the compatibility condition :

$$(2.4) \quad \nabla \cdot E^0 = n^{i0} - n^{e0} + b(x), \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{T}.$$

Now let  $(\bar{n}^\nu(x), \bar{u}^\nu(x), \bar{E}(x), \bar{B}(x))$  be an equilibrium solution of (2.2) with  $\bar{u}^\nu = 0$ . Then we have

$$(2.5) \quad \begin{cases} -\nabla h_e(\bar{n}^e) = \bar{E} = \nabla h_i(\bar{n}^i), \\ \nabla \times \bar{B} = 0, \quad \nabla \cdot \bar{E} = \bar{n}^i - \bar{n}^e + b(x), \\ \nabla \times \bar{E} = 0, \quad \nabla \cdot \bar{B} = 0, \quad \text{in } \mathbb{T}, \end{cases}$$

which implies that  $\bar{B}$  is a constant. Moreover, if we note  $\bar{\phi} = h_e(\bar{n}^e)$ , then

$$-\nabla h_i(\bar{n}^i) = \nabla \bar{\phi}.$$

It follows that

$$\bar{n}^e = h_e^{-1}(\bar{\phi}), \quad \bar{n}^i = h_i^{-1}(C_1 - \bar{\phi}),$$

where  $h_\nu^{-1}$  is the inverse function of  $h_\nu$  and  $C_1$  is a real constant. By the differential constraint of  $\bar{E}$ , namely,

$$\nabla \cdot \bar{E} = \bar{n}^i - \bar{n}^e + b(x),$$

we obtain the equation of  $\bar{\phi}$  :

$$(2.6) \quad \Delta \bar{\phi} = h_e^{-1}(\bar{\phi}) - h_i^{-1}(C_1 - \bar{\phi}) - b(x).$$

It is easy to see that the function  $f : \phi \mapsto h_e^{-1}(\phi) - h_i^{-1}(C_1 - \phi)$  is strictly increasing on  $(0, +\infty)$ . Therefore, by a similar argument to that of Proposition 1.1 in [56], we obtain the existence and uniqueness of a smooth periodic solution  $\bar{\phi}$  for (2.6). In order to establish  $\bar{n}^i > 0$  and  $\bar{n}^e > 0$ , let  $b_2 \geq b_1 > 0$  be two constants such that  $b_1 \leq b(x) \leq b_2$  on  $\mathbb{T}$ . By the maximum principle, the solution  $\bar{\phi}$  of (2.6) satisfies

$$f^{-1}(b_1) \leq \bar{\phi} \leq f^{-1}(b_2).$$

Hence,

$$\bar{n}^e = h_e^{-1}(\bar{\phi}) \geq h_e^{-1}(f^{-1}(b_1)), \quad \bar{n}^i = h_i^{-1}(C_1 - \bar{\phi}) \geq h_i^{-1}(C_1 - f^{-1}(b_2)).$$

Thus,  $\bar{n}^i > 0$  and  $\bar{n}^e > 0$  hold true if

$$h_e^{-1}(f^{-1}(b_1)) > 0, \quad h_i^{-1}(C_1 - f^{-1}(b_2)) > 0,$$

or equivalently,

$$b_1 > -h_i^{-1}(C_1 - h_e(0)), \quad b_2 < h_e^{-1}(C_1 - h_i(0)).$$

It is easy to see that these conditions are always satisfied if  $C_1$  is large enough.

The two-fluid Euler-Maxwell system (2.2) is nonlinear and symmetrizable hyperbolic for  $n^\nu > 0$  in the sense of Friedrichs. Therefore, it follows from the result of Kato [41] that the periodic problem (2.2)-(2.3) has a unique local smooth solution when the initial data are smooth.

**Proposition 2.1.** *(Local existence of smooth solutions for two-fluid Euler-Maxwell systems, see [41, 46]) Assume integer  $s \geq 3$  and (2.4) holds. Let  $\bar{B} \in \mathbb{R}^3$  be any given constant and  $(\bar{n}^\nu, 0, \bar{E}, \bar{B})$  be an equilibrium solution of (2.5) satisfying  $\bar{n}^\nu \geq \text{const.} > 0$ . Suppose  $(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0}, E^0 - \bar{E}, B^0 - \bar{B}) \in H^s(\mathbb{T})$  with  $n^{\nu 0} \geq 2\kappa$  for some given constant  $\kappa > 0$ . Then there exists  $T > 0$  such that problem (2.2)-(2.3) has a unique smooth solution satisfying  $n^\nu \geq \kappa$  in  $[0, T] \times \mathbb{T}$  and*

$$(n^\nu - \bar{n}^\nu, u^\nu, E - \bar{E}, B - \bar{B}) \in C^1([0, T]; H^{s-1}(\mathbb{T})) \cap C([0, T]; H^s(\mathbb{T})), \quad \nu = e, i.$$

Due to the velocity damping terms in the momentum equation, system (2.2) is partially dissipative. It is known that this system doesn't fulfill the so-called Kawashima condition (see [61]), which is a usual assumption in the study of global smooth solutions [31, 76, 4, 2].

There are mathematical investigations in the asymptotic limits with small parameters, numerical computation, the existence of solutions for the one-fluid Euler-Maxwell system. By using the compensated compactness argument, Chen-Jerome-Wang [9] proved the global existence of weak solutions in one space dimension. Peng-Wang [58] justified the convergence of the system to the incompressible Euler system for well-prepared smooth initial data. The existence of global smooth solutions with small amplitude to the Cauchy problem in  $\mathbb{R}^3$  and to the periodic problem in a torus is established by Peng-Wang-Gu [61], Ueda-Wang-Kawashima [68] and Xu [73]. The decay rate of the smooth solution when the time goes to infinity is obtained by Duan [17], Ueda-Kawashima [67], and Feng-Wang-Kawashima [24], independently. For the two-fluid Euler-Maxwell system, the global existence and the asymptotic behavior of the smooth solution are also obtained by Duan-Liu-Zhu [19], Peng [55] and Wang-Feng-Li [71]. In the case without velocity damping, an

initial assumption was made in [27] to yield such a global existence result. For numerical analysis, see [13] and references therein.

All these results hold when the solution is near a constant equilibrium state of the Euler-Maxwell system. Recently, with the help of an induction argument on the order of the derivatives of solutions, Peng [56] studies the stability of non constant equilibrium solutions for the one-fluid Euler-Maxwell system. However, there is no result on the stability of non constant equilibrium solutions for the two-fluid Euler-Maxwell system so far. The goal of this chapter is to consider this problem.

Now we state the main results of this chapter.

**Theorem 2.1.** *(Stability of smooth solutions for two-fluid Euler-Maxwell systems) Let the assumptions of Proposition 2.1 hold and*

$$(2.7) \quad \int_{\mathbb{T}} n^{\nu 0}(x) dx = \int_{\mathbb{T}} \bar{n}^{\nu}(x) dx, \quad \int_{\mathbb{T}} B^0(x) dx = \bar{B}, \quad \nu = e, i.$$

Then there exist constants  $\delta_0 > 0$  and  $C > 0$ , independent of any given time  $t > 0$ , such that if

$$\|(n^{\nu 0} - \bar{n}^{\nu}, u^{\nu 0}, E^0 - \bar{E}, B^0 - \bar{B})\|_s \leq \delta_0, \quad \nu = e, i,$$

where  $\|\cdot\|_s$  is the norm of  $H^s(\mathbb{T})$ , problem (2.2)-(2.3) admits a unique global smooth solution satisfying

$$(2.8) \quad (n^{\nu} - \bar{n}^{\nu}, u^{\nu}, E - \bar{E}, B - \bar{B}) \in \bigcap_{k=0}^s C^k(\mathbb{R}^+, H^{s-k}(\mathbb{T})),$$

$$(2.9) \quad \sup_{t \geq 0} \|(n^{\nu}(t) - \bar{n}^{\nu}, u^{\nu}(t), E(t) - \bar{E}, B(t) - \bar{B})\|_s \leq C \|(n^{\nu 0} - \bar{n}^{\nu}, u^{\nu 0}, E^0 - \bar{E}, B^0 - \bar{B})\|_s,$$

and

$$(2.10) \quad \int_0^{+\infty} \left( \sum_{\nu=e,i} \|(n^{\nu} - \bar{n}^{\nu}, u^{\nu})\|_s^2 + \|E - \bar{E}\|_{s-1}^2 + \|(\partial_t B, \nabla \times B)\|_{s-2}^2 \right) (t) dt \leq C \sum_{\nu=e,i} \|(n^{\nu 0} - \bar{n}^{\nu}, u^{\nu 0}, E^0 - \bar{E}, B^0 - \bar{B})\|_s^2.$$

Moreover, we have

$$(2.11) \quad \lim_{t \rightarrow \infty} \|(n^{\nu}(t) - \bar{n}^{\nu}, u^{\nu}(t))\|_{s-1} = 0,$$

$$(2.12) \quad \lim_{t \rightarrow \infty} \|E(t) - \bar{E}\|_{s-1} = 0,$$

and

$$(2.13) \quad \lim_{t \rightarrow \infty} \|B(t) - \bar{B}\|_{s-1} = 0,$$

where  $\|\cdot\|_s$  is defined in the next section.

Remark that condition (2.7) allows us to apply the Poincaré inequality, since  $n^\nu - \bar{n}^\nu$  and  $B - \bar{B}$  are two conservative variables.

In comparison with Euler-Maxwell systems, the Euler-Poisson system is another important class of equations arising from plasma physics ([1, 12, 29, 30]). Usually, it takes the form :

$$(2.14) \quad \begin{cases} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t (n^\nu u^\nu) + \nabla \cdot (n^\nu u^\nu \otimes u^\nu) + \nabla p_\nu(n^\nu) = -q_\nu n^\nu \nabla \phi - n^\nu u^\nu, \\ -\Delta \phi = n^i - n^e + b(x), \quad \nu = e, i, \quad \text{in } \mathbb{R}^+ \times \mathbb{T}. \end{cases}$$

It can be regarded formally as a particular case of (2.1) with  $E = -\nabla \phi$  and  $B = 0$ . Here,  $\phi$  is the electrostatic potential.

Then, system (2.14) becomes :

$$(2.15) \quad \begin{cases} \partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla h_\nu(n^\nu) = -q_\nu \nabla \phi - u^\nu, \\ -\Delta \phi = n^i - n^e + b(x), \quad \text{in } \mathbb{R}^+ \times \mathbb{T}. \end{cases}$$

Initial values are also given as :

$$(2.16) \quad (n^\nu, u^\nu)|_{t=0} = (n^{\nu 0}, u^{\nu 0}), \quad x \in \mathbb{T}.$$

Define  $\phi^0$  by

$$(2.17) \quad -\Delta \phi^0 = n^{i0} - n^{e0} + b(x), \quad x \in \mathbb{T}.$$

The equilibrium solution  $(\bar{n}^\nu, \bar{u}^\nu, \bar{\phi})$  with  $\bar{u}^\nu = 0$  is still given by (2.6) and  $\bar{\phi} = h_e(\bar{n}^e)$ . In order that  $\phi$  is uniquely determined, we add a restriction condition :

$$(2.18) \quad \int_{\mathbb{T}} \phi(\cdot, x) dx = \int_{\mathbb{T}} \bar{\phi}(x) dx.$$

It follows from (2.6) and (2.15) that

$$\Delta(\phi - \bar{\phi}) = (n^e - \bar{n}^e) - (n^i - \bar{n}^i).$$

Then for all integer  $s^* \geq 0$ , Lemma 2.1 together with (2.18) gives estimate

$$(2.19) \quad \|\nabla(\phi - \bar{\phi})\|_{s^*} \leq C \|(n^e - \bar{n}^e, n^i - \bar{n}^i)\|_{s^*}.$$

Then, regarding  $\nabla \phi$  as a function of  $n^\nu$ ,  $(n^\nu, u^\nu)$  still satisfies a symmetrizable hyperbolic system in which  $\nabla \phi$  appearing on the right-hand side of (2.15) is a low order



term of  $n^\nu$ . Following results of Kato [41], this implies that the periodic problem to (2.15) admits a unique local smooth solution, provided that the initial data  $(n^{\nu 0}, u^{\nu 0})$  are smooth. Moreover, estimate (2.19) implies that  $\phi \in C([0, T], H^{s^*+1}(\mathbb{T}^3))$  as soon as  $n^\nu \in C([0, T], H^{s^*}(\mathbb{T}^3))$  for some constant  $T > 0$  and integer  $s^* \geq 0$ .

The stability results for problem (2.15)-(2.16) is stated as follows.

**Theorem 2.2.** *(Stability of smooth solutions for two-fluid Euler-Poisson systems) Let  $s \geq 3$  be an integer,  $(n^{\nu 0}, u^{\nu 0}) \in H^s(\mathbb{T})$  and (2.7) holds for  $n^{\nu 0}$ . Then there exists a constant  $\delta_0 > 0$  small enough such that if  $\|(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0})\|_s \leq \delta_0$ , the periodic problem to the two-fluid Euler-Poisson system (2.15)-(2.16) has a unique global smooth solution  $(n^\nu, u^\nu, \phi)$  satisfying*

$$(2.20) \quad (n^\nu - \bar{n}^\nu, u^\nu, \nabla(\phi - \bar{\phi})) \in \bigcap_{k=0}^s C^k(\mathbb{R}^+, H^{s-k}(\mathbb{T})).$$

Furthermore, for all  $t > 0$ , we have

$$(2.21) \quad \sum_{\nu=e,i} \|\|(n^\nu - \bar{n}^\nu, u^\nu)(t)\|\|_s + \|\|\nabla(\phi(t) - \bar{\phi})\|\|_s \leq C e^{-\gamma t} \sum_{\nu=e,i} \|(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0})\|_s,$$

where positive constants  $C, \gamma$  are independent of time  $t$ .

We complete the proof of Theorems 2.1 and 2.2 by using the careful energy estimates and an induction argument on the order of the derivatives of solutions with respect to  $t$  and  $x$ . This argument, used by Peng [56] in the one-fluid case and then extended to the two-fluid case, can overcome the difficulty due to the appearance of non constant equilibrium solutions. It should be pointed out that the non constant equilibrium case is much more complex than the constant equilibrium case. For example, in the constant equilibrium case of the two-fluid Euler-Maxwell equations (2.15), energy estimates in usual Sobolev space  $H^s$  were used in [55, 19, 71] to get the global existence results. But this is no longer valid for the non constant equilibrium case, due to its complexity. In fact, the difficulty is that  $\partial_x \bar{n}^\nu \neq 0$ , therefore  $\partial_x n^\nu$  is not of order 1, and the remainders in the higher order energy estimates and the time dissipation estimates will be order two in stead of order three. This will not be sufficient to get the global existence of smooth solutions near a non constant equilibrium solution. Therefore, we introduce the standard technique of symmetrizer and the induction method to establish energy estimates. In fact, we separate the roles of each variable. The dissipation of  $\partial_t^k u^\nu$  in  $L^2$  is straightforward for all  $0 \leq k \leq s$ . Moreover, for  $s' \in \mathbb{N}$  with  $k + s' \leq s$  and  $s' \geq 1$ , the dissipation of  $\|\partial_t^k u^\nu\|_{s'}$  depends on that of  $\|\partial_t^k (n^\nu - \bar{n}^\nu)\|_{s'}$  and  $\|\partial_t^k u^\nu\|_{s'-1}$ , while the dissipation of  $\|\partial_t^k (n^\nu - \bar{n}^\nu)\|_{s'}$

depends only on that of  $\|\partial_t^k(n^\nu - \bar{n}^\nu, u^\nu)\|_{s'-1}$  and  $\|\partial_t^{k+1}u^\nu\|_{s'-1}$  (see Propositions 2.5-2.6). This implies a recurrence relation which allows us to get the energy estimates by induction for  $k$  and  $s$ .

Moreover, different from the one-fluid case in [56], we also need to overcome the difficulties caused by the coupling of two fluids when we establish the energy estimates. This can be done by introducing different multiplier functions. Indeed, we first introduce a new potential function  $\Psi$ , and then another function  $\eta_k^\nu = h_\nu'(\bar{n}^\nu)\partial_t^k(n^\nu - \bar{n}^\nu) + q_\nu\partial_t^k\Psi$  to achieve the estimate of  $|n^\nu - \bar{n}^\nu|$  indirectly (see Proposition 2.8).

Finally, we remark that the velocity relaxation term of the two-fluid Euler-Maxwell system (2.2) and the two-fluid Euler-Poisson system (2.15) plays a key role in the proof of Theorem 2.1 and Theorem 2.2. Our method also permits a steady state with non zero sufficiently small velocity  $u^{\nu 0} \neq 0$ .

The plan of this chapter is as follows. In Section 2, we recall some useful inequalities in Sobolev space and Banach space. Detailed energy estimates are established in Section 3. At last, in Section 4, we complete the proof of Theorem 2.1 and Theorem 2.2 by combing the estimates above.

## 2.2 Preliminaries

For later use in this thesis, let us introduce some notations. The expression  $f \sim g$  means  $\gamma g \leq f \leq \frac{1}{\gamma}g$  for a constant  $0 < \gamma < 1$ . We denote by  $\|\cdot\|_s$  the norm of the usual Sobolev space  $H^s(\mathbb{T})$ , and by  $\|\cdot\|$  and  $\|\cdot\|_\infty$  the norms of  $L^2(\mathbb{T})$  and  $L^\infty(\mathbb{T})$ , respectively. We also denote by  $\langle \cdot, \cdot \rangle$  the inner product over  $L^2(\mathbb{T})$ . For a multi-index  $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ , we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

For  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  and  $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$ ,  $\beta \leq \alpha$  stands for  $\beta_j \leq \alpha_j$  for  $j = 1, 2, 3$ , and  $\beta < \alpha$  stands for  $\beta \leq \alpha$  and  $\beta \neq \alpha$ . We also denote by  $\partial^m$  any derivative with respect to  $t$  or  $x$  of order  $m \in \mathbb{N}$  and  $\partial^1 = \partial$ . For  $T > 0$  and  $s \geq 1$ , we define the Banach space

$$B_{s,T}(\mathbb{T}) = \bigcap_{k=0}^s C^k([0, T], H^{s-k}(\mathbb{T})),$$

with the norm

$$|||f|||_s = \sqrt{\sum_{k+|\alpha|\leq s} \|\partial_t^k \partial^\alpha f\|^2}, \quad \forall f \in B_{s,T}(\mathbb{T}).$$

Obviously,  $\|\cdot\|_s \leq |||\cdot|||_s$ .

## The Leibniz formulas

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} C_\alpha^\beta \partial^{\alpha-\beta} f \partial^\beta g, \quad \forall \alpha \in \mathbb{N}^3, \quad \partial_t^k(fg) = \sum_{l \leq k} C_k^l \partial_t^{k-l} f \partial_t^l g, \quad \forall k \in \mathbb{N},$$

where  $C_\alpha^\beta > 0$  for  $\beta \leq \alpha$  are constants.

**Lemma 2.1.** (*Poincaré inequality, see [21].*) *Let  $1 \leq p < \infty$  and  $\Omega \subset \mathbb{R}^d$  be a bounded connected open domain with Lipschitz boundary. Then there exists a constant  $C > 0$  depending only on  $p$  and  $\Omega$  such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega),$$

where

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x) dx$$

is the average value of  $u$  over  $\Omega$ .

In the following, we borrow two Lemmas from [56], and assume integer  $s \geq 3$ .

**Lemma 2.2.** *Let  $u, v \in B_{s,T}(\mathbb{T})$ . It holds*

$$(2.22) \quad \|uv\|_s \leq C \|u\|_s \|v\|_s.$$

**Lemma 2.3.** *Let  $v \in B_{s,T}(\mathbb{T})$  satisfying  $\partial_t v = f(x, v, \partial_x v)$ , with  $f$  being a smooth function such that  $f(x, 0, 0) = 0$ . Then for all  $t \in [0, T]$ , we have*

$$(2.23) \quad \|\partial_t^k \partial^\alpha v(t, \cdot)\| \leq C \|v(t, \cdot)\|_s, \quad \forall k + |\alpha| \leq s,$$

where the positive constant  $C$  may depend continuously on  $\|v\|_s$ .

## 2.3 Energy estimates for Euler-Maxwell systems

It is well known that the global existence of smooth solutions follows from the local existence and uniform estimates of solutions with respect to  $t$ . Therefore, the goal of this section is to establish the uniform estimates for proving Theorem 2.1 and Theorem 2.2.

### 2.3.1 $L^2$ energy estimates

Suppose  $(n^\nu, u^\nu, E, B)$  is a smooth solution of the periodic problem for the two-fluid Euler-Maxwell system (2.2) with initial condition (2.3). Now, for  $\nu = e, i$ , set

$$(2.24) \quad n^\nu = \bar{n}^\nu + N^\nu, \quad E = \bar{E} + F, \quad B = \bar{B} + G.$$

Thus, we can rewrite problem (2.2)-(2.4) as :

$$(2.25) \quad \begin{cases} \partial_t N^\nu + u^\nu \cdot \nabla N^\nu + (\bar{n}^\nu + N^\nu) \nabla \cdot u^\nu + \nabla \bar{n}^\nu \cdot u^\nu = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla h_\nu (\bar{n}^\nu + N^\nu) - \nabla h_\nu (\bar{n}^\nu) + u^\nu = q_\nu (F + u^\nu \times (\bar{B} + G)), \\ \partial_t F - \nabla \times G = (\bar{n}^e + N^e) u^e - (\bar{n}^i + N^i) u^i, \quad \nabla \cdot F = N^i - N^e, \\ \partial_t G + \nabla \times F = 0, \quad \nabla \cdot G = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{T}, \end{cases}$$

with initial condition :

$$(2.26) \quad (N^\nu, u^\nu, F, G)|_{t=0} = (N^{\nu 0}, u^{\nu 0}, F^0, G^0), \quad \text{in } \mathbb{T},$$

which satisfies the compatibility condition :

$$(2.27) \quad \nabla \cdot F^0 = N^{i0} - N^{e0}, \quad \nabla \cdot G^0 = 0, \quad \text{in } \mathbb{T}.$$

Here  $N^{\nu 0} = n^{\nu 0} - \bar{n}^\nu$ ,  $F^0 = E^0 - \bar{E}$  and  $G^0 = B^0 - \bar{B}$ .

A direct computation gives

$$\nabla h_\nu (\bar{n}^\nu + N^\nu) - \nabla h_\nu (\bar{n}^\nu) = h'_\nu (\bar{n}^\nu + N^\nu) \nabla N^\nu + \nabla h'_\nu (\bar{n}^\nu) N^\nu + r_\nu (\bar{n}^\nu, N^\nu),$$

where

$$r_\nu (\bar{n}^\nu, N^\nu) = (h'_\nu (\bar{n}^\nu + N^\nu) - h'_\nu (\bar{n}^\nu) - h''_\nu (\bar{n}^\nu) N^\nu) \nabla \bar{n}^\nu.$$

Next, we also set

$$(2.28) \quad \begin{aligned} U^\nu &= \begin{pmatrix} N^\nu \\ u^\nu \end{pmatrix}, \quad U^{\nu 0} = \begin{pmatrix} N^{\nu 0} \\ u^{\nu 0} \end{pmatrix}, \quad U = \begin{pmatrix} U^e \\ U^i \end{pmatrix}, \quad U^0 = \begin{pmatrix} U^{e0} \\ U^{i0} \end{pmatrix}, \\ W &= \begin{pmatrix} U \\ F \\ G \end{pmatrix}, \quad W^0 = \begin{pmatrix} U^0 \\ F^0 \\ G^0 \end{pmatrix}. \end{aligned}$$

Then, the Euler equations in (2.25) can be rewritten in the following form :

$$(2.29) \quad \partial_t U^\nu + \sum_{j=1}^3 A_j^\nu (n^\nu, u^\nu) \partial_j U^\nu + L^\nu (x) U^\nu + M^\nu (W) = f_\nu (r_\nu),$$

with

$$(2.30) \quad A_j^\nu (n^\nu, u^\nu) = \begin{pmatrix} u_j^\nu & n^\nu e_j^T \\ h'_\nu (n^\nu) e_j & u_j^\nu I_3 \end{pmatrix}, \quad j = 1, 2, 3,$$

$$(2.31) \quad L^\nu(x) = \begin{pmatrix} 0 & (\nabla \bar{n}^\nu)^T \\ \nabla h'_\nu(\bar{n}^\nu) & 0 \end{pmatrix},$$

$$(2.32) \quad M^\nu(W) = \begin{pmatrix} 0 \\ u^\nu - q_\nu F - q_\nu (u^\nu \times (\bar{B} + G)) \end{pmatrix}, \quad f_\nu(r_\nu) = \begin{pmatrix} 0 \\ -r_\nu \end{pmatrix}.$$

Here and in the rest of this thesis,  $(e_1, e_2, e_3)$  denotes the canonical basis of  $\mathbb{R}^3$ ,  $I_3$  denotes the  $3 \times 3$  unit matrix and we use  $[\cdot]^T$  to denote the transpose of a vector.

It is easy to see that system (2.29) for  $U^\nu$  is symmetrizable hyperbolic when  $n^\nu = \bar{n}^\nu + N^\nu > 0$ . In fact, we can choose symmetrizer as

$$A_0^\nu(n^\nu) = \begin{pmatrix} h'_\nu(n^\nu) & 0 \\ 0 & n^\nu I_3 \end{pmatrix}.$$

Then we obtain

$$\tilde{A}_j^\nu(n^\nu, u^\nu) = A_0^\nu(n^\nu) A_j^\nu(n^\nu, u^\nu) = \begin{pmatrix} h'_\nu(n^\nu) u_j^\nu & p'_\nu(n^\nu) e_j^T \\ p'_\nu(n^\nu) e_j & n^\nu u_j^\nu I_3 \end{pmatrix}.$$

Let  $T > 0$  and  $W$  be a smooth solution of (2.2)-(2.3) defined on time interval  $[0, T]$ . We denote by

$$(2.33) \quad \omega_T = \sup_{0 \leq t \leq T} \| \|W(t)\| \|_s.$$

From the continuous embedding  $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$  for  $s \geq 3$ , there is a constant  $C_s > 0$  such that

$$\|f\|_\infty \leq C_s \|f\|_{s-1}, \quad \forall f \in H^{s-1}(\mathbb{T}).$$

If  $\omega_T$  is sufficiently small, it is easy to see from  $\bar{n}^\nu \geq \text{const.} > 0$  that  $n^\nu = \bar{n}^\nu + N^\nu \geq \text{const.} > 0$ . In the following, we always suppose that the conditions of Theorem 2.1 holds and  $\omega_T$  is sufficiently small independent of  $T$ .

**Lemma 2.4.** *For all  $t \in [0, T]$ , we have*

$$(2.34) \quad \|\partial_t N^\nu\|_\infty \leq C \| \|U\| \|_s, \quad \|\partial_t A_0^\nu(n^\nu)\|_\infty \leq C \| \|U\| \|_s$$

and

$$(2.35) \quad \left| \left\langle \sum_{j=1}^3 \partial_j \tilde{A}_j^\nu(n^\nu, u^\nu) - 2A_0^\nu(n^\nu) L^\nu V, V \right\rangle \right| \leq C \| \|U\| \|_s \|V\|^2, \quad \forall V \in \mathbb{R}^3.$$

**Proof.** Due to the continuous imbedding  $H^{s-1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ , we get

$$\|\partial_t N^\nu\|_\infty \leq C \|\partial_t N^\nu\|_{s-1} \leq C \|U\|_s.$$

Then, we obtain

$$\|\partial_t A_0^\nu(n^\nu)\|_\infty = \|(A_0^\nu)'(n^\nu) \partial_t N^\nu\|_\infty \leq C \|\partial_t N^\nu\|_\infty \leq C \|U\|_s.$$

Next, after directly computing, we find that the following matrix is anti-symmetric at  $(n^\nu, u^\nu) = (\bar{n}^\nu, 0)$ ,

$$\sum_{j=1}^3 \partial_j \tilde{A}_j^\nu(n^\nu, u^\nu) - 2A_0^\nu(n^\nu) L^\nu(x) = \begin{pmatrix} \nabla \cdot (h'_\nu(n^\nu) u^\nu) & (\nabla p'_\nu(n^\nu) - 2h'_\nu(n^\nu) \nabla \bar{n}^\nu)^T \\ \nabla p'_\nu(n^\nu) - 2n^\nu \nabla h'_\nu(\bar{n}^\nu) & \nabla \cdot (n^\nu u^\nu) I_3 \end{pmatrix},$$

which implies (2.35).  $\square$

**Proposition 2.2.** *It holds*

$$(2.36) \quad \frac{d}{dt} \left( \sum_{\nu=e,i} \langle A_0^\nu(n^\nu) U^\nu, U^\nu \rangle + \|F\|^2 + \|G\|^2 \right) + 2 \sum_{\nu=e,i} \langle n^\nu u^\nu, u^\nu \rangle \leq C \|U\|_s^3.$$

**Proof.** Multiplying (2.29) by  $2A_0^\nu(n^\nu)U^\nu$ , and then integrating over  $\mathbb{T}$ , we get

$$\frac{d}{dt} \langle A_0^\nu(n^\nu) U^\nu, U^\nu \rangle = \langle \operatorname{div} A^\nu(n^\nu, u^\nu) U^\nu, U^\nu \rangle + 2 \langle A_0^\nu(n^\nu) (f_\nu - L^\nu U^\nu - M^\nu(W)), U^\nu \rangle,$$

where

$$\operatorname{div} A^\nu(n^\nu, u^\nu) = \partial_t A_0^\nu(n^\nu) + \sum_{j=1}^3 \partial_j \tilde{A}_j^\nu(n^\nu, u^\nu).$$

It follows that

$$\begin{aligned} \frac{d}{dt} \langle A_0^\nu(n^\nu) U^\nu, U^\nu \rangle &= \langle \partial_t A_0^\nu(n^\nu) U^\nu, U^\nu \rangle + 2 \langle A_0^\nu(n^\nu) f_\nu, U^\nu \rangle \\ &+ \left\langle \left( \sum_{j=1}^3 \partial_j \tilde{A}_j^\nu(n^\nu, u^\nu) - 2A_0^\nu(n^\nu) L^\nu \right) U^\nu, U^\nu \right\rangle - 2 \langle A_0^\nu(n^\nu) M^\nu(W), U^\nu \rangle. \end{aligned}$$

From Lemma 2.4, we obtain

$$\left| \langle \partial_t A_0^\nu(n^\nu) U^\nu, U^\nu \rangle \right| \leq \|\partial_t A_0^\nu(n^\nu)\|_\infty \|U^\nu\|^2 \leq C \|U\|_s^3,$$

and

$$\left| \left\langle \left( \sum_{j=1}^3 \partial_j \tilde{A}_j^\nu(n^\nu, u^\nu) - 2A_0^\nu(n^\nu) L^\nu \right) U^\nu, U^\nu \right\rangle \right| \leq C \|U\|_s^3.$$

Since  $u^\nu \cdot (u^\nu \times (G + \bar{B})) = 0$  and the fact that  $|r_\nu| \leq C|N^\nu|^2$ , we have

$$- \langle A_0^\nu(n^\nu) M^\nu(W), U^\nu \rangle = - \langle n^\nu u^\nu, u^\nu \rangle + q_\nu \langle n^\nu u^\nu, F \rangle$$

and

$$|\langle A_0^\nu(n^\nu) f_\nu, U^\nu \rangle| = |\langle n^\nu r_\nu, u^\nu \rangle| \leq C \|U\|_s^3.$$

Therefore,

$$\frac{d}{dt} \langle A_0^\nu(n^\nu) U^\nu, U^\nu \rangle + 2 \langle n^\nu u^\nu, u^\nu \rangle \leq 2q_\nu \langle n^\nu u^\nu, F \rangle + C \|U\|_s^3,$$

provided that  $\omega_T$  is sufficiently small. On the other hand, an easy energy estimate for the Maxwell equations in (2.25) yields

$$(2.37) \quad \frac{d}{dt} (\|F\|^2 + \|G\|^2) = -2 \sum_{\nu=e,i} q_\nu \langle n^\nu u^\nu, F \rangle.$$

These last two relations imply (2.36).  $\square$

### 2.3.2 Higher order energy estimates

Assume  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $1 \leq k + |\alpha| \leq s$ . We establish a generalized energy estimate for  $\partial_t^k \partial^\alpha W$  in  $L^2(\mathbb{T})$  in order to get a recurrence relation on  $k$  and  $|\alpha|$ . Applying  $\partial_t^k \partial^\alpha$  to (2.29), we get

$$(2.38) \quad \partial_t (\partial_t^k \partial^\alpha U^\nu) + \sum_{j=1}^3 A_j^\nu \partial_j (\partial_t^k \partial^\alpha U^\nu) + L^\nu (\partial_t^k \partial^\alpha U^\nu) + \partial_t^k \partial^\alpha M^\nu = \partial_t^k \partial^\alpha f_\nu + g_\nu^{k,\alpha},$$

where

$$(2.39) \quad g_\nu^{k,\alpha} = \sum_{j=1}^3 (A_j^\nu \partial_j (\partial_t^k \partial^\alpha U^\nu) - \partial_t^k \partial^\alpha (A_j^\nu \partial_j U^\nu)) + L^\nu (\partial_t^k \partial^\alpha U^\nu) - \partial_t^k \partial^\alpha (L^\nu U^\nu).$$

For the Maxwell equations, we also have

$$(2.40) \quad \begin{cases} \partial_t (\partial_t^k \partial^\alpha F) - \nabla \times (\partial_t^k \partial^\alpha G) = \partial_t^k \partial^\alpha (n^\nu u^\nu), & \nabla \cdot (\partial_t^k \partial^\alpha F) = \partial_t^k \partial^\alpha (N^i - N^e), \\ \partial_t (\partial_t^k \partial^\alpha G) + \nabla \times (\partial_t^k \partial^\alpha F) = 0, & \nabla \cdot (\partial_t^k \partial^\alpha G) = 0. \end{cases}$$

For simplicity, in the following, we denote by  $U_{k,\alpha}^\nu = \partial_t^k \partial^\alpha U^\nu$  etc.

**Lemma 2.5.** *For all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $1 \leq k + |\alpha| \leq s$ , it holds*

$$(2.41) \quad \frac{d}{dt} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) \leq -2 \sum_{\nu=e,i} \langle A_0^\nu \partial_t^k \partial^\alpha M^\nu, U_{k,\alpha}^\nu \rangle \\ -2 \sum_{\nu=e,i} [q_\nu \langle \partial_t^k \partial^\alpha (n^\nu u^\nu), F_{k,\alpha} \rangle - \langle A_0^\nu g_\nu^{k,\alpha}, U_{k,\alpha}^\nu \rangle] + C \|U\|_s^2 \|W\|_s.$$

**Proof.** Taking the inner product of (2.38) with  $A_0^\nu U_{k,\alpha}^\nu$  in  $L^2(\mathbb{T})$  yields

$$\begin{aligned} \frac{d}{dt} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle &= \langle \operatorname{div} A^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle - 2 \langle A_0^\nu L^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle \\ &\quad + 2 \langle A_0^\nu \partial_t^k \partial^\alpha f_\nu, U_{k,\alpha}^\nu \rangle - 2 \langle A_0^\nu \partial_t^k \partial^\alpha M^\nu, U_{k,\alpha}^\nu \rangle + 2 \langle A_0^\nu g_\nu^{k,\alpha}, U_{k,\alpha}^\nu \rangle. \end{aligned}$$

We may write

$$\begin{aligned} &\langle \operatorname{div} A^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle - 2 \langle A_0^\nu L^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle \\ &= \langle \partial_t A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \left\langle \left( \sum_{j=1}^3 \partial_j \tilde{A}_j^\nu - 2A_0^\nu L^\nu \right) U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \right\rangle, \end{aligned}$$

On the other hand, a standard energy estimate for (2.40) gives

$$(2.42) \quad \frac{d}{dt} (\|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2) = -2 \sum_{\nu=e,i} q_\nu \langle \partial_t^k \partial^\alpha (n^\nu u^\nu), F_{k,\alpha} \rangle.$$

Then we have

$$\begin{aligned} &\frac{d}{dt} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) \\ &= \sum_{\nu=e,i} \left( \langle \partial_t A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \left\langle \left( \sum_{j=1}^3 \partial_j \tilde{A}_j^\nu - 2A_0^\nu L^\nu \right) U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \right\rangle + 2 \langle A_0^\nu \partial_t^k \partial^\alpha f_\nu, U_{k,\alpha}^\nu \rangle \right) \\ &\quad - 2 \sum_{\nu=e,i} (\langle A_0^\nu \partial_t^k \partial^\alpha M^\nu, U_{k,\alpha}^\nu \rangle + q_\nu \langle \partial_t^k \partial^\alpha (n^\nu u^\nu), F_{k,\alpha} \rangle - \langle A_0^\nu g_\nu^{k,\alpha}, U_{k,\alpha}^\nu \rangle). \end{aligned}$$

It is clear that the first three terms on the right hand side of the equality are bounded by  $C \|U\|_s^2 \| \|W\| \|_s$ . This proves (2.41).  $\square$

The following three Lemmas are similar to that of the one-fluid case established in [56]. Hence, their proofs are omitted for simplicity.

**Lemma 2.6.** *For all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \geq 1$  and  $k + |\alpha| \leq s$ , it holds*

$$(2.43) \quad \begin{aligned} &-2 \langle A_0^\nu \partial_t^k \partial^\alpha M^\nu, U_{k,\alpha}^\nu \rangle - 2q_\nu \langle \partial_t^k \partial^\alpha (n^\nu u^\nu), F_{k,\alpha} \rangle \\ &\leq - \langle n^\nu u_{k,\alpha}^\nu, u_{k,\alpha}^\nu \rangle + C \left( \|\partial_t^k u^\nu\|_{|\alpha|-1}^2 + \|\partial_t^k F\|_{|\alpha|-1}^2 + \| \|U\| \|_s^2 \| \|W\| \|_s \right). \end{aligned}$$

**Lemma 2.7.** *For all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \geq 1$  and  $k + |\alpha| \leq s$ , it holds*

$$(2.44) \quad \left| \langle A_0^\nu g_\nu^{k,\alpha}, U_{k,\alpha}^\nu \rangle \right| \leq \varepsilon \|u_{k,\alpha}^\nu\|^2 + C \|\partial_t^k u^\nu\|_{|\alpha|-1}^2 + C \|\partial_t^k N^\nu\|_{|\alpha|}^2 + C \| \|U\| \|_s^3.$$

where  $\varepsilon > 0$  is a small constant to be chosen and  $C$  may depend on  $\varepsilon$ .

**Lemma 2.8.** *For all  $1 \leq k \leq s$ , it holds*

$$(2.45) \quad -2 \langle A_0^\nu \partial_t^k M^\nu, U_k^\nu \rangle - 2q_\nu \langle \partial_t^k (n^\nu u^\nu), F_k \rangle \leq -2 \langle n^\nu u_k^\nu, u_k^\nu \rangle + C \| \|U\| \|_s^2 \| \|W\| \|_s,$$



and

$$(2.46) \quad \left| \langle A_0^\nu g_\nu^k, U_k^\nu \rangle \right| \leq C \|U\|_s^3.$$

where

$$g_\nu^k = g_\nu^{k,0}, \quad U_k^\nu = U_{k,0}^\nu = \partial_t^k U^\nu \text{ etc.}$$

From Lemmas 2.5-2.7, taking  $\varepsilon > 0$  sufficiently small, it is easy to obtain the following result.

**Proposition 2.3.** *There exists a positive constant  $c_0$  such that, for all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \geq 1$  and  $k + |\alpha| \leq s$ , it holds*

$$(2.47) \quad \begin{aligned} & \frac{d}{dt} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) + c_0 \sum_{\nu=e,i} \|u_{k,\alpha}^\nu\|^2 \\ & \leq C \sum_{\nu=e,i} \left( \|\partial_t^k u^\nu\|_{|\alpha|-1}^2 + \|\partial_t^k N^\nu\|_{|\alpha|}^2 \right) + C \|\partial_t^k F\|_{|\alpha|-1}^2 + C \|U\|_s^2 \|W\|_s. \end{aligned}$$

**Remark 2.1.** *Proposition 2.3 is valid for  $|\alpha| \geq 1$ . The next result concerns the  $L^2$  estimates for  $\partial_t^k W$  (i.e.  $\alpha = 0$ ), which is a starting point for applying the argument by induction.*

**Proposition 2.4.** *There exists a positive constant  $c_0$  such that, for all  $0 \leq k \leq s$ , it holds*

$$(2.48) \quad \frac{d}{dt} \left( \sum_{\nu=e,i} \langle A_0^\nu U_k^\nu, U_k^\nu \rangle + \|F_k\|^2 + \|G_k\|^2 \right) + c_0 \sum_{\nu=e,i} \|\partial_t^k u^\nu\|^2 \leq C \|U\|_s^2 \|W\|_s.$$

**Proof.** Recall that  $n^\nu \geq \text{const.} > 0$ . For  $k = 0$ , estimate (2.48) is given by Proposition 2.2. For  $1 \leq k \leq s$ , (2.48) follows from Lemma 2.5 with  $\alpha = 0$  and Lemma 2.8.  $\square$

We conclude the result of this subsection as follows.

**Proposition 2.5.** *There exists a positive constant  $c_0$  such that, for all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \geq 1$  and  $k + |\alpha| \leq s$ , it holds*

$$(2.49) \quad \begin{aligned} & \frac{d}{dt} \sum_{\beta \leq \alpha} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\beta}^\nu, U_{k,\beta}^\nu \rangle + \|F_{k,\beta}\|^2 + \|G_{k,\beta}\|^2 \right) + c_0 \sum_{\nu=e,i} \|\partial_t^k u^\nu\|_{|\alpha|}^2 \\ & \leq C \sum_{\nu=e,i} \left( \|\partial_t^k N^\nu\|_{|\alpha|}^2 + \|\partial_t^k u^\nu\|_{|\alpha|-1}^2 \right) + C \|\partial_t^k F\|_{|\alpha|-1}^2 + C \|U\|_s^2 \|W\|_s. \end{aligned}$$

**Proof.** Let  $k \geq 0$  be fixed. Remark that

$$\|\partial_t^k u^\nu\|^2 + \sum_{1 \leq |\beta|, \beta \leq \alpha} \|u_{k,\beta}^\nu\|^2 \sim \|\partial_t^k u^\nu\|_{|\alpha|}^2.$$

Hence, summing (2.47) for all indexes up to  $|\alpha| \leq s - k$  and combining the resulting inequality with Proposition 2.4 yields (2.49).  $\square$

### 2.3.3 Time dissipation estimates for $N^\nu$ and $F$

Estimate (2.49) contains a recurrence relation on the time dissipation of  $u^\nu$ . Obviously, this estimate is not sufficient to control the higher order term in (2.49) and time dissipation estimates for  $N^\nu$  and  $F$  are necessary. In the following, we establish these estimates.

**Proposition 2.6.** *For all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \geq 1$  and  $k + |\alpha| \leq s$ , it holds*

$$(2.50) \quad \sum_{\nu=e,i} \left\| \partial_t^k N^\nu \right\|_{|\alpha|}^2 + \left\| \partial_t^k (N^e - N^i) \right\|_{|\alpha|-1}^2 \\ \leq C \sum_{\nu=e,i} \left( \|u_k^\nu\|_{|\beta|}^2 + \|u_{k+1}^\nu\|_{|\beta|}^2 + \|N_k^\nu\|_{|\beta|}^2 \right) + C \|U\|_s^2 \|W\|_s,$$

and

$$(2.51) \quad \left\| \partial_t^k F \right\|_{|\alpha|}^2 \leq C \sum_{\nu=e,i} \left( \|u_k^\nu\|_{|\beta|}^2 + \|u_{k+1}^\nu\|_{|\beta|}^2 + \|N_k^\nu\|_{|\beta|}^2 \right) + C \|U\|_s^2 \|W\|_s.$$

**Proof.** For  $k \in \mathbb{N}$  and  $\beta \in \mathbb{N}^3$  with  $k + |\beta| \leq s - 1$ , applying  $\partial_t^k \partial^\beta$  to the momentum equations in (2.25) and using

$$\nabla (h_\nu (\bar{n}^\nu + N^\nu) - h_\nu (\bar{n}^\nu)) = \nabla (h_\nu' (\bar{n}^\nu) N^\nu) + (h_\nu' (\bar{n}^\nu + N^\nu) - h_\nu' (\bar{n}^\nu)) \nabla N^\nu + r_\nu,$$

we get

$$(2.52) \quad \partial^\beta \nabla (h_\nu' (\bar{n}^\nu) N_k^\nu) - q_\nu F_{k,\beta} = (q_\nu u_{k,\beta}^\nu \times \bar{B} - u_{k,\beta}^\nu - u_{k+1,\beta}^\nu) - R_{\nu 1}^{k,\beta},$$

where

$$R_{\nu 1}^{k,\beta} = \partial_t^k \partial^\beta \{ (u^\nu \cdot \nabla) u^\nu + (h_\nu' (\bar{n}^\nu + N^\nu) - h_\nu' (\bar{n}^\nu)) \nabla N^\nu + r_\nu - q_\nu u^\nu \times G \}.$$

By Lemma 2.2, we have

$$\left\| R_{\nu 1}^{k,\beta} \right\| \leq C \|U\|_s \|W\|_s.$$

Now we write

$$\partial^\beta \nabla (h_\nu' (\bar{n}^\nu) N_k^\nu) = h_\nu' (\bar{n}^\nu) \nabla N_{k,\beta}^\nu + R_{\nu 2}^{k,\beta},$$

where

$$R_{\nu 2}^{k,\beta} = \sum_{\gamma \leq \beta} d_\gamma(x) N_{k,\gamma}^\nu,$$

with  $d_\gamma$  being given smooth functions. It follows that

$$\left\| R_{\nu 2}^{k,\beta} \right\| \leq C \|N_k^\nu\|_{|\beta|}.$$

Taking the inner product of (2.52) with  $\nabla N_{k,\beta}^\nu$  in  $L^2(\mathbb{T})$  and noting that  $h'_\nu(\bar{n}^\nu) \geq \text{const.} > 0$  yields

$$\begin{aligned} & \left\| \sqrt{h'_\nu(\bar{n}^\nu)} \nabla N_{k,\beta}^\nu \right\|^2 - q_\nu \langle F_{k,\beta}, \nabla N_{k,\beta}^\nu \rangle \\ &= q_\nu \langle u_{k,\beta}^\nu \times \bar{B}, \nabla N_{k,\beta}^\nu \rangle - \langle u_{k,\beta}^\nu + u_{k+1,\beta}^\nu, \nabla N_{k,\beta}^\nu \rangle - \langle R_{\nu 1}^{k,\beta} + R_{\nu 2}^{k,\beta}, \nabla N_{k,\beta}^\nu \rangle \\ &\leq \frac{1}{2} \left\| \sqrt{h'_\nu(\bar{n}^\nu)} \nabla N_{k,\beta}^\nu \right\|^2 + C \left( \|u_k^\nu\|_{|\beta|}^2 + \|u_{k+1}^\nu\|_{|\beta|}^2 + \|N_k^\nu\|_{|\beta|}^2 + C \| \|U\|_s^2 \|W\|_s \right). \end{aligned}$$

Since

$$\begin{aligned} - \sum_{\nu=e,i} q_\nu \langle F_{k,\beta}, \nabla N_{k,\beta}^\nu \rangle &= - \langle F_{k,\beta}, \nabla(N_{k,\beta}^i - N_{k,\beta}^e) \rangle \\ &= \langle \nabla \cdot F_{k,\beta}, N_{k,\beta}^i - N_{k,\beta}^e \rangle \\ &= \left\| \partial_t^k \partial^\beta (N^i - N^e) \right\|^2 \geq 0, \end{aligned}$$

we obtain

$$\sum_{\nu=e,i} \left\| \nabla N_{k,\beta}^\nu \right\|^2 + \left\| N_{k,\beta}^e - N_{k,\beta}^i \right\|^2 \leq C \sum_{\nu=e,i} \left( \|u_k^\nu\|_{|\beta|}^2 + \|u_{k+1}^\nu\|_{|\beta|}^2 + \|N_k^\nu\|_{|\beta|}^2 \right) + C \| \|U\|_s^2 \|W\|_s.$$

Summing the inequalities for all indexes  $\beta$  yields

$$\sum_{\nu=e,i} \left\| \partial_t^k \nabla N^\nu \right\|_{|\beta|}^2 + \left\| \partial_t^k (N^e - N^i) \right\|_{|\beta|}^2 \leq C \sum_{\nu=e,i} \left( \|u_k^\nu\|_{|\beta|}^2 + \|u_{k+1}^\nu\|_{|\beta|}^2 + \|N_k^\nu\|_{|\beta|}^2 \right) + C \| \|U\|_s^2 \|W\|_s.$$

This shows (2.50) by replacing  $\beta$  by  $\alpha$  with  $|\alpha| = |\beta| + 1$  and using Lemma 2.1.

Finally, from (2.52), we have

$$\left\| \partial_t^k \partial^\beta F \right\|^2 \leq C \sum_{\nu=e,i} \left( \|N_k^\nu\|_{|\beta|+1}^2 + \|u_{k,\beta}^\nu\|^2 + \|u_{k+1,\beta}^\nu\|^2 + \|R_{\nu 1}^{k,\beta}\|^2 \right).$$

Summing the inequalities for all indexes  $\beta$  and combining the result with (2.50) yields (2.51).  $\square$

**Proposition 2.7.** *There exists a positive constant  $c_0$  such that, for all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \geq 1$  and  $k + |\alpha| \leq s$ , it holds*

$$\begin{aligned} (2.53) \quad & \frac{d}{dt} \sum_{\beta \leq \alpha} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\beta}^\nu, U_{k,\beta}^\nu \rangle + \|F_{k,\beta}\|^2 + \|G_{k,\beta}\|^2 \right) + c_0 \sum_{\nu=e,i} \left\| \partial_t^k U^\nu \right\|_{|\alpha|}^2 \\ & \leq C \sum_{\nu=e,i} \left( \left\| \partial_t^k U^\nu \right\|_{|\alpha|-1}^2 + \left\| \partial_t^{k+1} u^\nu \right\|_{|\alpha|-1}^2 \right) + C \| \|U\|_s^2 \|W\|_s. \end{aligned}$$

**Proof.** Estimate (2.53) is a direct consequence of Propositions 2.5-2.6 by noting  $U^\nu = (N^\nu, u^\nu)^T$  and  $U = (U^e, U^i)^T$ .  $\square$

Proposition 2.7 is valid only for all  $0 \leq k \leq s - 1$ . Besides, we need an estimate on  $\left\| \partial_t^k N^\nu \right\|$  as an initial value to use (2.53) by induction. The next result presents such an estimate and completes Proposition 2.7 for the case  $k = s$ .

**Proposition 2.8.** For all  $0 \leq k \leq s-1$ , we have

$$(2.54) \quad \|\partial_t^k N^\nu\|_1^2 \leq C \sum_{\nu=e,i} \left( \|\partial_t^k u^\nu\|^2 + \|\partial_t^{k+1} u^\nu\|^2 \right) + C \| \|U\|_s^2 \|W\|_s,$$

and

$$(2.55) \quad \begin{aligned} & \frac{d}{dt} \left( \sum_{\nu=e,i} \langle A_0^\nu \partial_t^s U^\nu, \partial_t^s U^\nu \rangle + \|\partial_t^s F\|^2 + \|\partial_t^s G\|^2 \right) + c_0 \sum_{\nu=e,i} \|\partial_t^s U^\nu\|^2 \\ & \leq C \sum_{\nu=e,i} \|\partial_t^{s-1} u^\nu\|_1^2 + C \| \|U\|_s^2 \|W\|_s. \end{aligned}$$

**Proof.** For  $k \in \mathbb{N}$  with  $k \leq s-1$ , applying  $\partial_t^k$  to the second equation of (2.25), we get

$$(2.56) \quad \nabla (h_\nu'(\bar{n}^\nu) N_k^\nu) - q_\nu F_k = (q_\nu u_k^\nu \times \bar{B} - u_k^\nu - u_{k+1}^\nu) - R_{\nu 1}^{k,0},$$

where

$$R_{\nu 1}^{k,0} = \partial_t^k \{ (u^\nu \cdot \nabla) u^\nu + (h_\nu'(\bar{n}^\nu + N^\nu) - h_\nu'(\bar{n}^\nu)) \nabla N^\nu + r_\nu - q_\nu u^\nu \times G \}.$$

Now, we define a potential function  $\nabla \Psi$  as

$$\nabla \cdot (\nabla \Psi) = \Delta \Psi = N^e - N^i, \quad \int_{\mathbb{T}} \Psi(t, x) dx = 0.$$

Then

$$\nabla \cdot (F + \nabla \Psi) = - (N^e - N^i) + N^e - N^i = 0,$$

and furthermore

$$\nabla \cdot (F_k + \nabla \Psi_k) = 0, \quad \forall 0 \leq k \leq s-1.$$

From (2.56), we have

$$(2.57) \quad \nabla \eta_k^\nu - q_\nu (F_k + \nabla \Psi_k) = (q_\nu u_k^\nu \times \bar{B} - u_k^\nu - u_{k+1}^\nu) - R_{\nu 1}^{k,0},$$

where

$$\eta_k^\nu = h_\nu'(\bar{n}^\nu) N_k^\nu + q_\nu \Psi_k.$$

Due to the fact that

$$\langle F_k + \nabla \Psi_k, \nabla \eta_k^\nu \rangle = - \langle \nabla \cdot (F_k + \nabla \Psi_k), \eta_k^\nu \rangle = 0,$$

we obtain

$$(2.58) \quad \|\nabla \eta_k^\nu\|^2 \leq \|\partial_t^k u^\nu\|^2 + \|\partial_t^{k+1} u^\nu\|^2 + \|R_{\nu 1}^{k,0}\|^2.$$

Since

$$\partial_t^k N^\nu = \frac{\eta_\nu^k}{h'_\nu(\bar{n}^\nu)} - \frac{q_\nu \partial_t^k \Psi}{h'_\nu(\bar{n}^\nu)}, \quad \Delta \Psi = N^e - N^i,$$

we have

$$-\Delta \partial_t^k \Psi = \partial_t^k (N^i - N^e) = \frac{\eta_i^k}{h'_i(\bar{n}^i)} - \frac{\partial_t^k \Psi}{h'_i(\bar{n}^i)} - \left( \frac{\eta_e^k}{h'_e(\bar{n}^e)} + \frac{\partial_t^k \Psi}{h'_e(\bar{n}^e)} \right).$$

Thus,

$$-\Delta \partial_t^k \Psi + \left( \frac{1}{h'_e(\bar{n}^e)} + \frac{1}{h'_i(\bar{n}^i)} \right) \partial_t^k \Psi = \frac{\eta_i^k}{h'_i(\bar{n}^i)} - \frac{\eta_e^k}{h'_e(\bar{n}^e)}.$$

Since  $h'_\nu(\bar{n}^\nu) \geq \text{const.} > 0$ , taking the inner product of the previous equality with  $\partial_t^k \Psi$  in  $L^2(\mathbb{T})$  and using an integration by parts, we get

$$(2.59) \quad \left\| \partial_t^k \nabla \Psi \right\|^2 + c_0 \left\| \partial_t^k \Psi \right\|^2 \leq \sum_{\nu=e,i} \left\| \eta_\nu^k \right\|^2 \leq \sum_{\nu=e,i} \left\| \nabla \eta_\nu^k \right\|^2,$$

where we have used Lemma 2.1.

From (2.56), (2.58)-(2.59) and the definition of  $\eta_\nu^k$ , we have

$$\left\| \nabla (h'_\nu(\bar{n}^\nu) N_k^\nu) \right\|^2 \leq \left\| \nabla \eta_k^\nu \right\|^2 + \left\| \partial_t^k \nabla \Psi \right\|^2 \leq \left\| \partial_t^k u^\nu \right\|^2 + \left\| \partial_t^{k+1} u^\nu \right\|^2 + \left\| R_{\nu 1}^{k,0} \right\|^2.$$

Since  $h'_\nu(\bar{n}^\nu) \geq \text{const.} > 0$ , by Lemma 2.1 again,  $\left\| \nabla (h'_\nu(\bar{n}^\nu) N_k^\nu) \right\| \sim \left\| N_k^\nu \right\|_1$ . This proves (2.54).

Next, from  $\partial_t N^\nu = -\nabla \cdot ((\bar{n}^\nu + N^\nu) u^\nu)$ , we get

$$\partial_t^s N^\nu = -\nabla \cdot (\partial_t^{s-1} (N^\nu u^\nu)) - \nabla \cdot (\bar{n}^\nu \partial_t^{s-1} u^\nu).$$

It follows from Lemma 2.2 that

$$\left\| \partial_t^s N^\nu \right\|^2 \leq C \left\| \partial_t^{s-1} u^\nu \right\|_1^2 + C \left\| \|U\| \right\|_s^3,$$

together with (2.48) for  $k = s$  implies (2.55).  $\square$

## 2.4 Proof of Theorem 2.1 and Theorem 2.2

### 2.4.1 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the following result which is a conclusion on the estimates established in the section 3.

**Proposition 2.9.** *Let  $s \geq 3$  be an integer. For all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^3$  with  $k + |\alpha| \leq s$ , there exist positive constants  $\lambda_{(k,|\alpha|)}$  such that*

$$(2.60) \quad \frac{d}{dt} \sum_{k+|\alpha| \leq s} \lambda_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) + \left\| \|U\| \right\|_s^2 \leq 0.$$

**Proof.** First, applying Proposition 2.7 with  $(k, |\alpha|) = (s-1, 1)$ , we have

$$(2.61) \quad \begin{aligned} & \frac{d}{dt} \sum_{\beta \leq 1} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{s-1,\beta}^\nu, U_{s-1,\beta}^\nu \rangle + \|F_{s-1,\beta}\|^2 + \|G_{s-1,\beta}\|^2 \right) + c_0 \sum_{\nu=e,i} \|\partial_t^{s-1} U^\nu\|_1^2 \\ & \leq C \sum_{\nu=e,i} \left( \|\partial_t^{s-1} U^\nu\|^2 + \|\partial_t^s u^\nu\|^2 \right) + C \| \|U\|_s^2 \|W\|_s. \end{aligned}$$

We find that the dissipation term  $\sum_{\nu=e,i} \|\partial_t^{s-1} U^\nu\|_1^2$  on the left hand side of (2.61) can control the term  $\sum_{\nu=e,i} \|\partial_t^{s-1} u^\nu\|_1^2$  on the right hand side of (2.55) through multiplying (2.55) by a small positive constant  $\lambda_{(s,0)}^* \ll 1$ .

Second, applying Proposition 2.7 with  $(k, |\alpha|) = (s-2, 1)$  and  $(k, |\alpha|) = (s-2, 2)$ , we obtain

$$(2.62) \quad \begin{aligned} & \frac{d}{dt} \sum_{\beta \leq 1} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{s-2,\beta}^\nu, U_{s-2,\beta}^\nu \rangle + \|F_{s-2,\beta}\|^2 + \|G_{s-2,\beta}\|^2 \right) + c_0 \sum_{\nu=e,i} \|\partial_t^{s-2} U^\nu\|_1^2 \\ & \leq C \sum_{\nu=e,i} \left( \|\partial_t^{s-2} U^\nu\|^2 + \|\partial_t^{s-1} u^\nu\|^2 \right) + C \| \|U\|_s^2 \|W\|_s, \end{aligned}$$

and

$$(2.63) \quad \begin{aligned} & \frac{d}{dt} \sum_{\beta \leq 2} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{s-2,\beta}^\nu, U_{s-2,\beta}^\nu \rangle + \|F_{s-2,\beta}\|^2 + \|G_{s-2,\beta}\|^2 \right) + c_0 \sum_{\nu=e,i} \|\partial_t^{s-2} U^\nu\|_2^2 \\ & \leq C \sum_{\nu=e,i} \left( \|\partial_t^{s-2} U^\nu\|_1^2 + \|\partial_t^{s-1} u^\nu\|_1^2 \right) + C \| \|U\|_s^2 \|W\|_s. \end{aligned}$$

We also find that the term  $\sum_{\nu=e,i} \|\partial_t^{s-1} u^\nu\|_1^2$  and the term  $\sum_{\nu=e,i} \|\partial_t^{s-2} U^\nu\|_1^2$  on the right side of (2.63) can be controlled by the dissipation term  $\sum_{\nu=e,i} \|\partial_t^{s-1} U^\nu\|_1^2$  on the left hand side of (2.61) and the term  $\sum_{\nu=e,i} \|\partial_t^{s-2} U^\nu\|_1^2$  on the left hand side of (2.62) through multiplying (2.63) by a small positive constant  $\lambda_{(s-2,2)}^* \ll 1$ .

In this way and by induction on  $(k, |\alpha|)$  with  $k$  decreasing and  $|\alpha|$  increasing, in (2.53) both the term  $\sum_{\nu=e,i} \|\partial_t^{k+1} u^\nu\|_{|\alpha|-1}^2$  and the term  $\sum_{\nu=e,i} \|\partial_t^k U^\nu\|_{|\alpha|-1}^2$  can be controlled by  $\sum_{\nu=e,i} \|\partial_t^k U^\nu\|_{|\alpha|}^2$  in the proceeding steps.

Now, set :

$$\left\{ \begin{array}{l} 0 < \lambda_{(s,0)}^* \ll 1, \\ \lambda_{(s-j,1)}^* = 1, \quad j \leq s, \\ 0 < \lambda_{(s-j,l)}^* \ll \lambda_{(s-j,l-1)}^* \ll 1, \quad 1 < l \leq j \leq s, \\ 0 < \lambda_{(s-j,l_1)}^* \ll \lambda_{(s-j+1,l_2)}^* \ll 1, \quad 1 < l_1, l_2 \leq j \leq s. \end{array} \right.$$

Then, summing (2.55)  $\times \lambda_{(s,0)}^*$ , (2.61)  $\times \lambda_{(s-1,1)}^*$  and  $\sum_{2 \leq j \leq s} \sum_{l \leq j} (2.53)_{(s-j,l)} \times \lambda_{(s-j,l)}^*$ , and noting (2.54), we deduce that there are positive constants  $\lambda_{(k,|\alpha|)}$  such that

$$\begin{aligned} & \frac{d}{dt} \sum_{k+|\alpha| \leq s} \lambda_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) + \sum_{k+|\alpha| \leq s} \sum_{\nu=e,i} \|\partial_t^k U^\nu\|_{|\alpha|}^2 \\ & \leq C \sum_{k=0}^{s-1} \sum_{\nu=e,i} \left( \|\partial_t^k U^\nu\|^2 + \|\partial_t^{k+1} u^\nu\|^2 \right) + C \|U\|_s^2 \|W\|_s. \end{aligned}$$

By Proposition 2.8 and noting that

$$\sum_{k+|\alpha| \leq s} \sum_{\nu=e,i} \|\partial_t^k U^\nu\|_{|\alpha|}^2 \sim \|U\|_s^2,$$

we have, after a modification of these constants (still denoted by  $\lambda_{(k,|\alpha|)}$ ),

$$\begin{aligned} & \frac{d}{dt} \sum_{k+|\alpha| \leq s} \lambda_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) + \|U\|_s^2 \\ & \leq C \sum_{k=0}^s \sum_{\nu=e,i} \|\partial_t^k u^\nu\|^2 + C \|U\|_s^2 \|W\|_s. \end{aligned}$$

Next, utilizing Proposition 2.4 and modifying again these constants  $\lambda_{(k,|\alpha|)}$ , we further obtain

$$\frac{d}{dt} \sum_{k+|\alpha| \leq s} \lambda_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) + \|U\|_s^2 \leq C \|U\|_s^2 \|W\|_s.$$

Finally, since  $\omega_T$  is small enough, (2.60) follows.  $\square$

The following result concerns the time dissipation of  $F$  and  $G$ .

**Lemma 2.9.** (see [56]) *It holds*

$$(2.64) \quad \|F\|_{s-1} \leq C \|U\|_s,$$

and

$$(2.65) \quad \|\partial_t G\|_{s-2} + \|\nabla G\|_{s-2} \leq C \|U\|_s.$$

**Proof of Theorem 2.1.** Combining Propositions 2.9 and Lemma 2.9 and modifying again the constants  $\lambda_{(k,|\alpha|)}$ , one has

$$\begin{aligned} & \frac{d}{dt} \sum_{k+|\alpha| \leq s} \lambda_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) \\ & \quad + \|U\|_s^2 + \|F\|_{s-1}^2 + \|\partial_t G\|_{s-2}^2 + \|\nabla G\|_{s-2}^2 \leq 0. \end{aligned}$$

Integrating the inequality over  $[0, t]$  and noting that

$$\sum_{k+|\alpha|\leq s} \lambda_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) \sim |||W|||_s^2,$$

we have

$$(2.66) \quad \begin{aligned} & |||W(t)|||_s^2 + \int_0^t (|||U(\tau)|||_s^2 + |||F(\tau)|||_{s-1}^2 + |||\partial_\tau G|||_{s-2}^2 + |||\nabla G(\tau)|||_{s-2}^2) d\tau \\ & \leq |||W(0)|||_s^2, \quad \forall t \in [0, T]. \end{aligned}$$

Since the Euler-Maxwell system (2.2) can be written as  $\partial_t W = f(x, W, \partial_x W)$  with a smooth function  $f$  such that  $f(x, 0, 0) = 0$ , it follows from Lemma 2.3 that

$$|||W(0)|||_s \leq C \|W^0\|_s.$$

This implies the global existence of solutions and estimates (2.8)-(2.10).

Furthermore, (2.66) implies that, for all  $k + |\beta| \leq s - 1$ ,

$$\partial_t^k \partial^\beta W \in L^\infty(\mathbb{R}^+, L^2(\mathbb{T}))$$

and

$$\partial_t(\partial_t^k \partial^\beta W) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{T})).$$

Then,

$$\partial_t^k \partial^\beta W \in W^{1,\infty}(\mathbb{R}^+, L^2(\mathbb{T})), \quad \forall k + |\beta| \leq s - 1.$$

Furthermore,

$$\partial_t^k \partial^\beta (n^\nu - \bar{n}^\nu, u^\nu, E - \bar{E}) \in L^2(\mathbb{R}^+, L^2(\mathbb{T})), \quad \forall k + |\beta| \leq s - 1.$$

We deduce that

$$\partial_t^k \partial^\beta (n^\nu - \bar{n}^\nu, u^\nu, E - \bar{E}) \in L^2(\mathbb{R}^+, L^2(\mathbb{T})) \cap W^{1,\infty}(\mathbb{R}^+, L^2(\mathbb{T})), \quad \forall k + |\beta| \leq s - 1,$$

which implies (2.11)-(2.12). Similarly, we have

$$\partial_t^k \partial^\beta B \in L^2(\mathbb{R}^+, L^2(\mathbb{T})) \cap W^{1,\infty}(\mathbb{R}^+, L^2(\mathbb{T})), \quad \forall 1 \leq k + |\beta| \leq s - 1,$$

which implies

$$(2.67) \quad \lim_{t \rightarrow \infty} (|||\partial_t B(t)|||_{s-2} + |||\nabla \times B(t)|||_{s-2}) = 0.$$

Remark that  $\bar{B}$  is a constant. Using  $\int_{\mathbb{T}} B^0(x) dx = \bar{B}$  and the conservative equations for  $B$ , we get

$$\int_{\mathbb{T}} B(t, x) dx = \int_{\mathbb{T}} B^0(x) dx = \bar{B}, \quad \forall t \geq 0.$$



It follows from the Poincaré inequality that

$$\|B(t) - \bar{B}\| \leq C\|\nabla B(t)\|.$$

Together with (2.67) yields (2.13).  $\square$

#### 2.4.2 Proof of Theorem 2.2

We still use notations in (2.24) and (2.28), with

$$(2.68) \quad \bar{E} = -\nabla\bar{\phi}, \quad \Phi = \phi - \bar{\phi}, \quad F = -\nabla\Phi.$$

Then the two-fluid Euler-Poisson system (2.15) can be written as :

$$(2.69) \quad \begin{cases} \partial_t N^\nu + u^\nu \cdot \nabla N^\nu + (\bar{n}^\nu + N^\nu)\nabla \cdot u^\nu + \nabla \bar{n}^\nu \cdot u^\nu = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla)u^\nu + \nabla h_\nu (\bar{n}^\nu + N^\nu) - \nabla h_\nu (\bar{n}^\nu) + q_\nu \nabla \Phi + u_\nu = 0, \\ \Delta \Phi = N^e - N^i, \quad \text{in } \mathbb{R}^+ \times \mathbb{T}, \end{cases}$$

in which the Euler equations are the special case of those of (2.25) with  $B = 0$ , while the Maxwell equations in (2.25) are replaced by  $\Delta \Phi = N^e - N^i$ .

Now we want to establish a similar energy estimate to (2.60) in the present case with  $G = 0$ . By checking all the steps before (2.60), we see that the Maxwell equations are concerned only in the proof of Propositions 2.2 and Lemmas 2.5-2.6 and 2.8. Essentially, we have to deal with the second order term  $\sum_{\nu=e,i} q_\nu \langle \partial_t^k \partial^\alpha (n^\nu u^\nu), F_{k,\alpha} \rangle$  for all  $k + |\alpha| \leq s$ , appeared in the proof due to the Maxwell equations (see (2.37) and (2.42)). It can be calculated as follows. From

$$F = -\nabla\Phi, \quad \partial_t N^\nu = -\nabla \cdot (n^\nu u^\nu) \quad \text{and} \quad \Delta \Phi = N^e - N^i,$$

we obtain

$$\begin{aligned} 2 \sum_{\nu=e,i} q_\nu \langle \partial_t^k \partial^\alpha (n^\nu u^\nu), \partial_t^k \partial^\alpha \nabla \Phi \rangle &= -2 \langle \partial_t^k \partial^\alpha (n^e u^e - n^i u^i), \partial_t^k \partial^\alpha \nabla \Phi \rangle \\ &= -2 \langle \partial_t^k \partial^\alpha \partial_t (N^e - N^i), \partial_t^k \partial^\alpha \Phi \rangle \\ &= \frac{d}{dt} \|\partial_t^k \partial^\alpha \nabla \Phi\|^2. \end{aligned}$$

This shows the validity of all the steps before (2.60). Consequently, there are positive constants  $b_{(k,|\alpha|)}$  such that

$$\frac{d}{dt} \sum_{k+|\alpha| \leq s} b_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|\partial_t^k \partial^\alpha \nabla \Phi\|^2 \right) + \|U\|_s^2 \leq 0.$$

On the other hand, the Poisson equation  $\Delta\Phi = N^e - N^i$  yields

$$|||\nabla\Phi|||_s \leq C \sum_{\nu=e,i} |||N^\nu|||_s \leq C|||U|||_s.$$

It follows that, after modifying the positive constants  $b_{(k,|\alpha|)}$ ,

$$\frac{d}{dt} \sum_{k+|\alpha|\leq s} b_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|\partial_t^k \partial^\alpha \nabla\Phi\|^2 \right) + |||U|||_s^2 + |||F|||_s^2 \leq 0.$$

Let us denote

$$\mathfrak{E}(t) = \sum_{k+|\alpha|\leq s} b_{(k,|\alpha|)} \left( \sum_{\nu=e,i} \langle A_0^\nu U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \rangle + \|\partial_t^k \partial^\alpha \nabla\Phi\|^2 \right).$$

Since  $b_{(k,|\alpha|)} > 0$  and  $A_0^\nu$  is positively definite, it is not difficult to see that

$$\mathfrak{E}(t) \sim |||U(t)|||_s^2 + |||\nabla\Phi(t)|||_s^2.$$

Thus, there exists a constant  $\gamma > 0$  such that

$$\frac{d}{dt} \mathfrak{E}(t) + \gamma \mathfrak{E}(t) \leq 0, \forall t \in [0, T],$$

which implies that

$$\mathfrak{E}(t) \leq \mathfrak{E}(0)e^{-\gamma t}, \forall t \in [0, T].$$

Since  $\mathfrak{E}(0) \leq C|||U^0|||_s$ , we obtain finally

$$|||U(t)|||_s^2 + |||\nabla\Phi(t)|||_s^2 \leq Ce^{-\gamma t} |||U^0|||_s^2, \forall t \in [0, T].$$

This shows the global existence of smooth solutions with the exponential decay estimate of Theorem 2.2.  $\square$



## Chapitre 3

# Asymptotic behavior of global smooth solutions for one-fluid non isentropic Euler-Maxwell systems

### 3.1 Introduction and main results

In this Chapter, we continue to study Euler-Maxwell systems in non isentropic case. Let us consider the Cauchy problem for system (1.21) with  $\theta_* = 1$  :

$$(3.1) \quad \begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \frac{1}{n} \nabla (n\theta) = -(E + u \times B) - u, \\ \partial_t \theta + u \cdot \nabla \theta + \frac{2}{3} \theta \nabla \cdot u + \frac{1}{3} |u|^2 + (\theta - 1) = 0, \\ \partial_t E - \nabla \times B = nu, \quad \nabla \cdot E = 1 - n, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3. \end{cases}$$

Initial values are given as :

$$(3.2) \quad (n, u, \theta, E, B)|_{t=0} = (n^0, u^0, \theta^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

with the compatibility condition :

$$(3.3) \quad \nabla \cdot E^0 = 1 - n^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

Similarly to the isentropic Euler-Maxwell system (2.2) in Chapter 2, the non isentropic Euler-Maxwell system (3.1) is also a symmetrizable hyperbolic system for  $n, \theta > 0$ . It follows that the Cauchy problem (3.1)-(3.2) has a local smooth solution when the initial data are smooth.

**Proposition 3.1.** *(Local existence of smooth solutions for non isentropic Euler-Maxwell systems, see [41, 46]) Assume integer  $s \geq 3$  and (3.3) holds. Suppose  $(n^0 - 1, u^0, \theta^0 - 1, E^0, B^0) \in H^s$  with  $n^0, \theta^0 \geq 2\kappa$  for some given constant  $\kappa > 0$ . Then there exists  $T_1 > 0$  such that problem (3.1)-(3.2) has a unique smooth solution satisfying  $n, \theta \geq \kappa$  in  $[0, T_1] \times \mathbb{R}^3$  and*

$$(n - 1, u, \theta - 1, E, B) \in C^1([0, T_1]; H^{s-1}) \cap C([0, T_1]; H^s),$$

where and in the following of this thesis,  $H^s = H^s(\mathbb{R}^3)$  and  $L^p = L^p(\mathbb{R}^3)$  with the norm  $\|\cdot\|_s$  and  $\|\cdot\|_{L^p}$ , respectively, and  $\|\cdot\| = \|\cdot\|_{L^2}$  in particular.

According to our knowledge, there is no analysis on the global existence of smooth solutions around an equilibrium solution  $(n, u, \theta, E, B) = (1, 0, 1, 0, 0)$  for the non isentropic Euler-Maxwell system in  $\mathbb{R}^3$  so far. The goal of the present Chapter is to establish such a result. The main results can be stated as follows.

**Théorème 3.1.** *Let  $s \geq 4$  and (3.3) hold. There exist  $\delta_0 > 0$  and  $C_0 > 0$  such that if*

$$\|(n^0 - 1, u^0, \theta^0 - 1, E^0, B^0)\|_s \leq \delta_0,$$

then, the Cauchy problem (3.1)- (3.2) admits a unique global solution  $(n, u, \theta, E, B)$  with

$$(n - 1, u, \theta - 1, E, B) \in C^1(\mathbb{R}^+; H^{s-1}) \cap C(\mathbb{R}^+; H^s)$$

and

$$\sup_{t \geq 0} \|(n(t) - 1, u(t), \theta(t) - 1, E(t), B(t))\|_s \leq C_0 \|(n^0 - 1, u^0, \theta^0 - 1, E^0, B^0)\|_s.$$

Moreover, there exist  $\delta_1 > 0$  and  $C_1 > 0$  such that if

$$\|(n^0 - 1, u^0, \theta^0 - 1, E^0, B^0)\|_{13} + \|(u^0, E^0, B^0)\|_{L^1} \leq \delta_1,$$

then the solution  $(n, u, \theta, E, B)$  satisfies that for any  $t \geq 0$ ,

$$\begin{aligned} \|(n(t) - 1, \theta(t) - 1)\|_{L^q} &\leq C_1(1+t)^{-\frac{1}{4}}, \quad \forall 2 \leq q \leq +\infty, \\ \|(u(t), E(t))\|_{L^q} &\leq C_1(1+t)^{-2+\frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty, \\ \|B(t)\|_{L^q} &\leq C_1(1+t)^{-\frac{3}{2}+\frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty. \end{aligned}$$

The proof of Theorem 3.1 is based on careful energy estimates and the Fourier multiplier technique. This is divided into three key steps.

The first key step is to establish the global a priori higher order Sobolev energy estimates in time by using the careful energy methods and the skew-symmetric dissipative

structure (see [68]) of Euler-Maxwell systems, which concludes the global existence results in Theorem 3.1. Due to the complexity of non isentropic case caused by the coupled energy equations, the technique of symmetrizer is used here to obtain the energy estimates. It is different from those used in [17, 68] to deal with the isentropic case of compressible Euler-Maxwell systems.

The second key step is to obtain the  $L^p - L^q$  time decay rate of the linearized operator for the non isentropic Euler-Maxwell system by using the Fourier technique. It is used by Kawashima in [42] and extended then to the other problems, see [32, 17, 67] and the references therein. Here we first apply energy methods in the Fourier space to obtain the basic  $L^\infty$  estimates for the Fourier transform of the solution. Then we solve the dissipative linear wave system of three order by the Fourier technique. We find that the 'error' functions  $\rho$ ,  $\Theta$ ,  $\nabla \cdot u$  satisfy the same dissipative linear wave equation which is of order three and is different from that of the isentropic Euler-Maxwell system. We also find an interesting phenomenon that the estimate on  $B$  depending on the temperature  $\Theta$  is different from that of the isentropic Euler-Maxwell equation. Since the linear system involved here is of order three, it is complex to obtain the time decay rate of the linear system by the Fourier analysis. Fortunately, we can obtain an elaborate spectrum structure of the eigenvalue equations of this linear system, which yields the desired time decay rate for the linearized system.

The third step of the proof is to obtain the time decay rate in Theorem 3.1 by combining the previous two steps and to apply the energy estimate technique to the nonlinear problem satisfied by the error functions, whose solutions can be represented by the semi-group operator for the linearized problem by using the Duhamel principle. This concludes the time asymptotic stability results in Theorem 3.1.

This Chapter is organized as follows. In Section 3.2, the transformation of system (3.1) is presented. In Section 3.3, we obtain the existence and uniqueness of global solutions. In Section 3.4, we study the linearized homogeneous system to get the  $L^p - L^q$  decay property and the explicit representation of solutions. Lastly, in Section 3.5, we investigate the decay rates of solutions of the nonlinear system (3.5) and complete the proof of Theorem 3.1.

## 3.2 Preliminaries

Besides the notations in Chapter 2, we also introduce some additional notations for later use. For an integrable function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , its Fourier transform is defined by

$$\hat{f}(k) = \int_{\mathbb{R}^3} e^{-ix \cdot k} f(x) dx, \quad x \cdot k := \sum_{j=1}^3 x_j k_j, \quad k \in \mathbb{R}^3,$$

where  $i = \sqrt{-1} \in \mathbb{C}$  is the imaginary unit. For two complex numbers or vectors  $a$  and  $b$ ,  $(a|b) := a \cdot \bar{b}$  denotes the dot product of  $a$  with the complex conjugate of  $b$ . We also use  $\Re(a)$  to denote the real part of  $a$ .

Next, we review Moser-type calculus and Hausdorff-Young inequalities, which are used in the rest of this thesis.

**Lemma 3.1.** (*Moser-type calculus inequalities, see [46, 43]*) *Let  $s \geq 1$  be an integer. Suppose  $u \in H^s$ ,  $\nabla u \in L^\infty$  and  $v \in H^{s-1} \cap L^\infty$ . Then for all multi-index  $\alpha$  with  $1 \leq |\alpha| \leq s$ , it holds*

$$\partial^\alpha(uv) - u\partial^\alpha v \in L^2,$$

and

$$\|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C_s (\|\nabla u\|_{L^\infty} \|D^{|\alpha|-1} v\| + \|D^{|\alpha|} u\| \|v\|_{L^\infty}),$$

where

$$\|D^{s'} u\| = \sum_{|\alpha|=s'} \|\partial^\alpha u\|.$$

Furthermore, if  $s \geq 3$ , then the embedding  $H^{s-1} \hookrightarrow L^\infty$  is continuous and we have

$$\|uv\|_{s-1} \leq C_s \|u\|_{s-1} \|v\|_{s-1}, \quad \forall u, v \in H^{s-1},$$

and for all  $u, v \in H^s$  and all smooth function  $f$ ,

$$\|\partial^\alpha f(u)\| \leq C_\infty (1 + \|u\|_s)^{s-1} \|u\|_s, \quad \|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C_s \|u\|_s \|v\|_{s-1}, \quad \forall |\alpha| \leq s.$$

**Lemma 3.2.** (*Hausdorff-Young inequalities, see [54]*) *For  $1 \leq p \leq 2$  and  $2 \leq q \leq +\infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , there exists a constant  $C > 0$  such that*

$$\|\hat{f}\|_{L^q} \leq C \|f\|_{L^p}, \quad \forall f \in L^p,$$

and

$$\|f\|_{L^q} \leq C \|\hat{f}\|_{L^p}, \quad \forall \hat{f} \in L^p.$$

Now, let  $(n, u, \theta, E, B)$  be a local smooth solution to the Cauchy problem of the non isentropic Euler-Maxwell system (3.1) with given initial values (3.2) satisfying (3.3). We introduce

$$(3.4) \quad n = 1 + \rho, \quad \theta = 1 + \Theta.$$

Then, system (3.1) becomes :

$$(3.5) \quad \begin{cases} \partial_t \rho + \nabla \cdot ((1 + \rho)u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \nabla \Theta + \frac{1 + \Theta}{1 + \rho} \nabla \rho = -(E + u \times B) - u, \\ \partial_t \Theta + u \cdot \nabla \Theta + \frac{2}{3}(1 + \Theta) \nabla \cdot u + \frac{1}{3}|u|^2 + \Theta = 0, \\ \partial_t E - \nabla \times B = (1 + \rho)u, \quad \nabla \cdot E = -\rho, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with the initial condition :

$$(3.6) \quad W|_{t=0} = (\rho, u, \Theta, E, B)|_{t=0} = W^0 := (\rho^0, u^0, \Theta^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfies the compatibility condition :

$$(3.7) \quad \nabla \cdot E^0 = -\rho^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3,$$

with  $\rho^0 = n^0 - 1$  and  $\Theta^0 = \theta^0 - 1$ .

In the following of this thesis, we always set  $s \geq 4$ . Besides, for  $W = (\rho, u, \Theta, E, B)$ , we use  $\mathfrak{E}_s(W(t))$ ,  $\mathfrak{E}_s^h(W(t))$ ,  $\mathfrak{D}_s(W(t))$ ,  $\mathfrak{D}_s^h(W(t))$  to denote the energy functionals, the higher order energy functionals, the dissipative rate and the higher order dissipative rate, respectively. Here,

$$(3.8) \quad \mathfrak{E}_s(W(t)) \sim \|(\rho, u, \Theta, E, B)\|_s^2,$$

$$(3.9) \quad \mathfrak{E}_s^h(W(t)) \sim \|\nabla(\rho, u, \Theta, E, B)\|_{s-1}^2,$$

$$(3.10) \quad \mathfrak{D}_s(W(t)) \sim \|(\rho, u, \Theta)\|_s^2 + \|E\|_{s-1}^2 + \|\nabla B\|_{s-2}^2,$$

and

$$(3.11) \quad \mathfrak{D}_s^h(W(t)) \sim \|\nabla(\rho, u, \Theta)\|_{s-1}^2 + \|\nabla E\|_{s-2}^2 + \|\nabla^2 B\|_{s-3}^2.$$

Then, for the Cauchy problem (3.5)-(3.6), we have the following global existence result.



**Proposition 3.2.** *Suppose (3.7) for given initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$ . Then, there exist  $\mathfrak{E}_s(\cdot)$  and  $\mathfrak{D}_s(\cdot)$  in the form of (3.8) and (3.10) such that the following holds true. If  $\mathfrak{E}_s(W^0) > 0$  is small enough, the Cauchy problem (3.5)-(3.6) admits a unique global solution  $W = (\rho, u, \Theta, E, B)$  satisfying*

$$(3.12) \quad W \in C^1(\mathbb{R}^+; H^{s-1}) \cap C(\mathbb{R}^+; H^s),$$

and

$$(3.13) \quad \mathfrak{E}_s(W(t)) + \int_0^t \mathfrak{D}_s(W(\tau)) d\tau \leq \mathfrak{E}_s(W_0), \quad \forall t \geq 0.$$

Furthermore, we study the time decay rates of solutions in Proposition 3.2 under some extra conditions on the given initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$ . For that, we define  $\omega_{s_*}(W^0)$  as

$$(3.14) \quad \omega_{s_*}(W^0) = \|W^0\|_{s_*} + \|(u^0, E^0, B^0)\|_{L^1},$$

for  $s_* \geq 4$ . Then we have two propositions as follows.

**Proposition 3.3.** *Suppose that  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfies (3.7). Then, if  $\omega_{s+2}(W^0)$  is small enough, the solution  $W = (\rho, u, \Theta, E, B)$  satisfies*

$$(3.15) \quad \|W(t)\|_s \leq C\omega_{s+2}(W^0)(1+t)^{-\frac{3}{4}}, \quad \forall t \geq 0.$$

Moreover, if  $\omega_{s+6}(W^0)$  is small enough, then the solution  $W$  also satisfies

$$(3.16) \quad \|\nabla W(t)\|_{s-1} \leq C\omega_{s+6}(W^0)(1+t)^{-\frac{5}{4}}, \quad \forall t \geq 0.$$

**Proposition 3.4.** *Let  $2 \leq q \leq \infty$ . Suppose that  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfies (3.7) and  $\omega_{13}(W^0)$  is small enough. Then, the solution  $W = (\rho, u, \Theta, E, B)$  satisfies that for any  $t \geq 0$ ,*

$$(3.17) \quad \|(\rho(t), \Theta(t))\|_{L^q} \leq C(1+t)^{-\frac{11}{4}},$$

$$(3.18) \quad \|(u(t), E(t))\|_{L^q} \leq C(1+t)^{-2+\frac{3}{2q}},$$

and

$$(3.19) \quad \|B(t)\|_{L^q} \leq C(1+t)^{-\frac{3}{2}+\frac{3}{2q}}.$$

Finally, it is easy to see that Theorem 3.1 follows from Propositions 3.2-3.4. Thus, the rest of this Chapter is to prove the stated-above three propositions.

### 3.3 Global solutions for nonlinear systems

#### 3.3.1 A priori estimates

In this subsection, we prove the following a priori estimates.

**Théorème 3.2.** *Let  $W = (\rho, u, \Theta, E, B) \in C^1([0, T]; H^{s-1}) \cap C([0, T]; H^s)$  be the solution to the Cauchy problem (3.5)-(3.6) for  $t \in (0, T)$  with  $T > 0$ . Then, if*

$$(3.20) \quad \sup_{0 \leq t \leq T} \|W(t)\|_s \leq \delta_0$$

with  $\delta_0$  sufficiently small, there exist  $\mathfrak{E}_s(\cdot)$  and  $\mathfrak{D}_s(\cdot)$  in the form of (3.8) and (3.10) such that for any  $0 \leq t \leq T$ ,

$$(3.21) \quad \frac{d}{dt} \mathfrak{E}_s(W(t)) + \mathfrak{D}_s(W(t)) \leq C \mathfrak{E}_s(W(t))^{\frac{1}{2}} \mathfrak{D}_s(W(t)).$$

**Proof.** We use five steps to prove the result. The step 1 is to estimate the Euler part and the Maxwell part of the Euler-Maxwell system, respectively. Steps 2-4 are to obtain the dissipative estimates for  $\rho$ ,  $E$  and  $B$  by using the skew-symmetric structure of the Euler-Maxwell system.

*Step 1.* It holds that

$$(3.22) \quad \frac{d}{dt} \sum_{|\alpha| \leq s} (\langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle + \|\partial^\alpha W_{II}\|^2) + \|u\|_s^2 + \|\Theta\|_s^2 \leq C \|W\|_s \|W_I\|_s^2,$$

where

$$W_I = \begin{pmatrix} \rho \\ u \\ \Theta \end{pmatrix}, \quad W_{II} = \begin{pmatrix} E \\ B \end{pmatrix}, \quad W = \begin{pmatrix} W_I \\ W_{II} \end{pmatrix}, \quad A_0^I(W_I) = \begin{pmatrix} \frac{1+\Theta}{1+\rho} & 0 & 0 \\ 0 & (1+\rho)I_3 & 0 \\ 0 & 0 & \frac{3}{2} \frac{1+\rho}{1+\Theta} \end{pmatrix}.$$

In fact, the Euler equations of (3.5) for  $W_I$  can be rewritten under the form :

$$(3.23) \quad \partial_t W_I + \sum_{j=1}^3 A_j^I(W_I) \partial_j W_I = K_1(W) + K_2(W),$$

with

$$A_j^I(W_I) = \begin{pmatrix} u_j & (1+\rho)e_j^T & 0 \\ \frac{1+\Theta}{1+\rho}e_j & u_j I_3 & e_j \\ 0 & \frac{2}{3}(1+\Theta)e_j^T & u_j \end{pmatrix}, \quad j = 1, 2, 3,$$

and

$$K_1(W) = - \begin{pmatrix} 0 \\ E + u \times B \\ 0 \end{pmatrix}, \quad K_2(W) = - \begin{pmatrix} 0 \\ u \\ \frac{1}{3}|u|^2 + \Theta \end{pmatrix}.$$

It is clear that system (3.23) for  $W_I$  is symmetrizable hyperbolic when  $1 + \rho, 1 + \Theta > 0$ . More precisely, since we consider small solutions defined in a time interval  $[0, T]$  with  $T > 0$ , (3.20) implies that, for  $s \geq 2$ ,

$$\|(\rho, \Theta)\|_{L^\infty} \leq C_s \|(\rho, \Theta)\|_s \leq C_s \delta_0 \leq \frac{1}{2}.$$

Then,

$$\frac{1}{2} \leq 1 + \rho, 1 + \Theta \leq \frac{3}{2}.$$

It follows that  $A_0^I(W_I)$  is symmetric positive definite and  $\tilde{A}_j^I(W_I) = A_0^I(W_I)A_j^I(W_I)$  are symmetric for all  $1 \leq j \leq 3$ .

Then, for  $|\alpha| \leq s$ , differentiating equations (3.23) with respect to  $x$  and multiplying the resulting equations by the symmetrizer matrix  $A_0^I(W_I)$ , we have

$$(3.24) \quad \begin{aligned} & A_0^I(W_I) \partial_t \partial^\alpha W_I + \sum_{j=1}^3 A_0^I(W_I) A_j^I(W_I) \partial_j \partial^\alpha W_I \\ &= A_0^I(W_I) \partial^\alpha (K_1(W) + K_2(W)) + J_\alpha, \end{aligned}$$

where  $J_\alpha$  is defined by

$$J_\alpha = - \sum_{j=1}^3 A_0^I(W_I) (\partial^\alpha (A_j^I(W_I) \partial_j W_I) - A_j^I(W_I) \partial^\alpha (\partial_j W_I)).$$

Applying Lemma 3.1 to  $J_\alpha$ , we get

$$(3.25) \quad \begin{aligned} \|J_\alpha\| &\leq C (\|\nabla A_j^I(W_I)\|_{L^\infty} \|\partial_j W_I\|_{s-1} + \|D^s A_j^I(W_I)\| \|\partial_j W_I\|_{L^\infty}) \\ &\leq C (\|W_I\|_s + \|W_I\|_s) \|W_I\|_s \leq C \|W_I\|_s^2. \end{aligned}$$

Taking the inner product of equations (3.24) with  $\partial^\alpha W_I$  in  $L^2$  and using the fact that the matrix  $\tilde{A}_j^I(W_I)$  is symmetric, we have

$$(3.26) \quad \begin{aligned} \frac{d}{dt} \langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle &= 2 \langle J_\alpha, \partial^\alpha W_I \rangle + \langle \operatorname{div} A^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle \\ &\quad + 2 \langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha K_1(W) + \partial^\alpha K_2(W) \rangle, \end{aligned}$$

where

$$\operatorname{div} A^I(W_I) = \partial_\rho A_0^I(W_I) \partial_t \rho + \partial_\Theta A_0^I(W_I) \partial_t \Theta + \sum_{j=1}^3 (\tilde{A}_j^I)'(W_I) \partial_j W_I.$$

By the first and third equations in (3.5) and Lemma 3.1, we have

$$(3.27) \quad \begin{cases} \|\partial_t \rho\|_{L^\infty} \leq C \|\partial_t \rho\|_{s-1} = C \|\nabla \cdot ((1 + \rho)u)\|_{s-1} \leq C(1 + \|\rho\|_s) \|u\|_s, \\ \|\partial_t \Theta\|_{L^\infty} \leq C \left( \|u \cdot \nabla \Theta\|_{s-1} + \|(1 + \Theta) \nabla \cdot u\|_{s-1} + \| |u|^2 \|_{s-1} + \|\Theta\|_{s-1} \right) \\ \leq C(1 + \|\Theta\|_s + \|u\|_s)(1 + \|u\|_s). \end{cases}$$

Then

$$(3.28) \quad \|\operatorname{div} A^I(W_I)\|_{L^\infty} \leq C(1 + \|W_I\|_s) \|W_I\|_s.$$

Now, let us estimate each term on the right hand side of (3.26). For the first two terms, by the Cauchy-Schwarz inequality and using estimates (3.25) and (3.28), we have

$$(3.29) \quad \langle J_\alpha, \partial^\alpha W_I \rangle + \langle \operatorname{div} A^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle \leq C(1 + \|W_I\|_s) \|W_I\|_s^3 \leq C \|W\|_s \|W_I\|_s^2.$$

For the third term of the right hand side of (3.26), by the definitions of  $A_0^I(W_I)$ ,  $K_1(W)$  and  $K_2(W)$ , we obtain

$$(3.30) \quad \begin{aligned} & 2\langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha K_1(W) + \partial^\alpha K_2(W) \rangle \\ &= -2\langle (1 + \rho) \partial^\alpha u, \partial^\alpha u \rangle - 2\langle \partial^\alpha u, \partial^\alpha E \rangle - 2\langle \partial^\alpha u, \partial^\alpha (u \times B) \rangle - 2\langle \rho \partial^\alpha u, \partial^\alpha E \rangle \\ &\quad - 2\langle \rho \partial^\alpha u, \partial^\alpha (u \times B) \rangle - \left\langle \frac{1 + \rho}{1 + \Theta} \partial^\alpha \Theta, \partial^\alpha (|u|^2) \right\rangle - 3\left\langle \frac{1 + \rho}{1 + \Theta} \partial^\alpha \Theta, \partial^\alpha \Theta \right\rangle \\ &\leq -\|\partial^\alpha u\|^2 - \|\partial^\alpha \Theta\|^2 - 2\langle \partial^\alpha u, \partial^\alpha E \rangle + C \|W\|_s \|W_I\|_s^2. \end{aligned}$$

Next we write the system for  $W_{II}$  as :

$$\partial_t W_{II} + \sum_{j=1}^3 A_j^{II} \partial_j W_{II} = ((1 + \rho)u, 0)^T,$$

where

$$A_j^{II} = \begin{pmatrix} 0 & L_j \\ L_j^T & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, similarly to (2.37) and (2.42) in Chapter 2, we have

$$(3.31) \quad \frac{d}{dt} \|\partial^\alpha W_{II}\|^2 = 2\langle \partial^\alpha E, \partial^\alpha u \rangle + 2\langle \partial^\alpha E, \partial^\alpha (\rho u) \rangle \leq 2\langle \partial^\alpha E, \partial^\alpha u \rangle + C \|W\|_s \|W_I\|_s^2.$$

Combining (3.26) and (3.29)-(3.31), summing up for all  $|\alpha| \leq s$ , we get (3.22).

*Step 2.* It holds that

$$(3.32) \quad \frac{d}{dt} \sum_{|\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle + c_0 \|\rho\|_s^2 \leq C \|W\|_s \|W_I\|_s^2 + C \|(u, \Theta)\|_s^2.$$

In fact, for  $|\alpha| \leq s - 1$ , differentiating the second equation of (3.5) with respect to  $x$  and taking the inner product of the resulting equation with  $\partial^\alpha \nabla \rho$  in  $L^2$ , we have

$$\begin{aligned}
(3.33) \quad & \left\langle \frac{1 + \Theta}{1 + \rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \right\rangle + \langle \partial^\alpha E, \partial^\alpha \nabla \rho \rangle \\
& = -\left\langle \partial^\alpha \left( \frac{1 + \Theta}{1 + \rho} \nabla \rho \right) - \frac{1 + \Theta}{1 + \rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \right\rangle - \langle \partial^\alpha \partial_t u, \partial^\alpha \nabla \rho \rangle \\
& \quad - \langle \partial^\alpha (u \cdot \nabla u + \nabla \Theta + u \times B), \partial^\alpha \nabla \rho \rangle - \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle.
\end{aligned}$$

Let us estimate each term in (3.33). First, noting that

$$\frac{1}{2} \leq 1 + \rho, 1 + \Theta \leq \frac{3}{2},$$

we have

$$\begin{aligned}
(3.34) \quad & \left\langle \frac{1 + \Theta}{1 + \rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \right\rangle + \langle \partial^\alpha E, \partial^\alpha \nabla \rho \rangle = \left\langle \frac{1 + \Theta}{1 + \rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \right\rangle + \langle \partial^\alpha \rho, \partial^\alpha \rho \rangle \\
& \geq C^{-1} (\|\partial^\alpha \nabla \rho\|^2 + \|\partial^\alpha \rho\|^2).
\end{aligned}$$

By Lemma 3.1, we obtain

$$\left\| \partial^\alpha \left( \frac{1 + \Theta}{1 + \rho} \nabla \rho \right) - \frac{1 + \Theta}{1 + \rho} \partial^\alpha \nabla \rho \right\| \leq C \|W_I\|_s^2.$$

Then,

$$(3.35) \quad -\left\langle \partial^\alpha \left( \frac{1 + \Theta}{1 + \rho} \nabla \rho \right) - \frac{1 + \Theta}{1 + \rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \right\rangle \leq C \|W\|_s \|W_I\|_s^2.$$

Obviously,

$$-\langle \partial^\alpha \partial_t u, \partial^\alpha \nabla \rho \rangle = \frac{d}{dt} \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle - \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \partial_t \rho \rangle.$$

Then, from (3.27) we have

$$\left| \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \partial_t \rho \rangle \right| \leq \|\partial^\alpha \nabla \cdot u\| \|\partial^\alpha \partial_t \rho\| \leq C \|u\|_s^2 + C \|W\|_s \|W_I\|_s^2.$$

Hence,

$$(3.36) \quad -\langle \partial^\alpha \partial_t u, \partial^\alpha \nabla \rho \rangle \leq \frac{d}{dt} \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle + C \|u\|_s^2 + C \|W\|_s \|W_I\|_s^2.$$

From (3.20), we have

$$\begin{aligned}
(3.37) \quad & \left| \langle \partial^\alpha (u \cdot \nabla u + \nabla \Theta + u \times B), \partial^\alpha \nabla \rho \rangle \right| \\
& \leq (\|\partial^\alpha (u \cdot \nabla u)\| + \|\partial^\alpha (u \times B)\| + \|\partial^\alpha \nabla \Theta\|) \|\partial^\alpha \nabla \rho\| \\
& \leq C (\|u\|_{s-1} \|u\|_s + \|u\|_{s-1} \|B\|_{s-1} + \|\Theta\|_s) \|\rho\|_s \\
& \leq C \|W\|_s \|W_I\|_s^2 + \varepsilon \|\rho\|_s^2 + C_\varepsilon \|\Theta\|_s^2.
\end{aligned}$$

Next, by the Cauchy-Schwarz inequality, we obtain the uniform estimates for the last term on the right hand side of (3.33)

$$(3.38) \quad -\langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle \leq \varepsilon \|\rho\|_s^2 + C_\varepsilon \|u\|_{s-1}^2.$$

Thus, combining (3.33)-(3.38), we get

$$\frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle + C^{-1} (\|\partial^\alpha \nabla \rho\|^2 + \|\partial^\alpha \rho\|^2) \leq C \|W\|_s \|W_I\|_s^2 + \varepsilon \|\rho\|_s^2 + C_\varepsilon (\|u\|_s^2 + \|\Theta\|_s^2).$$

Summing up this inequality for all  $|\alpha| \leq s-1$  and taking  $\varepsilon > 0$  small enough, we obtain (3.32).

*Step 3.* It holds that

$$(3.39) \quad \begin{aligned} & \frac{d}{dt} \sum_{|\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha E \rangle + c_0 \|E\|_{s-1}^2 \\ & \leq C \|W_{II}\|_s (\|u\|_s^2 + \|E\|_{s-1}^2) + C \|W_I\|_s^2 + C \|u\|_s \|\nabla B\|_{s-2} + C \|W\|_s \|W_I\|_s^2. \end{aligned}$$

In fact, for  $|\alpha| \leq s-1$ , applying  $\partial^\alpha$  to the second equation of (3.5), taking the inner product of the resulting equation with  $\partial^\alpha E$  in  $L^2$ , and then using the fourth equation of (3.5), we get

$$\begin{aligned} & \frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha E \rangle + \|\partial^\alpha E\|^2 \\ & = -\langle \partial^\alpha (u \cdot \nabla u), \partial^\alpha E \rangle - \left\langle \partial^\alpha \left( \frac{1+\Theta}{1+\rho} \nabla \rho \right), \partial^\alpha E \right\rangle - \langle \nabla \partial^\alpha \Theta, \partial^\alpha E \rangle - \langle \partial^\alpha u, \partial^\alpha E \rangle \\ & \quad - \langle \partial^\alpha (u \times B), \partial^\alpha E \rangle + \langle \partial^\alpha u, \nabla \times \partial^\alpha B \rangle + \|\partial^\alpha u\|^2 + \langle \partial^\alpha (\rho u), \partial^\alpha u \rangle. \end{aligned}$$

Furthermore, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha E \rangle + c_0 \|\partial^\alpha E\|^2 \\ & \leq C \|E\|_{s-1} \|u\|_s^2 + C \|W_I\|_s^2 + C \|u\|_s \|\nabla B\|_{s-2} + C \|B\|_s (\|u\|_s^2 + \|E\|_{s-1}^2) + C \|W\|_s \|W_I\|_s^2. \end{aligned}$$

Therefore, summing the previous estimate over  $|\alpha| \leq s-1$ , we obtain (3.39).

*Step 4.* It holds that

$$(3.40) \quad \frac{d}{dt} \sum_{|\alpha| \leq s-2} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle + c_0 \|\nabla B\|_{s-2}^2 \leq C \|(u, E)\|_{s-1}^2 + C \|\rho\|_s \|u\|_s^2.$$

In fact, for  $|\alpha| \leq s-2$ , applying  $\partial^\alpha$  to the fourth equation of (3.5), taking the inner product of the resulting equation with  $-\partial^\alpha \nabla \times B$  in  $L^2$ , and then using the fifth equation of (3.5) give

$$\frac{d}{dt} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle + \|\nabla \times \partial^\alpha B\|^2 = \|\nabla \times \partial^\alpha E\|^2 - \langle \partial^\alpha u, \nabla \times \partial^\alpha B \rangle - \langle \partial^\alpha (\rho u), \nabla \times \partial^\alpha B \rangle.$$

Furthermore, using the Cauchy-Schwarz inequality and taking summation over  $|\alpha| \leq s-2$ , we have (3.40), where we also used

$$(3.41) \quad \|\partial^\alpha \partial_i B\| = \left\| \partial_i \Delta^{-1} \nabla \times (\nabla \times \partial^\alpha B) \right\| \leq C \|\nabla \times \partial^\alpha B\|$$

for each  $1 \leq i \leq 3$ , due to the fact that  $\nabla \cdot B = 0$  and  $\partial_i \Delta^{-1} \nabla$  is bounded from  $L^p$  to  $L^p$  for  $1 < p < \infty$ , see [66].

*Step 5.* Finally, based on four previous steps, we prove (3.21). We define the energy functionals as :

$$(3.42) \quad \begin{aligned} \mathfrak{E}_s(W(t)) = & \sum_{|\alpha| \leq s} (\langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle + \|\partial^\alpha W_{II}\|^2) + \mathfrak{K}_1 \sum_{|\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle \\ & + \mathfrak{K}_2 \sum_{|\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha E \rangle + \mathfrak{K}_3 \sum_{|\alpha| \leq s-2} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle, \end{aligned}$$

where constants  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  are to be chosen later. It follows from the fact that  $A_0^I(W_I)$  is positive definite that  $\mathfrak{E}_s(W(t)) \sim \|W\|_s^2$  as soon as  $0 < \mathfrak{K}_j \ll 1$ ,  $j = 1, 2, 3$ , are sufficiently small. Furthermore, summing (3.22), (3.32)  $\times \mathfrak{K}_1$ , (3.39)  $\times \mathfrak{K}_2$  and (3.40)  $\times \mathfrak{K}_3$ , we get

$$\begin{aligned} & \frac{d}{dt} \mathfrak{E}_s(W(t)) + \|u\|_s^2 + \|\Theta\|_s^2 + c_0 \mathfrak{K}_1 \|\rho\|_s^2 + c_0 \mathfrak{K}_2 \|E\|_{s-1}^2 + c_0 \mathfrak{K}_3 \|\nabla B\|_{s-2}^2 \\ & \leq C \mathfrak{E}_s(W(t))^{\frac{1}{2}} \mathfrak{D}_s(W(t)) + C \mathfrak{K}_1 \|(u, \Theta)\|_s^2 + C \mathfrak{K}_2 \|W_I\|_s^2 + C \mathfrak{K}_3 \|(u, E)\|_{s-1}^2 \\ & \quad + C \mathfrak{K}_2 \|u\|_s \|\nabla B\|_{s-2} \\ & \leq C \mathfrak{E}_s(W(t))^{\frac{1}{2}} \mathfrak{D}_s(W(t)) + C \mathfrak{K}_1 \|(u, \Theta)\|_s^2 + C \mathfrak{K}_2 \|W_I\|_s^2 + C \mathfrak{K}_3 \|(u, E)\|_{s-1}^2 \\ & \quad + \frac{1}{2} C \left( \mathfrak{K}_2^{\frac{1}{2}} \|u\|_s^2 + \mathfrak{K}_2^{\frac{3}{2}} \|\nabla B\|_{s-2}^2 \right). \end{aligned}$$

By letting  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  be sufficiently small with  $\mathfrak{K}_2^{\frac{3}{2}} \ll \mathfrak{K}_3$ , we obtain (3.21). This finishes the proof of Theorem 3.2.  $\square$

### 3.3.2 Proof of Proposition 2.1

The global existence of smooth solutions follows from the standard argument by using the local existence result given in Proposition 3.1, the a priori estimates (3.21) given in Theorem 3.2 and the continuous extension argument, see [53]. This finishes the proof of Proposition 3.2.  $\square$

### 3.4 Linearized homogeneous systems

In this section, in order to study the time decay property of solutions to the nonlinear system (3.5) in the next section, we are concerned with the following Cauchy problem on the linearized homogeneous system of (3.5) :

$$(3.43) \quad \left\{ \begin{array}{l} \partial_t \rho + \nabla \cdot u = 0, \\ \partial_t u + \nabla \rho + \nabla \Theta + E + u = 0, \\ \partial_t \Theta + \frac{2}{3} \nabla \cdot u + \Theta = 0, \\ \partial_t E - \nabla \times B - u = 0, \quad \nabla \cdot E = -\rho, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{array} \right.$$

with the initial condition :

$$(3.44) \quad W|_{t=0} = W^0 := (\rho^0, u^0, \Theta^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfies the compatibility condition :

$$(3.45) \quad \nabla \cdot E^0 = -\rho^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

Here and throughout this section, we use  $W = (\rho, u, \Theta, E, B)$  to denote the solution of the linearized homogeneous system (3.43).

The rest of this section is arranged as follows. In Section 3.4.1, we derive a time frequency Lyapunov inequality, which leads to the pointwise time frequency upper bound of solutions. In Section 3.4.2, based on this pointwise upper bound, we obtain the elementary  $L^p - L^q$  time decay property of the linear solution operator for the Cauchy problem (3.43)-(3.44). In Section 3.4.3, we study the representation of the Fourier transform of solutions. In Section 3.4.4, we apply results of Section 3.4.3 to obtain the refined  $L^p - L^q$  time decay property for each component in the linear solution  $(\rho, u, \Theta, E, B)$  to the Cauchy problem (3.43)-(3.44).

#### 3.4.1 Pointwise time frequency estimates

In this subsection, we apply the energy method in the Fourier space to the Cauchy problem (3.43)-(3.45) to show that there exists a time frequency Lyapunov functional which is equivalent to  $|\hat{W}(t, k)|^2$  and furthermore its dissipative rate could also be represented by itself. The main result of this subsection is presented as follows.



**Théorème 3.3.** *Let  $W$  be a solution to the linearized homogeneous system (3.43). There exist a time frequency Lyapunov functional  $\mathfrak{E}(\hat{W}(t, k))$  with*

$$(3.46) \quad \mathfrak{E}(\hat{W}) \sim |\hat{W}|^2 := |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\Theta}|^2 + |\hat{E}|^2 + |\hat{B}|^2$$

and a constant  $\gamma > 0$  such that the Lyapunov inequality

$$(3.47) \quad \frac{d}{dt} \mathfrak{E}(\hat{W}(t, k)) + \frac{\gamma |k|^2}{(1 + |k|^2)^2} \mathfrak{E}(\hat{W}(t, k)) \leq 0$$

holds for  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$ .

**Proof.** It is based on the analysis of the system (3.43) in the Fourier space. For that, taking the Fourier transform in  $x$  for system (3.43), we have

$$(3.48) \quad \begin{cases} \partial_t \hat{\rho} + ik \cdot \hat{u} = 0, \\ \partial_t \hat{u} + ik \hat{\rho} + ik \hat{\Theta} + \hat{E} + \hat{u} = 0, \\ \partial_t \hat{\Theta} + \frac{2}{3} ik \cdot \hat{u} + \hat{\Theta} = 0, \\ \partial_t \hat{E} - ik \times \hat{B} - \hat{u} = 0, \quad ik \cdot \hat{E} = -\hat{\rho}, \\ \partial_t \hat{B} + ik \times \hat{E} = 0, \quad ik \cdot \hat{B} = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3. \end{cases}$$

First, it follows from system (3.48) that

$$(3.49) \quad \frac{d}{dt} \left( |\hat{\rho}|^2 + |\hat{u}|^2 + \frac{3}{2} |\hat{\Theta}|^2 + |\hat{E}|^2 + |\hat{B}|^2 \right) + 2|\hat{u}|^2 + 3|\hat{\Theta}|^2 = 0.$$

Multiplying the second equation of the system (3.48) by  $\overline{ik\hat{\rho}}$  and replacing  $\partial_t \hat{\rho}$  by the first equation of system (3.48), we obtain

$$(3.50) \quad \frac{d}{dt} (\hat{u} |ik\hat{\rho}|) + (1 + |k|^2) |\hat{\rho}|^2 = |k \cdot \hat{u}|^2 - |k|^2 \hat{\Theta} \bar{\rho} + ik \cdot \hat{u} \bar{\rho}.$$

Taking the real part after using the Cauchy-Schwarz inequality, we have

$$\frac{d}{dt} \Re(\hat{u} |ik\hat{\rho}|) + \gamma(1 + |k|^2) |\hat{\rho}|^2 \leq C(1 + |k|^2) (|\hat{u}|^2 + |\hat{\Theta}|^2).$$

Multiplying it by  $\frac{1}{1 + |k|^2}$ , we obtain

$$(3.51) \quad \frac{d}{dt} \frac{\Re(\hat{u} |ik\hat{\rho}|)}{1 + |k|^2} + \gamma |\hat{\rho}|^2 \leq C(|\hat{u}|^2 + |\hat{\Theta}|^2).$$

Similarly, multiplying the second equation of system (3.48) by  $\overline{\hat{E}}$ , replacing  $\partial_t \hat{E}$  by the fourth equation, we have

$$(3.52) \quad \frac{d}{dt} (\hat{u} |\hat{E}|) + (|\hat{E}|^2 + |k \cdot \hat{E}|^2) = |\hat{u}|^2 - \hat{\Theta} \bar{\rho} + (\hat{u} |ik \times \hat{B}|) - \hat{u} \cdot \overline{\hat{E}}.$$

Taking the real part of (3.52) and using the Cauchy-Schwarz inequality, we get

$$\frac{d}{dt} \Re(\hat{u} | \hat{E}) + \gamma \left( |k \cdot \hat{E}|^2 + |\hat{E}|^2 \right) \leq C |\hat{W}_I|^2 + \Re(\hat{u} | ik \times \hat{B}).$$

Multiplying it by  $\frac{|k|^2}{(1 + |k|^2)^2}$ , we obtain

$$(3.53) \quad \frac{d}{dt} \frac{|k|^2 \Re(\hat{u} | \hat{E})}{(1 + |k|^2)^2} + \frac{\gamma |k|^2 \left( |\hat{E}|^2 + |k \cdot \hat{E}|^2 \right)}{(1 + |k|^2)^2} \leq C |\hat{W}_I|^2 + \frac{|k|^2 \Re(\hat{u} | ik \times \hat{B})}{(1 + |k|^2)^2}.$$

Similarly, from the fourth and fifth equations of system (3.48), we have

$$\frac{d}{dt} \left( -ik \times \hat{B} | \hat{E} \right) + |k \times \hat{B}|^2 = |k \times \hat{E}|^2 - (ik \times \hat{B} | \hat{u}),$$

which after using the Cauchy-Schwarz inequality and multiplying it by  $\frac{1}{(1 + |k|^2)^2}$  gives

$$(3.54) \quad \frac{d}{dt} \frac{\Re(-ik \times \hat{B} | \hat{E})}{(1 + |k|^2)^2} + \gamma \frac{|k \times \hat{B}|^2}{(1 + |k|^2)^2} \leq \frac{|k|^2 |\hat{E}|^2}{(1 + |k|^2)^2} + C |\hat{u}|^2.$$

Finally, we define the time frequency Lyapunov functional as :

$$\begin{aligned} \mathfrak{E}(\hat{W}(t, k)) &= \left( |\hat{\rho}|^2 + |\hat{u}|^2 + \frac{3}{2} |\hat{\Theta}|^2 + |\hat{E}|^2 + |\hat{B}|^2 \right) + \mathfrak{K}_1 \frac{\Re(\hat{u} | ik \hat{\rho})}{1 + |k|^2} \\ &\quad + \mathfrak{K}_2 \frac{|k|^2 \Re(\hat{u} | \hat{E})}{(1 + |k|^2)^2} + \mathfrak{K}_3 \frac{\Re(-ik \times \hat{B} | \hat{E})}{(1 + |k|^2)^2}, \end{aligned}$$

where constants  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  are the same to that in (3.42) and will be determined later. Then, (3.46) follows as soon as  $0 < \mathfrak{K}_j \ll 1$ ,  $j = 1, 2, 3$ , are sufficiently small. Moreover, by setting  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  be sufficiently small with  $\mathfrak{K}_2^{\frac{3}{2}} \ll \mathfrak{K}_3$ , summing (3.49), (3.51)  $\times \mathfrak{K}_1$ , (3.53)  $\times \mathfrak{K}_2$  and (3.54)  $\times \mathfrak{K}_3$ , we have

$$(3.55) \quad \frac{d}{dt} \mathfrak{E}(\hat{W}(t, k)) + \gamma |\hat{W}_I|^2 + \frac{\gamma |k|^2}{(1 + |k|^2)^2} \left( |\hat{E}|^2 + |\hat{B}|^2 \right) \leq 0,$$

where we used the identity  $|k \times \hat{B}|^2 = |k|^2 |\hat{B}|^2$  due to  $k \cdot \hat{B} = 0$  and also used the following inequality :

$$\mathfrak{K}_2 \frac{|k|^2 \Re(-ik \times \hat{B} \cdot \hat{u})}{(1 + |k|^2)^2} \leq \frac{\mathfrak{K}_2^{\frac{1}{2}} |k|^4 |\hat{u}|^2}{2(1 + |k|^2)^2} + \frac{\mathfrak{K}_2^{\frac{3}{2}} |k|^2 |\hat{B}|^2}{2(1 + |k|^2)^2}.$$

Then, by (3.46), (3.55) and

$$\frac{\gamma |k|^2}{(1 + |k|^2)^2} |\hat{W}|^2 \leq \gamma |\hat{W}_I|^2 + \frac{\gamma |k|^2}{(1 + |k|^2)^2} \left( |\hat{E}|^2 + |\hat{B}|^2 \right),$$

we obtain (3.47). This ends the proof of Theorem 3.3.  $\square$

Based on Theorem 3.3, we obtain the pointwise time frequency estimate on  $\hat{W}(t, k)$  in terms of the initial data  $\hat{W}^0(k)$  as follows.

**Corollary 3.1.** *Let  $W$  be a solution to the Cauchy problem (3.43)-(3.45). Then, there exist constants  $\gamma > 0$  and  $C > 0$  such that*

$$(3.56) \quad |\hat{W}(t, k)| \leq C e^{-\frac{\gamma|k|^2 t}{(1+|k|^2)^2}} |\hat{W}_0(k)|$$

holds for  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$ .

### 3.4.2 $L^p - L^q$ time decay properties

Formally, the solution to the Cauchy problem (3.43)-(3.44) is denoted by

$$(3.57) \quad W(t) = e^{tL} W^0,$$

where  $e^{tL}$  denotes the linear solution operator [21] for  $t \geq 0$ . The main result of this subsection is stated as follows.

**Théorème 3.4.** *Let  $m \geq 0$  be an integer and  $1 \leq p, r \leq 2 \leq q \leq \infty, l \geq 0$ . Define*

$$(3.58) \quad [l + 3(\frac{1}{r} - \frac{1}{q})]_+ = \begin{cases} l, & \text{if } r = q = 2 \text{ and } l \text{ is an integer,} \\ [l + 3(\frac{1}{r} - \frac{1}{q})]_- + 1, & \text{otherwise,} \end{cases}$$

where and in Chapter 4, we use  $[\cdot]_-$  to denote the integer part of the argument. Then, if  $W^0$  satisfies (3.45), for  $t \geq 0$ ,  $e^{tL}$  satisfies the following time decay property :

$$(3.59) \quad \|\nabla^m e^{tL} W^0\|_{L^q} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|W^0\|_{L^p} + C(1+t)^{-\frac{l}{2}} \|\nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+} W^0\|_{L^r},$$

where  $C = C(p, q, r, l, m)$  is independent of any time  $t$ .

**Proof.** Take  $2 \leq q \leq \infty$  and an integer  $m \geq 0$ . Set  $W(t) = e^{tL} W^0$ . It follows from Lemma 3.2 with  $\frac{1}{q} + \frac{1}{q'} = 1$  that

$$(3.60) \quad \|\nabla^m W(t)\|_{L_x^q} \leq C \| |k|^m \hat{W}(t) \|_{L_k^{q'}} \leq C \| |k|^m \hat{W}(t) \|_{L^{q'}(|k| \leq 1)} + C \| |k|^m \hat{W}(t) \|_{L^{q'}(|k| \geq 1)}.$$

By (3.56), since

$$\begin{cases} \frac{|k|^2}{(1+|k|^2)^2} \geq \frac{|k|^2}{4}, & \text{over } |k| \leq 1, \\ \frac{|k|^2}{(1+|k|^2)^2} \geq \frac{1}{4|k|^2}, & \text{over } |k| \geq 1, \end{cases}$$

we have

$$|\hat{W}(t, k)| \leq \begin{cases} C e^{-\frac{\gamma}{4}|k|^2 t} |\hat{W}^0(k)|, & \text{over } |k| \leq 1, \\ C e^{-\frac{\gamma}{4|k|^2} t} |\hat{W}^0(k)|, & \text{over } |k| \geq 1. \end{cases}$$

Then

$$(3.61) \quad \left\| |k|^m \hat{W}(t) \right\|_{L^{q'}(|k| \leq 1)} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{q'} - \frac{1}{p'}\right) - \frac{m}{2}} \left\| \hat{W}^0 \right\|_{L^{p'}} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{m}{2}} \left\| W^0 \right\|_{L^p}$$

for any  $1 \leq p \leq 2$ . On the other hand, letting  $l \geq 0$ , we have

$$(3.62) \quad \begin{aligned} \left\| |k|^m \hat{W}(t) \right\|_{L^{q'}(|k| \geq 1)} &\leq \sup_{|k| \geq 1} \left( \frac{1}{|k|^l} e^{-\frac{\gamma}{4|k|^2} t} \right) \left\| |k|^{m+l} \hat{W}^0 \right\|_{L^{q'}(|k| \geq 1)} \\ &\leq C(1+t)^{-\frac{l}{2}} \left\| |k|^{m+l} \hat{W}^0 \right\|_{L^{q'}(|k| \geq 1)}. \end{aligned}$$

Now, take  $1 \leq r \leq 2$  and fix  $\varepsilon > 0$  sufficiently small. By the Hölder inequality with  $\frac{1}{r} + \frac{1}{r'} = 1$ ,

$$(3.63) \quad \begin{aligned} \left\| |k|^{m+l} \hat{W}^0 \right\|_{L^{q'}(|k| \geq 1)} &= \left\| |k|^{\left(\frac{1}{r'} - \frac{1}{q'}\right)(3+\varepsilon)} |k|^{m+l + \left(\frac{1}{q'} - \frac{1}{r'}\right)(3+\varepsilon)} \hat{W}^0 \right\|_{L^{q'}(|k| \geq 1)} \\ &\leq \left\| |k|^{-(3+\varepsilon)} \right\|_{L^1(|k| \geq 1)}^{\frac{1}{q'} - \frac{1}{r'}} \left\| |k|^{m+l + \left(\frac{1}{q'} - \frac{1}{r'}\right)(3+\varepsilon)} \hat{W}^0 \right\|_{L^{r'}(|k| \geq 1)} \\ &\leq C \left\| |k|^{m+l + \left(\frac{1}{r} - \frac{1}{q}\right)(3+\varepsilon)} \hat{W}^0 \right\|_{L^{r'}(|k| \geq 1)}. \end{aligned}$$

When  $r = q = 2$  and  $l$  is an integer, it follows from Lemma 3.2 that

$$\left\| |k|^{m+l + \left(\frac{1}{r} - \frac{1}{q}\right)(3+\varepsilon)} \hat{W}^0 \right\|_{L^{r'}(|k| \geq 1)} = \left\| |k|^{m+l} \hat{W}^0 \right\|_{L^2(|k| \geq 1)} \leq C \left\| \nabla^{m+l} W^0 \right\|.$$

Otherwise, by letting  $\varepsilon > 0$  sufficiently small, by Lemma 3.2 again, we get

$$\left\| |k|^{m+l + \left(\frac{1}{r} - \frac{1}{q}\right)(3+\varepsilon)} \hat{W}^0 \right\|_{L^{r'}(|k| \geq 1)} \leq \left\| |k|^{m+l + \left[\frac{1}{r} - \frac{1}{q}\right]_+ + 1} \hat{W}^0 \right\|_{L^{r'}(|k| \geq 1)} \leq C \left\| \nabla^{m+l + \left[\frac{1}{r} - \frac{1}{q}\right]_+} W^0 \right\|_{L^r}.$$

Combining (3.60)-(3.63) and the last two relations, we have (3.59).  $\square$

### 3.4.3 Representation of solutions

In this subsection, we study the explicit solution  $W = (\rho, u, \Theta, E, B) = e^{tL} W^0$  to the Cauchy problem (3.43)-(3.44) with the compatibility condition (3.45) or equivalently the system (3.48) in Fourier space. We show that  $\rho, \Theta, \nabla \cdot u$  satisfy the same equation which is of order three and is different from that of the isentropic Euler-Maxwell system [17]. The main task is to prove Theorem 3.5 presented in the end of this subsection.

From the first three equations of (3.43) and  $\nabla \cdot E = -\rho$ , we have

$$(3.64) \quad \partial_{ttt}\rho + 2\partial_{tt}\rho - \frac{5}{3}\partial_t\Delta\rho + 2\partial_t\rho + \rho - \Delta\rho = 0,$$

with the initial condition :

$$(3.65) \quad \begin{cases} \rho|_{t=0} = \rho^0 = -\nabla \cdot E^0, \\ \partial_t \rho|_{t=0} = -\nabla \cdot u^0, \\ \partial_{tt} \rho|_{t=0} = \Delta \rho^0 - \rho^0 + \nabla \cdot u^0 + \Delta \Theta^0. \end{cases}$$

Taking the Fourier transform in  $x$  for (3.64) and (3.65), we obtain

$$(3.66) \quad \partial_{ttt} \hat{\rho} + 2\partial_{tt} \hat{\rho} + \left(2 + \frac{5}{3}|k|^2\right) \partial_t \hat{\rho} + (1 + |k|^2) \hat{\rho} = 0,$$

with the initial condition :

$$(3.67) \quad \begin{cases} \hat{\rho}|_{t=0} = \hat{\rho}^0 = -ik \cdot \hat{E}^0, \\ \partial_t \hat{\rho}|_{t=0} = -ik \cdot \hat{u}^0, \\ \partial_{tt} \hat{\rho}|_{t=0} = -(1 + |k|^2) \hat{\rho}^0 + ik \cdot \hat{u}^0 - |k|^2 \Theta^0. \end{cases}$$

The characteristic equation of (3.66) is :

$$F(\mathfrak{X}) := \mathfrak{X}^3 + 2\mathfrak{X}^2 + \left(2 + \frac{5}{3}|k|^2\right) \mathfrak{X} + 1 + |k|^2 = 0.$$

For the roots of this equation and their properties, we have

**Lemma 3.3.** *Let  $|k| \neq 0$ . The equation  $F(\mathfrak{X}) = 0$ ,  $\mathfrak{X} \in \mathbb{C}$  has a real root  $\eta = \eta(|k|) \in (-1, -\frac{3}{5})$  and two conjugate complex roots  $\mathfrak{X}_{\pm} = \phi \pm i\psi$  with  $\phi = \phi(|k|) \in (-\frac{7}{10}, -\frac{1}{2})$  and  $\psi = \psi(|k|) \in (\frac{\sqrt{6}}{3}, +\infty)$  satisfying*

$$(3.68) \quad \phi = -1 - \frac{\eta}{2}, \quad \psi = \frac{1}{2} \sqrt{3\eta^2 + 4\eta + 4 + \frac{20}{3}|k|^2}.$$

Here,  $\eta, \phi, \psi$  are smooth over  $|k| > 0$ , and  $\eta(|k|)$  is strictly increasing in  $|k| > 0$  with

$$\lim_{|k| \rightarrow 0} \eta(|k|) = -1, \quad \lim_{|k| \rightarrow \infty} \eta(|k|) = -\frac{3}{5}.$$

Furthermore, the following asymptotic behaviors hold true :

$$\eta(|k|) = -1 + O(1)|k|^2, \quad \phi(|k|) = -\frac{1}{2} - O(1)|k|^2, \quad \psi(|k|) = \frac{\sqrt{3}}{2} + O(1)|k|$$

whenever  $|k| \leq 1$  is small, and

$$\eta(|k|) = -\frac{3}{5} - O(1)|k|^{-2}, \quad \phi(|k|) = -\frac{7}{10} + O(1)|k|^{-2}, \quad \psi(|k|) = O(1)|k|$$

whenever  $|k| \geq 1$  is large. Here and in the sequel of this thesis  $O(1)$  denotes a generic strictly positive constant independent of  $k$ .

**Proof.** Let  $|k| \neq 0$ . We first consider the possibly existing real root for equation  $F(\mathfrak{X}) = 0$ . From

$$F'(\mathfrak{X}) = 3\mathfrak{X}^2 + 4\mathfrak{X} + 2 + \frac{5}{3}|k|^2 = 3\left(\mathfrak{X} + \frac{2}{3}\right)^2 + \frac{2}{3} + \frac{5}{3}|k|^2 > 0,$$

and  $F(-1) = -\frac{2}{3}|k|^2 < 0$ ,  $F(-\frac{3}{5}) = \frac{38}{125} > 0$ , we see that equation  $F(\mathfrak{X}) = 0$  has one and only one real root denoted by  $\eta = \eta(|k|)$  which satisfies  $-1 < \eta < -\frac{3}{5}$ . By taking the derivative of  $F(\eta(|k|)) = 0$  in  $|k|$ , we have

$$\eta'(|k|) = \frac{-|k| \left(2 + \frac{10}{3}\eta\right)}{3\eta^2 + 4\eta + 2 + \frac{5}{3}|k|^2} > 0,$$

so that  $\eta(\cdot)$  is strictly increasing in  $|k| > 0$ . Since  $F(\eta) = 0$  can be rewritten as

$$\eta \left( \frac{\eta(\eta + 2)}{2 + \frac{5}{3}|k|^2} + 1 \right) = -\frac{1 + |k|^2}{2 + \frac{5}{3}|k|^2},$$

$\eta$  has limits  $-1$  and  $-\frac{3}{5}$  as  $|k| \rightarrow 0$  and  $|k| \rightarrow \infty$ , respectively. On the other hand,  $F(\eta(|k|)) = 0$  is also equivalent to

$$\eta + 1 = \frac{\frac{2}{3}|k|^2 + (\eta + 1)^2}{(\eta + 1)^2 + 1 + \frac{5}{3}|k|^2} \quad \text{or} \quad \eta + \frac{3}{5} = \frac{-\frac{1}{5} + \frac{3}{5}\eta(\eta + 2)}{\eta(\eta + 2) + 2 + \frac{5}{3}|k|^2}.$$

Therefore, it follows that

$$\eta(|k|) = -1 + O(1)|k|^2$$

whenever  $|k| < 1$  is small and

$$\eta(|k|) = -\frac{3}{5} - O(1)|k|^{-2}$$

whenever  $|k| \geq 1$  is large.

Next, we consider roots of  $F(\mathfrak{X}) = 0$  over  $\mathfrak{X} \in \mathbb{C}$ . Since  $F(\eta) = 0$  with  $\eta \in \mathbb{R}$ ,  $F(\mathfrak{X}) = 0$  can be factored as

$$F(\mathfrak{X}) = (\mathfrak{X} - \eta) \left( \left(\mathfrak{X} + 1 + \frac{\eta}{2}\right)^2 + \frac{3}{4}\eta^2 + \eta + \frac{5}{3}|k|^2 + 1 \right) = 0.$$

Then, there exist two conjugate complex roots  $\mathfrak{X}_{\pm} = \phi \pm i\psi$  which satisfy

$$\left(\mathfrak{X} + 1 + \frac{\eta}{2}\right)^2 + \frac{3}{4}\eta^2 + \eta + \frac{5}{3}|k|^2 + 1 = 0.$$

By solving the previous equation, we obtain  $\phi = \phi(|k|)$ ,  $\psi = \psi(|k|)$  in the form of (3.68). Moreover, it follows directly that the asymptotic behaviors of  $\phi(|k|)$ ,  $\psi(|k|)$  at  $|k| = 0$

and  $\infty$  from that of  $\eta(|k|)$ . We have finished the proof of Lemma 3.3.  $\square$

Based on Lemma 3.3, we define the solution of (3.66) as

$$(3.69) \quad \hat{\rho}(t, k) = c_1(k)e^{\eta t} + e^{\phi t} (c_2(k) \cos \psi t + c_3(k) \sin \psi t),$$

where  $c_j(k)$ ,  $1 \leq j \leq 3$ , are to be determined by (3.67) later. Again using  $\nabla \cdot E = -\rho$ , (3.69) implies

$$(3.70) \quad \tilde{k} \cdot \hat{E}(t, k) = i|k|^{-1} (c_1(k)e^{\eta t} + e^{\phi t} (c_2(k) \cos \psi t + c_3(k) \sin \psi t)),$$

where and in the following of both Chapter 3 and Chapter 4, we set  $\tilde{k} = \frac{k}{|k|}$ . Moreover, it follows from (3.69) that

$$(3.71) \quad \begin{bmatrix} \hat{\rho}|_{t=0} \\ \partial_t \hat{\rho}|_{t=0} \\ \partial_{tt} \hat{\rho}|_{t=0} \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ \eta & \phi & \psi \\ \eta^2 & \phi^2 - \psi^2 & 2\phi\psi \end{bmatrix}.$$

It is direct to check that

$$\det A = \psi (\psi^2 + (\eta - \phi)^2) = \psi \left( 3\eta^2 + 4\eta + 2 + \frac{5}{3}|k|^2 \right) > 0$$

and

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} (\phi^2 + \psi^2) \psi & -2\phi\psi & \psi \\ \eta(\eta - 2\phi) \psi & 2\phi\psi & -\psi \\ \eta(\phi^2 - \psi^2 - \eta\phi) & \psi^2 + \eta^2 - \phi^2 & \phi - \eta \end{bmatrix}.$$

From (3.67) and (3.71), we have

$$[c_1, c_2, c_3]^T = \frac{1}{3\eta^2 + 4\eta + 2 + \frac{5}{3}|k|^2} \begin{bmatrix} \phi^2 + \psi^2 - (1 + |k|^2) & -i|k|(2\phi + 1) & -|k|^2 \\ \eta^2 - 2\eta\phi + (1 + |k|^2) & i|k|(2\phi + 1) & |k|^2 \\ \frac{\eta(\phi^2 - \psi^2 - \eta\phi) - (\phi - \eta)(1 + |k|^2)}{\psi} & \frac{i|k|}{\psi} (\phi^2 - \eta^2 - \psi^2 + \phi - \eta) & \frac{\eta - \phi}{\psi} |k|^2 \end{bmatrix} \begin{bmatrix} \hat{\rho}^0 \\ \tilde{k} \cdot \hat{u}^0 \\ \hat{\Theta}^0 \end{bmatrix}.$$

Making further simplifications with the form of  $\phi$  and  $\psi$ , we have

$$(3.72) \quad [c_1, c_2, c_3]^T = \frac{1}{3\eta^2 + 4\eta + 2 + \frac{5}{3}|k|^2} \begin{bmatrix} (\eta + 1)^2 + \frac{2}{3}|k|^2 & -i|k|(\eta + 1) & -|k|^2 \\ 2\eta^2 + \eta + 1 + |k|^2 & i|k|(\eta + 1) & |k|^2 \\ \frac{\eta^2 + \frac{3}{2}\eta + (1 + |k|^2) - \frac{1}{6}\eta|k|^2}{\psi} & \frac{i|k|}{\psi} \left( \frac{3}{2}\eta^2 + \frac{3}{2}\eta + 1 + \frac{5}{3}|k|^2 \right) & \frac{1 + \frac{3}{2}\eta}{\psi} |k|^2 \end{bmatrix} \begin{bmatrix} \hat{\rho}^0 \\ \tilde{k} \cdot \hat{u}^0 \\ \hat{\Theta}^0 \end{bmatrix}.$$

Similarly, from the first three equations of (3.43) and  $\nabla \cdot E = -\rho$ , we also obtain

$$(3.73) \quad \partial_{ttt}\hat{\Theta} + 2\partial_{tt}\hat{\Theta} + \left(2 + \frac{5}{3}|k|^2\right) \partial_t\hat{\Theta} + (1 + |k|^2) \hat{\Theta} = 0,$$

with the initial condition :

$$(3.74) \quad \begin{cases} \hat{\Theta}|_{t=0} = \hat{\Theta}^0, \\ \partial_t\hat{\Theta}|_{t=0} = -\frac{2}{3}ik \cdot \hat{u}^0 - \hat{\Theta}^0, \\ \partial_{tt}\hat{\Theta}|_{t=0} = -\frac{2}{3}(1 + |k|^2)\hat{\rho}^0 + \frac{4}{3}ik \cdot \hat{u}^0 + \left(1 - \frac{2}{3}|k|^2\right) \hat{\Theta}^0. \end{cases}$$

From Lemma 3.3, we define the solution of (3.73) as

$$(3.75) \quad \hat{\Theta}(t, k) = c_4(k)e^{\eta t} + e^{\phi t} (c_5(k) \cos \psi t + c_6(k) \sin \psi t),$$

where  $c_j(k)$ ,  $4 \leq j \leq 6$ , are to be determined by (3.74) later. In fact, (3.74) together with (3.75) gives

$$(3.76) \quad [c_4, c_5, c_6]^T = \frac{1}{3\eta^2 + 4\eta + 2 + \frac{5}{3}|k|^2} \begin{bmatrix} \eta^2 + 2\eta + \frac{4}{3} + \frac{2}{3}|k|^2 & \frac{4}{3}|k| \left(2 + \frac{\eta}{2}\right) i & 1 - \frac{2}{3}|k|^2 \\ 2\eta^2 + 3\eta + \frac{8}{3} + \frac{2}{3}|k|^2 & -\frac{4}{3}|k| \left(2 + \frac{\eta}{2}\right) i & -\left(1 - \frac{2}{3}|k|^2\right) \\ \frac{-\frac{1}{2}\eta^2 + \eta - \frac{2}{3}\eta|k|^2 + \frac{2}{3}|k|^2}{\psi} & \frac{\frac{3}{2}\eta^2 - 3\eta - 2 + \frac{5}{3}|k|^2}{\psi} & \frac{-1 - \frac{3}{2}\eta + \frac{2}{3}|k|^2 + \eta|k|^2}{\psi} \end{bmatrix} \begin{bmatrix} \hat{\rho}^0 \\ \tilde{k} \cdot \hat{u}^0 \\ \hat{\Theta}^0 \end{bmatrix}.$$

In a similar way, we obtain the equation for  $\tilde{k} \cdot \hat{u}$  :

$$(3.77) \quad \partial_{ttt}(\tilde{k} \cdot \hat{u}) + 2\partial_{tt}(\tilde{k} \cdot \hat{u}) + \left(2 + \frac{5}{3}|k|^2\right) \partial_t(\tilde{k} \cdot \hat{u}) + (1 + |k|^2) (\tilde{k} \cdot \hat{u}) = 0.$$

Initial data are given as :

$$(3.78) \quad \begin{cases} (\tilde{k} \cdot \hat{u})|_{t=0} = (\tilde{k} \cdot \hat{u})_0, \\ \partial_t(\tilde{k} \cdot \hat{u})|_{t=0} = \frac{-i}{|k|} (1 + |k|^2) \hat{\rho}_0 - \tilde{k} \cdot \hat{u}_0 - i|k| \hat{\Theta}_0, \\ \partial_{tt}(\tilde{k} \cdot \hat{u})|_{t=0} = \frac{i}{|k|} (1 + |k|^2) \hat{\rho}_0 - \frac{5}{3}|k|^2 \tilde{k} \cdot \hat{u}_0 + 2i|k| \hat{\Theta}_0. \end{cases}$$

After tenuous computation, we have

$$(3.79) \quad \tilde{k} \cdot \hat{u}(t, k) = c_7(k)e^{\eta t} + e^{\phi t} (c_8(k) \cos \psi t + c_9(k) \sin \psi t),$$



with

$$(3.80) \quad [c_7, c_8, c_9]^T = \frac{1}{3\eta^2 + 4\eta + 2 + \frac{5}{3}|k|^2} \left( \begin{array}{ccc} \left[ \begin{array}{ccc} \eta^2 + 2\eta + 2 + \frac{5}{3}|k|^2 & -2 - \eta - \frac{5}{3}|k|^2 & -\eta|k|i \\ 2\eta^2 + 2\eta & 2 + \eta + \frac{5}{3}|k|^2 & \eta|k|i \\ \frac{\eta^2 - \frac{5}{3}\eta|k|^2}{\psi} & \frac{\frac{3}{2}\eta^2 + \frac{5}{2}\eta|k|^2}{\psi} & \frac{-\frac{3}{2}\eta^2 - 3\eta - 2 - \frac{5}{3}|k|^2}{\psi} \end{array} \right] \\ + \left[ \begin{array}{ccc} -(1+\eta)|k|^{-1}(1+|k|^2)i & 0 & 0 \\ (1+\eta)|k|^{-1}(1+|k|^2)i & 0 & 0 \\ -\left(\frac{3}{2}\eta^2 + \frac{3}{2}\eta + 1 + \frac{5}{3}|k|^2\right)|k|^{-1}(1+|k|^2)i & 0 & 0 \end{array} \right] \end{array} \right) \begin{bmatrix} \hat{\rho}^0 \\ \tilde{k} \cdot \hat{u}^0 \\ \hat{\Theta}^0 \end{bmatrix}.$$

Next, for  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$ , we get

$$\hat{u}_\perp := -\tilde{k} \times (\tilde{k} \times \hat{u}), \quad \hat{E}_\perp := -\tilde{k} \times (\tilde{k} \times \hat{E}), \quad \hat{B}_\perp := -\tilde{k} \times (\tilde{k} \times \hat{B}).$$

Taking the curl for the second, fourth and fifth equations of the system (3.43), we have

$$\begin{cases} \partial_t(\nabla \times u) + \nabla \times E + \nabla \times u = 0, \\ \partial_t(\nabla \times E) - \nabla \times (\nabla \times B) - \nabla \times u = 0, \\ \partial_t(\nabla \times B) + \nabla \times (\nabla \times E) = 0. \end{cases}$$

Taking the Fourier transform in  $x$  for the above system, it follows that

$$\begin{cases} \partial_t \hat{u}_\perp = -\hat{u}_\perp - \hat{E}_\perp, \\ \partial_t \hat{E}_\perp = \hat{u}_\perp + ik \times \hat{B}_\perp, \\ \partial_t \hat{B}_\perp = -ik \times \hat{E}_\perp. \end{cases}$$

Initial data are given as :

$$(\hat{u}_\perp, \hat{E}_\perp, \hat{B}_\perp)|_{t=0} = (\hat{u}_\perp^0, \hat{E}_\perp^0, \hat{B}_\perp^0),$$

where

$$\hat{u}_\perp^0 = -\tilde{k} \times (\tilde{k} \times \hat{u}^0), \quad \hat{E}_\perp^0 = -\tilde{k} \times (\tilde{k} \times \hat{E}^0), \quad \hat{B}_\perp^0 = -\tilde{k} \times (\tilde{k} \times \hat{B}^0).$$

Then, it is straightforward to get

$$(3.81) \quad \partial_{ttt} \hat{E}_\perp + \partial_{tt} \hat{E}_\perp + (1 + |k|^2) \partial_t \hat{E}_\perp + |k|^2 \hat{E}_\perp = 0,$$

with the initial condition :

$$(3.82) \quad \begin{cases} \hat{E}_\perp|_{t=0} = \hat{E}_\perp^0, \\ \partial_t \hat{E}_\perp|_{t=0} = \hat{u}_\perp^0 + ik \times \hat{B}_\perp^0, \\ \partial_{tt} \hat{E}_\perp|_{t=0} = -\hat{u}_\perp^0 - (1 + |k|^2) \hat{E}_\perp^0. \end{cases}$$

The characteristic equation of (3.81) is

$$F_*(\mathfrak{X}) := \mathfrak{X}^3 + \mathfrak{X}^2 + (1 + |k|^2)\mathfrak{X} + |k|^2 = 0.$$

For the roots of the above equation and their properties, we have

**Lemma 3.4.** *Let  $|k| \neq 0$ . The equation  $F_*(\mathfrak{X}) = 0$ ,  $\mathfrak{X} \in \mathbb{C}$  has a real root  $\eta_* = \eta_*(|k|) \in (-1, 0)$  and two conjugate complex roots  $\mathfrak{X}_\pm = \phi_* \pm i\psi_*$  with  $\phi_* = \phi_*(|k|) \in (-\frac{1}{2}, 0)$  and  $\psi_* = \psi_*(|k|) \in (\frac{\sqrt{6}}{3}, +\infty)$  satisfying*

$$(3.83) \quad \phi_* = -\frac{1}{2} - \frac{\eta_*}{2}, \quad \psi_* = \frac{1}{2}\sqrt{3\eta_*^2 + 2\eta_* + 3 + 4|k|^2}.$$

Here,  $\eta_*, \phi_*, \psi_*$  are smooth over  $|k| > 0$ , and  $\eta_*(|k|)$  is strictly decreasing in  $|k| > 0$  with

$$\lim_{|k| \rightarrow 0} \eta_*(|k|) = 0, \quad \lim_{|k| \rightarrow \infty} \eta_*(|k|) = -1.$$

Moreover, the following asymptotic behaviors hold true :

$$\eta_*(|k|) = -O(1)|k|^2, \quad \phi_*(|k|) = -\frac{1}{2} + O(1)|k|^2, \quad \psi_*(|k|) = \frac{\sqrt{3}}{2} + O(1)|k|$$

whenever  $|k| \leq 1$  is small, and

$$\eta_*(|k|) = -1 + O(1)|k|^{-2}, \quad \phi_*(|k|) = -O(1)|k|^{-2}, \quad \psi_*(|k|) = O(1)|k|$$

whenever  $|k| \geq 1$  is large.

**Proof.** The proof is similar to that of Lemma 3.3, we omit it here for simplicity.  $\square$

Based on Lemma 3.4, we obtain

$$(3.84) \quad \hat{u}_\perp(t, k) = -\frac{c_{10}(k)}{1 + \eta_*} e^{\eta_* t} - \frac{c_{11}(k)}{(1 + \phi_*)^2 + \psi_*^2} e^{\phi_* t} ((1 + \phi_*) \cos \psi_* t + \psi_* \sin \psi_* t) \\ - \frac{c_{12}(k)}{(1 + \phi_*)^2 + \psi_*^2} e^{\phi_* t} ((1 + \phi_*) \sin \psi_* t - \psi_* \cos \psi_* t),$$

$$(3.85) \quad \hat{E}_\perp(t, k) = c_{10}(k) e^{\eta_* t} + e^{\phi_* t} (c_{11}(k) \cos \psi_* t + c_{12}(k) \sin \psi_* t),$$

and

$$(3.86) \quad \hat{B}_\perp(t, k) = -ik \times \frac{c_{10}(k)}{\eta_*} e^{\eta_* t} - ik \times \frac{c_{11}(k)}{\phi_*^2 + \psi_*^2} e^{\phi_* t} (\phi_* \cos \psi_* t + \psi_* \sin \psi_* t) \\ - ik \times \frac{c_{12}(k)}{\phi_*^2 + \psi_*^2} e^{\phi_* t} (\phi_* \sin \psi_* t - \psi_* \cos \psi_* t)$$

with

$$(3.87) \quad [c_{10}, c_{11}, c_{12}]^T = \frac{1}{3\eta_*^2 + 2\eta_* + 1 + |k|^2} \begin{bmatrix} \eta_* I_3 & \eta_*(\eta_* + 1)I_3 & (\eta_* + 1)ik \times \\ -\eta_* I_3 & (2\eta_*^2 + \eta_* + |k|^2 + 1)I_3 & -(\eta_* + 1)ik \times \\ \frac{\frac{3}{2}\eta_*^2 + \frac{3}{2}\eta_* + 1 + |k|^2}{\psi_*} I_3 & \frac{(\eta_* + 1)(\eta_* + 1 + |k|^2)}{2\psi_*} I_3 & \frac{\frac{3}{2}\eta_*^2 + \frac{1}{2} + |k|^2}{\psi_*} ik \times \end{bmatrix} \begin{bmatrix} \hat{u}_\perp^0 \\ \hat{E}_\perp^0 \\ \hat{B}_\perp^0 \end{bmatrix}.$$

Now, let us summarize the above computations on the explicit representation of Fourier transforms of the solution as follows.

**Théorème 3.5.** *Let  $W = (\rho, u, \Theta, E, B)$  be the solution to the Cauchy problem (3.43)-(3.44) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfying (3.45). For  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$  with  $|k| \neq 0$ , we have the decomposition :*

$$(3.88) \quad \begin{bmatrix} \hat{\rho}(t, k) \\ \hat{u}(t, k) \\ \hat{\Theta}(t, k) \\ \hat{E}(t, k) \\ \hat{B}(t, k) \end{bmatrix} = \begin{bmatrix} \hat{\rho}(t, k) \\ \hat{u}_\parallel(t, k) \\ \hat{\Theta}(t, k) \\ \hat{E}_\parallel(t, k) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{u}_\perp(t, k) \\ 0 \\ \hat{E}_\perp(t, k) \\ \hat{B}_\perp(t, k) \end{bmatrix},$$

where and in the rest of this thesis,  $\hat{u}_\parallel$  is defined by

$$\hat{u}_\parallel = \tilde{k} \tilde{k} \cdot \hat{u} = \hat{u} - (-\tilde{k} \times (\tilde{k} \times \hat{u})) = \hat{u} - \hat{u}_\perp,$$

and likewise for  $\hat{E}_\parallel$ . Then, there exist matrices  $H_{8 \times 8}^I(t, k)$  and  $H_{9 \times 9}^{II}(t, k)$  such that

$$(3.89) \quad \begin{bmatrix} \hat{\rho}(t, k) \\ \hat{u}_\parallel(t, k) \\ \hat{\Theta}(t, k) \\ \hat{E}_\parallel(t, k) \end{bmatrix} = H_{8 \times 8}^I(t, k) \begin{bmatrix} \hat{\rho}_0(k) \\ \hat{u}_{0,\parallel}(k) \\ \hat{\Theta}_0(k) \\ \hat{E}_{0,\parallel}(k) \end{bmatrix}$$

and

$$(3.90) \quad \begin{bmatrix} \hat{u}_\perp(t, k) \\ \hat{E}_\perp(t, k) \\ \hat{B}_\perp(t, k) \end{bmatrix} = H_{9 \times 9}^{II}(t, k) \begin{bmatrix} \hat{u}_\perp^0(k) \\ \hat{E}_\perp^0(k) \\ \hat{B}_\perp^0(k) \end{bmatrix},$$

where  $H_{8 \times 8}^I$  is explicitly determined by representations (3.69), (3.79), (3.75), (3.70) for  $\hat{\rho}(t, k)$ ,  $\hat{u}_\parallel(t, k)$ ,  $\hat{\Theta}(t, k)$ ,  $\hat{E}_\parallel(t, k)$  with  $c_i(k)$ , ( $1 \leq i \leq 9$ ) defined by (3.72), (3.76), (3.80) in terms of  $\hat{\rho}^0(k)$ ,  $\hat{u}_\parallel^0(k)$ ,  $\hat{\Theta}^0(k)$ ,  $\hat{E}_\parallel^0(k)$  and  $H_{9 \times 9}^{II}$  is determined by the representations (3.84), (3.85) (3.86) for  $\hat{u}_\perp(t, k)$ ,  $\hat{E}_\perp(t, k)$ ,  $\hat{B}_\perp(t, k)$  with  $c_{10}(k)$ ,  $c_{11}(k)$  and  $c_{12}(k)$  defined by (3.87) in terms of  $\hat{u}_\perp^0(k)$ ,  $\hat{E}_\perp^0(k)$ ,  $\hat{B}_\perp^0(k)$ .

### 3.4.4 Refined $L^p - L^q$ time decay properties

In this subsection, we use Theorem 3.5 to refine the  $L^p - L^q$  time decay property for every component of the solution  $W = (\rho, u, \Theta, E, B)$ . For that, we first consider the delicate time frequency pointwise estimates on the Fourier transforms  $\hat{W} = (\hat{\rho}, \hat{u}, \hat{\Theta}, \hat{E}, \hat{B})$  as follows.

**Lemma 3.5.** *Let  $W = (\rho, u, \Theta, E, B)$  be the solution to the linearized homogeneous system (3.43) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfying (3.45). Then, there exist constants  $\gamma > 0$  and  $C > 0$  such that for all  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$ ,*

$$(3.91) \quad |\hat{\rho}(t, k)| \leq C e^{-\frac{t}{2}} |(\hat{\rho}^0(k), \hat{u}^0(k), \hat{\Theta}^0(k))|,$$

$$(3.92) \quad \begin{aligned} |\hat{u}(t, k)| &\leq C e^{-\frac{t}{2}} |(\hat{\rho}^0(k), \hat{u}^0(k), \hat{\Theta}^0(k), \hat{E}^0(k))| \\ &+ C |(\hat{u}^0(k), \hat{E}^0(k), \hat{B}^0(k))| \begin{cases} e^{-\gamma t} + |k| e^{-\gamma|k|^2 t}, & \text{if } |k| \leq 1, \\ e^{-\gamma t} + \frac{1}{|k|} e^{-\frac{\gamma t}{|k|^2}}, & \text{if } |k| > 1, \end{cases} \end{aligned}$$

$$(3.93) \quad |\hat{\Theta}(t, k)| \leq C e^{-\frac{t}{2}} |(\hat{\rho}^0(k), \hat{u}^0(k), \hat{\Theta}^0(k))|,$$

$$(3.94) \quad \begin{aligned} |\hat{E}(t, k)| &\leq C e^{-\frac{t}{2}} |(\hat{u}^0(k), \hat{\Theta}^0(k), \hat{E}^0(k))| \\ &+ C |(\hat{u}^0(k), \hat{E}^0(k), \hat{B}^0(k))| \begin{cases} e^{-\gamma t} + |k| e^{-\gamma|k|^2 t}, & \text{if } |k| \leq 1, \\ |k|^{-2} e^{-\gamma t} + e^{-\frac{\gamma t}{|k|^2}}, & \text{if } |k| > 1, \end{cases} \end{aligned}$$

and

$$(3.95) \quad |\hat{B}(t, k)| \leq C |(\hat{u}^0(k), \hat{E}^0(k), \hat{B}^0(k))| \begin{cases} |k| e^{-\gamma t} + e^{-\gamma|k|^2 t}, & \text{if } |k| \leq 1, \\ |k|^{-1} e^{-\gamma t} + e^{-\frac{\gamma t}{|k|^2}}, & \text{if } |k| > 1. \end{cases}$$

**Proof.** Recall the decomposition (3.88) of  $(\hat{\rho}, \hat{u}, \hat{\Theta}, \hat{E}, \hat{B})$ . We consider the upper bounds of each component in the first part  $(\hat{\rho}, \hat{u}_{||}, \hat{\Theta}, \hat{E}_{||})$  due to (3.89). For the upper bound of  $\hat{\rho}$  defined by (3.69), from Lemma 3.3 and (3.72), we obtain

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} O(1)|k|^2 & -O(1)|k|^3 i & -O(1)|k|^2 \\ O(1) & O(1)|k|^3 i & O(1)|k|^2 \\ O(1) & -O(1)|k| & -O(1)|k|^2 \end{bmatrix} \begin{bmatrix} \hat{\rho}^0 \\ \tilde{k} \cdot \hat{u}^0 \\ \hat{\Theta}^0 \end{bmatrix}, \quad \text{as } |k| \rightarrow 0,$$

and

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} O(1) & -O(1)|k|^{-1} i & -O(1) \\ O(1) & O(1)|k|^{-1} i & O(1) \\ O(1)|k|^{-1} & -O(1) i & O(1)|k|^{-1} \end{bmatrix} \begin{bmatrix} \hat{\rho}^0 \\ \tilde{k} \cdot \hat{u}^0 \\ \hat{\Theta}^0 \end{bmatrix}, \quad \text{as } |k| \rightarrow \infty.$$

Then, after plugging the previous computations into (3.69), we have

$$\begin{aligned}\hat{\rho}(t, k) &= (O(1)|k|^2\hat{\rho}^0 - O(1)|k|^3i\tilde{k} \cdot \hat{u}^0 - O(1)|k|^2\hat{\Theta}^0) e^{\eta t} \\ &\quad + (O(1)\hat{\rho}^0 + O(1)|k|^3i\tilde{k} \cdot \hat{u}^0 + O(1)|k|^2\hat{\Theta}^0) e^{\phi t} \cos \psi t \\ &\quad + (O(1)\hat{\rho}^0 - O(1)|k|i\tilde{k} \cdot \hat{u}^0 - O(1)|k|^2\hat{\Theta}^0) e^{\phi t} \sin \psi t, \quad \text{as } |k| \rightarrow 0,\end{aligned}$$

and

$$\begin{aligned}\hat{\rho}(t, k) &= (O(1)\hat{\rho}^0 - O(1)|k|^{-1}i\tilde{k} \cdot \hat{u}^0 - O(1)\hat{\Theta}^0) e^{\eta t} \\ &\quad + (O(1)\hat{\rho}^0 + O(1)|k|^{-1}i\tilde{k} \cdot \hat{u}^0 + O(1)\hat{\Theta}^0) e^{\phi t} \cos \psi t \\ &\quad + (O(1)|k|^{-1}\hat{\rho}^0 - O(1)i\tilde{k} \cdot \hat{u}^0 + O(1)|k|^{-1}\hat{\Theta}^0) e^{\phi t} \sin \psi t, \quad \text{as } |k| \rightarrow \infty.\end{aligned}$$

Thus, we obtain (3.91). Similarly, we get (3.93) and the first term on the right hand side of both (3.92) and (3.94).

Next, let us seek the upper bounds of the second part  $(\hat{u}_\perp, \hat{E}_\perp, \hat{B}_\perp)$  in terms of  $(\hat{u}^0(k), \hat{E}^0(k), \hat{B}^0(k))$  by (3.90). For the upper bound of  $\hat{u}_\perp$ , by Lemma 3.4 and (3.87), we obtain

$$\begin{bmatrix} c_{10} \\ c_{11} \\ c_{12} \end{bmatrix} = \begin{bmatrix} -O(1)|k|^2I_3 & -O(1)|k|^2I_3 & O(1)|k|i\tilde{k} \times \\ -O(1)|k|^2I_3 & O(1)I_3 & -O(1)|k|i\tilde{k} \times \\ O(1)I_3 & O(1)I_3 & O(1)|k|i\tilde{k} \times \end{bmatrix} \begin{bmatrix} \hat{u}_\perp^0 \\ \hat{E}_\perp^0 \\ \hat{B}_\perp^0 \end{bmatrix}, \quad \text{as } |k| \rightarrow 0,$$

and

$$\begin{bmatrix} c_{10} \\ c_{11} \\ c_{12} \end{bmatrix} = \begin{bmatrix} -O(1)|k|^{-2}I_3 & -O(1)|k|^{-4}I_3 & O(1)|k|^{-3}i\tilde{k} \times \\ -O(1)|k|^{-2}I_3 & O(1)I_3 & -O(1)|k|^{-3}i\tilde{k} \times \\ O(1)|k|^{-1}I_3 & O(1)|k|^{-3}I_3 & O(1)i\tilde{k} \times \end{bmatrix} \begin{bmatrix} \hat{u}_\perp^0 \\ \hat{E}_\perp^0 \\ \hat{B}_\perp^0 \end{bmatrix}, \quad \text{as } |k| \rightarrow \infty.$$

Moreover, it follows from Lemma 3.4 that

$$\frac{1 + \phi_*}{(1 + \phi_*)^2 + \psi_*^2} = \begin{cases} O(1), & \text{as } |k| \rightarrow 0, \\ O(1)|k|^{-2}, & \text{as } |k| \rightarrow \infty, \end{cases}$$

and

$$\frac{\psi_*}{(1 + \phi_*)^2 + \psi_*^2} = \begin{cases} O(1), & \text{as } |k| \rightarrow 0, \\ O(1)|k|^{-1}, & \text{as } |k| \rightarrow \infty. \end{cases}$$

Therefore, after plugging the previous computations into (3.84), we get

$$\begin{aligned}\hat{u}_\perp(t, k) &= -O(1) \left( -|k|^2\hat{u}_\perp^0 - |k|^2\hat{E}_\perp^0 + |k|i\tilde{k} \times \hat{B}_\perp^0 \right) e^{\eta_*(k)t} \\ &\quad - O(1) \left( -|k|^2\hat{u}_\perp^0 + \hat{E}_\perp^0 - |k|i\tilde{k} \times \hat{B}_\perp^0 \right) (\cos \psi_* t + \sin \psi_* t) e^{\phi_*(k)t} \\ &\quad - O(1) \left( \hat{u}_\perp^0 + \hat{E}_\perp^0 + |k|i\tilde{k} \times \hat{B}_\perp^0 \right) (\sin \psi_* t - \cos \psi_* t) e^{\phi_*(k)t}, \quad \text{as } |k| \rightarrow 0,\end{aligned}$$

and

$$\begin{aligned}\hat{u}_\perp(t, k) &= -O(1) \left( -|k|^{-2} \hat{u}_\perp^0 - |k|^{-4} \hat{E}_\perp^0 + |k|^{-3} i \tilde{k} \times \hat{B}_\perp^0 \right) |k|^2 e^{\eta_*(k)t} \\ &\quad - O(1) \left( |k|^{-2} \hat{u}_\perp^0 + \hat{E}_\perp^0 - |k|^{-3} i \tilde{k} \times \hat{B}_\perp^0 \right) \left( |k|^{-2} \cos \psi_* t + |k|^{-1} \sin \psi_* t \right) e^{\phi_*(k)t} \\ &\quad - O(1) \left( |k|^{-1} \hat{u}_\perp^0 + |k|^{-3} \hat{E}_\perp^0 + i \tilde{k} \times \hat{B}_\perp^0 \right) \left( |k|^{-2} \sin \psi_* t - |k|^{-1} \cos \psi_* t \right) e^{\phi_*(k)t}, \text{ as } |k| \rightarrow \infty.\end{aligned}$$

Notice that due to Lemma 3.4 again, there exists a constant  $\gamma > 0$  such that

$$\begin{cases} \eta_*(k) \leq -\gamma |k|^2, & \phi_*(k) = -\frac{1}{2} - \frac{\eta_*}{2} \leq -\gamma, & \text{over } |k| \leq 1, \\ \eta_*(k) \leq -\gamma, & \phi_*(k) = -\frac{1}{2} - \frac{\eta_*}{2} \leq -\frac{\gamma}{|k|^2}, & \text{over } |k| \geq 1. \end{cases}$$

Then we obtain, for  $|k| \leq 1$ ,

$$|\hat{u}_\perp(t, k)| \leq C \left( e^{-\gamma t} + |k| e^{-\gamma |k|^2 t} \right) |(\hat{u}_\perp^0, \hat{E}_\perp^0, \hat{B}_\perp^0)| \leq C \left( e^{-\gamma t} + |k| e^{-\gamma |k|^2 t} \right) |(\hat{u}_\perp^0, \hat{E}_\perp^0, \hat{B}_\perp^0)|,$$

and for  $|k| \geq 1$ ,

$$|\hat{u}_\perp(t, k)| \leq C \left( e^{-\gamma t} + \frac{1}{|k|} e^{-\frac{\gamma t}{|k|^2}} \right) |(\hat{u}_\perp^0, \hat{E}_\perp^0, \hat{B}_\perp^0)| \leq C \left( e^{-\gamma t} + \frac{1}{|k|} e^{-\frac{\gamma t}{|k|^2}} \right) |(\hat{u}_\perp^0, \hat{E}_\perp^0, \hat{B}_\perp^0)|,$$

which give the upper bound of  $\hat{u}_\perp(t, k)$  corresponding to the second term on the right hand side of (3.92). Hence, (3.92) is established. Finally, (3.94) and (3.95) can be proved in a similar way as for (3.92). This finishes the proof of Lemma 3.5.  $\square$

Based on Lemma 3.5, we refine the time decay property for the solution  $W = (\rho, u, \Theta, E, B)$  obtained in Theorem 3.4 as follows.

**Théorème 3.6.** *Let  $m \geq 0$  be an integer and  $1 \leq p, r \leq 2 \leq q \leq \infty, l \geq 0$ . Suppose  $W$  to be the solution to the Cauchy problem (3.43)-(3.44) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfying (3.45). Then, for any  $t \geq 0$ ,  $W$  satisfies the following time decay property :*

$$(3.96) \quad \|\nabla^m \rho(t)\|_{L^q} \leq C e^{-\frac{t}{2}} \left( \|(\rho^0, u^0, \Theta^0)\|_{L^p} + \left\| \nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+} (\rho^0, u^0, \Theta^0) \right\|_{L^r} \right),$$

$$(3.97) \quad \begin{aligned} \|\nabla^m u(t)\|_{L^q} &\leq C e^{-\frac{t}{2}} \left( \|(\rho^0, \Theta^0)\|_{L^p} + \left\| \nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+} (\rho^0, \Theta^0) \right\|_{L^r} \right) \\ &\quad + C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m+1}{2}} \|(u^0, E^0, B^0)\|_{L^p} \\ &\quad + C(1+t)^{-\frac{l+1}{2}} \left\| \nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+} (u^0, E^0, B^0) \right\|_{L^r}, \end{aligned}$$

$$(3.98) \quad \|\nabla^m \Theta(t)\|_{L^q} \leq C e^{-\frac{t}{2}} \left( \|(\rho^0, u^0, \Theta^0)\|_{L^p} + \left\| \nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+} (\rho^0, u^0, \Theta^0) \right\|_{L^r} \right),$$

$$(3.99) \quad \begin{aligned} \|\nabla^m E(t)\|_{L^q} &\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m+1}{2}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^p} \\ &\quad + C(1+t)^{-\frac{l}{2}} \|\nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+}(u^0, \Theta^0, E^0, B^0)\|_{L^r}, \end{aligned}$$

and

$$(3.100) \quad \begin{aligned} \|\nabla^m B(t)\|_{L^q} &\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|(u^0, E^0, B^0)\|_{L^p} \\ &\quad + C(1+t)^{-\frac{l}{2}} \|\nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+}(u^0, E^0, B^0)\|_{L^r}, \end{aligned}$$

where  $[l+3(\frac{1}{r}-\frac{1}{q})]_+$  is defined by (3.58).

**Proof.** We only give the estimate of  $\Theta$  for simplicity. Take  $1 \leq p, r \leq 2 \leq q \leq \infty$  and an integer  $m \geq 0$ . It follows from Lemma 3.2 with  $\frac{1}{q} + \frac{1}{q'} = 1$  and (3.93) that

$$(3.101) \quad \begin{aligned} \|\nabla^m \Theta(t)\|_{L_x^q} &\leq C \| |k|^m \hat{\Theta}(t) \|_{L_k^{q'}} \\ &\leq C e^{-\frac{t}{2}} \left( \| |k|^m (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \|_{L^{q'}(|k| \leq 1)} + \| |k|^m (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \|_{L^{q'}(|k| \geq 1)} \right). \end{aligned}$$

Now let us estimate each term on the right hand side of (3.101). Let  $\varepsilon > 0$  be a sufficiently small constant. For the first term, using the Hölder inequality with  $\frac{1}{q'} - \frac{1}{p'} = \frac{p' - q'}{p'q'}$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , we have

$$\begin{aligned} \| |k|^m (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \|_{L^{q'}(|k| \leq 1)} &= \left\| |k|^{-\frac{p'-q'}{p'q'}(3-\varepsilon)} |k|^{m+\frac{p'-q'}{p'q'}(3-\varepsilon)} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{q'}(|k| \leq 1)} \\ &\leq \| |k|^{-(3-\varepsilon)} \|_{L^1(|k| \leq 1)}^{\frac{p'-q'}{p'q'}} \left\| |k|^{m+\frac{p'-q'}{p'q'}(3-\varepsilon)} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{p'}(|k| \leq 1)} \\ &\leq C \left\| |k|^{m+\frac{p'-q'}{p'q'}(3-\varepsilon)} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{p'}(|k| \leq 1)} \\ &\leq C \| (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \|_{L^{p'}(|k| \leq 1)} \leq C \| (\rho^0, u^0, \Theta^0) \|_{L^p}. \end{aligned}$$

Similarly, for the second term, using the Hölder inequality with  $\frac{1}{q'} - \frac{1}{r'} = \frac{r' - q'}{r'q'}$  and  $\frac{1}{r} + \frac{1}{r'} = 1$ , we have

$$\begin{aligned} \| |k|^m (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \|_{L^{q'}(|k| \geq 1)} &= \left\| |k|^{-\frac{r'-q'}{r'q'}(3+\varepsilon)} |k|^{m+\frac{r'-q'}{r'q'}(3+\varepsilon)} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{q'}(|k| \geq 1)} \\ &\leq \| |k|^{-(3+\varepsilon)} \|_{L^1(|k| \geq 1)}^{\frac{r'-q'}{r'q'}} \left\| |k|^{m+\frac{r'-q'}{r'q'}(3+\varepsilon)} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{r'}(|k| \geq 1)} \\ &\leq C \left\| |k|^{m+\frac{r'-q'}{r'q'}(3+\varepsilon)} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{r'}(|k| \geq 1)} \\ &\leq C \left\| |k|^{m+[3(\frac{1}{r}-\frac{1}{q})]_-+1} (\hat{\rho}^0, \hat{u}^0, \hat{\Theta}^0) \right\|_{L^{r'}(|k| \geq 1)} \\ &\leq C \left\| \nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+} (\rho^0, u^0, \Theta^0) \right\|_{L^r}. \end{aligned}$$

These two estimates imply (3.98). We get (3.96), (3.97), (3.99) and (3.100) in a similar way.  $\square$

Based on Theorem 3.6, we list some particular cases as follows for later use.

**Corollary 3.2.** *Let  $W = (\rho, u, \Theta, E, B)$  be the solution to the Cauchy problem (3.43)-(3.44) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfying (3.45). Then, for any  $t \geq 0$ ,  $W$  satisfies the following time decay property :*

$$(3.102) \quad \left\{ \begin{array}{l} \|\rho(t)\| \leq C e^{-\frac{t}{2}} \|(\rho^0, u^0, \Theta^0)\|, \\ \|u(t)\| \leq C e^{-\frac{t}{2}} \|(\rho^0, \Theta^0)\| + C(1+t)^{-\frac{5}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2}, \\ \|\Theta(t)\| \leq C e^{-\frac{t}{2}} \|(\rho^0, u^0, \Theta^0)\|, \\ \|E(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^1 \cap \dot{H}^3}, \\ \|B(t)\| \leq C(1+t)^{-\frac{3}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2}, \end{array} \right.$$

$$(3.103) \quad \left\{ \begin{array}{l} \|\rho(t)\|_{L^\infty} \leq C e^{-\frac{t}{2}} \|(\rho^0, u^0, \Theta^0)\|_{L^2 \cap \dot{H}^2}, \\ \|u(t)\|_{L^\infty} \leq C e^{-\frac{t}{2}} \|(\rho^0, \Theta^0)\|_{L^1 \cap \dot{H}^2} + C(1+t)^{-2} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^5}, \\ \|\Theta(t)\|_{L^\infty} \leq C e^{-\frac{t}{2}} \|(\rho^0, u^0, \Theta^0)\|_{L^2 \cap \dot{H}^2}, \\ \|E(t)\|_{L^\infty} \leq C(1+t)^{-2} \|(u^0, \Theta^0, E^0, B^0)\|_{L^1 \cap \dot{H}^6}, \\ \|B(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^5}, \end{array} \right.$$

and

$$(3.104) \quad \left\{ \begin{array}{l} \|\nabla B(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^4}, \\ \|\nabla^s(E(t), B(t))\| \leq C(1+t)^{-\frac{5}{4}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^2 \cap \dot{H}^{s+3}}, \end{array} \right.$$

where and in the rest of this thesis,  $\dot{H}^s$  denotes the homogeneous Sobolev space of order  $s$  :

$$\dot{H}^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) \mid \partial^\alpha u \in L^2(\mathbb{R}^3), \text{ for } |\alpha| = s\}$$

with the norm

$$\|u\|_{\dot{H}^s} = \sqrt{\sum_{|\alpha|=s} \|\partial^\alpha u\|^2}.$$

### 3.5 Time decay rates for nonlinear systems

In this section, we give the proof of Propositions 3.3-3.4. For the solution  $W = (\rho, u, \Theta, E, B)$  of the nonlinear Cauchy problem (3.5)-(3.6), the first two subsections



are devoted to obtain the time decay rates of the energy  $\|W(t)\|_s^2$  and the higher order energy  $\|\nabla W(t)\|_{s-1}^2$ . In the last subsection, the time decay rates in  $L^q$  with  $2 \leq q \leq +\infty$  for every component  $\rho, u, \Theta, E$  and  $B$  of the solution  $W$  are established.

In the following, since we shall apply the linear  $L^p - L^q$  time decay property of the homogeneous system (3.43) studied in the previous section to nonlinear system (3.5), we rewrite (3.5) in the following form :

$$(3.105) \quad \begin{cases} \partial_t \rho + \nabla \cdot u = g_1, \\ \partial_t u + \nabla \rho + \nabla \Theta + E + u = g_2, \\ \partial_t \Theta + \frac{2}{3} \nabla \cdot u + \Theta = g_3, \\ \partial_t E - \nabla \times B - u = g_4, \quad \nabla \cdot E = -\rho, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with

$$(3.106) \quad \begin{cases} g_1 = -\nabla \cdot (\rho u), \\ g_2 = -(u \cdot \nabla) u - \left( \frac{1 + \Theta}{1 + \rho} - 1 \right) \nabla \rho - u \times B, \\ g_3 = -u \cdot \nabla \Theta - \frac{2}{3} \Theta \nabla \cdot u - \frac{1}{3} |u|^2, \\ g_4 = \rho u. \end{cases}$$

Then, by the Duhamel principle, the solution  $W$  can be formally written as

$$(3.107) \quad W(t) = e^{tL} W^0 + \int_0^t e^{(t-\tau)L} (g_1(\tau), g_2(\tau), g_3(\tau), g_4(\tau), 0) d\tau,$$

where  $e^{tL}$  is defined by (3.57).

### 3.5.1 Decay rates for the energy functionals

In this subsection we prove the time decay estimate (3.15) in Proposition 3.3 for the energy  $\|W(t)\|_s^2$ . We begin with the following Lemma which can be seen directly from the proof of Theorem 3.2.

**Lemma 3.6.** *Let  $W = (\rho, u, \Theta, E, B)$  be the solution to the Cauchy problem (3.5)-(3.6) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  which satisfy (3.7) in the sense of Proposition 3.2. Then, if  $\mathfrak{E}_s(W^0)$  is small enough, for any  $t \geq 0$ ,*

$$(3.108) \quad \frac{d}{dt} \mathfrak{E}_s(W(t)) + \mathfrak{D}_s(W(t)) \leq 0.$$

Based on Lemma 3.6, for  $p > 0$ , it holds that

$$\begin{aligned} & (1+t)^p \mathfrak{E}_s(W(t)) + \int_0^t (1+\tau)^p \mathfrak{D}_s(W(\tau)) d\tau \\ & \leq \mathfrak{E}_s(W^0) + Cp \int_0^t (1+\tau)^{p-1} (\|B(\tau)\|^2 + \mathfrak{D}_{s+1}(W(\tau))) d\tau, \end{aligned}$$

where the relation  $\mathfrak{E}_s(W(t)) \leq C \|B(t)\|^2 + C \mathfrak{D}_{s+1}(W(t))$  is used. By Lemma 3.6 again, we have

$$\mathfrak{E}_{s+2}(W(t)) + \int_0^t \mathfrak{D}_{s+2}(W(\tau)) d\tau \leq \mathfrak{E}_{s+2}(W^0)$$

and

$$\begin{aligned} & (1+t)^{p-1} \mathfrak{E}_{s+1}(W(t)) + \int_0^t (1+\tau)^{p-1} \mathfrak{D}_{s+1}(W(\tau)) d\tau \\ & \leq \mathfrak{E}_{s+1}(W^0) + C(p-1) \int_0^t (1+\tau)^{p-2} (\|B(\tau)\|^2 + \mathfrak{D}_{s+2}(W(\tau))) d\tau. \end{aligned}$$

Then, by iterating the previous estimates, we have

$$\begin{aligned} (3.109) \quad & (1+t)^p \mathfrak{E}_s(W(t)) + \int_0^t (1+\tau)^p \mathfrak{D}_s(W(\tau)) d\tau \\ & \leq C \mathfrak{E}_{s+2}(W^0) + C \int_0^t (1+\tau)^{p-1} \|B(\tau)\|^2 d\tau, \quad \forall 1 < p < 2. \end{aligned}$$

Now, let us estimate the integral term on the right hand side of (3.109). Applying the last linear estimate on  $B$  in (3.102) to (3.107), we have

$$(3.110) \quad \|B(t)\| \leq C(1+t)^{-\frac{3}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2} + C \int_0^t (1+t-\tau)^{-\frac{3}{4}} \|(g_2(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^2} d\tau.$$

It is direct to check that for any  $0 \leq \tau \leq t$ ,

$$\|(g_2(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^2} \leq C \|W(\tau)\|_3^2 \leq C \mathfrak{E}_s(W(\tau)) \leq C(1+\tau)^{-\frac{3}{2}} \mathfrak{E}_{s,\infty}(W(t)),$$

where and in the rest of both Chapter 3 and Chapter 4

$$\mathfrak{E}_{s,\infty}(W(t)) := \sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{3}{2}} \mathfrak{E}_s(W(\tau)).$$

Plugging this into (3.110) implies

$$(3.111) \quad \|B(t)\| \leq C(1+t)^{-\frac{3}{4}} \left( \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2} + \mathfrak{E}_{s,\infty}(W(t)) \right).$$

Next, we prove the uniform-in-time bound of  $\mathfrak{E}_{s,\infty}(W(t))$  which implies the decay rates of the energy functionals  $\mathfrak{E}_s(W(t))$  and thus  $\|W(t)\|_s^2$ . In fact, by choosing  $p = \frac{3}{2} + \varepsilon$  in (3.109) with  $\varepsilon > 0$  sufficiently small and using (3.111), we have

$$\begin{aligned} & (1+t)^{\frac{3}{2}+\varepsilon} \mathfrak{E}_s(W(t)) + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} \mathfrak{D}_s(W(\tau)) d\tau \\ & \leq C \mathfrak{E}_{s+2}(W^0) + C(1+t)^\varepsilon \left( \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2}^2 + (\mathfrak{E}_{s,\infty}(W(t)))^2 \right), \end{aligned}$$

which implies

$$(1+t)^{\frac{3}{2}} \mathfrak{E}_s(W(t)) \leq C \left( \mathfrak{E}_{s+2}(W^0) + \|(u^0, E^0, B^0)\|_{L^1}^2 + (\mathfrak{E}_{s,\infty}(W(t)))^2 \right),$$

and thus

$$\mathfrak{E}_{s,\infty}(W(t)) \leq C \left( (\omega_{s+2}(W^0))^2 + (\mathfrak{E}_{s,\infty}(W(t)))^2 \right),$$

since  $\omega_{s+2}(W^0) > 0$  is small enough, it holds that  $\mathfrak{E}_{s,\infty}(W(t)) \leq C(\omega_{s+2}(W^0))^2$  for any  $t \geq 0$ , which gives

$$\|W(t)\|_s \leq C \mathfrak{E}_s(W(t))^{\frac{1}{2}} \leq C \omega_{s+2}(W^0) (1+t)^{-\frac{3}{4}},$$

that is (3.15). □

### 3.5.2 Decay rates for the higher order energy functionals

In this subsection, we continue the proof of Proposition 3.3 for the second part (3.16), that is the time decay estimate of the higher order energy functionals  $\|\nabla W(t)\|_{s-1}^2$ . In fact, it can be reduced to the time decay estimates only on  $\|\nabla B\|$  and  $\|\nabla^s(E, B)\|$  with the help of the following Lemma.

**Lemma 3.7.** *Let  $W = (\rho, u, \Theta, E, B)$  be the solution to the Cauchy problem (3.5)-(3.6) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfying (3.7) in the sense of Proposition 3.2. Then, if  $\mathfrak{E}_s(W^0)$  is small enough, there exist the higher order energy functionals  $\mathfrak{E}_s^h(\cdot)$  and the corresponding higher order dissipative rate  $\mathfrak{D}_s^h(\cdot)$  in the form of (3.9) and (3.11) such that for any  $t \geq 0$ ,*

$$(3.112) \quad \frac{d}{dt} \mathfrak{E}_s^h(W(t)) + \mathfrak{D}_s^h(W(t)) \leq 0.$$

**Proof.** The proof is very similar to that of Theorem 3.2. In fact, by letting the energy estimates made only on the higher order derivatives, then corresponding to (3.22), (3.32), (3.39) and (3.40), it can be reverified that

$$\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s} (\langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle + \|\partial^\alpha W_{II}\|^2) + \|\nabla u\|_{s-1}^2 + \|\nabla \Theta\|_{s-1}^2 \leq C \|W\|_s \|W_I\|_s^2,$$

$$\frac{d}{dt} \sum_{1 \leq \alpha \leq s-1} \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle + c_0 \|\nabla \rho\|_{s-1}^2 \leq C \|W\|_s \|W_I\|_s^2 + C \|\nabla u\|_{s-1}^2 + C \|\nabla \Theta\|_{s-1}^2,$$

$$\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha E \rangle + c_0 \|\nabla E\|_{s-2}^2 \leq C (\|\nabla W_I\|_{s-1}^2 + \|W\|_s \|W_I\|_s^2 + \|\nabla u\|_{s-1} \|\nabla^2 B\|_{s-3}),$$

and

$$\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-2} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle + c_0 \|\nabla^2 B\|_{s-3}^2 \leq C (\|\nabla u\|_{s-1}^2 + \|\nabla E\|_{s-2}^2 + \|W_I\|_s^2 \|\nabla(\rho, u)\|_{s-1}^2).$$

Here, the details of proof are omitted for simplicity. Now, we define the higher order energy functionals as :

$$(3.113) \quad \begin{aligned} \mathfrak{E}_s^h(W(t)) &= \sum_{1 \leq |\alpha| \leq s} (\langle A_0^I(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle + \|\partial^\alpha W_{II}\|^2) \\ &+ \mathfrak{K}_1 \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle + \mathfrak{K}_2 \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha E \rangle \\ &+ \mathfrak{K}_3 \sum_{1 \leq |\alpha| \leq s-2} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle. \end{aligned}$$

Similarly, we can choose  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  to be sufficiently small with  $\mathfrak{K}_2^{\frac{3}{2}} \ll \mathfrak{K}_3$ , such that  $\mathfrak{E}_s^h(W(t)) \sim \|\nabla W(t)\|_{s-1}^2$ , that is,  $\mathfrak{E}_s^h(\cdot)$  is really a higher order energy functional satisfying (3.9), and moreover, the linear combination of the previously obtained four estimates with coefficients corresponding to (3.113) implies (3.112) with  $\mathfrak{D}_s^h(\cdot)$  defined as (3.11). We have finished the proof of Lemma 3.7.  $\square$

Then, from (3.9) and (3.11) for the definitions of  $\mathfrak{E}_s^h(W(t))$  and  $\mathfrak{D}_s^h(W(t))$ , we obtain

$$\frac{d}{dt} \mathfrak{E}_s^h(W(t)) + \mathfrak{E}_s^h(W(t)) \leq C (\|\nabla B\|^2 + \|\nabla^s(E, B)\|^2),$$

which implies

$$(3.114) \quad \mathfrak{E}_s^h(W(t)) \leq \mathfrak{E}_s^h(W^0) e^{-t} + C \int_0^t e^{-(t-\tau)} (\|\nabla B(\tau)\|^2 + \|\nabla^s(E(\tau), B(\tau))\|^2) d\tau.$$

To estimate the time integral term on the right hand side of (3.114), we have

**Lemma 3.8.** *Let  $W = (\rho, u, \Theta, E, B)$  be the solution of the Cauchy problem (3.5)-(3.6) with initial data  $W^0 = (\rho^0, u^0, \Theta^0, E^0, B^0)$  satisfying (3.7) in the sense of Proposition 3.2. Then, if  $\omega_{s+6}(W^0)$  is small enough, for any  $t \geq 0$ ,*

$$(3.115) \quad \|\nabla B(t)\|^2 + \|\nabla^s(E(t), B(t))\|^2 \leq C (\omega_{s+6}(W^0))^2 (1+t)^{-\frac{5}{2}}.$$

**Proof.** By applying the first linear estimate on  $\nabla B$  in (3.104) to (3.107), we have

$$\begin{aligned} \|\nabla B(t)\| &\leq C(1+t)^{-\frac{5}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^4} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_2(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^4} d\tau \\ &\leq C(1+t)^{-\frac{5}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^4} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|W(\tau)\|_{\max\{5, s\}}^2 d\tau \\ &\leq C(1+t)^{-\frac{5}{4}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^4} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} (\omega_{s+6}(W^0))^2 (1+\tau)^{-\frac{3}{2}} d\tau \\ &\leq C \omega_{s+6}(W_0) (1+t)^{-\frac{5}{4}}, \end{aligned}$$

where the smallness of  $\omega_{s+6}(W_0)$  is used. Similarly, by (3.15) and applying the second linear estimate on  $\nabla^s (E(t), B(t))$  in (3.104) to (3.107), we get

$$\begin{aligned}
& \|\nabla^s (E(t), B(t))\| \\
& \leq C(1+t)^{-\frac{5}{4}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^2 \cap \dot{H}^{s+3}} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_2(\tau), g_3(\tau), g_4(\tau))\|_{L^2 \cap \dot{H}^{s+3}} d\tau \\
& \leq C(1+t)^{-\frac{5}{4}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^2 \cap \dot{H}^{s+3}} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|W(\tau)\|_{s+4}^2 d\tau \\
& \leq C(1+t)^{-\frac{5}{4}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^2 \cap \dot{H}^{s+3}} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} (\omega_{s+6}(W^0))^2 (1+\tau)^{-\frac{3}{2}} d\tau \\
& \leq C\omega_{s+6}(W^0) (1+t)^{-\frac{5}{4}},
\end{aligned}$$

where we also used the smallness of  $\omega_{s+6}(W_0)$ . We have finished the proof of Lemma 3.8.  $\square$

Then, plugging (3.115) into (3.114), we have

$$\mathfrak{E}_s^h(W(t)) \leq \mathfrak{E}_s^h(W^0)e^{-t} + C(\omega_{s+6}(W^0))^2 (1+t)^{-\frac{5}{2}}.$$

Since  $\mathfrak{E}_s^h(W(t)) \sim \|\nabla W(t)\|_{s-1}^2$  holds true for any  $t \geq 0$ , (3.16) follows. Therefore, the proof of Proposition 3.5 is finished.  $\square$

### 3.5.3 Decay rates in $L^q$

In this subsection we prove Proposition 3.4 for the time decay rates of solutions  $W = (\rho, u, \Theta, E, B)$  to the Cauchy problem (3.5)-(3.6) in  $L^q$  with  $2 \leq q \leq +\infty$ . Throughout this subsection, we always suppose  $\omega_{13}(W^0) > 0$  to be small enough. In addition, for  $s \geq 4$ , Proposition 3.3 shows that if  $\omega_{s+2}(W^0)$  is small enough,

$$(3.116) \quad \|W(t)\|_s \leq C\omega_{s+2}(W^0)(1+t)^{-\frac{3}{4}},$$

and if  $\omega_{s+6}(W^0)$  is small enough,

$$(3.117) \quad \|\nabla W(t)\|_{s-1} \leq C\omega_{s+6}(W^0)(1+t)^{-\frac{5}{4}}.$$

Now, let us establish the estimates on  $B$ ,  $(u, E)$  and  $(\rho, \Theta)$  in turn as follows.

*Estimate on  $\|B\|_{L^q}$ .* For  $L^2$  rate, it is easy to see from (3.116) that

$$\|B(t)\| \leq C\omega_6(W^0)(1+t)^{-\frac{3}{4}}.$$

For  $L^\infty$  rate, by applying  $L^\infty$  estimate on  $B$  in (3.103) to (3.107), we have

$$\|B(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}} \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^5} + C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|(g_2(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^5} d\tau.$$

By (3.116), since

$$\|(g_2(t), g_4(t))\|_{L^1 \cap \dot{H}^5} \leq C \|W(t)\|_6^2 \leq C (\omega_8(W^0))^2 (1+t)^{-\frac{3}{2}},$$

we have

$$\|B(t)\|_{L^\infty} \leq C \omega_8(W^0) (1+t)^{-\frac{3}{2}}.$$

Thus, by  $L^2 - L^\infty$  interpolation

$$(3.118) \quad \|B(t)\|_{L^q} \leq C \omega_8(W^0) (1+t)^{-\frac{3}{2} + \frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

*Estimate on  $\|(u, E)\|_{L^q}$ .* For  $L^2$  rate, applying the  $L^2$  estimates on  $u$  and  $E$  in (3.102) to (3.107), we have

$$\begin{aligned} \|u(t)\| &\leq C(1+t)^{-\frac{5}{4}} \left( \|(\rho^0, \Theta^0)\| + \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2} \right) \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} (\|(g_1(\tau), g_3(\tau))\| + \|(g_2(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^2}) d\tau \end{aligned}$$

and

$$\|E(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(u^0, \Theta^0, E^0, B^0)\|_{L^1 \cap \dot{H}^3} + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_2(\tau), g_3(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^3} d\tau.$$

By (3.116), since

$$\|(g_1(t), g_3(t))\| + \|(g_2(t), g_3(t), g_4(t))\|_{L^1 \cap \dot{H}^3} \leq C \|W(t)\|_4^2 \leq C (\omega_6(W^0))^2 (1+t)^{-\frac{3}{2}},$$

we get

$$\|(u(t), E(t))\| \leq C \omega_6(W^0) (1+t)^{-\frac{5}{4}}.$$

For  $L^\infty$  rate, applying the  $L^\infty$  estimates on  $u$  and  $E$  in (3.103) to (3.107), we have

$$\begin{aligned} \|u(t)\|_{L^\infty} &\leq C(1+t)^{-2} \left( \|(\rho^0, \Theta^0)\|_{L^1 \cap \dot{H}^2} + \|(u^0, E^0, B^0)\|_{L^1 \cap \dot{H}^5} \right) \\ &\quad + C \int_0^t (1+t-\tau)^{-2} (\|(g_1(\tau), g_3(\tau))\|_{L^1 \cap \dot{H}^2} + \|(g_2(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^5}) d\tau \end{aligned}$$

and

$$\begin{aligned} \|E(t)\|_{L^\infty} &\leq C(1+t)^{-2} \|(u^0, \Theta^0, E^0, B^0)\|_{L^1 \cap \dot{H}^6} \\ &\quad + C \int_0^t (1+t-\tau)^{-2} \|(g_2(\tau), g_3(\tau), g_4(\tau))\|_{L^1 \cap \dot{H}^6} d\tau. \end{aligned}$$

Since

$$\|(g_1(t), g_3(t))\|_{\dot{H}^2} + \|(g_2(t), g_3(t), g_4(t))\|_{\dot{H}^5 \cap \dot{H}^6} \leq C \|\nabla W(t)\|_6^2 \leq (\omega_{13}(W^0))^2 (1+t)^{-\frac{5}{2}},$$

and

$$\|(g_1(t), g_2(t), g_3(t), g_4(t))\|_{L^1} \leq C \|W(t)\| (\|u(t)\| + \|\nabla W(t)\|) \leq (\omega_{10}(W^0))^2 (1+t)^{-2},$$

we obtain

$$\|(u(t), E(t))\|_{L^\infty} \leq C (\omega_{13}(W^0))^2 (1+t)^{-2}.$$

Thus, by  $L^2 - L^\infty$  interpolation

$$(3.119) \quad \|(u(t), E(t))\|_{L^q} \leq C \omega_{13}(W^0) (1+t)^{-2+\frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

*Estimate on  $\|(\rho, \Theta)\|_{L^q}$ .* For  $L^2$  rate, applying the  $L^2$  estimates on  $\rho$  and  $\Theta$  in (3.102) to (3.107), we have

$$(3.120) \quad \|(\rho(t), \Theta(t))\| \leq C e^{-\frac{t}{2}} \|(\rho^0, u^0, \Theta^0)\| + C \int_0^t e^{-\frac{t-\tau}{2}} \|(g_1(\tau), g_2(\tau), g_3(\tau))\| d\tau.$$

Since

$$(3.121) \quad \begin{aligned} \|(g_1(t), g_2(t), g_3(t))\| &\leq C \|u(t)\|_{L^\infty} (\|\nabla(\rho(t), u(t), \Theta(t))\| + \|(B(t), u(t))\|) \\ &\quad + C \|(\rho(t), \Theta(t))\| \|\nabla(\rho(t), u(t))\|_2 \\ &\leq C (\omega_{13}(W^0))^2 (1+t)^{-2}, \end{aligned}$$

(3.120) implies the slower decay estimate

$$(3.122) \quad \|(\rho(t), \Theta(t))\| \leq C \omega_{13}(W^0) (1+t)^{-2}.$$

Plugging (3.122) into (3.121) and re-estimating  $\|(g_1(t), g_2(t), g_3(t))\|$  implies

$$\|(g_1(t), g_2(t), g_3(t))\| \leq C (\omega_{13}(W^0))^2 (1+t)^{-\frac{11}{4}},$$

it follows from (3.120) that

$$\|(\rho(t), \Theta(t))\| \leq C \omega_{13}(W^0) (1+t)^{-\frac{11}{4}}.$$

For  $L^\infty$  rate, by applying the  $L^\infty$  estimates on  $\rho$  and  $\Theta$  in (3.103) to (3.107),

$$(3.123) \quad \begin{aligned} &\|(\rho(t), \Theta(t))\|_{L^\infty} \\ &\leq C e^{-\frac{t}{2}} \|(\rho^0, u^0, \Theta^0)\|_{L^2 \cap \dot{H}^2} + C \int_0^t e^{-\frac{t-\tau}{2}} \|(g_1(\tau), g_2(\tau), g_3(\tau))\|_{L^2 \cap \dot{H}^2} d\tau. \end{aligned}$$

It is straightforward to check

$$(3.124) \quad \begin{aligned} &\|(g_1(t), g_2(t), g_3(t))\|_{L^2 \cap \dot{H}^2} \\ &\leq C \|\nabla W(t)\|_4 (\|(\rho(t), \Theta(t))\| + \|(u(t), B(t))\|_{L^\infty} + \|\nabla(\rho(t), u(t), \Theta(t))\|_{L^\infty}) \\ &\leq C \omega_{13}(W^0) (1+t)^{-\frac{11}{4}}, \end{aligned}$$

which implies from (3.123) that

$$\|(\rho(t), \Theta(t))\|_{L^\infty} \leq C \omega_{13}(W_0) (1+t)^{-\frac{11}{4}}.$$

Therefore, by  $L^2 - L^\infty$  interpolation

$$(3.125) \quad \|(\rho(t), \Theta(t))\|_{L^q} \leq C\omega_{13}(W^0)(1+t)^{-\frac{11}{4}}, \quad \forall 2 \leq q \leq +\infty.$$

Therefore, (3.125), (3.119) and (3.118) give (3.17), (3.18) and (3.19), respectively. We have finished the proof of Proposition 3.4.  $\square$





## Chapitre 4

# Asymptotic behavior of global smooth solutions for two-fluid non isentropic Euler-Maxwell systems

### 4.1 Introduction and main results

Based on the results in Chapter 3, we continue to consider the Cauchy problem for the two-fluid non isentropic Euler-Maxwell system (1.18) with  $\theta_* = 1$  :

$$(4.1) \quad \left\{ \begin{array}{l} \partial_t n^e + \nabla \cdot (n^e u^e) = 0, \\ \partial_t u^e + (u^e \cdot \nabla) u^e + \frac{1}{n^e} \nabla (n^e \theta^e) = -(E + u^e \times B) - u^e, \\ \partial_t \theta^e + u^e \cdot \nabla \theta^e + \frac{2}{3} \theta^e \nabla \cdot u^e + \frac{1}{3} |u^e|^2 + (\theta^e - 1) = 0, \\ \partial_t n^i + \nabla \cdot (n^i u^i) = 0, \\ \partial_t u^i + (u^i \cdot \nabla) u^i + \frac{1}{n^i} \nabla (n^i \theta^i) = (E + u^i \times B) - u^i, \\ \partial_t \theta^i + u^i \cdot \nabla \theta^i + \frac{2}{3} \theta^i \nabla \cdot u^i + \frac{1}{3} |u^i|^2 + (\theta^i - 1) = 0, \\ \partial_t E - \nabla \times B = n^e u^e - n^i u^i, \quad \nabla \cdot E = n^i - n^e, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3. \end{array} \right.$$

Initial values are given as :

$$(4.2) \quad (n^\nu, u^\nu, \theta^\nu, E, B)|_{t=0} = (n^{\nu 0}, u^{\nu 0}, \theta^{\nu 0}, E^0, B^0), \quad \nu = e, i, \quad \text{in } \mathbb{R}^3,$$

with the compatibility condition :

$$(4.3) \quad \nabla \cdot E^0 = n^{i0} - n^{e0}, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

The main results of this Chapter can be stated as follows.

**Theorem 4.1.** *Let  $s \geq 4$  and (4.3) hold. Then, there are  $\delta_0 > 0$  and  $C_0 > 0$  such that if*

$$\left\| (n^{\nu 0} - 1, u^{\nu 0}, \theta^{\nu 0} - 1, E^0, B^0) \right\|_s \leq \delta_0,$$

*the Cauchy problem (4.1)-(4.2) admits a unique global solution  $(n^\nu, u^\nu, \theta^\nu, E, B)$  satisfying*

$$(n^\nu - 1, u^\nu, \theta^\nu - 1, E, B) \in C^1(\mathbb{R}^+; H^{s-1}) \cap C(\mathbb{R}^+; H^s)$$

*and*

$$\sup_{t \geq 0} \left\| (n^\nu(t) - 1, u^\nu(t), \theta^\nu(t) - 1, E(t), B(t)) \right\|_s \leq C_0 \left\| (n^{\nu 0} - 1, u^{\nu 0}, \theta^{\nu 0} - 1, E^0, B^0) \right\|_s.$$

*Moreover, there exist  $\delta_1 > 0$  and  $C_1 > 0$  such that if*

$$\left\| (n^{\nu 0} - 1, u^{\nu 0}, \theta^{\nu 0} - 1, E^0, B^0) \right\|_{13} + \left\| (n^{\nu 0} - 1, u^{\nu 0}, \theta^{\nu 0} - 1, E^0, B^0) \right\|_{L^1} \leq \delta_1,$$

*then, the solution  $(n^\nu, u^\nu, \theta^\nu, E, B)$  satisfies for any  $t \geq 0$ ,*

$$(4.4) \quad \left\| (n^e(t) - n^i(t), \theta^e(t) - \theta^i(t)) \right\|_{L^q} \leq C_1(1+t)^{-2-\frac{1}{q}}, \quad \forall 2 \leq q \leq +\infty,$$

$$(4.5) \quad \left\| (n^e(t) + n^i(t) - 2, \theta^e(t) + \theta^i(t) - 2) \right\|_{L^q} \leq C_1(1+t)^{-\frac{3}{2}+\frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty,$$

$$(4.6) \quad \left\| (u^e(t) \pm u^i(t), E(t)) \right\|_{L^q} \leq C_1(1+t)^{-\frac{3}{2}+\frac{1}{2q}}, \quad \forall 2 \leq q \leq +\infty,$$

$$(4.7) \quad \|B(t)\|_{L^q} \leq C_1(1+t)^{-\frac{3}{2}+\frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

Similarly to that for the one-fluid non isentropic Euler-Maxwell system in Chapter 3, we complete the proof of Theorem 4.1 by also using careful energy estimates and the Fourier multiplier method. For this purpose, we must overcome the difficulties caused by two-fluid particles and the temperature field to establish the energy estimates and the large time decay rate. This can be done by introducing the 'total functions' and 'difference functions' for the densities, the velocities and the temperatures. By studying the properties of both the functions' in Fourier space, we obtain the decay rate for two linearized systems. This concludes the decay rate results for the nonlinear two-fluid non isentropic Euler-Maxwell system by using the Duhamel principle and the energy estimates.

The rest of this Chapter is arranged as follows. In Section 4.2, the transformation of the Cauchy problem and the proof of the global existence and uniqueness of solutions are presented. In Section 4.3, we study the linearized homogeneous system to get the  $L^p - L^q$  decay property and the explicit representation of solutions. In the last Section 4.4, we investigate the decay rates of solutions to the transformed nonlinear system and complete the proof of Theorem 4.1.

## 4.2 Global solutions for nonlinear systems

### 4.2.1 Preliminaries

Let  $(n^\nu, u^\nu, \theta^\nu, E, B)$  be a local smooth solution to the Cauchy problem for system (4.1) with initial data (4.2) satisfying (4.3). Set

$$(4.8) \quad n^\nu = 1 + \rho^\nu, \quad \theta^\nu = 1 + \Theta^\nu, \quad \nu = e, i.$$

Then we can rewrite (4.1)-(4.3) as :

$$(4.9) \quad \left\{ \begin{array}{l} \partial_t \rho^e + \nabla \cdot ((1 + \rho^e)u^e) = 0, \\ \partial_t u^e + (u^e \cdot \nabla)u^e + \frac{1 + \Theta^e}{1 + \rho^e} \nabla \rho^e + \nabla \Theta^e = -(E + u^e \times B) - u^e, \\ \partial_t \Theta^e + u^e \cdot \nabla \Theta^e + \frac{2}{3}(1 + \Theta^e) \nabla \cdot u^e + \frac{1}{3}|u^e|^2 + \Theta^e = 0, \\ \partial_t \rho^i + \nabla \cdot ((1 + \rho^i)u^i) = 0, \\ \partial_t u^i + (u^i \cdot \nabla)u^i + \frac{1 + \Theta^i}{1 + \rho^i} \nabla \rho^i + \nabla \Theta^i = (E + u^i \times B) - u^i, \\ \partial_t \Theta^i + u^i \cdot \nabla \Theta^i + \frac{2}{3}(1 + \Theta^i) \nabla \cdot u^i + \frac{1}{3}|u^i|^2 + \Theta^i = 0, \\ \partial_t E - \nabla \times B - u^e + u^i = \rho^e u^e - \rho^i u^i, \quad \nabla \cdot E = \rho^i - \rho^e, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{array} \right.$$

with the initial condition :

$$(4.10) \quad W|_{t=0} = W^0 := (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfies the compatibility condition :

$$(4.11) \quad \nabla \cdot E^0 = \rho^{i0} - \rho^{e0}, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

Here,  $\rho^{\nu 0} = n^{\nu 0} - 1$  and  $\Theta^{\nu 0} = \theta^{\nu 0} - 1$ .

Similarly to Chapter 3, we usually assume  $s \geq 4$  in this Chapter. Moreover, for  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$ , we also use  $\mathfrak{E}_s(W(t))$ ,  $\mathfrak{E}_s^h(W(t))$ ,  $\mathfrak{D}_s(W(t))$  and  $\mathfrak{D}_s^h(W(t))$  to denote the energy functionals, the higher order energy functionals, the dissipative rate and the higher order dissipative rate for two-fluid particles. They satisfy

$$(4.12) \quad \mathfrak{E}_s(W(t)) \sim \sum_{\nu=e,i} \|(\rho^\nu, u^\nu, \Theta^\nu)\|_s^2 + \|(E, B)\|_s^2,$$

$$(4.13) \quad \mathfrak{E}_s^h(W(t)) \sim \sum_{\nu=e,i} \|\nabla(\rho^\nu, u^\nu, \Theta^\nu)\|_{s-1}^2 + \|\nabla(E, B)\|_{s-1}^2,$$

$$(4.14) \quad \mathfrak{D}_s(W(t)) \sim \sum_{\nu=e,i} (\|\nabla \rho^\nu\|_{s-1}^2 + \|(u^\nu, \Theta^\nu)\|_s^2) + \|E\|_{s-1}^2 + \|\nabla B\|_{s-2}^2 + \|\rho^e - \rho^i\|^2$$

and

$$(4.15) \quad \mathfrak{D}_s^h(W(t)) \sim \sum_{\nu=e,i} (\|\nabla^2 \rho^\nu\|_{s-2}^2 + \|\nabla(u^\nu, \Theta^\nu)\|_{s-1}^2) + \|\nabla E\|_{s-2}^2 + \|\nabla^2 B\|_{s-3}^2 + \|\nabla(\rho^e - \rho^i)\|^2,$$

respectively. Now, concerning the transformed Cauchy problem (4.9)-(4.10), we obtain the global existence result as follows.

**Proposition 4.1.** *Assume that  $W^0 = (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0)$  satisfies the compatibility condition (4.11). Then, if  $\mathfrak{E}_s(W^0)$  is small enough, the Cauchy problem (4.9)-(4.10) admits a unique global solution  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$  satisfying*

$$(4.16) \quad W \in C^1(\mathbb{R}^+; H^{s-1}) \cap C(\mathbb{R}^+; H^s),$$

and

$$(4.17) \quad \mathfrak{E}_s(W(t)) + \int_0^t \mathfrak{D}_s(W(\tau)) d\tau \leq \mathfrak{E}_s(W^0), \quad \forall t \geq 0.$$

Obviously, from Proposition 4.1, it is straightforward to get the existence result of Theorem 4.1. Furthermore, solutions of Proposition 4.1 really decay under some extra conditions on  $W^0 = (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0)$ . For that, we extend the definition of  $\omega_{s_*}(W^0)$  in (3.14) of Chapter 3 as :

$$(4.18) \quad \omega_{s_*}(W^0) = \|W^0\|_{s_*} + \|W^0\|_{L^1}$$

for  $s_* \geq 4$ . Then, we obtain the following decay results.

**Proposition 4.2.** *Let  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$  be the solution to the Cauchy problem (4.9)-(4.10) with initial data  $W^0 = (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0)$  satisfying (4.11) in the sense of Proposition 4.1. Then, if  $\omega_{s+2}(W^0)$  is sufficiently small,*

$$(4.19) \quad \|W(t)\|_s \leq C\omega_{s+2}(W^0)(1+t)^{-\frac{3}{4}}, \quad \forall t \geq 0.$$

Moreover, if  $\omega_{s+6}(W^0)$  is sufficiently small, then, the solution also satisfies

$$(4.20) \quad \|\nabla W(t)\|_{s-1} \leq C\omega_{s+6}(W^0)(1+t)^{-\frac{5}{4}}, \quad \forall t \geq 0.$$

Thus, we obtain the decay rates (4.4)-(4.7) through the method of bootstrap and Proposition 4.2.

## 4.2.2 Weighted energy estimates

In this subsection, we give the proof of Proposition 4.1 for the global existence and uniqueness of solutions to the Cauchy problem (4.9)-(4.10). Since hyperbolic system (4.9) is quasi-linear symmetrizable, we have the local existence of smooth solutions as follows.

**Lemma 4.1.** *(Local existence of smooth solutions, see [41, 46]) Assume integer  $s \geq 3$  and (4.3) holds. Suppose  $(n^{\nu^0} - 1, u^{\nu^0}, \theta^{\nu^0} - 1, E^0, B^0) \in H^s$  with  $n^{\nu^0}, \theta^{\nu^0} \geq 2\kappa$  for some given constant  $\kappa > 0$ . Then there exists  $T_1 > 0$  such that problem (4.1)-(4.2) has a unique smooth solution satisfying  $n^\nu, \theta^\nu \geq \kappa$  in  $[0, T_1] \times \mathbb{R}^3$  and*

$$(n^\nu - 1, u^\nu, \theta^\nu - 1, E, B) \in C^1([0, T_1]; H^{s-1}) \cap C([0, T_1]; H^s).$$

Then, with the help of the continuity argument, the global existence of solutions satisfying (4.16) and (4.17) follows by combining Lemma 4.1 and a priori estimate as follows.

**Theorem 4.2.** *Let  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B) \in C^1([0, T]; H^{s-1}) \cap C([0, T]; H^s)$  be the solution to the Cauchy problem (4.9)-(4.10) for  $t \in (0, T)$  with  $T > 0$ . Then, if*

$$(4.21) \quad \sup_{0 \leq t \leq T} \|W(t)\|_s \leq \delta_0$$

with  $\delta_0$  sufficiently small, there exist  $\mathfrak{E}_s(\cdot)$  and  $\mathfrak{D}_s(\cdot)$  in the form of (4.12) and (4.14) such that for any  $0 \leq t \leq T$ ,

$$(4.22) \quad \frac{d}{dt} \mathfrak{E}_s(W(t)) + \mathfrak{D}_s(W(t)) \leq C \mathfrak{E}_s(W(t))^{\frac{1}{2}} \mathfrak{D}_s(W(t)).$$

**Proof.** Similarly to that in Chapter 3, we also use five steps to finish the proof.

*Step 1.* It holds that

$$(4.23) \quad \frac{d}{dt} \|W\|_s^2 + \|(u^e, u^i, \Theta^e, \Theta^i)\|_s^2 \leq C \|W\|_s \left( \|(u^e, u^i, \Theta^e, \Theta^i)\|_s^2 + \|\nabla(\rho^e, \rho^i)\|_{s-1}^2 \right).$$

In fact, the Euler equations of (4.9), weighted energy estimates on  $\partial^\alpha \rho^\nu$ ,  $\partial^\alpha u^\nu$  and  $\partial^\alpha \Theta^\nu$  with  $|\alpha| \leq s$  imply

$$(4.24) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{\nu=e,i} \left( \left\langle \frac{1 + \Theta^\nu}{1 + \rho^\nu}, |\partial^\alpha \rho^\nu|^2 \right\rangle + \langle 1 + \rho^\nu, |\partial^\alpha u^\nu|^2 \rangle + \frac{3}{2} \left\langle \frac{1 + \rho^\nu}{1 + \Theta^\nu}, |\partial^\alpha \Theta^\nu|^2 \right\rangle \right) \\ & + \sum_{\nu=e,i} \left( \langle 1 + \rho^\nu, |\partial^\alpha u^\nu|^2 \rangle + \frac{3}{2} \left\langle \frac{1 + \rho^\nu}{1 + \Theta^\nu}, |\partial^\alpha \Theta^\nu|^2 \right\rangle \right) + \langle (1 + \rho^e) \partial^\alpha E, \partial^\alpha u^e \rangle \\ & - \langle (1 + \rho^i) \partial^\alpha E, \partial^\alpha u^i \rangle = - \sum_{\beta < \alpha} C_\beta^\alpha I_{\alpha, \beta}(t) + I_1(t), \end{aligned}$$

where

$$I_{\alpha,\beta}(t) = I_{\alpha,\beta}^e(t) + I_{\alpha,\beta}^i(t), \quad I_1(t) = I_1^e(t) + I_1^i(t),$$

and

$$\begin{aligned} I_{\alpha,\beta}^e(t) &= \left\langle \frac{1+\Theta^e}{1+\rho^e} \partial^{\alpha-\beta} \rho^e \nabla \partial^\beta u^e, \partial^\alpha \rho^e \right\rangle + \left\langle \frac{1+\Theta^e}{1+\rho^e} \partial^{\alpha-\beta} u^e \nabla \partial^\beta \rho^e, \partial^\alpha \rho^e \right\rangle \\ &\quad + \frac{3}{2} \left\langle \frac{1+\rho^e}{1+\Theta^e} \partial^{\alpha-\beta} u^e \nabla \partial^\beta \Theta^e, \partial^\alpha \Theta^e \right\rangle + \left\langle \frac{1+\rho^e}{1+\Theta^e} \partial^{\alpha-\beta} \Theta^e \nabla \partial^\beta u^e, \partial^\alpha \Theta^e \right\rangle \\ &\quad + \langle (1+\rho^e) \partial^{\alpha-\beta} u^e \nabla \partial^\beta u^e, \partial^\alpha u^e \rangle + \left\langle (1+\rho^e) \partial^{\alpha-\beta} \left( \frac{1+\Theta^e}{1+\rho^e} \right) \nabla \partial^\beta \rho^e, \partial^\alpha u^e \right\rangle \\ &\quad + \langle (1+\rho^e) \partial^{\alpha-\beta} u^e \times \partial^\beta B, \partial^\alpha u^e \rangle + \frac{1}{2} \left\langle \frac{1+\rho^e}{1+\Theta^e} \partial^{\alpha-\beta} u^e \partial^\beta u^e, \partial^\alpha \Theta^e \right\rangle, \end{aligned}$$

$$\begin{aligned} I_{\alpha,\beta}^i(t) &= \left\langle \frac{1+\Theta^i}{1+\rho^i} \partial^{\alpha-\beta} \rho^i \nabla \partial^\beta u^i, \partial^\alpha \rho^i \right\rangle + \left\langle \frac{1+\Theta^i}{1+\rho^i} \partial^{\alpha-\beta} u^i \nabla \partial^\beta \rho^i, \partial^\alpha \rho^i \right\rangle \\ &\quad + \frac{3}{2} \left\langle \frac{1+\rho^i}{1+\Theta^i} \partial^{\alpha-\beta} u^i \nabla \partial^\beta \Theta^i, \partial^\alpha \Theta^i \right\rangle + \left\langle \frac{1+\rho^i}{1+\Theta^i} \partial^{\alpha-\beta} \Theta^i \nabla \partial^\beta u^i, \partial^\alpha \Theta^i \right\rangle \\ &\quad + \langle (1+\rho^i) \partial^{\alpha-\beta} u^i \nabla \partial^\beta u^i, \partial^\alpha u^i \rangle + \left\langle (1+\rho^i) \partial^{\alpha-\beta} \left( \frac{1+\Theta^i}{1+\rho^i} \right) \nabla \partial^\beta \rho^i, \partial^\alpha u^i \right\rangle \\ &\quad - \langle (1+\rho^i) \partial^{\alpha-\beta} u^i \times \partial^\beta B, \partial^\alpha u^i \rangle + \frac{1}{2} \left\langle \frac{1+\rho^i}{1+\Theta^i} \partial^{\alpha-\beta} u^i \partial^\beta u^i, \partial^\alpha \Theta^i \right\rangle, \end{aligned}$$

$$\begin{aligned} I_1^e(t) &= \frac{1}{2} \left\langle \partial_t \left( \frac{1+\Theta^e}{1+\rho^e} \right), |\partial^\alpha \rho^e|^2 \right\rangle + \langle \nabla \Theta^e \partial^\alpha u^e, \partial^\alpha \rho^e \rangle + \langle \nabla \rho^e \partial^\alpha u^e, \partial^\alpha \Theta^e \rangle \\ &\quad + \frac{1}{2} \left\langle \nabla \cdot \left( \frac{1+\Theta^e}{1+\rho^e} u^e \right), |\partial^\alpha \rho^e|^2 \right\rangle + \frac{3}{4} \left\langle \partial_t \left( \frac{1+\rho^e}{1+\Theta^e} \right), |\partial^\alpha \Theta^e|^2 \right\rangle \\ &\quad + \frac{1}{2} \left\langle \nabla \cdot \left( \frac{1+\rho^e}{1+\Theta^e} u^e \right), |\partial^\alpha \Theta^e|^2 \right\rangle - \langle (1+\rho^e) u^e \times \partial^\alpha B, \partial^\alpha u^e \rangle \\ &\quad - \frac{1}{2} \left\langle \frac{1+\rho^e}{1+\Theta^e} u^e \partial^\alpha u^e, \partial^\alpha \Theta^e \right\rangle, \end{aligned}$$

$$\begin{aligned} I_1^i(t) &= \frac{1}{2} \left\langle \partial_t \left( \frac{1+\Theta^i}{1+\rho^i} \right), |\partial^\alpha \rho^i|^2 \right\rangle + \langle \nabla \Theta^i \partial^\alpha u^i, \partial^\alpha \rho^i \rangle + \langle \nabla \rho^i \partial^\alpha u^i, \partial^\alpha \Theta^i \rangle \\ &\quad + \frac{1}{2} \left\langle \nabla \cdot \left( \frac{1+\Theta^i}{1+\rho^i} u^i \right), |\partial^\alpha \rho^i|^2 \right\rangle + \frac{3}{4} \left\langle \partial_t \left( \frac{1+\rho^i}{1+\Theta^i} \right), |\partial^\alpha \Theta^i|^2 \right\rangle \\ &\quad + \frac{1}{2} \left\langle \nabla \cdot \left( \frac{1+\rho^i}{1+\Theta^i} u^i \right), |\partial^\alpha \Theta^i|^2 \right\rangle + \langle (1+\rho^i) u^i \times \partial^\alpha B, \partial^\alpha u^i \rangle \\ &\quad - \frac{1}{2} \left\langle \frac{1+\rho^i}{1+\Theta^i} u^i \partial^\alpha u^i, \partial^\alpha \Theta^i \right\rangle, \end{aligned}$$

where an integration by parts is used. When  $|\alpha| = 0$ , in view of (4.21), we have

$$I_1(t) = I_1^e(t) + I_1^i(t) \leq C \|(\rho^\nu, u^\nu, \Theta^\nu, B)\| (\|\nabla \rho^\nu\|_1^2 + \|u^\nu\|_2^2 + \|\nabla \Theta^\nu\|_1^2),$$

which can be bounded by the right hand side of (4.23). And when  $|\alpha| \geq 1$ , we obtain

$$I_{\alpha,\beta}(t) + I_1(t) \leq C \|(\rho^\nu, u^\nu, \Theta^\nu, B)\|_s (\|\nabla \rho^\nu\|_{s-1}^2 + \|(u^\nu, \Theta^\nu)\|_s^2),$$

which can be controlled by the right hand side of (4.23). On the other hand, for  $|\alpha| \leq s$ , standard energy estimates on  $\partial^\alpha E$  and  $\partial^\alpha B$  from (4.9) yield

$$(4.25) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\partial^\alpha E\|^2 + \|\partial^\alpha B\|^2) - \langle (1 + \rho^e) \partial^\alpha u^e - (1 + \rho^i) \partial^\alpha u^i, \partial^\alpha E \rangle \\ &= \langle \partial^{\alpha-\beta} \rho^e \partial^\alpha u^e - \partial^{\alpha-\beta} \rho^i \partial^\alpha u^i, \partial^\alpha E \rangle \leq C \|E\|_s (\|u^\nu\|_s^2 + \|\nabla \rho^\nu\|_{s-1}^2), \end{aligned}$$

which can also be bounded by the right hand side of (4.23). Then, with the help of (4.21), summing (4.24) and (4.25) over  $|\alpha| \leq s$  yields (4.23).

*Step 2.* It holds that

$$(4.26) \quad \begin{aligned} & \frac{d}{dt} \sum_{|\alpha| \leq s-1} \sum_{\nu=e,i} \langle \partial^\alpha u^\nu, \nabla \partial^\alpha \rho^\nu \rangle + c_0 \|\nabla (\rho^e, \rho^i)\|_{s-1}^2 + c_0 \|\rho^e - \rho^i\|^2 \\ & \leq C \sum_{\nu=e,i} (\|u^\nu\|_s^2 + \|\Theta^\nu\|_s^2) + C \sum_{\nu=e,i} \|(\rho^\nu, u^\nu, \Theta^\nu, B)\|_s (\|\nabla \rho^\nu\|_{s-1}^2 + \|(u^\nu, \Theta^\nu)\|_s^2). \end{aligned}$$

In fact, we rewrite system (4.9) as :

$$(4.27) \quad \left\{ \begin{array}{l} \partial_t \rho^e + \nabla \cdot u^e = g_{1e}, \\ \partial_t u^e + \nabla \rho^e + \nabla \Theta^e + u^e + E = g_{2e}, \\ \partial_t \Theta^e + \frac{2}{3} \nabla \cdot u^e + \Theta^e = g_{3e}, \\ \partial_t \rho^i + \nabla \cdot u^i = g_{1i}, \\ \partial_t u^i + \nabla \rho^i + \nabla \Theta^i + u^i - E = g_{2i}, \\ \partial_t \Theta^i + \frac{2}{3} \nabla \cdot u^i + \Theta^i = g_{3i}, \\ \partial_t E - \nabla \times B - u^e + u^i = g_{4e} - g_{4i}, \quad \nabla \cdot E = \rho^i - \rho^e, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{array} \right.$$

where

$$(4.28) \quad \left\{ \begin{array}{l} g_{1e} = -\rho^e \nabla \cdot u^e - u^e \nabla \rho^e, \\ g_{2e} = -(u^e \cdot \nabla) u^e - \left(\frac{1 + \Theta^e}{1 + \rho^e} - 1\right) \nabla \rho^e - u^e \times B, \\ g_{3e} = -\frac{2}{3} \Theta^e \nabla \cdot u^e - u^e \nabla \Theta^e - \frac{1}{3} |u^e|^2, \\ g_{4e} = \rho^e u^e, \\ g_{1i} = -\rho^i \nabla \cdot u^i - u^i \nabla \rho^i, \\ g_{2i} = -(u^i \cdot \nabla) u^i - \left(\frac{1 + \Theta^i}{1 + \rho^i} - 1\right) \nabla \rho^i + u^i \times B, \\ g_{3i} = -\frac{2}{3} \Theta^i \nabla \cdot u^i - u^i \nabla \Theta^i - \frac{1}{3} |u^i|^2, \\ g_{4i} = \rho^i u^i, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3. \end{array} \right.$$



Let  $|\alpha| \leq s - 1$ . Applying  $\partial^\alpha$  to the second equation of (4.27), taking the inner product of the resulting equation with  $\nabla \partial^\alpha \rho^e$  in  $L^2$ , and replacing  $\partial_t \rho^e$  from the first equation of (4.27), we get

$$\begin{aligned} & \frac{d}{dt} \langle \partial^\alpha u^e, \nabla \partial^\alpha \rho^e \rangle + \|\nabla \partial^\alpha \rho^e\|^2 + \|\partial^\alpha \rho^e\|^2 - \langle \partial^\alpha \rho^i, \partial^\alpha \rho^e \rangle + \langle \nabla \partial^\alpha \Theta^e, \nabla \partial^\alpha \rho^e \rangle \\ &= \|\partial^\alpha \nabla \cdot u^e\|^2 + \langle \partial^\alpha \nabla \rho^e, \partial^\alpha g_{2e} \rangle - \langle \partial^\alpha u^e, \nabla \partial^\alpha \rho^e \rangle - \langle \partial^\alpha \nabla \cdot u^e, \partial^\alpha g_{1e} \rangle. \end{aligned}$$

In a similar way, from the fourth and fifth equations of (4.27), we also have

$$\begin{aligned} & \frac{d}{dt} \langle \partial^\alpha u^i, \nabla \partial^\alpha \rho^i \rangle + \|\nabla \partial^\alpha \rho^i\|^2 + \|\partial^\alpha \rho^i\|^2 - \langle \partial^\alpha \rho^i, \partial^\alpha \rho^e \rangle + \langle \nabla \partial^\alpha \Theta^i, \nabla \partial^\alpha \rho^i \rangle \\ &= \|\partial^\alpha \nabla \cdot u^i\|^2 + \langle \partial^\alpha \nabla \rho^i, \partial^\alpha g_{2i} \rangle - \langle \partial^\alpha u^i, \nabla \partial^\alpha \rho^i \rangle - \langle \partial^\alpha \nabla \cdot u^i, \partial^\alpha g_{1i} \rangle. \end{aligned}$$

Then, summing the two equations above gives

$$\begin{aligned} & \frac{d}{dt} (\langle \partial^\alpha u^e, \nabla \partial^\alpha \rho^e \rangle + \langle \partial^\alpha u^i, \nabla \partial^\alpha \rho^i \rangle) + \|\nabla \partial^\alpha \rho^e\|^2 + \|\nabla \partial^\alpha \rho^i\|^2 + \|\partial^\alpha (\rho^e - \rho^i)\|^2 \\ &= \|\partial^\alpha \nabla \cdot u^e\|^2 + \|\partial^\alpha \nabla \cdot u^i\|^2 - \langle \nabla \partial^\alpha \Theta^i, \nabla \partial^\alpha \rho^i \rangle - \langle \nabla \partial^\alpha \Theta^e, \nabla \partial^\alpha \rho^e \rangle + \langle \partial^\alpha \nabla \rho^e, \partial^\alpha g_{2e} \rangle \\ & \quad - \langle \partial^\alpha u^e, \nabla \partial^\alpha \rho^e \rangle - \langle \partial^\alpha \nabla \cdot u^e, \partial^\alpha g_{1e} \rangle + \langle \partial^\alpha \nabla \rho^i, \partial^\alpha g_{2i} \rangle - \langle \partial^\alpha u^i, \nabla \partial^\alpha \rho^i \rangle - \langle \partial^\alpha \nabla \cdot u^i, \partial^\alpha g_{1i} \rangle, \end{aligned}$$

by the Cauchy-Schwarz inequality, we obtain

$$(4.29) \quad \begin{aligned} & \frac{d}{dt} (\langle \partial^\alpha u^e, \nabla \partial^\alpha \rho^e \rangle + \langle \partial^\alpha u^i, \nabla \partial^\alpha \rho^i \rangle) + c_0 (\|\nabla \partial^\alpha \rho^e\|^2 + \|\nabla \partial^\alpha \rho^i\|^2 + \|\partial^\alpha (\rho^e - \rho^i)\|^2) \\ & \leq C (\|\partial^\alpha \nabla \cdot u^\nu\|^2 + \|\partial^\alpha u^\nu\|^2 + \|\partial^\alpha \nabla \Theta^\nu\|^2 + \|\partial^\alpha g_{1\nu}\|^2 + \|\partial^\alpha g_{2\nu}\|^2). \end{aligned}$$

It follows from (4.21) and the definitions of  $g_{1\nu}$  and  $g_{2\nu}$  that

$$\|\partial^\alpha g_{1\nu}\|^2 + \|\partial^\alpha g_{2\nu}\|^2 \leq C \|(\rho^\nu, u^\nu, \Theta^\nu, B)\|_s (\|\nabla \rho^\nu\|_{s-1}^2 + \|u^\nu\|_s^2 + \|\Theta^\nu\|_s^2).$$

Plugging this estimate into (4.29) and summing the resulting equation over  $|\alpha| \leq s - 1$ , we obtain (4.26).

*Step 3.* It holds that

$$(4.30) \quad \begin{aligned} & \frac{d}{dt} \sum_{|\alpha| \leq s-1} \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + c_0 \|E\|_{s-1}^2 \\ & \leq C (\|(u^\nu, \Theta^\nu)\|_s^2 + \|\nabla \rho^\nu\|_{s-1}^2 + \|u^\nu\|_s \|\nabla B\|_{s-2} + \|W\|_s (\|\nabla \rho^\nu\|_{s-1}^2 + \|(u^\nu, \Theta^\nu)\|_s^2)). \end{aligned}$$

In fact, from the momentum equations of (4.27), we have

$$(4.31) \quad \partial_t (u^e - u^i) + \nabla (\rho^e - \rho^i) + \nabla (\Theta^e - \Theta^i) + 2E = g_{2e} - g_{2i} - (u^e - u^i).$$

For  $|\alpha| \leq s-1$ , applying  $\partial^\alpha$  to (4.31), taking the inner product of the resulting equation with  $\partial^\alpha E$  in  $L^2$ , and replacing  $\partial_t E$  from the seventh equation of (4.9), we have

$$\begin{aligned} & \frac{d}{dt} \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + \|\partial^\alpha (\rho^e - \rho^i)\|^2 + 2\|\partial^\alpha E\|^2 \\ &= -\langle \partial^\alpha (\Theta^e - \Theta^i), \partial^\alpha (\rho^e - \rho^i) \rangle + \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + \langle \partial^\alpha (u^e - u^i), \nabla \times \partial^\alpha B \rangle \\ & \quad + \|\partial^\alpha (u^e - u^i)\|^2 + \langle \partial^\alpha (u^e - u^i), \partial^\alpha (\rho^e u^e - \rho^i u^i) \rangle + \langle \partial^\alpha (g_{2e} - g_{2i}), \partial^\alpha E \rangle, \end{aligned}$$

by (4.21) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \frac{d}{dt} \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + c_0 \|\partial^\alpha E\|^2 \\ & \leq C \|\partial^\alpha (\nabla \rho^\nu, u^\nu, \Theta^\nu)\|^2 + C \|(u^e, u^i)\|_s \|\nabla B\|_{s-2} + C \|(\rho^\nu, u^\nu, \Theta^\nu, B)\|_s (\|\nabla \rho^\nu\|_{s-1}^2 + \|(u^\nu, \Theta^\nu)\|_s^2). \end{aligned}$$

Thus, summing the previous inequality over  $|\alpha| \leq s-1$  yields (4.30).

*Step 4.* It holds that

$$(4.32) \quad \frac{d}{dt} \sum_{|\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle + c_0 \|\nabla B\|_{s-2}^2 \leq C (\|(u^\nu, E)\|_{s-1}^2 + \|\nabla \rho^\nu\|_{s-1} \|u^\nu\|_s^2).$$

In fact, for  $|\alpha| \leq s-2$ , applying  $\partial^\alpha$  to the seventh equation of (4.9), taking the inner product of the resulting equation with  $-\partial^\alpha \nabla \times B$  in  $L^2$ , we have

$$\begin{aligned} & \frac{d}{dt} \sum_{|\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle + \|\nabla \times \partial^\alpha B\|^2 \\ &= \|\nabla \times \partial^\alpha E\|^2 - \langle \partial^\alpha (u^e - u^i), \nabla \times \partial^\alpha B \rangle + \langle \partial^\alpha (\rho^e u^e - \rho^i u^i), -\nabla \times \partial^\alpha B \rangle. \end{aligned}$$

Furthermore, with the help of the Cauchy-Schwarz inequality and summing the resulting equation over  $|\alpha| \leq s-2$ , we obtain (4.32), where we also used (3.41).

*Step 5.* Finally, based on four previous steps, we establish (4.22). We define the energy functionals as :

$$\begin{aligned} \mathfrak{E}_s(W(t)) &= \|W\|_s^2 + \mathfrak{K}_1 \sum_{|\alpha| \leq s-1} \sum_{\nu=e,i} \langle \partial^\alpha u^\nu, \nabla \partial^\alpha \rho^\nu \rangle \\ & \quad + \mathfrak{K}_2 \sum_{|\alpha| \leq s-1} \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + \mathfrak{K}_3 \sum_{|\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle, \end{aligned}$$

where constants  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  are to be chosen later. It follows that  $\mathfrak{E}_s(W(t)) \sim \|W\|_s^2$  as soon as  $0 < \mathfrak{K}_j \ll 1$ ,  $j = 1, 2, 3$ , are sufficiently small. Furthermore, by letting  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  be sufficiently small with  $\mathfrak{K}_2^{\frac{3}{2}} \ll \mathfrak{K}_3$ , summing (4.23), (4.26)  $\times \mathfrak{K}_1$ , (4.30)  $\times \mathfrak{K}_2$  and (4.32)  $\times \mathfrak{K}_3$ , we get (4.22), where we also used the following inequality :

$$\mathfrak{K}_2 \|u^\nu\|_s \|\nabla B\|_{s-2} \leq \mathfrak{K}_2^{\frac{1}{2}} \|u^\nu\|_s^2 + \mathfrak{K}_2^{\frac{3}{2}} \|\nabla B\|_{s-2}^2.$$

We have finished the proof of Theorem 4.2.  $\square$

### 4.3 Linearized homogeneous systems

In this section, in order to obtain the time decay properties of solutions for the non-linear system (4.9), we have to study the decay properties of solutions for the linearized system (4.27). For that, we introduce

$$(4.33) \quad \rho_1 = \frac{\rho^e - \rho^i}{2}, \quad u_1 = \frac{u^e - u^i}{2}, \quad \Theta_1 = \frac{\Theta^e - \Theta^i}{2}.$$

Then, from system (4.9),  $W_1 = (\rho_1, u_1, \Theta_1, E, B)$  satisfies :

$$(4.34) \quad \begin{cases} \partial_t \rho_1 + \nabla \cdot u_1 = \frac{1}{2} (g_{1e} - g_{1i}), \\ \partial_t u_1 + \nabla \rho_1 + \nabla \Theta_1 + E + u_1 = \frac{1}{2} (g_{2e} - g_{2i}), \\ \partial_t \Theta_1 + \frac{2}{3} \nabla \cdot u_1 + \Theta_1 = \frac{1}{2} (g_{3e} - g_{3i}), \\ \partial_t E - \nabla \times B - 2u_1 = g_{4e} - g_{4i}, \quad \frac{1}{2} \nabla \cdot E = -\rho_1, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with the initial condition :

$$W_1|_{t=0} = W_1^0 := (\rho_1^0, u_1^0, \Theta_1^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfies the compatibility condition :

$$\frac{1}{2} \nabla \cdot E^0 = -\rho_1^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3,$$

where  $(\rho_1^0, u_1^0, \Theta_1^0)$  is given from  $(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})$  due to (4.33). Moreover, we introduce

$$(4.35) \quad \rho_2 = \frac{\rho^e + \rho^i}{2}, \quad u_2 = \frac{u^e + u^i}{2}, \quad \Theta_2 = \frac{\Theta^e + \Theta^i}{2}.$$

Then  $W_2 = (\rho_2, u_2, \Theta_2)$  satisfies :

$$(4.36) \quad \begin{cases} \partial_t \rho_2 + \nabla \cdot u_2 = \frac{1}{2} (g_{1e} + g_{1i}), \\ \partial_t u_2 + \nabla \rho_2 + \nabla \Theta_2 + u_2 = \frac{1}{2} (g_{2e} + g_{2i}), \\ \partial_t \Theta_2 + \frac{2}{3} \nabla \cdot u_2 + \Theta_2 = \frac{1}{2} (g_{3e} + g_{3i}), \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with the initial condition :

$$W_2|_{t=0} = W_2^0 := (\rho_2^0, u_2^0, \Theta_2^0), \quad \text{in } \mathbb{R}^3,$$

where  $(\rho_2^0, u_2^0, \Theta_2^0)$  is given from (4.35). Therefore, we define the solution  $W_1 = (\rho_1, u_1, \Theta_1, E, B)$  and  $W_2 = (\rho_2, u_2, \Theta_2)$ , respectively, as follows

$$(4.37) \quad W_1(t) = e^{tL_1} W_1^0 + \frac{1}{2} \int_0^t e^{(t-\tau)L_1} (g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}, 2(g_{4e} - g_{4i}))(\tau) d\tau,$$

and

$$(4.38) \quad W_2(t) = e^{tL_2}W_2^0 + \frac{1}{2} \int_0^t e^{(t-\tau)L_2} (g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(\tau) d\tau,$$

where  $e^{tL_1}W_1^0$  and  $e^{tL_2}W_2^0$ , respectively, denote the solutions of the homogeneous Cauchy problems (4.39)-(4.40) and (4.42)-(4.43), which are given as follows.

The linearized homogeneous system of (4.34) is :

$$(4.39) \quad \begin{cases} \partial_t \rho_1 + \nabla \cdot u_1 = 0, \\ \partial_t u_1 + \nabla \rho_1 + \nabla \Theta_1 + E + u_1 = 0, \\ \partial_t \Theta_1 + \frac{2}{3} \nabla \cdot u_1 + \Theta_1 = 0, \\ \partial_t E - \nabla \times B - 2u_1 = 0, \quad \frac{1}{2} \nabla \cdot E = -\rho_1, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with the initial condition :

$$(4.40) \quad W_1|_{t=0} = W_1^0 := (\rho_1^0, u_1^0, \Theta_1^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfies the compatibility condition :

$$(4.41) \quad \frac{1}{2} \nabla \cdot E^0 = -\rho_1^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

And the linearized homogeneous system of (4.39) is :

$$(4.42) \quad \begin{cases} \partial_t \rho_2 + \nabla \cdot u_2 = 0, \\ \partial_t u_2 + \nabla \rho_2 + \nabla \Theta_2 + u_2 = 0, \\ \partial_t \Theta_2 + \frac{2}{3} \nabla \cdot u_2 + \Theta_2 = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with the initial condition :

$$(4.43) \quad W_2|_{t=0} = W_2^0 := (\rho_2^0, u_2^0, \Theta_2^0), \quad \text{in } \mathbb{R}^3.$$

In the sequel of this Chapter, we denote  $W_1 = (\rho_1, u_1, \Theta_1, E, B)$  as the solution to the Cauchy problem (4.39)-(4.40), and  $W_2 = (\rho_2, u_2, \Theta_2)$  as the one to the Cauchy problem (4.42)-(4.43).

Firstly, for the Cauchy problem (4.39)-(4.40), similarly to that in Chapter 3, we obtain the  $L^p - L^q$  decay property as follows.

**Proposition 4.3.** *Let  $W_1(t) = e^{tL_1}W_1^0$  be the solution to the Cauchy problem (4.39)-(4.40) with initial data  $W_1^0 = (\rho_1^0, u_1^0, \Theta_1^0, E^0, B^0)$  satisfying (4.41). Then, for any  $t \geq 0$ ,*

$W_1$  satisfies the following time decay property :

$$(4.44) \quad \begin{cases} \|(\rho_1(t), \Theta_1(t))\| \leq C e^{-\frac{t}{2}} \|(\rho_1^0, u_1^0, \Theta_1^0)\|, \\ \|u_1(t)\| \leq C e^{-\frac{t}{2}} \|(\rho_1^0, \Theta_1^0)\| + C(1+t)^{-\frac{5}{4}} \|(u_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2}, \\ \|E(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(u_1^0, \Theta_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^3}, \\ \|B(t)\| \leq C(1+t)^{-\frac{3}{4}} \|(u_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2}, \end{cases}$$

$$(4.45) \quad \begin{cases} \|(\rho_1(t), \Theta_1(t))\|_{L^\infty} \leq C e^{-\frac{t}{2}} \|(\rho_1^0, u_1^0, \Theta_1^0)\|_{L^2 \cap \dot{H}^2}, \\ \|u_1(t)\|_{L^\infty} \leq C e^{-\frac{t}{2}} \|(\rho_1^0, \Theta_1^0)\|_{L^1 \cap \dot{H}^2} + C(1+t)^{-2} \|(u_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^5}, \\ \|E(t)\|_{L^\infty} \leq C(1+t)^{-2} \|(u_1^0, \Theta_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^6}, \\ \|B(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}} \|(u_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^5}, \end{cases}$$

and

$$(4.46) \quad \begin{cases} \|\nabla B(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(u_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^4}, \\ \|\nabla^s(E(t), B(t))\| \leq C(1+t)^{-\frac{5}{4}} \|(u_1^0, \Theta_1^0, E^0, B^0)\|_{L^2 \cap \dot{H}^{s+3}}. \end{cases}$$

**Proof.** See the proof of Corollary 3.2 in Chapter 3. □

### 4.3.1 Explicit solutions

Next, we consider the explicit Fourier transform solution  $W_2 = (\rho_2, u_2, \Theta_2)$  of the Cauchy problem (4.42)-(4.43). It follows from (4.42) that

$$(4.47) \quad \partial_{ttt}\rho_2 + 2\partial_{tt}\rho_2 - \frac{5}{3}\Delta\partial_t\rho_2 + \partial_t\rho_2 - \Delta\rho_2 = 0,$$

with the initial condition :

$$(4.48) \quad \begin{cases} \rho_2|_{t=0} = \rho_2^0, \\ \partial_t\rho_2|_{t=0} = -\nabla \cdot u_2^0, \\ \partial_{tt}\rho_2|_{t=0} = \Delta\rho_2^0 + \nabla \cdot u_2^0 + \Delta\Theta_2^0. \end{cases}$$

Taking the Fourier transform on (4.47) and (4.48), we have

$$(4.49) \quad \partial_{ttt}\hat{\rho}_2 + 2\partial_{tt}\hat{\rho}_2 + (1 + \frac{5}{3}|k|^2)\partial_t\hat{\rho}_2 + |k|^2\hat{\rho}_2 = 0,$$

with the initial condition :

$$(4.50) \quad \begin{cases} \hat{\rho}_2|_{t=0} = \hat{\rho}_2^0, \\ \partial_t\hat{\rho}_2|_{t=0} = -i|k|\tilde{k} \cdot \hat{u}_2^0, \\ \partial_{tt}\hat{\rho}_2|_{t=0} = -|k|^2\hat{\rho}_2^0 + i|k|\tilde{k} \cdot \hat{u}_2^0 - |k|^2\hat{\Theta}_2^0. \end{cases}$$

The characteristic equation of (4.49) is :

$$F(\mathfrak{X}) := \mathfrak{X}^3 + 2\mathfrak{X}^2 + \left(1 + \frac{5}{3}|k|^2\right) \mathfrak{X} + |k|^2 = 0.$$

For the roots of this equation and their properties, we obtain

**Lemma 4.2.** *Assume  $|k| \neq 0$ . Then,  $F(\mathfrak{X}) = 0$ ,  $\mathfrak{X} \in \mathbb{C}$  has a real root  $\eta = \eta(|k|) \in (-\frac{3}{5}, 0)$  and two conjugate complex roots  $\mathfrak{X}_{\pm} = \phi \pm i\psi$  with  $\phi = \phi(|k|) \in (-1, -\frac{7}{10})$  and  $\psi = \psi(|k|) \in (0, +\infty)$  which satisfy the following properties :*

$$(4.51) \quad \phi = -1 - \frac{\eta}{2}, \quad \psi = \frac{1}{2} \sqrt{3\eta^2 + 4\eta + \frac{20}{3}|k|^2}.$$

Here,  $\eta, \phi, \psi$  are smooth in  $|k| > 0$ , and  $\eta(|k|)$  is strictly decreasing over  $|k| > 0$ , with

$$\lim_{|k| \rightarrow 0} \eta(|k|) = 0, \quad \lim_{|k| \rightarrow \infty} \eta(|k|) = -\frac{3}{5}.$$

Furthermore, the following asymptotic behaviors hold true :

$$\eta(|k|) = -O(1)|k|^2, \quad \phi(|k|) = -1 + O(1)|k|^2, \quad \psi(|k|) = O(1)|k|$$

whenever  $|k| \leq 1$  is small, and

$$\eta(|k|) = -\frac{3}{5} + O(1)|k|^{-2}, \quad \phi(|k|) = -\frac{7}{10} - O(1)|k|^{-2}, \quad \psi(|k|) = O(1)|k|$$

whenever  $|k| \geq 1$  is large.

**Proof.** The proof is similar to that of Lemma 3.3, we omit it here for simplicity. □

Based on Lemma 4.2, we define the solution of (4.49) as :

$$(4.52) \quad \hat{\rho}_2(t, k) = c_1(k)e^{\eta t} + e^{\phi t} (c_2(k) \cos \psi t + c_3(k) \sin \psi t),$$

where  $c_i(k)$ ,  $1 \leq i \leq 3$ , are to be determined by (4.50) later. In fact, (4.52) implies

$$(4.53) \quad \begin{bmatrix} \hat{\rho}_2|_{t=0} \\ \partial_t \hat{\rho}_2|_{t=0} \\ \partial_{tt} \hat{\rho}_2|_{t=0} \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

where matrix  $A$  is also defined by (3.71) in Chapter 3. Notice that (4.53) together with (4.50) gives

$$[c_1, c_2, c_3]^T = \frac{1}{3\eta^2 + 4\eta + 1 + \frac{5}{3}|k|^2} \begin{bmatrix} \phi^2 + \psi^2 - |k|^2 & i|k|(2\phi + 1) & -|k|^2 \\ \eta^2 - 2\eta\phi + |k|^2 & -i|k|(2\phi + 1) & |k|^2 \\ \frac{\eta(\phi^2 - \psi^2 - \eta\phi) - (\phi - \eta)|k|^2}{\omega} & \frac{i|k|}{\psi} (\phi^2 - \eta^2 - \psi^2 + \phi - \eta) & \frac{\eta - \phi}{\psi} |k|^2 \end{bmatrix} \begin{bmatrix} \hat{\rho}_2^0 \\ \tilde{k} \cdot \hat{u}_2^0 \\ \hat{\Theta}_2^0 \end{bmatrix}.$$

Substituting the form of  $\phi$  and  $\psi$  and making further simplifications, we obtain

$$(4.54) \quad [c_1, c_2, c_3]^T = \frac{1}{3\eta^2 + 4\eta + 1 + \frac{5}{3}|k|^2} \begin{bmatrix} (\eta + 1)^2 + \frac{2}{3}|k|^2 & -i|k|(\eta + 1) & -|k|^2 \\ 2\eta(\eta + 1) + |k|^2 & i|k|(\eta + 1) & |k|^2 \\ \frac{\eta(\eta+1) + (1-\frac{1}{6}\eta)|k|^2}{\psi} & -\frac{i|k|}{\psi} \left( \frac{3}{2}\eta^2 + \frac{3}{2}\eta + \frac{5}{3}|k|^2 \right) & \frac{1+\frac{3}{2}\eta}{\psi}|k|^2 \end{bmatrix} \begin{bmatrix} \hat{\rho}_2^0 \\ \tilde{k} \cdot \hat{u}_2^0 \\ \hat{\Theta}_2^0 \end{bmatrix}.$$

Next, again from (4.42), we obtain

$$(4.55) \quad \partial_{ttt}\hat{\Theta}_2 + 2\partial_{tt}\hat{\Theta}_2 + \left(1 + \frac{5}{3}|k|^2\right) \partial_t\hat{\Theta}_2 + |k|^2\hat{\Theta}_2 = 0,$$

with the initial condition :

$$(4.56) \quad \begin{cases} \hat{\Theta}_2|_{t=0} = \hat{\Theta}_2^0, \\ \partial_t\hat{\Theta}_2|_{t=0} = -\frac{2}{3}i|k|\tilde{k} \cdot \hat{u}_2^0 - \hat{\Theta}_2^0, \\ \partial_{tt}\hat{\Theta}_2|_{t=0} = -\frac{2}{3}|k|^2\hat{\rho}_2^0 + \frac{4}{3}i|k|\tilde{k} \cdot \hat{u}_2^0 + \left(1 - \frac{2}{3}|k|^2\right)\hat{\Theta}_2^0. \end{cases}$$

Based on Lemma 4.2, we set the solution of (4.55) as :

$$(4.57) \quad \hat{\Theta}_2(t, k) = c_4(k)e^{\eta t} + e^{\phi t} (c_5(k) \cos \psi t + c_6(k) \sin \psi t),$$

where  $c_i(k)$ ,  $4 \leq i \leq 6$ , are to be ascertained by (4.56) later. In fact, after tenuous computation, (4.56) implies

$$(4.58) \quad [c_4, c_5, c_6]^T = \frac{1}{3\eta^2 + 4\eta + 1 + \frac{5}{3}|k|^2} \begin{bmatrix} -\frac{2}{3}|k|^2 & -\frac{2}{3}i|k|\eta & (1 + \eta)(\eta - 1) + |k|^2 \\ \frac{2}{3}|k|^2 & \frac{2}{3}i|k|\eta & (1 + 2\eta)(1 + \eta) + \frac{2}{3}|k|^2 \\ \frac{\eta + \frac{2}{3}}{\psi}|k|^2 & -\frac{i|k|}{\psi}(\eta(\eta + 2) + \frac{2}{3} + \frac{10}{9}|k|^2) & -\frac{|k|^2 + \frac{1}{2}\eta(1 + \frac{4}{3}|k|^2 + \eta)}{\psi} \end{bmatrix} \begin{bmatrix} \hat{\rho}_2^0 \\ \tilde{k} \cdot \hat{u}_2^0 \\ \hat{\Theta}_2^0 \end{bmatrix}.$$

Similarly, again from (4.42), we also have

$$(4.59) \quad \partial_{ttt}(\tilde{k} \cdot \hat{u}_2) + 2\partial_{tt}(\tilde{k} \cdot \hat{u}_2) + \left(1 + \frac{5}{3}|k|^2\right) \partial_t(\tilde{k} \cdot \hat{u}_2) + |k|^2(\tilde{k} \cdot \hat{u}_2) = 0,$$

with the initial condition :

$$(4.60) \quad \begin{cases} \tilde{k} \cdot \hat{u}_2|_{t=0} = \tilde{k} \cdot \hat{u}_2^0, \\ \partial_t(\tilde{k} \cdot \hat{u}_2)|_{t=0} = -i|k|\hat{\rho}_2^0 - \tilde{k} \cdot \hat{u}_2^0 - i|k|\hat{\Theta}_2^0, \\ \partial_{tt}(\tilde{k} \cdot \hat{u}_2)|_{t=0} = -i|k|\hat{\rho}_2^0 - \left(1 - \frac{5}{3}|k|^2\right)\tilde{k} \cdot \hat{u}_2^0 - 2i|k|\hat{\Theta}_2^0. \end{cases}$$

From Lemma 4.2, we obtain

$$(4.61) \quad \tilde{k} \cdot \hat{u}_2(t, k) = c_7(k)e^{\eta t} + e^{\phi t} (c_8(k) \cos \psi t + c_9(k) \sin \psi t),$$

with

$$(4.62) \quad [c_7, c_8, c_9]^T = \frac{1}{3\eta^2 + 4\eta + 1 + \frac{5}{3}|k|^2} \begin{bmatrix} -i|k|(3+\eta) & \eta(1+\eta) - 2 & -i|k|(4+\eta) \\ i|k|(3+\eta) & 2\eta^2 + 3\eta + 3 + \frac{5}{3}|k|^2 & i|k|(4+\eta) \\ \frac{i|k|(\frac{3}{2}\eta(1-\eta)+2-\frac{5}{3}|k|^2)}{\psi} & \frac{4+\eta(5-\eta+\frac{5}{3}|k|^2)}{2\psi} & \frac{i|k|(3(1+\eta)-\frac{3}{2}\eta^2-\frac{5}{3}|k|^2)}{\psi} \end{bmatrix} \begin{bmatrix} \hat{\rho}_2^0 \\ \tilde{k} \cdot \hat{u}_2^0 \\ \hat{\Theta}_2^0 \end{bmatrix}.$$

Furthermore, taking the curl for the second equation of (4.42) and then taking the Fourier transform on the resulting equation, we have

$$(4.63) \quad \partial_t (\tilde{k} \times (\tilde{k} \times \hat{u}_2)) + \tilde{k} \times (\tilde{k} \times \hat{u}_2) = 0,$$

with the initial condition :

$$(4.64) \quad \tilde{k} \times (\tilde{k} \times \hat{u}_2) |_{t=0} = \tilde{k} \times (\tilde{k} \times \hat{u}_2^0).$$

It follows from (4.63)-(4.64) that

$$(4.65) \quad \tilde{k} \times (\tilde{k} \times \hat{u}_2) = e^{-t} (\tilde{k} \times (\tilde{k} \times \hat{u}_2^0)).$$

Now, from the above computations, we obtain the explicit Fourier transform solution  $\hat{W}_2 = (\hat{\rho}_2, \hat{u}_2, \hat{\Theta}_2)$  as follows.

**Theorem 4.3.** *Let  $W_2 = (\rho_2, u_2, \Theta_2)$  be the solution to the Cauchy problem (4.42)-(4.43). For  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$  with  $|k| \neq 0$ , we obtain*

$$(4.66) \quad \begin{bmatrix} \hat{\rho}_2(t, k) \\ \hat{u}_2(t, k) \\ \hat{\Theta}_2(t, k) \end{bmatrix} = \begin{bmatrix} \hat{\rho}_2(t, k) \\ \hat{u}_{2||}(t, k) \\ \hat{\Theta}_2(t, k) \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{u}_{2\perp}(t, k) \\ 0 \end{bmatrix},$$

where  $u_{2||}$  and  $u_{2\perp}$  are defined by (3.88) in Chapter 3. Then, there exist matrices  $H_{5 \times 5}^I(t, k)$  and  $H_{3 \times 3}^{II}(t, k)$  such that

$$(4.67) \quad \begin{bmatrix} \hat{\rho}_2(t, k) \\ \hat{u}_{2||}(t, k) \\ \hat{\Theta}_2(t, k) \end{bmatrix} = H_{5 \times 5}^I(t, k) \begin{bmatrix} \hat{\rho}_2^0(k) \\ \hat{u}_{2||}^0(k) \\ \hat{\Theta}_2^0(k) \end{bmatrix}$$

and

$$(4.68) \quad \hat{u}_{2\perp}(t, k) = H_{3 \times 3}^{II}(t, k) \hat{u}_{2\perp}^0(k),$$

where  $H_{5 \times 5}^I$  is explicitly determined by representations (4.52), (4.61), (4.57) for  $\hat{\rho}_2(t, k)$ ,  $\hat{u}_{2||}(t, k)$ ,  $\hat{\Theta}_2(t, k)$  with  $c_i(k)$ , ( $1 \leq i \leq 9$ ) defined by (4.54), (4.62), (4.58) in terms of  $\hat{\rho}_2^0(k)$ ,  $\hat{u}_{2||}^0(k)$ ,  $\hat{\Theta}_2^0(k)$ ; and  $H_{3 \times 3}^{II}$  is chosen by the representation (4.65) for  $\hat{u}_{2\perp}(t, k)$  in terms of  $\hat{u}_{2\perp}^0(k)$ .



### 4.3.2 $L^p - L^q$ decay properties

In this subsection, we use Theorem 4.3 to obtain the  $L^p - L^q$  decay property for every component of the solution  $W_2 = (\rho_2, u_2, \Theta_2)$ . For this aim, we consider the rigorous time frequency estimates on  $\hat{W}_2 = (\hat{\rho}_2, \hat{u}_2, \hat{\Theta}_2)$  as follows.

**Lemma 4.3.** *Let  $W_2 = (\rho_2, u_2, \Theta_2)$  be the solution to the Cauchy problem (4.42)-(4.43). Then, there are constants  $\gamma > 0$  and  $C > 0$  such that for all  $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$ ,*

$$(4.69) \quad |\hat{\rho}_2(t, k)| \leq C \left| (\hat{\rho}_2^0(k), \hat{u}_2^0(k), \hat{\Theta}_2^0(k)) \right| \begin{cases} e^{-\gamma t} + e^{-\gamma|k|^2 t}, & \text{if } |k| \leq 1, \\ e^{-\gamma t} + e^{\frac{-\gamma}{|k|^2} t}, & \text{if } |k| > 1, \end{cases}$$

$$(4.70) \quad |\hat{u}_2(t, k)| \leq C e^{-t} |\hat{u}_2^0(k)| + C \left| (\hat{\rho}_2^0(k), \hat{u}_2^0(k), \hat{\Theta}_2^0(k)) \right| \begin{cases} e^{-\gamma t} + |k| e^{-\gamma|k|^2 t}, & \text{if } |k| \leq 1, \\ |k|^{-1} e^{-\gamma t} + e^{\frac{-\gamma}{|k|^2} t}, & \text{if } |k| > 1, \end{cases}$$

and

$$(4.71) \quad |\hat{\Theta}_2(t, k)| \leq C \left| (\hat{\rho}_2^0(k), \hat{u}_2^0(k), \hat{\Theta}_2^0(k)) \right| \begin{cases} e^{-\gamma t} + e^{-\gamma|k|^2 t}, & \text{if } |k| \leq 1, \\ e^{-\gamma t} + e^{\frac{-\gamma}{|k|^2} t}, & \text{if } |k| > 1. \end{cases}$$

**Proof.** The proof is similar to that of Lemma 3.5, we omit it here for simplicity.  $\square$

From Lemma 4.3, it is straightforward to get the decay property for every component of the solution  $W_2 = (\rho_2, u_2, \Theta_2)$ .

**Theorem 4.4.** *Let  $m \geq 0$  be an integer and  $1 \leq p, r \leq 2 \leq q \leq \infty, l \geq 0$ . Suppose  $W_2(t) = e^{tL_2} W_2^0$  to be the solution to the Cauchy problem (4.42)-(4.43). Then, for any  $t \geq 0$ ,  $W_2$  satisfies the following time decay property :*

$$(4.72) \quad \begin{aligned} \|\nabla^m \rho_2(t)\|_{L^q} &\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \left\| (\rho_2^0, u_2^0, \Theta_2^0) \right\|_{L^p} \\ &\quad + C(1+t)^{-\frac{l}{2}} \left\| \nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+} (\rho_2^0, u_2^0, \Theta_2^0) \right\|_{L^r}, \end{aligned}$$

$$(4.73) \quad \begin{aligned} \|\nabla^m u_2(t)\|_{L^q} &\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m+1}{2}} \left\| (\rho_2^0, u_2^0, \Theta_2^0) \right\|_{L^p} \\ &\quad + C(1+t)^{-\frac{l}{2}} \left\| \nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+} (\rho_2^0, u_2^0, \Theta_2^0) \right\|_{L^r}, \end{aligned}$$

and

$$(4.74) \quad \begin{aligned} \|\nabla^m \Theta_2(t)\|_{L^q} &\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \left\| (\rho_2^0, u_2^0, \Theta_2^0) \right\|_{L^p} \\ &\quad + C(1+t)^{-\frac{l}{2}} \left\| \nabla^{m+[l+3(\frac{1}{r}-\frac{1}{q})]_+} (\rho_2^0, u_2^0, \Theta_2^0) \right\|_{L^r}. \end{aligned}$$

**Proof.** See the proof of Theorem 3.6 in Chapter 3. □

Based on Theorem 4.4, we list some particular cases as follows for later use.

**Corollary 4.1.** *Let  $W_2(t) = e^{tL_2}W_2^0$  be the solution to the Cauchy problem (4.42)-(4.43).*

*Then, for any  $t \geq 0$ ,  $W_2$  satisfies the following time decay property :*

$$(4.75) \quad \begin{cases} \|\rho_2(t)\| \leq C(1+t)^{-\frac{3}{4}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^2}, \\ \|u_2(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^3}, \\ \|\Theta_2(t)\| \leq C(1+t)^{-\frac{3}{4}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^2}, \end{cases}$$

$$(4.76) \quad \begin{cases} \|\nabla \rho_2(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^4}, \\ \|\nabla u_2(t)\| \leq C(1+t)^{-\frac{7}{4}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^5}, \\ \|\nabla \Theta_2(t)\| \leq C(1+t)^{-\frac{5}{4}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^4} \end{cases}$$

and

$$(4.77) \quad \begin{cases} \|\rho_2(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^5}, \\ \|u_2(t)\|_{L^\infty} \leq C(1+t)^{-2} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^6}, \\ \|\Theta_2(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}} \|(\rho_2^0, u_2^0, \Theta_2^0)\|_{L^1 \cap \dot{H}^5}. \end{cases}$$

## 4.4 Decay rates for nonlinear systems

### 4.4.1 Decay rates for energy functionals

In this subsection, we prove the decay rate (4.19) in Proposition 4.2 for the energy  $\|W(t)\|_s^2$ . We begin with the following Lemma which can be seen directly from the proof of Theorem 4.2.

**Lemma 4.4.** *Let  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$  be the solution to the Cauchy problem (4.9)-(4.10) with initial data  $W^0 = (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0)$  satisfying (4.11) in the sense of Proposition 4.1. Then, if  $\mathfrak{E}_s(W^0)$  is small enough, for any  $t \geq 0$ ,*

$$(4.78) \quad \frac{d}{dt} \mathfrak{E}_s(W(t)) + \mathfrak{D}_s(W(t)) \leq 0.$$

For  $p > 0$ , it follows from Lemma 4.4 that

$$\begin{aligned} & (1+t)^p \mathfrak{E}_s(W(t)) + \int_0^t (1+\tau)^p \mathfrak{D}_s(W(\tau)) d\tau \\ & \leq \mathfrak{E}_s(W^0) + Cp \int_0^t (1+\tau)^{p-1} \left( \|B(\tau)\|^2 + \|(\rho^e + \rho^i)(\tau)\|^2 + \mathfrak{D}_{s+1}(W(\tau)) \right) d\tau, \end{aligned}$$

where we used

$$\mathfrak{E}_s(W(t)) \leq \|B(t)\|^2 + \|(\rho^e + \rho^i)(t)\|^2 + \mathfrak{D}_{s+1}(W(t)).$$

By using (4.78) again, we obtain

$$\begin{aligned} & (1+t)^{p-1} \mathfrak{E}_{s+1}(W(t)) + \int_0^t (1+\tau)^{p-1} \mathfrak{D}_{s+1}(W(\tau)) d\tau \\ & \leq \mathfrak{E}_{s+1}(W^0) + C(p-1) \int_0^t (1+\tau)^{p-2} \left( \|B(\tau)\|^2 + \|(\rho^e + \rho^i)(\tau)\|^2 + \mathfrak{D}_{s+2}(W(\tau)) \right) d\tau. \end{aligned}$$

Then, by iterating the previous estimates, we have

$$(4.79) \quad \begin{aligned} & (1+t)^p \mathfrak{E}_s(W(t)) + \int_0^t (1+\tau)^p \mathfrak{D}_s(W(\tau)) d\tau \\ & \leq C \mathfrak{E}_{s+2}(W^0) + C \int_0^t (1+\tau)^{p-1} \left( \|B(\tau)\|^2 + \|(\rho^e + \rho^i)(\tau)\|^2 \right) d\tau, \quad \forall 1 < p < 2. \end{aligned}$$

Now, let us establish the estimates on the integral term on the right hand side of (4.79). Applying the estimate on  $B$  in (4.44) and the estimate on  $\rho_2$  in (4.75) to (4.37) and (4.38), respectively, we have

$$(4.80) \quad \begin{aligned} \|B(t)\| & \leq C(1+t)^{-\frac{3}{4}} \|(u_1^0, E^0, B^0)\|_{L^1 \cap \dot{H}^2} \\ & + C \int_0^t (1+t-\tau)^{-\frac{3}{4}} \|(g_{2e}(\tau) - g_{2i}(\tau), g_{4e}(\tau) - g_{4i}(\tau))\|_{L^1 \cap \dot{H}^2} d\tau, \end{aligned}$$

$$(4.81) \quad \begin{aligned} \|(\rho^e + \rho^i)(t)\| & \leq C \|\rho_2(t)\| \leq C(1+t)^{-\frac{3}{4}} \|(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})\|_{L^1 \cap \dot{H}^2} \\ & + C \int_0^t (1+t-\tau)^{-\frac{3}{4}} \|(g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(\tau)\|_{L^1 \cap \dot{H}^2} d\tau. \end{aligned}$$

It is direct to check that for any  $0 \leq \tau \leq t$ ,

$$\|(g_{2e}(\tau) - g_{2i}(\tau), g_{4e}(\tau) - g_{4i}(\tau))\|_{L^1 \cap \dot{H}^2} \leq C \mathfrak{E}_s(W(\tau)) \leq C(1+\tau)^{-\frac{3}{2}} \mathfrak{E}_{s,\infty}(W(t)),$$

$$\|(g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(\tau)\|_{L^1 \cap \dot{H}^2} \leq C \mathfrak{E}_s(W(\tau)) \leq C(1+\tau)^{-\frac{3}{2}} \mathfrak{E}_{s,\infty}(W(t)).$$

Plugging the two previous inequalities into (4.80) and (4.81) implies, respectively

$$(4.82) \quad \|B(t)\| \leq C(1+t)^{-\frac{3}{4}} \left( \|(u^{\nu 0}, E^0, B^0)\|_{L^1 \cap \dot{H}^2} + \mathfrak{E}_{s,\infty}(W(t)) \right)$$

and

$$(4.83) \quad \|(\rho^e + \rho^i)(t)\| \leq C(1+t)^{-\frac{3}{4}} \left( \|(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})\|_{L^1 \cap \dot{H}^2} + \mathfrak{E}_{s,\infty}(W(t)) \right).$$

Next, similarly to that in Section 3.5, choosing  $p = \frac{3}{2} + \varepsilon$  in (4.79) with  $\varepsilon > 0$  sufficiently small and using (4.82) and (4.83), we obtain

$$\|W(t)\|_s \leq C \mathfrak{E}_s(W(t))^{\frac{1}{2}} \leq C \omega_{s+2}(W^0) (1+t)^{-\frac{3}{4}}, \quad \forall t \geq 0,$$

that is (4.19).

#### 4.4.2 Decay rates for higher order energy functionals

In this subsection, we consider the decay estimate of the higher order energy  $\|\nabla W(t)\|_{s-1}^2$ , that is (4.20) in Proposition 4.2. We begin with the following Lemma.

**Lemma 4.5.** *Let  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$  be the solution to the Cauchy problem (4.9)-(4.10) with initial data  $W^0 = (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0)$  satisfying (4.11) in the sense of Proposition 4.1. Then, if  $\mathfrak{E}_s(W^0)$  is small enough, there exist the higher order energy functionals  $\mathfrak{E}_s^h(\cdot)$  and the higher order dissipative rate  $\mathfrak{D}_s^h(\cdot)$  in the form of (4.13) and (4.15) such that for any  $t \geq 0$ ,*

$$(4.84) \quad \frac{d}{dt} \mathfrak{E}_s^h(W(t)) + \mathfrak{D}_s^h(W(t)) \leq 0.$$

**Proof.** The proof is very similar to that of Theorem 4.2. In fact, by letting  $|\alpha| \geq 1$ , then corresponding to (4.23), (4.26), (4.30) and (4.32), it can also be checked that

$$\begin{aligned} & \frac{d}{dt} \|\nabla W\|_{s-1}^2 + \|\nabla(u^e, u^i, \Theta^e, \Theta^i)\|_{s-1}^2 \leq C \|W\|_s \|\nabla(\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i)\|_{s-1}^2, \\ & \frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-1} \sum_{\nu=e,i} \langle \partial^\alpha u^\nu, \nabla \partial^\alpha \rho^\nu \rangle + c_0 \|\nabla^2(\rho^e, \rho^i)\|_{s-2}^2 + c_0 \|\nabla(\rho^e - \rho^i)\|_{s-1}^2 \\ & \leq C (\|\nabla u^\nu\|_{s-1}^2 + \|W\|_s \|\nabla(\rho^\nu, u^\nu, \Theta^\nu)\|_{s-1}^2), \\ & \frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + c_0 \|\nabla E\|_{s-2}^2 \\ & \leq C (\|\nabla(u^\nu, \Theta^\nu)\|_{s-1}^2 + \|\nabla^2 \rho^\nu\|_{s-2}^2 + \|\nabla u^\nu\|_{s-1} \|\nabla^2 B\|_{s-3} + \|W\|_s \|\nabla(\rho^\nu, u^\nu, \Theta^\nu)\|_{s-1}^2), \end{aligned}$$

and

$$\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle + c_0 \|\nabla^2 B\|_{s-3}^2 \leq C (\|\nabla E\|_{s-2}^2 + \|\nabla u^\nu\|_{s-1}^2 + \|W\|_s \|\nabla(\rho^\nu, u^\nu)\|_{s-1}^2).$$

Now, let us define the higher order energy functionals as :

$$(4.85) \quad \begin{aligned} \mathfrak{E}_s(W(t)) &= \|\nabla W\|_{s-1}^2 + \mathfrak{K}_1 \sum_{1 \leq |\alpha| \leq s-1} \sum_{\nu=e,i} \langle \partial^\alpha u^\nu, \nabla \partial^\alpha \rho^\nu \rangle \\ &+ \mathfrak{K}_2 \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha (u^e - u^i), \partial^\alpha E \rangle + \mathfrak{K}_3 \sum_{1 \leq |\alpha| \leq s-2} \langle \partial^\alpha E, -\nabla \times \partial^\alpha B \rangle. \end{aligned}$$

Similarly, we choose  $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$  to be sufficiently small with  $\mathfrak{K}_2^{\frac{3}{2}} \ll \mathfrak{K}_3$ , such that  $\mathfrak{E}_s^h(W(t)) \sim \|\nabla W(t)\|_{s-1}^2$ , that is  $\mathfrak{E}_s^h(\cdot)$  is a higher order energy functionals which satisfies (4.13), and moreover, summing the four previously estimates with coefficients corresponding to (4.85) gives (4.84). This ends the proof of Lemma 4.5.  $\square$

Based on Lemma 4.5, we obtain

$$\frac{d}{dt} \mathfrak{E}_s^h(W(t)) + \mathfrak{E}_s^h(W(t)) \leq C (\|\nabla B\|^2 + \|\nabla^s(E, B)\|^2 + \|\nabla(\rho^e + \rho^i)\|^2),$$

which implies

$$(4.86) \quad \mathfrak{E}_s^h(W(t)) \leq e^{-t} \mathfrak{E}_s^h(W^0) + C \int_0^t e^{-(t-\tau)} (\|\nabla B\|^2 + \|\nabla^s(E, B)\|^2 + \|\nabla(\rho^e + \rho^i)\|^2)(\tau) d\tau.$$

Next, we estimate the time integral term on the right hand side of (4.86).

**Lemma 4.6.** *Let  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$  be the solution to the Cauchy problem (4.9)-(4.10) with initial data  $W^0 = (\rho^{e0}, \rho^{i0}, u^{e0}, u^{i0}, \Theta^{e0}, \Theta^{i0}, E^0, B^0)$  satisfying (4.11) in the sense of Proposition 4.1. Then, if  $\omega_{s+6}(W^0)$  is small enough, for any  $t \geq 0$ ,*

$$(4.87) \quad \|\nabla B(t)\|^2 + \|\nabla^s(E(t), B(t))\|^2 + \|\nabla(\rho^e + \rho^i)(t)\|^2 \leq C (\omega_{s+6}(W^0))^2 (1+t)^{-\frac{5}{2}}.$$

**Proof.** Applying the estimates on  $\nabla B(t)$  and  $\nabla^s(E, B)$  in (4.46) to (4.37), we get

$$\begin{aligned} \|\nabla B(t)\| &\leq C(1+t)^{-\frac{5}{4}} \|(u^{\nu 0}, E^0, B^0)\|_{L^1 \cap \dot{H}^4} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{2e}(\tau) - g_{2i}(\tau), g_{4e}(\tau) - g_{4i}(\tau))\|_{L^1 \cap \dot{H}^4} d\tau \\ &\leq C\omega_{s+6}(W^0) (1+t)^{-\frac{5}{4}} \end{aligned}$$

and

$$\begin{aligned} &\|\nabla^s(E(t), B(t))\| \\ &\leq C(1+t)^{-\frac{5}{4}} \|(u^{\nu 0}, \Theta^{\nu 0}, E^0, B^0)\|_{L^2 \cap \dot{H}^{s+3}} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{2e}(\tau) - g_{2i}(\tau), g_{3e}(\tau) - g_{3i}(\tau), g_{4e}(\tau) - g_{4i}(\tau))\|_{L^2 \cap \dot{H}^{s+3}} d\tau \\ &\leq C\omega_{s+6}(W^0) (1+t)^{-\frac{5}{4}}. \end{aligned}$$

Moreover, by (4.19) and applying the estimate on  $\rho_2$  in (4.76) to (4.38), we obtain

$$\begin{aligned} &\|\nabla(\rho^e + \rho^i)(t)\| \\ &\leq C(1+t)^{-\frac{5}{4}} \|(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})\|_{L^1 \cap \dot{H}^4} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{1e}(\tau) + g_{1i}(\tau), g_{2e}(\tau) + g_{2i}(\tau), g_{3e}(\tau) + g_{3i}(\tau))\|_{L^1 \cap \dot{H}^4} d\tau \\ &\leq C\omega_{s+6}(W^0) (1+t)^{-\frac{5}{4}}, \end{aligned}$$

where the smallness of  $\omega_{s+6}(W^0)$  is used. We have finished the proof of Lemma 4.6.  $\square$

Then, plugging (4.87) into (4.86), we have

$$\mathfrak{E}_s^h(W(t)) \leq e^{-t} \mathfrak{E}_s^h(W^0) + C (\omega_{s+6}(W^0))^2 (1+t)^{-\frac{5}{2}}.$$

Since  $\mathfrak{E}_s^h(W(t)) \sim \|\nabla W(t)\|_{s-1}^2$  holds true for any  $t \geq 0$ , (4.20) follows. This ends the proof of Proposition 4.2.  $\square$

#### 4.4.3 Decay rates in $L^q$

In this subsection, we consider the decay rates of solutions  $W = (\rho^e, \rho^i, u^e, u^i, \Theta^e, \Theta^i, E, B)$  to the Cauchy problem (4.9)-(4.10) in  $L^q$  with  $2 \leq q \leq +\infty$ , and prove the second part of Theorem 4.1. Throughout this subsection, we suppose  $\omega_{13}(W^0)$  to be small enough. Firstly, for  $s \geq 4$ , Proposition 4.2 shows that if  $\omega_{s+2}(W^0)$  is small enough,

$$(4.88) \quad \|W(t)\|_s \leq C\omega_{s+2}(W^0)(1+t)^{-\frac{3}{4}},$$

and if  $\omega_{s+6}(W^0)$  is small enough,

$$(4.89) \quad \|\nabla W(t)\|_{s-1} \leq C\omega_{s+6}(W^0)(1+t)^{-\frac{5}{4}}.$$

Now, let us establish the estimates on  $B$ ,  $(u^e - u^i, E)$ ,  $u^e + u^i$ ,  $(\rho^e - \rho^i, \Theta^e - \Theta^i)$  and  $(\rho^e + \rho^i, \Theta^e + \Theta^i)$  in turn as follows.

*Estimate on  $\|B\|_{L^q}$ .* For  $L^2$  rate, it follows from (4.88) that

$$\|B(t)\| \leq C\omega_6(W^0)(1+t)^{-\frac{3}{4}}.$$

For  $L^\infty$  rate, by applying  $L^\infty$  estimate on  $B$  of (4.45) to (4.37), we obtain

$$\begin{aligned} \|B(t)\|_{L^\infty} &\leq C(1+t)^{-\frac{3}{2}} \|(u^{\nu 0}, E^0, B^0)\|_{L^1 \cap \dot{H}^5} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|(g_{2e} - g_{2i}, g_{4e} - g_{4i})(\tau)\|_{L^1 \cap \dot{H}^5} d\tau. \end{aligned}$$

By (4.88), since

$$\|(g_{2e} - g_{2i}, g_{4e} - g_{4i})(t)\|_{L^1 \cap \dot{H}^5} \leq C \|W(t)\|_6^2 \leq C (\omega_8(W^0))^2 (1+t)^{-\frac{3}{2}},$$

we obtain

$$\|B(t)\|_{L^\infty} \leq C\omega_8(W^0)(1+t)^{-\frac{3}{2}}.$$

Therefore, by  $L^2 - L^\infty$  interpolation

$$(4.90) \quad \|B(t)\|_{L^q} \leq C\omega_8(W^0)(1+t)^{-\frac{3}{2} + \frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

*Estimate on  $\|(u^e - u^i, E)\|_{L^q}$ .* For  $L^2$  rate, by applying the  $L^2$  estimate on  $u^e - u^i$  and  $E$  in (4.44) to (4.37), we have

$$\begin{aligned} \|(u^e - u^i)(t)\| &\leq C(1+t)^{-\frac{5}{4}} \left( \|(\rho^{\nu 0}, \Theta^{\nu 0})\| + \|(u^{\nu 0}, E^0, B^0)\|_{L^1 \cap \dot{H}^2} \right) \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{1e} - g_{1i}, g_{3e} - g_{3i})(\tau)\| d\tau \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{2e} - g_{2i}, g_{4e} - g_{4i})(\tau)\|_{L^1 \cap \dot{H}^2} d\tau \end{aligned}$$

and

$$\begin{aligned} \|E(t)\| &\leq C(1+t)^{-\frac{5}{4}} \left\| (u^{\nu^0}, \Theta^{\nu^0}, E^0, B^0) \right\|_{L^1 \cap \dot{H}^3} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i})(\tau)\|_{L^1 \cap \dot{H}^3} d\tau. \end{aligned}$$

By (4.88), since

$$\begin{aligned} &\|(g_{1e} - g_{1i}, g_{3e} - g_{3i})(t)\| + \|(g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i})(t)\|_{L^1 \cap \dot{H}^3} \\ &\leq C \|W(t)\|_4^2 \leq C (\omega_6(W^0))^2 (1+t)^{-\frac{3}{2}}, \end{aligned}$$

we get

$$(4.91) \quad \|(u^e - u^i, E)(t)\| \leq C \omega_6(W^0) (1+t)^{-\frac{5}{4}}.$$

For  $L^\infty$  rate, by applying the  $L^\infty$  estimates on  $u^e - u^i$  and  $E$  in (4.45) to (4.37), we have

$$\begin{aligned} \|(u^e - u^i)(t)\|_{L^\infty} &\leq C(1+t)^{-2} \left( \left\| (\rho^{\nu^0}, \Theta^{\nu^0}) \right\|_{L^1 \cap \dot{H}^2} + \left\| (u^{\nu^0}, E^0, B^0) \right\|_{L^1 \cap \dot{H}^5} \right) \\ &\quad + C \int_0^t (1+t-\tau)^{-2} \|(g_{1e} - g_{1i}, g_{3e} - g_{3i})(\tau)\|_{L^1 \cap \dot{H}^2} d\tau \\ &\quad + C \int_0^t (1+t-\tau)^{-2} \|(g_{2e} - g_{2i}, g_{4e} - g_{4i})(\tau)\|_{L^1 \cap \dot{H}^5} d\tau \end{aligned}$$

and

$$\begin{aligned} \|E(t)\|_{L^\infty} &\leq C(1+t)^{-2} \left\| (u^{\nu^0}, \Theta^{\nu^0}, E^0, B^0) \right\|_{L^1 \cap \dot{H}^6} \\ &\quad + C \int_0^t (1+t-\tau)^{-2} \|(g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i})(\tau)\|_{L^1 \cap \dot{H}^6} d\tau. \end{aligned}$$

Since

$$\begin{aligned} &\|(g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i})(t)\|_{L^1} \\ &\leq C \|W(t)\| (\|(u^e - u^i)(t)\| + \|W(t)\| + \|\nabla W(t)\|) \leq (\omega_{10}(W^0))^2 (1+t)^{-\frac{3}{2}}, \end{aligned}$$

and

$$\|(g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i}, g_{4e} - g_{4i})(t)\|_{\dot{H}^5 \cap \dot{H}^6} \leq C \|\nabla W(t)\|_6^2 \leq (\omega_{13}(W^0))^2 (1+t)^{-\frac{5}{2}},$$

we obtain

$$\|(u^e(t) - u^i(t), E(t))\|_{L^\infty} \leq C \omega_{13}(W^0) (1+t)^{-\frac{3}{2}},$$

where the smallness of  $\omega_{13}(W^0)$  is used. Therefore, by  $L^2 - L^\infty$  interpolation

$$(4.92) \quad \|(u^e(t) - u^i(t), E(t))\|_{L^q} \leq C \omega_{13}(W^0) (1+t)^{-\frac{3}{2} + \frac{1}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

*Estimate on  $\|u^e + u^i\|_{L^q}$ .* For  $L^2$  rate, by applying the  $L^2$  estimates on  $u^e + u^i$  in (4.75) to (4.38), we have

$$\begin{aligned} \|(u^e + u^i)(t)\| &\leq C(1+t)^{-\frac{5}{4}} \left\| (\rho^{\nu^0}, u^{\nu^0}, \Theta^{\nu^0}) \right\|_{L^1 \cap \dot{H}^3} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(\tau)\|_{L^1 \cap \dot{H}^3} d\tau. \end{aligned}$$

By (4.88), since

$$\|(g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(t)\|_{L^1 \cap \dot{H}^3} \leq C \|W(t)\|_4^2 \leq (\omega_6(W^0))^2 (1+t)^{-\frac{3}{2}},$$

it follows that

$$\|(u^e + u^i)(t)\| \leq C \omega_6(W^0) (1+t)^{-\frac{5}{4}}.$$

For  $L^\infty$  rate, we use the  $L^\infty$  estimates on  $u^e + u^i$  in (4.77) to (4.38) to obtain

$$\begin{aligned} \|(u^e + u^i)(t)\|_{L^\infty} &\leq C(1+t)^{-2} \|(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})\|_{L^1 \cap \dot{H}^6} \\ &\quad + C \int_0^t (1+t-\tau)^{-2} \|(g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(\tau)\|_{L^1 \cap \dot{H}^6} d\tau. \end{aligned}$$

By (4.88), since

$$\|(g_{1e} + g_{1i}, g_{2e} + g_{2i}, g_{3e} + g_{3i})(t)\|_{L^1 \cap \dot{H}^6} \leq C \|W(t)\|_7^2 \leq (\omega_9(W^0))^2 (1+t)^{-\frac{3}{2}},$$

we get

$$\|u^e(t) + u^i(t)\|_{L^\infty} \leq C \omega_9(W^0) (1+t)^{-\frac{3}{2}}.$$

Therefore, by  $L^2 - L^\infty$  interpolation

$$(4.93) \quad \|u^e(t) + u^i(t)\|_{L^q} \leq C \omega_9(W^0) (1+t)^{-\frac{3}{2} + \frac{1}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

Then from (4.92) and (4.93) we have

$$(4.94) \quad \|u^\nu(t)\|_{L^q} \leq C \omega_{13}(W^0) (1+t)^{-\frac{3}{2} + \frac{1}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

*Estimate on  $\|(\rho^e - \rho^i, \Theta^e - \Theta^i)\|_{L^q}$  and  $\|(\rho^e + \rho^i, \Theta^e + \Theta^i)\|_{L^q}$ .* For  $L^2$  rate, by applying the  $L^2$  estimates on  $\rho^e - \rho^i$  and  $\Theta^e - \Theta^i$  in (4.44) to (4.37), we have

$$(4.95) \quad \begin{aligned} &\|(\rho^e - \rho^i, \Theta^e - \Theta^i)(t)\| \\ &\leq C e^{-\frac{t}{2}} \|(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})\| + C \int_0^t e^{-\frac{t-\tau}{2}} \|(g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i})(\tau)\| d\tau. \end{aligned}$$

Since

$$\begin{aligned} &\|(g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i})(t)\| \\ &\leq C \left( \|\nabla W(t)\|_1^2 + \|(u^e + u^i)(t)\| \|B(t)\|_{L^\infty} \right) \leq C (\omega_{10}(W^0))^2 (1+t)^{-\frac{5}{2}}, \end{aligned}$$

where (4.89), (4.90) and (4.93) are used. Then (4.95) implies the decay estimate

$$(4.96) \quad \|(\rho^e - \rho^i, \Theta^e - \Theta^i)(t)\| \leq C \omega_{10}(W^0) (1+t)^{-\frac{5}{2}}.$$

Similarly to that for  $\|(\rho^e - \rho^i, \Theta^e - \Theta^i)\|$ , by applying the  $L^2$  estimates on  $\rho^e + \rho^i, \Theta^e + \Theta^i$  in (4.75) to (4.38), we obtain the decay estimate

$$(4.97) \quad \|(\rho^e + \rho^i, \Theta^e + \Theta^i)(t)\| \leq C \omega_6(W^0) (1+t)^{-\frac{3}{4}}.$$



Combining (4.96) and (4.97), we obtain

$$(4.98) \quad \|(\rho^\nu, \Theta^\nu)(t)\| \leq C\omega_{10}(W^0)(1+t)^{-\frac{3}{4}}.$$

For  $L^\infty$  rate, by applying the  $L^\infty$  estimates on  $\rho^e - \rho^i, \Theta^e - \Theta^i$  in (4.45) to (4.37), we have the decay estimate

$$(4.99) \quad \begin{aligned} & \|(\rho^e - \rho^i, \Theta^e - \Theta^i)(t)\|_{L^\infty} \\ & \leq C e^{-\frac{t}{2}} \|(\rho^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0})\|_{L^2 \cap \dot{H}^2} + C \int_0^t e^{-\frac{t-\tau}{2}} \|(g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i})(\tau)\|_{L^2 \cap \dot{H}^2} d\tau. \end{aligned}$$

Notice that

$$(4.100) \quad \begin{aligned} & \|(g_{1e} - g_{1i}, g_{2e} - g_{2i}, g_{3e} - g_{3i})(t)\|_{L^2 \cap \dot{H}^2} \\ & \leq C \|\nabla W(t)\|_4 (\|(\rho^\nu, \Theta^\nu)\| + \|u^\nu\| + \|(u^\nu, B)\|_{L^\infty})(t) \leq C (\omega_{13}(W^0))^2 (1+t)^{-2}, \end{aligned}$$

where we have used (4.89), (4.90), (4.94) and (4.98). Together with (4.99) yields

$$\|(\rho^e - \rho^i, \Theta^e - \Theta^i)(t)\|_{L^\infty} \leq C\omega_{13}(W^0)(1+t)^{-2}.$$

Therefore, by  $L^2 - L^\infty$  interpolation

$$(4.101) \quad \|(\rho^e - \rho^i, \Theta^e - \Theta^i)\|_{L^q} \leq C\omega_{13}(W^0)(1+t)^{-2-\frac{1}{q}}, \quad \forall 2 \leq q \leq +\infty.$$

For  $\|(\rho^e + \rho^i, \Theta^e + \Theta^i)\|_{L^\infty}$ , by applying the  $L^\infty$  estimates on  $\rho^e + \rho^i, \Theta^e + \Theta^i$  in (4.77) to (4.38), we have the decay estimate

$$(4.102) \quad \|(\rho^e + \rho^i, \Theta^e + \Theta^i)(t)\|_{L^\infty} \leq C\omega_8(W^0)(1+t)^{-\frac{3}{2}}.$$

Then from (4.97) and (4.102) we have

$$(4.103) \quad \|(\rho^e + \rho^i, \Theta^e + \Theta^i)(t)\|_{L^q} \leq C\omega_8(W^0)(1+t)^{-\frac{3}{2}+\frac{3}{2q}}, \quad \forall 2 \leq q \leq +\infty.$$

Thus, (4.101), (4.103), (4.92)-(4.93) and (4.90) give (4.4), (4.5), (4.6) and (4.7), respectively. We have finished the proof of Theorem 4.1.  $\square$

## Chapitre 5

# Asymptotic behavior of global smooth solutions for non isentropic Navier-Stokes-Maxwell systems

### 5.1 Introduction and main results

Different from the three Chapters above, we consider the fluids with viscosity in this Chapter. Now, let us study the Cauchy problem for the non isentropic compressible Navier-Stokes-Maxwell system :

$$(5.1) \quad \left\{ \begin{array}{l} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t u + (u \cdot \nabla) u + \frac{1}{n} \nabla (n\theta) = - (E + u \times B) + \frac{1}{n} \Delta u, \\ \partial_t \theta + \frac{2}{3} \theta \nabla \cdot u + u \cdot \nabla \theta = -\frac{1}{3} |u|^2 - (\theta - \theta_*), \\ \partial_t E - \nabla \times B = nu, \quad \nabla \cdot E = 1 - n, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3. \end{array} \right.$$

Initial data are given as :

$$(5.2) \quad (n, u, \theta, E, B)|_{t=0} = (n^0, u^0, \theta^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfy the compatibility condition :

$$(5.3) \quad \nabla \cdot E^0 = 1 - n^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

The non isentropic compressible Navier-Stokes-Maxwell system (5.1) is a symmetrizable hyperbolic-parabolic system for  $n, \theta > 0$ . For the non isentropic compressible Navier-Stokes system, the local existence and uniqueness of classical solutions is known in [51, 63]

in the absence of vacuum. Then, according to the result of Kato [41] and the pioneering work of Matsumura-Nishida [49, 50], the Cauchy problem (5.1)-(5.2) has a unique local smooth solution when the initial data are smooth. Here we are concerned with stabilities of global smooth solutions to (5.1)-(5.2) around a constant state being a particular solution of (5.1). It is easy to see that this constant state is necessarily given by

$$(n, u, \theta, E, B) = (1, 0, \theta_*, 0, 0) \in \mathbb{R}^{11}.$$

**Proposition 5.1.** (*Local existence of smooth solutions, see [51, 63, 41, 46]*) *Assume (5.3) holds. Let  $s \geq 4$  be an integer,  $\theta_* > 0$  and  $\bar{B} \in \mathbb{R}^3$  be any given constants. Suppose  $(n^0 - 1, u^0, \theta^0 - \theta_*, E^0, B^0) \in H^s$  with  $n^0, \theta^0 \geq 2\kappa$  for some given constant  $\kappa > 0$ . Then there exists  $T_1 > 0$  such that problem (5.1)-(5.2) has a unique smooth solution  $(n, u, \theta, E, B)$  satisfying  $n, \theta \geq \kappa$  in  $[0, T_1] \times \mathbb{R}^3$  and*

$$\begin{aligned} u &\in C^1([0, T_1]; H^{s-2}) \cap C([0, T_1]; H^s), \\ (n - 1, \theta - \theta_*, E, B) &\in C^1([0, T_1]; H^{s-1}) \cap C([0, T_1]; H^s). \end{aligned}$$

There is no analysis on the global existence of smooth solutions around an equilibrium solution for the non isentropic Navier-Stokes-Maxwell equations so far. The goal of the present Chapter is to establish such a result.

The main result of this Chapter can be stated as follows.

**Theorem 5.1.** *Let  $s \geq 4$  be an integer. Assume (5.3) holds,  $\theta_* > 0$  be any given constant. Then there exist constants  $\delta_0 > 0$  small enough and  $C > 0$ , independent of any given time  $t > 0$ , such that if*

$$\|(n^0 - 1, u^0, \theta^0 - \theta_*, E^0, B^0)\|_s \leq \delta_0,$$

*the Cauchy problem (5.1)-(5.2) has a unique global solution  $(n, u, \theta, E, B)$  satisfying*

$$(5.4) \quad \begin{aligned} u &\in C^1(\mathbb{R}^+; H^{s-2}) \cap C(\mathbb{R}^+; H^s), \\ (n - 1, \theta - \theta_*, E, B) &\in C^1(\mathbb{R}^+; H^{s-1}) \cap C(\mathbb{R}^+; H^s), \end{aligned}$$

*and for all  $t > 0$ ,*

$$(5.5) \quad \begin{aligned} &\|(n - 1, u, \theta - \theta_*, E, B)\|_s^2 \\ &+ \int_0^t (\|(n - 1, \nabla u, \theta - \theta_*)(\tau)\|_s^2 + \|\nabla E(\tau)\|_{s-2}^2 + \|\nabla^2 B(\tau)\|_{s-3}^2) d\tau \\ &\leq C \|(n^0 - 1, u^0, \theta^0 - \theta_*, E^0, B^0)\|_s^2. \end{aligned}$$

*Moreover,*

$$(5.6) \quad \lim_{t \rightarrow +\infty} \|(n - 1, \theta - \theta_*)(t)\|_{s-1} = 0, \quad \lim_{t \rightarrow +\infty} \|\nabla u(t)\|_{s-3} = 0,$$

$$(5.7) \quad \lim_{t \rightarrow +\infty} \|\nabla E(t)\|_{s-2} = 0,$$

and

$$(5.8) \quad \lim_{t \rightarrow +\infty} \|\nabla^2 B(t)\|_{s-4} = 0.$$

**Remark 5.1.** *It should be emphasized that both the velocity viscosity term and the temperature relaxation term of the non isentropic Navier-Stokes-Maxwell equations (5.1) play a key role in the proof of global existence.*

We prove Theorem 5.1 by using careful energy estimates and a suitable choice of symmetrizer. It should be pointed out that the non isentropic Navier-Stokes-Maxwell system is much more complex than the isentropic Navier-Stokes-Maxwell system. For instant, Duan [18] introduced a new variable and reduced directly the isentropic Navier-Stokes-Maxwell system to a symmetric system by using a scaling technique. However, this technique doesn't work for the non isentropic Navier-Stokes-Maxwell system due to the complexity of the coupled energy equations. To overcome this difficulty, we choose a new symmetrizer.

Now, let us explain the main difference of proofs in the non isentropic Euler-Maxwell and non isentropic Navier-Stokes-Maxwell equations. From (5.1), it is easy to see that both  $\nabla u$  and  $\theta - \theta_*$  are dissipative. By using a classical  $H^s$  energy estimate, we obtain an energy estimate for  $\nabla u$  and  $\theta - \theta_*$  in  $L^2([0, T]; H^s)$ . In the non isentropic Euler-Maxwell system ( see [24]), this is achieved in estimate

$$(5.9) \quad \|w(t)\|_s^2 + \int_0^t D_s(w(\tau)) d\tau \leq C \|w(0)\|_s^2 + \int_0^t \|w(\tau)\|_s D_s(w(\tau)) d\tau,$$

provided that  $\sup_{0 \leq t \leq T} \|w(t)\|_s \leq C_1$ , where  $w = (n-1, u, \theta - \theta_*, E, B)$ ,

$$D_s(w(t)) = \|(n-1, u, \theta - \theta_*)(t)\|_s^2 + \|E(t)\|_{s-1}^2 + \|\nabla B(t)\|_{s-2}^2,$$

$C > 0$  and  $C_1 > 0$  are constants independent of  $T$ . In the non isentropic Navier-Stokes-Maxwell system, according to coupling viscosity term, the proof of such an estimate is more technical. It is divided into two steps. In the first step, we show a similar estimate as (5.9) (see (5.29) of Lemma 5.3) which is sufficient to prove the global existence and long time behavior for  $(n-1, u, \theta - \theta_*)$ . In the second step, we establish estimates for  $\nabla E$  in  $L^2([0, T]; H^{s-2})$  and for  $\nabla^2 B$  in  $L^2([0, T]; H^{s-3})$ , respectively. Thus, a classical argument yields the long time behavior for  $(E, B)$ .

The rest of this Chapter is arranged as follows. In Section 5.2, we deal with the global existence for smooth solutions. The main goal is to prove the first part of Theorem 5.1 by establishing energy estimates. In Section 5.3, the long time behavior of the solutions is presented, and we complete the second part of Theorem 5.1 by making further energy estimates.

## 5.2 Global existence of smooth solutions

According to [53], the global existence of smooth solutions follows from the local existence and uniform estimates of solutions with respect to  $t$ . The main task of this section is devoted to the uniform estimates for proving the first part of Theorems 5.1.

### 5.2.1 Preliminaries

The following Lemma is needed in the proof of Theorem 5.1.

**Lemma 5.1.** *For  $\nabla u \in H^1$ , there exists a constant  $C > 0$  such that*

$$\|u\|_{L^\infty} \leq C \|\nabla u\|_1.$$

**Proof.** From Morrey theorem [21], the imbedding  $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$  is continuous if  $p > d$ . Then for  $p = 6$  and  $d = 3$ , we have

$$\|u\|_{L^\infty} \leq C \|u\|_{W^{1,6}(\mathbb{R}^3)}, \quad \forall u \in W^{1,6}(\mathbb{R}^3).$$

By the Sobolev inequality [21], we obtain

$$\|u\|_{L^6} \leq C \|\nabla u\| \quad \text{and} \quad \|\nabla u\|_{L^6} \leq C \|\nabla u\|_1,$$

which imply the result of Lemma 5.1. □

Let  $(n, u, \theta, E, B)$  be a local smooth solution of Cauchy problem for the non isentropic Navier-Stokes-Maxwell system (5.1) with initial value (5.2) which satisfies (5.3). Set

$$(5.10) \quad n = 1 + N, \quad \theta = \theta_* + \Theta,$$

and

$$(5.11) \quad U = \begin{pmatrix} N \\ u \\ \Theta \end{pmatrix}, \quad W = \begin{pmatrix} U \\ E \\ B \end{pmatrix}.$$

Then, we can rewrite the system (5.1)-(5.3) as :

$$(5.12) \quad \begin{cases} \partial_t N + u \cdot \nabla N + (1 + N) \nabla \cdot u = 0, \\ \partial_t u + \frac{\theta_* + \Theta}{1 + N} \nabla N + (u \cdot \nabla) u + \nabla \Theta + E + u \times B - \frac{1}{1 + N} \Delta u = 0, \\ \partial_t \Theta + \frac{2}{3} (\theta_* + \Theta) \nabla \cdot u + u \cdot \nabla \Theta + \Theta + \frac{1}{3} |u|^2 = 0, \\ \partial_t E - \nabla \times B = (1 + N) u, \quad \nabla \cdot E = -N, \\ \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

with the initial condition :

$$(5.13) \quad W|_{t=0} = W^0 := (N^0, u^0, \Theta^0, E^0, B^0), \quad \text{in } \mathbb{R}^3,$$

which satisfies the compatibility conditions :

$$(5.14) \quad \nabla \cdot E^0 = -N^0, \quad \nabla \cdot B^0 = 0, \quad \text{in } \mathbb{R}^3.$$

Here,

$$N^0 = n^0 - 1, \quad \Theta^0 = \theta^0 - \theta_*.$$

Furthermore, the first three equations of (5.12) can be rewritten as :

$$(5.15) \quad \partial_t U + \sum_{j=1}^3 A_j(n, u, \theta) \partial_j U = K_I(W) + K_{II}(U),$$

with

$$(5.16) \quad A_j(n, u, \theta) = \begin{pmatrix} u_j & n e_j^T & 0 \\ \frac{\theta}{n} e_j & u_j I_3 & e_j \\ 0 & \frac{2}{3} \theta e_j^T & u_j \end{pmatrix}, \quad j = 1, 2, 3,$$

$$(5.17) \quad K_I(W) = - \begin{pmatrix} 0 \\ E + u \times B \\ \frac{1}{3} |u|^2 \end{pmatrix}, \quad K_{II}(U) = \begin{pmatrix} 0 \\ \frac{1}{n} \Delta u \\ -\Theta \end{pmatrix}.$$

Since  $\theta_* \geq \text{const.} > 0$  and we consider small solutions for which  $N, \Theta$  are close to zero, we have  $1 + N, \theta_* + \Theta \geq \text{const.} > 0$ . We choose the same symmetrizer as that in Chapter 3 :

$$A_0(n, \theta) = \begin{pmatrix} \frac{\theta}{n} & 0 & 0 \\ 0 & n I_3 & 0 \\ 0 & 0 & \frac{3n}{2\theta} \end{pmatrix}.$$

Then

$$\tilde{A}_j(n, u, \theta) = A_0(n, \theta) A_j(n, u, \theta) = \begin{pmatrix} \frac{\theta}{n} u_j & \theta e_j^T & 0 \\ \theta e_j & n u_j I_3 & n e_j \\ 0 & n e_j^T & \frac{3n}{2\theta} u_j \end{pmatrix}.$$

It is clear that  $A_0$  is symmetric positive definite and  $\tilde{A}_j$  is symmetric for all  $1 \leq j \leq 3$ .

Let  $T > 0$  and  $W$  be a smooth solution of (5.15) defined on time interval  $[0, T]$  with initial data  $W^0$ . This solution is given by Proposition 5.1. As in the previous Chapters, we define

$$(5.18) \quad \omega_T = \sup_{0 \leq t \leq T} \| \|W(t)\| \|_s,$$

and by  $C > 0$  various constants independent of any time  $t$  and  $T$ . From the continuous embedding  $H^s \hookrightarrow L^\infty$  for  $s \geq 2$ , there exists a constant  $C_m > 0$  such that

$$\|f\|_{L^\infty} \leq C_m \|f\|_s, \quad \forall f \in H^s, \quad s \geq 2.$$

If  $\omega_T \leq \frac{\min\{1, \theta_*\}}{2C_m}$ , from (5.18) it is easy to get

$$\|(N, \Theta)\|_\infty \leq \frac{\min\{1, \theta_*\}}{2} \quad \text{and} \quad \frac{\min\{1, \theta_*\}}{2} \leq n = 1 + N, \theta = \theta_* + \Theta \leq \frac{3 \max\{1, \theta_*\}}{2}.$$

Furthermore, by the embedding  $H^s \hookrightarrow L^\infty$ , for any smooth function  $g$  we have

$$\sup_{0 \leq t \leq T} \|g(W(t))\|_s \leq C.$$

Note that in the proof of Lemma 5.2-5.5, we always suppose  $\omega_T \leq \frac{\min\{1, \theta_*\}}{2C_m}$ .

## 5.2.2 Energy estimates

Now, let us establish the classical energy estimate for  $W$ .

**Lemma 5.2.** *Under the assumptions of Theorem 5.1, if  $\omega_T \leq \frac{\min\{1, \theta_*\}}{2C_m}$ , we have*

$$(5.19) \quad \begin{aligned} & \|W(t)\|_s^2 + \int_0^t \|(\nabla u, \Theta)(\tau)\|_s^2 d\tau \\ & \leq C \|W^0\|_s^2 + C \int_0^t \|W(\tau)\|_s \| (N, \nabla u, \Theta)(\tau) \|_s^2 d\tau, \quad \forall t \in [0, T]. \end{aligned}$$

**Proof.** For  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \leq s$ . Applying  $\partial^\alpha$  to (5.15) and multiplying the resulting equations by the symmetrizer matrix  $A_0(n, \theta)$ , we have

$$(5.20) \quad A_0(n, \theta) \partial_t \partial^\alpha U + \sum_{j=1}^3 \tilde{A}_j(n, u, \theta) \partial_j \partial^\alpha U = A_0(n, \theta) \partial^\alpha (K_I(W) + K_{II}(U)) + J_\alpha,$$

where

$$J_\alpha = - \sum_{j=1}^3 A_0(n, \theta) \{ \partial^\alpha (A_j(n, u, \theta) \partial_j U) - A_j(n, u, \theta) \partial^\alpha \partial_j U \}.$$

It is easy to see that  $J_\alpha$  will vanish when  $|\alpha| = 0$ .

Taking the inner product of (5.20) with  $2\partial^\alpha U$  in  $L^2$ , we have

$$(5.21) \quad \begin{aligned} \frac{d}{dt} \langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha U \rangle &= 2 \langle J_\alpha, \partial^\alpha U \rangle + \langle \operatorname{div} A(n, u, \theta) \partial^\alpha U, \partial^\alpha U \rangle \\ &\quad + 2 \langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha [K_I(W) + K_{II}(U)] \rangle, \end{aligned}$$

where

$$(5.22) \quad \operatorname{div} A(n, u, \theta) = \partial_t A_0(n, \theta) + \sum_{j=1}^3 \partial_j \tilde{A}_j(n, u, \theta).$$

Since

$$\partial_t A_0(n, \theta) = \begin{pmatrix} \frac{1}{n} \partial_t \Theta - \frac{\theta}{n^2} \partial_t N & 0 & 0 \\ 0 & \partial_t N I_3 & 0 \\ 0 & 0 & \frac{3}{2} \left( \frac{1}{\theta} \partial_t N - \frac{n}{\theta^2} \partial_t \Theta \right) \end{pmatrix},$$

using the first and third equations of (5.12), Lemma 5.1 and  $\omega_T \leq \frac{\min\{1, \theta_*\}}{2C_m}$ , we get

$$\|\partial_t N\| \leq C \|\nabla u\|_1, \quad \|\partial_t N\|_\infty \leq C \|\nabla u\|_2, \quad \|\partial_t \Theta\|_\infty \leq C (\|\nabla u\|_2 + \|\Theta\|_2).$$

Now, let us estimate each term on the right hand side of (5.21). For the first term, by Lemma (3.1), we obtain

$$\begin{aligned} &\langle A_0(n, \theta) (\partial^\alpha (A_j \partial_j U) - A_j \partial^\alpha \partial_j U), \partial^\alpha U \rangle \\ &= \left\langle \frac{\theta}{n} (\partial^\alpha (u_j \partial_j N) - u_j \partial^\alpha \partial_j N), \partial^\alpha N \right\rangle + \left\langle \frac{\theta}{n} (\partial^\alpha (n \partial_j u_j) - n \partial^\alpha \partial_j u_j), \partial^\alpha N \right\rangle \\ &\quad + \left\langle n \left( \partial^\alpha \left( \frac{\theta}{n} \partial_j N \right) - \frac{\theta}{n} \partial^\alpha \partial_j N \right), \partial^\alpha u_j \right\rangle + \langle n (\partial^\alpha (u_j \partial_j u) - u_j \partial^\alpha \partial_j u), \partial^\alpha u \rangle \\ &\quad + \left\langle \frac{n}{\theta} (\partial^\alpha (\theta \partial_j u_j) - \theta \partial^\alpha \partial_j u_j), \partial^\alpha \Theta \right\rangle + \left\langle \frac{n}{\theta} (\partial^\alpha (u_j \partial_j \theta) - u_j \partial^\alpha \partial_j \theta), \partial^\alpha \Theta \right\rangle \\ &\leq C \|(N, u, \Theta)\|_s \|(N, \nabla u, \Theta)\|_s^2, \end{aligned}$$

and then

$$(5.23) \quad 2 \langle J_\alpha, \partial^\alpha U \rangle \leq C \|(N, u, \Theta)\|_s \|(N, \nabla u, \Theta)\|_s^2.$$

For the second term on the right hand side of (5.21), we have

$$\begin{aligned} \langle \partial_t A_0(n, \theta) \partial^\alpha U, \partial^\alpha U \rangle &= \left\langle \frac{|\partial^\alpha N|^2}{n}, \partial_t \Theta \right\rangle - \left\langle \frac{\theta}{n^2} |\partial^\alpha N|^2, \partial_t N \right\rangle + \langle |\partial^\alpha u|^2, \partial_t N \rangle \\ &\quad + \frac{3}{2} \left\langle \frac{|\partial^\alpha \Theta|^2}{\theta}, \partial_t N \right\rangle - \frac{3}{2} \left\langle \frac{n}{\theta^2} |\partial^\alpha \Theta|^2, \partial_t \Theta \right\rangle. \end{aligned}$$



When  $|\alpha| = 0$ , it follows that

$$\begin{aligned}
& \langle \partial_t A_0(n, \theta) U, U \rangle \\
&= \left\langle \frac{|N|^2}{n}, \partial_t \Theta \right\rangle - \left\langle \frac{\theta}{n^2} |N|^2, \partial_t N \right\rangle + \langle |u|^2, \partial_t N \rangle \\
&\quad + \frac{3}{2} \left\langle \frac{|\Theta|^2}{\theta}, \partial_t N \right\rangle - \frac{3}{2} \left\langle \frac{n}{\theta^2} |\Theta|^2, \partial_t \Theta \right\rangle \\
&\leq C (\|\partial_t N\|_\infty + \|\partial_t \Theta\|_\infty) (\|N\|^2 + \|\Theta\|^2) + C \|u\|_\infty \|u\| \|\partial_t N\| \\
&\leq C (\|\nabla u\|_2 + \|\Theta\|_2) (\|N\|^2 + \|\Theta\|^2) + C \|u\| \|\nabla u\|_1^2 \\
&\leq C \|(N, u, \Theta)\| (\|N\|^2 + \|\nabla u\|_2^2 + \|\Theta\|_2^2),
\end{aligned}$$

where the Cauchy-Schwarz inequality is used. And when  $|\alpha| \geq 1$ , it can be controlled as follows

$$\begin{aligned}
\langle \partial_t A_0(n, \theta) \partial^\alpha U, \partial^\alpha U \rangle &\leq C (\|\partial_t N\|_\infty + \|\partial_t \Theta\|_\infty) (\|\partial^\alpha N\|^2 + \|\partial^\alpha u\|^2 + \|\partial^\alpha \Theta\|^2) \\
&\leq C \|(u, \Theta)\|_s \|\nabla(N, u, \Theta)\|_{s-1}^2.
\end{aligned}$$

Hence,

$$(5.24) \quad \langle \partial_t A_0(n, \theta) \partial^\alpha U, \partial^\alpha U \rangle \leq C \|(u, \Theta)\|_s \|\nabla(N, u, \Theta)\|_{s-1}^2, \quad \forall \alpha, \quad |\alpha| \leq s.$$

Next, from the definition of  $\tilde{A}_j(n, u, \theta)$ , we obtain

$$\partial_j \tilde{A}_j(n, u, \theta) = \begin{pmatrix} \partial_j \left( \frac{\theta}{n} u_j \right) & \partial_j \Theta e_j^T & 0 \\ \partial_j \Theta e_j & \partial_j (n u_j) I_3 & \partial_j N e_j \\ 0 & \partial_j N e_j^T & \frac{3}{2} \partial_j \left( \frac{n}{\theta} u_j \right) \end{pmatrix}.$$

Then,

$$\begin{aligned}
& \langle \partial_j \tilde{A}_j(n, u, \theta) \partial^\alpha U, \partial^\alpha U \rangle \\
&= \left\langle \frac{u_j}{n} |\partial^\alpha N|^2, \partial_j \Theta \right\rangle - \left\langle \frac{\theta}{n^2} u_j |\partial^\alpha N|^2, \partial_j N \right\rangle + \left\langle \frac{\theta}{n} |\partial^\alpha N|^2, \partial_j u_j \right\rangle + \langle |\partial^\alpha u_j|^2, \partial_j \Theta \rangle \\
&\quad + \langle |\partial^\alpha N|^2, \partial_j \Theta \rangle + \langle u_j |\partial^\alpha u|^2, \partial_j N \rangle + \langle n |\partial^\alpha u|^2, \partial_j u_j \rangle + \langle |\partial^\alpha \Theta|^2, \partial_j N \rangle \\
&\quad + \langle |\partial^\alpha u_j|^2, \partial_j N \rangle - \frac{3}{2} \left( \left\langle \frac{n}{\theta^2} u_j |\partial^\alpha \Theta|^2, \partial_j \Theta \right\rangle - \left\langle \frac{u_j}{\theta} |\partial^\alpha \Theta|^2, \partial_j N \right\rangle - \left\langle \frac{n}{\theta} |\partial^\alpha \Theta|^2, \partial_j u_j \right\rangle \right) \\
&\leq C \|(N, u, \Theta)\|_s \|\nabla(N, u, \Theta)\|_{s-1}^2.
\end{aligned}$$

This together with (5.24), we get

$$\begin{aligned}
(5.25) \quad \langle \operatorname{div} A(n, u, \theta) \partial^\alpha U, \partial^\alpha U \rangle &= \left\langle \left( \partial_t A_0(n, \theta) + \sum_{j=1}^3 \partial_j \tilde{A}_j(n, u, \theta) \right) \partial^\alpha U, \partial^\alpha U \right\rangle \\
&\leq C \|(N, u, \Theta)\|_s \|\nabla(N, u, \Theta)\|_{s-1}^2.
\end{aligned}$$

Now, for the last term on the right hand side of (5.21), from (5.17) it holds

$$\begin{aligned}
& 2 \langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha K_I(W) \rangle \\
&= -2 \langle \partial^\alpha(nu), \partial^\alpha E \rangle - \left\langle \frac{n}{\theta} \partial^\alpha \Theta, u \partial^\alpha u \right\rangle + 2 \sum_{\beta < \alpha} C_\alpha^\beta \langle \partial^{\alpha-\beta} N \partial^\beta u, \partial^\alpha E \rangle \\
&\quad - 2 \sum_{\beta < \alpha} C_\alpha^\beta \langle n \partial^\alpha u, \partial^\beta u \times \partial^{\alpha-\beta} B \rangle - \sum_{\beta < \alpha} C_\alpha^\beta \left\langle \frac{n}{\theta} \partial^\alpha \Theta, \partial^{\alpha-\beta} u \partial^\beta u \right\rangle \\
&\leq -2 \langle \partial^\alpha(nu), \partial^\alpha E \rangle + C \|(u, \Theta, E, B)\|_s \|(N, \nabla u, \Theta)\|_s^2,
\end{aligned}$$

and

$$\begin{aligned}
& 2 \langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha K_{II}(U) \rangle \\
&= 2 \langle \partial^\alpha u, \partial^\alpha \Delta u \rangle + 2 \sum_{\beta < \alpha} C_\alpha^\beta \left\langle n \partial^\alpha u, \partial^{\alpha-\beta} \left( \frac{1}{n} \right) \partial^\beta \Delta u \right\rangle - 3 \left\langle \frac{n}{\theta}, |\partial^\alpha \Theta|^2 \right\rangle \\
&\leq -2 \|\partial^\alpha \nabla u\|^2 - 3 \left\langle \frac{n}{\theta}, |\partial^\alpha \Theta|^2 \right\rangle + C \|N\|_s \|\nabla u\|_s^2.
\end{aligned}$$

Then,

$$\begin{aligned}
(5.26) \quad & 2 \langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha [K_I(W) + K_{II}(U)] \rangle \\
&\leq -2 \langle \partial^\alpha(nu), \partial^\alpha E \rangle - 2 \|\partial^\alpha \nabla u\|^2 - 3 \left\langle \frac{n}{\theta}, |\partial^\alpha \Theta|^2 \right\rangle \\
&\quad + C \|(N, \Theta, E, B)\|_s \|(N, \nabla u)\|_s^2.
\end{aligned}$$

On the other hand, an easy high order energy estimate for the Maxwell equations of (5.12) gives

$$(5.27) \quad \frac{d}{dt} (\|\partial^\alpha E\|^2 + \|\partial^\alpha B\|^2) = 2 \langle \partial^\alpha(nu), \partial^\alpha E \rangle.$$

It follows from (5.21)-(5.27) that

$$\begin{aligned}
(5.28) \quad & \frac{d}{dt} (\langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha U \rangle + \|\partial^\alpha E\|^2 + \|\partial^\alpha B\|^2) + 2 \|\partial^\alpha \nabla u\|^2 + 3 \left\langle \frac{n}{\theta}, |\partial^\alpha \Theta|^2 \right\rangle \\
&\leq C \|W\|_s \|(N, \nabla u, \Theta)\|_s^2.
\end{aligned}$$

Noting the fact that

$$\langle A_0(n, \theta) \partial^\alpha U, \partial^\alpha U \rangle + \|\partial^\alpha E\|^2 + \|\partial^\alpha B\|^2 \sim \|\partial^\alpha W\|^2 \quad \text{and} \quad \left\langle \frac{n}{\theta}, |\partial^\alpha \Theta|^2 \right\rangle \sim \|\partial^\alpha \Theta\|^2,$$

summing (5.28) for all  $\alpha$  with  $|\alpha| \leq s$ , and then integrating over  $[0, t]$ , we obtain (5.19).

□

Estimate (5.19) stands for the dissipation of  $\nabla u$  and  $\Theta$ . It is clear that this estimate is not sufficient to control the higher order term on the right hand side of (5.19) and the dissipation estimates of  $N$  is necessary.

**Lemma 5.3.** *Under the assumptions of Lemma 5.2, there exist positive constants  $C_1$  and  $C_2$ , independent of  $t$  and  $T$ , such that*

$$(5.29) \quad \begin{aligned} & \|W(t)\|_s^2 + \int_0^t \|(N, \nabla u, \Theta)(\tau)\|_s^2 d\tau \\ & \leq C_1 \|W^0\|_s^2 + C_2 \int_0^t \|W(\tau)\|_s \|(N, \nabla u, \Theta)(\tau)\|_s^2 d\tau, \quad \forall t \in [0, T]. \end{aligned}$$

**Proof.** For  $\alpha \in \mathbb{N}^3$  with  $|\alpha| \leq s-1$ , applying  $\partial^\alpha$  to the second equation of (5.12), and then taking the inner product of the resulting equation with  $\nabla \partial^\alpha N$  in  $L^2(\mathbb{R}^3)$  yields

$$(5.30) \quad \begin{aligned} \left\langle \frac{\theta}{n}, |\nabla \partial^\alpha N|^2 \right\rangle + \langle \nabla \partial^\alpha N, \partial^\alpha E \rangle &= -\frac{d}{dt} \langle \nabla \partial^\alpha N, \partial^\alpha u \rangle + \langle \nabla \partial^\alpha \partial_t N, \partial^\alpha u \rangle \\ &\quad - I_1(t) - \sum_{\beta < \alpha} C_\alpha^\beta I_2(t), \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= \langle u \nabla \partial^\alpha u, \nabla \partial^\alpha N \rangle + \langle \nabla \partial^\alpha \Theta, \nabla \partial^\alpha N \rangle - \left\langle \frac{\partial^\alpha \Delta u}{n}, \nabla \partial^\alpha N \right\rangle \\ &\quad + \langle \partial^\alpha u \times B, \nabla \partial^\alpha N \rangle, \\ I_2(t) &= \left\langle \partial^{\alpha-\beta} \left( \frac{\theta}{n} \right) \nabla \partial^\beta N, \nabla \partial^\alpha N \right\rangle + \langle \partial^{\alpha-\beta} u \nabla \partial^\beta u, \nabla \partial^\alpha N \rangle \\ &\quad + \langle \partial^\beta u \times \partial^{\alpha-\beta} B, \nabla \partial^\alpha N \rangle - \left\langle \partial^{\alpha-\beta} \left( \frac{1}{n} \right) \partial^\beta \Delta u, \nabla \partial^\alpha N \right\rangle. \end{aligned}$$

First, noting that  $n = 1 + N, \theta = \theta_* + \Theta \geq \frac{\min\{1, \theta_*\}}{2} \geq \text{const.} > 0$ , one has  $\frac{\theta}{n} \geq C^{-1}$ .

Hence,

$$(5.31) \quad \begin{aligned} \left\langle \frac{\theta}{n}, |\nabla \partial^\alpha N|^2 \right\rangle + \langle \nabla \partial^\alpha N, \partial^\alpha E \rangle &= \left\langle \frac{\theta}{n}, |\nabla \partial^\alpha N|^2 \right\rangle - \langle \partial^\alpha N, \partial^\alpha \nabla \cdot E \rangle \\ &= \left\langle \frac{\theta}{n}, |\nabla \partial^\alpha N|^2 \right\rangle + \|\partial^\alpha N\|^2 \\ &\geq C^{-1} (\|\partial^\alpha N\|^2 + \|\partial^\alpha \nabla N\|^2). \end{aligned}$$

By the first equation of (5.12) and an integration by parts, it follows that

$$(5.32) \quad \begin{aligned} \langle \nabla \partial^\alpha \partial_t N, \partial^\alpha u \rangle &= -\langle \partial^\alpha \partial_t N, \partial^\alpha \nabla \cdot u \rangle \\ &= \langle u \cdot \partial^\alpha \nabla N, \partial^\alpha \nabla \cdot u \rangle + \langle n, |\partial^\alpha \nabla \cdot u|^2 \rangle \\ &\quad + \sum_{\beta < \alpha} C_\alpha^\beta \langle \partial^{\alpha-\beta} u \partial^\beta \nabla N + \partial^{\alpha-\beta} N \partial^\beta \nabla \cdot u, \partial^\alpha \nabla \cdot u \rangle \\ &\leq C \|\nabla u\|_{s-1}^2 + C \|(N, u)\|_s \|(N, \nabla u)\|_s^2. \end{aligned}$$

When  $|\alpha| = 0$ , using an integration by parts, we get

$$(5.33) \quad \begin{aligned} |I_1(t)| &= \left| \langle u \nabla u, \nabla N \rangle + \langle \nabla \Theta, \nabla N \rangle - \left\langle \frac{\Delta u}{n}, \nabla N \right\rangle + \langle u \times B, \nabla N \rangle \right| \\ &\leq \|u\|_\infty \|\nabla N\| (\|\nabla u\| + \|B\|) + \varepsilon \|\nabla N\|^2 + C \|\nabla \Theta\|^2 + \varepsilon \|N\|^2 + C \|\nabla u\|_1^2 \\ &\leq \varepsilon \|N\|_1^2 + C \|(\nabla u, \Theta)\|_1^2 + C \|(N, B)\|_2 \|(N, \nabla u)\|_1^2. \end{aligned}$$

When  $|\alpha| \geq 1$ , we obtain similarly

$$(5.34) \quad |I_1(t) + I_2(t)| \leq \varepsilon \|N\|_s^2 + C \|(\nabla u, \Theta)\|_s^2 + C \|(N, u, \Theta, B)\|_s \|(N, \nabla u)\|_s^2.$$

Then, combining (5.30)-(5.34) yields

$$\begin{aligned} & C^{-1} (\|\partial^\alpha N\|^2 + \|\partial^\alpha \nabla N\|^2) + \frac{d}{dt} \langle \nabla \partial^\alpha N, \partial^\alpha u \rangle \\ & \leq \varepsilon \|N\|_s^2 + C \|(\nabla u, \Theta)\|_s^2 + C \|(N, u, \Theta, B)\|_s \|(N, \nabla u)\|_s^2. \end{aligned}$$

Summing up this inequality for all  $|\alpha| \leq s-1$  and choosing  $\varepsilon > 0$  small enough, so that the term  $\varepsilon \|N\|_s^2$  can be controlled by the left hand side. Hence, integrating the resulting equation over  $[0, t]$ , we have

$$\begin{aligned} \int_0^t \|N(\tau)\|_s^2 d\tau & \leq C \int_0^t \|(\nabla u, \Theta)(\tau)\|_s^2 d\tau + C \int_0^t \|(N, u, \Theta, B)(\tau)\|_s \|(N, \nabla u)(\tau)\|_s^2 d\tau, \\ & + \sum_{|\alpha| \leq s-1} [\langle \nabla \partial^\alpha N^0, \partial^\alpha u^0 \rangle - \langle \nabla \partial^\alpha N(t), \partial^\alpha u(t) \rangle]. \end{aligned}$$

Finally,

$$\langle \nabla \partial^\alpha N^0, \partial^\alpha u^0 \rangle \leq C \|u^0\|_{s-1} \|N^0\|_s \leq C \|W^0\|_s^2,$$

and

$$\langle \nabla \partial^\alpha N(t), \partial^\alpha u(t) \rangle \leq C \|u(t)\|_{s-1} \|N(t)\|_s \leq C \|W(t)\|_s^2.$$

Thus, together with (5.19), we obtain (5.29).  $\square$

### 5.2.3 Proof of the global existence of solutions in Theorem 5.1

By Lemma 5.3, we deduce that if  $C_2 \omega_T < 1$ , the integral term on the right hand side of (5.29) can be controlled by that of the left hand side. It follows that

$$\|W(t)\|_s \leq C_1^{\frac{1}{2}} \|W^0\|_s, \quad \forall t \in [0, T].$$

Then, it is sufficient to choose initial data  $\|W^0\|_s \leq \delta_0$  with the constant  $\delta_0$  satisfying

$$C_1^{\frac{1}{2}} \delta_0 < \min \left\{ \frac{\min\{1, \theta_*\}}{2C_m}, \frac{1}{C_2} \right\},$$

which ensures both  $\omega_T \leq \frac{\min\{1, \theta_*\}}{2C_m}$  and  $C_2 \omega_T < 1$ . Thus, the global existence of smooth solutions follows from the local existence result given in Proposition 5.1 and a standard argument on the continuous extension of local solutions. See [53].  $\square$

## 5.3 Long time behavior of smooth solutions

### 5.3.1 Dissipation of the electromagnetic fields

The long time behavior of smooth solutions follows from uniform energy estimates of  $N, \nabla u, \Theta, \nabla E$  and  $\nabla^2 B$  with respect to  $T$  in  $L^2([0, T]; H^{s'})$  for appropriate integers  $s' \geq 1$ . We will establish these estimates in the following two Lemmas.

**Lemma 5.4.** *Under the assumptions of Lemma 5.2, there exists a small constant  $\varepsilon > 0$ , such that for all  $t \in [0, T]$ ,*

$$(5.35) \quad \int_0^t \|\nabla E(\tau)\|_{s-2}^2 d\tau \leq C \|W^0\|_s^2 + \varepsilon \int_0^t \|\nabla^2 B(\tau)\|_{s-3}^2 d\tau \\ + C \int_0^t \|W(\tau)\|_s (\|(N, \nabla u, \Theta)(\tau)\|_s^2 + \|\nabla E(\tau)\|_{s-2}^2) d\tau.$$

**Proof.** For  $\alpha \in \mathbb{N}^3$  with  $1 \leq |\alpha| \leq s-1$ , applying  $\partial^\alpha$  to the second equation of (5.12) and taking the inner product of the resulting equations with  $\partial^\alpha E$  in  $L^2$ , we have

$$(5.36) \quad \|\partial^\alpha E\|^2 = -\frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha E \rangle + \langle \partial^\alpha u, \partial^\alpha \partial_t E \rangle - R_1(t) - \sum_{\beta < \alpha} C_\alpha^\beta R_2(t),$$

where

$$R_1(t) = \langle \partial^\alpha \nabla \Theta, \partial^\alpha E \rangle + \left\langle \frac{\theta}{n} \partial^\alpha \nabla N, \partial^\alpha E \right\rangle + \langle u \nabla \partial^\alpha u, \partial^\alpha E \rangle \\ + \langle \partial^\alpha u \times B, \partial^\alpha E \rangle - \left\langle \frac{\partial^\alpha \Delta u}{n}, \partial^\alpha E \right\rangle,$$

and

$$R_2(t) = \left\langle \partial^{\alpha-\beta} \left( \frac{\theta}{n} \right) \nabla \partial^\beta N, \partial^\alpha E \right\rangle + \langle \partial^{\alpha-\beta} u \nabla \partial^\beta u, \partial^\alpha E \rangle \\ + \langle \partial^\beta u \times \partial^{\alpha-\beta} B, \partial^\alpha E \rangle - \left\langle \partial^{\alpha-\beta} \left( \frac{1}{n} \right) \partial^\beta \Delta u, \partial^\alpha E \right\rangle.$$

From the fourth equation of (5.12), we deduce that

$$(5.37) \quad \langle \partial^\alpha u, \partial^\alpha \partial_t E \rangle = \langle \partial^\alpha u, \partial^\alpha \nabla \times B \rangle + \langle n, |\partial^\alpha u|^2 \rangle + \sum_{\beta < \alpha} C_\alpha^\beta \langle \partial^\alpha u, \partial^{\alpha-\beta} N \partial^\beta u \rangle \\ \leq \varepsilon \|\nabla^2 B\|_{s-3}^2 + C \|\nabla u\|_{s-1}^2 + C \|N\|_s \|\nabla u\|_s^2.$$

Similarly as before, we obtain

$$(5.38) \quad |R_1(t)| + \sum_{\beta < \alpha} C_\alpha^\beta |R_2(t)| \leq \varepsilon \|\partial^\alpha E\|^2 + C \|(N, \nabla u, \Theta)\|_s^2 \\ + C \|(E, B)\|_s (\|(N, \nabla u, \Theta)\|_s^2 + \|\nabla E\|_{s-2}^2).$$

It follows from (5.36)-(5.38) that

$$(5.39) \quad \|\partial^\alpha E\|^2 \leq -\frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha E \rangle + \varepsilon \|\nabla^2 B\|_{s-3}^2 + C \|(N, \nabla u, \Theta)\|_s^2 \\ + C \|(N, E, B)\|_s (\|(N, \nabla u, \Theta)\|_s^2 + \|\nabla E\|_{s-2}^2).$$

Note that for all  $t \in [0, T]$ ,

$$|\langle \partial^\alpha u, \partial^\alpha E \rangle| \leq C \|W(t)\|_s^2, \quad \forall \alpha, \quad 1 \leq |\alpha| \leq s-1.$$

Let  $\varepsilon > 0$  be small enough. Integrating (5.39) over  $[0, t]$  and summing for all  $1 \leq |\alpha| \leq s-1$ , together with (5.29), we obtain (5.35).  $\square$

**Lemma 5.5.** *Under the assumptions of Lemma 5.2, for all  $t \in [0, T]$ , it holds*

$$(5.40) \quad \int_0^t \left( \|\nabla E(\tau)\|_{s-2}^2 + \|\nabla^2 B(\tau)\|_{s-3}^2 \right) d\tau \\ \leq C \|W^0\|_s^2 + C \int_0^t \|W(\tau)\|_s \left( \|(N, \nabla u, \Theta)(\tau)\|_s^2 + \|\nabla E(\tau)\|_{s-2}^2 \right) d\tau.$$

**Proof.** For  $\alpha \in \mathbb{N}^3$  with  $1 \leq |\alpha| \leq s-2$ , applying  $\partial^\alpha$  to the fourth equation of (5.12) and taking the inner product of the resulting equation with  $-\nabla \times \partial^\alpha B$  in  $L^2$ , we have

$$(5.41) \quad \|\nabla \times \partial^\alpha B\|^2 = \frac{d}{dt} \langle \nabla \times \partial^\alpha B, \partial^\alpha E \rangle - \langle \nabla \times \partial^\alpha \partial_t B, \partial^\alpha E \rangle - \langle \nabla \times \partial^\alpha B, \partial^\alpha (nu) \rangle \\ = \frac{d}{dt} \langle \nabla \times \partial^\alpha B, \partial^\alpha E \rangle + \|\nabla \times \partial^\alpha E\|^2 - \langle \nabla \times \partial^\alpha B, n \partial^\alpha u \rangle \\ - \sum_{\beta < \alpha} C_\alpha^\beta \langle \nabla \times \partial^\alpha B, \partial^{\alpha-\beta} N \partial^\beta u \rangle \\ \leq \frac{d}{dt} \langle \nabla \times \partial^\alpha B, \partial^\alpha E \rangle + \|\nabla \times \partial^\alpha E\|^2 + \varepsilon \|\nabla \times \partial^\alpha B\|^2 \\ + C \|\partial^\alpha u\|^2 + C \|B\|_s \|(N, \nabla u)\|_s^2 \\ \leq \frac{d}{dt} \langle \nabla \times \partial^\alpha B, \partial^\alpha E \rangle + \|\nabla^2 E\|_{s-3}^2 + \varepsilon \|\nabla \times \partial^\alpha B\|^2 \\ + C \|\nabla u\|_{s-3}^2 + C \|B\|_s \|(N, \nabla u)\|_s^2.$$

Note that for all  $t \in [0, T]$ ,

$$|\langle \nabla \times \partial^\alpha B, \partial^\alpha E \rangle| \leq C \|W(t)\|_s^2, \quad \forall \alpha, \quad 1 \leq |\alpha| \leq s-2.$$

Let  $\varepsilon > 0$  be small enough, integrating (5.41) over  $[0, T]$  and summing for all  $1 \leq |\alpha| \leq s-2$ , together with (5.35), we get (5.40). In this estimate we have used (3.41).  $\square$

### 5.3.2 Proof of the long time behavior of solutions in Theorem 5.1

Now, we combine all the proceeding estimates to establish the long time behavior of solutions to complete the proof of Theorem 5.1.

From Lemma 5.13, there exists a constant  $\delta_0$  such that if  $\omega_T \leq \delta_0$ , it holds

$$(5.42) \quad \|W(t)\|_s^2 + \int_0^t \|(N, \nabla u, \Theta)(\tau)\|_s^2 d\tau \leq C \|W^0\|^2,$$

which implies

$$n - 1, \theta - \theta_* \in L^2(\mathbb{R}^+; H^s) \cap L^\infty(\mathbb{R}^+; H^s), \quad \nabla u \in L^2(\mathbb{R}^+; H^s) \cap L^\infty(\mathbb{R}^+; H^{s-1}).$$

Using the first three equations of (5.12), we obtain

$$\partial_t(n - 1), \partial_t(\theta - \theta_*) \in L^\infty(\mathbb{R}^+; H^{s-1}), \quad \partial_t \nabla u \in L^\infty(\mathbb{R}^+; H^{s-3}).$$

Therefore,

$$\begin{aligned} n - 1, \theta - \theta_* &\in L^2(\mathbb{R}^+; H^{s-1}) \cap W^{1,\infty}(\mathbb{R}^+; H^{s-1}), \\ \nabla u &\in L^2(\mathbb{R}^+; H^{s-3}) \cap W^{1,\infty}(\mathbb{R}^+; H^{s-3}), \end{aligned}$$

which imply (5.6).

Similarly as before, from (5.40) and (5.42), we get

$$\nabla E \in L^2(\mathbb{R}^+; H^{s-2}) \cap L^\infty(\mathbb{R}^+; H^{s-1}), \quad \nabla^2 B \in L^2(\mathbb{R}^+; H^{s-3}) \cap L^\infty(\mathbb{R}^+; H^{s-2}).$$

It follows from the Maxwell equations in (5.12) that

$$\partial_t \nabla E \in L^2(\mathbb{R}^+; H^{s-3}) \cap L^\infty(\mathbb{R}^+; H^{s-2}).$$

Therefore,

$$\nabla E \in L^2(\mathbb{R}^+; H^{s-2}) \cap W^{1,\infty}(\mathbb{R}^+; H^{s-2}),$$

which implies (5.7). We further deduce that

$$\partial_t B = -\nabla \times E \in L^2(\mathbb{R}^+; H^{s-2}) \cap W^{1,\infty}(\mathbb{R}^+; H^{s-2}).$$

Then

$$\partial_t \nabla^2 B \in L^2(\mathbb{R}^+; H^{s-4}) \cap W^{1,\infty}(\mathbb{R}^+; H^{s-4}),$$

which implies (5.8). □ —————

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